FUNCTION ALGEBRAS AND ABSTRACT HARDY SPACES

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STATEMENT

Except where otherwise indicated, this thesis is my own work.

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SUMMARY

In this thesis we consider certain subalgebras of L^{∞}, called weak-* Dirichlet algebras, which were first introduced by Sunivasan T.P and Wang J. (Srinivasan and Wang [1]). We consider the generalisation to these algebras of a portion of the theory of analytic functions in the unit disc.

In our development we follow the approach of Srinivasan and Wang, where the invariant subspace theorem, and not Szegö's Theorem, plays the central role. This theorem, for invariant subspaces of L^p, $1 \leq p \leq \infty$, is established in Chapter 1. We develop several important factorisation theorems in the process.

In Chapter 2 we show that H^{∞} is isomorphic to a logmodular algebra on the maximal ideal space of L^{∞}, and use this fact to prove the truth of Szegö's Theorem. However, the Kolmogorov-Krein theorem, which is a generalised version of Szegö's Theorem, is not true for a general weak-*Dirichlet algebra. In this chapter, also, we consider for which weak-*Dirichlet algebras the Kolmogorov-Krein Theorem does in fact hold.

In Chapter 3 we consider several generalisations of the classical F. and M. Riesz Theorem and the weak-* Dirichlet algebras for which these generalisations hold. We continue this theme in Chapter 5 where we develop a sequential F. and M. Riesz theorem and show the

connection between this and one generalised form of the F. and M. Riesz Theorem.

In Chapter 4 we use some of the results of Chapter 3 to show that there exists a unique extension of a weak-* continuous linear functional defined on a weak-* Dirichlet algebra to a weak-* continuous linear functional on L^{∞}. We generalise en-route a result of Gleason and Whitney. We conclude this chapter by considering the extension of certain positive linear functionals defined on a weak-* Dirichlet algebra.

Preliminaries

We shall begin with some necessary definitions.

Definition 0.1. A sup-norm algebra A on a compact Hausdorff space X is a complex linear subalgebra of C(X), the algebra (under pointwise operations) of continuous, complex-valued functions on X, such that

(i) A is closed under the norm $||f||_A = \sup_{x \in X} |f(x)|$;

(ii) $\underline{1} \in A$; and

(iii) A separates the points of X; that is, if x,y are distinct points of X, $\exists\, f \in A$ such that $f(x) \neq f(y)$.

We shall write V_A for the set of invertible elements in A; that is $V_A = \{f \in A : f \text{ and } 1/f \in A\}$;

$$\mathrm{Re}A = \{\mathrm{Re}f : f \in A\}; \quad |V_A| = \{|f| : f \in V_A\};$$

$$\log|V_A| = \{\log|f|; \ f \in V_A\}; \quad \overline{A} = \{\overline{f}; \ f \in A\},$$ where $\overline{f}$ denotes the complex conjugate of f. M_A (without subscript is no ambiguity ensures) denotes the maximal ideal space of A.

Unless specifically stated otherwise, by a measure on X we shall understand a finite, complex Baire measure on X.

We shall make use of the Reesz representation theorem in the form:

Every bounded (that is, continuous) linear functional Φ on

$C_R(X)$, the real-valued functions in $C(X)$, is induced by a real measure μ on X, that is,

$$\Phi(f) = \int f d\mu, \forall f \in C_R(X).$$

Similarly, every bounded linear functional on $C(X)$ is induced by a (complex) measure on X.

We shall also make frequent use of (Edwards [1], Chs 2,8) the Hahn-Banach Theorem and duality theory.

Definition 0.2. Let A be a sup-norm algebra on X. Let $\Phi \in M$, the maximal ideal space of A. A representing measure m, for Φ is a positive measure on X such that

$$\Phi(f) = \int f dm, \forall f \in A.$$

Definition 0.3. Let A be a sup-norm algebra on A. Let $\Phi \in M$. An Arens-Singer measure, m, for Φ is a positive measure on X such that

$$\log|\Phi(f)| = \int \log|f| dm, \forall f \in V_A.$$

Since $\Phi(\underline{1}) = 1$, both types of measures defined above satisfy $\int dm = 1$, and are, consequently, probability measures on X (positive measures of mass 1).

Definition 0.4. A Dirichlet algebra A on a compact Hausdorff space X is a sup-norm algebra on X such that the space $\mathrm{Re}A$ is uniformly dense in $C_R(X)$.

Hoffman (Hoffman [2]) extended the theory for Dirichlet algebras to a class of algebras he called logmodular algebras.

Definition 0.5. Let A be a sup-norm algebra on X. A is a logmodular algebra on X if the set of functions $\log|V_A|$ is uniformly dense in $C_R(X)$.

This leads to

Theorem 0.1. If A is a Dirichlet algebra on a compact Hausdorff space X, then A is a logmodular algebra on X.

Proof. Since, $\forall f \in A$, $\mathrm{Re}f = \log|\exp f|$, we have $\mathrm{Re}A \subset \log|V_A|$. But A is a Dirichlet algebra. Thus $\mathrm{Re}A$ is uniformly dense in $C_R(X)$ and so $\log|V_A| \supset \mathrm{Re}A$ is uniformly dense in $C_R(X)$ and A is a logmodular algebra.

Theorem 0.2. Let A be a logmodular algebra on X. Then, to every $\Phi \in M$, the maximal ideal space of A, corresponds a unique representing measure for Φ, and this measure is a Arens-Singer measure.

Proof. This follows directly from Hoffman [2], Theorems 2.1, 4.1, 4.2.

Srinivasan and Wang (Srinivasanand Wang [1]) extended most of the main theorems of logmodular algebra theory (the exceptions being the F. and M. Riesz theorem and the Kolmogorov-Krein Theorem) to a class of algebras they called weak-* Dirichlet.

Definition 0.6. Let (m,X) be a probability measure space. Let A be a subalgebra of $L^{\infty}(m)$ under pointwise operations, such that $\underline{1} \varepsilon A$. Then A is aweak -* Duichlet algebra if and only if the following conditions are satisfied:

(i) m is multiplicative on A; that is

$$\int fg\,dm = \int f\,dm \cdot \int g\,dm, \ \forall f,g \ \varepsilon\ A;$$

(ii) $A + \overline{A}$ is a dense subset of $L^{\infty}(m)$ in the $\sigma(L^{1},L^{\infty})$ topology.

To show that all logmodular algebras are also weak-* Dirichlet algebras we need the following three lemmas.

Lemma 0.1. Let m be a probability measure on a compact Hausdorff space X, and $g \ \varepsilon\ L^{1}_{\mathbb{R}}(m)$, the set of real-valued functions in $L^{1}(m)$. Then

$$\int (\exp o g)\,dm \geq \exp[\int g\,dm].$$

Proof. Since every $g \in L^1_{\mathbb{R}}(m)$ can be written in the form $g = f + \underline{c}$ where $f \in L^1_{\mathbb{R}}(m)$ such that $\int f dm = 0$, and $\underline{c}$ is a constant function, it suffices to prove this lemma for $g \in L^1_{\mathbb{R}}(m)$ such that $\int g dm = 0$. Since also, $\exp\circ g \geq \underline{1}+g$, we have

$$\int(\exp\circ g)dm \geq \int(\underline{1}+g)dm = 1 + \int g dm = 1 = \exp[\int g dm].$$

Lemma 0.2. Let m be a probability measure on a compact Hausdorff space X. Let A be a logmodular algebra on X. Let $g \in L^1(m)$ such that $\int fg dm = 0$, $\forall f \in A$. Then

$$\int \log|\underline{1} - g| dm \geq 0.$$

Proof. Let $f \in V_A$. Then $\int f dm = \int(\underline{1}-g) f dm$, and so

$$|\int f dm| \leq \int |f||\underline{1} - g| dm;$$

and

$$\log|\int f dm| \leq \log\int |f||1 - g| dm.$$

Since m is an Arens-Singer measure (Theorem 0.2), and $f \in V_A$,

$$\int \log|f| dm = \log|\int f dm| \leq \log\int |f||\underline{1} - g| dm, \forall f \in V_A$$

or

$$\exp[\int \log|f| dm] \leq \int |f||\underline{1} - g| dm, \forall f \in V_A.$$

Since A is a logmodular algebra, $\log|V_A|$ is uniformly dense in $C_R(X)$, and so

$$1 \leq \int(\exp o u)|\underline{1} - g|dm, \ \forall\, u \in C_R(X) \text{ such that } \int u dm = 0.$$

Thus, $1 \leq \inf_u \int(\exp o u)|\underline{1} - g|dm = \exp[\int \log|\underline{1} - g|dm]$, by Lemma 0.1, and the lemma follows.

Lemma 0.3. Let m be a probability measure on a compact Hausdorff space X. Let $g \in L^1_R(m)$ such that

$$\int \log|\underline{1}-tg|dm \geq 0,$$

for every real number t in some interval $|t| < \delta$. Then $g = $ a.e.(m).

Proof. This is the result of Hoffman [2], Lemma 6.6.

Thus we have,

Theorem 0.3. Let m be a probability measure on a compact Hausdorff space X. Let A be a logmodular algebra on X. Then A is a weak-* Dirichlet algebra.

Proof. It suffices to prove that, if $g \in L^1_R(m)$ such that

$$\int fg\,dm = 0, \forall f \in A,$$

then $g = 0$ a.e. (m).

Any such g satisfies Lemma 0.2 and so

$$\int \log|1 - g|\,dm \geq 0$$

and so, by Lemma 0.3, $g = 0$ a.e. (m). For $1 \leq p < \infty$, we define the space $H^p(m)$ by

$$H^p(m) = [A]_p,$$

the closure of A in the L^p-norm; and we define

$$H^\infty(m) = H^2(m) \cap L^\infty(m)$$

We shall show (Theorem 2.1.1) that $H^\infty = [A]_*$, the weak-* closure of A in $L^\infty(m)$.

Let $A_0 = \{f \in A : \int f\,dm = 0\}$ and define $H_0^p(m)$, $1 \leq p \leq \infty$ by

$$H_0^p(m) = \{f \in H^p : \int f\,dm = 0\}.$$

It is clear that for $1 \leq p < \infty$,

$$H_0^p = [A_0]_p.$$

CHAPTER 1.

INVARIANT SUBSPACE THEOREM

§1.1 L^2-invariant subspace Theorem.

For weak-* Dirichlet algebras the invariant subspace theorem is the basic one. For A a weak-* Dirichlet algebra on a compact Hausdorff space X, and m a probability measure multiplicative on A, we define a closed subspace M of $L^2(m)$ to be simply invariant if $[MA_0]_2 < M$, "<" denoting strict inclusion and $A_0 = \{f \in A : \int f dm = 0\}$.

For such subspaces we have

Theorem 1.1.1. Every simply invariant subspace M of $L^2(m)$ is of the form $M = qH^2(m)$, for some measurable q such that $|q| = 1$; and q is unique (modulo functions which are zero a.e) up to multiplication by a constant function with absolute value 1.

To prove this theorem we need the following four lemmas.

Lemma 1.1.1. Let $w \in L^1(m)$ be a real-valued function. If $\int fwdm = \int fdm, \forall f \in A$, then $w = \underline{1}$ a.e. (m).

Proof. Our assumption means that

$$\int f(\underline{1}-w)dm = 0 \ \forall f \in A,$$

But $w = \bar{w}$, so we have

$$\int \bar{f}(\underline{1}-w)dm = 0, \ \forall f \in A.$$

Hence $\underline{1} - w = \underline{0}$ a.e. (m) by the weak-* density of $A + \bar{A}$.

Remark 1.1.1. Since A contains all constant functions, the weak-* density of $A + \bar{A}$ is equivalent to that of $A + \bar{A}_0$.

We now prove

Lemma 1.1.2. $A + \bar{A}_0$ is norm dense in $L^2(m)$.

Proof. Take $f \in L^2(m)$ such that $\int fg dm = 0, \forall g \in A + \bar{A}_0$. Since $L^2(m) \subset L^1(m)$, Remark 1.1.1 shows that $f = \underline{0}$ a.e. (m). An appeal to the Hahn-Banach theorem gives result.

Lemma 1.1.3. Let A be a weak-* Dirichlet algebra on a compact Hausdorff space X. Let m be a probability measure on X, multiplicative on A. Then

(a) $\forall f,g \in H^2(m)$, $fg \in H^1(m)$ & $\int fg dm = \int f dm \cdot \int g dm$;

(b) $\forall f \ H^1(m)$, $g \in H^1(m) \ L^\infty$, $fg \in H^1(m)$ and $\int fg dm = \int f dm \cdot \int g dm$.

Proof. (a) Consider $f, g \in H^2(m) = [A]_2$. Then $\exists$ sequences $\{f_n\}, \{g_n\} \subset A$ which converge in the L^2-norm to f, g respectively. Now, $fg \in L^1(m)$ and

$$||fg-f_ng_n||_1 \leq ||fg-f_ng||_1 + ||f_ng + f_ng_n||_1$$

$$\leq ||f- f_n||_2||g||_2 + ||f_n||_2||g-g_n||_2$$

and so $\{f_ng_n\} \subset A$ converges to fg in the L'-norm, and so $fg \in H^1(m)$.

Since $|\int f_n dm - \int f dm| \leq ||f_n-f||_1 \leq ||f_n-f||_2$,

$$\lim_{n\to\infty} \int f_n dm = \int f dm.$$

Similarly, $\lim_{n\to\infty} \int g_n dm = \int g dm$;

and

$$\lim_{n\to\infty} \int f_n g_n dm = \int fg dm.$$

Also, since m is multiplicative on A,

$$\int f_n g_n dm = \int f_n dm \, . \int g_n dm.$$

Thus we have

$$\int fg dm = \lim_{n\to\infty}(\int f_n g_n dm)$$

$$= \lim_{n\to\infty}(\int f_n dm \, . \int g_n dm)$$

$$= (\lim_{n\to\infty}\int f_n dm).(\lim_{n\to\infty}\int g_n dm)$$

$$= \int f dm \, . \int g dm.$$

(b) Let $f \varepsilon H^1(m)$, $g \varepsilon H^1(m) L^\infty$. Since $f,g \ \varepsilon\ H^1(m)$ $\exists$ sequences $\{f_n\},\{g_k\} \subset A$ which converge in the L^1-norm to f,g respectively. Consider $\{f_n g_k\}_{k=1}^\infty \subset A$. Now,

$$||f_n g_k - f_n g||_1 \leq ||f_n||_\infty ||g_k - g||_1,$$

and so $\{f_n g_k\}_{k=1}^\infty \subset A$ converges in the L^1-norm to $f_n g$. Thus $f_n g \ \varepsilon\ H^1(m)$. Also,

$$||f_n g - fg||_1 \leq ||f_n - f||_1 ||g||_\infty,$$

and so $\{f_n g\}_{n=1}^\infty \subset H^1(m)$ converges in the L^1-norm to fg. But $H^1(m)$ is closed under the L^1-norm, and so $fg \ \varepsilon\ H^1(m)$. Now, since

$$|\int g_k dm - \int g dm| \leq ||g_k - g||_1,$$

$$\lim_{k\to\infty} \int g_k dm = \int g dm.$$

Similarly, $\lim_{n\to\infty} \int f_n dm = \int f dm$,

and $\lim_{n\to\infty} \lim_{k\to\infty} \int f_n g_k dm = \int fg dm$.

Also, since m is multiplicative on A,

$$\int f_n g_k dm = \int f_n dm \cdot \int g_k dm.$$

Thus we have

$$\int fg\,dm = \lim_{n\to\infty} \lim_{k\to\infty} \int f_n g_k\,dm$$

$$= \lim_{n\to\infty} \lim_{k\to\infty} \left(\int f_n\,dm \cdot \int g_k\,dm\right)$$

$$= \lim_{n\to\infty} \int f_n\,dm \cdot \lim_{k\to\infty} \int g_k\,dm$$

$$= \int f\,dm \cdot \int g\,dm .$$

Lemma 1.1.4. Let f be a real-valued funcjtion in $H^p(m)$, $1 \leq p < \infty$. Then f is equal a.e. (m) to a constant function.

Proof. Let $c = \int f\,dm$. Thus $\int (f-\underline{c})\,dm = 0$. Now m is multiplicative on A, and hence, by Lemma 1.1.3 (b)

$$\int (f-\underline{c})g\,dm = 0, \quad \forall\ g \in A.$$

Since f is real-valued, this implies that

$$\int (f-\underline{c})\bar{g}\,dm = 0, \quad \forall\ \bar{g} \in \bar{A}.$$

Thus $\int (f-\underline{c})h\,dm = 0, \ \forall\ h \in A + \bar{A}$,

and so $f - \underline{c} = \underline{0}$, a.e. (m) by the weak-* density of $A + \bar{A}$.

We now prove Theorem 1.1.1, henceforth omitting (m) in such terms as $L^2(m)$, when no ambiguity arises.

<u>Proof of Theorem 1.1.1.</u> Since $[MA_0]_2 < M$, $\exists\ q \neq \underline{0}$, $q \in M \ominus [MA_0]_2$. Without loss of generality, we may normalise q so that $\int|q|^2 dm = 1$. Let $f \in A$. Then $f - \underline{\int f dm} \in A_0$, and

$$\int f|q|^2 dm = (fq,q) = ([f-\underline{\int f dm}]q,q) + ([\underline{\int f dm}]q,q) = \int f dm$$

where the inner product is taken in L^2 as a Hilbert space. Hence, by Lemma 1.1.1, $|q| = 1$. Now $qA \subset MA \subset MA_0 + M \subset M + M = M$, and, since M is closed, $qH^2 = q[A]_2 = [qA]_2 \subset M$. Suppose $qH^2 < M$ and let $g \in M \ominus qH^2$. Then

$$\int g\bar{q} f dm = 0, \quad \forall\ f \in A.$$

Also, since $gA_0 \subset [MA_0]_2$, we have $q \perp gA_0$. That is

$$\int \bar{q} g f dm = 0, \quad \forall\ f \in A_0.$$

So $g\bar{q}\quad A + \bar{A}_0$ in L^2. Hence, by Lemma 1.1.2, $g\bar{q} = \underline{0}$ a.e. But $|q| = 1$, so $g = \underline{0}$ a.e. Hence, since qH^2 is closed $M = qH^2$.

That q is essentially unique follows immediately. For if $qH^2 = q'H^2$ ($|q| = |q'| = 1$), then both $\bar{q}q'$ and $\bar{q}'q \in H^2$ and so, by applying Lemma 1.1.4 separately to $\mathrm{Re}(\bar{q}q')$ and $\mathrm{Im}(\bar{q}q')$ we get $\bar{q}q' = \underline{c}$ a.e. where $\underline{c}$ is a constant function and $|\underline{c}| = 1$.

Remark 1.1.2. Since all logmodular algebras are also weak-* Dirichlet (Theorem 0.3), Theorem 1.1.1 implies Theorem 1.1.2, the invariant subspace theorem for A a logmodular algebra.

Theorem 1.1.2. Let A be a logmodular algebra on a compact Hausdorff space X. Let m be a probability measure on X, multiplicative on A. Suppose that M is a closed subspace of H^2 such that $MA \subset M$, and that $\exists$ at least one $g \in M$ such that $\int g\,dm \neq 0$. Then $\exists$ a function $q \in H^2$ such that

(i) $|q| = 1$

and (ii) $M = qH^2$.

The function q is unique (modulo functions which are zero a.e.) up to multiplication by a constant function with absolute value 1.

For A a logmodular algebra and M a closed subspace of H^2, we shall show the equivalence of the hypotheses of Theorem 1.1 Theorem 1.1.2. That is, we shall show

(i) If M is a closed subspace of H^2 and $[MA_0]_2 < M$, then $\exists$ $g \in M$ such that $\int g\,dm \neq 0$, and $MA \subset M$;

and (ii) If M is a closed subspace of H^2 and $MA \subset M$, and $\exists\, g \in M$ such that $\int g\,dm \neq 0$, then $[MA_0]_2 < M$.

In (i), since each function in A can be written as the sum of a function in A_0 and a constant function, it is easily seen that

$MA \subset M$. Lemma 1.1.3 ensures that m is multiplicative on H^2, and hence, on $[MA]_2$, and so the strict inclusion of $[MA_0]_2$ in M ensures that $\exists\ g \in M$ such that $\int g\,dm \neq 0$.

In (ii) we have $MA_0 \subset MA \subset M$. Since M is closed, this means that $[MA_0]_2 \subset M$. However, as in (i), we have $\int f\,dm = 0\ \forall f \in [MA_0]_2$.

But by hypothesis $\exists\ g \in M$ such that $\int g\,dm \neq 0$.

Thus $g \in M \setminus [MA_0]_2$ and so $[MA_0]_2 < M$.

<u>Remark 1.1.3.</u> In Theorem 1.1.2, the hypothesis that $\exists\ g \in M$ such that $\int g\,dm \neq 0$, is an essential one.

We shall give an example to illustrate this point. (Hoffman [1], p.102).

Let X be the torus Choose and fix an irrational number α, and let A be the algebra of all continuous functions f on X such that

$$a_{kn} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\theta,\psi)e^{-ik\theta}e^{-in\psi}d\theta\, d\psi$$

is zero for all pairs of integers (k,n), save perhaps those belonging to the half plane where $k + n\alpha \geq 0$. Now A is a Dirichlet algebra, and hence both a logmodular algebra and a weak-star-Duichlet algebra (Theorem 0.1.1 and Theorem 0.1.3).

If $dm = \frac{1}{4\pi^2}\, d\theta\, d\psi$,

$H^2(m)$ is the space of square summable functions on the torus with Fourier series

$$f = \sum_{k+n\alpha \geq 0} a_{kn} e_{kn}$$

where $e_{kn} : (e^{i\theta}, e^{i\psi}) \mapsto e^{ik\theta} e^{in\psi}$.

If we take M to be the subspace of functions for which $a_{00} = 0$, then $MA \subset M$ but M is not of the form $M = qH^2(m)$ for a measurable q such that $|q| = 1$. To see this suppose that $M = qH^2$. Since $|q| = 1$, $q\bar{q} = \underline{1}$ and so $H^2 = q^{-1}M = \bar{q}M$. Also, since $\underline{1} \in H^2$, $q \in M$. Thus, $\int q\,dm = 0$ and $\hat{q}(k,n) = 0$, if $k + n\alpha \leq 0$, where $\hat{g}(k,n)$ is the coefficient of e_{kn} in the Fourier series for q. Thus,

$$q = \sum_{\substack{k+n\alpha \geq 0 \\ (k,n) \neq (0,0)}} \hat{q}(k,n) e_{kn},$$

and so

$$\bar{q} = \sum_{\substack{k+n\alpha \leq 0 \\ (k,n) \neq (0,0)}} a_{kn} e_{kn} .$$

Let $f \in M$. Then

$$f = \sum_{\substack{k+n\alpha \geq 0 \\ (k,n) \neq (0,0)}} \hat{f}(k,n) e_{kn}.$$

Hence,

$$\int q f dm = \sum_{\substack{p+q\alpha\leq 0\\(p,q)\neq(0,0)\\k+n\alpha\geq 0\\(k,n)\neq(0,0)}} a_{pq}\hat{f}(k,n)\int e_{k+p,n+q}$$

$$\leq \sum_{\substack{p+q\alpha\leq 0\\(p,q)\neq(0,0)}} a_{pq}\hat{f}(-p,-q)$$

$$= \sum_{\substack{p+q\alpha=0\\(p,q)\neq(0,0)}} a_{pq}\hat{f}(-p,-q) ,$$

which is an empty sum since α is an irrational number. Hence

$$\int q f dm = 0, \quad \forall \; f \in M,$$

which contradicts $H^2 = \overline{qM}$ since $\underline{1} \in H^2$.

<u>Remark 1.1.4</u>. When A is the algebra of all continuous complex-valued functions on the unit circle which have analytic extensions to the interior of the unit disc, and m is the normalised Lebesgue measure, the situation reduced to the case of the shift operator "multiplication by j".

From this point on, we shall use A to refer to a weak-* Dirichletalgebra on a compact Hausdorff space X. We shall refer to Theorem 1.1.1 as the "L^2-invariant subspace theorem" and we shall generalise this theorem to L^p, $1 \leq p \leq \infty$.

§1.2. Some results concerning outer functions.

Before doing this, however, we need some preliminary results. First we need the concept of an outer function.

Definition 1.2.1. A function $h \in H^1 = [A]_1$ is said to be outer if $[hA]_1 = H^1$.

We note the following about outer functions.

Note 1.2.1. If h is outer, then $h \neq 0$ a.e. and $\int hdm \neq 0$. In particular, $h \notin [hA_0]_1$.

Proof. That $h \neq 0$ a.e. follows directly from Definition 1.2.1. If $\int hdm = 0$, Lemma 1.1.3 (b) ensures that $[hA]_1 < H^1$ which contradicts Definition 1.2.1.

Note 1.2.2. If h, h' are outer and $|h| = |h'|$, then $h = \underline{c}h'$ where $\underline{c}$ is a constant function such that $|\underline{c}| = 1$.

Proof. We first observe that, since $|h| = |h'|$, $h = qh'$ for some measurable q such that $|q| = 1$. Now, $H^1 = [hA]_1 = [qh'A]_1 = q[h'A]_1 = qH^1$, so both q and $\overline{q} \in H'$. Hence, by applying Lemma 1.1.4 separately to Req, Imq we have $q = \underline{c}$ a.e. and so $h = ch'$.

Note 1.2.3. Let $h \in H^2$. Then h is outer if and only if $[hA]_2 = H^2$.

Proof. (i) Let $[hA]_2 = H^2$. Then, $[hA]_1 = [[hA]_2]_1 = [H^2]_1 = H^1$, and so h is outer.

(ii) Let h be outer. Since $[[hA]_2 A_0]_2 = [hA_0]_2 < [hA]_2$ by Lemma 1.1.3 (b), Theorem 1.1.1 applied to $M = [hA]_2$ gives $[hA]_2 = qH^2$ for some measurable q such that $|q| = 1$. Now, since h is outer,

$$H^1 = [hA]_1 = [[hA]_2]_1 = [qH^2]_1 = q[H^2]_1 = qH^1.$$

Hence $q = \underline{1}$ and $[hA]_2 = H^2$.

From now on, we shall use q (with or without subscripts or superscripts) to refer to a measurable function everywhere of absolute value 1.

§1.3. Factorisation Theorems.

Theorem 1.3.1. If $f \in L^2$ and $f \notin [fA_0]_2$, then $f = qh$, where $h \in H^2$ is outer, and $q \in [fA]_2$.

Proof. Our assumption implies that $[fA]_2$ is a simply invariant outspace of L^2. Hence, by Theorem 1.1.1, $[fA]_2 = qH^2$. Thus, $f = qh$, some $h \in H^2$. Now,

$$q[hA]_2 = [qhA]_2 = [fA]_2 = qH^2;$$

thus we have $[hA]_2 = H^2$ and hence h is outer (by note 1.2.3). Also, since $\underline{1} \in H^2$, $q \in qH^2 = [fA]_2$.

Theorem 1.3.1 is actually a generalisation of a factorisation theorem due to Bernling and Nevanlinna (Benrling [1]), which applies to functions f in the Hardy space H^2 for which $\int f dm \neq 0$.

We now prove

Corollary 1.3.1. If $f \in L^1$ and $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$, then $f = qh^2$, where $h \in H^2$ is outer.

Proof. By theorem 1.3.1, $|f|^{1/2} = q_1h$, where $h \in H^2$ is outer. Thus we have, for

$$\text{sgn}f = \begin{cases} \frac{f}{|f|} & \text{if } f \neq \underline{0} \\ \underline{0} & \text{if } f = \underline{0} \end{cases}$$

$f = (\text{sgn}f)|f|^{1/2}|f|^{1/2} = (\text{sgn}f)q_1^2h^2 = qh^2$, where $q = (\text{sgn}f)q_1^2$.

<u>Corollary 1.3.2.</u> If $f \in L^1$, then $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$ if and only if $f \notin [fA_0]_1$.

<u>Proof.</u> Suppose first that $|f|^{1/2} \in [|f|^{1/2}A_0]_2$. Let $f_1 = (\text{sgn}f)|f|^{1/2}$. Then

$$f = f_1|f|^{1/2} \in f_1[|f|^{1/2}A_0]_2 \subset [f_1|f|^{1/2}A_0]_1 = [fA_0]_1.$$

Thus, if $f \notin [fA_0]_1$, then $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$. Now suppose $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$. Then, by Corollary 1.3.1 $f = qh^2$ by, where $h \in H^2$ is outer. We need to show that $h^2 \in H^2.H^2 \subset H^1$ (Lemma 1.1.3 (a)) is outer. Now, since h is outer,

$$hA \subset hH^2 = h[hA]_2 \subset [h^2A]_1.$$

Hence $H^1 = [[hA]_2]_1 \subset [h^2A]_1$. But, by Lemma 1.1.3 (a) $h^2 \in H'$, so $[h^2A]_1$ H', and so we have $[h^2A]_1 = H^1$ and $h^2 \in H'$ is outer. Now $[fA_0]_1 = [qh^2A_0]_1 = qH^1_0$, but $f = qh^2 \notin [fA_0]_1$, by Note 1.2.1, and hence result.

<u>Corollary 1.3.3.</u> If $f \in L^1$ and $f \notin [fA_0]_1$, then $f = Fh$, where $h \in H^2$ is outer, $|h|^2 = |f|$ and $F \in [fA]_1 \cap L^2$.

<u>Proof</u>. By Corollary 1.3.1 and Corollary 1.3.2, we have $f = qh^2$, where $h \varepsilon H^2$ is outer. Let $F = qh$. Then $F \varepsilon L^2$. Thus $f = Fh$ with $h \varepsilon H^2$ outer, and, since $|q| = 1$, $|h|^2 = |f|$. Further, $F \varepsilon FH^2 = F[hA]_2 \subset [FhA]_1 = [fA]_1$.

<u>Remark 1.3.1</u>. Since m is multiplicative on A, we have A and $\overline{A}_0$ as orthogonal subsets of L^2. It follows from Lemma 1.1.2 that $L^2 = H^2 \oplus \overline{H}^2_0$; or, equivalently,

$$H^2 = \{f \varepsilon L^2 : \int fgdm = 0 \ \forall \ g \varepsilon A_0\}. \qquad \text{(i)}$$

Since $f \varepsilon H^1$ implies $\int fgdm = 0 \ \forall \ g \varepsilon A_0$ (by Lemma 1.1.3 (b)), we see that $H^2 \supset H^1 \cap L^2$. Trivially, $H^2 \quad H^1 \quad L^2$. and so we have

$$H^2 = H' \quad L^2 \qquad \text{(ii)}$$

Since, by definition, $H^\infty = H^2 \cap L^\infty$ we have also

$$H^\infty = \{f \varepsilon L^\infty : \int fgdm = 0 \ \forall \ g \varepsilon A_0\}. \qquad \text{(iii)}$$

and

$$H^\infty = H \cap L^\infty. \qquad \text{(iv)}$$

We now prove

<u>Corollary 1.3.4</u>. $H^1 = \{f \varepsilon L^1 : \int fgdm = 0 \ \forall \ g \varepsilon A_0\}$.

<u>Proof</u>. Clearly, if $f \in H^1$ then Lemma 1.1.3 (b) ensures that $\int fgdm = 0 \ \forall \ g \in A_0$. Conversely, consider $f \in L^1$ such that $\int fgdm = 0, \ \forall \ g \in A_0$. By replacing f by $f + \underline{c}$ for some constant function $\underline{c}$ if necessary, we may assume that $\int fdm \neq 0$. Then $f \notin [fA_0]_1$ and so, by Corollary 1.3.3, $f = Fh$ where $F \in [fA]_1 \cap L^2$, and $h \in H^2$ is outer. Since $F \in [fA]_1$, it follows that $\int Fgdm = 0, \ \forall \ g \in A_0$. Since also $F \in L^2$, $F \in H^2$ (by (i), Remark 1.3.1). Thus, $f = Fh \in H^2.H^2 \subset H^1$, by Lemma 1.1.3 (a), and the proof is complete.

<u>Corollary 1.3.5</u>. If $f \in L^1$ and $f \notin [fA_0]_1$, then $f = qh_1$,, where $h_1 \in H$ is outer, and $q \in [fA]_1$. The converse is also true.

<u>Proof</u>. Suppose $f \notin [fA_0]_1$. Then, by Corollary 1.3.2, $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$ and so, by Corollary 1.3.1, $f = qh^2$, when $h \in H^2$ is outer. As in the proof of Corollary 1.3.2, if $h \in H^2$ is outer, then $h^2 \in H^1$ is also outer. Let $h_1 = h^2$. Then $f = qh_1$. Also,

$$q \in qH^1 = q[h,A]_1 = [qh,A]_1 = [fA]_1$$

and hence f is of the stated form.

The converse follows from the fact that, since h_1 is outer, $h_1 \notin [h_1 A_0]_1$ (Note 1.2.1), and so, $f = qh_1 \notin q[h_1 A_0]_1 = [qh_1 A_0]_1 = [fA_0]_1$.

§1.4. L^p-invariant subspace Theorem.

Consider p such that $1 \leq p < 2$. Define the number r by $\frac{1}{r} + \frac{1}{2} = 1/p$. Then $p/r + p/2 = 1$, and so r/p and $2/p$ are conjugate indices. We now prove the following two lemmas.

Lemma 1.4.1. If $f_n, f \in L^2$, and $g \in L^r$, then $f_n g, fg \in L^p$. If, further, $\{f_n\} \subset L^2$ converges to f in L^2, then $\{f_n g\} \subset L^p$ converges to fg in L^p.

Proof. Since $f \in L^2$ and $g \in L^r$ we have $f^p \in L^{2/p}$ and $g^p \in L^{r/p}$. Thus, by the Hölder inequality, $f^p g^p = (fg)^p \in L^1$, and hence $fg \in L^p$. Similarly, $f_n g \in L^p$. Again by the Hölder inequality, we have

$$||(f_n - f)g||_p^p \leq ||f_n - f||_2^p ||g||_r^p,$$

and hence

$$||f_n g - fg||_p = ||(f_n - f)g||_p \leq ||f_n - f||_2 ||g||_r$$

and so, since $\{f_n\} \subset L^2$ converges to f in L^2, we have $\{f_n g\} \subset L^p$ convergent to fg in L^p.

Lemma 1.4.2. Let $1 \leq p < 2$. Define r as before. If $f \in L^p$ and $f \notin [fA_0]_p$, then $f = Fh$, where $h \in H^2$ is outer, and $F \in [fA]_p \cap L^r$.

Proof. Let $f_1 = (\text{sgn} f)|f|^{p/r}$ and $f_2 = |f|^{p/2}$. Then $f_1 \in L^r$, $f_2 \in L$ and $f = f_1 f_2$. Also, $f_2 \notin [f_2 A_0]_2$, since, if $f_2 \in [f_2 A_0]_2$, then $f = f_1 f_2 \in f_1[f_2 A_0]_2 \subset [f_1 f_2 A_0]_p = [fA_0]_p$, by Lemma 1.4.1, which is contrary to hypothesis. Hence, by Theorem 1.3.1, we have $f_2 = qh$, where $q \in [f_2 A_0]_2$, and $h \in H^2$ is outer. Let $F = f_1 q$. Then, since $f_1 \in L^r$ and $|q| = 1$, $F \in L^r$. Also, $F \in f_1 q H^2 = f_1 q [hA]_2 \subset [f_1 q h A]_p = [fA]_p$, by Lemma 1.9.1. Clearly $f = Fh$ and hence result.

Corollary 1.4.1. If $1 \leq p \leq \infty$, then $H^p = H^1 \cap L^p$.

Proof. We have already shown in Remark 1.3.1 that the statement is true for $p = 2, \infty$. It is trivial for $p = 1$. We shall prove it to be true for $1 < p < 2$ by use of Lemma 1.4.2; and for $p > 2$ by a duality argument.

Let $1 < p < 2$. It is clear that $H^p \subset H^1 \cap L^p$. To show the reverse inclusion, consider any $f \in H^1 \cap L^p$. We may suppose, by considering $f + \underline{c}$ where $\underline{c}$ is a constant function, if necessary, that $\int fdm \neq 0$. Then $f \notin [fA_0]_p$ and so, by Lemma 1.4.2, $f = Fh$ when $h \varepsilon H^2$ is outer and $F \varepsilon [fA]_p \cap L^r$, r defined as before. Since $1 < p < 2$, $r > 2$, and so $F \varepsilon L^2$. Also, $F \varepsilon [fA]_p \subset H^1$ since $f \varepsilon H^1$. Thus, $F \varepsilon H^1 \cap L^2 = H^2$ ((i) Remark 1.3.1). In particular, since $1 < p < 2$, $F \varepsilon H^p$, and so $f = Fh \varepsilon FH^2 = F[A]_2$. But, since $F \varepsilon L^r$, $r > 2$, $F[A]_2 \subset [FA]_p \subset H^p$ and so $f \varepsilon H^p$. Therefore $H^p = H^1 \cap L^p$ for every p satisfying $1 < p < 2$. Now let $2 < p < \infty$. Again it is clear that $H^p \subset H^1 \cap L^p$. We wish to show equality here. It suffices to show that, if $g \varepsilon L^{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) is such that $g \perp H^p$, then $g \perp H^1 \cap L^p$. We have, by hypothesis, $\int gfdm = 0$, $\forall$ $f \varepsilon H^p$, and so, by Corollary 1.3.4, $\overline{g} \varepsilon H^1$. Since also $\underline{1} \varepsilon H^p$, $\overline{g} \varepsilon H_0^1$. Thus, since $1 < p' < 2$, what we have already proved shows that

$$\overline{g} \varepsilon H_0^1 \cap L^{p'} \subset H^1 \cap L^{p'} = H^{p'}.$$

Hence $\exists$ sequence $\{g_n\} \subset A_0$ which converges to $\overline{g}$ in $L^{p'}$, so that

$$\lim_{n \to \infty} \int fg_n dm = \int f\overline{g}dm, \quad \forall \ f \varepsilon H^1 \cap L^p.$$

Thus, making use of Lemma 1.1.3 (b), $g \perp H^1 \cap L^p$ and our proof is completed.

Corollary 1.4.2. If $1 \leq p \leq \infty$, then

$$H^p = \{f \in L^p : \int fg dm = 0 \ \forall \ g \in A_0\}.$$

Proof. This fact follows directly from Corollary 1.4.1 and Corollary 1.3.4 and is a generalisation of Corollary 1.3.4.

We now wish to prove the invariant subspace theorem for general p, $1 \leq p \leq \infty$. Here, a subspace M of $L^p(1 \leq p < \infty)$ is said to be simply invariant if M is norm closed and $[MA_0]_p < M$. (This agrees with our earlier definition for $p = 2$.) A subspace M of L^∞ is simply invariant if M is weak-* closed and $[MA_0]_* < M$. We now have

Theorem 1.4.1. (L^p-invariant subspace Theorem). Every simply invariant subspace M of L^p, $1 \leq p \leq \infty$, is of the form $M = qH^p$ for some measurable q such that $|q| = 1$ (and trivially convesely).

Proof. For $p = 2$, Theorem 1.4.1 reduces to Theorem 1.1.1.

For $1 \leq p < 2$ we shall use Lemma 1.4.2. For $p > 2$ we shall use a duality argument.

Let $1 \leq p < 2$. Put $N = M \cap L^2$. Clearly N is a closed subspace of L^2. We first show that N is non-zero. Since $[MA_0]_p < M$, $\exists$ $f \in M \setminus [MA_0]_p$. Thus $f \notin [fA_0]_p$ and so, by

Lemma 1.4.2, we can write $f = Fh$ where $h \in H^2$ is outer, and $F \in [fA]_p \cap L^r$, r defined as before. Now,

$$[fA]_p \cap L^r \subset [MA]_p \cap L^2 \subset M \cap L^2 = N$$

so $F \in N$ and N is non-zero. Also, $F \notin [NA_0]_2$, since if $F \in [NA_0]_2$,

$$f = Fh \in FH^2 = F[A]_2 \subset [FA]_p \subset [[NA_0]_2 A]_p \subset [NA_0]_p \subset [MA_0]_p$$

contrary to our assumption on f. Combined with the fact that $[NA_0]_2 \subset [MA_0]_2 \subset M \cap L^2 = N$, this shows that $[NA_0]_2 < N$ is simply invariant. Hence, by Theorem 1.1.1, $N = qH^2$ for some measurable q such $|q| = 1$. We shall now show that $M = qH^p$ with the same q. We have, already, $qA \subset qH^2 = N \subset M$, so $qH^p = q[A]_p = [qA]_p \subset [M]_p = M$. If we take $f \in M \setminus [MA_0]_p$ as before, we know that f can be written as $f = Fh$ where $h \in H^2$, and, as shown earlier, $F \in N = qH^2$. Thus, $\overline{q}F \in H^2$ and $\overline{q}f = \overline{q}Fh \in H^2.H^2 \in H^1$, by Lemma 1.1.3 (a). Hence $\overline{q}f \in H^1 \cap L^p = H^p$, by Corollary 1.4.1, and so $f \in qH^p$. This gives us the inclusion $M \setminus [MA_0]_p \subset qH^p$. Now, if $g \in [MA_0]_p$ and $f \in M \setminus [MA_0]_p$, then $f + g \in M \setminus [MA_0]_p$. From the preceeding discussion we then have $f, f + g \in qH^p$. Hence $g \in qH^p$ also. Thus $M = qH^p$ and the theorem is proved for $1 \leq p < 2$.

Now let $2 < p \leq \infty$. Let p' be the conjugate index of p. Thus $\frac{1}{p} + \frac{1}{p'} = 1$ and so $1 \leq p' < 2$.

Let $M_0 = [MA_0]_p$ or $[MA_0]_*$ according as $2 < p < \infty$ or $p = \infty$, and let

$$N = \{f \in L^{p'} : \int fg dm = 0, \ \forall \ g \in M_0\}.$$

Clearly N is a closed subspace of $L^{p'}$ and $NA \subset N$. We shall show that $[NA_0]_{p'} < N$. Since $M_0 < M$ $\exists$ a non-zero continuous (weak-* continuous if $p = \infty$) linear functional Φ on M which annihilates M_0. Φ is realised by a function $f \in L^{p'}$ and, by the choice of Φ, $f \in N$. Since Φ is non-zero on M, $\exists$ $g \in M$ such that $\int fg dm \neq 0$. This fact, combined with the definition of N implies that $f \notin [NA_0]_{p'}$. Hence N is simply invariant. Since $1 \leq p' < 2$, the first part of this proof applies and we can write $N = qH^{p'}$ for some measurable q such that $|q| = 1$.

By duality theory,

$$M_0 = \{g \in L^p : \int fg = 0, \ \forall \ f \in N\}.$$

This fact, combined with Corollary 1.4.2. leads to $M_0 = qH^p_0$. We shall show that $M = qH^p$. If $f \in \bar{q}M$ and $g \in A_0$, then

$$fg \in \bar{q}MA_0 \subset \bar{q}M_0 = H^p_0$$

and so, $\int fg\,dm = 0$, and hence, by Corollary 1.4.2 $f \in H^p$. This shows that $\bar{q}M \subset H^p$, or $M \subset qH^p$.

Now, since each $f \in H^p$ can be written in the form $f = g + \underline{c}$ where $g \in H_0^p$ and $\underline{c}$ is a constant function, H_0^p is a subspace of H^p of codimension 1, so that $M_0 = qH_0^p$ is a subspace of gqH^p of codimension 1. Hence, either $M = qH^p$ or $M = M_0$. The latter is impossible since M is simply invariant, and so $M = qH^p$ and our proof is completed.

We shall now add to our discussion of outer functions. We prove

Corollary 1.4.3. $h \in H^p$, $1 \leq p < \infty$ is outer if and only if $[hA]_p = H^p$.

Proof. (i) Let $[hA]_p = H^p$. Then

$$[hA]_1 = [[hA]_p]_1 = [H^p]_1 = H^1,$$

and so h is outer.

(ii) Let h be outer. Thus $[hA]_1 = H^1$. Since $h \notin [hA_0]_p$, $[hA]_p$ is an invariant subspace of L^p, and so, by Theorem 1.4.1, $[hA]_p = qH^p$, for some measurable q such that $|q| = 1$. But $H^1 = [hA]_1 = [[hA]_p]_1 = [qH^p]_1 = q[H^p]_1 = qH^1$.

Hence q and $\overline{q} = q^{-1}$ belong to H^1 and Lemma 1.1.4 implies that q = constant function = $\underline{1}$. Thus $[hA]_p = qH^p = H^p$.

<u>Corollary 1.4.4.</u> If $f \in L^p$, $1 \leq p < \infty$ and $f \notin [fA_0]_p$, then $f = qh$ where $h \in H^p$ is outer and $q \in [fA]_p$.

<u>Proof.</u> Our assumption implies that $[fA]_2$ is a simply invariant subspace of L^p. Hence, by Theorem 1.4.1, $[fA]_p = qH^p$, and so $f = gh$ for some $h \in H^p$. Now

$$q[hA]_p = [qhA]_p = [fA]_p = qH^p$$

and so we have $[hA]_p = H^p$, and hence, by Corollary 1.4.3, h is outer.

Since $\underline{1} \in H^p$, $q \in qH^p = [fA]_p$.

<u>§1.5. L^2-invariant subspace Theorem implies $A + \overline{A}$ weak* dense in L^∞</u>

We show first that the L^2-invariant subspace Theorem (Theorem 1.1.1) implies that $A + \overline{A}_0$ is norm dense in L^2; and then that $A + \overline{A}_0$ norm dense in L^2 implies $A + \overline{A}$ is weak -* dense in L^∞.

Assume Theorem 1.1.1 holds. We prove the following with this assumption.

Theorem 1.5.1. $L^2 = H^2 \oplus \overline{H}^2_0$.

Proof. Let $M = L^2 \ominus \overline{H}^2_0$. Then $[MA_0]_2 < M$ and so, by Theorem 1.1.1, $M = qH^2$, some measurable q such that $|q| = 1$. Since $\underline{1} \in M$, $\overline{q} \in H^2$. Let $\overline{c} = \int \overline{q}dm$. Then $\overline{q} - \overline{\underline{c}} \in H^2_0$ and so $q - \underline{c} \in \overline{H}^2_0$. But $q, \underline{c} \in M$, so $q - \underline{c} \in M$, and thus $q - \underline{c} \in \overline{H}^2_0$. It follows then that $q - \underline{c} = \underline{0}$ and $M = cH^2 = H^2$.

Corollary 1.5.1. That $A + \overline{A}_0$ is norm dense in L^2 follows directly from Theorem 1.5.1.

We need also

Corollary 1.5.2. $H^1 \cap L^2 = H^2$.

Proof. Clearly $H^2 \subset H^1 \cap L^2$. Now consider $f \in H^1 \cap L^2$. Since $f \in H^1$, $f \perp \overline{A}_0$ and so $f \perp \overline{H}^2_0$. Thus, by Theorem 1.5.1, $f \in H^2$ and so $H^1 \cap L^2 \subset H^2$.

Corollary 1.5.3. $H^1 = \{f \quad L^1 : \int fgdm = 0, \forall \quad g \in A_0\}$.

Proof. Clearly, if $f \in H^1$, then $\int fgdm = 0, \forall \quad g \in A_0$. Now consider $f \in L'$ such that $\int fgdm = 0, \forall \quad g \in A_0$. By replacing f by $f + \underline{c}$ for some constant function $\underline{c}$, if necessary, we may assume that $\int f dm \neq 0$. Then $f \in [fA_0]_1$ and so, by Corollary 1.3.3, $f = Fh$ where $F \in [fA]_1 \cap L^2$ and $h \in H^2$ is outer. Since $F \in [fA]_1$, it follows that $\int Fg dm = 0$, $\forall$ $g \in A_0$. Since also $F \in L^2$, we have, by Theorem 1.5.1 $F \in H^2$. Thus $f = Fh \in H^2.H^2 \subset H^1$ by Lemma 1.1.3 (a).

We are now in a position to show that the truth of Theorem 1.1.1 implies that $A + \overline{A}$ is weak-* dense in L^∞. In doing this we shall use the Jensen inequality (Corollary 2.1.3), our proof of which will depend only on Theorem 1.1.1, and Corollary 1.5.3 which we have just shown to follow from Theorem 1.1.1.

<u>Theorem 1.5.2.</u> $A + \overline{A}$ is weak-* dense in $L^\infty(m)$.

<u>Proof</u>. We need only show that any $g \in L^1_{\mathbb{R}}(m)$, such that $\int fg dm = 0$ for every $f \in A$. Suppose $g \neq 0$ a.e. (m). By Corollary 1.5.3, $g \in H^1$; and, since $\underline{1} \in A$, $\int g dm = 0$ and so $g \in H^1_0$. Thus $\underline{1} - tg \in H^1$, $\forall$ real t, and so, by Corollary 2.1.3 (Jensen inequality), we have

$$\int \log|\underline{1}-tg|dm \geq \log\left|\int(\underline{1}-tg)dm\right| = 0.$$

Thus, by Lemma 0.3 $g = \underline{0}$ a.e. (m).

From this we can see that the conditions necessary for A to be weak-* Dirichlet are the best possible such that the invariant subspace theorem is true for this class of algebras.

CHAPTER 2.

H^∞ AS A LOGMODULAR ALGEBRA; SZEGÖ'S THEOREM.

§2.1 H^∞ as a logmodular algebra.

We now look more closely at H^∞. We shall show that H^∞ is (isomorphic to) a logmodular algebra on the maximal ideal space, M, of L^∞, and then apply to H^∞ some known results about logmodular algebras. We first prove

Theorem 2.1.1. Let A be a weak-* Dirichlet algebra. Then $H^\infty = [A]_*$.

Proof. We first show that $H^\infty \supset [A]_*$. Consider any sequence $\{f_n\} \subset H^\infty$, which is weak-* convergent to $f \in L^\infty$. Then $\lim_{n\to\infty} \int g f_n dm = \int gfdm$, $\forall\ g \in L^1$. In particular, $\lim_{n\to\infty} \int g f_n dm = \int gfdm$, $\forall\ g \in A_0$. But, by Corollary 1.4.2, $\int g f_n = 0\ \forall\ g \in A_0$, for each f_n. Hence $\int gfdm = 0$, $\forall\ g \in A_0$, and, again by Corollary 1.4.2, $f \in H^\infty$.

To establish $H^\infty \subset [A]_*$ we shall show that every linear functional Φ on L^∞ given by an L^1 function f, which vanishes on A also vanishes on H^∞. Thus $\Phi(g) = \int gfdm = 0$, $\forall\ g \in A$, and so, by the use of Corollary 1.3.4, $f \in H^1$. Since also $\underline{1} \in A$, $f \in H_0^1$. Hence,

$$\forall \quad g \in H^\infty, \quad \int fg\,dm = \int f\,dm \cdot \int g\,dm = 0$$

and the theorem is proved.

<u>Corollary 2.1.1</u>. For every p satisfying $1 \leq p \leq \infty$,

$$H^\infty \subset H^p \quad \text{and} \quad [H^\infty]_p = H^p.$$

<u>Proof</u>. Since every norm closed subspace of L^p, $1 \leq p < \infty$ is also weakly closed, $H^\infty \subset H^p$, $1 \leq p \leq \infty$. Thus we have, since H^p is closed, $[H^\infty]_p \subset H^p$. Now $H^\infty \supset A$, so $[H^\infty]_p \supset [A]_p = H^p$, $1 \leq p < \infty$. Thus $[H^\infty]_p = H^p$ whenever $1 \leq p < \infty$. If $p = \infty$, H^∞ is weak-* closed (Theorem 2.1.1) and therefore norm closed and so $[H^\infty]_\infty = H$.

Now let $V = \{f : \frac{1}{f}$ and $f \in H^\infty\}$. Clearly, if $f \in V$, then $f^{\pm n} \in V$, $\forall\, n$. Let

$$\log|V| = \{\log|f| : f \in V\}.$$

Also, let L_R^∞ denote the set of all real-valued functions in L^∞. Then we have

<u>Lemma 2.1.1.</u> $\log|V| = L^\infty_R$.

<u>Proof.</u> Clearly $\log|V| \subset L^\infty_R$. Now consider $u \in L^\infty_R$. Then, since $\underline{1} \notin H^2_0$, $e^u \notin [e^u A_0]_2 = e^u H^2_0$. Hence, by Theorem 1.3.1, $e^u = qh$, where $h \in H^2$ and $q \in [e^u A]_2 = e^u H^2$. Thus $q = e^u h'$, and, since $|q| = 1$, $h, h' \in L^\infty$ and so $h, h' \in H^2 \cap L^\infty = H^\infty$. But, now, $e^u = qh = e^u h'h$ so that $h'h = \underline{1}$. Thus $h \in V$ and $u = \log|h| \in \log|V|$.

<u>Remark 2.1.1.</u> Under pointwise operations and the essential supremum norm, $L^\infty(m)$ is a Banach Algebra. Let $\mathcal{M}$ be its maximal ideal space. We know (Hoffman [1], p.169ff) that $\mathcal{M}$ is a compact Hausdorff space and that the Gelfand mapping $f \mapsto \hat{f}$ is an isometric isomorphism from $L^\infty(m)$ onto $C(\mathcal{M})$. the space of all continuous functions on $\mathcal{M}$, and this mapping preserves complex conjugation.

Since this isomorphism is onto, every function in $C(\mathcal{M})$ is of the form $\hat{f}$ for some $f \in L^\infty(m)$. If we let $\Phi(\hat{f}) = \int f dm$, we get a bounded linear functional on $C(\mathcal{M})$. Then $\exists$ a Radon measure $\hat{m}$ on $\mathcal{M}$ such that

$$\Phi(\hat{f}) = \int \hat{f} d\hat{m} = \int f dm, \quad \forall \ f \in L^\infty(m).$$

We now look at the relationship between $L^p(m)$ and $L^p(\hat{m})$, $1 \leq p < \infty$. First note that the Gelfand mapping preserves positive powers of non-negative elements, and, since it preserves complex conjugation, (Hoffman [1], p.170), it also preserves absolute values.

Consider $f \in L^p(m)$. Since $L^\infty(m)$ is dense in $L^p(m)$, $\exists$ $\{g_n\} \subset L^\infty(m)$ convergent to f in the $L^p(m)$- norm. Hence, we have that $\{g_n\}$ is a Cauchy sequence; that is

$$\lim_{n,k\to\infty} \int |g_n - g_k|^p dm = 0.$$

Thus

$$\lim_{n,k\to\infty} \int |\hat{g}_n - \hat{g}_k|^p d\hat{m} = 0$$

and so $\{\hat{g}_n\} \subset L^p(\hat{m})$ is a Cauchy sequence. Let $\hat{f}$ denote the limit of $\{\hat{g}_n\}$ in $L^p(\hat{m})$. Consider any other sequence $\{h_n\} \subset L^\infty(m)$ which converges to f in the $L^p(m)$-norm. Then we can show that

$$\lim_{n\to\infty} \int |g_n - h_n|^p dm = 0$$

and hence

$$\lim_{n\to\infty} \int |\hat{g}_n - \hat{h}_n|^p d\hat{m} = 0.$$

Thus $\hat{f}$ is uniquely determined by f (modulo functions equal to zero a.e. $(\hat{m})$.).

Conversely, consider $g \in L^p(\hat{m})$. Then $\exists$ $\{\hat{g}_n\} \subset C(\mathcal{M})$ such that $\{\hat{g}_n\}$ converges to g in the $L^p(\hat{m})$-norm. Hence

$$\lim_{n,k\to\infty} \int |\hat{g}_n - \hat{g}_k|^p d\hat{m} = 0,$$

and so $\{g_n\} \subset L^\infty(m)$ is a Cauchy sequence in $L^p(m)$. Suppose $\{g_n\}$ converges in the L^p-norm to $f \varepsilon L^p(m)$. Then, by our previous argument, $\{\hat{g}_n\}$ converges to $\hat{f} \varepsilon L^p(\hat{m})$ in the $L^p(\hat{m})$-norm, and so $g = \hat{f}$.

Thus, the mapping $f \mapsto \hat{f}$ sets up an isometry between $L^p(m)$ and $L^p(\hat{m})$ and this isometry is an extension of the mapping $f \mapsto \hat{f}$ from $L^\infty(m)$ onto $C(M)$.

Now suppose $\hat{f} \varepsilon L^\infty(\hat{m})$ is the image of $f \varepsilon L^p(m)$. Consider $g \varepsilon L^1(m)$. since $L^\infty(m)$ is dense in $L^1(m)$, $\exists$ $\{g_n\} \subset L^\infty(m)$ which converges to g in the $L^1(m)$-norm. Noting that $\hat{gf} = \hat{g}\hat{f}$, we have

$$\int |g_n f - g_k f| dm = \int |\hat{g}_n \hat{f} - \hat{g}_k \hat{f}| d\hat{m} \leq ||f||_\infty \int |g_n - g_k| dm$$

and so $\{g_n f\}$ converges to gf in the $L^1(m)$-norm. but

$$|\int g_n f dm| \leq \int |g_n f| dm = ||g_n f||_1 \leq ||\hat{f}||_\infty ||g_n||_1.$$

Hence $|\int gf dm| \leq ||\hat{f}||_\infty ||g||_1$, so that $f \varepsilon L^\infty(m)$ and hence $\hat{f} \varepsilon C(M)$. This shows that every function in $L^\infty(\hat{m})$ is equal a.e. $(\hat{m})$ to a continuous function.

Let $\hat{H}^\infty = \{\hat{f} : f \in H^\infty\}$. $\hat{H}^\infty$ is a subalgebra of $C(M)$ closed under the supremum norm, since H^∞ is a weak-* closed, and hence norm closed, subalgebra of $L^\infty(m)$.

We have already noted that the mapping $f \mapsto \hat{f}$ preserves absolute values and positive powers of non-negative elements. By the Stone-Weierstrass Theorem, for every $f \in V$, $\log|f|$ is the uniform limit of polynomials of the form

$$\sum_{k=0}^{n} a_k |f|^k.$$

But from above

$$\left(\sum_{k=0}^{n} a_k |f|^k\right)^\wedge = \sum_{k=0}^{n} \alpha_k |\hat{f}|^k,$$

and we may conclude that $(\log|f|)^\wedge = \log|\hat{f}|$.

Thus, we may restate Lemma 2.1.1 as

$$\log|V| = C_R(M);$$

where $\hat{V} = \{\text{set } \hat{f} : \hat{f} \text{ and } 1/\hat{f} \in \hat{H}^\infty\}$. Since $C_R(M)$ is a separating algebra of $C(M)$ so is $\log|\hat{V}|$, and hence also is $\hat{H}^\infty$.

Immediately we have

<u>Theorem 2.1.2.</u> $\hat{H}^\infty$ is a logmodular algebra on M, the maximal ideal space of L^∞.

Because H^∞ is isomorphic to $\hat{H}^\infty$ we can omit the "^" and consider H^∞ as a logmodular algebra on the maximal ideal space L^∞.

Appeal to Theorem 7.1 in Hoffman [2] now yields

Corollary 2.1.2. The maximal ideal space $\mathcal{M}$ of L^∞ can be embedded in that of H^∞ as its Shilov boundary.

We also have, by Theorem 0.2, that there exists an unique Arens-Singer measure $\hat{m}$ such that $\int \hat{f}d\hat{m}$ represents a complex homomorphism on $\hat{H}^\infty$. We can thus prove

Corollary 2.1.3. (Jensen inequality).

$$\log|\int fdm| \leq \int \log|f|dm, \ \forall \ f \in H^\infty.$$

Proof. Let $\hat{f} \in \hat{H}^\infty$ and $\varepsilon > 0$. Then $\log(|\hat{f}| + \underline{\varepsilon})$ $C_R(\mathcal{M})$. Hence, $\exists$ $u \in \log|\hat{V}|$ such that

$$w - \underline{\varepsilon} < \log(|\hat{f}| + \underline{\varepsilon}) < u + \underline{\varepsilon} \tag{1}$$

If $u = \log|\hat{g}|$, $\hat{g} \in \hat{V}$, let $\hat{h} = \hat{f}\hat{g}^{-1}$. Then $\hat{h} \in \tilde{\hat{H}}^\infty$, and, by the right hand side of (1), $\log|\hat{h}| < \varepsilon$, that is, $|\hat{h}| < \exp \circ \underline{\varepsilon}$ on $\mathcal{M}$ and so

$$|\int \hat{h}d\hat{m}| \leq \int |\hat{h}|d\hat{m} < \int(\exp \circ \underline{\varepsilon})d\hat{m} = e^{\varepsilon}.$$

Thus, $|\int \hat{f}d\hat{m}||\int \hat{g}d\hat{m}|^{-1} < e^{\varepsilon}$, and so

$$\log|\int \hat{f}d\hat{m}| - \log|\int \hat{g}d\hat{m}| < \varepsilon$$

Now, since $\hat{m}$ is an Arens-Singer measure and $\hat{g} \in \hat{V}$, we have

$$\log\left|\int \hat{g} d\hat{m}\right| = \int \log|\hat{g}| d\hat{m} = \int u \, d\hat{m}.$$

By the left hand side of (1)

$$\int u d\hat{m} < \varepsilon + \int \log(|\hat{f}| + \varepsilon) d\hat{m}.$$

Thus, we have

$$\log\left|\int \hat{f} d\hat{m}\right| < 2\varepsilon + \int \log(|\hat{f}| + \varepsilon) d\hat{m}.$$

Let ε tend monotonically to zero to obtain

$$\log\left|\int \hat{f} dm\right| \leq \int \log|\hat{f}| dm, \quad \forall \ \hat{f} \in H^\infty$$

and so

$$\log\left|\int f dm\right| \leq \int \log|f| dm, \quad \forall \ f \in H^\infty.$$

In particular, we have

$$\log|f dm| \leq \int \log|f| dm, \quad \forall f \in A.$$

Since m is an Arens-Singer measure we get equality for $f \in V$.

§2.2 More about Outer Functions.

Before using Cor 2.1.3 to help prove Szegö's theorem, we need some more facts about outer functions. We prove therefore, Lemma 2.2.1. Let $1 \leq p < \infty$. If $h \neq 0$ a.e. (m) and $\int h\,dm = \lambda \neq 0$, then $h \in H^p$ is outer if and only if $\underline{1} - \underline{\lambda}/h \in [A_0]_{L^p(|h|^p m)}$.

Proof. Suppose $\underline{1} - \underline{\lambda}/h \in [A_0]_{L^p(|h|^p m)}$. Then $\exists\ \{f_n\} \subset A_0$ such that

$$\lim_{n\to\infty} \int |f_n - (\underline{1} - \underline{\lambda}/h)|^p |h|^p dm = \lim_{n\to\infty} \int |(f_n - \underline{1})h + \underline{\lambda}|^p dm = 0 .$$

Thus $-\underline{\lambda} \in [hA]_p$ and, since $\lambda \neq 0$, $\underline{1} \in [hA]_p$. Hence $[hA]_p = H^p$ and so, by Corollary 1.4.3, h is outer. Since every implication in the proof is reversible the converse is also true.

§2.3 Szegö's Theorem.

Theorem 2.3.1. (Szegö). Let $1 \leq p < \infty$ and $w \in L^1(m)$, $w \geq 0$. Then

$$\inf_{f \in A_0} \int |\underline{1} - f|^p w\,dm = \exp[\int \log w\,dm],$$

where $\int \log w\,dm$ is defined to be $-\infty$ if $\log w \notin L^1(m)$.

Proof. Consider $f \in A_0$. Let $\delta > 0$ and apply Lemma 0.1 to $\log(|\underline{1} - f|^p w + \underline{\delta}) \in L^1_R(m)$ to get

$$\int (|\underline{1} - f|^p w + \underline{\delta})dm \geq \exp[\int \log(|\underline{1} - f|^p w + \underline{\delta})dm.$$

That is

$$\int |\underline{1} - f|^p w dm + \varepsilon \geq \exp[\int \log(|\underline{1} - f|^p w + \underline{\varepsilon}) dm].$$

Let ε tend monotonically to zero to obtain

$$\int |\underline{1} - f|^p w dm \geq \exp[\int \log |\underline{1} - f|^p dm + \int \log w dm].$$

By Jensen's inequality (Corollary 2.1.3),

$$\int \log |\underline{1} - f|^p dm \geq p \log |\int (\underline{1} - f) dm| = 0, \text{ since } f \in A_0.$$

Hence,

$$\int |\underline{1} - f|^p w dm \geq \exp[\int \log w dm], \ \forall \ f \in A_0,$$

and so

$$\inf_{f \in A_0} \int |\underline{1} - f|^p w dm \geq \exp[\int \log w dm],$$

which is one half of Szegö's Theorem.

To prove the reverse inequality we can assume that the infimum on the left hand side is positive. Then $w^{1/p} \notin [w^{1/p} A_0]_p$, so that, by Corollary 1.4.4, $w^{1/p} = qh$ where $h \in H^p$ is outer, and $w = |w| = |h|^p$. Since $h \in H^p$ is outer, we have, by Lemma 2.2.1, that

$$\underline{1} - \underline{\lambda}/h \in [A_0]_{L^p(|h|^p m)}$$

Now,

$$\inf_{f\varepsilon A_0} \int|\underline{1} - f|^p w dm = \inf_{f\varepsilon A_0} \int|\underline{1} - f|^p |h|^p dm$$

which is the distance of $\underline{1}$ from A_0 in $L^p(|h|^p m)$.

Hence,

$$\inf_{f\varepsilon A_0} \int|\underline{1} - f|^p w dm \leq \int|\underline{1} - (\underline{1} - \underline{\lambda}/h)|^p |h|^p dm$$

$$\leq |\lambda|^p = |\int h dm|^p = \exp(\log|\int h dm|^p)$$

$$\leq \exp(\int \log|h|^p dm) \text{ by the Jensen inequality}$$

$$= \exp(\int \log w dm).$$

That is,

$$\inf_{f\varepsilon A_0} \int|\underline{1} - f|^p w dm \leq \exp(\int \log w dm).$$

Hence result.

Corollary 2.3.1. If $f \varepsilon L^1$, then $f \notin [fA_0]_1$ if and only if $\int \log|f| dm > -\infty$.

Proof. By Corollary 1.3.2, $f \notin [fA_0]_1$ if and only if $|f|^{1/2} \notin [|f|^{1/2}A_0]_2$. That is, if and only if

$$\inf_{g \varepsilon A_0} \int \left| |f|^{1/2} - |f|^{1/2} g \right|^2 dm = \inf_{g \varepsilon A_0} \int |f| \, |\underline{1} - g|^2 dm > 0.$$

But, by Szegö's theorem (Theorem 2.3.1) for $p = 2$,

$$\inf_{g \varepsilon A_0} \int |f| \, |\underline{1} - g|^2 dm = \exp(\int \log |f| \, dm)$$

Hence, left hand side positive implies $\int \log |f| dm > -\infty$, and hence result.

<u>Corollary 2.3.2</u>. If $f \varepsilon L^p$ and $\int \log |f| dm > -\infty$, then $f = qh$, where $h \varepsilon H^p$ is outer; and conversely.

<u>Proof</u>. Let $f \varepsilon L^p$ and $\int \log |f| dm > -\infty$. Then, by Corollary 2.3.1, $f \notin [fA_0]_1$, and so, by Corollary 1.3.5, $f = qh$, where $h \varepsilon H'$ is outer. Since $|f| = |h|$ and $f \varepsilon L^p$, then $h \varepsilon L^p$. Thus $h \varepsilon H^1 \cap L^p = H^p$ by Corollary 1.4.1. Since each step in hthe proof is reversible the converse is true.

We are now in a position to prove a further Corollary of Theorem 1.4.1:

<u>Corollary 2.3.3</u>. Let $1 \leq p < \infty$. If $f \varepsilon L^1$, the following three conditions are equivalent:

(i) $|f|^{1/p} \notin [|f|^{1/p}A_0]_p$

(ii) $f \notin [fA_0]_1$

(iii) $\int \log|f|dm > -\infty$.

Proof. (ii) => (i).

Assume $|f|^{1/p} \in [|f|^{1/p}A_0]_p$. Let $f_1 = (\text{sgn}f)|f|^{1/p'}$, where $1/p + 1/p' = 1$. Then

$$f = f_1|f|^{1/p} \in f_1[|f|^{1/p}A_0]_p \subset [f_1|f|^{1/p}A_0]_1$$

and so

$$f \in [fA_0]_1.$$

(ii) <=> (iii)

This is simply Corollary 2.3.1.

(i) => (iii)

Assume $|f|^{1/p} \notin [|f|^{1/p}A_0]_p$. Now $f \in L^1$ as $|f|^{1/p} \in L^p$. Hence, by Corollary 1.4.4, $|f|^{1/p} = qh$ where $h \in H^p$ is outer, and so, by Corollary 2.3.2, $\log\int|f|^{1/p}dm = \frac{1}{p}\int\log|f|dm > -\infty$, and so $\int\log|f|dm > -\infty$.

§2.4 Kolmogorov-Krein Theorem.

For those weak-*Dirichlet algebras which are also logmodular, an extension of Szegö's Theorem (Theorem 2.3.1) holds. This extension is known as the Kolmogorov-Krein Theorem. In fact, Lumer (Lumer [1]) showed that this theorem holds for all sup-norm subalgebras of $C(X)$ such that $M = \{m\}$, $\forall\ \phi \in \mathcal{M}$, where M_ϕ is the set of representing measures for ϕ, and $\mathcal{M}$ is the maximal ideal space of A. We shall combine Theorem 2.3.1 (Szegö) with an adaptation of a result of Hoffman (Hoffman [2] Theorem 4.3) to show that the Kolmogorov-Krein Theorem holds for those weak star Dirichlet algebras which have the property that $M_\phi = \{m\}$, $\forall\ \phi \in \mathcal{M}$. We shall then show in §2.5 that this property is necessary for the truth of the Kolmogorov-Krein Theorem. We now prove

Theorem 2.4.1. Let A be a weak-* Dirichlet algebra on a compact Hausdorff space X, such that $M_\phi = \{m\}$, $\forall\ \phi \in \mathcal{M}$, where m is a probability measure on X, multiplicative on A. Let μ be a positive measure on X, not necessarily multiplicative on A. If $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m, then

$$\inf_{f \in A_0} \int |\underline{1} - f|^2 d\mu = \inf_{f \in A_0} \int |\underline{1} - f|^2 d\mu_a .$$

Proof. Let F be the orthogonal projection of $\underline{1}$ into $[A_0]_{L^2(\mu)}$. Then

$$\int|\underline{1}-F|^2 d\mu = \inf_{f\varepsilon A_0} \int|\underline{1} - f|^2 d\mu.$$

If $f \varepsilon A_0$, $(\underline{1}-F) \perp f$ in $L^2(\mu)$. Choose a sequence $\{f_n\} \subset A_0$ which converges to F in $L^2(\mu)$. If $f \varepsilon A_0$, then $(\underline{1}-f_n)f \varepsilon A_0$, since A_0 is an ideal in A. Since f is bounded $\{(\underline{1}-f_n)f\}$ converges to $(\underline{1}-F)f$ in $L^2(\mu)$, and so $(\underline{1}-F)f \varepsilon [A_0]_{L^2(\mu)}$. Hence $(\underline{1}-F) \perp (\underline{1}-F)f$. That is,

$$\int f(\underline{1}-F)(\underline{1}-\overline{F})d\mu = \int f|\underline{1}-F|^2 d\mu = 0, \ \forall \ f \varepsilon A_0 \qquad \text{(i)}$$

Let $K = \int|\underline{1}-F|^2 d\mu$. $K = 0$ if and only if $\underline{1} \varepsilon [A_0]_{L^2(\mu)}$. If $K > 0$, the measure $\mu_1 = K^{-1}|\underline{1}-F|^2\mu$ satisfies

$$\int f d\mu_1 = \int f dm, \ \forall \ f \varepsilon A.$$

In fact, this statement is true by (i) for $f \varepsilon A_0$; it is seen to be true for $f \varepsilon A$ by noting that any $f \varepsilon A$ can be written in the form $f = g + \underline{c}$, where $g \varepsilon A_0$ and $\underline{c}$ is a constant function. Hence, by our assumption that $M_\phi = \{m\}, \forall \ \phi \varepsilon M$, we have $\mu_1 = m$. Thus, for $K \geq 0$,

$$|\underline{1}-F|^2\mu = Km \qquad \text{(ii)}$$

Since $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m, we may rewrite (ii) as

$$|\underline{1}-F|^2\mu_s = Km - |\underline{1}-F|^2\mu_a.$$

Since the measure on the right hand side is absolutely continuous with respect to m, it follows that $|\underline{1}-F|^2\mu_s = 0$; and so $F = \underline{1}$ a.e. (μ_s). Thus,

$$f \varepsilon A_0, \quad \int(\underline{1}-\bar{F})fd\mu_a = \int(\underline{1}-\bar{F})fd\mu = 0 \qquad \text{(iii)}$$

where the last equality follows from the fact that F is the orthogonal projection of $\underline{1}$ into $[A_0]_{L^2(\mu)}$.

Now, for $\{f_n\} \subset A_0$ convergent to F in $L^2(\mu)$ as before, we have

$$\int|F-f_n|^2d\mu_a \leq \int|F-f_n|^2d\mu$$

and so $F \varepsilon [A_0]_{L^2(\mu_a)}$.

This fact, combined with (iii), gives us that F is the orthogonal projectionoof $\underline{1}$ into $[A_0]_{L^2(\mu_a)}$, and so

$$\inf_{f\varepsilon A_0} \int|\underline{1}-f|^2d\mu = \int|\underline{1}-F|^2d\mu = \int|\underline{1}-F|^2d\mu_a = \inf_{f\varepsilon A_0} \int|\underline{1}-f|^2d\mu_a.$$

From Theorem 2.3.1 and 2.4.1 we have at once

Theorem 2.4.2 (Kolmogorov-Krein).

Let A be a weak-* Dirichlet algebra on a compact Hausdorff space X, such that $M_\phi = \{m\}$, $\forall\ \phi \in \mathcal{M}$, where m is a probability measure on X, multiplicative on A.

Let μ be a positive measure on X and let $\mu = wm + \mu_s$ where $w \in L^1(m)$, be the Lebesgue decomposition of μ with respect to m. Then

$$\inf_{f \in A_0} \int |\underline{1} - f|^2 d\mu = \exp \int \log w\, dm,$$

where, as before, if $\log w \notin L^1(m)$, $\int \log w\, dm = -\infty$.

§2.5 Examples to illustrate the necessity of the hypothesis in Theorem 2.4.2 that $M_\phi = \{m\}$, $\forall\ \phi \in \mathcal{M}$.

We now give an example (Srinivasan and Wang [1]) to show the necessity of the condition in Theorem 2.4.2, that $M_\phi = \{m\}, \forall\ \phi \in \mathcal{M}$.

Let X be the unit circle and m the Haar measure on X. Let A be the algebra of those $f \in C(X)$ which have an analytic extension $\tilde{f}$ to the interior of the unit disc such that $\tilde{f}(0) = f(1)$. Then A is a uniformly closed separating subalgebra of $C(X)$, with X as the Shilov boundary of A and the support of m. We shall show that A is weak-* Dirichlet. Clearly $\underline{1} \in A$. Note that A may be considered as the set of functions f of the form

$f = j(j-\underline{1})g + \underline{c}$, where $\underline{c}$ is a constant function, and g is analytic in the interior of the unit disc, continuous on the closed unit disc, except possibly at 1, and

$g = 0\left(\frac{\underline{1}}{(j-\underline{1})}\right)$ near 1. In particular, the functions $j - j^2$, $j^2 - j^3, \ldots \in A$, and so $j^k - j^n \in A$, $\forall\ k \geq 0,\ n \geq 0$. By the Riemann-Lebesgue Lemma, $\{j^n\}$ converges to $\underline{0}$ in the $\sigma(L^1, L^\infty)$ topology. Hence, for fixed $k \geq 0$, $\{j^k - j^n\}_{n=0}^{\infty}$ converges to j^k in the $\sigma(L^1, L^\infty)$ topology and so $j^k \in [A]_*$, $k \geq 0$. Hence, by the complex version of the Stone-Weieistrass Theorem

$$C(X) \subset [A+\bar{A}]_*.$$

But $[C(X)]_* = L^\infty(m)$ (Edwards [1] Ex.8.5)

and so $[A+\bar{A}]_* = L^\infty(m)$ and A is weak-* Dirichlet

Now let μ be the unit point mass at 1. wWe have

$$\int f d\mu = f(1) = \tilde{f}(0) = \int f dm$$

so A does not have the property that $M_\phi = \{m\}$, $\forall\ \phi \in \mathcal{M}$.

In this case,

$$\inf_{f \in A_0} \int |\underline{1} - f|^2 d\mu = 1.$$

However, μ is completely singular with respect to m, so, in the notation of Theorem 2.4.2, $w = \underline{0}$, and so

$$\exp\int \log w\,dm = 0$$

and the conclusion of Theorem 2.4.2 fails to hold.

CHAPTER 3.

GENERALISATIONS OF THE F. AND M. RIESZ THEOREM

§3.1 Introduction.

One very important theorem in the theory of analytic functions in the unit disc, the F. and M. Riesz Theorem, [which provides a characterisation for the functions in the Hardy class H^1 (Hoffman [1] pp.50,51)], is not true for weak-* Dirichlet algebras. We shall show that it is not true even for Dirichlet algebras. However, for some subalgebras of $C(X)$, the set of continuous functions on a compact Hausdorff space X, we can prove a generalised F. and M. Riesz Theorem (Theorem 3.3.1) which implies the classical result.

For those weak-* Dirichlet algebras which are also logmodular algebras, we have a generalised F. and M. Riesz Theorem (Theorem 3.2.1) which was proved by Hoffman (Hoffman [2]). From this point on we write $A^{\perp}$ for the set of measures μ on X such that $\int f d\mu = 0, \forall f \in A$; as before, m is a probability measure on X, multiplicative on A, and A_0 is the set of $f \in A$ such that $\int f dm = 0$; and so $A_0^{\perp}$ is the set of measures μ on X such that $\int f d\mu = 0, \forall f \in A_0$.

§3.2 A generalised F. and M. Riesz Theorem for logmodular algebras.

Theorem 3.2.1. Let A be a logmodular algebra on a compact Hausdorff space X. Let μ be a complex measure on X such that $\mu \in A_0$. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m, where m is as above. Then $\mu_a, \mu_s \in A_0^{\perp}$, and $\int d\mu_s = 0$. Further, $\mu_a = hm$, where $h \in H^1(m)$.

Proof. Let $\mu_a = hm$ and let ρ be the positive measure on X defined by $\rho = (\underline{1}+|h|)m + |\mu_s|$, where $|\mu_s|$ denotes the total variation of μ_s. If $f \in A_0$, then

$$\int |\underline{1}-f|^2 d\rho \geq \int |\underline{1}-f|^2 dm \geq 1 . \qquad (\dot{\text{i}})$$

Let G be the orthogonal projection of $\underline{1}$ into $[A_0]_{L^2(\rho)}$. Now, by ($\dot{\text{i}}$),

$$\int |\underline{1}-G|^2 d\rho = \inf_{f \in A_0} \int |\underline{1}-f|^2 d\rho \geq 1,$$

and so $\underline{1} \notin [A_0]_{L^2(\rho)}$.

Choose a sequence $\{g_n\} \subset A_0$ which converges to G in $L^2(\rho)$. Let $f \in A_0$. Since A_0 is an ideal in A, $(\underline{1}-g_n)f \in A_0$; and, since f is bounded, $\{(\underline{1}-g_n)f\}$ converges to $(\underline{1}-G)f$ in the $L^2(\rho)$-norm, and so $(\underline{1}-G)f \in [A_0]_{L^2(\rho)}$. Thus $(\underline{1}-G)$ is orthogonal to $(\underline{1}-G)f$ in $L^2(\rho)$; that is,

$$\int f|\underline{1}-G|^2 d\rho = 0, \ \forall \ f \in A_0 \qquad \text{(ii)}$$

Let $K = \int |\underline{1}-G|^2 d\rho \geq 1$. Let $\rho_1 = K^{-1}|\underline{1}-G|^2\rho$. Then we have

$$\int f d\rho_1 = \int f dm, \ \forall \ f \in A.$$

In fact, this statement is true by (ii) for $f \in A_0$; it is seen to be true for $f \in A$ by noting than any $f \in A$ can be written in the form $f = g + \underline{c}$ where $g \in A_0$ and $\underline{c}$ is a constant function.

Since A is a logmodular algebra, Theorem 0.2 gives us that $\rho_1 = m$. Thus $|\underline{1}-G|^2\rho = Km$, and so $(\underline{1}-G) = \underline{0}$ a.e. (ρ_s); and, since $\rho_a = (\underline{1}+|h|)m$, we have

$$|\underline{1}-G|^2(\underline{1}+|h|)m = Km \ ; \qquad \text{(iii)}$$

that is,

$$|\underline{1}-G|^{-2}Km = (\underline{1}+|h|)m, \text{ and so}$$

$$(\underline{1}-G)^{-2} \in L^1(m) \text{ and hence } (\underline{1}-G)^{-1} \in L^2(m).$$

We now wish to show that $(\underline{1}G)^{-1}$ is in $H^2(m)$.

Let $f \in A_0$. Then

$$\int f(\underline{1}-G)^{-1}dm = \int f(\underline{1}-G)^{-1}d\rho_1$$

$$= \frac{1}{K}\int f(\underline{1}-G)^{-1}|\underline{1}-G|^2 d\rho$$

$$= \frac{1}{K}\int f(\underline{1}-\overline{G})d\rho$$

$$= 0 \quad \text{since} \quad (\underline{1}-G) \perp A_0 \quad \text{in} \quad L^2(\rho).$$

Thus we have

$$\int f(\underline{1}-G)^{-1}dm = 0, \quad \forall \quad f \in A_0,$$

and so, by Remark 1.3.1, $(1-G)^{-1} \in H^2(m)$.

From (iii) above we have

$$|\underline{1}-G|^2(\underline{1}+|h|) = K \text{ a.e.}(m)$$

which, with $(\underline{1}-G)^{-1} \in L^2(m)$, implies that $(\underline{1}-G)^{-1}(\underline{1}+|h|)$, and hence $(\underline{1}-G)^{-1}h$ also, is in $L^2(m)$.

We now wish to show that $\int(\underline{1}-G)fd\mu = 0 \ \forall \ f \in A_0$. Take $\{g_n\} \subset A_0$ convergent to G in $L^2(\rho)$ as before. Then $(\underline{1}-g_n)f \in A_0$. So, since $\mu << \rho$ and $d\mu/d\rho$ is bounded, while $\mu \in A_0^{\perp}$, we have

$$\int f(\underline{1}-G)d\mu = \lim_{n\to\infty} \int f(\underline{1}-g_n)d\mu = 0.$$

Also, since $(\underline{1}-G) = \underline{0}$ a.e.(ρ_s), we have

$(\underline{1}-G) = \underline{0}$ a.e.(μ_s) and so $(\underline{1}-G)\mu = (\underline{1}-G)hm$.

Thus,

$$0 = \int f(\underline{1}-G)d\mu = \int f(\underline{1}-G)hdm, \ \forall\ f \in A_0.$$

Since $(\underline{1}-G)^{-1} \in H^2(m)\ \exists\ \{f_n\} \subset A$ which converges to $(\underline{1}-G)^{-1}$ in $L^2(m)$. Since m is multiplicative on A

$$\int f_n f(\underline{1}-G)hdm = 0, \text{ for each } n. \qquad \text{(iv)}$$

Since also $(\underline{1}-G)h \in L^2(m)$, we may pass to the limit in (iv) to obtain

$$\int fhdm = 0, \ \forall\ f \in A.$$

By Corollary 1.3.4, $h \in H^1(m)$. Also, we have then that $\mu_a \in A_0^{\perp}$, and this combined with $\mu \in A^{\perp} \supset A_0^{\perp}$, gives us $\mu_s \in A_0^{\perp}$.

Since $\underline{1} \in [A_0]_{L^2(|\mu_s|)}$ (Hoffman [2], Theorem 4.3), we can choose $\{f_n\} \subset A_0$ which converges to $\underline{1}$ in $L^2(|\mu_s|)$. Then

$$\int d\mu_s = \lim_{n\to\infty} \int f_n d\mu_s = 0$$

since $\mu_s \in A_0^{\perp}$.

<u>Remark 3.2.1</u>. To express 3.2.1 in the same form as later generalisations of the F. and M. Riesz Theorem, we note that $\int d\mu_s = 0$,

together with the fact that each $f \in A$ can be written in the form $f = g + \underline{c}$, where $g \in A_0$ and $\underline{c}$ is a constant function, shows that $\mu_s \in A_0^{\perp} \Rightarrow \mu_s \in A^{\perp}$. Thus, if we assumed originally that $\mu \in A^{\perp}$, then we could conclude $\mu_s \in A^{\perp}$, and therefore that $\mu_a \in A^{\perp}$. Thus we may rewrite Theorem 3.2.1 in the form

Theorem 3.2.2. Let A be a logmodular algebra on a compact Hausdorff space X and μ a complex measure on X such that $\mu \in A^{\perp}$. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m, where m is as described in §3.1. Then $\mu_a, \mu_s \in A^{\perp}$.

We shall now show that Theorem 3.1.1 implies the classical F. and M. Riesz Theorem:

Theorem 3.2.3 (F. and M. Riesz). Let μ be a measure on the unit circle such that $\int e_n d\mu = 0$, $n = 1,2,3,\ldots$ where $e_n : e^{i\theta} \mapsto e^{in\theta}$. Then μ is absolutely continuous with respect to the Lebesgue measure on the unit circle.

Proof. We have assumed that $\mu \in A_0^{\perp}$, where A is the standard algebra on the unit circle. If $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ, then by Theorem 3.2.1, $\mu_a, \mu_s \in A_0^{\perp}$. Since $\underline{1} \in [A_0]_{L^2(|\mu_s|)}$, (Hoffman [2], Thm. 4.3) $\{f_n\} \subset A_0$ which converges to $\underline{1}$ in $L^2(|\mu_s|)$. Since $\mu_s \in A_0^{\perp}$, we have $\int d\mu_s = \lim_{n\to\infty} \int f_n d\mu_s = 0$. Thus μ_s is orthogonal to $\underline{1}$. The singular measure $e_{-1}\mu_s \in A_0^{\perp}$ is similarly orthogonal to $\underline{1}$. Repreating this process we conculde that

$$\int e_n d\mu_s = 0, \quad n = 0,\pm 1,\pm 2,\ldots$$

and so μ_s must be the zero measure; that is $\mu = \mu_a$.

For even a Dirichlet algebra, however, Theorem 3.2.3 (F. and M. Riesz) does not generalise directly, as one can have non-zero measures orthogonal to A_0 which are mutually singular with respect to m (Hoffman [1] p59. Ex 11).

§3.3 Another generalised F. and M. Riesz Theorem.

For a general sup-norm algebra A we have a generalisation of the F. and M. Riesz Theorem which is due to Ahern (Ahern [1]).

<u>Theorem 3.3.1</u>. (Ahern). Let A be a sup-norm algebra on a compact Hausdorff space X. Let M_ϕ be, as before, the set of representing measures for $\phi \in M$, the maximal ideal space of A, (each $m \in M_\phi$ is a probability measure on X, multiplicative on A). For every complex measure μ on X such that $\mu \in A^{\perp}$, we have $\mu_a, \mu_s \in A^{\perp}$ (where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m) if and only if $\rho \ll m$, $\forall\ \rho \in M_\phi$.

We prove first the following four lemmas.

Lemma 3.3.1. Let $\{v_n\}$ be a sequence of positive measures on X having m as a weak-* cluster point. Suppose $F \subset X$ is compact, and that $v_n(F) \geq \varepsilon_0 > 0$, $\forall$ n. Then $m(F) \geq \varepsilon_0$.

Proof. Since m is regular $\exists$ a decreasing sequence of open sets $\{\mathcal{O}_n\}$ such that $\mathcal{O}_n \supset F$ and $\lim_{n\to\infty} m(\mathcal{O}_n \setminus F) = 0$. By Urysohn's Lemma $\exists \{u_n\} \subset C_R(X)$ such that $u_n = \underline{1}$ on F, $u_n = \underline{0}$ on $X \setminus \mathcal{O}_n$, and $\underline{0} \leq u_n \leq \underline{1}$. From the construction $\{u_n\}$ converges a.e.(m) to χ_F, the characteristic function of F. Now,

$$m(F) = \int(\chi_F - u_k)dm + \int u_k dv_n + \int u_k(dm - dv_n)$$

and $\int(\chi_F - u_k)dm$ can be made small by choosing k large, and once k is fixed $\int u_k(dm - dv_n)$ can be made small by proper choice of n. Thus,

$$m(F) \geq \int u_k dv_n \geq v_n(F) \geq \varepsilon_0,$$

where k,n are as indicated.

Lemma 3.3.2. Let $u \in C_R(X)$. Write $\lambda(u)$ for $\int u d\lambda$. Then, for $\phi \in M$,

$$\sup\{\text{Re}\phi(f) : f \in A, \text{Re} f \leq u\} = \inf\{\lambda(u) : \lambda \in M_\phi\}.$$

Proof. Since $\lambda(\mathrm{Re}f) = \mathrm{Re}\lambda(f) = \mathrm{Re}\phi(f)$, $\forall$ $\lambda \in M_\phi$, $\forall$ $f \in A$, we have

$$\sup\{\mathrm{Re}\phi(f) : \mathrm{Re}f \leq u, f \in A\} \leq \inf\{\lambda(u) : \lambda \in M_\phi\} \qquad (1)$$

By the same equality, $F : \mathrm{Re}f \mapsto \mathrm{Re}\phi(f)$ is a well-defined non-negative linear functional on the subspace ReA of $C_R(X)$. We shall show that, for each $u \in C_R(X)$ $\exists$ a non-negative linear functional on the subspace of $C_R(X)$ spanned by $\mathrm{Re}A \cup \{u\}$ (call it E) such that

$$F_e(\mathrm{Re}f) = F(\mathrm{Re}f), \ \forall \ f \in A$$

and

$$F_e(u) = \sup\{\mathrm{Re}\phi(f) : \mathrm{Re}f \leq n, f \in A\}.$$

If $u \in \mathrm{Re}A$, our assertion is trivial:

$$\sup\{\mathrm{Re}\phi(f) : \mathrm{Re}f \leq u, f \in A\} = F(u).$$

If $u \notin \mathrm{Re}A$, each $v \in E$ can be written uniquely as $\mathrm{Re}f + \alpha u, f \in A$ and α a real number. Define

$$F_e(v) = F_e(\mathrm{Re}f + \alpha u) = F(\mathrm{Re}f) + \alpha \sup\{F(\mathrm{Re}f) : f \in A, \mathrm{Re}f \leq u\}$$

$$= \mathrm{Re}\phi(f) + \alpha \sup\{\mathrm{Re}\phi(f) : f \in A, \mathrm{Re}f \leq u\}$$

F_e is clearly linear with its restriction to ReA equal to F.

We now show that F_e is non-negative on E.

Let $\mathrm{Re} f + \alpha u \geq 0$.

<u>Case 1</u>. $\alpha = 0$. Trivial.

<u>Case 2</u>. $\alpha > 0$. Then $\frac{-\mathrm{Re} f}{\alpha} \leq u$, and

$$\sup\{\mathrm{Re}\phi(g) : g \in A, \mathrm{Re} g \leq u\} \geq F(-\frac{\mathrm{Re} f}{\alpha}) = -\frac{1}{\alpha}\mathrm{Re}\phi(f);$$

which implies that $F_e(\mathrm{Re} f + \alpha u) \geq 0$.

<u>Case 3</u>. $\alpha < 0$. Then $-\frac{\mathrm{Re} f}{\alpha} \geq u$. Hence, for each $g \in A$ with $\mathrm{Re} g \leq u$, we have $\mathrm{Re} g \leq -\mathrm{Re} f/\alpha$, and

$$\mathrm{Re}\phi(g) \leq -\frac{1}{\alpha}\mathrm{Re}\phi(f).$$

Hence,

$$\sup\{\mathrm{Re}\phi(g) : g \in A, \mathrm{Re} g \leq u\} \leq -\frac{1}{\alpha}\mathrm{Re}\phi(f),$$

and so

$$\mathrm{Re}\phi(f) + \alpha\sup\{\mathrm{Re}\phi(g) : g \in A, \mathrm{Re} g \leq u\} \geq \mathrm{Re}\phi(f) - \mathrm{Re}\phi(f) = 0.$$

Thus F_e is a non-negative extension of F, such that, since $\underline{1} \in A$, $F_e(\underline{1}) = 1$. It follows that

$$F_e(v) \le F_e(\sup v) = \sup v, \ \forall\ v \in E.$$

By the Hahn-Banach Theorem, there exists an extension of F_e to a linear functional F_{ee} on $C_R(X)$ such that

$$F_{ee}(w) \le \sup w, \ \forall\ w \in C_R(X).$$

Thus we have

$$-F_{ee}(w) = F_{ee}(-w) \le \sup(-w) = -\inf w.$$

$$\therefore \quad F_{ee}(w) \ge \inf w$$

and so F_{ee} is non-negative.

Since also $F_{ee}(\underline{1}) = F_e(\underline{1}) = 1$, F_{ee} is given by a probability measure λ such that

$$\lambda(\mathrm{Re} f) = F_{ee}(\mathrm{Re} f) = F(\mathrm{Re} f) = \mathrm{Re}\phi(f), \ \forall\ f \in A.$$

Then also

$$\lambda(\mathrm{Im} f) = -\lambda(\mathrm{Re}\, if) = -\mathrm{Re}\phi(if)$$

$$= -\mathrm{Re}\ i\ \phi(f) = \mathrm{Im}\ \phi(f), \ \forall\ f \in A.$$

$$\therefore \quad \lambda(f) = \lambda(\mathrm{Re} f) + i\lambda(\mathrm{Im} f)$$

$$= \mathrm{Re}\phi(f) + i\ \mathrm{Im}\ \phi(f)$$

$$= \phi(f), \ \forall\ f \in A.$$

and so $\lambda \in M_\phi$.

This, together with (i), gives the desired result.

The following lemma is an extension of a result of Fiorelli. (Fiorelli [1], Theorem 1).

Lemma 3.3.3. Let $F \subset X$ be a compact G_δ such that $\lambda(F) = 0$, $\forall \lambda \in M_\phi$. (We say F is "ϕ-null"). Then, for $\{n\}$ an increasing sequence of positive integers, $\exists \{f_n\} \subset$ ball A, the closed unit ball in A, such that

(1) $\phi(f_n) \geq e^{-2/n}$

(2) $|f_n| \leq \exp o(-\underline{n})$ on F.

<u>Proof</u>. Since F is a compact G_δ $\exists$ a sequence of open sets $\{\mathcal{O}_n\}$ such that $\overline{\mathcal{O}_{n+1}} \subset \mathcal{O}_n$ and $\bigcap_n \mathcal{O}_n = F$, where $\overline{\mathcal{O}_{n+1}}$ denotes the closure in X of $\mathcal{O}_{n+1}$. Let $\varepsilon > 0$ be given. Then there exists an integer N such that $\forall\ n \geq N$, $\forall\ \rho \in M_\phi$, $\rho(\mathcal{O}_n) < \varepsilon$. (If this were not so, there would exist $\varepsilon_0 > 0$ and sequences $\{\rho_k\}, \{\mathcal{O}_{n_k}\}$ with $\rho_k \in M_\phi$ and $\rho_k(\mathcal{O}_{n_k}) \geq \varepsilon_0$. Let $U_k = \mathcal{O}_{n_k}$; then $\rho_k(U_k) \geq \varepsilon_0 > 0$, and $\overline{U_{k+1}} \subset U_k$. Let ρ be a weak-* cluster point of $\{\rho_k\}$. Then $\rho \in M_\phi$ and so $\rho(F) = 0$. Fix k; then $\rho(U_k) \geq \rho(\overline{U_{k+1}})$. Now $\rho(U_k) \geq \rho_n(\overline{U_{k+1}}) \geq \rho_n(U_{k+1}) \geq \rho_n(U_n)$, $\forall\ n \geq k + 1$, and so, by Lemma 3.3.1, $\rho(U_k) \geq \varepsilon_0 > 0$, $\forall\ k$.

But this contradicts the fact that $\rho(F) = 0$.) Hence, by passage to a suitable subsequence, we may assume that $\rho(\mathcal{O}_n) < \frac{1}{n^2}$, $\forall\ \rho \in M_\phi$. Now, for each n, $\exists\ u_n \in C(X)$ such that $u_n = -\underline{n}$ on F, $u_n = \underline{0}$ on $X \setminus \mathcal{O}_n$ and $-\underline{n} \leq u_n \leq \underline{0}$ elsewhere. From Lemma 3.3.2 and the weak-* compactness of M_ϕ, there exists $\rho_n \in M_\phi$ such that

$$\sup\{\mathrm{Re}\phi(f) : \mathrm{Re}f \leq u_n, f \in A\} = \int u_n d\rho_n.$$

Hence, for each n, $\exists\ g_n \in A$ such that $\mathrm{Re}g_n \leq u_n$, and

$$\int \mathrm{Re}g_n dm \geq \int u_n d\rho_n - \frac{1}{n} \geq -n\rho_n(\mathcal{O}_n) - \frac{1}{n} \geq -2/n.$$

We may also assume that $\int \mathrm{Im}g_n dm = 0$. Now define $f_n = \exp \circ g_n$. $f_n \in A$ since $\underline{1} \in A$. Also,

$$|f_n| = \exp \circ \mathrm{Re}g_n \leq \exp \circ u_n \leq \underline{1};$$

and by multiplicativity of m,

$$f_n dm = \exp[\int g_n dm] = \exp[\int \mathrm{Re}g_n dm] \geq e^{-2/n};$$

and $|f_n| = \exp \circ \mathrm{Re}g_n \leq \exp \circ (-\underline{n})$ on F

<u>Lemma 3.3.4</u>. Suppose there exists $m \in M_\phi$ such that $\rho \ll m$, $\forall\ \rho \in M_\phi$; and suppose $F \subset X$ is compact and $m(F) = 0$. Then $\exists \{f_n\} \subset$ ball A satisfying (1) and (2) of Lemma 3.3.3.

Proof. $\exists$ a sequence of open set $\{\mathcal{O}_n\}$ such that $F \subset \mathcal{O}_{n+1} \subset \mathcal{O}_n$ and $\lim_{n\to\infty} m(\mathcal{O}_n) = 0$. For each n, $\exists$ a set F_n, which is a compact G_δ, such that $F \subset F_n \subset \mathcal{O}_n$. Let $S = \bigcap_n F_n$. Then $F \subset S$, S is a compact G, and $m(S) = 0$. Since $\rho << m$, $\forall\ \rho \in M_\phi$, we have $\rho(S) = 0$, $\forall\ \rho \in M_\phi$. We then apply Lemma 3.3.3 to the set S to obtain the desired result.

We can now prove Theorem 3.3.1.

Proof of Theorem 3.3.1.

Suppose first that there exists $m \in M_\phi$ such that $\rho << m$, $\forall\ \rho \in M_\phi$. Let S be a Barie set which carries μ_s (that is $\mu_s(T) = 0$ for every Baire subset T of $X \setminus S$) such that $m(S) = 0$. Then $\exists$ an increasing sequences $\{F_n\} \subset S$ of compact sets such that $\lim_{n\to\infty} |\mu_s|(F_n) = |\mu_s|(S)$, when $|\mu_s|$ denotes the total variation of μ_s. For each F_n we have, by Lemma 3.3.4, a sequence $\{f_{n,k}\}_{k=1}^{\infty} \subset$ ball A such that

(1) $\int f_{n,k} dm \geq e^{-2/k}$; and

(2) $|f_{n,k}| \leq e^{-k}$ on F_n.

Define $h_n = f_{n,n}$. Then we have $h_n \in$ ball A and

(1') $\int h_n dm = \int f_{n,n} dm \geq e^{-2/n}$;

and (2') $|h_n| = |f_{n,n}| \leq e^{-n}$ on F_n.

From (1') we see that $\lim_{n\to\infty} h_n = \underline{1}$ in $L^1(m)$ and so we have a subsequence $\{h_{n_k}\}$ which converges to $\underline{1}$ a.e. (m).

From (2') $\{h_{n_k}\}$ converges to $\underline{0}$ a.e. $(|\mu_s|)$.

Hence $\{g_k\} = \{h_{n_k}\}$ converges a.e. $(|\mu|)$ to $\chi_{X\setminus S}$.

If $f \in A$, then for each k, $g_k f \in A$ and we have $0 = \int g_k f d\mu \to \int_{X\setminus S} f d\mu = \int f d\mu_a$. That is, $\mu_a \in A^\perp$; and so since $\mu \in A^\perp$, we have $\mu_s \in A^\perp$.

To prove the "only if" part of Theorem 3.3.1, we assume $\exists\, v \in M_\phi$ which is not absolutely continuous with respect to m, and consider $\mu = v - m$. Since $v, m \in M_\phi$,

$$\int f dv = \int f dm, \quad \forall\, f \in A,$$

and so $\mu \in A^\perp$.

Now $\mu_a = v_a - m$, and $\mu_a \in A^\perp$ if and only if

$$\int f dv_a = \int f dm, \quad \forall\, f \in A.$$

But $\int f dv_a = \int f dm$, $\forall\, f \in A$ implies, since $\underline{1} \in A$, that $\int dv_a = 1 = \int dv$. Thus $v(X) = v_a(X)$, and hence $v_s = 0$. This contradicts our assumption, so $\mu_a \notin A^\perp$ and our proof is completed.

<u>Remark 3.3.1</u>. Theorem 3.2.1, where $M_\phi = \{m\}$, is a special case of Theorem 3.3.1.

§3.4 An abstract F. and M. Riesz Theorem.

Ahern's result is a particular case of a general result whose only special hypothesis is that A is a subalgebra of $C(X)$ which contains the constant functions on X. The extension arises from the idea of forming the Lebesgue decomposition of μ relative to the set M_ϕ, in the sense of the following definition (Glicksberg [1]).

Definition 3.4.1. The (complex) measure μ is singular with respect to a set M of probability measures ("μ is M-singular") if μ is carried by some Barie set F (that is $\mu(S) = 0$ for every Baire subset S of $X \setminus F$) of measure zero for all $m \in M$; such an F is called an M-null set. If μ vanishes on all M-null sets $F(\mu_F = 0)$, then μ is M-absolutely continuous ($\mu << M$).

When $M = M_\phi$, we frequently write "ϕ-singular" for "M_ϕ-singular".

Unlike our previous theory, where our choice of Baire measures rather than regular Borel measures was purely arbitrary, we consider Baire measures here to ensure the truth of the Choquet-Bishop-de Leeuw Theorem (Phelps [1], p24) which is necessary for the development of this theory. The full development will not be given here but may be found in Glicksberg [1] and Garnett and Glicksberg [1].

We note also that we always have a (unique) Lebesgue decomposition of any μ relative to M:

$$\mu = \mu_F + \mu_{F'}$$

where μ_F is M-singular and $\mu_{F'} << M$.

To do this choose an M-null set F which maximises $||\mu_F||$, so that if E (and so $E \cup F$) is M-null then $||\mu_{E \cup F}|| = ||\mu_F|| + ||(\mu_{F'})_E|| \leq ||\mu_F||$, and so $(\mu_{F'})_E = 0$.

We now prove the abstract F and M Riesz Theorem due to Glicksberg (Glicksberg [1]). The proof of this theorem follows that of Theorem 3.3.1 and both are closer in form to the original proof of F. and M. Riesz than the proof of Theorem 3.2.1.

<u>Theorem 3.4.1.</u> If $\mu \in A^{\perp}$ and $\phi \in M$, the maximal ideal space of A, and if $\mu = \mu_F + \mu_{F'}$ is the Lebesgue decomposition of μ relative to M_ϕ, then $\mu_F, \mu_{F'} \in A^{\perp}$.

We first prove an analogue of Lemma 3.3.3:

<u>Lemma 3.4.1.</u> If $F = \bigcup_n K_n$ is a ϕ-null union of compact Baire sets K_n, then $\exists$ sequence $\{f_n\} \subset$ ball A which converges to $\underline{0}$ on F and to $\underline{1}$ a.e. (λ), $\forall\ \lambda \in M_\phi$.

<u>Proof.</u> For n fixed we have a monotonic increasing sequence $\{u_k\} \subset C_R(X)$, which, since every compact Baire set is a G_δ,

(Berberian [1], p180 Ex.6.), converges pointwise to $-n\chi_{k_n}$. Thus, by monotone convergence $\lambda(u_k) \uparrow 0$, $\forall\ \lambda \in M_\phi$. Since M_ϕ is weak-* compact and $\lambda \mapsto \lambda(u_k)$ is weak* continuous, Dini's Theorem asserts that the convergence is uniform on M_ϕ. Thus,

$$\lambda(u_k) > -\frac{1}{2}n^{-4},\ \forall\ \lambda \in M_\phi, \text{ for some } k.$$

By Lemma 3.3.2 we have $g_n \in A$ such that $\mathrm{Re}g_n \leq -n\chi_{k_n}$ and so

$$\mathrm{Re}\phi(g_n) > -\frac{1}{2}n^{-4} - \frac{1}{2}n^{-4} = -n^{-4} .$$

Put $f_n = (\exp \circ g_n)\mathrm{sgn}(e^{\phi(g_n)})$. Since $\underline{1} \in A, f_n \in A$. Since $\mathrm{Re}g_n \leq 0$,

$$|f_n| = \exp \circ \mathrm{Re}g_n \leq \underline{1}.$$

Also, $|f| \leq \exp \circ -\underline{n}$ on K_n, and so $\{f_n\} \subset$ ball A and converges to $\underline{0}$ on F. Moreover, $\{x \in X : \overline{\lim}|f_n(x) - 1| > 0\}$ is M_ϕ-null, so that $\{f_n\}$ converges to $\underline{1}$ a.e. $(|\mu_{F'}|)$, since $\mu_{F'} << M_\phi$, and $\{f_n\}$ converges to $\chi_{F'}$ a.e. (μ). If $f \in A$, then, for each n, $f_n f \in A$ and, since $\mu \in A^\perp$,

$$0 = \int f_n f d\mu \to \int_{F'} f d\mu = \int f d\mu_{F'}.$$

That is, $\mu_{F'} \in A^\perp$ and hence also $\mu_F \in A^\perp$.

It is clear from the preceding proof that the only special property of A we require is that $\underline{1} \in A$ so that we may exponentiate. Thus, we may state Theorem 3.4.1 in a more general form:

<u>Corollary 3.4.1</u>. If B is any subalgebra of A such that $\underline{1} \in B$, and $\mu = \mu_F + \mu_{F'}$ is the Lebesgue decomposition of μ relative to $M_\psi(B)$, where $\psi \in M_B$, then $\mu \in A^\perp$ implies that $\mu_F, \mu_{F'} \in A^\perp$.

We note that if B is such a subalgebra of A, then $M = M_\phi = M_\phi(A) \subset M_\phi(B)$; so that, while $M_\phi(B)$-null sets are also M_ϕ-null, the converse is false. Thus $\mu << M_\phi$ is also $M_\phi(B)$-absolutely continuous, but an M_ϕ-singular measure μ may have a non-trivial decomposition relative to $M_\psi(B)$.

CHAPTER 4.

EXTENSION OF LINEAR FUNCTIONALS

§4.1 Unique norm-preserving extension of a weak-* continuous linear functional on a logmodular algebra A to a weak-* continuous linear functional on C(X).

We shall now make use of the abstract F. and M. Riesz Theorem in its form for logmodular algebras (Theorem 3.2.2) to prove the following result.

Theorem 4.1.1. Let A be a logmodular algebra on a compact Hausdorff space X. Let $g \in L^1(m)$, where m is a probability measure on X, multiplicative on A. Let Φ be a linear functional on A defined by

$$\Phi(f) = \int fg\,dm, \quad \forall\ f \in A.$$

Then Φ has a unique norm-preserving extension to a linear functional on $C(X)$, and this extension is weak-* continuous, considering $C(X)$ with respect to the topology induced on it by $\sigma(L^\infty(m), L^1(m))$.

Proof. (i) The existence of at least one norm-preserving extension of Φ is guaranteed by the Hahn-Banach Theorem. Let

ψ be any such extension of Φ to a linear functional on $C(X)$. Then ψ may be expressed in the form

$$\psi(f) = \int f d\mu, \ \forall \ f \in C(X),$$

where μ is a (complex) measure on X, and the total variation of μ equals $||\psi||$. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m. Then $\mu = \mu_a + \mu_s = hm + \mu_s$, where $h \in L^1(m)$. Since ψ is an extension of Φ, we have $\psi(f) = \Phi(f)$, $\forall$ $f \in A$. Thus, we have $\int f d\mu = \int fg dm$, $\forall$ $f \in A$, and hence the measure $\mu - gm \in A^{\perp}$. By Theorem 3.2.2 $\mu_a - gm$ and μ_s are both $\in A^{\perp}$. Thus,

$$\int f(h-g) dm = 0, \ \forall \ f \in A,$$

and so, by Corollary 1.3.4, $h - g \in H^1(m)$. But $\underline{1} \in A$, so $\int (h-g) dm = 0$ and so $h - g \in H_0^1(m)$. Also,

$$\int fg dm = \int f d\mu = \int f d\mu_a = \int fh dm, \ \forall \ f \in A.$$

We now have

$$\sup_{\substack{f \in A \\ ||f|| \leq 1}} \left|\int fh dm\right| = \sup_{\substack{f \in A \\ ||f|| \leq 1}} \left|\int fg dm\right| = ||\Phi|| = ||\psi||$$

and so $||\psi|| \leq ||h||_1$. But

$$||\psi|| = ||\mu|| = ||h||_1 + ||\mu_s||$$

and so we conclude that $\mu_s = 0$, and so $||h||_1 = ||\psi||$ and ψ is weak-* continuous.

(ii) Uniqueness. Let ψ and ψ_1 be norm-preserving extensions of Φ. By the above $\psi(f) = \int fhdm$, $\forall$ $f \varepsilon C(X)$, and $\psi_1(f) = \int f(h+h_1)dm$, $\forall$ $f \varepsilon C(X)$, where $||h + h_1||_1 = ||h||_1$. Also from above, $h - g \varepsilon H^1_0(m)$ and $(h+h_1) - g \varepsilon H^1_0(m)$. Thus $h \varepsilon H^1_0(m)$, Now

$$||h||_1 = ||\psi|| = ||\Phi|| = \sup_{\substack{f\varepsilon A \\ ||f||\leq 1}} |\int fhdm|,$$

and, since the unit ball in L^∞ is weak-* compact, and the unit ball in A is contained in the unit ball of $H^\infty(m)$, $\exists$ $f_1 \varepsilon H^\infty(m) = [A]_*$ (by Theorem 2.1.1), such that $||f_1||_\infty = 1$ and

$$||h||_1 = \sup_{\substack{f\varepsilon A \\ ||f||\leq 1}} |\int fhdm| = \int f_1 hdm.$$

Since $|f_1| \leq 1$, $f_1 h = |h|$. However, $\int f_1(h+h_1)dm = \int f_1 hdm = ||h||_1 = ||h + h_1||_1$, where we have used the fact that $h_1 \varepsilon H^1_0(m)$ and $f_1 \varepsilon H^\infty(m)$ to give $\int f_1 h_1 dm = 0$. Thus $f_1(h+h_1) = |h+h_1|$. Since $f_1 h$ and $f_1(h+h_1)$ are non-negative, $f_1 h_1$ is real-valued. But $f_1 \varepsilon H^\infty(m)$ and $h_1 \varepsilon H^1(m)$.

So $f_1h_1 \in H^1(m)$ and, by Lemma 1.1.4, is a constant function. But $\int f_1h_1dm = 0$, so $f_1h_1 = \underline{0}$ a.e. (m). Now, from $f_1h = |h|$, $f_1h_1 = \underline{0}$ a.e. (m), $|h| = |h + h_1|$, we see, by considering separately the points where $f_1 \neq 0$ and those where $f_1 = 0$, that $h_1 = \underline{0}$ a.e. (m). This prove uniqueness.

§4.2 Unique norm-preserving extension of a weak-* continuous linear functional on a weak-* Dirichlet algebra A to a weak * continuous linear functional on L^∞.

We have already shown (Theorem 2.1.1) that, for A any weak -* Dirichlet algebra, $H^\infty(m) = L^\infty(m) \cap [A]_1$ is isomorphic to a logmodular algebra on M, the maximal ideal space of $L^\infty(m)$. We have seen also (Hoffman [1], p.169) that $L^\infty(m)$ is isomorphic to C(M). Thus, we have, directly from Theorem 4.1.1, a generalisation of a result of Gleason and Whitney for H^∞ defined relative to the standard algebra on the unit circle. (Gleason and Whiteny [1]). That is, we have

Theorem 4.2.1. Let A be any weak-* Dirichlet algebra on a compact Hausdorff space X. Let $g \in L^1(m)$ where m is a probability measure, multiplicative on A. Let Φ be the linear functional on $H^\infty(m)$ defined by

$$\Phi(f) = \int fgdm, \ \forall f \in H^\infty(m).$$

Then Φ has a <u>unique</u> norm-preserving extension to $L^{\infty}(m)$, and this extension is weak-* continuous.

We can actually say more than this. Suppose we have a linear functional Φ defined on a weak-* Dirichlet algebra A by

$$\Phi(f) = \int fg\,dm, \quad \forall\ f \in A,$$

where $g \in L^1(m)$, m as in Theorem 4.2. Since we have shown (Theorem 2.1) that $H^{\infty}(m) = [A]_*$ we see that Φ can be extended to a unique linear functional Φ_e on $H^{\infty}(m)$ such that

$$\Phi_e(f) = \int fg\,dm, \quad \forall\ f \in H^{\infty}(m).$$

Because of its form we can refer to Φ_e by Φ also and combine this result with that of Theorem 4.2.1 to get

<u>Theorem 4.2.2.</u> Let A be a weak-* Dirichlet algebra on a compact Hausdorff space X. Let $g \in L^1(m)$, where m is a probability measure, multiplicative on A. Let Φ be the linear functional on A defined by

$$\Phi(f) = \int fg\,dm, \quad \forall\ f \in A.$$

Then Φ has a <u>unique</u> norm-preserving extension to $L^{\infty}(m)$, and this extension is weak-* continuous.

§4.3 Discussion of the hypothesis in §§4.1, 4.2 that Φ be weak-* continuous.

If, instead of assuming that Φ be defined by integration against a function $g \in L^1(m)$, we simply assume that Φ is a bounded linear functional, then it is possible to form more than one norm-preserving extension of Φ. Examples of non-unique norm-preserving extensions of a bounded linear functional defined on H^∞ [where H^∞ is that subspace of L^∞, the bounded measurable complex-valued functions on the unit circle, which consists of those functions which are boundary value funcjtions (existing a.e. by Fatou's Lemma) of bounded analytic functions in the interior of the unit disc] are given by Gleason and Whitney (Gleason and Whitney [1]).

Example 4.3.1. Suppose ψ is a non-zero bounded linear functional on L^∞ which vanishes on H^∞ and takes real values on L^∞_R. By the Hahn decomposition theorem ψ may be represented by the difference $\psi = \psi^+ - \psi^-$ of two non-negative linear functionals; here ψ^+ is defined on non-negative f by

$$\psi^+(f) = \sup\{\psi(g) : 0 \leq g \leq f\},$$

and is extended over the rest of L^∞ by linearity.

Since ψ^+ and ψ^- are real and non-negative, $||\psi^+|| = \psi^+(\underline{1})$, and $||\psi^-|| = \psi^-(\underline{1})$. Also, since $\psi^+ = \psi^-$ on H^∞, and $\underline{1} \in H^\infty$, we have

$$||\psi^+|| = \psi^+(\underline{1}) = \psi^-(\underline{1}) = ||\psi^-||.$$

Thus ψ^+ and ψ^- are distinct norm-preserving extensions over L^∞ of the linear functional Φ defined on H^∞ by

$$\Phi(f) = \psi^+(f) = \psi^-(f), \ \forall \ f \in H^\infty.$$

We must now construct a ψ with the required properties.

Let ν be a proper arc of the unit circle, and let ν' be its complement. Let f_0 be defined as equal to $\underline{1}$ on ν and equal to $-\underline{1}$ on ν'. Then

$$\inf_{f \in H^\infty_R} ||f - f_0||_\infty \geq 1.$$

Suppose this were not so; then $f_1 \in H^\infty_R$, $\mathcal{E} > 0$, such that $||f_1 - f_0||_\infty = 1 - \mathcal{E}$. Thus $f_1 \geq \mathcal{E}$ a.e. on ν, and $f_1 \leq -\mathcal{E}$ a.e. on ν'.

Now, let $\tilde{f}_1 = H * f_1$, where "*" denotes convolution; and, as in Edwards [2], p.86,

$$H = \sum_{n \in Z} -i.\operatorname{sgn} n.e^{inx}.$$

Note that, if a, b are the end-points of ν,

$$Df_1 = \delta_a - \delta_b$$

where Df_1 is the distributional derivative of f_1, and δ_x is the Dirac measure at x. Therefore,

$$\begin{aligned} D\tilde{f}_1 &= D(H*f_1) = H * Df_1 \\ &= (\delta_a - \delta_b) * H \\ &= 2(\delta_a - \delta_b) * D(\log|\sin\tfrac{1}{2}x|), \quad \text{(Edwards, [2], p.88, (12.8.6));} \end{aligned}$$

and so

$$\tilde{f}_1(x) = 2\log|\sin\tfrac{1}{2}(x-a)| - 2\log|\sin\tfrac{1}{2}(x-b)| + \text{constant a.e.}\ .$$

Thus $\tilde{f}_1$ is essentially unbounded in every neighbourhood of each of the points a,b. But, since $f_1 \in H^\infty_R$, and therefore $\tilde{f}_1 = -if_1$, we have a contradiction, and hence

$$\inf_{f \in H^\infty} ||f-f_0||_\infty \geq 1.$$

Thus, by the Hahn-Banach extension theorem (Edwards [1], §2.2.5) $\exists$ a bounded real-linear functional on L^∞_R which vanishes on H^∞_R but not at f_0. We can then extend this functional into a complex-linear functional over L^∞ to get the required linear functional ψ.

§4.4 Extension of multiplicative weak-* continuous linear functionals on a weak-* Dirichlet algebra.

We shall now consider further the extension of weak-* continuous linear functionals defined on a weak-* Dirichlet algebra A. In particular, we shall consider those functionals which are also multiplicative on A. We prove the following theorem, which is due to Hoffman and Rossi (Hoffman and Rossi [1]):

Theorem 4.4.1. Let A be a weak-* Dirichlet algebra. Let m be a probability measure multiplicative on A. Let Φ be a linear functional defined on A such that

(i) Φ is multiplicative on A,

and (ii) $\exists$ $g \in L^1(m)$ such that $\Phi(f) = \int fgdm, \ \forall\ f \in A$.

Then Φ can be extended to a positive weak-* continuous linear functional on $L^\infty(m)$, that is, $\exists$ a non-negative $k \in L^1(m)$ such that

$$\Phi(f) = \int fkdm, \ \forall\ f \in A.$$

Proof. Since, by Theorem 2.1.1, $H^\infty(m) = [A]_*$, we see as before that Φ has a unique extension, denoted by Φ also, to a linear functional on H^∞, namely

$$\Phi(f) = \int fg\,dm, \quad \forall\; f \in H^\infty.$$

We shall show that Φ is multiplicative on $H^\infty = [A[_*$. Consider $f,h \in H^\infty$. Then $\exists\ \{f_\mu\},\{h_\nu\}$ A which converge in the $\sigma(L^1,L^\infty)$ topology to f,h respectively. In particular,

$$\lim_\nu \Phi(f_\mu h_\nu) = \lim_\nu \int f_\mu h_\nu g\,dm = \int f_\mu hg\,dm = \Phi(f_\mu h);$$

and

$$\lim_\mu \Phi(f_\mu h) = \lim_\mu \int f_\mu hg\,dm = \int fhg\,dm = \Phi(fh).$$

Hence,

$$\lim_\mu \lim_\nu \Phi(f_\mu h_\nu) = \Phi(fh).$$

But $f_\mu h_\nu \in A$. Thus by (i),

$$\Phi(f_\mu h_\nu) = \Phi(f_\mu)\Phi(h_\nu),$$

and hence

$$\Phi(fh) = \lim_\mu \lim_\nu \Phi(f_\mu h_\nu) = \lim_\mu \Phi(f_\mu) \lim_\nu \Phi(h_\nu) = \Phi(f)\Phi(h)$$

which shows that Φ is multiplicative on H^∞.

Define $\sum$ to be the set of all $u \in L^\infty_R(m)$ such that for every positive real number t, $\exists\ h_t \in H^\infty$ such that

(a) $\quad tu \geq \log|h_t|$,

and (b) $\quad \Phi(h_t) = 1$.

Before proceeding further we need to prove the following two lemmas. The first of these is an extension of the Krein-Simulian Theorem. (Horvath [1], Ch.3 §10. Theorem 2.)

<u>Lemma 4.4.1</u>. Let K be a convex subset of $L^{\infty}(m)$. The following two conditions are equivalent.

(i) K is weak-* closed;

and (ii) If $\{f_n\} \subset K$ converges boundedly and pointwise a.e. to a function f, then $f \in K$.

<u>Proof</u>. (i) => (ii). Assume $K = [K]_*$ and that $\{f_n\} \subset K$ converges boundedly and pointwise a.e. to f. The latter condition implies that $\{f_n\} \subset K$ also converges weakly to f. So $f \in [K]_*$ and, since $[K]_* = K$, $f \in K$.

(ii) => (i). This follows directly from Edwards [1], 8.10.5 and Ex. 8.6.

<u>Lemma 4.4.2</u>. Let $\sum$ be as defined previously; namely the set of $n \in L^{\infty}_{R}(m)$ such that, for every positive real number t, $\exists\ h_t \in H^{\infty}$ such that

(a) $tu \geq \log|h_t|$;

and (b) $\Phi(h_t) = 1$.

Then $\sum$ is a convex cone which is weak-* closed in $L^{\infty}_{R}(m)$.

<u>Proof</u>. We first show that $\sum$ is a convex cone.

(i) $u_1, u_2 \in \sum \Rightarrow u_1 + u_2 \in \sum$.

Let $u_1, u_2 \in \sum$. Then, for every positive real number t, $\exists\, h_{1,t}$ and $h_{2,t}$, both $\in H^\infty$, such that $tu_1 \geq \log|h_{1,t}|$, $tu_2 \geq \log|h_{2,t}|$ and $\Phi(h_{1,t}) = \Phi(h_{2,t}) = 1$.

Consider $h_{1,t}\, h_{2,t} \in H^\infty$. Now, for every positive real number t,

$$t(u_1+u_2) \geq \log|h_{1,t}| + \log|h_{2,t}| = \log|h_{1,t}\, h_{2,t}|;$$

and

$$\Phi(h_{1,t}\, h_{2,t}) = \Phi(h_{1,t})\Phi(h_{2,t}) = 1.1 = 1.$$

Hence $u_1 + u_2 \in \sum$.

(ii) $u \in \sum$, $\alpha \geq 0 \Rightarrow \alpha u \in \sum$.

Let $u \in \sum$, $\alpha \geq 0$. Then, for every positive real number t, $\exists\, h_t \in H^\infty$ such that $tu \geq \log|h_t|$ and $\Phi(h_t) = 1$. Consider $h_t^\alpha \in H^\infty$. Now

$$t(\alpha u) = \alpha(tu) \geq \alpha \log|h_t| = \log|h_t|^\alpha = \log|h_t^\alpha|;$$

and

$$\Phi(h_t^\alpha) = [\Phi(h_t)]^\alpha = 1.$$

Hence $\alpha u \in \sum$.

We now show that $\sum$ is weak-* closed in $L_R^\infty(m)$. Using Lemma 4.4.1 we need to show that, if we consider $\{u_n\} \subset \sum$ such that $|u_n| \leq \underline{M}$, where $\underline{M}$ is a constant function, and $\{u_n\}$ converges to u pointwise a.e., then $u \in \sum$.

By the definition of $\sum$, for each n $\exists$ $h_n \varepsilon H^\infty$ such that $u_n \geq \log|h_n|$ and $\Phi(h_n) = 1$. In particular, $\{|h_n|\}$ is bounded by e^M. Let h be a weak-* cluster point of $\{h_n\}$ in $L^\infty(m)$. Since H^∞ is weak-* closed, $h \varepsilon H^\infty$; and, since Φ is weak-* continuous and $\Phi(h_n) = 1$, $\Phi(h) = 1$. Since h is a weak-* cluster point of $\{h_n\} \subset H^\infty$, then $\forall$ $g \varepsilon L^1(m)$ $\exists$ $\{h_{n_k}\} \subset \{h_n\}$ such that

$$\lim_{n_k \to \infty} \int h_{n_k} g\, dm = \int hg dm.$$

Thus

$$\lim_{n_k \to \infty} \left|\int h_{n_k} g\, dm\right| = \left|\int hg dm\right|,$$

and so

$$\limsup_{n_k} \left|\int h_{n_k} g\, dm\right| \geq \left|\int hg dm\right| \tag{1}$$

Since $|h_{n_k}|$ is bounded, for $g \varepsilon L^1(m)$ such that $|h_{n_k} g|$ is bounded, we can apply Fatou's Lemma; thus, for such $|h_{n_k} g|$,

$$\int \limsup_{n_k} |h_{n_k} g| \geq \limsup_{n_k} \int |h_{n_k} g|$$

$$\geq \limsup_{n_k} \left|\int h_{n_k} g\right| \tag{2}$$

Combine (1) and (2) to get

$$\int \limsup_{n_k} |h_{n_k} g| \, dm \geq \left| \int hg \, dm \right| .$$

Thus, if we have a set $S \subset X$ such that $m(S) > 0$, we can choose $g = \operatorname{sgn} h \cdot \chi_S / m(S)$ to get

$$\frac{1}{m(S)} \int_S \limsup_n |h_n| \, dm \geq \frac{1}{m(S)} \int_S |h| \, dm . \tag{3}$$

We need to show that (3) implies that

$$\limsup_n |h_n| \geq |h| \quad \text{a.e.} \tag{4}$$

Suppose (3) holds fut (4) does not hold. Then $\exists$ set $E \subset X$ such that $m(E) > 0$ and

$$\limsup_n |h_n| < |h| \quad \text{on } E.$$

Let E_k be the subset of X on which

$$\limsup_n |h_n| < |h| - \frac{1}{k} .$$

Then $E_K \subset E_{K+1}$ and $\bigcup_K E_K = E$. Since $m(E) > 0$, $\exists$ k_1 say, such that $m(E_{k_1}) > 0$. Choose $\delta = \frac{1}{k_1}$ and let $S = E_{k_1}$. Then

$$\limsup_{n\to\infty} |h_n| < |h| - \delta \quad \text{on } S,$$

and so

$$\int_S \limsup_n |h_n|\,dm < \int_S (|h|-\underline{\delta})\,dm = \int_S |h|\,dm - \underline{\delta}m(S).$$

Thus,

$$\frac{1}{m(S)}\int_S \limsup_n |h_n|\,dm < \frac{1}{m(S)}\int_S |h|\,dm - \underline{\delta}\ .$$

This contradicts (3). Hence (4) holds; and so

$$|h| \leq \limsup_n (\exp \circ u_n) = \exp \circ u.$$

That is, $u \geq \log|h|$. The same argument can be applied to tu, $\forall\ t \geq 0$. Thus $u \in \sum$.

We now continue with the proof of theorem 4.4.1.

<u>Proof (of Theorem 4.4.1) continued.</u>

$\sum$ is proper since $-\underline{1} \notin \sum$. (To show that $-\underline{1} \notin \sum$, suppose $-\underline{1} \in \sum$. Then, for every positive real number t, $\exists h_t \in H^\infty$ such that $-\underline{t} \geq \log|h_t|$ and $\phi(h_t) = 1$. However, since H^∞ is a Banach algebra and ϕ is a <u>multiplicative</u> linear functional on H^∞, $||\phi|| = 1$.

Since $-\underline{t} \geq \log|h_t|$, $||h_t||_\infty < 1$. But

$$1 = |\phi(h_t)| \leq ||\phi||\,||h_t||_\infty = ||h_t||_\infty;$$

which is a contradiction; and so $-\underline{1} \notin \sum$.)

Since $\sum$ is proper, and, by Lemma 4.4.2, weak-* closed, a corollary of the Hahn-Banach Theorem (Edwards [1], 2.2.3) ensures the existence of a non-zero weak-* continuous linear functional on L^{∞}_{R} which is greater than α on $\sum$, for some real number α. This linear functional must be non-negative on $\sum$, since, as $\sum$ is a cone, if it took a negative value at some point of $\sum$ then it would take arbitrary large negative values, thus contradicting the fact that it is bounded below by α. This functional may then be extended to ψ_1, a non-zero, weak-* continuous linear functional on L^{∞} which is non-negative on $\sum$. Thus, $\exists\, k_1 \in L^1(m)$ such that

$$\psi_1(f) = \int f k_1 dm, \quad \forall\ f \in L^{\infty}.$$

Let $k = [\int |k_1| dm]^{-1} k_1 \in L^1(m)$, so that

$$\int |k| dm = 1;$$

and form

$$\psi(f) = \int f k dm, \quad \forall\ f \in L^{\infty}.$$

Then ψ is a non-zero weak-* continuous linear functional on L^{∞} which is non-negative on $\sum$, and

$$||\psi|| = ||k||_1 = 1.$$

By taking $h_t = \underline{1}$, we see that $\sum$ contains every positive function in L^∞. Thus ψ is a positive functional and k is a non-negative function. Suppose now $f \in A$ such that $f \in \ker\Phi$. By taking $h_t = \exp o\ (tf)$, we see that $\text{Re}f \in \sum$. But, if $f \in \ker\Phi$, $-f \in \ker\Phi$ and so $\text{Re}(-f) = -\text{Re}f \in \sum$. Now ψ is non-negative on $\sum$, so $\psi(\text{Re}f) = 0$. By considering $(-if) \in \ker\Phi$ we get $\psi(\text{Im}f) = 0$. Thus $\psi(f) = 0$, $\forall\ f \in \ker\Phi$. Hence,

$$\psi(f) = \Phi(f), \quad \forall\ f \in A.$$

We have shown this for $f \in A$ such that $\Phi(f) = 0$. Consider $f \in A$ such that $\Phi(f) = c \neq 0$. Then

$$\Phi(f) - c = \Phi(f-\underline{c}) = 0.$$

Thus $\psi(f-\underline{c}) = \psi(f) - c = 0$, and so $\psi(f) = c$. Since also $||\psi|| = ||\Phi|| = 1$, ψ is a norm-preserving extension of Φ which is positive and weak-* continuous and takes the form

$$\psi(f) = \int fk\,dm, \quad \forall\ f \in L^\infty.$$

Thus, we may write

$$\Phi(f) = \int fk\,dm, \quad \forall\ f \in A,$$

where k is a non-negative function.

CHAPTER 5

SEQUENTIAL F. AND M. RIESZ THEOREM

§5.1 A sequential F. and M. Riesz Theorem.

Let A be the sup-norm Banach algebra of complex-valued functions on the unit circle whose Fourier coefficients, C_n say, are zero for $n < 0$. Then H^{∞} is the set of bounded complex-valued functions on the unit circle whose Fourier coefficients C_n, say are zero for $n < 0$. Let λ denote the Lebesgue measure on the unit circle.

Theorem 5.1.1. (Kahane [1]). Let $\{g_n\} \subset L^1(\lambda)$ be such that

$$\ell(f) = \lim_{n\to\infty} \int fg_n d\lambda$$

exists for every $f \in H^{\infty}$. Then $\exists$ $g \in L^1(\lambda)$ such that

$$\ell(f) = \int fgd\lambda, \quad \forall\ f \in A;$$

and every (complex) Baire measure μ which is a cluster point, in the $\sigma(A,A^*)$ topology, where A^* is the dual of A, of $\{g_n\lambda\}$, is such that $\mu << \lambda$.

Proof. We first show that a finite complex-valued Baire measure on the unit circle such that

$$\ell(f) = \int fd\mu, \quad \forall\ f \in A.$$

Define $\Phi_n(f) = \int fg_n d\lambda$, $\forall$ $f \varepsilon A$. For each $f \varepsilon A$, $\{\Phi_n(f)\}$ is bounded. Hence, by the principle of uniform boundedness, $\{||\Phi_n||\}$ is bounded. Denote by Φ_n also the norm-preserving extension of Φ_n to the continuous complex-valued functions on the unit circle. By the Riesz representation Theorem $\exists$ a finite Barie measure μ_n such that, for each n,

$$\Phi_n(f) = \int fd\mu_n, \quad \forall\ f \varepsilon A;$$

and the total variation of μ_n is equal to $||\Phi_n||$. Thus, by the weak-* compactness of measures, $\exists$ a finite Barie measure μ such that

$$\ell(f) = \lim_{n\to\infty} \int fd\mu_n = fd\mu, \quad \forall\ f \varepsilon A.$$

We now show that $\mu << \lambda$. Suppose it is not the case that $\mu << \lambda$. Let E be a closed set on the unit circle such that $\lambda(E) = 0$ and $\mu(E) \neq 0$. Such an E exists since μ is regular. Let $h \varepsilon A$ be such that $h = \underline{1}$ or E and $|h| < 1$ outside E. (The existence of such an $h \varepsilon A$ is established in Hoffman [1], p.81.)

We now have the following properties.

(1) $\lim_{m\to\infty} \int h^m d\mu = \mu(E)$

(2) $\lim_{m\to\infty} \int h^m g_n d\lambda = 0, \ \forall\ n.$

(3) $\lim_{n\to\infty} \int h^m g_n d\lambda = \int h^m d\mu, \ \forall\ m.$

If $\{m_j\}$ is rapidly increasing (meaning that m_{j+1} is sufficiently large when m_j is given), we have

$$f = \sum_{j=1}^{\infty} (-1)^j h^{m_j} \in H^\infty \qquad \text{(i)},$$

since, given m_j, we may define E_j as the set where $|h^{m_j} - 1| < 2^{-j}$, and, when m_{j+1} is large enough, we have $|h^{m_{j+1}}| < 2^{-j}$ on E'_j, the complement of E_j. But $2^{-j} < 1 - 2^{-(j+1)}$, and so $E'_j \cap E_{j+1} = \emptyset$. Thus, since also $E \subset E_j$, $\forall$ j, every $x \notin E$ belongs to E'_k for some k, chosen sufficiently large. By the method given in detail in the proof of Theorem 5.2.1, it now follows that $f = \sum_{j=1}^{\infty} (-1)^j h^{m_j}$ is the pointwise limit a.e. (λ) of a uniformly bounded sequence of functions in A and so $f \in H^\infty$.

Write (α) for the above condition on the $\{m_j\}$. We introduce here the formula,

$$\int f g_{n_j} d\lambda = \sum_{k=1}^{j-1} (-1)^k \int h^{m_k} g_{n_j} d\lambda + (-1)^j \int h^{m_j} g_{n_j} d\lambda + \sum_{k=j+1}^{\infty} \int h^{m_k} g_{n_j} d\lambda$$

$$= A_j + B_j + C_j$$

where we shall define by induction the sequences $\{m_j\}$ (satisfying (α)) and $\{n_j\}$ such that the following two conditions are satisfied:

(β) $\quad \sum_{k=j+1}^{\infty} \left|\int h^{m_k} g_{n_j} d\lambda\right| < \frac{1}{12}|\mu(E)|$

(γ) $\quad \left|\int h^{m_j} d\mu\right| > \frac{11}{12}|\mu(E)|$.

Choose n_1 any positive integer. Let m_1 be the least positive integer such that

$$|\int h^m d\mu| > \frac{11}{12}|\mu(E)|, \quad \forall\ m \geq m_1.$$

The existence of m_1 is guaranteed by (1).

Now suppose we have defined $n_1, m_1, \ldots, n_{j-1}, m_{j-1}$. Let $\mathfrak{m}_j$ be the least positive integer such that

$$|h^{m_{j-1}} - \underline{1}| < 2^{-(j-1)} \text{ on } E_{j-1} \Rightarrow |h^{\mathfrak{m}_j}| < 2^{-(j-1)} \text{ on } E'_{j-1}.$$

Let n_j^* be the least positive integer such that $n \geq n_j^*$ implies

$$\left|\sum_{k=1}^{j-1}(-1)^k \int h^{m_k} g_n d\lambda - \sum_{k=1}^{j-1}(-1)^k \int h^{m_k} d\mu\right| < \frac{1}{12}|\mu(E)|.$$

The existence of n_j^* is guaranteed by (3). Let $M(n,k)$ be the least positive integer such that

$$m \geq M(n,k) \Rightarrow \left|\int h^m g_n d\lambda\right| < 2^{-k} \cdot \frac{1}{12}|\mu(E)|.$$

The existence of $M(n,k)$ is guaranteed by (2). Put $m_j^* = \max(m_1, M(n_1,j), \ldots, M(n_{j-1},j), \mathfrak{m}_j)$. We define $A_j^\infty = \sum_{k=1}^{j-1}(-1)^k \int h^{m_k} d\mu$, and consider the following two cases:

(a) $|A_{j-1} + B_{j-1} - A_j^\infty| \leq 5/12|\mu(E)|$

(b) $|A_{j-1} + B_{j-1} - A_j^\infty| > 5/12|\mu(E)|$.

In case (a), define $m_j^* = m_j$, noting that (γ) is true, and choose $n_j \geq n_j^*$ so that $|B_j| \geq \frac{11}{12}|\mu(E)|$.

This is possible by (3) and (γ); by (3), with $m_j = m_j^*$, for any $\varepsilon > 0$, $\exists$ $n_j \geq n_j^*$ such that

$$\left|\int h^{m_j^*} d\mu - \int h^{m_j^*} g_{n_j} d\lambda\right| < \varepsilon.$$

Thus, $\left|\int h^{m_j} d\mu\right| - |B_j| < \varepsilon$, and, choosing

$$\varepsilon = \left|\int h^{m_j} d\mu\right| - \frac{11}{12}|\mu(E)|$$

(which is > 0 by (γ)), we get $|B_j| \quad \frac{11}{12}|\mu(E)|$.

In case (b), define $n_j = n_j^*$ and choose $m_j \geq m_j^*$ so that $|B_j| < \frac{1}{12}|\mu(E)|$. This is possible because of (2).

Note, that for $n_j \geq n_j^*$,

$$|A_j - A_j^\infty| < \frac{1}{12}|\mu(E)|.$$

We have, also, given n_j,

$$\left|\int h^m g_{n_j} d\lambda\right| < \frac{1}{2^k}\cdot\frac{1}{12}|\mu(E)|, \quad \forall\ m \geq M(j,k),$$

Now $m_k \geq M(j,k)$ $\forall k \geq j+1$, so if $m_k \geq m_k^*$,

$$\sum_{k=j+1}^{\infty} \left|\int h^{m_k} g_{n_j} d\lambda\right| < \frac{1}{12}|\mu(E)|.\sum_{k=j+1}^{\infty} \frac{1}{2^k} < \frac{1}{12}|\mu(E)|$$

and the sequences $\{n_j\},\{m_j\}$ in both cases (a) and (b) satisfy (β).

In case (a), we have

$$|A_{j-1} + B_{j-1} - A_j - B_j| = |(A_{j-1} + B_{j-1} - A_j^\infty| + (A_j^\infty - A_j) - B_j|$$

$$\geq |B_j| - |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j|$$

$$\geq \max(|B_j| - |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j|)$$

$$\geq \frac{11}{12}|\mu(E)| - 5/12|\mu(E)| - \frac{1}{12}|\mu(E)| = 5/12|\mu(E)| > \frac{3}{12}|\mu(E)|.$$

In case (b) we have

$$|A_{j-1} + B_{j-1} - A_j - B_j| > |A_{j-1} + B_{j-1} - A_j^\infty| - |A_j^\infty - A_j| - |B_j|$$

$$> |5/12 - \frac{1}{12} - \frac{1}{12})|\mu(E)| = 3/12|\mu(E)|.$$

Thus, in each case we have

$$|A_{j-1} + B_{j-1} - A_j - B_j| > 3/12|\mu(E)|.$$

Taking (β) into account we have $|C_{j-1}|$ and $|C_j|$ both majorised by $\frac{1}{12}|\mu(E)|$.

Therefore,

$$|\int fg_{n_{j-1}} d\lambda - \int fg_{n_j} d\lambda| > |A_{j-1} + B_{j-1} - A_j - B_j| - |C_{j-1}| - |C_j|$$

$$> (\frac{3}{12} - \frac{1}{12} - \frac{1}{12})|\mu(E)| = \frac{1}{12}|\mu(E)|$$

and so $\{\int fg_{n_j} d\lambda\}$ is not convergent, contrary to our initial assumption. This contradiction gives $\mu << \lambda$. That is,

$$\ell(f) = \int fd\mu = \int fgd\lambda, \; \forall \; f \in A, \text{ some } g \in L^1(\lambda).$$

Remark 5.1.1. Theorem 5.1.1 implies the classical F. and M. Riesz Theorem. To see this, suppose μ is a measure on the unit circle such that $\int e_n d\mu = 0$, $n \geq 1$. If $\{\sigma_n\} \subset L^1(\lambda)$ is the sequence of Cesarò means of the Fourier series of μ, then μ is the unique weak-* cluster point of $\{\sigma_n \lambda\}$ and

$$\lim_{n\to\infty} \int f\sigma_n d\lambda = \int f d\mu, \quad \forall\ f \in H^\infty$$

and so, by Theorem 5.1.1, $\mu << \lambda$.

The proof of Theorem 5.1.1 relies on the existence of $h \in A$ such that $h(E) = \underline{1}$ and $|h| < 1$ elsewhere, where $\lambda(E) = 0$. The existence of such an h is guaranteed by the classical F. and M. Riesz Theorem. Thus, Theorem 5.1.1 is equivalent to the classical F. and M. Riesz Theorem, and we shall refer to Theorem 5.1.1 as a sequential F. and M. Riesz Theorem.

§5.2. A relation between generalised and sequential F. and M. Riesz Theorems.

Elizabeth Heard (Heard [1]) considered the case where A is a subspace of $C(X)$, for X a compact Hausdorff space. She said that A and m, where $m \in M(X)$, the set of finite complex-valued Baire measures which form the dual of $C(X)$, satisfy a generalised F. and M. Riesz Theorem whenever $\mu \in A^{\perp} \Rightarrow \mu << m$, for μ any finite complex-valued Baire measure on X.

From our previous generalisations of the F. and M. Riesz Theorem, which are not nearly so strong, it would appear that, for A and m to satisfy such a theorem heavy restrictions would need to be placed on A. Bishop [1] claims there are at least three examples in the literature, one of which is given in Bishop [2]. Heard showed that, whenever A, m satisfy such a theorem, they also satisfy a sequential F. and M. Riesz Theorem.

<u>Theorem 5.2.1. (Heard [1])</u>. Let A be a closed subspace of $C(X)$. Let $m \varepsilon M(\lambda)$, where $M(X)$ is asddefined above. Let μ be any finite complex-valued Baire measure on X. Then (I) => (II)

(I) $\mu \varepsilon A^{\perp} \Rightarrow \mu << m$.

(II) If $\{g_n\} \subset L^1(m)$ and

$$\ell(f) = \lim_{n\to\infty} \int f g_n dm$$

exists for every $f \varepsilon [A]_*$, then any representative μ of a coset $\mu + A^{\perp}$ which is a cluster point in the $\sigma(A, A^*)$ topology of the set of cosets $\{\mu_n + A^{\perp} : \mu_n = g_n m\} \subset M(X)/A$ is absolutely continuous with respect to m $(\mu << m)$; and $\exists\ g \varepsilon L^1(m)$ such that

$$\ell(f) = \int fg dm, \quad \forall\ f \varepsilon A.$$

Proof. Suppose $\{g_n\} \subset L^1(m)$ such that

$$\ell(f) = \lim_{n\to\infty} \int f g_n \, dm$$

exists for every $f \in [A]_*$. Let μ_n be defined by $\mu_n = g_n m$. By a similar method to that used in Theorem 5.1.1 we can find a coset $\mu + A^\perp$ which is a cluster point in the $\sigma(A,A^*)$ topology of $\{\mu_n + A^\perp\}$. Let μ be a representative of the coset $\mu + A^\perp$. Then

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu = \ell(f), \quad \forall \ f \in A.$$

We shall now show that $\mu \ll m$. Suppose this is not the case. Then, since μ is regular, $\exists$ a closed set $E \subset X$ such that $m(E) = 0$ but $\mu(E) \neq 0$. We require $\{f_n\} \subset A$ with the following properties.

(i) $||f_n|| < 1 + 2^{-n}$

(ii) $f_n = \underline{1}$ on E

(iii) $\lim_{n\to\infty} f_n = 0$ a.e. (m)

(iv) $\lim_{n\to\infty} f_n = \chi_E$ a.e. $(|\mu|)$.

(v) $f = \sum_{j=1}^{\infty} (-1)^j f_{n(j)} \in [A]_*$ for every strictly increasing sequence of positive integers $\{n(j)\}$.

The construction of $\{f_n\}$ is by induction. Let $\{U_n\}$ be a sequence of open subsets of X such that $E \subset U_{n+1} \subset U_n$. Since $m(E) = 0$ we can suppose $m(U_n) < 2^{-n}$ and $|\mu|(U_n \setminus E) < 2^{-n}$ for all positive integers n.

Let $E_1 = U_1$. By Urysohn's Lemma $\exists\ h_1 \in C_R(X)$ such that $h_1(E) = \{1\}$, $h_1(X \setminus E_1) = \{0\}$, and $\underline{0} \leq h_1 \leq \underline{1}$. The assumption that (I) is true and the general Rudin-Carleson Theorem (Bishop [1]) allow us to choose $f_1 \in A$ such that

$$f_1(E) = \{1\} \text{ and } |f_1| < h_1 + \underline{2^{-1}} .$$

Now suppose $f_1, \ldots, f_n$ and the sets $E_1, \ldots, E_n$ which are open neighbourhoods of E, are chosen. Let V_{n+1} be the subset of X on which $|f_n - \underline{1}| < 2^{-n}$. Put $E_{n+1} = E_n \cap V_{n+1} \cap U_{n+1}$. Again by Urysohn's Lemma, $\exists\ h_{n+1} \in C_R(X)$ such that $h_{n+1}(E) = \{1\}$, $h_{n+1}(X \setminus E_{n+1}) = \{0\}$, and $\underline{0} \leq h_{n+1} \leq \underline{1}$. We can then choose $f_{n+1} \in A$ such that

$$f_{n+1}(E) = \{1\} \text{ and } |f_{n+1}| < h_{n+1} + \underline{2^{-(n+1)}} .$$

We now wish to show that the $\{f_n\}$ so defined satisfies (i) - (iv).

Properties (i) and (ii) are clearly satisfied. Now $\lim_{n\to\infty} f_n = \underline{0}$ except on the set $F = \bigcap_{j=1}^{\infty} E_j$. But $m(F) \leq m(E_n) \leq m(U_n) < 2^{-n}$, $\forall$ n, so $m(F) = 0$, and $\lim_{n\to\infty} f_n = \underline{0}$ a.e. (m), which is property (iii). Also, $\lim_{n\to\infty} f_n = \chi_E$ except on the set $F \setminus E$. But

$$|\mu|(F \setminus E) \leq |\mu|(E_n \setminus E) \leq |\mu|(U_n \setminus E) < 2^{-n}, \ \forall\ n,$$

so $|\mu|(F \setminus E) = 0$, and $\lim_{n\to\infty} f_n = \chi_E$ a.e. $|\mu|$, which is property (iv).

We now show that (v) is also satisfied. Let $\{n(j)\}$ be any strictly increasing sequence of positive integers. Let $E_{n(0)} = X$. Then $F = \bigcap_{j=1}^{\infty} E_j = \bigcap_{j=1}^{\infty} E_{n(j)}$; and

$X \setminus F = \bigcup_{j=1}^{\infty} (E_{n(j-1)} \setminus E_{n(j)})$, where the sets in this union are disjoint.

Consider $x \in X \setminus F$. Then, for some k, $x \in E_{n(k)} \setminus E_{n(k+1)}$; and hence $x \in \bigcap_{j=1}^{k} E_{n(j)}$ and $x \in X \setminus \bigcap_{j=k+1}^{\infty} E_{n(j)}$. Thus we have

$$\left| \sum_{j=1}^{k-1} (-1)^j f_{n(j)}(x) \right| \leq \sum_{j=1}^{k-1} \left| (f_{n(j)}(x)-1)(-1)^j \right| + \left| \sum_{j=1}^{k-1} (-1)^{j-1} \right|$$

(a) $$\leq \sum_{j=1}^{k-1} 2^{-n(j)} + 1;$$

(b) $$\left| \sum_{j=k+1}^{\infty} (-1)^j f_{n(j)}(x) \right| \leq \sum_{j=k+1}^{\infty} (h_{n(j)}(x) + 2^{-n(j)}) \leq 2^{-n(k)};$$

(c) $$||f_{n(k)}|| \leq 1 + 2^{-n(k)}.$$

Thus the series $\sum_{j=1}^{\infty} (-1)^j f_{n(j)}$ converges and (a), (b), (c) show that the sequences of partial sums,

$$\{s_n(x) : s_n(x) = \sum_{j=1}^{n} (-1)^j f_{n(j)}(x)\}$$

converges to $f(x)$ and $|s_n(x)| < 4$, $\forall$ positive integers n. Thus f is defined at every $x \in X \setminus F$. Let $y \in F$. Then $y \in \bigcap_j E_{n(j+1)}$

and

$$s_n(y) = \sum_{j=1}^{n} (-1)^j f_{n(j)}(y) = \sum_{j=2}^{n} (-1)^j (f_{n(j)}(y) - 1) + \sum_{j=1}^{n} (-1)^{j-1}$$

so that

$$|s_n(y)| \leq \sum_{j=1}^{n} |f_{n(j)}(y) - 1| + |\sum_{j=1}^{n} (-1)^{j-1}|$$

$$\leq \sum_{j=1}^{n} 2^{-n(j)} + 1 < 4.$$

Therefore $f = \sum_{j=1} (-1)^j f_{n(j)}$ is the pointwise limit a.e. (m) of a uniformly bounded sequence of functions in A. Thus $f \in [A]_*$ and (v) is established.

We complete the proof in the same manner as that of Theorem 5.1.1, showing that what we have just deduced leads to a contradiction of our original assumption regarding the convergence of $\{\int fg_n dm\}$, $\forall$ $f \in [A]_*$. Thus $\mu << m$ and $\exists$ $g \in L^1(m)$ such that

$$\ell(f) = \int fgdm, \quad \forall \ f \in A.$$

<u>§5.3 Description of more functions in H^∞ for which the limit relation in Theorem 5.1.1 holds.</u>

It is not known whether the limit relation in Theorem 5.1.1, namely

$$\ell(f) = \int fgd\lambda,$$

holds $\forall$ $f \in H^{\infty}$. Kahane himself (Kahane [1]) showed that it does hold for certain functions in $H^{\infty} \setminus A$.

Let L be the set of linear functionals ℓ on H^{∞} such that, for some (possibly ℓ-dependent) sequence $\{g_n\} \subset L^1(\lambda)$,

$$\ell(f) = \lim_{n \to \infty} \int f g_n \, d\lambda, \quad \forall \, f \in H^{\infty}.$$

Theorem 5.1.1 asserts that, to each $\ell \in L$ corresponds at least $g \in L^1(\lambda)$ such that

$$\ell(f) = \int fg\,d\lambda, \quad \forall \, f \in A \tag{i}$$

Denote by $G(\ell)$ the set of $g \in L^1(\lambda)$ such that (i) is true. Define $D_\ell = \{ f \in H^{\infty} : \ell(f) = \int fg\,d\lambda, \ g \in G(\ell)\}$, and let $D = \bigcap \{D_\ell : \ell \in L\}$. Then we have

<u>Theorem 5.3.1</u>. (a) D_ℓ is a closed subspace of H^{∞}; and, given any $f \in H^{\infty}$ almost all translates of f belong to D_ℓ.

(b) D is a closed subalgebra of H^{∞}, invariant under translation; it contains all $f \in H^{\infty}$ such that $fh \in D$, for some outer function h.

In particular, D contains all the $f \in H^{\infty}$ which are continuous on the unit circle except on a closed set of measure zero.

<u>Proof</u>. Define $\Phi_n(f) = \int f g_n \, d\lambda$, $\forall$ $f \in A$, where $\{g_n\} \subset L^1(\lambda)$. Since the trigonometric polynomials form a dense subset of $L^1(\lambda)$, we may suppose that each g_n is a trigonometric polynomial. For each $f \in A$, $\{\Phi_n(f)\}$ is bounded, and hence, by the principle of

uniform boundedness, $\{||\Phi_n||\}$ is bounded. Denote by $||\Phi_n||$ also the norm-preserving extension of Φ_n to the continuous complex-valued functions on the unit circle. By the Riesz Representation Theorem $\exists$ a finite Baire measure μ_n, such that, for each n,

$$\Phi_n(f) = \int f d\mu_n, \quad \forall\ f \in A,$$

and the total variation of μ_n is equal to $||\Phi_n||$. Thus we have

$$\int f(g_n d\lambda - d\mu_n) = 0, \quad \forall\ f \in A.$$

In particular,

$$\int f(g_n d\lambda - d\mu_n) = 0, \quad \forall\ f \in A_0,$$

where $A_0 = \{f \in A : \int f d\lambda = 0\}$. Hence, by the F. and M. Riesz Theorem.

$$g_n\lambda - \mu_n << \lambda,$$

and so $\mu_n << \lambda$. Therefore, we may suppose $\{||g_n||_1\}$ is bounded.

In order to prove (a) we may suppose $g = \underline{0}$. Clearly D_ℓ is a closed subspace of H^∞. Given $f \in H^\infty$, write $f_s : t \mapsto f(t-s)$ for the translate of f. Since f is bounded, $f*\psi \in A$ for every $\psi \in L^1(\lambda)$, where "*" is the operation of convolution. Hence, by Theorem 5.1.1,

$$\lim_{n\to\infty} \int g_n(t)\{\int f(t-s)\psi(s)d\lambda(s)\}d\lambda(t) = 0, \quad \forall\ \psi \in L^1(\lambda).$$

That is, by the Fubini-Tonelli Theorem,

$$\lim_{n\to\infty} \int\psi(s)\{\int f(t-s)g_n(t)d\lambda(t)\}d\lambda(s) = 0, \quad \forall\ \psi \in L^1(\lambda) \tag{1}$$

By hypothesis,

$$\lim_{n\to\infty} \int g_n(t)f(t-s)d\lambda(t) = \ell(f_s);$$

and, since $\{||g_n||_1\}$ is bounded, $\{\int g_n(t)f(t-s)d\lambda(t)\}$ is uniformly bounded with respect to n and s. Hence, (1) can be rewritten

$$\int\psi(s)\ell(f_s)d\lambda(s) = 0, \quad \forall\ \psi \in L^1(\lambda).$$

Therefore, $\ell(f_s) = 0$ for almost every s and (a) is established.

To prove (b) write

$$\lim_{n\to\infty} \int fhg_n d\lambda = \ell_f(h) = \ell_h(f) = \ell(fh), \quad \forall\ f,h \in H^\infty.$$

Then $\ell_f(h) = \int hg_f d\lambda$, $\forall\ h \in D$, where $g_f \in G(\ell_f)$.

Clearly Theorem 5.1.1 implies that $A \subset D$. Suppose $f \in A$. Then

$$fh \in A \subset D, \quad \forall\ h \in A \subset D.$$

Since $fh \in D$ and $h \in D$, we have

$$\int fhg\, d\lambda = \ell(fh) = \ell_f(h) = \int hg_f d\lambda, \quad \forall\ h \in A. \tag{i}$$

Thus,

$$\int h(fg-g_f)d\lambda = 0, \quad \forall \ h \in A;$$

and so, by Corollary 1.3.4, $fg - g_f \in H^1(\lambda)$. But $\underline{1} \in A$, so

$$\int (fg-g_f)d\lambda = 0$$

and we have $fg - g_f \in H_0^1(\lambda)$. Therefore,

$$fg = g_f \pmod{H_0^1}.$$

Now suppose $h \in D$. Taking $f \in A$, and using $fg = g_f \pmod{H_0^1}$ we have

$$\ell(fh) = \ell_f(h) = \int hg_f d\lambda = \int fhgd\lambda, \text{ for every } \ell \in L. \qquad \text{(iii)}.$$

Therefore, $fh \in D_\ell$, for every $\ell \in L$, and so $fh \in D$. Since $fh \in D$ and $h \in D$, we have

$$\int f(hg-g_h)d\lambda = 0, \quad \forall \ f \in A.$$

Hence, $hg = g_h \pmod{H_0^1}$.

If $f \in D$ and $h \in D$, $fg = g_f \pmod{H_0^1}$ and so we still have (iii) and, as a consequence, $fh \in D$. Therefore, D is a subalgebra of H^∞. It is closed because each D_ℓ is closed, and it is clearly invariant under translation.

Now, suppose $f \in H^\infty$, $h \in D$, where h is an outer function, and $fh \in D$. We have as before

$$\int fhg\,d\lambda = \ell(fh) = \ell_f(h) = \int hg_f\,d\lambda, \quad \forall\ h \in D \qquad \text{(i)'}.$$

Thus

$$\int h(fg-g_f)\,d\lambda = 0, \quad \forall\ h \in D \supset A, \qquad \text{(ii)'}$$

and so $fg - g_f \in H_0^1(\lambda)$.

Consequently,

$$\ell(f) = \ell_f(\underline{1}) = \int g_f\,d\lambda = \int fg\,d\lambda, \quad \text{for every } \ell \in \mathcal{L}.$$

That is, $f \in D_\ell$ for every $\ell \in \mathcal{L}$, and so $f \in D$.

Finally, if E is a closed subset of the unit circle such that $\lambda(E) = 0$, then $\exists$ a continuous outer function h such that $h(E) = \{0\}$ (Hoffman [1], p.80.)

Hence, if f is continuous except on E, $fh \in A$ and $f \in D$.

An alternative proof of the last part of Theorem 5.3.1 was suggested by J. Wells and is given in Heard [1].

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