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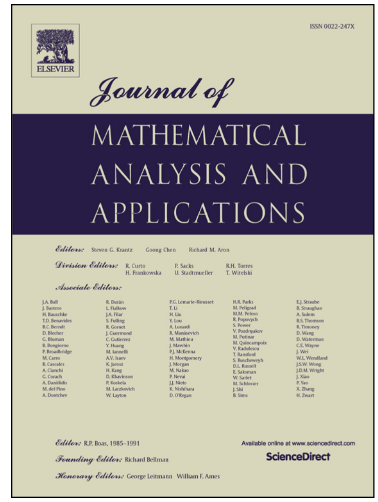
Alexander Isaev



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ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE KOBAYASHI HYPERBOLICITY OF TUBE DOMAINS IN $\mathbb{C}^2$

ALEXANDER ISAEV

Abstract. This note concerns tube domains in $\mathbb{C}^2$ with the envelope of holomorphy not equal to the entire space. We construct examples showing that for such domains the sufficient condition for Kobayashi hyperbolicity due to M. Jarnicki and P. Pflug cannot be replaced by its weaker "affine" variant, which is known to be a necessary condition for hyperbolicity. Thus, we arrive at the somewhat unexpected conclusion that the obstructions for a domain in the above class to be Kobayashi hyperbolic are not just "affine".

1. Introduction

A connected complex manifold X is said to be *Kobayashi-hyperbolic* (or simply *hyperbolic*) if the Kobayashi pseudodistance on X is in fact a distance (see [K1], [K2] for details). If X is endowed with a Riemannian metric, hyperbolicity is equivalent to the following property: for any point $x \in X$ there exist a neighborhood U of x and a constant $M > 0$ such that for all holomorphic maps $f : \Delta \to X$ with $f(0) \in U$ one has $||df(0)|| < M$, where Δ is the unit disk in $\mathbb{C}$ (see, e.g., [L] and [HK]). Verification of hyperbolicity may be quite hard even for very special classes of manifolds.

This short note is a follow-up to our earlier paper [I]. We discuss *tube domains* in $\mathbb{C}^n$, i.e., domains of the form $T_D := D + i\mathbb{R}^n$, where D is a domain in $\mathbb{R}^n$ called the *base* of T_D. Clearly, for a tube domain $T_D \subset \mathbb{C}^n$ hyperbolicity is equivalent to the following condition: for every point $x \in D$ there exist a neighborhood U of x in D and a constant $M > 0$ such that for all harmonic maps $f : \Delta \to D$ with $f(0) \in U$ one has $||df(0)|| < M$ (cf. [L] and [JP, Theorem 13.6.2]).

We assume that $n = 2$. Surprisingly, so far no easily verifiable criterion for the hyperbolicity of a tube domain has been found even in this situation. By Bochner's theorem, the envelope of holomorphy of T_D coincides with $T_{\widehat{D}}$, where $\widehat{D}$ is the convex hull of D (see, e.g., [V, Section 21]), and it is natural to investigate hyperbolicity separately in each of the cases: (i) $T_{\widehat{D}} \neq \mathbb{C}^2$ and (ii) $T_{\widehat{D}} = \mathbb{C}^2$. In [HI] we looked at several classes of hyperbolic domains in $\mathbb{C}^2$ falling in case (ii). For example, we showed that T_D is hyperbolic if D is a domain bounded by two spirals, where a spiral is a curve defined in polar coordinates in $\mathbb{R}^2$ by the equation $r = g(\varphi)$, with g being an increasing function of φ such that $\lim_{\varphi\to-\infty} g(\varphi) = 0$ and $\lim_{\varphi\to+\infty} g(\varphi) = \infty$. However, there is no comprehensive description of all hyperbolic domains covered by case (ii) (cf. [JP, p. 533, Question 13.6]).

On the other hand, for domains in $\mathbb{C}^2$ falling in case (i) certain progress towards finding a hyperbolicity criterion has been made. Firstly, such a criterion was proposed by J.-J. Loeb in [L, Théorème 6]. To state Loeb's result, let $D \subset \mathbb{R}^2$ be a domain with $\widehat{D} \neq \mathbb{R}^2$. Writing coordinates in $\mathbb{C}^2$ as $z_j = x_j + iy_j$, $j = 1, 2$, we may

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assume without loss of generality that D lies in the half-space $\{x_2 > 0\}$. In this situation, we say that D has Property (L) if the following holds:

> *there does not exist a point $a = (a_1, a_2) \in D$ for which there is a sequence of real numbers $\{b_k\}$ converging to a_2 such that the segment $[-k, k] \times \{b_k\}$ lies in D for all $k \in \mathbb{N}$.*

Then [L, Théorème 6] asserts that T_D is hyperbolic if and only if D has Property (L). The necessity implication is obvious, but, unfortunately, the nice argument by Loeb contains a flaw, and in [I] we were able to constructed counterexamples to the sufficiency implication (cf. [JP, Part (a) of Remark 13.6.7]).

In fact, as M. Jarnicki and P. Pflug observed in [JP, Part (b) of Theorem 13.6.6], the proof provided in [L] only yields a weaker statement. To formulate it, we say that a domain D lying in the half-space $\{x_2 > 0\}$ has Property (J-P) if

> *there does not exist a point $a = (a_1, a_2) \in D$ such that for every $k \in \mathbb{N}$ one can find a real-analytic function $\gamma_k(t)$ on $[-k, k]$, with $(t, \gamma_k(t)) \in D$ and $|\gamma_k(t) - a_2| \le 1/k$ for all t.*

Clearly, we have the implication Property (J-P)$\Rightarrow$Property (L). The result of Jarnicki and Pflug can now be stated as follows:

THEOREM 1.1. *Let $D \subset \mathbb{R}^2$ be a domain lying in the half-space $\{x_2 > 0\}$ that has Property* (J-P)*. Then the tube domain T_D is hyperbolic.*

Next, we introduce the "affine" variant of Property (J-P) (cf. [I]). Namely, we say that a domain D lying in the half-space $\{x_2 > 0\}$ has Property (J-P)$_{\mathrm{aff}}$ if

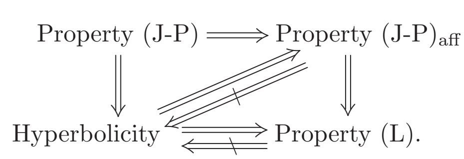
> *there does not exist a point $a = (a_1, a_2) \in D$ such that for every $k \in \mathbb{N}$ one can find an affine function $\gamma_k(t) = c_k t + d_k$, with $(t, \gamma_k(t)) \in D$ and $|\gamma_k(t) - a_2| \le 1/k$ for all $t \in [-k, k]$.*

We have the implications Property (J-P)$\Rightarrow$Property (J-P)$_{\mathrm{aff}}$$\Rightarrow$Property (L). It is not hard to see that Property (J-P)$_{\mathrm{aff}}$ is a necessary condition for the hyperbolicity of T_D (see [I, Theorem 1.3]).

The introduction of Property (J-P)$_{\mathrm{aff}}$ in [I] reflected the expectation, which has been around for some time now, that the obstructions for the hyperbolicity of a tube domain T_D with $T_{\widehat{D}} \ne \mathbb{C}^2$ should be "affine". The examples given in [I] show that Property (L) does not describe all the obstructions, so the only other natural "affine" condition appears to be the stronger Property (J-P)$_{\mathrm{aff}}$. In this note we demonstrate that Property (J-P)$_{\mathrm{aff}}$ is not sufficient for hyperbolicity either. Namely, we strengthen [I, Theorem 1.2] as follows:

THEOREM 1.2. *There exists a domain $D \subset \mathbb{R}^2$ lying in the half-space $\{x_2 > 0\}$ that has Property* (J-P)$_{\mathrm{aff}}$ *and for which T_D is not hyperbolic. Such a domain D can be chosen to have a C^∞-smooth boundary.*

Thus, hyperbolicity and the three properties discussed above are related as shown in the following diagram:



Although the examples provided below are elementary, they are nevertheless intriguing as they suggest that the search for an "affine" criterion for hyperbolicity should be abandoned. The problem of eliminating the gap between necessary and sufficient conditions remains open, but in order to solve this problem one must look beyond "affine" properties. The next most natural question to address is then whether Property (J-P) is a necessary condition for hyperbolicity.

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2. The examples

First, let

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 4\} \setminus$$

$$\Big(\{-3\pi/2\} \times [0,2] \cup \{-\pi/2\} \times [2,4] \cup \{\pi/2\} \times [0,2] \cup \{3\pi/2\} \times [2,4]\Big),$$

as shown in Fig. 1 below. Clearly, D has Property (J-P)$_{\text{aff}}$.

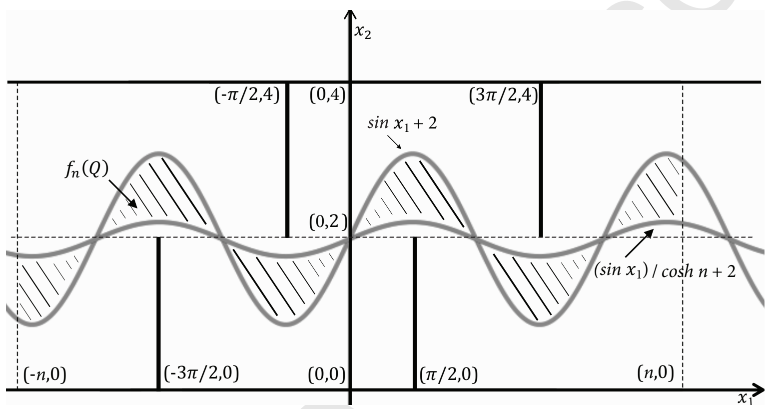

Figure 1. An example with rough boundary.

We will now prove that T_D is not hyperbolic. Let $a := (0,2) \in D$. We will construct a sequence of harmonic mappings $f_n : \Delta \to D$ such that $f_n(0) = a$ and $||df_n(0)|| \to \infty$ as $n \to \infty$. Define

$$f_n : \Delta \to \mathbb{R}^2, \quad z = x + iy \mapsto \left(nx, \frac{\sin(nx)\cosh(ny)}{\cosh n} + 2\right), \quad n \in \mathbb{N}. \tag{2.1}$$

Clearly, f_n is harmonic, $f_n(0) = a$ and $||df_n(0)|| \to \infty$ as $n \to \infty$.

It remains to see that f_n maps Δ into D for all $n \in \mathbb{N}$. Consider the closed unit square in $\mathbb{C}$

$$Q := \{z = x + iy \in \mathbb{C} \mid -1 \le x \le 1,\ -1 \le y \le 1\}.$$

For every fixed $-1 \le y_0 \le 1$, the map f_n takes the segment $[-1,1] \times \{y_0\} \subset Q$ into the curve

$$\Gamma_{y_0} := \left\{ \left(x_1, \frac{\sin x_1 \cosh(ny_0)}{\cosh n} + 2\right) \,\middle|\, -n \le x_1 \le n \right\}.$$

Therefore, the image

$$f_n(Q) = \bigcup_{-1 \le y_0 \le 1} \Gamma_{y_0}$$

is the closed set bounded by the graphs of $\sin x_1 + 2$ and $(\sin x_1)/\cosh n + 2$ on the segment $-n \le x_1 \le n$, which is shown as the shaded area in Fig. 1. Thus, $f_n(Q)$ lies in D and so does $f_n(\Delta) \subset f_n(Q)$.

The above example can be modified by choosing

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 4\} \setminus (S_1 \cup S_2 \cup S_3 \cup S_4),$$

where each of

$$S_1 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0,\, 0 \le x_2 \le 2\},$$

$$S_3 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0,\, 0 \le x_2 \le 2\}$$

is a closed region whose boundary contains a curve joining a pair of points on the line $\{x_2 = 0\}$ and passing through a point on the line $\{x_2 = 2\}$ and each of

$$S_2 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0,\, 2 \le x_2 \le 4\},$$

$$S_4 \subset \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0,\, 2 \le x_2 \le 4\}$$

is a closed region whose boundary contains a curve joining a pair of points on the line $\{x_2 = 4\}$ and passing through a point on the line $\{x_2 = 2\}$, as shown in Fig. 2 below. Furthermore, we require that none of the regions S_j intersects the graph of the function $\sin x_1 + 2$.

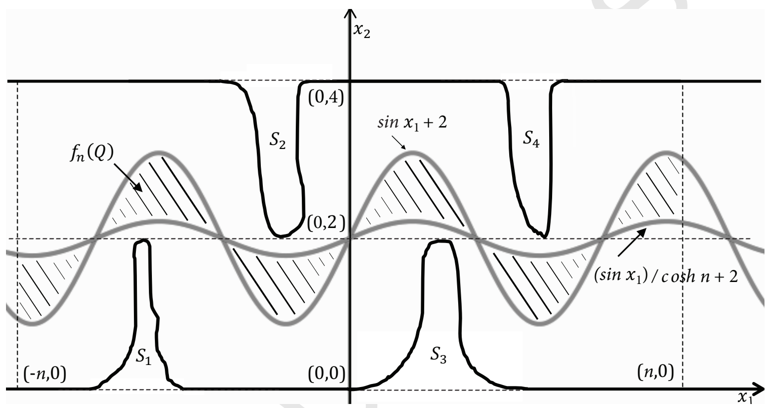

FIGURE 2. An example with smooth boundary.

Clearly, the regions S_j can be chosen to ensure that ∂D is smooth.

As above, we now see that D has Property (J-P)$_{\text{aff}}$ and that T_D is not hyperbolic. This yields the second statement of Theorem 1.2. □

Remark 2.1. Observe that in the examples given above Property (J-P) fails for $a = (0, 2)$ and

$$\gamma_k(t) = \frac{\sin t}{k} + 2, \quad t \in [-k, k].$$

Remark 2.2. It is easy to find harmonic conjugates to the components of the harmonic maps f_n defined in (2.1), which yields a sequence of holomorphic maps $g_n : \Delta \to T_D$ with $g_n(0) = a$ and $||dg_n(0)|| \to \infty$ as $n \to \infty$:

$$g_n(z) := \left(nz, \frac{\sin(nx)\cosh(ny) + i\cos(nx)\sinh(ny)}{\cosh n} + 2\right) = \left(nz, \frac{\sin(nz)}{\cosh n} + 2\right).$$

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Mathematical Sciences Institute, Australian National University, Canberra, Acton, ACT 2601, Australia

E-mail address: alexander.isaev@anu.edu.au