RELATIVE RELATION MODULES
OF FINITE GROUPS

by

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A thesis submitted for the degree of
Doctor of Philosophy
at the
Australian National University
Canberra
March 1983
STATEMENT

The work contained in this thesis is my own except where otherwise stated.

Mohammad Yamin
ACKNOWLEDGEMENTS

It is my great pleasure to thank my supervisor Dr John Cossey for his constructive advice and encouragement during the course of this thesis. I am very grateful to Dr Laci Kovacs who taught me many things during a number of long discussions. I also extend my thanks to the members of the Department of Mathematics who provided a cordial atmosphere to work in.

I am indebted and thankful to my friend Peter Kenne for his valuable comments during the writing of this thesis and for his help in proof reading.

I thank Barbara Geary, a very friendly lady, for her excellent typing.

I am grateful to the Australian National University for awarding me a scholarship.

Finally, I take this opportunity to thank my wife for her continual loving care, tolerance and moral support.
Let $G$ be a fixed finite group and consider a short exact sequence

$$1 \to S \to E \xrightarrow{\psi} G \to 1,$$

where $E$ is a finitely generated group. The abelian group $\overline{S} = S/S'$ may be regarded as a $\mathbb{Z}G$-module and, for a fixed prime $p$, the elementary abelian $p$-group $\hat{S} = S/S' p = \overline{S}/p \overline{S}$ may be regarded as an $\mathbb{F}^G_p$-module. If $E$ is a free group, $\overline{S}$ is called the relation module of $G$ determined by $\psi$, and $\hat{S}$ the relation module modulo $p$. In general we call $\overline{S}$ the relative relation module, and $\hat{S}$ the relative relation module modulo $p$. When the minimal number of generators of $G$ and $E$ is the same, $\overline{S}$ and $\hat{S}$ will be called minimal.

Gashütz, Gruenberg and others have studied relation modules and relation modules modulo $p$. The main aim of this thesis is to study relative relation modules modulo $p$ when $E$ is a free product of cyclic groups. To be more precise, let $X = \{g_i, 1 \leq i \leq d\}$ be a generating set of $G$, $G_i$ the cyclic group generated by $g_i$, $E$ the free product of the $G_i$, $1 \leq i \leq d$, $\psi$ the epimorphism whose restriction to each $G_i$ is the identity isomorphism, and $S$ the kernel of $\psi$.

Some of the results may be summarised as follows. $\hat{S}$ is embedded in the direct sum of the augmentation ideals of the $\mathbb{F}^G_p G_i$, $1 \leq i \leq d$, induced to $G$, and the resulting factor module is isomorphic to the augmentation ideal of $\mathbb{F}^G_p$. $\hat{S}$ may also be embedded in a free $\mathbb{F}^G_p$-module of rank $d - 1$.

Two relative relation modules, isomorphic as $\mathbb{F}^G_p$-spaces, are rarely isomorphic as $G$-modules; that is, $\hat{S}$ not only depends on $G$, $p$ and $d$
but also on $\psi$. Some cases when $\hat{S}$ does not depend on $\psi$ are established.

We say that $p$ is semicoprime to the order of $G$ if $p$ divides the order of $G$ and does not divide the orders of the $G_i$, $1 \leq i \leq d$. In the coprime and semicoprime cases a characterisation (including a criterion for counting projective summands) of $\hat{S}$ is given. Some relative relation modules (modulo $p$) of $\text{SL}(2, p)$ and $\text{PSL}(2, p)$ are described completely; the description may be useful in the study of the factor groups of $\text{PSL}(2, Z)$.

Given an unrefinable direct decomposition of a module, the direct sum of all the nonprojective summands is called the nonprojective part of the module. In the semicoprime case the nonprojective part of $\hat{S}$ is a uniquely determined, nonzero and indecomposable module (and is also the nonprojective part of $\hat{S}$ when $E$ is a free group). The nonprojective part of $\hat{S}$ in the nonsemicoprime case may be zero or decomposable (and may not be a homomorphic image of the nonprojective part of $\hat{S}$ when $E$ is free). When $G$ is a $p$-group, we prove that $\hat{S}$ is nonprojective and indecomposable.

Some of the above results can be generalised in the case when the cyclic factors of $E$ are not restricted to be the generators of $G$, however it is not known whether in this case the minimal relative relation modules of $p$-groups are also indecomposable.

Some results may also be extended to $\overline{S}$.

* for $\hat{S}$ minimal
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CHAPTER 1

INTRODUCTION

Let $G$ be a fixed finite group and consider a short exact sequence

$$1 \to R \to F \to G \to 1,$$

where $F$ is a finitely generated free group of rank $r(F)$. This exact sequence is often called a free presentation of $G$, and $R$ is a relation group. By the well known result of Nielsen and Schreier, $R$ is a free group. The free abelian group $\overline{R} = R/R'$ is a $\mathbb{Z}G$-module and, for a fixed prime $p$, the elementary abelian $p$-group $\hat{R} = R/R'p = \overline{R}/p\overline{R}$ is an $F_pG$-module. $\overline{R}$ is called the relation module of $G$ determined by $\pi$, and $\hat{R}$ the relation module modulo $p$.

A complete description of relation modules modulo $p$ is due to Gaschutz [3], whose main results may be summarised as follows.

(i) $\hat{R}$ may be embedded in a free $F_pG$-module of rank equal to $r(F)$. The resulting quotient module is isomorphic to the augmentation ideal of $F_pG$ and the resulting short exact sequence is called the relation sequence modulo $p$ of $G$ determined by $\pi$.

(ii) If $p$ does not divide the order of $G$, $\hat{R}$ is a direct sum of a trivial irreducible $F_pG$-module and a free $F_pG$-module of rank equal to $r(F) - 1$.

(iii) If $p$ divides the order of $G$, $\hat{R}$ is a direct sum of a projective $F_pG$-module (possibly zero) and a uniquely determined non-projective, nonzero and indecomposable $F_pG$-module.

(iv) If $\hat{R}_1$ and $\hat{R}_2$ are two relation modules modulo $p$ of $G$ with $r(F_2) \geq r(F_1)$, $\hat{R}_2$ is isomorphic to a direct sum of $\hat{R}_1$ and a free
A consequence of (iv) is that $\hat{R}$ does not depend on $\pi$ and only depends on $G$, $p$ and $r(F)$.

The study of relation modules was initiated by Gruenberg [5] and [6]. An account of the theory of relation modules, including that of the modulo $p$ case, may be seen in [5]. Some recent results may be found in [10] and [18].

Now consider a short exact sequence

$$1 \rightarrow S \rightarrow E \xrightarrow{\psi} G \rightarrow 1,$$

where $E$ is a finitely generated group. This exact sequence may be viewed as a homomorphic image of a suitable free presentation of $G$. As before, we define $\overline{S} = S/S'$ and $\hat{S} = S/S' \cap \overline{S}^p = \overline{S}/p\overline{S}$. We call $\overline{S}$ the relative relation module of $G$ determined by $\psi$, and $\hat{S}$ the relative relation module modulo $p$.

In [10], when $E$ is a free product of a free group and a subgroup of $G$, some results concerning whether a direct sum of a finite number of copies of $\overline{S}$ decomposes over $\mathbb{Z}G$ are proved. Apart from that, very little is known about $\overline{S}$ and $\hat{S}$.

The main aim of this thesis is to study the relative relation modules modulo $p$ of $G$ when $E$ is a free product of cyclic groups. To be more precise, suppose that a generating set $X = \{g_1, g_2, \ldots, g_d\}$ of $G$ is given. Let $G_i$ be the cyclic subgroup generated by $g_i$ and let $E$ be the free product of the $G_i$, $1 \leq i \leq d$. Define $\psi$ to be the epimorphism whose restriction to each $G_i$ is the identity isomorphism of $G_i$. Let $S$ be the kernel of $\psi$. It is known, for example see ([12], Corollary 4.9.2, p. 243), that $S$ is a free group.

In the rest of this introduction we shall only deal with relation and
relative relation modules modulo \( p \) (except for some comments where it should be clear from the context); for our convenience, reference to \( p \) will be dropped.

Our study of relative relation modules is along the lines of Gaschütz's theory of relation modules, and falls under three main headings:

(a) exact sequences associated with relative relation modules;
(b) decomposition properties;
(c) comparison of different relative relation modules of \( G \).

Much of our study depends on the relationship between relation and relative relation modules: when \( r(F) = d \), \( \hat{S} \) is a homomorphic image of \( \hat{R} \) and the kernel of the homomorphism is isomorphic to the direct sum of the trivial irreducible submodules of the \( \mathbb{F}_p G_i \), \( 1 \leq i \leq d \), induced to \( G \). As a consequence of this relationship, \( \hat{S} \) may be embedded in the direct sum of the augmentation ideals of the \( \mathbb{F}_p G_i \), \( 1 \leq i \leq d \), induced to \( G \); the resulting factor module is isomorphic to the augmentation ideal of \( \mathbb{F}_p G \) and the resulting short exact sequence will be called the relative relation sequence of \( G \) determined by \( \psi \). The relationship between \( \hat{R} \) and \( \hat{S} \) also yields a number of other important results, including a formula for the dimension of \( \hat{S} \) (and hence for the rank of \( S \)).

When \( p \) is coprime to the order of \( G \), the relative relation modules are easy to work with. Our attempts to describe the relative relation modules in the noncoprime case have been only partially successful. The difficulties in the noncoprime case are mainly due to the fact that the middle term of the relative relation sequence need not always be projective. (Compare this with the fact that the middle term of the relation sequence is always projective.) The middle term of the relative relation sequence of \( \hat{S} \) is projective precisely when \( p \) is coprime to the order of each \( G_i \). We say that \( p \) is semicoprime to the order of \( G \) if \( p \) divides the order of
Given an inrefinable direct decomposition of a module, the direct sum of all the projective summands is called the projective part of the module and the direct sum of the nonprojective summands is called the nonprojective part of the module. We know ((iii) above) that the projective part of any relation module $\hat{R}$ is nonzero and indecomposable, and (see (iv) above) does not depend even on $r(F)$. The same is true for relative relation modules in the semicoprime case only; in this case the nonprojective parts of $\hat{R}$ and $\hat{S}$ are isomorphic to each other. This makes it easy to describe the structure of relative relation modules in the semicoprime case. The nonprojective part of a relative relation module of $G$ in the nonsemicoprime case may be zero or decomposable, and need not be a homomorphic image of the nonprojective part of any relation module of $G$. Moreover the nonprojective parts, as well as the projective parts, of two relative relation modules of $G$ may differ considerably. A suitable description of relative relation modules in the nonsemicoprime case remains unknown.

$\hat{R}$ is called minimal if the minimal number of generators of $G$ is equal to $r(F)$. Similarly we say that $S$ is minimal if $d$ is the minimal number of generators of $G$. As a consequence of (iii) above, the minimal relation modules of finite $p$-groups are nonprojective and indecomposable. The minimal relative relation modules of finite $p$-groups are also nonprojective and indecomposable, a result far from being trivial.

We know ((iv) above) that any two relation modules of $G$ are either isomorphic to each other or differ by a direct summand isomorphic to a free $\mathbb{F}_p G$-module. An analogous result for relative relation modules is rarely true.

The thesis is divided into eight chapters, the first being this introduction. Chapter 2 contains some well known ideas and results about
groups and modules, which will be used for the main results of this thesis.

In Chapter 3, we introduce the notation and definitions of the relation and relative relation modules and using a result due to Smel'kin [16] and [17], we describe the relationship between relation and relative relation modules. The relative relation sequence and some other important exact sequences are also established in Chapter 3. It may be mentioned that the results of Chapter 3 also extend to $\overline{S}$.

In Chapter 4 we give a characterisation of relative relation modules in both the coprime case and the semicoprime case. Also in Chapter 4, by means of two separate examples we show that the nonprojective part of a relative relation module of $G$ in the nonsemicoprime case need not be either nonzero or a homomorphic image of the nonprojective part of any relation module of $G$.

Chapter 5 deals with the comparative study of two relative relation modules $\hat{S}_1$ and $\hat{S}_2$ corresponding to different generating sets $X_1$ and $X_2$ of $G$. If $X_1$ and $X_2$ contain the same number of elements, it does not necessarily mean that the dimensions of $\hat{S}_1$ and $\hat{S}_2$ are the same. Even if the dimensions of $\hat{S}_1$ and $\hat{S}_2$ are the same, $\hat{S}_1$ and $\hat{S}_2$ need not be isomorphic to each other. To show this we give an example which also shows that the projective parts, as well as the nonprojective parts, of $\hat{S}_1$ and $\hat{S}_2$ need not be isomorphic to each other. There are some cases where $\hat{S}_1$ and $\hat{S}_2$ may be isomorphic to each other. If $X_1$ is contained in $X_2$, we show that $\hat{S}_1$ may be embedded into $\hat{S}_2$ and give an example to show that $\hat{S}_2$ need not decompose over $\hat{S}_1$. However there are some cases where $\hat{S}_2$ may decompose over $\hat{S}_1$; a consequence of one of them is that the nonprojective part of a relative relation module need not be indecomposable.

The purpose of Chapter 6 is to study the semicoprime case in detail.
There we develop a criterion for counting the principal indecomposable summands of the relative relation modules, which is also applicable in the coprime case. Using this criterion, some relative relation modules of $\text{SL}(2, p)$ and $\text{PSL}(2, p)$ are completely described; the description of relative relation modules of $\text{PSL}(2, p)$ may be useful in the study of the factor groups of $\text{PSL}(2, \mathbb{Z})$.

In Chapter 7 we prove that the minimal relative relation modules of finite $p$-groups are nonprojective and indecomposable.

The aim of Chapter 8 is to examine whether the results of previous chapters generalise to the relative relation modules of $G$ when the cyclic factors of the free product $E$ are not restricted to be the generators of $G$. While most of the results of Chapters 3, 4, 5 and 6 generalise, it is not known whether the minimal relative relation modules of finite $p$-groups are still indecomposable.

In the development of the general theory of relation modules the importance of cohomological considerations can hardly be ignored. These considerations have motivated many of the results, providing natural applications for them. In this thesis we have been particularly concerned with the structure of relative relation modules modulo $p$. Here, as for relation modules modulo $p$, it is representation theory that provides the natural setting (even in Gruenberg [5], Chapter 2, cohomology only provides interpretations of some invariants). For this reason we have avoided the introduction of cohomological considerations. We hope to return to these considerations at a later stage.
CHAPTER 2

PRELIMINARIES

The purpose of this chapter is to define certain ideas and provide a list of associated results mainly about groups, representations and modules, which will be used frequently throughout the thesis, often without reference. Most of the results are well known; whenever convenient, reference is provided.

For the thesis, some knowledge of the theory of groups, including that of free groups and free products, representations and modules will already be assumed. We shall also assume familiarity with [3] and the first two lectures of [6]. Some knowledge of the homology of groups and modules (for example, [11]) will also be assumed, though our homological interpretation will be very limited; in particular, knowledge of diagram lemmas such as the $3 \times 3$ lemma ([11], Lemma 5.1, p. 49) will be assumed.

Throughout the thesis, all groups are finitely generated and all modules are finite dimensional vector spaces. In particular, $G$ is a finite group, $F$ a free group, $\mathbb{Z}$ the ring of integers, $p$ a fixed prime, and $\mathbb{F}_p$ the field of $p$ elements.

Depending on circumstances, an abelian group will be regarded as a $\mathbb{Z}$-module, and vice versa. Similarly an elementary abelian $p$-group will be regarded as a vector space over $\mathbb{F}_p$.

The rest of the chapter is divided into two sections, the first of which contains results mainly about groups, and the second about modules.

2.1 Groups and Extensions

The results of this section are drawn mainly from [7] and [12]. The last result is due to Šmel'kin [16] and [17].
Let $H$ be an arbitrary group, $H'$ the commutator subgroup of $H$, $H^p$ the subgroup generated by $p$th powers of the elements of $H$, $Z(H)$ the centre of $H$, and $\Phi(H)$ the Frattini subgroup of $H$.

Let $X$ be a generating set of $H$. $X$ is called minimal, or a minimal generating set of $H$, if the number of elements in $X$ is always less than or equal to the number of elements of any other generating set of $H$. A minimal generating set of $F$ is called free and $r(F)$, the number of free generators, is called the rank of $F$.

A subgroup $K$ of $H$ is called characteristic if $K$ is mapped onto itself by all automorphisms of $H$. A useful property of characteristic subgroups is: if $K$ is a normal subgroup of $H$, and $K_1$ a characteristic subgroup of $K$ then $K_1$ is normal in $H$. It is well known that $H'$ and $H^p$ are characteristic subgroups of $H$.

The minimal number of generators of $F/F'$, as well as that of $F/F'F^p$, is the same as the rank of $F$ (see [12], Theorem 2.4, p. 78).

If $G$ is a $p$-group, $\Phi(G)$ contains $G'$ ([7], Theorem 10.3.4, p. 156, and Theorem 10.4.3, p. 157), and so the minimal number of generators of $G$ and that of $G/G'$ is the same. Moreover $G' \cap Z(G) \neq \{1\}$ ([9], Kapitel I, Satz 6.9, p. 31), and so contains a cyclic group of order $p$, which is necessarily normal in $G$.

(2.1.1) **DEFINITION.** Let $B = \prod_{g \in G} H(g)$, the direct product of copies of $H$ indexed by $G$. Define $H \wr G$, the (complete) wreath product of $H$ by $G$ to be the semidirect product (split extension) of $B$ and $G$, where, for $g \in G$, $b \in B$, $b^g = g^{-1}bg$ is given by

(2.1.2) \[ b^g(g') = b\left(g'g^{-1}\right) \]

$B$ and $G$ are usually called the base group and the top group of the wreath product, respectively.
The following is a special case of an important and powerful embedding theorem due to Smel'kin [16] and [17]. This result has an important application in Chapter 3 below, and as it is not easily accessible, we give a proof. Smel'kin himself has given two proofs, one each in [16] and [17]. Here we present a simplified version of the proof of [17].

(2.1.3) THEOREM. Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be an exact sequence (a free presentation of $G$), where $F$ is freely generated by $\{f_1, f_2, \ldots, f_d\}$. Then there exists an embedding

$$\eta : F/RP \to F/F'P \wr G$$

such that

$$\left(f_iRP\right)\eta = \left(f_iF'P(1), f_i\pi\right).$$

The proof uses another embedding theorem, due to Kaloujnine and Krasner (see [13], Theorem 22.21, p. 46), which we describe first.

Let

$$\hat{W} = R/R'P \wr G,$$

and denote by $B^k$ the base group of $\hat{W}$.

Clearly, $\left(F/R'P\right) / \left(R/R'P\right) \cong F/R \cong G$.

Define

$$\pi : F/R'P \to G$$

such that $(fR'P)\pi = f\pi$, for all $f \in F$. Since $\pi$ is an epimorphism, $\pi$ is an epimorphism; and so, for some $x \in F/R'P$, an arbitrary element of $G$ may be expressed as $x\pi$. Let $\Gamma$ be a right transversal of $R/R'P$ in $F/R'P$ (a complete set of representatives of the right cosets of $R/R'P$ in $F/R'P$). Define a map $\rho$ of $G$ into $F/R'P$ such that, if $x\pi = g$, $g\rho$ is the chosen representative of the coset of $x$ (this is called a transversal map). $\rho$ is not a homomorphism, however it has the property that
\[\tau : F/R'F^P \to W^*\]
such that, for \(x \in F/R'F^P\),
\[\tau x = (\theta_x, \overline{x})\]
where, for all \(g \in G\),
\[\theta_x(g) = (g(x)\overline{x})^{-1}\rho_x(gp)^{-1} .\]

It may be verified that \(\theta_x(g) \in R/R'F^P\), and \(\tau\) is an embedding.

Proof of Theorem (2.1.3). Let \(W = F/F'F^P\ wr G\), and denote by \(B\) the base group of \(W\).

Let \(U\) be the homomorphism from \(F\) into \(W\) determined by
\[f \mapsto \left( fF'F^P(1), fF^P \right) .\]
Clearly, \(R'F^P \leq \ker U\), and so we may define a homomorphism
\[\eta : F/R'F^P \to W\]
such that \((fR'F^P)\eta = fU\), \(f \in F\). Then it only remains to check that \(\eta\) is one to one.

Define
\[\alpha : B \to B^*\]
such that \(\left( fF'F^P(1) \right) \alpha\) is the base group component of \(\left( fR'F^P \right) \tau\), that is
\[\left( fF'F^P(1) \right) \alpha = \left( fR'F^P \right) \tau \left( fF^P \right)^{-1},\]
where \(\tau\) is the Kaloujnine and Krasner embedding described above. \(B\) may be regarded as a free \(\mathbb{F}_p G\)-module with \(\left\{ fF'F^P(1), 1 \leq i \leq d \right\}\) as a free-generating set (see Section 3.2 below), and \(B\) as a submodule of \(B^*\). Therefore, \(\alpha\) determines a homomorphism of \(B\) into \(B^*\), which we also denote by \(\alpha\).

Now define
such that $(b, g)x = (ba, g)$, for all $b \in B$, and $g \in G$. Clearly, $\chi$ is a homomorphism. Moreover $\tau = \eta \chi$, because:

$$
\left( f_{i}R'\tilde{P}\right) \eta \chi = \left( f_{i}E'\tilde{P}(1), f_{i}\tau \right) \chi = \left( \left[ f_{i}E'\tilde{P}(1) \right] \alpha, f_{i}\tau \right)
$$

$$
= \left( \left[ f_{i}R'\tilde{P} \right] \tau \right) \left( f_{i}\tau \right)^{-1} \left( f_{i}\tau \right)
$$

$$
= \left( f_{i}R'\tilde{P} \right) \tau.
$$

Hence $\ker \eta \leq \ker \tau = \{1\}$, and so $\eta$ is one to one.

2.2 Group Representations and Modules

The main source of reference for the results of this section is [2]. Unless otherwise stated, we shall use the notation and ideas of [2].

Let $\mathbb{F}$ be an arbitrary field, and denote by $A$ the group algebra $\mathbb{F}G$. Unless otherwise stated, a module, or $G$-module, will always mean a right $A$-module. The (right) regular $A$-module will also be denoted by $A$. We shall assume familiarity with basic ideas and results about modules: for example, Chapter II of [2].

Let $V$ be a module. A subset $Y$ of $V$ is called a generating set (or $A$-generating set) of $V$, if every element of $V$ can be expressed as an $A$-linear combination of the elements of $Y$. A generating set $Y$ of $V$ is called minimal, if the number of elements of $Y$ is less than or equal to the number of elements of any other generating set of $V$. $V$ is called cyclic if $V$ can be generated by one element; thus $V$ is cyclic if and only if $V = vA$, for some element $v \in V$. A generating set $Y$ of $V$ is called free, or an $A$-basis, if every element of $V$ can be expressed uniquely as an $A$-linear combination of the elements of $Y$.

Not every module has a free generating set. If $V$ has a free generating set, $V$ is called a free module and the number of free
generators of $V$ is called the rank of $V$. If $V$ is a free module with \{v_i\} as a free generating set, then each $v_iA$ is isomorphic to the regular module $A$. In other words, $V$ is a free module of rank $d$ if and only if $V$ is isomorphic to a direct sum of $d$ copies of $A$. A property of free modules is that any mapping of free generators into a module uniquely extends to an $A$-homomorphism.

Within any module $V$, there are two important chains of submodules, which we describe next. Define $\phi^0V = V$. Suppose that, for $k \geq 1$, $\phi^{k-1}V$ has been defined. Then define $\phi^kV$ to be the smallest submodule of $V$ such that $\phi^{k-1}V/\phi^kV$ is completely reducible. Equivalently, $\phi^kV$ is the intersection of all maximal submodules of $\phi^{k-1}V$.

For the second chain, define $\sigma^0V = \{0\}$; and as above for $k \geq 1$, $\sigma^kV$ to be the largest submodule of $V$ such that $\sigma^kV/\sigma^{k-1}V$ is completely reducible. In other words:

$$\sigma^k(V/\sigma^{k-1}V) = \sigma^kV/\sigma^{k-1}V.$$ 

We adopt the convention that $\phi^1V = \phi V$, and $\sigma^1V = \sigma V$.

Usually \{\phi^kV\} is called the lower Loewy series of $V$, and $\phi^kV$ the radical of $\phi^{k-1}V$. Similarly \{\sigma^kV\} is called the upper Loewy series of $V$, and $\sigma(V/\sigma^{k-1}V)$ the socle of $V/\sigma^{k-1}V$. Since we are dealing with finite dimensional modules, there always exist integers $r$ and $s$ such that $\phi^rV = \{0\}$ and $\sigma^sV = V$; the smallest possible $r$ and $s$ are called the lower and the upper Loewy lengths of $V$, respectively. It is well known that both the upper and the lower Loewy lengths are the same, and so each will simply be called the Loewy length.

The following result is well known; we give a proof.
(2.2.1) **LEMMA.** Let $U$ be a submodule of $V$. Then

(i) $\varphi U \subseteq \varphi V$, and

(ii) $\sigma U \subseteq \sigma V$.

Moreover, if $V = \bigoplus_i V_i$,

(iii) $\varphi^k V = \bigoplus_i \varphi^k V_i$, and

(iv) $\sigma^k V = \bigoplus_i \sigma^k V_i$, for all $k \geq 0$.

**Proof.** (i) Let $M$ be a maximal submodule of $V$. Then we claim that $M \cap U$ is either $U$, or a maximal submodule of $U$. Suppose that $M \cap U$ is not $U$. Then $M + U = V$; and so, $V/M \cong (M+U)/M \cong U/(M \cap U)$.

Therefore, $U/(M \cap U)$ is irreducible, and hence the claim follows. This shows that $\varphi U$ is contained in every maximal submodule of $V$, and so $\varphi U$ must be contained in $\varphi V$.

(ii) Since $\sigma V$ is the largest completely reducible submodule of $V$, and $\sigma U$ is a completely reducible submodule of $V$, therefore $\sigma U \subseteq \sigma V$.

(iii) We first show that the result is true for $k = 1$. Clearly, $V/(\bigoplus_i \varphi V_i) \cong \bigoplus_i (V_i/\varphi V_i)$, and so $V/(\bigoplus_i \varphi V_i)$ is completely reducible. Therefore, $\bigoplus_i \varphi V_i$ must contain $\varphi V$. The other inclusion follows by (i).

If $\varphi^k V = \bigoplus_i \varphi^k V_i$, by the same argument it follows that $\varphi^{k+1} V = \bigoplus_i \varphi^{k+1} V_i$, and so the result follows by induction.

(iv) We prove this also by induction on $k$. By (ii) above, $\bigoplus_i \sigma V_i \subseteq \sigma V$. For the other inclusion, let $\alpha_i$ be the projection of $V$ onto $V_i$. Then $\alpha_i$ must map $\sigma V$ onto $\sigma V_i$, because $\sigma V$ is completely reducible, and $\alpha_i$ is a projection. Hence $\sigma V \subseteq \bigoplus_i \sigma V_i$, and so $\sigma V = \bigoplus_i \sigma V_i$. 

Suppose that $\sigma_{\frac{k-1}{1}} V = \bigoplus \sigma_{\frac{k-1}{i}} V_i$. Then

$$\sigma_{\frac{k}{1}} V / \sigma_{\frac{k-1}{1}} V = \sigma \left( \frac{V}{\sigma_{\frac{k-1}{1}} V} \right) = \bigoplus \left( \frac{V_i}{\sigma_{\frac{k-1}{1}} V_i} \right)$$

$$= \bigoplus \sigma \left( V_i / \sigma_{\frac{k-1}{1}} V_i \right) = \bigoplus \left( \sigma_{\frac{k}{1}} V_i / \sigma_{\frac{k-1}{1}} V_i \right)$$

$$= \left( \bigoplus \sigma_{\frac{k}{1}} V_i \right) / \sigma_{\frac{k-1}{1}} V.$$

Therefore, $\sigma_{\frac{k}{1}} V = \bigoplus \sigma_{\frac{k}{1}} V_i$, and so the proof is completed.

If $\varphi A = \{0\}$, $A$ is called semi-simple; otherwise $A$ is called non-semi-simple. $A$ is semi-simple if and only if the characteristic of $\mathbb{F}$ does not divide the order of $G$ (see [14], Theorem 1.3 B (ii), p. 12). In particular, $\mathbb{F} G$ is semi-simple if and only if $p$ does not divide the order of $G$.

A module $I$ is said to be indecomposable if $I \neq \{0\}$ and if it is impossible to express $I$ as a direct sum of two nonzero submodules. If $I$ is indecomposable, and $\varphi I = \{0\}$, then of course $I$ is an irreducible module. Let $V \cong \bigoplus_{i=1}^{r} V_i$ be an unrefinable decomposition of $V$ (that is, each $V_i$ is an indecomposable module). If $V$ has another unrefinable decomposition, $V \cong \bigoplus_{i=1}^{s} U_i$, say, then the Krull-Schmidt theorem ([2], Theorem 14.5, p. 83) gives $r = s$, and (possibly after reordering the $U_i$'s) $V_i$ and $U_i$ are pairwise isomorphic to each other. In other words, an unrefinable decomposition of a module is unique up to isomorphism. Each summand of an unrefinable direct decomposition of a module is called an indecomposable summand of the module. If $A$ is non-semi-simple, each indecomposable summand of $A$ is called principal indecomposable module.
The following result links irreducible and principal indecomposable modules.

(2.2.2) **THEOREM.** If $A_i$ is a principal indecomposable module, $A_i/\varphi A_i$ is irreducible. Two principal indecomposable modules $A_i$ and $A_j$ are isomorphic to each other if and only if $A_i/\varphi A_i$ is isomorphic to $A_j/\varphi A_j$. Every irreducible module is isomorphic to $A_i/\varphi A_i$, for some principal indecomposable module $A_i$.

For a proof, see ([2], Theorem (54.11), p. 372; and Theorem 54.13, p. 374).

Let $K$ be an extension field of $F$. For an $A$-module $V$, $V^K = K \otimes_F V$ is naturally a $KG$-module. $V$ is called absolutely irreducible, if $V^K$ is irreducible for each extension $K$ of $F$. $F$ is called a splitting field for $G$, if every irreducible module is absolutely irreducible.

(2.2.3) **DEFINITION.** Let $I$ be an indecomposable module. If, for an integer $r$, $I$ is isomorphic to $r$ indecomposable summands of an unrefinable decomposition of a module $V$, then we say that the multiplicity of $I$ in $V$ is $r$. If $I$ is not isomorphic to any of the indecomposable summands of $V$, the multiplicity of $I$ in $V$ is said to be zero.

(2.2.4) **THEOREM** ([2], Theorem (61.16), p. 419). Let $P$ be a principal indecomposable $\mathbb{F}_G$-module. Then the multiplicity of $P$ in $\mathbb{F}_G$ is equal to $\left(\dim P/\varphi P\right)/\left(\dim \left\{\text{Hom}_{\mathbb{F}_G}(P/\varphi P, P/\varphi P)\right\}\right)$.

Now we come to the ideas of projective and injective modules.

A module $P$ is called projective if every exact sequence $0 \to P \to V \to U \to 0$ splits ($V \cong P \oplus U$). A module $I$ is called injective if every diagram
in which the row is exact, can be completed to a commutative diagram
\[
\begin{array}{c}
0 \rightarrow I \rightarrow V \\
\downarrow \\
U
\end{array}
\]

It is perhaps desirable to mention that we do not consider projective and injective modules separately, because for a finite group $G$ they are equivalent concepts ([2], Theorem (62.3), p. 421).

\[(2.2.5) \text{THEOREM. The following statements are equivalent:}\]

(i) $P$ is a projective module;

(ii) $P$ is isomorphic to a direct summand of a free module;

(iii) any diagram
\[
\begin{array}{c}
P \\
\downarrow \\
U \rightarrow V \rightarrow 0
\end{array}
\]
in which the row is an exact sequence, can be completed to a commutative diagram
\[
\begin{array}{c}
P \\
\downarrow \\
U \rightarrow V \rightarrow 0 ;
\end{array}
\]

(iv) any exact sequence
\[
0 \rightarrow U \rightarrow V \rightarrow P \rightarrow 0
\]
splits.

See ([2], Theorem 56.6, p. 382; Exercise 7, p. 384; and Theorem 57.9, p. 389).

As a consequence of the above theorem, all principal indecomposable, irreducible (in the semi-simple case) and free modules are projective. Moreover, a direct sum of projective modules is projective.

Let $H$ be a subgroup of $G$. Since $\mathcal{F}H$ is a subalgebra of $A$, y
every $G$-module $V$ is also a $H$-module which we denote by $V_H$ and call the restriction of $V$ to $H$. Let $W$ be a $H$-module. Then $W^G = W \otimes_{W_H} A$ is a $G$-module, where the $G$-action is defined as $(w \otimes a)g = w \otimes ag$. $W^G$ is said to be induced from $W$ to $G$ (or to $A$). If $Y$ is an $WF$-basis of $W$, and $\Gamma$ a (right) transversal of $H$ in $G$, then $\{y \otimes t; y \in Y, t \in \Gamma\}$ is an $WF$-basis of $W^G$. If $Y$ is a $H$-generating set of $W$, $Y$ may also be regarded as a $G$-generating set of $W^G$. If the dimension of $W$ is $r$, obviously the dimension of $W^G$ is $r$ times the index of $H$ in $G$.

(2.2.6) REMARK. Let $G_i$ be a cyclic subgroup of $G$. In $WF_{G_i}$, let $t_i$ be the sum of all distinct elements of $G_i$; then $T_i$, the subspace spanned by $t_i$, is a trivial irreducible $G_i$-module. If $\Gamma_i$ is a transversal of $G_i$ in $G$, the subspace spanned by $\{t_i x : x \in \Gamma_i\}$ is isomorphic to $T_i^G$. This criterion will be used in Chapter 3.

The following are some of the consequences of the above definitions.

(2.2.7) LEMMA. Let $H$ be a subgroup of $G$. Then

(i) $(WF)_H^G = WF$, and

(ii) if $W$ is a projective $H$-module, $W^G$ is a projective $G$-module.

For a proof of (i), see ([2], Theorem (12.14), p. 67). For a proof of (ii), let $W'$ be a $H$-module such that $W \oplus W'$ is a free $H$-module (since $W$ is projective, $W'$ always exists). Then by ([2], Theorem 12.12, p. 64) $(W \oplus W')^G \cong W^G \oplus W'^G$. By (i) above, $(W \oplus W')^G$ is a free $G$-module, and so $W^G$ must be projective.

We shall assume familiarity with Mackey's subgroup theorem (for example, [2], Theorem (44.2), p. 324), as well as some other results on induced
modules.

Next we state a few other results we shall need

(2.2.8) SCHANUEL'S LEMMA. Let

\[ 0 \to U \to P \to V \to 0 \]

and

\[ 0 \to U_1 \to P_1 \to V \to 0 \]

be two short exact sequences of modules. Suppose that there exist homomorphisms

\[ f_1 : P \to P_1 \quad \text{and} \quad g_1 : P_1 \to P \]

such that \( f = f_1 g \) and \( g = g_1 f \). Then

\[ P_1 \oplus U \cong P \oplus U_1 . \]

For a proof, see ([5], Lemma 11, p. 152).

Note. If \( P \) and \( P_1 \) are projective, \( f_1 \) and \( g_1 \) always exist.

(2.2.9) PROPOSITION (Heller [8]). Let

\[ 0 \to U \to P \to V \to 0 \]

be an exact sequence of modules, where \( P \) is projective. Suppose \( U = U_0 \oplus U_1 \), \( V = V_0 \oplus V_1 \), where \( U_1 \), \( V_1 \) are projective, and \( U_0 \), \( V_0 \) are nonprojective and do not contain a projective direct summand. Then \( U_0 \) is indecomposable if and only if \( V_0 \) is indecomposable.

Finally let \( G \) be a \( p \)-group. It is well known for example ([2], Theorem (27.28), p. 189) that \( \varphi(F,G) \) is the unique maximal submodule of \( \mathbb{F}G \) and \( \sigma(F,G) \) is the unique minimal submodule of \( \mathbb{F}G \). Moreover every irreducible \( \mathbb{F}G \)-module is isomorphic to the trivial module \( \mathbb{F}^1 \).

Consequently, \( \mathbb{F}G \) is the unique principal indecomposable module. As \( \sigma(F,G) \) is the unique minimal submodule of \( \mathbb{F}G \) we have

(2.2.10) COROLLARY. If \( G \) is a \( p \)-group, every \( \mathbb{F}G \)-module isomorphic to a submodule of \( \mathbb{F}G \) is indecomposable.

The details of the following result will be omitted.
(2.2.11) **LEMMA.** Let $G$ be a $p$-group, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a generating set of an $\mathbb{F}_p G$-module $V$. Then the following statements are equivalent:

(i) $Y$ is a minimal generating set of $V$;

(ii) no proper subset of $Y$ generates $V$;

(iii) the dimension of $V|_{\mathfrak{p} V}$ is $\tilde{r}$, and $\{(y_i + \mathfrak{p} V); y_i \in Y\}$ is an $\mathbb{F}_p$-basis of $V|_{\mathfrak{p} V}$.

Apart from the results of this chapter, we shall also need some results which will be introduced as the need arises.
In this chapter, we introduce the concept of 'relative relation' modules, and establish a number of exact sequences associated with these modules. These exact sequences will play a central role throughout the rest of the thesis. The notation and terminology, as well as the results, of this chapter will be used repeatedly in the remaining chapters.

The chapter is divided into three sections. In the first section, 3.1, we define our notation and the concept of relative relation modules, and record a number of elementary results. In Section 3.2, we describe a short exact sequence for relation and relative relation modules, which plays a vital role for the other results of this thesis. The exact sequence of Section 3.2 is used in Section 3.3, where a number of other important exact sequences are established.

It is perhaps worth mentioning again that knowledge of [3], where the subject of relation modules and the theme of this thesis originates, is essential. Some knowledge of [6] (mainly Lecture 2 which contains an account of [3]) is also required.

3.1 Concept of relative relation modules

For the remainder of this chapter, $G$ will be a (fixed) finite group of order $n$, and $X = \{g_i; i \in \Lambda, |\Lambda| = d \geq 2\}$ a (fixed) generating set of $G$, where $\Lambda = \{1, 2, \ldots, d\}$. Corresponding to $i \in \Lambda$, let $G_i$ be the cyclic subgroup generated by $g_i$, and $n_i$ the order of $G_i$. Let $E_i$ be a cyclic group of order $n_i$ generated by $e_i$, $E = \prod E_i$, the free product...
of the \( E_i, i \in \Lambda \); and \( F \) a free group of rank \( d \), freely generated by \( \{f_i, i \in \Lambda\} \).

Let \( \pi \) be the natural epimorphism of \( F \) onto \( G \) determined by the mapping \( f_i \rightarrow g_i, i \in \Lambda \); and \( \psi \) the epimorphism of \( E \) onto \( G \) determined by \( e_i \rightarrow g_i \). Let \( R \) and \( S \) be the kernels of \( \pi \) and \( \psi \), respectively. Then

(3.1.1) \[ 1 \rightarrow R \rightarrow F \xrightarrow{\pi} G \rightarrow 1 \]

and

(3.1.2) \[ 1 \rightarrow S \rightarrow E \xrightarrow{\psi} G \rightarrow 1 \]

are short exact sequences.

(3.1.1) is also called a free presentation of \( G \), and \( R \) a relation group. By a well known result of O. Schreier, \( R \) is a free group and its rank is given by the formula

(3.1.3) \[ r(R) = n(d-1) + 1. \]

By ([12], Corollary 4.9.2, p. 243) \( S \) is also a free group.

Let \( p \) be a prime. Denote \( \overline{R} = R/R' \), \( \hat{R} = R/R'R^p = \overline{R}/p\overline{R} \),

and

\( \overline{S} = S/S' \), \( \hat{S} = S/S'S^p = \overline{S}/p\overline{S} \).

As usual we shall regard \( \overline{R}, \overline{S} \) as \( \mathbb{Z} \)-modules, and \( \hat{R}, \hat{S} \) as \( \mathbb{F}_p \)-modules.

Define

\( \overline{\pi} : F/R'R^p \rightarrow G \)

such that \( (fR'R^p)\overline{\pi} = f\overline{\pi} \), for all \( f \in F \); and

\( \overline{\psi} : E/S'S^p \rightarrow G \)

such that \( (eS'S^p)\overline{\psi} = e\overline{\psi} \), for all \( e \in E \). Clearly \( \overline{\pi}, \overline{\psi} \) are epimorphisms, and \( \operatorname{Ker} \overline{\pi} = \hat{R} \), \( \operatorname{Ker} \overline{\psi} = \hat{S} \). Thus
and

\[ 0 \to \hat{R} \to F/R'R'P \to \pi G \to 0, \]

are exact sequences (of groups).

For elements \( g \in G, r_{R'}R'P \in \hat{R} \), define

\[ (r_{R'}R'P)g = (f_{R'}R'P)^{-1}r_{R'}R'Pf_{R'}R'P = (f^{-1}r_f)R'R'P , \]

where \( (f_{R'}R'P)\pi = g \). This defines a \( G \)-action on \( \hat{R} \). Similarly,

\[ (sS'S')g = (eS'S')^{-1}sS'S'eS'S' , \]

where \( (eS'S')\psi = g \), defines a \( G \)-action on \( \hat{S} \).

Thus \( \hat{R} \) and \( \hat{S} \) may be regarded as \( \mathbb{F}_p G \)-modules. Similarly \( \overline{R} \) and \( \overline{S} \) may also be regarded as \( \mathbb{Z}_p G \)-modules.

\( \overline{R} \) is called the relation module determined by \( \pi \), or (3.1.1), and \( \hat{R} \) the relation module modulo \( p \). \( \overline{S} \) will be called the relative relation module determined by \( \psi \), or (3.1.2), and \( \hat{S} \) the relative relation module modulo \( p \).

Choosing different generating sets of \( G \), many relation and relative relation modules (modulo \( p \)) may be obtained. Relation modules modulo \( p \) have been completely described by Gaschütz [3], whose main results will be stated as the need arises. The aim of this thesis is to study relative relation modules modulo \( p \).

In the rest of the thesis, we shall only deal with relation and relative relation modules modulo \( p \); for our convenience, reference to \( p \) will be dropped.

In the remainder of this section we describe certain mappings, which will be used in the next section.

Let \( \phi \) be the natural epimorphism of \( F \) onto \( E \) determined by the mapping \( f_i \to e_i, i \in \Lambda \), and let \( Q \) be the kernel of \( \phi \). Then
(3.1.8) \[ 1 \to Q \to F \xrightarrow{\phi} E \to 1 \]
is an exact sequence. Clearly, \( \pi = \Phi \psi \), where \( \pi \) and \( \psi \) are the homomorphisms defined in (3.1.1) and (3.1.2), respectively. Moreover, \( Q \subseteq R \) because \( Q \pi = Q \Phi \psi = \{1\} \psi = \{1\} \). We claim that \( R \phi = S \). Since \( R \phi \psi = R \pi = \{1\} \), \( R \phi \subseteq \ker \psi = S \). For the other inclusion, let \( x \in S \).
Since \( \phi \) is onto, there exists \( f \in F \) such that \( f \phi = x \). Then
\[ 1 = x \psi = f \phi \psi = f \pi \text{, and so } f \in R \text{.} \]
Hence the claim follows. In other words, if \( \theta \) is the restriction of \( \phi \) to \( R \),
(3.1.9) \[ 1 \to Q \to R \xrightarrow{\theta} S \to 1 \]
is an exact sequence.

Let
\[ \varepsilon : F \to F/R' R'P \]
and
\[ \kappa : E \to E/S' S'P \]
be the natural epimorphisms. Then the product \( \Phi \kappa \) is an epimorphism of \( F \)
onto \( E/S' S'P \). Clearly, \( (R'R') \phi = S'S'P \); and so, \( \ker \varepsilon \subseteq \ker \phi \).
Therefore, there exists a homomorphism
\[ \phi' : F/R'R'P \to E/S'S'P \]
such that \( \varepsilon \phi' = \Phi \kappa \). Moreover \( \phi' \) is onto, because both \( \phi \) and \( \kappa \) are onto.

Let \( \hat{Q} = QR'R'R'R'P \). Since \( QR'R'R'P \) is a normal subgroup of \( F \), \( \hat{Q} \)
is a normal subgroup of \( F/R'R'R'P \). We claim that \( \hat{Q} = \ker \phi' \). Clearly \( \hat{Q} \subseteq \ker \phi' \).
For the other inclusion, let \( x \in \ker \phi' \). Then \( x = f \in R'R'R'P \),
for some \( f \in F \). Then \( S'S'P = x \phi' = f \in \phi' = f \phi = (f \phi) S'S'P \), and so
\( f \phi \in S'S'P \). Since \( (R'R'R') \phi = S'S'P \), there exists an element \( y \in R'R'R'P \) such that \( y \phi = f \phi \).
But then \( (fy^{-1}) \phi = f \phi(y \phi)^{-1} = 1 \), and so \( fy^{-1} \in Q \).
Therefore \( f \) is an element of \( QR' R^P \), and so \( x \in \hat{Q} \). Hence the claim follows. In other words, we have shown that

\[
0 \to \hat{Q} \to F/R'R^P \xrightarrow{\phi'} E/S'S^P \to 1
\]

is an exact sequence. (Note that \( \hat{Q} \) is contained in \( \hat{R} \).)

We conclude this section with the following

\[ (3.1.11) \] **Lemma.** Let \( \theta' \) be the restriction of \( \phi' \) to \( \hat{R} \). Then

\[
0 \to \hat{Q} \to \hat{R} \xrightarrow{\theta'} \hat{S} \to 0
\]

is an exact sequence of \( G \)-modules.

**Proof.** Clearly \( \bar{\pi} = \phi' \bar{\psi} \), where \( \bar{\pi} \) and \( \bar{\psi} \) are the homomorphisms defined in (3.1.4) and (3.1.5) respectively. The claim that (3.1.12) is exact follows in the same way as for (3.1.9). We only show that \( \theta \) is a \( G \)-homomorphism. Let \( x \) be an element of \( \hat{R} \), and \( g \) an element of \( G \).

By (3.1.6), \( xg = y^{-1}xy \), where \( y \) is an element of \( F/R'R^P \) such that \( y\bar{\pi} = g \). By (3.1.7),

\[
(xg)\theta' = (y^{-1}xy)\theta' = (y\theta')^{-1}x\theta'y\theta' = (y\phi')^{-1}x\phi'y\phi'
\]

\[
= (x\theta')g \quad \text{(because} \ y\phi'\bar{\psi} = y\bar{\pi} = g \text{)},
\]

and so \( \theta' \) is a \( G \)-homomorphism.

### 3.2 Relationship between relation and relative relation modules

Continuing our discussion of the last section, we now describe \( \hat{Q} \).

Let \( C \) be the subgroup of \( F \) generated by \( f_i^n, i \in \Lambda \); that is,

\[
C = \langle f_i^n, i \in \Lambda \rangle.
\]

Clearly \( C \subseteq \text{Ker} \, \phi = Q \). Moreover, \( Q \) is the normal closure of \( C \) in \( F \), that is,
Let
\[ Q = \left\langle f^{-1} f_i f; f \in F, i \in \Lambda \right\rangle. \]

Let
\[ \hat{C} = \left\langle f_i^{R'P}, i \in \Lambda \right\rangle. \]

Then, clearly, \( \hat{Q} \) is the normal closure of \( \hat{C} \) in \( F/R'P \). As \( \hat{Q} \) is an \( \mathbb{F}_p \) \( G \)-module, \( \hat{C} \) may be regarded as an \( \mathbb{F}_p \) \( G \)-generating set of \( \hat{Q} \).

Let \( W \) be the wreath product of \( F/F'R'P \) by \( G \) (see Definition (2.1.1)). As the base group \( B \) of \( W \) is abelian, we define
\[(3.2.1) \quad b(g')g = b(g'g)\]
(so as to correspond to the conjugation in \( B \) by the elements of \( G \), given by (2.1.2)). Accordingly, the product of elements of \( W \) is given by
\[(3.2.2) \quad (b, g)(b', g') = (bg' + b', gg').\]

In view of (3.2.1), \( B \) will be regarded as an \( \mathbb{F}_p \) \( G \)-module. Clearly
\[ \left\{ f_i^{F'R'P}(g); i \in \Lambda, g \in G \right\} \]
is an \( \mathbb{F}_p \)-basis of \( B \). By (3.2.1),
\[ f_i^{F'R'P}(1)g = f_i^{F'R'P}(g), \]
therefore \( \left\{ f_i^{F'R'P}(1), i \in \Lambda \right\} \) is an \( \mathbb{F}_p \) \( G \)-generating set of \( B \). In fact this is a free generating set because each \( f_i^{F'R'P}(g) \) has a unique expression in \( B \). Thus \( B \) is a free module of rank \( d \). To simplify the notation, we write
\[ b_i = f_i^{F'R'P}(1), \quad \text{and} \quad B = \bigoplus_{i=1}^d b_i \mathbb{F}_p G. \]

Let
\[ \eta : F/R'P \rightarrow W \]
be the embedding of Smel'kin's theorem (2.1.3). Then
\[ \left\{ f_i^{R'P} \right\} \eta = \left\{ f_i^{F'R'P}(1), f_i^{F'} \right\} \]
\[ = (b_i, g_i), \quad \text{for all} \quad i \in \Lambda. \]
Clearly \( \hat{\mathfrak{m}} \subseteq B \). Therefore \( \eta \) restricted to \( \hat{R} \) is an embedding of \( \hat{R} \) into \( B \). Moreover this restriction is a \( G \)-homomorphism.

By (3.2.2)

\[
\left( \left[ f \right]_{i}^{R'P} \right) \eta = \left( \left[ f \right]_{i}^{R'P} \right) \eta = (b, g)_{i}^{n_{1}}
\]

\[
= \left\{ b, \left( 1 + g + \ldots + g^{n-1} \right) \right\},
\]

which is an element of \( B \). Let

\[
t_{i} = b_{i} \left( 1 + g_{i} + \ldots + g_{i}^{n_{i}-1} \right)
\]

As \( \hat{Q} \) is a submodule of \( \hat{R} \) generated by \( \left\{ f_{i}^{i} R' P, i \in A \right\} \), \( \hat{Q} \eta \) is a submodule of \( B \) generated by \( \left\{ t_{i}, i \in A \right\} \), that is

\[
\hat{Q} \eta = \langle t_{i} g; g \in G, i \in A \rangle.
\]

Corresponding to \( i \in A \), let \( T_{i} \) be the subspace spanned by \( t_{i} \). Then, if \( g \in G_{i} \),

\[
t_{i} g = b_{i} \left( 1 + g_{i} + \ldots + g_{i}^{n_{i}-1} \right) g = b_{i} \left( 1 + g_{i} + \ldots + g_{i}^{n_{i}-1} \right)
\]

\[
= t_{i}.
\]

Therefore \( T_{i} \) is a trivial (irreducible) \( G_{i} \)-module. We claim that

\[
\hat{Q} \eta \cong \bigoplus_{i=1}^{d} T_{i}^{G}.
\]

Corresponding to \( i \in A \), let \( \Gamma_{i} \) be a (fixed) right transversal of \( G_{i} \) in \( G \). Clearly, for \( g \in G \), \( t_{i} x_{i} = t_{i} x_{i} \), where \( x_{i} \) is the chosen representative of \( g \). Therefore

\[
\hat{Q} \eta = \langle t_{i} x_{i}; i \in A, x_{i} \in \Gamma_{i} \rangle.
\]

If \( V_{i} = \langle t_{i} x_{i}, x_{i} \in \Gamma_{i} \rangle \), by Remark (2.2.6), \( V_{i} \) may be regarded as \( T_{i}^{G} \), and so the claim follows. Then
because \( \eta \) is an embedding.

In view of (3.1.12), our discussion so far may be summarised by

(3.2.3) **PROPOSITION.** For \( i \in \Lambda \), let \( T_i \) be a trivial irreducible \( \mathbb{F} G_i \)-module. Then

\[
0 \to \bigoplus_{i=1}^{d} T_i^G \to \hat{R} \to \hat{S} \to 0
\]

is an exact sequence of \( G \)-modules.

(3.2.5) **COROLLARY.** The dimension of \( \hat{S} \) is given by the formula

\[
\dim \hat{S} = n(d-1) - \left( \sum_{i=1}^{d} \frac{n}{n_i} \right) + 1.
\]

**Proof.** We know that the rank of \( R \) is equal to the dimension of \( \hat{R} \), and so by the Schreier formula (3.1.3), \( \dim \hat{R} = n(d-1) + 1 \).

We also know that the dimension of \( T_i^G \) is the index of \( G_i \) in \( G \), which is equal to \( n/n_i \). In view of (3.2.4),

\[
\dim \hat{S} = \dim \hat{R} - \dim \left( \bigoplus_{i=1}^{d} T_i^G \right),
\]

which gives the required formula (3.2.6).

**REMARK.** The rank of \( S \) is equal to the dimension of \( \hat{S} \), and so this gives a formula for the rank of \( S \).

As mentioned earlier, the sequence (3.2.4) is crucial for the development of other results. It has already given us the dimension of \( \hat{S} \). With the help of this sequence, we shall establish some more exact sequences in the next section.
Define
\[ \rho : \mathbb{F}_G \rightarrow \mathbb{F}_p \]
such that
\[ \left( \sum_{g \in G} k g \right) \rho = \sum k g, \quad k \in \mathbb{F}_p. \]

Let \( g = \text{Ker } \rho. \) Then

\[ (3.3.1) \quad 0 \rightarrow g \rightarrow \mathbb{F}_G \xrightarrow{\rho} \mathbb{F}_p \rightarrow 0 \]
is an exact sequence of \( G \)-modules. \( \rho \) is called the augmentation of \( \mathbb{F}_G \), and \( g \), which is a maximal submodule of \( \mathbb{F}_G \), is called the augmentation ideal of \( \mathbb{F}_G \). \( \{1-g, g \in G\} \) is an \( \mathbb{F}_p \)-basis and \( \{1-g_i, g_i \in X\} \) an \( \mathbb{F}_p \)-generating set of \( g \). The augmentation ideal of \( \mathbb{F}_G \) will be denoted by \( \mathbb{G}_i \).

Define
\[ \xi_i : b_i \mathbb{F}_G \rightarrow \mathbb{G}_i \]
such that
\[ (b_i x) \xi_i = (1-g_i)x, \quad \text{for all } x \in \mathbb{F}_G. \]

Clearly \( \xi_i \) is an epimorphism. Moreover \( T_{G}^i \subseteq \text{Ker } \xi_i \) because
\[
t_i \xi_i = \left( b_i \left[ 1 + g_i + \ldots + g_i^{n_i-1} \right] \right) (1-g_i) \left[ 1 + g_i + \ldots + g_i^{n_i-1} \right] = 0.
\]
In fact \( \text{Ker } \xi_i = T_{G}^i \) because \( \dim \mathbb{F}_G = \dim T_{G}^i + \dim \mathbb{G}_i \). Thus
\[ (3.3.2) \quad 0 \rightarrow \bigoplus_{i=1}^{d} T_{G}^i \rightarrow \bigoplus_{i=1}^{d} b_i \mathbb{F}_G \xrightarrow{\xi} \bigoplus_{i=1}^{d} \mathbb{G}_i \rightarrow 0 \]
is an exact sequence, where \( \xi \) restricted to each \( b_i \mathbb{F}_G \) is given by \( \xi_i \).
Next we define
\[ U_i : b_i \mathcal{F}_G \to T_{i}^G \]
such that
\[ (b_i x) U_i = t_i x, \quad \text{for all } x \in \mathcal{F}_G. \]
Then \( U_i \) is an epimorphism. Moreover, \( \text{Ker } U_i = T_i^G \) because
\[ (b_i (1-g_i)) U_i = (1-g_i) t_i = 0, \quad \text{and dim } \mathcal{F}_G = \text{dim } T_i^G + \text{dim } g_i^G. \]
Thus
\[ 0 \to \bigoplus_{i=1}^{d} \mathcal{F}_i^G \to \bigoplus_{i=1}^{d} b_i \mathcal{F}_G \xrightarrow{U} \bigoplus_{i=1}^{d} T_i^G \to 0 \]
is an exact sequence, where \( U \) restricted to each \( b_i \mathcal{F}_G \) is given by \( U_i \).

Remark. The existence of \( \xi_i \) and \( U_i \) does not imply that \( \mathcal{F}_G = \bigoplus_{i=1}^{d} T_i^G \circ g_i^G. \) For example, if \( G \) is a \( p \)-group, \( \mathcal{F}_G \) is indecomposable.

(3.3.4) Proposition (Gaschütz [3]). Given a free presentation (3.1.1),
\[ 0 \to \mathcal{F}_G \to \bigoplus_{i=1}^{d} b_i \mathcal{F}_G \xrightarrow{\pi'} g \to 0 \]
is an exact sequence, where \( \pi' \) is determined by the mapping \( b_i + (1-g_i) \), \( i \in \Lambda \).

For a proof see ([6], Proposition 2.3, p. 7).

(3.3.5) is called the relation sequence (modulo \( p \)) determined by (3.3.1).

Now we prove

(3.3.6) Proposition. Given an exact sequence (3.1.2), then
\[ 0 \to \mathcal{F}_G \to \bigoplus_{i=1}^{d} \mathcal{F}_i^G \xrightarrow{\psi'} g \to 0 \]
is an exact sequence, where
\[ \left( \sum \frac{1}{i} (1-g_i) x_i \right) \psi' = \sum \frac{1}{i} (1-g_i) x_i, \quad \text{for all } i \in \Lambda, \quad \text{and } x_i \in \mathcal{F}_i G. \]

Note. (3.3.7) will be called the relative relation sequence determined by (3.1.2).
Proof. Consider the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \oplus_{i=1}^{d} T_i^G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \hat{R} & \rightarrow & \oplus_{i=1}^{d} b_i P & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \hat{S} & \rightarrow & \oplus_{i=1}^{d} g_i^G & \rightarrow & 0 \\
\end{array}
\]

We know that all the rows and columns, except the bottom row, of the above diagram are exact (see (3.2.4), (3.3.2) and (3.3.6)). Since

\[\text{Ker } \xi = \oplus_{i=1}^{d} T_i^G \subseteq \hat{R} = \text{Ker } \pi',\]

there exists a homomorphism

\[\psi' : \oplus_{i=1}^{d} g_i^G \rightarrow g,\]

such that \(\pi' = \xi \psi'.\) Moreover, \(\psi'\) is onto because \(\pi'\) is onto.

Clearly \(\psi'\) is the same as described in the statement of Proposition (3.3.6). By the definition of \(\psi'\) the square \(\square\) is commutative, and so the commutativity of \(\square\) and \(\square\) follows. Then

\[\text{Ker } \Theta' = \oplus_{i=1}^{d} T_i^G = \text{Ker } \xi,\]

and so there exists a homomorphism

\[\nu : \hat{S} \rightarrow \oplus_{i=1}^{d} g_i^G,\]

such that \(\Theta' \nu = \psi'.\) In this way the square \(\square\) is also commutative, and so
the required result follows by the $3 \times 3$ lemma.

Next we define

$$
\lambda : \bigoplus_{i=1}^{d} b_i \mathbb{F}_p G \to \mathbb{F}_p G
$$

such that

$$
\left( \sum_i b_i x_i \right) \lambda = \sum_i x_i , \text{ for all } i \in \Lambda \text{ and } x_i \in \mathbb{F}_p G .
$$

Clearly $\lambda$ is an epimorphism and its kernel, which we denote by $L$, is a free module of rank $d - 1$. Thus

(3.3.8) \hspace{1cm} 0 \to L \xrightarrow{\bigoplus_{i=1}^{d} b_i \mathbb{F}_p G} \mathbb{F}_p G \xrightarrow{\lambda} \mathbb{F}_p G \to 0

is an exact sequence.

Finally, consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & S & \xrightarrow{\mu} & L & \xrightarrow{\lambda} & M & \xrightarrow{\beta} & 0 \\
0 & \downarrow & \boxed{1} & \quad & \boxed{2} & \quad & \boxed{3} & \quad & 0 \\
0 & \to & \bigoplus_{i=1}^{d} b_i \mathbb{F}_p G & \xrightarrow{\psi'} & \mathbb{F}_p G & \to & 0 \\
0 & \downarrow & \boxed{4} & \quad & \boxed{5} & \quad & \boxed{6} & \quad & 0 \\
0 & \to & \bigoplus_{i=1}^{d} \mathbb{F}_p G & \xrightarrow{\mu} & \mathbb{F}_p G & \to & 0 \\
0 & \downarrow & \boxed{7} & \quad & \boxed{8} & \quad & \boxed{9} & \quad & 0 \\
0 & \to & 0 & \xrightarrow{\beta} & \mathbb{F}_p G & \xrightarrow{\lambda} & 0 \\
0 & \downarrow & \boxed{10} & \quad & \boxed{11} & \quad & \boxed{12} & \quad & 0 \\
0 & \to & 0 & \xrightarrow{\beta} & 0 & \to & 0 \\
0 & \downarrow & \boxed{13} & \quad & \boxed{14} & \quad & \boxed{15} & \quad & 0 \\
0 & \to & 0 & \xrightarrow{\beta} & 0 & \to & 0
\end{array}
\]

We know that the top two rows (sequences (3.3.7) and (3.3.8)) and the right two columns (sequences (3.3.1) and (3.3.3)) are exact. Moreover, the squares 2 and 3 are commutative.

Let

$$
x = \sum_{g \in G} k_g g , \text{ where } k_g \in \mathbb{F}_p .
$$

Then, for $i \in \Lambda$, 


\[ (b_i (1-g_i)x) \lambda \pi = ((1-g_i)x) \rho = \left( \sum k_i g \right) - \sum k_i g (gg_i) \rho \]
\[ = \sum_{g \in G} k_i g - \sum_{g \in G} k_i g = 0. \]

Therefore, for \( a = \sum b_i (1-g_i)x, i \in \Lambda, x_i \in \mathbb{W}_G, a \lambda \rho = 0 \), and so \( a \in \text{Ker} \lambda \rho \). That is \( \text{Ker} \varnothing \subseteq \text{Ker} \lambda \rho \), and so there exists a homomorphism
\[
\beta : \bigoplus_{i=1}^{d} T_i^G \to \mathbb{W}_p
\]
such that \( \varnothing \beta = \lambda \rho \). Moreover, \( \beta \) is onto because both \( \lambda \) and \( \rho \) are onto. Denote by \( M \), the kernel of \( \beta \). Then the bottom row of the above diagram is exact. Let \( \varnothing \) be the restriction of \( \varnothing \) to \( L \). Then the commutativity of \( \varnothing \) is obvious, and so by the \( 3 \times 3 \) lemma, the above diagram is commutative with all rows and columns exact. In other words, we have proved

(3.3.9) PROPOSITION. Let \( L \) be a free \( \mathbb{W}_G \)-module of rank \( d - 1 \). Then

(3.3.10) \[ 0 \to \hat{S} \to L \xrightarrow{\varnothing} \bigoplus_{i=1}^{d} T_i^G \xrightarrow{\beta} \mathbb{W}_p \to 0 \]
is an exact sequence, where \( \beta \) and \( \varnothing \) are as defined above.

As an immediate consequence of the above proposition, \( \hat{S} \) is embedded in a free module of rank \( d - 1 \). It is perhaps appropriate to mention that an embedding of \( \hat{S} \) into a free module of rank \( d - 1 \) can also be achieved by using a result of Smel'kin (other than Theorem (2.1.3) above) given in [16].
CHAPTER 4

STRUCTURE OF RELATIVE RELATION MODULES

As we have already mentioned, the notation of Chapter 3 will be used throughout the rest of the thesis. It may be stressed that \( p \) will always be a fixed prime. Moreover, a relation module and a relative relation module, whenever used in the same context, will necessarily correspond to the same generating set \( X \) of \( G \). Unless otherwise stated, a module or a \( G \)-module will always mean an \( \mathbb{F}_p G \)-module.

Gaschütz [3] has completely described the structure of relation modules in terms of projective and nonprojective direct summands. If \( p \) is coprime to \( n \) (the order of \( G \)), that is, if \( \hat{R} \) is completely reducible, \( \hat{R} \) is isomorphic to the sum of \( \mathbb{F}_p \) and a free module of rank \( d - 1 \). If \( p \) is noncoprime to \( n \), \( \hat{R} \) is a direct sum of a projective (possibly zero) and a uniquely determined nonprojective, nonzero and indecomposable module (see Proposition (4.2.3) below).

In this chapter we attempt to describe relative relation modules in terms of projective and nonprojective direct summands. The coprime case is easy to deal with, as the modules are completely reducible; a characterisation (Proposition (4.1.2)) is given below. The noncoprime case is a complicated one, as the familiar tools, which enable us to describe relation modules, do not always enable us to describe relative relation modules. A suitable characterisation of relative relation modules in the noncoprime case remains unknown; however there are some cases where these modules can be described in a reasonable way.

The chapter is divided into two sections. The first section deals with the coprime case, and the second with the noncoprime case.
4.1 The coprime case

(4.1.1) **THEOREM** (Gaschütz [3]). Let \( \hat{R} \) be the relation module determined by the sequence (3.1.1). Then

\[
\hat{R} \cong \bigoplus_{i=1}^{d-1} b_i F \hat{G} \oplus F_p.
\]

This comes immediately by an application of Maschke's theorem to the relation sequence (3.3.5).

The following result characterises the relative relation modules.

(4.1.2) **PROPOSITION.** Let \( \hat{S} \) be the relative relation module determined by the sequence (3.1.2). Then

(i) \( \hat{S} \oplus \hat{G} \cong \bigoplus_{i=1}^{d} \hat{G}_i \),

(ii) \( \hat{S} \oplus \bigoplus_{i=1}^{d} T_i^G \cong \hat{R} \), and

(iii) \( \hat{S} \oplus U \cong \bigoplus_{j \neq i=1}^{d} \hat{G}_j \), where \( U \) is the sum of all nontrivial irreducible submodules of \( T_j^G \), for any (fixed) \( j \in \Lambda \).

**Proof.** (i) and (ii) follow by applying Maschke's theorem to the relative relation sequence (3.3.7) and the sequence (3.2.4). For (iii), clearly

\[
\hat{F}_p \oplus U \oplus \hat{G}_j \cong T_j^G \oplus \hat{G}_j \cong \hat{F}_p \hat{G} \cong \hat{F}_p \oplus \hat{G},
\]

and so the Krull Schmidt theorem gives

\[
\hat{G} \cong U \oplus \hat{G}_j.
\]

Then the result follows by (i) above.

(4.1.3) **COROLLARY.** \( \hat{S} \) does not contain any trivial irreducible submodules.

**Proof.** Since none of the \( \hat{G}_i \), \( i \in \Lambda \), contains any trivial irreducible modules, the result follows by (4.1.2) (iii).
Proposition (4.1.2) reduces the task of obtaining an unrefinable decomposition of \( \hat{S} \) to the study of irreducible submodules of \( T^G_i \), or those of \( G_i \). We shall return to these and other matters in Chapter 6, where we shall give a criterion for an unrefinable decomposition of \( \hat{S} \).

4.2 The noncoprime case

(4.2.1) **DEFINITION.** Let \( V \) be a module. Given an unrefinable (direct) decomposition of \( V \), let \( P(V) \) be the sum of all projective indecomposable summands of \( V \), and \( N(V) \) the sum of all nonprojective indecomposable summands of \( V \). By the Krull Schmidt theorem \( P(V) \) and \( N(V) \) are unique up to isomorphism. \( P(V) \) is called the projective part of \( V \), and \( N(V) \) the nonprojective part of \( V \). If \( N(V) \cong V \), we say that \( V \) does not contain a projective summand.

(4.2.2) **HYPOTHESIS.** \( 0 \to J \to V \xrightarrow{f} W \xrightarrow{g} P \to 0 \) is an exact sequence, where \( V \) and \( W \) are projective modules, \( \text{Ker } g = \varphi W \) and \( J \subseteq \varphi V \).

The following result describes relation modules completely.

(4.2.3) **THEOREM** (Gaschütz [3]). Suppose that Hypothesis (4.2.2) is true. Then

\[
\hat{R} \oplus V \cong J \bigoplus_{i=1}^{d-1} b_i G \oplus W,
\]

and \( J \) is a nonprojective, nonzero and indecomposable module.

For a proof, see ([6], Theorem 2.9, p. 9); also [3].

By Theorem (4.2.3) it is obvious that the non-projective part of \( \hat{R} \) is \( J \); that is, \( J \cong N(\hat{R}) \). Moreover, \( N(\hat{R}) \) does not depend on the choice of the generating set \( X \) of \( G \).

It is perhaps not unreasonable to say that a suitable description of \( N(\hat{S}) \) is a first step to describe an unrefinable decomposition of \( \hat{S} \).
Accordingly, we first ask whether $\hat{N}(S)$ is always nonzero. The answer is no, as shown by the following example.

(4.2.4) EXAMPLE. Let $G = S_3$, the symmetric group on three symbols; $p = 2$; and $X = \{g_1, g_2\}$, where $g_1^2 = g_2^3 = 1$. Then the relative relation sequence (3.3.7) may be written as

$$0 \to \hat{S} \to \mathbb{F}_2^G \oplus \mathbb{F}_2^G \xrightarrow{\psi'} \mathbb{F}_2 \to 0.$$  

It may be verified that $\mathbb{F}_2 S_3$ has only two distinct (nonisomorphic) principal indecomposables, $P_1$ and $P_2$ say, each of dimension two. Moreover, one of them, $P_1$, say, is irreducible and the other, namely $P_2$, has the property that

$$\varphi P_2 \cong \mathbb{F}_2 \cong \sigma P_2 \cong P_2/\varphi P_2.$$  

Clearly

$$\mathbb{F}_2 \cong P_1 \oplus P_1 \oplus \varphi P_2.$$  

Moreover, we have that

$$\mathbb{F}_2^G \cong P_1 \oplus \mathbb{F}_2 \quad \text{and} \quad \mathbb{F}_2^G \cong P_1 \oplus P_1.$$  

Thus $\hat{S} \cong P_1$, and so $\hat{S}$ is projective.

As $\hat{S}$ is a homomorphic image of $\hat{R}$, it is natural to ask whether $N(\hat{S})$ is always a homomorphic image of $N(\hat{R})$. Unfortunately, the answer is again no, as shown by the following example.

(4.2.5) EXAMPLE. Let $G = S_4$, the symmetric group on four symbols; $X$ a generating set of $G$ of any two elements of orders two and three respectively; and $p = 3$.

We know (see [15], Example 18.5, p. 153) that $\mathbb{F}_3 S_4$ has precisely four distinct indecomposable modules, each of dimension three. Moreover, two of the distinct principal indecomposables are irreducible and the other
remaining two, $P_1$ and $P_2$ say, have the following structure:

$$P_1/\phi P_1 \cong \mathbb{F}_p \cong \phi P_2 \cong \sigma P_1, \quad \phi P_2 /\phi^2 P_2 \cong U;$$

$$P_2/\phi P_2 \cong U \cong \phi^2 P_2 = \sigma P_2, \quad \phi P_2 /\phi^2 P_2 \cong \mathbb{F}_p;$$

where $U$ is the unique nontrivial, one dimensional and irreducible module corresponding to the parity representation of $S_4$. Clearly $P_2/\sigma P_2 \cong \phi P_1$, with the help of which we can establish an exact sequence

$$0 \rightarrow \sigma P_2 \rightarrow P_2 \rightarrow P_1 \rightarrow \mathbb{F}_p \rightarrow 0$$

satisfying all the conditions of Theorem (4.2.3). Thus $N(\hat{R}) \cong \sigma P_2$, whose dimension is of course one. By formula (3.2.6) the dimension of $\hat{S}$ is five. Therefore, the dimension of $N(\hat{S})$ is either two or five. In any case $N(\hat{S})$ cannot be a homomorphic image of $N(\hat{R})$. (REMARK. It may be verified that the dimension of $N(\hat{S})$ is two, and $P(\hat{S})$ is the projective irreducible (principal indecomposable) module corresponding to the parity representation of $S_4$.)

$N(\hat{R})$ and $N(\hat{S})$ are always isomorphic to each other provided that the given generating set $X$ of $G$ satisfies the condition of the following definition.

(4.2.6) DEFINITION. If $p$ does not divide $n_i$, the order of $g_i$, $g_i \in X$, $p$ is called semicoprime to $n$ with respect to $X$.

We adopt the convention that our usage of semicoprime will always be with respect to the given generating set $X$ of $G$. Thus, if $p$ is semicoprime to $n$, by (2.2.7) (ii) the $E_i^G$ and the $T_i^G$, $i \in \Lambda$, are projective.

(4.2.7) PROPOSITION. Suppose that $p$ is semicoprime to $n$, and Hypothesis (4.2.2) is true. Then
and \( J \) is a nonprojective, nonzero and indecomposable module.

Proof. The sequence of Hypothesis (4.2.2) gives the short exact sequences

\[
\begin{align*}
(i) & \quad 0 \to J \to V \xrightarrow{f'} Vf \to 0, \text{ and} \\
(ii) & \quad 0 \to Vf \to W \xrightarrow{G} W \to 0.
\end{align*}
\]

An application of Schanuel's lemma (2.2.8) to sequences (3.3.1) and (ii) gives

\[
(iii) \quad Vf \oplus \frac{G}{p} \cong \frac{S}{G} \oplus W.
\]

Sequence (i) above gives us an exact sequence

\[
(iv) \quad 0 \to J \to V \oplus \frac{G}{p} \to Vf \oplus \frac{G}{p} \to 0.
\]

The relative relation sequence gives an exact sequence

\[
0 \to \hat{S} \to \bigoplus_{i=1}^{d} \frac{G}{p_i} \oplus W \to \frac{S}{G} \oplus W \to 0,
\]

which in view of (iii) may be rewritten as

\[
(v) \quad 0 \to \hat{S} \to \bigoplus_{i=1}^{d} \frac{G}{p_i} \oplus W \to Vf \oplus \frac{G}{p} \to 0,
\]

and another application of Schanuel's lemma, this time to (iv) and (v), yields

\[
\hat{S} \oplus V \oplus \frac{G}{p} \cong J \oplus \bigoplus_{i=1}^{d} \frac{G}{p_i} \oplus W.
\]

The Krull Schmidt theorem now gives the required isomorphism. The rest of the assertions follow in the same way as in Theorem (4.2.3) (see the proof of Theorem 2.9, [6], p. 9).

REMARK. The isomorphism of Proposition (4.2.7) may also be derived from Theorem (4.2.3) and the sequence (3.2.4); however the above proof does highlight the importance of the relative relation sequence. It should be
mentioned that the main argument of the above proof is similar to the one used in the proof of Theorem 2.9 of [6].

By Theorem (4.2.3) and Proposition (4.2.7) we have

\[(4.2.9)\] \textbf{COROLLARY.} \textit{If} \( p \) \textit{is semicoprime to} \( n \),

\[N(\hat{R}) \cong N(\hat{S}) .\]

The semicoprime case will be studied further in Chapter 6, where a criterion for counting the multiplicities of principal indecomposables in \( \hat{S} \) will be established, and some relative relation modules of \( \text{SL}(2, p) \) and \( \text{PSL}(2, p) \) will be described completely.

We have already seen that \( N(\hat{S}) \) may not be a homomorphic image of \( N(\hat{R}) \). However, the case of minimal relation and relative relation modules (defined below) is entirely different.

\[(4.2.9)\] \textbf{DEFINITION.} \textit{If} \( X \) \textit{is a minimal generating set of} \( G \), the corresponding relation and the relative relation modules are called minimal.

\textit{(NOTE.} \textit{For this definition,} \( p \) \textit{need not divide} \( n \). \textit{)}

\textit{If} \( G \) \textit{is a} \( p \)-\textit{group, we know that} \( \mathbb{F}_p G \) \textit{is the unique principal indecomposable module. Therefore, if} \( \hat{R} \) \textit{is a minimal relation module of a} \( p \)-\textit{group} \( G \), \textit{it follows that}

\[0 \to \hat{R} \to \mathbb{F}_p G \to \mathbb{F}_p G \to \mathbb{F}_p \to 0\]

satisfies all the conditions of Theorem (4.2.3), where \( g \) \textit{is the augmentation and} \( f \) \textit{is the same as} \( \pi' \) \textit{of the relation sequence. Thus} \( \hat{R} \) \textit{itself is a nonprojective and indecomposable module. If the given generating set} \( X \) \textit{of} \( G \) \textit{contains only two elements, by Proposition (3.3.9) \( \hat{S} \) is embedded in} \( \mathbb{F}_p G \), \textit{so by Corollary (2.2.10) \( \hat{S} \) is also nonprojective and indecomposable.}

In Chapter 7 we shall prove that all minimal relative relation modules of \( p \)-\textit{groups are nonprojective and indecomposable.}

\textit{Coming back to the general case, we know that the nonprojective part of}
A suitable characterisation of the nonprojective parts of relative relation modules, and so the structure of relative relation modules, remains unknown.
Throughout this chapter, let \( p \) be a fixed prime; \( X_1 = \{ g_i, 1 \leq i \leq d_1 \} \), \( X_2 = \{ h_i, 1 \leq i \leq d_2 \} \) generating sets of \( G \); and \( G_i, H_i \) the cyclic subgroups of \( G \) generated by \( g_i, h_i \) respectively.

Corresponding to \( X_1 \) and \( X_2 \), let \( \hat{R}_1 \) and \( \hat{R}_2 \) be the relation modules of \( G \) with the relation sequences

\[
0 \rightarrow \hat{R}_1 \rightarrow \bigoplus_{i=1}^{d_1} G 
\xrightarrow{\psi_1} \mathbb{F}_p G \rightarrow 0
\]

and

\[
0 \rightarrow \hat{R}_2 \rightarrow \bigoplus_{i=1}^{d_2} G 
\xrightarrow{\psi_2} \mathbb{F}_p G \rightarrow 0
\]

and \( \hat{S}_1 \) and \( \hat{S}_2 \) the relative relation modules with the relative relation sequences

\[
0 \rightarrow \hat{S}_1 \rightarrow \bigoplus_{i=1}^{d_1} G 
\xrightarrow{\psi_1} \mathbb{F}_p G \rightarrow 0
\]

and

\[
0 \rightarrow \hat{S}_2 \rightarrow \bigoplus_{i=1}^{d_2} G 
\xrightarrow{\psi_2} \mathbb{F}_p G \rightarrow 0
\]

The following result comes immediately by an application of Schanuel's lemma (2.2.8) to the sequences (5.0.1) and (5.0.2).

(5.0.5) **Proposition.** Suppose that \( d_2 \geq d_1 \) and \( d_2 - d_1 = r \). Then

(i) if \( r = 0 \), \( \hat{R}_1 \cong \hat{R}_2 \);

(ii) if \( r \geq 1 \), \( \hat{R}_2 = \hat{R}_1 \oplus L \), where \( L \) is a free \( \mathbb{F}_p G \)-module of rank \( r \).

A result for \( \hat{S}_1 \) and \( \hat{S}_2 \) analogous to Proposition (5.0.5) is far from
being true, which we aim to analyse in this chapter. The chapter is divided into two sections. In the first section we choose \( \hat{S}_1 \) and \( \hat{S}_2 \) such that the dimensions of \( \hat{S}_1 \) and \( \hat{S}_2 \) are the same, and show by means of an example that \( \hat{S}_1 \) and \( \hat{S}_2 \) need not be isomorphic to each other. Some of the cases where \( \hat{S}_1 \) and \( \hat{S}_2 \) may be isomorphic to each other are also given in the first section. In the second section, considering \( X_1 \) a proper subset of \( X_2 \), we show that \( \hat{S}_1 \) may be embedded into \( \hat{S}_2 \) and give an example to show that \( \hat{S}_2 \) need not decompose over \( \hat{S}_1 \). Some cases where \( \hat{S}_2 \) does decompose over \( \hat{S}_1 \) are also given in the second section.

5.1 The isomorphism question

Throughout this section let \( d_1 = d_2 = d \) and \( |H_i| = |G_i| \), \( 1 \leq i \leq d \). As a result the dimensions of \( \hat{S}_1 \) and \( \hat{S}_2 \) are necessarily the same. The following example shows that \( \hat{S}_1 \) and \( \hat{S}_2 \) may not be isomorphic to each other.

(5.1.1) EXAMPLE. Let \( G = C_q \times C_q \), the direct product of two cyclic groups of order \( q \), where \( q \) is an integer greater than one; and \( x_1 = \{g_1, g_2\} \), \( x_2 = \{g_0, g_2\} \), where \( g_0 = g_1g_2 \). We consider the following two cases.

CASE (i). \( q \) is not divisible by \( p \).

Let

\[
0 \to T_1^G \oplus T_2^G \to \hat{R}_1 \to \hat{S}_1 \to 0
\]

and

\[
0 \to T_0^G \oplus T_2^G \to \hat{R}_2 \to \hat{S}_2 \to 0
\]

be the exact sequences analogous to (3.2.4). Since \( p \) does not divide the
order of $G$, by Maschke's theorem we have

$$
\hat{R}_1 \cong T_1^G \oplus T_2^G \oplus \hat{S}_1
$$

and

$$
\hat{R}_2 \cong T_0^G \oplus T_2^G \oplus \hat{S}_2
$$

Suppose that $\hat{S}_1 \cong \hat{S}_2$. Since $\hat{R}_1 \cong \hat{R}_2$ by Proposition (5.0.5), the Krull Schmidt theorem gives that

$$
T_0^G \cong T_1^G.
$$

Since $T_1$ is a trivial $\langle g_1 \rangle$-module and $G$ is an abelian group, $\langle g_1 \rangle$ acts trivially on $T_1^G$. Similarly $\langle g_1g_2 \rangle$ acts trivially on $T_0^G$. In view of the above isomorphism $\langle g_1 \rangle$ must also act trivially on $T_0^G$, and so $T_0^G$ must be a trivial $G$-module. We know that $T_0^G$ is not a trivial $G$-module, therefore $\hat{S}_1$ and $\hat{S}_2$ cannot be isomorphic to each other.

Case (ii). $p$ is an odd prime and $p = q$.

The relative relation sequences (5.0.3) and (5.0.4) may be rewritten as

$$
0 \to \hat{S}_1 \to b_{1g_1}^G \oplus b_{2g_2}^G \xrightarrow{\psi_1} G \to 0
$$

and

$$
0 \to \hat{S}_2 \to b_{0g_0}^G \oplus b_{2g_2}^G \xrightarrow{\psi_2} G \to 0.
$$

Let $x = (b_{1-b_2})(1-g_1)(1-g_2)$ and $y = (b_{0-b_2})(1-g_0)(1-g_2)$. Then $\hat{S}_1$ may be generated by $x$ and $\hat{S}_2$ by $y$ (see Lemma (7.1.2)). Suppose that there exists an $\mathbb{F}_p G$-isomorphism $\delta$ of $\hat{S}_1$ onto $\hat{S}_2$. Then $x\delta = ya$, for some $a \in \mathbb{F}_p G$. We may write $a = a + a'$, for some nonzero element $a$ of $\mathbb{F}_p$ and some element $a'$ of $G$. We know that $\{(1-g_1), (1-g_2)\}$ is a
generating set of \( g \); and so, \( a' = (1-g_1)a_1 + (1-g_2)a_2 \) for some \( a_1, a_2 \in \mathbb{F}_p G \). Then, using the commutativity of \( \mathbb{F}_p G \),
\[
y a = \alpha y + y (1-g_1)a_1 + y (1-g_2)a_2.
\]
Let \( b = (1-g_1)^{p-1}(1-g_2)^{p-3} \). We have
\[
(xy)\delta = (x\delta)b = (ya)b
= \alpha yb + y (1-g_1)ba_1 + y (1-g_2)ba_2.
\]
Clearly,
\[
(1-g_1)^{p-1}(1-g_1g_2) = \left(1 + g_1 + \ldots + g_1^{p-1}\right)(1-g_1g_2)
= \left(1 + g_1 + \ldots + g_1^{p-1}\right) - g_2 \left(1 + g_1 + \ldots + g_1^{p-1}\right)
= (1-g_1)^{p-1}(1-g_2).
\]
Therefore, using \( g_0 = g_1g_2 \),
\[
\alpha yb = \alpha (b_0-b_1)(1-g_1g_2)(1-g_2)(1-g_1)^{p-1}(1-g_2)^{p-3}
= \alpha (b_0-b_2)(1-g_1)^{p-1}(1-g_1g_2)(1-g_2)^{p-2}
= \alpha (b_0-b_2)(1-g_1)^{p-1}(1-g_2)^{p-1},
\]
and
\[
y (1-g_1)ba_1 = y (1-g_2)ba_2 = 0 \quad \text{(because } (1-g_2)^p = 0 \text{)}.
\]
Hence
\[
(xy)\delta = \alpha (b_0-b_1)(1-g_1)^{p-1}(1-g_2)^{p-1},
\]
which is a nonzero element of \( \hat{S}_2 \). But
\[
xy = (b_1-b_2)(1-g_1)(1-g_2)(1-g_1)^{p-1}(1-g_2)^{p-3} = 0.
\]
This is contrary to the assumption that \( \delta \) is an \( \mathbb{F}_p \)-isomorphism. Hence \( \hat{S}_1 \) and \( \hat{S}_2 \) can not be isomorphic.
Note. In case (ii) of the above example, if we suppose that $p = q = 2$ then $\hat{S}_1$ and $\hat{S}_2$ are isomorphic (one dimensional) modules. However, the minimal relative relation modules of $C_2 \times C_2$ in the coprime case need not be isomorphic as shown by Case (i) of the above example.

Remark. In the above example, the mapping $g_1 \mapsto g_0$, $g_2 \mapsto g_2$ naturally extends to an automorphism of $G$. This shows that $\hat{S}_1$ and $\hat{S}_2$ may not be isomorphic to each other even if there is an automorphism of $G$ mapping $X_1$ onto $X_2$.

$\hat{S}_1$ and $\hat{S}_2$ are projective in case (i) and nonprojective as well as indecomposable in case (ii) of Example (5.1.1). Case (i) is due to Maschke's theorem and case (ii) follows by Corollary (2.2.10) because $G$ is a $p$-group and both $\hat{S}_1$ and $\hat{S}_2$ are embedded in $\mathbb{F}_p G$ (see Proposition (3.3.9)). Thus in general the projective parts, as well as the nonprojective parts, of $\hat{S}_1$ and $\hat{S}_2$ need not be isomorphic to each other.

We prove

(5.1.2) PROPOSITION. Let $x$ be a fixed element of $G$ and suppose that $H_i = G^x_i$, $1 \leq i \leq d$. Then $\hat{S}_1 \cong \hat{S}_2$.

Proof. Clearly $h_i = \left( \alpha_i \right)^x$, $1 \leq i \leq d$, for some integer $\alpha_i$ such that $\langle h_i \rangle = G_i$. Then the statement of the proposition is equivalent to the following two statements:

(i) $\hat{S}_1 \cong \hat{S}_2$ if $h_i = g_i^\alpha$, $1 \leq i \leq d$;

(ii) $\hat{S}_1 \cong \hat{S}_2$ if $h_i = g_i^x$, $1 \leq i \leq d$.

Equivalently, we prove (i) and (ii).
(i) Clearly \( G_i^d = g_i^d \), \( 1 \leq i \leq d \). Therefore it is enough to show that the mappings \( \psi_1' \) and \( \psi_2' \) of (5.0.3) and (5.0.4) coincide with each other. Recall that

\[
\sum_i (1-g_i)x_i \psi_1' = \sum_i (1-g_i)x_i \psi_1 , \quad x_i \in \mathbb{F}^G_p ,
\]

and

\[
\sum_i (1-h_i)y_i \psi_2' = \sum_i \left( 1 \cdot \sum_{i=1}^{\alpha_{i-1}} g_i^\alpha_i y_i \right) \psi_2 = \sum_i \left( 1 \cdot \sum_{i=1}^{\alpha_{i-1}} g_i^\alpha_i y_i \right) , \quad y_i \in \mathbb{F}^G_p .
\]

Clearly,

\[
\sum_{i=1}^{\alpha_{i-1}} \left( 1-g_i^\alpha_i y_i \right) = \left( 1-g_i^\alpha_i \right) = \left( 1-h_i \right) .
\]

Therefore,

\[
\left( (1-h_i) y_i \right) \psi_1' = \left( \sum_{i=1}^{\alpha_{i-1}} \left( 1-g_i^\alpha_i y_i \right) \right) \psi_1
\]

\[
= \left( 1-h_i \right) y_i
\]

\[
= \left( (1-h_i) y_i \right) \psi_2' .
\]

Thus \( \psi_1' \) and \( \psi_2' \) coincide with each other.

(ii) Let \( \Gamma_i \) be a (right) transversal of \( G_i \) in \( G \). Then

\[
\{ t^x , t \in \Gamma_i \}
\]

is a transversal of \( H_i \) in \( G \). We know that

\[
\{ (1-z) , z \in G_i \}
\]

is an \( \mathbb{F}_p \)-basis of \( g_i^G \) and \( \{ (1-z^x) , z \in G_i \} \) an \( \mathbb{F}_p \)-basis of \( g_i^G \). Thus \( \{ (1-z)t ; z \in G_i , t \in \Gamma_i \} \) is an \( \mathbb{F}_p \)-basis of \( g_i^G \) and

\[
\{ (1-z^x)^t x ; z \in G_i , t \in \Gamma_i \}
\]

is an \( \mathbb{F}_p \)-basis of \( h_i^G \). Define
such that
\[
\xi_i : h_i G \rightarrow g_i G
\]

such that
\[
\left( \sum_{z \in G_i, t \in \Gamma_i} a_{zt} (1-z^x t^x) \right) \xi_i = \sum_{z \in G_i, t \in \Gamma_i} a_{zt} (1-z^x t^x) \xi_i,
\]

where \( a_{zt} \in \mathbb{F}_p \). It may be verified that \( \xi_i \) is an \( \mathbb{F}_p \)-isomorphism. To show that \( \xi_i \) is a \( G \)-isomorphism, let \( g \) be an element of \( G \) and \( t \) an element of \( \Gamma_i \). Then, for some \( g_1 \in G_i \) and \( t_1 \in \Gamma_i \),

and so
\[
((1-z^x t^x) g_1 t_1^x) \xi_i = \left( (1-z^x) (g_1 t_1^x) \right) \xi_i = \left( (1-z) g_1 \right) t_1^x \xi_i.
\]

Clearly, \( (1-z) g_1 = \sum_{z_j \in G_i} a_j (1-z_j) \), \( a_j \in \mathbb{F}_p \), and so

\[
((1-z^x t^x) g_1 t_1^x) \xi_i = \left( \sum_{z_j \in G_i} a_j (1-z_j) x_{t_1} \right) \xi_i = \left( \sum_{z_j \in G_i} a_j (1-z_j) t_1 \right) x = ((1-z) t_1) x = ((1-z) t_1) g
\]

Thus \( \xi_i \) is an \( \mathbb{F}_p G \)-isomorphism.

(Remark. It is a well known fact that \( \xi_i \) is an \( \mathbb{F}_p G \)-isomorphism.)

Next define
\[
U : G \rightarrow G
\]

such that
\[
\left( \sum_{g \in G} k_g (1-g^x) \right) U = \left( \sum_{g \in G} k_g (1-g) \right) x.
\]

It may also be verified that \( U \) is an \( \mathbb{F}_p G \)-isomorphism. Now consider
where \( \xi \) is the isomorphism whose restriction to \( \frac{G_i}{G_{i-1}} \) is the isomorphism \( \xi_i \) defined above. Then it follows that \( \psi_2' \psi_1 = \xi \psi_1' \). The existence of an isomorphism of \( \hat{S}_2 \) onto \( \hat{S}_1 \), namely the restriction of \( \xi \) to \( \hat{S}_2 \), is now immediate. This completes the proof.

It is not known whether there is an isomorphism between \( \hat{S}_1 \) and \( \hat{S}_2 \) if we only suppose that the elements of \( X_1 \) and \( X_2 \) are pairwise conjugate to each other. However this is true in the semicoprime case.

(5.1.3) COROLLARY. Let \( p \) be semicoprime to the order of \( G \) and suppose that \( h_i = g_i, x_i \in G, 1 \leq i \leq d \). Then \( \hat{S}_1 \cong \hat{S}_2 \).

This follows by Schanuel's lemma (2.2.8) as the \( \frac{G_i}{G_{i-1}} \), \( 1 \leq i \leq d \), are projective.

(5.1.4) Remark. For an isomorphism between \( \hat{S}_1 \) and \( \hat{S}_2 \) in the semicoprime case, it is only necessary to have an isomorphism between the middle terms of the sequences (5.0.3) and (5.0.4).

5.2 Adding generators

Throughout this section we suppose that \( X_2 = X_1 \cup \{g_0\} \).

We prove

(5.2.1) PROPOSITION. \( \hat{S}_1 \) is a submodule of \( \hat{S}_2 \) and \( \hat{S}_2/\hat{S}_1 \cong g_0^G \).

Proof. Let
Clearly the restriction of $\psi'_2$ to $V$ is precisely $\psi'_1$. Therefore,

$$\ker \psi'_2 \cap V = \ker \psi'_1,$$

that is

$$\hat{S}_2 \cap V = \hat{S}_1.$$ 

Moreover,

$$(\hat{S}_2/\hat{S}_1) \cap (V/\hat{S}_1) = (\hat{S}_2 \cap V)/\hat{S}_1 = \hat{S}_1/\hat{S}_1 = \{0\}.$$ 

Therefore the sum of $\hat{S}_2/\hat{S}_1$ and $V/\hat{S}_1$ is a direct sum. Then

$$(\hat{S}_2/\hat{S}_1) \oplus (V/\hat{S}_1) = (V \oplus W)/\hat{S}_1,$$

because $\hat{S}_2/\hat{S}_1$ and $V/\hat{S}_1$ are submodules of $(V \oplus W)/\hat{S}_1$ and

$$\dim(\hat{S}_2/\hat{S}_1) + \dim(V/\hat{S}_1) = \dim((V \oplus W)/\hat{S}_1).$$

Now

$$(V \oplus W)/\hat{S}_1 \cong (V/\hat{S}_1) \oplus W.$$ 

The Krull Schmidt theorem then gives the requires result that $\hat{S}_2/\hat{S}_1 \cong W$.

(5.2.2) COROLLARY. Suppose that $p$ does not divide the order of $\langle g_0 \rangle$. Then $\hat{S}_2 \cong \hat{S}_1 \oplus G$.

This follows because $G$ is projective.

The following example shows that $\hat{S}_2$ need not decompose over $\hat{S}_1$ and $G$.

(5.2.3) EXAMPLE. For this example, let $p = 2$, $G = C_2 \times C_2$, $X_1 = \{g_1, g_2\}$, $X_2 = \{g_0, g_1, g_2\}$, where $g_0 = g_1g_2$. Then the relative relation sequences (5.0.3) and (5.0.4) may be written as

$$0 \rightarrow \hat{S}_1 \rightarrow b_1G_1 \oplus b_2G_2 \rightarrow \psi' \rightarrow G \rightarrow 0$$

and
\[
0 \to \hat{S}_2 \to b_0 g_0^G \oplus b_1 g_1^G \oplus b_2 g_2^G \overset{\psi'}{\to} \hat{g}_2 \to 0.
\]

We know that \( b_i (1+g_i) \) (\( \leq b_i (1-g_i) \)) generates \( b_i g_i^G \), \( 0 \leq i \leq 2 \). Clearly \( b_i (1+g_i) = b_i (1+g_1 g_2) g_i = b_i (g_1 + g_2) \), for \( i = 1 \) or \( i = 2 \); and so,
\[
b_0 (1+g_0) g_i = b_0 (1+g_1 g_2) g_i = b_0 (g_1 + g_2), \quad \text{for } i = 1 \text{ or } i = 2;
\]
\[
y_1 = b_0 (g_1 + g_2) + b_1 (1+g_1) + b_2 (1+g_2),
\]
\[
y_2 = (b_1 + b_2) (1+g_1) (1+g_2) \quad \text{and } y_3 = b_0 (g_1 + g_2) (1+g_1) + b_1 (1+g_1) (1+g_2). \]

Then \( y_i \psi'_i = 0 \), \( 1 \leq i \leq 3 \), and so \( Y = \{y_1, y_2, y_3\} \) is a subset of \( \hat{S}_2 \). In fact \( Y \) is an \( \mathbb{F}_p \)-linearly independent set, and so must be a basis because \( \dim \hat{S}_2 = |Y| = 3 \). It is easy to check that the space spanned by \( \{y_2, y_3\} \) is the unique maximal submodule of \( \hat{S}_2 \), and so \( \hat{S}_2 \) is indecomposable.

Finally we prove

(5.2.4) **Proposition.** Suppose that \( g_0 \in X_1 \). Then \( \hat{S}_2 \cong \hat{S}_1 \oplus g_0^G \).

**Proof.** We suppose without loss of generality that \( g_0 = g_1 \). Then \( g_0^G = g_1^G \). Let \( V = \bigoplus_{i=1}^{d_1} g_i^G \), and define \( \eta_1 : V \to V \oplus g_0^G \) to be the identity isomorphism of \( V \) and \( \eta_2 : V \oplus g_0^G \to V \) to be the epimorphism whose restriction to \( V \) and \( g_0^G \) is the identity isomorphism. Then it is easy to check that \( \psi'_1 = \eta_1 \psi'_2 \) and \( \psi'_2 = \eta_2 \psi'_1 \). Now applying Schanuel's lemma to (5.0.3) and (5.0.4),
\[
\hat{S}_1 \oplus V \oplus g_0^G \cong \hat{S}_2 \oplus V.
\]

The Krull-Schmidt theorem then gives the required result.

**Remark.** For Proposition (5.2.4), it is only necessary to assume that \( g_0 \) is a power of an element of \( X_1 \). However it is not known whether Proposition (5.2.4) is true if we only suppose that \( g_0 \) is a conjugate of an element of \( X_1 \) in \( G \).
In Proposition (5.2.4) if we suppose that $G$ is a $p$-group, then $G^\wedge_{g_0}$ is nonprojective, and so the nonprojective part of $\hat{S}_2$ is isomorphic to the direct sum of $G^\wedge_{g_0}$ and the nonprojective part of $\hat{S}_1$. This substantiates our earlier remarks that the nonprojective part of a relative relation module (in the nonsemicoprime case) may be decomposable.
CHAPTER 6

RELATIVE RELATION MODULES IN THE SEMICOPRIME CASE

Let \( \mathcal{X} = \{ g_i, i \in \Lambda \} \) be a generating set of \( G \), and \( \hat{S} \) the relative relation module of \( G \) determined by (3.1.2). Throughout this chapter we assume that \( p \) is semicoprime to the order of \( G \), that is, \( p \) does not divide the orders of the cyclic groups \( G_i \) generated by \( g_i, i \in \Lambda \).

Proposition (4.2.7) has already outlined the structure of \( \hat{S} \) in terms of projective and nonprojective modules. Recall that the nonprojective part \( N(\hat{S}) \) of \( \hat{S} \) is isomorphic to the nonprojective part of relation modules, and so \( N(\hat{S}) \) is uniquely determined and does not depend on the choice of \( \mathcal{X} \). In the first of the two sections of this chapter, we give a criterion for counting the multiplicities of the principal indecomposable summands of \( \hat{S} \). The usefulness of this criterion is shown in the second section, where a complete description of some relative relation modules of \( SL(2,p) \) and \( PSL(2,p) \) is given.

6.1 Counting projective summands

Let \( V \) and \( W \) be \( \mathbb{F}G \)-modules, where \( \mathbb{F} \) is an arbitrary field. \( \text{Hom}_{\mathbb{F}G}(V,W) \), the set of all \( \mathbb{F}G \)-homomorphisms of \( V \) into \( W \), is a vector space over \( \mathbb{F} \). The dimension of this vector space, which is called the intertwining number of \( V \) and \( W \), will be denoted by \( I(V,W) \).

(6.1.1) LEMMA. \( I(\bigoplus_i V_i, \hat{W}) = \sum_i I(V_i, W) \).

Moreover, if \( \mathbb{F}G \) is semi-simple,

\[ I(V, W) = I(W, V). \]

For a proof, see ([2], Corollary (43.16), p. 320 and Exercise (43.6), p. 322).
(6.1.2) DEFINITION. Let $V$ be a $G$-module. Then
\{v \in V \mid vg = v \text{ for all } g \in G\} is a $G$-module, which we call the fixed point space of $V$. If $H$ is a (proper) subgroup of $G$, the fixed point space of $V_H$ will be called the $H$-fixed point space of $V$.

Now we prove

(6.1.3) PROPOSITION. Let $P$ be a principal indecomposable $W^G$-module with $P/\varphi P = U$, $r$ the dimension of $U$, and $s$ the dimension of the $G_i$-fixed point space of $U$. Then the multiplicity of $P$ in $T_i^G$ is $s/I(U, U)$ and the multiplicity of $P$ in $\mathbb{E}_i^G$ is $(r-s)/I(U, U)$.

Proof. Let $m$ be the multiplicity of $P$ in $T_i^G$. We claim that

(i) \[ m = \frac{I(T_i^G, U)}{I(U, U)}. \]

As $T_i^G$ is projective, we may write

\[ T_i^G \cong V_1 \oplus V_2 \oplus \cdots \oplus V_t, \]

where $V_j$, $1 \leq j \leq t$, are principal indecomposable modules. Clearly,

\[ I(V_j, U) = I(V_j/\varphi V_j, U); \]

and so

\[ I(V_j, U) = \begin{cases} I(U, U), & \text{if } V_j \cong P, \\ 0, & \text{if } V_j \not\cong P. \end{cases} \]

Without loss of generality, we may suppose that $V_1, V_2, \ldots, V_m, m \leq t$, are isomorphic to $P$. Then, by Lemma (6.1.1),

\[ I(T_i^G, U) = \sum_{j=1}^m I(V_j, U); \]

and so,
\[
\frac{I(T_i^G, U)}{I(U, U)} = \sum_{j=1}^{m} \frac{I(V_j, U)}{I(U, U)} = \sum_{j=1}^{m} \frac{I(V_j, U)}{I(V_j, U)} = m.
\]

Hence the claim follows.

Now we claim

(ii) \( I \left( T_i^G, U \right) = s. \)

By Nakayama's lemma ([9], Kapitel 5, Satz 16.6, p. 556),

\[
I \left( T_i^G, U \right) = I(T_i, U_{G_i}).
\]

Since \( p \) does not divide the order of \( G_i \), \( U_{G_i} \) is completely reducible, and so we suppose that

\[
U_{G_i} = T \oplus W,
\]

where \( T \) is the (direct) sum of all trivial irreducible submodules of \( U_{G_i} \), and \( W \) the sum of all nontrivial irreducible submodules of \( U_{G_i} \).

Clearly \( T \) is the \( G_i \)-fixed point space of \( U \), and so the dimension of \( T \) must be \( r \). Then by Lemma (6.1.1), \( I(T_i^G, T) = r \), because \( T_i^G \) is a trivial irreducible \( G_i \)-module. Moreover, \( I(T_i^G, W) = 0 \), because \( W \) does not contain any trivial irreducible modules. Then

\[
I \left( T_i^G, U \right) = I(T_i, U_{G_i}) = I(T_i, T) = s,
\]

and so the claim follows. Now the assertion that the multiplicity of \( P \) in \( T_i^G \) is \( s/I(U, U) \) follows from (i) and (ii).

The assertion that the multiplicity of \( P \) in \( g_i^G \) is \( (r-s)/I(U, U) \) follows from Theorem (2.2.4) and the Krull Schmidt theorem, because

\[
F \rightarrow T_i^G \oplus g_i^G.
\]
(6.1.4) COROLLARY. If in addition to the assumptions of Proposition
(6.1.3), \( \mathbb{F}_p \) is a splitting field for \( G \), the multiplicity of \( P \) in \( T_i^G \)
is \( s \) and the multiplicity of \( P \) in \( g_i^G \) is \( r - s \).

Remark. In Proposition (6.1.3), if we suppose that \( p \) is coprime to the
order of \( G \) then replacing \( P \) by \( U \) we get the multiplicities of the
irreducible submodules of \( T_i^G \) and \( g_i^G \). The same applies to Corollary
(6.1.4).

Propositions (4.2.7) and (6.1.3) provide us with a criterion for
finding an unrefinable decomposition of \( \hat{S} \). For this, we need to calculate
the dimensions of the \( G_i \)-fixed point spaces, as well as the dimensions of
the endomorphism rings, of the irreducible modules. This is not an easy
task in general. In cases when \( \mathbb{F}_p \) is a splitting field for \( G \), the task
is reduced to calculating the dimensions of the fixed point spaces alone,
which may still not be easy. In order to show that the above results may be
useful in some cases, we shall describe some of the relative relation
modules of \( \text{SL}(2, p) \) and \( \text{PSL}(2, p) \) in the next section.

We complete this section with the following well known result, which
will be needed in the next section.

(6.1.5) LEMMA. Let \( V \) be an \( \mathbb{F}G \)-module, where \( \mathbb{F} \) is an arbitrary field.
Then, for any extension field \( K \) of \( \mathbb{F} \), the dimensions of the fixed point
spaces of \( V \) and \( V^K \) are the same.

Proof. Let \( T \) be a trivial irreducible \( \mathbb{F}G \)-module.

By ([2], Theorem (43.14), p. 320) the dimension of the fixed point
space of \( V \) is equal to \( I(T, V) \), and that of \( V^K \) is \( I(T^K, V^K) \). By
([2], (29.5), p. 200),
\[(\text{Hom}_{FG}(T, V))^K \xrightarrow{\sim} \text{Hom}_{KG}(T^K, V^K)\],

and so

\[I(T, V) = I(T^K, V^K).\]

Hence the result follows.

6.2 Some relative relation modules of SL(2, p) and PSL(2, p)

Let \(p\) be a fixed prime other than 2 and 3, and \(G = \text{SL}(2, p)\), the group of all \(2 \times 2\) matrices over \(\mathbb{F}_p\) of determinant one. We know that \(G\) can be generated by

\[
g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},
\]

where

\[
g_1^4 = g_2^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \(\hat{S}\) be the relative relation module (modulo \(p\)) of \(G\) corresponding to \(X = \{g_1, g_2\}\). We wish to determine all indecomposable summands of \(\hat{S}\). To do so, of course, we must know the submodule structure of \(\mathbb{F}_p G\). The following description of all irreducible \(\mathbb{F}_p G\)-modules was first given by Brauer and Nesbitt [1].

For an integer \(m\) greater than or equal to one, let \(V_m\) be the set of all homogeneous polynomials in two indeterminates, \(x\) and \(y\) say, over \(\mathbb{F}_p\) of degree \(m - 1\). \(V_m\) is a vector space over \(\mathbb{F}_p\) and

\[\{x^j y^{m-1-j}, 0 \leq j \leq m-1\}\]

is an \(\mathbb{F}_p\)-basis of \(V_m\). \(V_m\) may be regarded as a \(G\)-module, where the \(G\)-action is defined by

\[(x^j y^{m-1-j})g = (a_{11}x + a_{12}y)^j (a_{21}x + a_{22}y)^{m-1-j},\]
where

\[ g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in G. \]

In fact \( \{V_m, 1 \leq m \leq p\} \) is a complete set of nonisomorphic irreducible \( G \)-modules. Moreover, each \( V_m \) is an absolutely irreducible module; and so, \( \mathbb{F}_p \) is a splitting field for \( G \). For details, see ([1], Section 30); also ([4], (3.1)).

We suppose that \( \{P_m, 1 \leq m \leq p\} \) is a complete set of all nonisomorphic principal indecomposable modules with \( P_m/\varphi_m = V_m \). The Loewy structure of each \( P_m \) is explicitly known (for example, see [4], Section 6), and is given below.

\[
\text{(6.2.2) (i) } \dim P_m = \begin{cases} p & \text{if } m = 1 \text{, or } m = p \\ 2p & \text{otherwise}; \end{cases}
\]

\[
\text{(ii) } \varphi P_1/\varphi^2 P_1 \cong V_{p-2}, \quad \varphi^2 P_1 \cong \sigma P_1 \cong \mathbb{F}_p \cong V_1;
\]

\[
\text{(iii) if } 2 \leq m \leq p-1,
\]

\[
\varphi P_m/\varphi^2 P_m \cong V_{p-m-1} \oplus V_{p-m+1},
\]

\[
\varphi^2 P_m \cong \sigma P_m \cong V_m;
\]

\[
\text{(iv) } P_1 = V_p.
\]

Now we are ready to determine the indecomposable summands of \( \hat{S} \).

First we determine the nonprojective part \( N(\hat{S}) \) of \( \hat{S} \).

Clearly there exists an epimorphism \( g : P_1 \twoheadrightarrow \mathbb{F}_p \) such that \( \text{Ker } g = \varphi P_1 \). Since

\[ V_{p-2} \cong \varphi P_1/\varphi^2 P_1 \]

and
\[ \phi_{p-2}/\phi_{p-2}^2 \cong V_1 \oplus V_3 \cong \phi_{p_1}^2 \oplus V_3 \cong \sigma P_1 \oplus V_3. \]

There exists an epimorphism \( f : P_{p-2} \rightarrow \phi_{p_1} \) such that \( \ker f \subseteq \phi_{p-2} \). Thus

\[
0 \rightarrow \ker f \rightarrow P_{p-2} \xrightarrow{f} P_1 \xrightarrow{g} \mathbb{F}_p \rightarrow 0
\]

is an exact sequence, which satisfies all the conditions of Proposition (4.2.7). Therefore we have

\[ (6.2.3) \quad N(\hat{S}) = \ker f, \quad \text{and} \quad \dim N(\hat{S}) = p + 1. \]

Now we proceed to determine the principal indecomposable summands of \( \hat{S} \). Our main task is to find the multiplicity of each principal indecomposable \( P_m \) in \( g_{11}^G \) and in \( g_{22}^G \). For \( i = 1, 2 \) and for \( m = 1, 2, \ldots, p \), let \( f_i(V_m) \) denote the dimension of the \( G_i \)-fixed point space of \( V_m \). By Corollary (6.1.4) the multiplicity of \( P_m \) in \( g_{ii}^G \) is equal to \( m - f_i(V_m) \), and so our need is to find the values of \( f_i(V_m) \).

As \( \mathbb{F}_p \) need not be a splitting field for \( G_i \), let \( \mathbb{F} \) be an extension of \( \mathbb{F}_p \) so that \( \mathbb{F} \) is a splitting field for \( G_i \). By Lemma (6.1.5), \( f_i(V_m) \) is the same as the dimension of the \( G_i \)-fixed point space of \( V_m^\mathbb{F} \), which is the same as the dimension of the subspace of \( V_m^\mathbb{F} \) fixed by \( g_i \). We may regard \( g_i \) as a matrix over \( \mathbb{F} \). Then, since \( p \) does not divide the order of \( G_i \), by Maschke's theorem we can find matrices

\[
\tilde{g}_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}, \quad i = 1, 2, \quad \alpha_i, \beta_i \in \mathbb{F},
\]

such that \( g_i \) is conjugate to \( \tilde{g}_i \) in \( \text{GL}(2, \mathbb{F}) \). In fact,

\[
\tilde{g}_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^3 \end{pmatrix}, \quad \tilde{g}_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^5 \end{pmatrix},
\]
where \( \omega \) and \( \mu \) are fourth and sixth roots of unity, because

\[
det \tilde{g}_1 = det g = 1, \quad \text{and} \quad g^4 = g^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We calculate the dimension of the subspace of \( V_m^F \) fixed by \( \tilde{g}_1 \), which is the same as \( f_1(V_m) \) because \( g \) and \( \tilde{g} \) are conjugate to each other. If we regard \( \{ x^j y^{m-1-j}, 0 \leq j \leq m-1 \} \) as an \( IF \)-basis of \( V_m^F \), (6.2.1) gives

\[
(a^j y^{m-1-j}) \tilde{g}_1 = (\omega x)^j (\omega^3 y)^{m-1-j} = \omega^{3(m-1)-2j} x^j y^{m-1-j},
\]

and

\[
(a^j y^{m-1-j}) \tilde{g}_2 = (\mu x)^j (\mu^5 y)^{m-1-j} = \mu^{5(m-1)-4j} x^j y^{m-1-j}.
\]

Clearly the dimension of the subspace of \( V_m^F \) fixed by \( \tilde{g}_1 \), and so \( f_1(V_m) \), is the number of \( x^j y^{m-1-j} \) fixed by \( \tilde{g}_1 \), which is the same as the number of equations (6.2.4) for which \( \omega^{3(m-1)-2j} = 1 \). In other words, \( f_1(V_m) \) is the number of equations (6.2.4) in which \( 3(m-1) - 2j \) is divisible by 4, because \( \omega \) is a fourth root of unity. Similarly, \( f_2(V_m) \) is the number of equations (6.2.5) in which \( 5(m-1) - 4j \) is divisible by 6. Firstly, we have

\[
(6.2.6) \quad f_1(V_1) = \dim V_1 = 1.
\]

Next we claim

\[
(6.2.7) \quad \text{If } m \text{ is even, } f_1(V_m) = 0.
\]

For this we need to show that

\[
3(m-1) - 2j \neq 1 \quad \text{and} \quad 5(m-1) - 4j \neq 1,
\]

for all \( m \) and for all \( j \). Since \( m \) is even, \( 3(m-1) - 2j \) is odd, and
so can not be a multiple of 4. For the same reason 5(m-1) - 4j can not be a multiple of 6. Hence the claim follows.

Now we come to the case when \( m \) is odd.

Let us calculate \( f_1(V_m) \) first. Suppose that 4 divides \( m - 1 \).

Then \( 3(m-1) - 2j \) is a multiple of 4, if \( j = 0 \pmod{2} \). If we suppose that 4 does not divide \( m - 1 \) then \( 3(m-1) - 2j \) is a multiple of 4 provided \( j = 1 \pmod{2} \). Then in the same way as before we have

(6.2.8) If \( m \) is odd,

\[
 f_1(V_m) = \begin{cases} 
 \frac{m+1}{2}, & \text{if } 4 \mid m-1, \\
 \frac{m-1}{2}, & \text{if } 4 \nmid m-1.
\end{cases}
\]

For odd \( m \), we now calculate \( f_2(V_m) \). 5(m-1) - 4j is even and so will be a multiple of 6 whenever it is a multiple of 3. Therefore

\[
 5^{(m-1)-4j} = 1, \text{ if } 3 \mid 5^{(m-1)-4j}.
\]

We need to consider the three cases:

(i) \( 3 \mid m-1 \),

(ii) \( 3 \mid m \), and

(iii) \( 3 \mid m+1 \).

Case (i). \( 3 \mid 5^{(m-1)-4j} \), if \( j = 0 \pmod{3} \).

Case (ii). \( 3 \mid 5^{(m-1)-4j} \), if \( j = 1 \pmod{3} \).

Case (iii). \( 3 \mid 5^{(m-1)-4j} \), if \( j = 2 \pmod{3} \).

The above three cases yield

(6.2.9) If \( m \) is odd,

\[
 f_2(V_m) = \begin{cases} 
 \frac{m+2}{3}, & \text{if } 3 \mid m-1, \\
 \frac{m}{3}, & \text{if } 3 \mid m, \\
 \frac{m-2}{3}, & \text{if } 3 \nmid m+1.
\end{cases}
\]
Now we come to the multiplicity of $P_m$ in $\hat{S}$. By Proposition (4.2.7),

$$\hat{S} \oplus P_{p-2} \oplus T_2^G \cong N(\hat{S}) \oplus G \oplus P_1,$$

where $N(\hat{S})$ is given by (6.2.3). Denote by $l_m$, the multiplicity of $P_m$ in $\hat{S}$. If $\alpha_m = m - f_1(V_m) - f_2(V_m)$, Proposition (6.1.3), together with the Krull Schmidt theorem, gives

$$l_m = \begin{cases} 
\alpha_m + 1, & \text{if } m = 1, \\
\alpha_m - 1, & \text{if } m = p - 2, \\
\alpha_m, & \text{otherwise}.
\end{cases}$$

Using (6.2.6), (6.2.7), (6.2.8) and (6.2.9), our discussion may be summarised by

(6.2.10) PROPOSITION. Let $l_m$ be the multiplicity of $P_m$ in $\hat{S}$, and $l_m P_m$ the direct sum of $l_m$ copies of $P_m$. Then

$$\hat{S} = N(\hat{S}) \bigoplus_{m=1}^{p} l_m P_m,$$

where $N(\hat{S})$ is given by (6.2.3), and

$$l_m = \begin{cases} 
0, & \text{if } m = 1, \\
m, & \text{if } m \text{ is even}, \\
\alpha_m - 1, & \text{if } m = p - 2, \\
\alpha_m, & \text{otherwise},
\end{cases}$$

where
\[
\alpha_m = \begin{cases} 
\frac{m-7}{6}, & \text{if } 12 \mid m-1, \\
\frac{m+1}{6}, & \text{if } 3 \mid m+1, 4 \mid m-1, \\
\frac{m-3}{6}, & \text{if } 3 \mid m, 4 \mid m-1, \\
\frac{m-1}{6}, & \text{if } 3 \mid m-1, 4 \mid m-1, \\
\frac{m+3}{6}, & \text{if } 3 \mid m, 4 \mid m-1, \\
\frac{m+7}{6}, & \text{if } 3 \mid m+1, 4 \mid m-1. 
\end{cases}
\]

Now let \( H = \text{PSL}(2, p) \). We know that \( H \) is the factor group \( G/Z(G) \), where the centre \( Z(G) \) of \( G \) is a cyclic subgroup of \( G \); in fact \( Z(G) = \langle z \rangle \), where \( z = g_1^2 = g_2^3 \). Corresponding to \( h_i = g_i \mod Z(G) \), \( i = 1, 2 \), let \( \hat{K} \) be the relative relation module of \( H \). We now describe \( \hat{K} \).

We know that \( |G| = (p-1)p(p+1) \), and \( |H| = \frac{1}{2}(p-1)p(p+1) \). We also know that \( \{P_1, P_2, ..., P_{p-2}, P_p\} \) is a complete set of nonisomorphic principal indecomposable \( \mathbb{F}^H \)-modules. Thus \( \mathbb{F}^G \cong \mathbb{F}^H \oplus V \) (as \( G \)-modules) where \( V \cong 2P_2 \oplus 4P_y \oplus \ldots \oplus (p-1)P_{p-1} \). By Proposition (6.2.10), \( V \) is also a summand of \( \hat{S} \). By the above description of \( N(\hat{S}) \) it follows that \( N(\hat{S}) = N(\hat{K}) \). In fact \( \hat{S} \cong \hat{K} \oplus V \) (as \( G \)-modules) because (by (3.2.6))

\[
\dim \hat{S} - \dim \hat{K} = \frac{7(p-1)p(p+1) - (p-1)p(p+1)}{12} = \frac{(p-1)p(p+1)}{2} = \dim V.
\]

This may be summarised by

(6.2.11) \textbf{COROLLARY.} \( \hat{K} \cong N(\hat{S}) \oplus W \), where \( W \) is the direct sum of all those principal indecomposable summands \( P_m \) of \( \hat{S} \) (given by Proposition (6.2.10)) for which \( m \) is odd.

\textbf{Note.} In the above description of \( \hat{S} \) we do not require specific generators...
of $G$ of order 4 and 6. That is, relative relation modules of $G$
corresponding to any two generators of order 4 and 6 are isomorphic. This is also in accordance with Remark (5.1.4), because if $g_i$ and $g_j$
are any two elements of order 4, or 6, $G_{g_i}^G \cong G_{g_j}^G$ as modules. Relative
relation modules of $H$ corresponding to any two generators of order 2 and 3 are also isomorphic.

Remark. Let $G$ be an arbitrary (finite) group, $Z$ a normal subgroup of $G$, and $H = G/Z$. Let $\hat{S}$ be a relative relation module of $H$
corresponding to a generating set $X$ of $G$, and $\hat{K}$ a relative relation
module of $H$ corresponding to a generating set $X'$ of $H$. Suppose that $p$ (the given (fixed) prime for $\hat{S}$ and $\hat{K}$) is semicoprime to the orders of $G$
and $H$ (with respect to $X$ and $X'$), and that $p$ does not divide the order of $\mathbb{Z}$. Then $N(\hat{S}) = N(\hat{K})$. Moreover, if $X' = \{Zx, x \in X\}$,
$\hat{S} \cong \hat{K} \oplus V$ (as $G$-modules), where $V$ is a projective module.

A final remark is concerned with $PSL(2, \mathbb{Z})$, which we denote by $E$.
We have an exact sequence

$$1 \rightarrow K \rightarrow E \rightarrow H \rightarrow 1,$$

where $H = PSL(2, p)$, which gives us another exact sequence

$$0 \rightarrow K/K'K \rightarrow E/K'K \rightarrow H \rightarrow 1.$$

We may regard $E$ as $\mathbb{Z}_2 \ast \mathbb{Z}_3$; this gives us $K/K'K$ as the above
relative relation module $\hat{K}$ of $H$. The above description of $\hat{K}$ may lead
to some useful information about $E/K'K$; the classification of factor
groups of $E$ is an interesting problem in its own right.
Throughout this chapter, let $p$ be a fixed prime, $G$ a $p$-group, $X$ a minimal generating set of $G$ and $\hat{R}, \hat{S}$ the relation and relative relation modules of $G$ corresponding to $X$. As a result $\hat{R}$ and $\hat{S}$ are minimal (see Definition (4.2.9)).

As mentioned in Chapter 4, $\hat{R}$ is nonprojective and indecomposable. In this chapter we shall prove that $\hat{S}$ too is nonprojective and indecomposable. (Recall that if $X$ contains only two elements, the result that $\hat{S}$ is nonprojective and indecomposable is a consequence of Proposition (3.3.9); however this result will not be needed in this chapter.)

The nonprojectivity of $\hat{S}$ follows easily; it is the indecomposability which occupies most of the discussion. The idea is to prove that $\hat{S}$ is indecomposable when $G$ is abelian and use this result to prove the indecomposability of $\hat{S}$ in the nonabelian case. It is perhaps desirable to mention that the indecomposability of $\hat{S}$ in the abelian case may be proved in two ways. The way which we adopt is longer but proves the result directly; as well it is more interesting and informative.

This chapter is also divided into two sections. The indecomposability of $\hat{S}$ in the abelian case is proved in Section 7.1; the nonabelian case is proved in Section 7.2. The nonprojectivity of $\hat{S}$ (in both cases) is also proved in Section 7.2.

7.1 Indecomposability in the abelian case

Throughout this section we shall assume that $G$ is the direct product of the $G_i$'s. This holds in particular if $G$ is elementary abelian.

The relative relation sequence of $G$ with respect to $X$ may be written as
For the rest of this section, let $A = \mathbb{F}_p G$, $B_i = b_i G$, and $B = \bigoplus_{i=1}^{d} B_i$.

We wish to construct a minimal generating set of $\hat{S}$, which will enable us to prove the indecomposability of $\hat{S}$. For the required generating set of $\hat{S}$, we need certain bases for the Loewy factors of $A$ and $B$ which we shall describe first.

We know that irreducible $A$-modules are isomorphic to the trivial module $\mathbb{F}_p$, and $\varphi A$ is the unique maximal submodule of $A$. Thus $\varphi A = G$, and each Loewy factor of an $A$-module is isomorphic to a direct sum of copies of $\mathbb{F}_p$. Some more facts about $A$-modules, which we shall need, are given by Lemma (2.2.11).

Let 

$$Y = \left\{ y \in A \mid y = \sum_{i=1}^{d} (1-g_i) x_i \cup_i ; g_i \in \mathcal{X}, 0 \leq x_i \leq n_i-1 \right\}.$$ 

Clearly $Y$ is a set of nonzero and distinct elements of $A$. (Note that $(1-g_i)^{n_i} = 0$, because the order $n_i$ of $g_i$ is a power of $p$.) Since $G$ is the direct product of the $G_i$'s, the number of elements of $Y$ is the same as the order of $G$. We claim that $Y$ is a basis of $A$. For this it is sufficient to show that $Y$ spans $A$.

Let $x$ be an arbitrary element of $A$. Clearly 

$$x = \alpha_0 + x_1,$$

for some $\alpha_0 \in \mathbb{F}_p$ and $x_1 \in A$. Since $\{1-g_i, g_i \in \mathcal{X}\}$ is a generating set...
of \( g \), \( x_1 \) may be written as

\[ x_1 = \sum_{i} (1-g_i) a_i , \quad a_i \in A. \]

Rewriting each \( a_i \) as a sum of an element of \( \mathbb{F}^p \) and an element of \( g \), \( x \) may be written as

\[ x = \alpha_0 + \sum_i \alpha_i (1-g_i) + \sum_{i,j} (1-g_i) (1-g_j) a_{ij}, \quad \alpha_i \in \mathbb{F}^p, \quad a_{ij} \in A. \]

Continuing in this way, \( x \) may be rewritten as

\[ x = \sum_{y \in Y} \alpha_y y + \sum_k \left( \prod_{i=1}^{d} (1-g_i) \right) \left( \prod_{i=1}^{d} n_i \right) \left( 1-g_k \right), \]

where \( \alpha_y \in \mathbb{F}^p \). In this expression of \( x \), the second sum must be zero because \( k = i \) for some \( i \in A = \{1, 2, \ldots, d\} \), and \( (1-g_i)^{n_i} = 0 \). Therefore

\[ x = \sum_{y \in Y} \alpha_y y, \]

and so the claim follows.

Now we describe the desired bases for the Loewy factors of \( A \). For an element \( y \) of \( Y \) given as \( y = \prod_{i=1}^{d} (1-g_i)^{n_i} \), define \( \sum_{i=1}^{d} v_i \) to be the length of \( y \). Let \( l = \sum_{i=1}^{d} (n_i-1) \). Clearly the length of an arbitrary element of \( Y \) ranges from 0 to \( l \). Corresponding to \( k \), \( 0 \leq k \leq l \), let \( Y_k \) be the set of all elements of \( Y \) of length \( k \). Clearly each \( Y_k \) is nonempty and \( Y \) is the disjoint union of the \( Y_k \), \( 0 \leq k \leq l \). (Note that \( Y_0 = \{1\} \), and \( Y_l = \left\{ \prod_{i=1}^{d} (1-g_i)^{n_i-1} \right\} \). Let \( M_k \) be the \( \mathbb{F}^p \)-space spanned by \( Y_k \cup Y_{k+1} \cup \ldots \cup Y_l \). As above, it is easy to check that \( M_k \)
is a submodule of $A$. Moreover $Y_k \cap M_{k+1} = \emptyset$. We claim that $M_k = \phi^k A$.

Clearly $M_0 = \phi^0 A = A$. Suppose that $M_{k-1} = \phi^{k-1} A$, $1 \leq k \leq 2$. Then $
M_k = M_{k-1}/M_k$. Clearly, for all $g_j \in X$ and $x \in \phi^{k-1} A$,

$x(1-g_j) \in M_k$. Therefore $G$ acts trivially on $\phi^{k-1} A/M_k$, and so

$\phi^k A \subseteq M_k$. For the other inclusion, let $x \in M_k$. Clearly

$$x = \sum_{i,j} y_i (1-g_j) a_{ij},$$

for some $y_i \in Y_{k-1}$ and $a_{ij} \in A$. Then $y_i \in \phi^{k-1} A$; and so, for all $i$ and $j$,

$$\left(y_i + \phi^k A\right)(1-g_j) = y_i (1-g_j) + \phi^k A = \phi^k A,$$

because $G$ acts trivially on $\phi^{k-1} A/\phi^k A$. Therefore $x \in \phi^k A$, and so

$M_k \subseteq \phi^k A$. Thus $M_k = \phi^k A$, and hence the claim follows by induction.

Now, clearly, $Y_k$ is a generating set of $\phi^k A$. Therefore

$$\dim(\phi^k A/\phi^{k+1} A) \leq |Y_k|.$$ 

In fact this is an equality because

$$\dim A = |Y| = \sum_{k=0}^2 |Y_k|.$$ 

Thus $\left\{y + \phi^k A, y \in Y_k\right\}$ is a basis of $\phi^k A/\phi^{k+1} A$. Equivalently (Lemma (2.2.11)), $Y_k$ is a minimal generating set of $\phi^k A$.

For a fixed $i \in A$, let

$$Z^i = \{z \mid z = b_i (1-g_i) y, y \in Y, (1-g_i) y \neq 0\}.$$ 

Clearly $(1-g_i)y = 0$ if and only if $y = (1-g_i)^{n_i-1} y'$ for some $y' \in Y$;
the number of such $y$ in $Y$ is exactly $n/n_i$, where $n = |Y| = |\mathcal{C}|$.

Therefore

$$|Z^i| = \frac{n(n_i-1)}{n_i} = \dim B_i.$$ 

Define the length of an element $b_i(1-g_i)y$ of $Z^i$ to be the same as the length of the element $(1-g_i)y$ of $Y$, and $Z_k^i$ the set of all elements of $Z^i$ of length $k$. (Note that the length of an arbitrary element of $Z_k^i$ ranges from 1 to $l$.) As before, it may be shown that $Z^i$ is a basis of $B_i$ and $Z_k^i$ is a minimal generating set of $\varphi^{k-1}B_i$. Let

$$Z = \bigcup_{i=1}^{d} Z^i,$$

and

$$Z_k = \bigcup_{i=1}^{d} Z_k^i.$$ 

Then, in view of Lemma (2.2.1) (iii), $Z$ is a basis of $B$ and $Z_k$ is a minimal generating set of $\varphi^{k-1}B$.

Remark. The Loewy length of $A$ is equal to $l + 1$ and the Loewy length of $B$ is $l$, where $l = \sum_{i=1}^{d} (n_i-1)$.

Now we prove

(7.1.2) **LEMMA.** Let

$$T = \{ t_{i,j} \mid t_{i,j} = (b_i-b_j)(1-g_i)(1-g_j) ; i, j \in \Lambda, i < j \}.$$ 

Then $T$ is a minimal generating set of $\hat{S}$.

(Note that $|T| = \frac{d}{2}d(d-1)$.)

**Proof.** We first show that $T$ generates $\hat{S}$. Let $V$ be the module generated by $T$. Clearly $V\psi' = \{0\}$, where $\psi'$ is the epimorphism of (7.1.1). Therefore $V \subseteq \hat{S}$. For the other inclusion, it is sufficient to show that an arbitrary element of $\hat{S}$ may be expressed as a sum of an element of $V$ and an element of $\varphi^kB$, $0 \leq k \leq l$, because $\varphi^kB = \{0\}$. 

This we show by induction on $k$. The result is trivially true for $k = 0$.

Let $x$ be an element of $\hat{S}$. For a fixed $k$, $0 \leq k \leq l-1$, suppose that

$$x = v + x' \quad ; \quad v \in V , \quad x' \in \phi^k_B .$$

We want to show that $x$ may also be expressed as a sum of an element of $V$ and an element of $\phi^{k+1}B$. $x'$ may be expressed as an $A$-linear combination of the elements of $Z_{k+1}^+, \hat{S}$, the given generating set of $\phi^k_B$; that is

$$x' = \sum_{\lambda_1} b_1 (1-g_1)y_{\lambda_1} \alpha_{\lambda_1} + \cdots + \sum_{\lambda_d} b_d (1-g_d)y_{\lambda_d} \alpha_{\lambda_d} ,$$

where $y_{\lambda_i} \in Y_k$ and $\alpha_{\lambda_i} \in A$. Rewriting each $\alpha_{\lambda_i}$ as a sum of an element of $W^\perp$ and an element of $\bar{A}$, for some $x_1 \in \phi^{k+1}B$, $x'$ may be expressed as

$$x' = x_0 + x_1 ,$$

where

$$x_0 = \sum_{\lambda_1} \alpha_{\lambda_1} b_1 (1-g_1)y_{\lambda_1} + \cdots + \sum_{\lambda_d} \alpha_{\lambda_d} b_d (1-g_d)y_{\lambda_d} ,$$

$\alpha_{\lambda_i} \in W^\perp$. Clearly, in the expression of $x_0$, the elements $(1-g_i)y_{\lambda_i}$, $i \in \Lambda$, belong to $Y_{k+1}^+$; however, these elements need not be distinct, for example $(1-g_i)y_{\lambda_i} = (1-g_j)y_{\lambda_j}$ if and only if $y_{\lambda_i} = (1-g_j)y$ and $y_{\lambda_j} = (1-g_i)y$ for some $y \in Y_k$. If

$$x_\lambda = (1-g_i)y_{\lambda_i} = (1-g_j)y_{\lambda_j} = \cdots = (1-g_r)y_{\lambda_r} = (1-g_s)y_{\lambda_s} ,$$

let

$$\mu_\lambda = \alpha_{\lambda_i} b_i + \alpha_{\lambda_j} b_j + \cdots + \alpha_{\lambda_r} b_r + \alpha_{\lambda_s} b_s ,$$

Then $x_0$ may be expressed as

$$x_0 = \sum_\lambda \mu_\lambda x_\lambda .$$
where the \( x_\lambda \) are distinct elements of \( \mathbb{F}_{k+1} \).

Then

\[ x_0' = \sum_\lambda \beta_\lambda x_\lambda, \]

where

\[ \beta_\lambda = \alpha_\lambda(i) + \alpha_\lambda(j) + \ldots + \alpha_\lambda(r) + \alpha_\lambda(s). \]

Clearly

\[ 0 = x_0' = x_1psi' + x_1psi' = x_1psi, \]

and

\[ (\phi^{k+1}B)psi' \subseteq \phi^{k+1} \mathbb{F} \] (in fact this is an equality).

Therefore

\[ \phi^{k+1} \mathbb{F} = (x_0'psi' + x_1psi') + \phi^{k+1} \mathbb{F} = x_0psi' + \phi^{k+1} \mathbb{F} \]

\[ = \sum_\lambda \left( (\beta_\lambda x_\lambda) \phi^{k+1} \mathbb{F} \right), \]

because \( x_1psi' \in \phi^{k+1} \mathbb{F} \). Since the \( x_\lambda \) are distinct elements of \( \mathbb{F}_{k+1} \), each \( \beta_\lambda \) must be zero. Therefore

\[ \alpha_\lambda = -\alpha_\lambda(i) - \alpha_\lambda(j) \ldots -\alpha_\lambda(r), \]

and so

\[ \mu_\lambda x_\lambda = (\alpha_\lambda(i-b) + \alpha_\lambda(j-b) + \ldots + \alpha_\lambda(r-b))x_\lambda, \]

which is an element of \( V \). Thus \( x_0 \in V \). We have shown that

\[ x = v + x_0 + x_1 = v_1 + x_1, \quad v_1 \in V, \quad x_1 \in \phi^{k+1}B. \]

By induction \( x \in V \), and so \( \hat{S} \subseteq V \). Hence \( V = \hat{S} \).

To complete the proof, we now establish the minimality of \( T \). In view of Lemma (2.2.11), we only need to show that no proper subset of \( T \)
generates \( \hat{S} \). On the contrary suppose that \( T \) is not minimal. Then an element of \( T \) may be expressed as an \( A \)-linear combination of the rest of the elements of \( T \). That is,
\[
(b - b)(1-g)(1-g) = \sum \frac{(b - b)(1-g)(1-g)}{\sum a_{i,j} \in A} (\mu, \nu) \neq (i, j)
\]
\[
= \alpha, \text{ say.}
\]
Then
\[
b(1-g)(1-g) = \alpha + b(1-g)(1-g).
\]
Note that \( \{b(1-g)(1-g), b(1-g)(1-g); i, j \in \Lambda, i < j\} \) is a set of distinct elements of \( Z_2 \), the given minimal generating set of \( \phi B \).

Therefore \( b(1-g)(1-g) \), which is an element of \( Z_2 \), is an \( A \)-linear combination of the rest of the elements of \( Z_2 \). It means that \( Z_2 \) is not a minimal generating set of \( \phi B \), which is a contradiction. Therefore \( T \) is a minimal generating set of \( \hat{S} \).

Remark. Using \( T \), bases for the Loewy factors of \( \hat{S} \) may be constructed.

It follows that the Loewy length of \( \hat{S} \) is one less than that of \( B \) (or two less than that of \( A \)). Moreover, \( \dim(\phi B) = d - 1 \), and so \( \hat{S} \) can not be embedded in a free \( G \)-module of rank \( d - 2 \) (we already know (Proposition (3.3.9)) that \( \hat{S} \) can be embedded in a free module of rank \( d - 1 \)).

Next let
\[
C = \left\{ c_{i,j} \mid c_{i,j} = \frac{d}{\sum_{\lambda=1}^{\eta \lambda - 1} (1-g)^{\lambda}} ; i, j \in \Lambda, i < j \right\}
\]
Then
\[
|C| = \frac{1}{2} d(d - 1).
\]
(In fact \( C \) is a subset of \( \sigma \hat{S} \) and a suitable subset of \( C \), for example \( \{c_{i,d}, 1 \leq i \leq d - 1\} \), is a basis of \( \sigma \hat{S} \).)

We prove
(7.1.3) LEMA. Let $V$ be a submodule of $\hat{S}$. Suppose that 
$$\dim((V+\phi\hat{S})/\phi\hat{S}) = r,$$
where $0 \leq r \leq \frac{1}{2}d(d-1)$. Then $V$ contains at least $r$ elements of $C$.

Proof. If $r = 0$, there is nothing to prove; so suppose that $r > 0$. We may choose a set \( \{v_m \mid v_m \in V, 1 \leq m \leq r\} \) such that \( \{v_m + \phi\hat{S}, 1 \leq m \leq r\} \) is a basis of \((V+\phi\hat{S})/\phi\hat{S}\). As $T$ is a (minimal) generating set of $\hat{S}$, each $v_m$ may be written as
\[
v_m = \sum_{t_{i,j} \in T} \alpha_{m,i,j} t_{i,j} + \omega,
\]
where $\alpha_{m,i,j} \in F$ and $\omega \in \phi\hat{S}$. Clearly, in the expression of each $v_m$, at least one $\alpha_{m,i,j}$ must be nonzero. Also, at least $r$ of the $t_{i,j}$ for which $\alpha_{m,i,j}$ are nonzero must be distinct. Suppose that $\alpha_{m,i,j} \neq 0$, and let
\[
y = \prod_{\lambda=1}^{d} (1-g_\lambda)^{\lambda}
\]
where
\[
u_\lambda = \begin{cases} 
  n_\lambda - 2, & \text{if } \lambda = i \text{ or } \lambda = j, \\
  n_\lambda - 1, & \text{otherwise.}
\end{cases}
\]
Then
\[
v_m y = \alpha_{m,i,j} (b_i - b_j) \prod_{\lambda=1}^{d} (1-g_\lambda)^{n_\lambda - 1} (wy = 0)
\]
\[
= \alpha_{m,i,j} c_{i,j};
\]
and so,
\[
(a_{m,i,j})^{-1} v_m y = c_{i,j} \in V.
\]
Since at least $r$ of the $t_{i,j}$ are distinct, $V$ contains at least $r$ distinct elements $c_{i,j}$ of $C$. This completes the proof.

(7.1.4) PROPOSITION. $\hat{S}$ is indecomposable.
Proof. Suppose that

\[ \mathcal{S} = V_1 \oplus V_2 , \]

and

\[ \dim(V_i/\phi V_i) = r_i, \quad i = 1, 2. \]

We know that \( T \) is a minimal generating set of \( \mathcal{S} \), and \( |T| = \frac{1}{2}d(d-1) \).

Therefore, by Lemmas (2.2.1) (iii) and (2.2.11),

\[ \dim(\mathcal{S}/\phi \mathcal{S}) = \frac{1}{2}d(d-1) = r_1 + r_2. \]

Since \( V_i/\phi V_i = V_i/(V_i \cap \phi \mathcal{S}) \cong (V_i + \phi \mathcal{S})/\phi \mathcal{S} \), \( V_i \) contains a set \( C_i \) of at least \( r_i \) elements of \( C \) by Lemma (7.1.3). In view of the above decomposition, \( C_i \) must contain precisely \( r_i \) elements. In fact

\[ C = C_1 \cup C_2, \quad \text{and} \quad C_1 \cap C_2 = \emptyset. \]

We pick the element \( a_{1d} \) of \( C \) and suppose without loss of generality that \( a_{1d} \in C_1 \). Then we claim that \( C \subseteq V_1 \). Clearly

\[ a_{1d} = a_{1k} + a_{kd}, \quad 1 < k < d. \]

There are four possibilities, namely:

(i) \( a_{1k}, a_{kd} \in V_1 \);

(ii) \( a_{1k}, a_{kd} \in V_2 \);

(iii) \( a_{1k} \in V_1, a_{kd} \in V_2 \);

(iv) \( a_{1k} \in V_2, a_{kd} \in V_1 \).

We establish (i) by eliminating the rest. Suppose that (ii) is true. Then \( a_{1d} \in V_2 \), which is a contradiction; so (ii) is not true. Suppose that (iii) is true. Then \( a_{kd} = a_{1d} - a_{1k} \), and so \( a_{kd} \in V_1 \). This is also a contradiction. Similarly (iv) is not true. Thus \( a_{1d}, a_{1k}, a_{kd}, \) \( 1 < k < d \), are elements of \( V_1 \). Using the above method, we may show that the other elements of \( C \) also belong to \( V_1 \), nevertheless the result
follows because the elements of $C$ are linear combinations of the elements of $C^* = \{c_{id}, 1 \leq i \leq d-1\}$. Thus $C \subseteq V_1$, and so $r_2 = 0$. That is, $V_2 = \emptyset V_2$; so $V_2 = \{0\}$. Therefore $V = V_1$, and hence $V$ is indecomposable.

Remark. In the above proof, using the fact that $C^*$ is a basis of $\sigma S$, we could conclude that $\sigma S = \sigma V_1$, and thereby complete the proof.

7.2 Nonprojectivity and indecomposability in the general case

In this section we return to the situation where $G$ is possibly nonabelian, and prove that $\hat{S}$ is indecomposable.

The sequence (3.3.10) is of great help, yielding the following two short exact sequences:

(7.2.1) $0 \rightarrow M \rightarrow \bigoplus_{i=1}^{d} T^G_i \xrightarrow{\beta} \mathbb{F}_p \rightarrow 0$,

where

$$\left(t_i \sum_{g \in G} k_g\right) \beta = \sum_{g \in G} k_g, \quad t_i = (1-g_i)^{n_i-1}, \quad i \in \Lambda;$$

(7.2.2) $0 \rightarrow \hat{S} \rightarrow L \xrightarrow{\mu} M \rightarrow 0$,

where $\mu$ is the same as in (3.3.10) and $L$ is a free $G$-module of rank $d-1$.

Note. Each $T^G_i$ is indecomposable by Corollary (2.2.10), because $T^G_i$ is embedded in $\mathbb{F}_p G$.

(7.2.3) **Lemma.** $\hat{S}$ and $M$ are nonprojective and do not contain any projective summands.

**Proof.** By Proposition (3.2.3), $\hat{S}$ is a homomorphic image of $\hat{R}$. Therefore, if $\hat{S}$ contains a projective summand, so does $\hat{R}$. We know that $\hat{R}$ is nonprojective and indecomposable. Therefore $\hat{S}$ is nonprojective and does not contain any projective summands.
Now suppose that $M$ contains a projective summand. Then $M$, and so
\[\bigoplus_{i=1}^{d} T_i^G,\]
contain $\mathbb{F} G$ as an indecomposable summand. This is not possible because the indecomposable summands of $\bigoplus_{i=1}^{d} T_i^G$ are $T_i^G$, and $\dim T_i^G < \dim \mathbb{F} G$. Therefore $M$ does not contain a projective summand.

In view of (7.2.2) and Lemma (7.2.3), Heller's Proposition (2.2.9) yields

(7.2.4) **LEMMA.** $S$ is indecomposable if and only if $M$ is indecomposable.

Proposition (7.1.4) yields

(7.2.5) **COROLLARY.** If $G$ is not abelian, $M$ is indecomposable.

The idea is to prove that $M$ is indecomposable for nonabelian $G$. First we describe a generating set of $M$ which will help us to prove the desired result.

(7.2.6) **LEMMA.** Let $\Gamma = \{ (t_i - t_u) \mid i, u \in \Lambda, u \text{ is fixed} \}$. Then $\Gamma$ is a generating set of $M$.

**Proof.** Let $V$ be the module generated by $\Gamma$. Clearly $V \subseteq \ker \beta = M$.

For the other inclusion, let $m \in M$. Since $\{ t_i, i \in \Lambda \}$ is a generating set of $\bigoplus_{i=1}^{d} T_i^G$, $m$ may be written as

\[m = \sum_{i} t_i a_i, \quad a_i \in \mathbb{F} G.\]

Rewriting each $a_i$ as a sum of an element of $\mathbb{F} G$ and an element of $\mathcal{g}$, we have

\[m = m_1 + m_2,\]

where

\[m_1 = \sum_{i} \alpha_i t_i, \quad \alpha_i \in \mathbb{F} G,\]
and

\[ m_2 = \sum_{i,j} t_i (1-g_j) a_{i,j}, \ a_{i,j} \in \mathbb{F} G. \]

We first show that \( m_2 \in V \). Since \( t_j (1-g_j) = (1-g_j)^n_j = 0 \), \( m_2 \) may be rewritten as

\[ m_2 = \sum_{i,j} (t_i - t_j) (1-g_j) a_{i,j}. \]

Since each \( (t_i - t_j) \) is an \( \mathbb{F} \)-linear combination of the elements of \( \Gamma \), \( m_2 \) is an element of \( V \). Now we show that \( m_1 \in V \). Clearly

\[ 0 = m_2 = m_1 \beta + m_2 \beta , \text{ and } m_2 \beta = 0. \]

Therefore

\[ m_1 \beta = \sum_i \alpha_i = 0, \]

and so

\[ \alpha_j = -\sum_i \alpha_i, \ j \neq i. \]

Thus

\[ m_1 = \sum_i a_i (t_i - t_j), \ i \neq j, \]

and so \( m_1 \in V \). Hence the result follows.

Remark. It may be shown that \( \Gamma \) is a minimal generating set of \( M \). In the abelian case, using \( \Gamma \), a result analogous to Lemma (7.1.2) may also be proved, which would also give us the indecomposability of \( M \).

Now we prove

(7.2.7) THEOREM. \( \hat{S} \) is indecomposable

Proof. Equivalently, we shall prove that \( M \) is indecomposable. In view of Corollary (7.2.5) it is only necessary to consider the case that \( G \) is not elementary abelian.
Let $Z$ be a normal subgroup of $G$ contained in $G/Z$, and

$$Y = \{h_i \mid h_i = zg_i, g_i \in X\}.$$ 

We know that $X$ is a minimal generating set of $G$ and $\Phi(G)$ is contained in $\Phi(G)$, therefore $Y$ is a minimal generating set of $H$. Let $H_i$ be the cyclic subgroup of $H$ generated by $h_i$, and $S_i$ the trivial irreducible submodule of $HF_i$. We know that $\{t_i x, x \in C_i\}$ is a basis of $S_i^{H}$, where $C_i$ is a (right) transversal of $G_i$ in $G$. Let $s_i$ be the sum of all distinct elements of $H_i$, and $\overline{C}_i$ a transversal of $H_i$ in $H$. Then

$$\{s_i \overline{x}, \overline{x} \in \overline{C}_i\}$$

is a basis of $S_i^H$. Let

$$(7.2.8) \quad 0 \to N \to \bigoplus_{i=1}^{d} S_i^H \to \mathbb{F} \to 0$$

be the analogue of (7.2.1), so that $M = N$ if $Z = \{1\}$.

The idea is to prove that $N$ is indecomposable for all $Z$, which would of course give us the required result. This will be proved by induction on $e = |\Phi G/Z|$. For $e = 1$, the result follows by Corollary (7.2.5) because $Z = \Phi G$ and $H$ is abelian. Suppose that the result is true for all $e$ less than $|\Phi G|$. In other words, $N$ is indecomposable for all $Z$ other than $Z = \{1\}$. In particular, let $Z$ be a fixed cyclic group of order $p$ contained in $G \cap Z(G)$ (recall that such a $Z$ always exists). Accordingly, we fix $H$ and $N$. By the induction hypothesis $N$ is indecomposable. To complete the proof, we need to show that $M$ is indecomposable. For this, we first describe a relationship between $M$ and $N$. We need to consider the following two cases.

Case (i). $G_i \cap Z = Z$. Let $C_i$ be a fixed transversal of $G_i$ in $G$. Then $\overline{C}_i = \{\overline{x} \mid \overline{x} = Zx, x \in C_i\}$ is transversal of $H_i$ in $H$, because $G_iZ = G_i$. Define $ug = u\overline{g}$, where $u \in S_i^H$, $g \in G$, and $\overline{g} = Zg$. This
is a $G$-action on $S^H_i$ and
\[ \delta : T^G_i \to S^H_i, \]
such that
\[ \left( \sum_{x \in C_i} a_i x \right) \delta = \sum_{x \in C_i} a_i s_{i, x}, \quad a_i \in \mathbb{F}_p, \]
is a $G$-isomorphism.

Case (ii). $G_i \cap Z = \{1\}$ . Let $D_i$ be a complete set of $(Z, G_i)$ double coset representatives in $G$ . By definition
\[ G = \bigcup_{x \in D_i} G_i x Z, \]
and
\[ x_1 = x_2 \text{ if and only if } G_i x_1 Z = G_i x_2 Z, \quad x_1, x_2 \in D_i. \]

We show that $C_i = \{ xz \mid x \in D_i, z \in Z \}$ is a transversal of $G_i$ in $G$ .

Let $g \in G$ . Clearly $g = g'xz$ for some $g' \in G_i, \ x \in D_i$ and $z \in Z$ .

For some $x_1, x_2 \in D_i$ and $z_1, z_2 \in Z$ , suppose that $G_i x_1 z_1 = G_i x_2 z_2$ .

Then $G_i x_1 z = G_i x_2 z$ , and so $x_1 = x_2$ . Thus $x_1 z_1 = g'' x_2 z_2$ , for some $g'' \in G_i$ . Then $z_1^{-1} z_2^{-1} = x_1^{-1} g'' x_1 \in Z \cap G_i = \{1\}$ , and so $z_1 = z_2$ .

Therefore $G_i x_1 z_1 = G_i x_2 z_2$ if and only if $x_1 = x_2$ and $z_1 = z_2$ . Hence $C_i$ is a transversal of $G_i$ in $G$ . Now it follows that
\[ \bar{C}_i = \{ \bar{x} \mid \bar{x} = Zx, \ x \in D_i \} \]
is a transversal of $H_i$ in $H$ . Define a $G$-action on $S^H_i$ as in Case (i) and define
\[ \lambda : T^G_i \to S^H_i \]
such that
\[
\left( \sum_{x \in D_i} \alpha_t xz \right) \lambda = \sum_{x \in D_i} \alpha_s xz, \quad \alpha_t, \alpha_s \in \mathbb{IF}_p.
\]

Clearly \( \lambda \) is an \( \mathbb{IF}_p \)-epimorphism; we show that \( \lambda \) is also a \( G \)-homomorphism. Let \( x \in G \). Then \( (t_i xz)g = t_i xzg = t_i z g \cdot xz = t_i xz \), where \( xz = g_1 xz_1, \quad z_2 = z_2, \quad g_1 \in G, \quad xz \in D_i \) and \( z_2 \in Z \). Clearly

\[
x_1 = z_1^{-1} g_1 xz, \quad \text{and} \quad z_1 = z_1 = zg_1 xz = h_1 xz,
\]

where \( h_1 = zg_1 \in H \). Therefore \( s \cdot x_1 = s \cdot xz \), and so

\[
(t_i xz)g) \lambda = (t_i xz)\lambda = (s \cdot xz)g = (s \cdot xz)g = ((t_i xz)\lambda)g.
\]

Hence \( \lambda \) is a \( G \)-epimorphism. We wish to find the kernel of \( \lambda \). By Mackey's subgroup theorem ([2], Theorem (44.2), p. 324),

\[
\left\{ \begin{array}{c}
T^G_i \\
Z
\end{array} \right\} = \bigoplus_{x \in D_i} \left\{ \begin{array}{c}
T_i \otimes x \\
G \otimes Z
\end{array} \right\}.
\]

Since \( G \cap Z = (G \cap Z)^x = \{ 1 \} \),

\[
\left\{ \begin{array}{c}
T^G_i \\
Z
\end{array} \right\} = \bigoplus_{x \in D_i} \left\{ \begin{array}{c}
T_i \otimes x \\
\{ 1 \}
\end{array} \right\} = \bigoplus_{x \in D_i} t_i Ux, \text{ say.}
\]

Since \( Z \) is a cyclic group of order \( p \), each \( t_i Ux \) is a free \( \mathbb{IF}_p \)-module of rank one and the Loewy series of \( t_i Ux \) is uniserial of length \( p \), with each factor a trivial irreducible module. Consequently,

\[
\varphi(t_i Ux) = \sigma^{p-1}(t_i Ux) \cong \varphi(\mathbb{IF}_p Z) = \mathbb{IF}_p Z,
\]

and so

\[
T^G_i (1-Z) = \bigoplus_{x \in D_i} \varphi(t_i Ux) = \bigoplus_{x \in D_i} \sigma^{p-1}(t_i Ux).
\]
Since $T_i^G$ is a $G$-module and $Z$ is central, $T_i^G(1-z)$ is also a $G$-module with $t_i(1-z)$ as its generator. In fact $T_i^G(1-z) = \text{Ker } \lambda$, because $(t_i(1-z)\alpha)\lambda = 0$, for all $\alpha \in W^G$, and
\[
\dim T_i^G - \dim \left( T_i^G(1-z) \right) = p|D_i| - ((p-1)|D_i|)
\]
\[
= \dim S_i^H.
\]

Now we are ready to associate $M$ and $N$ in an exact sequence.

Without loss of generality, we may suppose that
\[
G_i \cap Z = \begin{cases} 
\{1\}, \text{ if } i = 1, 2, ..., r, \ r \leq d, \\
Z, \text{ if } i = r+1, ..., d.
\end{cases}
\]

Define
\[
\theta : \bigoplus_{i=1}^d T_i^G \rightarrow \bigoplus_{i=1}^d S_i^H
\]
such that $\theta$ restricted to $T_i^G$ is

(a) the homomorphism $\lambda$ of Case (ii), if $1 \leq i \leq r$,

(b) the isomorphism $\delta$ of Case (i), if $r+1 \leq i \leq d$.

Let $W$ be the kernel of $\theta$. Clearly
\[
W = \bigoplus_{i=1}^r T_i^G(1-z).
\]

Moreover, using the sequences (7.2.1) and (7.2.8), it is easy to check that $M\theta = N$ and that $W$ is also the kernel of $\theta$ restricted to $M$. Thus
\[
M/W \cong N.
\]

The main argument of the proof is in the following two cases.

Case (1). $r < d$. Since $Z$ is central in $G$, it follows easily that $M(1-z)$ is a $G$-module with $\Gamma(1-z) = \{y(1-z), y \in \Gamma\}$ a generating set, where $\Gamma = \{(t_i-t_d); 1 \leq i \leq d-1\}$ is a minimal generating set of $M$ by
Lemma (7.2.6). As $r < d$, we have $G_d \cap Z = Z$. Then $t_d^{(1-z)} = 0$, and so

$$
\left( t_i^{(1-z)} - t_d^{(1-z)} \right) = \begin{cases} 
  t_i^{(1-z)}, & \text{if } 1 \leq i \leq r, \\
  0, & \text{otherwise}.
\end{cases}
$$

Thus $\Gamma^{(1-z)} = \{ t_i^{(1-z)}, 1 \leq i \leq r \}$, which is also a generating set of $W$. Therefore $M^{(1-z)} = W$.

Now suppose that $M$ is decomposable, that is

$$M = M' \oplus M''$$

Then $M^{(1-z)} = M'(1-z) \oplus M''(1-z)$ (as $G$-modules), and so

$$N \cong M/W = M/M^{(1-z)} = M'/M'(1-z) \oplus M''/M''(1-z).$$

Since $N$ is indecomposable, one of the above summands of $N$ must be zero. Suppose without loss of generality that $M' = M'(1-z)$. Then

$M' = M'(1-z)^P = \{0\}$. Hence $M = M''$, and so $M$ is indecomposable.

Case (2). $r = d$. Let

$$V = \bigoplus_{i=1}^{d} t_i^{(1-z)}$$

Then

$$V_Z = \bigoplus_{i=1}^{d} \left( \bigoplus_{x \in D_i} (t_i^{(1-z)}x) \right),$$

and

$$\varphi(V_Z) = W_Z = \bigoplus_{i=1}^{d} \left( T_i^{(1-z)} \right)$$

$$= \bigoplus_{i=1}^{d} \left( \bigoplus_{x \in D_i} \varphi(t_i^{(1-z)}x) \right)$$

$$= \bigoplus_{i=1}^{d} \left( \bigoplus_{x \in D_i} \sigma^{p-1}(t_i^{(1-z)}x) \right)$$

$$= \sigma^{p-1}(V_Z) \text{ (by Lemma (2.2.1) (iv)).}$$
(Note that if $\sigma^{p-1}(V_Z)$ is regarded as $G$-module, the indecomposable $G$-summands (up to isomorphism) of $\sigma^{p-1}(V_Z)$ are $T_G^{T_0}(1-\alpha)$, $i \in \Lambda$; this fact will be needed later in the proof.)

Let

$$k = \sum_{i=1}^{d} |D_i| .$$

Clearly $V_Z$ is a free $\mathbb{Z}$-module of rank $k$, with $\{t_i x; x \in D_i, i \in \Lambda\}$ as a free $\mathbb{Z}$-generating set. Let $\theta$ be the $\mathbb{F}_p$-$\mathbb{Z}$-epimorphism of $V_Z$ onto $\mathbb{F}_p$ determined by the mapping $t_i x \mapsto 1$, for all $t_i x$, and let $U$ be the kernel of $\theta$. Clearly, $U$ is a free $\mathbb{Z}$-module of rank $k-1$. Let $\rho$ be the augmentation of $\mathbb{F}_Z$. With the epimorphism $\beta$ of (7.2.1) it may be verified that $\beta = \theta \rho$. Then it follows that $(M_Z)\theta = \mathbb{Z}$, and $U$ is the kernel of $\theta$ restricted to $M_Z$. That is, $M_Z/U \cong \mathbb{Z}$. In fact

$$M_Z = U \oplus V_0, \quad V_0 \cong \mathbb{Z},$$

because $U$ is a free $\mathbb{Z}$-module. Then

$$\sigma^{p-1}(M_Z) = \sigma^{p-1}(V_Z),$$

because $\sigma^{p-1}(M_Z) \subseteq \sigma^{p-1}(V_Z)$ and (by Lemma (2.2.1) (iv)),

$$\dim(\sigma^{p-1}(M_Z)) = \dim(\sigma^{p-1}U) + \dim(\sigma^{p-1}V_0)$$

$$= (k-1)(p-1) + \dim V_0$$

$$= (k-1)(p-1) + (p-1)$$

$$= k(p-1)$$

$$= \dim(\sigma^{p-1}(V_Z)).$$

Now suppose that
\[ M = M' \oplus M'' \text{ (as } G\text{-modules)}. \]

Then

\[ M_Z = M'_Z \oplus M''_Z. \]

By the Krull Schmidt theorem we may suppose that \( M'_Z \) is a free \( Z \)-module of rank \( k' \) and

\[ M''_Z = U_1 \oplus V_1, \quad V_1 \cong V_0, \]

where \( U_1 \) is a free \( Z \)-module of rank \( (k-1-k') \). By Lemma (2.2.1) (iv),

\[ \sigma^{p-1}(M'_Z) = \sigma^{p-1}(M'_Z) \oplus \sigma^{p-1}(M''_Z). \]

Clearly

\[ \sigma^{p-1}(M'_Z) = \{ m \in M' \mid m(1-z)^{p-1} = 0 \}. \]

Moreover, for any \( m \in \sigma^{p-1}M'_Z \) and \( g \in G \), \( mg \in \sigma^{p-1}(M'_Z) \) because

\[ mg(1-z)^{p-1} = m(1-z)^{p-1}g = 0. \]

Therefore \( \sigma^{p-1}(M'_Z) \) is a \( G \)-module.

Similarly, \( \sigma^{p-1}(M''_Z) \) is a \( G \)-module. Then

\[ N \cong M/W = M/\sigma^{p-1}(M'_Z) \]

\[ = M'/\sigma^{p-1}(M'_Z) \oplus M''/\sigma^{p-1}(M''_Z). \]

Since \( N \) is an indecomposable \( G \)-module, one of the following must be true:

(a) \( M' = M'_Z = \sigma^{p-1}(M'_Z) \);

(b) \( M'' = M''_Z = \sigma^{p-1}(M''_Z) \).

Clearly (a) is only possible when \( M'_Z = \{0\} \). Therefore suppose that (b) is true. Then

\[ \sigma^{p-1}(M''_Z) = \sigma^{p-1}U_1 \oplus V_1 = M''_Z \]

\[ = U_1 \oplus V_1. \]
Again this is only possible when $U_1 = \{0\}$. Therefore

$$M'' = M''_Z = \sigma^{p-1}(M''_Z) = V_1.$$  

Then, clearly, $M''$ is an indecomposable $Z$-summand of $\sigma^{p-1}(M''_Z) = \sigma^{p-1}(V_Z)$. Therefore $M''$ must also be an indecomposable $G$-summand of $\sigma^{p-1}(V_Z)$. We know that the indecomposable $G$-summands of $\sigma^{p-1}(V_Z)$ are isomorphic to $T^{G}_{\lambda}(1-z)$, $i \in \Lambda$, and so $\dim M'' = \dim T^{G}_{\lambda}(1-z)$, for some $i \in \Lambda$. We know that $\dim M'' = p-1$ and $\dim T^{G}_{\lambda}(1-z) = (p-1)|D_i|$, where $D_i$ is a complete set of $(Z, G_i)$ double coset representatives in $G$. Therefore $D_i$ must contain only one element, and so $G = G_iZ$. Then

$$G = G_i \times Z,$$

and so $Z \nsubseteq \Phi G$. This is, of course, a contradiction. Therefore $M'' = \{0\}$, and hence $M = M'$. Thus $M$ is indecomposable. This completes the proof of the theorem.
CHAPTER 8

EPILOGUE

In this chapter we shall examine the way in which some of the results of previous chapters generalise for relative relation modules in the case when the cyclic factors of the free product $E$ of (3.1.2) are no longer restricted to be the generators of $G$. The main reason for considering this generalisation separately in this chapter is that we have not been able to find a suitable generalisation of some of the main results of previous chapters (Chapter 7 in particular).

Unless otherwise stated, we shall use the notation and definitions of previous chapters. Recall that, in Section 3.1, $X = \{g_i, 1 \leq i \leq d\}$ is a generating set of $G$, $G_i$ the cyclic group generated by $g_i$, and $n_i$ the order of $G_i$.

Throughout this chapter, let $E_i$ be a cyclic group of order $m_i$, generated by $e_i$, where $m_i = n_i k_i$, $1 \leq k_i \leq \infty$, and let $E$ be the free product of the $E_i$, $1 \leq i \leq d$. (Of course, if each $E_i$ is infinite, $E$ is a free group of rank $d$.) Let $\psi$ be the epimorphism of $E$ onto $G$ determined by the mapping $e_i \mapsto g_i$, $1 \leq i \leq d$, and let

(8.1) \[ 1 \to S \to E \xrightarrow{\psi} G \to 1 \]

be the resulting short exact sequence. For a fixed prime $p$, let $\hat{R}$ be the relation module (modulo $p$) determined by (3.1.1) and $\hat{S}$ the relative relation module (modulo $p$) determined by (8.1).

The results of Section 3.1 are still true except that here $E$ and $S$ are different groups from what we had there and that $S$ in the present case need not be a free group.
In order to describe the structure of $\hat{S}$, we suppose without loss of generality that

(a) $k_i$ is finite and $p$ does not divide $k_i$, if $i \leq \delta$,

$\delta \leq d$; and

(b) $k_i$ is infinite or $p$ divides $k_i$, if $\delta + 1 \leq i \leq d$.

Let $\theta'$ be the epimorphism of $\hat{R}$ onto $\hat{S}$ as in (3.1.12). Then the kernel $\hat{Q}$ of $\theta'$ is the normal closure of $\left\langle f_i^{m_i}R' R^p, 1 \leq i \leq \delta \right\rangle$. As in Section 3.2, it may be shown that $\hat{Q} \cong \bigoplus_{i=1}^{\delta} T_i^G$. Then

$$0 \to \bigoplus_{i=1}^{\delta} T_i^G \to R \xrightarrow{\theta'} S \to 0$$

is an exact sequence, a generalisation of (3.2.4). Now the dimension of $\hat{S}$ is given by the formula

$$\text{dim} \hat{S} = n(d-1) - \left( \sum_{i=1}^{\delta} \frac{n_i}{n} \right) + 1 \quad (n = |G|).$$

In the same way as (3.3.7), it may be shown that

$$0 \to \hat{S} \to \bigoplus_{i=1}^{\delta} G_i^G \oplus \bigoplus_{i=\delta+1}^{d} b_i \otimes G \xrightarrow{\psi'} \hat{G} \to 0$$

is an exact sequence, where $\psi'$ is the $\hat{G}$-extension of the mapping

$$(1-g_i) \to (1-g_i), \text{ if } 1 \leq i \leq \delta;$$

$$b_i \to (1-g_i), \text{ if } \delta + 1 \leq i \leq d.$$ Of course (8.4) reduces to the relation sequence (3.3.5) if $\delta = 0$, and to the relative relation sequence (3.3.7) if $\delta = d$. (8.4) will be called the relative relation sequence determined by (8.1).

In the case when $\delta = 0$, $\hat{S}$ being a relation module can not be embedded in a free $\hat{G}$-module of rank $d - 1$ (see (8.3)). However, in the case when $\delta = d$, we have a sequence (3.3.10) in which $\hat{S}$ is embedded in a
free $\mathbb{F}_p G$-module of rank $d - 1$. The proof of (3.3.10) does not generalise
in the case when $0 < \delta < d$ (because the square analogous to $\mathbb{F}$ (in the
proof of (3.3.10)) does not commute), and we do not know in this case
whether there exists a sequence analogous to (3.3.10) (or even an embedding
of $\hat{S}$ into a free module of rank $d - 1$).

In the coprime case $\hat{S}$ may be characterised by

(8.5). Suppose that $p$ is coprime to the order of $G$. Then

(i) $\hat{S} \oplus \mathbb{F}_p^G \cong \bigoplus_{i=1}^{\delta} \mathbb{F}_p^G \oplus b_i \mathbb{F}_p^G$,

(ii) $\hat{S} \oplus T_i^G \cong \hat{R}$, and

(iii) if $\delta < d$, $\hat{S} \cong \bigoplus_{i=1}^{\delta} \mathbb{F}_p^G \oplus b_i \mathbb{F}_p^G$ (if $\delta = d$, $\hat{S}$ may be
described as in (4.1.2) (iii))

(i) and (ii) follow by application of Maschke's theorem to (8.4) and (8.2),
and (iii) follows from (i). Note that $\hat{S}$ in the coprime case contains
exactly $d - \delta$ copies of the trivial module $\mathbb{F}_p^G$; this follows from
(8.5) (i). Also in this case if $\delta \geq 1$, $\hat{S}$ is embedded in a free $\mathbb{F}_p G$
module of rank $d - 1$, which also follows from (8.5) (i).

If $p$ divides the order of $G$ but does not divide the orders of the
$G_i$, $1 \leq i \leq \delta$, we say that $p$ is semicoprime to the order of $G$
(compare this with definition (4.2.6)).

The following result characterises $\hat{S}$ in the semicoprime case, and, by
using (8.4), follows as for Proposition (4.2.7).

(8.6). Suppose that $p$ is coprime to the order of $G$, and suppose that
the hypothesis (4.2.2) is true. Then

$$\hat{S} \oplus V \oplus T_j^G \cong \bigoplus_{i=1}^{\delta} \mathbb{F}_p^G \oplus \bigoplus_{i=1}^{d-1} b_i \mathbb{F}_p^G \oplus \mathbb{F}_p^G \oplus W,$$
for any fixed $j \leq \delta$; and $J$ is nonprojective, nonzero and indecomposable. (If $\delta < d$, the above isomorphism reduces to

$$\hat{S} \oplus V \cong J \oplus \bigoplus_{i=1}^{\delta} G \oplus \bigoplus_{i=1}^{d-1} b_i \mathbb{F} G .$$

Of course, (8.6) reduces to Theorem (4.2.3) if $\delta = 0$, and to Proposition (4.2.7) if $\delta = d$. (8.6) also shows that the nonprojective part (see definition (4.2.9)) of $\hat{S}$ in the semicoprime case is isomorphic to the nonprojective part of $\hat{R}$.

An unrefinable decomposition of $\hat{S}$ in the semicoprime and coprime case may be obtained by Proposition (6.1.3) and Corollary (6.1.4).

Let $X_1 = \{g_i, 1 \leq i \leq d_1\}$, $X_2 = \{h_i, 1 \leq i \leq d_2\}$ be generating sets of $G$, $G_i$, $H_i$ the cyclic groups generated by $g_i$ and $h_i$ respectively, and $\hat{S}_1, \hat{S}_2$ the relative relation modules with the relative relation sequences

$$0 \rightarrow \hat{S}_1 \rightarrow \bigoplus_{i=1}^{\delta_1} G \oplus \bigoplus_{i=\delta_1+1}^{d_1} b_i \mathbb{F} G \xrightarrow{\psi_1} \mathbb{F} G \rightarrow 0 ,$$

$$0 \rightarrow \hat{S}_2 \rightarrow \bigoplus_{i=1}^{\delta_2} G \oplus \bigoplus_{i=\delta_2+1}^{d_2} b_i \mathbb{F} G \xrightarrow{\psi_2} \mathbb{F} G \rightarrow 0 .$$

A generalisation of Proposition (5.1.2) may be given as (8.9). Suppose that $d_1 = d_2 = d$, and $\delta_1 = \delta_2 = \delta$. Let $x$ be a fixed element of $G$ such that $H_i = G_i^x$, $1 \leq i \leq \delta$. Then $\hat{S}_1 \cong \hat{S}_2$.

As for Proposition (5.1.2), we divide the proof of (8.9) into two parts:

(i) $\hat{S}_1 \cong \hat{S}_2$ if $h_i = g_i^x$, $1 \leq i \leq \delta$; \left\{g_i^x \right\} = G_i$;

(ii) $\hat{S}_1 = \hat{S}_2$ if $h_i = g_i^x$, $1 \leq i \leq \delta$. 
The proof of (8.9) (i) is the same as that of (5.1.2) (i). For the proof of
(8.9) (ii), define
\[
\xi : \bigoplus_{i=1}^{\delta} \frac{G}{h_i^i} \oplus b_i \frac{G}{P} \to \bigoplus_{i=1}^{\delta} \frac{G}{h_i^i} \oplus b_i \frac{G}{P}
\]
such that \(\xi\) restricted to \(\frac{G}{h_i^i}\) is the same as for (5.1.2) and \(\xi\)
restricted to \(b_i \frac{G}{P}\) is the isomorphism given by \(b_i a + b_i ax\), for all
\(a \in \frac{G}{P}\). Then the result follows as for Proposition (5.1.2).

In the same way as Proposition (5.2.4) and Corollary (5.2.2), we have
(8.10). Let \(X'_1 = \{g_i^i, 1 \leq i \leq \delta_1\}\), \(X'_2 = \{h_i^i, 1 \leq i \leq \delta_2\}\) and suppose
that \(d_2 = d_1 + 1\). Then

(i) \(\hat{S}_2 \cong \hat{S}_1 \oplus \frac{G}{P}\) if \(X'_1 = X'_2\), and

(ii) \(\hat{S}_2 \cong \hat{S}_1 \oplus \frac{G}{h}\) if \(X'_2 = X'_1 \cup \{h\}\) and if either \(h \in X'_1\) or
\(p\) does not divide the order of \(\langle h \rangle\).

For the rest of this chapter we suppose that \(G\) is a \(p\)-group and \(\hat{S}\)
is minimal (see Definition (4.2.9)).

Recall that, if \(\delta = 0\), \(\hat{S}\) being a relation module is nonprojective
and indecomposable, and if \(\delta = d\), the same result is proved in Chapter 7.
It is true that even in the general case \(\hat{S}\) is nonprojective and does not
contain any projective summands; the result follows as in Lemma (7.2.3).
However it is not known in the general case whether \(\hat{S}\) is indecomposable as
well. Recall that when \(\delta = d\), the proof of the indecomposability of \(\hat{S}\)
given by Theorem (7.2.7) relies on both the indecomposability in the abelian
case and the sequence (3.3.10), and the proof of the abelian case depends on
a minimal generating set of \(\hat{S}\). In the case when \(\delta < d\), we neither have
a minimal generating set of \(\hat{S}\) (which even in the abelian case may be
cumbersome to construct) nor do we know of any exact sequence analogous to
(3.3.10). Also, we do not have any counter examples to show that \(\hat{S}\) may be
decomposable.
We close this chapter and this thesis by the following

QUESTION. Let $\hat{S}$ be a minimal relative relation module of a $p$-group $G$ determined by (8.1) where $E$ is an arbitrary finitely generated group. Is $\hat{S}$ indecomposable?

(Note that $\hat{S}$ is always nonprojective and does not contain any projective summands (see Lemma (7.2.3).))
REFERENCES


