FUNCTION ALGEBRAS AND ABSTRACT HARDY SPACES

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STATEMENT

Except where otherwise indicated, this thesis is my own work.

Lynette M. Butler
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SUMMARY

In this thesis we consider certain subalgebras of $L^\infty$, called weak-*Dirichlet algebras, which were first introduced by
Srinivasan T.P and Wang J. (Srinivasan and Wang [1]). We consider
the generalisation to these algebras of a portion of the theory of
analytic functions in the unit disc.

In our development we follow the approach of Srinivasan and
Wang, where the invariant subspace theorem, and not Szegö's
Theorem, plays the central role. This theorem, for invariant
subspaces of $L^p$, $1 \leq p \leq \infty$, is established in Chapter 1. We
develop several important factorisation theorems in the process.

In Chapter 2 we show that $H^\infty$ is isomorphic to a logmodular
algebra on the maximal ideal space of $L^\infty$, and use this fact to prove
the truth of Szegö's Theorem. However, the Kolmogorov-Krein theorem,
which is a generalised version of Szegö's Theorem, is not true for
a general weak-*Dirichlet algebra. In this chapter, also, we
consider for which weak-*Dirichlet algebras the Kolmogorov-Krein
Theorem does in fact hold.

In Chapter 3 we consider several generalisations of the classical
F. and M. Riesz Theorem and the weak-* Dirichlet algebras for which
these generalisations hold. We continue this theme in Chapter 5
where we develop a sequential F. and M. Riesz theorem and show the
connection between this and one generalised form of the F. and M. Riesz Theorem.

In Chapter 4 we use some of the results of Chapter 3 to show that there exists a unique extension of a weak-* continuous linear functional defined on a weak-*Dirichlet algebra to a weak-* continuous linear functional on $L^\infty$. We generalise en-route a result of Gleason and Whitney. We conclude this chapter by considering the extension of certain positive linear functionals defined on a weak-*Dirichlet algebra.
1.

Preliminaries

We shall begin with some necessary definitions.

Definition 0.1. A sup-norm algebra $A$ on a compact Hausdorff space $X$ is a complex linear subalgebra of $C(X)$, the algebra (under pointwise operations) of continuous, complex-valued functions on $X$, such that

(i) $A$ is closed under the norm $||f||_A = \sup_{x \in X} |f(x)|$;

(ii) $1 \in A$; and

(iii) $A$ separates the points of $X$; that is, if $x, y$ are distinct points of $X$, $\exists f \in A$ such that $f(x) \neq f(y)$.

We shall write $V_A$ for the set of invertible elements in $A$; that is $V_A = \{ f \in A : f$ and $1/f \in A \}$.

$ReA = \{ Re(f) : f \in A \}$; $|V_A| = \{ |f| : f \in V_A \}$;

$\log |V_A| = \{ \log |f| : f \in V_A \}$; $\overline{A} = \{ \overline{f} : f \in A \}$, where $\overline{f}$ denotes the complex conjugate of $f$. $M_A$ (without subscript is no ambiguity ensures) denotes the maximal ideal space of $A$.

Unless specifically stated otherwise, by a measure on $X$ we shall understand a finite, complex Baire measure on $X$.

We shall make use of the Reesz representation theorem in the form:

Every bounded (that is, continuous) linear functional $\phi$ on
$C_R(X)$, the real-valued functions in $C(X)$, is induced by a real
measure $\mu$ on $X$, that is,

$$\phi(f) = \int fd\mu, \forall f \in C_R(X).$$

Similarly, every bounded linear functional on $C(X)$ is
induced by a (complex) measure on $X$.

We shall also make frequent use of (Edwards [1], Chs 2,8) the
Hahn-Banach Theorem and duality theory.

**Definition 0.2.** Let $A$ be a sup-norm algebra on $X$. Let $\phi \in M$, the maximal ideal space of $A$. A representing measure $m$, for $\phi$
is a positive measure on $X$ such that

$$\phi(f) = \int fdm, \forall f \in A.$$

**Definition 0.3.** Let $A$ be a sup-norm algebra on $A$. Let
$\phi \in M$. An Arens-Singer measure, $m$, for $\phi$ is a positive measure
on $X$ such that

$$\log|\phi(f)| = \int \log|f| dm, \forall f \in V_A.$$

Since $\phi(1) = 1$, both types of measures defined above satisfy
$\int dm = 1$, and are, consequently, probability measures on $X$
(positive measures of mass 1).
Definition 0.4. A Dirichlet algebra $A$ on a compact Hausdorff space $X$ is a sup-norm algebra on $X$ such that the space $\text{Re}A$ is uniformly dense in $C_R(X)$.

Hoffman (Hoffman [2]) extended the theory for Dirichlet algebras to a class of algebras he called logmodular algebras.

Definition 0.5. Let $A$ be a sup-norm algebra on $X$. $A$ is a logmodular algebra on $X$ if the set of functions $\log|V_A|$ is uniformly dense in $C_R(X)$.

This leads to

Theorem 0.1. If $A$ is a Dirichlet algebra on a compact Hausdorff space $X$, then $A$ is a logmodular algebra on $X$.

Proof. Since, $\forall f \in A$, $\text{Re} f = \log|\exp f|$, we have $\text{Re}A \subset \log|V_A|$. But $A$ is a Dirichlet algebra. Thus $\text{Re}A$ is uniformly dense in $C_R(X)$ and so $\log|V_A| \supset \text{Re}A$ is uniformly dense in $C_R(X)$ and $A$ is a logmodular algebra.

Theorem 0.2. Let $A$ be a logmodular algebra on $X$. Then, to every $\phi \in \mathcal{M}$, the maximal ideal space of $A$, corresponds a unique representing measure for $\phi$, and this measure is a Arens-Singer measure.
Proof. This follows directly from Hoffman [2], Theorems 2.1, 4.1, 4.2.

Srinivasan and Wang (Srinivasan and Wang [1]) extended most of the main theorems of logmodular algebra theory (the exceptions being the F. and M. Riesz theorem and the Kolmogorov-Krein Theorem) to a class of algebras they called weak-* Dirichlet.

Definition 0.6. Let \((m, X)\) be a probability measure space. Let \(A\) be a subalgebra of \(L^\infty(m)\) under pointwise operations, such that \(1 \in A\). Then \(A\) is a weak-* Dirichlet algebra if and only if the following conditions are satisfied:

(i) \(m\) is multiplicative on \(A\); that is
\[
\int fg dm = \int f dm \cdot \int g dm, \forall f, g \in A;
\]

(ii) \(A + \overline{A}\) is a dense subset of \(L^\infty(m)\) in the \(\sigma(L^1, L^\infty)\) topology.

To show that all logmodular algebras are also weak-* Dirichlet algebras we need the following three lemmas.

Lemma 0.1. Let \(m\) be a probability measure on a compact Hausdorff space \(X\), and \(g \in L^1_R(m)\), the set of real-valued functions in \(L^1(m)\). Then
\[
\int (\exp g) dm \geq \exp[\int g dm].
\]
Proof. Since every \( g \in L^1_R(m) \) can be written in the form \( g = f + c \) where \( f \in L^4_R(m) \) such that \( \int f dm = 0 \), and \( c \) is a constant function, it suffices to prove this lemma for \( g \in L^1_R(m) \) such that \( \int g dm = 0 \). Since also, \( \exp g \geq 1 + g \), we have

\[
\int (\exp g) dm \geq \int (1 + g) dm = 1 + \int g dm = 1 = \exp[\int g dm].
\]

**Lemma 0.2.** Let \( m \) be a probability measure on a compact Hausdorff space \( X \). Let \( A \) be a logmodular algebra on \( X \). Let \( g \in L^1(m) \) such that \( \int fg dm = 0 \), \( \forall f \in A \). Then

\[
\int |\log|1 - g|| dm > 0.
\]

**Proof.** Let \( f \in V_A \). Then \( \int f dm = \int (1 - g) f dm \), and so

\[
|\int f dm| \leq \int |f| |1 - g| dm;
\]

and

\[
\log|\int f dm| \leq \log\int |f| |1 - g| dm.
\]

Since \( m \) is an Arens-Singer measure (Theorem 0.2), and \( f \in V_A \),

\[
\int \log|f| dm = \log|\int f dm| \leq \log\int |f| |1 - g| dm, \forall f \in V_A;
\]

or

\[
\exp[\int \log|f| dm] \leq \int |f| |1 - g| dm, \forall f \in V_A.
\]
Since $A$ is a logmodular algebra, $\log|V_A|$ is uniformly dense in $C_R(X)$, and so
\[ 1 \leq \int (\exp u)|l - g| \, dm, \forall u \in C_R(X) \text{ such that } \int u \, dm = 0. \]

Thus,
\[ 1 \leq \inf_{u} \int (\exp u)|l - g| \, dm = \exp\left[\int \log|l - g| \, dm\right], \text{ by Lemma 0.1, } \]
and the lemma follows.

**Lemma 0.3.** Let $m$ be a probability measure on a compact Hausdorff space $X$. Let $g \in L^1_R(m)$ such that
\[
\int \log|1 - tg| \, dm > 0,
\]
for every real number $t$ in some interval $|t| < \delta$. Then $g = a.e(m)$.

**Proof.** This is the result of Hoffman [2], Lemma 6.6.

Thus we have,

**Theorem 0.3.** Let $m$ be a probability measure on a compact Hausdorff space $X$. Let $A$ be a logmodular algebra on $X$. Then $A$ is a weak-$^\ast$Dirichlet algebra.

**Proof.** It suffices to prove that, if $g \in L^1_R(m)$ such that
\[
\int fg\,dm = 0, \forall f \in A,
\]

then \( g = 0 \) a.e. \((m)\).

Any such \( g \) satisfies Lemma 0.2 and so

\[
\int |1 - g| \,dm \geq 0
\]

and so, by Lemma 0.3, \( g = 0 \) a.e. \((m)\). For \( 1 \leq p < \infty \), we define the space \( H^p(m) \) by

\[
H^p(m) = \{ A \}_{p}\n\]

the closure of \( A \) in the \( L^p \)-norm; and we define

\[
H^\infty(m) = H^2(m) \cap L^\infty(m)
\]

We shall show (Theorem 2.1.1) that \( H^\infty = [A]_\star \), the weak-\( \star \) closure of \( A \) in \( L^\infty(m) \).

Let \( A_0 = \{ f \in A : \int fdm = 0 \} \); and define

\[
H^p_0(m), \quad 1 \leq p \leq \infty
\]

by

\[
H^p_0(m) = \{ f \in H^p : \int fdm = 0 \}.
\]

It is clear that for \( 1 \leq p < \infty \),

\[
H^p_0 = [A_0]\}_{p}.
\]
CHAPTER 1.

INVARIANT SUBSPACE THEOREM

§1.1 $L^2$-invariant subspace Theorem.

For weak-* Dirichlet algebras the invariant subspace theorem is the basic one. For $A$ a weak-* Dirichlet algebra on a compact Hausdorff space $X$, and $m$ a probability measure multiplicative on $A$, we define a closed subspace $M$ of $L^2(m)$ to be simply invariant if $[MA_0]^2 < M$, "$<$" denoting strict inclusion and $A_0 = \{f \in A : \int fdm = 0\}$.

For such subspaces we have

Theorem 1.1.1. Every simply invariant subspace $M$ of $L^2(m)$ is of the form $M = qH^2(m)$, for some measurable $q$ such that $|q| = 1$; and $q$ is unique (modulo functions which are zero a.e.) up to multiplication by a constant function with absolute value 1.

To prove this theorem we need the following four lemmas.

Lemma 1.1.1. Let $w \in L^1(m)$ be a real-valued function. If $\int fwdm = \int fdm$, $\forall f \in A$, then $w = 1$ a.e. $(m)$.

Proof. Our assumption means that
\[ \int f(1-w)dm = 0 \quad \forall f \in A, \]

But \( w = \bar{w} \), so we have

\[ \int \bar{f}(1-w)dm = 0, \quad \forall f \in A. \]

Hence \( 1 - w = 0 \) a.e. \((m)\) by the weak-* density of \( A + \bar{A} \).

**Remark 1.1.1.** Since \( A \) contains all constant functions, the weak-* density of \( A + \bar{A} \) is equivalent to that of \( A + \bar{A}_0 \).

We now prove

**Lemma 1.1.2.** \( A + \bar{A}_0 \) is norm dense in \( L^2(m) \).

**Proof.** Take \( f \in L^2(m) \) such that \( \int fgdm = 0, \forall g \in A + \bar{A}_0 \). Since \( L^2(m) \subset L^1(m) \), Remark 1.1.1 shows that \( f = 0 \) a.e. \((m)\).

An appeal to the Hahn-Banach theorem gives result.

**Lemma 1.1.3.** Let \( A \) be a weak-* Dirichlet algebra on a compact Hausdorff space \( X \). Let \( m \) be a probability measure on \( X \), multiplicative on \( A \). Then

(a) \( \forall f, g \in H^2(m), \; fg \in L^1(m) \) & \( \int fgdm = \int fdm \cdot \int gdm; \)

(b) \( \forall f \in H^1(m), \; g \in L^1(m) \) & \( g \in H^1(m) \) \( \Rightarrow \) \( \int fgdm = \int fdm \cdot \int gdm. \)
Proof. (a) Consider \( f, g \in H^2(\mu) = [A]_2 \). Then \( \{ f_n \}, \{ g_n \} \subset \mathcal{A} \) which converge in the \( L^2 \)-norm to \( f, g \) respectively.

Now, \( fg \in L^1(\mu) \) and

\[
\left| \int f g - f g \right|_1 \leq \left| \int f g - f g_n \right|_1 + \left| \int f g + f g_n \right|_1
\]

\[
\leq \left| \int f - f \right|_2 \left| g \right|_2 + \left| \int f \right|_2 \left| g - g_n \right|_2
\]

and so \( \{ f g_n \} \subset \mathcal{A} \) converges to \( fg \) in the \( L^1 \)-norm, and so \( fg \in H^1(\mu) \).

Since

\[
\left| \int f_n d\mu - \int f d\mu \right| \leq \left| \int f_n - f \right|_1 \leq \left| \int f - f \right|_2
\]

\[
\lim_{n \to \infty} \int f_n d\mu = \int f d\mu.
\]

Similarly,

\[
\lim_{n \to \infty} \int g_n d\mu = \int g d\mu;
\]

and

\[
\lim_{n \to \infty} \int f g_n d\mu = \int f g d\mu.
\]

Also, since \( \mu \) is multiplicative on \( A \),

\[
\int f g_n d\mu = \int f d\mu \cdot \int g_n d\mu.
\]

Thus we have

\[
\int f g d\mu = \lim_{n \to \infty} \left( \int f g_n d\mu \right)
\]

\[
= \lim_{n \to \infty} \left( \int f d\mu \cdot \int g_n d\mu \right)
\]

\[
= \left( \lim_{n \to \infty} \int f d\mu \right) \cdot \left( \lim_{n \to \infty} \int g_n d\mu \right)
\]

\[
= \int f d\mu \cdot \int g d\mu.
\]
(b) Let \( f \in H^1(m), \ g \in H^1(m) \). Since \( f, g \in H^1(m) \) sequences \( \{f_n\}, \{g_k\} \subset A \) which converge in the \( L^1 \)-norm to \( f, g \) respectively. Consider \( \{f_n g_k\}_{k=1}^{\infty} \subset A \). Now,

\[
|f_n g_k - f_n g|_1 < |f_n|_\infty |g_k - g|_1,
\]

and so \( \{f_n g_k\}_{k=1}^{\infty} \subset A \) converges in the \( L^1 \)-norm to \( f_n g \). Thus \( f_n g \in H^1(m) \). Also,

\[
|f_n g - fg|_1 < |f_n - f|_1 |g|_\infty,
\]

and so \( \{f_n g\}_{n=1}^{\infty} \subset H^1(m) \) converges in the \( L^1 \)-norm to \( fg \). But \( H^1(m) \) is closed under the \( L^1 \)-norm, and so \( fg \in H^1(m) \). Now, since

\[
|\int g_k dm - \int g dm| \leq |g_k - g|_1,
\]

\[
\lim_{k \to \infty} \int g_k dm = \int g dm.
\]

Similarly, \( \lim_{n \to \infty} \int f_n dm = \int f dm \),

and \( \lim_{n \to \infty} \lim_{k \to \infty} \int f_n g_k dm = \int fg dm \).

Also, since \( m \) is multiplicative on \( A \),

\[
\int f_n g_k dm = \int f_n dm \cdot \int g_k dm.
\]

Thus we have
\[ fgdm = \lim_{n \to \infty} \lim_{k \to \infty} \int f_n g_k \, dm \]

\[ = \lim_{n \to \infty} \lim_{k \to \infty} (\int f_n \, dm \cdot \int g_k \, dm) \]

\[ = \lim_{n \to \infty} \int f_n \, dm \cdot \lim_{k \to \infty} \int g_k \, dm \]

\[ = \int f \, dm \cdot \int g \, dm \]

**Lemma 1.1.4.** Let \( f \) be a real-valued function in \( H^p(m), 1 \leq p < \infty \). Then \( f \) is equal a.e. (\( m \)) to a constant function.

**Proof.** Let \( c = \int f \, dm \). Thus \( \int (f-c) \, dm = 0 \). Now \( m \) is multiplicative on \( A \), and hence, by Lemma 1.1.3 (b)

\[ \int (f-c) g \, dm = 0, \quad \forall \ g \in A. \]

Since \( f \) is real-valued, this implies that

\[ \int (f-c) g \, dm = 0, \quad \forall \ g \in \overline{A}. \]

Thus \( \int (f-c) h \, dm = 0, \quad \forall \ h \in A + \overline{A}, \)

and so \( f - c = 0 \), a.e. (\( m \)) by the weak-* density of \( A + \overline{A} \).

We now prove Theorem 1.1.1, henceforth omitting (\( m \)) in such terms as \( L^2(m) \), when no ambiguity arises.
Proof of Theorem 1.1.1. Since $[\text{MA}_0]_2 < M$, $\exists q \neq 0$, $q \in M \cap [\text{MA}_0]_2$.

Without loss of generality, we may normalise $q$ so that $\int |q|^2 dm = 1$.

Let $f \in A$. Then $f - \int fdm \in A_0$, and

$$\int |f|^2 dm = (fq, q) = ([f - \int fdm]q, q) + ([\int fdm]q, q) = \int fdm$$

where the inner product is taken in $L^2$ as a Hilbert space. Hence, by Lemma 1.1.1, $|q| = 1$.

Now $qA \subseteq MA \subseteq MA_0 + M \subseteq M + M = M$, and, since $M$ is closed, $qH^2 = q[A]_2 = [qA]_2 \subseteq M$. Suppose $qH^2 < M$ and let $g \in M \cap qH^2$. Then

$$\int g\bar{q}fdm = 0, \ \forall \ f \in A.$$

Also, since $gA_0 \subseteq [\text{MA}_0]_2$, we have $q \perp gA_0$. That is

$$\int q\bar{g}fdm = 0, \ \forall \ f \in A_0.$$ 

So $\bar{g}q \perp A + A_0$ in $L^2$. Hence, by Lemma 1.1.2, $\bar{g}q = 0$ a.e.

But $|q| = 1$, so $g = 0$ a.e. Hence, since $qH^2$ is closed $M = qH^2$.

That $q$ is essentially unique follows immediately. For if $qH^2 = q'H^2$ ($|q| = |q'| = 1$), then both $\bar{qq}'$ and $\bar{q}'q \in H^2$ and so, by applying Lemma 1.1.4 separately to $\text{Re}(\bar{qq}')$ and $\text{Im}(\bar{qq}')$ we get $\bar{qq}' = c$ a.e. where $c$ is a constant function and $|c| = 1$. 
Remark 1.1.2. Since all logmodular algebras are also weak-* Dirichlet (Theorem 0.3), Theorem 1.1.1 implies Theorem 1.1.2, the invariant subspace theorem for A a logmodular algebra.

Theorem 1.1.2. Let A be a logmodular algebra on a compact Hausdorff space X. Let m be a probability measure on X, multiplicative on A. Suppose that M is a closed subspace of $\mathcal{H}^2$ such that $MA \subseteq M$, and that there exists at least one $g \in M$ such that $\int g \, dm \neq 0$. Then there exists a function $q \in \mathcal{H}^2$ such that

(i) $|q| = 1$

and

(ii) $M = q\mathcal{H}^2$.

The function $q$ is unique (modulo functions which are zero a.e.) up to multiplication by a constant function with absolute value 1.

For A a logmodular algebra and M a closed subspace of $\mathcal{H}^2$, we shall show the equivalence of the hypotheses of Theorem 1.1

Theorem 1.1.2. That is, we shall show

(i) If $M$ is a closed subspace of $\mathcal{H}^2$ and $[MA_0]_2 < M$, then there exists $g \in M$ such that $\int g \, dm \neq 0$, and $MA \subseteq M$;

and

(ii) If $M$ is a closed subspace of $\mathcal{H}^2$ and $MA \subseteq M$, and there exists $g \in M$ such that $\int g \, dm \neq 0$, then $[MA_0]_2 < M$.

In (i), since each function in $A$ can be written as the sum of a function in $A_0$ and a constant function, it is easily seen that
Lemma 1.1.3 ensures that \( m \) is multiplicative on \( \mathbb{H}^2 \), and hence, on \([MA]_2\), and so the strict inclusion of \([MA_0]_2 \) in \( M \) ensures that \( \exists g \in M \) such that \( \int gdm \neq 0 \).

In (ii) we have \( MA_0 \subset MA \subset M \). Since \( M \) is closed, this means that \( [MA_0]_2 \subset M \). However, as in (i), we have \( \int fdm = 0 \ \forall f \in [MA_0]_2 \).

But by hypothesis \( \exists g \in M \) such that \( \int gdm \neq 0 \).

Thus \( g \in M \setminus [MA_0]_2 \) and so \([MA_0]_2 \subset M \).

Remark 1.1.3. In Theorem 1.1.2, the hypothesis that \( \exists g \in M \) such that \( \int gdm \neq 0 \), is an essential one.

We shall give an example to illustrate this point. (Hoffman [1], p.102).

Let \( X \) be the torus. Choose and fix an irrational number \( \alpha \), and let \( A \) be the algebra of all continuous functions \( f \) on \( X \) such that

\[
a_{kn} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta,\psi)e^{-ik\theta}e^{-in\psi}d\theta d\psi
\]

is zero for all pairs of integers \( (k,n) \), save perhaps those belonging to the half plane where \( k + \alpha n \geq 0 \). Now \( A \) is a Dirichlet algebra, and hence both a logmodular algebra and a weak-star-Duichlet algebra. (Theorem 0.1.1 and Theorem 0.1.3).

If \( dm = \frac{1}{4\pi^2} d\theta d\psi \),
$H^2(m)$ is the space of square summable functions on the torus with Fourier series

$$f = \sum_{k+n\alpha \geq 0} a_{kn} e_{kn},$$

where $e_{kn} : (e^{i\theta}, e^{i\psi}) \mapsto e^{ik\theta} e^{in\psi}.$

If we take $M$ to be the subspace of functions for which $a_{00} = 0$, then $M \subset \mathbb{C}$, but $M$ is not of the form $M = qH^2(m)$ for a measurable $q$ such that $|q| = 1$. To see this suppose that $M = qH^2$. Since $|q| = 1$, $q\overline{q} = 1$ and so $H^2 = q^{-1}M = \overline{q}M$. Also, since $1 \in H^2$, $q \in M$. Thus, $\int qdm = 0$ and $\hat{q}(k,n) = 0$, if $k + n\alpha \leq 0$, where $\hat{g}(k,n)$ is the coefficient of $e_{kn}$ in the Fourier series for $q$. Thus,

$$q = \sum_{k+n\alpha > 0} \hat{q}(k,n)e_{kn},$$

and so

$$\overline{q} = \sum_{k+n\alpha < 0} a_{kn} e_{kn}.$$

Let $f \in M$. Then

$$f = \sum_{k+n\alpha \geq 0} \hat{f}(k,n)e_{kn},$$

Hence,
\[ \int q f \, d m = \sum_{p+q \alpha \leq 0} a_{pq} \hat{f}(k,n) e_{k+p,n+q} \]

\[ (p,q) \neq (0,0) \]

\[ k+n\alpha > 0 \]

\[ (k,n) \neq (0,0) \]

\[ \leq \sum_{p+q \alpha \leq 0} a_{pq} \hat{f}(-p,-q) \]

\[ (p,q) \neq (0,0) \]

\[ = \sum_{p+q \alpha = 0} a_{pq} \hat{f}(-p,-q) \]

\[ (p,q) \neq (0,0) \]

which is an empty sum since \( \alpha \) is an irrational number. Hence

\[ \int q f \, d m = 0, \quad \forall \ f \in M, \]

which contradicts \( H^2 = qM \) since \( 1 \in H^2 \).

**Remark 1.1.4.** When \( A \) is the algebra of all continuous complex-valued functions on the unit circle which have analytic extensions to the interior of the unit disc, and \( m \) is the normalised Lebesgue measure, the situation reduced to the case of the shift operator "multiplication by \( j \)."
From this point on, we shall use $A$ to refer to a weak-* Dirichlet algebra on a compact Hausdorff space $X$. We shall refer to Theorem 1.1.1 as the "$L^2$-invariant subspace theorem" and we shall generalise this theorem to $L^p$, $1 \leq p \leq \infty$.

§1.2. Some results concerning outer functions.

Before doing this, however, we need some preliminary results. First we need the concept of an outer function.

**Definition 1.2.1.** A function $h \in H^1 = [A]_1$ is said to be outer if $[hA]_1 = H^1$.

We note the following about outer functions.

**Note 1.2.1.** If $h$ is outer, then $h \neq 0$ a.e. and $\int h \, dm \neq 0$. In particular, $h \notin [hA_0]_1$.

**Proof.** That $h \neq 0$ a.e. follows directly from Definition 1.2.1. If $\int h \, dm = 0$, Lemma 1.1.3 (b) ensures that $[hA]_1 < H^1$ which contradicts Definition 1.2.1.

**Note 1.2.2.** If $h, h'$ are outer and $|h| = |h'|$, then $h = ch'$ where $c$ is a constant function such that $|c| = 1$. 
Proof. We first observe that, since $|h| = |h'|$, $h = qh'$ for some measurable $q$ such that $|q| = 1$. Now, $H^1 = [hA]_1 = [qh'A]_1 = q[h'A]_1 = qH^1$, so both $q$ and $-q \in H'$. Hence, by applying Lemma 1.1.4 separately to $R_{eq}$, $Imq$ we have $q = c$ a.e. and so $h = ch'$.

Note 1.2.3. Let $h \in H^2$. Then $h$ is outer if and only if $[hA]_2 = H^2$.

Proof. (i) Let $[hA]_2 = H^2$. Then, $[hA]_1 = [[hA]_2]_1 = [H^2]_1 = H^1$, and so $h$ is outer.

(ii) Let $h$ be outer. Since $[[hA]_2]_0 = [hA]_2 < [hA]_2$ by Lemma 1.1.3 (b), Theorem 1.1.1 applied to $M = [hA]_2$ gives $[hA]_2 = qH^2$ for some measurable $q$ such that $|q| = 1$. Now, since $h$ is outer,

$$H^1 = [hA]_1 = [[hA]_2]_1 = [qH^2]_1 = q[H^2]_1 = qH^1.$$ 

Hence $q = 1$ and $[hA]_2 = H^2$.

From now on, we shall use $q$ (with or without subscripts or superscripts) to refer to a measurable function everywhere of absolute value 1.
§1.3. Factorisation Theorems.

Theorem 1.3.1. If \( f \in L^2 \) and \( f \notin [fA_0]_2 \), then \( f = qh \), where \( h \in H^2 \) is outer, and \( q \in [fA]_2 \).

Proof. Our assumption implies that \([fA]_2\) is a simply invariant outspace of \( L^2 \). Hence, by Theorem 1.1.1, \([fA]_2 = qH^2\). Thus, \( f = qh \), some \( h \in H^2 \). Now,

\[
q[ha]_2 = [qha]_2 = [fA]_2 = qH^2;
\]

thus we have \([ha]_2 = H^2\) and hence \( h \) is outer (by note 1.2.3).

Also, since \( 1 \in H^2 \), \( q \in qH^2 = [fA]_2 \).

Theorem 1.3.1 is actually a generalisation of a factorisation theorem due to Bernling and Nevanlinna (Bernling [1]), which applies to functions \( f \) in the Hardy space \( H^2 \) for which \( \int |f| dm \neq 0 \).

We now prove

Corollary 1.3.1. If \( f \in L^1 \) and \( |f|^{1/2} \notin [|f|^{1/2}A_0]_2 \), then \( f = qh^2 \), where \( h \in H^2 \) is outer.

Proof. By theorem 1.3.1, \( |f|^{1/2} = q_1h \), where \( h \in H^2 \) is outer.

Thus we have, for

\[
\text{sgnf} = \begin{cases} 
\frac{f}{|f|} & \text{if } f \neq 0 \\
0 & \text{if } f = 0 
\end{cases}
\]
Corollary 1.3.2. If \( f \in L^1 \), then \( |f|^{1/2} \not\in [|f|^{1/2} A_0]_2 \) if and only if \( f \not\in [f A_0]_1 \).

Proof. Suppose first that \( |f|^{1/2} \in [|f|^{1/2} A_0]_2 \). Let \( f_1 = (sgn f)|f|^{1/2} \). Then

\[
f = f_1 |f|^{1/2} \in f_1 [|f|^{1/2} A_0]_2 \subseteq [f_1 |f|^{1/2} A_0]_1 = [f A_0]_1.
\]

Thus, if \( f \not\in [f A_0]_1 \), then \( |f|^{1/2} \not\in [|f|^{1/2} A_0]_2 \). Now suppose \( |f|^{1/2} \not\in [|f|^{1/2} A_0]_2 \). Then, by Corollary 1.3.1 \( f = qh^2 \) by , where \( h \in H^2 \) is outer. We need to show that \( h^2 \in H^2, H^2 \subseteq H^1 \) (Lemma 1.1.3 (a)) is outer. Now, since \( h \) is outer,

\[
hA \subseteq hH^2 = h[hA]_2 \subseteq [h^2 A]_1.
\]

Hence \( H^1 = [[hA]_2]_1 \subseteq [h^2 A]_1 \). But, by Lemma 1.1.3 (a)
\( h^2 \in H' \), so \( [h^2 A]_1 \subseteq H' \), and so we have \( [h^2 A]_1 = H^4 \) and \( h^2 \in H' \) is outer. Now \( [f A_0]_1 = [qh^2 A_0]_1 = qH^1_0 \), but \( f = qh^2 \not\in [f A_0]_1 \), by Note 1.2.1, and hence result.

Corollary 1.3.3. If \( f \in L^1 \) and \( f \not\in [f A_0]_1 \), then \( f = Fh \), where \( h \in H^2 \) is outer, \( |h|^2 = |f| \) and \( F \in [f A]_1 \cap L^2 \).
Proof. By Corollary 1.3.1 and Corollary 1.3.2, we have \( f = qh^2 \), where \( h \in H^2 \) is outer. Let \( F = qh \). Then \( F \in L^2 \). Thus \( f = Fh \) with \( h \in H^2 \) outer, and, since \( |q| = 1 \), \( |h|^2 = |f| \).

Further, \( F \in FH^2 = F[hA]_2 \subset [PhA]_1 = [fA]_1 \).

Remark 1.3.1. Since \( m \) is multiplicative on \( A \), we have \( A \) and \( \overline{A_0} \) as orthogonal subsets of \( L^2 \). It follows from Lemma 1.1.2 that \( L^2 = H^2 \oplus \overline{H_0} \); or, equivalently,

\[
H^2 = \{ f \in L^2 : \int fgdm = 0 \ \forall \ g \in A_0 \}. \tag{i}
\]

Since \( f \in H^1 \) implies \( \int fgdm = 0 \ \forall \ g \in A_0 \) (by Lemma 1.1.3 (b)), we see that \( H^2 \supseteq H^1 \cap L^2 \). Trivially, \( H^2 \cap H^1 = L^2 \), and so we have

\[
H^2 = H^1 \cap L^2. \tag{ii}
\]

Since, by definition, \( H^\infty = H^2 \cap L^\infty \) we have also

\[
H^\infty = \{ f \in L^\infty : \int fgdm = 0 \ \forall \ g \in A_0 \}. \tag{iii}
\]

and

\[
H^\infty = H \cap L^\infty. \tag{iv}
\]

We now prove

Corollary 1.3.4. \( H^1 = \{ f \in L^1 : \int fgdm = 0 \ \forall \ g \in A_0 \} \).
Proof. Clearly, if $f \in H^1$ then Lemma 1.1.3 (b) ensures that
\[ \int_{\mathcal{A}} f g \, dm = 0 \quad \forall \ g \in A_0. \]
Conversely, consider $f \in L^1$ such that
\[ \int_{\mathcal{A}} f g \, dm = 0, \quad \forall \ g \in A_0. \]
By replacing $f$ by $f + c$ for some constant function $c$ if necessary, we may assume that $\int_{\mathcal{A}} f \, dm \neq 0$. Then
\[ f \notin [fA_0]_1 \quad \text{and so, by Corollary 1.3.3,} \quad f = Fh \quad \text{where} \quad F \in [fA]_1 \quad \text{L}^2, \]
and $h \in H^2$ is outer. Since $F \in [fA]_1$, it follows that
\[ \int_{\mathcal{A}} Fg \, dm = 0, \quad \forall \ g \in A_0. \]
Since also $F \in L^2, F \in H^2$ (by (i), Remark 1.3.1). Thus, $f = Fh \in H^2, H^2 \subset H^1$, by Lemma 1.1.3 (a), and the proof is complete.

Corollary 1.3.5. If $f \in L^1$ and $f \notin [fA_0]_1$, then $f = qh_1$, where $h_1 \in H$ is outer, and $q \in [fA]_1$. The converse is also true.

Proof. Suppose $f \notin [fA_0]_1$. Then, by Corollary 1.3.2,
\[ |f|^{1/2} \notin [|f|^{1/2} A_0]_2 \quad \text{and so, by Corollary 1.3.1,} \quad f = qh^2, \]
when $h \in H^2$ is outer. As in the proof of Corollary 1.3.2, if $h \in H^2$ is outer, then $h^2 \in H^1$ is also outer. Let $h_1 = h^2$. Then $f = qh_1$. Also,
\[ q \in qh_1 = q[h,A]_1 = [qh,A] = [fA]_1 \]
and hence $f$ is of the stated form.
The converse follows from the fact that, since $h_1$ is outer, $h_1 \notin [h_1 A_0]_1$ (Note 1.2.1), and so, $f = qh_1 \notin q[h_1 A_0]_1 = [qh_1 A_0]_1 = [f A_0]_1$.

§1.4. $L^p$-invariant subspace Theorem.

Consider $p$ such that $1 \leq p < 2$. Define the number $r$ by $\frac{1}{r} + \frac{1}{2} = 1/p$. Then $p/r + p/2 = 1$, and so $r/p$ and $2/p$ are conjugate indices. We now prove the following two lemmas.

**Lemma 1.4.1.** If $f_n, f \in L^2$, and $g \in L^r$, then $f_n g, fg \in L^p$.

If, further, $(f_n) \subset L^2$ converges to $f$ in $L^2$, then $(f_n g) \subset L^p$ converges to $fg$ in $L^p$.

**Proof.** Since $f \in L^2$ and $g \in L^r$ we have $f^p \in L^{2/p}$ and $g^p \in L^{r/p}$. Thus, by the Hölder inequality, $(fg)^p = (fg) \in L^1$, and hence $fg \in L^p$. Similarly, $f_n g \in L^p$. Again by the Hölder inequality, we have

$$\| (f_n - f) g \|_p \leq \| f_n - f \|_2 \| g \|_r,$$

and hence

$$\| f_n g - fg \|_p = \| (f_n - f) g \|_p \leq \| f_n - f \|_2 \| g \|_r$$

and so, since $(f_n) \subset L^2$ converges to $f$ in $L^2$, we have $(f_n g) \subset L^p$ convergent to $fg$ in $L^p$. 

Lemma 1.4.2. Let \( 1 \leq p < 2 \). Define \( r \) as before. If \( f \in L^p \) and \( f \notin [fA_0]^p \), then \( f = Fh \), where \( h \in H^2 \) is outer, and \( F \in [fA]^p \cap L^r \).

**Proof.** Let \( f_1 = (\text{sgn} f)|f|^{p/r} \) and \( f_2 = |f|^{p/2} \). Then \( f_1 \in L^r \), \( f_2 \in L \) and \( f = f_1 f_2 \). Also, \( f_2 \in [f_2 A_0]^2 \), since, if \( f_2 \in [f_2 A_0]^2 \), then \( f = f_1 f_2 \in f_1 [f_2 A_0]^2 \subseteq [f_1 f_2 A_0]^p = [fA_0]^p \), by Lemma 1.4.1, which is contrary to hypothesis. Hence, by Theorem 1.3.1, we have \( f_2 = qh \), where \( q \in [f_2 A_0]^2 \), and \( h \in H^2 \) is outer. Let \( F = f_1 q \). Then, since \( f_1 \in L^r \) and \( |q| = 1 \), \( F \in L^r \). Also, \( F \in f_1 q H^2 = f_1 q[hA]_2 \subseteq [f_1 q hA]^p = [fA]^p \), by Lemma 1.9.1. Clearly \( f = Fh \) and hence result.

Corollary 1.4.1. If \( 1 \leq p \leq \infty \), then \( H^p = H^1 \cap L^p \).

**Proof.** We have already shown in Remark 1.3.1 that the statement is true for \( p = 2, \infty \). It is trivial for \( p = 1 \). We shall prove it to be true for \( 1 < p < 2 \) by use of Lemma 1.4.2; and for \( p > 2 \) by a duality argument.

Let \( 1 < p < 2 \). It is clear that \( H^p \subseteq H^1 \cap L^p \). To show the reverse inclusion, consider any \( f \in H^1 \cap L^p \). We may suppose, by considering \( f + c \) where \( c \) is a constant function, if
necessary, that \( \int f \, dm \neq 0 \). Then \( f \notin [fA]_p \) and so, by Lemma 1.4.2, \( f = F h \) when \( h \in H^2 \) is outer and \( F \in [fA]_p \cap L^r \), \( r \) defined as before. Since \( 1 < p < 2 \), \( r > 2 \), and so \( F \in L^2 \). Also, \( F \in [fA]_p \subseteq H^1 \) since \( f \in H^1 \). Thus, \( F \in H^1 \cap L^2 = H^2 \) ((i)

Remark 1.3.1). In particular, since \( 1 < p < 2 \), \( F \in H^p \), and so \( f = F h \in F H^2 = F[A]_2 \). But, since \( F \in L^r \), \( r > 2 \),

\( F[A]_2 \subseteq [fA]_p \subseteq H^p \) and so \( f \in H^p \). Therefore \( H^p = H^1 \cap L^p \)
for every \( p \) satisfying \( 1 < p < 2 \). Now let \( 2 < p < \infty \).

Again it is clear that \( H^p \subseteq H^1 \cap L^p \). We wish to show equality here. It suffices to show that, if \( g \in L^p' \) (where \( \frac{1}{p} + \frac{1}{p'} = 1 \))
is such that \( g \perp H^p \), then \( g \perp H^1 \cap L^p \). We have, by hypothesis,

\[ \int g f \, dm = 0, \quad \forall \ f \in H^p, \quad \text{and so, by Corollary 1.3.4}, \quad \overline{g} \in H^1. \]

Since also \( \overline{1} \in H^p \), \( \overline{g} \in H^1 _0 \). Thus, since \( 1 < p' < 2 \), what we have already proved shows that

\[ \overline{g} \in H^1 _0 \cap L^p' \subseteq H^1 \cap L^p' = H^p'. \]

Hence \( \exists \) sequence \( \{g_n\} \subseteq A_0 \) which converges to \( \overline{g} \) in \( L^p' \),
so that

\[ \lim_{n \to \infty} \int f g_n \, dm = \int f \overline{g} \, dm, \quad \forall \ f \in H^1 \cap L^p. \]

Thus, making use of Lemma 1.1.3 (b), \( g \perp H^1 \cap L^p \) and our proof is completed.
Corollary 1.4.2. If $1 \leq p \leq \infty$, then

$$H^p = \{ f \in L^p : \int fgdm = 0 \forall g \in A_0 \}.$$ 

Proof. This fact follows directly from Corollary 1.4.1 and Corollary 1.3.4 and is a generalisation of Corollary 1.3.4.

We now wish to prove the invariant subspace theorem for general $p$, $1 \leq p \leq \infty$. Here, a subspace $M$ of $L^p(1 \leq p < \infty)$ is said to be simply invariant if $M$ is norm closed and $[MA_0]^p_M < M$. (This agrees with our earlier definition for $p = 2$.) A subspace $M$ of $L^\infty$ is simply invariant if $M$ is weak-* closed and $[MA_0]^* < M$. We now have

Theorem 1.4.1. ($L^p$-invariant subspace Theorem). Every simply invariant subspace $M$ of $L^p$, $1 \leq p \leq \infty$, is of the form $M = qH^p$ for some measurable $q$ such that $|q| = 1$ (and trivially conversely).

Proof. For $p = 2$, Theorem 1.4.1 reduces to Theorem 1.1.1.

For $1 \leq p < 2$ we shall use Lemma 1.4.2. For $p > 2$ we shall use a duality argument.

Let $1 \leq p < 2$. Put $N = M \cap L^2$. Clearly $N$ is a closed subspace of $L^2$. We first show that $N$ is non-zero. Since $[MA_0]^p_N < M$, $\exists f \in M \backslash [MA_0]^p_M$. Thus $f \notin [fA_0]^p_M$ and so, by
Lemma 1.4.2, we can write $f = Fh$ where $h \in H^2$ is outer, and $F \in [fA]_p \cap L^r$, $r$ defined as before. Now,

$$[fA]_p \cap L^r \subset [MA]_p \cap L^2 \subset M \cap L^2 = N$$

so $F \in N$ and $N$ is non-zero. Also, $F \notin [NA^*]_2$, since if $F \in [NA^*]_2$,

$$f = Fh \in FH^2 = F[A]_2 \subset [FA]_p \subset [[NA^*]_2 A]_p \subset [NA^*]_p \subset [MA^*]_p$$

contrary to our assumption on $f$. Combined with the fact that $[NA^*]_2 \subset [MA^*]_2 \subset M \cap L^2 = N$, this shows that $[NA^*]_2 \subset N$ is simply invariant. Hence, by Theorem 1.1.1, $N = qH^2$ for some measurable $q$ such that $|q| = 1$. We shall now show that $M = qH^p$

with the same $q$. We have, already, $qA \subset qH^2 = N \subset M$, so $qH^p = q[A]_p = [qA]_p \subset [M]_p = M$. If we take $f \in M \setminus [MA^*]_p$ as before, we know that $f$ can be written as $f = Fh$ where $h \in H^2$, and, as shown earlier, $F \in N = qH^2$. Thus, $\overline{qF} \in H^2$ and $\overline{qf} = \overline{qFh} \in H^2 \cdot H^2 \subset H^1$, by Lemma 1.1.3 (a). Hence $\overline{qf} \in H^1 \cap L^p = H^p$, by Corollary 1.4.1, and so $f \in qH^p$. This gives us the inclusion $M \setminus [MA^*]_p \subset qH^p$. Now, if $g \in [MA^*]_p$ and $f \in M \setminus [MA^*]_p$, then $f + g \in M \setminus [MA^*]_p$. From the preceding discussion we then have $f$, $f + g \in qH^p$. Hence $g \in qH^p$ also. Thus $M = qH^p$ and the theorem is proved for $1 \leq p < 2$. 
Now let $2 < p \leq \infty$. Let $p'$ be the conjugate index of $p$. Thus $\frac{1}{p} + \frac{1}{p'} = 1$ and so $1 \leq p' < 2$.

Let $M_0 = [M_{A_0}]_p$ or $[M_{A_0}]_*$ according as $2 < p < \infty$ or $p = \infty$, and let

$$N = \{ f \in L^{p'} : \int fg dm = 0, \forall \ g \in M_0 \}.$$ 

Clearly $N$ is a closed subspace of $L^{p'}$ and $NA \subset N$. We shall show that $[M_{A_0}]_p', \subset N$. Since $M_0 \subset M$ there exists a non-zero continuous (weak-* continuous if $p = \infty$) linear functional $\phi$ on $M$ which annihilates $M_0$. $\phi$ is realised by a function $f \in L^{p'}$ and, by the choice of $\phi$, $f \in N$. Since $\phi$ is non-zero on $M$, there exists an $g \in M$ such that $\int fg dm \neq 0$. This fact, combined with the definition of $N$ implies that $f \notin [M_{A_0}]_p'$. Hence $N$ is simply invariant. Since $1 \leq p' < 2$, the first part of this proof applies and we can write $N = qH^{p'}$ for some measurable $q$ such that $|q| = 1$.

By duality theory,

$$M_0 = \{ g \in L^p : \int fg = 0, \forall f \in N \}.$$ 

This fact, combined with Corollary 1.4.2, leads to $M_0 = qH^{p'}$.

We shall show that $M = qH^{p}$. If $f \in \overline{qM}$ and $g \in \Lambda_0$, then

$$fg \in \overline{qM_{A_0}} \subset \overline{qM_0} = H^{p}_0.$$
and so, \( \int fgdm = 0 \), and hence, by Corollary 1.4.2 \( f \in H^p \). This shows that \( qM \subset H^p \), or \( M \subset qH^p \).

Now, since each \( f \in H^p \) can be written in the form \( f = g + c \) where \( g \in H^p_0 \) and \( c \) is a constant function, \( H^p_0 \) is a subspace of \( H^p \) of codimension 1, so that \( M_0 = qH^p_0 \) is a subspace of \( gqH^p \) of codimension 1. Hence, either \( M = qH^p \) or \( M = M_0 \). The latter is impossible since \( M \) is simply invariant, and so \( M = qH^p \) and our proof is completed.

We shall now add to our discussion of outer functions. We prove

**Corollary 1.4.3.** \( h \in H^p, \ 1 \leq p < \infty \) is outer if and only if

\[ [hA]_p = H^p. \]

**Proof.** (i) Let \([hA]_p = H^p\). Then

\[ [hA]_1 = [[hA]_p]_1 = [H^p]_1 = H^1, \]

and so \( h \) is outer.

(ii) Let \( h \) be outer. Thus \([hA]_1 = H^1\).

Since \( h \notin [hA^0]_p \), \([hA]_p \) is an invariant subspace of \( L^p \), and so, by Theorem 1.4.1, \([hA]_p = qH^p \), for some measurable \( q \) such that \( |q| = 1 \). But \( H^1 = [hA]_1 = [[hA]_p]_1 = [qH^p]_1 = q[H^p]_1 = qH^1. \)
Hence $q$ and $\overline{q} = q^{-1}$ belong to $H^1$ and Lemma 1.1.4 implies that $q = \text{constant function} = 1$. Thus $[hA]_p = qH^p = H^p$.

**Corollary 1.4.4.** If $f \in L^p$, $1 \leq p < \infty$ and $f \notin [fA_0]^p$, then $f = qh$ where $h \in H^p$ is outer and $q \in [fA]^p$.

**Proof.** Our assumption implies that $[fA]^2$ is a simply invariant subspace of $L^p$. Hence, by Theorem 1.4.1, $[fA]^p = qH^p$, and so $f = gh$ for some $h \in H^p$. Now

$$q[hA]_p = [qhA]_p = [fA]^p = qH^p$$

and so we have $[hA]_p = H^p$, and hence, by Corollary 1.4.3, $h$ is outer.

Since $1 \in H^p$, $q \in qH^p = [fA]^p$.

§1.5. $L^2$-invariant subspace Theorem implies $A + \overline{A}$ weak* dense in $L^\infty$.

We show first that the $L^2$-invariant subspace Theorem (Theorem 1.1.1) implies that $A + \overline{A}_0$ is norm dense in $L^2$; and then that $A + \overline{A}_0$ norm dense in $L^2$ implies $A + \overline{A}$ is weak-* dense in $L^\infty$.

Assume Theorem 1.1.1 holds. We prove the following with this assumption.
Theorem 1.5.1. \( L^2 = H^2 \oplus \overline{H^2}_0 \).

Proof. Let \( M = L^2 \ominus \overline{H^2}_0 \). Then \([MA_0]_2 < M\) and so, by Theorem 1.1.1, \( M = qH^2 \), some measurable \( q \) such that \( |q| = 1 \).

Since \( \frac{1}{2} \in M \), \( \frac{1}{2} \in H^2 \). Let \( \overline{c} = \int qdm \). Then \( q - \overline{c} \in H^2_0 \) and so \( q - c \in \overline{H^2}_0 \). But \( q, c \in M \), so \( q - c \in M \), and thus \( q - c \in \overline{H^2}_0 \).

It follows then that \( q - c = 0 \) and \( M = cH^2 = H^2 \).

Corollary 1.5.1. That \( A + \overline{A}_0 \) is norm dense in \( L^2 \) follows directly from Theorem 1.5.1.

We need also

Corollary 1.5.2. \( H^1 \cap L^2 = H^2 \).

Proof. Clearly \( H^2 \subseteq H^1 \cap L^2 \). Now consider \( f \in H^1 \cap L^2 \).

Since \( f \in H^1 \), \( f \perp \overline{A}_0 \) and so \( f \perp \overline{H^2}_0 \). Thus, by Theorem 1.5.1, \( f \in H^2 \) and so \( H^1 \cap L^2 \subseteq H^2 \).

Corollary 1.5.3. \( H^1 = \{ f \in L^1 : \int fgdm = 0, \forall g \in A_0 \} \).

Proof. Clearly, if \( f \in H^1 \), then \( \int fgdm = 0, \forall g \in A_0 \). Now consider \( f \in L^1 \) such that \( \int fgdm = 0, \forall g \in A_0 \). By replacing \( f \) by \( f + c \) for some constant function \( c \), if necessary, we may
assume that \( \int f \, dm \neq 0 \). Then \( f \in [fA_0]_1 \) and so, by Corollary 1.3.3, \( f = Fh \) where \( F \in [fA]_1 \cap L^2 \) and \( h \in H^2 \) is outer. Since \( F \in [fA]_1 \), it follows that \( \int fg \, dm = 0 \), \( \forall \ g \in A_0 \). Since also \( F \in L^2 \), we have, by Theorem 1.5.1 \( F \in H^2 \). Thus \( f = Fh \in H^2. H^2 \subset H^1 \) by Lemma 1.1.3 (a).

We are now in a position to show that the truth of Theorem 1.1.1 implies that \( A + \overline{A} \) is weak-* dense in \( L^\infty \). In doing this we shall use the Jensen inequality (Corollary 2.1.3), our proof of which will depend only on Theorem 1.1.1, and Corollary 1.5.3 which we have just shown to follow from Theorem 1.1.1.

**Theorem 1.5.2.** \( A + \overline{A} \) is weak-* dense in \( L^\infty(m) \).

**Proof.** We need only show that any \( g \in L^1(m) \), such that \( \int fg \, dm = 0 \) for every \( f \in A \). Suppose \( g \neq 0 \) a.e. (m).

By Corollary 1.5.3, \( g \in H^1 \); and, since \( 1 \in A \), \( \int g \, dm = 0 \) and so \( g \in H^1_0 \). Thus \( 1 - tg \in H^1 \), \( \forall \) real \( t \), and so, by Corollary 2.1.3 (Jensen inequality), we have

\[
\int \log |1-tg| \, dm \geq \log \left| \int (1-tg) \, dm \right| = 0.
\]

Thus, by Lemma 0.3 \( g = 0 \) a.e. (m).

From this we can see that the conditions necessary for \( A \) to be weak-* Dirichlet are the best possible such that the invariant subspace theorem is true for this class of algebras.
CHAPTER 2.

H\textsuperscript{∞} AS A LOGMODULAR ALGEBRA; SZEGO'S THEOREM.

§2.1 H\textsuperscript{∞} as a logmodular algebra.

We now look more closely at H\textsuperscript{∞}. We shall show that H\textsuperscript{∞} is (isomorphic to) a logmodular algebra on the maximal ideal space, \( \mathcal{M} \), of \( L\textsuperscript{∞} \), and then apply to H\textsuperscript{∞} some known results about logmodular algebras. We first prove

Theorem 2.1.1. Let A be a weak-\( \star \) Dirichlet algebra. Then H\textsuperscript{∞} = [A]\textsuperscript{*}.

Proof. We first show that H\textsuperscript{∞} \supseteq [A]\textsuperscript{*}. Consider any sequence \( \{f_n\} \subseteq H\textsuperscript{∞} \), which is weak-\( \star \) convergent to \( f \in L\textsuperscript{∞} \). Then

\[
\lim_{n \to \infty} \int g f_n \, dm = \int g f \, dm, \quad \forall \ g \in L^1.
\]

In particular,

\[
\lim_{n \to \infty} \int g f_n \, dm = \int g f \, dm, \quad \forall \ g \in A_{0}.
\]

But, by Corollary 1.4.2, \( \int g f_n = 0 \), \( \forall \ g \in A_{0} \), for each \( f_n \). Hence \( \int g f \, dm = 0 \), \( \forall \ g \in A_{0} \), and, again by Corollary 1.4.2, \( f \in H\textsuperscript{∞} \).

To establish H\textsuperscript{∞} \subseteq [A]\textsuperscript{*} we shall show that every linear functional \( \phi \) on L\textsuperscript{∞} given by an L\textsuperscript{1} function \( f \), which vanishes on A also vanishes on H\textsuperscript{∞}. Thus \( \phi(g) = \int g f \, dm = 0 \), \( \forall \ g \in A \),
and so, by the use of Corollary 1.3.4, \( f \in H^1 \). Since also
\( 1 \in A, \ f \in H^1_0 \). Hence,
\[
\forall \ g \in H^\infty, \ \int f g dm = \int f dm \cdot \int g dm = 0
\]
and the theorem is proved.

**Corollary 2.1.1.** For every \( p \) satisfying \( 1 \leq p < \infty \),
\[
H^\infty \subseteq H^p \quad \text{and} \quad [H^\infty]_p = H^p.
\]

**Proof.** Since every norm closed subspace of \( L^p \), \( 1 \leq p < \infty \) is
also weakly closed, \( H^\infty \subseteq H^p \), \( 1 \leq p < \infty \). Thus we have, since
\( H^p \) is closed, \( [H^\infty]_p \subseteq H^p \). Now \( H^\infty \supseteq A \), so \( [H^\infty]_p \supseteq [A]_p = H^p \),
\( 1 \leq p < \infty \). Thus \( [H^\infty]_p = H^p \) whenever \( 1 \leq p < \infty \). If \( p = \infty \),
\( H^\infty \) is weak-* closed (Theorem 2.1.1) and therefore norm closed
and so \( [H^\infty]_\infty = H \).

Now let \( V = \{ f : \frac{1}{f} \text{ and } f \in H^{\infty} \} \). Clearly, if \( f \in V \),
then \( f^{+n} \in V, \forall n \). Let
\[
\log |V| = \{ \log |f| : f \in V \}.
\]
Also, let \( L^\infty_n \) denote the set of all real-valued functions in
\( L^\infty \). Then we have
Lemma 2.1.1. \( \log |V| = L^\infty_R \).

**Proof.** Clearly \( \log |V| \subseteq L^\infty_R \). Now consider \( u \in L^\infty_R \). Then, since \( 1 \in H^2_0 \), \( e^u \notin [e^u A_0]_2 = e^u H^2_0 \). Hence, by Theorem 1.3.1, \( e^u = q_h \), where \( h \in H^2 \) and \( q \in [e^u A]_2 = e^u H^2 \). Thus \( q = e^u h' \), and, since \( |q| = 1 \), \( h, h' \in L^\infty \) and so \( h, h' \in H^2 \cap L^\infty = H^\infty \).

But, now, \( e^u = q_h = e^u h'h \) so that \( h'h = 1 \). Thus \( h \in V \) and \( u = \log |h| \in \log |V| \).

**Remark 2.1.1.** Under pointwise operations and the essential supremum norm, \( L^\infty(m) \) is a Banach Algebra. Let \( M \) be its maximal ideal space. We know (Hoffman [1], p.169ff) that \( M \) is a compact Hausdorff space and that the Gelfand mapping \( f \mapsto \hat{f} \) is an isometric isomorphism from \( L^\infty(m) \) onto \( C(M) \), the space of all continuous functions on \( M \), and this mapping preserves complex conjugation.

Since this isomorphism is onto, every function in \( C(M) \) is of the form \( \hat{f} \) for some \( f \in L^\infty(m) \). If we let \( \hat{\phi}(f) = \int f dm \), we get a bounded linear functional on \( C(M) \). Then \( \exists \) a Radon measure \( \hat{m} \) on \( M \) such that

\[
\hat{\phi}(f) = \int \hat{f} \hat{m} = \int f dm, \quad \forall f \in L^\infty(m).
\]
We now look at the relationship between $L^p(m)$ and $L^p(\hat{m})$, $1 \leq p < \infty$. First note that the Gelfand mapping preserves positive powers of non-negative elements, and, since it preserves complex conjugation, (Hoffman [1], p.170), it also preserves absolute values.

Consider $f \in L^p(m)$. Since $L^\infty(m)$ is dense in $L^p(m)$, there exists a sequence $\{g_n\} \subset L^\infty(m)$ convergent to $f$ in the $L^p(m)$-norm. Hence, we have that $\{g_n\}$ is a Cauchy sequence; that is

$$\lim_{n,k \to \infty} \int |g_n - g_k|^p dm = 0.$$  

Thus

$$\lim_{n,k \to \infty} \int |\hat{g}_n - \hat{g}_k|^p dm = 0$$  

and so $\{\hat{g}_n\} \subset L^p(\hat{m})$ is a Cauchy sequence. Let $\hat{f}$ denote the limit of $\{\hat{g}_n\}$ in $L^p(\hat{m})$. Consider any other sequence $\{h_n\} \subset L^\infty(m)$ which converges to $f$ in the $L^p(m)$-norm. Then we can show that

$$\lim_{n \to \infty} \int |g_n - h_n|^p dm = 0$$  

and hence

$$\lim_{n \to \infty} \int |\hat{g}_n - \hat{h}_n|^p dm = 0.$$  

Thus $\hat{f}$ is uniquely determined by $f$ (modulo functions equal to zero a.e. $\hat{m}$).

Conversely, consider $g \in L^p(\hat{m})$. Then there exists a sequence $\{g_n\} \subset C(\hat{M})$ such
that \{g_n\} converges to \(g\) in the \(L^p(m)\)-norm. Hence

\[
\lim_{n, k \to \infty} \int |g_n - g_k|^p dm = 0,
\]

and so \(\{g_n\} \subseteq L^\infty(m)\) is a Cauchy sequence in \(L^p(m)\). Suppose \(\{g_n\}\) converges in the \(L^p\)-norm to \(f \in L^p(m)\). Then, by our previous argument, \(\{\hat{g}_n\}\) converges to \(\hat{f} \in L^p(\hat{m})\) in the \(L^p(\hat{m})\)-norm, and so \(g = \hat{f}\).

Thus, the mapping \(f \mapsto \hat{f}\) sets up an isometry between \(L^p(m)\) and \(L^p(\hat{m})\) and this isometry is an extension of the mapping \(f \mapsto \hat{f}\) from \(L^\infty(m)\) onto \(C(\hat{m})\).

Now suppose \(\hat{f} \in L^\infty(\hat{m})\) is the image of \(f \in L^p(m)\). Consider \(g \in L^1(m)\). Since \(L^\infty(m)\) is dense in \(L^1(m)\), \(\exists \\{g_n\} \subseteq L^\infty(m)\) which converges to \(g\) in the \(L^1(m)\)-norm. Noting that \(\hat{gf} = \hat{gf}\), we have

\[
\int |g_n f - g_k f| dm = \int |\hat{g}_n \hat{f} - \hat{g}_k \hat{f}| d\hat{m} \leq ||f||_\infty \int |g_n - g_k| dm
\]

and so \(\{g_n f\}\) converges to \(gf\) in the \(L^1(m)\)-norm. but

\[
\int |g_n f| dm \leq \int |g_n f| dm = ||g_n f||_1 \leq ||\hat{f}||_\infty ||g_n||_1.
\]

Hence \(\int |gf| dm \leq ||\hat{f}||_\infty ||g||_1\), so that \(f \in L^\infty(m)\) and hence \(\hat{f} \in C(\hat{m})\). This shows that every function in \(L^\infty(\hat{m})\) is equal a.e. \((m)\) to a continuous function.
Let \( \hat{\mathcal{H}}_\infty = \{ \hat{f} : f \in \mathcal{H}_\infty \} \). \( \mathcal{H}_\infty \) is a subalgebra of \( \mathcal{C}(\mathcal{M}) \) closed under the supremum norm, since \( \mathcal{H}_\infty \) is a weak-* closed, and hence norm closed, subalgebra of \( L_\infty(\mathcal{M}) \).

We have already noted that the mapping \( f \mapsto \hat{f} \) preserves absolute values and positive powers of non-negative elements. By the Stowe-Weierstrass Theorem, for every \( f \in V \), \( \log|f| \) is the uniform limit of polynomials of the form

\[
\sum_{k=0}^{n} a_k |f|^k.
\]

But from above

\[
(\sum_{k=0}^{n} a_k |\hat{f}|^k)^\sim = \sum_{k=0}^{n} a_k |\hat{f}|^k,
\]

and we may conclude that \( (\log|f|)^\sim = \log|\hat{f}| \).

Thus, we may restate Lemma 2.1.1 as

\[
\log|V| = \mathcal{C}_R(\mathcal{M});
\]

where \( \hat{V} = \{ \text{set } \hat{f} : \hat{f} \text{ and } 1/\hat{f} \in \hat{\mathcal{H}}_\infty \} \). Since \( \mathcal{C}_R(\mathcal{M}) \) is a separating algebra of \( \mathcal{C}_R(\mathcal{M}) \) so is \( \log|\hat{V}| \), and hence also is \( \hat{\mathcal{H}}_\infty \).

Immediately we have

**Theorem 2.1.2.** \( \hat{\mathcal{H}}_\infty \) is a logmodular algebra on \( \mathcal{M} \), the maximal ideal space of \( L_\infty \).

Because \( \mathcal{H}_\infty \) is isomorphic to \( \hat{\mathcal{H}}_\infty \) we can omit the "\( \hat{\mathcal{H}}_\infty \)" and consider \( \mathcal{H}_\infty \) as a logmodular algebra on the maximal ideal space \( L_\infty \).
Appeal to Theorem 7.1 in Hoffman [2] now yields

**Corollary 2.1.2.** The maximal ideal space $M$ of $L^\infty$ can be embedded in that of $H^\infty$ as its Shilov boundary.

We also have, by Theorem 0.2, that there exists an unique Arens-Singer measure $\hat{m}$ such that $\int \hat{f} \hat{m}$ represents a complex homomorphism on $\hat{H}^\infty$. We can thus prove

**Corollary 2.1.3.** (Jensen inequality).

$$ \log |\int f \hat{m}| \leq \int \log |f| \hat{m}, \forall f \in H^\infty. $$

**Proof.** Let $\hat{f} \in \hat{H}^\infty$ and $\hat{g} > 0$. Then $\log(|\hat{f}| + \hat{g}) \in C_R(\hat{U})$. Hence, $\exists u \in \log |\hat{V}|$ such that

$$ w - \hat{g} < \log(|\hat{f}| + \hat{g}) < u + \hat{g} \quad (1) $$

If $u = \log|\hat{g}|, \hat{g} \in \hat{V}$, let $\hat{h} = \hat{f}^{-1}$. Then $\hat{h} \in \hat{H}^\infty$, and,

by the right hand side of (1), $\log |\hat{h}| \leq \hat{g}$, that is,

$$ |\hat{h}| \leq \exp o \hat{g} \quad \text{on } M \text{ and so} $$

$$ |\int \hat{h} \hat{m}| \leq \int |\hat{h}| \hat{m} < \int (\exp o \hat{g}) \hat{m} = e^{\hat{g}}. $$

Thus,

$$ |\int \hat{f} \hat{m}| |\int \hat{g} \hat{m}|^{-1} < e^{\hat{g}}, \quad \text{and so} $$

$$ \log |\int \hat{f} \hat{m}| - \log |\int \hat{g} \hat{m}| < \hat{g}. $$
Now, since \( \hat{m} \) is an Arens-Singer measure and \( \hat{g} \in \hat{V} \), we have

\[
\log|\int \hat{g} \, \hat{d}m| = \int \log|\hat{g}| \, \hat{d}m = \int u \, \hat{d}m.
\]

By the left hand side of (1)

\[
\int u \, \hat{d}m < \varepsilon + \int \log(|\hat{f}| + \delta) \, \hat{d}m.
\]

Thus, we have

\[
\log|\int \hat{f} \, \hat{d}m| < 2\varepsilon + \int \log(|\hat{f}| + \delta) \, \hat{d}m.
\]

Let \( \varepsilon \) tend monotonically to zero to obtain

\[
\log|\int \hat{f} \, \hat{d}m| < \int \log|\hat{f}| \, \hat{d}m, \quad \forall \hat{f} \in \hat{H}^\infty
\]

and so

\[
\log|\int f \, d\hat{m}| < \int \log|f| \, dm, \quad \forall f \in H^\infty.
\]

In particular, we have

\[
\log|f \, dm| < \int \log|f| \, dm, \quad \forall f \in A.
\]

Since \( m \) is an Arens-Singer measure we get equality for \( f \in V \).

§2.2 More about Outer Functions.

Before using Cor 2.1.3 to help prove Szegö's theorem, we need some more facts about outer functions. We prove therefore,
Lemma 2.2.1. Let \( 1 \leq p < \infty \). If \( h \neq 0 \) a.e. \((m)\) and \( \int h dm = \lambda \neq 0 \), then \( h \in H^p \) is outer if and only if
\[
1 - \frac{\lambda}{h} \in [A_0]_{L^p(|h|^{-p}m)}.
\]

Proof. Suppose \( 1 - \frac{\lambda}{h} \in [A_0]_{L^p(|h|^{-p}m)} \). Then \( \exists \{f_n\} \subset A_0 \) such that
\[
\lim_{n \to \infty} \int |f_n - (1 - \frac{\lambda}{h})|^p |h|^{-p} dm = 0.
\]
Thus \( -\lambda \in [hA]_p \) and, since \( \lambda \neq 0 \), \( 1 \in [hA]_p \). Hence \( [hA]_p = H^p \) and so, by Corollary 1.4.3, \( h \) is outer. Since every implication in the proof is reversible the converse is also true.

§2.3 Szegö's Theorem.

Theorem 2.3.1. (Szegö). Let \( 1 \leq p < \infty \) and \( w \in L^1(m), w > 0 \). Then
\[
\inf_{f \in A_0} \int |1-f|^p w dm = \exp[\int \log w dm],
\]
where \( \int \log w dm \) is defined to be \(-\infty\) if \( \log w \notin L^1(m) \).

Proof. Consider \( f \in A_0 \). Let \( \delta > 0 \) and apply Lemma 0.1 to \( \log(|1-f|^p w + \delta) \in L^1_R(m) \) to get
\[
\int (|1-f|^p w + \delta) dm \geq \exp[\int \log(|1-f|^p w + \delta) dm].
\]
That is

$$\int |1 - f|^p \, w \, d\mu + \delta \geq \exp[\int \log(1 - f)^p \, w \, d\mu + \delta] \, d\mu].$$

Let $\delta$ tend monotonically to zero to obtain

$$\int |1 - f|^p \, w \, d\mu \geq \exp[\int \log(1 - f)^p \, w \, d\mu + \int \log w \, d\mu].$$

By Jensen's inequality (Corollary 2.1.3),

$$\int \log(1 - f)^p \, w \, d\mu \geq p \log \int (1 - f) \, w \, d\mu = 0, \text{ since } f \in A_0.$$

Hence,

$$\int |1 - f|^p \, w \, d\mu \geq \exp[\int \log w \, d\mu], \forall f \in A_0,$$

and so

$$\inf_{f \in A_0} \int |1 - f|^p \, w \, d\mu \geq \exp[\int \log w \, d\mu],$$

which is one half of Szegö's Theorem.

To prove the reverse inequality we can assume that the infimum on the left hand side is positive. Then

$$w^{1/p} \not\in [w^{1/p} A_0]_p,$$ so that, by Corollary 1.4.4, $w^{1/p} = qh$ where $h \in H^p$ is outer, and $v = |w| = |h|^p$. Since $h \in H^p$ is outer, we have, by Lemma 2.2.1, that

$$\frac{1}{h} - \frac{\lambda}{h} \in [A_0]_{L^p(|h|^p m)}$$

Now,
\[
\inf \int |1 - f|^p \, d\mu = \inf \int |1 - f|^p |h|^p \, dm
\]
which is the distance of 1 from \( A_0 \) in \( L^p(|h|^p) \).

Hence,

\[
\inf \int |1 - f|^p \, d\mu \leq \int |1 - (1 - \lambda/h)|^p |h|^p \, dm
\]

\[
\leq |\lambda|^p = \int h^p = \exp(\int \log |h| \, dm)
\]

by the Jensen inequality

\[
\leq \exp(\int \log |h|^p \, dm)
\]

That is,

\[
\inf \int |1 - f|^p \, d\mu \leq \exp(\int \log \, dm).
\]

Hence result.

**Corollary 2.3.1.** If \( f \in L^\lambda \), then \( f \not\in [fA_0]^1 \) if and only if

\[
\int \log |f| \, dm > -\infty.
\]
inf \int |f|^{1/2} - |f|^{1/2}g\,dm = inf \int |f|\,1 - g\,dm > 0.

But, by Szegö's theorem (Theorem 2.3.1) for \( p = 2 \),

\[ \inf \int |f|\,1 - g\,dm = \exp(\int \log|f|\,dm) \]

Hence, left hand side positive implies \( \int \log|f|\,dm > -\infty \), and hence result.

**Corollary 2.3.2.** If \( f \in L^p \) and \( \int \log|f|\,dm > -\infty \), then \( f = qh \), where \( h \in H^p \) is outer; and conversely.

**Proof.** Let \( f \in L^p \) and \( \int \log|f|\,dm > -\infty \). Then, by Corollary 2.3.1, \( f \notin [fA_0]' \), and so, by Corollary 1.3.5, \( f = qh \), where \( h \in H' \) is outer. Since \( |f| = |h| \) and \( f \in L^p \), then \( h \in L^p \). Thus \( h \in H \cap L^p = H^p \) by Corollary 1.4.1. Since each step in the proof is reversible the converse is true.

We are now in a position to prove a further Corollary of Theorem 1.4.1:

**Corollary 2.3.3.** Let \( 1 < p < \infty \). If \( f \in L^1 \), the following three conditions are equivalent:
(i) \[ |f|^{1/p} \leq \left[ |f|^{1/p_A} \right]_p \]

(ii) \[ f \in [f_{A_0}]_1 \]

(iii) \[ \int \log |f| \, dm > -\infty \]

**Proof.** (ii) \(\Rightarrow\) (i).

Assume \( |f|^{1/p} \in \left[ |f|^{1/p_A} \right]_p \). Let \( f_1 = (\text{sgn}) |f|^{1/p} \), where \( 1/p + 1/p' = 1 \). Then

\[ f = f_1 |f|^{1/p} \in f_1 \left[ |f|^{1/p_A} \right]_p \leq [f_1 |f|^{1/p_A}]_1 \]

and so

\[ f \in [f_{A_0}]_1. \]

(ii) \(\iff\) (iii)

This is simply Corollary 2.3.1.

(i) \(\Rightarrow\) (iii)

Assume \( |f|^{1/p} \notin \left[ |f|^{1/p_A} \right]_p \). Now \( f \in L^1 \) as \( |f|^{1/p} \in L^p \).

Hence, by Corollary 1.4.4, \( |f|^{1/p} = qh \) where \( h \in H^p \) is outer, and so, by Corollary 2.3.2,

\[ \log \int |f|^{1/p} \, dm = \frac{1}{p'} \log \int |f| \, dm > -\infty, \]

and so

\[ \int \log |f| \, dm > -\infty. \]
§2.4 Kolmogorov-Krein Theorem.

For those weak-*Dirichlet algebras which are also logmodular, an extension of Szegö's Theorem (Theorem 2.3.1) holds. This extension is known as the Kolmogorov-Krein Theorem. In fact, Lumer (Lumer [1]) showed that this theorem holds for all sup-norm subalgebras of \( C(X) \) such that \( M = \{m\}, \forall \phi \in \hat{M} \), where \( \hat{M}_\phi \) is the set of representing measures for \( \phi \), and \( \hat{M} \) is the maximal ideal space of \( A \). We shall combine Theorem 2.3.1 (Szegö) with an adaptation of a result of Hoffman (Hoffman [2] Theorem 4.3) to show that the Kolmogorov-Krein Theorem holds for those weak star Dirichlet algebras which have the property that \( M_\phi = \{m\}, \forall \phi \in \hat{M} \). We shall then show in §2.5 that this property is necessary for the truth of the Kolmogorov-Krein Theorem. We now prove

Theorem 2.4.1. Let \( A \) be a weak-* Dirichlet algebra on a compact Hausdorff space \( X \), such that \( M_\phi = \{m\}, \forall \phi \in \hat{M} \), where \( m \) is a probability measure on \( X \), multiplicative on \( A \). Let \( \mu \) be a positive measure on \( X \), not necessarily multiplicative on \( A \). If \( \mu = \mu_a + \mu_s \) is the Lebesgue decomposition of \( \mu \) with respect to \( m \), then

\[
\inf_{f \in A_0} \int |1 - f|^2 \, d\mu = \inf_{f \in A_0} \int |1 - f|^2 \, d\mu_a.
\]
Proof. Let $F$ be the orthogonal projection of $1$ into $[A_0]_{L^2(\mu)}$. Then

$$
\int |1-F|^2d\mu = \inf_{f \in A_0} \int |1-f|^2d\mu.
$$

If $f \in A_0$, $(1-F) \perp f$ in $L^2(\mu)$. Choose a sequence $\{f_n\} \subset A_0$ which converges to $F$ in $L^2(\mu)$. If $f \in A_0$, then $(1-f_n)f \in A_0$, since $A_0$ is an ideal in $A$. Since $f$ is bounded

$$
\{ (1-f_n)f \}
$$

converges to $(1-F)f$ in $L^2(\mu)$, and so

$$(1-F)f \in [A_0]_{L^2(\mu)}.$$  Hence $(1-F) \perp (1-F)f$. That is,

$$
\int f(1-F)(1-F)d\mu = \int f|1-F|^2d\mu = 0, \forall f \in A_0. \tag{i}
$$

Let $K = \int |1-F|^2d\mu$. $K = 0$ if and only if $1 \in [A_0]_{L^2(\mu)}$. If $K > 0$, the measure $\mu_1 = K^{-1}|1-F|^2\mu$ satisfies

$$
\int fd\mu_1 = \int fd\mu, \forall f \in A.
$$

In fact, this statement is true by $(i)$ for $f \in A_0$; it is seen to be true for $f \in A$ by noting that any $f \in A$ can be written in the form $f = g + c$, where $g \in A_0$ and $c$ is a constant function. Hence, by our assumption that $M = \{m\}, \forall \phi \in \mathcal{M}$, we have $\mu_1 = m$. Thus, for $K > 0$,

$$
|1-F|^2\mu = Km \tag{ii}
$$
Since \( \mu = \mu_a + \mu_s \) is the Lebesgue decomposition of \( \mu \) with respect to \( m \), we may rewrite (ii) as

\[
|1-F|^2 \mu_s = K m - |1-F|^2 \mu_a.
\]

Since the measure on the right hand side is absolutely continuous with respect to \( m \), it follows that \( |1-F|^2 \mu_s = 0 \); and so \( F = 1 \) a.e. \((\mu_s)\). Thus,

\[
f \in A_0, \quad \int (1-F) \, d\mu_a = \int (1-F) \, d\mu = 0 \tag{iii}
\]

where the last equality follows from the fact that \( F \) is the orthogonal projection of \( 1 \) into \([A_0]_{L^2(\mu)}\).

Now, for \( \{f_n\} \subset A_0 \) convergent to \( F \) in \( L^2(\mu) \) as before, we have

\[
\int |F - f_n|^2 \, d\mu_a \leq \int |F - f_n|^2 \, d\mu
\]

and so \( F \in [A_0]_{L^2(\mu_a)} \).

This fact, combined with (iii), gives us that \( F \) is the orthogonal projection of \( 1 \) into \([A_0]_{L^2(\mu_a)}\), and so

\[
\inf_{f \in A_0} \int |1-f|^2 \, d\mu = \int |1-F|^2 \, d\mu = \int |1-F|^2 \, d\mu_a = \inf_{f \in A_0} \int |1-f|^2 \, d\mu_a.
\]

From Theorem 2.3.1 and 2.4.1 we have at once
Theorem 2.4.2 (Kolmogorov-Krein).

Let $A$ be a weak-* Dirichlet algebra on a compact Hausdorff space $X$, such that $M_\phi = \{m\}$, $\forall \phi \in \mathcal{M}$, where $m$ is a probability measure on $X$, multiplicative on $A$.

Let $\mu$ be a positive measure on $X$ and let $\mu = w\lambda + \mu_s$ where $w \in L^1(m)$, be the Lebesgue decomposition of $\mu$ with respect to $m$. Then

$$\inf_{f \in A_0} \int |1-f|^2 d\mu = \exp \int \log w d\mu,$$

where, as before, if $\log w \notin L^1(m)$, $\int \log w d\mu = -\infty$.

§2.5 Examples to illustrate the necessity of the hypothesis in Theorem 2.4.2 that $M_\phi = \{m\}$, $\forall \phi \in \mathcal{M}$.

We now give an example (Srinivasan and Wang [1]) to show the necessity of the condition in Theorem 2.4.2, that $M_\phi = \{m\}$, $\forall \phi \in \mathcal{M}$.

Let $X$ be the unit circle and $m$ the Haar measure on $X$.

Let $A$ be the algebra of those $f \in C(X)$ which have an analytic extension $\tilde{f}$ to the interior of the unit disc such that $\tilde{f}(0) = f(1)$.

Then $A$ is a uniformly closed separating subalgebra of $C(X)$, with $X$ as the Shilov boundary of $A$ and the support of $m$. We shall show that $A$ is weak-* Dirichlet. Clearly $1 \in A$. Note that $A$ may be considered as the set of functions $f$ of the form
51.

\[ f = j(j-1)g + c, \] where \( c \) is a constant function, and \( g \) is analytic in the interior of the unit disc, continuous on the closed unit disc, except possibly at \( 1 \), and

\[ g = 0 \left( \frac{1}{1-1} \right) \] near \( 1 \). In particular, the functions \( j - j^2, j^2 - j^3, \ldots \in A \), and so \( j^k - j^n \in A, \forall k \geq 0, n \geq 0 \). By the Riemann-Lebesgue Lemma, \( \{j^n\} \) converges to \( 0 \) in the \( \sigma(L^1,L^\infty) \) topology. Hence, for fixed \( k \geq 0 \),

\[ \{j^k - j^n\}_{n=0}^\infty \] converges to \( j^k \) in the \( \sigma(L^1,L^\infty) \) topology and so \( j^k \in [A]_\#, k \geq 0 \). Hence, by the complex version of the Stone-Weierstrass Theorem

\[ C(X) \subseteq [A+A]_\#. \]

But \( [C(X)]_\# = L^\infty(m) \) (Edwards [1] Ex.8.5)

and so \( [A+A]_\# = L^\infty(m) \) and \( A \) is weak-* Dirichlet

Now let \( \mu \) be the unit point mass at \( 1 \). We have

\[ \int fd\mu = f(1) = \tilde{f}(0) = \int fdm \]

so \( A \) does not have the property that \( M_\phi = \{m\}, \forall \phi \in M \).

In this case,

\[ \inf_{f \in A_0} \int |1-f|^2 d\mu = 1. \]

However, \( \mu \) is completely singular with respect to \( m \).
so, in the notation of Theorem 2.4.2, \( w = 0 \), and so

\[
\exp \sqrt{\log w} \delta m = 0
\]

and the conclusion of Theorem 2.4.2 fails to hold.
§3.1 Introduction.

One very important theorem in the theory of analytic functions in the unit disc, the E and M. Riesz Theorem, [which provides a characterisation for the functions in the Hardy class $H^1$ (Hoffman [1] pp. 50, 51)], is not true for weak-* Dirichlet algebras. We shall show that it is not true even for Dirichlet algebras. However, for some subalgebras of $C(X)$, the set of continuous functions on a compact Hausdorff space $X$, we can prove a generalised F. and M. Riesz Theorem (Theorem 3.3.1) which implies the classical result.

For those weak-* Dirichlet algebras which are also logmodular algebras, we have a generalised F. and M. Riesz Theorem (Theorem 3.2.1) which was proved by Hoffman (Hoffman [2]). From this point on we write $\mathfrak{A}$ for the set of measures $\mu$ on $X$ such that $\int fd\mu = 0$, $\forall f \in \mathfrak{A}$; as before, $m$ is a probability measure on $X$, multiplicative on $\mathfrak{A}$, and $\mathfrak{A}_0$ is the set of $f \in \mathfrak{A}$ such that $\int fdm = 0$; and so $\mathfrak{A}_0^\perp$ is the set of measures $\mu$ on $X$ such that $\int fd\mu = 0$, $\forall f \in \mathfrak{A}_0$. 
§3.2 A generalised F. and M. Riesz Theorem for logmodular algebras.

**Theorem 3.2.1.** Let $A$ be a logmodular algebra on a compact Hausdorff space $X$. Let $\mu$ be a complex measure on $X$ such that $\mu \in A^1_0$. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of $\mu$ with respect to $m$, where $m$ is as above. Then $\mu_a, \mu_s \in A^1_0$ and $\int d\mu_s = 0$. Further, $\mu_a = hm$, where $h \in H^1(m)$.

**Proof.** Let $\mu_a = hm$ and let $\rho$ be the positive measure on $X$ defined by $\rho = (1+|h|)m + |\mu_s|$, where $|\mu_s|$ denotes the total variation of $\mu_s$. If $f \in A^1_0$, then

$$\int |1-f|^2 d\rho > \int |1-f|^2 dm > 1.$$ (1)

Let $G$ be the orthogonal projection of $1$ into $[A^1_0]_{L^2(\rho)}$. Now, by (1),

$$\int |1-G|^2 d\rho = \inf_{f \in A^1_0} \int |1-f|^2 d\rho > 1,$$

and so $1 \notin [A^1_0]_{L^2(\rho)}$.

Choose a sequence $\{g_n\} \subseteq A^1_0$ which converges to $G$ in $L^2(\rho)$. Let $f \in A^1_0$. Since $A^1_0$ is an ideal in $A$, $(1-g_n)f \in A^1_0$; and, since $f$ is bounded, $\{(1-g_n)f\}$ converges to $(1-G)f$ in the $L^2(\rho)$-norm, and so $(1-G)f \in [A^1_0]_{L^2(\rho)}$. Thus $(1-G)$ is orthogonal to $(1-G)f$ in $L^2(\rho)$; that is,
\[ \int |1-G|^2 \, d\rho = 0, \quad \forall \ f \in A_0 \quad \text{(ii)} \]

Let \( K = \int |1-G|^2 \, d\rho \geq 1 \). Let \( \rho_1 = K^{-1} |1-G|^2 \). Then we have

\[ \int f \, d\rho_1 = \int f \, dm, \quad \forall \ f \in A. \]

In fact, this statement is true by (ii) for \( f \in A_0 \); it is seen to be true for \( f \in A \) by noting that any \( f \in A \) can be written in the form \( f = g + c \) where \( g \in A_0 \) and \( c \) is a constant function.

Since \( A \) is a logmodular algebra, Theorem 0.2 gives us that \( \rho_1 = m \). Thus \( |1-G|^2 \rho_1 = Km \), and so \( (1-G) = 0 \) a.e. \( (\rho_\sigma) \); and, since \( \rho_a = (1+|h|)m \), we have

\[ |1-G|^2 (1+|h|)m = Km; \quad \text{(iii)} \]

that is,

\[ |1-G|^{-2} \rho_m = (1+|h|)m, \quad \text{and so} \]

\[ (1-G)^{-2} \in L^1(m) \quad \text{and hence} \quad (1-G)^{-1} \in L^2(m). \]

We now wish to show that \( (1G)^{-1} \) is in \( H^2(m) \).

Let \( f \in A_0 \). Then
\[
\int f(l-G)^{-1} \, dm = \int f(l-G)^{-1} \, dp_1
\]
\[
= \frac{1}{K} \int f(l-G)^{-1} |l-G|^2 \, dp
\]
\[
= \frac{1}{K} \int f(l-G) \, dp
\]
\[
= 0 \text{ since } (1-G) \bot A_0 \text{ in } L^2(\rho).
\]
Thus we have
\[
\int f(l-G)^{-1} \, dm = 0, \quad \forall \ f \in A_0,
\]
and so, by Remark 1.3.1, \((1-G)^{-1} \in \mathbb{H}^2(m)\).

From (iii) above we have
\[
|1-G|^2 (1+|h|) = K \text{ a.e. (m)}
\]
which, with \((1-G)^{-1} \in L^2(m)\), implies that \((1-G)^{-1}(1+|h|)\),
and hence \((1-G)^{-1}h\) also, is in \(L^2(m)\).

We now wish to show that \(\int (1-G)f \, d\mu = 0 \forall f \in A_0\).
Take \(\{g_n\} \subset A_0\) convergent to \(G\) in \(L^2(\rho)\) as before. Then
\((1-g_n)f \in A_0\). So, since \(\mu \ll \rho\) and \(d\mu/d\rho\) is bounded, while
\(\mu \in A_0\), we have
\[
\int f(1-G) \, d\mu = \lim_{n \to \infty} \int f(1-g_n) \, d\mu = 0.
\]
Also, since \((1-G) = 0\) a.e. \((\rho_s)\), we have
Thus,

$$0 = \int f(1-G)\,d\mu = \int f(1-G)h\,dm, \quad \forall f \in A_0.$$ 

Since \((1-G)^{-1} \in H^2(m) \exists \{f_n\} \subseteq A\) which converges to 
\((1-G)^{-1}\) in \(L^2(m)\). Since \(m\) is multiplicative on \(A\)

\[
\int f_n f(1-G)h\,dm = 0, \text{ for each } n. \tag{iv}
\]

Since also \((1-G)h \in L^2(m)\), we may pass to the limit in \(\text{(iv)}\)
to obtain

$$\int fh\,dm = 0, \quad \forall f \in A.$$

By Corollary 1.3.4, \(h \in H^1(m)\). Also, we have then that
\(\mu_a \in A_0^\perp\), and this combined with \(\mu \in A^\perp \supset A_0^\perp\), gives us \(\mu_s \in A_0^\perp\).

Since \(1 \in [A_0^\perp] L^2(|\mu_s|)\) (Hoffman [2], Theorem 4.3), we

can choose \(\{f_n\} \subseteq A_0\) which converges to \(1\) in \(L^2(|\mu_s|)\). Then

$$\int d\mu_s = \lim_{n \to \infty} \int f_n \, d\mu_s = 0$$

since \(\mu_s \in A_0^\perp\).

**Remark 3.2.1.** To express 3.2.1 in the same form as later general-

isations of the F. and M. Riesz Theorem, we note that \(\int d\mu_s = 0,\)
together with the fact that each \( f \in A \) can be written in the form
\[
f = g + c,
\]
where \( g \in A_0 \) and \( c \) is a constant function, shows that
\[
\mu_s \in A_0 \Rightarrow \mu_s \in A_\perp.
\]
Thus, if we assumed originally that \( \mu \in A_\perp \), then we could conclude \( \mu_s \in A_\perp \), and therefore that \( \mu_a \in A_\perp \). Thus we may rewrite Theorem 3.2.1 in the form

**Theorem 3.2.2.** Let \( A \) be a logmodular algebra on a compact Hausdorff space \( X \) and \( \mu \) a complex measure on \( X \) such that \( \mu \in A_\perp \).

Let \( \mu = \mu_a + \mu_s \) be the Lebesgue decomposition of \( \mu \) with respect to \( m \), where \( m \) is as described in §3.1. Then \( \mu_a, \mu_s \in A_\perp \).

We shall now show that Theorem 3.1.1 implies the classical F. and M. Riesz Theorem:

**Theorem 3.2.3 (F. and M. Riesz).** Let \( \mu \) be a measure on the unit circle such that
\[
\int e_n d\mu = 0, \quad n = 1, 2, 3, \ldots
\]
where \( e_n : e^{i\theta} \mapsto e^{in\theta} \).

Then \( \mu \) is absolutely continuous with respect to the Lebesgue measure on the unit circle.

**Proof.** We have assumed that \( \mu \in A_0^\perp \), where \( A \) is the standard algebra on the unit circle. If \( \mu = \mu_a + \mu_s \) is the Lebesgue decomposition of \( \mu \), then by Theorem 3.2.1, \( \mu_a, \mu_s \in A_0^\perp \). Since
\[
\perp \in [A_0^\perp L^2(|\mu_s|)], \quad (Hoffman [2], Thm. 4.3)
\]
\[
\{f_n\} \subset A_0 \quad \text{which}
\]

converges to 1 in $L^2(\mu_s)$. Since $\mu_s \in A_0^\perp$, we have
\[ \int d\mu_s = \lim_{n \to \infty} \int d\mu_s_n = 0. \]
Thus $\mu_s$ is orthogonal to $1$. The singular measure $e^{-1}\mu_s \in A_0^\perp$ is similarly orthogonal to $1$. Repetating this process we conclude that
\[ \int e_n d\mu_s = 0, \quad n = 0, \pm 1, \pm 2, \ldots \]
and so $\mu_s$ must be the zero measure; that is $\mu = \mu_a$.

For even a Dirichlet algebra, however, Theorem 3.2.3 (F. and M. Riesz) does not generalise directly, as one can have non-zero measures orthogonal to $A_0$ which are mutually singular with respect to $m$ (Hoffman [1] p59. Ex 11).

§3.3 Another generalised F. and M. Riesz Theorem.

For a general sup-norm algebra $A$ we have a generalisation of the F. and M. Riesz Theorem which is due to Ahern (Ahern [1]).

Theorem 3.3.1. (Ahern). Let $A$ be a sup-norm algebra on a compact Hausdorff space $X$. Let $M_\phi$ be, as before, the set of representing measures for $\phi \in \hat{A}$, the maximal ideal space of $A$, (each $m \in M_\phi$ is a probability measure on $X$, multiplicative on $A$). For every complex measure $\mu$ on $X$ such that $\mu \in A^\perp$, we have $\mu_a^\perp \mu_s \in A^\perp$ (where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of $\mu$ with respect to $m$) if and only if $\rho \ll m$, $\forall \; \rho \in M_\phi$. 
We prove first the following four lemmas.

Lemma 3.3.1. Let \( \{v_n\} \) be a sequence of positive measures on \( X \) having \( m \) as a weak-\(^*\) cluster point. Suppose \( F \subset X \) is compact, and that \( v_n(F) \geq \delta_0 > 0, \forall n \). Then \( m(F) \geq \delta_0 \).

Proof. Since \( m \) is regular \( \exists \) a decreasing sequence of open sets \( \{\emptyset \} \) such that \( \emptyset_n \supset F \) and \( \lim_{n \to \infty} m(\emptyset_n \setminus F) = 0 \). By Urysohn's Lemma \( \exists \{u_n\} \subset C(X) \) such that \( u_n = 1 \) on \( F \), \( u_n = 0 \) on \( X \setminus \emptyset_n \), and \( 0 \leq u_n \leq 1 \). From the construction \( \{u_n\} \) converges a.e.(\( m \)) to \( \chi_F \), the characteristic function of \( F \). Now,

\[
m(F) = \int (\chi_F - u_k) \, dm + \int u_k \, dv_n + \int u_k (dm - dv_n)
\]

and \( \int (\chi_F - u_k) \, dm \) can be made small by choosing \( k \) large, and once \( k \) is fixed \( \int u_k (dm - dv_n) \) can be made small by proper choice of \( n \). Thus,

\[
m(F) \geq \int u_k \, dv_n > v_n(F) \geq \delta_0,
\]

where \( k, n \) are as indicated.

Lemma 3.3.2. Let \( u \in C_\infty(X) \). Write \( \lambda(u) \) for \( \int u \, d\lambda \). Then, for \( \phi \in M \),

\[
\sup \{ \Re \phi(f) : f \in A, \Re \phi \leq u \} = \inf \{ \lambda(u) : \lambda \in M_\phi \}.
\]
Proof. Since \( \lambda(\text{Ref}) = \text{Re}\lambda(f) = \text{Re}\phi(f), \ \forall \lambda \in \mathcal{M}_\phi, \ \forall f \in A, \)
we have

\[
\sup\{\text{Re}\phi(f) : \text{Ref} \leq u, f \in A\} \leq \inf\{\lambda(u) : \lambda \in \mathcal{M}_\phi\}
\]

(1)

By the same equality, \( F : \text{Ref} \mapsto \text{Re}\phi(f) \) is a well-defined non-negative linear functional on the subspace \( \text{Re}A \) of \( C_R(X) \). We shall show that, for each \( u \in C_R(X) \exists \) a non-negative linear functional on the subspace of \( C_R(X) \) spanned by \( \text{Re}A \cup \{u\} \) (call it \( E \)) such that

\[
F_e(\text{Ref}) = F(\text{Ref}), \ \forall f \in A
\]

and

\[
F_e(u) = \sup\{\text{Re}\phi(f) : \text{Ref} \leq u, f \in A\}.
\]

If \( u \in \text{Re}A \), our assertion is trivial:

\[
\sup\{\text{Re}\phi(f) : \text{Ref} \leq u, f \in A\} = F(u).
\]

If \( u \notin \text{Re}A \), each \( v \in E \) can be written uniquely as \( \text{Ref} + \alpha u, f \in A \) and \( \alpha \) a real number. Define

\[
F_e(v) = F_e(\text{Ref} + \alpha u) = F(\text{Ref}) + \alpha \sup\{F(\text{Ref}) : f \in A, \text{Ref} \leq u\}
\]

\[
= \text{Re}\phi(f) + \alpha \sup\{\text{Re}\phi(f) : f \in A, \text{Ref} \leq u\}
\]

\( F_e \) is clearly linear with its restriction to \( \text{Re}A \) equal to \( F \).
We now show that $F_e$ is non-negative on $E$.

Let $Ref + au \geq 0$.

**Case 1.** $\alpha = 0$. Trivial.

**Case 2.** $\alpha > 0$. Then $-\frac{Ref}{\alpha} \leq u$, and

$$\sup \{Re\phi(g) : g \in A, Reg \leq u \} \geq F(-\frac{Ref}{\alpha}) = -\frac{1}{\alpha} Re\phi(f),$$

which implies that $F_e(Ref + au) \geq 0$.

**Case 3.** $\alpha < 0$. Then $-\frac{Ref}{\alpha} \geq u$. Hence, for each $g \in A$ with $Reg \leq u$, we have $Reg \leq -Ref/\alpha$, and

$$Re\phi(g) \leq -\frac{1}{\alpha} Re\phi(f).$$

Hence,

$$\sup \{Re\phi(g) : g \in A, Reg \leq u \} \leq -\frac{1}{\alpha} Re\phi(f),$$

and so

$$Re\phi(f) + \alpha \sup \{Re\phi(g) : g \in A, Reg \leq u \} \geq Re\phi(f) - Re\phi(f) = 0.$$ 

Thus $F_e$ is a non-negative extension of $F$, such that, since $\frac{1}{\alpha} \in A$, $F_e(1) = 1$. It follows that
\[ F_e(v) \leq F_e(\sup v) = \sup v, \quad \forall \ v \in E. \]

By the Hahn-Banach Theorem, there exists an extension of \( F_e \) to a linear functional \( F_{ee} \) on \( C_R(X) \) such that

\[ F_{ee}(w) \leq \sup w, \quad \forall \ w \in C_R(X). \]

Thus we have

\[ -F_{ee}(w) = F_{ee}(-w) \leq \sup(-w) = -\inf w. \]

\[ \therefore \quad F_{ee}(w) \geq \inf w \]

and so \( F_{ee} \) is non-negative.

Since also \( F_{ee}(1) = F_e(1) = 1 \), \( F_{ee} \) is given by a probability measure \( \lambda \) such that

\[ \lambda(\text{Ref}) = F_{ee}(\text{Ref}) = F(\text{Ref}) = \text{Re}\phi(f), \quad \forall \ f \in A. \]

Then also

\[ \lambda(\text{Imf}) = -\lambda(\text{Reif}) = -\text{Re}\phi(if) \]

\[ = -\text{Re} i \phi(f) = \text{Im} \phi(f), \quad \forall \ f \in A. \]

\[ \therefore \quad \lambda(f) = \lambda(\text{Ref}) + i\lambda(\text{Imf}) \]

\[ = \text{Re}\phi(f) + i \text{Im} \phi(f) \]

\[ = \phi(f), \quad \forall \ f \in A. \]

and so \( \lambda \in M_{\phi} \).
This, together with (1), gives the desired result.

The following lemma is an extension of a result of Fiorelli. (Fiorelli [1], Theorem 1).

Lemma 3.3.3. Let $F \subseteq X$ be a compact $G_\delta$ such that $\lambda(F) = 0$, $\forall \lambda \in M_\phi$. (We say $F$ is "$\phi$-null"). Then, for $\{n\}$ an increasing sequence of positive integers, $\exists \{f_n\}$ ball $A$, the closed unit ball in $A$, such that

(1) $\phi(f_n) \geq e^{-2/n}$

(2) $|f_n| \leq \exp o(-n)$ on $F$.

Proof. Since $F$ is a compact $G_\delta$ $\exists$ a sequence of open sets $\{0\}_n$ such that $0_{n+1} \subseteq 0_n$ and $\bigcap_n 0_n = F$, where $\overline{0_{n+1}}$ denotes the closure in $X$ of $0_{n+1}$. Let $\delta > 0$ be given. Then there exists an integer $N$ such that $\forall n \geq N$, $\forall \rho \in M_\phi$, $\rho(0_n) < \delta$.

(If this were not so, there would exist $\delta_0 > 0$ and sequences $\{0_n\}, \{0_{n_k}\}$ with $\rho_{n_k} \in M_\phi$ and $\rho_{n_k}(0_{n_k}) \geq \delta_0$. Let $U_{n_k} = \overline{0_{n_k}}$;

then $\rho_{n_k}(U_{n_k}) \geq \delta_0 > 0$, and $\overline{U_{n_k+1}} \subseteq U_{n_k}$. Let $\rho$ be a weak-* cluster point of $\{\rho_{n_k}\}$. Then $\rho \in M_\phi$ and so $\rho(F) = 0$.

Fix $k$; then $\rho(U_{n_k}) \geq \rho(\overline{U_{n_k+1}})$. Now

$\rho(U_{n_k}) \geq \rho \left( \overline{U_{n_k+1}} \right) \geq \rho_n(U_{n_k+1}) \geq \rho_n(U_{n_k})$, $\forall n > k + 1$, and so, by Lemma 3.3.1, $\rho(U_{n_k}) \geq \delta_0 > 0$, $\forall n_k$. 

But this contradicts the fact that \( \rho(F) = 0 \). Hence, by passage to a suitable subsequence, we may assume that \( \rho(\mathcal{O}_n) < \frac{1}{n^2}, \forall \rho \in M_\phi \).

Now, for each \( n \), \( \exists u_n \in C(X) \) such that \( u_n = -n \) on \( F \), \( u_n = 0 \) on \( X \setminus \mathcal{O}_n \) and \(-n \leq u_n \leq 0\) elsewhere. From Lemma 3.3.2 and the weak-\( * \) compactness of \( M_\phi \), there exists \( \rho_n \in M_\phi \) such that

$$\sup\{\text{Re}\phi(f) : \text{Re}f < u_n, f \in A\} = \int u_n d\rho_n.$$ 

Hence, for each \( n \), \( \exists g_n \in A \) such that \( \text{Re}g_n \leq u_n \), and

$$\int \text{Re}g_n dm \geq \int u_n d\rho_n - \frac{1}{n} \geq -n \rho_n(\mathcal{O}) - \frac{1}{n} \geq -2/n.$$ 

We may also assume that \( \int \text{Im}g_n dm = 0 \). Now define \( f_n = \exp \circ g_n \).

\( f_n \in A \) since \( 1 \in A \). Also,

$$|f_n| = \exp \circ \text{Re}g_n \leq \exp \circ u_n \leq 1;$$

and by multiplicativity of \( m \),

$$f_n dm = \exp[\int g_n dm] = \exp[\int \text{Re}g_n dm] \geq e^{-2/n};$$

and \( |f_n| = \exp \circ \text{Re}g_n \leq \exp \circ (-n) \) on \( F \).

**Lemma 3.3.4.** Suppose there exists \( m \in M_\phi \) such that \( \rho \ll m \), \( \forall \rho \in M_\phi \); and suppose \( F \subset X \) is compact and \( m(F) = 0 \). Then \( \exists \{f_n\} \subset \text{ball } A \) satisfying (1) and (2) of Lemma 3.3.3.
Proof. \( \exists \) a sequence of open set \( \{ \mathcal{O}_n \} \) such that \( F \subseteq \mathcal{O}_{n+1} \subseteq \mathcal{O}_n \) and \( \lim_{n \to \infty} m(\mathcal{O}_n) = 0 \). For each \( n \), \( \exists \) a set \( F_n \), which is a compact \( G_\delta \), such that \( F \subseteq F_n \subseteq \mathcal{O}_n \). Let \( S = \bigcap F_n \). Then \( F \subseteq S \), \( S \) is a compact \( G \), and \( m(S) = 0 \). Since \( \rho << \mu \), \( \forall \rho \in M_\phi \), we have \( \rho(S) = 0 \), \( \forall \rho \in M_\phi \). We then apply Lemma 3.3.3 to the set \( S \) to obtain the desired result.

We can now prove Theorem 3.3.1.

**Proof of Theorem 3.3.1.**

Suppose first that there exists \( m \in M_\phi \) such that \( \rho << \mu \), \( \forall \rho \in M_\phi \). Let \( S \) be a Baric set which carries \( \mu_s \) (that is \( \mu_s(T) = 0 \) for every Baire subset \( T \) of \( X \setminus S \)) such that \( m(S) = 0 \). Then \( \exists \) an increasing sequences \( \{ F_n \} \subseteq S \) of compact sets such that \( \lim_{n \to \infty} |\mu_s|(F_n) = |\mu_s|(S) \), when \( |\mu_s| \) denotes the total variation of \( \mu_s \). For each \( F_n \) we have, by Lemma 3.3.4, a sequence \( \{ f_{n,k} \}_{k=1}^{\infty} \subseteq \text{ball } A \) such that

1. \( \int f_{n,k} \ dm > e^{-2/k} \); and
2. \( |f_{n,k}| \leq e^{-k} \) on \( F_n \).

Define \( h_n = f_{n,n} \). Then we have \( h_n \in \text{ball } A \) and

1'. \( \int h_n \ dm = \int f_{n,n} \ dm > e^{-2/n} \); and
2'. \( |h_n| = |f_{n,n}| \leq e^{-n} \) on \( F_n \).
From (1') we see that \( \lim_{n \to \infty} h_n = 1 \) in \( L^1(m) \) and so we have a subsequence \( \{h_{n_k}\} \) which converges to \( 1 \) a.e. \( (m) \).

From (2') \( \{h_{n_k}\} \) converges to \( 0 \) a.e. \( (|\mu_s|) \).

Hence \( \{g_{n_k}\} = \{h_{n_k}\} \) converges a.e. \( (|\mu|) \) to \( \chi_{\mathbb{X} \setminus S} \).

If \( f \in A \), then for each \( k \), \( g_k f \in A \) and we have

\[
0 = \int g_k f d\mu + \int_{\mathbb{X} \setminus S} f d\mu = \int f d\mu.
\]

That is, \( \mu_a \in A^1 \); and so since \( \mu \in A^1 \), we have \( \mu_s \in A^1 \).

To prove the "only if" part of Theorem 3.3.1, we assume \( \forall v \in M_\phi \) which is not absolutely continuous with respect to \( m \), and consider \( \mu = v - m \). Since \( v, m \in M_\phi \),

\[
\int f d\nu = \int f dm, \quad \forall f \in A,
\]

and so \( \mu \in A^1 \).

Now \( \mu_a = v_a - m \), and \( \mu_a \in A^1 \) if and only if

\[
\int f d\nu_a = \int f dm, \quad \forall f \in A.
\]

But \( \int f d\nu_a = \int f dm, \quad \forall f \in A \) implies, since \( 1 \in A \), that

\[
\int d\nu_a = 1 = \int d\nu.
\]

Thus \( v(X) = v_a(X) \), and hence \( v_s = 0 \). This contradicts our assumption, so \( \mu_a \notin A^1 \) and our proof is completed.

Remark 3.3.1. Theorem 3.2.1, where \( M_\phi = \{m\} \), is a special case of Theorem 3.3.1.
§3.4 An abstract F. and M. Riesz Theorem.

Ahern's result is a particular case of a general result whose only special hypothesis is that $A$ is a subalgebra of $C(X)$ which contains the constant functions on $X$. The extension arises from the idea of forming the Lebesgue decomposition of $\mu$ relative to the set $M_\phi$, in the sense of the following definition (Glicksberg [1]).

**Definition 3.4.1.** The (complex) measure $\mu$ is singular with respect to a set $M$ of probability measures ("$\mu$ is $M$-singular") if $\mu$ is carried by some Baire set $F$ (that is $\mu(S) = 0$ for every Baire subset $S$ of $X \setminus F$) of measure zero for all $m \in M$; such an $F$ is called an $M$-null set. If $\mu$ vanishes on all $M$-null sets $F(\mu_F = 0)$, then $\mu$ is $M$-absolutely continuous ($\mu \ll M$).

When $M = M_\phi$, we frequently write "$\phi$-singular" for "$M_\phi$-singular".

Unlike our previous theory, where our choice of Baire measures rather than regular Borel measures was purely arbitrary, we consider Baire measures here to ensure the truth of the Choquet-Bishop-de Leeuw Theorem (Phelps [1], p24) which is necessary for the development of this theory. The full development will not be given here but may be found in Glicksberg [1] and Garnett and Glicksberg [1].

We note also that we always have a (unique) Lebesgue decomposition
of any $\mu$ relative to $M$:

$$\mu = \mu_F + \mu_{F'}$$

where $\mu_F$ is $M$-singular and $\mu_{F'} \ll M$.

To do this choose an $M$-null set $F$ which maximises $||\mu_F||$, so that if $E$ (and so $E \cup F$) is $M$-null then

$$||\mu_{E \cup F}|| = ||\mu_F|| + ||(\mu_{F'})_E|| \leq ||\mu_F||,$$

and so $(\mu_{F'})_E = 0$.

We now prove the abstract $F$ and $M$ Riesz Theorem due to Glicksberg (Glicksberg [1]). The proof of this theorem follows that of Theorem 3.3.1 and both are closer in form to the original proof of $F$, and $M$. Riesz than the proof of Theorem 3.2.1.

**Theorem 3.4.1.** If $\mu \in A^\perp$ and $\phi \in M$, the maximal ideal space of $A$, and if $\mu = \mu_F + \mu_{F'}$ is the Lebesgue decomposition of $\mu$ relative to $M_\phi$, then $\mu_F, \mu_{F'} \in A^\perp$.

We first prove an analogue of Lemma 3.3.3:

**Lemma 3.4.1.** If $F = \bigcup_{n} K_n$ is a $\phi$-null union of compact Baire sets $K_n$, then $\exists$ sequence $\{f_n\} \subseteq$ ball $A$ which converges to $0$ on $F$ and to $\frac{1}{2}$ a.e. $(\lambda)$, $\forall \lambda \in M_\phi$.

**Proof.** For $n$ fixed we have a monotonic increasing sequence $\{u_k\} \subseteq C_R(X)$, which, since every compact Baire set is a $G_\delta$,
(Berberian [1], p180 Ex.6.), converges pointwise to \(-n\chi_k\). Thus, by monotone convergence \(\lambda(u_k) \uparrow 0\), \(\forall \lambda \in M_\phi\). Since \(M_\phi\) is weak\(^*\) compact and \(\lambda \mapsto \lambda(u_k)\) is weak\(^*\) continuous, Dini's Theorem asserts that the convergence is uniform on \(M_\phi\). Thus,

\[
\lambda(u_k) > -\frac{1}{2^n}, \quad \forall \lambda \in M_\phi, \text{ for some } k.
\]

By Lemma 3.3.2 we have \(g_n \in A\) such that \(\text{Reg}_n \leq -n\chi_k\) and so

\[
\text{Reg}_n < 0, \quad \phi(g_n) > 0.
\]

Put \(f_n = (\exp \circ g_n)\text{sgn}(e)\). Since \(1 \in A, f_n \in A\). Since

\[
|f_n| = \exp \circ \text{Reg}_n \leq 1.
\]

Also, \(|\hat{f}| \leq \exp \circ -n\) on \(K_n\), and so \(\{f_n\} \subseteq \text{ball } A\) and converges to 0 on \(F\). Moreover, \(\{x \in X : \lim_{n \to \infty} |f_n(x)| - 1 > 0\}\) is \(M_\phi\)-null, so that \(\{f_n\}\) converges to 1 a.e. (\(|\mu_F,|\)), since \(\mu_F, << M_\phi\), and \(\{f\}\) converges to \(\chi_F, a.e. (\mu)\). If \(f \in A\), then, for each \(n\), \(f_f \in A\) and, since \(u \in A^\perp\),

\[
0 = \int f_n f d\mu \to \int f d\mu = \int f d\mu_F = 0.
\]

That is, \(\mu_F, \in A^\perp\) and hence also \(\mu \in A^\perp\).
It is clear from the preceding proof that the only special property of $A$ we require is that $1 \in A$ so that we may exponentiate. Thus, we may state Theorem 3.4.1 in a more general form:

**Corollary 3.4.1.** If $B$ is any subalgebra of $A$ such that $1 \in B$, and $\mu = \mu_F + \mu_F'$ is the Lebesgue decomposition of $\mu$ relative to $M_\psi(B)$, where $\psi \in \mathcal{M}_B$, then $\mu \in A^\perp$ implies that $\mu_F, \mu_F' \in A^\perp$.

We note that if $B$ is such a subalgebra of $A$, then $M = M_\phi = M_\phi(A) \subseteq M_\phi(B)$; so that, while $M_\phi(B)$-null sets are also $M_\phi$-null, the converse is false. Thus $\mu \ll M_\phi$ is also $M_\phi(B)$-absolutely continuous, but an $M_\phi$-singular measure $\mu$ may have a non-trivial decomposition relative to $M_\psi(B)$. 

CHAPTER 4.

EXTENSION OF LINEAR FUNCTIONALS

§4.1 Unique norm-preserving extension of a weak-* continuous linear functional on a logmodular algebra $A$ to a weak-* continuous linear functional on $C(X)$.

We shall now make use of the abstract F. and M. Riesz Theorem in its form for logmodular algebras (Theorem 3.2.2) to prove the following result.

Theorem 4.1.1. Let $A$ be a logmodular algebra on a compact Hausdorff space $X$. Let $g \in L^1(m)$, where $m$ is a probability measure on $X$, multiplicative on $A$. Let $\phi$ be a linear functional on $A$ defined by

$$\phi(f) = \int fg\,dm, \forall f \in A.$$\[1.5ex]

Then $\phi$ has a unique norm-preserving extension to a linear functional on $C(X)$, and this extension is weak-* continuous, considering $C(X)$ with respect to the topology induced on it by $\sigma(L^\infty(m),L^1(m))$.

Proof. (i) The existence of at least one norm-preserving extension of $\phi$ is guaranteed by the Hahn-Banach Theorem. Let
\( \psi \) be any such extension of \( \phi \) to a linear functional on \( C(X) \). Then \( \psi \) may be expressed in the form

\[
\psi(f) = \int f \, d\mu, \quad \forall \ f \in C(X),
\]

where \( \mu \) is a (complex) measure on \( X \), and the total variation of \( \mu \) equals \( \|\psi\| \). Let \( \mu = \mu_a + \mu_s \) be the Lebesgue decomposition of \( \mu \) with respect to \( m \). Then

\[
\mu = \mu_a + \mu_s = hm + \mu_s, \quad \text{where} \quad h \in L^1(m).
\]

Since \( \psi \) is an extension of \( \phi \), we have \( \psi(f) = \phi(f), \quad \forall \ f \in A \). Thus, we have \( \int f d\mu = \int f g dm, \quad \forall \ f \in A \), and hence the measure \( \mu - gm \in A \). By Theorem 3.2.2 \( \mu_a - gm \) and \( \mu_s \) are both \( \in A \). Thus,

\[
\int f(h-g) dm = 0, \quad \forall \ f \in A,
\]

and so, by Corollary 1.3.4, \( h - g \in H^1(m) \). But \( \mathbb{1} \in A \), so \( \int (h-g) dm = 0 \) and so \( h - g \in H^1_0(m) \). Also,

\[
\int f g dm = \int f d\mu = \int f d\mu_a = \int f h dm, \quad \forall \ f \in A.
\]

We now have

\[
\sup_{f \in A} \| f h dm \| = \sup_{f \in A} \| f g dm \| = \| \phi \| = \| \psi \|
\]

\[
\| f \| \leq 1, \quad \| f \| \leq 1
\]

and so \( \| \psi \| \leq \| h \|_1 \). But
\[ ||\psi|| = ||\mu|| = ||h||_1 + ||\mu_s|| \]

and so we conclude that \( \mu_s = 0 \), and so \( ||h||_1 = ||\psi|| \) and \( \psi \) is weak-\( * \) continuous.

(ii) **Uniqueness.** Let \( \psi \) and \( \psi_1 \) be norm-preserving extensions of \( \phi \). By the above \( \psi(f) = \int fh \, dm, \forall f \in C(X) \), and \( \psi_1(f) = \int f(h+h_1) \, dm, \forall f \in C(X) \), where \( ||h + h_1||_1 = ||h||_1 \).

Also from above, \( h - g \in H^1_0(m) \) and \( (h+h_1) - g \in H^1_0(m) \).

Thus \( h \in H^1_0(m) \). Now

\[ ||h||_1 = ||\psi|| = ||\phi|| = \sup_{f \in \Lambda} \left| \int fh \, dm \right|, \]

and, since the unit ball in \( L^\infty \) is weak-\( * \) compact, and the unit ball in \( \Lambda \) is contained in the unit ball of \( H^\infty(m) \), \( \exists \ f_1 \in H^\infty(m) = [\Lambda]_* \) (by Theorem 2.1.1), such that \( ||f_1||_\infty = 1 \) and

\[ ||h||_1 = \sup_{f \in \Lambda} \left| \int fh \, dm \right| = \int f_1 h \, dm. \]

Since \( |f_1| \leq 1 \), \( f_1 h = |h| \). However,

\[ \int f_1 (h+h_1) \, dm = \int f_1 h \, dm = ||h||_1 = ||h + h_1||_1, \]

where we have used the fact that \( h_1 \in H^1_0(m) \) and \( f_1 \in H^\infty(m) \) to give \( \int f_1 h_1 \, dm = 0 \).

Thus \( f_1 (h+h_1) = |h+h_1| \). Since \( f_1 h \) and \( f_1 (h+h_1) \) are non-negative, \( f_1 h_1 \) is real-valued. But \( f_1 \in H^\infty(m) \) and \( h_1 \in H^1(m) \).
So \( f_1 h_1 \in H^1(m) \) and, by Lemma 1.1.4, is a constant function. But \( \int f_1 h_1 \, dm = 0 \), so \( f_1 h_1 = 0 \) a.e. (m). Now, from \( f_1 h = |h| \), \( f_1 h_1 = 0 \) a.e. (m), \(|h| = |h + h_1|\), we see, by considering separately the points where \( f_1 \neq 0 \) and those where \( f_1 = 0 \), that \( h_1 = 0 \) a.e. (m). This prove uniqueness.

§4.2 Unique norm-preserving extension of a weak-* continuous linear functional on a weak-* Dirichlet algebra \( A \) to a weak * continuous linear functional on \( L^\infty \).

We have already shown (Theorem 2.1.1) that, for \( A \) any weak -* Dirichlet algebra, \( H^\infty(m) = L^\infty(m) \cap [A]_1 \) is isomorphic to a logmodular algebra on \( \mathcal{M} \), the maximal ideal space of \( L^\infty(m) \). We have seen also (Hoffman [1], p.169) that \( L^\infty(m) \) is isomorphic to \( C(\mathcal{M}) \). Thus, we have, directly from Theorem 4.1.1, a generalisation of a result of Gleason and Whitney for \( H^\infty \) defined relative to the standard algebra on the unit circle. (Gleason and Whitney [1]). That is, we have

Theorem 4.2.1. Let \( A \) be any weak-* Dirichlet algebra on a compact Hausdorff space \( X \). Let \( g \in L^1(m) \) where \( m \) is a probability measure, multiplicative on \( A \). Let \( \phi \) be the linear functional on \( H^\infty(m) \) defined by

\[
\phi(f) = \int fg \, dm, \quad \forall f \in H^\infty(m).
\]
Then \( \phi \) has a **unique** norm-preserving extension to \( L^\infty(m) \), and this extension is weak-* continuous.

We can actually say more than this. Suppose we have a linear functional \( \phi \) defined on a weak-* Dirichlet algebra \( A \) by

\[
\phi(f) = \int f g \, dm, \quad \forall f \in A,
\]

where \( g \in L^1(m) \), \( m \) as in Theorem 4.2. Since we have shown (Theorem 2.1) that \( H^\infty(m) = [A]^* \), we see that \( \phi \) can be extended to a unique linear functional \( \phi_e \) on \( H^\infty(m) \) such that

\[
\phi_e(f) = \int f g \, dm, \quad \forall f \in H^\infty(m).
\]

Because of its form we can refer to \( \phi_e \) by \( \phi \) also and combine this result with that of Theorem 4.2.1 to get

**Theorem 4.2.2.** Let \( A \) be a weak-* Dirichlet algebra on a compact Hausdorff space \( X \). Let \( g \in L^1(m) \), where \( m \) is a probability measure, multiplicative on \( A \). Let \( \phi \) be the linear functional on \( A \) defined by

\[
\phi(f) = \int f g \, dm, \quad \forall f \in A.
\]

Then \( \phi \) has a **unique** norm-preserving extension to \( L^\infty(m) \), and this extension is weak-* continuous.
§4.3 Discussion of the hypothesis in §§4.1, 4.2 that $\phi$ be weak-* continuous.

If, instead of assuming that $\phi$ be defined by integration against a function $g \in L^1(m)$, we simply assume that $\phi$ is a bounded linear functional, then it is possible to form more than one norm-preserving extension of $\phi$. Examples of non-unique norm-preserving extensions of a bounded linear functional defined on $H^\infty$ [where $H^\infty$ is that subspace of $L^\infty$, the bounded measurable complex-valued functions on the unit circle, which consists of those functions which are boundary value functions (existing a.e. by Fatou's Lemma) of bounded analytic functions in the interior of the unit disc] are given by Gleason and Whitney (Gleason and Whitney [1]).

Example 4.3.1. Suppose $\psi$ is a non-zero bounded linear functional on $L^\infty$ which vanishes on $H^\infty$ and takes real values on $L^\infty_R$. By the Hahn decomposition theorem $\psi$ may be represented by the difference $\psi = \psi^+ - \psi^-$ of two non-negative linear functionals; here $\psi^+$ is defined on non-negative $f$ by

$$\psi^+(f) = \sup\{\psi(g) : 0 \leq g \leq f\},$$

and is extended over the rest of $L^\infty$ by linearity.
Since $\psi^+$ and $\psi^-$ are real and non-negative,

$$||\psi^+|| = \psi^+(1), \text{ and } ||\psi^-|| = \psi^-(1).$$

Also, since

$$\psi^+ = \psi^- \text{ on } H^\infty, \text{ and } 1 \in H^\infty,$$

we have

$$||\psi^+|| = \psi^+(1) = \psi^-(1) = ||\psi^-||.$$

Thus $\psi^+$ and $\psi^-$ are distinct norm-preserving extensions over $L^\infty$ of the linear functional $\phi$ defined on $H^\infty$ by

$$\phi(f) = \psi^+(f) = \psi^-(f), \forall f \in H^\infty.$$

We must now construct a $\psi$ with the required properties.

Let $\nu$ be a proper arc of the unit circle, and let $\nu'$ be its complement. Let $f_0$ be defined as equal to $1$ on $\nu$ and equal to $-1$ on $\nu'$. Then

$$\inf_{f \in H^\infty} ||f-f_0||_\infty > 1.$$

Suppose this were not so; then $f_1 \in H^\infty$, $\delta > 0$, such that $||f_1-f_0||_\infty = 1 - \delta$. Thus $f_1 > \delta$ a.e. on $\nu$, and $f_1 < -\delta$ a.e. on $\nu'$.

Now, let $f_1 = H \ast f_1$, where "$\ast$" denotes convolution; and, as in Edwards [2], p.86,

$$H = \sum_{n \in \mathbb{Z}} -i.\text{sgn } n.e^{inx}.$$
Note that, if \( a, b \) are the end-points of \( \nu \),

\[
Df_1 = \delta_a - \delta_b
\]

where \( Df_1 \) is the distributional derivative of \( f_1 \), and \( \delta_x \) is the Dirac measure at \( x \). Therefore,

\[
Df_1 = D(H \ast f_1) = H \ast Df_1
\]

\[
= (\delta_a - \delta_b) \ast H
\]

\[
= 2(\delta_a - \delta_b) \ast D(\log |\sin \frac{1}{2}x|), \quad \text{(Edwards, [2], p. 88, (12.8.6))}
\]

and so

\[
f_1(x) = 2\log |\sin \frac{1}{2}(x-a)| - 2\log |\sin \frac{1}{2}(x-b)| + \text{constant a.e.}
\]

Thus \( f_1 \) is essentially unbounded in every neighbourhood of each of the points \( a, b \). But, since \( f_1 \in H^\infty_R \), and therefore \( f_1 = -if_1 \), we have a contradiction, and hence

\[
\inf_{f \in H^\infty} ||f-f_0||_\infty \geq 1.
\]

Thus, by the Hahn-Banach extension theorem (Edwards [1], §2.2.5)

\[\exists \text{ a bounded real-linear functional on } L^\infty_R \text{ which vanishes on } H^\infty_R \text{ but not at } f_0.\]

We can then extend this functional into a complex-linear functional over \( L^\infty \) to get the required linear functional \( \psi \).
§4.4 Extension of multiplicative weak-* continuous linear functionals on a weak-* Dirichlet algebra.

We shall now consider further the extension of weak-* continuous linear functionals defined on a weak-* Dirichlet algebra $A$. In particular, we shall consider those functionals which are also multiplicative on $A$. We prove the following theorem, which is due to Hoffman and Rossi (Hoffman and Rossi [1]):

Theorem 4.4.1. Let $A$ be a weak-* Dirichlet algebra. Let $m$ be a probability measure multiplicative on $A$. Let $\phi$ be a linear functional defined on $A$ such that

(i) $\phi$ is multiplicative on $A$,

and (ii) $\exists g \in L^1(m)$ such that $\phi(f) = \int fgdm$, $\forall f \in A$.

Then $\phi$ can be extended to a positive weak-* continuous linear functional on $L^\infty(m)$, that is, $\exists$ a non-negative $k \in L^1(m)$ such that

$$\phi(f) = \int fkdgm, \forall f \in A.$$ 

Proof. Since, by Theorem 2.1.1, $H^\infty(m) = [A]_\ast$, we see as before that $\phi$ has a unique extension, denoted by $\phi$ also, to a linear functional on $H^\infty$, namely
\[ \phi(f) = \int fg \, dm, \forall f \in H^\infty. \]

We shall show that \( \phi \) is multiplicative on \( H^\infty = \mathbb{A}_\mu^\infty \).

Consider \( f, h \in H^\infty \). Then \( \exists \{f\}, \{h\} \) \( A \) which converge in the \( \sigma(L^1, L^\infty) \) topology to \( f, h \) respectively. In particular,

\[
\lim_{\mu} \phi(f, h) = \lim_{\mu} \int f \, h \, g \, dm = \int f \, h \, g \, dm = \phi(f) \, \phi(h);
\]
and

\[
\lim_{\mu} \phi(f) \phi(h) = \lim_{\mu} \int f \, h \, g \, dm = \int f \, h \, g \, dm = \phi(f) \phi(h).
\]

Hence,

\[
\lim_{\mu} \lim_{\nu} \phi(f, h) = \phi(fh).
\]

But \( f, h \in A \). Thus by (i),

\[
\phi(f, h) = \phi(f) \phi(h),
\]
and hence

\[
\phi(fh) = \lim_{\mu} \lim_{\nu} \phi(f, h) = \lim_{\mu} \phi(f) \lim_{\nu} \phi(h) = \phi(f) \phi(h)
\]

which shows that \( \phi \) is multiplicative on \( H^\infty \).

Define \( \sum \) to be the set of all \( u \in L^\infty_R(m) \) such that for every positive real number \( t \), \( \exists h_t \in H^\infty \) such that

(a) \( tu \geq \log |h_t| \),

and (b) \( \phi(h_t) = 1 \).

Before proceeding further we need to prove the following two lemmas. The first of these is an extension of the Krein-Smulian Theorem. (Horvath [1], Ch.3 §10. Theorem 2.)
Lemma 4.4.1. Let $K$ be a convex subset of $L^\infty(m)$. The following two conditions are equivalent.

(i) $K$ is weak-$*$ closed;

and (ii) If $\{f_n\} \subseteq K$ converges boundedly and pointwise a.e. to a function $f$, then $f \in K$.

Proof. (i) $\Rightarrow$ (ii). Assume $K = [K]_*$ and that $\{f_n\} \subseteq K$ converges boundedly and pointwise a.e. to $f$. The latter condition implies that $\{f_n\} \subseteq K$ also converges weakly to $f$. So $f \in [K]_*$ and, since $[K]_* = K$, $f \in K$.

(ii) $\Rightarrow$ (i). This follows directly from Edwards [1], 8.10.5 and Ex. 8.6.

Lemma 4.4.2. Let $\sum$ be as defined previously; namely the set of $n \in L^\infty_R(m)$ such that, for every positive real number $\tau$,

$\exists \ h_\tau \in H^\infty$ such that

(a) $tu \geq \log|h_\tau|$;

and (b) $\phi(h_\tau) = 1$.

Then $\sum$ is a convex cone which is weak-$*$ closed in $L^\infty_R(m)$.

Proof. We first show that $\sum$ is a convex cone.

(i) $u_1, u_2 \in \sum \Rightarrow u_1 + u_2 \in \sum$. 

Let $u_1, u_2 \in \mathcal{L}$. Then, for every positive real number $t$, 
exists $h_1, t$ and $h_2, t'$ both $\in H^\infty$, such that $t|u_1| > \log|h_1, t|$, 
$t|u_2| > \log|h_2, t|$ and $\phi(h_1, t) = \phi(h_2, t') = 1$.

Consider $h_1, t, h_2, t' \in H^\infty$. Now, for every positive real number $t$,

$$t(u_1 + u_2) \geq \log|h_1, t| + \log|h_2, t| = \log|h_1, t| h_2, t|;$$

and

$$\phi(h_1, t, h_2, t') = \phi(h_1, t) \phi(h_2, t') = 1.1 = 1.$$

Hence $u_1 + u_2 \in \mathcal{L}$.

(ii) $u \in \mathcal{L}$, $\alpha > 0 \Rightarrow au \in \mathcal{L}$.

Let $u \in \mathcal{L}$, $\alpha > 0$. Then, for every positive real number $t$, 
exists $h_t \in H^\infty$ such that $t|u| > \log|h_t|$ and $\phi(h_t) = 1$. Consider $h_t^\alpha \in H^\infty$. Now

$$t(au) = a(tu) \geq a \log|h_t| = \log|h_t|^\alpha = \log|h_t^\alpha|;$$

and

$$\phi(h_t^\alpha) = [\phi(h_t)]^\alpha = 1.$$

Hence $au \in \mathcal{L}$.

We now show that $\mathcal{L}$ is weak-* closed in $L^\infty_R(m)$. Using Lemma 4.4.1 we need to show that, if we consider $\{u_n\} \subset \mathcal{L}$ such that $|u_n| \leq M$, where $M$ is a constant function, and $\{u_n\}$ converges to $u$ pointwise a.e., then $u \in \mathcal{L}$. 


By the definition of $\sum$, for each $n \exists h_n \in H^\infty$ such that $u_n > \log|h_n|$ and $\phi(h_n) = 1$. In particular, $\{|h_n|\}$ is bounded by $e^M$. Let $h$ be a weak-* cluster point of $\{h_n\}$ in $L^\infty(m)$. Since $H^\infty$ is weak-* closed, $h \in H^\infty$; and, since $\phi$ is weak-* continuous and $\phi(h_n) = 1$, $\phi(h) = 1$. Since $h$ is a weak-* cluster point of $\{h_n\} \subset H^\infty$, then $\forall g \in L^1(m)$ $\exists \{h_n\} \subset \{h_n\}$ such that

$$\lim \int h_n g \, dm = \int h g \, dm.$$ 

Thus

$$\lim_{n_k \to \infty} \int h_n g \, dm = \int h g \, dm,$$

and so

$$\limsup_{n_k} \int h_n g \, dm \geq \int h g \, dm.$$  \hspace{1cm} (1)

Since $|h_n|_{n_k}$ is bounded, for $g \in L^1(m)$ such that $|h_n g|_{n_k}$ is bounded, we can apply Fatou's Lemma; thus, for such $|h_n g|_{n_k}$,

$$\int \limsup_{n_k} h_n g \, dm \geq \int h g \, dm \geq \int \limsup_{n_k} h_n g \, dm.$$  \hspace{1cm} (2)
Combine (1) and (2) to get

\[ \lim_{n_k} \sup |h_n g| \, dm > \left| \int h g \, dm \right|. \]

Thus, if we have a set \( S \subseteq X \) such that \( m(S) > 0 \), we can choose \( g = \text{sgn} \, h \cdot \chi_S / m(S) \) to get

\[ \frac{1}{m(S)} \int_S \lim_{n} \sup |h_n| \, dm > \frac{1}{m(S)} \int_S |h| \, dm. \]  

(3)

We need to show that (3) implies that

\[ \lim_{n} \sup |h_n| > |h| \quad \text{a.e.} \]  

(4)

Suppose (3) holds but (4) does not hold. Then \( \exists \) set \( E \subseteq X \) such that \( m(E) > 0 \) and

\[ \lim_{n} \sup |h_n| < |h| \quad \text{on } E. \]

Let \( E_k \) be the subset of \( X \) on which

\[ \lim_{n} \sup |h_n| < |h| - \frac{1}{k}. \]

Then \( E_k \subseteq E_{k+1} \) and \( \bigcup_{k} E_k = E \). Since \( m(E) > 0 \), \( \exists \) \( k \) say, such that \( m(E_k) > 0 \). Choose \( \delta = \frac{1}{k} \) and let \( S = E_1 \). Then

\[ \lim_{n \to \infty} \sup |h_n| < |h| - \delta \quad \text{on } S, \]  

and so
\[ \int_S \limsup_{n} |h_n| \, dm < \int_S (|h| - \delta) \, dm = \int_S |h| \, dm - \delta m(S). \]

Thus,
\[ \frac{1}{m(S)} \int_S \limsup_{n} |h_n| \, dm < \frac{1}{m(S)} \int_S |h| \, dm - \delta. \]

This contradicts (3). Hence (4) holds; and so
\[ |h| < \limsup_{n} (\exp \circ u_n) = \exp \circ u. \]

That is, \( u \geq \log |h| \). The same argument can be applied to \( tu, \ \forall t > 0 \).
Thus \( u \in \mathcal{L} \).

We now continue with the proof of theorem 4.4.1.

Proof (of Theorem 4.4.1) continued.

\( \mathcal{L} \) is proper since \( -1 \notin \mathcal{L} \). (To show that \( -1 \notin \mathcal{L} \),
suppose \( -1 \in \mathcal{L} \). Then, for every positive real number \( t \),
\[ \exists h_t \in H^\infty \text{ such that } -t > \log |h_t| \text{ and } \phi(h_t) = 1. \]
However, since \( H^\infty \) is a Banach algebra and \( \phi \) is a multiplicative linear functional
on \( H^\infty \), \( ||\phi|| = 1. \)

Since \( -t > \log |h_t| \), \( ||h_t||_\infty < 1. \) But
\[ 1 = ||\phi(h_t)|| < ||\phi|| ||h_t||_\infty = ||h_t||_\infty; \]
which is a contradiction; and so \( -1 \notin \mathcal{L} \).
Since $\Sigma$ is proper, and, by Lemma 4.4.2, weak-$\ast$ closed, a corollary of the Hahn-Banach Theorem (Edwards [1], 2.2.3) ensures the existence of a non-zero weak-$\ast$ continuous linear functional on $L^\infty_R$ which is greater than $a$ on $\Sigma$, for some real number $a$. This linear functional must be non-negative on $\Sigma$, since, as $\Sigma$ is a cone, if it took a negative value at some point of $\Sigma$ then it would take arbitrary large negative values, thus contradicting the fact that it is bounded below by $a$. This functional may then be extended to $\psi_1$, a non-zero, weak-$\ast$ continuous linear functional on $L^\infty$ which is non-negative on $\Sigma$. Thus, there exists $k_1 \in L^1(m)$ such that

$$\psi_1(f) = \int f k_1 \, dm, \quad \forall \ f \in L^\infty.$$  

Let $k = [\int |k_1| \, dm]^{-1} k_1 \in L^1(m), \text{ so that}$

$$\int |k| \, dm = 1;$$

and form

$$\psi(f) = \int f k \, dm, \quad \forall \ f \in L^\infty.$$  

Then $\psi$ is a non-zero weak-$\ast$ continuous linear functional on $L^\infty$ which is non-negative on $\Sigma$, and

$$||\psi|| = ||k||_1 = 1.$$
By taking \( h_t = \frac{1}{t} \), we see that \( \Sigma \) contains every positive function in \( L^\infty \). Thus \( \psi \) is a positive functional and \( k \) is a non-negative function. Suppose now \( f \in A \) such that \( f \in \ker \phi \). By taking \( h_t = \exp o (tf) \), we see that \( \text{Re} f \in \Sigma \). But, if \( f \in \ker \phi \), \( -f \in \ker \phi \) and so \( \text{Re}(-f) = -\text{Re} f \in \Sigma \). Now \( \psi \) is non-negative on \( \Sigma \), so \( \psi(\text{Re} f) = 0 \). By considering \( (-if) \in \ker \phi \) we get \( \psi(\text{Im} f) = 0 \). Thus \( \psi(f) = 0, \forall f \in \ker \phi \). Hence,

\[
\psi(f) = \phi(f), \forall f \in A.
\]

We have shown this for \( f \in A \) such that \( \phi(f) = 0 \). Consider \( f \in A \) such that \( \phi(f) = c \neq 0 \). Then

\[
\phi(f) - c = \phi(f-c) = 0.
\]

Thus \( \psi(f-c) = \psi(f) - c = 0 \), and so \( \psi(f) = c \). Since also \( ||\psi|| = ||\phi|| = 1 \), \( \psi \) is a norm-preserving extension of \( \phi \) which is positive and weak-* continuous and takes the form

\[
\psi(f) = \int f k \text{d}m, \forall f \in L^\infty.
\]

Thus, we may write

\[
\phi(f) = \int f k \text{d}m, \forall f \in A,
\]

where \( k \) is a non-negative function.
CHAPTER 5

SEQUENTIAL F. AND M. RIESZ THEOREM

§5.1 A sequential F. and M. Riesz Theorem.

Let $A$ be the sup-norm Banach algebra of complex-valued functions on the unit circle whose Fourier coefficients, $C_n$ say, are zero for $n < 0$. Then $H^\infty$ is the set of bounded complex-valued functions on the unit circle whose Fourier coefficients $C_n$, say are zero for $n < 0$. Let $\lambda$ denote the Lebesgue measure on the unit circle.

Theorem 5.1.1. (Kahane [1]). Let \( \{g_n\} \subseteq L^1(\lambda) \) be such that

\[
\ell(f) = \lim_{n \to \infty} \int f g_n \, d\lambda
\]

exists for every $f \in H^\infty$. Then \( \exists g \in L^1(\lambda) \) such that

\[
\ell(f) = \int f g \, d\lambda, \quad \forall f \in A;
\]

and every (complex) Baire measure $\mu$ which is a cluster point, in the $\sigma(A,A^*)$ topology, where $A^*$ is the dual of $A$, of \( \{g_n, \lambda\} \), is such that $\mu << \lambda$.

Proof. We first show that a finite complex-valued Baire measure on the unit circle such that

\[
\ell(f) = \int f \, d\mu, \quad \forall f \in A.
\]
Define \( \phi_n(f) = \int f g_n \, d\lambda \), \( \forall f \in A \). For each \( f \in A \), \( \{\phi_n(f)\} \) is bounded. Hence, by the principle of uniform boundedness, \( \{|\phi_n|\} \) is bounded. Denote by \( \phi_n \) also the norm-preserving extension of \( \phi_n \) to the continuous complex-valued functions on the unit circle. By the Riesz representation theorem, \( \exists \) a finite Barie measure \( \mu_n \) such that, for each \( n \),

\[
\phi_n(f) = \int f d\mu_n, \quad \forall f \in A;
\]

and the total variation of \( \mu_n \) is equal to \( |\phi_n|\). Thus, by the weak-* compactness of measures, \( \exists \) a finite Barie measure \( \mu \) such that

\[
\lambda(f) = \lim_{n \to \infty} \int f d\mu_n = \int f d\mu, \quad \forall f \in A.
\]

We now show that \( \mu << \lambda \). Suppose it is not the case that \( \mu << \lambda \). Let \( E \) be a closed set on the unit circle such that \( \lambda(E) = 0 \) and \( \mu(E) \neq 0 \). Such an \( E \) exists since \( \mu \) is regular. Let \( h \in A \) be such that \( h = 1 \) on \( E \) and \( |h| < 1 \) outside \( E \).

(The existence of such an \( h \in A \) is established in Hoffman [1], p.81.)

We now have the following properties.

1. \( \lim_{m \to \infty} \int h^m d\mu = \mu(E) \)
2. \( \lim_{m \to \infty} \int h^m g_n \, d\lambda = 0, \quad \forall n. \)
3. \( \lim_{n \to \infty} \int h^m g_n \, d\lambda = \int h^m d\mu, \quad \forall m. \)
If \( \{m_j\} \) is rapidly increasing (meaning that \( m_{j+1} \) is sufficiently large when \( m_j \) is given), we have

\[
f = \sum_{j=1}^{\infty} (-1)^j h^m \in H^\infty
\]

since, given \( m_j \), we may define \( E_j \) as the set where \( |h^{j-1}| < 2^{-j} \), and, when \( m_{j+1} \) is large enough, we have \( |h^{j+1}| < 2^{-j} \) on \( E_j \), the complement of \( E_j \). But \( 2^{-j} < 1 - 2^{-(j+1)} \), and so \( E_j \cap E_{j+1} = \emptyset \). Thus, since also \( E \subseteq E_j, \forall j \), every \( x \in E \) belongs to \( E_k \) for some \( k \), chosen sufficiently large. By the method given in detail in the proof of Theorem 5.2.1, it now follows that \( f = \sum_{j=1}^{\infty} (-1)^j h^m \) is the pointwise limit a.e. \((\lambda)\) of a uniformly bounded sequence of functions in \( A \) and so \( f \in H^\infty \).

Write \((\alpha)\) for the above condition on the \( \{m_j\} \). We introduce here the formula,

\[
\int fg \, d\lambda = \sum_{k=1}^{j-1} (-1)^k h^m g_n \, d\lambda + (-1)^j h^m g_n \, d\lambda + \sum_{k=j+1}^{\infty} h^m g_n \, d\lambda
\]

\[
= A_j + B_j + C_j
\]

where we shall define by induction the sequences \( \{m_j\} \) (satisfying \((\alpha)\)) and \( \{n_j\} \) such that the following two conditions are satisfied:

\[
(\beta) \quad \sum_{k=j+1}^{\infty} |\int h^m g_n \, d\lambda| < \frac{1}{12} |\mu(E)|
\]

\[
(\gamma) \quad |\int h^m d\mu| > \frac{11}{12} |\mu(E)|.
\]
Choose \( n_1 \) any positive integer. Let \( m_1 \) be the least positive integer such that

\[
|\int h^m d\mu| > \frac{11}{12} |\mu(E)|, \quad \forall \ m \geq m_1.
\]

The existence of \( m_1 \) is guaranteed by (1).

Now suppose we have defined \( n_1, m_1, \ldots, n_{j-1}, m_{j-1} \). Let \( \ell_j \) be the least positive integer such that

\[
|h^{m_{j-1}+1}| < 2^{-(j-1)} \text{ on } E_{j-1} = |h^{\ell_j}| < 2^{-(j-1)} \text{ on } E'_{j-1}.
\]

Let \( n^*_j \) be the least positive integer such that \( n \geq n^*_j \) implies

\[
|\sum_{k=1}^{j-1} (-1)^k \int h^m g_n d\lambda| < \frac{1}{2} |\mu(E)|.
\]

The existence of \( n^*_j \) is guaranteed by (3). Let \( M(n,k) \) be the least positive integer such that

\[
m \geq M(n,k) = |\int h^m g_n d\lambda| < 2^{-k} \frac{1}{12} |\mu(E)|.
\]

The existence of \( M(n,k) \) is guaranteed by (2). Put

\[
m^*_j = \max(m_1, M(n_1, j), \ldots, M(n_{j-1}, j), \ell_j).
\]

We define

\[
A^\infty_j = \sum_{k=1}^{m^*_j} (-1)^k \int h^m d\mu, \quad \text{and consider the following two cases:}
\]

(a) \( |A_{j-1} + B_{j-1} - A^\infty_j| \leq 5/12 |\mu(E)| \)

(b) \( |A_{j-1} + B_{j-1} - A^\infty_j| > 5/12 |\mu(E)| \).

In case (a), define \( m^*_j = m_j \), noting that (\( \gamma \)) is true, and
choose \( n_j \geq n_j^* \) so that \( |B_j| \geq \frac{11}{12}|u(E)| \).

This is possible by (3) and (\( \gamma \)); by (3), with \( m_j = m_j^* \), for any \( \varepsilon > 0 \), \( \exists \; n_j \geq n_j^* \) such that

\[
|\int h_j^m d\mu - \int h_j^m g_n d\lambda| < \varepsilon.
\]

Thus, \( |\int h_j^m d\mu| - |B_j| < \varepsilon \), and, choosing

\[
\varepsilon = |\int h_j^m d\mu| - \frac{11}{12}|u(E)|
\]

(which is \( > 0 \) by (\( \gamma \))), we get \( |B_j| \leq \frac{11}{12}|u(E)| \).

In case (b), define \( n_j = n_j^* \) and choose \( m_j \geq m_j^* \) so that \( |B_j| < \frac{1}{12}|u(E)| \). This is possible because of (2).

Note, that for \( n_j \geq n_j^* \),

\[
|A_j - A_j^\infty| < \frac{1}{12}|u(E)|.
\]

We have, also, given \( n_j \),

\[
|\int h_j^m g_n d\lambda| < \frac{1}{2k} \cdot \frac{1}{12}|u(E)|, \quad \forall \; m \geq M(j,k),
\]

Now \( m_k \geq M(j,k) \quad \forall k \geq j+1 \), so if \( m_k \geq m_k^* \),

\[
\sum_{k=j+1}^{\infty} |\int h_j^m g_n d\lambda| < \frac{1}{12}|u(E)|, \quad \sum_{k=j+1}^{\infty} \frac{1}{2k} < \frac{1}{12}|u(E)|
\]

and the sequences \( \{n_j\}, \{m_j\} \) in both cases (a) and (b) satisfy (\( \varepsilon \)).

In case (a), we have
\[ \begin{align*}
|A_{j-1} + B_{j-1} - A_j - B_j| &= |(A_{j-1} + B_{j-1} - A_j^\infty) + (A_j^\infty - A_j) - B_j| \\
&\geq |B_j| - |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j| \\
&\geq \max(|B_j| - |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j|) \\
&\geq \frac{11}{12} |\mu(E)| - 5/12 |\mu(E)| - \frac{1}{12} |\mu(E)| - 5/12 |\mu(E)| - \frac{3}{12} |\mu(E)|.
\end{align*} \]

In case (b) we have
\[ |A_{j-1} + B_{j-1} - A_j - B_j| > |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j| - |B_j| \]
\[ > |5/12 - \frac{1}{12} - \frac{1}{12}|\mu(E)| = 3/12 |\mu(E)|. \]

Thus, in each case we have
\[ |A_{j-1} + B_{j-1} - A_j - B_j| > 3/12 |\mu(E)|. \]

Taking (\(\beta\)) into account we have \(|C_{j-1}|\) and \(|C_j|\) both majorised by \(\frac{1}{12} |\mu(E)|\).

Therefore,
\[ \|f g_n\|_{L^1} \xrightarrow{n \to \infty} \|f g_d\|_{L^1} > |f g_n|_{L^1} \xrightarrow{n \to \infty} |f g_d|_{L^1} = \|f g_d\|_{L^1} \]
\[ > \left(\frac{3}{12} - \frac{1}{12} - \frac{1}{12}\right) |\mu(E)| = \frac{1}{12} |\mu(E)| \]

and so \(\{f g_n\} \) is not convergent, contrary to our initial assumption. This contradiction gives \(\mu << \lambda\). That is,
\[ \lambda(f) = \int f d\mu = \int f g d\lambda, \quad \forall \ f \in A, \ some \ g \in L^1(\lambda). \]
Remark 5.1.1. Theorem 5.1.1 implies the classical F. and M. Riesz Theorem. To see this, suppose \( \mu \) is a measure on the unit circle such that \( \int e_n \, d\mu = 0, \quad n \geq 1 \). If \( \{\sigma_n\} \subseteq L^1(\lambda) \) is the sequence of Cesàro means of the Fourier series of \( \mu \), then \( \mu \) is the unique weak-* cluster point of \( \{\sigma_n\} \) and

\[
\lim_{n \to \infty} \int f \sigma_n \, d\lambda = \int f \, d\mu, \quad \forall f \in H^\infty
\]

and so, by Theorem 5.1.1, \( \mu \ll \lambda \).

The proof of Theorem 5.1.1 relies on the existence of \( h \in A \) such that \( h(E) = 1 \) and \( |h| < 1 \) elsewhere, where \( \lambda(E) = 0 \). The existence of such an \( h \) is guaranteed by the classical F. and M. Riesz Theorem. Thus, Theorem 5.1.1 is equivalent to the classical F. and M. Riesz Theorem, and we shall refer to Theorem 5.1.1 as a sequential F. and M. Riesz Theorem.

§5.2. A relation between generalised and sequential F. and M. Riesz Theorems.

Elizabeth Heard (Heard [1]) considered the case where \( A \) is a subspace of \( C(X) \), for \( X \) a compact Hausdorff space. She said that \( A \) and \( m \), where \( m \in M(X) \), the set of finite complex-valued Baire measures which form the dual of \( C(X) \), satisfy a generalised F. and M. Riesz Theorem whenever \( \mu \in A \Rightarrow \mu \ll m \), for \( \mu \) any finite complex-valued Baire measure on \( X \).
From our previous generalisations of the F. and M. Riesz Theorem, which are not nearly so strong, it would appear that, for \( A \) and \( m \) to satisfy such a theorem heavy restrictions would need to be placed on \( A \). Bishop [1] claims there are at least three examples in the literature, one of which is given in Bishop [2]. Heard showed that, whenever \( A, m \) satisfy such a theorem, they also satisfy a sequential F. and M. Riesz Theorem.

**Theorem 5.2.1.** (Heard [1]). Let \( A \) be a closed subspace of \( C(X) \). Let \( m \in M(\lambda) \), where \( M(X) \) is as defined above. Let \( \mu \) be any finite complex-valued Baire measure on \( X \). Then (I) \( \Rightarrow \) (II):

(I) \( \mu \in \mathcal{A} \Rightarrow \mu \ll m \).

(II) If \( \{g_n\} \subseteq L^1(m) \) and

\[
\varphi(f) = \lim_{n \to \infty} \int f g_n \, dm
\]

exists for every \( f \in [A]_\chi \), then any representative \( \mu \) of a coset \( \mu + \mathcal{A} \) which is a cluster point in the \( \sigma(A, A^\ast) \) topology of the set of cosets \( \{\mu_n + \mathcal{A} : \mu_n = g_nw_n\} \subseteq M(X)/\mathcal{A} \) is absolutely continuous with respect to \( m \) (\( \mu \ll m \)); and \( \exists g \in L^1(m) \) such that

\[
\varphi(f) = \int fg \, dm, \quad \forall f \in A.
\]
Proof. Suppose \( \{g_n\} \subset L^1(m) \) such that

\[
\lambda(f) = \lim_{n \to \infty} \int f g_n \, dm
\]

exists for every \( f \in [A]_* \). Let \( \mu_n \) be defined by \( \mu_n = g_n m \).

By a similar method to that used in Theorem 5.1.1 we can find a coset \( \mu + A^\perp \) which is a cluster point in the \( \sigma(A, A^\star) \) topology of \( \{\mu_n + A^\perp\} \). Let \( \mu \) be a representative of the coset \( \mu + A^\perp \). Then

\[
\lim_{n \to \infty} \int f d\mu_n = \int f d\mu = \lambda(f), \quad \forall f \in A.
\]

We shall now show that \( \mu \ll m \). Suppose this is not the case.

Then, since \( \mu \) is regular, \( \exists \) a closed set \( E \subset X \) such that \( m(E) = 0 \) but \( \mu(E) \neq 0 \). We require \( \{f_n\} \subset A \) with the following properties.

(i) \( ||f_n|| < 1 + 2^{-n} \)

(ii) \( f_n = 1 \) on \( E \)

(iii) \( \lim_{n \to \infty} f_n = 0 \) a.e. \( (m) \)

(iv) \( \lim_{n \to \infty} f_n = \chi_E \) a.e. \( (|\mu|) \).

(v) \( f = \sum_{j=1}^{\infty} (-1)^j f_n(j) \in [A]_* \) for every strictly increasing sequence of positive integers \( \{n(j)\} \).

The construction of \( \{f_n\} \) is by induction. Let \( \{U_n\} \) be a sequence of open subsets of \( X \) such that \( E \subset U_{n+1} \subset U_n \).

Since \( m(E) = 0 \) we can suppose \( m(U_n) < 2^{-n} \) and \( |\mu|(U_n \setminus E) < 2^{-n} \) for all positive integers \( n \).
Let $E_1 = U_1$. By Urysohn's Lemma $\exists h_1 \in C_R(X)$ such that $h_1(E) = \{1\}$, $h_1(X \setminus E_1) = \{0\}$, and $0 \leq h_1 \leq 1$. The assumption that (I) is true and the general Rudin-Carleson Theorem (Bishop [1]) allow us to choose $f_1 \in A$ such that

$$f_1(E) = \{1\} \text{ and } |f_1| < h_1 + 2^{-1}.$$

Now suppose $f_1, \ldots, f_n$ and the sets $E_1, \ldots, E_n$ which are open neighbourhoods of $E$, are chosen. Let $V_{n+1}$ be the subset of $X$ on which $|f_n - 1| < 2^{-n}$. Put $E_{n+1} = E_n \cap V_{n+1} \cap U_{n+1}$. Again by Urysohn's Lemma, $\exists h_{n+1} \in C_R(X)$ such that $h_{n+1}(E) = \{1\}$, $h_{n+1}(X \setminus E_{n+1}) = \{0\}$, and $0 \leq h_{n+1} \leq 1$. We can then choose $f_{n+1} \in A$ such that

$$f_{n+1}(E) = \{1\} \text{ and } |f_{n+1}| < h_{n+1} + 2^{-n}.$$

We now wish to show that the $\{f_n\}$ so defined satisfies (i) - (iv).

Properties (i) and (ii) are clearly satisfied. Now

$$\lim_{n \to \infty} f_n = 0 \text{ except on the set } F = \bigcap_{j=1}^{\infty} E_j. \text{ But }$$

$$m(F) \leq m(E_n) \leq m(U_n) < 2^{-n}, \forall n, \text{ so } m(F) = 0, \text{ and } \lim_{n \to \infty} f_n = 0 \text{ a.e. (m), which is property (iii). Also, } \lim_{n \to \infty} f_n = \chi_E \text{ except on the set } F \setminus E. \text{ But }$$

$$|\mu|(F \setminus E) \leq |\mu|(E_n \setminus E) \leq |\mu|(U_n \setminus E) < 2^{-n}, \forall n,$$

so $|\mu|(F \setminus E) = 0$, and $\lim_{n \to \infty} f_n = \chi_E$ a.e. $|\mu|$, which is property (iv).
We now show that (v) is also satisfied. Let \( \{n(j)\} \) be any strictly increasing sequence of positive integers. Let
\[
E_n(0) = X. \quad \text{Then} \quad F = \bigcap_{j=1}^{\infty} E_j = \bigcap_{j=1}^{\infty} E_n(j); \quad \text{and}
\]
\[
X \setminus F = \bigcup_{j=1}^{\infty} (E_{n(j)-1} \setminus E_{n(j)}), \quad \text{where the sets in this union are disjoint.}
\]

Consider \( x \in X \setminus F. \) Then, for some \( k, \ x \notin E_{n(k)} \setminus E_{n(k+1)}; \) and hence \( x \in \bigcap_{j=1}^{k} E_{n(j)} \) and \( x \notin \bigcap_{j=k+1}^{\infty} E_{n(j)}. \) Thus we have
\[
\sum_{j=k+1}^{\infty} (-1)^j f_{n(j)}(x) \leq \sum_{j=1}^{k-1} \left| f_{n(j)}(x) - 1 \right| + \sum_{j=1}^{k-1} (-1)^{j-1} \left| f_{n(j)}(x) \right|,
\]
(a) \( \leq \sum_{j=1}^{k-1} 2^{-n(j)} + 1; \)

(b) \( \sum_{j=k+1}^{\infty} (-1)^j f_{n(j)}(x) \leq \sum_{j=k+1}^{\infty} (h_{n(j)}(x) + 2^{-n(j)}) \leq 2^{-n(k)}; \)

(c) \( |f_{n(k)}| \leq 1 + 2^{-n(k)}. \)

Thus the series \( \sum_{j=1}^{\infty} (-1)^j f_{n(j)} \) converges and (a), (b), (c) show that the sequences of partial sums,
\[
\{s_n(x) : s_n(x) = \sum_{j=1}^{n} (-1)^j f_{n(j)}(x)\}
\]
converges to \( f(x) \) and \( |s_n(x)| < 4, \forall \) positive integers \( n. \)

Thus \( f \) is defined at every \( x \in X \setminus F. \) Let \( y \in F. \) Then \( y \in \bigcap_{j=1}^{\infty} E_{n(j)+1} \)
and
\[ s_n(y) = \sum_{j=1}^{n} (-1)^j f_{n(j)}(y) = \sum_{j=2}^{n} (-1)^j (f_{n(j)}(y) - 1) + \sum_{j=1}^{n} (-1)^{j-1} \]
so that
\[ |s_n(y)| \leq \sum_{j=1}^{n} |f_{n(j)}(y) - 1| + \sum_{j=1}^{n} (-1)^{j-1} \]
\[ \leq \sum_{j=1}^{n} 2^{-n(j)} + 1 < 4. \]

Therefore \( f = \sum_{j=1}^{\infty} (-1)^j f_{n(j)} \) is the pointwise limit a.e. (m) of a uniformly bounded sequence of functions in \( A \). Thus \( f \in [A]_{**} \) and (v) is established.

We complete the proof in the same manner as that of Theorem 5.1.1, showing that what we have just deduced leads to a contradiction of our original assumption regarding the convergence of \( \{f_{g_{n}} \} \), \( \forall f \in [A]_{**} \). Thus \( \mu \ll m \) and \( \exists g \in L^1(m) \) such that
\[ \lambda(f) = \int fg \, dm, \quad \forall f \in A. \]

§5.3 Description of more functions in \( H^\infty \) for which the limit relation in Theorem 5.1.1 holds.

It is not known whether the limit relation in Theorem 5.1.1, namely
\[ \lambda(f) = \int fg \, d\lambda, \]
holds \( \forall f \in H^\infty \). Kahane himself (Kahane [1]) showed that it does hold for certain functions in \( H^\infty \setminus \mathcal{A} \).

Let \( L \) be the set of linear functionals \( \ell \) on \( H^\infty \) such that, for some (possibly \( \lambda \)-dependent) sequence \( \{g_n\} \subset L^1(\lambda) \),

\[
\ell(f) = \lim_{n \to \infty} \int f \cdot g_n \, d\lambda, \quad \forall f \in H^\infty.
\]

Theorem 5.1.1 asserts that, to each \( \ell \in L \) corresponds at least \( g \in L^1(\lambda) \) such that

\[
\ell(f) = \int fg \, d\lambda, \quad \forall f \in \mathcal{A} \tag{1}
\]

Denote by \( G(\ell) \) the set of \( g \in L^1(\lambda) \) such that (1) is true. Define \( D_\ell = \{ f \in H^\infty : \ell(f) = \int fg \, d\lambda, \ g \in G(\ell) \} \), and let \( D = \bigcap \{ D_\ell : \ell \in L \} \). Then we have

**Theorem 5.3.1.** (a) \( D_\ell \) is a closed subspace of \( H^\infty \); and, given any \( f \in H^\infty \) almost all translates of \( f \) belong to \( D_\ell \).

(b) \( D \) is a closed subalgebra of \( H^\infty \), invariant under translation; it contains all \( f \in H^\infty \) such that \( fh \in D \), for some outer function \( h \).

In particular, \( D \) contains all the \( f \in H^\infty \) which are continuous on the unit circle except on a closed set of measure zero.

**Proof.** Define \( \phi_n(f) = \int f g_n \, d\lambda, \ \forall f \in \mathcal{A} \), where \( \{g_n\} \subset L^1(\lambda) \).

Since the trigonometric polynomials form a dense subset of \( L^1(\lambda) \), we may suppose that each \( g_n \) is a trigonometric polynomial. For each \( f \in \mathcal{A} \), \( \{\phi_n(f)\} \) is bounded, and hence, by the principle of
uniform boundedness, \( \{ \| \phi_n \| \} \) is bounded. Denote by \( \| \phi_n \| \) also the norm-preserving extension of \( \phi_n \) to the continuous complex-valued functions on the unit circle. By the Riesz Representation Theorem, there exists a finite Baire measure \( \mu_n \) such that, for each \( n \),

\[
\phi_n(f) = \int f \, d\mu_n, \quad \forall f \in A,
\]

and the total variation of \( \mu_n \) is equal to \( \| \phi_n \| \). Thus we have

\[
\int f(g_n \, d\lambda - d\mu_n) = 0, \quad \forall f \in A.
\]

In particular,

\[
\int f(g_n \, d\lambda - d\mu_n) = 0, \quad \forall f \in A_0,
\]

where \( A_0 = \{ f \in A : \int f \, d\lambda = 0 \} \). Hence, by the F. and M. Riesz Theorem,

\[
g_n \lambda - \mu_n \ll \lambda,
\]

and so \( \mu_n \ll \lambda \). Therefore, we may suppose \( \{ \| g_n \| \} \) is bounded.

In order to prove (a) we may suppose \( g = 0 \). Clearly \( D \) is a closed subspace of \( H^\infty \). Given \( f \in H^\infty \), write \( f_s : t \mapsto f(t-s) \) for the translate of \( f \). Since \( f \) is bounded, \( f^\ast \psi \in A \) for every \( \psi \in L^1(\lambda) \), where \( \ast \) is the operation of convolution. Hence, by Theorem 5.1.1,

\[
\lim_{n \to \infty} \int g_n(t) \{ \int f(t-s) \psi(s) \, d\lambda(s) \} \, d\lambda(t) = 0, \quad \forall \psi \in L^1(\lambda).
\]
That is, by the Fubini-Tonelli Theorem,

\[
\lim_{n \to \infty} \int \psi(s) \{ \int f(t-s) g_n(t) d\lambda(t) \} d\lambda(s) = 0, \quad \forall \psi \in L^1(\lambda)
\]  \hspace{1cm} (1)

By hypothesis,

\[
\lim_{n \to \infty} \int g_n(t)f(t-s) d\lambda(t) = \langle f \rangle_s;
\]

and, since \( \{ \| g_n \|_1 \} \) is bounded, \( \{ \int g_n(t)f(t-s) d\lambda(t) \} \) is uniformly bounded with respect to \( n \) and \( s \). Hence, (1) can be rewritten

\[
\int \psi(s) \langle f \rangle_s d\lambda(s) = 0, \quad \forall \psi \in L^1(\lambda).
\]

Therefore, \( \langle f \rangle_s = 0 \) for almost every \( s \) and (a) is established.

To prove (b) write

\[
\lim_{n \to \infty} \int f h g_n d\lambda = \langle f \rangle_h = \langle f \rangle_f = \langle f \rangle(h), \quad \forall f, h \in H^\infty.
\]

Then \( \langle f \rangle_h = \int h g_f d\lambda \), \( \forall h \in D \), where \( g_f \in G(\langle f \rangle) \).

Clearly Theorem 5.1.1 implies that \( A \subset D \). Suppose \( f \in A \).

Then

\[ fh \in A \subset D, \quad \forall h \in A \subset D. \]

Since \( fh \in D \) and \( h \in D \), we have

\[
\int fh g d\lambda = \langle fh \rangle = \langle f \rangle_h = \int h g_f d\lambda, \quad \forall h \in A. \]

Thus,
\[ \int h(\text{fg} - \text{g}_f) \, d\lambda = 0, \quad \forall \ h \in A; \]

and so, by Corollary 1.3.4, \( \text{fg} - \text{g}_f \in H^1(\lambda). \) But \( 1 \in A, \) so

\[ \int (\text{fg} - \text{g}_f) \, d\lambda = 0 \]

and we have \( \text{fg} - \text{g}_f \in H^1_0(\lambda). \) Therefore,

\[ \text{fg} = \text{g}_f \pmod{H^1_0}. \]

Now suppose \( h \in D. \) Taking \( f \in A, \) and using \( \text{fg} = \text{g}_f \pmod{H^1_0} \)

we have

\[ \chi(fh) = \chi_f(h) = \int h \text{g}_f \, d\lambda = \int fhg \, d\lambda, \quad \text{for every } \chi \in L. \quad (\text{iii}). \]

Therefore, \( fh \in D_{\chi}, \) for every \( \chi \in L, \) and so \( fh \in D. \) Since \( fh \in D \) and \( h \in D, \) we have

\[ \int f(hg - g_h) \, d\lambda = 0, \quad \forall \ f \in A. \]

Hence, \( hg = g_h \pmod{H^1_0}. \)

If \( f \in D \) and \( h \in D, \) \( \text{fg} = \text{g}_f \pmod{H^1_0} \) and so we still have

(\text{iii}) and, as a consequence, \( fh \in D. \) Therefore, \( D \) is a subalgebra of \( H. \) It is closed because each \( D_{\chi} \) is closed, and it is clearly invariant under translation.

Now, suppose \( f \in H, \) \( h \in D, \) where \( h \) is an outer function, and \( fh \in D. \) We have as before
\[ \int fhgd\lambda = \ell(fh) = \ell_f(h) = \int hgd\lambda, \quad \forall \ h \in D. \quad (i)' \]

Thus

\[ \int h(fg-g_f)d\lambda = 0, \quad \forall \ h \in D \supset A, \quad (ii)' \]

and so \( fg - g_f \in H^1_0(\lambda) \).

Consequently,

\[ \ell(f) = \ell_l(1) = \int g_f d\lambda = \int g d\lambda, \quad \text{for every } l \in L. \]

That is, \( f \in D_\lambda \) for every \( l \in L \), and so \( f \in D \).

Finally, if \( E \) is a closed subset of the unit circle such that \( \lambda(E) = 0 \), then \( \exists \) a continuous outer function \( h \) such that \( h(E) = \{0\} \) (Hoffman [1], p.80.)

Hence, if \( f \) is continuous except on \( E \), \( fh \in A \) and \( f \in D \).

An alternative proof of the last part of Theorem 5.3.1 was suggested by J. Wells and is given in Heard [1].
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