DISPERSION THEORETIC APPROACH TO GRAPH THEORIES
OF CHARGED PARTICLES OF SPIN 0, 1/2, 1.

A thesis submitted to
The Australian National University
for the degree of
Doctor of Philosophy

by

Harvey Alan Cohen

May, 1965.
μήτι τοι δρυτόμος μέγ' ἀμέλων ἢ βίηφι·
μήτι ο' αὐτε κυβερνήτης ἐνι ὀξυνοι κόντω
νὴα θοὴν ἱεύνει ἐφεχθομένην ἀνέμοισι.

ΟΜΗΡΟΥ Ψ τε'—τις'
The woodcutter is far better for skill than he is for brute strength.

It is by skill that the sea captain holds his rapid ship on its course, though torn by winds, over the wine-blue water.

Iliad 23, 315-17.
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The initial stimulus to the work undertaken in this thesis was a suggestion by Professor Peaslee that I should examine the question of developing a finite theory of charged particles of spin one.

The thesis itself comprises of necessity much well known matter; some is the work of others, duly acknowledged; but otherwise in essence and in detail is the contribution of the undersigned.

It will perhaps be helpful to specify the particular major original elements in the thesis:

1. A formulation of the calculation of quantum electrodynamics of particles of various spin is given that does not involve the concept of renormalisation and yet does not reject graph theory [See Chapter II and subsequent Chapters].
2. Improved methods of handling infra-red divergences are suggested by our dispersion theoretic treatment of spin $1/2$ [See Chapter III].
3. The first account of the quantum electrodynamics of "vector spin zero" is to be found introduced in Chapter I, Section 2, and in Chapter 4, Sections 2, 3.
4. The first order radiative correction to photon propagator in Lee and Yang's theory of charged spin one is calculated (for $\xi > 0$) in Chapter VI.
5. A new theory of the vector spin one which provides asymptotically correct values of the charged particle and photon spectral weights is given in Chapter VII.
6. In Chapter V a new theorem relating to $10 \times 10$ Duffin-Kemmer $\beta$ matrices is proved, and shown to lead to a very convenient method of computing the trace of $10 \times 10$ $\beta$ matrices.
7. In Chapter VIII we examine, in a dispersion theoretic manner, a counterexample to Ward's Identity.

None of the material of this thesis has been used by me in any previous thesis.
ACKNOWLEDGEMENTS

I would like to thank Professor D.C. Peaslee, who suggested the problem whose full exploration led to the various elements of this thesis, and provided much needed encouragement at difficult periods.

From Professor Peaslee I have learned an overall approach to research in Physics for which I am particularly grateful.

At various times I also had very stimulating discussions with Professor K.J. Le Couteur.

Mrs. Robertson tackled the actual typing of the thesis with her usual efficiency.
NOTATION

\( k = c = 1 \)

We write a four vector \( k \) with Greek subscripts and specify components thus:

\[
k_\mu = (k_0, k_i) = (k^0, k^i) \quad ,
\]

and

\[
k_\mu = (k_0, k_i) = (k^0, k^i) \quad ,
\]

where

\( k_0 = -ik_4 \) is real ;

\[
\begin{align*}
x_\mu &= (t, x = r) = (x, it) , \\
\partial_\mu &= (-\partial \partial t, \partial \partial x) = (\partial \partial x, -i\partial \partial t) .
\end{align*}
\]

**NOTICE:** Unless indicated to the contrary, repeated Greek indices are summed from 1 to 4, repeated Roman indices from 1 to 3.

Explicitly,

\[
a_\mu b_\mu = -a_\alpha b_\alpha + a_1 b_1 + a_2 b_2 + a_3 b_3 ,
\]

\[
= a_1 b_1 + a_4 b_4 .
\]

Adjacent dummy indices are dropped without causing ambiguity:

\[
\begin{align*}
y_p &= y_\mu p_\mu , \quad \gamma_p = y_\mu p_\mu , \\
p_q &= p_\mu q_\mu , \quad p_p = p_\mu q_\mu , \\
y_p y_q &= y_\mu p_\mu y_\nu q_\nu ; \quad p_p p_q = p_\lambda q_\lambda p_\mu q_\mu ,
\end{align*}
\]

\[
x_\mu y_\nu = x_\mu y_\nu .
\]

Bracket pairs bearing a numerical affix are dropped wherever this can be done without causing any ambiguity:

\[
\begin{align*}
p_\mu^2 &= (\beta^\lambda p_\lambda)^2 , \\
p_\mu^2 &= (p_\mu p_\nu)^2 ,
\end{align*}
\]

where a free index must be absent on the LHS.
We write \( \Phi^k = \Phi_0 \Phi_1 \Phi_2 \Phi_3 \)

and \( \Phi^k = \Phi_1 \Phi_2 \Phi_3 \).

We generally omit the comma in \( V_\lambda (p'', p') \) to write \( V_\lambda (p''p') \), the implicit indices on \( p'' \) and \( p' \) being considered non-adjacent.
H. A. COHEN

DISPERSION THEORETIC APPROACH TO GRAPH THEORIES
OF CHARGED PARTICLES OF SPIN 0, 1/2, 1.

ERRATUM

Page 0, and throughout Chapter 1

Remove the bracket pairs about equation numbers
wherever they appear.

Page 1, line below 1.8b

Replace the sentence commencing "A wave function..." by:
"Such irreducible representations are utilised in
quantum mechanics to describe free particles of mass
and of spin j."

Page 5, line 5 from bottom

Replace the last five lines of page 5 by:
"Before making some first steps in a more satisfactory
explicandum of R,S theory we recall that one equation
suffices to describe spin one-particles: the Proca-
Maxwell equation 2.60 entails the supplementary
condition equation 2.9. In analogy we write down
the single equation
\[ \left( i\lambda + \mu \right) \nu + A(\gamma_\mu \mu \gamma_\nu) + B \gamma_\mu \mu \gamma_\nu + C \gamma_\mu \gamma_\nu \right] \psi^{3/2} = 0 \quad 4.3

in which the coefficients A, B, C are to be chosen such
that our equation 4.3 entails the condition 4.1b."

Page 11, line 11

Delete lines 11-14 inclusive, replace by:
"Principle B asserts that the graph elements derived in the usual perturbation theory of the interaction of the quantized electromagnetic field with the charged quantized Dirac spin 1/2, scalar spin zero, and 5 x 5 β spin zero field be taken as axiomatically true, i.e. we neither affirm nor deny the usual theoretical foundation, namely the Lagrangian formalism and Dyson-Wick procedures. In the case of theories of charged particles where the Dyson-Wick procedure fails to yield a finite covariant algorithm for electrodynamic processes, we propose to follow the Dyson-Wick procedure as far as possible, and make a minimum number of speculations in order to determine the sought graph elements."
Chapter I
FREE FIELDS OF DEFINITE SPIN

1. Scope of this Chapter

The requirements of invariance of quantum mechanics with respect to some group of transformations of space-time impose quite rigid restrictions on possible wave functions (state vectors). The wave function must transform according to one of the linear representations of the particular group. The most general group that has been treated is the inhomogeneous Lorentz group, consisting of all four-dimensional translations and rotations. The reader is referred to the papers of Shirokov (JETP 1958) for a lucid introduction to the classification of the irreducible representations of the inhomogeneous Lorentz group. We wish to probe the problem of determining relativistic wave equations, using the work of Wigner as a basis. Under the infinitesimal transformation

\[ x_{\mu} \rightarrow x'^{\mu}_{\mu} = x_{\mu} + \xi_{\mu} + \epsilon_{\mu\nu} x_{\nu}, \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \quad \text{I (1.1)} \]

a wave function \( \phi \) which is a representation of the inhomogeneous Lorentz group undergoes the transformation

\[ \phi \rightarrow \phi' = (1 + i \xi_{\mu} p_{\mu} + \frac{1}{2} \epsilon_{\mu\nu} M_{\mu\nu}) \phi, \quad \text{I (1.2)} \]

where \( p_{\mu} \) and \( M_{\mu\nu} \) are independent of \( \xi_{\mu} \), \( \epsilon_{\mu\nu} \).

Defining

\[ e_{\mu} = M_{\mu\nu} p_{\nu}, \quad \text{I.3} \]

\[ \Gamma_{\mu} = \frac{1}{2i} \epsilon_{\lambda\mu\nu\rho} M_{\lambda\mu\nu} p_{\rho}, \quad \text{I.4} \]

an analysis of the commutation rules satisfied by these quantities shows that we can make the identifications:
\( p_\mu \) : Momentum operator.

\( \sigma_\mu \) : Centre of inertia operator.

\( \Gamma_\mu \) : Intrinsic spin operator.

The intrinsic spin operator commutes with the momentum operator, i.e., it determines a translationally invariant quantity. The operators \( p^2 \) and \( \Gamma^2 \) both commute with \( p_\mu \) and \( M_{\mu\nu} \), i.e., are invariant operators for the \( \phi \) representation of the inhomogeneous Lorentz group.

Wigner showed [see Shirokov] that the physically interesting irreducible representations of the inhomogeneous Lorentz group may be characterised by the invariants

\[
\begin{align*}
-\mathbf{P}^2 &= m^2 \geq 0 , \\
\Gamma^2 &= j(j+1)m^2 , \quad 2j \text{ integral .}
\end{align*}
\]

A wave function, or in quantum field theory a field, \( \phi \), which is an irreducible representation as specified by \((1.8 \text{ a, b})\), is used to describe a free particle of mass \( m \) and of spin \( j \).

In practice such a \( \phi \) is always taken to be of the form of a representation of the direct product of (Homogeneous Lorentz Group) x (Translation Group) e.g., \( A_\mu(x) \) is a vectorial representation of the homogeneous Lorentz group, and also a representation of the translation group (the translation operator \( p_\mu = -i\partial_\mu \) for this Schrödinger 'picture'). The equations \((1.5)\) and \((1.6)\) mean that \( \phi(x) \) :

(i) shall satisfy the Klein-Gordon equation for mass \( m \),

(ii) be an eigenket of \( \Gamma^2 \) operator i.e., \( \Gamma^2 \phi(x) = j(j+1)m^2 \phi(x) \).

Below we determine the squared spin operator \( \Gamma^2 \) for various representations to find that in the case of bosons of spin \( \leq 1 \), \((1.8 \text{ a, b})\) are sufficient to determine the equation of motion, and for the Kemmer field the appropriate are precisely specified. We also find the squared spin operator for Dirac spin 1/2 field.
Rarita-Schwinger field we give an account that is analogous to our treatment of the vector field, and sheds some further insight thereon.

[In this Chapter we at times explicitly work in the Schrödinger picture.]

However all field equations not involving \( a_\mu \) hold good in Heisenberg representation. We conclude this section by quoting Shirokov once more:

"The unsatisfactory nature of the present scheme [of description of elementary particles of higher spin, e.g., Bhabha] is obvious simply from the fact that it gives not a single equation, which can be used for a completely correct configuration description of a single particle, and which leads to a positive definite normalisation and energy. We wish to focus attention on the need for 'correct configuration description', which we limit to mean that we seek wave functions describing particles of definite spin."

2. **Vector Particles**

The vectorial representation of the homogenous Lorentz group \( \phi \), under Lorentz transformation \((\mathbf{H})\) transforms thus:

\[
\phi'_{a} = (\delta_{a\beta} + \epsilon_{\mu\rho} \delta_{a\beta} + \epsilon_{a\beta})\phi_{\beta}
\]

whence

\[
M_{\mu\nu} = -i (\delta_{\mu\nu} \delta_{\rho\alpha} - \delta_{\mu\rho} \delta_{\nu\alpha})
\]
so that by (2.4) the intrinsic spin operator for vector particles is given by

\[ [ \Gamma^\lambda ]_{\alpha\beta} = \epsilon_{\lambda\alpha\beta\gamma} \Gamma_\gamma \cdot \]

The squared spin operator is then

\[ \Gamma^2_{\alpha\beta} = -2 (p^2 \delta_{\alpha\beta} - p_{\alpha} p_{\beta}) \cdot \]

For a vector field \( \phi_{\alpha} \) satisfying \((p^2 + m^2) \phi_{\alpha} = 0\), the eigenvalue equation

\[ \Gamma^2_{\alpha\beta} \phi_{\beta} = s(s + 1) m^2 \phi_{\alpha} , s > 0 \]

which reduces to

\[ p_{\alpha} p_{\beta} \phi_{\beta} = \frac{1}{2} [s^2 + s - 2] m^2 \phi_{\alpha} \]

can be seen on multiplication by \( p_{\alpha} p_{\alpha} \) to have the solutions \( s = 0, 1 \). Thus we can distinguish two classes of what we call vector particles:

Spin one vector particles: \[ \Gamma^2 \phi^1 = 2 m^2 \phi^1 \], \hspace{1cm} (2.6a)

Spin zero vector particles: \[ \Gamma^2 \phi^0 = 0 \], \hspace{1cm} (2.7)

whence by (2.4)

\[ [(p^2 + m^2) \delta_{\alpha\beta} - p_{\alpha} p_{\beta}] \phi^1_{\beta} = 0 \], \hspace{1cm} (2.6a)

and

\[ [p^2 \delta_{\alpha\beta} - p_{\alpha} p_{\beta}] \phi^0_{\beta} = 0 \]. \hspace{1cm} (2.7a)

Operating for a wave function, \( \phi \), describing a particle of mass \( m \), momentum \( p \),

\[ (p^2 + m^2) \phi = 0 \],

so that

\[ (p_{\alpha} p_{\beta} + m^2 \delta_{\alpha\beta}) \phi^0_{\beta} = 0 \]. \hspace{1cm} (2.8)
Equation (2.6a) is the Proca-Maxwell equation; we note that it implies that

$$\mathbf{p}_a \varphi^a = 0,$$

(2.9)

while equation (2.7a) likewise implies that

$$\mathbf{p}_a \rho^s \rho^a = \mathbf{p}^2 \rho^a.$$

(2.10)

3. Dirac Particles

These massive particles are described by a dyadic spinor which transforms homomorphically with (1.1): \(\mathbf{M}\)

$$\psi \rightarrow \psi' = (1 + \gamma_\mu \gamma_\mu + \frac{1}{4} \gamma_\rho \gamma_\sigma)\psi.$$  

(3.1)

so that

$$M^\mu_\nu = \frac{1}{4i} [\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu],$$

(3.2)

and

$$\gamma_\lambda = -\frac{1}{4} \varepsilon_{\lambda \mu \nu \rho} \gamma_\mu \gamma_\nu \gamma_\rho.$$  

(3.3)

so that

$$\gamma_\lambda^2 = -\frac{1}{16} \varepsilon_{\lambda \mu \nu \rho} \varepsilon_{\lambda \mu \nu \rho} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\rho = -\frac{3}{4} \mathbf{p}^2.$$  

(3.4)

It follows that a dyadic spinor satisfying

$$(\mathbf{p}^2 + m^2)\psi = 0,$$

(3.5)

does describe a particle of spin \(\frac{1}{2}\), i.e., satisfying

$$\Gamma^2 \psi = \frac{3}{4} m^2 \psi.$$  

(3.6)
We can restrict the field equation further by postulating, consistent with (3.5), the Bhabha form

\[(i\gamma \! \! \! / + m)\psi = 0\]  \hspace{1cm} (3.7)

which is of course the Dirac equation for neagtorn of mass \(m\).

4. Generalised RS Theory

Rarita and Schwinger (1941) wrote down, as an alternative to Fierz and Pauli (1939) description of particles of half-integral spin, two equations, which for the case of spin \(3/2\) are

\[ (i\gamma \! \! \! / + m)\psi_{\mu}^{3/2} = 0 \] \hspace{1cm} (4.1a)
\[ \gamma^{\mu}\psi_{\mu}^{3/2} = 0 \] \hspace{1cm} (4.1b)

The wave function \(\psi_{\mu}\) has both dyadic spinor and tensor indices, \(\gamma_{\mu}\) are usual Dirac matrices. These equations taken together imply

\[ p_{\mu}\psi_{\mu}^{3/2} = 0 \] \hspace{1cm} (4.2)

Rarita and Schwinger noted that this equation has an appropriate number of independent plane wave solutions, and (b) the square of the non-covariant intrinsic angular momentum has value \(\frac{15}{4}\) in rest frame of particle.

Before making some first steps in a more satisfactory explicandum of R-S theory, we replace (4.1a) and (4.1b) by a single equation in analogy to the way the Proca-Maxwell equation (2.6a) entails the supplementary condition (2.9) for spin one. This equation we take as

\[ [(ix+m)\delta_{\mu\nu} + A(\gamma_{\mu} + ip_{\nu} + ip_{\mu}\gamma_{\nu}) + B\gamma_{\mu} + \gamma_{\nu} + \alpha\gamma_{\mu}\gamma_{\nu}]\psi_{\nu}^{3/2} = 0 \] \hspace{1cm} (4.3)
On contraction with \( \gamma_\mu \) one deduces
\[
2(1+2A) i p_\mu \psi^{3/2} = [(A+4B-1) i p^0 + (1+4C) m] \gamma_\mu \psi^{3/2}
\]
and on contraction of (4.3) with \( i p_\mu \),
\[
[(A+1) ip^0 m] i p_\mu \psi^{3/2} = [-(A+B)p^2 + Cm ip^0] \gamma_\nu \psi^{3/2}.
\]
Then, provided \( A \neq -\frac{1}{2} \),
\[
(3A^2+2A+1-2B) p^2 \gamma_\nu \psi^{3/2} + (2A+4B+2C) m i p^0 \gamma_\mu \psi^{3/2} + m(4C+1) \gamma_\mu \psi^{3/2} = 0
\]
Setting
\[
B = \frac{3}{2} A^2 + A + \frac{1}{2}
\]
\[
C = -3A^2 - 3A - 1
\]
yields equation (4.1b) at once. Hence (4.2) and ultimately (4.1a) is seen to hold for a \( \psi^{3/2} \) satisfying (4.3).

Moldauer and Case (1956) first wrote down the Lagrangian for equation (4.3) – though unfortunately with a misprint for \( B \).

We now our first step is to compute the intrinsic spin operator for a quantity \( \psi_\mu \) which transforms under inhomogenous Lorentz (to first order in infinitesimals);
\[
\psi_\mu \rightarrow \psi'_\mu = [\delta_{\mu\nu} + \varepsilon_{\alpha} \gamma_\alpha \delta_{\mu\nu} + \varepsilon_{\mu\nu} + \frac{1}{4} \varepsilon_{\alpha\beta} \gamma_\alpha \gamma_\beta \delta_{\mu\nu}] \psi_\nu
\]
whence
\[
\psi_{\lambda\mu\nu} = \frac{1}{2i} \varepsilon_{\lambda\alpha\beta\gamma} \gamma_{\mu\nu} \gamma_\alpha \gamma_\beta \gamma_\gamma \psi_\nu = \varepsilon_{\lambda\mu\nu\delta} \gamma_\delta - \frac{1}{4} \varepsilon_{\lambda\alpha\beta\gamma} \gamma_\alpha \gamma_\beta \gamma_\gamma \delta_{\mu\nu}.
\]
Then the squared spin operator for this representation is

\[ \Gamma_{\mu\nu}^2 = -2 \left( p_{\delta\mu}^{2} - p_{\mu} p_{\nu} \right) - \frac{3}{4} p^2 - \frac{1}{2} \epsilon_{\lambda\mu\nu\rho} \epsilon_{\lambda\alpha\beta\gamma} p_\rho p_\gamma \gamma_\alpha \gamma_\beta , \]

\[ = - \frac{11}{4} p^2 \delta_{\mu\nu} + 2 p_\mu p_\nu - \frac{1}{2} p^2 (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) + p_\mu (\gamma_\nu \gamma_\mu - \gamma_\mu \gamma_\nu) - p_\nu (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \]

\[ = - \frac{11}{4} p^2 \delta_{\mu\nu} + 2 p_\mu p_\nu - \frac{1}{2} 2 p^2 \gamma_\mu \gamma_\nu - 2 \gamma^2_{\mu\nu} + 4 p_\mu p_\nu -- 2 p_\mu \gamma_\nu - 2 p_\nu \gamma_\mu \]

\[ = - \frac{7}{4} p^2 \delta_{\mu\nu} - p^2 \gamma_\mu \gamma_\nu + p_\mu \gamma_\nu + p_\nu \gamma_\mu . \tag{4.6} \]

We note for future use that

\[ p_\mu \Gamma_{\mu\nu}^2 = - \frac{7}{4} p^2 p_\nu - p^2 \gamma_\nu + p^2 \gamma_\nu + p^2 p_\nu = - \frac{3}{4} p^2 p_\nu, \tag{4.7} \]

\[ \gamma_\mu \Gamma_{\mu\nu}^2 = - \frac{7}{4} p^2 \gamma_\nu - 4 p^2 \gamma_\nu + p^2 \gamma_\nu + 4 p_\nu \gamma_\nu \]

\[ = - \frac{19}{4} p^2 \gamma_\nu + 4 p_\nu \gamma_\nu . \tag{4.8} \]

The commutator of \( \gamma^2 \) and \( \Gamma^2 \) vanishes:

\[ [\gamma^2, \Gamma_{\mu\nu}^2] = - p^2 [\gamma^2, \gamma_\mu \gamma_\nu] + p_\mu [\gamma^2, \gamma_\nu \gamma_\mu] + p_\nu [\gamma^2, \gamma_\mu \gamma_\nu] \]

\[ = - p^2 2 (p_\mu \gamma_\nu - p_\nu \gamma_\mu) + p_\mu (2p^2 \gamma_\nu - 2p_\nu \gamma_\mu) + p_\nu (-2p^2 \gamma_\mu + 2p_\mu \gamma_\nu) \tag{4.9} \]

\[ = 0 . \]

We also note that the matrix \( \bar{F}, \bar{F}_{\mu\nu} = p_\mu p_\nu \) commutes with \( \Gamma^2 \), i.e.,

\[ [\Gamma^2, \bar{F}] = 0 . \tag{4.10} \]

Thus one can consistently require both

\[ \Gamma_{\mu\nu}^2 \psi^j_\nu = j(j + 1) m^2 \psi^j_\mu . \tag{4.11} \]
and \((i\gamma p + m) \psi_v^j = 0\) \hspace{1cm} (4.12)

We now consider the eigenvalue equation for spin \(1/2\):
\[ \Gamma_{\mu\nu}^2 \psi_v^1 = \frac{3}{4} m^2 \psi_v^1 \hspace{1cm} (4.13) \]

Contraction with \(p\mu\), shows that \((p^2 + m^2) \psi_v = 0\)
contracting with \(\gamma\mu\) yields
\[ [-\frac{19}{4} p\gamma_v + \frac{5}{4} p\gamma_v] \psi_v^1 = \frac{3}{4} m^2 \gamma_v \psi_v^1 \hspace{1cm} (4.14) \]
i.e., \[ \not{p} \psi_v^1 = -m^2 \gamma_v \psi_v^1 \]
Thence \[ \not{p} \gamma_v \psi_v^1 = p\psi_v^1 \hspace{1cm} (4.15) \]

Now this equation is consistent with the formula
\[ \psi_v^1 = p\mu \psi_v \hspace{1cm} \] (4.16)
whence \[ \not{p} \gamma_v \psi_v^1 = \not{p} \not{p} \psi_v = p^2 \psi_v = p\mu p\mu \psi_v = p\psi_v^1 \hspace{1cm} (4.17) \]

The R-S wave function describing spin \(3/2\) must satisfy
\[ \Gamma_{\mu\nu}^2 \psi_v^3 = \frac{15}{4} \psi_v^3 \hspace{1cm} (4.18) \]

On contraction with \(p\mu\) we obtain
\[ (p^2 - 5m^2) p\mu \psi_v^3 = 0 \hspace{1cm} (4.19) \]

Contraction with \(\gamma\mu\) yields
\[ (19 p^2 + 15 m^2) \gamma \psi_v^3 = 16 \not{p} \psi_v^3 \hspace{1cm} (4.20) \]
The restriction to definite mass \( m \) for the state function, i.e., \( p^2 + m^2 = 0 \) shows that (4.19) implies
\[
p_\nu \psi_\nu^2 = 0,
\]
and further (4.20) and (4.21) yield
\[
\gamma_\mu \psi_\mu^2 = 0.
\]

5. The Kemmer Equation

The field equation
\[
(i\beta p + m)\phi = 0,
\]
for \( \beta_\mu \) satisfying
\[
\beta_\mu \lambda \nu + \beta_\nu \lambda \mu = \beta_\mu \delta \nu \lambda - \beta_\nu \delta \mu \lambda,
\]
was written down by Kemmer (1939). What follows is very much in the spirit of Kemmer's paper where (5.1) and (5.2) are taken as axioms and the properties of the field \( \phi \) subsequently deduced. Now as by (5.2),
\[
[\beta_\mu \nu - \beta_\nu \mu, \beta_\lambda] = \beta_\mu \delta \nu \lambda - \beta_\nu \delta \mu \lambda,
\]
we have
\[
M_{\mu \nu} = -i (\beta_\mu \nu - \beta_\nu \mu),
\]
whence
\[
\Gamma_\lambda = -\frac{1}{2} \epsilon_{\lambda \mu \nu \rho} \beta_\mu \nu \beta_\rho.
\]

Before calculating the square of the intrinsic spin operator \( \Gamma_\lambda \), we note that, by (5.2),
\[
defining \quad M = \beta_\mu \beta_\nu \quad \text{(summed)},
\]
the following contractions hold:
\[
\beta_\mu \beta_\nu \beta_\mu = \beta_\nu, \quad \beta_\mu \beta_\rho \sigma_\mu = (1-M) \beta_\sigma + M \delta_\rho \sigma,
\]
and \[ M_\mu + \beta_\mu M = 5M \] \[ (5.8) \]

Then \[ \Gamma^2 = p^2(\beta_\mu \beta_\rho \beta_\mu \rho - \beta_\mu \beta_\rho \beta_\mu \rho - \beta_\mu \beta_\rho \beta_\mu \rho - \beta_\mu \beta_\rho \beta_\mu \rho)^2 + \beta_\mu \beta_\rho \beta_\mu \rho - \beta_\mu \beta_\rho \beta_\mu \rho \]

\[ (5.9) \]

\[ = - p^2 M(3-\lambda) - 2(\beta_\rho^2)(\lambda-2). \] \[ (5.10) \]

The form of this expression is rather interesting. We see that if \( \lambda \) field \( \phi \) satisfying (5.1) has spin \( S \),

i.e., \[ \Gamma^2 \phi = - S(S + 1)m^2 \phi, \] \[ (5.11) \]

then \[ M(3-\lambda) + 2(\lambda-2) = S(S + 1). \] \[ (5.12) \]

Thus for spin one:

\[ (\lambda-2)(3-\lambda) = 0 \] \[ (5.13) \]

while for spin zero:

\[ (\lambda-1)(\lambda-4) = 0 \] \[ (5.14) \]

An examination of the representations of the Kemmer algebra shows that (5.13) is an equation valid in the 10 x 10 irreducible representation, while (5.14) holds for the 5 x 5 representations. The key point is that there are in the \( \beta \) algebra three linearly independent quantities that commute with all other elements. One of the quantities is the identity element, another in terms of

\[ \gamma_\mu = 2(\beta_\mu)^2 - 1 \]

is

\[ \Sigma \gamma_\mu - \Sigma \gamma_\mu \gamma_\nu = - 2m^2 + 10M - 16. \] \[ (5.15) \]

As this must be a scalar matrix in a particular irreducible representation.

We see from \[ M^2 = M + 2 \Sigma \beta_\mu \beta_\nu \gamma_\mu \gamma_\nu, \] \[ (5.16) \]

that \( \Sigma \gamma_\mu - \Sigma \gamma_\mu \gamma_\nu \phi = (2m^2 - 8)\phi \),

showing the relationship between \( \Gamma^2 \) and the quantity of (5.15).
CHAPTER II

A NEW APPROACH TO CALCULATIONS IN THE QUANTUM FIELD THEORY OF CHARGED PARTICLES.

The calculations we perform in subsequent chapters constitute the first exposition of quantum electrodynamics that does not involve the concept of renormalisation and yet does not reject graph theory.

Two principles guide us:

**Principle A** asserts the validity of the dispersion relations appropriate to the quantities to be calculated; some of these DR are well known, other postulated by us for the first time. This principle is used to help us ascertain the graph elements for new theories, as well as providing our basic computational device.

**Principle B** asserts the correctness of the well-known graph elements for Dirac spin 1/2, scalar spin zero, and 5 x 5 \( \beta \) spin zero; and prescribes that as much of Lagrangian formalism and Dyson-Wick procedures as seems substantive be utilised to determine the graph elements for previously unconsidered theories.

We first address attention to principle A. There are extant so-called derivations of dispersion relations based on the basic postulates of field theory. Such proofs always use a relation of the form

\[ f(x) = \exp(-ipx)f(0)\exp(ipx) \tag{1.1} \]

where the operator \( p \) is clearly the generator of infinitesimal translations.

Without justification, these proofs (e.g., Lehman (1954)) identify \( p \) with 4 momentum \( mv \). Now for a charged particle the "canonical" momentum \( p_c = mv - eA \); this \( p_c \) is not even gauge independent, so does not seem suitable to fill the role of generator of infinitesimal translations, but just what operator fills this role we must insist is an open question [see Tassie (1964)].

Now the various quantities we evaluate [using the graph elements specified by
Principle $B$ could be evaluated in normal renormalisation theory and then shown to satisfy the DR we use. However, the very point of our method is to circumvent renormalisation theory by use of the DR, so we can scarcely claim the results of unperformed calculations [in renormalisation theory] to bolster our procedures.

As to principle $B$: Its adoption means that we are concerned with the evaluation of the usual graphs. Landau (1959) considered the graph with $N$ internal lines and $n$ independent loops,

$$ G = \int \frac{d^4 k_1}{2\pi^4} \cdots \frac{d^4 k_n}{2\pi^4} B/A_1 \cdots A_2, $$

where

$$ A_i = M_i^2 + q_i^2. $$

The $q_i$ are linear combinations of the $k_i$ and the external momenta $p_i$. On introducing a Feynman parametrisation one gets

$$ G = (N-1)! \int \frac{d\alpha_1}{2\pi} \cdots \frac{d\alpha_N}{2\pi} \frac{d^4 k_1}{2\pi^4} \cdots \frac{d^4 k_N}{2\pi^4} B D^N \delta(1-\alpha), $$

where

$$ D = \sum a_i A_i, \quad \bar{\alpha} = \sum a_i, $$

As one can eliminate terms linear in $k_i$ (by the usual axes shifting), it is clear that on its principal branch $G$ has no singularities for sufficiently small, real $p_i$. Landau showed that the first singularity of $G$ reached as the $p_i^2$ are increased occurs when, for each $i$,

$$ a_i A_i = 0 \quad (\text{no sum}), $$

and for each closed loop,

$$ \sum a_i q_i = 0. $$

Landau pointed out that a singularity of $G$ exists when 1.7 is satisfied with arbitrary $a$, so that, in general, the 'first singularity' is a branch point. The
discontinuity of $G$ across a branch cut starting from a point specified by 1.6 and 1.7 is

$$\text{disc } G = G_{-i\varepsilon} - G_{+i\varepsilon}.$$  

$G_{-i\varepsilon}$ is the 'physical value' of the graph for real $p_i - p_j$ and is value obtained by giving the masses small negative imaginary parts. The graph obtained by giving the masses small positive imaginary parts is $G_{+i\varepsilon}$.

The discontinuity, disc $G$, across a branch cut starting from a point satisfying Landau's conditions 1.6 and 1.7 and the further specification determined on relabelling lines

$$A_i = 0 \text{ for } i < m; \quad a_i = 0 \text{ for } i > m$$  

was shown by Gutkosky (1960) to be:

$$\text{disc } G = (2\pi)^m \int d^4k_1 \ldots d^4k_n \frac{\delta_p (q_1^2 + M_1^2) \ldots \delta_p (q_m^2 + M_m^2)}{A_m + 1 \ldots A_N}.$$  

The subscript $p$ on the delta functions means that only the contribution of the "proper" root of $q_1^2 + M_1^2 = 0$ is to be taken - this being readily found from 1.7, which must be satisfied with non-negative $a$'s. The direction of the vector $q_1$ corresponds to the direction of the contour around which the summation is taken.

The D.R. specified by principle A likewise give an expression for disc $G$ in terms of spectral weights. Thus principle A and principle B, combined with 1.10 which we refer to as Gutkosky's prescription for disc $G$, determines the spectral weights for the quantities sought.

In this thesis we in fact only compute the lowest order radiative corrections to propagators and $\gamma$-vertex. For this purpose we need the various integrals:

$$I_{mm}[f] = \int d^4p^1 d^4p^2 \delta(k - p^1 + p^2) \delta_p [p^1^2 + m^2] \delta_p [p^2^2 + m^2] f(p^1, p^2).$$  

1.11
The 'basic' such integral is calculated in Appendix 1, various $I_{mn}[f]$ in Appendix 2, $I_{md}[f]$ in Appendix 3-4, $V[f]$ in Appendix 5. The calculation of these various integrals is sometimes difficult, but is performed once and for all to apply to a whole class of graphs: e.g., the spectral weights for the propagator of all charged particles is very expeditiously determined in any photon gauge by use of Cutkosky's prescription leading to a particular $I_{md}[f]$.  

Two other dispersion-theoretic approaches to quantum electrodynamics are alluded to at times in the text. These are:

(i) Kallen's approach, as is to be found outlined in his Handbuch der Physik article [Kallen (1958)]. This approach dispenses with graph theory and retains renormalisation as a procedure albeit in dispersion theoretic dress.

(ii) A possible approach utilising dispersion relations, the spectral weights being determined by unitarity. Some information re the nature of the interaction must be further supplied. Salam and Delbourgo (1964) have such an approach and they use Ward's Identity to relate $3$-vertex charged particle propagator.

For both (i) and (ii) the integrals calculated in the appendices to this Chapter are an essential adjunct to the calculation of two body unitarity contributions. We give in Appendix I and in Appendix 5 examples of the trivially modified integrals that occur in these other approaches.

The first formulation of quantum field theory that did not require a renormalisation procedure was the LSZ theory [see e.g., Schweber (1961)]. However the LSZ theory is far from amenable to application to quantum electrodynamics, and research
done in this field has been formal and added little insight into the real problems of quantum electrodynamics. [See Pugh (1964) for a summary of such recent investigations].

The calculations we perform in subsequent chapters constitute the first exposition of quantum electrodynamics that does not involve the concept of renormalisation and yet does not reject graph theory. Our own conviction is that those graph theories which are found by our methods to be finite, involve integrals which on suitable redefinition of the integration operation - i.e., an extension past the definition of Lebesgue - would converge. Caianiello in various papers [references are given in Schweber (1961)] has reached this particular conclusion with regard to ultraviolet divergences.
II APPENDIX 1

DETAILED CALCULATION OF A PHASE INTEGRAL

In this Appendix we give a somewhat detailed calculation of the integral

$$I(t) = \int d^4k \delta_p[(t-k)^2 + m^2] \delta_p[k^2 + \mu^2].$$  \hspace{1cm} \text{II Al.1}

We have followed the notation of Cutkosky (1960) in using the subscript $p$ on the delta functions to denote that the "proper" root is to be taken. This "proper" root is the root appropriate to an analysis of the Feynman integral

$$F(t) = \int d^4k [(t-k)^2 + m^2 - i\varepsilon]^{-1} [k^2 + \mu^2 - i\varepsilon]^{-1},$$  \hspace{1cm} \text{Al.2}

for which the branch cut discontinuity is given by Cutkosky's prescription:

$$\text{disc } F(t) = (2\pi i)^2 I(t).$$  \hspace{1cm} \text{Al.3}

The Feynman integral of Al.2 is invariant under $t \rightarrow -t$, i.e., $F(t) = F(-t)$, \hspace{1cm} \text{Al.4}

and thence appropriately defining physical values, \hspace{1cm} \text{Al.5}

$$\text{disc } F(-t) = \text{disc } F(t),$$

which prescribes that

$$I(t) = I(-t).$$  \hspace{1cm} \text{Al.6}

It is possible to then calculate $I(t)$ in a rather pedestrian way.

Applying a Feynman parametrization to $F(t)$ - as is done by Cutkosky for a general graph - gives [Eden (1951)] -

$$F(t) = \int d^4k \int_0^1 dx \left( x [(t-k)^2 + m^2 - i\varepsilon] + (1-x)[k^2 + \mu^2 - i\varepsilon] \right)^{-2},$$  \hspace{1cm} \text{Al.7}

After shifting the origin,

$$F(t) = \int_0^1 dx \int d^4k [k^2 + L - i\varepsilon]^{-2},$$  \hspace{1cm} \text{Al.8}

where,

$$L = x m^2 + (1-x)\mu^2 + (x-x^2) t^2,$$  \hspace{1cm} \text{Al.9}

This expression for $F(t)$ is logarithmic divergent. In the spirit of subtraction physics (renormalization), we can make the formal subtraction - compare formal
Subtractions in the derivation of DR -

\[ F(t) - F(t_s) = -\frac{1}{2} \int_0^1 dx \int_{L_s}^L dL \int d^4k [k^2 + L - i\varepsilon]^{-3} \quad \text{A1.10} \]

Then by equation A5.12 of [Jauch and Rohrlich (1955)] the "renormalised" graph is

\[ F(t) - F(t_s) = -\frac{1}{2} \int_0^1 dx \int_{L_s}^L dL \frac{4\pi^2}{iL-i\varepsilon} B(2, 1) \quad \text{A1.11} \]

\[ = \frac{-i\pi^2}{4} \int_0^1 dx \left[ \log(L - i\varepsilon) - \log(L_s - i\varepsilon) \right] \quad \text{A1.12} \]

This equation together with A1.9 determines \( F(t) \) so that after some careful analysis \( I(t) \) can be found by use of A1.3. However it is for more instructive to evaluate the integral \( I(t) \) directly from Cutkosky's formula II 1.1. Our method was suggested by that used by Kallen [(1955) Section 29], where in the course of a calculation of vacuum polarization he implicitly calculated

\[ I^K(t) = \frac{1}{2} \int d^4k \delta[k^2 + m^2] \delta[(t-k)^2 + m^2] [\varepsilon(k) + \varepsilon(t-k)] \quad \text{A1.13} \]

\[ = \frac{\pi\varepsilon(t)[1 + 4m^2/t^2]^{1/2}}{2t^2 - 4m^2} \quad \text{A1.14} \]

for which our analysis shows

\[ I^K(t) = I(t) \quad \text{for } t_o > 0 \quad m = \mu \quad \text{A1.15} \]

Salam and Delbourgo [(1964) Appendix I] state the value of the two particle unitary phase integral,

\[ I^S(t) = \int d^4k \delta_+ [k^2 + \mu^2] \delta_+ [(t-k)^2 + m^2] \quad \text{A1.16} \]

where

\[ \delta_+ [k^2 + m^2] = \Theta(k_o) \delta[k^2 + m^2] \quad \text{A1.17} \]

It follows from the definition that

\[ I^S(t) = 0 \quad \text{for } t_o < 0 \quad \text{A1.18} \]

On the other hand

\[ I^S(t) = I(t) \quad \text{for } t_o > 0 \quad \text{A1.19} \]
Thus the calculation of $I(t)$ gives directly the magnitude of $I^K(t)$ and $I^S(t)$. We proceed to calculate $I(t)$ for $t_0 > 0$. The non-vanishing contribution to $I(t)$ comes from the confluence of the proper roots of the delta functions:

$$t_0 - k_0 = [(\vec{k} - \vec{t})^2 + m^2]^{1/2} \quad \text{Al.20}$$

$$k_0 = [\vec{k}^2 + \mu^2]^{1/2} \quad \text{Al.21}$$

As $I(t)$ is an invariant function of the four vector $t$ we can determine its behaviour for timelike, null, and space-like $t$ by taking $t$ as $(t_0, 0), (t, \vec{t})$ and $(0, \vec{t})$ respectively - corresponding to calculating in a particular reference frame in the first and third cases. With $t = (t_0, 0)$, Al.20 and Al.21 imply $t_0 > m + \mu$ or $-t^2 \geq (m + \mu)^2$,

with $t = (0, \vec{t})$, Al.20 and Al.21 are consistent only if $\vec{t} = 0$ as well as $m = \mu = 0$.

With $t = (1, \vec{t}, 1, \vec{t})$, this type of argument fails for $I(t)$, although Kallen's integral $I^K(t)$ is seen to vanish for such timelike $t$. We recall that Landau (1959) and Taylor (1960) showed that the integral

$$F(t) = \int d^4k \frac{[(t-k)^2 + m^2 - is]^{-1} [k^2 + \mu^2 - is]^{-1}}{[(\vec{k} - \vec{t})^2 + m^2]^{1/2}} \quad \text{Al.22}$$

has a branch cut which can be taken to be on the real axis $-t^2 \geq (m + \mu)^2$; the magnitude of the branch cut discontinuity is calculated by Cutkosky's analysis and the magnitude prescribed is $(2\pi i)^2 I(t)$: although some further notational refinement seems desirable we are thus happy to simply take $I(t)$ as zero for both space-like and null $t$. Then in the special reference frame where $\vec{t} = 0$,

$$I(t) = \int d^3k \frac{\delta[t_0 - (\vec{k}^2 + \mu^2)^{1/2} - (\vec{t}^2 + m^2)^{1/2}]}{4(\vec{k}^2 + \mu^2)^{1/2}(\vec{t}^2 + m^2)^{1/2}} \quad \text{Al.23}$$

In terms of new integration variable

$$y = (\vec{k}^2 + \mu^2)^{1/2} + (\vec{t}^2 + m^2)^{1/2} \quad \text{Al.24}$$

$$|\vec{k}| = \frac{1}{2y} [y^4 - 2(m^2 + \mu^2)y^2 + (m^2 - \mu^2)^2]^{1/2} \quad \text{Al.25}$$
and the integral becomes

\[ I(t) = \pi \int_{m+\mu}^{\infty} \frac{dy}{y} k \delta(t_0 - y) \]

\[ = \frac{\pi}{2t_0^2} \left[ t_0^4 - 2(m^2 + \mu^2)t_0^2 + (m^2 - \mu^2)^2 \right]^{1/2} \Theta[t - (m + \mu)]. \] \text{A1.26}

Hence in any reference frame

\[ I(t) = \frac{\pi}{2} \left\{ 1 + \frac{2(m^2 + \mu^2)}{t^2} + \frac{(m^2 - \mu^2)^2}{t^4} \right\}^{1/2} \Theta[-t^2 - (m + \mu)^2]. \] \text{A1.27}
VACUUM POLARIZATION PHASE INTEGRALS

In this Appendix we calculate for various commonly occurring functions $f$ of the integration variables the value of the phase integral $I_{\text{mm}}[f]$ which occurs in the calculation of vacuum polarization:

$$I_{\text{mm}}[f] = \int \frac{d^4 p}{p^2} \frac{d^4 p'}{p'^2} \delta(p-p'+p'') \delta(p'^2 + m^2) \delta(p''^2 + m^2) f(p'p''). \quad \text{(II A2.1)}$$

The value of $I_{\text{mm}}[f]$ for $f = 1$ is a special case of the integral evaluated in Appendix I. We write

$$I_{\text{mm}} = I_{\text{mm}}[1] = \frac{\pi}{2} \left(1 + \frac{4m^2}{p^2}\right)^{1/2} \Theta(-p^2 - 4m^2). \quad \text{(II A2.2)}$$

For covariant $f$, $I_{\text{mm}}[f]$ is a Lorentz covariant function of the four vector $p$. The application of invariance principles makes the calculation of the various $I_{\text{mm}}[f]$ tabulated below very easy, when one takes into account the fact that within the integral the following equivalences hold:

$$p'^2 = p''^2 = -m^2; \quad q = p' - p''; \quad A2.4$$

$$p'p'' = -\frac{1}{2} [p'^2 + p''^2 - (p' - p'')^2] = -\frac{1}{2} (2m^2 + p^2); \quad A2.5$$

$$p'p = -p''p = \frac{1}{2} p^2. \quad A2.6$$

As an example we calculate $I_{\text{mm}}[p'_{\mu} p'_{\nu}]$: covariance requires

$$I_{\text{mm}}[p'_{\mu} p'_{\nu}] = (p^2 \delta_{\mu\nu} A + p_{\mu} p_{\nu} B)I_{\text{mm}}. \quad A2.7$$
Contraction with $\delta_{\mu\nu}$ utilising A2.4 gives
\[-m^2 = 4p^2A + p^2B\]  \hspace{1cm} (A2.8)

Contraction with $p_\mu p_\nu$ using A2.6 gives
\[\frac{1}{4}p^4 = p^4A + p^4B\]  \hspace{1cm} (A2.9)

whence $A = -\frac{1}{3} \left( \frac{1}{4} + \frac{m^2}{p^2} \right)$, \hspace{1cm} $B = \frac{1}{3} \left( 1 + \frac{m^2}{p^2} \right)$  \hspace{1cm} (A2.10)

We proceed to write down the various $I[f]$:

\[I_{nm} = \frac{\pi}{2} \left( 1 + \frac{4m^2}{p^2} \right)^{1/2} \Theta(-p^2 - 4m^2).\]  \hspace{1cm} (A2.11)

\[I_{nm}[p'] = -I_{nm}[p''] = \frac{1}{2} p_\mu I_{nm}.\]  \hspace{1cm} (A2.12)

\[I_{nm}[p'_\mu p'_\nu] = -\frac{1}{3} \left[ \left( \frac{1}{4} + \frac{m^2}{p^2} \right) p^2 \delta_{\mu\nu} - (1 + \frac{m^2}{p^2}) p_\mu p_\nu \right] I_{nm}.\]  \hspace{1cm} (A2.13)

\[I_{nm}[p'_\mu p''_\nu] = -\frac{1}{3} \left[ \left( \frac{1}{4} + \frac{m^2}{p^2} \right) p^2 \delta_{\mu\nu} - \left( -\frac{1}{2} + \frac{m^2}{p^2} \right) p_\mu p_\nu \right] I_{nm}.\]  \hspace{1cm} (A2.14)

\[I_{nm}[p''_\mu p'_\nu] = I_{nm}(p'_\mu p''_\nu).\]  \hspace{1cm} (A2.15)

\[I_{nm}[p''_\mu p''_\nu] = I_{nm}(p''_\mu p'_\nu).\]  \hspace{1cm} (A2.16)

\[I_{nm}[(p'_\mu + p''_\mu)(p'_\nu + p''_\nu)] = -\frac{1}{3} \left( 1 + \frac{4m^2}{p^2} \right) (p^2 \delta_{\mu\nu} - p_\mu p_\nu) I_{nm}.\]  \hspace{1cm} (A2.17)

As all such $I_{nm}[f]$ are of the form
\[I_{nm}[f(p'_\mu p''_\nu)] = h(p) I_{nm},\]  \hspace{1cm} (A2.18)

we may meaningfully state that for such $f$,
\[f(p'_\mu p''_\nu) = h(p)\]  \hspace{1cm} (A2.19)

on the mass shell specified by A2.4.
II APPENDIX 3

SELF ENERGY PHASE INTEGRALS

Table

In this Appendix we calculate for various commonly occurring functions f of the integration variables q, k the two particle unitarity phase integral

$$I_{\text{mo}}[f(q,k)] = \int d^4q \, d^4k \, \delta(p-q-k) \delta_p (q^2 + m^2) \delta_p (k^2) f(q,k). \quad \text{II A3.1}$$

The value of $I_{\text{mo}}[f]$ for $f = 1$ is a special case of the integral evaluated in Appendix I. All other $I_{\text{mo}}[f]$ tabulated below may be computed by the method applied in the previous Appendix to $I_{\text{mm}}[f]$, in this case, of course, using the mass shell relations:

$$q^2 + m^2 = 0, \quad k^2 = 0, \quad \text{and} \quad p = q - k \quad \text{A3.2}$$

which imply

$$k \cdot p = k \cdot q = -\frac{1}{2} p^2 (1 + m^2 / p^2), \quad \text{A3.3}$$

$$p \cdot q = \frac{1}{2} p^2 (1 - m^2 / p^2). \quad \text{A3.4}$$

After a little labour one finds that:

$$I_{\text{mo}} = I_{\text{mo}}[1] = \frac{\pi}{2} (1 + m^2 / p^2) \Theta (-p^2 - m^2). \quad \text{A3.5}$$

In terms of $I_{\text{mo}}$:

$$I_{\text{mo}}[q] = \frac{1}{2} (1 - m^2 / p^2) p \mu I_{\text{mo}}. \quad \text{A3.6}$$

$$I_{\text{mo}}[k] = - \frac{1}{2} (1 + m^2 / p^2) p \mu I_{\text{mo}}. \quad \text{A3.7}$$

$$I_{\text{mo}}[q, q_v] = \left[ \frac{1}{3} \left(1 - \frac{m^2}{p^2} + \frac{m^4}{p^4}\right) p \mu p_v - \frac{1}{12} \left(1 + \frac{m^2}{p^2}\right)^2 \delta_{\mu v} I_{\text{mo}} \right]. \quad \text{A3.8}$$

$$I_{\text{mo}}[q, k_v] = \left[ \frac{1}{6} \left(-1 + \frac{m^2}{p^2} + \frac{2m^4}{p^4}\right) p \mu p_v - \frac{1}{12} \left(1 + \frac{m^2}{p^2}\right)^2 \delta_{\mu v} I_{\text{mo}} \right]. \quad \text{A3.9}$$
\[ I_{\infty} [k_{\mu k_{\nu}}] = \frac{1}{3} \left( 1 + \frac{m^2}{p^2} \right)^2 \left[ p_{\mu} p_{\nu} - \frac{\delta_{\mu \nu}}{4} p^2 \right] I_{\infty}. \quad (A3.10) \]

\[ I_{\infty} [k_{\mu k_{\nu} k_{\lambda}}] = -\frac{1}{4} \left( 1 + \frac{m^2}{p^2} \right)^3 \left[ p_{\mu} p_{\nu} p_{\lambda} - \frac{1}{6} p^2 (p_{\mu} \delta_{\nu \lambda} + p_{\nu} \delta_{\mu \lambda} + p_{\lambda} \delta_{\mu \nu}) \right] I_{\infty}. \quad (A3.11) \]

Writing \( k_{\mu \nu \rho \sigma} = k_{\mu} k_{\nu} k_{\rho} k_{\sigma} \) etc.,

\[ I_{\infty} [k_{\mu \nu \rho \sigma}] = \left( 1 + \frac{m^2}{p^2} \right)^4 \left( \frac{1}{5} p_{\mu} p_{\nu} p_{\rho} p_{\sigma} - \frac{p^2}{40} \delta_{\mu \nu} \delta_{\rho \sigma} + \frac{4}{240} \delta_{\mu \nu} \delta_{\rho \sigma} \right) I_{\infty} \quad (A3.12) \]

where by \( \Sigma [x] \) we mean the sum of the \( n \) terms like \( x \).
II APPENDIX 4

DIFFICULT SE PHASE INTEGRALS

We define
\[ I_{\mu \nu} = \int d^4k \left[ k^2 \right] \delta_p \left[ (p-k)^2 + m^2 \right] \frac{k \cdot k}{k^2}. \]  \hspace{1cm} (II A4.1)

Such integrals occur when the lowest order radiative corrections to the propagators of charged particles are determined in gauges other than the Fermi.

In the notation of the previous appendix,
\[ I_{\mu \nu} = I_{\mu \nu} \left[ k, k \sqrt{k^2} \right] \]  \hspace{1cm} (A4.2)

but the methods used to evaluate the \( I_{\mu \nu}[f] \) tabled above are not applicable to \( I_{\mu \nu} \). We must proceed by an extension of the method of Appendix 1. Therein we satisfied ourselves that such phase integrals vanish unless \( p \) is timelike.

We therefore first calculate \( I_{\mu \nu} \) in the reference frame where \( p = (p_0, 0, 0) \), and arbitrarily set \( p_0 > 0 \).

We take \( [k] = x \) and polar angles \( \theta, \phi \) so that
\[ d^4k = dk_0 x^2 dx d\theta d\phi. \]  \hspace{1cm} (A4.3)

Then
\[ I_{33} = \int d^4k \left[ k^2 \right] \delta_p \left[ (p-k)^2 + m^2 \right] \frac{k_z^2}{k^2} \]  \hspace{1cm} (A4.4)

\[ = \int d^4k \frac{\delta(k_0-x) \delta(p_0 - k_0 - \sqrt{x^2 + m^2}) x^2 \cos^2 \theta}{2x} \left( \frac{1}{k_0 + x} - \frac{1}{k_0 - x} \right) \]  \hspace{1cm} (A4.5)
\[ I_{33} = \frac{1}{6} \int_0^{2\pi} \, d\varphi \int_{-1}^1 \, d(\cos \vartheta) \cos^2 \vartheta \int_0^{\infty} \frac{\mathrm{d}x}{x^2 + m^2} \delta(p_o - x^2 + m^2) \]

\[
\left( \frac{1}{p_o - x^2 + m^2 + x} - \frac{1}{p_o - x^2 + m^2 - x} \right).
\]

we introduce:

\[ y = x^2 + m^2 + x, \]

whence

\[ x = (y^2 - m^2)/2y; \quad x^2 + m^2 = (y^2 + m^2)/2y, \]

\[ \mathrm{d}y = y\mathrm{d}x/ (x^2 + m^2)^{1/2}. \]

Writing

\[ I_{33}^A = I_{33}^A + I_{33}^B. \]

\[ I_{33}^A = \frac{\pi}{6} \int_m^{\infty} \frac{\mathrm{d}y}{y} \frac{1}{4y^2} \frac{1}{(y^2 - m^2)^2} \delta(p_o - y) \frac{1}{p_o - y + (y^2 - m^2)/y} \]

\[ = \frac{1}{12} \frac{\pi}{2} \frac{1 - m^2}{p_o^2} \theta(p_o - m). \]

\[ I_{33}^B = \frac{\pi}{6} \int_m^{\infty} \frac{\mathrm{d}y}{y} \frac{1}{4y^2} \frac{1}{(y^2 - m^2)^2} \delta(p_o - y) \frac{1}{p_o - y}. \]
To evaluate this integral we need the Taylor expansions
\[
y^3 = p_0^{-3} \left[ 1 + \frac{3(p_0 - y)}{p_0} + \ldots \right]
\]  
A4.13

\[
(y^2 - m^2)^2 = (p_0^2 - m^2)^2 \left[ 1 - \frac{4p_0}{p_0^2 - m^2} \right] \ldots
\]  
A4.14

The part of \( I_{33}^B \) of the form \( \int dz \frac{\delta(z)}{z} \) [even function of \( z \)] vanishes. The non-zero part is

\[
I_{33}^B = -\frac{\pi}{24} \int_m^\infty dy p_0^{-3} (p_0^2 - m^2)^2 \left[ \frac{3}{p_0} - \frac{4p_0}{p_0^2 - m^2} \right] \delta(p_0 - y)
\]  
A4.15

\[
= -\frac{\pi}{24} \left[ 3 \left(1 - \frac{m^2}{p_0^2}\right)^2 - 4 \left(1 - \frac{m^2}{p_0^2}\right) \right] \Theta(p_0 - m)
\]  
A4.16

\[
= +\frac{1}{12} \left(1 + \frac{3m^2}{p_0^2}\right) \left[1 - \frac{m^2}{p_0^2}\right] \Theta(p_0 - m)
\]  
A4.17

Thus in the chosen reference frame

\[
I_{33} = \frac{1}{12} \left(2 + \frac{3m^2}{p_0^2}\right) \left[1 - \frac{m^2}{p_0^2}\right] \Theta(p_0 - m)
\]  
A4.18

Lorentz invariance prescribes the form of \( I_{\mu\nu} \) as

\[
I_{\mu\nu} = A \delta_{\mu\nu} + B p_\mu p_\nu \sqrt{p^2}
\]  
A4.19

where \( A \) and \( B \) are Lorentz scalars.

Equation A4.18 specifies \( A \) in a particular reference frame. \( B \) is determined by contracting \( I_{\mu\nu} \) with \( \delta_{\mu\nu} \) when one obtains

\[
I_{\mu\nu} \delta_{\mu\nu} = 4A + B
\]  
A4.20
\[
\int d^4k \delta_p \left[ k^2 \right] \delta_p \left[ (p-k)^2 + m^2 \right] = I_{\mu\nu} \\
\frac{m^2}{2} \left( 1 - \frac{m^2}{p^2} \right) \Theta \left( p_0^2 - m^2 \right).
\]

We can now write down for arbitrary \( \epsilon(p_0) \) the covariant expression for \( I_{\mu\nu} \):

\[
I_{\mu\nu} = \frac{\delta_{\mu\nu}}{12} \left( 2 - \frac{3m^2}{p^2} \right) + \frac{1}{3} \left( 1 + \frac{3m^2}{p^2} \right) \frac{p_\mu p_\nu}{p^2} \frac{m^2}{4} \left( 1 + \frac{m^2}{p^2} \right) \Theta \left( -p^2 - m^2 \right).
\]

\( I_{\mu\nu} [1/k^2] \) and \( I_{\mu\nu} [k^2/k^2] \) may be evaluated in the same manner as

\[
I_{\mu\nu} [k,k/k^2] = I_{\mu\nu}. \quad \text{We merely state the results:}
\]

\[
I_{\mu\nu} [1/k^2] = \frac{m^2}{4} \left( 1 + \frac{m^2}{p^2} \right)^{-2} I_{\mu\nu}.
\]

\[
I_{\mu\nu} [k,k/k^2] = -\frac{p_\mu}{2p^2} \left( 1 + \frac{2m^2}{p^2} \right) \left( 1 + \frac{m^2}{p^2} \right)^{-1} I_{\mu\nu}.
\]

\[
I_{\mu\nu} [k,k/k^2] = \left\{ \left( 2 - \frac{3m^2}{p^2} \right) \frac{\delta_{\mu\nu}}{12} + \frac{1}{3} \left( 1 + \frac{3m^2}{p^2} \right) \frac{p_\mu p_\nu}{p^2} \right\} I_{\mu\nu}.
\]

The relation \( p.k = -\frac{1}{2} \left[ p^2 + k^2 - (p.k)^2 \right] \),

leads to the consistency checks

\[
-p_\mu I_{\mu\nu} [k,k^2] = \frac{1}{2} \left( p^2 + m^2 \right) I_{\mu\nu} [1/k^2] + \frac{1}{2} I_{\mu\nu},
\]

and

\[
p_\mu I_{\mu\nu} [k,k/k^2] = -\frac{1}{2} \left( p^2 + m^2 \right) I_{\mu\nu} [k^2/k^2] + \frac{1}{2} I_{\mu\nu} [k],
\]

which are satisfied.
We define

\[ V[f] = \rho \int \frac{d^4k}{k^2} \delta_p [(p''-k)^2 + m^2] \delta_p [(p'-k)^2 + m^2] f(k) . \]

These integrals occur in the calculation of two body unitarity contributions to vertex function in Q.E.D. We limit ourselves to an evaluation on the mass shells

\[ m^2 + m^2 = 0 , \]
\[ p^2 + m^2 = 0 . \]

Define \( Q = p'' - p' \).

The intimately related integrals

\[ \int \frac{d^4k}{k^2} \delta [(p''-k)^2 + m^2] \delta [(p'-k)^2 + m^2] [1 - \varepsilon (p''-k) \varepsilon (p'-k)] f(k) \]

may be found tabulated in Källén (1957). Salam and Delbourgo (1964) give an explicit formula Eq. (A3) which is rather rich in misprints, but which we correct to read in our notation as

\[ \int \frac{d^4k}{k^2} \delta [(p''-k)^2 + \mu^2] \delta [(p'-k)^2 + m^2] f(k^2) = \frac{\pi \theta[-Q^2 - (m+\mu)^2]}{2[(Q^2 + m^2 + \mu^2)^2 - 4m^2 \mu^2]^{1/2}} \]
\[ \times \int \frac{d^2k}{2m^2 + 2\mu^2 + Q^2} f(k^2) . \]

This integral may be readily calculated by a small extension of the method of Appendix I.

(i) \( f(k) = k^2 \); In this case we may use the result of Appendix I, for
(ii) \( f(k) = k^\mu \): By relativistic invariance we must have

\[
V[k^2] = \int \delta[p - q - k] \left[(p'^2 - k^2) + m^2\right] \delta[p - q] \left[(q - k)^2 + m^2\right]
\]

\[
= \int \delta[k - p] \left[(q - k)^2 + m^2\right] \delta[k^2 + m^2]
\]

\[
= \frac{\pi}{2} \frac{1}{\sqrt{\frac{Q^2}{Q^2} - \frac{4m^2}{Q^2}}} \theta(-Q^2 - 4m^2).
\]

Then, as within the integrand

\[
(p'' - k)^2 + m^2 = 0,
\]

\[
(p' - k)^2 + m^2 = 0,
\]

we have, by A5.1,

\[
(p' - p'') \cdot k = 0,
\]

and

\[
(p' + p'') \cdot k = k^2.
\]

Thus contraction of A3.5 with \( (p' - p'') \) establishes that \( Y = 0 \), whilst contraction with \( (p' + p'') \) using the relation

\[
(p' + p'')^2 = -Q^2 (1 + 4m^2/Q^2),
\]

yields

\[
-Q^2(1 + 4m^2/Q^2)X = V(k^2).
\]

Hence

\[
V[k^2] = - (p' + p'') \cdot \frac{\pi}{2} \frac{\theta(-Q^2 - 4m^2)}{\sqrt{1 + 4m^2/Q^2}^{3/2}}.
\]

(iii) \( f(k) = k^\mu k^\nu \).
Using principles of relativistic invariance, and noting the symmetry in \( p' \) and \( p'' \) enables one to set

\[
V(k, k_J) = [A Q^2 \delta_{\mu \nu} + B(p'^I_{\mu} p'^\nu_I + p'^{II}_{\mu} p'^{II}_{\nu}) + C(p'^I_{\mu} p'^{II}_{\nu} + p'^{II}_{\mu} p'^I_{\nu})]k_J,
\]

with

\[K = \frac{\pi Q^2 (-Q^2 - 4m^2)}{-4 Q^2 (1 + 4m^2/Q^2)^{1/2}} = \frac{1}{2} W.\]

Contracting A5.12 with \( p^I_\nu \) and using A5.7 and A5.8 leads on substituting the independent result A5.25 to the equation

\[
[Q^2 A - m^2 B - \frac{1}{2} (Q^2 + 2m^2) 0] p^I_\mu + \left[ - \frac{1}{2} (2m^2 + Q^2) B - m^2 C \right] p^{II}_\mu = \frac{1}{2} \gamma(k^2 k_\mu) = \frac{1}{2} (p'^I + p'^{II})_\mu Q^2 (1 + 4m^2/Q^2).
\]

Equating coefficients on both sides of this equation, leads, after some trivial manipulation, to the determination of \( A, B, C \) as

\[
A = -\frac{1}{2} (1 + 4m^2/Q^2), \quad \text{A5.15a}
\]

\[
B = 1 + 2m^2/Q^2, \quad \text{A5.15b}
\]

\[
C = -2m^2/Q^2. \quad \text{A5.15c}
\]

A check on this calculation is obtained by contracting A5.12 with \( \delta_{\mu \nu} \), when one obtains

\[
-2Q^2 (1 + 4m^2/Q^2) = 4Q^2 A - 2m^2 B - (Q^2 + 2m^2) C,
\]

which equation is satisfied by the solutions A5.15.

(iv) \( f = 1 \)

For this \( f \), the phase integral diverges – this is an infra\( \nu \) divergence. Introducing a photon mass \( \mu \rightarrow 0 \), we have using A5.3 and the ad hoc definition

\[
V[1] = V[k^2/(k^2 + \mu^2)]; \quad \mu^2 \rightarrow 0,
\]

\[
\text{A5.17}
\]
that
\[
V[1] = \int \frac{d^4k}{k^2 + \mu^2} \delta_p \left[ (p_k^\mu - k)^2 + m^2 \right] \delta_p \left[ (p_k'\mu - k)^2 + m^2 \right]
\]

\[
= \frac{\pi \Theta(-Q^2 - 4m^2)}{2Q^2(1 + 4m^2/Q^2)^{1/2}} \int_0^\infty \frac{dk^2}{k^2 + \mu^2} + \frac{m^2}{4m^2 + Q^2} \Theta(-Q^2 - 4m^2)\]

\[
= \log \left[ 1 - \frac{Q^2 + 4m^2}{\mu^2} \right] \frac{\pi \Theta(-Q^2 - 4m^2)}{2Q^2(1 + 4m^2/Q^2)^{1/2}} .
\]

(v) \( f = k^2k' \)

[We have already utilised this integral to determine \( V[k_k'k'\mu] \)]

\[
V[k_k'k'] = \int d^4k k' \delta_p \left[ (p_k^\mu - k)^2 + m^2 \right] \delta_p \left[ (p_k'\mu - k)^2 + m^2 \right]
\]

\[
= \frac{1}{2} \int d^4k \delta_p \left[ (p_k^\mu - k)^2 + m^2 \right] \delta_p \left[ (p_k'\mu - k)^2 + m^2 \right] \left[ (p_k'' + p_k'^\prime) - (p_k^\mu) - (p_k'\mu) \right] .
\]

Then by A2.11, A2.12 as \( Q = (p_k'' - k) - (p_k' - k) \),

\[
V[k_k'k'] = \frac{1}{2} \left[ (p_k'' + p_k'^\prime) - \frac{1}{2} Q_k^\mu + \frac{1}{2} Q_k'^\mu \right] \left( 1 + 4m^2/Q^2 \right)^{1/2} \Theta(-Q^2 - 4m^2)
\]

\[
= -\frac{1}{2} \frac{(p_k'' + p_k'^\prime)\lambda}{2} \left( Q^2 + 4m^2 \right) \frac{\pi \Theta(-Q^2 - 4m^2)}{2Q^2(1 + 4m^2/Q^2)^{1/2}} .
\]

We note that \( V[k_k'k'] = \frac{1}{2} (p_k'' + p_k'^\prime) \lambda \ V[k^2] \).

(vi) Summary of Calculated \( V[f] \)

We define \( W = \frac{\pi \Theta(-Q^2 - 4m^2)}{-2Q^2(1 + 4m^2/Q^2)^{1/2}} \).

\[
V[1] = \log \left[ 1 - (Q^2 + 4m^2)/\mu^2 \right] W .
\]

\[
V[k_k'k'] = (p_k'' + p_k'^\prime) W .
\]

\[
V[k^2] = -\left( Q^2 + 4m^2 \right) W .
\]
$$V[k_2 k_\alpha] = -\frac{1}{2} (p^n + p^\mu) (Q^2 + 4m^2) W.$$  

$$V[k_\mu k_\nu] = \frac{1}{2} \left[ A Q^2 \delta_{\mu\nu} + B (p_\mu p_\nu^* + p_\mu^* p_\nu) + C (p_\mu^* p_\nu + p_\mu p_\nu^*) \right] W,$$  

where

$$A = - (q^2 + 4m^2) / 2Q^2,$$  

$$B = 1 + m^2 / Q^2,$$  

$$C = - 2m^2 / Q^2.$$
CHAPTER III
THEORY OF CHARGED PARTICLES OF SPIN 1/2

1. Introduction and Summary

In this Chapter we compute the lowest order radiative corrections to the propagators of the photon and Dirac spinor particle (herein called electron), and to the 3 vertex. We find that our method, which was described in general terms in Chapter II, leads to very simple computations in any gauge. The following special features emerge:

Section 2. The gauge invariance of \( \Pi_{\mu\nu}(k) \) is manifest, and does not need to be artificially introduced.

Section 3. An understanding of the origin of the infra-red divergence in previous calculations of \( S(p) \) is gained, and it is shown that the correct expression is free from such divergencies for \( p^2 \neq -m^2 \).

Section 4. We prove the gauge invariance of the 3-vertex as a sidepoint to a treatment that is the dispersion theoretic analogue of the usual approach. This approach leads to an infra-red divergence in the coefficient of \( \gamma_\mu \) that is eliminated as usual by introducing a non-zero photon mass. We outline a more sophisticated calculation to circumvent the IR problem for \( p^2 \neq -m^2 \).

The major omission from this Chapter is a calculation of the (Mandelstam) spectral weights for the box diagram

III 1.1

Higher order corrections to photon and electron propagators have not been
calculated. In the case of higher order corrections to electron propagator we have found that the appropriate phase integrals involve complete elliptic integrals in rather cumbersome formulae.

As a preliminary to the calculations of Sections 2, 3, 4 we table below some basic information, including a rather practical arrangement of graph elements.

Field equations:

\[(\gamma_\mu \partial_\mu + m_0)\psi = -ie \gamma_\mu A_\mu \psi, \tag{1.2}\]
\[\bar{\psi} (\gamma_\mu \partial_\mu - m) = i e \bar{\psi} \gamma_\mu A_\mu, \tag{1.3}\]
\[\partial_\mu \partial_\lambda A_\lambda = -J_\lambda \tag{1.4}\]
\[= \frac{ie}{2} \left[ \bar{\psi} \gamma_\lambda \psi - \bar{\psi} \gamma_\lambda \psi^c \right]. \tag{1.5}\]

We call the particle described by \(\psi\) satisfying 1.2 a 'negaton'; its charge conjugate a 'position'. The negaton has charge \(-e\).

Notation:
\[\kappa = \gamma_\mu k_\mu. \tag{1.6}\]

Algebra of \(\gamma_\mu:\)
\[\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu}, \tag{1.7}\]
\[\kappa \kappa = k^2, \tag{1.8}\]
\[\gamma_\mu \gamma_\rho \gamma_\mu = -2 \gamma_\rho, \tag{1.9}\]
\[\kappa \gamma_\lambda \kappa = -2k_\lambda \kappa - k^2 \gamma_\lambda. \tag{1.10}\]

Traces:
\[\text{Tr} \gamma_{\mu\nu} = 4 \delta_{\mu\nu}, \tag{1.11}\]
\[\text{Tr} \gamma_{\mu\nu\rho\sigma} = 4 [\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\rho\sigma}] \tag{1.12}\]

Ward's Identity: as \(S^{-1}_F(p) = i (ip + m)\).  \(\tag{1.13}\)
\[ S_F^{-1}(p^n) - S_F^{-1}(p^{'}) = -(p^n - p^{'})_\mu \gamma_\mu \]  \hspace{1cm} 1.14

i.e., \[ S_F^{-1}(p^n) - S_F^{-1}(p^{'}) = -e^{-1}(p^n - p^{'})_\mu V_\mu^0(p^n p^{'}) \]  \hspace{1cm} 1.15

[The (-) occurs in 1.15 to correspond to the negaton having charge \(-e\).]

As \( (\partial/\partial p_\mu)S_F(p) = -S_F(p) \left[(\partial/\partial p_\mu)S_F^{-1}(p)\right]S_F(p) \), another form Ward's identity takes is

\[ (\partial/\partial p_\mu)S_F(p) = e^{-1}S_F(p) V_\mu^0(pp) S_F(p) \]  \hspace{1cm} 1.16
### TABLE III T 1

Spin 1/2 graph elements in Momentum Representation

<table>
<thead>
<tr>
<th>Element</th>
<th>Graph</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal photon line</td>
<td>$\mu \rightarrow k \rightarrow \nu$</td>
<td>$D_{\mu\nu} = \frac{-i[\delta_{\mu\nu}+(\mathbf{q}-\mathbf{p})\cdot \mathbf{k}<em>\mu \mathbf{k}</em>\nu/k^2]}{k^2 - i\varepsilon}$</td>
</tr>
<tr>
<td>Internal electron line</td>
<td>$\mu \rightarrow p$</td>
<td>$S(p) = \frac{i(\varepsilon - m)}{p^2 + m^2 - i\varepsilon}$</td>
</tr>
<tr>
<td>Vertex</td>
<td>$\mu \rightarrow p'' \rightarrow p'$</td>
<td>$\gamma^0_{\mu}(p''p') = c\gamma_\mu$</td>
</tr>
<tr>
<td>External photon line</td>
<td>(polarization $\lambda$)</td>
<td>$\frac{1}{(2\pi)^{3/2}} \frac{\epsilon_\lambda(k)}{</td>
</tr>
<tr>
<td>Incoming negaton</td>
<td>$\mu \rightarrow p$</td>
<td>$\frac{1}{(2\pi)^{3/2}} \frac{m}{</td>
</tr>
<tr>
<td>Outgoing position</td>
<td>$\mu \rightarrow p$</td>
<td>$\frac{1}{(2\pi)^{3/2}} \frac{m}{</td>
</tr>
</tbody>
</table>

Electron loop factor

- 1

The diagrams are evaluated by the integration $(2\pi)^{-4} \int d^3k \, dk_0$ over each independent internal momentum $k$. A complete diagram, as distinct from a diagram part, acquires a factor $(2\pi)^4 \delta(P_1 - P_f)\delta_p$, where $\delta_p$ is the signature of the permutation of final electron states.
2. Radiative Corrections to Photon Propagator

The second order correction to the photon propagator is given by the graph

\[ \Pi_{\mu\nu}(k) = \begin{array}{c} \mu \\ \downarrow p' \\ \leftarrow \\ \leftarrow \\ \rightarrow \\ \uparrow p'' \\ \nu \end{array} \]

We mean that to second order the photon propagator \( D_{\mu\nu}(k) \) is given by

\[ D_{\mu\nu}(k) = D_{\mu\nu}(k) + i D_{\rho\mu}(k) \Pi_{\rho\sigma}(k) D_{\rho\sigma}(k) \]

The Feynman integral expression for \( \Pi_{\mu\nu}(k) \) is

\[ \Pi_{\mu\nu}(k) = \frac{i e^2}{(2\pi)^4} \int d^4 p' d^4 p'' \delta(k - p' + p'') \frac{\text{Tr} [(i\gamma'^\mu - m)\gamma_\mu (i\gamma''^\nu - m)\gamma_\nu]}{(p'^2 + m^2 - \text{i}\epsilon)(p''^2 + m^2 - \text{i}\epsilon)} \]

wherein we have carefully chosen the sign of \( p', p'' \), such that when \( k_0 > 0 \), for the negative \( p' > 0 \), for the positive \( -p'' > 0 \). We note that the corresponding formula [9.41] in the book of Jauch and Rohrlich [Jauch and Rohrlich (1955)] – due allowance being made for a different definition of \( D_{\mu\nu} \) – differs by a factor \(-1\) from this expression. Our observation is confirmed by the remark to be found in [Cheng and Bludman (1964)] that the explicit formula for \( \Pi_{\mu\nu} \) [9.66] is incorrect by a factor \(-1\). The reader will find our table of S matrix elements far more convenient for this check than the cumbersome – though equivalent – table of Jauch and Rohrlich. We should further state that the expressions given by Jauch and Rohrlich for the graphs

\[ \begin{array}{c} \bullet \\ \rightarrow \end{array}, \quad \begin{array}{c} \rightarrow \\ \leftarrow \end{array}, \quad \begin{array}{c} \leftarrow \\ \rightarrow \end{array}, \quad \begin{array}{c} \square \end{array} \]

are correct.
\[ \Pi_{\mu\nu}(k) = -\frac{4ie^2}{(2\pi)^4} \int d^4p' d^4p'' \delta(k-p'+p') \frac{[\delta_{\mu\nu}(m^2+p'^2)-p'_\mu p''\nu - p''_\mu p'_\nu]}{(p'^2+m^2-i\epsilon)(p''^2+m^2-i\epsilon)}, \quad 2.5 \]

One would like to be able to prove directly that this expression implied

\[ k^\mu \Pi_{\mu\nu}(k) = 0, \quad k^\nu \Pi_{\mu\nu}(k) = 0. \quad 2.6 \]

Now Kallen, in his article in Handbuch der Physik, calculated the polarization of the vacuum due to an external field by integrating field equations directly in the Heisenberg representation, to obtain an expression which we transform to the form

\[ \Pi_{\mu\nu}^N(k) = \frac{e^2}{4\pi^3} \int d^4p' d^4p'' \delta(k-p'+p') \left[ \delta_{\mu\nu}(p'^2+m^2) - p'_\mu p''\nu - p''_\mu p'_\nu \right] \times \left\{ \delta(p'^2+m^2)[p' \frac{1}{p'^2+m^2} - i\pi\epsilon(p'^2+m^2)] + \delta(p''^2+m^2)[p'' \frac{1}{p''^2+m^2} - i\pi\epsilon(p'') \right. \]

\[ \left. \times \delta(p'^2+m^2)] \right\}, \quad 2.7 \]

for which expression the results

\[ k^\mu \Pi_{\mu\nu}^N(k) = 0, \quad k^\nu \Pi_{\mu\nu}^N(k) = 0, \quad 2.8 \]

follow in a trivial manner. However the usual Feynman integral expression 2.5 includes very ill-defined terms of the form

\[ \int d^4p' d^4p'' \frac{1}{p'^2+m^2} \times P \frac{1}{p''^2+m^2}, \quad 2.9 \]

which Khana and Rohrlich [P.R. 131, 2721 (1963)] rather blithely ignore in their most inadequate analysis of this problem. Khana and Rohrlich claim in this paper that Kallen proved in Handbuch der Physik the gauge invariance of the imaginary part of \( \Pi_{\mu\nu}^N \), this is true, à fortiori, yet irrelevant to their purposes.

The gauge invariance of the theory however manifests itself when one uses Cutkosky's prescription of replacing \( (p^2+m^2-i\epsilon) \) denominators by \( 2\pi \delta_p (p^2+m^2) \) to determine the discontinuity of \( \Pi_{\mu\nu}(k) \) across the branch cut Landau branch point starting from the
\[ k^2 = -4m^2 \] i.e., the jump discontinuity as one goes from the physical to unphys-
ical plane is
\[
\text{disc } \Pi_{\mu\nu}(k) = \frac{i e^2}{\pi^2} \int d^4 p' d^4 p'' \delta(k-p'+p'') \delta_p(p'^2-m^2) \delta_p(p''^2-m^2) \delta_{\mu\nu}(m^2+p'^2) \delta_{\mu\nu}(m^2+p''^2).
\]

Then by IIA 2.5, IIA 2.14 we have at once
\[
\text{disc } \Pi_{\mu\nu}(k) = \frac{i e^2}{\pi^2} \left[ -\delta_{\mu\nu} \frac{k^2}{2} + \frac{2}{3} \left( \frac{m^2}{k^2} \right) k^2 \delta_{\mu\nu} - \frac{2}{3} \left( \frac{1}{2} + \frac{m^2}{k^2} \right) k_{\mu} k_{\nu} \right] \text{Im}
\]
\[
= -\frac{i e^2}{3\pi^2} \left( 1 - \frac{2m^2}{k^2} \right) (k^2 \delta_{\mu\nu} - k_{\mu} k_{\nu}),
\]
where \( \text{Im} = \frac{\pi}{2} \left( 1 + 4m^2/k^2 \right)^{1/2} \theta(-k^2-4m^2). \)

The gauge independence of \( \text{disc } \Pi_{\mu\nu}(k) \) is rather cogently exhibited in 2.11. We
then see that to second order the jump discontinuity in \( D_{\mu\nu}(k) \) is independent of
\( \mu \) gauge where
\[
(k^2-\imath\epsilon)D_{\mu\nu}(k) = -i\left[ \delta_{\mu\nu} - k_{\mu} k_{\nu}/k^2 + a k_{\mu} k_{\nu}/k^2 \right],
\]
as given by 2.2 is independent of \( \lambda \):
\[
\text{disc } D_{\mu\nu}(k) = -i \left[ \delta_{\mu\nu} - k_{\mu} k_{\nu}/k^2 \right] \frac{e^2}{6\pi} \left( \frac{k^2-2m^2}{k^4} \right) \left( 1+4m^2/k^2 \right)^{1/2} \theta(-k^2-4m^2).
\]
The Lehman-Kallen D.R. for \( D_{\mu\nu}(k) \) is
\[
D_{\mu\nu}(k) = -i \left[ \delta_{\mu\nu} - k_{\mu} k_{\nu}/k^2 \right] \frac{e^2}{k^2-\imath\epsilon} \left( 1 + k^2 \right) \int_{0}^{\infty} \frac{d \mu^2 \rho_{\mu}(\mu^2)}{k^2+\mu^2-\imath\epsilon},
\]
whence the spectral weight \( \rho_{\mu} \) given by
\[
\text{disc } D_{\mu\nu}(k) = i \left[ \delta_{\mu\nu} - k_{\mu} k_{\nu}/k^2 \right] 2\pi \rho_{\mu} (-k^2),
\]
is found by 2.15 to be
\[
\rho_{\mu}(\mu^2) = \frac{e^2}{12\pi^2} \left( \frac{k^2+2m^2}{\mu^4} \right) \left( 1 - 4m^2/\mu^2 \right)^{1/2} \theta(\mu^2-4m^2).
\]
Note that this spectral weight is positive:

\[ \rho_3 > 0 . \]  

The problem is solved at this stage, the calculated value of the spectral weight given in 2.18 together with the D.R. 2.16, constituting a most satisfactory specification of the second order radiative correction. However, it is amusing to add the explicit formulas (for Landau gauge):

\[ D_{\mu \nu}(k) = D_{F\mu \nu} \left(1 + \frac{e^2}{12\pi^2} \left[ \frac{5}{3} - \frac{4m^2}{k^2} - \left(1 - \frac{2m^2}{k^2} \right) \log \frac{1 + \xi}{1 - \xi} \right] \right), \]  

2.20a

where \( \xi = (1 + 4m^2/k^2)^{1/2} \) real.  

We write (Landau gauge)

\[ D_{\mu \nu}(k) = D_{F\mu \nu} \left[1 + \Pi(k^2) \right], \]  

2.21a

i.e., \( \Pi_{\mu \nu}(k^2) = (k^2 \delta_{\mu \nu} - k_{\mu} k_{\nu}) \Pi(k^2) \).  

Then the derivative at the origin,

\[ \Pi'(0) = \lim_{k^2 \to 0} \frac{\Pi(k^2)}{k^2}, \]  

2.22

\[ = \int_0^\infty \frac{d\mu^2 \rho_3(\mu^2)}{\mu^2}, \]  

2.23

\[ = \frac{e^2}{12\pi^2 m} \int_4^\infty \frac{dx(x+2)(x-4)^{1/2}}{x^{9/2}}, \]  

2.24

\[ = \frac{e^2}{60\pi^2 m^2}, \]  

2.25
In the notation of Jauch and Rohrlich (1955; Eq. 9-63)

\[ k^2 \Pi_f(k^2) = \Pi(k^2), \]

The expressions written down in their book for \( \Pi_f(k^2) \) and \( \Pi_f(o) = \Pi'(o) \) are in error by a factor \((-1)\).
3. Radiative Corrections to Electron Propagator

To second order, the electron propagator is

\[ S(p) = S_F(p) + S_F(p) \Sigma(p) S_F(p) \quad 3.1 \]

where the SE part \( \Sigma(p) \) has graph

\[ \Sigma(p) = \begin{pmatrix} \frac{k}{q = p + k} \end{pmatrix} \quad 3.2 \]

Landau's analysis applied directly to this graph [Landau (1960)] shows that \( \Sigma(p) \) has a branch cut on the real axis of the \(-p^2\) plane along \( m^2 < -p^2 < \infty \). The discontinuity across the branch cut is given by the Cutkosky prescription of replacing \((p^2 + m^2 - i\varepsilon)\) denominators by \(2\pi \delta_p(p^2 + m^2)\) as

\[
\text{disc } \Sigma(p) = \text{physical } \Sigma(p) - \text{unphysical } \Sigma(p) \quad 3.3
\]

\[
= \frac{e^2}{(2\pi)^4} \int d^4k \frac{2\pi i \epsilon_p [(p+k)^2 + m^2] 2\pi i \delta_p[k^2]} \quad 3.4
\]

where, in gauge \( a \),

\[ X = \gamma_\mu (i\gamma^\nu - m) \gamma_\nu [\delta_{\mu\nu} + (a-1)k_\mu k_\nu / k^2] \quad 3.5 \]

Within the integrand, by IIA 3.6,

\[ \gamma_\mu (i\gamma^\nu - m) \gamma_\nu \epsilon_{\mu\nu} = -2i\gamma^\nu 4m = -i\gamma^\nu (1 - m^2/p^2) - 4m \quad 3.6 \]

while by IIA 4.26 and IIA 3.7,

\[ \gamma_\mu (i\gamma^\nu - m) \gamma_\nu \epsilon_{\mu\nu} \frac{k_\mu k_\nu / k^2}{k^2} = k^{-2} \epsilon_{\mu\nu} + i\mu \epsilon_{\mu\nu} + i\nu \epsilon_{\mu\nu} - m \quad 3.7 \]

\[ = \gamma_\mu i\epsilon_{\mu\nu} \left( \frac{\delta_{\mu\nu}}{12} (2 - \frac{3m^2}{p^2}) + \frac{\epsilon_{\mu\nu}}{3p^2} (1 + \frac{3m^2}{p^2}) \right) - \frac{1}{2} i\epsilon_{\mu\nu} (1 + \frac{m^2}{p^2}) - m \quad 3.8 \]

\[ = -\frac{1}{2} i\epsilon_{\mu\nu} (1 - 2m^2/p^2) - m \quad 3.9 \]

Adding 3.6 and 3.9, using the definition 3.5,

\[ X = -i\gamma^\nu \left[ \frac{a+1}{2} - am^2/p^2 \right] - (a+3)m \quad 3.10 \]
where by IIA 3.5, we can express the integral as

$$\text{disc } \Sigma(p) = \frac{-e^2}{(2\pi)^2} \times I_{\text{mo}} ,$$  \hspace{1cm} 3.11

and

$$I_{\text{mo}} = \frac{m}{2p^2} (p^2 + m^2) \Theta(-p^2 - m^2).$$  \hspace{1cm} 3.12

In the Handbuch der Physik article by Kallen [Kallen (1958)] there is a somewhat sketchy proof of a dispersion relation for a quantity $F(p)$ (Kallen Eq 31.19), which appears – to a phase factor – to be the value ascribed to $\Sigma(p)$ in this formulation.

Explicitly, in terms of spectral weights $\Sigma_1^o(p^2)$ and $\Sigma_2^o(p^2)$, Kallen has the following subtracted D.R. [$p_0 > 0$],

$$F(p) = -i(p^2 + m^2) \int_0^\infty d\mu^2 \frac{m \Sigma_1^o(-\mu^2) + (i\gamma p + m) \Sigma_2^o(-\mu^2)}{(p^2 + \mu^2 - i\epsilon)(-\mu^2 + m^2)} \hspace{1cm} 3.13$$

[One replaces $-ie$ by $+ie$ when $p_0 < 0$ in this expression for $F(p)$]. The branch cut discontinuity $\text{disc } F(p)$ is

$$\text{disc } F(p) = 2\pi [i\gamma p \Sigma_2^o(p^2) + m \Sigma_1^o(p^2) - m \Sigma_2^o(p^2)] . \hspace{1cm} 3.14$$

If one equates $F(p)$ and $\Sigma(p)$, for $p_0 > 0$, then

$$\Sigma_1^o(p^2) = \frac{e^2}{16\pi^2} \frac{1}{2} (a + 5 + 2am^2/p^2)(1 + m^2/p^2) \Theta(-p^2 - m^2) , \hspace{1cm} 3.15$$

$$\Sigma_2^o(p^2) = \frac{e^2}{16\pi^2} \frac{1}{2} (a + 1 - 2am^2/p^2)(1 + m^2/p^2) \Theta(-p^2 - m^2) , \hspace{1cm} 3.16$$

and putting $a = 1$ (Fermi gauge) we get exactly the values ascribed by Kallen to $\Sigma_1^o(p^2)$ and $\Sigma_2^o(p^2)$ [Kallen : 31.15, 31.17] [We note en passant that the explicit method Kallen uses in determining these spectral weights – the taking of particular traces – is false and only fortuitously gives the same answer as does the correct procedure]. However we do assert that
\[ \Sigma(p) = \frac{i}{2 \pi} \int_{m^2}^{\infty} \frac{d\mu^2}{\mu^2} \frac{i\gamma p \sigma_1(\mu^2) + m \sigma_2(\mu^2)}{p^2 + \mu^2 - i\epsilon} \]

wherein we have chosen to write the D.R. in the simplest form. The spectral weights are determined from 3.11, as

\[ \sigma_1(\mu^2) = \frac{e^2}{16\pi^2} \frac{(a + 1)\mu^2 + 2am}{2\mu^2} \]

\[ \sigma_2(\mu^2) = \frac{e^2}{16\pi^2} \frac{a + 3}{\mu^2} \]

Consequently, as

\[ [S_F(p)]^2 = (p^2 + m^2)^{-2} \left[ p^2 + m^2 + 2m(i\gamma p - m) \right] \]

\[ [S_F(p)]^2 i\gamma p = (p^2 + m^2)^{-2} \left[ -m(p^2 + m^2) + (p^2 - m^2)(i\gamma p - m) \right] \]

the radiative correction to second order to \( S(p) \) is \( S_F(p) \Sigma(p) S_F(p) \)

\[ = \frac{e^2 m}{16\pi^2} \left\{ \frac{am^2}{p^4} + \left[ \frac{a+5}{2p^2} + \frac{am^2}{p^2} \right] \log \left[ \frac{m^2}{(p^2 + m^2 - i\epsilon)} \right] \right\} \]

\[ + \frac{i(i\gamma p - m)}{16\pi^2} \left\{ \frac{am^2}{p^4} + \left[ -\frac{(a+1)}{2p^2} + \frac{am^2}{p^2} \right] \log \left[ \frac{m^2}{(p^2 + m^2 - i\epsilon)} \right] \right\} \]

\[ - \frac{i(i\gamma p - m)}{p^2 + m^2 - i\epsilon} \frac{e^2}{16\pi^2} \left\{ \frac{am^2}{p^4} + \left[ \frac{(a+5)}{2p^2} + \frac{am^2}{p^2} \right] \log \left[ \frac{m^2}{(p^2 + m^2 - i\epsilon)} \right] \right\} \]

Thus we have determined an expression for the radiative correction that is free from infra red (IR) divergences for \( p^2 \neq -m^2 \).
On the other hand, if we use the Lehman D.R. for spin 1/2 propagator

\[ S(p) = \frac{-i}{i\gamma^\mu + \gamma^5 - i\epsilon} \left( 1 + (i\gamma^\mu + m) \int_{m^2}^{\infty} \text{d} \mu^2 \frac{i\gamma^\mu \rho_1(\mu^2) + m \rho_2(\mu^2)}{\mu^2 + \mu^2 - i\epsilon} \right) \]  

the spectral weights being determined from 3.11 as

\[ \text{disc } S(p) = 2\pi \left[ i\gamma^\mu \rho_1(-p^2) + m \rho_2(-p^2) \right] \Theta(-p^2 - m^2) \]

\[ = \frac{-e^2}{(2\pi)^2} \times \left[ S_p(p) \right]^2 \text{Im} \]

we find after a little algebra using 3.19, that

\[ \rho_1(\mu^2) = \frac{e^2}{16\pi^2} \frac{- (a + 1) \mu^2 + (a + 1) m^2 - 2am^4/\mu^2}{2\mu^2 (\mu^2 - m^2)} \]  

\[ \rho_2(\mu^2) = \frac{e^2}{16\pi^2} \frac{- 4\mu^2 + (2a - 6)m^2}{2\mu^2 (\mu^2 - m^2)} \]

\[ \rho_1(\mu^2) + \rho_2(\mu^2) = \frac{e^2}{16\pi^2} \frac{- (a + 5) + 2am^2/\mu^2}{2\mu^2} = \sigma_1(\mu^2) - \sigma_2(\mu^2) \]

We rewrite Lehman's DR 3.18 as

\[ S(p) = S_F(p) - i (i\gamma^\mu - m) \int_{m^2}^{\infty} \frac{\text{d} \mu^2 \rho_2(\mu^2)}{\mu^2 + \mu^2 - i\epsilon} - \text{Im} \int_{m^2}^{\infty} \frac{\text{d} \mu^2 [\rho_1(\mu^2) + \rho_2(\mu^2)]}{\mu^2 + \mu^2 - i\epsilon} \]

On examination of the RHS side of 3.25, 3.26 we see that in all gauges the first integral has an IR divergence, whilst the second is finite and has, by 3.27, the same magnitude as the first term of 3.21 a. Explicitly we have in Fermi gauge (a = 1):
which is the same - apart from appropriate phase factors - as the value determined by Karplus and Kroll, and reported in the book [Jauch and Rohrlich (1955)] equation 9.26.

The situation is now clear. We correctly determine the branch cut discontinuity of $\Sigma(p)$ and $S(p)$ by use of Cutkosky's prescription. The usual magnitude of $S(p)$ - as determined by Karplus and Kroll - results if one then applies Lehman's D.R. - the expression involving an IR divergence. The correct magnitude is determined only by use of our D.R. 3.17 - this magnitude as given by 3.21 is free of IR divergence - the IR divergences being clearly SPURIOUS and arising only from an invalid application of Cauchy's theorem.

To illustrate the last point we consider a scalar function $\sigma(p^2)$ that is real

$$\sigma(p^2) = \sigma^x(p^2 x)$$

3.30

everywhere in the complex $p^2$ plane cut \(m^2 \leq -p^2 \leq \infty\). We assume that $\sigma(p^2)$ is of order of $p^2$. Then if we prescribe for small $\epsilon > 0$,

$$\sigma(-m^2) = 0 \quad ; \quad \frac{\partial}{\partial p}\sigma(-m^2 i\epsilon) \neq 0$$

3.31

we can apply Cauchy's formula to the analytic function $\sigma(p^2)/(p^2 + m^2)$ to get

$$\sigma(p^2) = (p^2 + m^2) \int_{m^2}^{\infty} \frac{dp^2 \text{Im} \sigma(-p^2 i\epsilon)/p}{(p^2 + m^2 - i\epsilon)(-p^2 + m^2)}$$

3.32

[In 3.29 $\sigma(-p^2) = \sigma(-p^2 + i\epsilon)$; compare 3.17]
But we cannot in general apply Cauchy's formula to the function

\[ s(p^2) = \frac{\sigma(p^2)}{(p^2 + m^2)^2} \quad 3.33 \]

to get the D.R. (compare 3.22),

\[ s(p^2) = \int_{m^2}^{\infty} \frac{d\mu^2 \text{Im} \frac{s(-\mu^2)}{\mu}}{p^2 + \mu^2 - i\epsilon} \quad 3.34 \]

\( s(p^2) \) has a pole at the branch point \( p^2 = -m^2 \) which gives a non vanishing contribution to the end bit of the contour

```
\[ \text{"end-bit"} \]
```

so that the DR 3.31 is invalid.

To restate our conclusions: The graph

```
\[ \text{---} \]
```

is correctly given in any gauge by equation 3.21 for the quantity \( [S_p(p)]^2 \Sigma(p) \) where \( \Sigma(p) \) was calculated from our D.R. 3.17. Previous calculations involving an IR divergence (other than for \( p^2 = m^2 \)) correspond to the equation 3.28, which gives part of this graph correctly but has a spurious divergence due to the implicit (our case explicit) use of false analytic assumptions.

Footnote: We have defined the spectral weights \( \rho_1, \rho_2 \) so as to conform to the notation of Strathdee [Strathdee (1964)] for the Lehman form of the D.R. for \( S(p) \) — equation 3.22 being readily transformed into Strathdee's form on the replacement

\[ \rho(\mu) = \varepsilon(\mu)[\rho_1(\mu^2 i\gamma_p + \rho_2(\mu^2)m] \]
However the expressions Strathdee writes down for $\rho_1$ and $\rho_2$ differ markedly from the correct values above. It is however possible that these incorrect values arise from Strathdee's procedure wherein he took the photon propagator as

$$\mathcal{D}_{\mu\nu} = -i\left\{ \frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + \mu^2 - i\epsilon} + \frac{\lambda^2}{\mu^2} \frac{k_\mu k_\nu / k^4}{k^2 + \lambda^2 i\epsilon} \right\} \gamma$$

and took the simultaneous limits

$$\lambda^2 \to 0, \quad \mu^2 \to 0, \quad \lambda^2 / \mu^2 \to a \quad \gamma$$

at the conclusion of the calculation.
4. Radiative correction to 3-Vertex

We are concerned with the evaluation in an arbitrary gauge of the graph

\[ e \text{ rad } V_\lambda (p''p') = \frac{i}{p'' - k - p' - k} \]

4.1

So that we need to deal with the function of one variable only, namely the variable

\[ Q = p'' - p' \]

4.2

we evaluate \( V_\lambda (p''p') \) for

\[ p'^2 = p''^2 = p^2 \]

4.3a

and make our dispersive theoretic calculation conform to usual approach by setting

\[ p^2 = -m^2 \]

4.3b

As this vertex part is inserted between electron lines we can replace \( i\gamma^n \) on left hand side (LHS) of \( V_\lambda (p''p') \) by \(-m\), that is

- LHS : \( i\gamma^n = -m \)

4.4

and

RHS : \( i\gamma^n = -m \)

4.5

More formally, one can introduce "coverings" and calculate \( \bar{u}(p'') \) \( V_\lambda (p''p') u(p') \), the \( u(p) \) to correspond to mass \((-p^2)^{1/2}\).

We apply Cutkosky's prescription of replacing the appropriate \( (p'^2 + m^2 - i\varepsilon) \) denominators by \( 2\pi\delta_p (p'^2 + m^2) \) to determine the discontinuity of \( V_\lambda \) for a jump from the physical \( Q^2 \) plane - or equivalently we determine the two body unitarity contribution to \( V_\lambda \) as determined from

\[ \text{disc rad } V_\lambda (p''p') = \frac{i\varepsilon^2}{(2\pi)^4} P \int \frac{dk}{k^2} 2\pi i \delta_p [(p''-k)^2 + m^2] 2\pi i \delta_p [(p'-k)^2 + m^2] W_\lambda \]

4.6

where in gauge a

\[ W_\lambda = \gamma_\mu [i\gamma^n - i\gamma - m] \gamma_\nu [i\gamma^m - i\gamma - m] \gamma_\nu [\delta_\mu^\nu + (a-1)k_k^2] \]

4.7

Now on LHS by 4.3,

\[ \gamma_\mu i\gamma^n = -i\gamma^n \gamma_\mu + 2i\gamma_\mu = m\gamma_\mu + 2i\gamma_\mu \]

4.8
so the outer factors in $\mathbb{W}_\lambda$ may be replaced by

$$\gamma_\mu (i\not{p}'' - i\not{k} - m) = 2ip_\mu'' - \gamma_\mu i\not{k}. \quad 4.9$$

Within the integrand of 4.6,

$$(p'' - k)^2 = -m^2 \quad 4.10$$

whence

$$2p''k - k^2 = 0. \quad 4.11$$

Inserting this result into the contraction of 4.9:

$$\gamma_\mu (i\not{p}'' - i\not{k} - m) k_\mu = 2ip''k - ik^2 = 0 \quad 4.12$$

we see that the term in disc rad $V_\lambda$ proportional to $(a - l)$ vanishes, i.e., the vertex correction is independent of gauge for $p^2 = -m^2$. This new result may be likewise demonstrated for the other 3-vertex computed in this thesis.

We can therefore replace $\mathbb{W}_\lambda$ in the integral by

$$\mathbb{W}_\lambda = [2i\not{p}''_\mu - \gamma_\mu i\not{k}]\gamma_\lambda [2i\not{p}''_\nu - i\not{k}\gamma_\nu]\delta_{\mu\nu} \quad 4.13$$

$$= -4p''p''\gamma_\lambda - 2ip''i\not{k}\gamma_\lambda - 2\gamma_\lambda i\not{k}i\not{p}'' - \gamma_\mu k\gamma_\lambda k\gamma_\mu \quad 4.14$$

$$= -4p''p''\gamma_\lambda + 2im[k\gamma_\lambda + i\not{k}\gamma_\lambda] - 4k(p''p'')\gamma_\lambda + 4(p''p'')k\gamma_\lambda + 4k^2\gamma_\lambda \quad 4.15$$

$$= -4p''p''\gamma_\lambda + 4(p' + p'')k\gamma_\lambda - 2k^2\gamma_\lambda + 4k\not{k} + 4im k\gamma_\lambda. \quad 4.16$$

We can now utilise the integrals of II Appendix 5 – recall that we replaced the infinite $V(1)$ by $V[k^2/(k^2 + \mu^2)]$ by giving the photon a vanishing mass $\mu$:

$$\text{disc rad } V_\lambda (p''p') = \frac{-ie^3}{(2\pi)^2} \frac{\pi}{-2q^2} \frac{\Theta(-q^2 - 4m^2)}{(1 + 4m^2/q^2)^{1/2}} \mathbb{W}_\lambda \quad 4.17$$
where $\bar{\lambda} = \frac{\lambda}{\lambda} \log \left[ 1 - \frac{(q^2 + 4m^2)/\mu^2}{\lambda} \right] + 2(p^1 + p^2)^2 \gamma_\lambda$

$$+ 4 \left[ \frac{1}{\lambda} \left( 1 + 2m^2/q^2 \right) \log \left[ 1 - \frac{(q^2 + 4m^2)/\mu^2}{\lambda} \right] + \frac{1}{\lambda} \gamma_\lambda q^2 \right] - \frac{m^2}{2q^2} \left[ \text{im} (p^1 + p^2) \gamma_\lambda + \frac{1}{\lambda} \gamma_\lambda q^2 \right]$$

$$- 8 \text{im} (p^1 + p^2)_\lambda = 4 \text{im} (p^1 + p^2)_\lambda$$

$$= 2q^2 \gamma_\lambda \left\{ \left( 1 + 2m^2/q^2 \right) \log \left[ 1 - \frac{(q^2 + 4m^2)/\mu^2}{\lambda} \right] - \frac{3}{2} \left( 1 + 4m^2/q^2 \right) \right\}$$

$$+ 4q^2 \left[ i (p^1 + p^2)_\lambda / 2m \right] (- m^2/q^2)$$


The vertex function by 4.4 has the form (invariant)

$$\nu_\mu (p'^p) = \gamma_\mu R(q^2) + \frac{i (p'^p)\mu}{2m} s(q^2)$$

To second order,

$$\nu_\mu (p'^p) = e[\gamma_\mu + \text{rad} \nu_\mu (p'^p)]$$

Then assuming the D.R. below 4.19, determines the spectral weights

$$r(q^2) = \frac{-e^2}{2\pi^2} \left[ \left( 1 + 2m^2/q^2 \right) \log \left[ 1 - \frac{(q^2 + 4m^2)/\mu}{\lambda} \right] + \frac{3}{2} \left( 1 + 4m^2/q^2 \right) \right] \text{e}^{(-q^2 - 4m^2)}$$

$$s(q^2) = \frac{-e^2}{4\pi^2} \frac{m^2}{q^2} \left( 1 + 4m^2/q^2 \right) \text{e}^{(-q^2 - 4m^2)}$$

to be substituted in the D.R.

$$R(q^2) = -q^2 \int_0^\infty \frac{r(-x)dx}{[q^2 + x - i\epsilon]}$$

$$S(q^2) = \int_0^\infty \frac{s(-x)dx}{q^2 + x - i\epsilon}$$

We note

(a) $\nu_\mu (p'^p)$ is the proper vertex - i.e. sum of proper parts

(b) The D.R. 4.22, 4.23 have been proved to any order of perturbation theory for $\nu_\mu (p'^p)$ when $p^1 = p^2 = \epsilon p^2 \ll m^2$. However, as Ya. Fainberg (JETP 1960) proved, in fact such and D.R. is not satisfied by the Feynman
integral expression for $R(Q^2)$ when $p^2 = -m^2$. However the device used
[following Kallen (1958)] of introducing a small photon mass $\mu$ circumvents
this difficulty though in an unattractive manner.

(c) The DR 4.22, 4.23 have been established outside of perturbation theory
only for $p^2 < 0$ [cf statement in V. Ya. Fainberg (JETP 1960)]

(d) The term in the vertex

$$S(Q^2)i[p'_\mu + p''_\mu]/2m$$

corresponds to an additional magnetic moment ascribable to the electron :
expressing

$$S(Q^2) = S(0) - Q^2 \int_0^\infty \frac{s(-x)dx}{x[q^2 + x - i\epsilon]} \quad 4.26$$

where the second term vanishes for $Q^2 = 0$, we see that the gyromagnetic
ratio is given by

$$g - 2 = 2 S(0) \quad 4.27$$

$$= \frac{2\alpha^2}{4\pi^2} \int_{4\pi m^2}^\infty \frac{m^2}{x} \frac{(1 - 4m^2/x)^{-1/2}}{x} \, dx \quad 4.28$$

i.e.,

$$g = 2 \left(1 + \frac{e^2}{8\pi^2}\right) \quad 4.29$$

to this approximation.

In conclusion we would suggest that rather than do as we have done in present­ing
the dispersion theoretic reformulation of the usual calculation of $3\text{-}\text{Vertex},$
that one should instead calculate the radiative correction

$$e \text{ rad } V_{\lambda} (p''p') = \frac{p'' - k}{p' - k} \quad 4.30$$
on the mass shelf \(-p'^2 = -p''^2 = -p^2 \ll m^2\). Then the D.R. 4.24, 4.25 are unambiguously valid, the spectral weights \(r(p^2)\) and \(s(p^2)\) being finite. We would note that to perform this calculation we must need to calculate a new series of phase integrals to replace the \(V[p]\). Note in particular that in place of \(V[1]\) there is

\[
\int \frac{d^4 k}{k^2} \delta_p [(p''^2 - k^2 + m^2)] \delta_p [(p'^2 - k^2 + m^2)]
\]

\[\propto (1 + 4m^2/q^2)^{1/2} [\log (-q^2 - 4m^2) - \log (p^2 + m^2)] \theta(-q^2 - 4m^2).
\]
CHAPTER IV

THEORY OF CHARGED PARTICLES OF SPIN ZERO

1. Introduction

The subject matter of this chapter is the equivalence, or otherwise, of various graph theories of the quantum electrodynamics of zero spin particles: uncharged spin zero particles can be described by a variety of different fields (cf. Chapter I) all spin zero representations of the inhomogeneous Lorentz group; on making the usual ad hoc prescription \( \delta_\mu \rightarrow \pi_\mu = \delta_\mu - i e A_\mu \) etc. [but see re spin 3/2 Pauli Fierz (1939)] and following the usual procedures one may derive a plentitude of graph theories claiming to describe the interaction of spin zero particles with quanta of the electromagnetic field. Now by the equivalence of classical field theories we would mean the equality of the values calculated for such quasi-observables as energy-momentum tensor and probability current. By the equivalence of graph theories we likewise mean the equality of matrix elements describing the same process: such equality to hold irrespective of the assignment of different values to the graph elements and perhaps also despite different rules for constructing graphs. [On this last point note the absence of seagull vertices in \( \beta \) theory of spin zero]. It is our belief that all well-behaved graph theories of spin zero shall turn out to be equivalent in this sense: that is, we subscribe to the "representational invariance" of good theories of charged spin zero.

Although we are ultimately only concerned with the equivalence of graph theories, we will clarify the meaning of representational invariance in terms of the equations for charged matter field. Charged spin zero in usual \( 5 \times 5 \) \( \beta \) formalism is described by a 5 component field \( \psi \) satisfying

\[
(\beta \pi + m)\psi = 0.
\]

Writing out \( \frac{1}{1.1} \) for usual \( \beta \) matrices and \( \psi = [-m^{-1} \psi, \psi] \)
gives
\[ \pi_\mu \psi - \dot{\psi}_\mu = 0 , \]  \hspace{1cm} (1.1a)
\[ - \frac{1}{m} \pi_\mu \dot{\psi}_\mu + m \dot{\psi} = 0 , \]  \hspace{1cm} (1.1b)

which establishes the equivalence of equation 1.1 with the equation
\[ (\pi^2 - m^2) \phi = 0 , \]

where \( \phi \) is an (homogeneous Lorentz) scalar field. These \( \psi \) and \( \phi \) are two different representations (in our sense) of spin zero fields: no one has ever doubted the equivalence of the usual graph theories derived from perturbation theory \( \tau \alpha \) (1.1 and 1.2). On the other hand, within the context of the 5 component \( \beta \) representation of spin zero, a different graph theory can be formulated, to correspond to the field equation
\[ (\beta \pi + m + \epsilon K \beta_{\mu\nu} F_{\mu\nu}) \psi = 0 \]  \hspace{1cm} (1.3)

This graph theory gives for \( K \neq 0 \) infinite (i.e., "non-renormalizable") second order radiative corrections, this supplying us with an instance of a "bad" graph theory inequivalent to the "good" usual \( \beta \) graph theory.

In Section 2 of this Chapter we give a dispersion - theoretic account of the usual graph theory of charged scalar spin zero. We simply write down in IV T 2 the usual graph elements for this theory and then proceed to determine the spectral weights of the photon and meson propagators and of the meson 3-vertex.

In Section 3 we discuss the derivation of the graph elements for the vector spin zero meson, i.e. for the graph theory corresponding to the field equation
\[ \pi_\mu \pi^\nu \phi^\circ - m^2 \phi^\circ = 0 . \]  \hspace{1cm} (1.4)

It is worth noting that when we originally decided to analyse this theory we had thought that the value of internal meson lines was...
and that this theory would serve as a counterexample to the "fundamental criterion" of Salam and Delbourgo \((1964)\) Section 13 which states that at least asymptotically

\[
G_{\nu\nu}^{1/2} = 0 \left(1/p^2\right) \quad (1.6)
\]

However the theoretical considerations of section 3 lead us to specify that the value of an internal spin zero line is

\[
G_{\alpha\beta} = \frac{i p_\alpha p_\beta}{p^2 + m^2 - i\varepsilon} \quad (1.7)
\]

The graph elements of this theory are to be found in IV T 4. We use them to calculate the same quantities as was determined in section 2 and we do in fact get the same values for the spectral weights of the photon and meson propagators and of the 3-vertex. Our happy experience with these lower order quantities leads us to postulate the equivalence of the scalar and our vector spin zero theories to all orders, an assertion which satisfies our demand of "representational invariance" of all spin zero theories.

As an interesting side point and check on the considerations of Section 2, in Section 3 we also evaluated the same diagrams using \(G_{\alpha\beta}^Y\) and another 'possibility', \(G_{\alpha\beta}^X\) in place of \(G_{\alpha\beta}\). We found that though these values gave the same magnitude to photon propagator and 3-vertex - in the case of meson propagator they proved totally recalcitrant and intractable quantities.

The usual 5 \(\times\) 5 matrix theory of charged spin zero is in some senses intermediate between the scalar and vector theories, and it was therefore of some interest to explore the graph theory using dispersion theoretic methods. We have therefore done this in Section 5. No difficulty was encountered in using DR methods in \(\beta\).
theory, but unlike the vector spin zero theory there is some difficulty in seeing
the relationship between the spectral weights of the \( \beta \) theory and of the scalar
theory.
2. **Scalar Spin Zero**

Without further ado we write down the well known graph elements for the scalar theory.

### TABLE IV T 2

<table>
<thead>
<tr>
<th>Element</th>
<th>Graph</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal photon line [in Fermi gauge]</td>
<td><img src="image1" alt="Graph" /></td>
<td>[D_{\mu \nu} = \frac{-i \delta_{\mu \nu}}{k^2 - i\epsilon}]</td>
</tr>
<tr>
<td>Internal meson line</td>
<td><img src="image2" alt="Graph" /></td>
<td>[G_m(p) = \frac{-i}{p^2 + m^2 - i\epsilon}]</td>
</tr>
<tr>
<td>3-Vertex</td>
<td><img src="image3" alt="Graph" /></td>
<td>[V_\mu(p''p') = i\varepsilon (p'' + p') \mu]</td>
</tr>
<tr>
<td>4 Vertex (&quot;Seagull&quot;)</td>
<td><img src="image4" alt="Graph" /></td>
<td>[U = 2i\varepsilon^2 \delta_{\mu \nu}]</td>
</tr>
<tr>
<td>External photon line (polarization (\lambda))</td>
<td><img src="image5" alt="Graph" /></td>
<td>[\frac{\varepsilon_\mu(\lambda)(k)}{(2\pi)^{3/2}(2</td>
</tr>
<tr>
<td>External meson line</td>
<td><img src="image6" alt="Graph" /></td>
<td>[\frac{1}{(2\pi)^{3/2}(2</td>
</tr>
<tr>
<td>Additional loop factor</td>
<td><img src="image7" alt="Graph" /></td>
<td>1/2</td>
</tr>
</tbody>
</table>
Each diagram is evaluated by the integration $(2\pi)^{-4} \int d^3 k \, d k_0$ over each independent internal momenta $k$. Complete diagrams, as distinct from diagram parts, acquire a factor $2n i \delta(p_i - p_f)$ which expresses overall momentum conservation.

The vertex functions in the above table are those appropriate to a charged (anti-) particle of charge $(-)e$; these functions and $G_F(p)$ satisfy the generalised Ward's identities [Ward (1951); Takahashi (1957)]

$$G_F^{-1}(p'') - G_F^{-1}(p') = e^{-1}(p'' - p') \, \mu \, V_\mu (p'' - p')$$  \hspace{1cm} \text{IV 2.1}$$

and

$$V_{\nu} (p'' + k, p' + k) - V_{\nu}(p''p') = -e^{-1} k_\mu U_{\mu\nu} [p'' + k, p'; k]$$  \hspace{1cm} 2.2$$

We are to calculate the lowest order radiative corrections to photon and meson propagators and to the 3-vertex function; these corrections are well-known but our use of dispersion-theoretic techniques is an innovation. First we recall the D.R. for the gauge-invariant part of the photon propagator

$$D_{\mu\nu}(k) = -i \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \left( 1 + k^2 \int_0^{\infty} \frac{\rho_3(m^2) \, d \, m^2}{k^2 + m^2 - i\epsilon} \right), \hspace{1cm} 2.3$$

which specifies the branch-cut discontinuity of $D_{\mu\nu}$ as

$$\text{disc } D_{\mu\nu} = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \, 2\pi \, \rho_3(-k^2). \hspace{1cm} 2.4$$

Then the graphical formula

$$D_{\mu\nu}(k)$$
may be evaluated by use of the above table to determine the second order contribution to \( D_{\mu \nu} \) as

\[
-\frac{e^2}{(2\pi)^4 k^4} \int \frac{d^4 p \cdot d^4 p''}{(p^2 + m^2 - i\epsilon) (p''^2 + m^2 - i\epsilon)} \delta(k-p+p'') \delta(p''+m^2) \mu \mu (p'+p'') \nu
\]

The Cutkosky prescription of replacing \((p^2 + m^2 - i\epsilon)\) denominators by

\[
2\pi i \delta_p (p^2 + m^2)
\]

gives the unitarity discontinuity

\[
\text{disc } D_{\mu \nu} (k) = -\frac{e^2}{(2\pi)^2 k^4} \int \frac{d^4 p \cdot d^4 p''}{(p^2 + m^2 - i\epsilon) (p''^2 + m^2 - i\epsilon)} \delta(k-p+p'') \delta(p''+m^2) \delta(p'+m^2) \mu \mu (p'+p'') \nu
\]

Then by II A 2.17,

\[
\text{disc } D_{\mu \nu} (k) = -\frac{e^2}{24\pi k^2} (\delta_{\mu \nu} - k_{\mu} k_{\nu} / k^2) (1 + 4m^2 / k^2)^{3/2} \sigma(-k^2 - 4m^2).
\]

Thus we have determined to second order the Lehmann-Kallen spectral weight \( \rho_3 \)

\[
\rho_3 (\mu^2) = \frac{e^2}{48 \pi^2 \mu^2} (1 - \frac{4m^2}{\mu^2})^{3/2} \sigma(\mu^2 - 4m^2)
\]

which can be substituted in 2.3 to give an explicit expression for \( D_{\mu \nu} \).

In Chapter II we discussed Lehman's derivation of the D.R. for the scalar spin zero propagator \( G(p) \):

\[
G(p) = G_F(p) [1 + (p^2 + m^2)] \int_0^{\infty} \frac{\sigma(\mu^2) d\mu^2}{p^2 + \mu^2 - i\epsilon}
\]

Thence the branch cut discontinuity of \( G(p) \) is

\[
\text{disc } G(p) = 2\pi \sigma(-p^2)
\]
and this is determined to second lowest order by analysis of the graph

\[ \begin{aligned}
\text{We have in Fermi gauge} \\
G(p) &= G_F(p) - \frac{e^2}{(p^2 + m^2)^2} \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{\delta(q-p-k)(p+q)(\rho(p+q))_\mu \delta_{\mu\nu}}{(q^2 + m^2 - i\epsilon)(k^2 - i\epsilon)}, \tag{2.13}
\end{aligned} \]

so the Cutkosky prescription gives,

\[ \text{disc } G(p) = \frac{1}{(p^2 + m^2)^2} \frac{e^2}{(2\pi)^2} \int d^4q d^4k \delta(q-p-k)\delta(p^2 + m^2)\delta(k^2)(p+q)^2. \tag{2.14} \]

On the mass shell,

\[ (p+q)^2 = (p+q)^2 + (p-q)^2 - k^2 = 2p^2 (1 - m^2/p^2) \tag{2.15} \]

The integral in 2.14 is given by IIA 3.5, whereby

\[ \text{disc } G(p) = \frac{e^2}{4\pi} \frac{(1 - m^2/p^2)}{p^2 + m} \Theta(-p^2 - m^2), \tag{2.16} \]

thence
\[ \sigma(\mu^2) = \frac{e^2}{\delta\pi^2} \frac{(\mu^2 + m^2)}{(\mu^2 - m^2)\mu^2} \Theta(\mu^2 - m^2). \tag{2.17} \]

To determine \( \sigma(\mu^2) \) in an arbitrary gauge specified by the parameter \( a \) [of Feldman and Mathews (1963)] we must replace \( \delta_{\mu\nu} \) in 2.13 as follows

\[ \delta_{\mu\nu} \rightarrow \delta_{\mu\nu} + (a - 1)k\mu k^2. \tag{2.18} \]

Then as by IIA 4.25 the mass shell value of
\[ k^{-2}[(p+q)\cdot k]^2 = k^{-2} [k^4 + 4k^2 p\cdot k + 4(p\cdot k)^2] \tag{2.19} \]

\[ = 0-2(p^2+m^2) + 4p^2 \left[ \frac{1}{12} \left( 2 - \frac{3m^2}{p^2} \right) + \frac{1}{3} \left( 1 + \frac{3m^2}{p^2} \right) \right] \tag{2.20} \]

\[ = m^2 \tag{2.21} \]

the magnitude of the spectral weight in gauge \( a \) is

\[ \sigma (\mu^2) = \frac{-e^2}{16\pi^2} \frac{[2\mu^2 + (\gamma-a)m^2]}{\mu^2 (\mu^2-m^2)} \quad \Theta (\mu^2-m^2). \tag{2.22} \]

We are perturbed to note that the asymptotic value of \( \sigma (\mu^2) \) is gauge independent and negative - in contradiction to the Lehman theorem:

\[ \sigma (\mu^2) > 0 \tag{2.23} \]

Of course for a particular fixed \( \mu^2 \), \( \sigma (\mu^2) \) becomes positive as \( a \to \infty \), which corresponds [Feldman and Mathews (1963)] to timelike photons becoming infinitely massive and therefore no longer making negative contributions to \( \sigma (\mu^2) \).

The radiative correction to the \( \gamma \)-vertex in second order is \( \text{rad } V_\lambda (p'p) \):

\[ \text{rad } V_\lambda (p'p) = \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{gamma_vertex}\end{array} \tag{2.24} \]

\[ \text{e.g., } \text{rad } V_\lambda (p'p') = \frac{e^3}{(2\pi)^3} \int d^4k \quad \frac{\delta_{\mu\nu} (2p'\cdot k) \mu (2p\cdot k) \nu (p+p'\cdot 2k) \lambda}{[k^2-\text{i}\varepsilon][ (p'\cdot k)^2+m^2-\text{i}\varepsilon][ (p\cdot k)^2+m^2-\text{i}\varepsilon]} \tag{2.25} \]

We evaluate this expression on the mass shell \( p'^2 = p^2 = -m^2 \); in view of the discussion of spin 1/2 case in Chapter III, we have in mind the taking of the limit
\[ p^2 = p'^2 \rightarrow -m^2 \] from the unphysical region, thereby validating Cutkosky's prescription to determine the discontinuity

\[
disc V_\mu(p',p) = \frac{-e^3}{(2\pi)^2} \int \frac{dk^2}{k^2} \frac{\delta [(p' - k)^2 + m^2] \delta [(p - k)^2 + m^2] (2p' - k) \cdot \delta \mu \nu (p + p' - 2k)}{(p' - k)^2 + m^2 (2p' - k) \cdot \delta \mu \nu (p + p' - 2k)}. \tag{2.26}
\]

As in the spin \(1/2\) case this expression is independent of the gauge of the intermediate photon: for on the mass shell \((p' - k)^2 = p'^2 = -m^2\), \tag{2.27}

implying \(k \cdot (2p' - k) = 0\) \tag{2.28}

so hence the integral in 2.26 vanishes on the replacement of \(\delta \mu \nu \) by \(k \cdot k /k^2\).

Referring to Appendix 5 of Chapter II we may determine - note especially II A 5.23 -

\[
V [(2p - k) \cdot (2p' - k) (p + p' - 2k)] = V [(-2q^2 - 4m^2 - k^2) (p + p' - 2k)] \tag{2.29}
\]

\[
= -(2q^2 + 4m^2)(p + p') V(k) + 4(q^2 + 2m^2) V(k) + (p + p') V(k^2) + 2V(k^2 k) \tag{2.30}
\]

\[
= (q^2 + 2m^2) V(1) + 4 V(k) \tag{2.31}
\]

Setting \(disc V_\mu(p',p) = e(p + p') \cdot p \cdot V(q^2 q^2)\), \tag{2.32}

the computations of II A give (note \(\mu^2 \rightarrow 0\))

\[
V(q^2) = \frac{e^2}{(2\pi)^2} \left( 1 - \frac{1}{2} \log \left(1 - \frac{q^2 + 4m^2}{\mu^2}\right) \right) \frac{(1 + 2m^2/q^2) q^2 (q^2 - 4m^2)}{q^2 (1 + 4m^2/q^2)^{1/2}} \tag{2.33}
\]

the D.R. (herein unproven) for the vertex function being

\[
V_\mu(p',p) = ie(p + p') \cdot p \left( 1 - q^2 \int_0^{\infty} \frac{dx \cdot V(-x)}{q^2 + x - i\epsilon} \right) \tag{2.34}
\]
We have not as yet discussed the other radiative correction to 3-vertex viz.

\[ \mathcal{W}_\mu (p^*p) = \mathcal{W}_\mu (p^*) + \mathcal{W}_\mu (p) \]  

where the graph for \( \mathcal{W}_\mu (p) \) is

\[ \mathcal{W}_\mu (p) = \frac{p^*}{p + k} - \frac{p}{p} \]  

Applying Cutkosky's principle determines the discontinuity of \( \mathcal{W}_\mu (p) \) in gauge a as

\[ \text{disc } \mathcal{W}_\mu (p) = \frac{-2i e^3}{(2\pi)^2} \int \frac{d^4k}{p^2} \delta^4(p+k)[\delta (p^2 + m^2) \delta (k^2) + \delta (k^2) \delta (p^2 + m^2)] \]  

As within the integrand

\[ (2p + k) \cdot k = -(p^2 + m^2) \]  

We can utilise \( \text{II A 3.5, II A 4.25} \) to write

\[ \text{disc } \mathcal{W}_\mu (p) = \int \frac{d^4k}{p^2} \left[ (3m^2/p^2) + (a-1)(1+2m^2/p^2) \right] I_{\mu \nu} \]  

in which

\[ I_{\mu \nu} = \frac{\pi}{2} \left( 1 + m^2/p^2 \right) \Theta (-p^2 - m^2) \]  

vanishes on mass shell \( p^2 = -m^2 \). Then we see that

\[ \text{disc } \mathcal{W}_\mu (p^*^2 = -m^2; p^2 = -m^2) = 0 \]  

It is apparent that just like the electron S.E. part \( \Sigma(p) \) of Chapter III, that \( \mathcal{W}_\mu (p) \) could satisfy a subtracted dispersion relation for which

\[ \mathcal{W}_\mu (p; p^2 = -m^2) = 0 \]  

Thus on mass shell

\[ \mathcal{W}_\mu (p^*^2 = -m^2; p^2 = -m^2) = 0 \]
However considerations of relativistic invariance make the notion of separate
DR for $W_{\mu}(p)$ and $W_{\mu}(p')$ unpalatable; we prefer to assert the D.R. 2.34 for the
3-vertex and note that 2.41 implies that $W_{\mu}(p'p)$ makes no additions to the spectral
weight $\nu(q^2)$; a statement equivalent to 2.42.
3. Vector Spin Zero Graph Elements

The starting point of this theory is the free wave equation for the field 
\( \phi_\mu^0 \) which is a vectorial representation of the Homogeneous Lorentz Group:

\[
\ddot{\phi}_\nu^0 - m^2 \phi_\mu^0 = 0 .
\]  \( 3.1 \)

We showed in Section 2 of Chapter I that a \( \phi_\mu^0 \) satisfying 3.1 is a zeroth eigenket of the invariant intrinsic spin operator \( \Gamma^2 \). We also noted that 3.1 implies (by contraction)

\[
\ddot{\phi}_\nu^0 = \ddot{\phi}_\mu^0 .
\]  \( 3.2 \)

Perhaps because others have only been concerned with the vectorial representation of the Homogeneous Lorentz group as providing the simplest representation of spin one, and the vector spin zero as merely a nuisance eliminatable by the supplementary condition

\[
\frac{\partial}{\partial \mu} \phi_\mu^0 = 0 .
\]  \( 3.2 \)

(Note analogy with 3.2!) serious attention has not been given to the vector spin zero field: even the basic equation 3.1 does not appear in the literature.

Polubarinov and Ogievsky in several papers give a careful treatment of the \( \phi_\mu^1 \) field, and stress the importance of the \( \Gamma^2 \) operator, but never explicitly write down the field equation for \( \phi_\mu^0 \).

It is worthwhile to point out a curious reciprocity between the vector spin zero field satisfying 3.1 and the vector spin one field satisfying

\[
[\left(\ddot{\phi}_\rho^\rho - m^2\right)\delta_{\mu\nu} - \ddot{\phi}_\mu^0]\phi_\nu^1 = 0 ,
\]  \( 3.3 \)
which becomes apparent by comparing 3.1 and 3.3 with the identity

\[
[m^2 \delta_{\mu \nu} + p_{\mu} p_{\nu}] [(p^2 + m^2) \delta_{\nu \lambda} - p_{\nu} p_{\lambda}] = m^2 (p^2 + m^2) \delta_{\mu \lambda} \quad .
\]

This identity also links the "natural" Green's functions of the two fields.

In terms of \( \pi_\mu = \partial_\mu - ie A_\mu \),

We write the field equation for charged vector spin zero as

\[
\pi_\mu \pi_\nu \phi^o_\nu - m^2 \phi^o_\mu = 0 \quad ,
\]

which implies

\[
\pi_\mu \pi_\nu \phi^o_\nu = \pi_\rho \pi_\rho \phi^o_\mu \quad .
\]

our choice of field equation being limited by the requirement that \( \pi_\mu f \) should satisfy it, when \( f \) is a charged scalar field obeying

\[
(p^2 - m^2)f = 0 \quad .
\]

The zero affix on \( \phi^o_\mu \) is omitted in the rest of this section.

By taking the usual lagrangian for the scalar field \( f \),

\[
L_s = - (\pi_\mu f)^* \pi_\mu f - m^2 f^* f \quad ,
\]

and making the replacements

\[
f \rightarrow m^{-1} \pi_\mu \phi_\mu \quad ; \quad \pi_\mu f \rightarrow m \phi_\mu \quad .
\]
one obtains
\[ -L_v = -m^2 \partial_{\mu}^2 \phi_{\mu} - (\pi_{\mu} \phi_{\mu}) \pi_{\rho} \phi_{\rho} \]
showing that \( L_v \) is a Lagrangian for \( \phi_{\mu} \) satisfying \( 3.6 \). We take the 3 vertex and 4 vertex functions as the matrix elements of
\[ -i [L_v(e = 0) - L_v(e)] = -e (A_{\mu} \phi_{\mu}^* \phi_{\nu} - \delta_{\mu\nu} \phi_{\mu}^* \phi_{\mu}) + ie A_{\mu} A_{\nu} \phi_{\mu}^* \phi_{\nu} \]
where all operators are regarded as free fields - the Fourier expansion of \( \phi_{\mu} \) being written below.

The conventional pragmatic argument for fixing the propagator for vector spin one as
\[ -i (\delta_{\mu\nu} + m^2 p_{\mu} p_{\nu})/(p^2 + m^2 - i\epsilon) \]
and 3.4 and further insist on a Feynman-Stueckelberg denominator: we apply the same argument, comparing 3.1 and 3.4 to obtain as the propagator for vector spin zero
\[ G^{X}_{F\mu\nu} = -i \left[ p_{\mu} p_{\nu} - (p^2 + m^2)\delta_{\mu\nu} \right] / m^2 p^2 + m^2 - i\epsilon \]

Therefore, if we have cause for rejecting \( G^{X}_{F} \), we likewise have good cause for rejecting the usual spin one propagator. In many places we find that the vector spin zero propagator is implicitly taken as
\[ G^{X}_{F\mu\nu} = \frac{-i p_{\mu} p_{\nu}}{p^2 + m^2 - i\epsilon} \]

The more sophisticated argument for determining the propagator is to calculate the vacuum expectation value \( \langle T \phi_{\mu}^* (x) \phi_{\nu} (0) \rangle \). We refer our readers to the parallel argument for spin one to be found in Appendix A2 of Lee and Yang. Briefly,
in our case we would take the Fourier decomposition of the second quantised field as
\[
\Phi(X) = (2\pi)^{-3/2} \int d^3k (2\pi)^{-1/2} \left[ a(k) \exp(ikr - i\omega t) + b(k) \exp(ikr + i\omega t) \right] \frac{k}{m} ,
\]
\[
\Phi_\mu(X) = (2\pi)^{-3/2} \int d^3k (2\pi)^{-1/2} \left[ a(k) \exp(ikr - i\omega t) - b(k) \exp(ikr + i\omega t) \right] (-i\omega) / m ,
\]
in which \( w = (k^2 + m^2)^{1/2} \),

and \( a(k), b(k) \) are annihilation operators for which

\[
[a(k), a^*(l)] = [b(k), b^*(l)] = \delta^3(k-l) ,
\]

and other commutators vanish. Dyson's T product is defined for \( X = (r,t) \) by

\[
T \Phi_\mu(X) \Phi_\nu^*(o) = \Phi_\mu(X) \Phi_\nu^*(o) \text{ for } t > o
\]

\[
= \Phi_\nu^*(o) \Phi_\mu(X) \text{ for } t < o
\]

\[
= \frac{1}{2} (\Phi_\mu(X) \Phi_\nu^*(o) + \Phi_\nu^*(o) \Phi_\mu(X)) \text{ for } t = o
\]

has the vacuum expectation value

\[
<T \Phi_\mu(X) \Phi_\nu^*(o)> = (2\pi)^{-3} \int d^3k (2\pi)^{-1} \exp(ikr - i\omega t) q_\mu q_\nu / m^2 \text{ for } t > o
\]

\[
= (2\pi)^{-3} \int d^3k (2\pi)^{-1} \exp(ikr + i\omega t) q_\mu^* q_\nu^* / m^2 \text{ for } t < o
\]

\[
= (2\pi)^{-3} \int d^3k (2\pi)^{-1} \exp(ikr) [q_\mu q_\nu + q_\mu^* q_\nu^*] / 2m^2 \text{ for } t = o ,
\]

where \( q_\mu = i\omega, q_j = k_j (j = 1,2,3) \). One can replace the right hand side of 3.21 with the following Feynman type integral

\[
\Xi = (2\pi)^{-3} \int d^3k (2\pi 1)^{-1} \int_{-\infty}^{\infty} \frac{dk}{k} \frac{k^2}{k^2 + m^2 - i\epsilon} \exp ik_\mu x^\mu x_\nu
\]

The appropriate integration contours for \( t > o, \ t < o \) being indicated by the
following sketch

For $t = 0$, except in the case $\mu = 4, \nu = j$ either upper or lower contours can be taken, while for $\mu = 4, \nu = j$ the $k_0$ integration along the contour

vanishes, as is required by the symmetric definition of the $T$ product for $t = 0$.

[Lee and Yang, ibid., have made a faulty analysis, in spin one theory, of the similar determination of the vacuum expectation value of the $T$ product for $t = 0$].

One could proceed to take $G^T_{F\mu\nu}$ as the vector spin zero propagator if one now determined the Hamiltonian in the interaction representation, and determine the 3- and 4-vertex functions as well as a large number of non-covariant (i.e., non-manifestly-covariant) vertices. Presumably one could prove equivalence theorems whereby some of the non-covariant vertices could be eliminated simultaneously with the dropping of the non-covariant term in the Fourier transform of $\Xi$ (which leaves $G^X_{F\mu\nu}$). We have most deliberately abstained from such proceedings, as we have noted that the purpose of the $\delta_{\mu 4} \delta_{\nu 4}$ term is merely to give the integrand in 3.22 correct asymptotic behavior which is achieved more simply by omitting the $\delta$ term and replacing $1/m^2$ by the principal part of $1/k_\alpha k_\alpha$. Note in amplification
that
\[
\frac{k_0^2}{(k^2 - k_0^2)(k^2 + m^2 - k_0^2)} = \frac{(k^2 + m^2)/m^2}{k^2 + m^2 - k_0^2} - \frac{k^2/m^2}{k - k_0^2}
\]

explicitly showing how pole and asymptotic behaviour are just what is needed for 3.21 to be equivalent to

\[
T \phi_\mu(x) \phi_\nu^*(0) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 + m^2 - i\varepsilon} \frac{k_\mu k_\nu /k^2}{k^2 + m^2 - i\varepsilon} \exp ikx
\]

provided we do not count in the contribution from the pole of the integrand at \( k^2 = 0 \). There are several devices available for this purpose, such as specifying that

\[
\frac{1}{k^2} \rightarrow \left( \frac{1}{k^2} \right)' = \frac{1/2|k|}{-k_0 + |k| + i\varepsilon(t)} + \frac{1/2|k|}{k_0 + |k| - i\varepsilon(t)}
\]

where \( \varepsilon(t) \rightarrow 0 \) and \( \varepsilon(t) \) is the signature of \( t \). We don't seriously propose the specification 3.25, nor equivalent manipulations of the \( k_0 \) contour. Failing our future finding of an elegant specification, we shall examine the calculation of the Fourier transform \( \langle T \phi_\mu(x) \phi_\nu^*(0) \rangle \) using the modern mathematical methods of the theory of distributions, [See Lighthill (1958), and Guelfand (1962)]. For the moment, without giving a precise meaning to \( 1/p^2 \), we take the value of vector spin lines as

\[
G_{\alpha\beta}(p) = \frac{p_\alpha p_\beta /p^2}{p^2 + m^2 - i\varepsilon}
\]

The ultimate justification for this "choice" lies in the results of calculations performed in the next Section, wherein it is determined that on this \( G \) leads to a theory manifestly isomorphic with scalar spin zero (the spectral weights of vector spin zero being identical with the spectral weights of scalar spin zero).
4. Vector Spin Zero Calculations

In the previous section we detailed the considerations that lead to the following specification of graph elements for the vector spin zero theory.

<table>
<thead>
<tr>
<th>Element</th>
<th>Graph</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal photon line [in Fermi gauge]</td>
<td><img src="image" alt="Graph" /></td>
<td>$D_{\mu \nu} = \frac{-i \delta_{\mu \nu}}{k^2 - i\epsilon}$</td>
</tr>
<tr>
<td>Internal meson line</td>
<td><img src="image" alt="Graph" /></td>
<td>$G_{\alpha \beta} = \frac{i p_\alpha p_\beta / p^2}{p^2 + m^2 - i\epsilon}$</td>
</tr>
<tr>
<td>3-Vertex</td>
<td><img src="image" alt="Graph" /></td>
<td>$V_\mu (p''<em>\beta, p'</em>\alpha) = -i\epsilon (p'<em>\alpha \delta</em>{\beta \mu} + p''<em>\beta \delta</em>{\alpha \mu})$</td>
</tr>
<tr>
<td>4-Vertex</td>
<td><img src="image" alt="Graph" /></td>
<td>$U = i\epsilon^2 (\delta_{\alpha \mu} \delta_{\beta \nu} + \delta_{\alpha \nu} \delta_{\beta \mu})$</td>
</tr>
<tr>
<td>External photon line (polarization $\lambda$)</td>
<td><img src="image" alt="Graph" /></td>
<td>$\frac{\epsilon_\mu (\lambda)}{(2\pi)^{3/2}</td>
</tr>
<tr>
<td>External meson line</td>
<td><img src="image" alt="Graph" /></td>
<td>$\frac{p_\alpha / m}{(2\pi)^{3/2}</td>
</tr>
<tr>
<td>Addition loop factor</td>
<td><img src="image" alt="Graph" /></td>
<td>1/2</td>
</tr>
</tbody>
</table>

Each diagram is evaluated by the integration $\int (2\pi)^{-4} \, d^3 k \, d k_0$ over each
independent internal momenta \( k \). Complete diagrams, as distinct from diagram parts, acquire a factor \( 2\pi \delta(p_i - p_f) \) which expresses overall momentum conservation.

By equation 3.4, the putative propagator \( G^X_{\alpha\beta} \) has the inverse

\[
G^X_{\alpha\beta}^{-1}(p) = -i (p_\alpha p_\beta + m^2 \delta_{\alpha\beta})
\]

so that

\[
G^X_{\alpha\beta}^{-1}(p'') - G^X_{\alpha\beta}^{-1}(p') = e^{-1}(p''-p')^\mu [-ie (p_\alpha \delta_{\beta\mu} + p''_\beta \delta_{\alpha\mu})]
\]

\[
= e^{-1}(p''-p)^\mu V\mu(p''_\beta, p'_\alpha)
\]

Thus \( G^X \) satisfies Ward's identity. In passing we note that (omitting meson indices)

\[
V\mu(p''+k, p'+k) - V\nu(p''p') = -e^{-1} k^\mu U_{\mu\nu}
\]

so that the generalised Ward's identity is satisfied by ever 3, 4-vertices.

The reader familiar with the interesting paper by Ward and Delbourgo (1964) will note that at this early stage of the analysis it is apparent that our theory of vector spin zero is in contradiction to at least one of two basic axioms adopted in that paper:

(i) If the appropriate propagator is \( G^X \) or \( G^Y \) and yet the theory is finite, then we have produced a counterexample to "the fundamental criterion for the stability of any approximation for computing \( M \) [arbitrary graph] which bases itself on the Dyson formalism" postulated by Salam and Delbourgo, namely

\[
G^V D^{1/2} = 0 (1/k^2)
\]

for large \( k \).

(ii) Salam and Delbourgo take Ward's identity as fundamental to quantum electrodynamics. However neither \( G^Y \) or \( G \) - both putative zero-order approximation, to
the meson propagator - an inverse so neither can satisfy Ward's identity, whereas \( G^X \) does [4.3]. Nor do \( G^Y \) or \( G \) satisfy Ward's identity in the differentiated form not involving inverses.

\[
\theta S(p)/\partial p_{\mu} = e^{-1} S(p)\gamma_\mu(p,p)S(p)
\]

Of course \( G^X \) does satisfy 4.6.

We now proceed to outline calculations performed in this theory. First we note that on the mass shell \( p^2 + m^2 = 0 \), the numerators of \( G(p) \), \( G^X(p) \), \( G^Y(p) \) are all equal. Thus, provided that in the case of \( G^X(p) \) one applies Cutkosky's analysis in a simple-minded manner, merely replacing the denominator by delta functions irrespective of numerator, one will determine the same discontinuity - spectral weight - for the graphs

and all three putative propagators will give the same magnitude for the vertex function spectral weight

and for the Mandelstam double spectral functions of scattering diagrams

We save ourselves some trivial computation by noting that
\[ p^\beta V_\mu (p'^\beta, p'^\alpha) p'^\alpha = -ie \left[ p'^2 p'^\mu + p'^2 p''^\mu \right] , \]  

so that on the mass shells \( p'^2 = p''^2 = -m^2 \),

\[ m^{-2} p''^\beta V_\mu (p'^\beta, p'^\alpha) p'^\alpha = ie(p'^\mu + p''^\mu) , \]

so that the graphs depicted in 4.7 and 4.9 that involve closed meson loops, and in addition the vertex function 4.8, all have the same discontinuities when evaluated in vector spin zero theory as when evaluated in the scalar spin zero theory. In particular the correction to the photon propagator is the same for vector spin zero as for scalar spin zero, namely that (to second order) the Lehman-Kallen spectral weight \( \rho_3 \) is

\[ \rho_3(\mu^2) = \frac{e^2}{48 \pi^2 \mu^2} \left( 1 - \frac{4m^2}{\mu^2} \right)^{3/2} \Theta (\mu^2 - 4m^2) , \]

which can be substituted in 2.4 to give an explicit expression for \( D_{\mu \nu} \).

Applying Cutkosky's prescription to the graph

\[ \Sigma_{\gamma^0}(p) = \gamma \quad \quad \gamma \quad \quad \gamma \]

the branch cut discontinuity of \( \Sigma(p) \) is in Fermi gauge, for all three trial \( G \) as noted previously

\[ \text{disc} \Sigma_{\gamma^0}(p) = \frac{-e^2}{m^2} \int \frac{dk}{(2\pi)^2} \delta_p[k^2] \delta_p[(p-k)^2 + m^2] \left\{ p_\gamma \delta_{\mu \nu} + q_\gamma \delta_{\mu \nu} \right\} \left( q_\mu \delta_{\nu \rho} + p_\mu \delta_{\nu \rho} \right) \],

where the expression in curly brackets has the mass shell value (cf II Appendix 3)

\[ p^\gamma p^\delta q^2 + p_\gamma p^\delta q^2 + p^\gamma q^\delta q^2 = -m^2[p^\gamma q^\delta + p^\gamma q^\delta - m^2 \delta_{\gamma \delta}] , \]

\[ = -m^2[p^\gamma p^\delta (1 - m^2/p^2) + p^\gamma p^\delta - m^2 \delta_{\gamma \delta}] . \]
Thus

\[
\text{disc } \Sigma_{\gamma \delta}(p) = \frac{\alpha^2}{8\pi} \left[ p \gamma \rho \left( 2 - m^2/p^2 \right) - \delta_{\gamma \delta} m^2 \right] \left[ 1 + m^2/p^2 \right] \Theta \left( -p^2 - m^2 \right).
\]

The branch cut discontinuity of the corrected propagator

\[
G_{\alpha \beta}(p) = G_{F\alpha \beta}(p) + G_{F\alpha \gamma}(p) \Sigma_{\gamma \delta}(p) G_{F\gamma \delta}(p),
\]

\[
= G_{F\alpha \beta}(p) + [G \Sigma G]_{\alpha \beta}(p),
\]

is then given by

\[
\text{disc } [G \Sigma G]_{\alpha \beta} = \frac{p_\alpha p_\beta}{p^2} \frac{\alpha^2}{4\pi} \frac{\left( 1 - m^2/p^2 \right)}{p^2 + m^2},
\]

\[
\text{disc } \left[ G^X \Sigma G^X \right]_{\alpha \beta} = \frac{-p_\alpha p_\beta (1 + \frac{4m^2}{p^2} - \frac{m^4}{4}) - (p^2 + m^2) \delta_{\alpha \beta} \left( 1 + \frac{m^2}{2} \right)}{m^2(p^2 + m^2)} \Theta \left( -p^2 - m^2 \right),
\]

\[
\text{disc } \left[ G^Y \Sigma G^Y \right]_{\alpha \beta} = \frac{-p_\alpha p_\beta}{m^2} \frac{\alpha^2}{4\pi} \frac{(p^2 - m^2)}{m^2(p^2 + m^2)} \Theta \left( -p^2 - m^2 \right).
\]

These expressions give us our ultimate confidence in \( G_{F\alpha \beta} \) being the appropriate propagator for vector spin zero in Table IV T 4. For the asymptotic behavior of \( \text{disc } [G^X \Sigma G^X] \) and \( \text{disc } [G^Y \Sigma G^Y] \) hardly makes either suitable for determining spectral weights; and one is at a total loss to envisage the relationship to the scalar theory. As to our theory, we note the presence of \( p_\alpha p_\beta / p^2 \) in \( G_{F\alpha \beta} \) ensures that to any order the corrected propagator \( G_{\alpha \beta} \) shall not involve \( \delta_{\alpha \beta} \); we then postulate the D.R.
\[ G_{\alpha\beta}(p) = \frac{-\hbar \gamma_{\alpha} \gamma_{\beta}}{p^2} \frac{i}{p^2 + m^2 - i\epsilon} \left\{ 1 + \left( \frac{p^2 + m^2}{p^2 + m^2 - i\epsilon} \right) \int_{0}^{\infty} \frac{\sigma(V^2) d\mu^2}{p^2 + m^2 - i\epsilon} \right\} , \]  

which to

and see from (4.20) that in second order the vector spin zero spectral weight is

\[ \sigma^V(\mu^2) = \frac{e^2}{8\pi^2} \frac{\mu^2 + m^2}{(\mu^2 - m^2)\mu^2} \Theta(\mu^2 - m^2) . \]

Referring back to 2.17 we see that in second order

\[ \sigma^V(\mu^2) = \sigma(\mu^2) . \]

We have confidence in suggesting that this equation (4.25) holds to all orders.

In regard to the D.R. 4.23, we must positively decry simple minded attempts at derivation. The problems involved can be seen from the following scheme:

The vacuum expectation value of the \( T \) product for a charged scalar field \( f(x) \)

\[ \mathcal{I} \mathcal{S} \left\langle Tf(x)f^{\dagger}(a) \right\rangle = \int \frac{d^4k}{(2\pi)^4} \exp(i k x) \left\{ \frac{-i}{p^2 + m^2 - i\epsilon} \left[ 1 + \left( \frac{p^2 + m^2}{p^2 + m^2 - i\epsilon} \right) \int_{0}^{\infty} \frac{\sigma(\mu^2) d\mu^2}{p^2 + m^2 - i\epsilon} \right] \right\} , \]

Also

\[ \left\langle A_{\mu}(x)A_{\nu}(o) \right\rangle = \int \frac{d^4k}{(2\pi)^4} \exp(i k x) \left[ \frac{-i\delta_{\mu\nu}}{k^2 - i\epsilon} + \text{order } e^2 \right] . \]

As usual we define

\[ \text{Then in terms of } \partial_{\mu} \equiv \partial_{\mu} - ieA_{\mu} . \]

One might anticipate that 4.23 would arise from determining the vacuum expectation of

\[ m^{-2} T \pi f(x) \pi f(y) \text{ or } \Gamma^{-1} T\pi f(x) \pi f(y) . \]

We have to second order,
\[ \langle T \pi_\mu f(x) \pi_\nu f^*(y) \rangle = \langle T(\partial f(x)/\partial x_\mu)(\partial f^*(y)/\partial y) \rangle + e^2 \langle T A_{\mu}(x) A_{\nu}(y) f(x) f^*(y) \rangle. \]  

4.29

The second term in 4.27 is easily overlooked: if one chooses to ignore this term then presuming that differentiation within the integrand is permitted one will at once arrive at the integral in 4.23 (though perhaps with \( p^{-2} \) replaced by \( m^{-2} \)). The neglected term can however be readily determined. It is in Fermi gauge

\[ e^2 \langle T A_{\mu}(x) A_{\nu}(o) f(x) f^*(o) \rangle = e^2 \langle T A_{\mu}(x) A_{\nu}(o) \rangle \langle T f(x) f^*(o) \rangle \]

4.28

\[ = e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i\delta_{\mu\nu}}{k^2-i\varepsilon} \exp(ikx) \int \frac{d^4l}{(2\pi)^4} \frac{-i}{l^2+m^2-i\varepsilon} \exp(ilx) \]

4.29

\[ = e^2 \int \frac{d^4k}{(2\pi)^4} \exp ikx \left[ \frac{d^4l}{(2\pi)^4} \frac{-i\delta_{\mu\nu}}{(l-k)^2-i\varepsilon} \frac{-i}{l^2+m^2-i\varepsilon} \right] \delta_{\mu\nu} \]

4.30

[The step from 4.28 to 4.29 is a Fourier convolution: we treated the resultant graph type integral in our usual manner-].
Finally we write down an explicit formula for the radiative correction to the 3-vertex in vector spin zero theory. For the case considered, both mesons on the mass shell, \( p^2 = p'^2 = -m^2 \), the radiative correction is given by the graph 4.8; and by virtue of 4.12 and ensuing discussion we can write it down at once in terms of the radiative correction to 3-vertex in scalar theory as

\[
\text{rad } V_{\mu}(p''\beta, p'^a) = \frac{-p''_\beta p'^a}{m^2} \text{ rad } V_{\mu}(p''p')
\]

\[
= i e \left( p'_a p''^\beta (p'^i + p'^n) \mu \right) \frac{Q^2}{m^2} \int_0^\infty \frac{dx v(-x)}{Q^2 + x - i\epsilon} \quad 4.32
\]

where

\[
Q = (p'' - p') \quad 4.33
\]

and the spectral weight \( v(Q^2) \) calculated in Section 2 is (note \( \mu^2 \to 0; \) infrared divergence)

\[
v(Q^2) = \frac{e^2}{(2\pi)^2} \left( 1 - \frac{1}{2} \log \left( 1 - \frac{Q^2 + 4m^2}{\mu^2} \right) \right) \frac{(1 + 2m^2/Q^2) v(-Q^2 - 4m^2)}{Q^2 (1 + 4m^2/Q^2)^{1/2}} \quad 2.33
\]

In summary: in this section calculations were made in the vector spin zero graph theory, using the elements of Table IV T 4, of the lowest order radiative corrections to photon propagator, meson propagator, 3-vertex; the quantities determined being expressed as dispersion integrals. We failed to prove the dispersion integral for the meson propagator, but showed that an equivalence [isomorphism] with the scalar theory would hold for the adopted form of D.R.: this particular D.R. is the most interesting outcome of this analysis. We noted that Ward's identity is not satisfied by our choice of meson propagator and 3-Vertex. We saw that diagram parts incorporating closed meson loops only would have same magnitude in this theory as in the scalar theory.
5. **β Matrix Theory of Charged Spin Zero**

The graph elements for this theory are as follows:

<table>
<thead>
<tr>
<th>Element</th>
<th>Graph</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal photon line [Fermi gauge]</td>
<td>(- \cdots \ - \cdots \ - \cdots )</td>
<td>(D_{\mu \nu} = \frac{-i \delta_{\mu \nu}}{k^2 - i \epsilon})</td>
</tr>
<tr>
<td>Internal meson line</td>
<td>(\mu \quad \nu)</td>
<td>(S(p) = \frac{i [i \beta \cdot p - m - 1 (\beta \cdot p)^2]}{p^2 + m^2 - i \epsilon})</td>
</tr>
<tr>
<td>3-Vertex</td>
<td>(\mu)</td>
<td>(\psi(p, p', p'') = -e \beta_\mu)</td>
</tr>
<tr>
<td>4-Vertex</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>External photon line (polarization (\lambda))</td>
<td>(\mu \quad \cdots \quad \mu)</td>
<td>(\frac{e^\lambda \mu(k)}{(2\pi)^{3/2}} \sqrt{\frac{m}{</td>
</tr>
<tr>
<td>External meson lines (of charge + e)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Incoming)</td>
<td></td>
<td>(\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{</td>
</tr>
<tr>
<td>(Outgoing)</td>
<td></td>
<td>(\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{</td>
</tr>
</tbody>
</table>
Peaslee (1949) showed that the above set of graph elements gave in lowest order the same value of matrix elements as the theory based on the interaction Hamiltonian density [Neuman and Purry (1949)]

\[ H = \left[ -i e \beta \right] \beta \bar{\phi} + \frac{e^2}{m} \beta_\mu (1 + \beta_4^2) \beta_\mu \bar{A}_\nu A_\mu \right] \]

where \( \beta = \beta_4 \), \( \beta_4 = 2\beta_4^2 - 1 \).

We are unaware of any proof to "all orders of perturbation theory," that the non-covariant 4-vertex can be omitted. With regard to Ward's identity, it is apparent that the relative \( \omega \)-potential \( S_p^X(p) \) defined by

\[ S_p^X(p) = \frac{i \beta p - m + m^{-1} (\beta p^2 - p^2)}{p^2 + m^2 - ie} \]

satisfies this identity, i.e.,

\[ [S_p^X(p''')^{-1} - S_p^X(p'')]^{-1} = e^{-1} (p'' - p')_\mu V_\mu (p''p') \]

by virtue of the identity

\[ [i \beta p - m + m^{-1} (\beta p^2 - p^2)] [i \beta p + m] = -p^2 - m^2 . \]

The \( S_p \) of the table does not satisfy

\[ \delta S_p^X(p)/\delta p_\mu = e^{-1} S_p^X(p) V_\mu (pp') S_p^X(p) \]

and having no inverse, it cannot possibly satisfy Ward's identity in the strong form (5.4). It is interesting to see that just as in the previous Section where we discussed alternate forms of \( G(p) \), in this case both \( S_F^X \) and \( S_p \) give the same magnitudes to the diagrams

\[ \text{[Generic diagrams]} \]

which we evaluate below.

So far the discussion really covers both spin zero and spin one for which
\[ \beta_{\mu\nu} + \beta_{\rho\nu\mu} = \beta_{\mu\delta\nu} + \beta_{\rho\delta\nu\mu} \]

We specialise to the 5 x 5 irreducible representation appropriate to spin zero when we require

\[ (M - 1)\beta_{\mu\nu} = (M - 1)\delta_{\mu\nu} \]

\[ M = \beta_{\mu\nu} \quad \text{(summed)} \]

a theorem that is especially easy to verify for the 5 x 5 \( \beta \) that are written down in Appendix I (These same \( \beta \) are to be found in Roman[(1961) p. 147]). This theorem yields on contraction with \( \delta_{\mu\nu} \)

\[ (M - 1)(4 - M) = 0 \]

so that

\[ \left( \frac{M - 1}{3} \right)^2 = \frac{M - 1}{3} \quad ; \quad \left( \frac{4 - M}{3} \right)^2 = \frac{4 - M}{3} \]

As by 5.8

\[ M\beta_{\mu} + \beta_{\mu} M = 5\beta_{\mu} \]

whence

\[ M\beta_{\mu\nu} = \beta_{\mu\nu} M \]

and also

\[ \text{Tr} (M\beta_{\mu}\beta_{\mu} + \beta_{\mu}M\beta_{\mu}) = 5 \text{Tr} \beta_{\mu\nu} \]

i.e., \[ 2 \text{Tr} M^2 = 5 \text{Tr} M \]

So by 5.11,

\[ 5 \text{Tr} M = 4 \text{Tr} 1 \]

thence

\[ \text{Tr} M = 4 \]

The trace

\[ \text{Tr} \frac{M - 1}{3} = 1 \]

taken together with 5.9 enables one to determine

\[ \text{Tr} \frac{(M - 1)}{3} \beta_{\mu_1\nu_1}\beta_{\mu_2\nu_2}\cdots\beta_{\mu_n\nu_n} = \delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\cdots\delta_{\mu_n\nu_n} \]

then as by 5.13, \[ (M - 1)\beta_{\mu} = \beta_{\mu} (4 - M) \]
\[ \text{Tr} \left( \frac{4 - \beta_1}{3} \beta_1 \beta_2 \cdots \beta_{n-1} \right) = \text{Tr} \left( \frac{1}{3} \beta_1 \beta_2 \cdots \beta_{n-1} \right) \]

\[ = \delta_{12} \delta_{23} \cdots \delta_{n1} \]

This rather neat formulation is an exact parallel to our treatment of 10 x 10 β matrices in Chapter V, theorem 5.9 corresponding to the theorem proved therein in the sense that \((\frac{4 - \beta_1}{3}) \beta_{\mu\nu}/3\) can replace \(\hat{H}_{\mu\nu}\). See especially V Section 4.

So much for algebra. We now apply Cutkosky's prescription to determine the discontinuity of the graph giving the lowest order radiative correction to the photon propagator, viz.,

\[ \text{Disc} \; D_{\mu\nu}(k) = \frac{\alpha^2}{(2\pi)^2} \int d^4 p' d^4 p'' \delta(k-p'+p'') \delta_p(p'^2+m^2) \delta_p(p''^2+m^2) \text{Tr}_{\mu\nu}(k) \]

where \(\text{Tr}_{\mu\nu}(k)\) is the mass shell value of

\[ \text{Tr}_{\mu\nu}(k) = \text{Tr}[ip_{p'} + m^{-1}p'^2] \beta_\nu [ip_{p''} + m^{-1}p''^2] \]

\[ = -(p'_{\mu}p''_\nu + p''_{\mu}p'_\nu) + m^{-2}(p'_{\mu}p'_\nu p''^2 + p''_{\mu}p''_\nu p'^2) \]

\[ = -(p'_{\mu} + p''_{\mu})(p'_{\nu} + p''_{\nu}) \]

Thus disc \(D_{\mu\nu}(k)\) has the same value as in the scalar and vector spin zero theories presented in Sections 2, 3 and 4. We refer the reader to these calculations for the explicit formula for \(D_{\mu\nu}\).

The radiative correction to the meson propagator depends on the SE part

\[ Z(p) = \frac{1}{q = p + k} \]
which has the branch cut discontinuity in Fermi gauge

\[
\text{disc } Z(p) = \frac{e^2}{(2\pi)^4} \int \frac{d^4k d^4\delta(p-q+k)2\pi i^4 (k^2) 2\pi i^6 (q^2 + m^2) \beta^\mu [i\beta q + m^{-1} \beta q^2]}{\beta^\nu \delta^\nu_{\mu \nu}}
\]

in which we may reduce

\[
\beta^\mu (i\beta q + m^{-1} \beta q^2) \beta^\mu = i\beta q + m^{-1}(1 - M) \beta q^2 + m^{-1} M q^2
\]

\[
= i\beta q + m^{-1} q^2
\]

Then by II A 3.5, II A 3.6

\[
\text{disc } Z(p) = \frac{e^2}{(2\pi)^2} \left[ i\beta p (p^2 - m^2)/2p^2 - m \right] \frac{\pi}{2} (1 + m^2/p^2) \theta (-p^2 - m^2)
\]

Note that there are no terms in \(\beta_{\mu \nu}\) in 5.31. We can therefore proceed exactly as in spin 1/2 case (where we wrote down the DR III 3.17 for \(Z(p)\)) to postulate the D.R.

\[
Z(p) = -i(p^2 + m^2) \int d\mu^2 \frac{i\beta p z_1(\mu^2) + mz_2(\mu^2)}{p^2 + \mu^2 - i\varepsilon}
\]

The spectral weights \(z_1\) and \(z_2\) are then readily determined from 5.31 and the formula

\[
\text{disc } Z(p) = 2\pi (p^2 + m^2)[i\beta p z_1(-p^2) + m z_2(-p^2)]
\]

as

\[
z_1(\mu^2) = \frac{e^2}{32\pi^2} \frac{\mu^2 + m^2}{\mu^4}
\]

and

\[
z_2(\mu^2) = -\frac{e^2}{16\pi^2} \frac{1}{\mu^2}
\]

The corrected propagator is then given by

\[
S(p) = S_F(p) + S_F(p) Z(p) S_F(p)
\]

where, explicitly for \(p^2 + m^2 > 0\),

\[
Z(p) = -\frac{ie^2}{32\pi^2} \frac{(p^2 + m^2)}{p^2} \left\{ i\beta p \left[ 1 + \frac{(p^2 - m^2)}{p^2} \log \frac{p^2 + m^2}{m^2} \right] + m \log \frac{p^2 + m^2}{m^2} \right\}
\]
We complete this Section by calculating the radiative correction to the 3-vertex:

\[-e \text{rad } V, (p^n p') = \frac{q'' = p'' - k}{q' = p' - k} \]

As usual we set \( p''^2 = p'^2 = -m^2 \) and handle the IR divergence as in Chapter II Section 4. Then Cutkosky's prescription applied to this diagram gives irrespective of gauge of photon line

\[
\text{disc rad } V, (p^n p') = \frac{-ie^2}{(2\pi)^2} \int \frac{d^4 k}{k^2} \delta_p [q''^2 + m^2] \delta_p [q' + m^2] \tilde{W}, \]

where \( \tilde{W}, = \beta_\mu [i \beta q'' + m^{-1} \beta q''^2] \beta_\lambda [i \beta p' + m^{-1} \beta p'^2] \beta_\nu \delta_{\mu \nu} \),

and

\[q'' = p' - k, \quad q' = p' - k\]

Expanding

\[\tilde{W}, = - \beta_\mu \beta q'' \beta q' \beta_\mu + m^{-2} \beta_\mu \beta q''^2 \beta q' \beta_\mu + \frac{i}{m} [\beta_\mu \beta q'' \beta_\lambda \beta q' \beta_\mu + \beta_\mu \beta q''^2 \beta_\lambda \beta q' \beta_\mu] \]

The value of \( \tilde{W}, \) is easily found by using 5.2. We have

\[
\frac{M-1}{3} \tilde{W}, = \frac{M-1}{3} \left[ -q'' q' \beta q'' + m^{-2} q'' q' \beta q' + \frac{i}{m} (q'' q'' q' q' + q' q'' q' q'') \right] \]

and as \( (4 - M) \beta_\mu = \beta_\mu (1 - M) \),

\[
\frac{4-M}{3} \tilde{W}, = \frac{4-M}{3} \left[ -q'' q' \beta q'' + m^{-2} q'' q' \beta q' + \frac{i}{m} (q'' q'' q' q'' + q'' q' q'' q'' q' q' q' q' q'' q') \right] - \]

As within the integrand

\[q'^2 = q''^2 = -m^2, \quad 2q' q'' = -2m^2 - q^2 \]

and the coverings are

LHS \( i \beta p'' = -m \), \quad RHS \( i \beta p' = -m \)

we can further simplify 5.43, 5.44;
\[
\frac{\mu-1}{3} \tilde{W}_\lambda = \frac{\mu-1}{3} (q' + q'')_\lambda \beta k - \text{im} \, \frac{4-\mu}{3} (q' + q'')_\lambda + \frac{\mu-1}{3} (q' + q'')_\lambda \text{im} q''/m \quad 5.47
\]

\[
\frac{4-\mu}{3} \tilde{W}_\lambda = \frac{4-\mu}{3} (q' + q'')_\lambda \beta k - \text{im} \, \frac{4-\mu}{3} (q' + q'')_\lambda - \text{im} \, \frac{4-\mu}{3} (q' + q'')_\lambda \quad 5.48
\]

We separate out the part of \( \tilde{W}_\lambda \) that gives an IR divergence:

\[
\tilde{W}_\lambda = \tilde{W}_\lambda^{\text{IR}} + \tilde{W}_\lambda^{\text{OK}} \quad 5.49
\]

\( \tilde{W}_\lambda^{\text{IR}} \) is the part of \( \tilde{W}_\lambda \) independent of \( k \):

\[
\tilde{W}_\lambda^{\text{IR}} = \text{im} (p' + p'')_\lambda \left\{ - \frac{3 \mu}{3} \frac{4-\mu}{3} + \frac{2m^2 + Q^2}{2m} - \frac{M-1}{3} \right\} \quad 5.50
\]

\[
\tilde{W}_\lambda^{\text{OK}} = \text{im} (p' + p'')_\lambda \beta k + \text{im} \lambda \left[ \frac{3 \mu}{3} \frac{4-\mu}{3} - \frac{2m^2 + Q^2}{m} - \frac{M-1}{3} \right] - 2k \lambda \beta k \quad 5.51
\]

Corresponding we have, by II Appendix 5,

\[
\text{disc rad } V_\lambda^{\text{IR}} = (p' + p'')_\lambda \left\{ - \frac{3 \mu}{3} \frac{4-\mu}{3} + \frac{2m^2 + Q^2}{2m} - \frac{M-1}{3} \right\} \frac{e^2}{(2\pi)^2} \nu[1] \quad 5.52
\]

\[
\nu[1] = 2 \log \left[ 1 - \frac{Q^2 + 4m^2}{\mu^2} \right] \quad 5.52
\]

where \( \mu = -\pi \phi(-Q^2 - 4m^2)/4q^2 (1 + 4m^2/q^2)^{1/2} \).

Now, as by II A 5.11,

\[
\beta \mu \nu(k'\mu) = 2\beta \mu (p' + p')_K = 4 \text{im} k \quad 5.53
\]

and by II A 5.12

\[
\beta \mu \nu(k'k) = [A Q^2 \beta \lambda + \text{im} \, (B + C)(p' + p'')_\lambda]_K \quad 5.54
\]

wherein

\[
A Q^2 = -Q^2 - 4m^2 \quad 5.55
\]

\[
B + C = 1 \quad 5.56
\]

the part of the radiative correction free from IR divergences

\[
\text{disc rad } V_\lambda^{\text{OK}} (p''p') = -\frac{i e^2}{(2\pi)^2} \nu \left[ \tilde{W}_\lambda^{\text{OK}} \right] \quad 5.57
\]
The spectral weights for substitution in the dispersion integrals for the coefficients of

$$\frac{1-M}{3} (p^1 + p^m)_{\lambda}; \quad \frac{4-M}{3} (p^1 + p^m)_{\lambda}; \quad \beta_{\lambda}$$

are then expeditiously determined. It is too trivial to actually write down these various DR and corresponding spectral weights. Note however that the coefficients of $\beta_{\lambda}$ satisfies a subtracted DR, which is free of IR divergence, just like the coefficient $S(Q^2)$ of $(p^1 + p^m)_{\lambda}$ in the spin 1/2 case. We have not as yet determined a simple physical consequence (akin to electrons magnetic moment).
A SET OF 5 x 5 \( \beta \) MATRICES

\[
\beta_1 = \begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
\beta_2 = \begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
\beta_3 = \begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot \\
\end{array}
\]

\[
\beta_4 = \begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot \\
\end{array}
\]

\[
\bar{\beta}_{\mu\nu} = \begin{array}{cccccc}
(11) & (12) & (13) & (14) & \cdot \\
(21) & (22) & (23) & (24) & \cdot \\
(31) & (32) & (33) & (34) & \cdot \\
(41) & (42) & (43) & (44) & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

where \( (a \beta) = \delta_{\mu a} \delta_{\nu \beta} \)

and \( \delta = (11) + (22) + (33) + (44) = \delta_{\mu\nu} \)

\[
M = \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = \text{diag} [1, 1, 1, 1, 4]
\]

i.e.,\( M = \begin{array}{ccccc}
1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 4 \\
\end{array} \)
CHAPTER V

ON THE TRACES OF 10 x 10 DUFFIN-KEMMER \( \beta \) MATRICES

1. Introduction

We confine our attention to the 10 x 10 irreducible representations of the Kemmer algebra

\[
\beta_{\mu\nu\lambda} + \beta_{\lambda\nu\mu} = \beta_{\mu} \delta_{\nu\lambda} + \beta_{\lambda} \delta_{\nu\mu}.
\]

(A particular set of \( \beta_{\mu} \) matrices and corresponding \( \mathbb{M} \) is to be found in Appendix 5)

In Section 3 we prove that the quantity \( H_{\mu\nu} \) defined in terms of the operator \( \mathbb{M} = \beta_{\mu} \beta_{\mu} \) as

\[
H_{\mu\nu} = (\lambda-2)(\beta_{\mu\nu} - \delta_{\mu\nu}),
\]

has the trace

\[
\text{Tr} H_{\mu\nu} = -\delta_{\mu\nu},
\]

and satisfies the following reduction theorem

\[
H_{\mu\nu} H_{\rho\sigma} = -\delta_{\mu\sigma} H_{\rho\nu}.
\]

This theorem leads to an explicit convenient method for computing the trace of an arbitrary product of 10 x 10 \( \beta \) matrices (Section 4). Moreover, it is found that our theorem is particularly convenient and invaluable for determining the traces of combinations of \( \beta \) products that occur in such calculations - this we demonstrate in this Chapter in Appendix 3 in reference to the calculation in the usual perturbation theory, for the charged Kemmer field, of the finite part of vacuum polarization. A further example of the value of this theorem is seen in constant occurrence of traces of \( H_{\mu\nu} \) products in the calculation of the traces evaluated in VI Appendix 2 in the course of the calculation.
Previous determinations of such traces as are to be found in Appendix 3 would use tabled values of \( S_p_n \), where
\[
S_p_n = \text{Tr} \beta_{\mu_1 \nu_2} \cdots \beta_{\mu_n} ,
\]
vanishes for odd \( n \) [Kemmer (1939)]. Now such traces have been tabulated before — see e.g. Neuman and Furry (1949) who tabulate \( S_p_n \) for \( n \leq 6 \) — but the methods used to compute \( S_p_n \) for \( n > 4 \) lacking knowledge of our theorem were exceedingly laborious [as is confirmed, for instance by a remark in paper of Peaslee (1949)]. We set down in Section 4 an explicit method to compute \( S_p_n \) that is convenient and short for all \( n \). As an example we compute \( S_p_4 \); the reader may compare this method with two methods expounded in Appendix 1 and 2.

As the calculation of Appendix 3 and calculations in VI Appendix 2 show that the particular trace
\[
S_p (M^2)_n = \text{Tr} (M^2) \beta_{\mu_1 \nu_2} \cdots \beta_{\mu_n} ,
\]
occurs frequently in evaluating loop diagrams, we tabulate this trace for \( n = 0, 2, 4, 6 \) in Appendix 4. Also therein we tabulate the corresponding
\[
S_p (3-M)_n = \text{Tr} (3-M) \beta_{\mu_1 \nu_2} \cdots \beta_{\mu_n} ,
\]
and we write down the previously unpublished trace \( S_p_8 \) whose 24 terms demonstrate the increasing complexity of \( S_p_n \) as \( n \) increases.

We interpose before the proof of our theorem a section giving a summary account of the algebraic properties of the 10 x 10 \( \beta \) matrices.

2. The 10 x 10 \( \beta \) matrices
\[
\beta_{\mu \nu \lambda} + \beta_{\lambda \nu \mu} = \beta_{\mu} \delta_{\nu \lambda} + \beta_{\lambda} \delta_{\nu \mu} ,
\]
implies that the operator \( M \) [Peaslee (1949)] defined by
\[ M = \beta \mu \beta_{\mu} \text{ (summed)}, \]

has (in any representation of 2.1) the property

\[ \beta_{\mu} M + M \beta_{\mu} = 5 \beta_{\mu}, \]

and

\[ \beta_{\mu \nu} M - M \beta_{\mu \nu} = 0. \]

The property of \( M \) specific to 10 x 10 irreducible representation that

\[ (M-2)(3-M) = 0; \]

is easily verified in a particular representation of \( \beta \) matrices. We noted in I Section 5 that this is a rather fundamental property, insofar as the condition that \( \phi \) satisfying

\[ (i \beta p + m)\phi = 0; \]

(where \( \beta_{\mu} \) satisfy 2.1) shall be a spin one field, that is that \( \phi \) be an eigenket of the intrinsic spin operator,

\[ \Gamma^2 = -p^2 M(3-M) - 2(\beta p)^2 (M-2), \]

of eigenvalue \( 2m^2 \):

\[ \Gamma^2 \phi = 2m^2 \phi; \]

entails equation 2.5.

Equation 2.5 implies that \( M-2 \) and \( 3-M \) are projection operators

\[ (M-2)^2 = M-2; \quad (3-M)^2 = 3-M. \]

Note \( (M-2) + (3-M) = 0 \).

As \( M-2 \) and \( 3-M \) commute, there is a representation in which both are diagonal and of the form - see 2.10, 2.11, 2.5 -

\[ M-2 = 0' \oplus 1, \quad 3-M = 1' \oplus 0; \]

\( \oplus \) denoting a direct sum.
Now by 2.3
\[ \text{Tr} \beta_{\mu} \beta_{\mu} + \text{Tr} M \beta_{\mu} \beta_{\mu} = 5 \text{Tr} \beta_{\mu} \beta_{\mu}, \]
\[ i.e., \quad 2\text{Tr} M^2 = 5 \text{Tr} M. \]

Then by 2.5 and the statement of the dimension of this representation,

\[ \text{viz.,} \quad \text{Tr} 1 = 10. \]

We deduce \[ \text{Tr} M = 24; \]

thence \[ \text{Tr} (M-2) = 4; \]

and \[ \text{Tr} (3-M) = 6; \]

so that the matrices 0 and 1 in 2.13 are 4 x 4, the primed matrices 0' and 1' are 6 x 6.

To determine \( \text{Tr} \beta_{\mu \nu} \) we observe that 2.1 implies

\[ \text{for} \quad \mu \neq \nu:\quad \beta_{\mu \nu} \beta_{\mu} = 0, \quad \text{(no sum)} \]

\[ \beta_{\mu} = \beta_{\mu}. \]

\[ \text{For } \mu = \nu: \quad \text{Tr} \beta_{\mu \nu} = 0, \]

\[ \text{Tr} (M-2) = 4; \]

and \[ \text{Tr} (3-M) = 6; \]

so that the matrices 0 and 1 in 2.13 are 4 x 4, the primed matrices 0' and 1' are 6 x 6.

The following contractions follow immediately from 2.1 (and therefore hold for the 5 x 5 representation):

\[ \beta_{\mu \lambda} \beta_{\mu} = \beta_{\lambda}; \]

\[ \beta_{\mu \rho \sigma} \beta_{\mu} = \beta_{\rho \sigma} + M \rho \sigma; \]

\[ \beta_{\mu \rho \lambda} \beta_{\mu} = - \beta_{\rho \lambda} + \beta_{\lambda \rho \sigma} + \beta_{\rho} \delta \lambda. \]
\[ \beta_\mu (\beta_\rho \sigma + \beta_\sigma \rho) \beta_\lambda \beta_\mu = \beta_\lambda (\beta_\rho \sigma + \beta_\sigma \rho), \]

3. Proof of Theorem

We verify the theorem

\[ H_{\mu \nu} H_{\rho \sigma} = -\delta_{\mu \sigma} H_{\rho \nu}, \]

in a particular representation of 10 x 10 \( \beta \) matrices. The representation need not be explicitly written down for our purposes but instead may be elegantly specified by introducing a 10 element column vector \( \phi \) with six elements comprising the independent elements of the antisymmetric tensor \( \phi_{a \beta} \) and four elements comprising the components of the four vector \( \phi_a \). We adopt the shorthand

\[ \phi = [\phi_{a \beta}, \phi_a], \]

and then specify the \( \beta \) matrices we are to use as being square matrices such as that for any \( b_\mu \),

\[ \beta_\mu b_\mu = [b_\alpha \phi_\beta - b_\beta \phi_\alpha, b_\rho \phi_\rho]. \]

The specification of order of indices \( \mu \omega b_j \) by the formula

\[ \phi^m = [\phi_{14} \phi_{24} \phi_{34} \phi_{23} \phi_{31} \phi_{12} ; \phi_1 \phi_2 \phi_3 \phi_4], \]

fixes our representation as that to be found in Appendix 5 and in the book of Roman [Roman (1960) p. 151; misprint for \( \beta_4 \)]. We continue by applying a second time the defining formula 3.3:

\[ \beta_{\mu \nu} a_\mu b_\nu = \beta_\mu a_\mu (\beta_\mu b_\nu). \]

\[ = [a_\alpha b_\rho \phi_\beta - a_\beta b_\rho \phi_\alpha ; a_\rho b_\rho \phi_\alpha - a_\rho b_\rho \phi_\alpha], \]

in particular

\[ M \phi = \beta_{\mu \nu} \delta_{\mu \nu} \phi = [2 \phi_{a \beta}, 3 \phi_a]. \]

The result in this representation \( M \) is diagonal with "tensor-tensor" components

\[ M_{a \beta, \gamma \delta} = 2 \delta_{a \beta, \gamma \delta} \phi. \]
and "vector-vector" components

\[ M_{\alpha,\beta} = 3 \delta_{\alpha\beta} \]  

3.8b

The operators \( M-2 \) and \( 3-M \) are also diagonal in this representation

\[ [M-2]_{\alpha\beta} = \delta_{\alpha\beta}; \quad [3-M]_{\alpha\beta}, \gamma^\delta = \delta_{\alpha\beta}, \gamma^\delta \]  

3.9

all other elements being zero. These two operators are manifestly orthogonal projection operators; as an aside we note that this verifies equation 2.5. Equation 3.6 shows that the matrices \( \beta_{\mu\nu} \) are the direct sum of two matrices acting separately on \( \phi_{\alpha\beta} \) and \( \phi_\alpha \). The components of \( \beta_{\mu\nu} \) acting only on the vector part of \( \phi \) being

\[ [\beta_{\mu\nu}]_{\alpha\beta} = \delta_{\mu\nu} \delta_{\alpha\beta} - \delta_{\mu\beta} \delta_{\nu\alpha} \]  

3.10

which may be written using 3.9 as

\[ [(M-2)\beta_{\mu\nu}]_{\alpha\beta} = \delta_{\mu\nu} \delta_{\alpha\beta} - \delta_{\mu\beta} \delta_{\nu\alpha} \]  

3.11

Thus

\[ [H_{\mu\nu}]_{\alpha\beta} = - \delta_{\mu\beta} \delta_{\nu\alpha} \]  

3.12

while the "tensor-tensor" components of \( H_{\mu\nu} \) are zero ,

i.e.,

\[ [H_{\mu\nu}]_{\alpha\beta}, \gamma^\delta = 0 \]  

3.13

Our theorem - equation 3.1 - follows immediately. We also see from 3.13 that

\[ \text{Tr} H_{\mu\nu} = - \delta_{\mu\nu} \]  

3.14

As arbitrary 10 x 10 \( \beta \) matrices are of the form \( S \beta S^{-1} \) where \( S \) is a non-singular matrix the algebraic formula 3.1 is valid in all representations, while the trace in 3.14 is a representational invariant.

We append to this proof the corollary that

\[ \text{Tr} \beta_{\mu_1\nu_1} \beta_{\mu_2\nu_2} \cdots \beta_{\mu_n\nu_n} = (-)^n \delta_{\nu_1\mu_2} \delta_{\nu_2\mu_3} \cdots \delta_{\nu_n\mu_1} \]  

3.15

where

\[ \beta_{\mu\nu} = H_{\nu\mu} = (M-2)(\beta_{\nu\mu} - \delta_{\nu\mu}) \]  

3.16
4. New method of computing traces

Our theorem 1.3 recast in terms of

\[ \hat{H}_{\mu \nu} = (M-2)(\beta_{\nu \mu} - \delta_{\nu \mu}) , \]

is

\[ \hat{H}_{\mu \nu} \hat{H}_{\rho \sigma} = - \delta_{\nu \rho} \hat{H}_{\mu \sigma} . \]

Thence we may reduce any product of \( n \) \( \hat{H} \) factors to the product of \( (n-1) \) \( \delta \) functions and one \( \hat{H} \) factor:

\[ \hat{H}_{\rho_1 \sigma_1} \hat{H}_{\rho_2 \sigma_2} \ldots \hat{H}_{\rho_n \sigma_n} = (-)^{n-1} \delta_{\rho_1 \rho_2} \delta_{\rho_2 \rho_3} \ldots \delta_{\rho_{n-1} \rho_n} \hat{H}_{\rho_1 \sigma_n} . \]

As \( (M-2) \) is a projection operator:

\[ (M-2)^2 = M - 2 \]

we can write

\[ (M-2)\beta_{\mu \nu} = (M-2)(\hat{H}_{\nu \mu} + \delta_{\nu \mu}) ; \]

and moreover as \( M-2 \) commutes with the product of 2 \( \beta \) matrices,

\[ (M-2)\beta_{\mu \nu} = \beta_{\mu \nu} (M-2) ; \]

we can expand the product

\[ (M-2)\beta_{\mu_1 \mu_2} \ldots \beta_{\mu_{2n}} = (M-2)(\hat{H}_{\mu_2 \mu_1} + \delta_{\mu_2 \mu_1}) \ldots (\hat{H}_{\mu_{2n} \mu_{2n-1}} + \delta_{\mu_{2n} \mu_{2n-1}}) ; \]

and use the traces

\[ \text{Tr} \hat{H}_{\mu \nu} = - \delta_{\mu \nu} ; \quad \text{Tr} (M-2) = 4 . \]
to expeditiously determine

\[ \text{Sp} (M-2)_{2n} = \text{Tr} (M-2) \beta_{\mu_1 \mu_2 \ldots \mu_{2n}}. \quad 4.9 \]

Further, as a variation of 2.3

\[ (3-M) \beta_{\mu} = \beta_{\mu} (M-2), \quad 4.10 \]

thence the trace

\[ \text{Sp}(3-M)_{2n} = \text{Tr} (3-M) \beta_{\mu_1 \mu_2 \ldots \mu_{2n}} \quad 4.11 \]

\[ = \text{Tr} (M-2) \beta_{\mu_2 \mu_3 \ldots \mu_{2n+1}} \quad 4.12 \]

is determined by a simple permutation of the indices of \( \text{Sp} (M-2)_{2n} \). Then as

\[ (M-2) + (3-M) = 1, \quad 4.13 \]

the trace

\[ \text{Sp}_{2n} = \text{Tr} \beta_{\mu_1 \mu_2 \ldots \mu_{2n}}, \quad 4.14 \]

is determined by the addition

\[ \text{Sp}_{2n} = \text{Sp} (M-2)_{2n} + \text{Sp} (3-M)_{2n} \quad 4.15 \]

As an example we compute \( \text{Tr} \beta_{\mu \nu \rho \sigma} \). First

\[ \text{Tr} (M-2) \beta_{\mu \nu \rho \sigma} = \text{Tr} (M-2) (\delta_{\mu \nu} + \delta_{\mu \nu}) (\delta_{\rho \sigma} + \delta_{\rho \sigma}) \quad 4.16 \]

\[ = \text{Tr} [\delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\rho \sigma} \delta_{\mu \nu} + (M-2) \delta_{\mu \nu} \delta_{\rho \sigma}] \quad 4.17 \]

\[ = \delta_{\mu \sigma} \delta_{\nu \rho} - 2\delta_{\mu \nu} \delta_{\rho \sigma} + 4\delta_{\mu \nu} \delta_{\rho \sigma} \quad 4.18 \]

\[ = \delta_{\mu \sigma} \delta_{\nu \rho} + 2\delta_{\mu \nu} \delta_{\rho \sigma} \quad 4.19 \]
Then \( \text{Tr} (3-M) \beta_{\mu \nu \rho \sigma} = \text{Tr} (M-2) \beta_{\nu \rho \sigma \mu} \)

\[ = \delta_{\mu \nu} \delta_{\rho \sigma} + 2 \delta_{\nu \rho} \delta_{\sigma \mu}. \]

So by addition

\[ \text{Tr} \ \beta_{\mu \nu \rho \sigma} = 3(\delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\mu \sigma} \delta_{\nu \rho}). \]

This same trace is computed in Appendices 1 and 2 by the previous methods. The tremendous simplicity and convenience of the method in calculating \( S_{n} \) becomes increasingly apparent as \( n \) further increases.
V APPENDIX I

INDUCTIVE METHOD OF COMPUTING TRACES

This method of computing the trace of products of $10 \times 10 \beta$ matrices utilises the following formula in an inductive manner:

$$\text{Tr} \beta_{\mu \lambda \nu} P + \text{Tr} \beta_{\nu \lambda \mu} P = \delta_{\mu \lambda} \text{Tr} \beta_{\nu} P + \delta_{\lambda \nu} \text{Tr} \beta_{\mu} P$$  \hspace{1cm} V Al.1

where $P$ is the product of an odd number of $\beta$ matrices. Except for the case of example (i) the formula is insufficient by itself for determining the trace of $n$ $\beta$ matrices from the trace of $n-2$. However by careful examination of the symmetries of the product, and note of such features as that $\text{Tr} \beta_{\mu \lambda \nu} P$ does not include a term in $\delta_{\mu \nu}$, traces can be, with ever increasing labour, determined by Al.1 from the starting point

$$\text{Tr} \beta_{\mu \nu} = 6 \delta_{\mu \nu}$$  \hspace{1cm} Al.2

We give two examples of this method. The writer has found it possible to determine the trace of up to 8 $\beta$ matrices by this method, but the labour is enormous. We do present in (ii) the trace of 8 $\beta$ matrices in the special case where by the contraction $\delta_{\mu \nu} \beta_{\mu \nu} = M$ there are only 6 free indices.

(i) $\text{Tr} \beta_{\mu \nu \rho \sigma}$

Take $P = \beta_{\rho \sigma}$, then by Al.1 and Al.2

$$\text{Tr} \beta_{\mu \nu \rho \sigma} + \text{Tr} \beta_{\rho \nu \mu \sigma} = \delta_{\mu \nu} \text{Tr} \beta_{\rho \sigma} + \delta_{\rho \nu} \text{Tr} \beta_{\mu \nu}$$  \hspace{1cm} Al.3

$$= 6(\delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\rho \nu} \delta_{\mu \sigma})$$  \hspace{1cm} Al.4

While in general the relationship between $\text{Tr} \beta_{\mu \nu \rho \sigma} P$ and $\text{Tr} \beta_{\rho \nu \mu \sigma} P$, where $P$ is a product of $\beta$ matrices, is complicated, in this case

$$\text{Tr} \beta_{\rho \nu \mu \sigma} = \text{Tr} \beta_{\sigma \rho \nu \mu} = \text{Tr} \beta_{\mu \nu \rho \sigma}$$  \hspace{1cm} Al.5

Thence

$$\text{Tr} \beta_{\mu \nu \rho \sigma} = 3(\delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\mu \sigma} \delta_{\rho \nu})$$  \hspace{1cm} Al.6
(ii) $\text{Tr } M_{\mu\nu\rho\sigma\alpha\beta}$: Owing to the definition

$$M = \beta_\rho \beta_\rho,$$

This quantity is in fact the product of 8 $\beta$ matrices. The determination proceeds neatly after applying the symmetry consideration

$$M_{\mu\nu} = \beta_{\mu\nu} M,$$

when set

$$\text{Tr } M_{\mu\nu} \beta_{\rho\sigma} = x K + y L + z M + q N,$$

where

$$K = \delta_{\mu\nu} \delta_{\rho\sigma} \delta_{\alpha\beta},$$

$$L = \delta_{\mu\nu} \delta_{\rho\beta} \delta_{\sigma\alpha} + \delta_{\rho\sigma} \delta_{\mu\beta} \delta_{\nu\alpha} + \delta_{\alpha\beta} \delta_{\nu\rho} \delta_{\mu\sigma},$$

$$M = \delta_{\mu\nu} \delta_{\sigma\alpha} \delta_{\rho\beta},$$

$$N = \delta_{\nu\rho} \delta_{\sigma\alpha} \delta_{\beta\mu}.$$

By Al.7 and Al.1,

$$\text{Tr } M_{\mu\nu\rho\sigma\alpha\beta} + \text{Tr } M_{\nu\rho\sigma\alpha\mu\beta} = 5 \text{Tr } \beta_{\mu\nu\rho\sigma\alpha\beta},$$

where the L.H.S. - the trace of 8 $\beta$ matrices - is taken as known:

$$\text{Tr } \beta_{\mu\nu\rho\sigma\alpha\beta} = K + 2L - 2M + N;$$

as also

$$\text{Tr } M_{\alpha\beta} = 15 \delta_{\alpha\beta}; \quad \text{Tr } \beta_{\alpha\beta} = 6 \delta_{\alpha\beta}.$$

Then trivially from Al.7,

$$\text{Tr } M_{\nu\rho\sigma\alpha\beta\mu} = q K + y L + z M + x N.$$
so that \( x + q = 5, \ y = 5, \ z = -5 \) \quad (10 \times 10) \quad (A1.16)

Since \( M^2 = 5M - 6 \)
\( M^3 = 19M - 30 \) \quad (A1.17)
\( (A1.18)

So \( \text{Tr} \ M^3 \beta_{\alpha \beta} = \text{Tr} \ (19M - 30) \beta_{\alpha \beta} \) \quad (A1.19)
\[ = 3 \cdot 5 \cdot 7 \delta_{\alpha \beta} \quad \text{[by (A1.14)]} \quad (A1.20)

But by (A1.7)
\[ \text{Tr} \ M^3 \beta_{\alpha \beta} = (4^3 x + 3 \cdot 4 y + z + q) \delta_{\alpha \beta} \quad (A1.21)
\]

thence \( x = 3, \ q = 2 \) and the trace has been determined.
V APPENDIX 2

APPLICATION OF COMPLETE REPRESENTATION

This method of computing the trace of $10 \times 10$ $\beta$ matrices we find in Peaslee (1949). The method is most straightforward in principle as it utilises the $16 \times 16$ reducible representation of the $\beta$ algebra

$$\beta^\mu_\mu = \frac{1}{2} (\gamma^\mu_\mu I^\dagger + \gamma^\mu\mu I),$$

A2.1

which is decomposable into the direct sum $(\uparrow)$ of the irreducible $10 \times 10$, $5 \times 5$, and trivial $(1 \times 1)$ representations

$$\beta^\mu_\mu = \beta^\mu_\mu + \beta^\mu_\mu \uparrow 0.$$ A2.2

Because of lack of interference the trace sought may be found from 2.1 using the formula

$$\text{Tr} \beta^\mu_\mu \uparrow \beta^\mu_\mu = \delta^\mu_\mu \delta^\mu_\mu + \delta^\mu_\mu \delta^\mu_\mu + \delta^\mu_\mu \delta^\mu_\mu,$$ A2.3

and the well known explicit formula for the trace of $\gamma^\nu$ products.

We illustrate the method by reference to the calculation of

$$\text{Tr} \beta^\mu_\mu \gamma^\rho \gamma^\sigma = \text{Tr} \beta^\mu_\mu \beta^\rho \beta^\sigma - \text{Tr} \beta^\mu_\mu \beta^\rho \beta^\sigma.$$ A2.4

This determination of these traces needs all of the following traces:

$$\text{Tr} I = 4;$$ A2.5

$$\text{Tr} \gamma^\mu_\mu \gamma^\nu_\nu = 4 \delta^\mu_\nu;$$ A2.6

$$\text{Tr} \gamma^\mu_\mu \gamma^\nu_\nu \gamma^\rho_\rho = 4 \left( \delta^\mu_\nu \delta^\rho_\rho - \delta^\mu_\rho \delta^\nu_\rho \delta^\nu_\rho \delta^\nu_\rho \right);$$ A2.7

$$\text{Tr} \beta^\rho_\rho \beta^\sigma_\sigma = \delta^\rho_\rho \delta^\rho_\rho \delta^\sigma_\sigma.$$ A2.8

plus the information that the trace of an odd number of $\gamma$ matrices is zero.

Then

$$\text{Tr} \beta^\mu_\mu \beta^\rho_\rho \beta^\sigma_\sigma = \frac{2}{16} \text{Tr} \gamma^\mu_\mu \gamma^\nu_\nu \text{Tr} I + \frac{2}{16} \text{Tr} \gamma^\nu_\nu \text{Tr} \rho^\rho_\rho \text{Tr} \gamma^\rho_\rho \gamma^\sigma_\sigma + \frac{2}{16} \text{Tr} \gamma_\nu \gamma_\nu \gamma_\nu.$$ A2.9
\[ = 4 (\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) . \tag{A2.10} \]

Hence \[ \text{Tr} \beta_{\mu\nu\rho\sigma} = 3 (\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) . \tag{A2.11} \]

In conclusion we may say that the method has the theoretical attraction of utilizing an explicit formula for the trace; but is lengthy.
EXAMPLE OF IMPROVED TRACE CALCULATION

We have mentioned in the text above that there is a tremendous saving in the labour involved in calculating the sort of traces that occur in practical computation through the direct application of our theorem.

\[
\text{Tr} \left( H_{\mu_1 \nu_1} H_{\mu_2 \nu_2} \cdots H_{\mu_n \nu_n} \right) = (-)^n \delta_{\nu_1 \mu_2} \delta_{\nu_2 \mu_3} \cdots \delta_{\nu_n \mu_1} ,
\]

We recall the definitions

\[
H_{\mu \nu} = (M-2) (\beta_\mu \beta_\nu - \delta_{\mu \nu}) ,
\]

\[
M = \beta_\mu \beta_\mu \text{ (summed)} ,
\]

in terms of the 10 x 10 \( \beta \) matrices. We substantiate this claim with regard to the calculation of the finite contribution to the imaginary part of the vacuum polarization in the conventional non-renormalizable theory of spin one mesons of magnetic moment \( K \). Such a particle has the field equation in 10 x 10 \( \beta \) formalism

\[
[\beta \psi + m + \frac{i e \gamma}{m} (M-2) \beta_\mu \gamma_{\mu \nu} \bar{\phi} = 0 ,
\]

\[
\pi_\mu = \delta_\mu - i e A_\mu ,
\]

equivalent to the vectorial form

\[
\pi_\mu G_{\mu \nu} - m^2 \phi_\nu + i e \gamma_\rho \bar{\phi} F_{\rho \nu} = 0 ,
\]

where

\[
G_{\mu \nu} = \pi_\mu \phi_\nu - \pi_\nu \phi_\mu .
\]
Expanding the gauge invariant part of the photon propagator as

\[ D_{\mu\nu}(q) = -i \left( \delta_{\mu\nu} - q^2 q_\mu q_\nu \right) \left( 1 + \Pi(q^2) \right) / (q^2 - i\epsilon) \] \quad A3.8

Then "to lowest order perturbation theory"

\[ \text{Im} \Pi(q^2) = \frac{\alpha^2}{4\pi q^2} \left( \frac{q^2 + 4m^2}{q^2} \right)^{1/2} \text{Tr}(q^2) \Theta(-q^2 - 4m^2), \] \quad A3.9

in which the trace

\[ \text{Tr}(q^2) = \text{Tr} \left[ N(p) V_\mu(p'^n) N(p') V_\mu(p'^n) \right], \] \quad A3.10

where \( N(p) = i(p - m + m^{-1}(\beta p^2 - p^2)) \) \quad A3.11

and \( V_\mu(p'^n) = -\beta_\mu - \frac{K}{m} (M-2) (\beta_\mu - \beta_\mu^i) i(p' - p)^i \) \quad A3.12

must be evaluated on the mass shell

\[ p'^2 = p'^2 = -m^2; \quad q = p' - p'' \] \quad A3.13

The evaluation of \( \text{Tr}(q^2) \) using the tables of traces of \( \beta \) matrices is obviously most tedious, as failing the making of many algebraic reductions, the trace of the product of up to \( 10 \beta \)-matrices must be determined.

We note first that that on expansion

\[ V_\mu(p''p') = -\beta_\mu - \frac{K}{m} [H_{\rho\mu}p_\rho + H_{\mu\sigma}p_\sigma] + \frac{iK}{m} [H_{\rho\mu}p''_\rho + H_{\mu\sigma}p''_\sigma], \] \quad A3.14

so that as

\[ \beta_{p''} H_{\mu\rho} p_\rho = 0 \] \quad A3.15

\[ \beta_{p''} V_\mu(p''p') \beta_{p'} = -\beta_{p''} \beta_\mu \beta_{p'} + i\frac{K}{m} \beta_{p''} [H_{\rho\mu}p''_\rho + H_{\mu\sigma}p''_\sigma] \beta_{p'}, \] \quad A3.16
The computational convenience we claim is sufficiently shown in the computation of that part of $\text{Tr}(q^2)$ proportional to $K^2$, which we denote by $\text{Tr}^2(q^2)$.

When $\beta_p (M-2) = 0$, \hspace{1cm} A3.17

$$\text{Tr}^2(q^2) = -m^4 K^2 \text{Tr} \left[ \beta_p^2 \left( H_{\mu\nu} p_{\mu} p_{\nu} + H_{\mu\rho} p_{\mu} p_{\rho} \right) \beta_p^2 \left( H_{\sigma\mu} p_{\sigma} + H_{\mu\sigma} p_{\sigma} \right) \right], \hspace{1cm} A3.18$$

Equating terms symmetric in $p', p''$

$$\text{Tr}^2(q^2) = -2m^4 K^2 \text{Tr} \left[ \beta_p^2 H_{\mu\nu} p_{\mu} p_{\nu} \beta_p^2 H_{\sigma\mu} p_{\sigma} + \beta_p^2 H_{\mu\nu} p_{\mu} p_{\nu} \beta_p^2 H_{\mu\sigma} p_{\sigma} \right], \hspace{1cm} A3.19$$

On making the substitutions

$$\beta_p^2 = \beta_p^2 \hspace{1cm} A3.21$$

$$\beta_p^2 = \beta_p^2 \hspace{1cm} A3.20a$$

$$\beta_p^2 = \beta_p^2 \hspace{1cm} A3.20b$$

where we have made the usual abbreviations

$$\beta_p^2 = (\beta_p)^2; \hspace{1cm} p_{a\beta} = p_{a\beta} \hspace{1cm} A3.21$$

$\text{Tr}^2(q^2)$ is expressed as the sum of 8 traces of products of $\hat{H}$ matrices immediately evaluated by A3.1 as

$$\text{Tr}^2(q^2) = -2m^4 K^2 \left[ p_{1}^{2} p_{2}^{2} p_{3}^{2} p_{4}^{2} + p_{1}^{2} p_{3}^{2} p_{4}^{2} \right]$$

$$- p_{2}^{2} (p_{1}^{2} p_{3}^{2} p_{4}^{2}) - p_{3}^{2} (4p_{1}^{2} p_{3}^{2} p_{4}^{2})$$

$$- p_{1}^{2} (p_{2}^{2} p_{3}^{2} p_{4}^{2}) - p_{2}^{2} (p_{2}^{2} p_{3}^{2} p_{4}^{2})$$

$$+ p_{1}^{2} p_{2}^{2} (p_{1}^{2} p_{3}^{2}) + p_{1}^{2} p_{2}^{2} (4p_{1}^{2} p_{2}^{2}) \hspace{1cm} A3.22$$
\[= -2m^{-2} k^2 \left[ (p \cdot p^n)^3 - 3p^n (p \cdot p^n)^2 - p^n (p \cdot p^n) + 3p^n (p \cdot p^n) \right] \quad A3.23\]

As on the mass shell \( p \cdot p^n = -\frac{1}{2} (q^2 + 2m^2) \)

\[\text{Tr}^2(q^2) = \frac{k^2 q^2}{4m^2} - 4k^2 q^2 \quad A3.25\]

Likewise one can determine

\[\text{Tr}^1(q^2) = 4 \text{Tr}(\beta_\mu i \beta_\rho (i m^{-1})(\rho \mu \rho + H_\mu \rho \sigma)) (m^{-1} \beta_\mu \beta_\rho \beta_\sigma) \quad A3.26\]

\[= 4k^2 m^{-2} \text{Tr} (\gamma_\rho \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_{\rho \mu \nu \alpha} \gamma_{\rho \mu \nu \alpha}) \quad A3.27\]

\[= 4k^2 m^{-2} [-4p \cdot p^n \cdot p \cdot p^n + p \cdot p^n (p \cdot p^n) + p \cdot p^n (p \cdot p^n)] \quad A3.28\]

\[= 4k^2 m^{-2}(-3p \cdot p^n \cdot p \cdot p^n + 3p \cdot p^n (p \cdot p^n) \quad A3.29\]

\[= \frac{-3k q^2}{m^2} - 12k q^2 \quad A3.30\]

while in the same simple manner one computes

\[\text{Tr}^0(q^2) = \text{Tr} i \beta_\mu i \beta_\rho i \beta_\sigma + m^{-2} \text{Tr} \beta_\rho (p \cdot p^n) + m^{-2} \text{Tr} \beta_\rho (p \cdot p^n) \quad A3.31\]

\[= - \text{Tr} (\beta_\mu \beta_\rho \beta_\sigma + 2m^{-2} \text{Tr} (\gamma_\rho \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_{\rho \mu \nu \alpha} \gamma_{\rho \mu \nu \alpha}) \quad A3.32\]

\[= -6p \cdot p^n + 2m^{-2} [-4p \cdot p^n \cdot p \cdot p^n + 2p \cdot (p \cdot p^n) \quad A3.33\]

\[= -4m^{-2} (p \cdot p^n)^2 - 14p \cdot p^n + 2m^2 \quad A3.34\]

\[= \frac{-q^2}{m^2} - q^2 + 12m^2 \quad A3.35\]
Adding A3.25, A3.30, A3.35 we find

\[ \text{Tr} (q^2) = \frac{K^2 q^2}{4m} - \left(1 + 3k \right) \frac{q^2}{m^2} - (1 + 12k + 4k^2) q^2 + 12m^2. \]  

We note again that

\[ \text{Im} \Pi(q^2) = \frac{e^2}{48\pi q^2} \left( \frac{q^2 + 4m^2}{q^2} \right)^{1/2} \text{Tr}(q^2) \Theta (-q^2 - 4m^2). \]
### TABLE OF TRACES OF 10 x 10 β MATRICES

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Tr}(M-2) )</td>
<td>4</td>
</tr>
<tr>
<td>( \text{Tr}(M-2) \beta_{\mu\nu} )</td>
<td>3 ( \delta_{\mu\nu} )</td>
</tr>
<tr>
<td>( \text{Tr}(M-2) \beta_{\mu\nu\rho\sigma} )</td>
<td>2 ( \delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\nu\rho} \delta_{\sigma\mu} )</td>
</tr>
<tr>
<td>( \text{Tr}(M-2) \beta_{\mu\nu\rho\sigma\alpha\beta} )</td>
<td>( \delta_{\mu\nu} \delta_{\rho\sigma} \delta_{\alpha\beta} + (\delta_{\mu\nu} \delta_{\rho\beta} \delta_{\sigma\alpha} + \delta_{\rho\sigma} \delta_{\mu\beta} \delta_{\nu\alpha} + \delta_{\alpha\beta} \delta_{\nu\rho} \delta_{\mu\sigma}) - \delta_{\mu\sigma} \delta_{\nu\rho} \delta_{\alpha\beta} )</td>
</tr>
<tr>
<td>( \text{Tr}(3-M) )</td>
<td>6</td>
</tr>
<tr>
<td>( \text{Tr}(3-M) \beta_{\mu\nu} )</td>
<td>3 ( \delta_{\mu\nu} )</td>
</tr>
<tr>
<td>( \text{Tr}(3-M) \beta_{\mu\nu\rho\sigma} )</td>
<td>( \delta_{\mu\nu} \delta_{\rho\sigma} + 2 \delta_{\nu\rho} \delta_{\sigma\mu} )</td>
</tr>
<tr>
<td>( \text{Tr}(3-M) \beta_{\mu\nu\rho\sigma\alpha\beta} )</td>
<td>( \delta_{\nu\rho} \delta_{\sigma\alpha} \delta_{\beta\mu} + (\delta_{\mu\nu} \delta_{\rho\beta} \delta_{\sigma\alpha} + \delta_{\rho\sigma} \delta_{\mu\beta} \delta_{\nu\alpha} + \delta_{\alpha\beta} \delta_{\nu\rho} \delta_{\mu\sigma}) - \delta_{\mu\sigma} \delta_{\nu\rho} \delta_{\alpha\beta} )</td>
</tr>
<tr>
<td>( \text{Tr} \hat{H}_{\mu\nu} )</td>
<td>(- \delta_{\mu\nu} )</td>
</tr>
<tr>
<td>( \text{Tr} \hat{H}<em>{\mu\nu} \hat{H}</em>{\rho\sigma} )</td>
<td>( \delta_{\nu\rho} \delta_{\sigma\nu} )</td>
</tr>
<tr>
<td>( \text{Tr} \hat{H}<em>{\mu\nu} \hat{H}</em>{\rho\sigma} \hat{H}_{\alpha\beta} )</td>
<td>(- \delta_{\nu\rho} \delta_{\sigma\alpha} \delta_{\beta\mu} )</td>
</tr>
<tr>
<td>( \text{Tr} \hat{H}<em>{\mu\nu} \hat{H}</em>{\rho\sigma} \hat{H}<em>{\alpha\beta} \hat{H}</em>{\gamma\delta} )</td>
<td>( \delta_{\nu\rho} \delta_{\sigma\alpha} \delta_{\beta\gamma} \delta_{\delta\mu} )</td>
</tr>
</tbody>
</table>

\[ S_{\beta} = \text{Tr} \beta_{\mu_1 \mu_2 \ldots \mu_8} = \delta_{12} \delta_{34} \delta_{56} \delta_{78} + \delta_{12} \delta_{34} \delta_{56} \delta_{78} + \delta_{16} \delta_{34} \delta_{52} \delta_{76} + \delta_{16} \delta_{34} \delta_{52} \delta_{76} + \delta_{12} \delta_{32} \delta_{58} \delta_{76} + \delta_{16} \delta_{32} \delta_{58} \delta_{76} + \delta_{16} \delta_{32} \delta_{54} \delta_{78} + \delta_{16} \delta_{32} \delta_{54} \delta_{78} + \delta_{12} \delta_{38} \delta_{54} \delta_{76} + \delta_{16} \delta_{38} \delta_{54} \delta_{76} + \delta_{12} \delta_{38} \delta_{54} \delta_{76} + \delta_{16} \delta_{38} \delta_{54} \delta_{76} + \delta_{16} \delta_{38} \delta_{52} \delta_{74} + \delta_{16} \delta_{38} \delta_{52} \delta_{74} + \delta_{12} \delta_{36} \delta_{58} \delta_{76} + \delta_{16} \delta_{36} \delta_{58} \delta_{76} + \delta_{12} \delta_{38} \delta_{56} \delta_{76} + \delta_{16} \delta_{38} \delta_{56} \delta_{76} + \delta_{16} \delta_{36} \delta_{52} \delta_{78} + \delta_{16} \delta_{36} \delta_{52} \delta_{78} + \delta_{12} \delta_{36} \delta_{52} \delta_{78} + \delta_{16} \delta_{36} \delta_{52} \delta_{78} + \delta_{16} \delta_{36} \delta_{52} \delta_{78} + \delta_{16} \delta_{36} \delta_{52} \delta_{78} \]
Where \( \delta_{rs} = \delta_{rs}^\mu \).

\[
\text{Tr} \ H_{\mu\nu} = - \delta_{\mu\nu} .
\]

\[
\text{Tr} \ H_{\mu\nu} H_{\rho\sigma} = \delta_{\nu\rho} \delta_{\mu\sigma} .
\]

\[
\text{Tr} \ H_{\mu\nu} H_{\rho\sigma} H_{a\beta} = - \delta_{\mu\sigma} \delta_{\nu\alpha} \delta_{\rho\beta} .
\]

\[
\text{Tr} \ H_{\mu\nu} H_{\rho\sigma} H_{a\beta} H_{\gamma\delta} = \delta_{\mu\sigma} \delta_{\nu\gamma} \delta_{\rho\beta} \delta_{\alpha\delta} .
\]

**Note**

\[
H_{\mu\nu} = (\kappa - 2)(\beta_{\mu\nu} - \delta_{\mu\nu}) .
\]

\[
\hat{H}_{\mu\nu} = (\kappa - 2)(\beta_{\nu\mu} - \delta_{\nu\mu}) .
\]
A PARTICULAR SET OF $10 \times 10 \beta_\mu$

$\beta_1 = \begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{array}$

$\beta_2 = \begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{array}$

$\beta_3 = \begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{array}$

$\beta_4 = \begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{array}$

$M = \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = \begin{array}{cccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}$
CHAPTER VI

CRITIQUE OF LEE AND YANG THEORY OF CHARGED SPIN ONE

1. On the Graph Elements of LY Theory.

This Section serves to give a preliminary account of the $\xi$-Limiting theory of charged spin one [Lee and Yang (1962)]. We sketch the derivation of the Lee and Yang (LY) graph elements and make some interpretative criticism.

In attempting to devise a graph theory of charged spin one, on assigning to spin one lines the value

$$S_{a\beta}^{-1}(p) = -i \frac{\delta_{a\beta} + p_{a}p_{\beta}/m^2}{p^2 + m^2 - is}$$

VI 1.1

it is readily found that for no non zero $3$-vertex of order $p$ can acceptable lower order radiative corrections be determined. For example, see V Appendix 3 where an unacceptable value for the imaginary part of vacuum polarization was calculated. The calculation of VA 3 was performed in $10 \times 10 \beta$ formalism but is equivalent to using $S^{-1}$ of 1.1 and the 3 vertex

$$V_{\mu}(p''\beta, p'\alpha) = i e [\delta_{\alpha\beta}(p'_{\mu} + p'_{\nu}) - \delta_{\alpha\mu}(-K_{\beta}p''_{\mu} + p''_{\beta} + K_{\mu}'') - \delta_{\beta\mu}(-K_{\alpha}p''_{\mu} + p''_{\alpha} + K_{\mu}'')]$$

VI.2

leading to

$$\text{Im} \Pi(q^2) = \frac{e^2}{48\pi q^2} \left( \frac{q^2 + 4m^2}{q^2} \right)^{1/2} \text{Tr}(q^2)\theta(-q^2 - 4m^2)$$

VI.3a

where

$$\text{Tr}(q^2) = \frac{K_{\beta}q_{\alpha}}{4m^2} - (1 + 2K) \frac{q^2}{m^2} - (1 + 12K + 4K^2)q^2 + 12m^2$$

VI.3b

If this expression for $\text{Im} \Pi(q^2)$ was substituted in the usual DR for $\Pi(q^2)$ the integral would diverge.

It is rather obvious that the failure that the failure of the spin one graph theories based on $S_{a\beta}^{-1}(p)$ is due to the numerator term in $p_{a}p_{\beta}/m^2$. Lee and Yang
apparently recognized this and sought to make a rather minimal innovation to produce a finite theory. On the basis of the expectation that $S_{a\beta}(p)$ is the correct value for spin one lines, but leads to ultra-violet divergences, they took instead the charged line value

$$S_{F a\beta}(p) = S_{a\beta}(p) + S_{a\beta}^0(p)$$

where

$$S_{a\beta}^0(p) = \frac{p \cdot a_{\beta} / m^2}{p^2 + M^2 - i\varepsilon}$$

and

$$M^2 = M_o^2 = \frac{1}{\xi-1} m^2 \gg m^2$$

For small $p$, as $\xi \to \infty$

$$S_{F a\beta} \to S^1$$

For large $p$,

$$S_{F a\beta} \to -i \frac{\delta_{a\beta}}{(p^2 + m^2 - i\varepsilon)}$$

The charged line value $S^L_{F a\beta}$ and the value of other corresponding graph elements is immediately deducible by Dyson-Wick procedures from the Lagrangian density

$$L_\xi = -\xi(\pi_{\mu} \phi^\dagger \phi_{\mu}) - \frac{1}{2} (\phi_{\mu} A_{\mu})^2 - \frac{1}{2} G_{\mu\nu}^x G_{\mu\nu}^x - m^2 \phi^\dagger \phi_{\mu} - i e K F_{\mu\nu} \phi^\dagger \phi_{\mu\nu}$$

where

$$\phi_{\mu}^\dagger = \eta^{-1} \phi_{\mu}^\dagger \eta$$

$$G_{\mu\nu}^x = \eta^{-1} G_{\mu\nu}^x \eta$$

A field $\phi_{\mu}$ described by such a Lagrangian has four degrees of freedom and corresponds to, at least for $\varepsilon = 0$, a system containing spin one particles of mass $m$ and vector spin zero of mass $M$. The metric $\eta$ is chosen (in the interaction representation) so that for any state $\psi$

$$\eta \psi = (-)^N \psi$$
where \( N_0 \) is the number of spin zero particles in the state \( \psi \).

It is not at all surprising that Hilbert's space has to be endowed with a negative metric in this theory: of old-fashioned regulator theory [Villars (1960)].

We have an important interpretative criticism to make of \( \psi \) theory. The field equations deducible from Lagrangian density \( L_\xi \) of 1.9 are

\[
\xi \pi \mu \phi^\mu + \pi \mu \Gamma^\mu_{\nu\mu} - m^2 \phi^\nu + i e K \phi^\mu \Gamma^\mu_{\mu
u} = 0 \quad \text{1.13}
\]

wherin

\[
G^\mu_{\nu\mu} = \pi^\mu \phi^\nu - \pi^\nu \phi^\mu \quad \text{1.14}
\]

Contracting with \( \pi^\nu \) using the relation \( \pi^\mu \pi^\nu - \pi^\nu \pi^\mu = -i e \Gamma^\mu_{\mu
u} \) gives

\[
\xi \pi^\nu \phi^\mu = -m^2 \pi^\nu \phi^\nu = \frac{-i e}{2} (K - 1) \Gamma^\mu_{\mu
u} G^\mu_{\nu\mu} \quad \text{1.15}
\]

We write

\[
\phi^\mu = \phi^\mu_1 + \phi^0_\mu \quad \text{1.16}
\]

setting

\[
m^2 \phi^0_\mu = \xi \pi^\mu \phi^\mu_1 \quad \text{1.17}
\]

and fix

\[
K = +1 \quad \text{1.18}
\]

Then

\[
\pi^\mu \pi^\nu \phi^\mu + m^2 \phi^0_\mu = 0 \quad \text{1.19}
\]

Defining

\[
G^\mu_1_{\nu\mu} = \pi^\mu \phi^\nu - \pi^\nu \phi^\mu_1 \quad \text{1.20}
\]

We see that

\[
G^\mu_{\nu\mu} = G^\mu_1_{\nu\mu} + \xi m^2 (-ie \Gamma^\mu_{\mu
u}) \phi^\mu_1 \quad \text{1.21}
\]

Hence

\[
\pi G^\mu_1_{\nu\mu} = \pi G^\mu_{\nu\mu} + ie \Gamma^\mu_{\nu\mu} \phi^\mu_1 \quad \text{1.22}
\]

Substituting \( \text{1.21} \) in \( \text{1.13} \) for \( K = +1 \) gives

\[
\pi G^\mu_1_{\nu\mu} = m^2 \phi^\nu + ie \Gamma^\mu_{\nu\mu} \phi^\mu_1 = 0 \quad \text{1.23a}
\]

Now \( \text{1.19} \) is the field equation for vector spin zero of mass \( M \). As equation
1.23a implies
\[ \pi \phi^1_\mu = 0 \] 

the identification of this equation as being appropriate for vector spin one seems reasonable. [The free field equations are to be found in I Section 2]. Thus assuredly the field equation 1.13 for \( K = +1 \) describes a decoupled system wherein there is no direct interaction of the form

\[ \text{(spin one)} \rightarrow \text{(photon)} + \text{(spin zero)} \]

As a footnote we comment that the above calculation is trivially extendable to establish that the field \( \phi_v \) satisfying

\[ \pi \phi_v - m^2 \phi_v + 2ie \phi_v \mathcal{F}_{\mu \nu} = -j_v \]  

is the charged analogue of the neutral field \( \phi_v \) of Polubarinov and Ogievsky [JETP (1962)]. For provided the external current is conserved

\[ \partial_{\mu} j_{\mu} = 0 \]

one can set

\[ \phi^1_\mu = \phi_v - \phi^0_v = \phi_v - m^{-2} \pi_v \pi_\rho \phi_\rho \]

and

\[ \pi \pi_v \phi^0_v - m^2 \phi^0_v = 0 \]

Our assignment of labels seems appropriate as \( \pi \phi^1_\mu = 0 \), and \( \phi^0_v \) is the charged spin zero vector field.

Yet we find that the LY graph elements are such that for any \( K \),

\[ S^{\alpha}(p^n) \mathcal{V}_{\mu}(p'^n \beta, p'^1 \gamma) \mathcal{S}^{0}_{\gamma \delta}(p'^1) \neq 0 \]

as is shown in our later calculations of the quantity \( \text{Tr}^{101}_0 \) which would be zero for
some $K$, were 1.24 false. The equations 1.23 and 1.24 are incompatible with the interpretation that $S^1$ and $S^0$ propagate spin one and (the unphysical) spin zero respectively. (Lee and Yang call $S^1$ the "spin-one part" of $S^{LY}$). Although it may be argued that the unphysical indefinite metric has led to the anomaly, we maintain this objection in view of our analysis of vector spin zero of IV Sections 3, 4, wherein we determined the value of charged spin zero lines as
\[
G_{a\beta}(p) = \frac{i}{p^2 + m^2 - i\epsilon} \frac{p \cdot p_{a\beta}}{p^2} .
\]

Now the multiplication of $G$ by a factor such as $\xi^{-1}$ would only trivially affect the graph theory (correspondingly, $V_3$-vertex would be multiplied by $\xi$ etc.). Despite this freedom, the different numerators make $G$ and $S^0$ totally distinct, and we must reject the LY interpretation of $S^1$, and $S^0$.

In the calculations we are to perform in LY theory we find it very convenient to work with matrices rather than with tensors. For this purpose we utilise the calculations of V Section 3, where we showed that the $10 \times 10$ Duffin-Kemmer matrices $\beta$ of V Appendix 5 have products $\beta_{\mu}$ which are of the form of the direct sum of a $6 \times 6$ matrix and a $4 \times 4$ matrix. The $4 \times 4$ matrix we call $\tilde{\beta}_{\mu}$: i.e.,
\[
(M - 2)\beta_{\mu} = \delta_{6x6} + \tilde{\beta}_{\mu} .
\]

The components of $\tilde{\beta}_{\mu}$ are by V 3.11
\[
[\tilde{\beta}_{\mu\nu}]_{a\beta} = \delta_{\mu\nu} \delta_{a\beta} - \delta_{\mu\beta} \delta_{va} .
\]

We also introduce the matrix $\bar{H}_{\mu\nu}$ such that
\[
[H_{\mu\nu}]_{a\beta} = -\delta_{\mu\beta} \delta_{va} .
\]

The special virtue of this unusual procedure is that when we calculate the traces $(Tr_{ab})$ which occur in the calculation of vacuum polarization we shall be evaluating
traces like

\[
\text{Tr } \beta_{\mu \nu} \beta_{\rho \sigma} H_{\gamma \delta} = \text{Tr } (M-2) \beta_{\mu \nu} \beta_{\rho \sigma} H_{\gamma \delta}
\]

\[
= \text{Tr } (M-3) \beta_{\nu} \beta_{\rho} H_{\gamma \delta} \beta_{\mu}
\]

where the second or third form is usually easier to calculate than the initial trace. In any case we have already tabled in V Appendix 4 all the traces we shall need (and others).

In matrix notation the field equation for LX field, i.e. alternate form of 1.13, is

\[
-\xi H_{\mu \nu} \pi_{\nu} \pi_{\mu} \phi + \bar{\beta}_{\mu \nu} \pi_{\mu} \pi_{\nu} \phi - m^2 \phi - i e K \bar{\beta}_{\mu \nu} \phi F_{\mu \nu} = 0
\]

Without further ado we now write down the graph elements of the LY theory.
### Table VI T 1

**LY Graph Elements in Momentum Space**

<table>
<thead>
<tr>
<th>Element</th>
<th>Graph</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal photon line</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\mathcal{D}_{F\mu\nu}(q)$</td>
</tr>
<tr>
<td>Internal meson line</td>
<td><img src="image" alt="Diagram" /></td>
<td>$S^L_Y(p) = S^1_p(p) + S^0_p(p)$</td>
</tr>
<tr>
<td>3-Vertex</td>
<td><img src="image" alt="Diagram" /></td>
<td>$e^2 U_{\mu\nu}(p'p''q'q'')$</td>
</tr>
<tr>
<td>4-Vertex</td>
<td><img src="image" alt="Diagram" /></td>
<td></td>
</tr>
</tbody>
</table>

\[
\mathcal{D}_{F\mu\nu}(q) = -i \frac{\delta_{\mu\nu}}{(q^2 - i\epsilon)} \quad \text{ (Fermi gauge)}
\]

\[
S^1_p(p) = i \frac{p^2 - (p^2 + m^2)}{m^2 [p^2 + m^2 - i\epsilon]} ; \quad S^1_{F\alpha\beta}(p) = -i \frac{\delta_{\alpha\beta}}{p^2 + m^2 - i\epsilon}
\]

\[
S^0_p(p) = -i \frac{p^2 - p^2}{m^2 [p^2 + m^2 - i\epsilon]} ; \quad S^0_{F\alpha\beta}(p) = i \frac{p^2 \delta_{\alpha\beta}}{p^2 + m^2 - i\epsilon}
\]

\[
\mathcal{V}_\mu(p''p') = i(1 + K)[\overline{p}^\mu \gamma^\nu p'^\nu - i(\gamma^\nu)[\overline{p}^\mu \gamma^\nu p'^\nu - i(P' + P'')^\nu] - i(\gamma^\nu)[\overline{p}^\mu \gamma^\nu p'^\nu - i(P' + P'')^\nu]
\]

\[
\mathcal{V}_\mu(p''p'a) = i \delta_{\alpha\beta}(p' + p'')^\mu - \delta_{\alpha\mu}(Kp' + p' + \xi P'')^\beta - \delta_{\beta\mu}(Kp' + p' + \xi P'')^\alpha
\]

\[
\mathcal{V}_{\nu\mu}(p''p'aq''q') = i(\overline{p}_{\mu\nu} + \overline{p}_{\nu\mu}) + i\xi[\overline{p}_{\mu\nu} + \overline{p}_{\nu\mu} - 2\delta_{\mu\nu}]
\]

\[
\mathcal{U}(p''p'aq''q') = i[2 \delta_{\mu\nu} \delta_{\alpha\beta} - (1 - \xi)\delta_{\alpha\mu} \delta_{\beta\nu} - (1 - \xi)\delta_{\alpha\nu} \delta_{\beta\mu}]
\]
Auxiliary Notation: \[ M_0 = M = \xi^{-1}m \]
\[ M_1 = m \]

The graphs are evaluated by the integration \( (2\pi)^{-4} \int \, d^2k \) over each independent internal momentum \( k \).

A complete diagram, as distinct from a diagram part acquires a factor \( (2\pi)^{4} \delta(P_i - P_f) \) where \( P_i - P_f \) expresses momentum conservation.
2. The Calculation of Vacuum Polarization

The photon propagator as calculated by graph methods is

\[ D_{\mu\nu}(q) = D_{\rho\mu\nu}(q) - i (\delta_{\mu\nu} - q_{\mu} q_{\nu}/q^2) \Pi(q^2) \]  \hspace{1cm} \text{(2.1)}

We suppose that \( \Pi(q^2) \) satisfies the dispersion relation

\[ \Pi(q^2) = -\frac{q^2}{\pi} \int_0^\infty \frac{\text{Im} \Pi(\mu^2)}{\mu^2} \frac{d\mu^2}{q^2 + \mu^2 - i\epsilon} \]  \hspace{1cm} \text{(2.2)}

We discuss the validity of this DR in Section 3. In terms of the Lehman-Kallen spectral weight \( \rho^3 \) used in previous chapters,

\[ \text{Im} \Pi(q^2) = \pi q^2 \rho^3 (-q^2) \]  \hspace{1cm} \text{(2.3)}

It is most expeditious, particularly in view of the considerations outlined in Section 3, to actually calculate

\[ \Pi'(0) = \lim_{q^2 \to 0} q^2 \Pi(q^2)/q^2 \]  \hspace{1cm} \text{(2.4)}

which is given by the DR as

\[ \Pi'(0) = -\frac{1}{\pi} \int_0^\infty \frac{\text{Im} \Pi(-\mu^2)}{\mu^4} d\mu^2 \]  \hspace{1cm} \text{(2.5)}

\[ = \int_0^\infty \frac{d\mu^2 \rho^3 (\mu^2)}{\mu^2} \]  \hspace{1cm} \text{(2.6)}

We can now write in terms of the graph elements given in Table VI T 1

\[ \Pi(q^2) = \sum_{ab} \Pi^{ab}(q^2) = \Pi^{11} + \Pi^{01} + \Pi^{10} + \Pi^{00} \]  \hspace{1cm} \text{(2.7)}

where the graphical expression for \(-i(\delta_{\mu\nu} - q_{\mu} q_{\nu}/q^2)\Pi^{ab}(q^2)/q^2\) is

\[ \begin{array}{c}
\text{spin } b, p'' \\
\text{spin } a, p'
\end{array} \]
The vacuum polarization contribution is

\[ \Pi_{ab}(q^2) = \frac{-i}{3q^2} \frac{e^2}{(2\pi)^4} \int \! \! d^4p \! \! d^4p'' \delta(q-p''+p') \delta_{\mu\nu} \text{Tr}[S_a(p') \bar{\nu}_\mu(p''p)S_b(p''p')\bar{\nu}_\nu(p''p')] \]  

2.8

Henceforth, we refer to the mass of the particle of spin a (0 or 1) by \( m_a \) where

\[ m_a = m + a(m - M) \]  

2.9

The specification by Lee and Yang of a negative metric in Hilbert space has lead to the denominators of both \( S^a(p) \) being \((p^2 + m_a^2 - i\epsilon)\); we can therefore at once apply Cutkosky's analysis [Cutkosky (1960)] to determine the jump discontinuity across the branch cuts of each \( \Pi_{ab}(q^2) \), which start at the Landau branch points at \( q^2 = -(M_a + M_b)^2 \).

The discontinuity is

\[ 2i\text{Im}\Pi_{ab} = \frac{-ie^2}{3q^2(2\pi)^4} \int \! \! d^4p \! \! d^4p'' \delta(q-p''+p') \delta_p(p''^2 + M_a^2) \delta_p(p''^2 + M_b^2) \frac{\text{Tr}^{ab}}{m^4} \]  

2.10

[Note that our requirement that \( D_{\mu\nu} \) have form of equation 2.1 is not an arbitrary procedure to "force" gauge invariance: see VI Appendix 1].

The two particle unitary phase integral has been evaluated in II Appendix 1.

We write

\[ \text{Im}\Pi_{ab} = -\frac{e^2}{4\delta m^4 q^2} f_{ab} \text{Tr}^{ab} \Theta[-q^2 - (M_a + M_b)^2] \]  

2.11

where

\[ f_{ab}(q^2) = (1 + \frac{2(m_a^2 + m_b^2)}{q^2} + \frac{(m_a^2 - m_b^2)^2}{4}) \]  

2.12

The reader will note that \( a(\cdot) \) has been artificially introduced into the definition of the \( \text{Tr}^{ab} \) which are as follows:

\[ \text{Tr}^{11} = -\delta_{\mu\nu} \text{Tr}[i\bar{\nu}_\mu(p''p)\bar{\nu}^\mu(p''p') \bar{\nu}_\nu(p''p')] \]  

2.13a

\[ \text{Tr}^{10} = -\delta_{\mu\nu} \text{Tr}[i\bar{\nu}_\mu(p''p)\bar{\nu}^\mu(p''p')(-1)\bar{\nu}^\nu(p''p')] \]  

2.13b

\[ \text{Tr}^{01} = \text{Tr}^{10} \]  

2.13c

\[ \text{Tr}^{00} = -\delta_{\mu\nu} \text{Tr}[(-1)\bar{\nu}^\mu(p''^2-p_1^2)\nu_\nu(p''p')(-1)\bar{\nu}^\nu(p''^2-p_2^2)\nu_\nu(p''p')] \]  

2.13d

and which, by virtue of the \( \delta \) functions, are to be evaluated on the mass shell
specified by
\[ q = p' - p''; \quad p'^2 = -\lambda_a^2; \quad p''^2 = -\lambda_b^2. \]
\[ q = p' - p'', \quad p'^2 = -\lambda_a^2, \quad p''^2 = -\lambda_b^2. \]

Then, for example,
\[ p'p'' = -\frac{1}{2}(q^2 + \lambda_a^2 + \lambda_b^2). \]
\[ \frac{1}{2}(q^2 + \lambda_a^2 + \lambda_b^2). \]

The \( \text{Tr}^{ab} \) (\( q^2 \)) are functions of \( q^2 \). We name the quantity in curly brackets in \( 2.16 \) \( M^{ab} \):
\[ \text{Tr}^{ab} = \text{Trace} \ M^{ab}. \]
\[ \text{Tr}^{ab} = \text{Trace} \ M^{ab}. \]

\( \text{Tr}^{11} \) is very readily calculated by treating \( M^{11} \) as the product of the usual \( \beta \) matrices, when algebraic reduction simplifies \( M^{11} \) markedly. (See Appendix 2, calculation of IV). The 4 x 4 matrix \( M^{00} \) comprises sums of products of \( H_{\mu\nu} \) matrices, so that \( \text{Tr}^{00} \) can be easily found (Appendix 2, calc. of I). \( M^{01} \) is more complex, and so the determination of \( \text{Tr}^{01} \) requires the calculation of the trace of several matrices like \( M^{00} \) and \( M^{11} \) but on different mass shells. We have therefore found it expeditious to define and calculate in Appendix 2 the 6 traces \( I, II_A, II_B, III, IV, V \) of which as
\[ II_A \ (M_a, M_b) = II_B \ (M_a, M_a), \]
\[ I + II_A + II_B + III = IV, \]
\[ I + II_A = V. \]

only 3 are in fact independent. As a consistency check, in Appendix 2, the 4 traces; \( I, II_A, III, IV \) are explicitly calculated from their defining formulas using the tables of traces of 10 x 10 \( \beta \) matrices of the preceding Chapter. All 6 traces are then tabulated in VI TA 2, and on making the trivial substitutions
\[ \text{Tr}^{11} = IV (M_a = M_b = m), \]
\[ \text{Tr}^{01} = V (M_a = m, M_b = M), \]
\[ \text{Tr}^{00} = I (M_a = M_b = M), \]

the three sought traces are found and written down in full in Appendix 3.
A few comments on the form of the $\text{Tr}^{ab}$ is in order here. In the notation of V A 3.37,

$$\text{Tr}^{11}(q^2) = \frac{m^4}{4} \text{Tr}(q^2),$$

i.e., $\text{Tr}^{11}$ is exactly the expression given in the usual "renormalisable" theory of charged particles of spin one mass $m$, anomalous moment $K$. The dispersion integral for $\Pi^{11}(q^2)$ diverges very badly, and we see that only by subtracting from $\text{Im} \Pi^{11}$ an expression of like asymptotic behaviour, can a finite theory be developed.

For $K = 0$, $\text{Tr}^{00}$ likewise has the usual value: the contribution to vacuum polarization being exactly that ascribable to charged spin zero particles of mass $M$. $K \neq 0$, $\text{Im} \Pi^{00}(q^2)$ has the same asymptotic behaviour as $\text{Im} \Pi^{11}(q^2)$.

$2\text{Im} \Pi^{01}(q^2)$ is the contribution to vacuum polarization due to pairs of particles of spin one and spin zero. It is negative as such a state includes one spin zero particle.

The first question to be asked at this stage is whether in fact the dispersion integrals for $\Pi(q^2)$ converge. This is ascertained by calculating in Appendix IV the asymptotic value of

$$U = f_{11} \text{Tr}^{11} + 2f_{01} \text{Tr}^{01} + f_{00} \text{Tr}^{00},$$

the asymptotic value of $f_{ab}$ being given by 2.12 as

$$f_{ab} \sim \frac{M_a^2 + M_b^2}{q^2} - \frac{2 M_a^2 M_b^2}{q^4} + \frac{(M_a^2 + M_b^2)^3}{2q^6}.$$

Hence the asymptotic value ($-q^2 \to \infty$)

$$\text{Im} \Pi(q^2) = -\frac{e^2}{4\pi m^2} \frac{U}{q^2}$$

$$= -\frac{e^2}{4\pi m^2} \left[ \frac{1}{3} \left(1-K^2\right) \xi^{-1} + 5 - 12K - 3K^2 \right].$$

Thus the dispersion integrals do converge. We can now use 2.5 to evaluate

$$\Pi'(0) = \lim_{\lambda \to 0} \frac{e^2}{4\pi m^2} \left[ \frac{1}{3} \left(1-K^2\right) \xi^{-1} + 5 - 12K - 3K^2 \right].$$
where \[ \frac{e^2}{48\pi^2m^2} \pi^{ab}(\lambda) = -\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dx}{x^2} \Im \Pi^{ab}(-x) \] 2.25

\[ = -\frac{e^2}{48\pi^2m^2} \int_{(M_a^2+m_b^2)^2}^{\infty} \frac{dx}{x^2} f^{ab}(-x) \text{Tr}^{ab}(-x)m^2. \] 2.26

To evaluate \( \pi^{ab} \) we need the elementary integrals:

(A) "Equal Mass"
\[ \int_{4}^{X} \frac{dx}{x} \left[ x^2 - 4x \right]^{1/2} \] 2.27
with upper bound \( X = \lambda^{-1} \), \( \xi^{-1} \).

(B) "Unequal Mass"
\[ \int_{Y}^{\lambda^{-1}} \frac{dx}{x} \left[ x^2 - 2(1 + \xi)x + (1 - \xi)^2 \right]^{1/2} \] 2.28
with lower bound \( Y = (1 + \xi^{1/2})^2 \).

These integrals are to be found in Appendix 5, the \( \pi^{ab} \) then being determined in Appendix 6 using the \( \text{Tr}^{ab} \) of Appendix 3. We find that

\[ \pi^{11}(\lambda) = \frac{K^2}{4} \xi^{-1} \lambda^{-1} + (1 + 3K - \frac{K^2}{2}) \log \lambda^{-1} + (1 + 3K - \frac{K^2}{2}) \log \xi^{-1} - \frac{71}{30} \xiK - \frac{7K^2}{6}, \] 2.29

\[ 2\pi^{01}(\lambda) = -\frac{K^2}{2} \xi^{-1} \lambda^{-1} + \left( \frac{3K^2}{2} \xi^{-1} - 4K + \frac{K^2}{2} \right) \log \lambda^{-1} - \frac{3}{4} K^2 \xi^{-1} + \frac{11}{6} + \frac{16}{3} K - \frac{23}{12} K^2. \] 2.30

\[ \pi^{00}(\lambda) = \frac{K^2}{4} \xi^{-1} \lambda^{-1} + \left( -\frac{3}{2} K^2 \xi^{-1} + K \right) \log \lambda^{-1} + \frac{3}{2} K^2 \xi^{-1} - \frac{8}{3} K . \] 2.31

Thence for small \( \xi \) the Lee and Yang theory predicts

\[ \Pi'(0) = \frac{e^2}{48\pi^2m^2} \left\{ \frac{3}{4} K^2 \xi^{-1} + (1 + 3K - \frac{K^2}{2}) \log \xi^{-1} - \frac{16}{15} - \frac{16}{3} K - \frac{37}{12} K^2 \right\}. \] 2.32
3. Discussion of Calculation of Vacuum Polarization

In the previous section we used the DR

\[ \Pi(q^2) = q^2 \int_0^\infty \frac{d\mu^2 \rho_2(\mu^2)}{p^2 + \mu^2 - i\varepsilon} \]  \hspace{1cm} (3.1)

to calculate

\[ \Pi'(0) = \int_0^\infty \frac{d\mu^2 \rho_3(\mu^2)}{\mu^2} \]  \hspace{1cm} (3.2)

We refer the reader to the paper of Lehman (1954) where DR are derived from the "basic principles" of field theory [note, re our remarks in Chapter II, that the photon field is uncharged], the derivation being independent of metric, but in case Hilbert space has positive metric, the spectral weights involved satisfy certain inequalities. Thus in Lee and Yang theory the DR 3.1 is valid but the theorem

\[ \rho_3 \geq 0 \]  \hspace{1cm} (3.3)

no longer holds as the various contributions to \( \rho_3 \) are no longer positive definite. This theorem was first proved by Kallen (1952), who utilised the theory of the electromagnetic field due to Guypa (1950) and Bleuler (1950). In the Guypa-Bleuler theory the metric operator \( \gamma \) has the property

\[ \gamma \psi = (-)^{N_T} \psi \]  \hspace{1cm} (3.4)

where \( N_T \) is the number of timelike photons in the state \( \psi \). Allowable states are limited by the constraint

\[ \partial_\mu A_\mu^+ \psi = 0 \]  \hspace{1cm} (3.5)

which ensures that any negative contribution to \( \rho_3(-q^2) \) due to a state including
timelike photons, is exactly cancelled by the contribution of a similar state with a longitudinal photon.

Referring to equation 3.2 we see that Kallen's theorem, $\rho_3 \geq 0$, has the consequence that $\Pi'(0)$ is positive definite:

$$\Pi'(0) \geq 0.$$  \hspace{1cm} 3.6

We propose this theorem (3.6) as a criterion which must be satisfied by any theory of charged particles. Our theorem (3.6) has not yet been proved in a manner that is totally to our satisfaction: within the context of quantum field theory a satisfactory proof would not depend on the vagaries of the Gupta-Bleuler schema. Another approach to a proof of 3.6 would come from considerations of the macro properties of the vacuum, for which the dielectric constant is simply given by

$$n(q^2) = 1 + \Pi(q^2).$$  \hspace{1cm} 3.7

[see e.g. Kallen (1958)]. We strongly suspect that a vacuum state for which $\Pi'(0)$ is negative would be unstable; in amplification we remark that no energy need be supplied to form a pair comprising one positive energy and one negative energy particle.

In the original paper [Lee and Yang (1962)] Lee and Yang seem to have satisfied themselves of the physical reasonableness of their theory in the limit $\xi \to 0$: we quote their only statement in this connection: "if the limit $\xi \to 0$ exists, the limiting $S$ matrix becomes completely unitary." Later within the broader realm of weak interactions it was found by Bernstein and Lee (1963) that for $K \neq 0$, the Lee and Yang theory, taken together with Lee's prescription of calculation methods [Lee (1963)] leads to a neutrino charge radius independent of the electric charge. Restricting consideration to purely electromagnetic phenomena we have found that a second order calculation gives a $\Pi'(0) \geq 0$. Explicitly we found
\[
\Pi^*(0) = \frac{e^2}{48\pi^2 m^2} \left( \frac{3}{4} K^2 \xi^{-1} + (1 + \xi - \frac{K^2}{2}) \log \xi^{-1} - \frac{16}{15} K - \frac{37}{12} K^2 \right)
\]

on neglect of terms of order \(\xi\) and higher powers of \(\xi\). The satisfaction of the criterion 3.6 is a favourable aspect of the Lee and Yang theory.

Before proceeding we must mention that a nominal calculation of \(\Pi^*(0)\) in Lee and Yang theory has been performed by Beg [Beg (1964)]. Beg cut off the dispersion integral 3.1 at the threshold where the metric in Hilbert space ceases to be positive definite for meson states of zero total charge, that is at \((m + M)^2\). Thus Beg's method ensures satisfaction of the criterion \(\Pi^*(0) > 0\), a justification not adduced by Beg, but we are not aware of any other theoretical arguments which justify his method. Surely the convergence of the dispersion integral for \(\Pi(q^2)\) is the very raison d'être of the Lee and Yang theory, and Beg's cut off procedure is quite foreign to the theory. Beg's results are in our notation

\[
\Pi^*(0)_{\text{Beg}} = -\frac{1}{\pi} \int_{4m^2}^{(m + M)^2} \frac{dx}{x^2} \Im \Pi(-x)
\]

\[
= -\frac{1}{\pi} \int_{4m^2}^{(m + M)^2} \frac{dx}{x^2} \Im \Pi^{11}(-x)
\]

\[
\frac{e^2}{48\pi^2 m^2} \frac{K^2}{4} \xi^{-1} + \text{less singular terms } K \neq 0 \quad 3.10a
\]

\[
= \frac{e^2}{48\pi^2 m^2} \log \xi^{-1} + \text{finite terms } K = 0 \quad 3.10b
\]

Thus, quite fortuitously, Beg's results agree with ours for \(K = 0\), and differ by a factor 3 for \(K \neq 0\).
Now as to the interpretation we place on the calculated value of vacuum polarization: We feel that this value, that for any \( K \) diverges as \( \xi \to 0 \) gives us sufficient ground for rejecting this putative theory of charged spin one. Bég (1964) makes some sense of his results by invoking the procedures of Lee (1962) and Bernstein and Lee (1963), but these procedures are far too ad hoc to be acceptable; the \( \xi \) limiting procedure is ad hoc enough, but we would accept it as merely an extension of regularization theory if it gave a finite limit; the appending to this theory of Lee and Yang the doctrinal belief that the calculations that one hasn't performed of higher order graphs eliminate the infinities in the calculations of lower order graphs is unreasonable.
VI. APPENDIX 1

REMARKS ON GAUGE INVARiance OF $D_{\mu \nu}$

This Appendix serves as a footnote to our calculation of the discontinuity of $D_{\mu \nu}$ (equation 2.10). The method used lacks the elegance of our treatment in earlier Chapters: in the scalar spin zero case we could write - ignoring factors -

$$\text{disc } D_{\mu \nu}(q) = \int \frac{d^4p'}{2\pi^2} \frac{d^4p''}{2\pi^2} \delta(p'^2 + m^2) \delta(p''^2 + m^2) (p'_\mu p''^\nu - p''_\mu p'_\nu) \nu$$ \hspace{1cm} VI Al.1

$$= (q^2 \mu \nu - q^\mu q^\nu) \theta(q^2 - 4m^2) \hspace{1cm} \text{Al.2}$$

where we applied II A 2. Thus the gauge invariance of disc $D_{\mu \nu}$ was explicitly demonstrated for spin zero, and for spin 1/2 case likewise. In Lee and Yang theory the integrand is far too complex for this direct approach to be convenient.

The G.I. follows directly from the observation that

$$(p'' - p')^\mu \nu (p''p') = i(1 - \xi)(p''^2 - p'^2) + i\xi(p''^2 - p'^2) \hspace{1cm} \text{Al.3}$$

vanishes in each of three relevant cases:

(i) $p'^2 = p''^2 = -m^2; \hspace{1cm} p'_\mu p'^\mu = -m^2 \hspace{1cm} \text{Al.4}$

(ii) $p'^2 = -m^2, p''^2 = -m^2 = -\xi m^2; \hspace{1cm} p'_\mu p'^\mu = -m^2, p''_\mu p''^\mu = 0 \hspace{1cm} \text{Al.5}$

(iii) $p'^2 = p''^2 = -m^2; \hspace{1cm} p'_\mu p''^\mu = -m^2 \hspace{1cm} \text{Al.6}$

Then as disc $D_{\mu \nu}$ is the mass shell value of

$$\text{disc } D_{\mu \nu} = V_\mu (p''^\nu) V_\nu (p'^\mu)$$ \hspace{1cm} \text{Al.7}

we must have

$$\text{disc } D_{\mu \nu} = \frac{1}{2}(\delta_{\mu \nu} - q_\mu q_\nu / q^2) \text{disc } D_{pp} \hspace{1cm} \text{Al.8}$$
VI APPENDIX 2

CALCULATION OF VARIOUS TRACES

This appendix contains the direct calculation of the following traces; where by direct calculation we imply an evaluation using the table of the traces of $\beta$ matrices in chapter V

\begin{align*}
I &= \text{Tr} \left\{ [\bar{\beta} p^2 - p'^2] \bar{\nu}_\mu (p' p^\mu) \left[ \bar{\beta} p^n^2 - p''^2 \right] \bar{\nu}_\mu (p'' p^\mu) \right\} \\
II_A &= \text{Tr} \left\{ p'^2 \bar{\nu}_\mu (p' p^n) \left[ \bar{\beta} p^n^2 - p''^2 \right] \bar{\nu}_\mu (p'' p^\mu) \right\} \\
II_B &= \text{Tr} \left\{ [\bar{\beta} p^2 - p'^2] \bar{\nu}_\mu (p' p^n) p^n^2 \bar{\nu}_\mu (p'' p^\mu) \right\} \\
III &= \text{Tr} \left\{ p'^2 \bar{\nu}_\mu (p' p^n) p^n^2 \bar{\nu}_\mu (p'' p^\mu) \right\} \\
IV &= \text{Tr} \left\{ [\bar{\beta} p^2 - p'^2] \bar{\nu}_\mu (p' p^n) \bar{\beta} p^n^2 \bar{\nu}_\mu (p'' p^\mu) \right\}
\end{align*}

The traces are evaluated

for \( q = p' - p'' \)

with \( p'^2 = -M_a^2 \)

and \( p''^2 = -M_b^2 \).

Thus \( p' p'' = -\frac{1}{2} (q^2 + M_a^2 + M_b^2) \).

We note once more for completeness that

\[
\bar{\nu}_\mu (p' p^n) = i(1+K) \left[ \beta p^\mu \beta + \beta^\mu p^\nu \beta \right] - i(K+\xi) \left[ \beta p^n^\nu \beta + \beta^\nu p^n^\mu \beta \right] + \xi \left[ p^n p^\mu \right] \bar{\nu}_\mu \]

We have calculated one more trace than sufficient for our needs in order to be able to carry out a consistency check.
The reader on examining the table and having in mind the relation of $II_B$ to $II_A$ will readily be able to see that the relation

$$I + II_A + II_B + III = IV$$

does in fact hold. We also write down in the table

$$V_{df} = \text{Tr} \left\{ \bar{p}_f 2\gamma \mu (p'p'\nu) [\bar{p}_f p'^2 - p'^2] \gamma \mu (p''p''\nu) \right\} = I + II_A$$

$$I \text{ Tr} [\bar{p}_f p'^2] \gamma \mu (p'p'\nu) [\bar{p}_f p'^2] \gamma \mu (p''p''\nu)$$

In terms of $H_{\mu\nu} = \beta_{\mu\nu} - \delta_{\mu\nu}$, the vertex function

$$V_{\mu}(p'p'\nu) = i(1+K)(\beta_{\mu\nu} \beta_{\mu\nu} + \beta_{\mu\nu} \beta_{\mu\nu}) - i(K+\xi) (\beta_{\mu\nu} H_{\rho\nu} p'\rho + H_{\rho\nu} p'\rho) - iK(p' + p''\nu) \beta_{\mu\nu}.$$

As $\beta_{\mu\nu} H_{\rho\nu} p'\rho = 0 = H_{\alpha\beta} p'\alpha p'\beta$, the terms proportional to $(1+K)$ vanish from the trace which becomes, noting the symmetry in $p', p''$ of the cross terms,

$$- (K+\xi)^2 \delta_{\mu\nu} \text{Tr} [(M-2)H_{\alpha\beta} p'\alpha p'\beta (H_{\mu\nu} p'\rho + H_{\rho\nu} p'\rho) H_{\gamma\delta} (H_{\mu\nu} p'\rho + H_{\rho\nu} p'\rho) - iK(p' + p'\nu) \beta_{\mu\nu}.$$

The traces involved are tabulated in Chapter V. Applying them we obtain

$$- (K+\xi)^2 \delta_{\mu\nu} \left[ p' p'^2 p'^2 p'\mu p'\nu + p' p'^2 p'^2 p'\nu + p' p'^2 p'\mu p'\nu + p'^2 p' p'^2 p'\mu p'\nu \right]$$

$$+ 2K(K+\xi) \left[ p' p'^2 (p' p'\nu)^2 + p'^2 (p' p'\nu)^2 \right] - K^2 (p' p'^2 (p' p' p'\nu)^2) - \left[ p' p'^2 (p' p' p'\nu)^2 \right]$$

$$= (K+\xi)^2 m_a^2 m_b^2 \left[ q^2 + 2 (m_a^2 + m_b^2) \right]$$

$$- \frac{K}{2} (K+\xi) \left[ q^2 + (m_a^2 + m_b^2) \right] \left[ (m_a^2 + m_b^2) q^2 + m_a^4 + 6 m_a^2 m_b^2 + m_b^4 \right]$$

$$+ \frac{K^2}{4} \left[ q^2 + m_a^2 + m_b^2 \right] \left[ q^2 + 2 (m_a^2 + m_b^2) \right]$$
\[
\begin{align*}
&= \frac{1}{4} \left[ K^2 + 2 K \xi + \xi^2 \right] \left[ 4 M_a^2 M_b^2 q^2 + 8 M_a^2 M_b^2 (M_a^2 + M_b^2) \right] \\
&\quad - \frac{1}{4} \left[ K^2 + K \xi \right] \left[ 2(M_a^2 + M_b^2) q^4 + 4 (M_a^2 + M_b^2) q^2 + 8 M_a^2 M_b^2 q^4 + (M_a^2 + M_b^2) \left[ 2(M_a^2 + M_b^2) q^2 + 8 M_a^2 M_b^2 \right] \right] \\
&\quad + \frac{K^2}{4} \left[ q^6 + 4 (M_a^2 + M_b^2) q^4 + 5 (M_a^2 + M_b^2) q^2 + 2 (M_a^2 + M_b^2)^3 \right] \\
&= \frac{K^2}{4} q^6 + (K^2 - K \xi) \left( \frac{M_a^2 + M_b^2}{2} \right) q^4 \\
&\quad + \left[ (K^2 - 4 K \xi) \left( \frac{M_a^2 + M_b^2}{2} \right) + (-K^2 + \xi^2) M_a^2 M_b^2 \right] q^2 \\
&\quad + \left[ -4 K \xi \left( \frac{M_a^2 + M_b^2}{2} \right) + (4 K \xi + \xi^2) M_a^2 M_b^2 \right] \frac{M_a^2 + M_b^2}{2}.
\end{align*}
\]

\[ \Pi_A \, \text{Tr} \left[ p^\sigma \bar{V}_\mu (p^\nu p^{\nu}) \right] \left[ \bar{p} p^{\sigma 2} - p^{\nu 2} \right] V^{\nu}_{\mu} (p^{\nu} p^{\nu}) \]

We recall the definition of \( H_{\mu\nu} \):

\[
\bar{H}_{\mu\nu} = \bar{\beta}_{\mu\nu} - \delta_{\mu\nu}.
\]

Then \( V_{\mu} (p^\nu p^{\nu}) = \ii (1+K) \left[ H_{\mu \rho} p^{\rho} p^{\nu} + H_{\mu \sigma} p^{\rho} p^{\sigma} \right] - \ii (K+\xi) \left[ H_{\mu \rho} p^{\rho} p^{\nu} + H_{\mu \sigma} p^{\rho} p^{\sigma} \right] + \ii (p^\rho p^{\nu}) \nu_{\mu}, \)

\[
\beta p^{\nu 2} - p^{\nu 2} = H_{\alpha \beta} p^{\alpha} p^{\beta},
\]

while

\[
V_{\mu} (p^{\nu} p^{\nu}) = \ii (1+K) \left[ H_{\mu \rho} p^{\rho} p^{\nu} + H_{\mu \sigma} p^{\rho} p^{\sigma} \right] - \ii (K+\xi) \left[ H_{\mu \rho} p^{\rho} p^{\nu} + H_{\mu \sigma} p^{\rho} p^{\sigma} \right] + \ii (p^\rho p^{\nu}) \nu_{\mu}.
\]

Hence \( \text{Tr} \left[ V_{\mu} (p^\nu p^{\nu}) \left( \bar{p} p^{\nu 2} - p^{\nu 2} \right) \bar{V}_{\nu} (p^{\nu} p^{\nu}) \right] \)

\[
= (p^\nu + p^{\nu} \nu_{\mu} (p^\sigma + p^{\nu}), (p^\nu + p^{\nu}) \nu_{\mu} p^{\nu 2}
\]

\[
+ (K+\xi) \left( p^{\nu 2} + p^{\nu} p^{\nu} \right) \left[ (p^\rho + p^{\nu}), \nu_{\mu} p^{\nu 2} + (p^\rho + p^{\nu}), \nu_{\mu} p^{\nu 2} \right]
\]

\[
- (1+K) \left( p^{\nu 2} + p^{\nu} p^{\nu} \right) \left[ (p^\rho + p^{\nu}), \nu_{\mu} p^{\nu 2} + (p^\rho + p^{\nu}), \nu_{\mu} p^{\nu 2} \right]
\]

+ (K+\xi)^2 \left[ p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \right] \\
- (1+K)^2 \left[ p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \right] \\
+ (1+K) (K+\xi) \left[ -p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \delta_{\mu\nu} - p^{\mu} p^{\nu} \right] \\
+ (1+K) (K+\xi) \left[ -p^{\mu} p^{\nu} \delta_{\mu\nu} + p^{\mu} p^{\nu} \delta_{\mu\nu} - p^{\mu} p^{\nu} \right] \\
so that \ Tr \ p^2 \left[ \bar{\nu}_{\mu} (p' p') \left[ \bar{\nu}_{\nu} (p' p') \right] \right] \\
= \frac{\nu_{a}^2 \nu_{b}^2}{M_a} \left[ - q^2 - 2 M_a^2 - 2 M_b^2 \right] \\
+ (1-\xi) M_a^2 \left[ \frac{1}{2} \left( q^2 + M_a^2 + 3 M_b^2 \right)^2 \right] \\
+ (K+\xi)^2 M_a^2 M_b^2 \left[ - \left( q^2 + M_a^2 + M_b^2 \right) - 4 M_b^2 - M_a^2 \right] \\
+ (1+K)^2 M_a^2 \left[ - \left( q^2 + M_a^2 + M_b^2 \right)^2 - M_a^2 \left( q^2 + M_a^2 + M_b^2 \right) - M_b^2 \right] \\
+ (1+K) (K+\xi) M_a^2 \left[ \frac{1}{2} \left( q^2 + M_a^2 + M_b^2 \right)^2 + 5 M_b^2 \left( q^2 + M_a^2 + M_b^2 \right) + 2 M_b^2 \right] \\
= \frac{M_a^2}{2} q^4 \left[ 1 - \xi - 2 (1+K)^2 + (1+K) (K+\xi) \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) q^2 \left[ 4 - 4 \xi - 8 (1+K)^2 + 4 (1+K) (K+\xi) \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) q^2 \left[ - 1 + 2 (1-\xi) - (K+\xi)^2 + (1+K)^2 + 5 (1+K) (K+\xi) \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) \left[ 8 (1-\xi) - 16 (1+K)^2 + 8 (1+K) (K+\xi) \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) \left[ - 4 + 4 (1-\xi) - 4 (1+K)^2 - 2 (1+K)^2 + 10 (1+K) (K+\xi) \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) \left[ - 4 + 8 (1-\xi) - 10 (K+\xi)^2 - 4 (1+K)^2 + 14 (1+K) (K+\xi) \right] \\
= \frac{M_a^2}{2} q^4 \left[ 1 - \xi - K^2 + K \xi \right] \\
+ \frac{M_a^2}{2} \left( \frac{M_a^2 + M_b^2}{2} \right) q^2 \left[ - 4 - 12 K - 4 K^2 + 3 K \xi \right]
\[ + M_a^2 M_b^2 a^2 [3K + 3 K^2 + \xi + 3 \xi K - \xi^2] \]
\[ + \frac{M^2}{2} (\frac{M_a^2 + M_b^2}{2}) [ - 4 - 12 K - 4 K^2 + 4 \xi K] \]
\[ + \frac{M^2}{2} M_a M_b [ - 2 + 6 K + 4 K^2 + 5 \xi + 2 \xi K - 4 \xi^2] \]
\[ + \frac{M^2}{2} M_a M_b [6K + 6 \xi - 6 \xi K - 10 \xi^2]. \]

III \[ \text{Tr} \ p_i^2 \bar{\nu}_\mu (p^*p^\mu) p_i^* p_i^\mu \bar{\nu}_\mu (p^*p^\mu) \]

Now
\[ \bar{\nu}_\mu (p^*p^\mu) = i(1+K)(\beta p^* \beta \mu + \beta \mu \beta p^* \bar{\nu}_\mu) - i(K+\xi)(\beta p^* \beta \mu + \beta \mu \beta p^*) + i \xi (p^*+p^\mu) \bar{\nu}_\mu, \]
and
\[ \bar{\nu}_\mu (p^*p^\mu) = i(1+K)(\beta p^* \beta \mu + \beta \mu \beta p^* \bar{\nu}_\mu) - i(K+\xi)(\beta p^* \beta \mu + \beta \mu \beta p^*) + i \xi (p^*+p^\mu) \bar{\nu}_\mu. \]

Making the expansion
\[ \text{Tr} [\bar{\nu}_\mu (p^*p^\mu) \bar{\nu}_\mu (p^*p^\mu)] \]

\[ = - [(1+K)^2 + (K+\xi)^2] \text{Tr}_r + 2(1+K)(K+\xi) \text{Tr}_s - 2[(1+K) - (K+\xi)] \text{Tr}_t - \xi^2 \text{Tr}_u \]
\[ = - (1+2K+2K^2+2K+\xi^2) \text{Tr}_r - (2K-2K^2-2\xi-2K\xi) \text{Tr}_s - (2\xi-2\xi^2) \text{Tr}_t - \xi^2 \text{Tr}_u, \]
we have
\[ \text{Tr}_r = \text{Tr} (M-2)(\beta p^* \beta \mu + \beta \mu \beta p^* \bar{\nu}_\mu) \]
\[ = \text{Tr} (M-2)(\beta p^* \beta \mu \beta p^* \beta \mu + \beta \mu \beta p^* \beta \mu \beta p^* \bar{\nu}_\mu) \]
\[ = \text{Tr} (2 \beta p^* \beta p^* + 2 \beta p^* \beta p^* \bar{\nu}_\mu) \]
\[ = \text{Tr} (2 \beta p^* \beta p^* + 2 \beta p^* \beta p^* \bar{\nu}_\mu). \]
\[
= 6 \, p'p'' + 5 \, p' + 5 \, p''^2
\]

\[
= -3 \, q^2 - 9 \left( M_a^2 + M_b^2 \right)
\]

\[
\text{Tr}_s = \text{Tr}(M-2) \left( \beta_\mu \beta_\mu + \beta_\mu \beta_{p''} \right) \left( \beta_\mu \beta_\mu + \beta_\mu \beta_{p''} \right)
\]

\[
= \text{Tr}(M-2) \left( 2 \beta_\mu \beta_\mu \beta_{p''} \beta_{p''} + \beta_\mu \beta_\mu \beta_{p''} \beta_{p''} + \beta_\mu \beta_{p''} \beta_{p''} \beta_{p''} \right)
\]

\[
= \text{Tr} \left( 4 \beta_{p''}^4 + \beta_{p''}^2 + \beta_{p''}^2 \right)
\]

\[
= 12 \, p'p'' + 3 \, p' + 3 \, p''^2
\]

\[
= -6 \, q^2 - 9 \left( M_a^2 + M_b^2 \right)
\]

\[
\text{Tr}_t = \text{Tr} \left[ 2 \beta_{p'p''} + \beta_{p'p''}^2 + \beta_{p''}^2 \right]
\]

\[
= 6 \, p'p'' + 3 \, p' + 3 \, p''^2
\]

\[
= -3 \, q^2 - 6(M_a^2 + M_b^2)
\]

\[
\text{Tr}_u = (p' + p'')^2 \text{Tr} (M-2) = -4 \, q^2 - 8(M_a^2 + M_b^2)
\]

so that \[
\text{Tr} \left[ \bar{\nu}_\mu (p'p'') \bar{\nu}_\mu (p''p') \right]
\]

\[
= (3 - 6k - 6k^2 - 6k^2 + 4k^2)q^2 + (9 - 6k + 5k^2)(M_a^2 + M_b^2)
\]

For completeness we write down

\[
\text{Tr} \left[ p'^2 \bar{\nu}_\mu (p'p'')p''^2 \bar{\nu}_\mu (p''p') \right]
\]

\[
= (3 - 6k - 6k^2 - 6k^2 + 4k^2)M_a^2 M_b^2 q^2 + (18 - 12k + 10k^2)M_a^2 M_b^2 \left( \frac{M_a^2 + M_b^2}{2} \right)
\]

IV \[
\text{Tr} \left[ \bar{p}_{p'}^2 \bar{\nu}_\mu (p'p'')p''^2 \bar{\nu}_\mu (p''p') \right]
\]

Now in terms of 10 \times 10 \beta matrices

\[
\text{Tr} \left[ \bar{p}_{p'}^2 \bar{\nu}_\mu (p'p'')p''^2 \bar{\nu}_\mu (p''p') \right]
\]
\[ = \text{Tr} \left( (3-M) (\beta^p \gamma^\mu (p' \gamma^n) \beta^p n) (\beta^p n' (p'' \gamma^\mu) \beta^p p') \right) \]

As \( V_\mu (p' \gamma^n) = i(1+K)(\beta^p' \gamma^\mu + \beta^p \gamma^\mu) - i(K+\xi)(\beta^p n' \gamma^\mu + \beta^p p') + i\xi (p' \gamma^n) \),

it follows that

\[ \beta^p V_\mu (p' \gamma^n) \beta^p n = i(1+K)(\beta^p' \gamma^\mu \beta^p n + \beta^p \gamma^\mu \beta^p p') - i K(p' \gamma^n) \beta^p \gamma^\mu \beta^p p', \]

while

\[ \beta^p n V_\mu (p' \gamma^n \beta^p p') = i(1+K)(\beta^p n' \gamma^\mu \beta^p p' + \beta^p n \gamma^\mu \beta^p p') - i K(p' \gamma^n) \beta^p n' \beta^p p', \]

We therefore conveniently expand the trace as

\[ \omega \ H \ e \ \epsilon \]

\[ S_{p_x} = \text{Tr} (3-M) (\beta^p n^2 \gamma^\mu \beta^p p' + \beta^p n^2 \gamma^\mu \beta^p p') (\beta^p p' \gamma^\mu \beta^p n + \beta^p p' \gamma^\mu \beta^p p') \]

\[ = \text{Tr} (3-M) \left[ p' (p' n^2 \beta^p p' + (p' n^2 + p' p^2) \beta^p n^2) \right] \]

\[ = p' p' n^2 \text{Tr}(\beta^p p'^2) + p' p' n^2 \text{Tr}(\beta^p p'^2) \]

\[ + 2 p' p' n^2 \text{Tr}(\beta^p p'^2 + \beta^p p'^2)(p' p^2 + p' p^2) \text{Tr}(3-M) \beta^p p'^2 \beta^p n^2 \]

\[ = p' p' n^2 \left( 6 p' p'' + 6 p' p^2 + 6 p' n^2 \right) - (p' p^2 + p' n^2)(2 p' p'' + p' p^2 n^2) \]

\[ = \frac{1}{2} (M_a^2 + M_b^2) q^2 + 4 \left( \frac{M_a^2 + M_b^2}{2} \right) q^2 - 3 M_a^2 M_b^2 q^2 \]

\[ + 4 \left( \frac{M_a^2 + M_b^2}{2} \right) - 16 \left( \frac{M_a^2 + M_b^2}{2} \right) M_a^2 M_b^2, \]

\[ S_{p_y} = \text{Tr} (3-M) (2 \beta^p n^2 \beta^p p' + (p' p^2 + p' n^2) \beta^p n' \beta^p p') (\beta^p p' \beta^p n') \]

\[ = 6 p' p'' n^2 \beta^p p'' + (p' p^2 + p' p^2) \left[ (p' p'' + 2 p' p' n^2) \right] \]

\[ (p' p' n^2) + 2 p' p^2 p' n^2 \]
\[ \begin{align*}
(p', p'')^2 (p', p'')^2 + 6 p', p'' (p', p'')^2 + 2 p', p''^2 (p', p'')^2
\end{align*} \]

\[ = -\frac{1}{4} (\mu_a^2 + \mu_b^2) q^4 - \left[ \frac{1}{2} (\mu_a^2 + \mu_b^2)^2 + 3 \mu_a^2 \mu_b^2 \right] q^2 \]

\[ - (\mu_a^2 + \mu_b^2) \left[ \frac{1}{4} (\mu_a^2 + \mu_b^2)^2 + 5 \mu_a^2 \mu_b^2 \right] \]

\[ = -\frac{1}{2} \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) q^4 + [-2 \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) - 3 \mu_a^2 \mu_b^2] q^2 \]

\[ + \left[ -2 \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) - 10 \mu_a^2 \mu_b^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) \cdot \]

\[ \text{Sp}_z = (p', p'')^2 \text{Tr} (3-M) \beta p'' \beta p' \beta p' \beta p'' \]

\[ = (p', p'')^2 [(p', p'')^2 + 2 p', p''^2] \]

\[ = 2 (p', p'')^3 + (p', p'') (p', p'')^2 + 4 p', p'' (p', p'')^2 + 2 p', p''^2 (p', p'')^2 \]

\[ = -\frac{6}{4} - 2 \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) q^4 + [-5 \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) + 2 \mu_a^2 \mu_b^2] q^2 \]

\[ + \left[ -4 \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) - 8 \mu_a^2 \mu_b^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) \cdot \]

Thus

\[ \text{Tr} \bar{p}_\mu \gamma^2 \bar{p}_\mu (p', p'')^2 \bar{p}_\mu (p'' p') \]

\[ = \frac{K^2}{4} q^6 + \left[ - (1+K)^2 - K(1+K) + 2K^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) q^4 \]

\[ + \left[ -4(1+K)^2 - 4K(1+K) + 8K^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) q^2 \]

\[ + \left[ 3(1+K)^2 - 6K(1+K) + 2K^2 \right] \mu_a^2 \mu_b^2 q^2 \]

\[ + \left[ -4(1+K)^2 - 2K(1+K) + 4K^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) \]

\[ + \left[ 16(1+K)^2 - 20K(1+K) + 8K^2 \right] \left( \frac{\mu_a^2 + \mu_b^2}{2} \right) \mu_a^2 \mu_b^2 \]
\[
\frac{K^2}{4} q^6 + (-1 - 3K) \left(\frac{M_a^2 + M_b^2}{2}\right) q^4 \\
+ (-4 - 12K - 3K^2) \left(\frac{M_a^2 + M_b^2}{2}\right)^2 q^2 + (3 - K^2) M_a^2 M_b^2 q^2 \\
+ (-4 - 12K - 4K^2) \left(\frac{M_a^2 + M_b^2}{2}\right)^3 \\
+ (16 + 12K + 4K^2) \left(\frac{M_a^2 + M_b^2}{2}\right)^2 M_a^2 M_b^2 
\]
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<td>( 18 )</td>
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<td>( 20 )</td>
<td>( 21 )</td>
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</tbody>
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**Table VI**

Trace of Correlations of Traces of Terms Evaluated in \( K \) at \( T \).
**APPENDIX 3**

**CALCULATION OF $\text{Tr}^{ab}$**

We use the traces IV, V, I evaluated and tabled in Appendix 2, Page 138

\[ \text{Tr}^{11} = \text{IV} (M_a = M_b = m) \]
\[ = \frac{K^2}{4} q^6 + (-1-3K)m^2 q^4 + (-1-12K-4K^2)m^4 q^2 + 12 m^6 , \]

\[ \text{Tr}^{01} = \text{-V} (M_a^2 = m^2 = \xi M^2, \quad M_b^2 = M^2) \]
\[ = c_1 q^6 + c_2 q^4 M^2 + c_3 q^2 M^4 + c_4 M^6 , \]

where the table gives the values

\[ c_1 = \frac{-K^2}{4} , \]

\[ c_2 = \frac{-1}{2} (-K^2 + \xi + 4K \xi) , \]

\[ c_3 = \frac{-1}{4} [(K^2 - 4\xi - 16K\xi - 3K^2 \xi)(1 + \xi) \]
\[ + 4\xi (3K + 2K^2 + 3\xi + 3K^2 \xi)] \]
\[ = \frac{1}{4} [-K^2 + 4\xi + 4K\xi - 6K^2 \xi - 6\xi^2 + 4K^2 \xi^2 + 3K^2 \xi^2] , \]

\[ c_4 = \frac{-1}{2} [(- \xi - 4K\xi - K^2 \xi)(1 + \xi)^2 \]
\[ + \xi^2 (-2 + 6K + 4K^2 + 6K + 6K^2) \]
\[ + \xi (6K + 6\xi - 2K\xi - 6\xi^2)] \]
\[ = \frac{1}{2} [\xi - 2K\xi + K^2 \xi - 2K^2 \xi + 4K^2 \xi^2 - 2K^2 \xi^2 + \xi^3 - 2K^3 \xi^3 + K^2 \xi^3] , \]

\[ \text{Tr}^{00} = \text{I} (M_a = M_b = M) \]
\[ = \frac{K^2}{4} q^6 + (K^2-2K\xi)M^2 q^4 + (-4K^2\xi^2)M^4 q^2 + 4\xi^2 M^6 . \]
**APPENDIX 4**

**ASYMPTOTIC FORM OF \( \text{Im} \pi(q^2) \)**

We present as a tabular calculation the determination of

\[
U = f_{11} \text{Tr}^{11} + 2f_{01} \text{Tr}^{01} + f_{00} \text{Tr}^{00}
\]

where

\[
\begin{align*}
    f_{11} &= 1 + 2 \frac{m^2}{q^2} - 2 \frac{m^4}{q^4}, \\
    f_{01} &= 1 + (1 + \xi^{-1}) \frac{m^2}{q^2} - 2 \xi^{-1} \frac{m^4}{q^4}, \\
    f_{00} &= 1 + \xi^{-1} \frac{m^2}{q^2} - 2 \xi^{-2} \frac{m^4}{q^4}.
\end{align*}
\]

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<th>Quantity</th>
<th>( \frac{q^6}{4} )</th>
<th>( \frac{1}{2} m^2 \frac{q^4}{q} )</th>
<th>( \frac{1}{2} m^4 \frac{q^2}{q} )</th>
</tr>
</thead>
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<td>\text{Tr}^{11}</td>
<td>( K^2 )</td>
<td>(-2-6K )</td>
<td>(-2-24K-6K^2 )</td>
</tr>
<tr>
<td>( f_{11} \text{Tr}^{11} )</td>
<td>( K^2 )</td>
<td>(-2-6K+K^2 )</td>
<td>(-6-36K-9K^2 )</td>
</tr>
<tr>
<td>( 2 \text{Tr}^{01} )</td>
<td>(-2K^2 )</td>
<td>(-2K^2+2+8K )</td>
<td>(-2K^2+2+8K+3K^2 )</td>
</tr>
<tr>
<td>( 2 f_{01} \text{Tr}^{01} )</td>
<td>(-2K^2 )</td>
<td>(-3K^2+2+8K-K^2 )</td>
<td>(-3K^2+2+6K-1+2K^2 )</td>
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<tr>
<td>\text{Tr}^{00}</td>
<td>( K^2 )</td>
<td>( 2K^2+1-2K )</td>
<td>(-8K\xi^{-1}+2 )</td>
</tr>
<tr>
<td>( f_{00} \text{Tr}^{00} )</td>
<td>( K^2 )</td>
<td>( 3K^2-1-2K )</td>
<td>( 3K^2-2-12K\xi^{-1}+2 )</td>
</tr>
<tr>
<td>( U )</td>
<td></td>
<td></td>
<td>(-6\xi^{-1}-6K\xi^{-1}-10-2qK )</td>
</tr>
</tbody>
</table>

\( q \)
(A) Equal Mass Integrals

In this section we evaluate the integrals

\[ \int_4^{\lambda^{-1}} \frac{dx}{x^n \sqrt{x(x-4)}} \quad n = 1, 2, 3, 4 \]

In evaluating these integrals we bear in mind that we are to let \( \lambda \to 0 \) before taking the limit \( \xi \to 0 \). Thus we proceed by

(i) firstly dropping terms in \( \lambda \) and higher powers of \( \lambda \)

(ii) then dropping higher power terms in \( \xi \).

Consequent on this limit process, the integral

\[ \int_4^{\xi^{-1} \lambda^{-1}} \frac{dx}{x^n \sqrt{x(x-4)}} \quad n = 1, 2, 3, 4 \]

may be determined by the substitutions

\[ \lambda^{-1} \to \xi^{-1} \lambda^{-1}, \quad \log \lambda \to \log \xi \lambda \]

in the expressions below.
As a preliminary to the 'equal mass' integrals we note that

\[ \left[ \frac{\lambda^{-1}}{x(x-4)} \right]_{4}^{\infty} = \lambda^{-1} (1 - 2\lambda - 2\lambda^2) \]

and

\[ \left[ \log \left( \sqrt{\frac{x}{x-4}} + \sqrt{\frac{x-4}{x}} \right) \right]_{4}^{\infty} = \log \lambda^{-1/2} [1 - \lambda - \lambda^2] \]

\[ = -\frac{1}{2} \log \lambda - \lambda \]

\[ \int_{4}^{\infty} \frac{dx}{x} \sqrt{x(x-4)} = \left[ \sqrt{x(x-4)} - 4 \log \left( \sqrt{x} + \sqrt{x-4} \right) \right]_{4}^{\infty} \]

\[ = \lambda^{-1} + 2 \log \lambda - 2 \]

\[ \int_{4}^{\infty} \frac{dx}{x} \frac{\sqrt{x(x-4)}}{x} = \left[ -2 \frac{\sqrt{x-4}}{x} + 2 \log \left( \sqrt{x} + \sqrt{x-4} \right) \right]_{4}^{\infty} \]

\[ = -\log \lambda - 2 \]

\[ \int_{4}^{\infty} \frac{dx}{x^2} \frac{\sqrt{x(x-4)}}{x} = \left[ -\frac{2}{3} \frac{\sqrt{x(x-4)}}{x^2} + \frac{\sqrt{x(x-4)}}{x} \right]_{4}^{\infty} \]

\[ = \frac{1}{6} \]

\[ \int_{4}^{\infty} \frac{dx}{x^4} \frac{\sqrt{x(x-4)}}{x} = \left[ \frac{1}{10} \frac{(x-4)^{3/2}}{x^{7/2}} - \frac{\sqrt{x(x-4)}}{15x^2} + \frac{1}{60} \frac{\sqrt{x(x-4)}}{x} \right]_{4}^{\infty} \]

\[ = \frac{1}{60} \]

These integrals may be checked by reference to the easily derived recurrence formula

\[ \int \frac{dx}{x^n} \sqrt{x(x-4)} = \frac{1}{4n-6} \left[ \frac{(x-4)^{n/2}}{x^{n/2}} \sqrt{x(x-4)} + (n-3) \int \frac{dx}{x^{n-1}} \sqrt{x(x-4)} \right] \]
(B) 'Unequal Mass' Integrals

In this section we evaluate the integrals

\[ \int \frac{dx}{x^{n-1}} \left( 1 - 2(1+\xi)x^{-1} + (1-\xi)^2x^{-2} \right)^{1/2} \quad (n = 1, 2, 3, 4) . \]

The remarks at the beginning of the previous section apply here too.

We omit the subscripts and call

\[ f = \left( 1 - 2(1+\xi)x^{-1} + (1-\xi)^2x^{-2} \right)^{1/2} . \]

Then \( f[(1 + \xi^{1/2})^2] = 0 \) while the asymptotic forms of \( f, f^3 \) are

\[ f \sim 1 - (1+\xi)x^{-1} - 2\xi x^{-2} , \]
\[ f^3 \sim 1 - \frac{3}{2}(1+\xi)x^{-1} + \frac{3}{2}(1+\xi^2)x^{-2} . \]

We first note two auxiliary integrals. (Therein as elsewhere in this section we omit the limits of integration ((1 + \xi^{1/2})^2, \lambda^{-1}) from integrals):

\[ \int \frac{dx}{xf} = \left[ \log (xf + x - 1 - \xi) \right] \]
\[ = \log \frac{2\lambda^{-1}(1 - (1+\xi)\lambda - \xi^2 \lambda^2)}{2\xi^{1/2}} \]
\[ = - \log \lambda - \frac{1}{2} \log \xi - (1+\xi)\lambda . \]

\[ \int \frac{dx}{x^2f} = \frac{1}{1 - \xi} \left[ \log \left( x(1+\xi)(1-\xi)^{-1} - (1-\xi) + xf \right) - \log x \right] \]
\[ = \frac{1}{1 - \xi} \log \left( \frac{(1+\xi)(1-\xi)^{-1} - (1-\xi)\lambda + 1 - \frac{1}{2} \xi \lambda}{(1-\xi)^{-1}(4\xi + 2\xi^{3/2} + 2\xi^{5/2})(1+\xi^{1/2})^{-2}} \right) \]
\[ = (1-\xi)^{-1} (- \frac{1}{2} \log \xi - \lambda) \]
\[ = - \frac{1}{2} \log \xi - \frac{1}{2} \xi \log \xi - \frac{1}{2} \xi^2 \log \xi . \]
(i) \[ \int \frac{dx}{xf} = \int \frac{dx}{xf} \frac{1}{x} [x(x-1-\xi) - (1+\xi)x + (1-\xi)^2] \]
\[ = [xf] - (1+\xi) \int \frac{dx}{xf} + (1-\xi)^2 \int \frac{dx}{x^2f} \]
\[ = \lambda^{-1} - (1+\xi) - (1+\xi)(-\log \lambda - \frac{1}{2} \log \xi) + (1-\xi)(- \frac{1}{2} \log \xi) \]
\[ = \lambda^{-1} - (1+\xi) \log \lambda - (1+\xi) + \xi \log \xi . \]

(ii) \[ \Delta s \frac{d}{dx} f = \frac{x^2 - (1+\xi)x}{x^2 (xf)} - \frac{(xf)}{x^2} = \frac{(1+\xi)x - (1-\xi)^2}{x^3f} \]
\[ \int \frac{dx}{x} f = \int \frac{dx}{x^3f} [(1-\xi)^2 - (1+\xi)x - (1+\xi)x + x^2] \]
\[ = - [f] - (1+\xi) \int \frac{dx}{x^2f} + \int \frac{dx}{xf} \]
\[ = -1 + (1+\xi)(1-\xi)^{-1} \left( \frac{1}{2} \log \xi \right) - \log \lambda - \frac{1}{2} \log \xi \]
\[ = - \log \lambda - 1 + \xi \log \xi + \xi^2 \log \xi . \]

(iii) \[ \int \frac{dx}{x^2} f = \frac{-1}{2(1-\xi)^2} \left\{ [xf^3] - (1+\xi) \int \frac{dx}{x^2} f - \int \frac{dx}{xf} f \right\} \]
\[ = \frac{-1}{2(1-\xi)^2} \left\{ [xf(1 - \frac{2(1+\xi)}{x} + \frac{(1-\xi)^2}{x^2})] \right\} \]
\[ + (1+\xi) [f] + (1+\xi)^2 \int \frac{dx}{x^2f} - (1+\xi) \int \frac{dx}{xf} \]
\[ - [xf] - (1-\xi)^2 \int \frac{dx}{x^2f} + (1+\xi) \int \frac{dx}{xf} \right\} \]
\[ = \left[ \frac{1}{2} f(\frac{1+\xi}{(1-\xi)^2} - \frac{1}{x}) \right] - \frac{2\xi}{(1-\xi)^2} \int \frac{dx}{x^2f} \]
\[ = \frac{1}{2} \frac{1+\xi}{(1-\xi)^2} - \frac{2\xi}{(1-\xi)^3} \left( - \frac{1}{2} \log \xi \right) \]
\[ = \xi \log \xi + 2 \xi^2 \log \xi + \frac{1}{2} + \frac{3}{2} \xi + \frac{5}{2} \xi^2 . \]
\[
\begin{align*}
\text{(iv)} \int \frac{dx}{x^3} f &= \frac{-1}{2(1-\xi)^2} \left\{ [f^2] - 3 (1+\xi) \int \frac{dx}{x^4} f \right\} \\
&= \frac{-1}{3(1-\xi)^2} \left\{ \frac{1}{\xi} (1+\xi) (1-\xi)^{-3} \log \xi + \frac{3}{1} (1+\xi) (1-\xi)^{2} - 1 \right\} \\
&= \xi \log \xi + 5 \xi^2 \log \xi + \frac{1}{6} + 2 \xi
\end{align*}
\]
APPENDIX 6

**CALCULATION OF $\pi^{ab}(\lambda)$**

The definite integrals utilised in this appendix are to be found in Appendix 5.

\begin{equation}
(A) \quad \pi^{11}(\lambda) = -\frac{1}{m} \int \frac{dx}{x^3} f_{11}(-x) \text{ Tr}^{11}(-x)
\end{equation}

\begin{equation}
= \int \frac{\xi^{-1} \lambda^{-1}}{4m^2} \ dy \ f_{11}(-m^2 y) [-y^{-3} m^{-6} \text{ Tr}^{11}(-m^2 y)]
\end{equation}

wherein \( f_{11}(-m^2 y) = y^{-1} \sqrt{y(y-4)} \), (Note \( y > 0 \))

whilst

\begin{equation}
\pi^{11}(\lambda)
= \frac{K^2}{4} + (1 + \xi \log \lambda - 2) + (1 + \xi)(- \log \xi \log \lambda - 2) + (-1 - 2\xi - 4\xi^2\lambda^{-2}) y^{-3}
\end{equation}

So \( -\pi^{11}(\lambda) \)

\begin{equation}
= \frac{K^2}{4} (\xi^{-1} \lambda^{-1} + 2 \log \xi \log \lambda - 2) + (1 + \xi)(- \log \xi \log \lambda - 2) + (-1 - 2\xi - 4\xi^2\lambda^{-2}) y^{-3}
\end{equation}

\begin{equation}
= \frac{K^2}{4} \xi^{-1} \lambda^{-1} + (-1 - 2\xi - 4\xi^2\lambda^{-2}) \log \lambda
\end{equation}

\begin{equation}
+ (-1 - 2\xi - 4\xi^2\lambda^{-2}) \log \xi
\end{equation}

\begin{equation}
= \frac{-7}{3\lambda} - 8\xi - \frac{7}{6} K^2
\end{equation}

\begin{equation}
(B) \quad 2 \pi^{01}(\lambda) = -\frac{1}{m^2} \int \frac{dx}{(m^2 + k^2)^2} f_{01}(-x) 2 \text{ Tr}^{01}(-x)
\end{equation}

\begin{equation}
= \int \frac{\lambda^{-1}}{(1 + \xi^2)^2} \ dy \ f_{01}(-m^2 y) [-2 \xi^{-1} y^{-3} m^{-6} \text{ Tr}^{01}(-m^2 y)]
\end{equation}
where \( f_{01}(-m^2y) = y^{-1} \left[ y^2 - 2 (1+x)y + (1-x)^2 \right] \),

and

\[
-2 \xi^{-1} y^{-3} M^2 \Tr^{01}(-m^2y) = \frac{K^2}{2} \xi^{-1} + \left( \frac{K^2}{2} \xi^{-1} - 1 - 4K \right) y^{-1} \left[ \frac{K^2}{2} \xi^{-1} + 2 + 2K - 3K^2 + \text{O}(\xi) \right] y^{-2} + \left[ -1 + 2K^2 + \text{O}(\xi) \right] y^{-3},
\]

so

\[
2 \pi^{01}(\lambda) = \frac{\lambda^{-1}}{2} \left[ (1+\xi) \log \lambda - (1+\xi) \log \xi + \frac{K^2}{2} \xi^{-1} - 1 - 4K \right] (\log \lambda - 1 + \xi \log \xi) + \left( \frac{K^2}{2} \xi^{-1} + 2 + 2K - 3K^2 \right) (\xi \log \xi + \frac{1}{2}) + (-1 + 2K^2) (\xi \log \xi + \frac{1}{\xi})
\]

\[
= \frac{K^2}{2} \xi^{-1} \lambda^{-1} + \left( \frac{3}{2} K^2 \xi^{-1} + 1 + 4K - \frac{K^2}{2} \right) \log \lambda
\]

\[
- \frac{3}{4} K^2 \xi^{-1}
\]

\[
+ \frac{11}{6} + \frac{16K - 23}{144} K^2
\]

(c) \( \pi^{00}(\lambda) = \frac{1}{M^2} \int_{4M^2}^{M^2} \frac{dx}{x} f_{00}(-x) \Tr^{00}(-x)
\]

\[
= \int_{4}^{\lambda^{-1}} dy f_{00}(-m^2y) \left[ -\xi^{-1} y^{-3} M^2 \Tr^{00}(-m^2y) \right]
\]

Now \( f_{00}(-m^2y) = \frac{\sqrt{y(y-4)}}{y} \),

while \( C \)

\[
-\xi^{-1} y^{-3} M^2 \Tr^{00}(-m^2y) = \frac{K^2}{4} \xi^{-1} + \left( \frac{K^2}{2} \xi^{-1} + 2K \right) y^{-1} + (-4K+\xi) y^{-2} - 4\xi y^{-3}
\]
\pi^{00}(\lambda) = \frac{k^2}{4} \xi^{-1}(\lambda^{-1} + 2 \log \lambda - 2) + (-k^2 \xi^{-1} + k)(-\log \lambda - 2) + (-4k^2\xi)(\frac{1}{6}) - 4\xi(\frac{1}{60})

= \frac{k^2}{4} \xi^{-1}(\lambda^{-1} + \frac{3}{2} k^2 \xi^{-1} - K) \log \lambda

+ \frac{3}{2} k^2 \xi^{-1}

- \frac{8}{3} K.
CHAPTER VII

AN ASYMPTOTICALLY CORRECT THEORY OF CHARGED PARTICLES OF SPIN ONE.

1. The Theory

Lee and Yang (1962) showed that insisting \textit{a priori} that internal spin one lines have the value

\[ S_{F_{\alpha\beta}} = -i \frac{\delta_{\alpha\beta} + p_{\alpha} p_{\beta}}{p^2 + m^2 - i\epsilon} \]

that the usual Dyson Wick formalism applied to the Lagrangian density

\[ L = -\frac{1}{2} (\partial_{\nu} A_{\mu})(\partial_{\nu} A_{\mu}) - \frac{1}{2} G_{\mu\nu} \ast \ast \ast - m^2 \rho_{\mu} \rho_{\mu} - i\epsilon F_{\mu\nu} \rho_{\mu} \rho_{\nu} \]

where

\[ G_{\mu\nu} = \pi_{\mu} \rho_{\nu} - \pi_{\nu} \rho_{\mu} \]

fails to determine a set of graph elements for non zero K. Our phraseology is most carefully considered: any graph theory that works - witness spin 1/2 and scalar and vector spin zero - has a covariant set of graph elements - and we simply do not regard the non-covariant entities [see their Fig. 1] produced by Lee and Yang as constituting a set of graph elements. On the other hand, as is well known, the graph theory deducible by Dyson-Wick methods for K = 0 is not a finite one. [See our own calculation of Im\Pi(q^2) in V Appendix 3].

Our own experience with vector spin zero field suggests that the value of internal spin zero lines will be correctly determined by calculating the vacuum expectation value of the T product defined for \( x = (t, r) \) as

\[ T \rho_{\mu}(x) \rho_{\nu}(o) = \rho_{\mu}(x) \rho_{\nu}(o) \] for \( t > 0 \)
the calculation of vacuum expectation value to use the usual Fourier expansion
for vector spin one quantum field:

\[ \varphi(x) = (2\pi)^{-3/2} \int d^3k \left( \frac{1}{2\pi} \right)^{1/2} \sum_{s}(a_{ks}\exp(ikr - iwt) + b_{-ks}^\dagger\exp(ikr + iwt)) \right] \delta(k) \]

1.6a

\[ \varphi_4(x) = (2\pi)^{-3/2} \int d^3k \left( \frac{1}{2\pi} \right)^{1/2} \left[ a_{k3}\exp(ikr - iwt) + b_{-k3}^\dagger\exp(ikr + iwt) \right] \left( \frac{k}{m} \right) \]

1.6b

In this expansion the sum is over the three states of polarization, \( s = 1, 2, 3 \);
\( e_{k1}, e_{k2}, \) and \( k/|k| \) form a right-handed orthogonal set of three unit vectors;
while \( e_{k3} = \frac{w\vec{k}}{|k|} \), and \( w = (k^2 + m^2)^{1/2} \). \( a_{ks}, b_{ks} \) are annihilation
operators for which

\[ [a_{ks}, a_{lt}^\dagger] = [b_{ks}, b_{lt}^\dagger] = \delta_{st} \delta^3(k - l) \]

1.7

and all other commutators vanish.

Then the vacuum expectation value

\[ \langle \varphi(x) \varphi_4^\dagger(o) \rangle \]

\[ = (2\pi)^{-3} \int d^3k \left( \frac{1}{2\pi} \right)^{-1} \exp(ikr - iwt) \left[ \delta_{\mu\nu} + q_{\mu}q_{\nu} / m^2 \right] t > 0 \]

1.8a

\[ = (2\pi)^{-3} \int d^3k \left( \frac{1}{2\pi} \right)^{-1} \exp(ikr) \left[ \delta_{\mu\nu} + (q_{\mu}q_{\nu} + q_{\mu}q_{\nu}^\dagger) / 2m^2 \right] t = 0 \]

1.8b

\[ = (2\pi)^{-3} \int d^3k \left( \frac{1}{2\pi} \right)^{-1} \exp(ikr + iwt) \left[ \delta_{\mu\nu} + q_{\mu}q_{\nu}^\dagger / m^2 \right] t \leq 0 \]

1.8c

wherein \( q_\mu = iw, q_\mu = k_\mu (j = 1, 2, 3) \). Equation 1.8 is equivalent to the formula
\[ \langle T \phi(x)\phi^*(o) \rangle = (2\pi)^{-4} \int d^4k \frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + m^2 - i\varepsilon} \exp(ikx) , \quad \text{1.9} \]

where as usual it is understood that the integration \( \int dk_0 \) is performed first.

[Compare the vector spin zero case discussed below equation IV 3.20]. Note that

1. If \( \mu \neq 4, \nu \neq 4 \) we may make the replacement in numerator: \( k_\mu k_\nu / k^2 \rightarrow -k^2 / m^2 \).

2. If \( \mu = 4, \nu \neq 4 \) equation 1.9 agrees with 1.8b if \( t = 0 \).

3. If \( \mu = 4, \nu = 4 \)

\[
\frac{\delta_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 + m^2 - i\varepsilon} = \frac{k^2/(k^2 - k_0^2)}{k^2 - k^2 - m^2 + i\varepsilon} , \quad \text{1.10}
\]

so that 1.9 is in agreement with 1.8 in this key case.

4. The numerator term \( 1/k^2 \) is specified to have the same meaning as in vector spin zero case discussed below IV 3.24.

We are thus lead to the following assignment of the value of the internal spin one line

\[ S_{F\alpha\beta}(p) = -i \frac{\delta_{\alpha\beta} - p_\alpha p_\beta / p^2}{p^2 + m^2 - i\varepsilon} , \quad \text{1.11} \]

Note that we put forward this value so confidently on the basis of our experience with the vector spin zero field, for which we found that the specification

\[ G_{F\alpha\beta}(p) = \frac{p_\alpha p_\beta / p^2}{p^2 + m^2 - i\varepsilon} \]

lead to a theory of spin zero manifestly isomorphic with the usual scalar spin zero theory.
The next question is what are the value of 3- and 4-vertex in the correct theory of vector spin one. The simple minded procedure that worked for vector spin zero of merely using the matrix elements of $-i \left[ L(e = 0) - L(e) \right]$ where $L$ was a Lagrangian density giving desired field equations, does not work for vector spin one. The reader will note that the numerator of $S_{\alpha\beta}(p)$ is the same "on the mass shelf" as the numerator of the usual $S_{\alpha\beta}(p)$, so that for example the spectral weights for photon and spin one have the usual unacceptable values.

We confess our inability to specify the value of 3- and 4-vertex but instead provide a strategem to calculate at least some graphs. The strategem is based on the observation that the usual Dyson-Wick formalism goes through without a hitch for the Lagrangian density adopted by Lee and Yang in their theory

$$L_{\xi} = -\xi \left( \pi_{\mu} \phi_{\mu} \right) \left( \pi \phi \right) - \frac{1}{2} (\partial_{\mu} A_{\nu})^{2} - \frac{1}{2} G_{\mu\nu} \phi_{\mu} \phi_{\nu} - m^{2} \phi_{\mu} \phi_{\mu} - ieK_{\mu\nu} \phi_{\mu} \phi_{\nu},$$

$$\phi_{\mu} = \eta^{-1} \phi_{\mu} \eta; \quad G_{\mu\nu} \phi_{\mu} = \eta^{-1} G_{\mu\nu} \eta,$$

for which with metric chosen appropriately, the value of internal lines is

$$S_{L_{\xi}}^{\alpha\beta}(p) = S_{\alpha\beta}^{\alpha\beta}(p) + S_{\alpha\beta}^{\alpha\beta}(p),$$

where

$$S_{\alpha\beta}^{\alpha\beta}(p) = -i \frac{\delta_{\alpha\beta} + p_{\alpha} p_{\beta}}{p^{2} + M_{2}^{2} - i\epsilon},$$

and

$$S_{\alpha\beta}^{\alpha\beta}(p) = i \frac{p_{\alpha} p_{\beta}}{p^{2} + M_{2}^{2} - i\epsilon},$$

on defining

$$M_{a}^{2} = a m^{2} + (1 - a) \xi^{-1} m^{2},$$

$$M_{o}^{2} = M^{2},$$

We therefore regard the graph elements as given by Lee and Yang and tabled in
VI T 1 as providing an acceptable mathematical starting point. Now for \( p^2 \gg M^2 \),

\[
S_{\alpha \beta} (p) = -i \frac{\delta_{\alpha \beta} - p_\alpha p_\beta / p^2}{p^2 + m^2} + i \frac{m^2}{p^2 + m^2} \frac{p_\alpha p_\beta / p^2}{p^2 + M^2} .
\]

Then if we let \( M = M_0 \rightarrow 0 \), i.e. \( \xi \rightarrow \infty \)

\[
S_{\alpha \beta} (p) \rightarrow S(p) ,
\]

i.e., for sufficiently large \( p^2 \) the Lee and Yang particle line becomes the spin one particle line as \( \xi \rightarrow \infty \). We can therefore calculate those graphs which had we a complete set of graph elements would be determined for all spin one lines \( p \) satisfying \( p^2 \gg M^2 \) [note \( m^2 \gg M^2 \)], our calculation of such graphs to be performed using the graph elements of Lee and Yang and setting \( \xi \rightarrow \infty \) at the conclusion of the calculation. (Our position is the very opposite of that taken by Lee and Yang who used the property that for small \( p^2 \), \( S^0 (p) \) becomes negligible as \( \xi \rightarrow 0 \)). Thus in particular we can calculate the spectral weights for the graphs

\[
\begin{align*}
\text{---} & \quad \text{---} \\
\text{\quad} \quad \text{\quad}
\end{align*}
\]

We calculate the values of the spectral weights for photon and meson propagators in Sections 2, 3 respectively. As we have not as yet determined the appropriate set of phase integrals

\[
V_{\alpha \beta} [f] = \int d^4 k \, \delta_\alpha \delta [ (p^{-k})^2 + m^2 ] \delta_\beta \delta [(p'^{-k})^2 + M^2 ] f(k) ,
\]

we are unable to calculate the radiative correction to the \( \lambda \)-vertex. These calculations of photon and spin one spectral weights are of the nature of calculations to explore the theory. We are at once forced to limit our claims for the theory and merely claim that it gives the correct value for these spectral weights at \( - p^2 \) well
removed from the branch point. Equivalently we describe our theory as an asymptotically correct theory. The calculation of $\rho_3$ shows that the requirement that

$$\rho_3 \geq 0$$

leads to limits on the possible values of the parameter $K$ which are satisfied by setting

$$K = -1$$

Although we can't determine the spectral weights other than asymptotically, it seems reasonable to guess the low $\mu^2$ behaviour by insisting on the usual functional form. This principle of simplest analytic continuation enables us to also calculate $\Pi'(0)$ in Section 2.

We conclude this Section with a tentative discussion of the magnetic moment of the particle dealt with in this theory. The field equation of Lee and Yang $\phi_\mu$ is

$$\xi \pi_\mu \pi_\nu \phi_\mu + \pi_\mu G_{\mu\nu} - m^2 \phi_\nu + i.e. K \phi_\mu F_{\mu\nu} = 0$$

Multiplying this equation by $\xi^{-1}$ and letting $\xi \to \infty$ leads to

$$\pi_\mu \pi_\rho \phi_\rho = 0$$

a relation deducible from the field equation

$$\pi_\mu G_{\mu\nu} - m^2 \phi_\nu + i.e. \phi_\mu F_{\mu\nu} = 0$$

which describes charged spin one particle of magnetic moment $\mu$ [e.g., Young (1964)]

$$\mu = 2(e^2/2m)S$$

In Lee and Yang theory where the contrary limit $\xi \to 0$ is taken, the specification $K = -1$ leads to

[Field equation]
\[ \pi_{\mu}^{\nu} - m^2 \phi_{\nu} - \text{i.e.} \phi_{\mu} F_{\mu\nu} = 0 \] \hspace{1cm} \text{1.26a}

which can be elegantly written as

\[ \pi_{\mu}^{\pi_{\mu}} - \pi_{\mu}^{\mu} \phi_{\mu} - m^2 \phi_{\nu} = 0 \] \hspace{1cm} \text{1.26b}

1.26 a implies (contraction with \( \pi_{\nu} \))

\[ m^2 \pi_{\nu} \phi_{\mu} = \text{i.e.} F_{\mu\nu} G_{\mu\nu} \] \hspace{1cm} \text{1.26c}

The magnetic moment of spin one particle to correspond to 1.26 a is

\[ M = 0 \] \hspace{1cm} \text{1.26d}

A summary of our results in this theory of charged spin one is given in Section 4.
2. RADIATIVE CORRECTION TO PHOTON PROPAGATOR

The radiative correction to photon propagator due to charged spin one is given by the graph

\[ - i(\delta_{\mu\nu} - k_\mu k_\nu / k^2) \Pi(k^2/k^2) = \frac{\mu^2}{k^2} - \frac{\nu^2}{k^2} \],  \hspace{1cm} 2.1

where in terms of the Lehman-Kallen spectral weight \( \rho_3 \),

\[ \Pi(k^2) = k^2 \int_{4m^2}^{\infty} \frac{d\mu^2 \rho_3(\mu^2)}{k^2 + \mu^2 - i\epsilon} \], \hspace{1cm} 2.2

and

\[ \text{Im} \Pi(k^2) = \pi k^2 \rho_3(-k^2) \].  \hspace{1cm} 2.3

We have already determined \( \text{Im} \Pi(k^2) \) in Chapter VI, Section 2, using the Lee and Yang graph element as (omitting the \( \Theta \) functions)

\[ \text{Im} \Pi = - \frac{e^2}{48\pi m^2 k^2} \left[ f_{11} \text{Tr}^{11} + 2f_{01} \text{Tr}^{01} + f_{00} \text{Tr}^{00} \right] \], \hspace{1cm} 2.4

where

\[ f_{11} = (1 + 4m^2/k^2)^{1/2}, \hspace{1cm} 2.5 \]

\[ f_{01} = (1 + 2(m^2 + M^2)/k^2 + (m^2 - M^2)^2/k^2)^{1/2}, \hspace{1cm} 2.6 \]

\[ f_{00} = (1 + 4M^2/k^2)^{1/2}. \hspace{1cm} 2.7 \]

The \( \text{Tr}^{ab} \) are tabulated in VI Appendix 3. They are

\[ \text{Tr}^{11} = \frac{k^2}{4} + k^6 + (-1-3k)m^2k^4 + (-1-12k-4k^2)m^4k^2 + 12m^6, \hspace{1cm} 2.8 \]

\[ \text{Tr}^{01} = b_1 k^6 + b_2 m^2k^4 + b_3 m^4k^2 + b_4 m^6, \hspace{1cm} 2.9 \]

where

\[ b_1 = -k^2/4; \hspace{1cm} b_2 = (-k^2/4 + 1 + 4k^2)/2. \hspace{1cm} 2.9a \]
\[ b_3 = \frac{1}{4} (-K^2 \xi^{-2} + 4 \xi^{-1} + 4K \xi^{-1} - 6K^2 \xi^{-1} - 8 + 4K + 3K^2), \]  
\[ b_4 = \frac{1}{2} (\xi^{-2} - 2K \xi^{-2} + K^2 \xi^{-2} - 4K \xi^{-1} - 2K^2 \xi^{-1} + 1 - 2K + K^2), \]  
\[ \text{Tr}^{00} = \frac{K^2}{4} k^6 + (K^2 - K) m^2 k^4 + (-4K \xi^{-1} + 1)m^4 k^2 + 4K \xi^{-1} m^6. \]  

From these explicit expressions it is clear that each \( f^{ab} \) \( \text{Tr}^{ab} \) is finite as \( \xi \to 0 \) i.e. \( \xi \to \infty \). Behaviour at low \( q^2 \) of this expression for \( \text{Im} \Pi(q^2) \) does manifest features which will certainly be absent from the correct theory. The asymptotic value of \( \text{Im} \Pi(q^2) \) is

\[ \text{Im} \Pi(k^2) = \frac{-e^2}{4\pi m^2 k^2} \eta(k^2), \]  

where \( \eta(q^2) \) was calculated in VI Appendix 4:

\[ \eta(k^2) = [3(1 - K^2) \xi^{-1} - 5 - 12K - 3K^2] m^4 k^2. \]  

Thence \( \rho_3(\mu^2, K) = \frac{e^2}{4\pi^2} [3(1 - K^2) \xi^{-1} - 5 - 12K - 3K^2] \mu^{-2}. \)

The requirement that

\[ \rho_3 \geq 0, \]  

after the limit \( \xi \to \infty \) has been taken, amounts to requiring that

\[ 3K^2 + 12K + 5 \leq 0, \]  

i.e.,

\[ -2 - \sqrt{\frac{7}{3}} \leq K \leq -2 + \sqrt{\frac{7}{3}}. \]  

It seems most acceptable, therefore, to fix \( K \) within this range:

\[ K = -1. \]

The absence of a term in \( \xi \) in the asymptotic expression of \( \eta \) for this value of \( K \) is a particularly pleasing feature.
We now have asymptotically,

\[ \rho_3 (\mu^2) = \frac{e^2}{12\pi^2 \mu^2} \]  \hspace{1cm} 2.17

Let us now suppose that \( \rho_3 (\mu^2) \) has at \( \mu^2 = 4m^2 \) the usual form; then the simplest such \( \rho_3 \) satisfying this requirement is

\[ \rho_3 (\mu^2) = \frac{e^2}{12\pi^2 \mu^2} \left( 1 - \frac{4m^2}{\mu^2} \right)^{1/2} \Phi (\mu^2 - 4m^2) \]  \hspace{1cm} 2.18

and corresponding to this value of \( \rho_3 (\mu^2) \),

\[ \Pi'(o) = \int_{4m^2}^{\infty} \frac{d\mu^2 \rho_3 (\mu^2)}{\mu^2} \]  \hspace{1cm} 2.19

\[ = \frac{e^2}{72\pi^2 m^2} \]  \hspace{1cm} 2.20
We postulate the DR for spin one propagator:

\[ S_{\alpha\beta}(p) = \frac{-i [\delta_{\alpha\beta} - p_{\alpha}p_{\beta} / p^2]}{p^2 + m^2 - i\epsilon} \left\{ 1 + \left( p^2 + m^2 \right) \int \frac{d\mu^2 \rho(\mu^2)}{p^2 + \mu^2 - i\epsilon} \right\} . \quad (3.1) \]

Thus were the value of 3 Vertex known, we could determine the magnitude of the branch cut singularity

\[ \text{disc } S_{\alpha\beta}(p) = 2\pi [\delta_{\alpha\beta} - p_{\alpha}p_{\beta} / p^2] \rho (\mu^2) , \quad (3.2) \]

from an analysis of

\[ S_{F_{\alpha\beta}} S_{F_{\gamma\delta}} = \frac{\alpha}{\gamma} \delta (p), \quad (3.3) \]

and thus determine \( \rho(\mu^2) \) to second order. Calculating \( S_{\alpha\beta} \) using the graph elements of Lee and Yang, we may write

\[ S_{\alpha\beta} = S_{\alpha\beta}^1(p) + S_{\alpha\beta}^0(p) , \quad (3.4) \]

where in Fermi gauge:

\[ S_{\alpha\beta}^a(p) = \frac{e^2}{(2\pi)^4} \int \frac{d^4k}{\gamma} V_{\mu}(p,q)S^{\beta}(q)V_{\nu}(q\bar{\nu})(-i\delta_{\mu\nu})/(k^2 - i\epsilon) . \quad (3.5) \]

The branch cut discontinuity of each \( S_{\alpha\beta}^a(p) \) may then be determined in the usual manner by making the replacements

\[ \left( q^2 + M_a^2 - i\epsilon \right)^{-1} \rightarrow 2\pi i \frac{\delta(p^2 + M_a^2)}{p^2 + M_a^2} , \quad (3.6a) \]

\[ \left( k^2 - i\epsilon \right)^{-1} \rightarrow 2\pi i \frac{\delta(k^2)}{k^2} . \quad (3.6b) \]

As we wish to calculate the graph 3.3 we may drop such terms in \( p_{\alpha} \), \( p_{\beta} \) as occur at any stage of the calculation. Thus we may take the 3 vertex of Lee and Yang, for
\[ K = -1 \text{ as} \]

\[ V_\mu (p^a, q^\gamma) = i \delta_\mu^\gamma (p + q)_\mu - (1 - \xi) i q_\gamma \delta_\mu^\gamma \]

\[ V_\mu (q^\delta, p_\beta) = i \delta_\beta^\mu (p + q)_\mu - (1 - \xi) i q_\delta \delta_\beta^\mu \]

and

\[ \text{disc } \Sigma_{a\beta} = \text{disc } \Sigma^1_{a\beta} + \text{disc } \Sigma^0_{a\beta} \]

where as \[ q^\gamma \left[ \delta_\gamma^\delta + q_\gamma q_\delta / m^2 \right] = 0 \] on the mass shell we have the simple expression

\[ \text{disc } \Sigma^1_{a\beta} = -\frac{e^2}{(2\pi)^2} \left. \text{Im} \left[ \delta_\gamma^\delta (p + q)_\mu (\delta_\gamma^\delta + q_\delta q_\delta / m^2) \delta_\beta^\mu (p + q)_\mu \right] \right. \]

Then by II A 3.8 and as \((p + q)^2 = 2(p^2 - m^2)\) on mass shell,

\[ \text{disc } \Sigma^1_{a\beta} = -\frac{e^2}{(2\pi)^2} \delta_\alpha^\beta 2(p^2 - m^2)[1 - (p^2/12m^2)(1 + m^2/p^2)^2] \text{Im} \]

where

\[ \text{Im} = \frac{\pi}{2} (1 + m^2/p^2) \vartheta (-p^2 - m^2) \]

The other component of \( \text{disc } \Sigma_{a\beta} \) is

\[ \text{disc } \Sigma^0_{a\beta} = \frac{e^2}{(2\pi)^2} \left. \text{Im} \left[ (\delta_\gamma^\delta (p + q)_\mu - (1 - \xi) q_\gamma \delta_\mu^\gamma) q_\delta q_\delta / m^2 (\delta_\beta^\mu (p + q)_\mu - (1 - \xi) q_\delta \delta_\beta^\mu) \right] \right. \]

\[ = \frac{e^2}{4\pi^2 m^2} \left. \text{Im} \left[ (p + q)^2 q_\alpha q_\beta - 2(1 - \xi) q_\gamma q_\delta q_\alpha q_\beta + (1 - \xi)^2 q_\gamma q_\delta q_\alpha q_\beta \right] \right. \]

\[ = \frac{e^2}{4\pi^2 m^2} \delta_\alpha^\beta \left[ 2(p^2 - 2m^2 + 2m^2(1 - \xi))(p^2/12)(1 + m^2/p^2)^2 + m^4(1 - \xi)^2 \right] \text{Im} \]

where

\[ \text{Im} = \frac{\pi}{2} (1 + m^2/p^2) \vartheta (-p^2 - m^2) \]

As \( M^2 = \xi^{-1} m^2 \) the RHS of 3.12 is finite as \( \xi \to \infty \). Then for \(-p^2 \gg m^2\)

we may take the limit \( \xi \to \infty \), to get - remember we have dropped terms in \( p_\alpha p_\beta \)

\[ \text{disc } \Sigma^0_{a\beta} (p) = \frac{e^2}{2\pi^2 m^2} \delta_\alpha^\beta \left[ 2(p^2 - m^2)(p^2/12m^2) + m^2 \right] \vartheta /2 \]

3.16
Then again for \(-p^2 \gg m^2\),

\[
\text{disc } \Sigma_{a\beta}(p) = \frac{e^2}{2\pi^2 m^2} \delta_{a\beta} \left[ -2 \frac{(p^2-m^2)(1/12)(3+3m^2/p^2 + m^4/p^4)}{2} - m^2/p \right] \pi/2 \quad 3.17
\]

\[
= \frac{e^2}{4\pi^2 m^2} \delta_{a\beta} \left[ -(p^2-m^2)(p^2+m^2)/2p^2 - 7m^4/6p^2 + m^6/6p^4 \right] \quad 3.18
\]

Dropping all but the two leading terms,

\[
\text{disc } \Sigma_{a\beta}(p) = \frac{-e^2}{8\pi m^2} (p^2 - m^2)(1 + m^2/p^2)^2 \delta_{a\beta} \quad 3.17
\]

Thence from 3.2 and 3.3, we have, for large \(\mu^2\),

\[
2\pi [\delta_{a\beta} - p_a p_\beta / p^2] \rho (-p^2) = \frac{e^2}{8\pi m^2} \frac{(p^2 - m^2)}{p^2 + m^2} [\delta_{a\beta} - p_a p_\beta / p^2] \quad 3.18
\]

or \(\rho(\mu^2) = \frac{e^2}{16\pi^2 \mu^2} \frac{\mu^2 + m^2}{\mu^2 - m^2}, \mu^2 \gg m^2\), Fermi gauge \(3.19\)

It would be reasonable to take

\[
\rho(\mu^2) = \frac{e^2}{16\pi^2 \mu^2} \frac{\mu^2 + m^2}{\mu^2 - m^2} \Theta (\mu^2 - m^2) \quad 3.20
\]

Such a \(\rho(\mu^2)\) leads to an IR divergence in the propagator. Had we not dropped a couple of terms small in the asymptotic limit expression for \(\rho\) this IR divergence would be far worse. However, the dispersion integral does converge nicely at infinity.
4. Conclusion

Finite and reasonable expressions for the asymptotic values of the spectral weights of charged spin one theory are determined by performing the calculations using the Lee and Yang graph elements for $K = -1$ and letting $\xi \to \infty$.

Explicitly we found,

$$\rho_3(\mu^2) = \frac{e^2}{12\pi^2\mu^2}$$

and $\rho(\mu^2)$ of the DR

$$S_{\alpha\beta}(p) = \frac{-i [\delta_{\alpha\beta} - \frac{p_{\alpha}p_{\beta}}{p^2}]}{p^2 + m^2 - i\epsilon} \left( 1 + \frac{p^2 + m^2}{p^2 + \mu^2 - i\epsilon} \right)$$

was calculated in Section 3 to be

$$\rho(\mu^2) = \frac{e^2}{16\pi^2\mu^2} \frac{\mu^2 + m^2}{\mu^2 - m^2}$$

Assuming the simplest possible relationship between low energy and high energy forms of $\rho_3(\mu^2)$ such that $\rho_3$ has the usual form about $\mu^2 = 4m^2$ lead to

$$\Pi'(0) = \frac{e^2}{72\pi^2m^2}$$

a value of vacuum polarization that seems far too small for a $m$ of order of 1 Bev to be measurable in present day experiments.

As an appendum to our description of the theory as giving correct asymptotic information:

We are aware that at the very highest momentum quantum electrodynamics expressions must be 'damped' so that for instance $\pi(k^2)/k^2$ is finite. For our purposes, we merely claim the calculation method of using Lee and Yang graph elements and letting $\xi \to \infty$ gives the correct form of spectra weights well away from the branch points e.g. up at $(6 \text{ Bev})^2$ for $m = 1 \text{ Bev}$. 
CHAPTER VIII

THE VALIDITY OF WARD'S IDENTITY FOR DIRAC SPIN 1/2

1. Counterexample.

We are concerned with the quantum electrodynamics of Dirac spin 1/2 particles. The graph elements for this theory are given in III T 1.

Ward (1949) noted that

\[ \left( \partial / \partial p_\mu \right) S_\mu(p) = e^{-1} S_\mu(p) V_\mu(p, p) S_\mu(p) \quad \text{VIII 1.1} \]

and assumed that differentiation commuted with the integration over the independent internal momenta of graphs so that

\[ \partial \Sigma^\#(p) / \partial p_\mu = e^{-1} V_\mu^\#(p, p) \quad \text{1.2} \]

where \( \Sigma^\# \) is the sum of all proper SE parts, \( V_\mu^\#(p_\mu p'') \) is the sum of all proper vertex parts with \( p_\mu^2, p''^2 \) unspecified. (A 'proper' graph cannot be separated into two disconnected parts by the omission of a single line). The calculation methods in use at the time introduced infinite quantities such that the finite parts were given by

\[ V_\mu^\#(p, p) = L V_\mu^\#(p, p) + \text{Finite} \left[ V_\mu^\#(p, p) \right] \quad \text{1.3} \]

and

\[ \Sigma^\#(p) = A - S_\mu^{-1}(p) B + \text{Finite} \left[ \Sigma^\#(p) \right] \quad \text{1.4} \]

Ward substituted 1.3 and 1.4 in 1.2 to 'prove' that \( B = L \). Of far greater import is the other formula derivable relating the finite parts. [Ward's terminology the "renormalised" parts]-

\[ \left( \partial / \partial p_\mu \right) \text{Finite } \Sigma^\#(p) = e^{-1} \text{Finite } V_\mu^\#(p, p) \quad \text{1.5} \]

The coefficients of \( e^2 \) in these expressions are

\[ \text{Finite } \Sigma^\#(p) ; \quad \Sigma(p) \quad \text{1.6} \]
Finite $\phi^{\ast}_{\mu}(pp)$: $\text{rad} \phi^{\ast}_{\mu}(pp) = \phi_{\mu} R(0) + i(p_{\mu}/m)S(0)$, \hspace{1cm} 1.7

where the quantities $\Sigma(p)$ and $\text{rad} \phi^{\ast}_{\mu}(p^{\ast}p')$ for $p''^{2} = p'^{2} = p^{2}$ were calculated in Chapter III by dispersion theoretic methods, in the photon gauge $a$, where 'a' specifies the value of photon lines as

$$D_{\mu \nu}(k) = -i \left( \frac{\delta_{\mu \nu} + (a-1) k_{\mu} k_{\nu}}{k^{2} - m^{2} - i\epsilon} \right)$$ \hspace{1cm} 1.8

We write

$$\text{rad} \phi^{\ast}_{\mu}(p^{\ast}p') = \phi_{\mu} R(Q^{2}) + i\frac{p_{\mu} + p_{\mu}^{\ast}}{2m} S(Q^{2})$$ \hspace{1cm} 1.9

We proved [see III 4.12] that as $p^{2} \rightarrow -m^{2}$ rad $\phi^{\ast}_{\mu}(p^{\ast}p')$ is independent of a. We also calculated

$$S(0) = \frac{e^{2}}{8\pi^{2}}$$ \hspace{1cm} 1.11

and noted that one can circumvent the infra-red (IR) divergence in $R$ by introducing photon mass $\mu$ so that

$$\text{Lim} [\mu \rightarrow 0] R(0) = S(0); \hspace{0.5cm} p^{2} = -m^{2}$$ \hspace{1cm} 1.12

or otherwise more elegantly, and without ambiguity:

$$\text{Lim} [M \rightarrow m] R(0) = S(0); \hspace{0.5cm} -p^{2} = M^{2} < m^{2}$$ \hspace{1cm} 1.13

In III Section 3 we proved that $\Sigma(p)$ is

(i) free of IR divergence for $p^{2} \neq -m^{2}$, and

(ii) a gauge dependent quantity. Explicitly we determined the finite spectral weights $\sigma, \sigma_2$ of the DR

$$\Sigma(p) = i(p^{2} + m^{2}) \int_{m^{2}}^{\infty} d\mu^{2} \frac{i\gamma p \sigma_2(\mu^{2}) + m \sigma_2(\mu^{2})}{p^{2} + \mu^{2} - i\epsilon}$$ \hspace{1cm} 1.14
The derivative of $\Sigma(p)$ is

$$\frac{\partial \Sigma(p)}{\partial p^\mu} = \gamma^\mu \mathbb{F}(p) + 2i\gamma^\mu \mathbb{G}(p) + 2i\gamma^\mu \mathbb{H}(p)$$  \hspace{1cm} (1.15)

where

$$\mathbb{F}(p) = - (p^2 + m^2) \int_{m^2}^{\infty} \frac{d\mu^2}{\mu^2} \frac{\sigma_1(\mu^2)}{p^2 + \mu^2 - i\epsilon}$$ \hspace{1cm} (1.16)

$$\mathbb{G}(p) = \int_{m^2}^{\infty} \frac{d\mu^2}{\mu^2} \frac{i\gamma^\mu \sigma_1(\mu^2) + m \sigma_2(\mu^2)}{p^2 + \mu^2 - i\epsilon}$$ \hspace{1cm} (1.17)

$$\mathbb{H}(p) = \int_{m^2}^{\infty} \frac{d\mu^2}{\mu^2} \frac{i\gamma^\mu \sigma_1(\mu^2) + m \sigma_2(\mu^2)}{p^2 + m^2}$$ \hspace{1cm} (1.18)

We have restricted consideration to $p^2$ on the uncut portion of the real axis

i.e., $-p^2 = M^2 < m^2$ \hspace{1cm} (1.19)

In this region the above differentiation is strictly valid.

Now the vertex function $\mathbb{V}_\mu(p^\mu p')$ is defined between 'coverings' $\mathbb{u}(p^n)$, $\mathbb{u}(p')$ such that

$$(i\gamma^\mu + M)\mathbb{u}(p') = 0$$ \hspace{1cm} (1.20)

$$(\mathbb{u}(p^n)(i\gamma^\mu + M) = 0$$ \hspace{1cm} (1.21)

where $-p'^2 = -p^n^2 = -p^2 = M^2$ \hspace{1cm} (1.22)

Hence in order to test 1.17 using the above $\mathbb{V}_\mu(p^\mu)$ we must likewise give $\mathbb{F}$, $\mathbb{G}$, and $\mathbb{H}$ coverings, i.e., put

$$i\gamma^\mu = -M$$ \hspace{1cm} (1.23)
First, by 3.18b

\[ F(p) = -(p^2 + m^2) \int \frac{e^{2/16\pi^2}}{p^2 + \mu^2} \, dp \]

\[ = -(a+3)[e^{2/16\pi^2}](p^2 + m^2) \log (p^2 + m^2)/m^2 \]

\[ \longrightarrow 0 \quad \text{as} \quad p^2 + m^2 \rightarrow 0 \]

Thus the coefficients of \( y \) on both sides of 1.7 are equal in this limit.

A brief examination shows that in general \( A \) and \( B \) diverge as \( M \rightarrow m \). For convenience in checking this point, we have put down the appropriate partial fractions in Appendix 1. However, in the gauge \( a = 5 \), in which by III 3.18 the spectral weights are

\[ \sigma_1(\mu^2) = \sigma_2(\mu^2) = \frac{e^2}{16\pi^2} - \frac{5\mu^2 + 5m^2}{\mu} \]

\[ \sigma_2(\mu^2) = \frac{e^2}{16\pi^2} \frac{8\mu^2}{\mu} \]

the integrals \( G(p) \) and \( H(p) \) have a finite limit as \( m \rightarrow M \). In fine -

\[ G(p) = M \int_{m^2}^{\infty} dp \, \frac{\sigma_1(\mu^2) + \sigma_2(\mu^2)}{p^2 + \mu^2} + \delta \]

where

\[ \delta = -(m - M) \int_{m^1}^{\infty} dp \, \frac{\sigma_2(\mu^2)}{p^2 + \mu^2} \]

As

\[ m - M = m - (m^2 + (m^2 + p^2)^{1/2}) \]

\[ = \frac{1}{2} (m^2 + p^2) + \cdots \]

\[ \delta \approx (m^2 + p^2) \log [(p^2 + m^2)/m^2] \]
So that the non zero part of $G(p)$ is (see A 1.5)

$$G(p) = \frac{-5e^2}{16\pi^2} M \int_{m^2}^{\infty} d\mu^2 \frac{\mu^2 - m^2}{\mu^2 (p^2 + \mu^2)}$$

$$= \frac{-5e^2}{16\pi^2} \left\{ \frac{-1}{p^2} + \frac{p^2 + m^2}{p^2} \log \frac{p^2 + m^2}{m^2} \right\}$$

$$\rightarrow 0 \text{ as } p^2 + m^2 \rightarrow 0$$

1.33

The other coefficient to be determined in this gauge is

$$H(p) = M(p^2 + m^2) \int_{m^2}^{\infty} d\mu^2 \frac{\sigma_1(\mu^2) - \sigma_2(\mu^2)}{(p^2 + \mu^2)^2} + \xi$$

1.34

$$\xi = (m-M)(p^2 + m^2) \int_{m^2}^{\infty} d\mu^2 \frac{\sigma_2(\mu^2)}{(p^2 + \mu^2)^2}$$

1.35a

$$= (m-M)(p^2 + m^2) \left[ (p^2 + m^2)^{-1} + \log \frac{p^2 + m^2}{m^2} \right]$$

1.35b

$$\rightarrow 0 \text{ as } p^2 + m^2 \rightarrow 0$$

1.36

Thus $H(p)$ can be evaluated using the partial fraction A 1.5 as

$$H(p) = \frac{-5e^2 M}{16\pi^2} \left\{ \frac{-(p^2 + m^2)/p^4}{p^2 + m^2} + \frac{p^2 - 2m^2}{p^2} \log \frac{p^2 + m^2}{m^2} + \frac{m^2/p^4}{m^2} \right\}$$

$$\rightarrow 0 \text{ as } p^2 + m^2 \rightarrow 0$$

1.37

Thus to conclude: In the gauge $a = 5, i\gamma_5 = M \gamma_5 M \rightarrow m$ we calculated:

$$\frac{\partial \Sigma(p)}{\partial p_\mu} = \frac{2ip_\mu}{2m} \frac{e^2}{16\pi^2} (-10)$$

1.38
\[
\text{rad} \, V_\mu (pp) = \frac{2ip_0}{2m} e^2 \frac{1}{16\pi^2} (2).
\]

There has been no ambiguity with regard to the limit process used. Thus we are led to deny Ward's identity, when expressed in the form
\[
\frac{\partial \Sigma (p)}{\partial p_\mu} = \text{rad} \, V_\mu (pp)
\]
which is an instance of the general form
\[
(\partial/\partial p_\mu) \text{Finite } \Sigma^* (p) = e^{-1} \text{Finite } \text{rad} \, V^*_\mu (pp).
\]

2. Counterexample Re-evaluated.

The reader may suspect that the calculation of the previous Section depended critically on our analysis of III Section 3 where we showed that the usual IR divergence in electron SE part \(\Sigma (p)\) was spurious. To show that this is not the case, we re-evaluate \(\partial \Sigma (p)/\partial p_\mu\) using the Lehman form of DR
\[
S(p) = S_F(p) - i \int_{m^2}^{\infty} d\mu^2 \frac{i\gamma p_\mu (\mu^2) + mp_\mu (\mu^2)}{p^2 + \mu^2 - i\varepsilon}, \quad \text{VIII 2.1}
\]
the spectral weights \(\rho_1, \rho_2\) are given in III 3.25. We showed that 2.1 is exactly the dispersion theoretic analogue of the usual approach: the coefficient of 
\((i\gamma \mu - m)\) has an IR divergence, while the remainder is free of IR divergence and has the usual value. What we are going to show is that the coefficient of \(p_\mu\) in 
\(\partial \Sigma(p)/\partial p_\mu\) in fact depends only on the part of \(S(p)\) that is free of IR divergence.

Taking 2.1 as a starting point,
\[
\Sigma(p) = -i \int_{m^2}^{\infty} d\mu^2 \frac{S_F^{-1}(p)i\gamma pS_F^{-1}(p)\rho_1 (\mu^2) + mp_\mu (\mu^2)}{p^2 + \mu^2 - i\varepsilon}, \quad 2.2
\]

We recall \(\leftrightarrow \quad \quad \quad \quad \quad \quad \quad S_F^{-1}(p) = i(i\gamma p + m) \quad 2.3\)
\[
\frac{3}{\delta p_\mu} S_F^{-1}(p) i \gamma_\mu S^{-1}(p) = -\gamma_\mu i \gamma_\mu S^{-1}(p) - S^{-1}(p) i \gamma_\mu \gamma_\mu + i p_\mu S_F^{-2}(p) \\
= 2i p_\mu + i p^2 \gamma_\mu + i p_\mu S_F^{-2}(p)
\]

\[
\frac{3}{\delta p_\mu} S_F^{-2}(p) = -(i \gamma_\mu + m) i \gamma_\mu - i \gamma_\mu (i \gamma_\mu + m)
\]

\[
= 2i p_\mu - 2im \gamma_\mu
\]

On mass shell \( p^2 = -m^2 \), \( S_F^{-1}(p) = 0 \)

Hence on this mass shell [section 1 gives a more detailed and explicit formulation]

\[
\frac{\partial \Sigma(p)}{\partial p_\mu} = -i \int_{m^2}^{\infty} d\mu^2 \frac{2i p_\mu [p_1(\mu^2) + p_2(\mu^2)] + i \gamma_\mu [p_1 p_2(\mu^2) - 2m^2 p_2(\mu^2)]}{p^2 + \mu^2 - i\varepsilon}
\]

In this expression the coefficient of \( \gamma_\mu \) has an IR divergence. However, the coefficient of \( p_\mu \) involves \( p_1(\mu^2) + p_2(\mu^2) = \sigma_1(\mu^2) - \sigma_2(\mu^2) \)

[III 3.27] so that in the notation of the previous Section, especially 1.37,

\[
\frac{\partial \Sigma(p)}{\partial p_\mu} = 2ip_\mu G(p)
\]

\( G(p) \) is free of IR divergence in gauge \( a = 5 \), whence

\[
\frac{\partial \Sigma(p)}{\partial p_\mu} = -2ip_\mu \frac{5e^2}{16\pi^2 m} + \gamma_\mu (\infty)
\]

We found in III Section 4 that for \( p^2 = -m^2 \),

\[
\text{rad} \ V_\mu(\bar{p}p) = -ip_\mu \frac{e^2}{8\pi^2 m}
\]

The relationship in question is: \( \frac{\partial \Sigma(N)}{\partial p_\mu} = \text{rad} \ V_\mu(\bar{p}p) \)

So that once more we see that even those terms on both sides which are free of IR divergence differ substantially.
3. Summary and Conclusion.

We proved that Ward's identity for finite SE and V parts:

\[ \text{Finite } \frac{\partial \Sigma(p)}{\partial p_\mu} = \text{Finite } e^{-1} V_\mu(p\bar{p}), \quad \text{VIII 3.1} \]

does not hold, by considering the particular counterexample

\[ \frac{\partial \Sigma(p)}{\partial p_\mu} \neq \text{rad } V_\mu(p\bar{p}). \quad \text{3.2} \]

A sufficient proof of 3.2 came from our determination that \(-p^2 \rightarrow m^2\),
rad \(V_\mu(p\bar{p})\) is gauge independent, while \(\Sigma(p)\) and hence also \(\partial \Sigma(p)/\partial p_\mu\) is gauge dependent. As this argument might not seem conclusive by virtue of the infra-red divergences in \(\partial \Sigma(p)/\partial p_\mu\), we showed in Section I that in the gauge \(a = 5\) where, between "coverings" \(\partial \Sigma(p)/\partial p_\mu\) was finite for \(-p^2 = M^2 < m^2\) in the limit \(M \rightarrow m\)
the finite expressions \(\partial \Sigma(p)/\partial p_\mu\) and rad \(V_\mu(p\bar{p})\) differed in the coefficient of \(p_\mu\). As a pedantic exercise, we showed in Section 2 that the same result can be derived in the dispersion - theoretic analogue of the usual treatment of electron SE part wherein a spurious IR divergence is present.

It is very interesting to note that Salem and Delbourgo (1964) and Strathdee (1964) used the Ward-Takahashi identity,

\[ S_F^{-1}(p''') - S_F^{-1} = -e^{-1}(p''-p')\mu V_\mu(p''p'), \quad \text{3.3} \]
in their calculations outside of graph theory, of electrodynamics.

It must finally be stressed that it is not necessary that Ward's identity in form 3.2 should always fail. It would be interesting to examine the two graphs

\[ \frac{\partial}{\partial p_\mu}, \quad \text{Lim } [Q^2 \rightarrow 0], \quad \text{3.4} \]

where a double line is an electron, single: electron. The first of these is gauge invariant due to the gauge independence of \(D_{\mu\nu}\) and in particular the two electron loop. Its branch cut starts at \((M_\mu + 2M)^2\), so that its exact behaviour would be interesting to examine. The second is well known and gives one of the terms of the anomalous
magnetic moment of the meson calculated by Peterman (1957) and by Suura and Wickman (1957).

Footnote:

A close analysis of the paper by Sommerfield [Ann Phys (N.Y.) 5 26 (1957)] may shed some considerable light on our calculation: Sommerfield calculated Schwinger's mass operator for an electron in a constant external field, and identified the magnetic moment as that part of the expectation value of that operator linear in the external field. This procedure seems very much akin to differentiating the SE part $\Sigma(p)$. 
VIII APPENDIX I

SOME PARTIAL FRACTIONS

\[
\frac{1}{p^2(p^2 + \mu^2)} = -\frac{1/p^2}{p^2 + \mu^2} + \frac{1/p^2}{\mu^2} \quad \text{VIII A1.1}
\]

\[
\frac{m^2}{\mu^4(p^2 + \mu^2)} = \frac{m^4/p^4}{p^2 + \mu^2} - \frac{m^2/p^2}{\mu^2} + \frac{m^2/p^4}{\mu^2} \quad \text{Al.2}
\]

\[
\frac{1}{\mu^2(p^2 + \mu^2)^2} = -\frac{1/p^2}{(p^2 + \mu^2)^2} - \frac{1/p^4}{p^2 + \mu^2} + \frac{1/p^4}{\mu^2} \quad \text{Al.3}
\]

\[
\frac{m^2}{\mu^4(p^2 + \mu^2)^2} = \frac{m^4/p^4}{(p^2 + \mu^2)^2} + \frac{2m^2/p^6}{(p^2 + \mu^2)} - \frac{2m^2/p^6}{\mu^2} + \frac{m^2/p^4}{\mu^2} \quad \text{Al.4}
\]

\[
\frac{\mu^2 - m^2}{\mu^2(p^2 + \mu^2)} = \frac{(p^2 + m^2)/p^2}{p^2 + \mu^2} + \frac{(p^2 + m^2)/p^2}{\mu^2} - \frac{m^2/p^2}{\mu^4} \quad \text{Al.5}
\]

\[
\frac{\mu^2 - m^2}{\mu^4(p^2 + \mu^2)} = -\frac{(p^2 + m^2)/p^4}{(p^2 + \mu^2)^2} + \frac{(p^2 - 2m^2)/p^4}{\mu^2(p^2 + \mu^2)} + \frac{m^2/p^4}{\mu^4} \quad \text{Al.6}
\]
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