USE OF VARIATIONAL METHODS IN THE THEORY OF STELLAR OSCILLATIONS.

by

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CHAPTER I

INTRODUCTION AND SUMMARY.

Variational methods, which permit very simple calculation of eigenvalues and eigenfunctions, have previously been used to obtain the fundamental mode for both radial [1] and non-radial [2] oscillations of certain stellar models. In this thesis variational methods are used also to determine higher modes. In the radial case they are very successful. Numerical results agree very well with those obtained by the much slower method of numerical integration of the equations.

In the non-radial case, the calculations in this thesis show a very interesting result. Previous work [3] has shown that there are two families of modes in this case: the p-modes (with eigenvalues tending to $+\infty$ as the number of zeros in the eigenfunction increases) and the g-modes (eigenvalues $\longrightarrow +0$). Calculations using variational methods give good results for the fundamental mode and the p-modes but fail even to detect the g-modes. Half the eigenvalues of the characteristic equation yielded by the Ritz method correspond to f- or p-modes. The other half seem to be completely spurious.

This thesis develops new techniques for the numerical calculations and discusses their uses. Theoretical properties of the differential equations and of the methods of solution used

are discussed in the light of the numerical results and in particular the failure of the methods to detect the g-modes. The effects of assumptions and simplifications made in deriving the differential equations are considered briefly.

Chapters 2 and 3 develop the mathematical techniques used for the calculations. Chapter 2 begins with a review of the Ritz method (which reduces the solution of variational problems to the solution of matrix equations) and then suggests a technique which leads to very simple characteristic matrix equations. This enables a large number of parameters to be used, as is necessary for accuracy with the higher modes. The method is examined in detail. It is seen to satisfy the requirements of the Ritz method and the precise form of the matrix equations is obtained.

Chapter 3 develops efficient methods for solving numerically the matrix equations obtained in Chapter 2 and shows that these methods may also be used in more general cases. Accuracy is discussed. Some of the results of Chapter 3 have been discovered independently by Wilkinson [4] but were not published until after the present work had been done.

Chapter 4 describes the stellar model used in the calculations. It is free from some of the approximations made in the models to which variational methods have previously been applied. ([1] and [2])

Chapter 5, after a review of existing theory of non-radial oscillations, investigates the properties of the differential

equations and their variational formulation, and discusses the failure of variational methods to detect the g-modes.

Chapter 6 considers the simpler case of purely radial oscillations. Much of the discussion is relevant to more general oscillations.

Chapter 7 describes the numerical calculations and results in detail and compares these with the results of other workers. The effect of the choice of coordinate functions in the Ritz method is apparent here.

A number of unsolved questions are suggested by the discussion.

The most important numerical results are summarised in tables and some of the terms used are defined in the appendix.

CHAPTER 2

DIRECT VARIATIONAL TECHNIQUES USED IN THIS STUDY

§1. Preliminary Review of Theory.

A problem which often arises in Mathematical Physics is that of finding a function which will make the value of some integral an extremum. Some principles are stated directly in this form (e.g. the Principle of Least Action). Also the problem of solving a certain differential equation with given boundary conditions is sometimes equivalent to the problem of finding that function from a given set of "allowable" functions (satisfying the same boundary conditions) which extremises some integral.

Problems of this sort can be discussed naturally in the context of Hilbert spaces. Briefly a Hilbert space is a vector space with an inner product and which is complete with respect to the norm induced by the inner product.

If A is some symmetric operator in Hilbert space, finding an element u of the space to satisfy equations of the form $Au = f$ or $Au = \lambda u$, where f is a given element of the space and λ is an unspecified scalar, is often equivalent to extremising realvalued functionals of the form $(Au,u) - (u,f) - (f,u)$ or $(Au,u) - \lambda(u,u)$ respectively.

Important examples of Hilbert spaces are

(a) The set of all real scalar-valued functions defined almost everywhere on some interval (a,b) (not necessarily finite) of the

real line and quadratically summable (in the Lebesgue sense) on this interval.

(b) The set of all real n-vector valued functions $(u_1,\ldots,u_n)$ defined almost everywhere on some interval (a,b) of the real line and quadratically summable on this interval.

(c) The set of all (possibly complex) n-vector valued functions defined almost everywhere on some subset Ω of a Euclidean m-space and quadratically summable on Ω.

The inner product of two elements u,v of the spaces mentioned would be

$$\text{(a)} \qquad (u,v) = \int_a^b uv$$

$$\text{(b)} \qquad (\underset{\sim}{u},\underset{\sim}{v}) = \int_a^b \sum_{i=1}^n u_i\, v_i$$

$$\text{(c)} \qquad (\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega \sum_{i=1}^n u_i\, \overline{v}_i$$

where $\{u_i\},\{v_i\}$ are the components of the vectors $\underset{\sim}{u},\underset{\sim}{v}$ in some orthogonal coordinate system.

Here we shall be concerned mainly with the spaces (a) and (b) above and subspaces of these but most of the results stated hold also for other Hilbert spaces, sometimes with appropriate modifications.

A very good account of the Calculus of Variations, the study of how to find the element of some Hilbert space which extremises some real-valued functional, is given by Mikhlin [5] who also gives a

good bibliography of some of the extensive work recently done on the subject by Soviet writers. A more elementary treatment, giving references to the better known English works is given by Sagan [6]. Here the relevant terms will mainly be used in the sense defined in [5]. Some of the principal terms are defined in the appendix. In particular a symmetric operator A will be called positive-bounded-below (abbreviated to p.b.b.) if there exists a real number $\gamma \neq 0$ such that $(Au,u) \geq \gamma^2(u,u)$ for all u in D_A where the notation D_A is used for the domain of the operator A. Note that a positive operator is always bounded below but not necessarily p.b.b.

If A and B are two p.b.b. operators in some Hilbert space H, so that $D_A \subset D_B \subset H$ and if there is a u_1 in D_A such that $a(u_1) = d = \inf_{u \in D_A} a(u)$ where the functional $a(u)$ is defined by

$$a(u) = \frac{(Au,u)}{(Bu,u)}$$

then d is the lowest eigenvalue of

$$Au = \lambda Bu \tag{2.1}$$

and u_1 is the corresponding eigenfunction of (2.1). If further $\lambda_1,\ldots,\lambda_n$ are (in increasing order) the first n eigenvalues of (2.1) and $u_1,\ldots,u_n$ the corresponding eigenfunctions

and if there is a u_{n+1} in D_A which minimises $a(u)$ under the supplementary conditions

$$(Bu,u_i) = 0 , \qquad i = 1,\ldots,n \tag{2.2}$$

then u_{n+1} is the eigenfunction of (2.1) corresponding to the eigenvalue $\lambda_{n+1} = a(u_{n+1})$. This eigenvalue is the next after λ_n .

Historically interest lay mainly in finding a differential equation that must be satisfied in order to extremise some functional. Equation (2.1) is a necessary condition that $a(u)$ be extremised and is often called the Euler-Lagrange equation associated with the problem of extremising $a(u)$. This approach is given in e.g.[6] or Courant and Hilbert [7].

Of the many practical methods for determining which element of a Hilbert space minimises a given functional, one of the most important is the Ritz method. A full description of this method, and a proof that under fairly general conditions the method will determine the eigenvalues of (2.1) is given in [5]. Briefly the method is as follows. Take a sequence $\{\varphi_i\}$ of linearly independent elements (called the coordinate elements) from D_A (it is assumed $D_A \subset D_B$) which is complete in D_A in the energy of A . The idea is to define a sequence $\{u_n\}$ where u_n is that member of the set of linear combinations of the first n terms of $\{\varphi_i\}$ which minimises $a(u)$. The space of all linear combinations

of the φ_i , the new space of allowable functions, is a subspace of the original space of allowable functions. Now

$$u_n = \sum_{i=1}^{n} \alpha_i \varphi_i \tag{2.3}$$

where

$$\sum_{i=1}^{n} \alpha_i [(A\varphi_i, \varphi_j) - \lambda(B\varphi_i, \varphi_j)] = 0, \quad j = 1,\ldots,n \tag{2.4}$$

This has a non-zero solution if and only if

$$\det(M(n)) = 0 \tag{2.5}$$

where the matrix $M(n) = (m(n)_{ij})$ is given by

$$m(n)_{ij} = (A\varphi_i, \varphi_j) - \lambda(B\varphi_i, \varphi_j) \quad .$$

The kth eigenvalue $\Lambda_k(n)$ of (2.5) will be an upperbound for the kth eigenvalue λ_k of (2.1) , and for a given k , $\lim_{n\to\infty} \Lambda_k(n) = \lambda_k$.

In general the rate of convergence is very good for the lower modes, especially if the coordinate functions are chosen carefully. For higher modes a space of allowable functions of much higher dimension n (determined by a number n of arbitrary parameters α_i) must be considered. A rough idea of the accuracy of the solution obtained for a given n may be gained by observing the effect of increasing n . More precise methods are considered in [5] and in Gould [8] .

§2. Choice of Coordinate Elements.

Let U be the set of all real, scalar valued differentiable functions defined on $[0,1]$, whose derivatives are uniformly continuous except (possibly) at a finite number of points. Suppose it is required to minimise the functional $\lambda(u)$ given by

$$\lambda(u)C(u) = I(u) \tag{2.6}$$

where

$$I(u) = \int_0^1 f_1 u^2 + f_2 uu' + f_3 (u')^2 \tag{2.6a}$$

and

$$C(u) = \int_0^1 f_4 u^2 + f_5 uu' + f_6 (u')^2 \tag{2.6b}$$

where the f_i are given piecewise continuous functions defined on $(0,1)$ and u is an arbitrary function from the set U satisfying prescribed boundary conditions at 0 and 1. (More general f_i and a more general set of allowable functions could be considered. Also there is no need to restrict the interval of integration to $(0,1)$ as is done here for convenience.) The Hilbert space U is a subspace of a space of the type (a) of §1. The functionals $I(u), C(u)$ may be regarded as inner products $(I^* u, u), (C^* u, u)$ where I^*, C^* are the (linear differential) operators associated with I and C. With suitable boundary conditions, I^* and C^* will be symmetric if $f_2 = f_5 = 0$. The problem could be solved by the Ritz method using a sequence of coordinate functions such as polynomials satisfying the boundary conditions.

Alternatively u may be restricted to belong to the set of continuous functions, satisfying the boundary conditions, whose derivatives in each of the intervals (a_i, a_{i+1}) are constant where

$$0 = a_0 < a_1 < \ldots < a_n = 1 \tag{2.7}$$

This method has many advantages and although similar methods are mentioned in the standard works (e.g. [7] p 177) I have not found in the literature either a theoretical account of this method showing that the standard results of the Ritz method apply, or any mention of the properties of the resulting characteristic equations.

The method is essentially a case of the Ritz method. Let $P_n = \{a_i\}$ be any partition of $[0,1]$ satisfying (2.7) . Define the first $n + 1$ coordinate functions as follows.

$$\varphi_0 = 1$$

$$\varphi_i(x) = \left\{ \begin{array}{ll} 0 & \text{on } [0, a_{i-1}] \\ \dfrac{x - a_{i-1}}{a_i - a_{i-1}} & \text{on } [a_{i-1}, a_i] \\ 1 & \text{on } [a_i, 1] \end{array} \right\} \quad i = 1, \ldots, n \tag{2.8a}$$

Define the remaining φ_i as follows. Let P_{n+1} be a refinement of P_n which contains exactly one extra number a_{n+1} .

Suppose $a_j < a_{n+1} < a_{j+1}$. Define φ_{n+1} by

$$\varphi_{n+1}(x) = \begin{cases} 0 & \text{on } [0,a_j] \\ \dfrac{x - a_j}{a_{n+1} - a_j} & \text{on } [a_j,a_{n+1}] \\ 1 & \text{on } [a_{n+1},1] \end{cases} \qquad (2.8b)$$

(The points $a_{n+1}, \ldots , a_n$ could be renamed so that with the new notation, $a_j < a_{j+1} < \ldots < a_n < a_{n+1}$.) The remaining φ_i are defined similarly. Thus defined, the φ_i are linearly independent. (Take any linear combination and equate to zero at the points a_i . This shows the coefficients to be zero.) Although not orthogonal they can be orthogonalised.

For a given P_n , the set of all functions of the form $\sum_{i=0}^{n} \alpha_i \varphi_i$ is precisely the set of all continuous functions on $[0,1]$ whose derivatives are constant in each of the intervals (a_i,a_{i+1}) .

Let $\|P_n\| = \max_i |a_{j+1} - a_i|$. Then if $\|P_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, the sequence $\{\varphi_i\}$ is complete in U in the energy of I^* .

The φ_i can be modified so that they satisfy any given boundary conditions, and the set of all linear combinations $\Sigma \alpha_i \varphi_i$ represents the set of all continuous functions, satisfying the given boundary conditions, whose derivatives are constant in the prescribed intervals. e.g. If it is required that $u(0) = u(1) = 0$,

φ_i can be defined by

$$\varphi_i(x) = \begin{cases} 0 & \text{on } [0, a_{i-1}] \\ \dfrac{x - a_{i-1}}{a_i - a_{i-1}} & \text{on } [a_{i-1}, a_i] \quad i = 1, \ldots, n-1 \\ \dfrac{1 - x}{1 - a_i} & \text{on } [a_i, 1] \end{cases} \tag{2.9}$$

In this case only $n - 1$ coordinate functions are required for P_n. However it is often possible to use a change of variable so that the boundary conditions are satisfied automatically. This was done in the calculations described in Chapter7 .

Provided $\|P_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, the sequence $\{\varphi_i\}$ defined by (2.8) satisfies all the conditions required for the Ritz method. Its use in that method will determine the eigenvalues and eigenfunctions of (2.6) provided that I^* and C^* are such that the Ritz method is applicable.

If $f_1 f_3 \geq f_2^2$ and $f_1 \geq 0$, I^* is positive. If $f_2 = 0$, $f_3 \geq 0$ and $f_1 \geq \varepsilon > 0$ where ε is constant, I^* is p.b.b. With appropriate boundary conditions I^* may be p.b.b. under more general conditions. Conditions for C^* are similar. If both I^* and C^* are p.b.b. operators the Ritz method will be applicable. This is a sufficient condition but not a necessary one.

The theorems of [5] may be applied directly if the sequence

$\{P_i\}$ and hence $\{\varphi_i\}$ are regarded as fixed. If arbitrary P_i are considered, so that any continuous function with piecewise constant derivative defined on $(0,1)$ may be written in the form $\Sigma \alpha_i \varphi_i$, then the theorems must be modified to consider limits of nets instead of limits of sequences. (The set of all partitions is partially ordered.)

For a given P_n, the space of all $\Sigma \alpha_i \varphi_i$ (i.e. all linear combinations of $\varphi_0,\ldots,\varphi_n$) is isomorphic with the $(n+1)$ dimensional space defined by all $(d_0,\ldots,d_n)$ where d_i is the value of u_n at a_i. In the interval (a_i,a_{i+1}),

$$u_n(x) = d_i + \frac{x - a_i}{a_{i+1} - a_i}\,(d_{i+1} - d_i) \tag{2.10a}$$

and

$$u_n'(x) = \frac{d_{i+1} - d_i}{a_{i+1} - a_i} \tag{2.10b}$$

Define functions I_n and C_n on R^{n+1} by

$$I_n(d_0,d_1,\ldots,d_n) = I(u_n)$$

$$= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \{ f_1(x)[d_{i-1} + \frac{x - a_{i-1}}{a_i - a_{i-1}}\,(d_i - d_{i-1})]^2$$

$$+ f_2(x)[d_{i-1} + \frac{x-a_{i-1}}{a_i - a_{i-1}}\,(d_i - d_{i-1})]\,(\frac{d_i - d_{i-1}}{a_i - a_{i-1}})$$

$$+ f_3(x)(\frac{d_i - d_{i-1}}{a_i - a_{i-1}})^2\}\,dx \tag{2.11a}$$

where the expression for $I(u_n)$ is obtained by substituting (2.10) in (2.6a), and

$$C_n(d_o,d_1,\ldots,d_n) = C(u_n) \quad \text{etc} . \tag{2.11b}$$

The Ritz method can now be carried out by solving the $(n + 1)$ equations

$$\frac{\partial}{\partial d_i}(I_n - \lambda C_n)(d_o,\ldots,d_n) = 0 , \qquad i = 0,\ldots,n \tag{2.12}$$

where λ is an eigenvalue. The eigenvalues λ of (2.12), which give upper bounds to the eigenvalues $\lambda(u)$ of (2.6), are the roots of

$$\det(P - \lambda Q) = 0 \tag{2.13}$$

where $P = (p_{ij})$, $Q = (q_{ij})$ and p_{ij} , q_{ij} are the coefficients of d_{j-1} in $\frac{\partial}{\partial d_{i-1}} I_n(d_o,\ldots,d_n)$ and $\frac{\partial}{\partial d_{i-1}} C_n(d_o,\ldots,d_n)$ respectively.

Now I_n and C_n are quadratic expressions in the d_i in which the coefficient of $d_i d_j$ is zero whenever $|i - j| > 1$. Thus the matrices P and Q are triple diagonal, i.e. $p_{ij} = q_{ij} = 0$ whenever $|i - j| > 1$. Also if $p = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} p_{ij} x_i x_j$ is any quadratic expression in the x_i , the coefficient of x_j in $\frac{\partial p}{\partial x_i}$ and the coefficient of x_i in $\frac{\partial p}{\partial x_j}$ are both equal to

$p_{ij} + p_{ji}$. Thus since I_n and C_n are quadratic expressions in the d_i , the matrices P and Q are symmetric.

A matrix $Q = (q_{ij})$ is said to be positive definite if its principal minors are all positive. A sufficient condition for Q to be positive definite is that the quadratic form $\Sigma \Sigma q_{ij}x_i x_j$ is positive definite. (If Q is symmetric this is also a necessary condition.) Now if $C(u) > 0$ for all non-zero u in D_c (and this is usually the case as C usually represents some form of kinetic energy), then clearly for $u_n \neq 0$, $C(u_n) > 0$ where u_n is restricted to a subset of D_c . As $C(u) > 0$ for all non-zero u in D_c if $4f_4f_6 > f_5^2$ and $f_4 > 0$, almost everywhere on $(0,1)$, this is a sufficient condition (but not a necessary one) for Q to be positive definite.

Thus in most cases, with the choice of coordinate functions given by (2.8), application of the Ritz method to (2.6) leads to a matrix equation of the form

$$Px = \lambda Qx$$

where both P and Q are symmetric and triple diagonal, and Q is positive definite. A very simple method for solving such a system is described in Chapter 3 §1. Most other choices of coordinate functions lead to matrix equations which are much more difficult to solve. Also the method described gives immediately a picture of the eigenfunctions, showing clearly

the position of their zeros and their general shape, without further calculation.

If P_n is given by $a_i = {}^{i}/_{n}$, $i = 0,\ldots,n$, substitution of (2.11) in (2.12) shows that the coefficients p_{ij} are given by

$$p_{1,1} = \int_0^{\frac{1}{n}} [(1-nx)^2 f_1(x) - n(1-nx)f_2(x) + n^2 f_3(x)]dx$$

$$p_{n+1,n+1} = \int_{1-\frac{1}{n}}^{1} [(nx+1-n)^2 f_1(x) + (nx+1-n)nf_2(x) + n^2 f_3(x)]dx$$

$$p_{i+1,i+1} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} [(nx+1-i)^2 f_1(x) + (nx+1-i)nf_2(x) + n^2 f_3(x)]dx$$

$$+ \int_{\frac{i}{n}}^{\frac{i+1}{n}} [(i-nx+1)^2 f_1(x) + (nx-1-i)nf_2(x) + n^2 f_3(x)]dx , \qquad (2.14a)$$

$$i = 1,\ldots,n-1$$

$$p_{i+1,i} = p_{i,i+1} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} [(i-nx)(nx+1-i)f_1(x) + (i-\tfrac{1}{2}-nx)nf_2(x)$$

$$- n^2 f_3(x)]dx , \quad i = 1,\ldots,n$$

Expressions (2.14b) for q_{ij} can be obtained by replacing f_1, f_2, f_3 in (2.14a) by f_4, f_5, f_6 respectively.

§3. More General Hilbert Spaces.

In the case of a Hilbert space of type (b) from §1, where $\underset{\sim}{u}$ is a real vector-valued function, again defined on $(0,1)$, the analogue of the functional given by (2.6), the integral of a homogeneous quadratic expression in the components of $\underset{\sim}{u}$ and their derivatives, is more complicated. Let $\underset{\sim}{u}$ have components $(v_1,\ldots,v_m)$. The relevant forms are

$$I(u) = \int_o^1 \sum_{i=1}^m \sum_{j=1}^m (f_{ij_1}v_iv_j + f_{ij_2}v_iv_j' + f_{ij_3}v_i'v_j') \qquad (2.15a)$$

$$C(u) = \int_o^1 \sum_{i=1}^m \sum_{j=1}^m (f_{ij_4}v_iv_j + f_{ij_5}v_iv_j' + f_{ij_6}v_i'v_j') \qquad (2.15b)$$

For $k \neq 2$ or 5 we can, without loss of generality, consider $f_{ijk} = f_{jik}$. The f_{ijk} are otherwise arbitrary, given, piecewise continuous functions defined on $(0,1)$ and $\underset{\sim}{u}$ may be restricted to the space U_m of m-vector valued functions whose components belong to U. Then U_m is a subspace of a Hilbert space of the type (b) of §1. With suitable boundary conditions, I^* and C^* will generally be symmetric if for all i and j, f_{ij_2} and f_{ij_5} respectively are zero. A more complicated case arises in Chapter 5 where $\underset{\sim}{u}$ is a 2 dimensional vector but the inner product is defined in terms of a 3 dimensional vector.

The method of §2 may be modified for the case when I and C have the form of (2.15). Corresponding to a partition P_n of $(0,1)$, take as coordinate functions the $m(n+1)$ functions

φ_{ij} , $i = 0,\ldots,n$; $j = 1,\ldots,m$ where φ_{ij} is the m-vector valued function whose jth component is φ_i defined by (2.8a), and whose other components are zero. The set of all linear combinations $\Sigma\Sigma\, \alpha_{ij}\varphi_{ij}$ of the φ_{ij} is precisely the set of all continuous m-vector valued functions whose derivatives are constant in each of the intervals (a_i, a_{i+1}) . Again the coordinate functions could be modified to suit boundary conditions. If it is required e.g. that $\underset{\sim}{u}(0) = \underset{\sim}{u}(1) = 0$ then φ_{ij} can be defined as the m-vector valued function whose jth component is φ_i as defined in (2.9), and whose other components are zero. In this case there would be $m(n - 1)$ coordinate functions. Again if $\|P_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ the φ_{ij} are complete in U_n in the energy of I^* . Thus the φ_{ij} satisfy all the conditions required for the Ritz method.

The space of all $\underset{\sim}{u}_n = \Sigma_{i=0}^{n}\, \Sigma_{j=1}^{m}\, \alpha_{ij}\varphi_{ij}$ where the α_{ij} are real is isomorphic with the space of real $(n + 1)$ X m matrices $D = (d_{ij})$ where d_{ij} is the value of the jth component of $\underset{\sim}{u}_n$ at a_{i-1}. Define I_{nm} , C_{nm} by $I_{nm}(D) = I(\underset{\sim}{u}_n)$ (2.16a)

$$C_{nm}(D) = C(\underset{\sim}{u}_n) \qquad (2.16b)$$

where $I(\underset{\sim}{u}_n)$, $C(\underset{\sim}{u}_n)$ may be expanded by substituting expressions analogous to (2.10) in (2.15). The Ritz method may be applied by solving the $m(n + 1)$ equations:

$$\frac{\partial}{\partial d_{ij}}(I_{nm} - \lambda C_{nm})(D) = 0 , \quad i = 0,\ldots,n ; \quad j = 1,\ldots,m \qquad (2.17)$$

This may be written in matrix form as

$$\underline{P}\,\underset{\sim}{x} = \lambda \underline{Q}\,\underset{\sim}{x} \qquad (2.18)$$

and the eigenvalues λ are the zeros of $\det(\underline{P} - \lambda \underline{Q})$. Again $\underline{P}$ and $\underline{Q}$ are symmetric (since I_{nm} and C_{nm} are quadratic forms) and $\underline{Q}$ is positive definite if (but not only if) C is positive. This time $\underline{P}$ and $\underline{Q}$ are not triple diagonal but of the form

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} & \ldots & P_{1m} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{m1} & \ldots\ldots & & P_{mm} \end{pmatrix} , \quad \underline{Q} = \begin{pmatrix} Q_{11} & \ldots & Q_{1m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Q_{m1} & \ldots & Q_{mm} \end{pmatrix} \qquad (2.19)$$

where each of the P_{ij} is an $(n+1) \times (n+1)$ triple diagonal matrix. The method of Chapter 3 §1 cannot be used in this case, but a method is suggested in Chapter 3 §3 which for large n is considerably better than standard methods.

When P_n is given by $a_i = \frac{i}{n}$, $i = 0,\ldots,n$. (2.10), (2.15) and (2.16) show that equations (2.17) are of the form

$$\sum_{j=1}^{m} (d_{jk-1} F_1(i,j,k) + d_{jk}(F_2(i,j,k) + F_3(i,j,k+1)) + d_{jk+1} F_1(j,i,k+1))$$

$$= \lambda \sum_{j=1}^{m} (d_{jk-1} F_4(i,j,k) + d_{jk}(F_5(i,j,k) + F_6(i,j,k+1))$$

$$+ d_{jk+1} F_4(j,i,k+1)) \qquad i = 1,\ldots,m ; \quad k = 1,\ldots,n-1 \qquad (2.20a)$$

$$\sum_{j=1}^{m} (d_{jo}F_3(i,j,1) + d_{j1}F_1(j,i,1)) = \lambda \sum_{j=1}^{m} (d_{jo}F_6(i,j,1)$$

$$+ d_{j1}F_4(j,i,1)) \qquad i = 1,\ldots,m \ . \tag{2.20b}$$

$$\sum_{j=1}^{m} (d_{jn-1}F_1(i,j,n) + d_{jn}F_2(i,j,n)) = \lambda \sum_{j=1}^{m} (d_{jn-1}F_4(i,j,n)$$

$$+ d_{jn}F_5(i,j,n)) \qquad i = 1,\ldots,m \ . \tag{2.20c}$$

where

$$F_1(i,j,k) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f_{ij1}(x)(nx+1-k)(k-nx)+n(k-1-nx)\frac{f_{ij2}(x)+f_{ji2}(x)}{2}$$

$$+ \frac{n}{2}f_{ji2}(x) - n^2 f_{ij3}(x)]dx \qquad i,j = 1,\ldots m \ ; \quad k = 1,\ldots,n \ . \tag{2.21a}$$

$$F_2(i,j,k) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f_{ij1}(x)(nx+1-k)^2 + n(nx+1-k)\frac{f_{ij2}(x)+f_{ji2}(x)}{2}$$

$$+ n^2 f_{ij3}(x)]dx \qquad i,j = 1,\ldots,m \ ; \quad k = 1,\ldots,n \tag{2.21b}$$

$$F_3(i,j,k) = \int_{\frac{k-1}{n}}^{\frac{k}{n}} [(k-nx)^2 f_{ij1}(x) + n(nx-k)\frac{f_{ij2}(x)+f_{ji2}(x)}{2}$$

$$+ n^2 f_{ij3}(x)]dx \qquad i,j,= 1,\ldots,m \ ; \quad k = 1,\ldots n \tag{2.21c}$$

Note that (2.14a) is a special case of this. Expressions for F_4 , F_5 , F_6 may be obtained from F_1 , F_2 , F_3 respectively by replacing f_{ij1} , f_{ij2} , f_{ji2} , f_{ij3} by f_{ij4} , f_{ij5} , f_{ji5} ,

$f_{ij\delta}$ respectively in (2.21).

This type of choice of coordinate functions may be modified for some other integrands. If the functions are defined on an m-vector domain (as with Hilbert space type (c) of §1), φ_i may be chosen which have constant derivatives on m-dimensional simplexes instead of on intervals. If the integrands contain mth order derivatives, φ_i may be chosen in which the mth order derivative is piecewise constant. In all cases the resulting matrices will, for large n, consist mainly of zeros and this fact may be used to facilitate the solution.

CHAPTER 3

SOLUTION OF MATRIX EQUATIONS

§1. A Modification of Givens' Method.

In Chapter 2 §2 it was required to solve the eigenvalue problem

$$P\underset{\sim}{x} = \lambda Q\underset{\sim}{x} \tag{3.1}$$

where P and Q are symmetric triple diagonal matrices and Q is positive definite.

Givens [9] has developed a very efficient method for calculating eigenvalues of a symmetric triple diagonal matrix, and shown that any symmetric matrix can be converted to triple diagonal form by a finite series of orthogonal rotations. Since Q is symmetric positive definite there is a symmetric matrix C such that $C^2 = Q$ and (3.1) could be solved by applying standard techniques to $C^{-1}PC^{-1}$. However $C^{-1}PC^{-1}$ will not in general have the simple form of P and Q, and in this case standard methods will be very inefficient. They will require more time and more computer storage than necessary and by increasing greatly the number of operations will tend to increase rounding errors. It turns out that Givens' method for solving (3.1) in the case $Q = I$, where I is the unit matrix, can readily be generalised to give a direct method for solving (3.1) with more

general Q . Even the condition that P must be symmetric can be relaxed slightly.

Let $P = (p_{ij})$ and $Q = (q_{ij})$ be real $n \times n$ matrices defined by

$$\left.\begin{aligned} &p_{ii} = a_i \ , \ q_{ii} = d_i \ , \quad i = 1,\ldots,n \\ &p_{i,i+1} = b_i \ , \ p_{i+1,i} = c_i \ , \ q_{i,i+1} = e_i \ , \ q_{i+1,i} = f_i \ , \\ &\qquad\qquad i = 1,\ldots,n-1 \\ &p_{ij} = q_{ij} = 0 \quad \text{for} \ |i - j| > 1 \end{aligned}\right\} \tag{3.2}$$

For any real number λ define a sequence $\{p_i(\lambda)\}$ as follows

$$p_0(\lambda) = 1$$

$$p_1(\lambda) = a_1 - \lambda d_1 \tag{3.3}$$

$$p_i(\lambda) = (a_i - \lambda d_i)p_{i-1}(\lambda) - (b_{i-1} - \lambda e_{i-1})(c_{i-1} - \lambda f_{i-1})p_{i-2}(\lambda)$$
$$2 \leq i \leq n$$

It can be shown by induction that the $p_i(\lambda)$, $i = 1,\ldots,n$, are the principal minors of $P - \lambda Q$ and that $p_n(\lambda) = \det(P - \lambda Q)$.

Define condition (1) to be the condition that for all i and all real λ , $(b_i - \lambda e_i)(c_i - \lambda f_i) > 0$ whenever $p_i(\lambda) = 0$. In the classical case where $Q = I$ and P is symmetric, this is clearly satisfied if the b_i are all non-zero. The case where some $b_i = 0$ presents no difficulty for (3.1) may then

be written in the form $\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where P_1 satisfies condition (1) and if P_2 does not satisfy condition (1) it may be split further and the argument continued. The eigenvalues of P will be those of P_1 together with those of P_2 . This argument does not extend for more general Q , but if for some i and λ_o , $(b_i - \lambda_o e_i) = (c_i - \lambda_o f_i) = p_i(\lambda_o) = 0$, then λ_o is an eigenvalue of (3.1) .

Define $S(\lambda)$ = the number of sign changes in the sequence $\{p_i(\lambda)\}$, $i = 0,\ldots,n$ where a zero element is taken to have the same sign as the preceding element.

Theorem 1.

Let P , Q be two triple diagonal matrices satisfying condition (1) and let Q be positive definite. Then for each real number λ_o , $S(\lambda_o)$ is the number of eigenvalues of (3.1) less than λ_o .

Proof.

(i) For all i and real numbers λ_o , if $p_i(\lambda_o) = 0$ then $p_{i-1}(\lambda_o) \neq 0$. For if $p_i(\lambda_o) = p_{i-1}(\lambda_o)$ then, by (3.3) and condition (1) , $p_{i-2}(\lambda_o)$ would also be zero and so by induction would $p_o(\lambda_o)$ which is not so. Also if $i < n$, (3.3) and condition (1) show that

$$p_{i+1}(\lambda_o)p_{i-1}(\lambda_o) < 0 \quad \text{whenever} \quad p_i(\lambda_o) = 0 .$$

(ii) Whenever $S(\lambda)$ changes, at least one of the $p_i(\lambda)$ must change sign, and hence pass through a zero value $p_i(\lambda_1)$

say, as the p_i are continuous functions (polynomials in fact.) Thus by (i), if $i < n$, $p_{i+1}(\lambda_1)p_{i-1}(\lambda_1) < 0$, say $p_{i+1}(\lambda_1) > 0$, $p_{i-1}(\lambda_1) < 0$. By continuity there is a number $\delta > 0$ such that in $(\lambda_1 - \delta, \lambda_1 + \delta)$, $p_{i+1} > 0 > p_{i-1}$. Then the change of sign in p_i will cause no change in $S(\lambda)$. Thus $S(\lambda)$ can change only when $p_n(\lambda) = 0$, i.e. at an eigenvalue of (3.1).

(iii) At each eigenvalue S can change by at most 1. For by (i) when $p_n(\lambda_o) = 0$, $p_{n-1}(\lambda_o) \neq 0$, say $p_{n-1}(\lambda_o) > 0$. Then by continuity there is a $\delta > 0$ such that $p_{n-1} > 0$ in $(\lambda_o - \delta, \lambda_o + \delta)$, so that the change in sign of p_n causes S to change by exactly one.

(iv) Now S changes at least n times. For as $p_i(\lambda)$ is a polynomial of degree i in λ in which the coefficient of $(-\lambda)^i$ is the ith principal minor of Q, and as all the principal minors of Q are positive (since Q is positive definite) then $p_i(-\infty) > 0$ and $p_i(+\infty)$ has sign $(-)^i$ so that $S(-\infty) = 0$ and $S(+\infty) = n$. Thus by (iii) S changes at least n times.

(v) As p_n is a polynomial of degree n it has at most n distinct roots. Thus by (iii) and (iv) p_n has exactly n real distinct roots (so that the eigenvalues of (3.1) are real and distinct) and S increases by one at each root. The result follows. Q.E.D.

Applying Theorem 1 to the principal minors, it follows that each

p_i has i distinct real roots and that the number of sign changes in the sequence $p_0(\lambda_0)$, $p_1(\lambda_0),\ldots,p_i(\lambda_0)$ is the number of roots of the equation $\det(P_i - \lambda Q_i) = 0$ less than λ_0, where P_i, Q_i are the ith principal submatrices of P and Q respectively. It follows that the zeros of p_i separate those of p_{i+1}. This may also be proved directly by induction.

Practical Procedure.

The method used for finding eigenvalues is as follows. Suppose upper and lower bounds for an eigenvalue are known. Bisect the interval thus defined. The value of S at the midpoint of the interval shows in which half of the interval the required eigenvalue lies. The bisection may be continued until the eigenvalue is known to the required degree of accuracy. The ultimate limit to accuracy depends on rounding errors (see §2). It remains to find a method for determining upper and lower bounds for eigenvalues.

Now $\det(P - \lambda Q)$ is a polynomial of degree n in which the coefficient of $(-\lambda)^n$ is $\det Q$ and the constant term is $\det P$. Thus, since Q is non-singular, the product of the eigenvalues of (3.1) is $\frac{\det P}{\det Q}$, and the geometric mean of their magnitudes is $\left|\frac{\det P}{\det Q}\right|^{\frac{1}{n}}$. Thus in the case described in Chapter 7 when P and Q are both positive definite it is easy to calculate the eigenvalues systematically. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of (3.1). Then $0 \leq \lambda_1 \leq \left(\frac{\det P}{\det Q}\right)^{\frac{1}{n}}$. Calculate λ_1 by the method of bisections and then use

$$\lambda_1 \leq \lambda_2 \leq \left(\frac{\det P}{\lambda_1 \det Q}\right)^{\frac{1}{n-1}} .$$

After m eigenvalues have been calculated use

$$\lambda_m \leq \lambda_{m+1} \leq \left(\frac{\det P}{\det Q \prod_{i=1}^{m} \lambda_i}\right)^{\frac{1}{n-m}} .$$

This method was used for the calculations described in Chapter 7. The method may still be used when P is not positive definite although in this case the bounds are weaker. For λ_{m+1} the bounds are

$$\pm \left| \frac{\det P}{\det Q \prod_{i=1}^{m} \lambda_i} \right|^{\frac{1}{n-m}} \quad \text{where}$$

$$|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n| .$$

These bounds are rather wide and some devices may be used to speed convergence. Two were used in the present calculations. If say the mth eigenvalue was being sought and it was found that there were $(m + r)$ eigenvalues less than some number α, then α was stored in computer memory and used as an upper bound in determining also the eigenvalues from λ_{m+1} to λ_{m+r}. Also if it was found that there were $(m + r)$ eigenvalues below α and $(m - 1)$ below β, then $\beta + \frac{\alpha - \beta}{r + 2}$ was used as the next trial value instead of $\frac{\alpha + \beta}{2}$.

It can readily be shown that the magnitudes of the eigenvalues of a matrix $P = (p_{ij})$ are bounded by

$$\sup_r |\lambda_r| \leq \|P\| \tag{3.4}$$

where the norm $\|P\|$ is defined by

$$\|P\| = \max_i \sum_j |p_{ij}| \tag{3.5}$$

This bound is generally used in the standard case $Q = I$ but cannot readily be generalised for (3.1) with $Q \neq I$. Indeed eigenvalues of (3.1) can be made arbitrarily large when Q is in the neighbourhood of a singular matrix, and in such a neighbourhood, arbitrarily small alterations in the q_{ij} may make arbitrarily large alterations in some eigenvalues of (3.1). But an upper bound for $|\lambda|$ can be found for the case considered. Let λ_{Qn}, λ_{Q1} respectively be the numerically greatest and numerically smallest eigenvalues of Q with similar notation for P. As an upper bound for $|\lambda_{Qn}|$ is given by (3.4), and the product of the eigenvalues is $\det Q$, a positive lower bound for

$$|\lambda_{Q1}| \text{ is } \frac{|\det Q|}{\|Q\|^{n-1}}$$

Theorem 2.

If P and Q are real symmetric matrices (not necessarily triple diagonal) and Q is positive definite, the magnitude of the eigenvalues of (3.1) cannot exceed $|\lambda_{Pn}/\lambda_{Q1}|$.

Proof.

Denote the magnitude of the numerically greatest eigenvalue of (3.1) by λ_n . Then denoting the transpose of $\underset{\sim}{x}$ by $\underset{\sim}{x}^T$

$$\lambda_n = \sup_{\underset{\sim}{x}\neq 0} \left| \frac{\underset{\sim}{x}^T P \underset{\sim}{x}}{\underset{\sim}{x}^T Q \underset{\sim}{x}} \right| .$$

This is analagous to the well-known result that

$$|\lambda_{Pn}| = \sup_{\underset{\sim}{x}\neq 0} \left| \frac{\underset{\sim}{x}^T P \underset{\sim}{x}}{\underset{\sim}{x}^T \underset{\sim}{x}} \right| ,$$

and may be proved in a similar way. For since Q is symmetric there is an orthogonal matrix R_1 such that $R_1^T Q R_1$ is pure diagonal, and since Q is positive definite there is a real non-singular diagonal matrix H such that $H^T R_1^T Q R_1 H = I$. Since $H^T R_1^T P R_1 H$ is clearly symmetric there is an orthogonal matrix R_2 such that $D = R_2^T H^T R_1^T P R_1 H R_2$ is diagonal. Then $R_2^T H^T R_1^T Q R_1 H R_2 = I$. Then

$$\det(D - \lambda I) = \det R_2^T \det(H^T R_1^T P R_1 H - H^T R_1^T Q R_1 H) \det R_2$$

$$= \det H^T \det(R_1^T P R_1 - \lambda R_1^T Q R_1) \det H$$

$$= \det H^T \det R_1^T \det(P - \lambda Q) \det R_1 \det H$$

$$= \det H^T \det(P - \lambda Q) \det H$$ which, since H is non-singular, vanishes if and only if $\det(P - \lambda Q) = 0$ so that the eigenvalues of

D are the eigenvalues of (3.1). Then if $x = R_1 H R_2 y$,

$$\lambda_n = \max_{y \neq 0} \left| \frac{y^T D y}{y^T y} \right|$$

$$= \max_{x \neq 0} \left| \frac{y^T D y}{y^T y} \right| \quad \text{(since } R_1 H R_2 \text{ is non-singular)}$$

$$= \max_{x \neq 0} \left| \frac{x^T P x}{x^T Q x} \right| = \max_{x \neq 0} \left| \frac{x^T P x}{x^T x} \cdot \frac{x^T x}{x^T Q x} \right|$$

$$\leq \max_{x \neq 0} \left| \frac{x^T P x}{x^T x} \right| \max_{x \neq 0} \left| \frac{x^T x}{x^T Q x} \right| = \left| \frac{\lambda_{Pn}}{\lambda_{Q1}} \right| \qquad \text{Q.E.D.}$$

Equality holds only in the exceptional case when

$\left| \frac{x^T P x}{x^T x} \right|$ and $\left| \frac{x^T x}{x^T Q x} \right|$ attain maximum value for the same x .

e.g. If $P = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ then $Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives inequality and

$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ gives equality. This bound also follows immediately from the properties of the Hilbert norm ([11] p. 4) .

The Meaning of "Condition (1)"

Consider the cases in which condition (1) is satisfied. In Chapter 1 we obtained a system in which P and Q were both symmetric. In this case condition (1) is satisfied provided only that for all i and all real λ , $p_i(\lambda)$ and $(b_i - \lambda e_i)$ never vanish simultaneously. Call this weaker requirement condition (2).

It turns out that this condition is very seldom violated but it is a little difficult to prove that it is satisfied in a particular case as it is the exact values of $p_i(\lambda)$ and $b_i - \lambda e_i$ which are significant and not the computed ones. In the case $e_i = 0$, condition (2) is satisfied unless $b_i = 0$ and p_i has at least one real root. Error bounds for $p_i(b_i/e_i)$ were calculated in the case $e_i \neq 0$ but were not very helpful. If $|p_{ij}|$ and $|q_{ij}|$ are bounded above by m it was found that the error could grow as

$$i \left(\frac{m(1 + |b_i/e_i|)(1 + \sqrt{5})}{2} \right)^i \varepsilon \quad \text{where} \quad \varepsilon$$

is a small constant depending on the computer used. With an IBM 1620 (which was used for the present calculations) $\varepsilon \sim 10^{-8}$. The position is much better than these wide bounds might suggest.

If $i = 1$, the condition is $\frac{b_1}{e_1} \neq \frac{a_1}{d_1}$ which is easily checked. And if $i > 1$ the zeros of p_i will generally be irrational whereas b_i/e_i must be rational. Even if p_i had a rational zero the probability that condition (2) might not hold for $i > 1$ would be extremely small as the elements of the matrices will usually contain about 8 significant decimals so that b_i/e_i and the zeros of p_i are likely to contain even more.

Note also that condition (2) though sufficient is by no means necessary for the method to give good results. If condition (2)

does not hold it is possible for consecutive p_i to vanish simultaneously, but (3.3) shows that if $p_{i-1}(\alpha) = p_i(\alpha)$ then $p_j(\alpha) = 0$, $j = i + 1,\ldots,n$, so that α must be an eigenvalue of (3.1). Define condition (3) to be the condition that $b_i f_i = c_i e_i$ and $b_i c_i > 0$, for all i . This is clearly satisfied when P and Q are symmetric provided the b_i are non-zero. When this condition holds, (3.3) shows that if $p_i(\alpha) = 0$ and $i < n$ then $p_{i-1}(\alpha)p_{i+1}(\alpha) \leq 0$. Hence, as before , S can change only at an eigenvalue of (3.3). Further if all p_i which vanish at α change sign at α then S can change by only one at α . If some p_i vanishes but does not change sign at α then $p_i'(\alpha) = 0$ and it can be shcwn using condition (3) and considering the derivatives of (3.3) that if S changes by m at α , then (3.3) has an eigenvalue of multiplicity at least m at α . Argument similar to that of Theorem 1 shows that except at eigenvalues of (3.3) , S has the value predicted by Theorem 1. At eigenvalues of (3.3) this is not so, as is shown by considering $S(1)$ for

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} , \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \tag{3.6}$$

This difficulty cannot be overcome by changing the convention of the sign of 0 as shown by other simple examples. In practice the p_i will contain rounding errors and the signs of those which should be zero will be randomly distributed. The effect may be

felt even for trial values differing very slightly from true eigenvalues. Thus if condition (3) holds but condition (2) does not, the method should still give good results but the accuracy may be slightly less than if condition (2) held, and difficulties in programming may arise as the value of S for λ near an eigenvalue may be misleading.

Another important case in which condition (1) is satisfied is the case in which Q is purely diagonal and for all i, $b_i c_i > 0$. This case could easily be modified so that $Q = I$, but is not covered by Givens' result as P is not required to be symmetric. This case, which does not even need condition (2), can arise when derivatives are replaced by finite differences in eigenvalue problems. An example from the theory of stellar oscillations arises in [10] (see Chapter 6). In that example P is not symmetric but condition (1) is satisfied.

§2. The Role of Rounding Errors.

Numbers stored in a computer must be limited to a certain number of digits. When the number resulting from an arithmetic operation contains more than the allowable number of digits, the last significant digits are lost. The effect of those rounding errors is one of the most important considerations when assessing the merit of a numerical method.

As well as a detailed discussion of the mathematical principles involved and technical suggestions for programming, Givens in [9] gives a careful analysis of the effect of rounding errors in calculating the eigenvalues of a triple diagonal matrix by his method. Wilkinson [12] in an article on the principles governing rounding errors in floating point calculations gives, as one of his illustrations, a simpler treatment of the problem and shows that provided all the elements of the matrix lie in the interval $[-\frac{1}{4},\frac{1}{4}]$ then calculation of the eigenvalues of a symmetric triple diagonal matrix by Givens' method on a computer retaining t binary places gives an error of less than $13 \times 2^{-t-2}$. This remarkable result is independent of the dimension of the matrix.

Wilkinson shows that when $Q = I$ in (3.1) the computed values of $p_i(\lambda)$ defined by (3.3) are the exact values for a symmetric triple diagonal matrix P_1 and obtains (very small) upper bounds for $\|P - P_1\|$ (where the norm is defined by (3.5)). An exactly similar argument shows that this is also true when

$Q \neq I$. i.e. The computed values of the sequence defined by (3.3) where P and Q are any triple diagonal matrices is the exact sequence for some triple diagonal matrices P_1 , Q_1 (which will be symmetric if P and Q are symmetric) and similar small bounds may be found for $\|P - P_1\|$ and $\|Q - Q_1\|$.

Wilkinson concludes his proof by using a result of Lidskii [13] also used in [9] where Lidskii's main theorem is quoted in full. It follows from this theorem that when the eigenvalues of two symmetric matrices P and P_1 are arranged in order of magnitude, the corresponding eigenvalues of P and P_1 differ by less than the numerically greatest eigenvalue of $P - P_1$, which cannot exceed $\|P - P_1\|$.

Unfortunately the position when $Q \neq I$ is more complicated. If Q is sufficiently ill-conditioned, arbitrarily small changes in the elements of Q may make arbitrarily large changes in some eigenvalues of (3.1). If however Q is not ill-conditioned, and

$$\|Q_1 - Q\| << |\det Q| \tag{3.7}$$

then Q_1 will be positive definite. Then if Q is symmetric there will exist symmetric matrices C and C_1 such that $C^2 = Q$ and $C_1^2 = Q_1$. Bounds can be obtained for $\|P_1 - P\|$ and $\|Q_1 - Q\|$. If bounds could be deduced for $\|C_1^{-1} P_1 C_1^{-1} - C^{-1} P C^{-1}\|$, Lidskii's Theorem would give error bounds for the process. These should be small if (3.7) is

satisfied. If Q is ill-conditioned most techniques will prove unsatisfactory.

Note that Wilkinson's bound is a bound for absolute error. In general no bound can be found for the relative error in any calculation involving more than one consecutive addition. The computed value of $a + b + c$ may be zero when the true value is not (or vice versa) if c and $a + b$ are of opposite sign and both much smaller numerically than $|a| + |b|$.

§3. A Method for Triangulation of Certain Matrices.

In Chapter 2 §3 it was required to solve an equation of the type (3.1) where in this case P and Q were given by

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{m1} & \cdots\cdots & & P_{mm} \end{pmatrix} \qquad Q = \begin{pmatrix} Q_{11} & \cdots & Q_{1m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Q_{m1} & & Q_{mm} \end{pmatrix} \tag{3.8}$$

where the P_{ij} , Q_{ij} were triple diagonal matrices. This section describes a method for solving equations of this type when $m = 2$ (as is the case with the calculations described in Chapter 7 where this method was used) which is readily generalised for other m .

Theoretically the equation could have been solved by applying Givens' method to $C^{-1} P C^{-1}$ where $C^2 = Q$, but to do this with matrices of the size used in Chapter 7 would have required more storage than is available on an IBM 1620 . Also the number of calculations would have been excessively large which would be wasteful of time and tend to increase rounding errors. Furthermore the method described, unlike Givens' method, does not require P and Q to be symmetric.

First $\det(P - \lambda Q)$ was computed for increasing λ until a change of sign was obtained. Interpolation was then used to obtain the eigenvalue λ , and the corresponding eigenvector was calculated by the method of §4 of this Chapter. Calculation of the eigenvector and evaluation of the determinant involved

triangulation, which for a matrix of the form (3.8) can be done as follows.

Let $R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$ where

$$R_i = \begin{pmatrix} p_{i1} & q_{i1} \cdots\cdots & & \\ r_{i1} & p_{i2} & \ddots & \\ \cdot & & & \\ \cdot & & p_{in-1} & q_{in-1} \\ \cdot & & r_{in-1} & p_{in} \end{pmatrix} \quad i = 1,\ldots,4 .$$

First eliminate r_{11} and r_{21} simultaneously by subtracting from the second row an appropriate linear combination of the first and $(n+1)$th rows of R. This can always be done provided

$$p_{11}p_{41} - p_{21}p_{31} \neq 0 \qquad (3.9)$$

The appropriate combination of the first and $(n+1)$th rows is then given by Cramer's rule. The restriction (3.9) is analogous to the condition $p_{11} \neq 0$ required to eliminate r_{11} separately by subtracting a suitable multiple of the first row from the second. In the standard case this restriction is overcome simply by an interchange of rows. Here the problem is more difficult. Fortunately in practice (3.9) is seldom violated. (In the calculations required for this thesis a condition of this kind was required well over 100,000 times and was always satisfied. This is not surprising as $p_{11}\,p_{41}$ and $p_{21}\,p_{31}$ contained 8 significant figures and

were not necessarily of the same order of magnitude.) In the programme used in this case, where $R = P - \lambda Q$ and the relevant elements of Q (i.e. those corresponding to r_{ij}, p_{ij}, q_{ij}) are all non-zero, the following device was used. If (3.9) (or corresponding conditions required later) were not satisfied, the trial λ was to be changed slightly and a message printed on the typewriter of the computer. If necessary this procedure could be repeated. The slight change in λ would not have mattered as the method described in §4 calculates very accurate eigenvectors from quite crude eigenvalues, and the final calculation of accurate eigenvalues was made from these vectors.

Having eliminated r_{11} and r_{21}, eliminate r_{31} and r_{41} simultaneously again using a combination of the first and $(n+1)$th rows. Then eliminate r_{12} and r_{22} using the second and $(n+2)$th rows. Then eliminate r_{32} and r_{42} and so on, until all r_{ij} have been eliminated. Next eliminate $q_{1,n-1}$ and $q_{2,n-1}$ simultaneously by subtracting a linear combination of the nth and 2nth rows from the $(n-1)$th. Then eliminate $q_{3,n-1}$ and $q_{4,n-1}$, then $q_{1,n-2}$ and $q_{2,n-2}$ and so on until all the q_{ij} have been eliminated (in decreasing order of j). Then eliminate the p_{3j} in the usual manner. The calculations become increasingly simple as the number of zeros increases. The final matrix is of the form $\begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}$ where the D_i are pure diagonal and this simple form facilitates the calculation of eigenvectors.

With the method of calculating $\det(P - \lambda Q)$ for arbitrary increases in λ there is a risk that some pair of consecutive eigenvalues may be missed. The following devices were used to prevent this.

(a) The step length $\lambda_{i+1} - \lambda_i$ was increased with λ_i as lower eigenvalues were more closely grouped. The choice of step length could be changed by use of a sense switch and further change was possible by changing a single instruction in the programme.

(b) By means of a sense switch a negative step length could also be used if it was felt an eigenvalue had been missed.

(c) By means of a sense switch any new trial value of λ could be read in and steps started again from that point. / As the number of eigenvalues and their product are known, it is easy to check where an isolated pair of eigenvalues may have been omitted.

§4. Calculation of Eigenvectors.

The calculation of the eigenvectors of (3.1) when λ is known approximately (even to 8 decimal places the maximum possible accuracy with single precision arithmetic on an IBM 1620) can give trouble with large matrices, unless some care is taken. Wilkinson [14] considered the case $Q = I$ and P symmetric triple diagonal and showed that the obvious method of solving $(n - 1)$ of the equations $\Sigma_j p_{ij}x_j = \lambda \Sigma_j q_{ij}x_j$ and hoping that the other one is satisfied (as it would if the exact λ were used and there were no rounding errors) could lead to hopelessly inaccurate results even for quite simple-looking matrices. He showed that even in the absence of rounding errors, very accurate approximate eigenvalues could give hopeless results by this method, and stated that in his experience such inaccuracy occurred with most large matrices. He suggested an alternative method which is easily modified for more general P and Q.

The only condition is that the eigenvectors $\underset{\sim}{v}_i$ of (3.1) must span the space of n-vectors, where P and Q are $n \times n$ matrices. Then, since (3.1) has at most n eigenvectors, they form a basis for this space and any n-vector $\underset{\sim}{x}_o$ can be expressed uniquely in the form

$$\underset{\sim}{x}_o = \sum_{i=1}^{n} \alpha_i \underset{\sim}{v}_i \qquad (3.10)$$

where the α_i are scalars.

This condition is not very restrictive. The condition that P

and Q be symmetric (not necessarily triple diagonal) and Q positive definite, as happens in the case considered, is sufficient. For as Q is symmetric and positive definite there is a non-singular symmetric matrix C such that $C^2 = Q$ Then $P\underset{\sim}{v}_i = \lambda_i Q \underset{\sim}{v}_i$ can be written as $C^{-1} P C^{-1}(C\underset{\sim}{v}_i) = \lambda_i(C\underset{\sim}{v}_i)$. As $C^{-1} P C^{-1}$ is a symmetric matrix with eigenvectors $C\underset{\sim}{v}_i$, the $C\underset{\sim}{v}_i$ span the space of n-vectors. Then for any n-vector $\underset{\sim}{x}$, $C\underset{\sim}{x}$ may be written in the form $C\underset{\sim}{x} = \Sigma_{i=1}^{n} \beta_i(C\underset{\sim}{v}_i)$ and hence, as C is non-singular, any n-vector $\underset{\sim}{x}$ can be written in the form $\underset{\sim}{x} = \Sigma_{i=1}^{n} \beta_i \underset{\sim}{v}_i$. Thus the $\underset{\sim}{v}_i$ span the space of n-vectors. Q.E.D.

Another sufficient condition for the $\underset{\sim}{v}_i$ to span the space is that the eigenvalues of (3.1) be distinct. Consideration of the polynomial $\det(P - \lambda Q)$ shows that as (3.1) has n distinct eigenvalues, Q is non-singular. Hence $Q^{-1} P$ has n distinct eigenvalues and hence its eigenvectors span the space of n-vectors ([15] p. 95) . But the eigenvectors of $Q^{-1} P$ are precisely those of (3.1) . Q.E.D.

Theorem 3.

Let P and Q be any $n \times n$ matrices such that the eigenvectors of (3.1) span the space of n-vectors. Then for any n-vector $\underset{\sim}{x}_0$, the sequence $\{\underset{\sim}{x}_r\}$ defined by

$$(P - \lambda_0 Q)\underset{\sim}{x}_r = Q\underset{\sim}{x}_{r-1} \qquad (3.11)$$

converges to the eigenvector $\underset{\sim}{v}_m$ (which is arbitrary to within a

scalar multiple) corresponding to whichever eigenvalue (λ_m say) is closest to λ_o, provided only that α_m defined by (3.10) is non-zero.

Proof.

Since the eigenvectors $\underset{\sim}{v}_i$ of (3.1) span the space of n-vectors

$$\underset{\sim}{x}_o = \sum_{i=1}^{n} \alpha_i \underset{\sim}{v}_i \quad . \tag{3.10}$$

Now for all i, $(P - \lambda_o Q)\underset{\sim}{v}_i = (\lambda_i - \lambda_o)Q\underset{\sim}{v}_i$ and hence

$$(\lambda_i - \lambda_o)^{-1}\underset{\sim}{v}_i = (P - \lambda_o Q)^{-1}Q\underset{\sim}{v}_i \quad . \tag{3.12}$$

Make the inductive hypothesis that

$$\underset{\sim}{x}_r = \sum_{i=1}^{n} \frac{\alpha_i}{(\lambda_o - \lambda_i)^r} \underset{\sim}{v}_i \quad . \tag{3.13}$$

Then by (3.11)

$$\underset{\sim}{x}_{r+1} = (P - \lambda_o Q)^{-1}Q\underset{\sim}{x}_r$$

$$= \sum_{i=1}^{n} \frac{\alpha_i}{(\lambda_o - \lambda_i)^r} (P - \lambda_o Q)^{-1}Q\underset{\sim}{v}_i \quad , \text{ by (3.13)}$$

$$= \sum_{i=1}^{n} \frac{\alpha_i}{(\lambda_o - \lambda_i)^{r+1}} \underset{\sim}{v}_i \quad \text{by (3.12)} \quad .$$

Now (3.13) holds for $r = 0$, by (3.10), and hence by induction for all positive integers r. But by hypothesis $\alpha_m \neq 0$, and also $|\lambda_o - \lambda_m| < |\lambda_o - \lambda_i|$, for all $i \neq m$. Thus for all positive ε, there exists an integer R such that for all $i \neq m$ and $r > R$, $\left| \frac{\alpha_i}{(\lambda_o - \lambda_i)^r} \right| < \varepsilon \left| \frac{\alpha_m}{(\lambda_o - \lambda_m)^r} \right|$. Thus as $r \longrightarrow \infty$, $x_r \longrightarrow v_m$. Q.E.D.

Note.

(1) Provided $|\lambda_o - \lambda_m| << |\lambda_o - \lambda_i|$ for all $i \neq m$, convergence will be rapid. For example see Table 5. Similar convergence was obtained in general.

(2) Even if (as is extremely unlikely) $\alpha_m = 0$ this does not matter a great deal in practice, as rounding errors will soon introduce a component of $\underset{\sim}{x}_r$ along $\underset{\sim}{v}_m$. The only disadvantage in this case is that a few more iterations will be required than with a more fortunate choice of $\underset{\sim}{x}_o$.

Practical Procedure.

Start with any $Q\underset{\sim}{x}_o$ (note that $\underset{\sim}{x}_o$ need not be calculated) and solve (3.11) by triangulation where λ_o is an approximate eigenvalue. For the calculation of each $\underset{\sim}{x}_r$ the triangulated form of $P - \lambda_o Q$ will be the same, so that for subsequent $\underset{\sim}{x}_r$ all that is required are the row operations on $\underset{\sim}{x}_{r-1}$ (the same each time) and matrix multiplication by Q. Since repeated iterations are easy, careful choice of $\underset{\sim}{x}_o$ is of secondary importance. In our calculations $Q\underset{\sim}{x}_o$ was always taken as the vector whose elements are all equal.

Before calculating $\underset{\sim}{x}_{r+1}$ it is advisable to normalise $\underset{\sim}{x}_r$. This enables convergence to show more clearly and avoids the production of very large numbers.

Although this method eliminates errors due to inaccurate λ_o, there remains the problem of rounding errors in solving (3.11). The computed values of $\underset{\sim}{x}_r$ may converge (as the equation for $\underset{\sim}{x}_{r+1}$ is little different from that for $\underset{\sim}{x}_r$) and yet be in error. As a check on the accuracy of the method the ratios $r_i = (\Sigma_j p_{ij}x_j)/(\Sigma_j q_{ij}x_j)$ were calculated in the calculations described in Chapter 7 §2. Eigenvalues calculated from these ratios are shown in Tables 7 and 8. In almost all cases the r_i were so close that λ_o lay outside the interval $(\inf_i r_i, \sup_i r_i)$. Hence the method is good for improving crude estimates of eigenvalues. The calculations used 8 digit floating point arithmetic. An approximate solution to a matrix equation need not always be a good approximation to a true solution (and may exist when a true solution does not exist.) But in general good agreement of the r_i suggests that the vectors are fairly accurate. Indeed small errors in the vectors can cause large errors in the r_i, as the proportionate error in the sum of two slightly inaccurate numbers of nearly equal magnitude can be large.

A further useful refinement was suggested by Wilkinson to reduce the effect of rounding errors when $Q = I$. This may also be used for more general Q. After sufficient convergence

has been obtained define $\underset{\sim}{\delta}$ by

$$\underset{\sim}{\delta} = (Q\underset{\sim}{x}^*_{n-1} - (P - \lambda_o Q)\underset{\sim}{x}^*_n)^* \qquad (3.14)$$

where the asterisk * indicates the computed value of a quantity. In general the vector $\underset{\sim}{\delta}$, which is due to rounding errors in computing $\underset{\sim}{x}^*_n$ from $\underset{\sim}{x}^*_{n-1}$, will be very small. Solve

$$(P - \lambda_o Q)\underset{\sim}{y} = \underset{\sim}{\delta} \qquad (3.15)$$

Then $\underset{\sim}{x}^*_n + \underset{\sim}{y}$ will be a more accurate eigenvector than $\underset{\sim}{x}^*_n$. Since in general $\underset{\sim}{y}$ will be very small compared with $\underset{\sim}{x}^*_n$, rounding errors in solving (3.15) are not important. This refinement was not used in the present calculations.

§5. The Work of Wilkinson.

After the writing of this thesis was nearly complete I learned that the main results of Theorems 1 and 3 of this Chapter had already been discovered by Wilkinson ([4]p. 340) although I had used the techniques described in December 1964, before his book was published. Wilkinson's discussion does not contain all the detail mentioned here and he makes no mention of condition (2). He mentions the main case where condition (1) holds for non-symmetric P on p.336. It is possible that Theorem 2 is also known but I have not seen a proof of it anywhere.

Although Wilkinson does not claim his treatise is comprehensive it gives an excellent account of the main results in the field with considerable emphasis on error analysis and other practical criteria. It also contains a good bibliography.

CHAPTER 4

THE STELLAR MODEL.

The techniques described in the preceding chapters will now be applied to the study of oscillations of stars.

The calculations used a stellar model, constructed by Van der Borght [16], of a massive star of uniform composition, but the same techniques could be used for any model.

A very good account of the theory of stellar structure is given by Schwarzschild [17]. This describes both observational and theoretical knowledge, gives some numerical examples, and has a good bibliography. A more mathematical account is given by Chandrasekhar [18]. This also has good bibliographical notes. The system of differential equations used in [16] is derived from the principles outlined in these books.

In addition to the gas pressure given by

$$p_G = \rho \mathcal{R} T/\mu \quad , \tag{4.1}$$

where ρ = density, T = temperature, μ = molecular weight and $\mathcal{R}$ is the gas constant, it is necessary to consider pressure due to radiation. (An atom loses momentum when it emits a photon) Although negligible for small stars, radiation pressure is quite important for massive stars and is given by

$$p_R = \frac{a}{3} T^4 \qquad (4.2)$$

where a is the Stefan-Boltzman constant. (See [17] p. 39.)
Total pressure is given by

$$p = p_G + p_R \qquad (4.3)$$

Chandrasekhar ([18] p. 55) shows that the well-known adiabatic relation between pressure and density may be generalised to the case $p_R \neq 0$ where it becomes

$$\frac{1}{p}\frac{dp}{dt} = \frac{\Gamma_1}{\rho}\frac{d\rho}{dt} \qquad (4.4)$$

where t is time and

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)} \quad , \qquad (4.5)$$

$$\beta = \frac{p_G}{p} \qquad (4.6)$$

and γ is the ratio of specific heats which in this model has the value $\frac{5}{3}$, as gases will be monatomic at the temperatures considered. When $\beta = 1$, (4.4) assumes the familiar form. Chandrasekhar gives similar generalisations for adiabatic relations between temperature and pressure or density.

The model considered consists of two layers: a core in which energy is transported entirely by convection and an envelope in which the energy is transported entirely by radiation. The core is assumed to contain all the nuclear energy generation, so that the luminosity equation can be solved independently of the other three equations of the system. The model is spherically symmetric and the condition for radiative equilibrium, which holds throughout the envelope, is

$$A < 0 \tag{4.7}$$

where

$$A = \frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{\Gamma_1 p}\frac{dp}{dr} \tag{4.8}$$

and r is the distance from the centre. The traditional justification of (4.7) due to K. Schwarzschild is given in [17]p.44. Throughout the core $A = 0$.

At the temperatures prevailing in massive stars, the contributions to opacity of bound-free and free-free transitions are small compared with the contribution of electron scattering. Neglecting these first two small effects, opacity is given by $0{\cdot}19(1 + X)$ (see [17]p. 71) where X is the proportion by mass of hydrogen (constant in this case). Opacity is therefore taken as constant in this model. Explicit reference to μ may then be eliminated from the equations [19] and one set of solutions covers a whole family of models. Nevertheless

Boury [20] has warned that the effects of small changes in opacity may not be negligible.

In [16] the following variables are used

$$\bar{p} = R^4 \mu^4 p/M_o^2 \qquad \bar{t} = R\mu T/M_o$$

$$\bar{m}(x) = \mu^2 m(r)/M_o \qquad x = r/R \tag{4.9}$$

where $m(r)$ is the mass contained within a radius r, M_o is the mass of the sun and R the (unknown) radius of the star. This choice of variables ensures that μ and R do not enter into the differential equations. (But M_o is merely a convenient scaling factor to prevent overflow in the floating decimals.)

In a massive star composed of pure hydrogen, $\mu = \frac{1}{2}$ effectively, as at the temperatures prevailing, except at the surface, the gas is completely ionised. In this model μ is assumed constant. The mass of the star is not specified in this model but $\bar{m}(1)$ is taken as 10, so that if the star is pure hydrogen, the mass is 40 times that of the sun.

From (4.1), (4.2), (4.3), (4.6) and (4.9)

$$\bar{p} = \frac{a M_o^2 \bar{t}^4}{3(1-\beta)} \tag{4.10}$$

and

$$\rho = \frac{M_o}{R^3 \mu^2 \mathcal{R}} \frac{\beta \bar{p}}{\bar{t}} \tag{4.11}$$

Van der Borght used the basic equations in the following form. Throughout the star

$$\frac{d\bar{m}}{dx} = 4\pi x^2 \frac{\beta\bar{p}}{\Re\,\bar{t}} \tag{4.12a}$$

from continuity and

$$\frac{d\bar{p}}{dx} = -\frac{\beta\bar{p}\,G\bar{m}}{\Re\bar{t}\,x^2} \tag{4.13}$$

from the hydrostatic equation. In the core there are the further equations

$$\frac{d\bar{t}}{dx} = -\frac{2(4-3\beta)\beta G\bar{m}}{(32-3\beta^2-24\beta)\Re x^2} \tag{4.14a}$$

from the energy transport equation and

$$\frac{d\beta}{dx} = \frac{3G\bar{m}\beta^3(1-\beta)}{\Re\bar{t}\,x^2(32-24\beta-3\beta^2)} \tag{4.15a}$$

where G is the gravitational constant. Only three of the last four equations are independent and (4.15a) was not used in the calculations. In the envelope the equations corresponding to (4.14a) and (4.15a) are

$$\frac{d\bar{t}}{dx} = -\frac{G\bar{m}(1)\,(1-\beta_R)\beta}{4\Re(1-\beta)x^2} \tag{4.14b}$$

and

$$\frac{d\beta}{dx} = \frac{\beta\,G}{\Re\bar{t}x^2}\left((1-\beta_R)\bar{m}(1) - (1-\beta)\bar{m}(x)\right) \tag{4.15b}$$

where β_R is the value of β at the surface. Again only three equations are independent and (4.13) was not used in the envelope. Since it varies more slowly in the envelope than $\bar{p}$, β proved more satisfactory to use as independent variable in the envelope where (4.12a) was used in the equivalent form

$$\frac{d\bar{m}}{dx} = 4\pi x^2 \frac{aM_0^2 \bar{t}^3 \beta}{3\mathcal{R}(1-\beta)} \qquad (4.12b)$$

The above equations were solved with the boundary conditions

$$\bar{m}(0) = 0 \qquad (4.16a)$$

$$\bar{t} = 0\,, \quad \bar{m} = \bar{m}(1) \text{ and } \beta = \beta_R \text{ at } x = 1 \qquad (4.16b)$$

where $\bar{m}(1)$ is a prescribed constant and β_R an eigenvalue. Since ρ is bounded and continuous, (4.11), (4.12a) and (4.16a) show that $\bar{m}/x^3$ is bounded.

The boundary between the core and the envelope was given by the smallest radius for which (4.14b) gave a lower value for $|\frac{d\bar{t}}{dx}|$ than (4.14a). (See [18] p. 224.) In this model $x = 0{:}391641$ at the boundary. Continuity of $\bar{m}$, β, $\bar{t}$ (and hence by (4.10) of $\bar{p}$) across the boundary sufficed to determine the eigenvalue β_R. In this model $\beta_R = 0{:}903555$ and at the centre $\beta = 0{\cdot}794155$.

Van der Borght determined the values of $\bar{m}$, $\bar{p}$, $\bar{t}$ and β to 5 significant figures at points distance $x = 0{\cdot}01$ apart, from

$x = 0$ to $x = 0{\cdot}99$ and also at $x = 0{\cdot}999$. He used series expansion at the boundaries and a Runge-Kutta integration. As the detailed results are unpublished, a complete listing of x , $\overline{m}$, $\overline{p}$, $\overline{t}$, β for the model used is given in Table 1. It is these results that were used in the calculations described in the following chapters.

Most previous work on stellar oscillations and, as far as I know, all numerical work using variational methods has been concerned with polytropic models. A good account of such models is given in [18] (pp84-182). An important case of a polytropic model is Eddington's "standard model" ([18] p. 228). These models are quite good approximations but do not give a good picture of the relative importance of convective and radiative energy transport in different regions of the star. The model used here is probably more accurate in this respect.

CHAPTER 5

NON-RADIAL OSCILLATIONS OF STARS

§1. Introductory Remarks and Derivation of the Equations.

The theory of small oscillations occupies a central role in the theory of stellar stability [21] . If an oscillation of the form $f(\underset{\sim}{r})e^{i\sigma t}$ is assumed, linearised theory gives an equation of the form $\sigma^2 X = Y$. Clearly if σ^2 can be negative the configuration is unstable and most workers have not enquired much further. Most work has been done on the fundamental mode. However, as well as contributing to a more complete picture of observed variations of certain stars, higher modes prove useful for investigation of non-linear [22] and non-adiabatic effects which become important if instability exists. There has also been very little investigation of non-exponential growth of disturbances.

This thesis investigates the feasibility of using variational methods, whose application so far has been mainly confined to the fundamental mode, to determine higher modes, especially when quick estimates are required.

A very good account of the theory of stellar oscillations is given by Ledoux and Walraven [23] . As this also has a good bibliography, only papers directly relevant to the present work will be cited here. A more recent account of the theory of stellar stability and the role of small oscillations in this theory is

given by Ledoux in [24] (pp499-574). This again has a good bibliography. Other chapters of [24] give good accounts of other important topics in the theory of stellar structure.

Here we shall be concerned with the linearised theory of non-radial adiabatic oscillations of a non-viscous, non-rotating (in fact non-accelerating) gaseous sphere on which the only external force is its own gravity, and where magnetic fields and relativistic effects are ignored. These conditions simplify the equations considerably and have been assumed by most workers, although some work has been done to estimate the errors introduced by the simplifications. Some of this is reviewed in [23].

As the differential equations of motion are partial, the usual method of separation of variables is used in [23]. Let ρ_o , p_o , ϕ_o denote the equilibrium values of density ρ , pressure p and gravitational potential ϕ , and $\delta'\rho$, $\delta'p$, $\delta'\phi$ the Eulerian variation of these quantities resulting from a Lagrangian displacement $\delta'\underset{\sim}{r}$, so that $\rho = \rho_o + \delta'\rho$ etc. Although we do not here assume that oscillations are radial, we shall, following [23], consider only a rather limited class of non-radial oscillations, namely those described by

$$\begin{aligned}
\delta'\underset{\sim}{r}(\underset{\sim}{r},t) &= \underset{\sim}{\delta r}(r)Y_\ell^m(\theta,\varphi)e^{i\sigma t} \\
\delta'\rho(\underset{\sim}{r},t) &= \delta\rho(r)Y_\ell^m(\theta,\varphi)e^{i\sigma t} \\
\delta'p(\underset{\sim}{r},t) &= \delta p(r)Y_\ell^m(\theta,\varphi)e^{i\sigma t} \\
\delta'\phi(\underset{\sim}{r},t) &= \delta\phi(r)Y_\ell^m(\theta,\varphi)e^{i\sigma t}
\end{aligned} \tag{5.1}$$

where r , θ , φ are spherical polar coordinates of position, t is time and $Y_\ell^m(\theta,\varphi) = P_\ell^m(\cos\theta)e^{im\varphi}$ satisfies

$$\frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2 Y_\ell^m}{\partial\varphi^2} + \frac{1}{\mathrm{Sin}\theta}\frac{\partial}{\partial\theta}\left(\mathrm{Sin}\theta\frac{\partial Y_\ell^m}{\partial\theta}\right) + \ell(\ell+1)Y_\ell^m = 0 \qquad (5.2)$$

The P_ℓ^m are associated Legendre functions.

As stated in [23] (p. 509) , $\ell = 0$ gives purely radial oscillations which are dealt with in Chapter 6, and $\ell = 1$ corresponds to a shifting of the centre of gravity. Cases $\ell \geq 2$ are thus more interesting.

For adiabatic oscillations (4.4) holds, and when linearised this gives, since p_o , ρ_o are functions of r only,

$$\delta p + \delta r\frac{dp_o}{dr} = \frac{\Gamma_1 p_o}{\rho_o}\left(\delta\rho + \delta r\frac{d\rho_o}{dr}\right) \qquad (5.3)$$

Also Poisson's equation for gravitational potential holds for both the equilibrium and the perturbed states. Subtraction and linearisation then gives

$$\nabla^2\delta'\phi = 4\pi G\delta'\rho \qquad (5.4)$$

and hence by (5.1) ,

$$\frac{d^2}{dr^2}\delta\phi + \frac{2}{r}\frac{d}{dr}\delta\phi - \frac{\ell(\ell+1)}{r^2}\delta\phi = 4\pi G\delta\rho \ . \qquad (5.5)$$

The equation of continuity, which depends only on the conservation

of mass and is independent of the other assumptions listed above, becomes when linearised

$$\delta'\rho + \nabla.(\rho_o \delta' \underset{\sim}{r}) = 0 \quad . \tag{5.6}$$

There remains the Euler momentum equation which, when linearised becomes, by (5.1)

$$\sigma^2 \delta' \underset{\sim}{r} = \nabla \delta'\phi - \frac{\delta'\rho}{\rho_o^2} \nabla p_o + \frac{1}{\rho_o} \nabla \delta' p \tag{5.7}$$

Using (5.3) and (5.6) , (5.7) can, as shown in [23] , be written

$$\sigma^2 \delta' \underset{\sim}{r} = \nabla \left(\delta'\phi + \frac{\delta' p}{\rho_o}\right) - \underset{\sim}{A} \frac{\Gamma_1 p_o}{\rho_o} \nabla.\delta' \underset{\sim}{r} \tag{5.8}$$

where $\underset{\sim}{A}$ is a vector with radial component only and $|\underset{\sim}{A}| = A$ defined by (4.8). Since $\delta' \underset{\sim}{r}$ is the sum of a conservative function and a purely radial vector function, its transverse component can be represented by a single scalar function, and (5.8) suggests that $\delta\phi + \frac{\delta p}{\rho_o}$ would be a suitable choice of second dependent variable. The independent variables chosen in [23] are

$$\psi = r^3 \delta r_r \tag{5.9}$$

which represents the radial component and

$$y = \frac{\delta p}{\rho_o} \tag{5.10}$$

where δr_r , δr_θ and δr_φ are the components of $\delta\underset{\sim}{r}$ in the directions of r , θ and φ respectively increasing.

Assuming $\delta\rho$ and the first two derivatives of $\delta\phi$ are bounded at $r = 0$, (5.5) shows that $\delta\phi = 0$ at $r = 0$. Assuming $\delta\underset{\sim}{r}$ is bounded at $r = 0$, then since $A = 0$ throughout the core, it follows from the transverse components of (5.8) that $y = 0$ also at $r = 0$. From the continuity of ϕ it is shown in [23] that $\frac{d}{dr}\delta\phi + \frac{\ell+1}{R}\delta\phi = 0$ at $r = R$. Also as the Lagrangian variation in pressure is zero at the surface,

$$\delta p = -\delta r \frac{\partial p_o}{\partial r} \quad \text{at} \quad r = R \tag{5.11}$$

In our model this becomes $\delta p = 0$ at $r = R$.

Using the fact that $\delta\underset{\sim}{r}$, $\delta\rho$, δp , $\delta\phi$, p_o and ρ_o are functions of r alone and writing

$$g = \frac{Gm}{r^2} \left(= \frac{1}{\rho_o}\frac{dp_o}{dr} \quad \text{when} \quad \rho_o \neq 0\right) \tag{5.12}$$

Ledoux and Walraven obtain from the preceding equations the following fourth order system of linear differential equations:

$$\frac{d\psi}{dr} - \frac{\rho_o g}{\Gamma_1 p_o}\psi = \left(\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho_o r^2}{\Gamma_1 p_o}\right)y + \frac{\ell(\ell+1)}{\sigma^2}\delta\phi \tag{5.13a}$$

$$\frac{dy}{dr} + yA = \frac{\sigma^2 + Ag}{r^2}\psi - \frac{d}{dr}\delta\phi \tag{5.13b}$$

$$\frac{d^2}{dr^2}\delta\phi + \frac{2}{r}\frac{d}{dr}\delta\phi - \frac{\ell(\ell+1)}{r^2}\delta\phi = 4\pi G\rho_o\left(\frac{\rho_o y}{\Gamma_1 p_o} - \frac{\psi A}{r^2}\right) \tag{5.13c}$$

to be solved with the boundary conditions

$$y = 0 , \quad \delta\phi = 0 \quad \text{at} \quad r = 0 \tag{5.14a}$$

$$y = \frac{g\psi}{R^2} , \quad \frac{d}{dr}\delta\phi + \frac{\ell+1}{R}\delta\phi = 0 \quad \text{at} \quad r = R \tag{5.14b}$$

Since $r^2\frac{dy}{dr} = 0$ at $r = 0$, it follows from (5.13b) and (5.14a) that provided $\sigma^2 \neq 0$, $\psi = 0$ at $r = 0$. This corresponds to the physical fact that δr_r must be bounded at $r = 0$.

If $\delta\phi$ is neglected, (5.13) simplifies considerably. Although this simplification may seem a little drastic it has been made by most workers. Cowling [3] justified the simplification on the grounds that since the mass is usually strongly concentrated near the centre of the star, variations in density do not produce great variations in ϕ in the outer regions, although these regions have considerable effect on the period. He suggested that (5.13,14) first be solved with $\delta\phi = 0$ and that the true solution be obtained by imposing small perturbations on this initial solution. If σ_n , $\underset{\sim}{\delta r}_n$ etc. represent the true solutions of (5.13,14) corresponding to the nth mode and $\bar{\bar{\sigma}}_n$, $\underset{\sim}{\bar{\bar{\delta r}}}_n$ etc. the solutions obtained by putting $\delta\phi = 0$ he showed that to the first order

$$(\sigma_n^2 - \bar{\bar{\sigma}}_n^2)\int_V \rho_o \, |\underset{\sim}{\bar{\bar{\delta r}}}_n|^2 \, dV = \int_V \bar{\bar{\delta\rho}}_n \, \delta\phi_n \, dV \tag{5.15a}$$

or using the notation introduced in §3 ,

$$(\sigma_n^2 - \bar{\bar{\sigma}}_n^2)\int_0^R \rho_o\left(\frac{\bar{\bar{\psi}}^2}{r^2} + \frac{(\bar{\bar{\chi}}')^2}{\ell(\ell+1)}\right)dr$$

$$= -4\pi G\int_0^R r^{2\ell}\left(\int_r^R \frac{d}{ds}(\rho_o\bar{\bar{\psi}}) - \rho_o\bar{\bar{\chi}}'\right)\frac{ds}{s^{\ell+1}}\Big)^2 dr \qquad (5.15b)$$

to the first order.

Van der Borght and Wan [25] have applied this correction to solutions of (5.13,14) obtained for this model with $\delta\phi = 0$ for the 9 lowest modes with $\ell = 2$. In all cases the result differed by less than 4% from the exact solution of (5.13,14). Each time the estimated correction was too small and in fact if (5.15b) indicated a ratio x of error to true value then the empirical correction $x + 3x^2$ gave a final error of less than 0.2%. This suggests a satisfactory higher order correction may exist.

Cowling noted that the error would decrease as the mode increased as the right hand side of (5.15a) will be smaller when $\overline{\overline{\delta\rho}}_n$ contains more zeros. In [23] it is noted that as $\nabla^2\delta\rho \sim \ell(\ell+1)\delta\rho$ and $\nabla^2\delta\phi \sim \delta\rho$, the error will decrease as ℓ increases. These two effects were noted by Sauvenier-Goffin [26] who obtained numerical solutions for the (unstable) compressible homogeneous model. She observed that the former effect was greater than the latter. Cowling's reasoning suggests also that error should decrease as central condensation increases.

In the case where $\delta\phi = 0$, (5.13,14) are simplified in [23] by putting

$$v = \psi p_o^{1/\Gamma_1} \tag{5.16a}$$

$$w = y\rho_o p_o^{-1/\Gamma_1} = \delta p\, p_o^{-1/\Gamma_1} \tag{5.16b}$$

Equation (5.13) then becomes

$$\frac{dv}{dr} = \left(\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho_o r^2}{\Gamma_1 p_o}\right) \frac{p_o^{2/\Gamma_1}}{\rho_o} w \tag{5.17a}$$

$$\frac{dw}{dr} = \frac{(\sigma^2+Ag)}{r^2} \frac{\rho_o}{p_o^{2/\Gamma_1}} v \tag{5.17b}$$

The boundary conditions become

$$v = 0 \quad (\text{and } w = 0) \quad \text{at} \quad r = 0 \tag{5.18a}$$

$$\frac{v}{w} \sim \frac{R^2}{g} \frac{p_o^{2/\Gamma_1}}{\rho_o} \; (\longrightarrow 0) \quad \text{as} \quad r \longrightarrow R \tag{5.18b}$$

Equations (5.17) are equivalent to either of the second order equations

$$\frac{d}{dr}\left(\frac{\rho_o}{p_o^{2/\Gamma_1}\left(\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho_o r^2}{\Gamma_1 p_o}\right)} \frac{dv}{dr}\right) = \frac{(\sigma^2+Ag)}{r^2} \frac{\rho_o}{p_o^{2/\Gamma_1}} v \tag{5.19a}$$

$$\frac{d}{dr}\left(\frac{p_o^{2/\Gamma_1} r^2}{\rho_o(\sigma^2+Ag)} \frac{dw}{dr}\right) = \left(\frac{\ell(\ell+1)}{\sigma^2} - \frac{\rho_o r^2}{\Gamma_1 p_o}\right) \frac{p_o^{2/\Gamma_1}}{\rho_o} w \tag{5.19b}$$

§2. Properties of the Equations.

It is clear from (5.19) that the eigenvalue σ^2 occurs in a non-linear fashion. Cowling [3] solved numerically a system of equations essentially equivalent to (5.17,18), but using different variables, for a polytropic model. He obtained two distinct families of eigenvalues. The oscillations associated with the first family, which he called p- or pressure modes, were mainly radial. In this case σ^2 tended monotonically to $+\infty$ as the number of zeros in $\underset{\sim}{\delta r}$ increased. With the second family, which he called g- or gravity modes, and which represented mainly transverse oscillations, σ^2 tended monotonically to $+0$ as the number of zeros in $\underset{\sim}{\delta r}$ increased. Between these two families was a single eigenvalue whose corresponding $\underset{\sim}{\delta r}$ had no zeros. This he termed the f- or fundamental mode. Numerical solutions of (5.17,18) and also of (5.13,14) with $\delta\phi \neq 0$ by subsequent workers using various models have also given these two families of eigenvalues. A review of some of this is given by Smeyers [27] who gives further numerical results.

There does not appear to be much theoretical work on equations of this type. Langer [28] has studied second order eigenvalue problems of the type

$$\underset{\sim}{v}'(x) - \Big(\sum_{n=1}^{m} \frac{1}{\lambda - a_n} Q_n(x) + Q_o(x)\Big)\underset{\sim}{v}(x) = 0 \qquad (5.20)$$

where the a_n are constants and the elements of the matrices

Q_n are functions of x only, with boundary conditions of the form

$$W_1(\lambda)\underset{\sim}{y}(0) + W_2(\lambda)\underset{\sim}{y}(1) = 0 \qquad (5.21)$$

where the elements of the matrices W_i are rational functions in λ with poles, if any, only at the a_n. He derived from (5.20) m "associated" equations by using transformations of the type

$$\rho_n = \frac{\beta_n}{\lambda - a_n} \qquad (5.22)$$

and then omitting terms of order $\frac{1}{\rho_n}$. He then showed under certain conditions (the $Q_n(x)$ must be sufficiently differentiable, the eigenvalues of each $Q_n(x)$, which are themselves functions of x, must be distinct, nonvanishing and of constant argument) that when the associated equations all had families of eigenvalues tending to ∞, then (5.20) had an infinite set of eigenvalues with cluster points at and only at the poles a_n of the coefficient matrix. A transformation of the type (5.22) with $n = 1$ transforms (5.20) with $a_a = 0$ into an equation similar to (5.17), but for (5.17) the Q_n do not have the required properties.

When higher order terms in σ^2 or in $\frac{1}{\sigma^2}$ are omitted from (5.19a) we obtain

$$\frac{d}{dr}\left(\frac{\rho_o}{\ell(\ell+1)} p_o^{-2/\Gamma_1} \frac{dv}{dr}\right) - \rho_o p_o^{-2/\Gamma_1}\left(1 + \frac{Ag}{\sigma^2}\right)\frac{v}{r^2} = 0 \qquad (5.23a)$$

$$\frac{d}{dr}\left(\frac{\Gamma_1}{r^2}\, p_o^{(1-2/\Gamma_1)}\, \frac{dv}{dr}\right) + \rho_o p_o^{-2/\Gamma_1}(Ag + \sigma^2)\frac{v}{r^2} = 0 \qquad (5.23b)$$

With the given boundary conditions equations (5.23a), (5.23b) each form a singular Sturm-Liouville system. A Sturm-Liouville system on a finite interval is an equation of the form $(f_1 x')' + (f_2 + \lambda f_3)x = 0$ to be solved with certain linear homogeneous boundary conditions in x and x' at a and b where f_1', f_2 and f_3 are real and continuous and $f_1 > 0$, $f_3 > 0$ on (a,b). If these conditions hold on $[a,b]$ the system is termed regular. A regular system has an infinite sequence $\lambda_o < \lambda_1 < \ldots$ of eigenvalues, all real and distinct, $\Sigma_{i=0}^{\infty} \frac{1}{\lambda_i}$ is convergent and the eigenfunction associated with λ_n has exactly n zeros in (a,b). There is an extensive literature on such systems. (See e.g. Ince [29], Tricomi [30] or [7].)

Cowling [3], without proof, gave the Sturm-Liouville form of (5.23) as the explanation of the two families of eigenvalues. Since the derivation of (5.23) from (5.19a) is similar to the derivation of Langer's "associated" equations from (5.20), this seems very plausible. And the properties of both sets of eigenfunctions of (5.17,18), which have been obtained numerically, are similar to those of Sturm-Liouville systems.

The position is complicated in this case by the singularities in (5.23) and in (5.17). The coefficients in (5.23) become zero or unbounded at the boundaries. The question of singularities in Sturm-Liouville systems arises again in simpler context in

Chapter 6. Much of the more detailed discussion given there is also relevant to (5.23). Unfortunately most papers do not state clearly what numerical methods were used and in particular how the singularities at the boundaries were treated. Van der Borght and Wan [25] used truncated series expansions at the boundaries and Runge-Kutta integration.

Another complication when our model is used is that throughout the core, which contains more than half the mass of the star, $A = 0$, so that one term in each of the equations is zero throughout this region. For this reason, an equation comparable to (5.23a) cannot be derived from (5.19b). An equation with parameter σ^2 as with (5.23b) can be obtained by neglecting Ag in comparison, but the approximation $Ag + \sigma^2 \simeq Ag$ for small σ^2, required to obtain the second system, is clearly not satisfactory. Also this would give the identically zero quantity Ag in the denominator. In Cowling's model, A is uniformly bounded above by a negative quantity throughout the model so that the assumption $\sigma^2 << |Ag|$ seems more reasonable.

When $A = 0$, (5.19b) gives

$$\frac{d}{dr}\left(\frac{p_0}{\rho_0}^{2/\Gamma_1} r^2 \frac{dw}{dr}\right) = \left(\ell(\ell+1) - \frac{\sigma^2 \rho_0 r^2}{\Gamma_1 p_0}\right) \frac{p_0}{\rho_0}^{2/\Gamma_1} w \qquad (5.24)$$

With appropriate boundary conditions this would form a Sturm-Liouville system with a single family of eigenvalues tending to $+\infty$. This does not happen with our model as $A < 0$ in the envelope and we

do not have boundary conditions of the required type at the boundaries of the core. However with the convective model the system is of the Sturm-Liouville type so that in the case $\delta\phi = 0$ there are no g-modes for that model.

Van der Borght and Wan [25] have solved the equations for the model used here, both with and without the simplification $\delta\phi = 0$, and have found both p- and g- modes in both cases. Their results are discussed in Chapter7 .

§3. Variational Formulation.

A variational formulation of the equations governing non-radial oscillations of the stars has been given by Chandrasekhar [31]. Instead of using y as dependent variable, he used $\chi' = \frac{d\chi}{dr}$ given by

$$\delta r_\theta = \frac{\chi'(r)}{\ell(\ell+1)r} \frac{\partial Y_\ell^m(\theta,\varphi)}{\partial \theta} \tag{5.25a}$$

From (5.1) and (5.8) it follows that

$$\delta r_\varphi = \frac{\chi'(r)}{\ell(\ell+1)r\sin\theta} \frac{\partial Y_\ell^m(\theta,\varphi)}{\partial \varphi} \tag{5.25b}$$

and

$$\chi' = \frac{\ell(\ell+1)}{\sigma^2} (y + \delta\phi) \tag{5.26}$$

Chandrasekhar used the same basic assumptions and equations as in [23]. By showing that

$$r^2\delta\rho + \rho_0(\psi' - \chi') + \rho_0'\psi = 0 \tag{5.27}$$

and using the fact that $\delta'\phi$ satisfies Poisson's equation and hence must be given by Poisson's integral formula, and using the expansion of $|\underset{\sim}{r} - \underset{\sim}{r}^1|^{-1}$ in spherical harmonics, he showed that a solution of the system must extremise σ^2 given by

$$\sigma^2(I\underset{\sim}{\delta r},\underset{\sim}{\delta r}) = (B\underset{\sim}{\delta r},\underset{\sim}{\delta r}) + (C\underset{\sim}{\delta r},\underset{\sim}{\delta r}) \tag{5.28}$$

where

$$(I\,\underset{\sim}{\delta r}_1, \underset{\sim}{\delta r}_2) = \int_0^R \rho_0 \left(\frac{\psi_1\psi_2}{r^2} + \frac{\chi_1'\chi_2'}{\ell(\ell+1)}\right) dr \qquad (5.29a)$$

$$(B\,\underset{\sim}{\delta r}_1, \underset{\sim}{\delta r}_2) = \int_0^R \left(\Gamma_1 p_0 \left(\frac{dp_0}{dr}\frac{\psi_1}{\Gamma_1 p_0} + \psi_1' - \chi_1'\right)\left(\frac{dp_0}{dr}\frac{\psi_2}{\Gamma_1 p_0} + \psi_2' - \chi_2'\right) + A\frac{dp_0}{dr}\psi_1\psi_2\right)\frac{dr}{r^2} \qquad (5.29b)$$

$$(C\,\underset{\sim}{\delta r}_1, \underset{\sim}{\delta r}_2) = \int_V \delta\rho_1\delta\phi_2 dv = \int_V \delta\rho_2\delta\phi_1 dv$$

$$= -4\pi G\int_0^R r^{2\ell}\left(\int_r^R \left((\rho_0\psi_1)'(s) - \rho_0\chi_1'(s)\right)\frac{ds}{s^{\ell+1}}\right)\left(\int_r^R (\rho_0\psi_2)'(s) - \rho_0\chi_2'(s)\frac{ds}{s^{\ell+1}}\right)dr \qquad (5.29c)$$

and ψ_i , χ_i' , $\delta\rho_i$, $\delta\phi_i$ correspond to an eigenfunction $\underset{\sim}{\delta r}_i$. Hence if $\underset{\sim}{\delta r}_1$, $\underset{\sim}{\delta r}_2$ are eigenfunctions corresponding to distinct eigenvalues

$$(I\,\underset{\sim}{\delta r}_1, \underset{\sim}{\delta r}_2) = 0 . \qquad (5.30)$$

This formulation requires ρ_0 and $\frac{dp_0}{dr}$ to vanish at $r = R$ and uses $\rho_0 y = -\frac{dp_0}{dr}\frac{\psi}{R^2}$ instead of $y = \frac{g\psi}{R^2}$ as boundary condition. Chandrasekhar wrote the expression $(B\,\underset{\sim}{\delta r}, \underset{\sim}{\delta r})$ differently but the above form is used as it makes clearer the positive nature of B. (In our model $A \leq 0$ and $\frac{dp_0}{dr} \leq 0$ throughout.)

Putting $\delta\phi = 0$ gives rise to the variational problem of extremising σ^2 given instead by

$$\sigma^2(I\,\underset{\sim}{\delta r},\underset{\sim}{\delta r}) \;=\; (B\,\underset{\sim}{\delta r},\underset{\sim}{\delta r}) \tag{5.31}$$

The Euler-Lagrange equations obtained from (5.31) by considering variations in χ' and ψ are

$$\frac{\sigma^2}{\ell(\ell+1)}\,\chi' \;=\; -\frac{1}{\rho_o r^2}\left(\frac{dp_o}{dr}\,\psi + \Gamma_1 p_o(\psi' - \chi')\right) \tag{5.32a}$$

$$\frac{\sigma^2}{r^2}\,\psi \;=\; -\frac{d}{dr}\left(\frac{1}{\rho_o r^2}\left(\frac{dp_o}{dr}\,\psi + \Gamma_1 p_o(\psi' - \chi')\right)\right) - \frac{A p_o \Gamma_1}{\rho_o r^2}\,(\psi' - \chi') \tag{5.32b}$$

Substituting for $\frac{1}{\rho_o r^2}\left(\frac{dp_o}{dr}\,\psi + \Gamma_1 p_o(\psi' - \chi')\right)$ from (5.32a) in (5.32b) and equating the expression so obtained for $\frac{d\psi}{dr}$ with that from (5.32a) gives an expression for $\frac{d\chi'}{dr}$ as a linear combination of ψ and χ'. The equations may be written in standard matrix form

$$\begin{pmatrix} \psi \\ \chi' \end{pmatrix}' \;=\; \begin{pmatrix} -\frac{1}{\Gamma_1 p_o}\frac{dp_o}{dr} & 1 - \frac{\sigma^2 \rho_o r^2}{\Gamma_1 p_o \ell(\ell+1)} \\ \frac{\ell(\ell+1)}{r^2}\left(1 - \frac{A}{\rho_o \sigma^2}\frac{dp_o}{dr}\right) & -A \end{pmatrix} \begin{pmatrix} \psi \\ \chi' \end{pmatrix} \tag{5.33}$$

Denote the equations represented by the first and second elements of these vectors by (5.33a) and (5.33b) respectively. In this second order system, σ^2 occurs in a non-linear manner. Equation (5.26) shows (5.33) to be equivalent to (5.17). Thus the apparently linear occurrence of σ^2 in (5.31) is deceptive.

The corresponding equations for (5.28) are more complicated. Using (5.1), (5.3), (5.6), (5.8), (5.25) and (5.27) it can be shown that

$$\frac{\sigma^2}{\ell(\ell+1)}\chi' = \delta\phi - \frac{1}{\rho_o r^2}\left(\frac{dp_o}{dr}\psi + \Gamma_1 p_o(\psi' - \chi')\right) \qquad (5.34a)$$

and

$$\frac{\sigma^2}{r^2}\psi = \frac{d}{dr}\left(\delta\phi - \frac{1}{\rho_o r^2}\left(\frac{dp_o}{dr}\psi + \Gamma_1 p_o(\psi' - \chi')\right)\right) - \frac{Ap_o\Gamma_1}{\rho_o r^2}(\psi' - \chi') \qquad (5.34b)$$

These must be considered with Poisson's equation

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\delta\phi\right) - \ell(\ell+1)\delta\phi + 4\pi G\left(\frac{d\rho_o}{dr}\psi + \rho_o(\psi' - \chi')\right) = 0 \qquad (5.34c)$$

Simplifying as before we may write these equations as

$$\underset{\sim}{x}' = M\,\underset{\sim}{x} \qquad (5.35)$$

where $\underset{\sim}{x}^T = (\psi, \chi', \delta\phi, r^2\frac{d}{dr}\delta\phi)$ and

$$M = \begin{pmatrix} -\frac{1}{\Gamma_1 p_o}\frac{dp_o}{dr} & 1 - \frac{\sigma^2\rho_o r^2}{\Gamma_1 p_o \ell(\ell+1)} & \frac{\rho_o r^2}{\Gamma_1 p_o} & 0 \\ \frac{\ell(\ell+1)}{r^2}\left(1 - \frac{A}{\rho_o\sigma^2}\frac{dp_o}{dr}\right) & -A & \frac{A\ell(\ell+1)}{\sigma^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \\ -4\pi G\rho_o A & \frac{\sigma^2 4\pi G\rho_o^2 r^2}{\Gamma_1 p_o \ell(\ell+1)} & \ell(\ell+1) - \frac{4\pi G\rho_o^2 r^2}{\Gamma_1 p_o} & 0 \end{pmatrix}$$

This is equivalent to (5.13).

Chandrasekhar and Lebovitz [2] have used (5.28) to obtain the fundamental mode in the case $\ell = 2$ and also in the radial case $(\ell = 0)$ for certain polytropic models. Neverthless, as far as I know, the calculations in this thesis represent the first use of variational methods to obtain higher modes and their first use with a model of the type considered here. It is also the first use in this field of the techniques described in Chapter 2 which make possible accurate calculation of higher modes with quite limited computing facilities.

In [2], Γ_1 is regarded as constant. This is a good approximation when β is near 1 or 0 (so that Γ_1 is near γ or $\frac{4}{3}$). But, except when β is exactly 1 or 0, Γ_1 is constant only in the standard model (a polytrope with index 3).

Guided by the fact that for the homogeneous model [26] the eigenfunctions are of the form $x^{\ell} \sum_{i=0}^{k} c_{2i} x^{2i}$, and the condition [32] that $\operatorname{div} \underset{\sim}{\delta r} = 0(r^{\ell})$ as $r \longrightarrow 0$, Chandrasekhar and Lebovitz chose as trial functions

$$\psi = ar^{\ell+1} + br^{\ell+3}, \quad \chi = ar^{\ell+1} + cr^{\ell+3} \tag{5.36}$$

(Note that they chose a form for χ, whereas in the calculations in this thesis a form was chosen instead for χ' as χ never occurs explicitly in the equations.) In the radial case they chose

$$\psi = ar^3 + br^5 . \tag{5.37}$$

This had been anticipated in [1] . They give the form of the characteristic equation and list the fundamental eigenvalues for polytropes with indices 1 , $1\frac{1}{2}$, 2 , $2\frac{1}{2}$, 3 , $3\frac{1}{2}$ with (constant) $\Gamma_1 = 1{\cdot}55$, $1{\cdot}6$, $1{\cdot}65$, $\frac{5}{3}$. Also for each index they give the (constant) value of Γ_1 which first produces accidental degeneracy between the fundamental modes of radial oscillations and non-radial oscillations with $\ell = 2$.

They also give an alternative variational formulation of the problem which does not assume that ρ_o vanishes at $r = R$ or that it is continuous. This is

$$\sigma^2 \int_0^R \rho_o \left(\frac{\psi^2}{r^2} + \frac{\chi'^2}{\ell(\ell+1)}\right)dr = \int_0^R \left(\frac{\Gamma_1 p_o}{r^2} (\psi'-\chi')^2 + \frac{2\psi}{r^2}\frac{dp_o}{dr}\left(\frac{2\psi}{r} - \chi'\right)\right)dr$$

$$+ \frac{8\pi G}{2\ell+1} \int_0^R \rho_o r^\ell \left(\frac{\ell\psi}{r} + \chi'\right)\left(\int_r^R \frac{\rho_o(s)}{s^{\ell+1}}\left((\ell+1)\frac{\psi(s)}{s} - \chi'(s)\right)ds\right)dr \qquad (5.38)$$

Lebovitz [32] has used (5.28) to give a more rigorous proof of Schwarzschild's stability criterion (4.7), for displacements of the type described by (5.1). Under certain conditions he showed that $(B + C)$ is positive whenever $A < 0$ throughout the star and hence, since I is positive, that exponential instability could not occur for oscillations of this type. He noted that, without a completeness theorem for allowable displacements, stability still does not follow, but at least this approach considers the reaction of the star as a whole.

The variational approach may be modified to consider more general situations. Clement [33] has modified (5.28) to consider the effect of small rotations. His approach is similar to that of [31].

§4. Failure to Detect g-Modes.

Consider what happens when values of σ^2 given by (5.28) or (5.31) are extremised by the Ritz method. Results obtained for such a calculation with (5.31) are discussed in detail in Chapter 7. The most notable fact is that while the method gave approximations to the f-mode and the first 25 p-modes, it gave no indication of the existence of g-modes. The remaining 26 eigenvalues of the characteristic equation were spurious.

I have not been able to find references in the literature to variational formulation of systems of differential equations where the eigenvalue occurs in a non-linear manner. Many standard theorems apply only to p.b.b. operators (see Chapter 1). In this case although B (and in fact B + C) are positive, they are not p.b.b. For ψ, χ' could be given by

$$\psi = \chi' = 0 \text{ , throughout the envelope}$$

$$(\psi p_0^{1/\Gamma_1})' = \chi' p_0^{1/\Gamma_1} \text{ , } \quad |\psi| + |\chi'| \neq 0 \text{ throughout the core} \tag{5.39}$$

This gives σ^2 defined by (5.31) its minimum value of zero. These values of ψ, χ' satisfy (5.33a) and ψ and χ' may be made continuously differentiable while satisfying (5.39). But as A/σ^2 is not defined in this case it does not appear that (5.33b) can be satisfied. Nevertheless it might be expected that one of the spurious solutions obtained by the Ritz method would

correspond to (5.39), since it gives the true minimum of σ^2 . To show this is not the case, the eigenvector corresponding to the smallest eigenvalue (which is spurious) is listed in Table 8 . This could perhaps be expected as (5.27) shows that, since $A = 0$ throughout the core, (5.39) implies that $\delta\rho = 0$ throughout the star and so represents a highly degenerate "oscillation".

If in a problem $L_1 u = \lambda L_2 u$, the operator L_2 is p.b.b., then provided L_2 is bounded below, say $(L_1 u, u) \geq k(u,u)$ for all u , then even if L_2 is not p.b.b., the problem is equivalent to $\tilde{L}_1 u = \tilde{\lambda} L_2 u$, where $\tilde{L}_1 u = L_1 u + c L_2 u$, $\tilde{\lambda} = \lambda + c$ and $c > |k|$, so that $\tilde{L}_1$ and $\tilde{L}_2$ are both p.b.b. But in this case even I is not p.b.b., as ψ and χ' may be zero except in an arbitrarily small region near $r = R$, where $\rho_0 \longrightarrow 0$ as $(R - r)^3$. However it seems possible that this is not the only reason why the Ritz method breaks down. The fact that as $\sigma^2 \longrightarrow 0$, ψ and χ' do not tend to the values given by (5.39), which anyway do not satisfy (5.33b), the fact that $(B\underset{\sim}{\delta r}, \underset{\sim}{\delta r})$ could be regarded as a functional of the functions $(p_0'\psi + \Gamma_1 p_0(\psi' - \chi'))$ and ψ , and the fact that the Euler-Lagrange equations contain a non-linear parameter are also probably relevant.

From the requirement that (5.28) and (5.31) must extremise σ^2 with respect to variations in the relative magnitudes of ψ and χ' , Chandrasekhar [31] deduced that σ^2 must be a root of a certain quadratic equation, whose coefficients are definite integrals involving ψ and χ' . The validity of this argument is queried in [2]. In any case, the physical significance of the second

root of the quadratic is less clear. Chandrasekhar suggested that the two roots might correspond to a p-mode and a g-mode. Although the form of the g-modes is different from that of the p-modes the integrals might be equal. In the case $\delta\phi = 0$, Van der Borght and Wan [25] substituted eigenfunctions ψ and χ' into this equation. In the case of both the p-modes and the g-modes, one of the values so obtained for σ^2 was very nearly equal to the correct eigenvalue corresponding to the given eigenfunction, while the other value appeared to be completely spurious. (Cf [23] p. 522.) In the case of the p-modes the two roots of the determinant differed by only a few percent. However the fact that this equation is satisfied by g-modes as well as p-modes, and the fact that the Euler-Lagrange equations corresponding to (5.28) and (5.31) are essentially the same as (5.13) and (5.17) respectively, seem to some extent to vindicate the variational formulation, although the Euler-Lagrange equations are merely necessary and not sufficient conditions for the existence of an extremum.

The result [5] for a p.b.b. operator L, that the nth eigenvalue of L is the minimum value of $(Lu,u)/(u,u)$ where u is a non-zero function such that if $n > 1$, $(u,u_i) = 0$, $i = 1,\ldots,n-1$, where the u_i are the first $n-1$ eigenvalues of L, is important in the formulation of the Ritz method. (See Chapter 1 §1). Clearly this result does not hold in the present case with two families of eigenvalues. Thus it is not

altogether surprising that the Ritz method does not yield both families of eigenvalues. In fact I do not know of any standard theory that says it should yield even the p-modes. The fundamental mode does not give the minimum of σ^2, defined by either (5.31) or (5.28), nor does the nth p-mode yield a minimum of σ^2 when ψ and χ' are subject to the appropriate linear restraints. The position seems even less promising with the g-modes where σ^2 decreases as n increases. It would seem there is scope for investigating the validity of the Ritz method in such cases not covered by the standard theory.

Since, as shown in Chapter 2, our choice of coordinate functions satisfies all the requirements for validity of the Ritz method, but, as shown in this section, the differential equations do not satisfy the usual requirements for the use of variational techniques, it would seem reasonable to conjecture that with any choice of coordinate functions the Ritz method would fail to detect the g-modes. A transformation of the differential equations might be more successful. Since B + C seems to have properties similar to those of B, it seems likely that the Ritz method would not detect the g-modes when applied to (5.28).

This discussion poses more questions than it answers. In what circumstances do eigenvalue problems have more than one cluster point of eigenvalues? In what circumstances may they be posed as variational problems of the form $\lambda(L_2 u,u) = (L_1 u,u)$? Which families of eigenvalues, if any, will be obtained by

the Ritz method? Will more subtle direct methods yield additional eigenvalues? However it is hoped that at least the preceding discussion will provide some pointers for further investigation.

More details of calculations are given in Chapter 7 . But first it would seem useful to discuss the important special case of purely radial oscillations.

CHAPTER 6

PURELY RADIAL OSCILLATIONS

§1. The Differential Equations.

The mathematical description of purely radial oscillations is much simpler than that of non-radial oscillations, and for this simpler type of oscillation much work has been done on the effect of relaxing the restrictions imposed in Chapter 5. Many of the conclusions of this work should apply also to non-radial oscillations. Much of this work is reviewed in [23] which also derives (p. 455) the well-known equation for purely radial oscillations under the same restrictions as in Chapter 5. This equation is linearised and describes adiabatic oscillationsof a non-viscous, non-rotating gaseous sphere on which the only external force is its own gravity, and magnetic fields and relativistic effects are ignored.

The oscillations considered are described by

$$\begin{aligned} \delta'\underset{\sim}{r}(\underset{\sim}{r},t) &= \underset{\sim}{r}\xi(r)e^{i\sigma t} \\ \delta'\rho(\underset{\sim}{r},t) &= \delta\rho(r)e^{i\sigma t} \\ \delta'p(\underset{\sim}{r},t) &= \delta p(r)e^{i\sigma t} \\ \delta'\phi(\underset{\sim}{r},t) &= \delta\phi(r)e^{i\sigma t} \end{aligned} \tag{6.1}$$

where as before $\delta'\rho$, $\delta'p$, $\delta'\phi$ are the Eulerian variations in

density ρ , pressure p and gravitational potential ϕ caused by a (radial) Lagrangian displacement $\delta\underset{\sim}{r}$. Using the same four basic equations as in Chapter 5 (energy, Poisson, continuity and momentum) and linearising gives

$$L\xi = \sigma^2 \rho_o r^4 \xi \tag{6.2}$$

where the operator L is defined by

$$L = -\frac{d}{dr}\left(r^4 \Gamma_1 p_o \frac{d}{dr} \ldots\right) - r^3 \frac{d}{dr}\left((3\Gamma_1 - 4)p_o\right) \tag{6.3}$$

The simplification $\delta\phi = 0$ made in the non-radial case is not made here. Boundary conditions are

$$r\xi = 0 \quad \text{at} \quad r = 0 \tag{6.4a}$$

$$\delta p = -\Gamma_1 p_o\left(3\xi + r\frac{d\xi}{dr}\right) = 0 \quad \text{at} \quad r = R \tag{6.4b}$$

Since $p_o = 0$ at $r = R$ it is sufficient that ξ and $\frac{d\xi}{dr}$ be bounded.

In all realistic models, including the one considered here, $r^4\Gamma_1 p_o$ and $\rho_o r^4$ are positive for $0 < r < R$ and non-negative throughout the star. Since $\gamma = \frac{5}{3}$ in this case, (4.5) shows that $(3\Gamma_1 - 4) = (4 - 3\beta)\beta/(8 - 7\beta)$. From (4.10) and the method used for determining the boundary between the core and the envelope it follows that $\frac{d\beta}{dx}$ nowhere exceeds the value given by (4.15a). Hence, since $0 \leq \beta \leq 1$ and since $\frac{dp_o}{dr}$ is negative

for $r < R$ and non-positive throughout the star, it follows from (4.9) (4.13)and (4.15a) that $\frac{d}{dr}((3\Gamma_1 - 4)p_o)$ is negative for $r < R$ and non-positive throughout the star. Hence the operator L is positive. Since also $\frac{d\Gamma_1}{dr}$, $\frac{dp_o}{dr}$ and ρ_o are continuous in our model, (6.2,4) differs from the standard Sturm-Liouville problem only by the existence of singularities at the boundaries.

These show more clearly when (6.2) is written as

$$\frac{d^2\xi}{dr^2} + \left(\frac{4}{r} + \frac{1}{\Gamma_1}\frac{d\Gamma_1}{dr} - \frac{\rho_o}{p_o}\frac{Gm}{r^2}\right)\frac{d\xi}{dr}$$

$$+ \left(\frac{\sigma^2\rho_o}{\Gamma_1 p_o} + \frac{3}{r\Gamma_1}\frac{d\Gamma_1}{dr} - \frac{3\Gamma_1 - 4}{\Gamma_1 r p_o}\frac{Gm\rho_o}{r^2}\right)\xi = 0 \qquad (6.5)$$

Since $\frac{m}{r^3}$, $\frac{d\Gamma_1}{dr}$ and $\frac{1}{\Gamma_1}$ are bounded and, in our model, $\frac{\rho_o}{p_o} \sim \left(\frac{r}{R} - 1\right)^{-1}$ as $r \longrightarrow 1$, the only singularities are simple poles at $r = 0$ and $r = R$. These are regular singularities ([29] p. 160). Much of the standard theory of Sturm-Liouville systems applies only when there are no singularities but, as stated in [23] p. 461, "if as is the case here the singularities are regular, and the boundary conditions are automatically satisfied by the regular part of the solution, the spectrum in general remains purely discrete and the eigenvalues and eigenfunctions have the same properties as in the case of the Sturm-Liouville problem." This remark is supported by the numerical solutions.

The singularity at $r = R$ depends on the model used.

If the boundary condition $\bar{t}(1) = 0$ from (4.16b) were replaced [34] by $\bar{t}(1) = T_R > 0$ say, then (4.10), (4.11) show that the singularities at $r = R$ in (6.5) and also in (5.13), (5.17), (5.19) , (5.23) and (5.24) would be eliminated. Since in practice the boundary of a star will not be clearly defined it would seem that only an incomplete physical picture could be obtained from a perfect mathematical analysis of the effect of the singularity at $r = R$ and that perhaps the results of the common method (used in [25]), of using, without rigorous justification, a truncated series at this boundary and starting numerical integration a slight distance in, may be just as reliable.

§2. Variational Formulation.

Equation (6.2) is the Euler-Lagrange equation (necessary condition for an extremum of σ^2) of the variational problem

$$\sigma^2 = \min_{\xi} \frac{\int_0^R \xi L \xi \, dr}{\int_0^R \rho_0 r^4 \xi^2 dr} \tag{6.6}$$

Also for $n > 0$

$$\sigma_n^2 = \max_{(\xi_0,\ldots,\xi_{n-1})} \min_{\xi_n} \frac{\int_0^R \xi_n L \xi_n dr}{\int_0^R \rho_0 r^4 \xi_n^2 dr}$$

where ξ_n is subject to the n auxilliary conditions $\int_0^R \rho_0 \xi_i \xi_n = 0$, $i = 0,\ldots,n-1$. Boundary conditions (ξ, ξ' bounded, $\rho_0(R) = 0$) give

$$\int_0^R \xi L \xi dr = \int_0^R (\Gamma_1 p_0 r^4 (\frac{d\xi}{dr})^2 - r^3 \xi^2 \frac{d}{dr}((3\Gamma_1 - 4)p_0))dr \tag{6.6a}$$

In [23], (6.6) is also derived directly from Hamilton's principle.

Equation (6.6a) shows that (6.6) is of the form of (2.6) so that the method of Chapter 2 §2 may be used and hence higher modes determined quite efficiently. This also applies to a variational formulation by Chandrasekhar [35] of the problem of radial oscillations in the context of general relativity. Tooper [36] has used Chandrasekhar's formulation [35] to determine the fundamental mode for certain very massive stars as part of a study of stability.

If $\chi' = 0$ in (5.28), Chandrasekhar [31] showed the case $\ell = 0$ to reduce to

$$\sigma^2 \int_0^R \rho_0 \psi^2 \frac{dr}{r^2} = \int_0^R \left(\Gamma_1 p_0 \left(\frac{d\psi}{dr}\right)^2 + \frac{4}{r} \frac{dp_0}{dr} \psi^2\right) \frac{dr}{r^2} \qquad (6.7)$$

Putting $\xi = \psi/r^3$, expanding and integrating by parts using $[\Gamma_1 p_0 r^3 \xi^2]_0^R = 0$ and expanding $\frac{d}{dr}(\Gamma_1 p_0 r^3)$ reduce this to (6.6) which is thus seen to be a special case of (5.28).

As both integrals in (6.6) are positive (as noted in §1), all values of σ are real. This rules out exponential instability for small oscillations of the type considered here but does not ensure stability.

The operators L and $\rho_0 r^4$. are not p.b.b (see Chapter 2) as we could make $\xi = 0$ except in arbitrarily small neighbourhoods of $r = 0$ and $r = R$. However this will make both integrals small. The minimum eigenvalue is strictly positive and the Ritz method is applicable. Considerable discussion on (6.6) is given in [23] including alternative forms and physical interpretation.

The Ritz method has been applied to (6.6) by Ledoux and Pekeris [1] in the case of the "standard model". The paper begins with a physical discussion on the variational principle, its use to obtain qualitative results (e.g. σ_0 increases with central condensation) and analytical expressions for upper bounds.

Since ξ is clearly an even function of r , Ledoux and Pekeris assumed (cf. (5.37)) that

$$\xi = \sum_{i=0}^{n} a_i r^{2i} \qquad (6.8)$$

The matrices obtained when

the usual procedures are carried out with this form of ξ have certain symmetry properties and for large n contain a considerable number of zeros, but far less than those obtained in Chapter 2 §2. Furthermore the matrices obtained in this case cannot be dealt with by special simple methods such as those described in Chapter 3. Thus, for comparable computer time and storage, n, the number of parameters, must be kept much lower with this method than with that of Chapter 2. The method is therefore less well suited to the determination of higher modes which requires very large n for accuracy. However even with $n = 2$ they obtained very good convergence for the fundamental mode and conjectured that their result was probably more accurate than some obtained by numerical integration. Higher n were not considered in [1] and consequently no accurate estimates were obtained for higher modes.

As far as I know, the calculations in this thesis represent the first use of variational methods to obtain higher modes of oscillation. They also represent the first use of variational methods for the present model.

Another treatment of radial oscillations which reduces to the solution of matrix equations is described by Whitney and Ledoux [10]. Further results are given in [37]. Following Zhevakin [38] they regarded the mass of the star as concentrated in n spherical surfaces of radius r_i, where $r_1 < r_2 < \ldots < r_n$. The matrix equation can be dealt with by the method of Chapter 3 §1 although the matrices are not symmetric. The authors state that

the problem can be approached in the same manner as an application of Rayleigh's principle. Zhevakin's method has also been used [39] for non-radial, non-adiabatic oscillations.

Since the publication of [1] the limitations of direct integration have been greatly reduced by the improvement of high-speed computing facilities. For the model used here, and for several others, Van der Borght [40] has obtained values for the first 13 modes using a fourth order Runge-Kutta integration with 50 steps, starting with series expansion at the centre. This method required an initial guess of σ^2 for each integration and although this was done systematically the method is much slower than the variational method. Van der Borght [41] later calculated a further 7 eigenvalues and eigenfurctions and recalculated some of the lower ones, using the same Runge-Kutta method with 500 steps. The values $\overline{m}$, $\overline{p}$, $\overline{t}$, β at the extra points required were obtained by linear interpolation. The results of these later calculations sometimes differed from those of [40] in the fourth significant decimal. This error may be largely due to the linear interpolation which is unsatisfactory near the surface. The eigenvalues obtained for this thesis are compared with Van der Borght's in Table 2. The values for the fundamental agree to 6 significant decimals. This suggests that the results obtained for that mode by the methods described in this thesis are correct to at least 6 places. It is doubtful that the Runge-Kutta method is more accurate than this as rounding errors limit its accuracy.

CHAPTER 7

THE NUMERICAL CALCUIATIONS

§1. The Radial Case.

Using the notation of the last three chapters, and defining λ and the dimensionless ω by

$$\sigma^2 = \frac{M_0 \bar{m}(1) G \omega^2}{R^3 \mu^2} \tag{7.1a}$$

$$\lambda = \frac{G \bar{m}(1) \omega^2}{R} \tag{7.1b}$$

we can write (6.6) as

$$\lambda = \min_{\xi} \frac{\int_0^1 (\Gamma_1 \bar{p} x^4 (\frac{d\xi}{dx})^2 - x^3 \frac{d}{dx}((3\Gamma_1 - 4)\bar{p})\xi^2)dx}{\int_0^1 \frac{\bar{p}\beta}{\bar{t}} x^4 \xi^2 dx} \tag{((7.2)}$$

This is of the form of (2.6). The methods of Chapters 2 and 3 were used to find eigenvalues λ (and hence ω) and eigenfunctions ξ. As in Chapter 2 §2, the interval $(0,1)$ was divided into n equal subintervals and a form of ξ was considered which was linear in each subinterval. Since ξ and ξ' $(= \frac{d\xi}{dx})$ were assumed finite at $r = 0$ and at $r = R$ the boundary conditions (6.4) were satisfied automatically. Differentiation with respect to the values of ξ at the ends of the subintervals, to extremise λ, gave, as in Chapter 2, an equation $Px = \lambda Qx$ where P and Q are $(n + 1) \times (n + 1)$

matrices whose elements are given by (2.14) where the f_i are obtained from (7.2). The matrices are triple diagonal.

Three calculations were made using $n = 10$, 25 and 50 respectively. For a given n only $n + 1$ eigenvalues can be found. Eigenvalues obtained with these three values of n are compared with those obtained by Van der Borght in Table 2.

The elements of P and Q, which are all integrals, were calculated from the values of $\bar{m}$, $\bar{p}$, $\bar{t}$, β listed in Table 1, using Simpson's rule with three points. The values of $\frac{d\bar{p}}{dx}$, $\frac{d\beta}{dx}$ were calculated from (4.13) and (4.15). The elements in the last row and column of P and Q involve an integral over the interval $(1 - \frac{1}{n}, 1)$. Instead of the values of $\bar{m}$, $\bar{p}$, $\bar{t}$, β at $x = 1$, their values at $x = 0{\cdot}999$ were used in the evaluation of the integrals. The integrands, although some are not defined at $x = 1$, all have limit zero at that point and the values at $x = 0{\cdot}999$ are very small. As Simpson's rule is only approximate it is not even certain that this approximation will increase the error. Also the matrices are not ill-conditioned and a very small change in their elements should make only a small change in eigenvalues. As Van der Borght also used values of $\bar{m}$, $\bar{p}$, $\bar{t}$, β at $x = 0{\cdot}999$ instead of at $x = 1$, comparisons would seem more valid when this is done with the variational method also.

In order to avoid overflow in the floating decimals, a hazard with large matrices, the equation $Px = \lambda Qx$ was transformed to $(10^4 P)x = \Lambda(10^{-7} Q)x$ where $\Lambda = 10^{11}\lambda$ so that the elements were

of the order of 1 . Even so, overflow can again become a problem for larger λ , but this can be overcome by rescaling. Thus if values of $\bar{m}$, $\bar{p}$, $\bar{t}$, β were known at all points of the star, very large n could be used as the storage of the computer will not impose severe limits.

There will be errors in the numerical processes but, as suggested in Chapter 3, they will be small compared with those arising in some methods. Consideration of the form of the eigenfunctions obtained in the calculations described here, and of the rapid decrease in accuracy as the mode increases, suggests that in this case, except perhaps with the fundamental, the value of n , and not rounding errors, was the main factor limiting accuracy.

The eigenfunctions obtained by Van der Borght [41] and other workers show two important characteristics. (1) The zeros of ξ are not evenly distributed but become much denser near $x = 1$. (2) The magnitude of ξ is much greater very near $x = 1$ than in the rest of the star. These two effects increase as the mode increases. There is some difficulty approximating a function whose derivative changes so rapidly by one of piecewise constant derivative. With the coordinate functions used here separations of less than $\frac{1}{n}$ between zeros cannot be made clearly. Thus it is not surprising that zeros of the approximate eigenfunctions obtained by this method are slightly further from $x = 1$ than those of the true eigenfunctions. Nevertheless all the approximate eigenfunctions obtained by variational methods for radial

oscillations had the correct number of zeros.

As the approximate eigenfunctions have piecewise constant derivative, the exact positions of their zeros may be determined by linear interpolation from the eigenvectors of the matrix equation. In order to move the last zero as close as possible to $x = 1$ the value of ξ at $x = 1$ must become very small. This makes it difficult for the approximate eigenfunction to satisfy the second property of the true eigenfunctions (i.e. ξ largest at $x = 1$). The first property, the position of the zeros, turned out to be much better satisfied than the second. Even for quite high modes, the nodes and antinodes are very little displaced from their correct position. But the ratio ξ_R/ξ_o, of the value of ξ at the surface to its value at the centre, although it increases for the first ten modes obtained here, levels off and ultimately decreases almost exponentially reaching a value 0·16 for the 23rd mode. A third property of the true eigenfunctions is rather better satisfied by the approximate functions, but this property does not require rapid change of derivative. This is that, although ξ_o is much less than ξ_R, for higher modes it is still appreciably greater than the value of ξ at the intermediate maxima.

In Table 3 the positions of the first two zeros and the value of ξ_R/ξ_o for the first 20 eigenfunctions obtained by the variational method are compared with those of Van der Borght. The agreement of the shape of the eigenfunctions is rather better than indicated by these figures as the discrepancy in ξ_R/ξ_o is mainly due to a very sharp increase in ξ at the surface in Van der Borght's results. Until very near the surface the amplitudes agree quite well.

Also the displacement of the zeros is greater for the second than for the latest ones. A complete listing of the first 20 eigenvectors of the matrix equation with $n = 50$ is given in Table 4 . (These give the values of the approximate eigenfunctions obtained by variational methods at the 51 points $^{i}/50$, $i = 0,\ldots,50$.)

Unfortunately Van der Borght did not list his eigenfunctions in [40] but eigenfunctions for similar calculations are available in the literature. Schwarzschild [42] lists the first 5 eigenfunctions he obtained for the standard model and Boury and Hustin-Breton [43] show graphically the 1st, 5th, 9th and 13th eigenfunctions. The first 6 eigenfunctions of a model similar to the present one are shown in [22] where they are used as a basis for a non-linear theory.

Ledoux has studied the asymptotic behaviour of both eigenvalues [44] and eigenfunctions [45] considering dominant terms of the equation near both singularities. Van der Borght [40] modified Ledoux's result for the eigenvalues and his asymptotic values are given in Table 2. His asymptotic eigenvalues seem to be slightly too small. Both Van der Borght and Ledoux fitted the solutions obtained near the two singularities in a rather arbitrary manner and it would seem that with greater refinement even better results could be obtained by asymptotic methods.

Because of the difficulty caused by the uneven clustering of the nodes of the true eigenfunctions it might seem that when using coordinate functions with piecewise constant derivative it is

better to choose a partition (see Chapter 2) whose points are closer together near $x = 1$ rather than one where they are evenly spaced. This is probably so but experiments with this modification were inconclusive.

In this case the main factor limiting n and hence limiting accuracy was lack of information on the distribution of $\bar{m}$, $\bar{p}$, $\bar{t}$ and β. Their values were available only at intervals of $0{\cdot}01$. As many parts of the integrands which occur in (2.14) vanish at the end-points of the intervals, it was felt that at least three points were needed to evaluate the integrals by Simpson's rule, so that n could not easily be taken as greater than 50.

With $n = 25$, in order to check the error caused by inaccurate evaluation of the integrals, eigenvalues were calculated using integrals computed using five points for Simpson's rule as well as those using only three. The difference obtained by this refinement was generally a fraction of 1%. The first four eigenvalues were obtained more accurately with the five points than with three, but this was not the case for the higher eigenvalues. However the negative result for higher eigenvalues is probably not significant as with higher eigenvalues there is a relatively large error due to the restricted form assumed for ξ and there may be some cancellation of errors. With both methods of evaluating the integrals eigenvalues obtained with $n = 25$ were all less accurate than the corresponding ones with $n = 50$.

Simpson's rule is probably quite accurate for evaluating these integrals as each term in the integrands of (2.14) is a quadratic in the distance from the end of the interval of integration (of length $\frac{1}{n}$) multiplied by a function f_i which, except near $x = 0$ and $x = 1$, is fairly constant over that interval.

An attempt was made to increase n by using extra values of $\bar{m}$, $\bar{p}$, $\bar{t}$, β obtained by linear interpolation, but this was not very successful. Accuracy could be increased by obtaining more detailed solutions to the equations of Chapter 4, but results would no longer be comparable with those of Van der Borght.

With $n = 50$, 38 eigenvalues were calculated. The value obtained for the 38th was 83·8. All 26 eigenvalues were calculated with $n = 25$, and all 11 with $n = 10$. The results are summarised in Table 2. Eigenvectors were calculated only in the case $n = 50$, using the method of Chapter 3 §4. The fourth approximation was taken and the rapidity of convergence of the method is shown by Table 5.

One of the main advantages of the variational method is its speed. With $n = 50$, using a Fortran programme, an IBM 1620 calculated 38 eigenvalues in less than two hours. For the fundamental the method is probably as accurate as numerical integration, and for the lower overtones the accuracy is fair. For the higher modes quite good estimates may be made by asymptotic methods. If a rapid estimate is required (e.g. as an initial guess for integration or as a check) variational methods

can be used for the first few modes and an asymptotic expression for the others. Also the analysis required for variational and asymptotic methods may give information about the nature of the equations not given by routine numerical integration.

§ 2. The Non-radial Case.

Writing $\chi' = r^2\eta$ and using the same notation as for (7.2) we may write (5.31) as

$$\lambda = \min_{\xi,\eta} \frac{\int_0^1 f_1 \xi^2 + 2f_2 \xi\xi' + f_3(\xi')^2 + f_4 \eta^2 + 2f_5 \xi\eta + 2f_6 \eta\xi'}{\int_0^1 f_7\xi^2 + f_8 \eta^2} \tag{7.3}$$

where

$$f_1(x) = 9x^2\Gamma_1\bar{p} + 6x^3 \frac{d\bar{p}}{dx} + x^4 \left(\frac{1}{\beta} \frac{d\beta}{dx} + \frac{1}{\bar{p}} \frac{d\bar{p}}{dx} - \frac{1}{\bar{t}} \frac{d\bar{t}}{dx}\right) \frac{d\bar{p}}{dx}$$

$$f_2(x) = 3x^3\Gamma_1\bar{p} + x^4 \frac{d\bar{p}}{dx}, \quad f_3(x) = x^4\Gamma_1 \bar{p}$$

$$f_4(x) = x^2\Gamma_1\bar{p}, \quad f_5(x) = -3x^2\Gamma_1\bar{p} - x^3 \frac{d\bar{p}}{dx} \tag{7.4}$$

$$f_6(x) = -x^3\Gamma_1\bar{p}, \quad f_7(x) = x^4 \frac{\beta\bar{p}}{\bar{t}}, \quad f_8(x) = \frac{f_7(x)}{\ell(\ell+1)}$$

This is of the form of (2.15) and the methods of Chapters 2 and 3 were used to find eigenvalues λ (and hence ω) and eigenfunctions ξ, η. Again the boundary conditions are automatically satisfied as ξ, η and their first derivatives are assumed bounded. This time only one partition was considered: that with (n =) 25 equal intervals. Partial differentiation with respect to the values of ξ and η at the points of the partition leads to a matrix equation of the form

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \underset{\sim}{x} = \lambda \begin{pmatrix} D_5 & 0 \\ 0 & D_6 \end{pmatrix} \underset{\sim}{x} \qquad (7.5)$$

where the D_i are 26 X 26 triple diagonal matrices. The elements of the matrices are given by (2.21). The eigenvalues of (7.5) were evaluated by the method of Chapter 3 §3 and the method of §4 of that chapter was used to evaluate the eigenvectors. All 52 eigenvalues and eigenvectors were calculated for the case corresponding to $\ell = 2$. The matrix elements, which are again integrals, were calculated by Simpson's rule, using five points, from the values of $\overline{m}$, $\overline{p}$, $\overline{t}$, β in Table 1. Again the values of $\overline{m}$, $\overline{p}$, $\overline{t}$, β at $x = 0{\cdot}999$ were used instead of values at $x = 1$, which again should have very little effect. Again the matrices were scaled to avoid overflow in the floating decimals.

Similarity of several of the f_i simplifies calculations. As the ratio f_7/f_8 is constant for a given spherical harmonic, the ratio of the elements of D_5 to the corresponding elements of D_6 also equals this constant. Thus D_6 need not be calculated separately. The programme was written so that, in order to find eigenvalues and eigenfunctions corresponding to a different spherical harmonic, only a single instruction (of the form $\ell = \ldots$) needed to be changed. As the main purpose of this project was to study methods and not to obtain numerical results, little use was made of this, but the fundamental was calculated in the case $\ell = 3$, as described later.

Even when $\delta\phi$ is neglected the problem of non-radial oscillations is more difficult than the special case of radial oscillations and it is perhaps not surprising that variational methods were also less successful in the former case. Although variational methods were still much faster than direct integration they were slower than in the radial case where the eigenvalues can be found by the very efficient method of Chapter 3 §1 .

Despite the fact that 52 parameters were used in this case (which made the equations more difficult) it seems to be the fact that the partition divides the interval into only 25 sections that is significant for determining accuracy. Accuracy is more comparable with that of the radial case with $n = 25$ than with $n = 50$.

As before, the zeros of the true eigenfunctions are denser near $x = 1$, and the magnitudes of the eigenfunctions increase rapidly near $x = 1$. (See [27].) With the p-modes, the uneven distribution of zeros is especially important with η (representing transverse displacement) for, as noted by Cowling [3], the first zero of η is further from $x = 0$ than the first zero of ξ . As with the radial case, the zeros of the approximate eigenfunctions obtained in this study are displaced slightly towards $x = 0$ (first effect). Again, while the computed values of the ratios ${}^{\xi}R/_{\xi_o}$, ${}^{\eta}R/_{\eta_o}$, of the values of ξ , η at $x = 1$ to their values at $x = 0$, initially increase with the mode, they fall off rapidly for the higher modes (second effect). Perhaps because a smaller n was used, the behaviour of these ratios is less regular than in the radial case. Again the first

effect is less marked than the second effect. (In the true eigenfunctions both ξ_R/ξ_o and η_R/η_o increase indefinitely with the mode.) The two effects are illustrated in Tables 9 and 10 which compare the results obtained by variational methods with the results obtained by Van der Borght and Wan [25]. Unfortunately most of the work of Van der Borght and Wan is unpublished but some results for the case $\delta\phi \neq 0$ are given in [46]. Eigenfunctions are not described in [46] but Smeyers [27] represents graphically the lower modes he obtained by numerical integration in the case $\delta\phi \neq 0$ for a model of Ottelet. The general shape is the same as in [25].

The mth p-mode should have m zeros in ξ and m zeros in η. For $m = 1,\ldots,25$, ξ has m zeros in the approximation obtained for the mth p-mode. For $m \leq 8$ this is also true for η. But for $m > 8$, except for $m = 11$ which gives a full complement of zeros, η has only $m - 1$ zeros in the approximation obtained for the mth p-mode. This is also the effect of the concentration of zeros near $x = 1$ in the true eigenfunctions. The ratio η_R/η_o obtained by the variational method also falls off more rapidly than ξ_R/ξ_o.

Other properties of the f-mode and p-modes obtained in these calculations are as expected. Within the limits of numerical accuracy the zeros of the mth mode separate those of the $(m + 1)$th and the zeros of ξ separate those of η for the same mode. A

complete listing of the f-mode and the first 20 p-modes is given in Table 7.

The eigenvalues ω obtained by variational means for the fundamental and lower p-modes agree quite well with those of [25] (see Table 6) but, as expected, the discrepancy becomes steadily greater as the mode increases, the increment $\omega_m - \omega_{m-1}$ in the computed values increasing very regularly. Again there would seem to be scope for investigating the asymptotic behaviour of the higher modes.

The main failure of the variational method is the fact, discussed in Chapter 5, that while it found the f-mode and the first 25 p-modes it showed no trace of the g-modes. The remaining 26 eigenvalues and eigenvectors of the characteristic equation (2.20) seem to be completely spurious. In general it is easy to distinguish the true eigenvalues from the spurious ones by the properties of the corresponding eigenvectors. The one possible exception is the largest eigenvalue which could conceivably be a very bad approximation to the 26th p-mode, but in any case this eigenvalue is not very useful.

The spurious eigenvalues seem to be almost randomly distributed. They include the 17 lowest eigenvalues of the matrix equation. The sign-changes of the spurious vectors, which should correspond to the zeros of eigenfunctions, seem also to be almost randomly distributed, but for each spurious vector (including the last) η has more sign-changes than ξ. This was also true for the few spurious

vectors calculated with $\ell = 3$. There is an increase but not a monotonic increase in the number of sign changes in the spurious vectors as the eigenvalue increases. Also the ratios ξ_R/ξ_o and η_R/η_o seem to decrease as the eigenvalue increases but this effect is also irregular. Other tendencies can be noted but none seem very significant. Three of the spurious eigenvectors are listed in Table 8. They correspond to the lowest eigenvalue, to $\omega = 0{\cdot}777$ and to $\omega = 2{\cdot}93$ (the next highest eigenvalue after the fundamental). The first of these disproves a conjecture made in Chapter 5.

The first 17 spurious eigenvalues are $\omega = 0{\cdot}187$, $0{\cdot}257$, $0{\cdot}263$, $0{\cdot}321$, $0{\cdot}388$, $0{\cdot}460$, $0{\cdot}533$, $0{\cdot}598$, $0{\cdot}691$, $0{\cdot}777$, $0{\cdot}822$, $0{\cdot}969$, $1{\cdot}15$, $1{\cdot}37$, $1{\cdot}64$, $1{\cdot}97$, $2{\cdot}39$. The eigenvalues obtained by Van der Borght and Wan for the first 5 g-modes are $0{\cdot}781$, $0{\cdot}501$, $0{\cdot}371$, 0.295 , $0{\cdot}246$. The first of these is very close to the spurious value $0{\cdot}777$ but as shown in Table 8, the vector corresponding to $\omega = 0{\cdot}777$ in no way resembles the first g-mode which has one zero in both ξ and η . The remaining 9 spurious eigenvalues are shown in Table 6 , which shows how they are distributed amongst the true eigenvalues, and indicates the number of sign changes in the corresponding vectors.

For $\ell = 3$, three spurious eigenvalues ($\omega = 2{\cdot}70$, $3{\cdot}27$, $8{\cdot}07$) were obtained and also the fundamental mode ($\omega_o = 2{\cdot}8454$). As theory shows that ω_o increases with ℓ , the value of ω_o for $\ell = 2$ was used as a first trial value for eigenvalues of the

matrix equation (using the method of Chapter 3 §3.) The other eigenvalues could easily have been obtained by the same method.

We may add to the questions posed in Chapter 5, two more. What, if any, is the physical significance of the spurious eigenvalues? Is there any regular pattern in their distribution among the eigenvalues of the characteristic equation? In this case exactly half the eigenvalues of that equation are spurious. However although the method of Chapter 2 always yields an even number of eigenvalues there are other methods which yield an odd number. Perhaps it is only a coincidence that exactly half the eigenvalues are spurious in the present case. But again it is hoped this work will help as a guide for further investigation.

APPENDIX.

The terms below are defined in the sense in which they are used in this thesis.

Bounded below. A symmetric operator A in a Hilbert space is termed bounded below iff (i.e. if and only if) there exists a real number k such that for all u in the domain D_A of A, $(Au,u) > k(u,u)$.

Complete sequence. A sequence $\{\varphi_i\}$ of elements in some Hilbert space is said to be complete in a subset U of the space iff for all elements u in U and all positive numbers ε, there exists an integer N and a set of N real numbers $\alpha_1,\ldots,\alpha_N$ such that $(u - \Sigma_{i=1}^{N} \alpha_i \varphi_i\ ,\ u - \Sigma_{i=1}^{N} \alpha_i \varphi_i) < \varepsilon$. The sequence $\{\varphi_i\}$ is said to be complete in U in the energy of some symmetric operator A where $U \subset D_A$ iff for all u in U and all $\varepsilon > 0$ there is an integer N and a set of N real numbers $\alpha_1,\ldots,\alpha_N$ such that $(A(u - \Sigma_{i=1}^{N} \alpha_i \varphi_i),\ u - \Sigma_{i=1}^{N} \alpha_i \varphi_i) < \varepsilon$.

Complete set. A normed set S is termed complete iff each Cauchy sequence of elements in S has a limit in S.

Functional. A functional is a linear mapping from the elements of a Hilbert space into the set of scalars of that space.

Positive. A positive operator A in Hilbert space is a symmetric operator such that for all non-zero u in D_A, $(Au,u) > 0$.

Positive-bounded-below. (abbreviated in the text to p.b.b.). A positive-bounded-below operator in Hilbert space is a symmetric operator with the property that there exists a real number $\gamma \neq 0$ such that for all u in D_A $(Au,u) > \gamma^2(u,u)$.

Positive definite. A positive definite matrix is a matrix whose principal minors are all positive. A positive definite quadratic form $\underset{\sim}{x}^T A\underset{\sim}{x}$ is a quadratic form which is strictly positive for all non-zero vectors $\underset{\sim}{x}$, such that $\underset{\sim}{x}^T A\underset{\sim}{x}$ is defined.

Quadratically summable. A set of functions is said to be quadratically summable iff the product of any two members of the set is integrable.

Symmetric. (= self-adjoint). A linear operator A in some Hilbert space is termed symmetric iff for all u, v in D_A $(Au,v) = (u,Av)$.

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KEY TO THE TABLES.

Table 1.

This gives the values of $\overline{m}$, $\overline{p}$, $\overline{t}$, β in columns 2 , 3 , 4 , 5 respectively against x in column 1 for the stellar model used in the present calculations. (See Chapter 4.) The E-FORMAT of Fortran is used. A number written as "aEb" in this format is to be read as "$a \times 10^{b}$".

Table 2.

This gives in columns 2 , 3 and 4 some of the eigenvalues ω found for radial oscillations in the present study using $n = 50$, 25 and 10 intervals respectively. (See Chapter 7 §1.) Column 5 gives the values obtained by Van der Borght ([40] and [41]) using direct integration (Runge-Kutta method). Column 6 shows the values given by his asymptotic formula. Column 1 gives the number of the mode, using 1 for the fundamental, 2 for the first overtone, etc.

Table 3.

This compares the eigenfunctions ξ for radial oscillations obtained by variational means in this study with those obtained by direct integration by Van der Borght. Columns 2 , 4 and 6 describe the eigenfunctions obtained in this study. Column 2 gives the ratio ξ_R/ξ_o of the value of ξ at the outside to its

value at the centre. Columns 4 and 6 give the values of $x(= {}^{r}/_{R})$ at the first and second zeros respectively of ξ. Columns 3, 5 and 7 respectively give the corresponding information $({}^{\xi_R}/_{\xi_o}$, position of first two zeros) about the eigenfunctions obtained by Van der Borght. Column 1 gives the number of the mode, again using 1 for the fundamental, etc.

Table 4.

This lists in columns the values obtained, by the variational method, for the eigenfunctions ξ, for radial oscillations, at the 51 positions defined by $x = {}^{i}/_{50}$, $i = 0,\ldots,50$. The value of i is given in the 6th column and the first 5 columns of each page list the values of 5 consecutive eigenfunctions. The first 20 eigenfunctions are listed and the number of the mode is given at the top of each column.

Table 5.

This demonstrates the rate of convergence of the numerical method of calculating eigenvectors derived from Theorem 3 (Chapter 3). Let the norm of $\underset{\sim}{x}_i = (x_{i1},\ldots,x_{in})$ be defined by $\|\underset{\sim}{x}_i\| = (\sum_{j=1}^{n} x_{ij}^2)^{\frac{1}{2}}$. Successive approximations to $\underset{\sim}{x}$ given by (3.11) were normalised in this manner. Columns 2, 3, 4 show $\|\underset{\sim}{x}_{i+1} - \underset{\sim}{x}_i\|$ for $i = 1,2,3$ respectively where $\underset{\sim}{x}_i$ is the normalised ith approximation to $\underset{\sim}{x}$. Column 1 gives the number of the vector being approximated, where again 1 is used for the fundamental, etc.

As eigenvectors are arbitrary to within a scalar multiple, which

may be negative, the appropriate measure of convergence is sometimes $\|\underset{\sim}{x}_{i+1} + \underset{\sim}{x}_i\|$. As this was not realised when the lower modes were calculated, the information on convergence was not obtained for all eigenvectors, but the results should be similar in all cases. This very good convergence was obtained with eigenvalues which were calculated to only 3 significant figures, although at least 7 figures could easily have been obtained, and in fact more accurate eigenvalues (those in Table 2) were calculated later.

Table 6.

This compares the eigenvalues ω for non-radial oscillations obtained in this study with those obtained by numerical integration by Van der Borght and Wan [25]. Column 1 gives the number of the mode where 0 denotes the f-mode, 1 the first p-mode and so on. (Note that the numbering system used here in the non-radial case is not quite the same as that used in the radial case.) Column 2 gives the values of the eigenvalues obtained in this study (with the approximation $\delta\phi = 0$) . Column 3 shows those spurious eigenvalues which are greater than that for the f-mode. The 17 lower spurious eigenvalues are listed in the text (Chapter 7 §2). Column 4 shows the eigenvalues obtained by Van der Borght and Wan with the approximation $\delta\phi = 0$ and column 5 those they obtained without this approximation. Columns 6 and 7 show the number of zeros in ξ and η respectively corresponding to the eigenvalues of columns 2 and 3 as obtained in this study. For true eigenfunctions they should

equal the numbers in column 1.

Table 7.

This lists the values of ξ and η obtained in this study for the f-mode and the first 20 p-modes, with 3 modes per page. Values are given at the 26 points $x = {}^{i}/_{25}$, $i = 0,\ldots,25$. The last column of each page shows the value of i . Columns 1 and 2 give the values of ξ and η respectively for one mode, columns 3 and 4 for the next and columns 5 and 6 for the next.

The three numbers at the top of each pair of columns represent values for the eigenvalue calculated from these solutions, using $r_i = (\Sigma_j p_{ij} x_j)/(\Sigma_j q_{ij} x_j)$ as described in Chapter 3 §4 . The number at the top is calculated from the mean value of r_i and is the best estimate obtained for the eigenvalue. The two numbers underneath this are calculated from the lowest and greatest values of r_i and thus give lower and upper bounds for the eigenvalues of the matrix equation. The closeness of these numbers indicates the accuracy of the numerical process. They do not necessarily give bounds of the eigenvalues of the differential equation (as the Ritz method is an approximate one) but give bounds of the values the method would yield if the matrix equation were solved exactly.

Table 8.

This lists 3 of the spurious eigenvectors yielded by the Ritz method. They are discussed in Chapter 7 §2. The format is exactly the same as for Table 7.

Tables 9 and 10.

These compare the eigenfunctions obtained in this study with those of Van der Borght and Wan [25]. Their eigenfunctions showed a slight rise at the origin which was later found to be a spurious effect due to the numerical approximations used. Consequently it was decided to compare the values of ξ and η at $x = \frac{1}{25}$ and $x = 1$ instead of at $x = 0$ and $x = 1$ as in the radial case. Table 9 concerns ξ. Columns 2, 4 and 6 concern the results of this study. Column 2 gives the ratio $\xi_R/10\xi_1$ of the value of ξ at $x = 1$ to ten times its value at $x = \frac{1}{25}$. Column 4 gives the value of x at the first zero of ξ and column 6 its value at the second zero. Columns 3, 5 and 7 respectively give the corresponding results obtained by Van der Borght and Wan by direct integration. Column 1 gives the number of the mode (0 for f-mode etc.) Table 10 gives the same information about η, using the same format, except that in this case columns 2 and 3 represent η_R/η_1 and not $\eta_R/10\eta_1$.

TABLE 1.
THE STELLAR MODEL (PART 1).

1	2	3	4	5
.0000E-99	.0000E-99	2.3325E-05	4.6908E-15	7.9415E-01
1.0000E-02	2.0084E-04	2.3293E-05	4.6890E-15	7.9420E-01
2.0000E-02	1.6025E-03	2.3196E-05	4.6832E-15	7.9435E-01
3.0000E-02	5.3935E-03	2.3036E-05	4.6738E-15	7.9459E-01
4.0000E-02	1.2735E-02	2.2816E-05	4.6606E-15	7.9493E-01
5.0000E-02	2.4752E-02	2.2536E-05	4.6438E-15	7.9536E-01
6.0000E-02	4.2516E-02	2.2198E-05	4.6233E-15	7.9590E-01
7.0000E-02	6.7040E-02	2.1806E-05	4.5992E-15	7.9653E-01
8.0000E-02	9.9261E-02	2.1363E-05	4.5715E-15	7.9725E-01
9.0000E-02	1.4003E-01	2.0872E-05	4.5404E-15	7.9807E-01
1.0000E-01	1.9014E-01	2.0337E-05	4.5060E-15	7.9899E-01
1.1000E-01	2.5024E-01	1.9763E-05	4.4682E-15	8.0000E-01
1.2000E-01	3.2090E-01	1.9154E-05	4.4272E-15	8.0110E-01
1.3000E-01	4.0260E-01	1.8514E-05	4.3832E-15	8.0230E-01
1.4000E-01	4.9569E-01	1.7848E-05	4.3361E-15	8.0359E-01
1.5000E-01	6.0042E-01	1.7161E-05	4.2861E-15	8.0498E-01
1.6000E-01	7.1692E-01	1.6458E-05	4.2334E-15	8.0646E-01
1.7000E-01	8.4524E-01	1.5741E-05	4.1781E-15	8.0803E-01
1.8000E-01	9.8524E-01	1.5018E-05	4.1202E-15	8.0969E-01
1.9000E-01	1.1367E-00	1.4290E-05	4.0600E-15	8.1144E-01
2.0000E-01	1.2996E-00	1.3563E-05	3.9975E-15	8.1329E-01
2.1000E-01	1.4733E-00	1.2840E-05	3.9329E-15	8.1522E-01
2.2000E-01	1.6574E-00	1.2124E-05	3.8663E-15	8.1724E-01
2.3000E-01	1.8514E-00	1.1420E-05	3.7978E-15	8.1935E-01
2.4000E-01	2.0547E-00	1.0729E-05	3.7276E-15	8.2155E-01
2.5000E-01	2.2667E-00	1.0055E-05	3.6558E-15	8.2384E-01
2.6000E-01	2.4865E-00	9.3999E-06	3.5826E-15	8.2621E-01
2.7000E-01	2.7134E-00	8.7652E-06	3.5080E-15	8.2867E-01
2.8000E-01	2.9466E-00	8.1528E-06	3.4322E-15	8.3121E-01
2.9000E-01	3.1852E-00	7.5642E-06	3.3553E-15	8.3383E-01
3.0000E-01	3.4285E-00	7.0005E-06	3.2775E-15	8.3654E-01
3.1000E-01	3.6754E-00	6.4625E-06	3.1988E-15	8.3933E-01
3.2000E-01	3.9252E-00	5.9507E-06	3.1194E-15	8.4220E-01
3.3000E-01	4.1769E-00	5.4655E-06	3.0394E-15	8.4515E-01
3.4000E-01	4.4297E-00	5.0070E-06	2.9589E-15	8.4819E-01
3.5000E-01	4.6827E-00	4.5750E-06	2.8779E-15	8.5130E-01
3.6000E-01	4.9351E-00	4.1694E-06	2.7967E-15	8.5449E-01
3.7000E-01	5.1860E-00	3.7895E-06	2.7155E-15	8.5775E-01
3.8000E-01	5.4347E-00	3.4350E-06	2.6347E-15	8.6109E-01
3.9000E-01	5.6804E-00	3.1050E-06	2.5541E-15	8.6451E-01
4.0000E-01	5.9224E-00	2.7988E-06	2.4711E-15	8.6788E-01
4.1000E-01	6.1598E-00	2.5157E-06	2.3918E-15	8.7096E-01
4.2000E-01	6.3916E-00	2.2551E-06	2.3143E-15	8.7384E-01
4.3000E-01	6.6177E-00	2.0159E-06	2.2386E-15	8.7647E-01
4.4000E-01	6.8368E-00	1.7974E-06	2.1645E-15	8.7888E-01
4.5000E-01	7.0485E-00	1.5983E-06	2.0922E-15	8.8110E-01
4.6000E-01	7.2524E-00	1.4177E-06	2.0217E-15	8.8314E-01
4.7000E-01	7.4481E-00	1.2543E-06	1.9528E-15	8.8502E-01
4.8000E-01	7.6355E-00	1.1070E-06	1.8857E-15	8.8674E-01
4.9000E-01	7.8139E-00	9.7466E-07	1.8204E-15	8.8831E-01
5.0000E-01	7.9830E-00	8.5605E-07	1.7569E-15	8.8976E-01

TABLE 1.
THE STELLAR MODEL (PART 2).

1	2	3	4	5
5.1000E-01	8.1445E-00	7.5009E-07	1.6941E-15	8.9108E-01
5.2000E-01	8.2960E-00	6.5570E-07	1.6335E-15	8.9230E-01
5.3000E-01	8.4400E-00	5.7184E-07	1.5745E-15	8.9341E-01
5.4000E-01	8.5747E-00	4.9755E-07	1.5171E-15	8.9442E-01
5.5000E-01	8.7009E-00	4.3190E-07	1.4611E-15	8.9535E-01
5.6000E-01	8.8189E-00	3.7405E-07	1.4066E-15	8.9619E-01
5.7000E-01	8.9289E-00	3.2315E-07	1.3536E-15	8.9696E-01
5.8000E-01	9.0312E-00	2.7855E-07	1.3020E-15	8.9766E-01
5.9000E-01	9.1261E-00	2.3948E-07	1.2518E-15	8.9829E-01
6.0000E-01	9.2138E-00	2.0540E-07	1.2030E-15	8.9887E-01
6.1000E-01	9.2947E-00	1.7571E-07	1.1554E-15	8.9940E-01
6.2000E-01	9.3691E-00	1.4992E-07	1.1092E-15	8.9987E-01
6.3000E-01	9.4374E-00	1.2757E-07	1.0642E-15	9.0030E-01
6.4000E-01	9.4999E-00	1.0825E-07	1.0204E-15	9.0068E-01
6.5000E-01	9.5569E-00	9.1584E-08	9.7777E-16	9.0103E-01
6.6000E-01	9.6088E-00	7.7245E-08	9.3627E-16	9.0134E-01
6.7000E-01	9.6558E-00	6.4936E-08	8.9588E-16	9.0162E-01
6.8000E-01	9.6984E-00	5.4405E-08	8.5656E-16	9.0187E-01
6.9000E-01	9.7367E-00	4.5414E-08	8.1828E-16	9.0210E-01
7.0000E-01	9.7712E-00	3.7765E-08	7.8101E-16	9.0229E-01
7.1000E-01	9.8020E-00	3.1273E-08	7.4470E-16	9.0247E-01
7.2000E-01	9.8295E-00	2.5784E-08	7.0934E-16	9.0263E-01
7.3000E-01	9.8540E-00	2.1158E-08	6.7489E-16	9.0278E-01
7.4000E-01	9.8756E-00	1.7275E-08	6.4132E-16	9.0288E-01
7.5000E-01	9.8947E-00	1.4024E-08	6.0860E-16	9.0299E-01
7.6000E-01	9.9114E-00	1.1319E-08	5.7671E-16	9.0308E-01
7.7000E-01	9.9260E-00	9.0760E-09	5.4562E-16	9.0316E-01
7.8000E-01	9.9386E-00	7.2257E-09	5.1530E-16	9.0323E-01
7.9000E-01	9.9495E-00	5.7078E-09	4.8573E-16	9.0329E-01
8.0000E-01	9.9589E-00	4.4700E-09	4.5687E-16	9.0334E-01
8.1000E-01	9.9668E-00	3.4674E-09	4.2872E-16	9.0338E-01
8.2000E-01	9.9735E-00	2.6612E-09	4.0124E-16	9.0341E-01
8.3000E-01	9.9792E-00	2.0100E-09	3.7441E-16	9.0344E-01
8.4000E-01	9.9838E-00	1.5104E-09	3.4821E-16	9.0347E-01
8.5000E-01	9.9876E-00	1.1152E-09	3.2262E-16	9.0349E-01
8.6000E-01	9.9907E-00	8.0842E-10	2.9762E-16	9.0350E-01
8.7000E-01	9.9932E-00	5.7260E-10	2.7320E-16	9.0352E-01
8.8000E-01	9.9951E-00	3.9742E-10	2.4932E-16	9.0353E-01
8.9000E-01	9.9966E-00	2.6811E-10	2.2598E-16	9.0353E-01
9.0000E-01	9.9977E-00	1.7514E-10	2.0316E-16	9.0354E-01
9.1000E-01	9.9986E-00	1.0999E-10	1.8083E-16	9.0354E-01
9.2000E-01	9.9991E-00	6.5708E-11	1.5899E-16	9.0355E-01
9.3000E-01	9.9995E-00	3.6868E-11	1.3762E-16	9.0355E-01
9.4000E-01	9.9998E-00	1.9077E-11	1.1671E-16	9.0355E-01
9.5000E-01	1.0000E+01	8.8191E-12	9.6256E-17	9.0355E-01
9.6000E-01	1.0000E+01	3.4641E-12	7.6187E-17	9.0355E-01
9.7000E-01	1.0000E+01	1.0515E-12	5.6551E-17	9.0355E-01
9.8000E-01	1.0000E+01	1.9950E-13	3.7310E-17	9.0355E-01
9.9000E-01	1.0000E+01	1.1964E-14	1.8469E-17	9.0355E-01
9.9990E+00	1.0000E+01	1.1964E-18	1.8469E-18	9.0355E-01

TABLE 2
Eigen Values (Radial Case)

1	2	3	4	5	6
1	2.050678	2.05086	2.0518	2.050683	3.063
2	3.65223	3.6628	3.729	3.6486	4.253
3	4.99386	5.0433	5.286	4.9762	5.444
4	6.3084	6.4368	6.651	6.2584	6.634
5	7.6239	7.864	7.906	7.5158	7.824
6	8.9541	9.279	10.03	8.7578	9.015
7	10.3050	10.51	12.91	9.9904	10.205
8	11.6715	11.71	16.7	11.2169	11.395
9	13.0246	13.28	22.0	12.4390	12.586
10	14.2892	15.10	29.6	13.6569	13.776
11	15.4482	17.10	46.3	14.8703	14.966
12	16.7251	19.23		16.0794	16.156
13	18.1879	21.58		17.2853	17.347
14	19.7661	24.10		18.501	18.537
15	21.4226	26.84		19.706	19.727
16	23.145	29.85		20.910	20.918
17	24.928	33.16		22.113	22.108
18	26.773	36.8		23.315	23.298
19	28.68	40.9		24.517	24.489
20	30.65	45.6		25.717	25.636
21	32.69	51.0			26.785
22	34.80	57.3			27.933
23	36.99	64.7			29.081
24	39.26	73.5			30.229
25	41.61	84.1			31.377

TABLE 3

Comparison of Eigen Functions (Radial Case)

1	2	3	4	5	6	7
1	2.537	2.567				
2	-26.51	- 28.05	.603	.603		
3	61.3	69.6	.520	.520	.803	.803
4	-99.7	-125.7	.462	.463	.725	.726
5	135.5	194.5	.417	.420	.654	.659
6	-167.3	-276.7	.380	.384	.596	.604
7	198.3	361.8	.347	.354	.548	.558
8	-236.4	-460.5	.320	.328	.508	.520
9	295.7	566.5	.297	.306	.473	.487
10	-372.0	-677.6	.279	.286	.445	.458
11	326.8	792	.260	.265	.422	.432
12	-174.8	-908	.246	.253	.400	.409
13	85.2	1028	.230	.239	.375	.388
14	- 43.8	-1150	.215	.226	.353	.370
15	23.5	1273	.200	.215	.332	.352
16	-12.8	-1396	.188	.205	.312	.337
17	7.1	1515	.177	.195	.294	.322
18	- 3.9	-1629	.166	.187	.277	.309
19	2.1	1737	.157	.179	.278	.296
20	- 1.2	-1841	.148	.171	.248	.285

TABLE 4.

RADIAL EIGENFUNCTIONS (PART 1).

1	2	3	4	5	
.09766058	-.01959079	.01091701	-.00779169	.00631785	0
.09768025	-.01958821	.01091095	-.00778315	.00630671	1
.09774544	-.01957949	.01089070	-.00775465	.00626960	2
.09784841	-.01956541	.01085837	-.00770932	.00621073	3
.09799018	-.01954541	.01081320	-.00764628	.00612918	4
.09817180	-.01951871	.01075419	-.00756445	.00602390	5
.09839432	-.01948436	.01068015	-.00746257	.00589366	6
.09865890	-.01944113	.01058960	-.00733911	.00573709	7
.09896685	-.01938752	.01048075	-.00719229	.00555264	8
.09931969	-.01932167	.01035148	-.00702003	.00533855	9
.09971913	-.01924137	.01019924	-.00681992	.00509297	10
.10016716	-.01914391	.01002108	-.00658925	.00481389	11
.10066602	-.01902607	.00981351	-.00632495	.00449927	12
.10121829	-.01888398	.00957248	-.00602364	.00414713	13
.10182694	-.01871300	.00929331	-.00568160	.00375564	14
.10249538	-.01850758	.00897055	-.00529480	.00332326	15
.10322748	-.01826100	.00859793	-.00485896	.00284900	16
.10402771	-.01796522	.00816819	-.00436961	.00233264	17
.10490133	-.01761047	.00767302	-.00382223	.00177516	18
.10585452	-.01718489	.00710280	-.00321240	.00117911	19
.10689470	-.01667397	.00644653	-.00253608	.00054927	20
.10803409	-.01606032	.00569169	-.00179004	-.00010670	21
.10928866	-.01532321	.00482468	-.00097284	-.00077701	22
.11067328	-.01443745	.00383105	-.00008607	-.00144444	23
.11220192	-.01337273	.00269605	.00086436	-.00208527	24
.11388790	-.01209279	.00140540	.00186662	-.00266831	25
.11574426	-.01055445	-.00005367	.00290099	-.00315429	26
.11778386	-.00870656	-.00169084	.00393770	-.00349586	27
.12001947	-.00648882	-.00351045	.00493439	-.00363855	28
.12246400	-.00383025	-.00550865	.00583356	-.00352343	29
.12513051	-.00064771	-.00766933	.00656018	-.00309186	30
.12803246	.00315616	-.00995912	.00701996	-.00229329	31
.13118358	.00769498	-.01232046	.00709880	-.00109688	32
.13459788	.01310061	-.01466262	.00666465	.00049247	33
.13829021	.01952693	-.01685087	.00557307	.00241371	34
.14227583	.02715216	-.01869158	.00367890	.00452460	35
.14657071	.03618275	-.01991403	.00085687	.00657536	36
.15119150	.04685733	-.02014721	-.00296439	.00818543	37
.15615583	.05945178	-.01889075	-.00774940	.00883360	38
.16148149	.07428212	-.01547887	-.01327475	.00787828	39
.16718780	.09171374	-.00903519	-.01901267	.00463721	40
.17329490	.11216562	.00158223	-.02396115	-.00142916	41
.17982393	.13611790	.01784815	-.02640656	-.01036187	42
.18679693	.16411866	.04165039	-.02359633	-.02107520	43
.19423662	.19679020	.07538276	-.01129995	-.03031921	44
.20216703	.23483849	.12205356	.01676288	-.03106260	45
.21061207	.27905094	.18538608	.06962008	-.01012076	46
.21959455	.33028601	.26988648	.15979323	.05499416	47
.22913006	.38940975	.38072962	.30371088	.19934581	48
.23918058	.45689032	.52265988	.51996998	.47037266	49
.24778159	.51927402	.66877911	.77651538	.85590997	50

TABLE 4.
RADIAL EIGENFUNCTIONS (PART 2).

6	7	8	9	10	
-.00546527	.00481736	-.00415354	.00336779	-.00268622	0
-.00545125	.00480044	-.00413442	.00334821	-.00266724	1
-.00540461	.00474429	-.00407111	.00328350	-.00260467	2
-.00533086	.00465577	-.00397166	.00318227	-.00250718	3
-.00522913	.00453424	-.00383584	.00304481	-.00237559	4
-.00509850	.00437915	-.00366371	.00287194	-.00221140	5
-.00493802	.00419011	-.00345577	.00266513	-.00201696	6
-.00474674	.00396692	-.00321291	.00242653	-.00179546	7
-.00452364	.00370960	-.00293660	.00215905	-.00155100	8
-.00426778	.00341849	-.00262892	.00186651	-.00128866	9
-.00397831	.00309440	-.00229278	.00155372	-.00101461	10
-.00365460	.00273876	-.00193207	.00122669	-.00073612	11
-.00329637	.00235382	-.00155189	.00089273	-.00046160	12
-.00290392	.00194290	-.00115880	.00056054	-.00020051	13
-.00247828	.00151069	-.00076097	.00024029	.00003679	14
-.00202150	.00106347	-.00036841	-.00005654	.00023953	15
-.00153701	.00060950	.00000695	-.00031734	.00039694	16
-.00103000	.00015936	.00035125	-.00052882	.00049913	17
-.00050787	-.00027383	.00064883	-.00067775	.00053819	18
.00001924	-.00067416	.00088271	-.00075209	.00050956	19
.00053791	-.00102282	.00103541	-.00074245	.00041366	20
.00103085	-.00129848	.00109035	-.00064415	.00025767	21
.00147634	-.00147796	.00103402	-.00045987	.00005730	22
.00184788	-.00153773	.00085900	-.00020238	-.00016225	23
.00211463	-.00145657	.00056771	.00010309	-.00036684	24
.00224270	-.00121931	.00017648	.00041778	-.00051635	25
.00219754	-.00082174	-.00028065	.00069055	-.00057068	26
.00194789	-.00027650	-.00074962	.00086272	-.00049906	27
.00147143	.00038071	-.00115688	.00087752	-.00029196	28
.00076250	.00108570	-.00141489	.00069448	.00002677	29
-.00015839	.00174295	-.00143445	.00030789	.00039185	30
-.00123400	.00222822	-.00114523	-.00023450	.00069862	31
-.00235996	.00239964	-.00052397	-.00081921	.00082207	32
-.00337756	.00212100	.00037305	-.00127083	.00065692	33
-.00407427	.00129981	.00138249	-.00138393	.00017546	34
-.00419753	-.00006011	.00222562	-.00099072	-.00051343	35
-.00349050	-.00179774	.00254101	-.00006164	-.00113818	36
-.00175841	-.00354180	.00198024	.00118734	-.00131840	37
.00102717	-.00469695	.00037965	.00224071	-.00073838	38
.00458550	-.00451710	-.00201030	.00237529	.00057277	39
.00816384	-.00232951	-.00432089	.00099495	.00198142	40
.01040515	.00203549	-.00506356	-.00178045	.00229048	41
.00936526	.00761176	-.00261503	-.00449618	.00043876	42
.00290596	.01163789	.00351617	-.00435987	-.00305619	43
-.01013700	.00943069	.01065838	.00107263	-.00480437	44
-.02737983	-.00393604	.01128872	.00972929	.00001332	45
-.03801747	-.02775616	-.00469052	.01026349	.01012205	46
-.01473642	-.04199344	-.03573311	-.01349159	.00714619	47
.09741388	.01750153	-.03024979	-.04440414	-.03191720	48
.38953590	.29003549	.18264388	.07747098	-.01509827	49
.91408526	.95520776	.98180614	.99573626	.99924930	50

TABLE 4.

RADIAL EIGENFUNCTIONS (PART 3).

11	12	13	14	15	
.00304591	-.00563035	.01130966	-.02129854	.03789149	0
.00302059	-.00557518	.01117800	-.02100467	.03727580	1
.00293731	-.00539426	.01074772	-.02004810	.03528080	2
.00280809	-.00511490	.01008734	-.01859036	.03226503	3
.00263470	-.00474265	.00921505	-.01668455	.02836838	4
.00242006	-.00428621	.00815807	-.01440731	.02378664	5
.00216846	-.00375768	.00695287	-.01185799	.01876568	6
.00188549	-.00317240	.00564425	-.00915467	.01358775	7
.00157810	-.00254882	.00428429	-.00642952	.00855545	8
.00125463	-.00190831	.00293089	-.00382281	.00397217	9
.00092480	-.00127483	.00164587	-.00147534	.00011937	10
.00059963	-.00067426	.00049205	.00048120	-.00276985	11
.00029127	-.00013352	-.00047042	.00193559	-.00453831	12
.00001263	.00032080	-.00118887	.00280987	-.00513148	13
-.00022314	.00066440	-.00162374	.00307265	-.00461926	14
-.00040354	.00087754	-.00175499	.00275115	-.00320709	15
-.00051772	.00094793	-.00158840	.00193946	-.00123014	16
-.00055790	.00087381	-.00116096	.00079995	.00087583	17
-.00052103	.00066700	-.00054396	-.00044577	.00263164	18
-.00041051	.00035541	.00015812	-.00153993	.00360000	19
-.00023793	-.00001590	.00081505	-.00222775	.00349145	20
-.00002434	-.00038624	.00128664	-.00231219	.00227234	21
.00019946	-.00068433	.00144831	-.00171839	.00024246	22
.00039457	-.00083898	.00122528	-.00055121	-.00196874	23
.00051862	-.00079512	.00062904	.00088081	-.00353411	24
.00053367	-.00053361	-.00021611	.00210505	-.00370252	25
.00041708	-.00009006	-.00107300	.00260896	-.00216232	26
.00017378	.00043474	-.00163434	.00204236	.00064559	27
-.00015365	.00088244	-.00161706	.00044353	.00347658	28
-.00048239	.00107183	-.00089591	-.00162059	.00468860	29
-.00070041	.00086289	.00036740	-.00314013	.00312310	30
-.00069625	.00023889	.00167633	-.00307155	-.00087442	31
-.00040680	-.00062487	.00231998	-.00100803	-.00501407	32
.00012803	-.00135606	.00170555	.00221098	-.00599284	33
.00072925	-.00149705	-.00017666	.00450273	-.00193286	34
.00109828	-.00074988	-.00241796	.00363132	.00501002	35
.00092758	.00071696	-.00338306	-.00079217	.00862990	36
.00010325	.00209959	-.00171151	-.00576390	.00345240	37
-.00107456	.00221176	.00215634	-.00602947	-.00777066	38
-.00183186	.00033154	.00515481	.00096418	-.01223714	39
-.00125849	-.00266117	.00326266	.00946467	.00038859	40
.00085676	-.00393800	-.00383768	.00738460	.01796023	41
.00308760	-.00067331	-.00878552	-.00845568	.00953765	42
.00252554	.00556831	-.00140701	-.01692432	-.02477205	43
-.00241872	.00625681	.01393904	.00737176	-.01929003	44
-.00702619	-.00584259	.00860178	.03191146	.04824925	45
-.00004326	-.01552365	-.02781186	-.02257388	.01128894	46
.01741996	.01406059	-.01060811	-.05949853	-.12819481	47
-.00353734	.04080759	.10128764	.16995569	.23778781	48
-.09270184	-.16870708	-.24255525	-.30587633	-.35387727	49
.99546229	.98442290	.96375826	.93237432	.88895515	50

TABLE 4.

RADIAL EIGENFUNCTIONS (PART 4).

16	17	18	19	20	
.06473009	.10640441	.16670812	.24441101	.32927927	0
.06350028	.10405695	.16246420	.23727163	.31829850	1
.05953577	.09653267	.14894693	.21469053	.28383458	2
.05359692	.08537501	.12912740	.18199371	.23461977	3
.04602463	.07135907	.10464283	.14234752	.17617490	4
.03728267	.05551046	.07759875	.09970311	.11516843	5
.02793493	.03903410	.05037820	.05835024	.05850653	6
.01860429	.02320429	.02537697	.02234967	.01226450	7
.00992564	.00924309	.00472433	-.00502965	-.01929184	8
.00249198	-.00180981	-.00999155	-.02180699	-.03444692	9
-.00320380	-.00922479	-.01796018	-.02763799	-.03437894	10
-.00682893	-.01269505	-.01931002	-.02393399	-.02297763	11
-.00826454	-.01241308	-.01514819	-.01367776	-.00607196	12
-.00764528	-.00908959	-.00744064	-.00090233	.00987543	13
-.00537326	-.00389292	.00129064	.01013852	.01939643	14
-.00209371	.00170808	.00844547	.01598923	.01958702	15
.00137726	.00619638	.01193587	.01497116	.01110932	16
.00417796	.00833274	.01077400	.00781821	-.00187025	17
.00557231	.00749716	.00547942	-.00236030	-.01309876	18
.00514359	.00395374	-.00189519	-.01100651	-.01683527	19
.00296571	-.00107958	-.00828118	-.01387231	-.01080319	20
-.00031272	-.00567124	-.01067706	-.00912935	.00192194	21
-.00354720	-.00778215	-.00756424	.00111253	.01363864	22
-.00540433	-.00615440	-.00007343	.01107581	.01607307	23
-.00488021	-.00116070	.00792273	.01407050	.00629302	24
-.00188017	.00489819	.01140287	.00695745	-.00943997	25
.00242737	.00853421	.00723952	-.00633352	-.01828564	26
.00582532	.00692541	-.00284899	-.01573426	-.01070052	27
.00602656	.00015487	-.01193458	-.01197000	.00878412	28
.00218647	-.00772927	-.01192183	.00409151	.02139989	29
-.00387946	-.01046923	-.00069448	.01839290	.01077787	30
-.00797918	-.00430124	.01322472	.01428953	-.01533716	31
-.00607498	.00724223	.01537129	-.00809447	-.02550964	32
.00185543	.01363556	-.00018341	-.02444868	-.00040555	33
.00976564	.00600231	-.01904812	-.00967289	.03093555	34
.00901017	-.01093862	-.01623237	.02359815	.01723171	35
-.00266880	-.01779482	.01169115	.02637683	-.03170864	36
-.01435297	-.00064981	.02867255	-.01770194	-.03157672	37
-.00937802	.02304434	.00003571	-.04084613	.03662188	38
.01208096	.01465142	-.03864702	.01557736	.04222607	39
.02103315	-.02535490	-.00955125	.05689071	-.05798385	40
-.00668896	-.02835455	.05457975	-.03115936	-.03767163	41
-.03368818	.03505647	.00642462	-.07220506	.10845384	42
.00648032	.03953438	-.08729278	.09437198	-.03817925	43
.05520739	-.07453279	.04939583	.02842650	-.13052040	44
-.03776513	-.01344650	.10301431	-.20491357	.27684398	45
-.07659655	.16526472	-.25689993	.32118344	-.33204265	46
.20696669	-.28161994	.33409949	-.34610508	.31045186	47
-.29606174	.33564411	-.34679416	.32206301	-.26432218	48
.38318041	-.38991403	.36951311	-.31974868	.24745890	49
-.83072600	.75301283	-.65027378	.52148067	-.37889155	50

TABLE 5

Convergence of Eigen Vectors

1.	2.	3.	4.
	$\| x_2 - x_1 \|$	$\| x_3 - x_2 \|$	$\| x_4 - x_3 \|$
1	3.1×10^{-3}	1.3×10^{-7}	0
2	3.0×10^{-3}	2.0×10^{-7}	3.1×10^{-8}
3	1.4×10^{-3}	9.5×10^{-8}	0
12	3.1×10^{-5}	2.0×10^{-8}	1.4×10^{-9}
13	1.4×10^{-4}	4.2×10^{-8}	1.8×10^{-9}
16	7.9×10^{-3}	1.0×10^{-6}	3.4×10^{-8}
18	1.3×10^{-1}	3.1×10^{-5}	2.6×10^{-7}
19	5.2×10^{-2}	1.9×10^{-6}	2.0×10^{-7}
20	8.2×10^{-2}	1.2×10^{-5}	1.3×10^{-8}
21	1.9×10^{-2}	3.9×10^{-7}	6.4×10^{-8}
22	2.3×10^{-2}	1.5×10^{-6}	4.3×10^{-8}
23	2.0×10^{-2}	7.9×10^{-7}	5.7×10^{-8}
24	2.2×10^{-2}	1.7×10^{-6}	1.5×10^{-8}
25	8.1×10^{-3}	2.0×10^{-7}	0

TABLE 6

Eigen Values (Non-radial Case)

1	2	3	4	5	6	7
0	2.6132		2.6120	2.278	0	0
		2.93			8	14
		3.64			10	14
1	3.9031		3.8847	3.603	1	1
		4.61			9	15
2	5.221		5.1481	4.867	2	2
		5.98			13	17
3	6.576		6.3911	6.114	3	3
4	7.987		7.6217	7.355	4	4
		8.05			9	15
5	9.457		8.891		5	5
6	10.965	11.53			6	6
7	12.44				13	15
8	13.85				7	7
9	15.41				8	8
10	17.25				9	8
		18.76			10	9
11	19.32				13	16
12	21.59				11	11
13	24.06				12	11
14	26.75				13	12
15	29.70				14	13
16	32.93				15	14
17	36.52				16	15
18	40.53				17	16
		41.9			18	17
19	45.1				19	21
20	50.3				19	18
21	56.4				20	19
22	63.6				21	20
23	72.2				22	21
24	82.8				23	22
25	105.4				24	23
		154.6			25	24
					24	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 1).

2.613205		3.903144		5.220990		
2.613137	2.613229	3.903067	3.903178	5.220789	5.221048	
1	2	3	4	5	6	
.01901287	.05726450	-.00972218	-.02925080	.00570351	.01714016	0
.02045251	.06234028	-.00980103	-.02931794	.00582411	.01753819	1
.02179154	.06603611	-.00986084	-.02902767	.00592764	.01778498	2
.02308052	.06825404	-.00993343	-.02851913	.00602357	.01792511	3
.02432251	.06892696	-.01005797	-.02813102	.00611310	.01802778	4
.02555781	.06846282	-.01030334	-.02842117	.00619880	.01821754	5
.02693367	.06826161	-.01073879	-.02972612	.00627330	.01852668	6
.02875044	.07064478	-.01135192	-.03132155	.00629605	.01871348	7
.03137579	.07689394	-.01201177	-.03192211	.00619968	.01867070	8
.03497110	.08401466	-.01261944	-.03253579	.00592032	.01870894	9
.03934572	.08720018	-.01319779	-.03470680	.00535601	.01832143	10
.04444255	.09068908	-.01360444	-.03483906	.00436183	.01779230	11
.05059290	.09746804	-.01361708	-.03615570	.00275920	.01690514	12
.05776743	.09561258	-.01297206	-.03659305	.00032988	.01555556	13
.06612894	.10269343	-.01121071	-.03711755	-.00314792	.01351525	14
.07608315	.10531876	-.00769089	-.03696831	-.00782885	.01054657	15
.08784030	.11291008	-.00146238	-.03591042	-.01365353	.00641487	16
.10185165	.12113920	.00882654	-.03318840	-.02008032	.00098547	17
.11853036	.13265380	.02508522	-.02786875	-.02562024	-.00560959	18
.13840812	.14687956	.04996810	-.01858182	-.02706488	-.01269973	19
.16207992	.16462963	.08711479	-.00348024	-.01826803	-.01862981	20
.19023238	.18621105	.14143431	.01991794	.01168403	-.02008893	21
.22364037	.21215527	.21939328	.05484389	.08097042	-.01118010	22
.26313176	.24290748	.32911163	.10529604	.21772381	.01764446	23
.30941567	.27880625	.47946082	.17534383	.45993286	.07916636	24
.35995168	.31756160	.66782504	.26335833	.82880473	.17942324	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 2).

6.576454		7.987316		9.457030		
6.576305	6.579225	7.987233	7.987367	9.456820	9.457109	
1	2	3	4	5	6	
-.00442098	-.01326677	.00815236	.02440133	-.00367594	-.01098698	0
-.00448019	-.01350453	.00645227	.01776216	-.00357169	-.01072571	1
-.00452967	-.01373993	.00495467	.01235184	-.00347741	-.01076478	2
-.00456248	-.01395304	.00392351	.01001896	-.00334975	-.01091338	3
-.00455332	-.01403933	.00354966	.01170518	-.00309944	-.01060890	4
-.00446702	-.01390612	.00347131	.01289352	-.00265576	-.00970432	5
-.00427713	-.01367618	.00305631	.00993895	-.00204998	-.00905962	6
-.00396017	-.01344581	.00244592	.01086720	-.00127609	-.00789299	7
-.00345777	-.01286475	.00167926	.00910086	-.00032691	-.00661156	8
-.00270149	-.01214798	.00064577	.00817861	.00075551	-.00504885	9
-.00161673	-.01103827	-.00061881	.00650209	.00187815	-.00330738	10
-.00012679	-.00965523	-.00208068	.00474154	.00285520	-.00151315	11
.00181076	-.00790886	-.00358113	.00272614	.00338833	.00018375	12
.00415492	-.00571240	-.00480881	.00058985	.00308785	.00157599	13
.00669178	-.00303313	-.00525116	-.00148399	.00159644	.00236429	14
.00891317	.00004928	-.00423306	-.00313457	-.00112819	.00225142	15
.00988609	.00325331	-.00113043	-.00387008	-.00437337	.00111253	16
.00818924	.00600464	.00410472	-.00318195	-.00627841	-.00074934	17
.00209772	.00737856	.01003282	-.00084792	-.00409063	-.00239611	18
-.00960462	.00618467	.01265953	.00257174	.00386805	-.00241339	19
-.02547935	.00139515	.00535310	.00519636	.01369316	.00010379	20
-.03680103	-.00673381	-.01711707	.00388396	.01112542	.00361221	21
-.01891340	-.01436833	-.04392106	-.00361659	-.02127693	.00284986	22
.08210485	-.01018182	-.01378056	-.01181321	-.05425136	-.00596340	23
.36131475	.02788166	.23730904	.00397202	.11560428	-.00513531	24
.91762606	.11967301	.96562987	.07800779	.98963556	.04834610	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 3).

10.964933		12.442457		13.854610		
10.964838	10.964966	12.442397	12.442495	13.854579	13.854631	
1	2	3	4	5	6	
.00478269	.01425626	-.00232259	-.00690666	.00408929	.01212202	0
.00397262	.01132837	-.00257763	-.00817959	.00379994	.01150095	1
.00331001	.00950352	-.00274147	-.00944567	.00352370	.01168329	2
.00286262	.00932815	-.00259527	-.00940130	.00304660	.01140683	3
.00250542	.00959169	-.00204355	-.00774196	.00218333	.00941706	4
.00193165	.00796080	-.00132317	-.00708442	.00110851	.00792990	5
.00117954	.00722344	-.00050176	-.00558245	-.00003645	.00586509	6
.00031328	.00575747	.00037922	-.00415660	-.00111473	.00381378	7
-.00063132	.00430041	.00119272	-.00257865	-.00191042	.00180963	8
-.00153444	.00267087	.00177601	-.00108143	-.00218334	.00012475	9
-.00221950	.00107177	.00192349	.00018995	-.00172911	-.00100724	10
-.00243882	-.00032377	.00144776	.00102750	-.00051964	-.00140136	11
-.00192826	-.00131235	.00030477	.00128644	.00110151	-.00104016	12
-.00054780	-.00169028	-.00121535	.00091637	.00229698	-.00014662	13
.00147444	-.00131956	-.00235020	.00007999	.00197870	.00075512	14
.00328459	-.00028829	-.00203645	-.00077668	-.00031107	.00101008	15
.00341604	.00094646	.00029777	-.00103986	-.00314097	.00030558	16
.00056381	.00158227	.00340762	-.00034795	-.00307013	-.00078878	17
-.00454695	.00088796	.00368040	.00082689	.00196976	-.00095397	18
-.00713186	-.00093113	-.00189817	.00111199	.00654370	.00038205	19
-.00002602	-.00203437	-.00825883	-.00036626	-.00065824	.00131448	20
.01481566	-.00010925	-.00010907	-.00165272	-.01332884	-.00056114	21
.00872056	.00318872	.02005545	.00076026	.00791756	-.00158377	22
-.04810088	-.00055663	-.01409828	.00217819	.03091983	.00262756	23
.00951054	-.00677559	-.07458433	-.00513112	-.13848502	-.00240169	24
.99777161	.02684450	.99652093	.01142896	.98929470	.00022618	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 4).

15.413231		17.248298		19.315678		
15.413204	15.413248	17.248284	17.248308	19.315659	19.315700	
1	2	3	4	5	6	
-.01037817	-.03066309	.03689172	.10856548	.05339848	.15720048	0
-.00864176	-.02538281	.02551986	.07002746	-.00915540	-.08902777	1
-.00723928	-.02366861	.01751508	.05365990	-.04004610	-.20431916	2
-.00575960	-.02299636	.01238689	.05602387	-.02638815	-.09771611	3
-.00357537	-.01812120	.00602213	.03977791	-.00647906	-.09118750	4
-.00104057	-.01420968	-.00079544	.02875466	.00941253	-.04996674	5
.00138770	-.00944416	-.00630151	.01597660	.01962420	-.02199802	6
.00327965	-.00499593	-.00922849	.00544552	.02046230	.00030048	7
.00411865	-.00116295	-.00850133	-.00208895	.01173119	.01108722	8
.00350526	.00147196	-.00408148	-.00546371	-.00261983	.01111809	9
.00140549	.00253682	.00237911	-.00471648	-.01420238	.00380914	10
-.00151644	.00202190	.00733290	-.00127581	-.01375952	-.00415906	11
-.00372713	.00043985	.00666567	.00227713	.00078201	-.00650370	12
-.00332154	-.00118558	-.00071718	.00326812	.01664633	-.00165957	13
.00037724	-.00165867	-.00898255	.00086117	.01248838	.00459240	14
.00479283	-.00049515	-.00734196	-.00240840	-.01350925	.00393231	15
.00441530	.00123450	.00678202	-.00233756	-.02293534	-.00309002	16
-.00317976	.00139237	.01432755	.00146612	.01462482	-.00443665	17
-.00887689	-.00063804	-.00601306	.00285266	.03365703	.00369538	18
.00177988	-.00179389	-.02342493	-.00177629	-.03737967	.00340376	19
.01602767	.00079944	.01932415	-.00269386	-.02833085	-.00660203	20
-.00957282	.00198252	.02982281	.00422233	.11095142	.00364533	21
-.02833582	-.00281224	-.09028547	-.00199557	-.16348347	.00112982	22
.08466807	.00142268	.14315852	-.00125086	.19096628	-.00461030	23
-.19302345	.00093774	-.23687844	.00461046	-.25871247	.00776690	24
.97493031	-.00928674	.94038459	-.01746720	.86594240	-.02304607	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 5).

21.585465		24.058650		26.750820		
21.585415	21.585496	24.058627	24.058661	26.750499	26.750845	
1	2	3	4	5	6	
.08622474	.25004784	.17126772	.49365641	.22208829	.63627907	0
.08922366	.29626349	.13769572	.43261111	.15103676	.45975156	1
.07835308	.32309316	.09813091	.41545885	.08895032	.40522839	2
.03969631	.21669689	.03454572	.26803239	.01405748	.23800960	3
-.00217000	.14662040	-.02337647	.15329404	-.04139003	.10988427	4
-.03204684	.07047655	-.05350570	.05219048	-.05545210	.01583522	5
-.04196536	.01456300	-.04802111	-.00964978	-.03004661	-.02587634	6
-.02933777	-.01711085	-.01467710	-.03005007	.01059721	-.02414108	7
-.00186479	-.02274357	.02211419	-.01849898	.03161995	-.00242039	8
.02297379	-.01029712	.03317499	.00340722	.01448274	.01214297	9
.02580812	.00578334	.00779235	.01382646	-.01972874	.00737678	10
.00160310	.01152374	-.02725108	.00569723	-.02387610	-.00540533	11
-.02642687	.00348482	-.02436589	-.00758625	.01323444	-.00673454	12
-.02023667	-.00737535	.02073670	-.00709830	.02873522	.00365875	13
.02141521	-.00615595	.03417910	.00523103	-.01827769	.00527242	14
.03309077	.00512492	-.02605884	.00625968	-.03025713	-.00476546	15
-.02536069	.00619574	-.03936192	-.00631279	.04240883	-.00223498	16
-.04295975	-.00608628	.05754556	-.00300481	.00374450	.00609898	17
.05802349	-.00372721	.00894085	.00839059	-.06704622	-.00431934	18
.02062455	.00903176	-.10211506	-.00583209	.10426044	.00041636	19
-.12650263	-.00576473	.15787282	.00022463	-.10578609	.00231043	20
.18981797	-.00050983	-.16239972	.00370328	.08629814	-.00314079	21
-.20051930	.00484488	.13874484	-.00500845	-.06336323	.00285498	22
.19248770	-.00671911	-.11582187	.00498673	.04763587	-.00236163	23
-.22719001	.00865947	.12362630	-.00551873	-.04712591	.00236228	24
.67215637	-.02261353	-.33374264	.01329232	.11867064	-.00538125	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 6).

29.695026		32.932688		36.518606		
29.695025	29.695049	32.932679	32.932734	36.518592	36.518626	
1	2	3	4	5	6	
.25539925	.72577581	.28328112	.79763815	.30965013	.86304778	0
.14828329	.43822036	.13697558	.38819699	.11617040	.29950978	1
.06925814	.35568735	.04637290	.29106233	.02304337	.21383104	2
-.00722521	.18251709	-.02350690	.12346911	-.03080306	.06810045	3
-.04938200	.06077998	-.04656880	.02163505	-.03367410	-.00240987	4
-.04281411	-.01037914	-.02282687	-.02089240	-.00325523	-.01790530	5
-.00558722	-.02601689	.01276847	-.01584837	.01765327	-.00383310	6
.02472270	-.00944194	.02163339	.00277191	.00722250	.00655641	7
.01966647	.00833909	-.00074789	.00852757	-.01184063	.00252445	8
-.01030673	.00849861	-.01802352	.00015768	-.00621055	-.00357362	9
-.02177612	-.00265850	-.00168007	-.00512450	.01143405	-.00096210	10
.00516880	-.00607921	.01757355	.00025452	.00201765	.00257950	11
.02309729	.00186449	-.00286290	.00355092	-.01300165	-.00060990	12
-.00984759	.00437521	-.01815288	-.00171658	.00843134	-.00144151	13
-.02338714	-.00311353	.01692050	-.00169142	.00606250	.00174704	14
.02805802	-.00187643	.00526479	.00282441	-.01865903	-.00086166	15
.00379484	.00420314	-.02898716	-.00169367	.02324455	-.00009615	16
-.04359609	-.00283381	.04028158	.00005777	-.02083195	.00058765	17
.06544212	.00027028	-.03830299	.00092779	.01530540	-.00064823	18
-.06484415	.00143740	.02934746	-.00114593	-.00981402	.00050404	19
.05133628	-.00190802	-.01954045	.00094602	.00573623	-.00032850	20
-.03554834	.00165234	.01195670	-.00065112	-.00318587	.00019465	21
.02327637	-.00121484	-.00715878	.00041632	.00177589	-.00011253	22
-.01613522	.00088827	.00465373	-.00027851	-.00109634	.00007040	23
.01504861	-.00082657	-.00414265	.00024557	.00094011	-.00005957	24
-.03590309	.00180837	.00947145	-.00052127	-.00207743	.00012356	25

TABLE 7
NON-RADIAL EIGENFUNCTIONS (PART 7).

40.525325		45.083606		50.289862		
40.525261	40.525368	45.083573	45.083637	50.289719	50.289943	
1	2	3	4	5	6	
.33683295	.92847940	.02965657	.07733224	.29647891	.79233972	0
.06999180	.08724747	.18231071	.93372764	.12179883	.49575721	1
.00224120	.10291483	-.01640142	.26924924	-.03623502	.13457899	2
-.01967927	.01773406	-.08601438	.03119155	-.04836132	-.00870760	3
-.01183022	-.00718676	-.01727801	-.03883325	.01022349	-.01905395	4
.00539960	-.00625371	.03671384	-.00702249	.01929898	.00511614	5
.00748777	.00190312	.00694912	.01287561	-.01187746	.00455999	6
-.00313317	.00250399	-.02507323	.00033595	-.00659119	-.00384066	7
-.00521845	-.00114535	.00400077	-.00608891	.01286381	.00002922	8
.00382553	-.00106649	.01804629	.00262910	-.00548922	.00188668	9
.00295646	.00106791	-.01678530	.00176751	-.00544220	-.00161509	10
-.00557910	.00006965	-.00085951	-.00295054	.01216727	.00061926	11
.00194498	-.00075206	.01816288	.00180609	-.01329801	.00012083	12
.00387826	.00065831	-.02598620	-.00029797	.01093202	-.00040785	13
-.00776647	-.00023491	.02469055	-.00058258	-.00753345	.00040108	14
.00850380	-.00010579	-.01877618	.00079967	.00457275	-.00028776	15
-.00707150	.00024212	.01226853	-.00066373	-.00251879	.00017379	16
.00492865	-.00023299	-.00716410	.00044011	.00128633	-.00009367	17
-.00302838	.00016837	.00384313	-.00025437	-.00062018	.00004662	18
.00169838	-.00010371	-.00193865	.00013427	.00028730	-.00002199	19
-.00089627	.00005794	.00094188	-.00006705	-.00013049	.00001008	20
.00046058	-.00003081	-.00045415	.00003283	.00005971	-.00000462	21
-.00024228	.00001650	.00022774	-.00001656	-.00002878	.00000222	22
.00014341	-.00000980	-.00013020	.00000944	.00001599	-.00000122	23
-.00011930	.00000803	.00010563	-.00000754	-.00001271	.00000096	24
.00025646	-.00001636	-.00022200	.00001514	.00002622	-.00000190	25

TABLE 8.
SPURIOUS EIGENVECTORS.

.187229		.777121		2.931532		
.185725	.188271	.776562	.777792	2.929235	2.931653	
1	2	3	4	5	6	
.00000400	.00001206	.08769492	.26386062	.23833779	.71637384	0
.00000367	.00001057	.07328164	.20343175	.18116248	.48235374	1
.00000332	.00000890	.05883469	.14096125	.12532495	.25344263	2
.00000297	.00000713	.04453313	.07931301	.07303275	.05112691	3
.00000263	.00000532	.03046768	.02151642	.02858295	-.08779643	4
.00000229	.00000354	.01682377	-.02886198	-.00165153	-.11998262	5
.00000194	.00000184	.00393321	-.06760216	-.01234247	-.03548669	6
.00000161	.00000029	-.00768769	-.08992982	-.00572077	.08050059	7
.00000128	-.00000104	-.01733538	-.09122633	.00434432	.06712081	8
.00000095	-.00000211	-.02421114	-.06898778	.00324490	-.07559099	9
.00000064	-.00000283	-.02770745	-.02687095	-.00303052	-.04166215	10
.00000036	-.00000314	-.02794405	.02045794	.00037808	.11651298	11
.00000010	-.00000313	-.02614178	.05240894	.00106010	-.11239279	12
-.00000014	-.00000306	-.02430943	.05744548	-.00093668	.07458965	13
-.00000040	-.00000295	-.02398259	.05290367	.00067600	-.03992840	14
-.00000067	-.00000193	-.02420850	.09473220	-.00021614	.01864232	15
-.00000083	.00000310	-.02097496	.21891819	.00022225	-.00784830	16
-.00000043	.00001915	-.01218920	.30193103	.00006553	.00321674	17
.00000168	.00005614	-.00481847	.14247739	.00015934	-.00112573	18
.00000756	.00010968	-.00498753	.07149282	.00016504	.00057135	19
.00001863	.00008258	.00065846	.50791754	.00021364	-.00003447	20
.00002633	-.00045656	.00840008	-.09260054	.00025855	.00024748	21
-.00002360	-.00264092	.00723350	.45054652	.00031968	.00019161	22
-.00033191	-.00579802	.01528999	.02513026	.00039229	.00028982	23
-.00134831	.03622624	.01492714	.36210678	.00048125	.00032618	24
-.00274232	.99931847	.02135617	-.15717612	.00058078	.00042918	25

TABLE 9

Comparison of Eigen Functions (ξ_i)

1	2	3	4	5	6	7
0	1.759	1.787				
1	-6.813	-8.193	.646	.648		
2	14.23	18.30	.524	.529	.824	.830
3	-20.48	-31.53	.443	.452	.727	.737
4	14.96	47.61	.380	.395	.649	.665
5	-27.70		.332	.350	.583	.606

TABLE 10

Comparison of Eigen Functions (η)

1	2	3	4	5	6	7
0	5.094	5.258				
1	-8.982	-10.89	.806	.808		
2	10.23	13.83	.686	.691	.896	.901
3	-8.86	-15.41	.599	.610	.807	.816
4	4.39	16.28	.531	.548	.730	.746
5	-4.51		.476	.496	.664	.686