COMPACTNESS-TYPE PROBLEMS IN TOPOLOGICAL VECTOR SPACES

A treatment mainly from the viewpoint of Non-Standard Analysis.

by

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A thesis presented for the degree of Doctor of Philosophy in the Australian National University

Canberra, January, 1971.

STATEMENT

The results presented in this thesis are my own except where stated otherwise.

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ACKNOWLEDGEMENTS

During the time in which the work for this thesis was done, I held a Commonwealth Postgraduate Award, which was supplemented by the Australian National University.

My thanks are due to Dr R.W. Cross, my supervisor during the past three years. I am most grateful to him for his friendly criticism and encouragement, and his constant readiness to discuss my problems.

Secondly, my thanks go to many other people for their comments and suggestions related to the work of this thesis. In particular I wish to thank Drs John Giles and John Price and Messrs Jim Chadwick and Greg O'Brien. I wish also to thank Miss Toni Purdy for her excellent typing.

My final thanks go to my wife Jill for her patience and encouragement over the past months.

CONTENTS

INTRODUCTION

This thesis derives from an attempt to apply the techniques of Non-Standard Analysis to problems in the theory of topological vector spaces; in particular, to compactness-type problems in these spaces. However, because these methods are not always employed the thesis divides into two parts: Chapters 1 to 4 and Chapters 5 and 6. The initial chapters are written essentially from the viewpoint of Non-Standard Analysis whilst the later work uses only standard techniques.

As Non-Standard Analysis is still a relatively recent development we include an outline of the non-standard theory in Chapter 1. Then in Chapter 2 the basic non-standard concepts which we find useful in functional analysis are introduced and a number of preliminary theorems are established. In Chapter 3 we consider a class of generalizations of weak compactness for subsets of locally convex spaces. These generalized concepts are useful in allowing us, for example, to overcome the difficulties arising from completeness assumptions. Chapter 4 deals with continuous linear maps between topological vector spaces. Several theorems of Grothendieck [1] and Ringrose [1], [2] are re-proved and generalized.

The main purpose of Chapter 5 is to show that if X is a smooth Banach space with a certain property, its conjugate space X' is isomorphic to a rotund space. This result clarifies an observation of Day [1].

Finally in Chapter 6 some problems related to almost reflexivity are considered.

We comment that the main result of Chapter 5 has been published; see Tacon [1]. Also the results of Chapter 3 (when restricted to subsets of normed linear spaces) have been accepted for publication; see Tacon [2]. We abbreviate topological vector space and locally convex topological vector space to TVS and LCTVS respectively, and assume that these spaces are separated (i.e., Hausdorff).

CHAPTER 1

NON-STANDARD ANALYSIS

1.0 Introduction and Background.

In 1934 Thoralf Skolem [1] published a paper which showed the existence of proper extensions of the natural number system which have, in a certain sense, "the same properties" as the natural numbers. The purpose of Skolem's work was to prove that no axiom system specified in a formal language (in particular, the lower predicate calculus) characterizes the natural numbers categorically. An interest in the properties of these structures which are now known as *non-standard models of arithmetic* came only at a much later time.

Abraham Robinson extended these ideas to analysis in 1960. This recent development led to the establishment of new structures which are also proper extensions of the real number system. Robinson was able to provide a logical foundation for the nonarchimedean approach to the Differential and Integral Calculus which was strongly advocated by Leibniz and which enjoyed popularity until the middle of the last century, when it was replaced by the ε,δ method of Weierstrass. Furthermore, Robinson showed that his approach was sufficiently general to make it applicable to other mathematical objects. The resulting subject was called by Robinson *Non-Standard Analysis*.

1.1 Enlargements.

In this section we give an informal outline of the framework which is required for our subsequent arguments. We refer the reader to Robinson [2] for a complete account; Luxemburg [2], Machover and Hirschfeld [1] and Fenstad [1] also contain more detailed discussions of the non-standard theory. We need presuppose a certain background from logic. The basic concepts in this connection are: structure (or model), language and the notion of satisfaction (or interpretation). For our purposes we need have the ideas of higher-order structures and higher-order languages. However, before we can properly introduce these general notions we need another concept due to Robinson.

The class T of *types* is defined inductively as follows: (a) 0 is a type; (b) if $\tau_1,\tau_2,\ldots,\tau_n$ are types, then $(\tau_1,\tau_2,\ldots,\tau_n)$ is also a type; (c) T is the smallest class satisfying (a) and (b).

A *higher-order structure* or simply a *structure* M is a set $\{A_\tau : \tau \in T\}$ of sets indexed in T such that A_0 is non-empty and such that for every $\tau \neq 0$, $\tau = (\tau_1,\tau_2,\ldots,\tau_n)$, A_τ is a set of subsets of $A_{\tau_1} \times A_{\tau_2} \times \ldots \times A_{\tau_n}$. The elements of M are called the *entities* of M and those entities of type 0 are called the *individuals*. If

$R \in A_\tau$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, then R is called a *relation* of M . Thus R is a set of n-tuples $\langle a_1, a_2, \ldots, a_n \rangle$ where $a_i \in A_{\tau_i}$. We say M is a *full structure* if for each $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, A_τ is the set of all subsets of $A_{\tau_1} \times A_{\tau_2} \times \ldots \times A_{\tau_n}$.

The *higher-order formal language* L is introduced in the following way. The atomic symbols of L are:

(a) The usual *connectives*, $\neg$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\supset$ (implication), $\equiv$ (equivalence).

(b) The *variables*, an infinite sequence of symbols, usually denoted by $x, y, z, \ldots$.

(c) The *basic predicates*, a sequence $\Phi_n(\ ,\ldots,\ )$, $n = 1,2,\ldots$, Φ subscript n followed by round brackets enclosing $n + 1$ spaces.

(d) The *type predicates*. For every $\tau \in T$, a symbol $T_\tau(\ )$.

(e) The *quantifiers* $(\forall\ )$ - universal, and $(\exists\ )$ - existential.

(f) *Brackets* for grouping formulae.

(g) *Extralogical constants*. This is a set of symbols of which there are at least as many as to be put in one-to-one correspondence with the entities of a structure. When a basic predicate $\Phi_n(\ ,\ldots,\ )$ is filled with constants $a, a_1, \ldots, a_n$ we shall read $\Phi_n(a, a_1, \ldots, a_n)$ as "$a_1, \ldots, a_n$ satisfies a" or "a holds for $a_1, \ldots, a_n$" .

The set of well-formed formulae (wff) of L are obtained in the usual manner. A basic predicate or type predicate whose empty spaces have been filled with variables and or constants is called an *atomic well-formed formula*. If W is a wff, then $\neg(W)$ is a wff; if W_1 and W_2 are wff, then $(W_1 \wedge W_2)$, $(W_1 \vee W_2)$, $W_1 \supset W_2$, and $W_1 \equiv W_2$ are wff; if W is a wff and if W does not already contain a particular variable x under a quantifier, then $(\forall x)W$ and $(\exists x)W$ are wff. The class of wff is the smallest class that satisfies these rules. A wff is called a *sentence* whenever every variable is under the scope of a quantifier; otherwise a wff is called a *predicate*.

Suppose now that a subset of the set of constants of the language L has been put in one-to-one correspondence with the entities of M. A sentence of L is defined in M whenever all the constants contained in it denote entities of M. A sentence of L defined in M may be true or false in M according to the following rules. (a) An atomic sentence $T_\tau(a)$ defined in M holds in M if and only if the entity of M denoted by a (under the given correspondence) is of type τ. (b) An atomic sentence of the form $\Phi(a,a_1,\ldots,a_n)$ defined in M holds in M if and only if the corresponding entity a in M contains the n-tuple $\langle a_1,a_2,\ldots,a_n\rangle$. This can be the case if and only if the type $\tau = (\tau_1,\ldots,\tau_n)$ can be assigned to the entity a where the entities $a_1,\ldots,a_n$ are of type $\tau_1,\ldots,\tau_n$, respectively. (c) If a sentence in M is of the form $\neg(W)$, then it holds in M if and only if W does not hold in M. The sentence $(W_1 \vee W_2)$ holds in M if and only if at least one of W_1 and W_2 holds in M ; $(W_1 \wedge W_2)$ holds in M if and only if both W_1 and W_2 hold in M ; $(W_1 \supset W_2)$ holds in M if and only if $(\neg(W_1) \vee W_2)$ holds in M ; $W_1 \equiv W_2$ holds in M if and only if $((W_1 \wedge W_2) \vee (\neg(W_1) \wedge \neg(W_2)))$ holds in M .

(d) $(\forall x)(W(x))$ holds in M if and only if W(a) holds in M for all entities a of M , and $(\exists x)(W(x))$ holds in M if and only if W(a) holds in M for at least one entity of M .

If the entities of a structure M are in one-to-one correspondence with a subset of the extralogical constants of L , then M is called an L-structure. We denote by K the set of all sentences of L which are defined in M and which furthermore hold in M . An L-structure *M is called a *higher-order non-standard model* of an L-structure M whenever all the sentences of K hold in *M . A higher-order non-standard model *M may be regarded as an extension of M , for if the sentence $T_\tau(a)$ belongs to K , it also holds in *M . Thus to a there corresponds an entity *a of type τ in *M . The mapping a → *a of the entities of M into the entities of *M is one-to-one and defines an embedding of M into *M . In our next chapters we frequently identify the entities a of M with the corresponding entity *a of *M .

A non-standard model *M of an L-structure M need not be full even if M is full. The entities of *M are *internal* and the relations of the full structure based on $^*A_0$ which are not in *M are said to be *external*. An entity a of $^*M$ which is $^*b$ for some b belonging to M is called a *standard entity* of $^*M$.

The non-standard models that concern us are called enlargements. Let M be an L-structure and let b be a binary relation of M , say of type (τ_1, τ_2). The domain of b is the set of those x in A_{τ_1} for which there is a y in A_{τ_2} such that $\Phi_3\ (b,x,y)$ holds in M . We say that b is *concurrent* (or *finitely satisfiable*) if, for every finite set $\{x_1,\ldots,x_n\}$ of entities in the domain of b , there is an entity y in A_{τ_2} such that $\Phi_3\ (b,x_i,y)$ holds simultaneously in M for $i = 1,\ldots,n$.

A higher-order non-standard model $^*M$ of an L-structure M is called an *enlargement* of M whenever for every concurrent binary relation b of M there exists an entity y in $^*M$ such that $\Phi_3\ (^*b,^*x,y)$ holds in $^*M$, for all x belonging to the domain of b .

Robinson [2] established the existence of enlargements as a consequence of the general compactness principle of model theory (strictly speaking, we need assume that L contain enough extralogical constants to allow a one-to-one map from $^*M$ into its constants). It was also observed by Robinson and Luxemburg independently that there exist ultrapowers which are enlargements (see Luxemburg [3]).

We comment that when applying Non-Standard Analysis to mathematical structures; e.g., to a topological vector space E , it is essential to consider not only an enlargement *E of E but to enlarge simultaneously all other mathematical structures which occur in the argument; e.g., the real numbers, R . This can be done by taking for M some structure which includes both E and R . We then work in an enlargement *M of M which contains simultaneous enlargements *E and *R of E and R respectively.

CHAPTER 2

PRELIMINARY RESULTS

2.0 Introduction.

The main concepts of non-standard analysis which we find useful in functional analysis are the related notions of monad and near-standardness. In Section 2.1 we introduce and discuss these and related definitions and establish a number of characterizations of these properties.

Next we prove a simple, but nonetheless useful, embedding theorem which allows us to obtain a non-standard version of Helly's theorem. This theorem is basic to the non-standard portion of our work.

We utilize Helly's theorem in Section 2.3 to obtain the usual compactness theorems from other non-standard results. These theorems, and often their proofs, are central to the remainder of this thesis; particularly to Chapters 3 and 4.

In the remainder of the chapter we establish a number of standard theorems using the techniques and results we have already developed. These methods are intended to be essentially illustrative, although the results themselves generally have applications later.

In this chapter, as well as in the next two, we take it for granted that our object space is embedded in some full structure M (together with the appropriate scalar field whenever necessary). We develop the non-standard theory of the space in an enlargement $^*M$ of M. Our convention is to denote non-standard entities in $^*M$ by underlining, as in, for example, $\underline{x} \in {}^*X$ and $\underline{F} \subset {}^*X$. Furthermore, when there is no confusion, we omit the asterisk from standard entities in $^*M$; for example, we write $|f(x - \underline{x})| < \varepsilon$, for $f \in S'$ when we should properly write $^*|{}^*f({}^*x - \underline{x})| \; {}^*\!< {}^*\varepsilon$, $^*f \; {}^*\!\in {}^*S'$.

For the time being we allow the scalar field to be either the real numbers or complex numbers. We assume for simplicity that our TVSs are separated.

2.1 The concepts of monad and near-standardness.

Let T denote a topological space and let x denote any (standard) point in T. Suppose that Ω_x denotes the set of all open neighbourhoods of x. The following two definitions are due to Robinson [2, p. 90 and p. 93].

2.1.1 DEFINITION. The *monad* of x, which we denote by $\mu(x)$, is the intersection of all standard sets in $^*T$ which are open neighbourhoods of x; i.e.,

$$\mu(x) = \cap\{{}^*U : U \in \Omega_x\} .$$

2.1.2 DEFINITION. A point $\underline{x} \in {}^*T$ is said to be *near-standard* if there exists a (standard) point $x \in T$ such that $\underline{x} \in \mu(x)$; if T is a Hausdorff space we then say x is the *standard part* of $\underline{x}$.

We denote the standard part of $\underline{x}$ by ${}^\circ\underline{x}$ and, if $\underline{x},\underline{y} \in {}^*C$ where C denotes the complex numbers, we write $\underline{x} \simeq \underline{y}$ whenever $|\underline{x} - \underline{y}|$ is infinitesimal. It is easily seen that if $\mathcal{S}$ is a sub-base of neighbourhoods of x , $\mu(x) = \cap\{{}^*S : S \in \mathcal{S}\}$. We refer the reader to Robinson for a discussion of these concepts. Although Definition 2.1.1 is generally sufficient for our needs we remark that Luxemburg [3] has generalized the concept of monad to arbitrary filters of subsets: in particular, if $\mathcal{F}$ is a filter of subsets of T , the *intersection monad* of $\mathcal{F}$ is defined by

$$\mu(\mathcal{F}) = \cap\{{}^*F : F \in \mathcal{F}\} .$$

Definition 2.1.2 is the basic non-standard notion for our purposes. It is therefore important that we obtain characterizations of near-standardness for points in the enlargements of TVSs equipped with the common topologies. We do this first for vector spaces forming a duality and derive as corollaries the specific cases of interest to us.

We consider a duality $\langle E,F\rangle$ between two vector spaces E and F (we do not assume the duality to be separated). If $\mathcal{S}$ denotes a family of

$\sigma(F,E)$-bounded subsets of F, the (absolute) polars S^o of the sets $S \in \mathcal{S}$ generate a linear topology on E, called the $\mathcal{S}$-topology; i.e., the topology of uniform convergence on sets belonging to $\mathcal{S}$ (see, for example, Horvath [1, p. 195]).

2.1.3 THEOREM. *Let* $\langle E,F\rangle$ *be a duality and let* $\mathcal{S}$ *be a family of* $\sigma(F,E)$*-bounded subsets of* F. *Then the point* $\underline{x} \in {}^*E$ *is near-standard in the* $\mathcal{S}$*-topology if and only if there is a point* $x \in E$ *such that for each* $S \in \mathcal{S}$

$$\langle \underline{x},\underline{f}\rangle \simeq \langle x,\underline{f}\rangle \text{ for every } \underline{f} \in {}^*S .$$

PROOF. We first suppose that $\underline{x}$ is near-standard in the $\mathcal{S}$-topology. This implies there is an $x \in E$ such that

$$\underline{x} \in x + {}^*(\lambda S^o) \text{ for each } S \in \mathcal{S} \text{ and scalar } \lambda .$$

Therefore, for an arbitrary $S \in \mathcal{S}$ and scalar λ,

$$\langle \underline{x} - x,\underline{f}\rangle \leq |\lambda| \text{ whenever } \underline{f} \in {}^*S .$$

This establishes the necessity of the condition.

Let us now suppose the condition is true. Then we have that

$$\underline{x} \in x + {}^*S^o \text{ for each } S \in \mathcal{S} .$$

But the set $\{x + S^o : S \in \mathcal{S}\}$ forms a sub-base of neighbourhoods of x

in the *S*-topology. Therefore by our previous comment x belongs to the monad of $\underline{x}$ in the *S*-topology. //

We now note some consequences.

2.1.4 COROLLARY. *Let* X *be a normed vector space. Then* $\underline{x} \in {}^*X$ *is near-standard if and only if there is an* $x \in X$ *such that* $\|\underline{x} - x\| \simeq 0$.

PROOF. If $\|\underline{x} - x\| \simeq 0$ then $\underline{f}(\underline{x} - x) \simeq 0$ whenever $\|\underline{f}\|$ is finite. The result is implied by Theorem 2.1.3.

Otherwise, suppose x is the standard part of $\underline{x}$. Then $\underline{f}(\underline{x} - x) \simeq 0$ whenever $\|\underline{f}\|$ is finite. The result is then a consequence of the Hahn-Banach theorem. //

2.1.5 COROLLARY. *Let* $\langle E,F\rangle$ *be a duality. A point* $\underline{x} \in {}^*E$ *is near-standard in the* $\sigma(E,F)$*-topology if and only if there exists an* $x \in E$ *such that*

$$\langle \underline{x},f\rangle \simeq \langle x,f\rangle \quad \textit{for all} \quad f \in E' .$$

PROOF. The weak topology on E is generated by the polars of finite sets in F . Theorem 2.1.3 yields the result once it is remembered that ${}^*\{f_1,\ldots,f_n\} = \{{}^*f_1,\ldots,{}^*f_n\}$ $(= \{f_1,\ldots,f_n\}$ in our notation). //

2.1.6 COROLLARY. *Let* $\langle E,F\rangle$ *be a separated duality. A point* $\underline{x} \in {}^*E$ *is near-standard in the* $\tau(E,F)$*-topology (i.e., Mackey topology) if and only if there is an* $x \in E$ *such that, for each circled, convex,* $\sigma(F,E)$*-compact subset* S *of* F *,*

$$\langle \underline{x},\underline{f}\rangle \simeq \langle x,\underline{f}\rangle \quad \textit{for every} \quad \underline{f} \in {}^*S .$$

2.1.7 COROLLARY. *Let* $\langle E,F\rangle$ *be a duality. A point* $\underline{x} \in {}^*E$ *is near-standard in the* $\beta(E,F)$*-topology (i.e., strong topology) if and only if there is an* $x \in E$ *such that for each* $\sigma(F,E)$*-bounded subset* S *of* F

$$\langle \underline{x},\underline{f}\rangle \simeq \langle x,\underline{f}\rangle \quad \textit{for every} \quad \underline{f} \in {}^*S .$$

It is often possible to give more detailed characterizations of near-standardness. The following is a result which is sometimes useful. We do not refer to this theorem again however so we only sketch the proof.

2.1.8 THEOREM. *Let* $\{b_n\}$ *be a (Schauder) basis for a Banach space* X . *A point* $\underline{x} = \sum_{i=1}^{\infty} \underline{t}_i b_i$ *in* ${}^*X$ *is near-standard if and only if* $\|\underline{x}\|$ *is finite and* $\|\sum_{i=\omega+1}^{\infty} \underline{t}_i b_i\| \simeq 0$ *for every infinite integer* ω .

PROOF. As monads are invariant under homeomorphisms we can replace the norm $\|\cdot\|$ by an equivalent norm $|\cdot|$ such that $\{b_n\}$ is monotone (see Wilansky [1, p. 207]).

We suppose first that $\underline{x}$ is near-standard; let $x = \sum_{i=1}^{\infty} t_i b_i$ be the standard part of $\underline{x}$. Then $|\sum_{i=1}^{\infty}(t_i - \underline{t}_i)b_i| \simeq 0$ so that $|\sum_{i=n+1}^{\infty}(t_i - \underline{t}_i)b_i| \simeq 0$ for every integer n . Now $|\sum_{i=\omega+1}^{\infty} t_i b_i| \simeq 0$ for each infinite ω so that $|\sum_{i=\omega+1}^{\infty} \underline{t}_i b_i| \simeq 0$ as required. That $|\underline{x}|$ is finite is immediate.

The condition is also sufficient. As $|\underline{x}|$ is finite $\underline{t}_i$ is finite whenever i is finite. We wish to define a point $x \in X$ by

$x = \sum_{i=1}^{\infty} t_i b_i$, letting $t_i = {}^\circ t_i$ for each finite i . Let us show that x is well-defined. If ε is a (standard) positive number there is a finite integer k such that $|\sum_{i=k+1}^{\infty} \underline{t}_i b_i| < \varepsilon$ since $\sum_{i=\omega+1}^{\infty} \underline{t}_i b_i \simeq 0$ whenever ω is infinite and there is a smallest k satisfying the previous inequality. In particular $|\sum_{i=k+1}^{n} \underline{t}_i b_i| < \varepsilon$ for all finite $n > k$. Thus $|\sum_{i=k+1}^{n} t_i b_i| < \varepsilon$ for each (finite) $n > k$. Hence $\{\sum_{i=1}^{n} t_i b_i\}$ is a Cauchy sequence and x is well-defined. Next we show that x is the standard part of $\underline{x}$. As $|\sum_{i=1}^{n}(t_i - \underline{t}_i)b_i| \simeq 0$ for each finite n there is an infinite integer ω such that $|\sum_{i=1}^{\omega}(t_i - \underline{t}_i)b_i| \simeq 0$ (see Theorem 3.3.20, Robinson [2,p. 65]). Now

$$|x - \underline{x}| = |\sum_{i=1}^{\infty} (t_i - \underline{t}_i)b_i|$$

$$\leq |\sum_{i=1}^{\omega} (t_i - \underline{t}_i)b_i| + |\sum_{i=\omega+1}^{\infty} t_i b_i| + |\sum_{i=\omega+1}^{\infty} \underline{t}_i b_i| ,$$

and the result then follows from the second assumption and Corollary 2.1.4. //

The concept of near-standardness may be generalized in the following way.

2.1.9 DEFINITION. Let E be a TVS. A point $\underline{x} \in {}^*E$ is *pre-near-standard* if, for each 0-neighbourhood V in E, there is an $x \in E$ such that $\underline{x} - x \in {}^*V$.

An analogous definition has been introduced by Luxemburg [3, p. 76] for points in the enlargements of uniformities; see also Machover and Hirschfeld [1, p. 54].

Definition 2.1.9 allows us to give the following characterization of completeness; see Luxemburg [3, p. 78] and Machover and Hirschfeld [1, p. 55].

2.1.10 THEOREM. *Let* A *be a subset of a* TVS E . *Then* A *is complete if and only if each pre-near-standard point in* ${}^*A$ *is near-standard.*

PROOF. Suppose that A is complete and that $\underline{x}$ is pre-near-standard. Consider a 0-neighbourhood V in E. Then there exists a circled 0-neighbourhood U such that $U + U \subset V$. If $\underline{x} \in x_U + {}^*U$ then $x_U \in \underline{x} + {}^*U$. This implies that the sentence

$$\exists x\ ((x \in A) \wedge (x_U \in x + U))$$

holds in ${}^*M$. It therefore holds in M and so there is an $x_V \in A$ such that $x_U \in x_V + {}^*U$. Hence $\underline{x} \in x_V + {}^*U + {}^*U$ so that $\underline{x} \in x_V + {}^*V$. Thus for each 0-neighbourhood V in E there is a point $x_V \in A$ such that $\underline{x} \in x_V + {}^*V$. If the neighbourhoods $\{V\}$ are ordered by inclusion $\{x_V\}$ becomes a Cauchy net in A. As A is complete, $\{x_V\}$ has a limit x. It is easily checked that x is the standard part of $\underline{x}$.

We now prove the converse. Suppose that A is not complete. Then there exists a Cauchy net $\{x_\lambda : \lambda \in \Lambda\}$ in A which is not convergent. As Λ is a directed set there is a $\underline{\lambda} \in {}^*\Lambda$ such that $\underline{\lambda} > {}^*\lambda$ for each $\lambda \in \Lambda$. But then $x_{\underline{\lambda}}$ is pre-near-standard and is not near-standard. //

There are two more definitions which we find useful.

2.1.11 DEFINITION. Let E be a TVS. A point $\underline{x} \in {}^*E$ is *bounded* if there is a bounded subset B of E such that $\underline{x} \in {}^*B$.

If X is a normed vector space a point $\underline{x} \in {}^*X$ is bounded if and only if $\|\underline{x}\|$ is finite; i.e., $\underline{x}$ is *finite* in the sense of Robinson [2, p. 118].

Definition 2.1.12 is due to Luxemburg [3, p. 67].

2.1.12 DEFINITION. Let T be a topological space. A point $\underline{x} \in {}^*T$ is *compact* if there exists a compact set C in T such that $\underline{x} \in {}^*C$.

2.2 Helly's theorem.

An important result in the theory of TVSs is that if E is a LCTVS then the strong bidual E" is the union of the $\sigma(E'^{\#},E')$-closures in $E'^{\#}$ of all bounded subsets of E (see Schaefer [1, p. 143]). We find it more convenient to work from a non-standard variant of Helly's theorem.

We first establish our general version of Helly's theorem, which is a generalization of that given in Wilansky [1, p. 103] for normed spaces.

2.2.1 THEOREM. *Let* $\langle E,F\rangle$ *be a duality and let* $\phi \in F^{\#}$. *Suppose* S *is an arbitrary finite dimensional subspace of* F *and let* $\varepsilon > 0$. *Then if* B *is a circled, convex,* $\sigma(E,F)$*-bounded subset of* E *and* ϕ *is bounded on* B° *by unity, there is an* $x \in (1 + \varepsilon)B$ *such that*

$$\phi(f) = \langle x,f\rangle \quad \textit{for every} \quad f \in S .$$

PROOF. As ϕ is linear we have

$$|\phi(f)| \le \sup\{|f(x)| : x \in B\} \text{ for each } f \in B^o .$$

As B is bounded B^o is absorbing and consequently

$$|\phi(f)| \le \sup\{|f(x)| : x \in B\} \text{ for every } f \in F .$$

In particular if $\{f_1,\ldots,f_n\}$ is a basis for S and $a_1,\ldots,a_n$ are arbitrary scalars

$$|\phi(a_1f_1 + \ldots + a_nf_n)| \le \sup\{|(a_1f_1 + \ldots + a_nf_n)x| : x \in B\} .$$

By Helly's condition (Kelley and Namioka [1, p. 151]) this last inequality guarantees the existence of the x of the theorem. //

Before proving our non-standard version of Theorem 2.2.1 we need note an embedding theorem; this result should be compared with the *star-finiteness principle* of Luxemburg [3, p. 27].

2.2.2 THEOREM. *Let* E *be a vector space. Then there exists a "finite dimensional" subspace* $\underline{E}$ *of* $^*E$ *such that* $E \subset \underline{E}$; *i.e., such that* $\{^*x : x \in E\} \subset \underline{E}$.

PROOF. If E is finite dimensional there is no difficulty; we may clearly take $^*E$ for $\underline{E}$.

If E is infinite dimensional we construct a concurrent relation R in the following way: we define the relation $R(x,S)$ to hold between x and S in M if S is a finite dimensional subspace of E and $x \in S$.

It follows then that there exists a "finite dimensional" subspace $\underline{E}$ of ${}^*E$ such that

$$\{{}^*x : x \in E\} \subset \underline{E} .$$

We remark that if $\underline{E}$ is "finite-dimensional" there exists an (infinite) integer ω such that the dimension of $\underline{E}$ is ω . //

2.2.3 THEOREM. *Let* $\langle E,F\rangle$ *be a duality, let* $\phi \in F^{\#}$ *and let* $\underline{\delta}$ *be a positive infinitesimal. Then if* B *is a circled, convex,* $\sigma(E,F)$*-bounded subset of* E *and* ϕ *is bounded by unity on* B° *, there is an* $\underline{x} \in (1 + \underline{\delta}){}^*B$ *such that*

$$\phi(f) = \langle \underline{x},f\rangle \quad \textit{for every} \quad f \in F .$$

PROOF. Let us write Helly's theorem in our higher order language. We will then reinterpret its statement in the enlargement ${}^*M$ for a suitable space S and positive real ε .

We have:

$$\forall S\ (\ S \text{ is a finite dimensional subspace of } F\ ) \wedge \forall \varepsilon\ (\varepsilon > 0)$$
$$\exists x\ ((x \in (1 + \varepsilon)B) \wedge (\forall f\ ((f \in S) \supset (\phi(f) = \langle x,f\rangle)))) .$$

In $^*M$ we take for ε , $\underline{\delta}$, and for S , a subspace $\underline{F}$ of $^*F$ which is "finite dimensional" and which contains F in the sense of Theorem 2.2.2. Then interpreting the statement we have that there exists an $\underline{x} \in (1 + \underline{\delta})^*B$ such that

$$\phi(\underline{f}) = \langle \underline{x}, \underline{f} \rangle \quad \text{whenever} \quad \underline{f} \in \underline{F} .$$

But we have chosen $\underline{F}$ so that $F \subset \underline{F}$. Therefore

$$\phi(f) = \langle \underline{x}, f \rangle \quad \text{for all} \quad f \in F .$$

This establishes the result. //

Let ℓ_1 denote the space of absolutely summable sequences and let m denote its dual space of bounded sequences. There is no simple representation of the dual of m . We do have however as an immediate consequence of Theorem 2.2.3 the following result of Robinson [1, Theorem 4.1].

2.2.4 COROLLARY. *Let* x'' *belong to the dual of* m . *Then there exists a point* $\underline{x} \in {}^*\ell_1$ *such that* $\|\underline{x}\| \simeq \|x''\|$ *and such that*

$$f(\underline{x}) = x''(f) \quad \textit{for every} \quad f \in m .$$

2.3 Compactness arguments.

It is frequent that results in the theory of TVSs depend on one or more of three general methods. They depend primarily on convexity arguments, on compactness arguments and on category results. The principal compactness results which find use are Tychonoff's theorem, the Banach-Alaoglu theorem and to a lesser extent Šmulian's compactness criterion. The purpose of this section is to show how these and similar compactness results may be proved, and often efficiently replaced, by non-standard methods.

The next few results are basic to this section and to the next two chapters.

Theorem 2.3.1 is well-known and is due to Robinson [2, p. 93].

2.3.1 THEOREM. *A topological space* T *is compact if and only if every point of* $^*T$ *is near-standard.*

PROOF. Suppose there is a point $\underline{x}$ in $^*T$ which is not near-standard. Then for each $x \in T$ there is an open neighbourhood U_x of x such that $\underline{x} \notin {}^*U_x$. The family $\{U_x : x \in T\}$ is a covering of T so that, if T is compact, it contains a finite subcover $\{U_1,\ldots,U_n\}$. Thus

$$U_1 \cup U_2 \cup \ldots \cup U_n = T .$$

As this equation can be formulated in our higher order language we may interpret its statement in M. We obtain

$$^*U_1 \cup {}^*U_2 \cup \ldots \cup {}^*U_n = {}^*T .$$

But then $\underline{x} \in {}^*T$ whilst it does not belong to any of the sets ${}^*U_i$, $i = 1,\ldots,n$. This is a contradiction.

The condition is also sufficient. Suppose that T is not compact. Then there exists a covering Ψ of open sets of T such that Ψ contains no finite subcover. We define a concurrent relation $R(U,x)$ to hold between U and x if $U \in \Psi$, $x \in T$ and $x \notin U$. Thus there exists a point $\underline{x} \in {}^*T$ such that $\underline{x} \notin {}^*U$ for every $U \in \Psi$. If x is any point in T, then $x \in V$ for some $V \in \Psi$. But $\underline{x} \notin {}^*V$ so that $\underline{x} \notin \mu(x)$. //

For subsets of TVSs we may obtain the following result.

2.3.2 THEOREM. *Let* E *be a* TVS *and let* A *be a subset of* E. *Then* A *is compact if and only if each point* $\underline{x} \in {}^*A$ *belongs to the monad of a (standard) point in* A.

A more useful result is the following.

2.3.3 THEOREM. *Let* E *be a* TVS *and let* A *be a subset of* E. *Then* A *is relatively compact if and only if each point* $\underline{x} \in {}^*A$ *is near-standard.*

PROOF. As E is a TVS, E is regular (see Wilansky [1, p. 175]).

The result is therefore implied by Theorem 5.5.3 of Machover and Hirschfeld [1, p. 31]. //

We recall that a subset A of a (separated) TVS E is *precompact* (or *totally bounded*) if, for each 0-neighbourhood V in E, there is a finite subset $\{x_1, \ldots, x_n\}$ of E such that $A \subset \bigcup_{i=1}^{n} (x_i + V)$.

The following result has been established more generally by Luxemburg [3, p. 77] for uniformities; see also Machover and Hirschfeld [1, p. 55].

2.3.4 THEOREM. *Let* E *be a* TVS *and let* A *be a subset of* E. *Then* A *is precompact if and only if each point* $\underline{x} \in {}^*A$ *is pre-near-standard.*

PROOF. We prove the necessity of the condition first. Let $\underline{x} \in {}^*A$ and suppose that V is a 0-neighbourhood in E. As A is precompact there exists a finite set $\{x_1, \ldots, x_n\} \subset E$ such that

$$A \subset \bigcup_{i=1}^{n} (x_i + V) .$$

Reinterpreting this statement in ${}^*M$ yields the result that $\underline{x} \in x_i + {}^*V$ for some i.

Let us now suppose that A is not precompact. Then there exists a 0-neighbourhood V in E such that there is no finite subset $\{x_1, \ldots, x_n\}$ in E such that A is contained in $\bigcup_{i=1}^{n} (x_i + V)$. We denote the family $\{x + V : x \in E\}$ by Ψ and define a binary relation $R(U,x)$ to hold in M if $U \in \Psi$, $x \in A$ and if $x \notin U$. By assumption R is concurrent and therefore there exists an $\underline{x} \in {}^*A$ such that $\underline{x} \notin {}^*U$ for each $U \in \Psi$. This implies that $\underline{x}$ is not pre-near-standard. //

Our next characterization is due to Luxemburg [3, Theorem 3.7.1]. If T is a non-compact space the *compact Fréchet filter* is the filter generated by the complements of the compact subsets of T; we denote this filter by Fc.

2.3.5 THEOREM. *A topological space* T *is locally compact if and only if every near-standard point in* ${}^*T$ *is compact.*

PROOF. Let us suppose that T is locally compact. If $\underline{x}$ is near-standard there exists an $x \in T$ such that $\underline{x} \in \mu(x)$. If V is a compact neighbourhood of x then $\underline{x} \in {}^*V$ so that $\underline{x}$ is compact.

Otherwise suppose each near-standard point is compact. We may assume that T is not compact. Then, for each standard point x, $\mu(x) \cap \mu(Fc) = \emptyset$. Hence for every $x \in T$ there exists a neighbourhood V_x and a set $F \in Fc$ such that $V_x \cap F = \emptyset$. This implies that $X - F$ is a compact neighbourhood of x. //

We now give a short non-standard proof of Alexander's sub-base theorem (see Kelley [1, p. 139]).

2.3.6 THEOREM. *Suppose that* S *is a sub-base for a topological space* T *and suppose that every cover of* T *by members of* S *has a finite subcover. Then* T *is compact.*

PROOF. Suppose that T is not compact. Theorem 2.3.1 implies there exists a point $\underline{x} \in {}^*T$ which is not near-standard. Then for every $x \in T$ there is an $S_x \in S$ such that $x \in S_x$ but such that $\underline{x} \notin {}^*S_x$. But $\cup\{S_x : x \in T\}$ covers T and so, by assumption, there is a finite subcover $\{S_{x_1}, \ldots, S_{x_n}\}$ of X. This implies that $\underline{x} \in {}^*S_{x_i}$ for some i. This is a contradiction and so T must be compact. //

A non-standard proof of Tychonoff's theorem is to be found in Robinson [2, p. 95].

2.3.7 THEOREM. *The topological product of a family of compact topological spaces is compact.*

PROOF. Let $T = X\{T_\alpha : \alpha \in A\}$ where each T_α is a compact topological space and where T has the product topology. We suppose that $\underline{x} \in {}^*T$. By Theorem 2.3.1 it is sufficient to show that $\underline{x}$ is near-standard. Now $\underline{x}(\alpha) \in {}^*T_\alpha$ for each $\alpha \in A$ and so, since each T_α is compact, $\underline{x}(\alpha)$ is near-standard for each $\alpha \in A$. Hence, by using the axiom of choice, we determine a point $x = (x(\alpha))$ in T such that $\underline{x}(\alpha)$ belongs to the monad of $x(\alpha)$ for all α. It is not difficult to check that this implies that $\underline{x}$ belongs to the monad of x (Robinson [2, Theorem 4.1.17]). //

The Banach-Alaoglu theorem is a fundamental result in the theory of TVSs. Here we prove a more general result which does not seem to be stated explicitly in the literature. We will use Theorem 2.3.8 in Chapter 5.

We suppose that E is a TVS and we denote by E^α the space of homogeneous functionals on E which are continuous at zero. Hence $f \in E^\alpha$ if f is a functional on E, $f(ax) = af(x)$ for all scalars a, and if f is bounded on a 0-neighbourhood in E (see Wilansky [1, p. 186]). If S is a subset of E we extend the notion of polar by defining the general polar S°_α to be the subset $\{f \in E^\alpha : |f(x)| \leq 1 \text{ for all } x \in S\}$.

2.3.8 THEOREM. *Let* E *be a* TVS. *Then for any* 0-*neighbourhood* V *in* E, V°_α *is* $\sigma(E^\alpha,E)$-*compact.*

PROOF. Suppose that $\underline{f} \in {}^*(V^\circ_\alpha)$. By Theorem 2.3.2 it is sufficient to show that $\underline{f}$ belongs to the monad of some $f \in V^\circ_\alpha$, when E^α is equipped with the $\sigma(E^\alpha,E)$-topology.

We define a functional f on E by

$$f(x) = {}^\circ[\underline{f}(x)] \quad \text{for each } x \in E .$$

Then f is well-defined. Furthermore f is homogeneous and is continuous at zero; indeed $f \in V^{\circ}_{\alpha}$. A slight extension to Corollary 2.1.5 implies that f is the standard part of $\underline{f}$ in the $\sigma(E^{\alpha},E)$-topology. //

2.3.9 COROLLARY. *Let* E *be a* TVS. *Then for any* 0*-neighbourhood* V *in* E , V° *is* $\sigma(E',E)$*-compact.*

PROOF. It is easily checked that V° is $\sigma(E^{\alpha},E)$-closed as a subset of E^{α} . Consequently V° is $\sigma(E^{\alpha},E)$-compact and is therefore $\sigma(E',E)$-compact. //

We now establish Šmulian's criterion for weak compactness. Kelley and Namioka [1, p. 142] provide a standard proof of this result.

2.3.10 THEOREM. *Let* $\langle E,F \rangle$ *be a duality and let* B *be a* $\sigma(E,F)$*-closed, circled convex subset of* E . *Then* B *is* $\sigma(E,F)$*-compact if and only if* B° *is absorbing and each linear functional on* F *which is bounded on* B° *is represented by some member of* E .

PROOF. We show the necessity of the condition first. If B is $\sigma(E,F)$-compact then B is $\sigma(E,F)$-bounded so that B° is absorbing.

Now suppose that ϕ is a linear functional on F which is bounded on B° . Without loss of generality we may assume that $|\phi(B^{\circ})| < 1$.

Then, by Theorem 2.2.3, there exists an $\underline{x} \in {}^*B$ such that

$$\phi(f) \simeq \langle \underline{x}, f \rangle \quad \text{for every} \quad f \in F \ .$$

As $\underline{x} \in {}^*B$, $\underline{x}$ is near-standard in the $\sigma(E,F)$-topology. By Theorem 2.1.3 there exists an $x \in B$ such that

$$\langle \underline{x}, f \rangle \simeq \langle x, f \rangle \quad \text{for every} \quad f \in F \ .$$

It follows that

$$\phi(f) = \langle x, f \rangle \quad \text{for every} \quad f \in F \ ,$$

thus establishing the necessity.

On the other hand the condition is sufficient, for suppose that $\underline{x} \in {}^*B$. Then we may define a linear functional ϕ on F by

$$\phi(f) = {}^\circ\langle \underline{x}, f \rangle \quad \text{for each} \quad f \in F \ .$$

As B° is absorbing B is $\sigma(E,F)$-bounded so that ϕ is well-defined. Furthermore ϕ is bounded by unity on B° . Therefore, by assumption, there is an $x \in E$ such that

$$\phi(f) = \langle x, f \rangle \quad \text{for every} \quad f \in F \ .$$

But then

$$\langle x, f \rangle \simeq \langle x, f \rangle \quad \text{whenever} \quad f \in F \ ,$$

so that $\underline{x}$ is near-standard in the $\sigma(E,F)$-topology. Hence B is

$\sigma(E,F)$-compact by Theorem 2.3.3. //

2.4 Non-standard proofs of some standard theorems.

The theorems we prove here are chosen generally because they have some relation to the following chapters. We refer the reader in particular to the relevant parts of Robinson [2] and Machover and Hirschfeld [1] for many more applications.

We begin by outlining a proof of the Hahn-Banach theorem. A non-standard proof of this result appeared in Luxemburg [1]; see also Luxemburg [4] for a more detailed and interesting discussion of this and related results.

2.4.1 THEOREM. *Let* E *be a real vector space, let* p *be a sub-linear functional on* E*, and let* M *be a linear subspace of* E*. If* f *is a linear functional on* M *such that* $f(x) \leq p(x)$ *for all* $x \in M$*, there exists a linear functional* f_1 *on* E *such that* $f_1(x) = f(x)$ *for all* $x \in M$ *and* $f_1(x) \leq p(x)$ *for all* $x \in E$.

PROOF. The first step in Banach's proof shows that if N is a linear subspace in E containing M with $\dim N/M < \infty$, f can be extended to a linear functional g on N satisfying the condition $g(x) \leq p(x)$ for all $x \in N$. We start with this assumption. Suppose then that $\underline{E}$ is a "finite dimensional" subspace containing E as in Theorem 2.2.2.

Then there exists a linear functional $\underline{g}$ defined on $\underline{E}$ which satisfies the conditions $\underline{g}(\underline{x}) = f(\underline{x})$ for every $\underline{x} \in {}^*M$ and $\underline{g}(\underline{x}) \leq p(\underline{x})$ for every $\underline{x} \in \underline{E}$. We define f_1 by

$$f_1(x) = {}^\circ[\underline{g}(x)] \text{ for every } x \in E .$$

It is easily checked that f_1 is a suitable functional. //

2.4.2 THEOREM. *Let* A *be a subset of a* TVS E . *Then* A *is compact if and only if* A *is precompact and complete* (see Luxemburg [3, p. 79]).

PROOF. The result is an immediate consequence of Theorem 2.1.10, Theorem 2.3.3 and Theorem 2.3.4. //

We frequently use the technique involved in the next proof.

2.4.3 THEOREM. *The (weakly) bounded sets in a* LCTVS E *are weakly precompact.*

PROOF. Let B be a bounded set in E and let $\underline{x} \in {}^*B$ (so that $\underline{x}$ is a bounded point in E). We then define $x'' \in E''$ by

$$x''(f) = {}^\circ[f(\underline{x})] \text{ for each } f \in E' .$$

That $x'' \in E''$ follows as x'' is bounded on the 0-neighbourhood B° in E' . Let S be a finite set in E' . The statement

$$\exists x\,(\forall f\,(f \in S) \supset (|x''(f) - f(x)| < 1))$$

holds in $^*M$ (for $\underline{x}$ satisfies the condition). Thus it also holds in M and so there is an $x_S \in E$ such that

$$|x''(f) - f(x_S)| < 1 \quad \text{for all} \quad f \in S\,.$$

But then $\underline{x} \in x_S + {}^*S^{\circ}$ so that $\underline{x}$ is pre-near-standard. The result follows by Theorem 2.3.4. //

Next we prove the converse of the Banach-Alaoglu theorem in a Banach space setting. The result is due to Dixmier [1, p. 1069]; we comment that Dixmier uses "compact" in the sense of Bourbaki for separated spaces only.

2.4.4 THEOREM. *Let* X *be a Banach space. Suppose that* V *is a closed, total subspace of* X' *and that the unit ball of* X *is relatively* $\sigma(X,V)$*-compact. Then* X *is isomorphic to* V' .

PROOF. We denote the unit ball by B . Let π be the natural map from X into V' . Suppose that $\phi \in V'$ and that $\|\phi\| < 1$. By the Hahn-Banach theorem we may suppose that $\phi \in X''$. Therefore, by Theorem 2.2.3, there exists an $\underline{x} \in {}^*B$ satisfying

$$\phi(f) = f(\underline{x}) \quad \text{for all} \quad f \in X'\,.$$

As B is relatively $\sigma(X,V)$-compact Theorems 2.3.1 and 2.1.3 combine to imply the existence of an $x \in X$ such that

$$f(\underline{x}) \simeq f(x) \quad \text{for all} \quad f \in V .$$

Therefore,

$$\phi(f) = f(x) \quad \text{for all} \quad f \in V$$

and consequently π is onto. As V is total π is one-to-one. That π is an isomorphism then follows from the interior mapping principle (Dunford and Schwartz [1, p. 57]) as π is continuous. This establishes the result. //

CHAPTER 3

GENERALIZATIONS OF WEAK COMPACTNESS

3.0 Introduction.

The notion of weak compactness plays a central role in the theory of LCTVSs. However in the statement of many theorems, completeness of the space, or at least quasi-completeness of the space in the Mackey topology, is an important assumption.

In this chapter we extend the concept of weak compactness in a general manner and obtain a number of interesting particular cases. If we replace weak compactness by one of these generalizations we can drop the completeness assumption from the statement of many theorems. Using non-standard techniques we are able to prove a generalized version of Eberlein's classical theorem. We then consider generalizations of semi-reflexivity and reflexivity and characterize these properties in terms of our new notions as well as in terms of known concepts.

3.1 Notation and Definitions.

Suppose E is a (separated) LCTVS with topological dual E' and that S is a family of $\sigma(E',E)$-bounded subsets of E' which cover E'. Let

E_S denote E equipped with the S-topology and let $S_1 = \{S\}$ be a family of subsets of $(E_S)'$.

Corresponding to a map ϕ from E into the set of finite subsets of $S \in S_1$ we define a $\sigma(E,(E_S)')$-neighbourhood of each point $x \in E$ by

$$U_x(\phi,S) = \{y : |f(y - x)| \leq 1 \text{ for all } f \in \phi(x)\} .$$

The system of $\sigma(E,(E_S)')$-neighbourhoods $\{U_x(\phi,S) : x \in E\}$ forms a covering of E, which we call the (ϕ,S)-*covering of* E.

Using this notation we introduce the following definition.

3.1.1 DEFINITION. Let A be a subset of a LCTVS E and let S and S_1 be two families of sets as described. Then we say A is $S_1 - \sigma(E,(E_S)')$-*compact* if, for each $S \in S_1$ and each map ϕ previously described, the (ϕ,S)-covering of E contains a finite subcover of A; i.e., there exists a finite subset $\{x_1,\ldots,x_n\}$ of E such that

$$A \subset U_{x_1}(\phi,S) \cup \ldots \cup U_{x_n}(\phi,S) .$$

We will be mostly interested in $S_1 - \sigma(E,(E_S)')$-compactness when S_1 generates a topology on E, specifically the S-topology. If the S-topology is consistent with duality and the S_1-topology is the Mackey topology $\tau(E,E')$, we introduce another definition.

3.1.2 DEFINITION. Suppose $\mathcal{S}_1$ is the family $\mathcal{K}$ of all circled, convex, $\sigma(E',E)$-compact subsets of E'. If a subset A of E is $\mathcal{K}$ - $\sigma(E,E')$-compact we say that A is *nearly* $\sigma(E,E')$-*compact* (or *nearly weakly compact*).

Although our proofs need only be altered slightly in the complex case we restrict our attention to real spaces. As in Chapter 2 we find it convenient to use the term "polar" in the sense of "absolute polar". We emphasise that, if E is a LCTVS, $E''_{\mathcal{S}}$ denotes the bidual E'' equipped with the $\mathcal{S}$-topology. Most of the standard texts on TVSs (for example, Schaefer [1], Horvath [1] or Köthe [1]) are suitable references for this chapter.

3.2 $\tilde{\mathcal{S}}$ - $\sigma(E,(E_{\mathcal{S}})')$-compactness.

Let E be a LCTVS. It is an immediate consequence of Corollary 2.1.5 that a point $\underline{x} \in {}^*E$ is weak near-standard (i.e., near-standard in the weak topology) if and only if there is an $x \in E$ such that $f(\underline{x}) \simeq f(x)$ for all $f \in E'$. We generalize this property of points of ${}^*E$ in the following way.

3.2.1 DEFINITION. Let E be a LCTVS and let $\mathcal{S}$ and $\mathcal{S}_1$ be defined as in 3.1. We say that the point $\underline{x} \in {}^*E$ is $\mathcal{S}_1$ - $\sigma(E,(E_{\mathcal{S}})')$-*near-standard* if, for each $S \in \mathcal{S}_1$, there is an $x \in E$ such that

$$|f(\underline{x} - x)| \leq 1 \text{ for all } f \in S .$$

This allows us to generalize Theorem 2.3.2.

3.2.2 THEOREM. *Let* E *be a* LCTVS. *A subset* A *of* E *is* $S_1 - \sigma(E,(E_S)')$*-compact if and only if each point* $\underline{x} \in {}^*A$ *is* $S_1 - \sigma(E,(E_S)')$*-near-standard.*

PROOF. Suppose there exists an $\underline{x} \in {}^*E$ which is not $S_1 - \sigma(E,(E_S)')$-near-standard. Then there is an $S \in S_1$ so that given $x \in E$ there is $f \in S$ such that $|f(\underline{x} - x)| > 1$. That is to say, there is a map ϕ such that $\underline{x} \notin {}^*U_x(\phi,S)$ for each $x \in E$. Now A is $S_1 - \sigma(E,(E_S)')$-compact, and so there is a finite subset $\{x_1,\ldots,x_n\}$ of E such that

$$A \subset U_{x_1}(\phi,S) \cup \ldots \cup U_{x_n}(\phi,S) .$$

This equation can be formulated as a sentence of K, which interpreted in *M, yields

$${}^*A \subset {}^*U_{x_1}(\phi,S) \cup \ldots \cup {}^*U_{x_n}(\phi,S) .$$

We know that $\underline{x}$ does not belong to any of the sets on the right hand side and consequently it does not belong to *A .

Now on the other hand suppose A is not $S_1 - \sigma(E,(E_S)')$-compact. Then there exists an $S \in S_1$ together with a map ϕ such that the (ϕ,S)-covering Ψ of E has no finite subcover of A . We define a binary relation $R(U,y)$ to hold in M if and only if $U \in \Psi$ and $y \in A$ but $y \notin U$. By assumption $R(U,y)$ is concurrent, so that by definition of $^*M$, there is a point $\underline{x} \in {}^*A$ such that $\underline{x} \notin {}^*U_x(\phi,S)$ for all $x \in E$. This then means that $\underline{x}$ is not $\mathcal{S}_1 - \sigma(E,(E_{\mathcal{S}})')$-near-standard (for remember that $^*\{f_1,\ldots,f_n\} = \{^*f_1,\ldots,{}^*f_n\}$). //

3.2.3 REMARK. It is interesting to note that an analysis of the previous proof reveals that when defining $\mathcal{S}_1 - \sigma(E,(E_{\mathcal{S}})')$-compact sets it suffices to consider only those (ϕ,S)-coverings of E for which ϕ maps E into singletons of S.

As a consequence of Theorem 3.2.2 we note the following.

3.2.4 COROLLARY. *Suppose* $\mathcal{S}$ *and* $\mathcal{S}_1$ *both equal the family of finite subsets* $\mathcal{F}$ *of* E. *Then a subset* A *of* E *is* $\mathcal{F} - \sigma(E,E')$*-compact if and only if* A *is weakly precompact.*

PROOF. Suppose A is $\mathcal{F} - \sigma(E,E')$-compact. Then given an $\underline{x} \in {}^*A$ and $S \in \mathcal{F}$ there exists an $x \in E$ such that

$$|f(\underline{x} - x)| \leq 1 \quad \text{for all} \quad f \in S .$$

But $^*S = {}^*\{f_1,\ldots,f_n\} = \{^*f_1,\ldots,{}^*f_n\}$ so that

$$|\underline{f}(\underline{x} - x)| \leq 1 \quad \text{for all} \quad \underline{f} \in {}^*S .$$

This means that $\underline{x}$ is pre-near-standard in the weak topology and hence, by Theorem 2.3.4, that A is weakly precompact. The converse is similarly established. //

We find it convenient to denote the family of equicontinuous subsets of $(E_S)'$ by $\tilde{S}$. For the remainder of this section we intend to examine properties of $\tilde{S} - \sigma(E,(E_S)')$-compactness.

Before continuing it is important to know to what extent this notion is independent of our choice of $\tilde{S}$. Theorem 3.2.5 clarifies this situation. We let S_1 be another family of subsets which cover $(E_S)'$ such that the polars of its sets form a basis of 0-neighbourhoods in E_S. (This requires that S_1 satisfies the two conditions:

(S_I) <u>If $S_1, S_2 \in S_1$, there is an $S_3 \in S_1$ such that $S_1 \cup S_2 \subset S_3$,</u>

(S_{II}) <u>If λ is a real number and $S \in S_1$ there is an $S_1 \in S_1$ such that $\lambda S \subset S_1$</u> .)

3.2.5 THEOREM. *Let* A *be a subset of a* LCTVS E . *Then* A *is* $S_1 - \sigma(E,(E_S)')$-*compact if and only if* A *is* $\tilde{S} - \sigma(E,(E_S)')$-*compact.*

PROOF. By Theorem 3.2.2 it suffices to show that a point $\underline{x} \in {}^*A$ is $S_1 - \sigma(E,(E_S)')$-near-standard if and only if $\underline{x}$ is $\tilde{S} - \sigma(E,(E_S)')$-

near-standard. We suppose that $\underline{x}$ is $\mathcal{S}_1 - \sigma(E,(E_{\mathcal{S}})')$-near-standard. For any $S \in \mathcal{S}_1$ there exists an x_S such that

$$|f(\underline{x} - x_S)| \leq 1 \text{ for all } f \in S .$$

It follows that $\{x_S\}$ is a Cauchy net in $E_{\mathcal{S}}$ if the sets $\{S\}$ are ordered by containment. Therefore given a set $\tilde{S} \in \tilde{\mathcal{S}}$ there exists an $S_0 \in \mathcal{S}_1$ such that whenever $S_1, S_2 \supset S_0$

$$|f(x_{S_1} - x_{S_2})| \leq 1/2 \text{ for all } f \in \tilde{S} .$$

Consider an arbitrary functional $g \in \tilde{S}$. As $\mathcal{S}_1$ covers $(E_{\mathcal{S}})'$ and satisfies condition $(\mathcal{S}_{II})$ there exists an $S_3 \supset S_0$ such that

$$|g(\underline{x} - x_{S_4})| \leq 1/2 \text{ whenever } S_4 \supset S_3 .$$

Consequently,

$$|g(\underline{x} - x_{S_0})| \leq |g(\underline{x} - x_{S_4})| + |g(x_{S_4} - x_{S_0})| \leq 1 .$$

But S_0 was chosen independently of g and so it follows that $\underline{x}$ is $\tilde{\mathcal{S}} - \sigma(E,(E_{\mathcal{S}})')$-near-standard. The converse is immediate since $\mathcal{S}_1 \subset \tilde{\mathcal{S}}$. //

We denote the natural embeddings of x and A in the bidual by $\hat{x}$ and $\hat{A}$; <u>and in this and the next section we use H as an abbreviation for $E_{\mathcal{S}}$</u>.

3.2.6 LEMMA. *Suppose* $\underline{x}$ *is a bounded point in* $^*H$. *Then* $\underline{x}$ *is* $\tilde{S} - \sigma(E,(E_S)')$*-near-standard if and only if* $\hat{\underline{x}}$ *is weak near-standard in* $^*H''_{\tilde{S}}$.

PROOF. We prove the necessity of the condition first. As in Theorem 2.4.3 we define $x'' \in H''$ by

$$x''(f) = {}^{\circ}[f(\underline{x})] \text{ for all } f \in H' . \quad (1)$$

Let $x''' \in (H''_{\tilde{S}})'$. The restriction of x''' to $\hat{H}$ may be assumed to be an element of H' which we denote by g . If $\underline{x}$ is $\tilde{S} - \sigma(E,(E_S)')$-near-standard, given an equicontinuous subset S of H' there is a point $x_S \in E$ such that

$$|f(\underline{x} - x_S)| \leq 1 \text{ for all } f \in S .$$

Therefore (1) implies that $x'' - \hat{x}_S \in S^{\circ}$. Consequently, if the sets $\{S\}$ are ordered by containment, $\{\hat{x}_S\}$ is a net convergent to x'' in the $\tilde{S}$-topology. In particular, $g(x_S) \to x''(g)$ and $x'''(\hat{x}_S) \to x'''(x'')$ so that $x'''(x'') = x''(g)$. Therefore,

$$x'''(x'') = x''(g) \simeq g(\underline{x}) = x'''(\hat{\underline{x}})$$

and $\hat{\underline{x}}$ is weak near-standard as a consequence of Corollary 2.1.5.

Now let us suppose that $\hat{\underline{x}}$ is weak near-standard. Then there exists an $x'' \in H''$ such that

$$x'''(x'') \simeq x'''(\hat{\underline{x}}) \quad \text{for all} \quad x''' \in (H''_{\tilde{S}})' .$$

It follows (see Robinson [2, p. 91]) that x'' belongs to the weak closure of $\hat{H}$ in $H''_{\tilde{S}}$, and thus to the closure of $\hat{H}$ in $H''_{\tilde{S}}$ (Schaefer [1, p. 130]). It is then an easy consequence that $\underline{x}$ is $\tilde{S} - \sigma(E,(E_S)')$-near-standard. //

Of course if the S-topology on E is compatible with the duality $\langle E,E'\rangle$, Lemma 3.2.6 simplifies to the extent that H and $H''_{\tilde{S}}$ may be replaced by E and $E''_{\tilde{S}}$ respectively. Indeed a similar substitution can be made if S is a family of strongly bounded sets (so that the S-topology on E'' is a linear topology) as we now show.

3.2.7 LEMMA. *Suppose* S *is a family of strongly bounded subsets of* E' *and that* $\underline{x}$ *is a bounded point in* ${}^*E_S$. *Then* $\underline{x}$ *is* $\tilde{S} - \sigma(E,(E_S)')$*-near-standard if and only if* $\hat{\underline{x}}$ *is weak near-standard in* ${}^*E''_S$.

PROOF. The proof is similar to that of Lemma 3.2.6. We suppose that $\underline{x}$ is $\tilde{S} - \sigma(E,(E_S)')$-near-standard and define $x'' \in H''$ as in the previous lemma; we let $y'' \in E''$ be the restriction of x'' to H' . If $y''' \in (E''_S)'$ the restriction g of y''' to $\hat{E}$ may be assumed to be an element of H' . Then we find, extending the method of Lemma 3.2.6 slightly, that

$$y'''(y'') = x''(g) \simeq g(\underline{x}) = y'''(\hat{\underline{x}}) .$$

Corollary 2.1.5 implies that $\hat{\underline{x}}$ is weak near-standard in $*E''_S$.

We now prove the converse. Suppose that there exists a $y'' \in E''$ such that

$$y'''(y'') \simeq y'''(\hat{\underline{x}}) \quad \text{for all} \quad y''' \in (E''_S)' .$$

Then y'' belongs to the $\sigma(E'',(E''_S)')$-closure of $\hat{E}$ in E'' and thus to the $\tau(E'',(E''_S)')$-closure of $\hat{E}$ in E'' . We may assume that $\mathcal{S}$ satisfies the general conditions $\mathcal{S}_I$ and $\mathcal{S}_{II}$ (see Schaefer [1, p. 81]). Then if $\tilde{S} \in \tilde{\mathcal{S}}$ there is a set $S \in \mathcal{S}$ such that each $f \in \tilde{S}$ is bounded by unity on the polar S^o of S in E . The polar S^o of S in E'' is a 0-neighbourhood in E''_S and so, by the Banach-Alaoglu theorem, S^{oo} is $\sigma((E''_S)',E'')$-compact in $(E''_S)'$. Therefore as y'' belongs to the $\tau(E'',(E''_S)')$-closure of $\hat{E}$ there is an $x \in E$ such that

$$|y'''(y'') - y'''(\hat{x})| < 1 \quad \text{for all} \quad y''' \in S^{oo} ,$$

which implies

$$|y'''(\hat{\underline{x}}) - y'''(\hat{x})| \le 1 \quad \text{for all} \quad y''' \in S^{oo} .$$

It is an easy consequence of the Hahn-Banach theorem that each $f \in \tilde{S}$ can be extended to a functional $y''' \in S^{oo}$ (if p is the gauge of S^o , then $|f(x)| \le p(x)$ for all $x \in E$). Thus

$$|f(\underline{x} - x)| \le 1 \quad \text{for all} \quad f \in \tilde{S}$$

which implies $\underline{x}$ is $\tilde{S}$ - $\sigma(E,(E_S)')$-near-standard. //

As consequences of the previous two lemmas we obtain the following results. Since the proofs are very similar we only give that of the first theorem.

3.2.8 THEOREM. *Let* A *be a subset of a* LCTVS E . *Then* A *is* $\tilde{S}$ - $\sigma(E,(E_S)')$*-compact if and only if* $\hat{A}$ *is relatively weakly compact as a subset of* $H''_{\tilde{S}}$.

PROOF. Suppose that A is $\tilde{S}$ - $\sigma(E,(E_S)')$-compact. As $\tilde{S}$ covers H' it follows that A is $\sigma(H,H')$-bounded and hence bounded (Schaefer [1, p. 132]). By Theorem 3.2.2 each point $\underline{x} \in {}^*A$ is $\tilde{S}$ - $\sigma(E,(E_S)')$-near-standard and so, by Lemma 3.2.6, each $\hat{\underline{x}}$ is weak near-standard in $H''_{\tilde{S}}$. Theorem 2.3.3 then establishes the necessity of the condition.

Conversely suppose that $\hat{A}$ has the stated property. Theorem 2.3.3 implies that each point $\hat{\underline{x}} \in {}^*\hat{A}$ is weak near-standard in $H''_{\tilde{S}}$. It follows in turn that $\underline{x}$ is $\tilde{S}$ - $\sigma(E,(E_S)')$-near-standard and thus A is $\tilde{S}$ - $\sigma(E,(E_S)')$-compact by Theorem 3.2.2. //

3.2.9 THEOREM. *Suppose* E *is a* LCTVS *and that* S *is a family of strongly bounded subsets of* E' . *Then* A *is* $\tilde{S}$ - $\sigma(E,(E_S)')$*-compact if and only if* $\hat{A}$ *is relatively weakly compact in* E''_S .

3.2.10 LEMMA. *Let* A *be a bounded subset of* E_S . *A point* $\underline{x} \in {}^*A$ *is* $\tilde{S}$ - $\sigma(E,(E_S)')$*-near-standard if and only if for every* $S \in \tilde{S}$ *there exists an* x *belonging to the convex hull of* A *, such that*

$$|f(\underline{x} - x)| \le 1 \text{ for all } f \in S .$$

PROOF. It is immediate that the condition is sufficient.

Therefore we suppose that $\underline{x}$ is $\tilde{S}$ - $\sigma(E,(E_S)')$-near-standard. Then by Lemma 3.2.6 there exists an $x'' \in F''$ such that

$$x'''(\hat{\underline{x}}) \simeq x'''(x'') \text{ for all } x''' \in (H''_{\tilde{S}})' .$$

This implies that x'' belongs to the weak closure of $\hat{A}$ in $H''_{\tilde{S}}$ and thus to the closure of its convex hull (Schaefer [1, p. 130]). Hence given $S \in \tilde{S}$ there is a point x belonging to the convex hull of A such that

$$|f(x) - x''(f)| < 1 \text{ for all } f \in S .$$

Therefore

$$|f(\underline{x} - x)| \le 1 \text{ for all } f \in S . \quad //$$

3.2.11 THEOREM. *Let* A *be a subset of a* LCTVS E . *If* A *is* $\tilde{S}$ - $\sigma(E,(E_S)')$*-compact and the closed convex hull of* A *in* E_S *is complete, then* A *is relatively* $\sigma(E,(E_S)')$*-compact.*

PROOF. By Lemma 3.2.10 given an $\underline{x} \in {}^*A$ and an $S \in \tilde{S}$ there is a point x_S belonging to the convex hull of A such that

$$|f(x_S - \underline{x})| \leq 1 \quad \text{for all} \quad f \in S .$$

Thus $\{x_S\}$ is a Cauchy net in the convex hull of A (where the sets $\{S\}$ are ordered by containment). By the completeness assumption $\{x_S\}$ has a limit x which is the standard part of $\underline{x}$ in the $\sigma(E,(E_S)')$-topology. By Theorem 2.3.3 A is relatively $\sigma(E,(E_S)')$-compact. //

3.2.12 REMARK. It is a simple consequence of Lemma 3.2.10 and the proof of Theorem 3.2.2 that, when defining $\tilde{S}$ - $\sigma(E,(E_S)')$-compactness, we may require that the finite subset $\{x_1,\ldots,x_n\}$ in Definition 3.1.1, be chosen in the convex hull of A .

3.3 Eberlein's theorem.

The main purpose of this section is to give a non-standard proof of Theorem 3.3.2. From this result we derive Eberlein's theorem. As in Section 3.2 S denotes a family of $\sigma(E',E)$-bounded subsets of E' unless the contrary is stated; we use H as before to denote E_S .

3.3.1 LEMMA. *Let* E *be a* LCTVS *and suppose that* $\underline{x}$ *is a bounded point in* ${}^*E_S$. *Then* $\underline{x}$ *is* $\tilde{S}$ - $\sigma(E,(E_S)')$*-near-standard if and only if, for each* $S \in \tilde{S}$ *, there is a finite subset* C_S *of* E *such that for* *each* $f \in S$, *there is an* $x \in C_S$ *satisfying* $f(x - \underline{x}) \leq 1$.

PROOF. The necessity of the condition is immediate.

It therefore only remains to prove the sufficiency of the condition. As $\underline{x}$ is bounded we define, as in Lemma 3.2.6, $x'' \in H''$ by

$$x''(f) = {}^{\circ}[f(\underline{x})] \text{ for all } f \in H' .$$

We let C equal $\cup\{C_S : S \in \mathcal{S}\}$ and let D equal the closed convex hull of $\hat{C}$ in $H''_{\tilde{S}}$. We claim that $x'' \in D$ and establish this claim by contradiction. If $x'' \notin D$ the separation theorem (Schaefer [1, p. 65]) implies there is a continuous functional $x''' \in (H''_{\tilde{S}})'$ and real number c such that

$$x'''(D) < c - 2 < c < x'''(x'') .$$

As $x''' \in (H''_{\tilde{S}})'$ x''' is bounded on a 0-neighbourhood of $H''_{\tilde{S}}$ (Schaefer [1, p. 74]). Hence we may assume that x''' is bounded by unity on a polar S° in H'' of a convex, circled, equicontinuous set $S \in \tilde{S}$. Now S is strongly bounded and so, by Theorem 2.2.1, for an arbitrarily small $\varepsilon > 0$, there is an $f \in (1 + \varepsilon)S$ such that $x''' = \hat{f}$ on $\hat{C}_S = \{\hat{x}_1, \ldots, \hat{x}_n\}$ and x'' . Then we have

$$f(x_i) + 2 < x''(f) \text{ for } i = 1, \ldots, n ,$$

so that

$$f(\underline{x}) - f(x_i) > 2 \text{ for } i = 1, \ldots, n .$$

This is a contradiction and so $x'' \in D$. But then $\underline{x}$ is $\tilde{S}$ - $\sigma(E,(E_S)')$-near-standard. For suppose $S \in \tilde{S}$, then there is an x belonging to the convex hull of C such that

$$|x''(f) - f(x)| < 1 \quad \text{for all} \quad f \in S .$$

But then

$$|f(\underline{x} - x)| \leq 1 \quad \text{for all} \quad f \in S . \quad //$$

3.3.2 THEOREM. *Let* E *be a* LCTVS *and let* A *be a subset of* E . *If* $\hat{A}$ *is relatively countably weakly compact in* $H''_{\tilde{S}}$ *then the* $\sigma(E,(E_S)')$-*closure of* A *is* $\tilde{S}$ - $\sigma(E,(E_S)')$-*compact.*

PROOF. Let $\bar{A}$ denote the closure of A in the $\sigma(E,(E_S)')$-topology. Then $\bar{A}$ is bounded in E_S so that, if $\underline{x} \in {}^*\bar{A}$, $\underline{x}$ is a bounded point in ${}^*E_S$. Let us suppose $\bar{A}$ is not $\tilde{S}$ - $\sigma(E,(E_S)')$-compact. Then, by Theorem 3.2.2 and Lemma 3.3.1, there is a set $S \in \tilde{S}$ such that for each finite set C of E there is an $f \in S$ such that for all $x \in C$, $f(\underline{x} - x) > 1$.

We construct three sequences $\{x_n\} \subset \bar{A}$, $\{y_n\} \subset A$ and $\{f_n\} \subset S$ in the following manner. We choose $x_0 = y_0$ arbitrarily in A , then $f_0 \in S$ such that $f_0(y_0 - \underline{x}) > 1$. Now the statement

$$\exists x \, (x \in \bar{A} \wedge (f_0(y_0 - x) > 1))$$

holds in $^*M$ (for $\underline{x}$ satisfies both conditions), and so it is true in M. Hence there is an $x_1 \in \bar{A}$ such that $f_0(y_0 - x_1) > 1$. As $x_1 \in \bar{A}$ there exists $y_1 \in A$ such that $|f_0(x_1 - y_1)| < 1/2$. Suppose now that we have chosen y_k, x_k for $k = 0,1,\ldots,n-1$, and f_j for $j = 0,1,\ldots,n-2$ satisfying

$$f_j(y_i - \underline{x}) > 1 , \quad i = 0,1,\ldots,j ,$$

$$f_j(y_i - x_k) > 1 , \quad 0 \le i \le j < k \le n-1 ,$$

$$|f_j(y_i - x_i)| < 1/2 , \quad j = 0,1,\ldots,i-1 .$$

Then we choose $f_{n-1} \in S$ such that

$$f_{n-1}(y_j - \underline{x}) > 1 , \quad j = 0,1,\ldots,n-1 .$$

The abbreviated statement

$$\exists x\, ((x \in \bar{A}) \wedge (f_i(y_j - x) > 1 , \ 0 \le j \le i < n))$$

is true in $^*M$ (for again $\underline{x}$ satisfies these conditions) and so it is true in M. This means we can choose $x_n \in \bar{A}$ such that

$$f_i(y_j - x_n) > 1 , \quad 0 \le j \le i < n ,$$

and in turn $y_n \in A$ such that

$$|f_k(x_n - y_n)| < 1/2 , \quad k = 0,1,\ldots,n-1 .$$

Therefore we can choose sequences $\{y_n\}$ in A, $\{f_n\}$ in S satisfying

$$f_k(y_i - y_n) > 1/2 , \quad 0 \le i \le k < n .$$

As $\hat{A}$ is relatively countably weakly compact in $H''_{\tilde{S}}$, $\{\hat{y}_n\}$ has a weak limit point x'' in $H''_{\tilde{S}}$. Subsequently,

$$(f_k(y_i) - x''(f_k)) \geq 1/2 \ , \quad 0 \leq i \leq k \ .$$

Now, because S^o is a 0-neighbourhood in $H''_{\tilde{S}}$, the Banach-Alaoglu theorem implies S^{oo} is $\sigma((H''_{\tilde{S}})', H'')$-compact. Consequently $\{\hat{f}_k\}$ has a limit point x''' in the $\sigma((H''_{\tilde{S}})', H'')$-topology. But then

$$x'''(\hat{y}_i) - x'''(x'') \geq 1/2 \ , \quad i = 1,2,\ldots \ ,$$

contradicting the assumption that x'' is a weak limit point of $\{\hat{y}_n\}$ in $H''_{\tilde{S}}$. //

If S is a family of strongly bounded subsets of E' we may replace $H''_{\tilde{S}}$ by E''_S (cf. Theorems 3.2.8 and 3.2.9). We outline the proof for completion.

3.3.3 THEOREM. *Suppose* S *is a family of strongly bounded subsets of* E'. *If* $\hat{A}$ *is relatively countably weakly compact in* E''_S, *then the* $\sigma(E,(E_S)')$*-closure of* A *is* $\tilde{S} - \sigma(E,(E_S)')$*-compact.*

PROOF. Following the method of Theorem 3.3.2 we suppose the closure of A is not $\tilde{S} - \sigma(E,(E_S)')$-compact and we then construct sequences $\{y_n\}$ in A and $\{f_n\}$ in a set $S \in \tilde{S}$ satisfying

$$f_k(y_i - y_n) > 1/2 \ , \quad 0 \le i \le k < n \ .$$

As in the proof of Lemma 3.2.7 there is a $\sigma((E''_S)', E'')$-compact set S_1 in $(E''_S)'$ such that each f_k has an extension y'''_k to E''_S which belongs to S_1. Rewriting the previous equation we obtain

$$y'''_k(\hat{y}_i - \hat{y}_n) > 1/2 \ , \quad 0 \le i \le k < n \ .$$

If the sequence $\{\hat{y}_n\}$ has a weak limit point y'' in E''_S it follows that

$$y'''_k(\hat{y}_i) - y'''_k(y'') > 1/2 \ , \quad 0 \le i \le k \ .$$

But the sequence $\{y'''_k\}$ has a limit point y''' in the $\sigma((E''_S)', E'')$-topology and so

$$y'''(\hat{y}_i) - y'''(y'') \ge 1/2 \ , \quad i = 1,2,\ldots \ ,$$

thus contradicting the assumption that y'' is a weak limit point of $\{\hat{y}_n\}$ in E''_S. //

3.3.4 COROLLARY (Eberlein's theorem). *Let* E *be a* LCTVS *and let* A *be a subset of* E . *If* A *is relatively countably weakly compact then* A *is nearly weakly compact. Furthermore, if the closed convex hull of* A *is complete in the Mackey topology, then* A *is relatively weakly compact.*

PROOF. By Theorem 3.3.3 A is nearly weakly compact. The end remark is a consequence of Theorem 3.2.11. //

3.3.5 REMARKS. Recent standard proofs of Eberlein's theorem in a Banach space context have been obtained by Pelczynski [3] and Whitley [1]; see also Dunford and Schwartz [1, p. 466] for a discussion of the history of the theorem. We comment too that Theorem 3.2.8 ensures that the converse of Theorem 3.3.2 holds. It would be interesting to obtain the natural generalization of Krein's theorem (Schaefer [1, p. 189]) by non-standard methods. We give the result here as a corollary to Krein's theorem and Theorem 3.2.8.

3.3.6 COROLLARY. *Let* E *be a* LCTVS *and* A *be a subset of* E . *If* A *is* $\tilde{S} - \sigma(E,(E_S)')$*-compact then its convex hull is also* $\tilde{S} - \sigma(E,(E_S)')$*-compact.*

PROOF. Suppose B is the convex hull of A . Now every Cauchy net from $\hat{B}$ in $H''_{\tilde{S}}$ has a limit point and so the closure of $\hat{B}$ is complete. As A is $\tilde{S} - \sigma(E,(E_S)')$-compact $\hat{A}$ is relatively weakly compact by Theorem 3.2.8. Therefore Krein's theorem ensures that $\hat{B}$ is relatively weakly compact in $H''_{\tilde{S}}$ and consequently B is $\tilde{S} - \sigma(E,(E_S)')$-compact using Theorem 3.2.8 once more. //

3.4 Generalizations of semi-reflexive spaces.

We intend now to consider a class of generalizations of semi-reflexive spaces. We find again that non-standard techniques are helpful in the investigation of these properties.

In the following work S denotes a covering of E' by strongly bounded subsets which satisfy conditions S_I and S_{II} mentioned in Section 3.2: thus the polars S° of the sets $S \in S$ form a basis of 0-neighbourhoods in E". Initially we do not assume that the S-topology on E is consistent with the duality $\langle E,E' \rangle$. With these further restrictions on S we introduce the following definition.

3.4.1 DEFINITION. Let E be a LCTVS. We say that E is *S-semi-reflexive* if $\hat{E}$ is dense in E''_S.

Thus if F is the family of finite subsets of E', then E is F-semi-reflexive (for recall result 5.4 of Schaefer [1, p. 143]). It is well-known (Schaefer [1, p. 144]) that *a* LCTVS E *is semi-reflexive if and only if each bounded subset of* E *is relatively weakly compact.* The following theorem generalizes this result.

3.4.2 THEOREM. *Let* E *be a* LCTVS. *Then* E *is* *S-semi-reflexive if and only if each bounded set of* E *is* $S - \sigma(E,E')$*-compact.*

PROOF. Suppose first that E is S-semi-reflexive and let B be a bounded set of E. If $\underline{x} \in {}^*B$ it is sufficient, by Theorem 3.2.2, to show that $\underline{x}$ is $S - \sigma(E,E')$-near-standard. We define $x'' \in E''$ by

$$x''(f) = {}^\circ[f(\underline{x})] \quad \text{for all} \quad f \in E' .$$

Now let $S \in \mathcal{S}$. By assumption there exists an $x \in E$ such that

$$|f(x) - x''(f)| < 1 \quad \text{for all} \quad f \in S \ .$$

But this implies that

$$|f(x) - f(\underline{x})| \leq 1 \quad \text{for all} \quad f \in S \ ,$$

and consequently that $\underline{x}$ is $\mathcal{S}$ - $\sigma(E,E')$-near-standard.

Conversely, let us suppose each bounded set of E is $\mathcal{S}$ - $\sigma(E,E')$-compact. Accordingly, by Theorem 3.2.2, each bounded point $\underline{x} \in {}^*E$ is $\mathcal{S}$ - $\sigma(E,E')$-near-standard. Consider an arbitrary element x'' of E'' . By Theorem 2.2.3, there exists a bounded point $\underline{x} \in {}^*E$ such that

$$f(\underline{x}) = x''(f) \quad \text{for all} \quad f \in E' \ .$$

Let $S \in \mathcal{S}$. As $\underline{x}$ is $\mathcal{S}$ - $\sigma(E,E')$-near-standard there exists an $x \in E$ such that

$$|f(\underline{x} - x)| \leq 1 \quad \text{for all} \quad f \in S \ .$$

This implies that

$$|f(x) - x''(f)| \leq 1 \quad \text{for all} \quad f \in S \ .$$

Therefore, as S was chosen arbitrarily, and the family of polars $\{S^o : S \in \mathcal{S}\}$ forms a basis of 0-neighbourhoods in $E''_{\mathcal{S}}$, $\hat{E}$ is dense in $E''_{\mathcal{S}}$. //

Suppose now that the $\mathcal{S}$-topology on E is consistent with the duality $\langle E,E' \rangle$. Theorem 3.2.5 then implies that $\mathcal{S}$ - $\sigma(E,E')$-compact sets are $\tilde{\mathcal{S}}$ - $\sigma(E,E')$-compact. It is therefore an easy consequence of Lemma 3.2.10 that if E is $\mathcal{S}$-semi-reflexive and $x'' \in E''$ we can choose a bounded net $\{x_S\}$ in E convergent to x'' in the $\mathcal{S}$-topology. It follows that if E is $\mathcal{S}$-semi-reflexive and quasi-complete in the $\mathcal{S}$-topology then E is semi-reflexive. It is not difficult to check that if the quasi-completion of $E_{\mathcal{S}}$ is $\mathcal{S}$-semi-reflexive E is semi-reflexive. The converse seems more difficult. It is at least true if E is distinguished.

3.4.3 THEOREM. *Suppose the* LCTVS E *is distinguished. Then* E *is* $\mathcal{S}$*-semi-reflexive if and only if the quasi-completion of* $E_{\mathcal{S}}$ *is semi-reflexive.*

PROOF. We only prove the necessity of the condition. We note that as E is distinguished E" is the quasi-completion of E_σ (Köthe [1, p. 306]). Therefore, since E is $\mathcal{S}$-semi-reflexive, E" is the quasi-completion of $E_{\mathcal{S}}$. Furthermore the strong topologies $\beta(E',E)$ and $\beta(E',E'')$ are identical on E' for E' is barrelled (Köthe [1, p. 306]). Thus E" is semi-reflexive establishing that the quasi-completion of $E_{\mathcal{S}}$ is semi-reflexive. //

Let S be a subset of E' and F be a subspace of E . Suppose the set of restrictions of functionals in S to F is denoted by $\underset{\sim}{S}$. Then the $\mathcal{S}$-topology on E induces a topology on F which is the $\underset{\sim}{\mathcal{S}}$-topology, where $\underset{\sim}{\mathcal{S}} = \{\underset{\sim}{S} : S \in \mathcal{S}\}$. If F is $\underset{\sim}{\mathcal{S}}$-semi-reflexive we agree to say that F is $\mathcal{S}$-semi-reflexive. With this notation we prove the following generalization of a result of Fleming [1, Theorem 4.1].

3.4.4 THEOREM. *Let* E *be a* LCTVS. *Then* E *is* $\mathcal{S}$*-semi-reflexive if and only if every separable subspace is* $\mathcal{S}$*-semi-reflexive.*

PROOF. We prove the necessity of the condition first. Suppose that F is any subspace of E. Let B be a bounded set in F and let $\underline{x} \in {}^*B$. As E is $\mathcal{S}$-semi-reflexive $\underline{x}$ is $\mathcal{S} - \sigma(E,E')$-near-standard and so, by Lemma 3.2.10, for each $S \in \mathcal{S}$ there is a point x belonging to the convex hull of B such that

$$|f(\underline{x} - x)| \leq 1 \quad \text{for all} \quad f \in S .$$

This implies that $\underline{x}$ is $\underset{\sim}{\mathcal{S}} - \sigma(F,F')$-near-standard so that B is $\underset{\sim}{\mathcal{S}} - \sigma(F,F')$-compact. That F is $\mathcal{S}$-semi-reflexive therefore follows by Theorem 3.4.2.

Next we prove the sufficiency of the condition. Suppose in fact that E is not $\mathcal{S}$-semi-reflexive. Hence there is a bounded subset B of E which is not $\mathcal{S} - \sigma(E,E')$-compact. Thus Theorem 3.2.2 implies there is a sequence $\{x_n\}$ in B such that $\{\hat{x}_n\}$ has no weak limit point in

$E''_{\underset{\sim}{S}}$. Let F be the linear span of $\{x_n\}$. Then F is a separable space and $\{x_n\}$ is a bounded sequence in F . Suppose that $\{\hat{x}_n\}$ has a weak limit point y'' in $F''_{\underset{\sim}{S}}$. We define an element $x'' \in E''$ by

$$x''(f) = y''(f/F) \quad \text{for all} \quad f \in E' .$$

It follows that x'' is a weak limit point of $\{\hat{x}_n\}$ in $E''_{\underset{\sim}{S}}$, which is a contradiction. //

If $\mathcal{S}$ generates the Mackey topology the notion of $\mathcal{S}$-semi-reflexivity is of special interest.

3.4.5 DEFINITION. Let $\mathcal{K}$ be the family of circled, convex, $\sigma(E',E)$-compact subsets of E' . If E is $\mathcal{K}$-semi-reflexive we say E is *nearly semi-reflexive*.

As a consequence of Theorem 3.4.4 we have the following.

3.4.6 COROLLARY. *Let* E *be a* LCTVS *and suppose that* E *is quasi-complete in the Mackey topology. Then* E *is semi-reflexive if and only if each separable subspace is nearly semi-reflexive.*

PROOF. The necessity of the condition is obvious. The sufficiency is an immediate consequence of Theorem 3.4.4 and the comment preceding Theorem 3.4.3. //

3.4.7 EXAMPLE. We refer the reader to Day [2, p. 28] for a discussion of the following spaces. Let Γ be an arbitrary set and define

$m(\Gamma)$ to be the space of all bounded real functions on Γ with norm defined by $\|x''\| = \sup\{|x''(\gamma)| : \gamma \in \Gamma\}$,

$m_0(\Gamma)$ to be the subspace of all those x'' in $m(\Gamma)$ which vanish except on a countable set, and

$\ell_1(\Gamma)$ to be the space of real functions f on γ for which $\|f\| = \sum_{\gamma \in \Gamma} |f(\gamma)| < \infty$.

If E denotes $m_0(\Gamma)$ equipped with the $\sigma(m_0(\Gamma), \ell_1(\Gamma))$-topology E is nearly semi-reflexive (see Corollary 6.2.2). However, if Γ is uncountable, E is not semi-reflexive since $E'_\beta = \ell_1(\Gamma)$ so that $E'' = m(\Gamma)$. On the other hand if G is a separable subspace of E the set $\{\gamma \in \Gamma : x''(\gamma) \neq 0 \text{ for some } x'' \in G\}$ is countable. From this observation it is an easy consequence that a closed separable subspace of E is quasi-complete and hence semi-reflexive. This example, together with our previous results, clarifies the comment made by Fleming after the proof of Theorem 4.1 [1, p. 77].

The following is a useful characterization of nearly semi-reflexive spaces.

3.4.8 THEOREM. *Let* E *be a* LCTVS. *Then* E *is nearly semi-reflexive if and only if the topology* $\tau(E'', E')$ *coincides on* E *with the* *topology* $\tau(E,E')$.

PROOF. Let us suppose firstly that E is nearly semi-reflexive. Then every circled, convex, $\sigma(E',E)$-compact set S is $\sigma(E',E'')$-compact and it therefore follows that the topologies are equivalent.

Conversely, we know that E'' is obtained from E by taking the $\sigma(E'',E')$-closure points of the bounded sets in E . Since these can be taken to be circled and convex it is sufficient to consider the $\tau(E'',E')$-closure points. But by assumption this implies that E is nearly semi-reflexive. //

So far we have only considered generalizations of semi-reflexivity. There is an obvious generalization of reflexivity too.

3.4.9 DEFINITION. Let E be a LCTVS. We say E is *nearly reflexive* if E is nearly semi-reflexive and E'' induces the topology on E ; i.e., if E is nearly semi-reflexive and E is infrabarrelled (see Schaefer [1, p. 144]).

It is possible to extend a number of known results using Definition 3.4.9. We prove one here.

3.4.10 THEOREM. *Suppose the strong dual of a* LCTVS E *is semi-reflexive.*

Then E_τ *is nearly reflexive.*

PROOF. Let B be a strongly bounded set in E'. It follows from the semi-reflexivity of E'_β that B is relatively $\sigma(E',E'')$-compact, and thus relatively $\sigma(E',E)$-compact. This implies that E is infrabarrelled.

We complete the proof once we show that E is nearly semi-reflexive. Suppose S is a circled, convex, $\sigma(E',E)$-compact set in E', then, as S is strongly bounded, S is $\sigma(E',E'')$-compact. Therefore E is nearly semi-reflexive by Theorem 3.4.8. //

CHAPTER 4

LINEAR MAPPINGS BETWEEN TOPOLOGICAL VECTOR SPACES

4.0 Introduction.

In *Non-Standard Analysis* Robinson gave non-standard characterizations of bounded and compact linear operators on normed spaces. It was by using these characterizations that Bernstein and Robinson [1] showed that a linear operator in Hilbert space which has a compact square, possesses a non-trivial invariant subspace.

In this chapter we characterize certain linear maps between (separated) TVSs by non-standard properties. We use these characterizations to obtain and generalize results of Grothendieck [1] which extended results of Schauder and Gantmacher (see Dunford and Schwartz [1, p. 485]). We also give non-standard proofs of two theorems of Ringrose (see [1], [2]) and generalize one of his results. We find the main concepts of Chapter 3 useful in this work. Some examples of linear maps are included to clarify the results.

The author would like to comment that he became aware of the relevant work of Grothendieck only after he had developed much of this chapter;

indeed it seems very likely that Ringrose was also unaware of Grothendieck's results at the time he wrote his papers ([1] and [2]).

4.1 Notation and Definitions.

Throughout this chapter E,F and G denote either (separated) TVSs or LCTVSs. The scalar field may be assumed to be either the real numbers or the complex numbers (provided it is the same for all spaces mentioned in any result). Generally we will be concerned with continuous linear maps between pairs of the spaces E,F and G .

In Definitions 4.1.1 to 4.1.3 we consider a linear map T from a TVS E into a TVS F .

4.1.1 DEFINITION. The map T is *boundedly precompact* (*boundedly compact*) if the set T(B) is precompact (relatively compact) in F whenever B is a bounded subset in E .

4.1.2 DEFINITION. The map T is *precompact* (*compact*) if there is a 0-neighbourhood V in E such that T(V) is a precompact (relatively compact) set in F .

4.1.3 DEFINITION. The map T is *bounded* if there is a 0-neighbourhood V in E such that T(V) is a bounded subset in F .

In the remaining definitions we consider a linear map T from a LCTVS E into a LCTVS F .

4.1.4 DEFINITION. The map T is *boundedly weakly compact* if the set T(B) is relatively weakly compact in F whenever B is a bounded subset in E .

More generally, if S is a covering of F' by weakly bounded sets and S_1 is a family of subsets of $(F_S)'$ we may introduce the following definition (where the notation is that of Chapter 3).

4.1.5 DEFINITION. The map T is *boundedly* $S_1 - \sigma(F(F_S)')$*-compact* if T(B) is $S_1 - \sigma(F,(F_S)')$-compact in F whenever B is a bounded set in E .

4.2 <u>Non-standard characterizations of linear maps.</u>

Our first theorem is a restatement of a result of Robinson [2, p. 98]. We include it for completeness although we do not offer a proof.

4.2.1 THEOREM. *Let* E *and* F *be* TVSs *and let* T *be a linear map from* E *into* F . *Then* T *is continuous if and only if* Tx *is the standard part of* $T\underline{x}$ *in* *F *whenever* x *is the standard part of* $\underline{x}$ *in* *E .

4.2.2 REMARK. It is not true in general that continuity of T is characterized by T preserving bounded points, as is the case when T is an operator on a normed space (Robinson [2, p. 178]). The following simple example illustrates this. Let X be an infinite dimensional Banach space and let T be the identity map of X_σ into X. Then it is clear that T preserves bounded points but that T is not continuous.

4.2.3 THEOREM. *Let* E *and* F *be* TVSs *and* T *be a linear map from* E *into* F. *Then* T *is closed if and only if whenever* x *is the standard part of* $\underline{x}$ *in* $^*E$, *and* y *is the standard part of* $T\underline{x}$ *in* $^*F$ *it follows that* $y = Tx$.

PROOF. Let T be a closed map. Suppose that x is the standard part of $\underline{x}$ and that y is the standard part of $T\underline{x}$. Then (x,y) is the standard part of $(\underline{x},T\underline{x})$ in $E \times F$. Since T is closed (x,y) belongs to the graph $G(T)$ of T, so consequently $y = Tx$. Conversely, let T satisfy the given condition. We show that $G(T)$ is closed. Suppose the point $u \in E \times F$ is the standard part of $\underline{u} \in {}^*G(T)$. Let $\underline{u} = (\underline{x},T\underline{x})$ and let $u = (x,y)$. Then x is the standard part of $\underline{x}$ in $^*E$ and y is the standard part of $T\underline{x}$ in $^*F$. By hypothesis, $y = Tx$ so that $u \in G(T)$ and the result follows by Robinson [2, p. 91]. //

4.2.4 THEOREM. *Let* E *and* F *be* TVSs *and let* T *map* E *into* F. *Then* T *is boundedly precompact (boundedly compact) if and only if* $T\underline{x}$ *is pre-near-standard (near-standard) whenever* $\underline{x}$ *is a bounded point in* $^*E$.

PROOF. Suppose T is boundedly precompact and that $\underline{x}$ is a bounded point in $^*E$. Then $\underline{x} \in {}^*B$ where B is a bounded set in E. By assumption $T(B)$ is precompact and therefore, by Theorem 2.3.4, $T\underline{x}$ is pre-near-standard. To establish the converse we consider a bounded set B in E. By assumption each point of $^*(T(B))$ is pre-near-standard and so $T(B)$ is precompact again by Theorem 2.3.4. //

We list some similar results for reference omitting the proofs.

4.2.5 THEOREM. *Let* E *and* F *be* TVSs *and let* T *map* E *into* F. *Then* T *is precompact (compact) if and only if there exists a* 0*-neighbourhood* V *in* E *such that* $T\underline{x}$ *is pre-near-standard (near-standard) for each* $\underline{x} \in {}^*V$.

4.2.6 THEOREM. *Let* E *and* F *be* LCTVSs *and let* T *map* E *into* F. *Then* T *is boundedly weakly compact if and only if* $T\underline{x}$ *is weak near-standard whenever* $\underline{x}$ *is a bounded point in* $^*E$.

4.2.7 THEOREM. *Let* E *and* F *be* LCTVSs *and let* T *map* E *into* F. *Then* T *is boundedly* $S_1 - \sigma(F,(F_S)')$*-compact if and only if* $T\underline{x}$ *is* $S_1 - \sigma(F,(F_S)')$*-near-standard for each bounded point* $\underline{x} \in {}^*E$.

4.3 Properties of the adjoint map.

Let E,F be two TVSs and T be a linear map of E into F. We denote the algebraic adjoint of T by $T^{\#}$ and, if T is continuous, the adjoint map by T'. The reader is referred to Edwards [1, p. 514] for a discussion of these concepts. We intend to investigate properties of T'.

It is well-known that if T is a compact linear map of E into itself the adjoint map T' of E'_β into itself is not necessarily precompact. Conversely, it is also true that if the adjoint T' is compact T is not necessarily precompact. Examples illustrating this type of behaviour were first published by Ringrose [1, p. 585]. However, Ringrose did publish an interesting positive result [2, Theorem 4.2]. Theorem 4.3.1 generalizes this result of Ringrose.

4.3.1 THEOREM. *Let* E,F *and* G *be* TVSs *with* $\mathcal{S}$ *a family of weakly bounded subsets of* E. *Suppose that* T *is a continuous linear map of* E *into* F *which maps sets of* $\mathcal{S}$ *into precompact sets in* F *and suppose that* T_1 *is a bounded linear map from* F *into* G. *Then* $T'T'_1$ *is a compact map from* G'_β *into* $E'_{\mathcal{S}}$.

PROOF. As the quasi-completion of F has the same dual as F we may suppose that T maps sets of $\mathcal{S}$ into compact sets. Since T_1 is bounded there is a 0-neighbourhood U in F such that $T_1(U)$ is bounded in G. It follows that $V = (T_1(U))^\circ$ is a 0-neighbourhood in G'_β. Let us show that $T'T'_1(V)$ is relatively compact in $E'_{\mathcal{S}}$. This is true if, for each $\underline{f} \in {}^*V$, $T'T_1'\underline{f}$ is near-standard in the S-topology. Let us define g by

$$g(x) = {}^\circ[T_1'\underline{f}(x)] \quad \text{for all} \quad x \in F \; .$$

As $T_1'\underline{f} \in {}^*U^\circ$ the Banach-Alaoglu theorem ensures that $g \in U^\circ$. Now consider a set $S \in \mathcal{S}$ and suppose $\underline{x} \in {}^*S$. Since T maps S into a compact set $T\underline{x}$ is near-standard in ${}^*F$. Therefore, if y is the standard part of $T\underline{x}$

$$\underline{g}(T\underline{x}) \simeq \underline{g}(y) \quad \text{for all} \quad \underline{g} \in {}^*U^\circ \; .$$

Consequently we have

$$(T'T_1'\underline{f})\underline{x} = T_1'\underline{f}(T\underline{x}) \simeq (T_1'\underline{f})y \simeq g(y) \simeq g(T\underline{x}) = (T'g)\underline{x} \; .$$

Theorem 2.1.3 thus implies that $T'T_1'\underline{f}$ is near-standard in the S-topology. //

4.3.2 COROLLARY (Ringrose). *Let* E *be a* TVS *and* T *be a precompact linear map of* E *into itself. Then* $(T')^2$ *is a compact map from* E'_β *into itself.*

PROOF. Since T is precompact, it is bounded and maps the family of (weakly) bounded subsets of E into precompact sets. //

4.3.3 REMARK. Let X be an infinite dimensional Banach space, and let

T be the identity map of X_σ into itself. It is easily checked that T is boundedly precompact but that $(T')^2 = T'$ is not compact. This shows it is not sufficient to assume in Corollary 4.3.2 that T is only boundedly precompact.

The natural continuation of this aspect of the work of Ringrose is to alter the initial topologies on E and F and to consider topologies other than the strong on the dual spaces. Our next result is due to Grothendieck [1, Lemme 1].

4.3.4 THEOREM. *Let* E *and* F *be* LCTVSs, S *be a family of weakly bounded subsets of* E *and* S' *be a family of weakly bounded subsets of* F' . *Suppose* T *is a continuous map from* E *into* F . *Then* T *maps sets of* S *into precompact sets in* $F_{S'}$ *if and only if* T' *maps the sets of* S' *into precompact sets in* E'_S .

PROOF. Let us prove the necessity of the condition first. Consider a set $S' \in S'$ and suppose that $\underline{f} \in {}^*S'$. It suffices, by Theorem 2.3.4, to show that $T'\underline{f}$ is pre-near-standard in ${}^*E'_S$. For such an $\underline{f}$ we define $f \in F^\#$ by

$$f(y) = {}^\circ[\underline{f}(y)] \quad \text{for all } y \in F .$$

Now consider a set $S \in S$ and a point $\underline{x} \in {}^*S$. As $T(S)$ is a precompact set in $F_{S'}$ $T\underline{x}$ is pre-near-standard in the S'-topology. Thus given an arbitrary $\varepsilon > 0$, there exists $y \in F$ such that

$$|\underline{f}(T\underline{x} - y)| \leq \varepsilon \quad \text{for all} \quad \underline{f} \in {}^*S' .$$

It follows too that $|f(T\underline{x} - y)| \leq \varepsilon$, so that as $f(y) \simeq \underline{f}(y)$, $|\underline{f}(T\underline{x}) - f(Tx)| < 3\varepsilon$. But $\varepsilon > 0$ was chosen arbitrarily so that

$$(T'\underline{f})\underline{x} \simeq (T^{\#}f)\underline{x} \quad \text{for each} \quad \underline{x} \in {}^*S .$$

This equation ensures that $T'\underline{f}$ is pre-near-standard in E'_{S}. The sufficiency of the condition now follows by symmetry. //

If S' is a family of $\sigma(F',F)$-compact subsets of F' it is easily checked that T' maps the sets of S' into compact sets in E'_{S}. This is because the $f \in F^{\#}$ defined in the previous proof then belongs to S'. Using this observation we note three consequences of Theorem 4.3.4. We point out that in Corollary 4.3.6 $F_{\beta*}$ denotes F equipped with the topology $\beta^*(F,F')$ of uniform convergence on strongly bounded subsets of F'.

4.3.5 COROLLARY. *Let* E,F *be* LCTVSs *and let* T *be a continuous linear map from* E *into* F. *Then* T *is boundedly precompact as a map from* E *into* F_{τ} *if and only if the adjoint* T' *maps circled, convex,* $\sigma(F',F)$*-compact sets into compact sets in* E'_{β}.

4.3.6 COROLLARY. *Let* E,F *be* LCTVSs *and let* T *be a continuous linear map from* E *into* F. *Then* T *is boundedly precompact as a map from* E *into* $F_{\beta*}$ *if and only if the adjoint* T' *from* F'_{β} *into* E'_{β} *is* *boundedly precompact.*

4.3.7 COROLLARY. *Let* E,F *be* LCTVSs *and let* T *be a continuous linear map from* E *into* F . *Then* T *is boundedly compact as a map from* E *into* F_β *if and only if the adjoint map from* F'_τ (or F'_σ) *into* E'_β *is boundedly precompact.*

In Theorem 4.3.8 we suppose that S is a covering of E by weakly bounded subsets and that S' is a covering of F' by weakly bounded subsets. Furthermore for simplicity we assume that S' satisfies the conditions S_I and S_{II} of Section 3.2 and that it also contains the circled, convex hulls of its members. We use H to denote $F_{S'}$ and we denote the family of equicontinuous subsets of H by $\tilde{S}'$.

4.3.8 THEOREM. *Let* E *and* F *be* LCTVSs. *Suppose that* T *is a linear map from* E *into* F , *continuous from* E *into* $F_{S'}$. *Then* T *maps the sets of* S *into* $\tilde{S}'$ - $\sigma(F,(F_{S'})')$*-compact sets if and only if* T' *maps the sets of* S' *into relatively* $\sigma(E',(E'_S)')$*-compact sets.*

PROOF. We prove the necessity of the condition first. First note that we may assume that S satisfies conditions S_I and S_{II} . Furthermore Corollary 3.3.6 ensures that if $S \in S$ we may assume that its circled, convex hull $\Gamma(S)$ is also in S . We begin by considering a set $S' \in S'$ and an $\underline{f} \in {}^*S'$. We define $x''' \in (H''_{\tilde{S}'})'$ by

$$x'''(y'') = {}^{\circ}[y''(\underline{f})] \quad \text{for all} \quad y'' \in H'' .$$

Now suppose $x'' \in (E'_{\mathcal{S}})'$. Then, by Theorem 2.2.3, there exists an $\underline{x} \in {}^*S$, for some $S \in \mathcal{S}$, such that

$$x''(g) = g(\underline{x}) \quad \text{for all} \quad g \in E' .$$

Since $T(S)$ is $\tilde{\mathcal{S}}' - \sigma(F,(F_{\mathcal{S}'})')$-compact $T\underline{x}$ is $\tilde{\mathcal{S}}' - \sigma(F,(F_{\mathcal{S}'})')$-near-standard. Therefore Lemma 3.2.6 guarantees the existence of a $z'' \in H''$ such that

$$y'''(z'') \simeq y'''(\hat{T\underline{x}}) \quad \text{for all} \quad y''' \in (H''_{\tilde{\mathcal{S}}'})' .$$

We have then, for $f \in H'$,

$$x''(T'f) = (T'f)\underline{x} = f(T\underline{x}) \simeq z''(f)$$

so that consequently $T''x'' = z''$. Hence the following equation holds:

$$x''(T'\underline{f}) = (T''x'')\underline{f} \simeq x'''(T''x'') \simeq x'''(\hat{T\underline{x}}) .$$

As $T''x''$ belongs to the closure of $\hat{H}$ in $H''_{\tilde{\mathcal{S}}'}$ for each $x'' \in (E'_{\mathcal{S}})'$, $T'''x''' = T'f$ where f is the restriction of x''' to $\hat{H}$. Thus we have

$$x''(T'\underline{f}) \simeq (T'''x''')\hat{\underline{x}} = (T'''x''')x''$$

which ensures that $T'\underline{f}$ is $\sigma(E',(E'_{\mathcal{S}})')$-near-standard as required.

Conversely suppose that T' maps the sets of $\mathcal{S}'$ into relatively $\sigma(E',(E'_{\mathcal{S}})')$-compact sets. We first observe that if $\tilde{S} \in \tilde{\mathcal{S}}'$, $T'(\tilde{S})$ is also relatively $\sigma(E',(E'_S)')$-compact. For given such a set $\tilde{S}$ there is a circled, convex set $S' \in \mathcal{S}'$ such that each $f \in \tilde{S}$ is bounded by unity on S'°, the polar of S' in F. Theorem 2.2.3 thus implies that if $f \in \tilde{S}$ there is an $\underline{f} \in {}^*S'$ such that

$$f(x) \simeq \underline{f}(x) \quad \text{for all} \quad x \in F .$$

Because $T'(S')$ is relatively $\sigma(E',(E'_S)')$-compact $T'\underline{f}$ is near-standard in the $\sigma(E',(E'_S)')$-topology. It follows that $T'f$ is equal to the standard part of $T'\underline{f}$, from whence it is easily seen that $T'(\tilde{S})$ is relatively $\sigma(E',(E'_S)')$-compact. Now consider a set $S \in \mathcal{S}$ and take an $\underline{x} \in {}^*S$. We define $z'' \in (E'_S)'$ by

$$z''(f) = {}^{\circ}[f(\underline{x})] \quad \text{for all} \quad f \in E' .$$

If x''' is an arbitrary functional in $(H''_{\tilde{S}'})'$ by Theorem 2.2.3 there exists an $\underline{f} \in {}^*\tilde{S}$, for some $\tilde{S} \in \tilde{\mathcal{S}}'$, such that

$$x'''(x'') = x''(\underline{f}) \quad \text{for all} \quad x'' \in H'' .$$

Since $T'(\tilde{S})$ is relatively $\sigma(E',(E'_S)')$-compact $T'\underline{f}$ is $\sigma(E',(E'_S)')$-near-standard so that there exists a $g \in E'$ such that

$$y''(T'\underline{f}) \simeq y''(g) \quad \text{for all} \quad y'' \in (E'_S)' .$$

Hence we have

$$x'''(T''y'') = (T''y'')\underline{f} = y''(T'\underline{f}) \simeq y''(g) \quad \text{for all} \quad y'' \in (E'_S)' .$$

This implies that $T'''x''' = \hat{g}$ and consequently we have

$$x'''(T\hat{\underline{x}}) = x'''(T''\hat{\underline{x}}) = (T'''x''')\hat{x} \simeq (T'''x''')z'' = x'''(T''z'') .$$

Lemma 3.2.6 therefore implies that $T\underline{x}$ is $\tilde{S}' - \sigma(F,(F_{S'})')$-near-standard and thus $T(S)$ is $\tilde{S}' - \sigma(F,(F_{S'})')$-compact completing the proof. //

Theorem 4.3.8 should be compared with a result of Grothendieck [1, Lemme 1]; see also Edwards [1, Theorem 9.3.1]. Perhaps the most interesting consequences of the theorem occur when S is the family of weakly bounded subsets of E . We note four corollaries (recall Theorem 3.2.5).

4.3.9 COROLLARY. *Suppose that* E,F *are* LCTVSs *and that* $\mathcal{E}$ *denotes the family of equicontinuous subsets of* F' . *Let* T *be a continuous linear map from* E *into* F . *Then* T *is boundedly* $\mathcal{E} - \sigma(F,F')$*-compact if and only if* T' *maps members of* $\mathcal{E}$ *into relatively weakly compact sets in* E'_β .

4.3.10 COROLLARY. *Suppose* E,F *are* LCTVSs *and that* K *denotes the family of circled, convex,* $\sigma(F',F)$*-compact subsets of* F' . *Let* T *be a continuous linear map from* E *into* F . *Then* T *is boundedly* $K - \sigma(F,F')$*-compact if and only if* T' *maps sets of* K *into weakly compact sets in* E'_β .

4.3.11 COROLLARY. *Suppose* E,F *are* LCTVSs *and that* $\mathcal{B}$ *denotes the family of strongly bounded subsets of* F' . *Let* T *be a linear map from* E *into* F , *continuous from* E *into* $F_{\beta *}$. *Then* T *is boundedly* $\tilde{\mathcal{B}} - \sigma(F,(F_{\beta *})')$*-compact if and only if* T' *is a boundedly weakly compact map from* F'_β *into* E'_β .

4.3.12 COROLLARY. *Suppose* E,F *are* LCTVSs *and that* $\mathcal{B}_\sigma$ *denotes the family of weakly bounded subsets of* F' . *Let* T *be a linear map from* E *into* F , *continuous from* E *into* F_β . *Then* T *is boundedly* $\tilde{\mathcal{B}}_\sigma - \sigma(F,(F_\beta)')$*-compact if and only if* T' *maps sets of* $\mathcal{B}_\sigma$ *into relatively weakly compact sets in* E'_β .

4.4 Related results.

The next two results are due to Ringrose [2, Theorems 3.3 and 3.4]. Edwards [1, p. 619] attributes similar results to Grothendieck (see Grothendieck [1, Lemme 3]).

4.4.1 THEOREM. *Let* E,F *be* TVSs *and let* T *be a continuous linear map from* E *into* F . *Then*

(a) *If* T *is continuous as a mapping from each bounded set of* E *under the* E'*-topology into* F *it is boundedly precompact.*

(b) *The converse of* (a) *holds provided the* F'*-topology on* *the quasi-completion of* F *is separated.*

PROOF. We establish (a) first. Suppose that $\underline{x}$ is a bounded point in $^*E$. The proof of Theorem 2.4.3 implies that $\underline{x}$ is pre-near-standard in the $\sigma(E,E')$-topology (here Definition 2.1.9 is extended to non-separated spaces). Therefore, by the continuity condition, $T\underline{x}$ is a pre-near-standard point in $^*F$. Theorem 4.2.4 then implies that T is boundedly precompact. We now prove (b). If $\tilde{F}$ is the quasi-completion of F the map $T : E \to \tilde{F}$ is boundedly compact. Let B be a bounded set in E and suppose that $\underline{x}$ belongs to the monad of x in the $\sigma(E,E')$-topology where both $\underline{x}$ and $^*x$ belong to $^*B$. It follows by Theorem 4.2.4 that $T\underline{x}$ is near-standard in $^*\tilde{F}$; let it have standard part y . Since T is continuous as a map from E_σ into $\tilde{F}_\sigma$ it follows, using the separation assumption, that Tx = y . That T has the desired property now follows by a simple extension to Theorem 4.2.1. //

A similar technique allows us to establish:

4.4.2 THEOREM. *Let* E,F *be* TVSs *and let* T *be a continuous linear map from* E *into* F . *Then*

(a) *If* T *is continuous from a* 0*-neighbourhood* V *of* E *under the* E'*-topology into* F , T *is precompact.*

(b) *The converse of* (a) *holds provided the quasi-completion of* F *equipped with the* F'*-topology is separated.*

We now wish to consider a different type of problem. Consider a linear map $T : E \to F$. The *reduced mapping* $T_o : E \to R(T)$ is defined by $T_o x = Tx$ (here $R(T)$ denotes the range of T). It is well-known (see Goldberg and Thorp [1]) that if T is compact T_o is not necessarily compact. However we do have the following easy result.

4.4.3 THEOREM. *Let* T *be a boundedly compact (compact) map from* E *into* F. *Then* T_o *is boundedly precompact (precompact).*

PROOF. Let $\underline{x}$ be a bounded point in $^*E$ and suppose that y is the standard part of $T\underline{x}$ in $^*F$. Then y belongs to the closure of $T(E)$. Now given a 0-neighbourhood V in F there exists a 0-neighbourhood U such that $U + U \subset V$ so that if we choose a point $y_V \in T(E)$ such that $y - y_V \in U$ we have that $T\underline{x} \in y_V + {}^*V$. Consequently $T\underline{x}$ is pre-near-standard in $^*R(T)$ as required by Theorem 4.2.4. //

Indeed if T is weakly compact T_o need not be weakly compact (see Arterburn [1]). In the context of our work it is instructive to note the following result.

4.4.4 THEOREM. *Let* E,F *be* LCTVSs *and let* T *be a linear map from* E *into* F. *If* T *is boundedly weakly compact then* T_o *is boundedly* $K - \sigma(F,F')$*-compact.*

PROOF. Let $\underline{x}$ be a bounded point in $^*E$ and suppose that y is the standard part of $T\underline{x}$ in the $\sigma(F,F')$-topology. It follows that y belongs to the $\sigma(F,F')$-closure of $T(E)$, and thus to the $\tau(F,F')$-closure of $T(E)$. Let S be a circled, convex, $\sigma(F',F)$-compact subset of F'. Then there is a point $y_S \in T(E)$ such that

$$|f(y - y_S)| < 1 \quad \text{for all} \quad f \in S .$$

Consequently we have

$$|f(y_S - T\underline{x})| \leq 1 \quad \text{for all} \quad f \in S$$

so that $T\underline{x}$ is K - $\sigma(F,F')$-near-standard. The result therefore follows by Theorem 4.2.7. //

CHAPTER 5

THE CONJUGATE OF A SMOOTH BANACH SPACE

5.0 Introduction.

The preceding three chapters have been largely concerned with a non-standard treatment of compactness in TVSs. In this chapter compactness arguments by way of the Banach-Alaoglu and Tychonoff theorems again play a central role.

The main purpose of this chapter is to show that if X is a smooth Banach space with a certain property, X' is isomorphic to a rotund space. This follows from a mapping theorem which guarantees the existence of a set Γ and a continuous one-to-one linear map of X' into $c_0(\Gamma)$. The proof of this theorem occupies the major part of this chapter. We begin with a short outline of some pertinent results and end with a brief discussion of a number of related problems.

5.1 Definitions and Basic Results.

Throughout this chapter we consider a real infinite dimensional Banach space X.

5.1.1 DEFINITION. A Banach space X is *smooth* if at every point of the unit sphere there is only one supporting hyperplane of the unit ball.

5.1.2 DEFINITION. A Banach space X is *rotund*, or *strictly convex*, if the unit sphere contains no line segment.

We refer the reader to Day [2, pp 111-113] and to the relevant sections of Köthe [1] for a discussion of the above concepts. If X is smooth for $x \in X$ we denote by f_x the unique element of X' such that $\|f_x\| = \|x\|$ and $f_x(x) = \|f_x\|\|x\|$. Non-standard analysis is used to prove the following result of Cudia [2, p. 300].

5.1.3 PROPOSITION. *Let* X *be a smooth Banach space and let* $\{x_n\}$ *be a sequence convergent to* x *in the norm topology. Then* $f_{x_n} \to f_x$ *in the weak* topology.*

PROOF. Let ω be an infinite integer, so that x_ω belongs to the monad of x. We consider f_{x_ω} and define a linear functional $f \in X'$ by

$$f(x) = {}^\circ[f_{x_\omega}(x)] \quad \text{for each} \quad x \in X .$$

It is readily checked that $f(x) = \|f_x\|\|x\|$ and that $\|f\| = \|x\|$. Because X is smooth we have that $f = f_x$. The proposition therefore follows by Corollary 2.1.5 and Theorem 4.2.1. //

Let Y be a closed linear subspace of X. Then we denote the set of $f \in X'$ which attain their norm on the unit sphere S_Y of Y by $D_{X'}(Y)$; thus $f \in D_{X'}(Y)$ if there exists a point $y \in S_Y$ such that $f(y) = \|f\|$. When $D_{X'}(X)$ is norm dense in X', X is said to be subreflexive. Bishop and Phelps [1] have shown that all Banach spaces are subreflexive and often this result can be used in conjunction with smoothness with considerable effect; see, for example, Giles [1]. In the next proposition $K(X)$ denotes the weak* sequential closure of $\hat{X}$ in X'' ($K(X)$ is sometimes termed the *Baire subspace of class one*).

5.1.4 PROPOSITION. *Suppose* X *is a Banach space such that the conjugate space* X' *is smooth. Then* $K(X) = X''$.

PROOF. Suppose $F_f \in D_{X''}(X')$. As X is subreflexive there exists a sequence $\{f_n\} \subset D_{X'}(X)$ such that $f_n \to f$ in norm. By Proposition 5.1.3 $F_{f_n} \to F_f$ in the $\sigma(X'',X')$-topology. But $F_{f_n} = \hat{x}_n$ for $n = 1,2,\ldots$, so that $x_n \to F_f$ in the weak* topology. The subreflexivity of X' therefore implies that $K(X)$ is norm dense in X''. But $K(X)$ is closed in the norm topology (see McWilliams [1]) and thus $K(X) = X''$ as claimed. //

5.1.5 COROLLARY. *The space* m *of bounded sequences is not isomorphic to a smooth conjugate space.*

PROOF. The corollary is a consequence of the proposition since $K(\ell) = \ell$. //

In Section 5.2 we require a stronger property than smoothness. It is well-known (Day [2, p. 112]) that X is smooth if and only if its norm is Gateau (weakly) differentiable at each point except the origin. If the norm of X is Fréchet (strongly) differentiable X is sometimes said to be strongly smooth. With this stronger assumption on X in Proposition 5.1.3 we may assume that $f_{x_n} \to f_x$ in the norm topology (see Giles [1, Theorem 1]).

5.1.6 DEFINITION. We say that a Banach space has *property* A if it is smooth and if, whenever $x_n \to x$ in norm, f_x belongs to the closed linear span of $\{f_{x_n} : n = 1,2,...\}$.

Superficially at least, it seems that property A is a weaker condition than Fréchet differentiability of the norm. If X is a *Grothendieck* space (i.e., if weak* convergent sequences in X' are weakly convergent) which is smooth X has property A . This follows by a result of Mazur (Dunford and Schwartz [1, p. 422]) and Proposition 5.1.3. The next result, which generalizes a theorem of Šmulian (Giles [1, Theorem 2]), is relevant to Section 5.2.

5.1.7 PROPOSITION. *Suppose* X *is a Banach space and that its conjugate* *space* X' *has property* A . *Then* X *is reflexive.*

PROOF. Extending the proof of Proposition 5.1.4 we obtain that $F_f \in \overline{sp}\{\hat{x}_n\}$. The subreflexivity of X' therefore implies that $\hat{X}$ is dense in X" ; but X is complete and consequently is reflexive. //

5.2 Statement of the main result.

There are a number of basic problems connected with smoothness, rotundity, and the stronger properties which arise by imposing uniformity conditions on these concepts. One such problem is the extent of the duality between these concepts. Another is the degree of rotundity or smoothness of X which can be obtained by renorming the space without changing the topology.

For example it is well-known (see Day [2, p. 112]) that if X' is smooth (rotund) then X is rotund (smooth). This implies that, if X is reflexive, X is smooth (rotund) if and only if X' is rotund (smooth). Day [1] showed that these properties are not quite dual in general, by giving an example of a rotund space whose dual space is not smooth (in fact, whose dual space is not isomorphic to a smooth space). Klee [1] has produced a smooth space whose conjugate is not rotund. However there is no known example of a smooth space with conjugate not isomorphic to a rotund space (see Day [1, p. 518] and Cudia [1, p. 88]). We shall show that if X has property A its conjugate is isomorphic to a rotund space. This follows from the following mapping theorem, the proof of which we give in the next section.

5.2.1 THEOREM. *Let* X *be a Banach space with property* A . *Then there exist a set* Γ *and a bounded one-to-one linear map* T *from* X' *into* $c_0(\Gamma)$.

Let us recall that $c_0(\Gamma)$ is the Banach space consisting of the real-valued functions f on Γ which vanish at infinity; i.e., such that $\{\gamma : \gamma \in \Gamma , |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$.

Theorem 5.2.1 should be compared with the following powerful result of Lindenstrauss [2]: *If* X *is a reflexive Banach space, then there exist a set* Γ *and a continuous one-to-one linear map* T *of* X *into* $c_0(\Gamma)$. In fact Theorem 5.2.1 follows from this result if we assume X to be a conjugate space, for then X is reflexive as we observed in Proposition 5.1.7. More generally, Amir and Lindenstrauss [1] have shown that if X is *weakly compactly generated* (i.e., is the closed linear span of a weakly compact subset), then there exist such a set Γ and mapping T .

We prove our smoothness result as a corollary of Theorem 5.2.1 at this point.

5.2.2 COROLLARY. *Let* X *be a Banach space with property* A . *Then* X' *is isomorphic to a rotund space.*

PROOF. Theorem 5.2.1 implies the existence of a set Γ and a one-to-one bounded linear map T from X' into $c_0(\Gamma)$. But by Day [1, p. 523] $c_0(\Gamma)$ admits an equivalent strictly convex norm $|\cdot|$. We renorm X' by putting $|f| = \|f\| + |Tf|$. It is readily checked that $|\cdot|$ is an equivalent strictly convex norm on X' and so the result follows. //

We point out that though we consider spaces over the reals our results are equally valid for complex spaces.

5.3 <u>Proof of Theorem 5.2.1.</u>

The proof of Theorem 5.2.1 is based on techniques developed by Lindenstrauss [1 and 2]. It is long and is broken up by a series of lemmas.

The first result is due to Lindenstrauss [2, Lemma 1]. As the proof is not short we refer the reader to the above reference for a proof.

5.3.1 LEMMA. *Let* X *be a Banach space and let* B *be a finite dimensional subspace of* X . *Let* k *be an integer and suppose* $\varepsilon > 0$. *Then there is a finite dimensional subspace* Z *of* X *containing* B *such* *that for every subspace* Y *of* X *containing* B *with* $\dim Y/B = k$ *there is a linear map* $T : Y \to Z$ *with* $\|T\| \le 1 + \varepsilon$, *and such that* $Tb = b$ *for every* $b \in B$.

We denote by X^{α} the space of homogeneous functionals on X which are bounded on the unit ball of X . If $f \in X^{\alpha}$ we define $\|f\|$, the norm of f , by $\|f\| = \sup\{|f(x)| : \|x\| \le 1\}$. With this norm the unit ball of X^{α} is compact in the $\hat{X}$-topology as we established in Theorem 2.3.8. If C is a subspace of X and T is a map from C' into X' we find it convenient to denote the extension map of T from X' into X' by $\tilde{T}$; i.e., $\tilde{T}$ is defined by $\tilde{T}(f) = T(f/C)$, where f/C denotes the restriction of f to C .

5.3.2 LEMMA. *Let* X *be a Banach space and let* B *be a finite dimensional subspace of* X . *Then there exist a separable subspace* C *of* X *and a linear map* $T : C' \to X'$ *such that* $\|\tilde{T}\| = 1$ *and* $\tilde{T}'\hat{x} = \hat{x}$ *for each* $x \in B$.

PROOF. Let $C_n \supset B$, $n = 1,2,\ldots$ be the subspaces of X given by Lemma 5.3.1 for $k = n$ and $\varepsilon = 1/n$, and let $C = \overline{sp}[\bigcup_{n=1}^{\infty} C_n]$. If E is a subspace of X containing B and such that $\dim E/B = n$, there is a linear map $T_E : E \to C$ such that $\|T_E\| \le 1 + 1/n$ and $T_E x = x$ for every $x \in B$. We extend T_E to a map (not linear) $U_E : X \to C$ by defining $U_E x = 0$ if $x \in X \setminus E$. Consider the adjoint map $U_E' : C' \to X^{\alpha}$. In the space of all bounded linear maps from C' into X^{α} we take the pointwise topology, and on X^{α} the $\hat{X}$-topology. As the unit ball of X^{α} is $\hat{X}$-compact, Tychnonoff's theorem ensures that the net $\{U_E' : E \supset B\}$ (here we order the subspaces E by inclusion) has a limit point $T : C' \to X^{\alpha}$.

Let us show that T maps C' into X'. Suppose that $f \in C'$ and that $x,y \in X$. If $E \supset B \oplus x \oplus y$ then

$$(U_E'f)(x + y) = (U_E'f)x + (U_E'f)y .$$

Consequently the above equation holds for the limit point T. This implies that T maps C' into X'. Similarly we can show that $\|T\| \leq 1$. Therefore it suffices to show that $\tilde{T}'\hat{x} = \hat{x}$ whenever $x \in B$. We consider an arbitrary $f \in X'$ and $x \in B$. Given $\varepsilon > 0$ there exists a subspace E containing B such that $|(U_E'f/C)x - (Tf/C)x| < \varepsilon$. This implies $|(f/C)x - (Tf/C)x| < \varepsilon$ and in turn that $|\hat{x}(f) - (\tilde{T}'\hat{x})f| < \varepsilon$. But the f and $\varepsilon > 0$ were chosen arbitrarily; hence we have the result. //

5.3.3 LEMMA. *Let* X *be a smooth Banach space, let* x_i, $i = 1,\ldots,n$, *and* f_j, $j = 1,\ldots,m$, *be finite sets in* X *and* X' *respectively, and let* $\varepsilon > 0$. *Then there exist a separable subspace* C *of* X *and a linear map* $T : C' \to X'$ *such that* $\|\tilde{T}\| = 1$, $\tilde{T}'\hat{x}_i = \hat{x}_i$, $i = 1,\ldots,n$, *and* $\|\tilde{T}f_j - f_j\| < \varepsilon$, $j = 1,\ldots,m$.

PROOF. By subreflexivity there exist y_j, $j = 1,\ldots,m$, such that

$\|f_j - f_{y_j}\| < \varepsilon$, $j = 1,\dots,m$. By Lemma 5.3.2 there exist a separable subspace C of X and a linear map $T : C' \to X'$ such that $\|\tilde{T}\| = 1$, $\tilde{T}'\hat{x}_i = \hat{x}_i$, $i = 1,\dots,n$, and $\tilde{T}'\hat{y}_j = \hat{y}_j$, $j = 1,\dots,m$. As $\tilde{T}'\hat{y}_j = \hat{y}_j$, $j = 1,\dots,m$, we have $f_{y_j} = \tilde{T}f_{y_j}$, $j = 1,\dots,m$. This implies that $\|\tilde{T}f_j - f_j\| < \varepsilon$, $j = 1,\dots,m$. //

Before continuing we note an easy result.

5.3.4 LEMMA. *Let* Y *be a closed subspace of* X . *If* $\overline{D_{X'}(Y)}$ *is a linear subspace, then it is isometric to* Y' .

PROOF. Let $T : \overline{D_{X'}(Y)} \to Y'$ be the restriction map. T is a linear norm preserving map of $\overline{D_{X'}(Y)}$ into Y' . That T is onto follows from the Hahn-Banach theorem as Y is subreflexive. //

By the *density character* of a Banach space we mean the minimal cardinality of a dense subset.

5.3.5 LEMMA. *Let* X *be a smooth Banach space and let* M *be an infinite cardinal number. Suppose* Z,W *are subspaces of* X,X' *respectively of density character not greater than* M . *Then there exists a subspace* C *of* X *of density character not greater than* M *which contains* Z , *together with a linear map* $T : C' \to X'$ *such that* $P = \tilde{T}$ *is a bounded linear projection satisfying* $\|P\| = 1$, $Pf = f$ *for every* $f \in W$, $P'\hat{x} = \hat{x}$ *for every* $x \in C$, *and such that* $PX' = \overline{D_{X'}(C)}$. *In particular,* PX'

is isometric to the dual of C .

PROOF. The proof is by transfinite induction. Initially we assume that $\{f_j : j = 1,2,\ldots\}$ is dense in W, and that $\{x_j : j = 1,2,\ldots\}$ is dense in Z'. By Lemma 5.3.3 we can construct inductively a sequence $\{C_n : n = 1,2,\ldots\}$ of separable subspaces of X and a sequence $\{T_n : n = 1,2,\ldots\}$ of linear maps $T_n : C_n' \to X'$, such that

(i) $\|\tilde{T}_n\| = 1$, $\quad n = 1,2,\ldots$,

(ii) $\tilde{T}_n'\hat{x}_i = \hat{x}_i$, $\quad 1 \le i \le n$, $n = 1,2,\ldots$,

$\tilde{T}_n'\hat{x}_i^k = \hat{x}_i^k$, $\quad 1 \le i \le n$, $1 \le k \le n-1$, and

(iii) $\|\tilde{T}_n f_i - f_i\| < 1/n$, $\quad 1 \le i \le n$, $n = 1,2,\ldots$

where $\{x_i^k : i = 1,2,\ldots\}$ is dense in C_k for $k = 1,2,\ldots$.

We let $C = \overline{sp}[\bigcup_{n=1}^{\infty} C_n]$ and we consider the extensions of T_n, $\bar{T}_n : C' \to X'$, for $n = 1,2,\ldots$, defined by $\bar{T}_n(f) = T_n(f/C_n)$ where $f \in C'$. Following the technique of Lemma 5.3.2 we let T be a limit point in the $\hat{X}$-operator topology of the sequence $\{\bar{T}_n : n = 1,2,\ldots\}$ and let $P = \tilde{T}$. It follows then that $\|P\| = 1$, P is linear and that $P'\hat{x}_i^k = \hat{x}_i^k$ for every i,k. This last equation ensures that $P'\hat{x} = \hat{x}$ for each $x \in C$. This implies that $Pf = f$ for each $f \in D_{X'}(C)$ as X is smooth. Using the subreflexivity of C we easily obtain that $\overline{D_{X'}(C)} = PX'$ and that P is a projection. The last remark follows from Lemma 5.3.4.

We assume now that the lemma holds for all cardinals less than M. Suppose that Ω is the well-ordered set of ordinals less than M. Then there are closed subspaces $\{Z_\alpha : \alpha \in \Omega\}$ of Z, $\{W_\alpha : \alpha \in \Omega\}$ of W with $Z_\alpha \subset Z_\beta$, $W_\alpha \subset W_\beta$ for $\alpha < \beta$, such that the density characters of Z_α, W_α are at most the cardinality of α for infinite α, and such that $Z = \overline{\underset{\alpha \in \Omega}{\cup} Z_\alpha}$, $W = \overline{\underset{\alpha \in \Omega}{\cup} W_\alpha}$. By the induction hypothesis we can construct inductively for every $\alpha \in \Omega$ a subspace C_α of X whose density character is at most the cardinality of α for infinite α and such that $C_\alpha \supset Z_\alpha \cup \underset{\beta<\alpha}{\cup} C_\beta$. Together with each C_α we can construct a linear map $T_\alpha : C'_\alpha \to X'$ such that $P_\alpha = \tilde{T}_\alpha$ is a linear projection satisfying the conditions $\|P_\alpha\| = 1$, $P'_\alpha \hat{x} = \hat{x}$ for each $x \in C_\alpha$, $P_\alpha f = f$ for each $f \in W_\alpha$ and $P_\alpha X' = \overline{D_{X'}(C_\alpha)}$. We let $C = \overline{\underset{\alpha \in \Omega}{\cup} C_\alpha}$ and consider the extensions of T_α, $\bar{T}_\alpha : C' \to X'$ for each α. Again for T we take a limit point in the $\hat{X}$-operator topology of the net $\{\bar{T}_\alpha : \alpha \in \Omega\}$. We leave the reader to check that T and C satisfy the conditions of the lemma. //

Before proceeding we need to note two simple properties of Banach spaces with property A.

5.3.6 LEMMA. *Let* Y *be a Banach space with property* A. *Then the density character of* Y' *is that of* Y.

PROOF. It is sufficient to check that the density character of Y' is not greater than the density character M of Y. If Ω is the well-ordered set of ordinals less than M we may assume that $\{y_\alpha : \alpha \in \Omega\}$ is dense in Y. The set Φ consisting of all finite rational linear combinations of the elements f_{y_α} is a set of cardinality M. We show first that Φ is dense in $D_{Y'}(Y)$. If $y \in Y$ there is a sequence $\{y_{\alpha_n} : n = 1,2,\ldots\}$ such that $y_{\alpha_n} \to y$ in norm. Property A implies that $f_y \in \overline{sp}\{f_{y_{\alpha_n}}\}$ which ensures that f_y belongs to the closure of Φ. The lemma now follows as Y is subreflexive. //

5.3.7 LEMMA. *Suppose* X *is a Banach space with property* A, *and that* $Y_\alpha \subset Y_\beta \subset X$ *for* $\alpha < \beta < \gamma$. *Then*

$$\overline{D_{X'}(\overline{\bigcup_{\alpha<\gamma} Y_\alpha})} = \overline{\bigcup_{\alpha<\gamma} D_{X'}(Y_\alpha)},$$

provided

$$\overline{\bigcup_{\alpha<\gamma} D_{X'}(Y_\alpha)} \text{ is a subspace.}$$

PROOF. It suffices to show

$$D_{X'}(\overline{\bigcup_{\alpha<\gamma} Y_\alpha}) \subset \overline{\bigcup_{\alpha<\gamma} D_{X'}(Y_\alpha)}.$$

To establish this we consider a support functional f_y where $y \in \overline{\bigcup_{\alpha<\gamma} Y_\alpha}$. Then there exists a sequence $\{y_n : n = 1,2,\ldots\} \subset \bigcup_{\alpha<\gamma} Y_\alpha$ such that $y_n \to y$ in norm. We need only invoke property A once more to obtain the result. //

We are now in a position to prove a theorem whereby it will be possible to reduce the proof of Theorem 5.2.1 to the separable case.

5.3.8 THEOREM. *Let* X *be a Banach space with property* A . *Let* μ *be the first ordinal of cardinality the density character* M *of* X . *For every* α *satisfying* $\omega \leq \alpha \leq \mu$, *there is a subspace* X_α *of* X *of density character at most the cardinality of* α , *together with a linear map* $T_\alpha : X'_\alpha \to X'$ *such that* $P_\alpha = \tilde{T}_\alpha$ *is a bounded linear projection of* X' *into* X' *satisfying*

1. $\|P_\alpha\| = 1$,
2. $P_\alpha X' = \overline{D_{X'}(X_\alpha)}$, *and is isometric to* X'_α ,
3. $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ *where* $\beta < \alpha$,
4. $\bigcup_{\beta<\alpha} P_{\beta+1} X'$ *is dense in* $P_\alpha X'$, *for every* $\alpha > \omega$.

Moreover, $\bigcup_{\alpha<\mu} P_\alpha X'$ *is dense in* X' .

PROOF. By Lemma 5.3.6 we may assume that $\{f_\alpha : \alpha < \mu\}$ is a dense subset of X' . We construct $\{T_\alpha : \omega \leq \alpha \leq \mu\}$ by transfinite induction. If $M = \aleph_0$, $T_\omega = P_\omega = I$ has the required properties. Assume now that $M > \aleph_0$. By Lemma 5.3.5 there is a separable space X_ω together with a map T_ω such that $P_\omega = \tilde{T}_\omega$ satisfies $\|P_\omega\| = 1$, $P_\omega X' = \overline{D_{X'}(X_\omega)}$ and $P_\omega f_\alpha = f_\alpha$ for $\alpha < \omega$. We therefore assume that the subspaces X_β , and corresponding maps T_β have been defined for $\omega \leq \beta < \gamma$, and that they satisfy conditions 1 to 4 of the theorem.

Suppose that $\gamma = \alpha + 1$. Then we invoke Lemma 5.3.5 to obtain a subspace X_γ and a linear map T_γ so that $X_\alpha \subset X_\gamma$ and also so that $P_\gamma = \tilde{T}_\gamma$ restricted to $P_\alpha X' \cup \{f_\alpha\}$ is the identity. Lemma 5.3.5 is applicable by Lemma 5.3.6. It follows that $P_\gamma P_\beta = P_\gamma P_\alpha P_\beta = P_\alpha P_\beta = P_\beta$ for $\beta < \gamma$. Similarly $P'_\gamma P'_\beta = P'_\beta$ provided $\beta < \gamma$ may be establish using the fact that $\hat{X}_\alpha$ is weak* dense in X''_α (Dunford and Schwartz [1, p. 425]).

If on the other hand γ is a limiting ordinal, let $X_\gamma = \overline{\bigcup_{\alpha < \gamma} X_\alpha}$ and let $\bar{T}_\alpha : X'_\gamma \to X'$ be the extensions of T_α to X'_γ for $\omega \le \alpha < \gamma$. For T_γ we take a limit point in the $\hat{X}$-operator topology of the net $\{\bar{T}_\alpha : \omega \le \alpha < \gamma\}$. Properties 1, 2 and 3 follow without difficulty whilst 4 holds by virtue of Lemma 5.3.7.

The last statement now follows as $f_\alpha \in P_\omega X'$ for $\alpha < \omega$ and $f_\alpha \in P_{\alpha+1} X'$ for $\alpha \ge \omega$. //

5.3.9 LEMMA. *Let* X *be a Banach space with property* A *and let* $\{P_\alpha : \omega \le \alpha \le \mu\}$ *be the set of projections on* X' *as in Theorem* 5.3.8. *Then for every* $f \in X'$ *and* $\varepsilon > 0$ *the set* $\{\alpha : \|P_{\alpha+1} f - P_\alpha f\| \ge \varepsilon\}$ *is finite.*

PROOF. Suppose we assume on the contrary that there is an infinite sequence of ordinals $\omega \le \alpha_1 < \alpha_2 < \ldots < \mu$ such that $\|P_{\alpha_{i+1}} f - P_{\alpha_i} f\| \ge \varepsilon$ for $i = 1,2,\ldots$. Let us denote α_i by $2i - 1$, α_{i+1} by $2i$. Let

$X_\infty = \overline{\bigcup_{i=1}^{\infty} X_i}$ and consider the extensions of T_i, $\bar{T}_i : X'_\infty \to X'$, for $i = 1,2,\ldots$. Suppose that T_∞ is a limit point in the $\hat{X}$-operator topology of the sequence $\{\bar{T}_i : i = 1,2,\ldots\}$. Then $P_\infty = \tilde{T}_\infty$ is a projection of X' onto $\overline{\bigcup_{i=1}^{\infty} P_i X'}$ and $P_i P_\infty = P_i$ for $i = 1,2,\ldots$.

If $h \in P_\infty X'$, it follows that $\lim_i \|P_i h - h\| = 0$. For suppose that $g \in P_j X'$ and that $\|g - h\| < \delta/2$, then
$\|P_i h - h\| \le \|P_i h - P_i g\| + \|P_i g - g\| + \|g - h\| < \delta$ for $i > j$. Hence
$\lim \|P_i f - P_\infty f\| = \lim \|P_i P_\infty f - P_\infty f\| = 0$.

But this implies that $\{P_i f : i = 1,2,\ldots\}$ is a Cauchy sequence which contradicts our original assumption. //

Before establishing Theorem 5.2.1 we need observe an elementary result.

5.3.10 LEMMA. *Let* X *be a Banach space with property* A . *Then if* Y *is a closed subspace of* X, Y *is a Banach space with property* A .

PROOF. As X is smooth we have that Y is smooth. Let us therefore suppose that $\{y_n\}$ is a sequence in Y such that $y_n \to y$ in norm. Let g_{y_n}, g_y denote the support functionals in Y' and f_{y_n}, f_y the support functionals in X'.

Since $f_y \in \overline{sp}\{f_{y_n}\}$ in X' it follows that $g_y \in \overline{sp}\{g_{y_n}\}$ in Y' and this completes the proof. //

PROOF OF THEOREM 5.2.1. The proof is by transfinite induction on the density character M of X or X' (recall Lemma 5.3.6). If $M = \aleph_0$ the result is well-known: we may take $\{x_n\}$ to be a dense subsequence of the unit ball and define $(Tf)(n) = f(x_n)/n$, for $n = 1,2,\ldots$.

Let us assume now that the theorem has been proved for all cardinals smaller than M . Let $\{P_\alpha : \omega \le \alpha < \mu\}$ be the set of projections constructed in Theorem 5.3.8. We know by the previous lemma that $P_\alpha X'$ is isometric to X'_α the conjugate of a Banach space with property A . Furthermore the density character of X_α is less than M and so the induction hypothesis implies the existence of a set Γ_α and a one-to-one bounded linear map from $P_\alpha X'$ into $c_0(\Gamma_\alpha)$. We may assume that the Γ_α are pairwise disjoint and that $\|T_\alpha\| \le 1$ for α satisfying $\omega \le \alpha < \mu$. We let $\Gamma = N \cup \bigcup\{\Gamma_{\alpha+1} : \omega \le \alpha < \mu\}$ where N denotes the natural numbers. Then we define

$$(Tf)n = (T_\omega P_\omega f)n \quad \text{for } n \in N \text{, and}$$

$$(Tf)\gamma = 1/2\, T_{\alpha+1}(P_{\alpha+1}f - P_\alpha f)\gamma \quad \text{for } \gamma \in \Gamma_{\alpha+1} .$$

Lemma 5.3.9 guarantees that T maps X' into $c_0(\Gamma)$. It is also clear that T is linear and $\|T\| \le 1$. Furthermore if $Tf = 0$ we have $P_\omega f = 0$ and $P_{\alpha+1}f = P_\alpha f$ for α satisfying $\omega \le \alpha < \mu$. As $\bigcup_{\beta < \alpha} P_\beta X'$ is dense in $P_\alpha X'$ for every limiting ordinal $\alpha > \omega$, it follows by transfinite induction that $P_\alpha f = 0$ for all $\alpha < \mu$. But $\bigcup_{\alpha < \mu} P_\alpha X'$ is dense in X' so that $f = 0$. Hence T is one-to-one and the proof is completed. //

5.4 Remarks on the proof of Theorem 5.2.1.

There are a number of immediate questions concerning Theorem 5.2.1. We would of course like to establish the existence of a bounded one-to-one linear map from X into $c_0(\Gamma)$ for some set Γ . Then the technique of Corollary 5.2.2 would allow us to assert that if X is a Banach space with property A , it is isomorphic to a rotund space. It is not difficult to see that we could show the existence of such a map if the linear maps $\{T_\alpha : \omega \le \alpha \le \mu\}$ of Theorem 5.3.8 were adjoints. In Lemma 5.3.2 we began by considering a net of adjoints $\{U_E' : E \supset B\}$ and then taking a limit point T in the $\hat{X}$-operator topology. The following simple example shows that generally we could not expect T to be an adjoint also.

5.4.1 EXAMPLE. We consider a sequence of maps $P_n : m \to c_0$ defined by $P_n : (x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots, x_n, 0 \ldots)$ for $n = 1,2,\ldots$. Now the natural embedding π of c_0' into m' is a limit point of the sequence $\{P_n'\}$ in the $\hat{X}$-operator topology. However π is not an adjoint map for it is not continuous from $(c_0', \sigma(c_0', c_0))$ into $(m', \sigma(m', m))$.

5.4.2 REMARKS. We comment that one consequence of Lemma 5.3.6 is that not all smooth spaces have property A . For example, ℓ_1 is isomorphic to a smooth space as it is separable (see Day [1, Theorem 4]), yet $\ell_1' = m$ is non-separable. On the other hand it is interesting to observe that a result of Klee [2, Corollary 1.5] implies that a Banach space X with separable conjugate X' is isomorphic to a strongly smooth space (and hence to a Banach space with property A).

It is perhaps instructive to observe another consequence of Klee's result. Suppose we consider a Banach space X with separable second conjugate X" . Klee's corollary implies that X' is isomorphic to a strongly smooth space. If this space were a conjugate space we would have that X is reflexive by Proposition 5.1.7. But separable quasi-reflexive spaces (see Civin and Yood [1]) have separable second conjugates. Thus we have an example of a conjugate space with an equivalent norm which is not a dual norm. In fact there is a considerable history to this problem which was posed first by Dixmier [1, p. 1070]. Recently Williams [1] has shown that each equivalent norm on X' is a dual norm if and only if X is reflexive.

CHAPTER 6

ALMOST REFLEXIVITY AND RELATED PROPERTIES

6.0 Introduction.

In Chapter 3 we discussed a generalization of weak compactness for subsets of LCTVSs. Then we considered a class of generalizations of semi-reflexivity. In this final chapter we are chiefly concerned with a different type of generalization of reflexivity called almost reflexivity. This concept is of special relevance in the theory of Banach spaces and so our attention is again restricted to these spaces. We use only standard methods in this treatment: the author was unable to gain any advantage by the use of non-standard techniques.

6.1 Basic definitions and questions.

6.1.1 DEFINITION. Let X be a Banach space and suppose that A is a subset of X . We say that A is *weakly conditionally sequentially compact* if every sequence in A contains a weak Cauchy subsequence. If the unit ball is weakly conditionally sequentially compact X is said to be *almost reflexive*; i.e., X is almost reflexive if every bounded sequence contains a weak Cauchy subsequence.

We remark that in arbitrary Banach spaces every bounded sequence has a weak Cauchy subnet. This follows as bounded sets are weakly precompact for recall Theorem 2.4.3. Furthermore, if a bounded set is weakly metrizable it is weakly conditionally sequentially compact (for in metric spaces sequential compactness is equivalent to compactness). Eberlein's theorem implies that a weakly sequentially complete space which is almost reflexive is reflexive.

6.1.2 DEFINITION. A Banach space X is *quasi-separable* if every separable subspace of X has a separable conjugate space.

Lacey and Whitley [1, Theorem 3] show that a quasi-separable space X is almost reflexive. On the other hand they were not able to decide whether the converse also holds. There is another concept closely related to almost reflexivity and we find it convenient to name it for the purposes of this chapter.

6.1.3 DEFINITION. We say that a Banach space X is *nearly almost reflexive* if for each bounded sequence $\{x_n\}$ in X there is a weak Cauchy sequence of averages far out in $\{x_n\}$ (see Day [2, p. 40]).

An almost reflexive space is nearly almost reflexive. McWilliams [2, Theorem 2] shows that a Banach space X which is weak* sequentially dense in X'' (i.e., for which $K(X) = X''$) is nearly almost reflexive. Conversely, McWilliams shows that if X is almost reflexive then $K(X)$

need not equal X''. However the converse remains open for separable spaces. We comment that if X' is separable then $K(X) = X''$; for the unit ball B is weak* dense in B'' and B'' equipped with the X'-topology is metrizable (Dunford and Schwartz [1, p. 426]). These and other questions are closely related to Problem 5.4 of Bessaga and Pelczynski [2]. Let us recall it here.

Let X *be a separable Banach space. Are the following conditions equivalent:*

(a) *every bounded set in* X *is weakly conditionally sequentially compact,*

(b) *no subspace of* X *is isomorphic to* ℓ,

(c") X' *is separable?*

It is clear that the equivalence of these conditions implies the equivalence of almost reflexivity, quasi-separability and nearly almost reflexivity for spaces of arbitrary density character. Indeed if X were separable these conditions would then be equivalent to the condition that $K(X) = X''$.

Furthermore a number of other results would follow trivially from the equivalence of (a), (b) and (c"). For example we easily obtain that if X' is almost reflexive then X is almost reflexive (for assuming that X is not almost reflexive we would have that X contains a copy of ℓ_1 so that consequently, since m is not almost reflexive, X' would not be almost reflexive).

One important positive result is the following.

6.1.4 THEOREM. *Let* X *be a subspace of a Banach space with an unconditional basis. Then the following conditions are equivalent:*

(1) X *is almost reflexive,*

(2) X *is nearly almost reflexive,*

(3) X' *is separable,*

(4) $K(X) = X''$.

PROOF. The result is an immediate consequence of Theorem 1 of Bessaga and Pelczynski [2]. //

In this context it may prove useful to know that Pelczynski [2, p. 373] has shown that a Banach space X is almost reflexive if and only if every subspace of X with a basis is almost reflexive.

6.2 Almost reflexive spaces.

In this section we are concerned with almost reflexive spaces and weakly conditionally sequentially compact subsets of Banach spaces. As we remarked in the previous section it is not true that X almost reflexive implies $K(X) = X''$: the space $c_0(\Gamma)$ is almost reflexive but is not weak* sequentially dense in $m(\Gamma)$ when Γ is uncountable (McWilliams

[2, Example 2]). We do have however the following result.

6.2.1 THEOREM. *Let* X *be an almost reflexive Banach space. Then* $K(X)$ *equipped with the* X'*-topology is nearly semi-reflexive.*

PROOF. Let E denote $(K(X), \sigma(K(X),X'))$. It suffices by Theorem 3.4.2 to show that each bounded set in E is nearly weakly compact. If B is such a set it is bounded in the norm topology on $K(X)$ by the uniform boundedness principle. Consequently there exists a bounded set A in X such that B is contained in the closure of $\hat{A}$ considered as a subset of E . Now A is weakly conditionally sequentially compact, so that $\hat{A}$ is a weakly sequentially compact subset in E . Corollary 3.3.4 implies that B is nearly weakly compact which establishes the result. //

We observe that $K(X) = X''$ if and only if E is semi-reflexive. An interesting consequence of the previous theorem is obtained by considering the space $c_0(\Gamma)$.

6.2.2 COROLLARY. *Let* Γ *be an arbitrary set. Then the space* $m_0(\Gamma)$ *equipped with the* $\ell_1(\Gamma)$*-topology is nearly semi-reflexive.*

PROOF. The corollary follows from Theorem 6.2.1 as $c_0(\Gamma)$ is almost reflexive and $K(c_0(\Gamma)) = m_0(\Gamma)$. //

6.2.3 EXAMPLE. It is not generally true that if X is almost reflexive then the unit ball B'' of X'' is weak* sequentially compact. Let $X = c_0(\Gamma)$ where $\Gamma = [0,2\pi]$. Then X is almost reflexive but the unit ball of $X'' = m(\Gamma)$ is not weak* sequentially compact as we now show. We define a bounded sequence $\{x''_n\}$ in $m(\Gamma)$ by $x''_n(\gamma) = \sin n\gamma$ for all $\gamma \in [0,2\pi]$ and for $n = 1,2,\dots$. If $\{x''_n\}$ has a weak* Cauchy sequence $\{x''_{n_k}\}$, $\lim_k x''_{n_k}(f)$ exists for each $f \in \ell_1(\Gamma)$. This implies that $\lim_k \sin n_k\gamma$ exists for each $\gamma \in [0,2\pi]$. But this is impossible (see Rudin [1, p. 143]). The question remains whether if X is separable and almost reflexive, B'' is weak* sequentially compact (if, in fact, X' is then separable, B'' is weak* sequentially compact). The converse is always true. Thus if X' is weakly compactly generated X is almost reflexive for B'' is weak* sequentially compact by a result of Amir and Lindenstrauss [1, Corollary 2].

Let us now consider a weakly conditionally sequentially compact subset A of a Banach space X. It is of some interest and importance to know whether the closed, circled, convex hull of A, $\overline{\Gamma(A)}$, is also weakly conditionally sequentially compact. We do have the following result.

6.2.4 THEOREM. *Suppose that* $\{x_n\}$ *is a weak Cauchy sequence in* X. *Then the closed, convex, circled hull of* $\{x_n\}$ *is weakly conditionally sequentially compact.*

PROOF. We denote $(X'',\sigma(X'',X'))$ by E and consider $\hat{A} = \{\hat{x}_n\}$ as a subset of E. Then $\hat{A}$ is relatively compact and so Krein's theorem implies that $\Gamma(\hat{A})$ is compact. Now as $\hat{A}$ is the set of points of a Cauchy sequence in E, $\overline{\Gamma(\hat{A})}$ is metrizable (Edwards [1, p. 634]). Therefore $\overline{\Gamma(\hat{A})}$ is sequentially compact which in turn implies that $\overline{\Gamma(A)}$ is weakly conditionally sequentially compact in X. //

Garling [1, Proposition 2] has shown that if the unit ball is contained in the closed, circled, convex hull of a weak Cauchy sequence then the space is finite dimensional.

Let X,Y be Banach spaces. A linear map $T : X \to Y$ is *completely continuous* if it maps weakly convergent sequences into norm convergent sequences.

6.2.5 DEFINITION. Let X be a Banach space. If for every Banach space Y, each weakly compact map $T : X \to Y$ is completely continuous then X is said to have *property* D.P. (i.e., the *Dunford-Pettis property*).

The following result is essentially one of Grothendieck [1, Théorème 10]; see also Pelczynski [4, Proposition 1.2].

6.2.6 THEOREM. *Let* X *be an almost reflexive Banach space. Suppose that either* X *or* X' *has property* D.P. *Then every weak Cauchy sequence in* X' *converges in the norm topology of* X'.

PROOF. If X' has property D.P., X has property D.P. (Grothendieck [1, p. 136]). Therefore it suffices to establish the result under the assumption that X has property D.P. It is sufficient to show that every sequence which is weakly convergent to zero in X' converges in the norm topology of X'. This follows from the fact that in a Banach space a sequence $\{x_n\}$ is (weak) Cauchy if and only if $\{x_{n_i} - x_{n_{i+1}}\}$ is (weakly) convergent to zero for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

For such a sequence $\{f_n\}$ we define

$$Tx = (f_n(x)) \quad \text{for all} \quad x \in X .$$

T is a linear map from X into c_0 and is weakly compact as we now show. We first observe that $T'e'_n = f_n$ where $e'_n = (\delta_n^m)$ is the nth unit vector in the space $\ell = c_0'$. Since the unit ball B_ℓ of ℓ is the closed, circled, convex hull of the unit vectors, the set $T'B_\ell$ is the closed, circled, convex hull of a sequence which is weakly convergent to zero in X'. Hence by Krein's theorem $T'B_\ell$ is a weakly compact subset of X'. Thus T' is weakly compact and therefore so is T. As X has property D.P. this implies that TB_X is a relatively compact subset of c_0 since X is almost reflexive.

Therefore $\lim_n \|f_n\| = \lim_n \sup \{|e'_n Tx| : \|x\| \leq 1\} = 0$. //

6.2.7 COROLLARY. *Suppose* X *is an almost reflexive Grothendieck space which has property* D.P. *Then weak* convergent sequences in* X' *are norm* *convergent.*

6.2.8 REMARK. A simple generalization of Lemma 8 of Grothendieck [1] shows that the condition that weak* convergent sequences in X' be norm convergent is equivalent to the property that any continuous linear map from X into a separable Banach space Y is compact.

6.2.9 COROLLARY. *Suppose* X *is a separable, almost reflexive space which has property* D.P. *If* X *is isomorphic to a conjugate space then* X *is finite dimensional.*

PROOF. If X is isomorphic to a conjugate space weak Cauchy sequences in X converge in the norm topology. As X is almost reflexive this implies that the unit ball B_X is compact. Thus X is finite dimensional as claimed. //

6.3 <u>Nearly almost reflexive spaces.</u>

6.3.1 DEFINITION. Let X be a Banach space. Then X has *property* D (i.e., the *Dieudonné property*) if for every Banach space Y each linear map $T : X \to Y$ which maps weak Cauchy sequences into weakly convergent sequences is weakly compact.

6.3.2 THEOREM. *Let* X *be a Banach space. If* X *is nearly almost* *reflexive then* X *has property* D .

PROOF. Let $T : X \to Y$ and suppose T maps weak Cauchy sequences into weakly convergent sequences. Consider a bounded sequence $\{x_n\}$ in X . Then there is a weak Cauchy sequence of averages far out in $\{x_n\}$, say $\{w_n\}$. Therefore $\{Tw_n\}$ is a weakly convergent sequence in Y and so T is weakly compact by a result of Nishuira and Waterman [1, Theorem 2]. //

The following corollary should be compared with Proposition 15 of Grothendieck [1, p. 170].

6.3.3 COROLLARY. *Let* X *be nearly almost reflexive and let* T *be a linear map from* X *into* Y . *Then if* T *is completely continuous* T *is weakly compact.*

PROOF. If T is completely continuous T maps weak Cauchy sequences into norm convergent sequences. //

6.3.4 DEFINITION. A continuous linear map T is *strictly singular* if whenever the restriction of T to a closed subspace has a bounded inverse it follows that the subspace is finite dimensional.

The following result generalizes Theorem 7 of Lacey and Whitley [1].

6.3.5 THEOREM. *Let* Y *be nearly almost reflexive and suppose that* T *is a linear map from* X *into* Y . *If* T *is completely continuous then it is strictly singular.*

PROOF. Suppose that T is completely continuous and that the restriction of T to a closed subspace M has a bounded inverse. Then M is nearly almost reflexive and so, by Corollary 6.3.3, T/M is weakly compact. The result is implied by a theorem in Goldberg [1, p. 88]. //

6.3.6 REMARK. The result of Nishiura and Waterman [1, Theorem 2] implies that if a weakly sequentially complete space X is nearly almost reflexive then X is reflexive. Thus for weakly sequentially complete spaces we have that nearly almost reflexivity is equivalent to almost reflexivity, which in turn is equivalent to reflexivity.

6.3.7 DEFINITION. (a) The series Σx_n is *weakly unconditionally Cauchy* (*w.u.c.*) if for every permutation (k_n) the series Σx_{k_n} is weak Cauchy.

(b) The series Σx_n is *unconditionally convergent* (*u.c.*) if for every permutation (k_n) the series Σx_{k_n} converges.

6.3.8 THEOREM. *Suppose* X *is nearly almost reflexive. Then in* X' *every w.u.c. series is u.c..*

PROOF. Suppose there exists in X' a w.u.c. series which is not u.c.. By Theorem 5 of Bessaga and Pelczynski [1] X' contains a subspace isomorphic to c_0 . By Theorem 4 of the same paper this implies X contains a subspace isomorphic to ℓ_1 . But ℓ_1 is not nearly almost reflexive. //

6.3.9 DEFINITION. (a) Let X,Y be Banach spaces. A linear map $T : X \to Y$ is *unconditionally converging* (*u.c.*) if it maps every w.u.c. series in X into u.c. series in Y .

(b) A Banach space X has *property* (V) if every u.c. map $T : X \to Y$ is weakly compact.

(c) A Banach space X has *property* (u) if for every weak Cauchy sequence $\{x_n\}$ in X there exists a w.u.c. series Σu_k such that $\{x_n - \sum_{k=1}^{n} u_k\}$ is weakly convergent to zero.

6.3.10 THEOREM. *If* X *is a nearly almost reflexive Banach space which has property* (u) , *then* X *has property* (V) .

PROOF. Let Y be an arbitrary Banach space and let $T : X \to Y$ be u.c.. Let $\{x_n\}$ be an arbitrary sequence in X . Then, by assumption, there exists a weak Cauchy sequence of averages $\{w_n\}$ far out in $\{x_n\}$. Since X has property (u) there is a w.u.c. series Σu_k such that the sequence $\{w_n - \sum_{k=1}^{n} u_k\}$ converges weakly to zero. Thus as T is u.c., $\{Tw_n\}$ converges weakly to the element ΣTu_k of Y . That T is weakly compact follows by the result of Nishiura and Waterman [1]. //

6.3.11 COROLLARY. *If* X *is a nearly almost reflexive space which is isomorphic to a subspace of a space with an unconditional basis, then* X *has property* (V) .

PROOF. Corollary 2 of a paper of Pelczynski [1] implies such a space has property (u) . The result is therefore a consequence of Theorem 6.3.10. //

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