THE THEORY OF PARTIALLY IONISED AND NON-UNIFORM PLASMA

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Preface

This thesis is primarily concerned with the development and application of certain equations in the kinetic theory of partially ionised gases and plasma. In particular, the "13-moment" equations, which were originally developed by Grad for a simple gas and applied to mono-atomic gas mixtures by subsequent authors, have been further developed and extended to include partially ionised gases. In doing this, all possible types of binary collision phenomena have been included. The resulting equations are more general than certain other sets of moment equations (e. g. those of Chapman and Enskog), and adequately describe the behaviour of non-uniform and partially ionised gas mixtures.

The thesis is divided into an Introduction and three Parts. In the Introduction, the contents of the thesis are summarised, while the use of the Boltzmann equation and the associated moment equations or equations of change are briefly discussed. Grad's Hermite polynomial expansion of the distribution function and the 13-moment approximation to this function are also given. Chapter 1 gives the general form of the dynamical equations for a gas mixture as obtained by Herdan and Liley. The transport approximations of these equations for a several component plasma are discussed. In Chapter 2 the general collision integrals for the 13-moment equations are obtained in a form applicable to all types of binary collision processes. In terms of originality and applicability, this is possibly the most important result of the thesis. Chapters 3 and 4 comprise the second Part. In Chapter 3, the collision integrals derived in Chapter 2 are explicitly calculated for the processes involved in the formation of a hydrogenic-like plasma, while the fourth

Chapter is concerned with reducing the moment equations to normal form, and obtaining boundary conditions in a form appropriate to these equations. A finite difference scheme applicable to the resulting equations is also briefly considered. In the final Part, Chapters 5 and 6, the propagation of high frequency waves is discussed according to the 13-moment and Maxwell's equations. Chapter 5 applies to a homogeneous partially ionised plasma, with all possible types of binary collisions being (implicitly) included, and in Chapter 6, wave motion in inhomogeneous plasma is considered to the approximation of geometrical optics, but with collisions ignored.

References are given after each Chapter, and, where applicable, after the appropriate appendices. Equations have been numbered on the left hand side.

With regard to the originality of this thesis, the Introduction and Chapter 1 essentially summarise established theory. The general collision integrals given in Chapter 2 were obtained in cooperation with Dr B. S. Liley. The remaining Chapters, while principally the work of the author, owe much to the free exchange of ideas and discussions with Dr Liley. Again, the author has been fortunate to have access to the report (and thesis) "The Non-Equilibrium Theory of Mono-Atomic Gas Mixtures and Plasma" by B. S. Liley.

In making acknowledgements, the author owes grateful thanks to Dr Liley, without whose assistance and encouragement little may have been achieved. He also wishes to thank the other members of the Plasma Physics Group of the Department of Engineering Physics at the Australian National University for assistance given and for helpful discussions. Finally, the assistance of the heads of the Department,

formerly Professor M. L. Oliphant, followed by Mr. J. W. Blamey (interim head of Department for one year) and presently Professor G. Newstead, is acknowledged.

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Introduction

1. General

The basic object of this thesis is to extend the application of the 13-moment equations. These were originally developed by Grad[1] to describe the behaviour of a simple gas, and subsequently applied to gas mixtures and plasmas by a number of other authors (Kolodner[2], Herdan and Liley[3]).

In the first part of the thesis, the moment equations and the associated collision integrals are generalised to take full account of inelastic (binary) collisions, including excitation, ionisation, charge exchange, fusion, and particle annihilation and creation.

In the second part, these equations are then given in a form applicable to the behaviour of a simple hydrogenic-like plasma in a cylindrical geometry. Certain coefficients in the collision integrals are calculated explicitly, while in Chapter 4 the 13-moment equations are expressed in terms of the physical components of the moments, with particular reference to a cylindrical geometry. The associated characteristic equation is solved and the eigenvalues and eigenfunctions obtained. Using these, the equations are reduced to Jordan normal form, and to characteristic normal form for special cases. The boundary conditions relevant to this form of the moment equations are obtained, again with particular reference to cylindrical geometry, so that the equations are essentially in a form applicable to numerical solution.

In the third and final part of the thesis the propagation of high frequency waves in both homogeneous and inhomogeneous plasmas is considered. By using the 13-moment equations, collision effects such as resistivity, thermoelectric processes, viscosity etc., are fully taken into account in the homogeneous case. The propagation of "stress" waves is also discussed in detail for both cases.

Thus, briefly summarising, the general scope of this thesis involves the derivation of a closed system of moment equations, in which all types of binary collision phenomena relevant to a partially ionised gas are included, together with the application of these equations to certain particular problems.

Because of the large number of symbols and definitions involved, all except those used locally within a section have been listed at the end of the thesis with a reference to the page number on which they are defined. The notation is basically similar to that of Liley[4], a work that is followed closely in many respects although of course this thesis is intended to be self-contained. Wherever applicable, the m.k.s. system of rationalised units has been used.

2. The Boltzmann Equation

The physical derivation of this equation is well known. However, in attempting a rigorous derivation of the Boltzmann equation from the Louiville equation certain assumptions and approximations are necessary. In general, these are such that the adequacy of a binary collision type of integral has been severely questioned from time to time, particularly in the application to ionised gases. There is little doubt that the Boltzmann equation is valid for adequately rarefied neutral gases; however, for ionised gases the use of a binary collision

integral is obviously open to criticism with the total integral of the differential cross section (for Coulomb collisions) being infinite.

Nevertheless there are well established methods of circumventing this difficulty, and their use has been discussed by Grad[5]. His argument is, briefly, as follows. Taking the Debye length as a convenient cut-off distance (more strictly an exponential form of cut-off should be used, but it has been shown by Liboff[6] that there is little error in replacing it by the simple Debye cut-off), one may consider two regions within the Debye sphere, an inner "core" around the particle of interest within which the particles experience binary collisions, and the region outside this "core" where there are multiple collisions. Supposing that the multiple deflections in this region are random and independent, the mathematical model of this stochastic process is the same as that for the different physical process of independent random binary deflections; in other words, the binary collision term mathematically describes the process up to the Debye cut-off.

The question as to whether the simultaneous small deflections are completely independent, thus limiting the validity of this argument, has been examined by Koga[7] in a different manner. Using order of magnitude calculations, he was able to show that the contribution to the interaction term from the effects of correlations of particle motion is in general much less than the "binary" part, thus limiting the consequences of inadequate assumptions in this region. In fact, provided

(1) $$\frac{1}{n \lambda_D^3} < 10^{-2}$$

n being particle density and λ_D the Debye length, the effects of such correlations are negligible.

Other aspects of the Boltzmann equation, for example, quantum mechanical effects, are discussed in several books (e.g. Hirschfelder, Curtiss and Bird[8]). The full validity of the Boltzmann equation is assumed for the purposes of this thesis.

3. Enskog's Equation of Change

The derivation of the equations of change, or the moment equations, is well known (see, for example, Chapman and Cowling[9]). In particular, the equation of change for a molecular property $\psi_i(\underline{r}, \underline{w}_i, t)$ is obtained by multiplying the Boltzmann equation by $\psi_i \, d\underline{c}_i$ and integrating over velocity space $\underline{c}_i$, $\underline{c}_i$ being the molecular velocity. With the moments referred to a frame moving with velocity $\underline{V}$, for each particle type i present in a gas mixture, this equation is

$$(2) \quad \frac{d}{dt}(n_i \overline{\psi_i}) + n_i \overline{\psi_i} \nabla \cdot \underline{V} + \nabla \cdot (n_i \overline{\psi_i \underline{w}_i}) - n_i \left[\overline{\frac{d\psi_i}{dt}} + \overline{\underline{w}_i \cdot \nabla \psi_i} + \underline{F}_i \cdot \overline{\frac{\partial \psi_i}{\partial \underline{w}_i}} + \underline{b}_i \cdot \overline{\frac{\partial \psi_i}{\partial \underline{w}_i} \times \underline{w}_i} - \overline{\frac{\partial \psi_i}{\partial \underline{w}_i} \underline{w}_i} : \nabla \underline{V} \right] = \frac{1}{2} \sum_{j,k} I_{jk}(\psi_i) \, .$$

In this equation, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{V} \cdot \nabla$, $\underline{F}_i = \underline{a}_i + \underline{V} \times \underline{b}_i - \frac{d\underline{V}}{dt}$, and $\underline{w}_i = \underline{c}_i - \underline{V}$, where $\underline{c}_i$ is the particle velocity relative to a rest frame; $\underline{b}_i = \frac{e_i}{m_i} \underline{B}$ with e_i, m_i the charge and mass of particles of type i and $\underline{B}$ the macroscopic magnetic field; $\underline{a}_i + \underline{V} \times \underline{b}_i$ is the acceleration per unit mass due to macroscopic fields (e.g. for an electric field $\underline{E}$, $\underline{a}_i = \frac{e_i}{m_i} \underline{E}$); and n_i is the number density

of particles of type i . The roman subscripts denote particle type. The average of ψ_i over velocity space, $\overline{\psi_i}$, is defined by

$$\overline{\psi_i} = \frac{1}{n_i} \int_{\underline{w}_i} f_i \, \psi_i \, d\underline{w}_i \tag{3}$$

and the moment of ψ_i is $n_i \overline{\psi_i}$, where f_i is the velocity distribution function of particles of type i . The term on the right hand side of equation (2) is the collision term and will be discussed later.

Regarding notation, if $\underset{\sim}{x}$, $\underset{\sim}{y}$ are second rank tensors, $\underline{z}$ a vector and $\underline{e}_\alpha$, $\underline{e}^\beta$ covariant and contravariant base vector components respectively, $\underset{\sim}{g}$ the metric tensor:

$$\underset{\sim}{x} : \underset{\sim}{y} \equiv x_{\alpha\beta} \, \underline{e}^\alpha \underline{e}^\beta : y^{\alpha\beta} \underline{e}_\alpha \underline{e}_\beta = x_{\alpha\beta} \, y^{\alpha\beta} \, g^\alpha_\alpha \, g^\beta_\beta \tag{4}$$

is a double contraction (using summation convention), while the cross product of $\underset{\sim}{x}$ and $\underline{z}$ is

$$\underset{\sim}{x} \times \underline{z} \equiv x_{\alpha\beta} \, \underline{e}^\alpha \left(\underline{e}^\beta \times z_\gamma \underline{e}^\gamma \right) = x_{\alpha\beta} \, z_\gamma \, \underline{e}^\alpha \left(\underline{e}^\beta \times \underline{e}^\gamma \right). \tag{5}$$

It is usually found most convenient to put $\underline{V}$ equal to $\underline{V}_o$, the mean mass velocity of the system of particles of all types in the mixture, or else equal to $\underline{V}_i$, the mean mass velocity of the particles of type i in a rest frame, depending on the particular problem being considered. It is, however, important to appreciate that the temperature, stress and heat flux are defined with respect to a moving frame of reference (see below) so that the choice of frame affects the physical nature of these parameters. Thus Everett[10], using the 13-moment equations with the moments for each particle type i referred to a frame moving with $\underline{V}_i$ (mean mass velocity of the particular particle type i), on comparing his results with those of Herdan and Liley[3], who used moments referred to a frame moving with $\underline{V}_o$,

found "correction" terms. In fact such terms do not exist. The apparent discrepancy is due to the use of moments referred to a frame moving with $\underline{V}_i$ in the collision terms in equations written in terms of temperature, stress and heat flux referred to a frame for which $\underline{V} = \underline{V}_o$. With the correct reference velocities for the moments, the results are identical.

The moments, density, mean mass velocity, temperature, stress, energy flow, and heat flux for particles of type i are defined as follows:

(6) Density $\left(\psi_i = m_i\right)$

$$\rho_i = \int_{\underline{w}_i} f_i \, m_i \, d\underline{w}_i \quad .$$

(7) Mean mass velocity $\left(\psi_i = m_i \underline{w}_i\right)$

$$\underline{u}_i = \frac{1}{\rho_i} \int_{\underline{w}_i} f_i \, m_i \, \underline{w}_i \, d\underline{w}_i \quad .$$

(8) Temperature $\left(\psi_i = \frac{1}{2} m_i w_i^2\right)$

$$T_i = \frac{2}{3 n_i k} \int_{\underline{w}_i} f_i \, \frac{1}{2} m_i w_i^2 \, d\underline{w}_i \quad .$$

(9) Stress $\left(\psi_i = m_i \underline{w}_i \underline{w}_i\right)$

$$\underset{\sim}{S}_i = \int_{\underline{w}_i} f_i \, m_i \underline{w}_i \underline{w}_i \, d\underline{w}_i \quad .$$

(10) Energy flux $\left(\psi_i = \frac{1}{2} m_i \underline{w}_i w_i^2\right)$

$$\underline{Q}_i = \int_{\underline{w}_i} \frac{1}{2} m_i \underline{w}_i w_i^2 \, d\underline{w}_i \quad .$$

(11) Heat flux $\left(\psi_i = \frac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right)$

$$\underline{R}_i = \int_{\underline{w}_i} f_i \, \frac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right) d\underline{w}_i$$

$$= \underline{Q}_i - \frac{5}{4\alpha_i} \underline{u}_i \rho_i \quad .$$

k is Boltzmann's constant and

$$(12) \quad \alpha_i = \frac{m_i}{2kT_i} \quad .$$

The hydrostatic pressure, p_i, is defined by

$$(13) \quad p_i = n_i k T_i \quad ;$$

and $3p_i$ is the contraction of the stress tensor (9). Hence the non-hydrostatic component of the stress tensor, $\underset{\sim}{P}_i$, may be defined:

$$(14) \quad \underset{\sim}{P}_i \equiv \{\underset{\sim}{S}_i\} \equiv \underset{\sim}{S}_i - p_i \underset{\sim}{U}$$

where $\underset{\sim}{U}$ is the unit tensor. This bracket notation of (14) with a second rank tensor will be used to denote the symmetrised traceless form of a tensor, i.e. if $\underset{\sim}{A}'$ is the transpose of $\underset{\sim}{A}$,

$$(15) \quad \{\underset{\sim}{A}\} = \tfrac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}') - \tfrac{1}{3}(A_{11} + A_{22} + A_{33})\underset{\sim}{U} \quad .$$

The collision term in (2), $\frac{1}{2}\sum I_{jk}(\psi_i)$, is the sum over all possible collisions between particles of types j, k that affect the particles of type i (the factor $\frac{1}{2}$ is a formal factor necessary to avoid counting collisions twice, associated with making the integral term symmetrical in j and k). The collision integrals constituting this collision term are

$$(16) \quad I_{jk}(\psi_i) = \int f_j f_k \Delta_{jk}(\psi_i) g_{jk} d\sigma_{jk,i} d\underline{c}_j d\underline{c}_k$$

where $\Delta_{jk}(\psi_i)$ is the change in $\psi_i(\underline{r}, \underline{w}_i, t)$ in a collision between particles of types j, k ; $d\sigma_{jk,i}$ is the differential cross section for the collision in the sense that it is the probability that two particles of types j, k with asymptotic relative velocity

$$(17) \qquad \underline{g}_{jk} = \underline{c}_j - \underline{c}_k$$

will collide, producing particles of types j', k' with asymptotic relative veclocity in a solid angle, $d\Omega$ say, and with kinetic energy $\Delta\varepsilon$ absorbed from the system in the collision. This differential cross section is discussed in detail in Appendix I.

4. Closure of the Moment Equations

As is well known, the Enskog equations of change do not form a closed set, since the equations (2) for a dynamical variable ψ_i contain higher velocity moments of the form $\overline{\psi_i \underline{w}_i}$. In other words, the moment equations form an infinite set, because the sequence of N equations in the velocity moments always contains at least $N+1$ variables. Thus a suitable approximation must be chosen to obtain a solvable set of equations.

One of the best known methods is that of Chapman and Enskog (discussed in detail in Chapman and Cowling[9]). Esentially the method of these authors was to expand the distribution function, f , in the form

$$(18) \qquad f = f^{(0)} + \frac{1}{\nu} f^{(1)} + \frac{1}{\nu^2} f^{(2)} + \ldots ,$$

where ν is a parameter of the order of the collision frequency.

This approximation is valid only near to equilibrium, and the first approximation corresponds to a collision-dominant case. Using perturbation theory they obtain an explicit expression for f, and hence for the moments. It follows from their method that different gas components must have temperatures which are similar, and indeed, Grad[11] has shown that their solution represents only a very restricted subset of solutions of the Boltzmann equation, although it leads to correct asymptotic solutions.

More recently, several authors have proposed representing the distribution function as a series of orthogonal polynomials, with the closure approximation being obtained from a termination (somewhat arbitrarily) of the polynomial series. Wang-Chang and Uhlenbeck[12] used a series of Sonine polynomials, but these led to complicated results which have not been much extended. Ikenberry and Truesdell[13] have discussed the use of a type of spherical moment polynomial expansion so chosen to give simple results for Maxwellian molecules (inverse fifth power force law), but which is for other cases relatively intractable. Mott-Smith[14] has considered a Sonine expansion applicable only to special cases of extreme departures from equilibrium, in particular, for shocks.

However one of the most useful polynomial expansions for moderate departures from equilibrium is that used by Grad[1] to consider the non-equilibrium behaviour of a simple one-component gas. The distribution function is expanded as an infinite series of multi-dimensional Hermite polynomials in the (dimensionless) components of the particle velocity, multiplied by the appropriate weighting factor for the Hermite orthogonality relations. Thus the distribution function

for a given particle type (but ignoring the subscripts) may be written

$$(19)\quad f(\underline{r}, \underline{w}, t) = e^{-\alpha w^2} \sum_n a_n(\underline{r}, t)\, \mathcal{H}^{(n)}(w_1, w_2, w_3)$$

where w_1, w_2, w_3 are the particle velocity components relative to a frame moving with $\underline{V}$, and $\mathcal{H}^{(n)}$ is a Hermite polynomial of order n. Because of the convenient orthogonality relations, the coefficients α and a_n are obtained directly, in order corresponding to the order of the moment equations, by taking the moment of the corresponding dynamical function, ψ, with this expansion of the distribution function. To achieve a closed set of moment equations and to compromise between complexity and accuracy of dynamical description, this distribution function is approximated by the first four terms in the series (actually using a contracted form of the fourth term), since these contain the physically significant a_n. This gives (for particles of type i)

$$(20)\quad f_i = n_i \left(\frac{\alpha_i}{\pi}\right)^{\frac{3}{2}} e^{-\alpha_i w_i^2} \left[1 + 2\alpha_i \underline{u}_i \cdot \underline{w}_i + \frac{\alpha_i}{p_i} \underset{\sim}{P}_i : \underline{w}_i \underline{w}_i + \frac{4}{5} \frac{\alpha_i^2}{p_i} \underline{R}_i \cdot \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right]$$

This approximation (20) to the distribution function is expressed in terms of the 13 physically significant moments: ρ_i $(= n_i m_i)$, $\underline{u}_i$, p_i, $\underline{R}_i$, and the 5 independent components of $\underset{\sim}{P}_i$. Therefore with the use of this approximation for the distribution functions, all higher moments than those listed above can be expressed in terms of the listed ones, thus closing the equations with as much accuracy as the 13 variables allow. A further advantage of this method is that it places no restriction on the relative temperatures of different

component particle types.

Obviously, however, this approximation does involve a restriction on the departure of each gas component from its local equilibrium. At what degree of non-uniformity the approximation ceases to be reasonable is not clear, for it is difficult to understand the physical significance of the higher moments neglected. It may be possible to come to more definite conclusions after a comparison with suitable experimental results. Furthermore, it is not clear at this stage how much effect the choice of reference velocity has on this approximation, in the sense of "best" frame of reference for the moments temperature, stress and heat flux (bearing in mind that the algebraic results are independent of the choice of frame).

5. Some Remarks on General Moment Equations

The moment equations may be regarded as a means of obtaining a solution to the Boltzmann equation. However, they are probably better regarded simply as a set of equations for the moments concerned, in fact as possibly the "best" set of equations for such physical variables. Hence they are in no real way in competition with the Boltzmann equation itself. Although in general they do not contain Landau damping, which follows directly from the linearised Boltzmann or Vlasov equations, Sedlacek[15] has considered the solution of an infinite set of moment equations and concluded that under certain conditions this set does display such damping. The physics of this is not clear and until properly understood no definite reason can be given for the absence of such effects from a finite set of moment equations. Also the question of convergence of moment equations, although intuitively credible, is not conclusively settled.

On the other hand the Boltzmann equation itself is an extremely difficult equation to solve. When sufficient approximations are made to enable full account to be taken of (elastic) collisions, as in Chapman and Cowling[9], the solutions become very restricted, while attempts to obtain more general solutions usually require the neglect of collision terms to a greater or lesser extent. The moment equations are particularly effective in taking full account of collisions; indeed as shown in this thesis, they are able to take proper account even of (binary) inelastic processes. A further advantage of the 13-moment equations particularly, is their direct relationship to the physical variables of gaseous phenomena, and in particular, to the boundary conditions. Finally it is to be noted that certain phenomena such as the so-called finite Larmor radius effects, first recognised in obtaining solutions of the Vlasov equation, are automatically included in the moment equations (see for example Sedlacek[16]). As a matter of fact, the moment equations make the physical significance of ordering of terms in such considerations much clearer.

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PART I

Equations for a Partially Ionised Gas

Chapter 1

The Dynamical Equations

1.1 Introduction

The main purpose of this chapter is to give in detail the 13-moment equations. In formulating the dynamical equations the relevance of collision terms to other terms may be seen, although the detailed evaluation of the collision integrals is left to Chapter 2. The form of the equations with the moments referred to certain particular frames is discussed in some detail. Application of the moment equations to the derivation of transport relationships is considered, effects associated with inelastic collisions and particle mixtures being included.

1.2 The Dynamical Equations

Referring to the Enskog equations of change (2), given in the Introduction, the 13-moment equations are obtained by specialising the dynamical variable $\psi_i(\underline{r}, \underline{w}_i, t)$ to be

$$(1.1)\quad m_i \;;\; m_i \underline{w}_i \;;\; \tfrac{1}{2} m_i \underline{w}_i^2 \;;\; m_i \left\{ \underline{w}_i \underline{w}_i \right\} \;;$$

$$\tfrac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i} \right) ,$$

with corresponding moments

$$(1.2)\quad \rho_i \;;\; \rho_i \underline{u}_i \;;\; \tfrac{3}{2} n_i k T_i \;;\; \underset{\sim}{P}_i \;;\; \underline{R}_i \,.$$

Alternatively to heat flux vector the energy flux vector $\underline{Q}_i$,

corresponding to $\psi_i = \frac{1}{2} m_i \underline{w}_i w_i^2$, may be used. k is Boltzmann's constant.

In obtaining the moment equations in a closed form, the moments of "higher" order, which occur in the equations of change, are expressed in terms of these variables by using the 13-moment approximation, equation (20). The equations corresponding to the variables ρ_i , $\underline{u}_i$ and T_i are well known (Chapman and Cowling[1]), while for the equations corresponding to $\underset{\sim}{P}_i$ and $\underline{R}_i$, the relevant terms, including the "higher" moments, occurring in the equations of change are as follows (Liley[2]).

$$(1.3)\quad \nabla \cdot n_i \overline{m_i \left\{ \underline{w}_i \underline{w}_i \right\} \underline{w}_i} = \frac{4}{5} \left\{ \nabla \underline{R}_i \right\} + 2 \left\{ \nabla p_i \underline{u}_i \right\}$$

$$(1.4)\quad \frac{d}{dt} \overline{m_i \left\{ \underline{w}_i \underline{w}_i \right\}} = 0$$

$$(1.5)\quad \underline{w}_i \cdot \nabla \overline{m_i \left\{ \underline{w}_i \underline{w}_i \right\}} = 0$$

$$(1.6)\quad \underline{F}_i \cdot \overline{\frac{\partial}{\partial \underline{w}_i} m_i \left\{ \underline{w}_i \underline{w}_i \right\}} = 2 m_i \left\{ \underline{F}_i \underline{u}_i \right\}$$

$$(1.7)\quad \underline{b}_i \cdot \overline{\frac{\partial}{\partial \underline{w}_i} m_i \left\{ \underline{w}_i \underline{w}_i \right\} \times \underline{w}_i} = \frac{2}{n_i} \left\{ \underset{\sim}{P}_i \times \underline{b}_i \right\}$$

$$(1.8)\quad \overline{\frac{\partial}{\partial \underline{w}_i} \left(m_i \left\{ \underline{w}_i \underline{w}_i \right\} \right) \underline{w}_i} : \nabla \underline{V} = \frac{2}{n_i} \left(\left\{ \underset{\sim}{P} \cdot \nabla \underline{V} \right\} + p \left\{ \nabla \underline{V} \right\} \right)$$

$$(1.9)\quad \nabla \cdot n_i \overline{\frac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i} \right) \underline{w}_i} = \nabla \cdot \frac{k T_i}{m_i} \underset{\sim}{P}_i$$

$$(1.10)\quad \frac{d}{dt} \overline{\left(\frac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i} \right) \right)} = -\frac{5}{2} k \underline{u}_i \frac{dT_i}{dt}$$

$$(1.11)\quad \overline{\underline{w}_i \cdot \nabla\left(\tfrac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right)} = -\frac{5}{2}\frac{k}{\rho_i}\left(\underset{\sim}{P}_i \cdot \nabla T_i + p_i \nabla T_i\right)$$

$$(1.12)\quad \overline{\underline{F}_i \cdot \frac{\partial}{\partial \underline{w}_i}\left(\tfrac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right)} = \frac{1}{n_i}\underset{\sim}{P}_i \cdot \underline{F}_i$$

$$(1.13)\quad \overline{\underline{b}_i \cdot \frac{\partial}{\partial \underline{w}_i}\left(\tfrac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right) \times \underline{w}_i} = \frac{1}{n_i}\underline{R}_i \times \underline{b}_i$$

$$(1.14)\quad \overline{\frac{\partial}{\partial \underline{w}_i}\left(\tfrac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)\right)\underline{w}_i} : \nabla \underline{V} = \frac{1}{n_i}\underline{R}_i \cdot \nabla \underline{V}$$
$$+ \frac{2}{5 n_i}\left(\underline{R}_i \nabla \cdot \underline{V} + \underline{R}_i \cdot \nabla \underline{V} + \nabla \underline{V} \cdot \underline{R}_i\right) + kT_i\left(\underline{u}_i \nabla \cdot \underline{V}$$
$$+ \underline{u}_i \cdot \nabla \underline{V} + \nabla \underline{V} \cdot \underline{u}_i\right) .$$

Of these, (1.3), (1.9) and (1.14), involving the "higher" moments, have been determined by using the 13-moment approximation.

The collision terms in equation (2), which have the form

$$(1.15)\quad \tfrac{1}{2}\sum_{j,k} I_{jk}(\psi_i) = \tfrac{1}{2}\sum_{j,k}\int f_j f_k \Delta_{jk}(\psi_i)\, q_{jk}\, d\sigma_{jk,i}\, d\underline{w}_j\, d\underline{w}_k$$

are conveniently expressed in the notation:

$$(1.16)\quad \tfrac{1}{2}\sum_{j,k} I_{jk}(\psi_i) = \tfrac{1}{2}\sum_{j,k}\Big(\delta^{(0)}_{jk,i} + \delta^{(1)}_{jk,i}\underline{u}_j + \delta^{(2)}_{jk,i}\underline{u}_k$$
$$+ \delta^{(3)}_{jk,i}\underset{\sim}{P}_j + \delta^{(4)}_{jk,i}\underset{\sim}{P}_k + \delta^{(5)}_{jk,i}\underline{R}_j + \delta^{(6)}_{jk,i}\underline{R}_k\Big)$$

where it is clear that because of the inhomogeneous tensorial rank, certain of the $\delta_{jk,i}$ are necessarily zero for each case. It should be noted, however, that , i.e. , etc.

The dynamical equations can now be written in a convenient form. The equations are quoted basically as derived by Herdan and Liley[3], but without specifying $\underline{V}$ to be $\underline{V}_o$, and with a slightly more general form for the collision terms so that inelastic processes are included. The $\delta_{jk,i}$ in the following equations are of course different for each equation.

1.2.1 Continuity Equation

Taking $\psi_i = m_i$ in equation (2) gives the mass continuity equation. For inelastic collisions, the collision term is not necessarily zero.

$$(1.17)\quad \frac{\partial}{\partial t}\rho_i + \nabla\cdot(\rho_i \underline{u}_i + \rho_i \underline{V}) = \frac{1}{2}\sum_{j,k} \delta^{(0)}_{jk,i} \quad .$$

1.2.2 Momentum Equations

The momentum equations for particles of type i are obtained by taking $\psi_i = m_i \underline{w}_i$, and using the form of collision term in (1.16):

$$(1.18)\quad \frac{d}{dt}(\rho_i \underline{u}_i) + \rho_i \underline{u}_i \nabla\cdot\underline{V} + \nabla\cdot\underset{\sim}{P}_i + \nabla p_i - \rho_i \underline{F}_i - \rho_i \underline{u}_i \times \underline{b}_i + \rho_i \underline{u}\cdot\nabla\underline{V}$$

$$= \frac{1}{2}\sum_{j,k}\left(\delta^{(1)}_{jk,i}\,\underline{u}_j + \delta^{(2)}_{jk,i}\,\underline{u}_k + \delta^{(5)}_{jk,i}\,\underline{R}_j + \delta^{(6)}_{jk,i}\,\underline{R}_k \right).$$

1.2.3 Thermal Energy Equation

This equation is obtained by taking $\psi_i = \frac{1}{2} m_i w_i^2$. Since

$p_i = n_i k T_i$, it could alternatively be expressed in terms of T_i . The collision term involves elastic and inelastic collisions.

$$(1.19)\quad \frac{d}{dt}\left(\frac{3}{2} p_i\right) + \frac{3}{2} p_i \nabla\cdot\underline{V} + \nabla\cdot\underline{R}_i + \frac{5}{2} \nabla\cdot\left(p_i \underline{u}_i\right) - \rho_i \underline{F}_i\cdot\underline{u}_i + \underset{\sim}{P}_i : \nabla\underline{V} + p_i \underset{\sim}{U} : \nabla\underline{V} = \frac{1}{2} \sum_{J,k} \delta^{(0)}_{Jk,i} \quad .$$

1.2.4 Stress Tensor Equation

Throughout the thesis, apart from the introduction, the term "stress tensor" will be used for $\underset{\sim}{P}_i$, although this is actually only the non-hydrostatic component of the full stress tensor $\underset{\sim}{S}_i$, given in equations (7) and (12). Taking $\psi_i = m_i \{\underline{w}_i \underline{w}_i\}$, and using relations (1.3) to (1.8) in equation (2), gives the stress equation. The form of collision terms used is again from (1.16).

$$(1.20)\quad \frac{d}{dt}\underset{\sim}{P}_i + \underset{\sim}{P}_i \nabla\cdot\underline{V} + \frac{4}{5}\{\nabla\underline{R}_i\} + 2\{\nabla p_i \underline{u}_i\} - 2\rho_i\{\underline{F}_i \underline{u}_i\} - 2\{\underset{\sim}{P}_i \times \underline{b}_i\} + 2\{\underset{\sim}{P}_i \cdot \nabla\underline{V}\} + 2p\{\nabla\underline{V}\} = \frac{1}{2}\sum_{J,k}\left(\delta^{(3)}_{Jk,i}\underset{\sim}{P}_j + \delta^{(4)}_{Jk,i}\underset{\sim}{P}_k\right) \quad .$$

1.2.5 Heat Flux Equation

The heat flux equation is obtained by taking $\psi = \frac{1}{2} m_i \underline{w}_i \left(w_i^2 - \frac{5}{2\alpha_i}\right)$ in equation (2). It involves the use of relations (1.9) to (1.14) and of course the notation of (1.16) for the collision terms.

$$(1.21)\quad \frac{d}{dt}\underline{R}_i + \frac{k}{m_i}\left(\frac{7}{2}\underset{\sim}{P}_i\cdot\nabla T_i + \frac{5}{2}p_i\nabla T_i + T_i\nabla\cdot\underset{\sim}{P}_i\right) + \frac{5}{2}n_i k\underline{u}_i\frac{dT_i}{dt}$$

$$-\underset{\sim}{P}_i\cdot\underline{F}_i - \underline{R}_i\times\underline{b}_i + \frac{7}{5}\underline{R}_i\nabla\cdot\underline{V} + \frac{7}{5}\underline{R}_i\cdot\nabla\underline{V} + \frac{2}{5}\nabla\underline{V}\cdot\underline{R}_i$$

$$+ p_i\left(\underline{u}_i\nabla\cdot\underline{V} + \nabla\underline{V}\cdot\underline{u}_i + \underline{u}_i\cdot\nabla\underline{V}\right)$$

$$= \tfrac{1}{2}\sum_{j,k}\left(\delta_{jk,i}\,\underline{u}_j + \delta_{jk,i}\,\underline{u}_k + \delta_{jk,i}\,\underline{R}_j + \delta_{jk,i}\,\underline{R}_k\right) .$$

1.2.6 Conservation Equations

Since inelastic collisions are allowed, neither energy nor particle number is conserved. Over the whole system, however, there is still conservation of mass and momentum. Strictly speaking, the mass and momentum associated with photons should also be included; in general, though, these are insignificant and can be ignored.

Although not stated explicitly before, it is obvious from (7) and the definition of $\underline{w}_i$ (after (2)) that

$$(1.22)\quad \sum_i \rho_i\underline{V}_i = \rho\underline{V}_0$$

where $\rho = \sum_i \rho_i$. Using this relation and summing (1.17) over all particle types, since mass is conserved in collisions, it follows that

$$(1.23)\quad \frac{\partial}{\partial t}\rho + \nabla\cdot\rho\underline{V}_0 = 0 .$$

Similarly, on summing the momentum equations over all particle types (noting that $\underline{F}_i$ contains a term $-\frac{d\underline{V}}{dt}$), and on using (1.22),

(1.23) the momentum conservation equation is

$$(1.24)\quad \frac{\partial}{\partial t}\rho \underline{V}_o = \sum_i \rho_i(\underline{a}_i + \underline{V}\times\underline{b}_i) + \sum_i \rho_i \underline{u}_i \times \underline{b}_i$$
$$- \nabla\cdot\left(\rho \underline{V}_o\underline{V}_o - \rho(\underline{V}_o-\underline{V}_o)(\underline{V}_o-\underline{V}_o) + \underset{\sim}{P} + p\underset{\sim}{U}\right)$$

where $\underset{\sim}{P} = \sum_i \underset{\sim}{P}_i$, $p = \sum_i p_i$.

These equations (1.23) and (1.24), or rearrangements of them, are the only ones for which the collision integrals always vanish.

1.3 The Dynamical Equations with Moments Referred to a Frame Moving with the Mean Mass Velocity of the System

Obviously the equations of the previous section are not complete without specifying $\underline{V}$. As suggested earlier, the most useful choices for $\underline{V}$ are the mean mass velocity of the particles of type i , $\underline{V}_i$, or the mean mass velocity of the complete system, $\underline{V}_o$, and it is the latter which is considered in this section. The importance of this case is its convenience in dealing with a gas mixture, the moments for all gas components being referred to the same frame. As compared with any other value of $\underline{V}$ which is the same for all components, the choice of $\underline{V} = \underline{V}_o$ is physically the most realistic for a gas mixture at or near equilibrium.

The additional relations which determine $\underline{V}_o$ are the conservation equations. Since, when $\underline{V} = \underline{V}_o$,

$$(1.25)\quad \sum_i \rho_i \underline{u}_i = 0$$

(by definition), on putting $\underline{V} = \underline{V}_o$ in equations (1.23), (1.24), the

particular conservation equations are:

$$(1.26)\quad \frac{\partial}{\partial t}\rho + \nabla\cdot\rho\underline{V}_o = 0$$

and

$$(1.27)\quad \frac{\partial}{\partial t}\rho\underline{V}_o = \sum_i \rho_i(\underline{a}_i + \underline{V}_o\times\underline{b}_i) + \sum_i \rho_i \underline{u}_i\times\underline{b}_i - \nabla\cdot(\underline{\underline{P}} + p\underline{\underline{u}} + \rho\underline{V}_o\underline{V}_o)$$

where again $\underline{\underline{P}} = \sum_i \underline{\underline{P}}_i$, $p = \sum_i p_i$.

1.4 The Dynamical Equations with Moments Referred to a Frame with $\underline{V} = \underline{V}_i$

This reference velocity for the moments has the advantage of giving a much simpler set of equations for the dynamical variables, but the disadvantage that the temperature, stress and heat flux are referred to a different frame for each gas component. It is therefore generally unsatisfactory for dealing with gas mixtures, except when collisions can be completely ignored or (as will be shown later for a mixture of electrons and ions) where the gas components completely "decouple".

For this case, the velocity distribution functions f_j , f_k are referred to $\underline{V}_i$, and therefore, in the moment equations for particles of type i , the dynamical variables for the particles of types j, k in the collision terms are also referred to $\underline{V}_i$. But for consistency the dynamical variables temperature, stress, and heat flux for other particle types j, k,... should be referred to their own particle type

mean mass velocities, $\underline{V}_j, \underline{V}_k, \ldots$. Let the subscripted dynamical variable h_{ji} represent the variable h of particles of type j when referred to a frame moving with $\underline{V}_i$. Consider temperature first. By definition,

$$(1.28)\quad T_{jj} = \frac{2}{3kn_j} \int_{\underline{c}_j} f_j \tfrac{1}{2} m_j (\underline{c}_j - \underline{V}_j)^2 \, d\underline{c}_j$$

and so

$$(1.29)\quad T_{ji} = \frac{2}{3kn_j} \int_{\underline{c}_j} f_j \tfrac{1}{2} m_j (\underline{c}_j - \underline{V}_i)^2 \, d\underline{c}_j \quad .$$

On expanding these definitions it is clear that

$$(1.30)\quad T_{ji} = T_{jj} + \tfrac{1}{3} \frac{m_j}{k} (\underline{V}_j - \underline{V}_i)^2 \quad .$$

Similarly,

$$(1.31)\quad \underset{\sim}{P}_{ji} = \int_{\underline{c}_j} f_j m_j \{(\underline{c}_j - \underline{V}_i)(\underline{c}_j - \underline{V}_i)\} \, d\underline{c}_j$$

$$(1.32)\quad \underline{R}_{ji} = \int_{\underline{c}_j} f_j \tfrac{1}{2} m_j (\underline{c}_j - \underline{V}_i) \left[(\underline{c}_j - \underline{V}_i)^2 - \frac{5}{2\alpha_{ji}}\right] d\underline{c}_j$$

from which

$$(1.33)\quad \underset{\sim}{P}_{ji} = \underset{\sim}{P}_{jj} + \rho_j \{(\underline{V}_j - \underline{V}_i)(\underline{V}_j - \underline{V}_i)\}$$

$$(1.34)\quad \underline{R}_{ji} = \underline{R}_{jj} + \rho_j (\underline{V}_j - \underline{V}_i) \left[\tfrac{1}{2}(\underline{V}_j - \underline{V}_i)^2 - \frac{5}{4\alpha_{ji}}\right] + (\underline{V}_j - \underline{V}_i) \cdot \left(\underset{\sim}{P}_{jj} + \tfrac{5}{2} p_{jj} \underset{\sim}{U}\right) \quad .$$

In obtaining these expressions use has been made of the relations

$$(1.35)\quad n_j m_j \overline{c_j^2} = 3p_{jj} + \rho_j V_j^2$$

$$n_j m_j \overline{\underset{\sim}{c}_j \underset{\sim}{c}_j} = \underset{\sim}{S}_{jj} + \rho_j \underline{V}_j \underline{V}_j$$

and since

$$(1.36)\quad \frac{1}{\alpha_{ji}} = \frac{2kT_{ji}}{m_j} = \frac{1}{\alpha_{jj}} + \frac{2}{3}(\underline{V}_j - \underline{V}_i)^2$$

equation (1.34) may be rewritten as

$$(1.37)\quad \underline{R}_{ji} = \underline{R}_{jj} + \rho_j(\underline{V}_j - \underline{V}_i)\left[\frac{-5}{4\alpha_{jj}} - \frac{1}{3}(\underline{V}_j - \underline{V}_i)^2\right] + (\underline{V}_j - \underline{V}_i)\cdot\left(\underset{\sim}{P}_{jj} + \frac{5}{2}p_{jj}\underset{\sim}{U}\right) .$$

With the variables for each particle type j referred to the respective $\underline{V}_j$, it is convenient to write T_{jj}, $\underset{\sim}{P}_{jj}$, $\underline{R}_{jj}$, etc. simply as T_j, $\underset{\sim}{P}_j$, $\underline{R}_j$, etc. in the equations that follow. It should be mentioned that the collision coefficients $\delta_{jk,i}$ also involve the dynamical variables ρ_j, ρ_k, T_j and T_k, and corresponding changes should be made in these as well. With the above formulation, the 13-moment equations for the particles of type i, with the moments referred to the mean mass velocity of their own particle type, are:

1.4.1 Continuity Equation

This equation follows from (1.17).

$$(1.38)\quad \frac{\partial}{\partial t}\rho_i + \nabla\cdot\rho_i\underline{V}_i = \frac{1}{2}\delta^{(0)}_{jk,i} .$$

1.4.2 Momentum Equations

The momentum equations correspond to (1.18) .

$$(1.39)\quad \nabla\cdot\underset{\sim}{P}_i + \nabla p_i - \rho_i \underline{F}_i = \frac{1}{2}\sum_{j,k}\Big[\delta^{(1)}_{jk,i}(\underline{V}_j-\underline{V}_i) + \delta^{(2)}_{jk,i}(\underline{V}_k-\underline{V}_i) + \delta^{(5)}_{jk,i}\Big(\underline{R}_j + \rho_j(\underline{V}_j-\underline{V}_i)\Big(\frac{-5}{2\alpha_j} - \frac{1}{3}(\underline{V}_j-\underline{V}_i)^2\Big) + (\underline{V}_j-\underline{V}_i)\cdot\Big(\underset{\sim}{P}_j + \frac{5}{2}p_j\underset{\sim}{U}\Big)\Big) + \delta^{(6)}_{jk,i}\Big(\underline{R}_k + \rho_k(\underline{V}_k - \underline{V}_i)\Big(-\frac{5}{2\alpha_k} - \frac{1}{3}(\underline{V}_k-\underline{V}_i) + (\underline{V}_k-\underline{V}_i)\cdot\Big(\underset{\sim}{P}_k + \frac{5}{2}p_k\underset{\sim}{U}\Big)\Big)\Big]\Big].$$

1.4.3 Thermal Energy Equation

Corresponding to equation (1.19), this equation becomes

$$(1.40)\quad \frac{d}{dt}\Big(\frac{3}{2}p_i\Big) + \frac{5}{2}p_i\nabla\cdot\underline{V}_i + \nabla\cdot\underline{R}_i + \underset{\sim}{P}_i : \nabla\underline{V}_i = \frac{1}{2}\sum_{j,k}\delta^{(0)}_{jk,i}.$$

1.4.4 Stress Tensor Equation

This equation follows from (1.20) with the use of relations (1.33).

$$(1.41)\quad \frac{d}{dt}\underset{\sim}{P}_i + \underset{\sim}{P}_i\nabla\cdot\underline{V}_i + \frac{4}{5}\{\nabla\underline{R}_i\} - 2\{\underset{\sim}{P}_i\times\underline{b}_i\} + 2\{\underset{\sim}{P}_i\cdot\nabla\underline{V}_i\} + 2p\{\nabla\underline{V}_i\} = \frac{1}{2}\sum_{j,k}\Big[\delta^{(3)}_{jk,i}\Big(\underset{\sim}{P}_j + \rho_j\{(\underline{V}_j-\underline{V}_i)(\underline{V}_j-\underline{V}_i)\}\Big) + \delta^{(4)}_{jk,i}\Big(\underset{\sim}{P}_k + \rho_k\{(\underline{V}_k-\underline{V}_i)(\underline{V}_k-\underline{V}_i)\}\Big)\Big].$$

1.4.5 Heat Flux Equation

Using (1.37) this equation corresponds to (1.21).

$$(1.42)\quad \frac{d}{dt}\underline{R}_i + \frac{k}{m_i}\left(\frac{7}{2}\underset{\sim}{P}_i\cdot\nabla T_i + \frac{5}{2}p_i\nabla T_i + T_i\nabla\cdot\underset{\sim}{P}_i\right) - \underset{\sim}{P}_i\cdot\underline{F}_i$$
$$- \underline{R}_i\times\underline{b}_i + \frac{7}{5}\underline{R}_i\nabla\cdot\underline{V}_i + \frac{7}{5}\underline{R}_i\cdot\nabla\underline{V}_i + \frac{2}{5}\nabla\underline{V}_i\cdot\underline{R}_i$$
$$= \frac{1}{2}\sum_{jk}\Big[\delta^{(1)}_{jk,i}(\underline{V}_j-\underline{V}_i) + \delta^{(2)}_{jk,i}(\underline{V}_k-\underline{V}_i)$$
$$+ \delta^{(5)}_{jk,i}\Big(\underline{R}_j + \rho_j(\underline{V}_j-\underline{V}_i)\Big(-\frac{5}{4\alpha_j} - \frac{1}{3}(\underline{V}_j-\underline{V}_i)^2\Big) + (\underline{V}_j$$
$$- \underline{V}_i)\cdot\Big(\underset{\sim}{P}_j + \frac{5}{2}p_j\underset{\sim}{U}\Big)\Big) + \delta^{(6)}_{jk,i}\Big(\underline{R}_k + \rho_k(\underline{V}_k-\underline{V}_i)\Big(-\frac{5}{4\alpha_k}$$
$$- \frac{1}{3}(\underline{V}_k-\underline{V}_i)^2\Big) + (\underline{V}_k-\underline{V}_i)\cdot\Big(\underset{\sim}{P}_k + \frac{5}{2}p_k\underset{\sim}{U}\Big)\Big)\Big].$$

1.4.6 Conservation Equations

Conservation of mass simply gives (from 1.23)

$$(1.43)\quad \frac{\partial}{\partial t}\rho + \nabla\cdot\rho\underline{V}_o = 0\ ,$$

and from (1.24) it follows that the conservation of momentum equation is

$$(1.44)\quad \frac{\partial}{\partial t}\rho\underline{V}_o = \sum_j \rho_j(\underline{a}_j + \underline{V}_j\times\underline{b}_j)$$
$$- \nabla\cdot\Big(\underset{\sim}{P} + \sum_j \rho\underline{V}_j\underline{V}_j + p\underset{\sim}{U}\Big)$$

where $\underset{\sim}{P} = \sum_j \underset{\sim}{P}_{jj}$, $p = \sum_j p_{jj}$.

1.5 The Transport Relationships

Depending on the relative magnitudes of the collision frequencies, the rates of change of variables due to external sources, and of the applied electric or magnetic fields, different approximations to the preceding equations are possible. Generally speaking, of these, transport relationships correspond to approximations valid for low frequency phenomena. The inclusion of the transport relationships in this chapter is principally an illustration of the use of the 13-moment equations, and also to indicate how the transport parameters of viscosity, thermal conductivity and resistivity are affected by inelastic collisions and the presence of several gas components. The general method of deriving transport relationships from the 13-moment equations has been given elsewhere (Herdan and Liley[3]).

As shown in section 2.6 of Chapter 2, when certain inequalities involving the component temperatures are satisfied, the collision terms involving the variables $\underline{u}_i$, $\underset{\sim}{P}_i$, $\underline{R}_i$ in the electron component equations are dominated by the terms involving the electron component variables $\underline{u}_e$, $\underset{\sim}{P}_e$, $\underline{R}_e$. Similarly, in the "heavy" particle component moment equations the collision terms involving $\underline{u}_i$, $\underset{\sim}{P}_i$, $\underline{R}_i$ of particle types other than electrons are much larger than those involving the electron variables $\underline{u}_e$, $\underset{\sim}{P}_e$, $\underline{R}_e$. This will be termed "decoupling".

The transport equations are the equations relating the "secondary" variables $\underline{u}$, $\underset{\sim}{P}$ and $\underline{R}$ to the remaining variables ($\underline{V}$, $\underline{E}$, $\underline{B}$, ρ_i, T_i, ρ_j, T_j, ...). Often it is more convenient to use the current density $\underline{J}$ in preference to $\underline{u}_e$, the electron drift speed, where with

the neglect of terms of order of the electron-proton mass ratio,

$$(1.45) \quad \underline{J} = n_e e_e \underline{u}_e \ .$$

Since the transport relations will refer to a gas mixture, all moments will be referred to $\underline{V}_o$. It should be mentioned that, in view of the use of particle-type subscripts, the electron charge e_e always includes its sign, that is, $e_e = -1.60 \times 10^{-19}$ coulombs.

1.5.1 Transport Relationships for the Electrons

With the decoupling of electron component moment equations from the equations for the atomic particle types, the first approximations for the transport relationships for electrons are (neglecting subscripts)

$$(1.46) \quad -\frac{1}{ne}\nabla p + \frac{1}{ne}\underline{J}\times\underline{B} + \underline{E} + \underline{V}\times\underline{B} = \eta\underline{J} + \alpha\underline{R}$$

$$(1.47) \quad -2p\{\nabla\underline{V}\} + 2\{\underset{\sim}{P}\times\underline{b}\} + 2ne\{\underline{u}(\underline{E}+\underline{V}\times\underline{B})\} = \frac{\beta}{\tau}\underset{\sim}{P}$$

$$(1.48) \quad -\frac{5}{2}\frac{pk}{m}\nabla T + \underline{R}\times\underline{b} + \frac{e}{m}\underset{\sim}{P}\cdot(\underline{E}+\underline{V}\times\underline{B}) = \frac{1}{\tau}\underline{R} + \gamma\underline{J} \ .$$

The ordering of terms for which these equations are valid is apparent on comparing them with (1.18), (1.20) and (1.21) respectively.

The solutions of equations (1.46) to (1.48) are formally the same as given by Herdan and Liley[3]. Beginning with (1.47), define

$$(1.49) \quad \underset{\sim}{e} = \{\nabla\underline{V}\} - \frac{1}{p}\{\underline{J}(\underline{E}+\underline{V}\times\underline{B})\} \ .$$

Let a cartesian frame be oriented so that the magnetic field is in the

x-direction, i.e. $\underline{B} = (B, 0, 0)$. Then using the scalar cyclotron frequency $b = eB/m$, the components of the electron stress tensor are:

$$(1.50)\quad P^{xx} = -2\mu e^{xx}$$

$$P^{xy} = \frac{-2\mu}{1 + b^2\tau^2/\beta^2}\left[e^{xy} + \frac{b\tau}{\beta} e^{xz} \right]$$

$$P^{xz} = \frac{-2\mu}{1 + b^2\tau^2/\beta^2}\left[e^{xz} - \frac{b\tau}{\beta} e^{xy} \right]$$

$$P^{yy} = \frac{-2\mu}{1 + 4b^2\tau^2/\beta^2}\left[e^{yy} - \frac{2b^2\tau^2}{\beta^2} e^{xx} + 2\frac{b\tau}{\beta} e^{yz} \right]$$

$$P^{yz} = \frac{-2\mu}{1 + 4b^2\tau^2/\beta^2}\left[e^{yz} - \frac{b\tau}{\beta^2} e^{xx} - 2\frac{b\tau}{\beta} e^{yy} \right].$$

$\underset{\sim}{P}$ being a traceless tensor, P^{zz} is given by $-(P^{xx} + P^{yy})$.
μ is the coefficient of viscosity defined by

$$(1.51)\quad \mu = p\frac{\tau}{\beta}.$$

It is apparent that a magnetic field causes the collision coefficient, β/τ, to enter the solutions in a complex manner.
Since

$$(1.52)\quad \frac{\beta}{\tau}\underset{\sim}{P}_e \equiv -\frac{1}{2} I_{ee}(m_e\{\underline{w}_e \underline{w}_e\}) - \frac{1}{2}\sum_{j,k\ (\text{not } j=k=e)} I_{jk}(m_e\{\underline{w}_e \underline{w}_e\}),$$

the effects of inelastic, as well as elastic collisions, are contained within the last term, which of course only involves $\underset{\sim}{P}_e$ $\left(+O\left(\frac{m_e}{m_p}\right)\underset{\sim}{P}_p\right)$.

The transport equation (1.48) involving the heat flux vector also involves the stress tensor, so that in general the above solutions are required to obtain a final result. However, defining

$$(1.53)\quad \underline{D} = \nabla T + \frac{2m}{5pk}\left(\gamma \underline{J} - \frac{e}{m}\underset{\sim}{P}\cdot(\underline{E} + \underline{V}\times\underline{B})\right)$$

equation (1.48) can be solved to give

$$(1.54)\quad \underline{R} = \frac{-\lambda}{1+b^2\tau^2}\left(\underline{D} + \tau^2(\underline{b}\cdot\underline{D})\underline{b} - \tau\,\underline{b}\times\underline{D}\right)$$

(where as usual $\underline{b} = e\underline{B}/m$). This solution gives the usual properties of heat conduction parallel and perpendicular to the magnetic field, while the transverse term $\underline{b}\times\underline{D}$ is associated with the Righi-Leduc and Ettinghausen effects. The coefficient of thermal conduction is

$$(1.55)\quad \lambda = \frac{5}{2}\frac{pk}{m}\tau\ .$$

The effects of collisions enter through the terms τ and γ :

$$(1.56)\quad ne\gamma\underline{u}_e + \frac{1}{\tau}\underline{R}_e \equiv -\frac{1}{2}I_{ee}\left(\tfrac{1}{2}m_e\underline{w}_e\left(w_e^2 - \frac{5}{2\alpha_e}\right)\right) - \frac{1}{2}\sum_{j,k\ (\text{not } j=k=e)} I_{jk}\left(\tfrac{1}{2}m_e\underline{w}_e\left(w_e^2 - \frac{5}{2\alpha_e}\right)\right),$$

the last term again being associated with any inelastic, as well as elastic, collisions.

Equation (1.46) is usually called the generalised Ohm's law, or the first approximation to the generalised Ohm's law, for obvious

reasons. η of course has the dimensions of resistivity. In this equation the coupling between $\underline{J}$ and $\underline{R}$ leads to several well known effects. With the term involving $\underline{\underline{P}}$ ignored for mathematical tractability, equations (1.46) and (1.48) may be combined to yield

$$(1.57)\quad \eta_{\parallel}\,\underline{J} = \underline{A} + \underline{K} \times \underline{B}$$

where

$$(1.58)\quad \underline{A} = -\frac{1}{ne}\nabla p + \underline{E} - \alpha \underline{R}_D + \frac{\varkappa_T \tau^2}{1 + b^2\tau^2}(\underline{b}\cdot\nabla T)\,\underline{b}$$

$$\underline{K} = \frac{1}{ne}\underline{J} + \underline{V} - \frac{\alpha e \tau}{m}\underline{R}_D$$

$$\underline{R}_D = \frac{-\gamma\tau}{1 + b^2\tau^2}\left(\underline{J} \times \underline{b} + \frac{\lambda \nabla T}{\gamma \tau}\right) .$$

$\eta_{\parallel}$ is defined by

$$(1.59)\quad \eta_{\parallel} = \eta\left(1 - (\alpha\gamma\tau/\eta)\right)$$

and is the resistivity parallel to the magnetic field. The coefficient $\varkappa_T$ is the thermal diffusion coefficient defined by

$$(1.60)\quad \varkappa_T = \alpha\lambda \quad ,$$

λ being the thermal conductivity coefficient. Except for obvious special cases these results are very complicated; nevertheless, they are discussed in detail by Herdan and Liley[3]. The coefficients α, η are obviously defined by

$$(1.61)\quad n_e(\eta_e \underline{u}_e + \alpha \underline{R}_e) \equiv -\tfrac{1}{2} \sum_{j,k} I_{jk}(m_e \underline{w}_e)$$

1.5.2 The Transport Relationships for "Heavy" Particles

From equation (1.25) (since $\underline{V} = \underline{V}_o$), it is evident that the atomic particle drift speeds $|\underline{u}_i|$ are of order m_e/m_i of the electron drift speeds $|\underline{u}_e|$. The velocities imparted to the ions by the macroscopic accelerating fields in a collision time are also of the same order. For this reason the relevant approximations of the momentum equations (1.18) for "heavy" particles will be ignored in discussing the transport properties of such particles. It should be mentioned however that it is necessary to retain these equations and also to retain the $\underline{u}_i$ term in equations (1.20), (1.21), to account for the general diffusion of the atomic particles. These phenomena are not considered in this subsection, and for present purposes the "heavy" particle transport equations analogous to (1.47), (1.48) are:

$$(1.62)\quad -2p_i\{\nabla \underline{V}\} + 2\{\underline{P}_i \times \underline{b}_i\} = \sum_j \beta_{ij} \underline{P}_j$$

$$(1.63)\quad -\tfrac{5}{2} p_i \frac{k}{m_i} \nabla T_i + \underline{R}_i \times \underline{b}_i + \frac{e_i}{m_i} \underline{P}_i \cdot (\underline{E} + \underline{V} \times \underline{B}) = \sum_j \theta_{ij} \underline{R}_j \quad .$$

For ease of writing when treating particle mixtures, β_{ij} and θ_{ij} have been used for $(\beta/\tau)_{ij}$ and $(1/\tau)_{ij}$, respectively.

In order to examine the effect on the transport properties of a particular gas component by other components in a mixture, only two

"heavy" components, i, j will be considered. Furthermore the discussion will be restricted to three particular cases, namely (1) no applied magnetic field, (2) an applied magnetic field but with the particle concentrations such that $n_j << n_i$, and (3) an applied magnetic field but with type j particles neutral. These represent three important and entirely different situations.

Considering (1.62) first, when $\underline{B} = 0$, these equations take the simple form

$$(1.64) \quad -2p_i \underset{\sim}{e}_i = \beta_{ii} \underset{\sim}{P}_i + \beta_{ij} \underset{\sim}{P}_j$$

$$-2p_j \underset{\sim}{e}_j = \beta_{ji} \underset{\sim}{P}_i + \beta_{jj} \underset{\sim}{P}_j$$

where

$$(1.65) \quad \underset{\sim}{e}_i = \underset{\sim}{e}_j = \{\nabla \underline{V}\} \equiv \underset{\sim}{e} \quad .$$

The solutions are easily obtained:

$$(1.66) \quad \underset{\sim}{P}_i = -\frac{2\beta_{jj} p_i \underset{\sim}{e}}{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}} + \frac{2\beta_{ij} p_j \underset{\sim}{e}}{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}}$$

$$\underset{\sim}{P}_j = -\frac{2\beta_{ii} p_j \underset{\sim}{e}}{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}} + \frac{2\beta_{ji} p_j \underset{\sim}{e}}{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}} \quad .$$

By analogy with the single particle type situation, for instance the solution of the electron equation (1.47), a generalised coefficient of viscosity may be defined:

$$(1.67)\quad \mu_{ij} = \frac{\beta_{jj} p_i - \beta_{ij} p_j}{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}} \,.$$

As $n_j \to 0$, $\mu_{ij} \to \mu = p_i/\beta_{ii}$ which is the single particle type result. In terms of these definitions,

$$(1.68)\quad \underset{\sim}{P}_i = -2\mu_{ij}\underset{\approx}{e}$$

$$\underset{\sim}{P}_j = -2\mu_{ji}\underset{\approx}{e} \,.$$

For the second case, where $n_j << n_i$, take the magnetic field parallel to the x-axis of a rectangular cartesian frame. By considering the explicit form of the collision integrals of Chapter 2 with respect to n_i, n_j, it is evident that equation (1.62) becomes

$$(1.69)\quad -2p_i\underset{\approx}{e} + 2\{\underset{\sim}{P}_i \times \underline{b}_i\} = \beta_{ii}\underset{\sim}{P}_i + \beta_{ij}\underset{\sim}{P}_j$$

$$O(n_j/n_i) + 2\{\underset{\sim}{P}_j \times \underline{b}_j\} = \beta_{jj}\underset{\sim}{P}_j + O(n_j/n_i) \,.$$

That is, neglect of n_j/n_i compared with unity is equivalent to taking $\underset{\sim}{P}_j = 0$. The equation for $\underset{\sim}{P}_i$ in (1.68) then has the single particle type solution,

$$(1.70)\quad P_i^{xx} = -2\mu_i e^{xx}$$

$$P_i^{xy} = \frac{-2\mu_i}{1 + b_i^2/\beta_{ii}^2}\left(e^{xy} + \frac{b_i}{\beta_{ii}} e^{xz}\right)$$

$$P_i^{xz} = \frac{-2\mu_i}{1 + b_i^2/\beta_{ii}^2}\left(e^{xz} - \frac{b_i}{\beta_{ii}} e^{xy}\right)$$

$$P_i^{yy} = \frac{-2\mu_i}{1 + 4 b_i^2/\beta_{ii}} \left(e^{yy} - 2 \frac{b_i^2}{\beta_{ii}^2} e^{xx} + \frac{2 b_i}{\beta_{ii}} e^{yz} \right)$$

$$P_i^{yz} = \frac{-2\mu_i}{1 + 4 b_i^2/\beta_{ii}} \left(e^{yz} - \frac{b_i}{\beta_{ii}^2} e^{xx} - \frac{2 b_i}{\beta_{ii}} e^{yy} \right)$$

where $\mu_i = 2 p_i / \beta_{ii}$ is a single particle type viscosity. P^{zz} is of course given by $-(P^{xx} + P^{yy})$. To the order of approximation used for this solution, the particles of type j make no contribution to the stress tensor transport relationships for particles of type i.

The third case is when particles of type j are neutral. Equations (1.62) become

$$(1.71) \quad -2 p_i \underset{\sim}{e}_i + 2\{\underset{\sim}{P}_i \times \underset{}{b}_i\} = \beta_{ii} \underset{\sim}{P}_i + \beta_{ij} \underset{\sim}{P}_j$$

$$-2 p_j \underset{\sim}{e}_j = \beta_{ji} \underset{\sim}{P}_i + \beta_{jj} \underset{\sim}{P}_j \; .$$

The generalised viscosity (1.67) occurs in the solution of these equations. Defining

$$(1.72) \quad \epsilon = \frac{\beta_{ii}\beta_{jj} - \beta_{ij}\beta_{ji}}{\beta_{jj}}$$

and again taking the magnetic field in the x-direction, the components of $\underset{\sim}{P}_i$ are given by

$$(1.73) \quad P_i^{xx} = -2\mu_{ij} e^{xx}$$

$$P_i^{xy} = \frac{-2\mu_{ij}}{1 + b_i^2/\epsilon^2}\left(e^{xy} + \frac{b_i}{\epsilon} e^{xz}\right)$$

$$P_i^{xz} = \frac{-2\mu_{ij}}{1 + b_i^2/\epsilon^2}\left(e^{xz} - \frac{b_i}{\epsilon} e^{xy}\right)$$

$$P_i^{yy} = \frac{-2\mu_{ij}}{1 + 4b_i^2/\epsilon^2}\left(e^{yy} - 2\frac{b_i^2}{\epsilon^2} e^{xx} + \frac{2b_i}{\epsilon} e^{yz}\right)$$

$$P^{yz} = \frac{-2\mu_{ij}}{1 + 4b_i^2/\epsilon^2}\left(e^{yz} - \frac{b_i}{\epsilon} e^{xx} - \frac{2b_i}{\epsilon} e^{yy}\right) .$$

In terms of $\underset{\sim}{P}_i$, $\underset{\sim}{P}_j$ takes the simple form

$$(1.74)\quad \underset{\sim}{P}_j = -\frac{2p_j}{\beta_{jj}}\underset{\sim}{e} - \frac{\beta_{ji}}{\beta_{jj}}\underset{\sim}{P}_i$$

and may be obtained using (1.73).

The heat flux transport equations (1.63) are simplified by the neglect of the drift velocity terms discussed earlier, although of course they still contain the stress components, the values of which have been given above for the three cases to be considered. Beginning with the case $\underline{B} = 0$, the equations (1.63) become

$$(1.75)\quad -\frac{5}{2}\frac{p_i k}{m_i}\underline{D}_i = \Theta_{ii}\underline{R}_i + \Theta_{ij}\underline{R}_j$$

$$-\frac{5}{2}\frac{p_j k}{m_j}\underline{D}_j = \Theta_{ji}\underline{R}_i + \Theta_{jj}\underline{R}_j$$

where

$$(1.76)\quad \underline{D}_i = \nabla T_i - \frac{e_i}{m_i}\frac{2m_i}{5p_i k}\underset{\sim}{P}_i \cdot (\underline{E} + \underline{V} \times \underline{B})$$

and similarly for $\underline{D}_j$.

When $T_i = T_j$ and the terms in $\underline{D}_i$, $\underline{D}_j$ involving the stress tensor are ignored, it is possible to define a generalised coefficient of thermal conductivity. For this case,

$$(1.79)\quad \underline{D}_i = \underline{D}_j = \nabla T \equiv \underline{D}$$

with $T = T_i = T_j$, and the solutions are

$$(1.78)\quad \underline{R}_i = -\lambda_{ij}\,\underline{D}$$

$$\underline{R}_j = -\lambda_{ji}\,\underline{D} ,$$

the generalised thermal conductivity, λ_{ij} , being given by

$$(1.79)\quad \lambda_{ij} = \frac{5}{2}\,\frac{k(\Theta_{jj}\,p_i/m_i - \Theta_{ij}\,p_j/m_j)}{\Theta_{ii}\,\Theta_{jj} - \Theta_{ij}\,\Theta_{ji}} .$$

As $n_j \to 0$, $\lambda_{ij} \longrightarrow \lambda_i = \frac{5}{2}\,k p_i/m_i\,\Theta_{ii}$, the single component type thermal conductivity. However, more generally, when $T_i \neq T_j$, the solution of (1.75) is

$$(1.80)\quad \underline{R}_i = -\frac{5}{2}\,\frac{k\Theta_{jj}\,p_i/m_i}{\Theta_{ii}\,\Theta_{jj} - \Theta_{ij}\,\Theta_{ji}}\,\underline{D}_i + \frac{5}{2}\,\frac{k\,\Theta_{ij}\,p_j/m_j}{\Theta_{ii}\,\Theta_{jj} - \Theta_{ij}\,\Theta_{ji}}\,\underline{D}_j$$

$$\underline{R}_j = -\frac{5}{2}\,\frac{k\,\Theta_{ii}\,p_j/m_j}{\Theta_{ii}\,\Theta_{jj} - \Theta_{ij}\,\Theta_{ji}}\,\underline{D}_j + \frac{5}{2}\,\frac{k\,\Theta_{ji}\,p_i/m_i}{\Theta_{ii}\,\Theta_{jj} - \Theta_{ij}\,\Theta_{ji}}\,\underline{D}_i .$$

For the second case, taking $n_j \ll n_i$, the equations corresponding to (1.63) are

$$(1.81)\quad -\frac{5}{2}\frac{p_i k}{m_i}\underline{D}_i + \underline{R}_i \times \underline{b}_i = \theta_{ii}\underline{R}_i + \theta_{ij}\underline{R}_j$$

$$\underline{R}_j \times \underline{b}_j = \theta_{jj}\underline{R}_j \quad .$$

Similarly to (1.68), neglect of n_j/n_i is equivalent to taking $\underline{R}_j = 0$. The solution for $\underline{R}_i$ is then the same as for the single particle type case:

$$(1.82)\quad \underline{R}_i = -\frac{\lambda_i}{1 + b_i^2/\theta_{ii}^2}\left(\underline{D}_i + \frac{(\underline{b}_i \cdot \underline{D}_i)\,\underline{b}_i}{\theta_{ii}^2} - \frac{\underline{b}_i \times \underline{D}_i}{\theta_{ii}}\right).$$

It is evident that the transport properties of particles of type i are essentially unaffected by a small number of particles of another type.

The remaining case to be considered is that for which particles of type j have no net charge. The equations for the heat flux vectors become

$$(1.83)\quad -\frac{5}{2}\frac{p_i k}{m_i}\underline{D}_i + \underline{R}_i \times \underline{b}_i = \theta_{ii}\underline{R}_i + \theta_{ij}\underline{R}_j$$

$$-\frac{5}{2}\frac{p_j k}{m_j}\underline{D}_j = \theta_{ji}\underline{R}_i + \theta_{jj}\underline{R}_j$$

Defining

$$(1.84)\quad \underline{H} = \frac{p_i}{m_i}\underline{D}_i - \frac{p_j}{m_j}\theta_{ij}\underline{D}_j$$

and

$$(1.85)\quad \mathcal{S} = \frac{\theta_{ii}\theta_{jj} - \theta_{ji}\theta_{ij}}{\theta_{jj}}\ ,$$

the general solution of (1.82) is

$$(1.86)\quad \underline{R}_i = -\frac{5}{2}\frac{k}{\vartheta(1+b_i^2/\vartheta^2)}\left(\underline{H} + \frac{(\underline{b}_i\cdot\underline{H})\,\underline{b}_i}{\vartheta^2} - \frac{\underline{b}_i\times\underline{H}}{\vartheta}\right)$$

and

$$(1.87)\quad \underline{R}_j = -\frac{5}{2}\frac{p_j k}{m_j \theta_{jj}}\underline{D}_j - \frac{\theta_{ji}}{\theta_{jj}}\underline{R}_i \ .$$

In the case of equal temperatures, and neglecting the terms in the $\underline{D}$'s involving the stress tensor components, the solution for $\underline{R}_i$ is

$$(1.88)\quad \underline{R}_i = -\frac{\lambda_{ij}}{1+b_i^2/\vartheta^2}\left(\underline{D}_i + \frac{(\underline{b}_i\cdot\underline{D}_i)\,\underline{b}_i}{\vartheta^2} - \frac{\underline{b}_i\times\underline{D}_i}{\vartheta}\right)$$

where $\underline{D}_j = \underline{D}_i$.

The effects of inelastic collisions throughout these transport relationships cannot be separated from, for example, the elastic collisions, without considering in detail particular cases. However, for the sake of completeness, the β_{ij}, θ_{ij} are obtained in terms of the collision integrals, from:

$$(1.89)\quad \beta_{ii}\underset{\sim}{P}_i + \beta_{ij}\underset{\sim}{P}_j \equiv -\frac{1}{2}\sum_{k,\ell} I_{k\ell}\left(m_i\{\underline{w}_i\underline{w}_i\}\right)$$

$$(1.90)\quad \theta_{ii}\underline{R}_i + \theta_{ij}\underline{R}_j \equiv -\frac{1}{2}\sum_{k,\ell} I_{k\ell}\left(\frac{1}{2}m_i\underline{w}_i\left(w_i^2 - \frac{5}{2\alpha_i}\right)\right).$$

Included in the summation over particle type are the electrons, while in these definitive equations, terms involving $\underline{u}_i$, $\underline{u}_j$ have been neglected. The details of the collision integrals are given in Chapter 2.

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Chapter 2

The Collision Integrals

2.1 Introduction

In Chapter 1, the 13-moment equations were presented as a closed set of equations for the dynamical variables ρ_i, $\underline{u}_i$, T_i (or p_i) $\underset{\sim}{P}_i$ and $\underline{R}_i$. The general form of the collision terms was indicated, but the explicit determination of these was left to this Chapter. Since the evaluation of the relevant integrals is lengthy and somewhat cumbersome, this is presented in summary form only, leaving much of the explicit analysis to a report by Bydder and Liley[3]. These results are general enough to include all possible types of binary collisions and almost all types of interaction forces. The elastic collision integrals are obtained as a special case, being the same as those previously derived by Herdan and Liley (Liley[1]).

It is perhaps worthwhile noting here that an interesting application of methods devised for nuclear collision dynamics to the calculation of moment equation collision terms has been given by Kumar[2]. He finds that by a transformation to spherical polar coordinates, it is possible to use the algebra of irreducible tensorial sets and certain transformations known as Talmi transformations, to substantially reduce the work of calculating the collision integrals. Unfortunately, the method as discussed assumes elastic collisions and equal temperatures for all component particle types. This latter assumption is required since the application of the Talmi transformation depends on taking

$$e^{-\alpha_j c_j^2} \cdot e^{-\alpha_k c_k^2} = e^{-\Gamma_{jk} G_{jk}^2} \cdot e^{-\gamma_{jk} g_{jk}^2}$$

($\underline{g}_{jk}$ being the relative velocity in a binary collision, $\underline{G}_{jk}$ the c.m. velocity, $\Gamma_{jk} = \alpha_j + \alpha_k$, $\gamma_{jk} = \alpha_j \alpha_k / \Gamma_{jk}$) and this relation is of course invalid for unequal temperatures. While it should be possible to devise alternative transformations to enable this type of approach to be applied to the case of differing component temperatures and perhaps inelastic collisions, such considerations are not within the scope of this thesis.

2.2 The Dynamics of General Binary Collisions

The collision integrals for the 13-moment equations (1.17) to (1.21) for particles of type i are

$$(2.1) \quad I_{jk}(\psi_i) = \int f_j f_k \, \Delta_{jk}(\psi_i) \, g_{jk} \, d\sigma_{jk,i} \, d\underline{w}_i \, d\underline{w}_j$$

where the following values of ψ_i are considered: m_i ; $m_i \underline{w}_i$; $\frac{1}{2} m_i w_i^2$; $m_i \{ \underline{w}_i \underline{w}_i \}$; and $\frac{1}{2} m_i \underline{w}_i (w_i^2 - 5/2\alpha_i)$. The derivation of this integral is similar to that given by Chapman and Cowling[4] (pp 53-68 and 296). However the integration over $d\sigma_{jk,i}$ deserves some mention. The differential cross section is, in general, a relation involving various scattering angles and velocities for a given kinetic energy defect, either discrete as in excitation, or continuous as in ionisation, and the collision terms must be "summed" over all such possible energy losses.

The product $f_j \times f_k$ is simplified by neglecting the "quadratic" terms, as in Herdan and Liley[5]. (Kolodner[6], in treating elastic collisions, retained the "quadratic" terms, $f_j^{(o)} f_k^{(o)} s_j s_k$, but found that they were of no great significance.) This corresponds to writing

$$(2.2) \quad f_j f_k = f_j^{(o)} f_k^{(o)} \left(1 + s_j + s_k \right)$$

where $f_j^{(o)}$ is the Maxwellian distribution function for particles of type j,

$$(2.3) \quad f_j^{(o)} = n_j \left(\frac{\alpha_j}{\pi} \right)^{\frac{3}{2}} e^{-\alpha_j w_j^2}$$

and

$$(2.3) \quad s_j = 2 \alpha_j \underset{\sim}{u_j} \cdot \underset{\sim}{w_j} + \frac{\alpha_j}{p_j} \underset{\sim}{P_j} : \underset{\sim}{w_j} \underset{\sim}{w_j} + \frac{4}{5} \frac{\alpha_j}{p_j} \underset{\sim}{R_j} \cdot \underset{\sim}{w_j} \left(\alpha_j w_j^2 - \frac{5}{2} \right).$$

Similarly for $f_k^{(o)}$ and s_k.

$\Delta_{jk}(\psi_i)$ is the change in ψ_i due to a collision involving two particles of types j and k. Let any such collision produce particles of types j', k'. Then

$$(2.5) \quad \Delta_{jk}(\psi_i) = \delta_{ij'} \psi_{j'} + \delta_{ik'} \psi_{k'} - \delta_{ij} \psi_j - \delta_{ik} \psi_i ,$$

the δ's being the ordinary Kronecker deltas. $\Delta_{jk}(\psi_i)$ is a function of $\underset{\sim}{w_i}$, $\underset{\sim}{w_i}'$, where primes are used to denote values of a parameter after a collision. It is therefore necessary to express $\underset{\sim}{w_i}'$ in terms of $\underset{\sim}{w_i}$ and the scattering parameters that occur in the differential cross sections. In general, the particles j', k' may be composite

particles; for example, several independent particles which are products of the collision may form "particle" k' . They may also be excited-state particles. Strictly speaking, particles in different excited states should be regarded as different particle types. However, if the lifetime of the excited state is considerably less than the time between collisions, it is more satisfactory to regard excited states as of the same particle type, but this is of no consequence for the present chapter.

Defining

$$(2.6)\quad M_j = \frac{m_j}{m_o}\ ,\ M_k = \frac{m_k}{m_o}\ ,\ M_{j'} = \frac{m_{j'}}{m_o}\ ,\ M_{k'} = \frac{m_{k'}}{m_o}\ ,$$

where

$$(2.7)\quad m_o = m_j + m_k = m_{j'} + m_{k'}$$

the centre of mass velocity in a rest frame is given by

$$(2.8)\quad \underline{G}_o = M_j \underline{c}_j + M_k \underline{c}_k\ ,\ = M_{j'}\underline{c}_{j'} + M_k \underline{c}_{k'} = \underline{G}_o'\ ,$$

the last part of (2.8) resulting from the conservation of momentum in a collision. Again, defining the relative velocities

$$(2.9)\quad \underline{g} \equiv \underline{g}_{jk} = \underline{c}_j - \underline{c}_k = \underline{w}_j - \underline{w}_k$$

$$\underline{g}' \equiv \underline{g}_{j'k'} = \underline{c}_{j'} - \underline{c}_{k'} = \underline{w}_{j'} - \underline{w}_{k'}\ ,$$

on putting

$$(2.10)\quad \underline{G} = \underline{G}_0 - \underline{V} \ ,$$

it follows from these definitions that

$$(2.11)\quad \underline{w}_j = \underline{G} + M_k \underline{g} \ ; \quad \underline{w}_k = \underline{G} - M_j \underline{g} \ ;$$

$$\underline{w}_j' = \underline{G} + M_k' \underline{g}' \ ; \quad \underline{w}_k' = \underline{G} - M_j' \underline{g}' \ .$$

Furthermore, from the energy equation for a collision,

$$(2.12)\quad \tfrac{1}{2} m_0 G^2 + \tfrac{1}{2} \frac{m_j m_k}{m_0} g^2 = \tfrac{1}{2} m_0 G'^2 + \tfrac{1}{2} \frac{m_j' m_k'}{m_0} g'^2 + \Delta\mathcal{E}$$

it is possible to write

$$(2.13)\quad g' = \lambda g$$

where

$$(2.14)\quad \lambda^2 = \frac{m_j m_k}{m_j' m_k'} - \frac{2\Delta\mathcal{E}}{g^2 m_j' m_k'} \ .$$

$\Delta\mathcal{E}$ is the kinetic energy lost in the collision. In general, j' or k' may have "gains" in internal kinetic energy (that is, about their centres of mass), and this must be regarded as energy lost in the collision.

Let χ , ϵ be the polar and azimuthal angles which $\underline{g}'$ makes with $\underline{g}$, in the frame moving with the centre of mass of the colliding particles. Choosing mutually orthogonal unit vectors $\hat{\underline{i}}$, $\hat{\underline{j}}$ $\hat{\underline{k}}$ such that $\hat{\underline{i}} = \underline{g}/|\underline{g}|$, $\underline{g}'$ can be expressed in component form:

$$(2.15)\quad \underline{g}' = \lambda g \left(\hat{\underline{i}} \cos\chi , \hat{\underline{j}} \sin\chi \cos\epsilon , \hat{\underline{k}} \sin\chi \sin\epsilon \right) .$$

With the aid of equations (2.11) and (2.14), the $\Delta_{jk}(\psi_i)$ may now be given in terms of $\underline{G}$, $\underline{g}$, λ, χ and ϵ .

The writing of the $\Delta_{jk}(\psi_i)$, and the subsequent form of the collision integrals, is much simplified by the use of the following notation:

$$(2.16)\quad \begin{aligned} \delta(1) &= \delta_{ij'} + \delta_{ik'} - \delta_{ij} - \delta_{ik} \quad , \\ \delta(2) &= \delta_{ij'} - \delta_{ik'} - \delta_{ij} + \delta_{ik} \quad , \\ \delta(3) &= \delta_{ij'} - \delta_{ik'} \quad , \\ \delta(4) &= \delta_{ij'} + \delta_{ik'} \quad ; \end{aligned}$$

and

$$(2.17)\quad m_{ir} = \frac{m_i \, (m_o - m_i)}{m_o} \quad ,$$

m_{ir} being the reduced mass in the collision. The details being given in Bydder and Liley[3], with the use of this notation and the previous relations for $\underline{w}_j$, $\underline{w}_k$, $\underline{w}_{j'}$, $\underline{w}_{k'}$, $\underline{g}'$, the $\Delta_{jk}(\psi_i)$

become (in the following equations, terms of odd order in $\sin\epsilon$ or $\cos\epsilon$ have been ignored for reasons given in the next section):

(2.18) $$\Delta_{jk}(m_i) = m_i\,\delta(1)\ ;$$

(2.19) $$\Delta_{jk}(m_i\underline{w}_i) = m_i\,\delta(1)\,\underline{G} + m_{ir}\,\delta(2)\,\underline{g} - m_{ir}\,\delta(3)\,\underline{g}(1-\lambda\cos\chi);$$

(2.20) $$\Delta_{jk}\left(\tfrac{1}{2}m_i w_i^2\right) = \tfrac{1}{2}\Big[m_i\,\delta(1)\,G^2 + \frac{m_{ir}^2}{m_i}\,\delta(1)\,g^2 + 2m_{ir}\,\delta(2)\,\underline{G}\cdot\underline{g} - 2m_{ir}\,\delta(3)\,\underline{G}\cdot\underline{g}\,(1-\lambda\cos\chi) - \frac{m_{ir}^2}{m_i}\,\delta(4)\,g^2(1-\lambda^2)\Big];$$

(2.21) $$\Delta_{jk}\left(m_i\{\underline{w}_i\,\underline{w}_i\}\right) = \delta(1)\,m_i\{\underline{G}\,\underline{G}\} + \frac{m_{ir}^2}{m_i}\,\delta(1)\{\underline{g}\,\underline{g}\} + 2m_{ir}\,\delta(2)\{\underline{G}\,\underline{g}\} - 2m_{ir}\,\delta(3)\{\underline{G}\,\underline{g}\}(1-\lambda\cos\chi) - \frac{m_{ir}^2}{m_i}\,\delta(4)\{\underline{g}\,\underline{g}\} + \frac{m_{ir}^2}{m_i}\,\delta(4)\,\lambda^2 g^2\Big(\{\hat{\underline{\imath}}\,\hat{\underline{\imath}}\}\cos^2\chi + \{\hat{\underline{\jmath}}\,\hat{\underline{\jmath}}\}\sin^2\chi\cos^2\epsilon + \{\hat{\underline{k}}\,\hat{\underline{k}}\}\sin^2\chi\sin^2\epsilon\Big);$$

(2.22) $$\Delta_{jk}\left(\tfrac{1}{2}m_i\,\underline{w}_i\,w_i^2\right) = \tfrac{1}{2}\Big[m_i\,\delta(1)\,\underline{G}\,G^2 - 2m_{ir}\,\delta(3)\,\underline{G}\,\underline{G}\cdot\underline{g}\,(1-\lambda\cos\chi) + 2m_{ir}\,\delta(2)\,\underline{G}\,\underline{G}\cdot\underline{g} - \frac{m_{ir}^2}{m_i}\,\delta(4)\,\underline{G}\,g^2(1-\lambda^2) + \frac{m_{ir}^2}{m_i}\,\delta(1)\,\underline{G}\,g^2 - m_{ir}\,\delta(3)\,\underline{g}\,G^2(1-\lambda\cos\chi) + m_{ir}\,\delta(2)\,\underline{g}\,G^2 - \frac{m_{ir}^3}{m_i^2}\,\delta(3)\,g^2\underline{g}\,(1-\lambda^3\cos\chi) + \frac{m_{ir}^3}{m_i^2}\,\delta(2)\,g^2\underline{g} + \frac{2m_{ir}^2}{m_i}\,\delta(4)\,\lambda^2 g^2\Big(\hat{\underline{\imath}}\,\underline{G}\cdot\hat{\underline{\imath}}\cos^2\chi + \hat{\underline{\jmath}}\,\underline{G}\cdot\hat{\underline{\jmath}}\sin^2\chi\cos^2\epsilon + \hat{\underline{k}}\,\underline{G}\cdot\hat{\underline{k}}\sin^2\chi\sin^2\epsilon\Big) + \frac{2m_{ir}^2}{m_i}\,(\delta(4)-\delta(1))\,\underline{g}\,\underline{G}\cdot\underline{g}\Big].$$

2.3 Integration over Azimuthal Angle

Although at the present stage there is no need to consider the specific collision cross sections, considerable simplification of the collision terms is obtained by assuming that the differential cross sections are independent of ϵ, the azimuth of the scattering angle. Since the Jacobian of the transformation $d\underline{w}_j\, d\underline{w}_k$ to $d\underline{G}\, d\underline{g}$ is

$$\left| \frac{\partial(\underline{G}, \underline{g})}{\partial(\underline{w}_j, \underline{w}_k)} \right| = 1 , \tag{2.23}$$

the general collision integrals (2.1) become

$$I_{jk}(\psi_i) = \int f_j f_k \, \Delta_{jk}(\psi_i)\, g\sigma \sin\chi \, d\chi \, d\epsilon \, d\underline{G} \, d\underline{g} . \tag{2.24}$$

Here the differential cross section has been written as

$$d\sigma_{jk,i} = \sigma \sin\chi \, d\chi \, d\epsilon \tag{2.25}$$

with σ independent of ϵ, and subscripts associated with σ and χ being ignored. Since ϵ only occurs in the $\Delta_{jk}(\psi_i)$, while χ and λ only occur in this term and the differential cross section, integration over the azimuthal angle ϵ can be effected immediately. As the limits of integration over ϵ are $0, 2\pi$, terms odd in $\sin\epsilon$ or $\cos\epsilon$ integrate to zero, while square terms, $\sin^2\epsilon$ or $\cos^2\epsilon$, integrate to π.

Defining

$$\phi_{jk,i}(\mu,\nu) = \int_0^{\pi} (1 - \lambda^{\mu} \cos^{\nu}\chi)\, g\sigma \sin\chi \, d\chi \tag{2.26}$$

(and, to maintain a uniform notation, $\phi(-\infty,0)$ is also defined by this integral but with $\lambda^{-\infty}$ taken as zero, no matter what the magnitude of λ)

the collision integrals can be represented as integrals over $\underline{G}$ and $\underline{g}$ involving the ϕ's. With these considerations (and neglecting the subscripts of the $\phi_{jk,i}(\mu,\nu)$) the collision integrals corresponding to (2.18) to (2.22) are:

$$(2.27)\quad I_{jk}(m_i) = 2\pi m_i \delta(1) \int f_j f_k \phi(-\infty,0)\, d\underline{G}\, d\underline{g};$$

$$(2.28)\quad I_{jk}(m_i \underline{w}_i) = 2\pi m_i \delta(1) \int f_j f_k \phi(-\infty,0)\, \underline{G}\, d\underline{G}\, d\underline{g}$$
$$- 2\pi m_{ir} \delta(3) \int f_j f_k \phi(1,1)\, \underline{g}\, d\underline{G}\, d\underline{g}$$
$$+ 2\pi m_{ir} \delta(2) \int f_j f_k \phi(-\infty,0)\, \underline{g}\, d\underline{G}\, d\underline{g};$$

$$(2.29)\quad I_{jk}\left(\tfrac{1}{2} m_i w_i^2\right) = \pi m_i \delta(1) \int f_j f_k \phi(-\infty,0)\, G^2\, d\underline{G}\, d\underline{g}$$
$$- 2\pi m_{ir} \delta(3) \int f_j f_k \phi(1,1)\, \underline{G}\cdot\underline{g}\, d\underline{G}\, d\underline{g}$$
$$+ 2\pi m_{ir} \delta(2) \int f_j f_k \phi(-\infty,0)\, \underline{G}\cdot\underline{g}\, d\underline{G}\, d\underline{g}$$
$$- \pi \frac{m_{ir}^2}{m_i} \delta(4) \int f_j f_k \phi(2,0)\, g^2\, d\underline{G}\, d\underline{g}$$
$$+ \pi \frac{m_{ir}^2}{m_i} \delta(1) \int f_j f_k \phi(-\infty,0)\, g^2\, d\underline{G}\, d\underline{g};$$

$$(2.30)\quad I_{jk}(m_i \{\underline{w}_i \underline{w}_i\}) = 2\pi m_i \delta(1) \int f_j f_k \phi(-\infty,0) \{\underline{G}\,\underline{G}\}\, d\underline{G}\, d\underline{g}$$
$$- 4\pi m_{ir} \delta(3) \int f_j f_k \phi(1,1) \{\underline{G}\,\underline{g}\}\, d\underline{G}\, d\underline{g}$$

$$+ 4\pi m_{ir}\delta(2)\int f_j f_k \phi(-\infty,0)\{\underline{G}\,\underline{g}\}\,d\underline{G}\,d\underline{g}$$

$$- 3\pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,2)\{\underline{g}\,\underline{g}\}\,d\underline{G}\,d\underline{g}$$

$$+ \pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,0)\{\underline{g}\,\underline{g}\}\,d\underline{G}\,d\underline{g}$$

$$+ 2\pi\frac{m_{ir}^2}{m_i}\delta(1)\int f_j f_k \phi(-\infty,0)\{\underline{g}\,\underline{g}\}\,d\underline{G}\,d\underline{g};$$

(2.31) $$I_{jk}\left(\tfrac{1}{2}m_i\underline{w}_i w_i^2\right) = \pi m_i\delta(1)\int f_j f_k \phi(-\infty,0)\,\underline{G}\,G^2\,d\underline{G}\,d\underline{g}$$

$$- 2\pi m_{ir}\delta(3)\int f_j f_k \phi(1,1)\,\underline{G}\,\underline{G}\cdot\underline{g}\,d\underline{G}\,d\underline{g}$$

$$- \pi m_{ir}\delta(3)\int f_j f_k \phi(1,1)\,\underline{g}\,G^2\,d\underline{G}\,d\underline{g}$$

$$+ 2\pi m_{ir}\delta(2)\int f_j f_k \phi(-\infty,0)\,\underline{G}\,\underline{G}\cdot\underline{g}\,d\underline{G}\,d\underline{g}$$

$$+ \pi m_{ir}\delta(2)\int f_j f_k \phi(-\infty,0)\,\underline{g}\,G^2\,d\underline{G}\,d\underline{g}$$

$$- 2\pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,0)\,\underline{G}\,g^2\,d\underline{G}\,d\underline{g}$$

$$+ \pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,2)\,\underline{G}\,g^2\,d\underline{G}\,d\underline{g}$$

$$+ \pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,0)\,\underline{g}\,\underline{G}\cdot\underline{g}\,d\underline{G}\,d\underline{g}$$

$$- 3\pi\frac{m_{ir}^2}{m_i}\delta(4)\int f_j f_k \phi(2,2)\,\underline{g}\,\underline{G}\cdot\underline{g}\,d\underline{G}\,d\underline{g}$$

$$+ 2\pi\frac{m_{ir}^2}{m_i}\delta(1)\int f_j f_k \phi(-\infty,0)\,\underline{G}\,\underline{G}\cdot\underline{g}\,d\underline{G}\,d\underline{g}$$

$$+ \pi \frac{m_{ir}^2}{m_i} \delta(1) \int f_j f_k \phi(-\infty, 0) \underline{G} \, g^2 \, d\underline{G} \, d\underline{g}$$

$$- \pi \frac{m_{ir}^3}{m_i^2} \delta(3) \int f_j f_k \phi(3,1) \underline{g} \, g^2 \, d\underline{G} \, d\underline{g}$$

$$+ \pi \frac{m_{ir}^3}{m_i^2} \delta(2) \int f_j f_k \phi(-\infty, 0) \underline{g} \, g^2 \, d\underline{G} \, d\underline{g} .$$

2.4 Integration over Velocity Space

From (2.3), it follows that

$$(2.32) \quad f_j^{(0)} f_k^{(0)} = n_j n_k (\alpha_j \alpha_k)^{\frac{3}{2}} \pi^{-3} e^{-(\alpha_j w_j^2 + \alpha_k w_k^2)} .$$

Expressed in terms of $\underline{G}$ and $\underline{g}$, the exponent becomes

$$(2.33) \quad \alpha_j w_j^2 + \alpha_k w_k^2 = (\alpha_j + \alpha_k) G^2 + 2(\alpha_j M_k - \alpha_k M_j) \underline{G} \cdot \underline{g} + (\alpha_j M_k^2 + \alpha_k M_j^2) g^2$$

which is not a convenient form for integration with respect to $\underline{G}$ or $\underline{g}$. The transformation used by Liley[1] is to the dimensionless velocities $\underline{X}$, $\underline{Y}$ where

$$(2.34) \quad \underline{X} = \beta_{jk} (\underline{G} + \delta_{jk} \underline{g}) \; ; \quad \underline{Y} = \gamma_{jk} \underline{g} \quad ,$$

with

$$(2.35)\qquad \beta_{jk}^2 = \alpha_j + \alpha_k \quad ; \quad \gamma_{jk}^2 = \frac{\alpha_j \alpha_k}{\alpha_j + \alpha_k} \quad ;$$

$$\delta_{jk} = \frac{\alpha_j M_k - \alpha_k M_j}{\alpha_j + \alpha_k}$$

This reduces the exponent to $-(X^2+Y^2)$. The Jacobian of the transformation is

$$(2.36)\qquad \left| \frac{\partial(\underline{\xi}, \underline{\eta})}{\partial(\underline{X}, \underline{Y})} \right| = (\alpha_j \alpha_k)^{-\frac{3}{2}} .$$

Also, by elementary substitution in equations (2.11),

$$(2.37)\qquad \underline{w}_j = \frac{1}{\beta_{jk}} \underline{X} + \frac{1}{\gamma_{jk}} (M_k - \delta_{jk}) \underline{Y}$$

$$\underline{w}_k = \frac{1}{\beta_{jk}} \underline{X} - \frac{1}{\gamma_{jk}} (M_j + \delta_{jk}) \underline{Y} .$$

Using these relations to express the s_j, s_k of (2.4) in terms of $\underline{X}$, $\underline{Y}$, the approximation for $f_j f_k$ in equation (2.2) gives

$$(2.38)\qquad f_j f_k = n_i n_j (\alpha_j \alpha_k)^{\frac{3}{2}} \pi^{-3} D(\underline{X}, \underline{Y}) e^{-(X^2+Y^2)}$$

where

$$(2.39)\qquad D(\underline{X}, \underline{Y}) = 1 + s_j + s_k ,$$

the s_j, s_k now being expressed in terms of $\underline{X}$, $\underline{Y}$. The explicit form of $D(\underline{X}, \underline{Y})$ is given in Bydder and Liley[3].

The collision integrals were given at the end of the previous section in terms of integrals of the form

$$(2.40) \qquad \int f_j f_k \, \phi(\mu,\nu) \, \underline{\Phi}(\underline{G},\underline{g}) \, d\underline{G} \, d\underline{g}$$

where the $\underline{\Phi}(\underline{G},\underline{g})$ are obtained on comparing (2.40) with (2.27) to (2.31). By virtue of the relations (2.34), (2.36), and (2.38), these integrals (2.40) transform to

$$(2.41) \qquad \int n_j n_k \pi^{-2} \, D(\underline{X},\underline{Y}) \, e^{-(X^2+Y^2)} \, \phi(\mu,\nu) \, \underline{\Phi}(\underline{X},\underline{Y}) \, d\underline{X} \, d\underline{Y}$$

where of course $Y = |\underline{Y}|$. The $\underline{\Phi}(\underline{G},\underline{g})$ representing the functions of $\underline{G}$, $\underline{g}$ occurring in these integrals, are functions of $\underline{X}$, $\underline{Y}$. On the other hand, the $\phi(\mu,\nu)$ are functions of g and not $\underline{g}$ or $\underline{G}$, and hence become functions of Y . Therefore integration over $\underline{X}$ is immediately possible, so that all the collision integrals can be reduced to functions of $\underline{Y}$. Furthermore, since the $\phi(\mu,\nu)$ are the only unknown expressions, the collision integrals are reducible to integrals over Y . The integration over $\underline{X}$ and the reduction of the integrals over $\underline{Y}$ are achieved by using certain relations given in reference 3. Briefly, the integrals in the form (2.41) are subdivided according to the components of $\underline{\Phi}(\underline{X},\underline{Y})$, e. g. $\underline{X}$, $\underline{Y}$, $\underline{X}\cdot\underline{Y}$, X^2 , ... etc., there being 15 cases to consider. Each of these components is taken in turn with each of the components of $D(\underline{X},\underline{Y})$ (there are 12 such terms) and the component multiple integral of the form (2.41) evaluated with the appropriate reduction formulae. However, of the 180 integrals, 142 are zero by inspection, making the task less arduous than apparent at first sight.

In their final form the collision integrals reduce to expressions in terms of certain generalised collision "frequencies", $\Omega^{\mu,\nu}_{jk,i}(r)$ defined by

$$(2.42)\quad \Omega^{\mu,\nu}_{jk,i}(r) = \pi^{\frac{1}{2}} \int_0^\infty e^{-Y^2} Y^{2r+2} \phi_{jk,i}(\mu,\nu)\, dY .$$

These Ω^{l}s are a generalisation of those used by Chapman and Cowling[4]. It is only in the Ω^{l}s that the differential cross sections or equivalently, the interaction force laws, occur and for this reason the general results can be expressed in terms of such functions.

2.5 The Collision Integrals in Terms of the Ω^{l}s

There are in fact up to 20 Ω^{l}s occurring in the 13-moment collision integrals for each type of collision, the $\Omega^{\mu,\nu}_{jk,i}(r)$ being such that (μ,ν) takes the values* $(-\infty,0)$, $(1,1)$, $(2,0)$, $(2,2)$ and $(3,1)$, each with $r = 0, 1, 2,$ and 3. In writing out the collision integrals, the subscripts jk,i associated with the Ω^{l}s have been neglected for reasons of brevity. Expressed in terms of the Ω^{l}s, the binary collision integrals are as follows.

2.5.1 Continuity (or Mass) Integral

This integral (which is zero for elastic collisions) is

$$(2.43)\quad I_{jk}(m_i) = n_j n_k\, 8 m_i\, \delta(1)\, \Omega^{-\infty,0}(0) .$$

2.5.2 Momentum Integral

The final form of this integral is

$$(2.44)\quad I_{jk}(m_i \underline{w}_i) = n_j n_k \left[\left(c_1^{uu}(\alpha_j) + c_2^{uu}(\alpha_j) \right) \underline{u}_j \right.$$

---

* Note that $\phi(-\infty,0)$ is defined by taking $\lambda^{-\infty} \equiv 0$, see equation (2.26).

$$+\left(C_1^{uu}(\alpha_k) - C_2^{uu}(\alpha_k)\right)\underline{u}_k$$

$$+\left(C_1^{uq}(\alpha_j) + C_2^{uq}(\alpha_j)\right)\frac{\underline{\underline{R}}_j}{\rho_j}$$

$$+\left(C_1^{uq}(\alpha_k) - C_2^{uq}(\alpha_k)\right)\frac{\underline{\underline{R}}_k}{\rho_k}\Bigg].$$

The coefficients $C(\alpha)$ have been used in preference to the $\delta_{jk,i}$ of Chapter 1, partly to avoid confusion with the use of relations involving the Kronecker δ's, and also because of the symmetrical form in which they occur in the collision terms. Subscripts associated with the C's as with the Ω's are neglected. These coefficients are

$$(2.45)\quad C_1^{uu}(\alpha) = \frac{8\alpha}{\beta^2} m_i \delta(1) \Omega^{-\infty,0}(0),$$

$$(2.46)\quad C_2^{uu}(\alpha) = \frac{16}{3}\Big(-m_i \delta(1) S \Omega^{-\infty,0}(1) + m_{ir} \delta(2) \Omega^{-\infty,0}(1) - m_{ir} \delta(3) \Omega^{-\infty,0}(1)\Big),$$

$$(2.47)\quad C_1^{uq}(\alpha) = \frac{32}{3}\frac{\gamma^2\alpha}{\beta^2} m_i \delta(1)\left(\Omega^{-\infty,0}(1) - \frac{3}{2}\Omega^{-\infty,0}(0)\right),$$

$$(2.48)\quad C_2^{uq}(\alpha) = \frac{32}{3}\gamma^2\Big(m_i \delta(1) S\left(\Omega^{-\infty,0}(1) - \frac{2}{5}\Omega^{-\infty,0}(2)\right) + m_{ir} \delta(2)\left(\frac{2}{5}\Omega^{-\infty,0}(2) - \Omega^{-\infty,0}(1)\right)$$

$$+ m_{ir}\,\delta(3)\left(\Omega^{1,1}(1) - \tfrac{2}{5}\Omega^{1,1}(2)\right)\Bigg).$$

2.5.3 Energy Integral

This collision term is

$$(2.49)\quad I_{jk}\left(\tfrac{1}{2}m_i w_i^2\right) = n_j n_k\Bigg[\frac{6\,m_i\,\delta(1)}{\beta^2}\Omega^{-\infty,0}(0) + 4\left(\frac{m_i\,\vartheta^2}{\gamma^2}\delta(1) + \frac{m_{ir}^2}{m_i\gamma^2}\delta(1) - 2\frac{m_{ir}\,\vartheta}{\gamma^2}\delta(2)\right)\Omega^{-\infty,0}(1) + \frac{8\,m_{ir}\,\vartheta}{\gamma^2}\delta(3)\Omega^{1,1}(1) - \frac{4\,m_{ir}^2}{m_i\gamma^2}\delta(4)\Omega^{2,0}(1)\Bigg].$$

2.5.4 Stress Integral

The stress integral can be written as

$$(2.50)\quad I_{jk}\left(m_i\{\underset{\sim}{w}_i\underset{\sim}{w}_i\}\right) = n_j n_k\Big[\left(c_1^p(\alpha_j) + c_2^p(\alpha_j)\right)\underset{\sim}{P}_j/\rho_j + \left(c_1^p(\alpha_k) - c_2^p(\alpha_k)\right)\underset{\sim}{P}_k/\rho_k\Big].$$

Again subscripts in the coefficients c have been ignored. The values are

$$(2.51)\quad c_1^p(\alpha) = \frac{16}{3}\Bigg[m_i\,\delta(1)\left(\frac{3}{2}\frac{\alpha^2}{\beta^4}\Omega^{-\infty,0}(0) + \frac{2}{5}\left[\vartheta^2 + \frac{m_{ir}^2}{m_i^2}\right]\Omega^{-\infty,0}(2)\right) - m_{ir}\,\delta(2)\,\frac{4}{5}\,\vartheta^2\,\Omega^{-\infty,0}(2) \qquad \text{(ctd.)}$$

$$+ m_{ir}\,\delta(3)\,\frac{4}{5}\,\mathcal{S}^2\,\Omega^{1,1}(2)$$

$$+ \frac{m_{ir}^2}{m_i}\,\delta(4)\left(+\frac{1}{5}\,\Omega^{2,0}(2) - \frac{3}{5}\,\Omega^{2,2}(2)\right)\Bigg].$$

$$(2.52)\quad c_2^p(\alpha) = \frac{16}{3}\Bigg[-2\,m_i\,\delta(1)\,\frac{\mathcal{S}\alpha}{\beta^2}\,\Omega^{-\infty,0}(1)$$

$$+2\,m_{ir}\,\delta(2)\,\frac{\alpha}{\beta^2}\,\Omega^{-\infty,0}(1)$$

$$-2\,m_{ir}\,\delta(3)\,\frac{\alpha}{\beta^2}\,\Omega^{1,1}(1)\Bigg].$$

2.5.5 Heat Flux Integral

This integral can be separated into two parts:

$$(2.53)\quad I_{jk}\left(\tfrac{1}{2}\,m_i\,\underline{w}_i\left(w_i^2 - \frac{5}{2\alpha_i}\right)\right) = I_{jk}\left(\tfrac{1}{2}\,m_i\,\underline{w}_i\,w_i^2\right) - \frac{5}{4\alpha_i}\,I_{jk}\left(m_i\underline{w}_i\right).$$

As the last term involves the momentum integral, (2.41), only the first part of the right hand side of (2.53) need be given; this is in fact the energy flux integral. Similarly to the momentum integral, this has the form

$$(2.54)\quad I_{jk}\left(\tfrac{1}{2}m_i\,\underline{w}_i\,w_i^2\right) = n_j\,n_k\Big[\left(c_1^{qu}(\alpha_j) + c_2^{qu}(\alpha_j)\right)\underline{u}_j$$

$$+\left(c_1^{qu}(\alpha_k) - c_2^{qu}(\alpha_k)\right)\underline{u}_k$$

$$+\left(C_1^{qq}(\alpha_j) + C_2^{qq}(\alpha_j)\right)\underline{R}_j/\rho_j$$

$$+\left(C_1^{qq}(\alpha_k) - C_2^{qq}(\alpha_k)\right)\underline{R}_k/\rho_k\Big].$$

The coefficients C, the most complicated of those associated with the 13-moment equation collision integrals, are:

$$(2.55)\quad C_1^{qu}(\alpha) = m_i\,\delta(1)\left[\frac{10\alpha}{\beta^4}\Omega^{-\infty,0}(0) + \frac{20}{3}\frac{\alpha}{\beta^2\gamma^2}\left(\vartheta^2 + \frac{m_{ir}^2}{m_i^2}\right)\Omega^{-\infty,0}(1)\right]$$

$$- m_{ir}\,\delta(2)\,\frac{40}{3}\frac{\alpha\vartheta}{\beta^2\gamma^2}\,\Omega^{-\infty,0}(1)$$

$$+ m_{ir}\,\delta(3)\,\frac{40}{3}\frac{\alpha\vartheta}{\beta^2\gamma^2}\,\Omega^{1,1}(1)$$

$$- \frac{m_{ir}^2}{m_i}\,\delta(4)\,\frac{20}{3}\frac{\alpha}{\beta^2\gamma^2}\,\Omega^{2,0}(1).$$

$$(2.56)\quad C_2^{qu}(\alpha) = - m_i\,\delta(1)\left[\frac{20}{3}\frac{\vartheta}{\beta^2}\Omega^{-\infty,0}(1) + \frac{8}{3}\frac{\vartheta}{\gamma^2}\left(\vartheta^2 + \frac{3m_{ir}^2}{m_i^2}\right)\Omega^{-\infty,0}(2)\right]$$

$$+ m_{ir}\,\delta(2)\left[\frac{20}{3\beta^2}\Omega^{-\infty,0}(1) + \frac{8}{\gamma^2}\left(\vartheta^2 + \frac{1}{3}\frac{m_{ir}^2}{m_i^2}\right)\Omega^{-\infty,0}(2)\right]$$

$$- m_{ir}\,\delta(3)\left[\frac{20}{3\beta^2}\Omega^{1,1}(1) + \frac{8\vartheta^2}{\gamma^2}\Omega^{1,1}(2) + \frac{8}{3}\frac{m_{ir}^2}{\gamma^2 m_i^2}\Omega^{3,1}(2)\right]$$

$$+ \frac{m_{ir}^2}{m_i}\,\delta(4)\left[\frac{8}{3}\frac{\vartheta}{\gamma^2}\left(\Omega^{2,0}(2) + 2\,\Omega^{2,2}(2)\right)\right].$$

$$(2.57)\quad C_1^{qq}(\alpha) = m_i\,\delta(1)\Big[-\frac{20\alpha^2}{\beta^4}\Big(1-\frac{7}{5}\frac{\alpha}{\beta^2}\Big)\Omega^{-\infty,0}(0) + \frac{40}{3}\frac{\alpha}{\beta^2}\Big(\frac{\gamma^2}{\beta^2}-\vartheta^2 - \frac{m_{ir}^2}{m_i^2}\Big)\Omega^{-\infty,0}(1) + \frac{176}{15}\frac{\alpha}{\beta^2}\Big(\frac{m_{ir}^2}{m_i^2}+\vartheta^2\Big)\Omega^{-\infty,0}(2)\Big]$$
$$+ m_{ir}\,\delta(2)\Big[\frac{80}{3}\frac{\alpha\vartheta}{\beta^2}\Omega^{-\infty,0}(1) - \frac{352}{15}\frac{\alpha\vartheta}{\beta^2}\Omega^{-\infty,0}(2)\Big]$$
$$- m_{ir}\,\delta(3)\Big[\frac{16}{3}\frac{\alpha\vartheta}{\beta^2}\Big(5\Omega^{1,1}(1) - \frac{22}{5}\Omega^{1,1}(2)\Big)\Big]$$
$$- \frac{m_{ir}^2}{m_i}\delta(4)\Big[-\frac{40}{3}\frac{\alpha}{\beta^2}\Omega^{2,0}(1) + \frac{16}{15}\frac{\alpha}{\beta^2}\Big(7\Omega^{2,0}(2)+4\Omega^{2,2}(2)\Big)\Big].$$

$$(2.58)\quad C_2^{qq}(\alpha) = m_i\,\delta(1)\vartheta\Big[\frac{8}{3}\frac{\alpha}{\beta^2}\Big(5-11\frac{\alpha}{\beta^2}\Big)\Omega^{-\infty,0}(1) + \frac{16}{3}\Big(\vartheta^2-\frac{\gamma^2}{\beta^2} + \frac{3m_{ir}^2}{m_i^2}\Big)\Omega^{-\infty,0}(2) - \frac{32}{15}\Big(\vartheta^2+\frac{3m_{ir}^2}{m_i^2}\Big)\Omega^{-\infty,0}(3)\Big]$$
$$+ m_{ir}\,\delta(2)\Big[-\frac{8}{3}\frac{\alpha}{\beta^2}\Big(5-11\frac{\alpha}{\beta^2}\Big)\Omega^{-\infty,0}(1) + \frac{16}{3}\Big(\frac{\gamma^2}{\beta^2}-3\vartheta^2 - \frac{m_{ir}^2}{m_i^2}\Big)\Omega^{-\infty,0}(2) + \frac{32}{15}\Big(3\vartheta^2+\frac{m_{ir}^2}{m_i^2}\Big)\Omega^{-\infty,0}(3)\Big]$$
$$+ m_{ir}\,\delta(3)\Big[\frac{8}{3}\frac{\alpha}{\beta^2}\Big(5-11\frac{\alpha}{\beta^2}\Big)\Omega^{1,1}(1) - \frac{16}{3}\Big(\frac{\gamma^2}{\beta^2}-3\vartheta^2\Big)\Omega^{1,1}(2) - \frac{32}{5}\Omega^{1,1}(3) + \frac{16}{3}\frac{m_{ir}^2}{m_i^2}\Omega^{3,1}(2) - \frac{32}{15}\frac{m_{ir}^2}{m_i^2}\Omega^{3,1}(3)\Big]$$
$$- \frac{m_{ir}^2}{m_i}\delta(4)\Big[\frac{16}{3}\vartheta\Big(\Omega^{2,0}(2)+2\Omega^{2,2}(2)\Big) - \frac{32}{15}\vartheta\Big(\Omega^{2,0}(3) + 2\Omega^{2,2}(3)\Big)\Big].$$

It is important to mention that in all these terms the order of the subscripts associated with $\mathcal{S}$ are regarded as independent of the order of the subscripts of the c's . The $\mathcal{S}_{jk}$ are defined in (2.35), arising automatically out of the product $f_j f_k$.

Apart from the next section and Chapter 3, particular cases of these integrals will not be considered. However, it should be noted that much of the physics of a particular type of collision is contained within the δ's . For example, in elastic collisions,

$$\delta(1) = \delta(2) = 0 , \tag{2.59}$$

while

$$\delta(3) = 0 , \quad \delta(4) = 2 \tag{2.60}$$

for self-collisions. Again, for particle annihilation in a collision, $\delta(4)$ is zero. As well as neatly displaying the form of the collision integrals for special types of collisions, these δ's indicate which of the Ω's need be calculated for any specific case.

2.6 The General Collision Integrals for Electrons in a Gas Mixture

The collision integrals are relatively complicated, and in general depend on the specific consideration of each component in the gas mixture for reduction to a useful form. However for the case of electrons it is possible to take account of the small mass ratio, as compared with all other possible particles, to obtain some simplifications. Further, when the temperatures of the components in the mixture are such that

(2.61) $\alpha_e << \alpha_p, \alpha_q, \ldots$

and

(2.62) $T_e << \frac{m_p}{m_e} T_p, \frac{m_q}{m_e} T_q, \ldots$

(where the subscripts are e for electrons, and $p,q,\ldots$ for "heavy" particles, for instance atoms and protons), reduction of the electron collision integrals to a form where they only involve the $\underline{u}, \underset{\sim}{P}, \underline{R}$ of the electrons is possible. This was, in the previous Chapter, termed "decoupling".

From the above inequalities it follows that

(2.63) $$\beta_{ep}^2 \simeq \alpha_p \; ; \quad \beta_{pq}^2 \sim \alpha_p \; ;$$
$$\gamma_{ep}^2 \simeq \alpha_e \; ; \quad \vartheta_{ep} \sim \frac{m_e}{m_p} \; .$$

With the specialisation of the particles i in the collision integrals (2.43) to (2.58) to be electrons, the collision is considered, as before, between particles of types j and k producing particles of types j' and k'.

2.6.1 The Continuity Integral

This integral, from (2.43) is

(2.64) $$I_{jk}(m_e) = n_j n_k \, 8 m_e \, \delta(1) \, \Omega_{jk,e}^{-\infty,0}(0) \; .$$

This is zero for electron-electron collisions and for all elastic collisions. No immediate simplification for inelastic collisions is possible, without some consideration of the inelastic processes involved. Obviously this integral does not involve the dynamical parameters $\underline{u}$, $\underset{\sim}{P}$, $\underline{R}$ of any gas particles, although in general $\Omega^{(r)}_{jk,i}$ does involve both the temperatures, T_j, T_k.

2.6.2 The Momentum Integral

From (2.44) the momentum equation collision integral for electrons is

$$(2.65)\quad I_{jk}(m_e \underline{w}_e) = n_j n_k \Big[\big(c_1^{uu}(\alpha_j) + c_2^{uu}(\alpha_j) \big) \underline{u}_j + \big(c_1^{uu}(\alpha_k) - c_2^{uu}(\alpha_k) \big) \underline{u}_k + \big(c_1^{uq}(\alpha_j) + c_2^{uq}(\alpha_j) \big) \underline{R}_j/\rho_j + \big(c_1^{uq}(\alpha_k) - c_2^{uq}(\alpha_k) \big) \underline{R}_k/\rho_k \Big] .$$

Owing to the conservation of momentum in a collision, it is evident that the collision terms will be zero for electron self-collisions. There is, however, a contribution to this integral from collisions of electrons with "heavy" particles.

Since from (2.45),

$$(2.66)\quad c_1^{uu}(\alpha) = 8 \frac{m_e}{\beta^2} \alpha \delta(1) \, \Omega^{-\infty,0}_{jk,e}(0) ,$$

on considering the largest terms apart from the electron self-collision terms, when one of the particles j, k is an electron,

$$(2.67)\quad c_1^{uu}(\alpha) \sim m_e \, \Omega^{-\infty,0}_{ep,e}(0) ,$$

while if neither is an electron,

$$C_1^{uu}(\alpha) \sim m_e \Omega^{-\infty,0}_{pq,e}(0) \quad . \tag{2.68}$$

Similarly, from (2.46),

$$C_2^{uu}(\alpha) = \frac{16}{3}\left[\left(m_{er}\delta(2) - m_e\vartheta\delta(1)\right)\Omega^{-\infty,0}_{jk,e}(1) - m_{er}\delta(3)\Omega^{(1,1)}_{jk,e}(1)\right]. \tag{2.69}$$

When one of j,k is an electron,

$$C_2^{uu}(\alpha) \sim m_e \Omega^{\mu,\nu}_{ep,e}(r) \tag{2.70}$$

and when both are "heavy" particles,

$$C_2^{uu}(\alpha) \sim m_e \Omega^{\mu,\nu}_{pq,e}(r) \quad . \tag{2.71}$$

For elastic Coulomb collisions, for which the $\Omega^{l,s}$ are given in Chapter 3, it is apparent, on comparing relative magnitudes of these functions, that

$$\Omega^{\mu,\nu}_{ee}(r) \sim \Omega^{\mu,\nu}_{ep}(r), \tag{2.72}$$

$$\Omega^{\mu,\nu}_{ee}(r) \Big/ \Omega^{\mu,\nu}_{pq}(r) \sim \left(\frac{m_p}{m_e}\right)^2\left(\frac{\alpha_e}{\alpha_p}\right)^{\frac{3}{2}} = \left(\frac{m_p}{m_e}\right)^{\frac{1}{2}}\left(\frac{T_p}{T_e}\right)^{\frac{3}{2}} \quad .$$

Therefore, assuming the ratios of the electron-electron collision $\Omega^{l,s}$ to all other $\Omega^{l,s}$ involving the electrons to be at least of the same order as the ratio for elastic Coulomb collisions (evidently this is equivalent to assuming that the collision cross sections for other

processes involving electrons are not greater than elastic Coulomb cross sections for particles of the same types; and this is true for almost all such collision processes), by (2.61)

$$(2.73)\qquad \frac{m_p}{m_e}\,\Omega^{\mu,\nu}_{ep,e}(r) \;>>\; \Omega^{\mu,\nu}_{pq,e}(r)\ .$$

Also it was pointed out in Chapter 1 that in a gas mixture,

$$(2.74)\qquad |\underline{u}_p| \;\sim\; \frac{m_e}{m_p}\,|\underline{u}_e|\ .$$

Using these two relations in (2.67), (2.68), (2.69) and (2.71), to this order of approximation the "dominant" terms involving the $\underline{u}$'s are those containing $\underline{u}_e$. That is, there is "decoupling" of the electron drift speed collision terms from the corresponding "heavy" particle terms.

Again, for the terms involving $\underline{R}$'s, from (2.47)

$$(2.75)\qquad C_1^{uq}(\alpha) \;=\; \frac{32}{3}\,\frac{\gamma^2\alpha}{\beta^2}\,m_e\,\delta(1)\left(\Omega^{-\infty,0}(1) \;-\; \frac{3}{2}\,\Omega^{-\infty,0}(0)\right).$$

This term is zero for elastic collisions, including electron-electron collisions. When one of j,k is an electron,

$$(2.76)\qquad \frac{C_1^{uq}(\alpha_e)}{\rho_e} \;\sim\; \frac{\alpha_e^2}{\alpha_p}\,\frac{m_e}{\rho_e}\,\Omega^{-\infty,0}_{ep,e}(r)\ ;\ \frac{C_1^{uq}(\alpha_p)}{\rho_p} \;\sim\; \frac{\alpha_e}{\rho_p}\,m_e\,\Omega^{-\infty,0}_{ep,e}(r)$$

(the largest term being considered in each case), while when neither of j,k is an electron,

$$(2.77)\qquad \frac{c_1^{uq}(\alpha_p)}{\rho_p} \sim \frac{\alpha_p}{\rho_p} m_e \Omega_{pq,e}^{-\infty,0}(r) \ .$$

From (2.48),

$$(2.78)\qquad c_2^{uq}(\alpha) = \frac{32}{3}\gamma^2 \Big[m_e \delta(1) \varsigma \Big(\Omega^{-\infty,0}(1) - \frac{2}{5}\Omega^{-\infty,0}(2) \Big) + m_{er}\delta(2)\Big(\frac{2}{5}\Omega^{-\infty,0}(2) - \Omega^{-\infty,0}(1)\Big) + m_{er}\delta(3)\Big(\Omega^{1,1}(1) - \frac{2}{5}\Omega^{1,1}(2)\Big)\Big] .$$

The electron self-collisions can give no net contribution. When one of j, k is an electron,

$$(2.79)\qquad \frac{c_2^{uq}(\alpha_e)}{\rho_e} \sim \frac{\alpha_e m_e}{\rho_e}\Omega_{ep,e}^{\mu,\nu}(r) \ ; \quad \frac{c_2^{uq}(\alpha_p)}{\rho_p} \sim \frac{\alpha_e m_e}{\rho_p}\Omega_{ep,e}^{\mu,\nu}(r) .$$

When j, k are both "heavy" particles,

$$(2.80)\qquad \frac{c_2^{uq}(\alpha)}{\rho_p} \sim \frac{\alpha_p m_e}{\rho_p}\Omega_{pq,e}^{\mu,\nu}(r) .$$

Using (2.61) and (2.72) it immediately follows that the "dominant" term of the coefficients $c^{uq}(\alpha)$ is that occurring in the first expression of (2.79). From (2.65) it is evident that this is part of the coefficient of $\underline{R}_e$; from this and with a similar result regarding $\underline{u}_e$, it follows that the electron momentum equation collision terms involve $\underline{u}_e$, $\underline{R}_e$ and not $\underline{u}_p$, $\underline{R}_p$, $\underline{u}_q$, $\underline{R}_q$, $\cdots$.

Summarising, the momentum collision integral for electrons, with inelastic collisions allowed, but only the "dominant" contributary

terms included, is

$$(2.81)\quad I_{ep}(m_e \underline{w}_e) = n_e n_p \Big[\Big(\tfrac{16}{3}\delta(2)\,\Omega^{-\infty,0}_{(1)} - \tfrac{16}{3} m_e \Omega^{1,1}_{(1)} \Big) \underline{u}_e + \Big(-\tfrac{32}{3}\alpha_e m_e \delta(2)\,\Omega^{-\infty,0}_{(1)} + \tfrac{32}{3}\alpha_e m_e \delta(3)\,\Omega^{1,1}_{(1)} + \tfrac{64}{15}\alpha_e m_e \delta(2)\,\Omega^{-\infty,0}_{(2)} - \tfrac{64}{15}\alpha_e m_e \delta(3)\,\Omega^{1,1}_{(2)} \Big) \frac{\underline{\underline{R}}_e}{\rho_e} \Big].$$

2.6.3 The Energy Integral

$$(2.82)\quad I_{jk}\left(\tfrac{1}{2} m_e w_e^2\right) = n_j n_k \Big[\frac{6 m_e \delta(1)}{\beta^2}\,\Omega^{-\infty,0}_{(0)} + 4 \Big(\frac{m_e \vartheta^2 \delta(1)}{\gamma^2} + \frac{m_{er}^2 \delta(1)}{\gamma^2 m_e} - \frac{2\delta(2)\vartheta m_{er}}{\gamma^2} \Big) \Omega^{-\infty,0}_{(1)} - \frac{4 m_{er}^2 \delta(4)}{\gamma^2 m_e}\,\Omega^{2,0}_{(1)} + \frac{8 m_{er} \vartheta \delta(3)}{\gamma^2}\,\Omega^{1,1}_{(1)} \Big].$$

Although of course for self-collisions this is zero, in general there is a contribution from terms associated with collisions between electrons and particles of other types whenever the respective particle types have different temperatures, and also from any inelastic collisions. However, since no terms involving $\underline{u}$'s or $\underline{R}$'s of any particle type occur in the collision terms, this equation does not affect the "decoupling" of the electron moment equations. No appreciable simplification can be made without considering the nature of the collisions.

2.6.4 The Stress Integral

$$(2.83)\quad I_{jk}\left(m_e\left\{\underline{w}_e\underline{w}_e\right\}\right) = n_j n_k\Big[\left(c_1^p(\alpha_j) + c_2^p(\alpha_j)\right)\underset{\sim}{P}_j/\rho_j + \left(c_1^p(\alpha_k) - c_2^p(\alpha_k)\right)\underset{\sim}{P}_k/\rho_k\Big].$$

For this collision integral, the c^p's are obtained by taking particles of type i to be electrons in equations (2.51) and (2.52). For this case, it follows directly from (2.61) and (2.73) that the "dominant" terms in these collision integrals are those involving $\underset{\sim}{P}_e$. Included in these terms is a contribution from the electron self-collisions. The results are, for self-collisions

$$(2.84)\quad I_{ee}\left(m_e\left\{\underline{w}_e\underline{w}_e\right\}\right) = -\frac{16}{5}\frac{n_e^2 m_e}{\rho_e}\Omega^{2,2}_{(2)}\underset{\sim}{P}_e$$

(it should be noted, in connexion with (2.5), that the actual coefficient of $\underset{\sim}{P}_e$ in the electron 13-moment equations is one half of that given in (2.84), and similarly in the heat flux equations). For collisions with "heavy" particles

$$(2.85)\quad I_{ep}\left(m_e\left\{\underline{w}_e\underline{w}_e\right\}\right) = n_e n_p\Big[\frac{32}{15}m_e\delta(1)\Omega^{-\infty,0}_{ep,e(2)} + \frac{16}{15}m_e\delta(4)\Omega^{2,0}_{ep,e(2)} - \frac{16}{5}m_e\delta(4)\Omega^{2,2}_{ep,e(2)}\Big]\underset{\sim}{P}_e/\rho_e .$$

2.6.5 The Heat Flux Integral

As shown in (2.53), where the full form of the heat flux integral was given, it is only necessary to consider the energy flux term; the

remaining part involves the momentum integral, already given in (2.81). The energy flux integral is

$$(2.86)\quad \mathbf{I}_{jk}\left(\tfrac{1}{2} m_e \underline{w}_e w_e^2\right) = n_j n_k \Big[\left(C_1^{qu}(\alpha_j) + C_2^{qu}(\alpha_j) \right) \underline{u}_j + \left(C_1^{qu}(\alpha_k) - C_2^{qu}(\alpha_k) \right) \underline{u}_k + \left(C_1^{qq}(\alpha_j) + C_2^{qq}(\alpha_j) \right) \frac{\underline{R}_j}{\rho_j} + \left(C_1^{qq}(\alpha_k) - C_2^{qq}(\alpha_k) \right) \frac{\underline{R}_k}{\rho_k} \Big],$$

the C's being given by (2.55) to (2.58) on taking particles i to be electrons. These collision terms can be considered in the same way as the momentum and stress collision integrals, using (2.61) and (2.73). There is a non-zero electron self-collision term, and contribution from terms involving collisions with other particle types, but the "decoupling" found in the momentum and stress collision integrals is also found for this case. The "dominant" terms in the electron collision integrals are (where particles p are not electrons):

$$(2.87)\quad \mathbf{I}_{ee}\left(\tfrac{1}{2} m_e \underline{w}_e w_e^2\right) = -\frac{32}{15} n_e^2 m_e \Omega_{ee,e}^{2,2}(2) \frac{\underline{R}_e}{\rho_e} \quad ;$$

$$(2.88)\quad \mathbf{I}_{ep}\left(\tfrac{1}{2} m_e \underline{w}_e w_e^2\right) = n_e n_p \Big[\left(\frac{8}{3} \frac{m_e}{\alpha_e} \delta(2) \Omega_{ep,e}^{-\infty,0}(2) - \frac{8}{3} \frac{m_e}{\alpha_e} \delta(3) \Omega_{ep,e}^{3,1}(2) \right) \underline{u}_e + \left(-\frac{16}{3} m_e \delta(2) \Omega_{ep,e}^{-\infty,0}(2) + \frac{16}{3} m_e \delta(3) \Omega_{ep,e}^{3,1}(2) + \frac{32}{15} m_e \delta(2) \Omega_{ep,e}^{-\infty,0}(3) - \frac{32}{15} m_e \delta(3) \Omega_{ep,e}^{3,1}(3) \right) \frac{\underline{R}_e}{\rho_e} \Big].$$

It can be concluded from this discussion on the electron collision terms that, provided the component particle temperatures satisfy (2.61) and (2.62), irrespective of the type of collisions, the 13-moment

equations for electrons only involve the temperatures and number densities of other particle types, and no other dynamical variables. These considerations can be extended to show that under the same conditions, the moment equations of the "heavy" particle types do not involve the electron component variables $\underline{u}_e$, $\underset{\sim}{P}_e$, $\underline{R}_e$.

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PART II

Discussion on the Formation of a Hydrogenic-Like Plasma

Chapter 3

Evaluation of the Ω's for Certain Collision Processes Involving Hydrogen

3.1 Introduction

The final form of the collision integrals of the preceding Chapter are of little practical value without the Ω's being determined explicitly. In the limiting case of elastic collisions, there are only five such collision "frequencies" involved and these have already been obtained for many different types of interactions (e.g. Chapman and Cowling[1]). However since the general $\Omega^{\mu\nu}(r)$ as defined in Chapter 2 are peculiar to this thesis, obviously no equivalent calculations have been carried out for the inelastic case.

In general there are in all 20 Ω's to be determined for each type of collision for each particle type present. Therefore at first sight the task is a formidable one. However, as will be seen, reasonable and inter-related expressions can be obtained for certain specific cases.

In this Chapter and the following the application of the equations of the preceding Part to the formation of a laboratory hydrogenic-like plasma is considered. The Ω's are obtained explicitly, while the particular form of the equations and the imposition of associated boundary conditions, suitable for numerical integration, are discussed.

In attempting to follow the collision processes during the development of a fully ionised plasma from molecular hydrogen, collisions involving (Goodyear and Von Engel[2]) H_2, H, H_2^*, H^*, H_2^+, H^+,

and electrons should be taken into account. Because of the large number of possible types of collisions between these particles, a simplified model, restricted to H, H^*, H^+ and electrons, is considered. In all probability this is a realistic model for sufficiently high initial temperatures, and in general should give results suitable for comparison with experiment. Unfortunately, even for this simple case, the relevant expressions for the Ω's are not particularly amenable to integration, and in certain cases various approximations are necessary. Generally speaking, relatively exact expressions can be obtained for elastic collisions (Chapman and Cowling, loc. cit.), the major approximations applying to the inelastic processes. These approximations are such that the Ω's tend to be inaccurate near the threshold energy, which is unfortunate, but they do lead to reasonably accurate results for appreciably higher energies.

However, in the present calculations, it is possible that, due to the nature of the approximations, certain of the Ω's corresponding to the inelastic processes are not physically accurate. Furthermore, it is not unlikely that another author, in attempting such calculations, would prefer different approximations.

3.2 The Differential Cross Sections

Since there are not suitable analytical expressions for the differential (or for that matter, total) cross sections for all of the processes being considered, two main approaches are possible. The first would be to obtain empirical relations for the differential cross section for each process, in the relevant energy range, based on the use of accurate calculations and experimental results. Unfortunately there are relatively few accurate calculations for the differential

cross sections, and fewer experimental results. More data is available on total cross sections, and empirical relations for the differential cross sections could have been chosen, which integrate to the known total values. This would, however, be particularly tedious. The second approach is to use a consistent mathematical approximation for the theoretical evaluations of the differential cross sections, for example, the Born approximation or certain modifications of this approximation. It is the latter approach which is used in the subsequent calculations. It must be stated, however, that the inverse fifth power law and other such classical models provide a much better description of certain of the elastic collisions than any form of the Born approximation. $\Omega^{(l,s)}$ calculated for such classical cross sections are given in Chapman and Cowling[1]. For the present chapter the Born approximation has been applied to all collisions, being multiplied by a numerical factor to produce "reasonable" approximations to the total cross sections in the energy range 0 - 100 ev. The detailed derivation of the various cases and a comparison with known values of total cross section is given in Appendix A.

For notational convenience, in the following sections, $q^2/4\pi\epsilon_0$ is replaced by e^2 . q is the electron charge, and ϵ_0 is the free-space permittivity. Except where otherwise designated, $d\Omega$ is the solid angle into which the asymptotic relative velocity $\underline{g}$ of the colliding particles is deflected with polar angle χ ; m_r is the reduced mass of the colliding particles; $\hbar$ is Plank's constant divided by 2π ; and $\underline{k}$ is the (reduced) wave vector of the colliding particles, given by $\underline{p}/\hbar$. $\underline{p}$ is the reduced momentum $m_r \underline{g}$. Finally, a_0 is the Bohr radius, and the change in wave

vector in a collision, $\underline{K}$, is defined by

$$(3.1) \qquad \underline{K} = \underline{k}_f - \underline{k}_o \ ,$$

$\underline{k}_o$ and $\underline{k}_f$ being the wave vectors before and after the collision. The moduli of these wave vectors, k_o and k_f (the wave numbers), are of course equal for elastic collisions. The differential cross sections given in the following sub-sections are derived in Appendix A, and include the "correction" factors.

3.2.1 Elastic Collisions Between Electrons

The differential cross section for these Coulomb collisions is

$$(3.2) \qquad \sigma \, d\Omega = \frac{m_r^2 e^4}{4 \hbar^4 k_o^4} \operatorname{cosec}^4 \chi/2 \; d\Omega$$

with $\chi \geq \chi_o$. χ_o is the Debye cut-off angle (see Appendix A).

3.2.2 Elastic Collisions Between Electrons and Hydrogen Atoms

The adjusted Born differential cross section is

$$(3.3) \qquad \sigma \, d\Omega = \frac{(.45)\, 4 m_r^2 e^4 a_o^4 \left(K^2 a_o^2 + 8 \right)^2}{\hbar^4 \left(K^2 a_o^2 + 4 \right)^4} \, d\Omega \ .$$

3.2.3 Elastic Collisions Between Electrons and Ions

This Coulomb differential cross section has the same form as (3.2):

$$(3.4) \qquad \sigma \, d\Omega = \frac{m_r^4 e^4}{4 \hbar^4 k_o^4} \operatorname{cosec}^4 \chi/2 \; d\Omega$$

with $\chi \geq \chi_o$ (Debye cut-off).

3.2.4 Elastic Collisions Between Hydrogen Atoms

The adjusted Born cross section is

$$(3.5)\qquad \sigma\, d\Omega = \frac{(2.0\times 10^{-2})\, 4 m_r^2 e^4 a_o^8 K^4 (a_o^2 K^2 + 8)^4}{\hbar^4 (a_o^2 K^2 + 4)^8}\, d\Omega .$$

3.2.5 Elastic Collisions Between Hydrogen Atoms and Protons

Of the same general form as (3.3), the differential cross section is

$$(3.6)\qquad \sigma\, d\Omega = \frac{(2.0\times 10^{-2})\, 4 m_r^2 e^4 a_o^4 (K^2 a_o^2 + 8)^2}{\hbar^4 (K^2 a_o^2 + 4)^4}\, d\Omega .$$

3.2.6 Elastic Collisions Between Protons

Formally the same as (3.2) and (3.4),

$$(3.7)\qquad \sigma\, d\Omega = \frac{m_r^2 e^4 \operatorname{Cosec}^4 \chi/2}{4\hbar^4 k_o^4}\, d\Omega$$

with $\chi \geqslant \chi_o$ (Debye cut-off).

3.2.7 1s - 2p Excitation Collisions Between Electrons and Hydrogen Atoms

The adjusted Born approximation gives

$$(3.8)\qquad \sigma\, d\Omega = \frac{(.45)\, 3^2 2^{15} k_f m_r^2 e^4 a_o^2}{k_o \hbar^4 K^2 (4 a_o^2 K^2 + 9)^6}\, d\Omega .$$

3.2.8 Ionisation Collisions Between Electrons and Hydrogen Atoms

A simplified form of the Born approximation used for this case gives

(3.9) $$\sigma\, d\Omega_1\, d\Omega_2\, d\chi = \frac{(.14\times10^{-3})\, 2^6 \pi a_o^2 m_r^2 e^4 k_f \chi}{\hbar^4 k_o K^4 (a_o^2 K^2 + 1)^4}\, d\Omega_1\, d\Omega_2\, d\chi$$

where $d\Omega_1$ is the solid angle into which the incident electron is scattered (with wave vector $\underline{k}_f$), and $d\Omega_2$ the solid angle into which the atomic electron is ejected with wave vector $\underline{\chi}$. Details are given in Appendix A.

3.2.9 Charge Exchange Collisions Between Hydrogen Atoms and Protons

A simplified Born approximation gives

(3.10) $$\sigma\, d\Omega = \frac{(2.1\times10^{-2})\, 2^8 m_r^2 e^4 a_o^4}{\hbar^4 (K^2 a_o^2 + 1)^6}\, d\Omega$$

where, for this case, $\underline{K} = \underline{k}_o + \underline{k}_f$.

3.3 Evaluation of the ϕ's

3.3.1 Elastic Collisions

The general $\phi(\mu,\nu)$ of section 2.3 (Chapter 2) are simplified for elastic collisions, since $\lambda = 1$. Hence the $\phi(\mu,\nu)$ are independent of μ:

(3.11) $$\phi(\mu,\nu) = \int (1 - \cos^\nu \chi)\, g\sigma \sin\chi\, d\chi$$

$$\equiv \phi(\nu)$$

and these $\phi(\nu)$ are the same as those used by Chapman and Cowling[1].

(a) Electron-electron collisions

These results are given in reference 1.

$$(3.12)\quad \phi_{ee}(1) = \int_{\chi_0}^{\pi} \frac{(1-\cos\chi)\, e^4 g \operatorname{cosec}^4\chi/2 \sin\chi\, d\chi}{4 m_r^2 g^4} = \frac{e^4}{m_r^2 g^3} \log(1+\Lambda_{ee}^2)$$

where

$$(3.13)\quad \Lambda_{ee} = \frac{\lambda_D g^2 m_r}{e^2}$$

(λ_D being the Debye length). Similarly,

$$(3.14)\quad \phi_{ee}(2) = \frac{2e^4}{m_r g^3}\left(\log(1+\Lambda_{ee}^2) - \frac{\Lambda_{ee}^2}{1+\Lambda_{ee}^2}\right) \sim 2\phi_{ee}(1),$$

since Λ_{ee} is large.

(b) Electron-hydrogen atom collisions

According to (3.3),

$$(3.15)\quad \phi_{ea}(1) = \int_0^{\pi} \frac{(1-\cos\chi)\, g\, (.45)\, 4 m_r^2 e^4 a_o^4 (K^2 a_o^2 + 8)^2 \sin\chi\, d\chi}{\hbar^4 (4 + K^2 a_o^2)^4}$$

Since for elastic collisions

$$(3.16)\quad K^2 = 2k_o^2(1-\cos\chi),$$

using the substitution $y = 1 - \cos\chi$, after some manipulation the result is

$$(3.17)\quad \phi_{ea}(1) = \frac{(.45)\, 4 m_r^2 e^4 a_o^4}{\hbar^4 q^2} g \left(\log\frac{q+2}{2} + \frac{q^3 - 6q^2 - 24q}{6(q+2)^3}\right)$$

where $q^2 = 2k_o^2 a_o^2 = 2 m_r^2 g^2 a_o^2/\hbar^2$.

Similarly

$$(3.18)\quad \phi_{ea}(2) = \int_0^{\pi} \frac{(1-\cos^2\chi)\, g\,(.45)\, 4 m_r^2 e^4 a_o^4 (K^2 a_o^2 + 8)^2 \sin\chi \, d\chi}{\hbar^4 (4 + K^2 a_o^2)^4}$$

$$= 2\phi_{ea}(1) - \frac{(.45)\, 4 m_r^2 e^4 a_o^4}{\hbar^4}\, g\, \frac{(6q+16)}{3(q+2)^3} \quad .$$

Alternatively, for the same collisions considered according to an inverse power force law, the ϕ_{ea}^{l}s are given in Chapman and Cowling[1].

(c) Electron-proton collisions

These collisions give the same expressions as for electron-electron collisions.

$$(3.19)\quad \phi_{ep}(1) = \frac{e^4}{m_r^2 g^3} \log(1 + \Lambda_{ep}^2),$$

$$(3.20)\quad \phi_{ep}(2) = \frac{2e^4}{m_r^2 g^3} \left(\log(1+\Lambda_{ep}^2) - \frac{\Lambda_{ep}^2}{1+\Lambda_{ep}^2} \right) \sim 2\phi_{ep}^{(1)},$$

where $\Lambda_{ep} = \lambda_D g^2 m_r / e^2$.

(d) Hydrogen atom-hydrogen atom collisions

Using equation (3.5)

$$(3.21)\quad \phi_{aa}(1) = \int_0^{\pi} \frac{(1-\cos\chi)\, g\,(2\times 10^{-2})\, 4 m_r^2 e^4 a_o^8 K^4 (a_o^2 K^2 + 8)^4 \sin\chi \, d\chi}{\hbar^4 (a_o^2 K^2 + 4)^8}.$$

On substituting for K^2 (from (3.16)) the part of (3.21) to be integrated is

$$(3.22)\quad I = \int_0^{\pi} \frac{(1-\cos\chi)^3 (p - q\cos\chi)^4 \sin\chi \, d\chi}{(r - q\cos\chi)^8}$$

where $q = 2k_o^2 a_o^2$, $p = q+8$ and $r = q+4$. Elementary application of standard integrals provides the result:

$$(3.23)\quad I = \frac{1}{q^4}\left[\frac{24576}{7(qy+4)^7} - \frac{143360}{6(qy+4)^6} - \frac{3072}{5(qy+4)^5} - \frac{768}{4(qy+4)^4} + \frac{192}{3(qy+4)^3} + \frac{216}{2(qy+4)^2} - \frac{4}{qy+4} + \log(qy+4)\right]_0^2,$$

the limits being on y . For the greater part of the energy range being considered, $q >> 1$; taking this into account, the approximation

$$(3.24)\quad I = \frac{1}{q^4}\log\frac{q+2}{2}$$

will be used. In this case,

$$(3.25)\quad \phi_{aa}(1) = g\frac{(2\times10^{-2})16m_r^2e^4k_o^4a_o^8}{\hbar^4 q^4}\log\frac{q+2}{2} .$$

Similarly, with the same approximation,

$$(3.26)\quad \phi_{aa}(2) = g\frac{(2\times10^{-2})32m_r^2e^4k_o^4a_o^8}{\hbar^4 q^4}\log\frac{q+2}{2} = 2\phi_{aa}(1) .$$

Again, possibly better alternative results, based on inverse power force laws, are given in reference 1.

(e) Hydrogen atom-proton collisions

Using the differential cross section (3.6), for this case

$$(3.27)\quad \phi_{ap}(1) = \int_0^{\pi} \frac{(1-\cos\chi)\, g\,(2\times10^{-2})\, 4 m_r^2 e^4 a_0^4 (K^2 a_0^2 + 8)^2 \sin\chi \, d\chi}{\hbar^4 (K^2 a_0^2 + 4)^4}.$$

The integration is similar to the previous cases. Taking $q = 2k_0^2 a_0^2 \gg 1$,

$$(3.28)\quad \phi_{ap}(1) = \frac{(2\times10^{-2})\, 4 g m_r^2 e^4 a_0^4}{\hbar^4 q^2} \log\frac{q+2}{2}.$$

Again, it may be shown that

$$(3.29)\quad \phi_{ap}(2) = (2\times10^{-2})\, 8 g m_r^2 e^4 a_0^4 \log\frac{q+2}{2}$$

$$= 2\,\phi_{ap}(1).$$

(As mentioned with certain other cases, reference may also be made to Chapman and Cowling[1]).

(f) Proton-proton collisions

The calculation for this case is formally the same as for electron-electron collisions, the results being

$$(3.30)\quad \phi_{pp}(1) = \frac{e^4}{m_r^2 g^3} \log(1+\Lambda_{pp}^2)$$

where $\Lambda_{pp} = \frac{\lambda_D g^2 m_r}{e^2}$; and similarly,

$$(3.31)\quad \phi_{pp}(2) = \frac{2e^4}{m_r^2 g^3}\left(\log(1+\Lambda_{pp}^2) - \frac{\Lambda_{pp}^2}{1+\Lambda_{pp}^2}\right) \sim 2\,\phi_{pp}(1).$$

3.3.2 Inelastic Collisions

For inelastic collisions involving particles j, k colliding to produce particles j', k',

$$(3.32)\qquad \lambda^2 = \frac{m_j m_k}{m_{j'} m_{k'}} - \frac{2 m_o \Delta\mathcal{E}}{m_{j'} m_{k'} g^2} .$$

Therefore it is necessary to use the general form for the ϕ's which, from Chapter 2, is

$$(3.33)\qquad \phi(\mu,\nu) = \int_o^\pi (1 - \lambda^\mu \cos^\nu\chi)\, g\sigma \sin\chi \, d\chi .$$

For excitation collisions where the nature of the colliding particles does not change (the energy being carried away by photons, for example), the following ϕ's are required to evaluate the collision terms: $\phi(-\infty,0)$, $\phi(1,1)$, $\phi(2,0)$ and $\phi(2,2)$. In more general inelastic collisions it is also necessary to calculate $\phi(3,1)$.

(a) Electron-hydrogen atom 1s - 2p excitation collisions

Using the differential cross section (3.8),

$$(3.34)\qquad \phi(\mu,\nu) = \int_o^\pi \frac{(1-\lambda^\mu \cos^\nu\chi)\, g\, (.45)\, 3^2 2^{15} k_f m_r^2 e^4 a_o^2 \sin\chi \, d\chi}{\hbar^4 k_o K^2 (4K^2 a_o^2 + 9)^6}$$

and since

$$(3.35)\qquad K^2 = k_o^2 + k_f^2 - 2 k_o k_f \cos\chi ,$$

ignoring the constant terms in (3.34), this becomes

$$(3.36)\qquad I(\mu,\nu) = \int_o^\pi \frac{(1-\lambda^\mu \cos^\nu\chi)\sin\chi \, d\chi}{(p - q\cos\chi)(r - s\cos\chi)^6}$$

where

(3.37) $p = k_o^2 + k_f^2 \quad ; \quad q = 2k_o k_f \; ; \; r = 9 + 4a_o^2(k_o^2 + k_f^2);$

$s = 8a_o^2 k_o k_f \; ; \quad t = ps - qr = -18 k_o k_f \; ;$

$r \pm s = 9 + 4a_o^2 (k_o \pm k_f)^2.$

Using the substitution $y = r - s\cos\chi$,

(3.38) $$I(\mu,\nu) = \int_{r-s}^{r+s} \frac{(1 - \lambda^{\mu}[r/s - y/s]^{\nu})\, dy}{(t + qy)\, y^6}$$

and it follows that $I(\mu,\nu)$ can be expressed in terms of integrals P , Q and R , where (since ν takes the values $0, 1$ and 2):

(3.39) $$P = \int_{r-s}^{r+s} \frac{dy}{(t+qy)\, y^6} \; ,$$

(3.40) $$Q = \int_{r-s}^{r+s} \frac{dy}{(t+qy)\, y^5} \; ,$$

(3.41) $$R = \int_{r-s}^{r+s} \frac{dy}{(t+qy)\, y^4} \; .$$

These integrals are readily evaluated. Considering (3.39), it is found that

(3.42) $$P = \frac{1}{t^6}\left[-\frac{1}{5}\left(\frac{t+qy}{y}\right)^5 + \frac{5}{4}\left(\frac{t+qy}{y}\right)^4 q - \frac{10}{3}\left(\frac{t+qy}{y}\right)^3 q^2 \right.$$
$$\left. + \frac{10}{2}\left(\frac{t+qy}{y}\right)^2 q^3 - 5\left(\frac{t+qy}{y}\right) q^4 + q^5 \log \frac{t+qy}{y} \right]_{r-s}^{r+s}.$$

After inserting the limits of integration in the general term

$\left[q^{5-n} \left((t+qy)/y \right)^n \right]_{r-s}^{r+s}$ and substituting for $r+s$ and $r-s$, except near threshold it is found that this can be approximated by

$$(3.43) \qquad \left[q^{5-n} \left(\frac{t+qy}{y} \right)^n \right]_{r-s}^{r+s} \sim (2k_0 k_f)^5 = q^5 .$$

Similarly, considering the term in (3.42) involving the logarithm, away from the threshold,

$$(3.44) \qquad \left[q^5 \log \frac{t+qy}{y} \right]_{r-s}^{r+s} \sim q^5 \log \frac{q}{4a_0^2 (k_0 - k_f)^2} .$$

On the other hand, near threshold, it is found that both of these terms (3.43) and (3.44) are of the same order. From a physical point of view, it is the near threshold region that is usually of particular importance despite the small total cross section, since during the excitation and ionisation stages the mean particle energies are within this region. However, it is only with the simplification achieved by discarding all terms in (3.42) except the logarithm term that a comprehensible form for the Ω's can be obtained. In this case, the integral (3.42) becomes

$$(3.45) \qquad P = \frac{q^5}{t^6} \log \frac{q}{4a_0^2 (k_0 - k_f)^2}$$

$$= \frac{1}{3^{12} k_0 k_f} \log \frac{3}{2a_0 (k_0 - k_f)} ,$$

while similar approximations in calculating Q, R give

$$(3.46) \qquad Q = qP ,$$

(3.47) $R = 81 P$.

Taking the relevant particular values of ν ,

(3.48) $$I(\mu,0) = (1 - \lambda^{\mu}) P$$

$$I(\mu,1) = (1 - \frac{\lambda^{\mu} r}{s}) P + \frac{\lambda^{\mu}}{s} Q$$

$$I(\mu,2) = (1 - \frac{\lambda^{\mu} r^2}{s}) P + \frac{2\lambda^{\mu} r}{s^2} Q - \frac{\lambda^{\mu}}{s^2} R .$$

Writing

(3.49) $$A = \frac{(.45)\, 3^2\, 2^{15}\, g\, m_r^2\, k_f\, e^4 a_o^2}{\hbar^4\, k_o}$$

and using (3.46), (3.47), the ϕ's become

(3.50) $$\phi(\mu,0) = A(1 - \lambda^{\mu}) P$$

(3.51) $$\phi(\mu,1) = A\left(1 - \lambda^{\mu} \frac{k_o^2 + k_f^2}{2 k_o k_f}\right) P$$

(3.52) $$\phi(\mu,2) = A\left(1 - \lambda^{\mu}\left(\frac{k_o^2 + k_f^2}{2 k_o k_f}\right)^2\right) P .$$

Again it is necessary to make assumptions corresponding to a non-threshold region. Since for the excitation collisions being considered,

(3.53) $$\lambda^2 = 1 - \frac{2\Delta\mathcal{E}}{m_r g^2} = 1 - \frac{2 m_r \Delta\mathcal{E}}{\hbar^2 k_o^2} ,$$

(3.54) $$\lambda \sim 1 - \frac{m\Delta\mathcal{E}}{\hbar^2 k_o^2} \equiv \frac{\alpha}{k_o^2} .$$

Also from the energy conservation, (since $\Delta\mathcal{E}$ is the excitation energy)

(3.55) $$k_o^2 = k_f^2 + \frac{2 m_r \Delta\mathcal{E}}{\hbar^2} ;$$

$$k_o^2 + k_f^2 = 2k_o^2 - \frac{2 m_r \Delta\mathcal{E}}{\hbar^2} = 2\alpha ;$$

$$k_f \sim \frac{\alpha}{k_o} .$$

Using these relations in equations (3.50) to (3.52), giving the ϕ's, it follows that

(3.56) $$\phi(-\infty, o) = AP ,$$

(3.57) $$\phi(1,1) = A\left(1 - \frac{2\alpha}{2\alpha}\lambda\right) P$$

$$\sim \frac{m_r \Delta\mathcal{E}}{\hbar^2 k_o^2} AP ,$$

(3.58) $$\phi(2,2) = A\left(1 - \left(\frac{2\alpha}{2\alpha}\right)^2 \lambda^2\right) P = \frac{2 m_r \Delta\mathcal{E}}{\hbar^2 k_o^2} AP$$

$$= 2\phi(1,1) ,$$

(3.59) $$\phi(2,o) = A(1 - \lambda^2) P$$

$$= \phi(2,2) ,$$

$$= 2\,\phi(1,1) \quad .$$

Hence for the Ω^{l}s the following relationships apply :

$$(3.60) \quad \Omega^{2,2}(r) = \Omega^{2,0}(r) = 2\,\Omega^{1,1}(r) .$$

Again, using (3.55) in (3.45), AP simplifies to

$$(3.61) \quad AP = \frac{(\cdot 45)\, 2^{15}\, q\, m_r^2\, e^4\, a_o^2}{3^{10}\, \hbar^4\, k_o^2} \log \frac{3\,\hbar^2 k_o^2}{2\, m_r\, \Delta\mathcal{E}} \quad .$$

Equations (3.56) to (3.59) enable the ϕ^{l}s to be expressed in terms of (3.61).

(b) Electron-hydrogen atom ionising collisions

An ionisation collision is essentially an encounter in which the atomic electron is excited to the continuum state. For this reason, for the incident electron scattered in the collision, the process may be treated as excitation with variable $\Delta\mathcal{E}$. On the other hand, owing to the stripping of the atom and the production of new particles, for the atom, the proton produced, and the atomic electron ejected, the collisions need to be considered in more detail. To some extent the physics of these collisions is contained within the δ -functions (Chapter 2); nevertheless the Ω^{l}s must be calculated for each different particle type for which the 13-moment equations are required. In general, for each particle considered, the $\Delta\mathcal{E}$ are different, and the λ^{l}s must be expressed in terms of these for integration over the range of $\Delta\mathcal{E}$.

Let $d\Omega_1$ be the solid angle into which the incident electron (with incident (reduced) wave vector $\underline{k}_o$) is scattered with wave vector $\underline{k}_f$, and $d\Omega_2$ the solid angle into which the atomic electron is ejected with wave vector $\underline{\kappa}$. It is shown in Appendix A that it is satisfactory to refer all these parameters to a frame moving with the centre of mass of the two colliding particles. The subscripts e_1 , e_2 , a , p will refer respectively to the scattered electron, atomic electron, atom, and proton produced. For the collision dynamics (but not the general equations for the electron component) the scattered and atomic electrons are treated separately.

(i) Scattered electron, e_1

Using equation (3.33) and the differential cross section (3.9), (χ being the polar angle of $\underline{k}_f$ with respect to $\underline{k}_o$)

$$(3.62)\qquad \phi_{e_1 a, e_1}(\mu,\nu) = \int \frac{(1-\lambda_{e_1}^{\mu}\cos^{\nu}\chi)\, g(\cdot 14\times 10^{-3})\, 2^4\pi m_r^2 e^4 a_o^2 k_f \kappa \sin\chi \, d\chi \, d\Omega_2 d\kappa}{\hbar^4 k_o K^4 (a_o^2K^2+1)^4}$$

where

$$(3.63)\qquad K^2 = k_o^2 + k_f^2 - 2k_o k_f \cos\chi \, .$$

It is almost impossible to calculate the Ω^1s from (3.62) as such; however, by approximating the term $(a_o^2K^2+1)^4$ in the denominator by $(a_o^2k_o^2+1)^4$, it becomes possible to evaluate the sequence of integrals required to obtain the Ω^1s . Although by no means an accurate approximation, the K^4 term, also in the denominator, nullifies the effect of this approximation.

From the conservation of energy, it follows that

$$(3.64)\quad k_o^2 = k_f^2 + \chi^2 + \frac{2m_r I}{\hbar^2}$$

where I is the ionisation energy of the atom. Integration over $d\Omega_2$ yields a factor 4π. The limits of integration over χ are $(0, \pi)$, while from (3.64) the limits on χ are $(0, (k_o^2 - 2m_r I/\hbar^2)^{\frac{1}{2}})$. The equation (3.62) becomes

$$(3.65)\quad \phi_{ea,e_1}(\mu,\nu) = \frac{4\pi(\cdot 14\times 10^{-3})\, 2^6\pi g m_r^2 a_o^2 e^4}{\hbar^4 k_o (a_o^2 k_o^2 + 1)^4}\, J_1(\mu,\nu)$$

where

$$(3.66)\quad J_1(\mu,\nu) = \int_0^{\pi}\int_0^{(k_o^2 - 2m_r I/\hbar^2)^{\frac{1}{2}}} \frac{(1-\lambda^\mu \cos^\nu\chi)\, k_f \sin\chi\, d\chi\ \chi\, d\chi}{(k_o^2 + k_f^2 - 2k_o k_f \cos\chi)^2}\ ,$$

this being evaluated in Appendix B. For this case, λ is given by

$$(3.67)\quad \lambda^2 = 1 - \frac{\chi^2 + 2m_r I/\hbar^2}{k_o^2}\ .$$

Using the values from this appendix for the appropriate μ,ν, on putting

$$(3.68)\quad \alpha^2 = k_o^2 - \frac{2m_r I}{\hbar^2}$$

and

$$(3.69)\quad A = \frac{(\cdot 14\times 10^{-3})\, 2^6\pi g a_o^2 m_r^2 e^4\, 4\pi}{\hbar^4 k_o (a_o^2 k_o^2 + 1)^4}\, \frac{2\alpha^3}{3(2m_r/\hbar^2)^2}$$

the results are:

$$\phi_{ea,e_1}(-\infty,0) = A \,, \tag{3.70}$$

$$\phi_{ea,e_1}(1,1) = A\left(1 - \frac{k_0^2+\alpha^2}{6k_0^2}\right), \tag{3.71}$$

$$\phi_{ea,e_1}(2,2) = A\left(1 - \frac{(k_0^2+\alpha^2)^2}{4k_0^4}\right), \tag{3.72}$$

$$\phi_{ea,e_1}(2,0) = A\left(1 - \frac{3}{5}\frac{\alpha^2}{k_0^2}\right), \tag{3.73}$$

$$\phi_{ea,e_1}(3,1) = A\left(1 - \frac{3}{10}\frac{\alpha^2(k_0^2+\alpha^2)}{k_0^4}\right). \tag{3.74}$$

Finally, $2m_r I/\hbar^2$ is neglected compared with k_0^2 in the parethesised terms after substituting for α from (3.59). (Again this corresponds to an approximation valid away from threshold. However, apart from being consistent with previous approximations taken, this type of approximation in the present calculations is necessary to obtain workable results). The results then become

$$\phi_{ea,e_1}(1,1) = \frac{2}{3}\phi_{ea,e_1}(-\infty,0) \,, \tag{3.75}$$

$$\phi_{ea,e_1}(2,2) = \frac{2m_r I}{\hbar^2 k_0^2}\phi_{ea,e_1}(-\infty,0), \tag{3.76}$$

$$\phi_{ea,e_1}(2,0) = \frac{2}{5}\phi_{ea,e_1}(-\infty,0), \tag{3.77}$$

$$\phi_{ea,e_1}(3,1) = \frac{2}{5}\phi_{ea,e_1}(-\infty,0) \,. \tag{3.78}$$

(ii) The atomic electron, e_2

Similarly to (3.62), but for this case with χ being the polar angle of $\underline{\kappa}$ with respect to $\underline{k}_o$, and χ_1 the deflection angle of $\underline{k}_f$,

$$(3.79)\quad \phi_{e_a,e_2}(\mu,\nu) = \int (1-\lambda^\mu \cos^\nu\chi)\, g(.14\times10^{-3})\,\pi m_r^2 e^4 2^6 a_o^2 \times \frac{k_f \kappa \sin\chi\, d\chi\, d\Omega_1\, d\kappa}{\hbar^4 k_o K^4 (a_o^2K^2+1)^4}$$

where now K^2 is in terms of χ_1 (being defined in fact from the differential cross section):

$$(3.80)\quad K^2 = k_o^2 + k_f^2 - 2k_o k_f \cos\chi_1 .$$

For this case (Appendix B)

$$(3.81)\quad \lambda^2 = 1 - \frac{k_o^2 - \kappa^2}{k_o^2} .$$

Integration over χ is straighforward, owing to the cross sectional independence of this angle. Defining

$$(3.82)\quad J_2(\mu,\nu) = \int_o^\pi (1-\lambda^\mu\cos^\nu\chi)\sin\chi\, d\chi ,$$

elementary integration gives

$$(3.83)\quad J_2(\mu,\nu) = 2 - \frac{\lambda^\mu(1+(-1)^\nu)}{\nu+1} .$$

Integration of (3.79) over the solid angle $d\Omega_1$ involves only terms in M where

$$(3.84)\quad M = \int \frac{\sin\chi_1 \, d\chi_1 \, d\epsilon_1}{K^4 (a_o^2 K^2 + 1)^4} .$$

As in the previous case, the term $(a_o^2 K^2 + 1)^4$ is replaced by $(a_o^2 k_o^2 + 1)^4$. Integration over the azimuthal angle gives 2π , and with the above approximation, integration over χ_1 may be effected, giving

$$(3.85)\quad M = \frac{4\pi}{(a_o^2 k_o^2 + 1)^4 (k_o^2 - k_f^2)^2} .$$

Returning to (3.79), $\phi(\mu,\nu)$ becomes

$$(3.86)\quad \phi_{e a_1, e_2}(\mu,\nu) = \int_0^{\infty} \frac{\pi m_r^2 M J_2(\mu,\nu) \, q \, (.14 \times 10^{-3}) \, 2^6 a_o^2 k_f \, K \, dK}{\hbar^4 k_o}$$

$$= \frac{4\pi B}{(a_o^2 k_o^2 + 1)^4} L(\mu,\nu)$$

where

$$(3.87)\quad B = \frac{(.14 \times 10^{-3}) \, 2^6 a_o^2 \pi m_r^2 e^4}{\hbar^4 k_o} .$$

α is defined by (3.68) and $L(\mu,\nu)$ is calculated in Appendix B. Taking the values of $L(\mu,\nu)$ from this Appendix, and since A , already defined by (3.69), is

$$(3.88)\quad A = \frac{4\pi B}{(a_o^2 k_o^2 + 1)^4} \frac{2\alpha^3}{3(2m_r I/\hbar)^2} ,$$

the $\phi(\mu,\nu)$ are given by

(3.89) $$\phi_{e_a,e_2}(-\infty,0) = A ,$$

(3.90) $$\phi_{e_a,e_2}(1,1) = A ,$$

(3.91) $$\phi_{e_a,e_2}(2,2) = A\left(\frac{2}{3} + \frac{\alpha^2}{5k_0^2}\right),$$

(3.92) $$\phi_{e_a,e_2}(2,0) = \frac{3}{5}\frac{\alpha^2}{k_0^2} A ,$$

(3.93) $$\phi_{e_a,e_2}(3,1) = A .$$

Since

(3.94) $$\left(\frac{2}{3} + \frac{\alpha^2}{5k_0^2}\right) \sim \frac{13}{15}$$

the $\phi(\mu,\nu)$ may be expressed as follows in terms of $\phi(-\infty,0)$:

(3.95) $$\phi_{e_a,e_2}(1,1) = \phi_{e_a}(-\infty,0) ,$$

(3.96) $$\phi_{e_a,e_2}(2,2) = \frac{13}{15}\phi_{e_a}(-\infty,0),$$

(3.97) $$\phi_{e_a,e_2}(2,0) = \frac{3}{5}\frac{\alpha^2}{k_0^2}\phi_{e_a}(-\infty,0) \sim \frac{3}{5}\phi_{e_a}(-\infty,0),$$

(3.98) $$\phi_{e_a,e_2}(3,1) = \phi_{e_a}(-\infty,0) .$$

In any case it is obvious from (3.33) that $\phi_{e_a}(-\infty,0)$ is simply the total cross section for the collision divided by $\frac{2\pi}{g}$; and therefore for

all particles involved in a particular (inelastic) collision, the $\phi(-\infty,0)$ are identical. Thus (3.89) is identical with (3.70).

(iii) The atoms

In the part of Appendix B dealing with the calculation of the λ's it was pointed out that for atoms as a particle type in this type of ionising collision, the only Ω's involved are the $\Omega^{-\infty,0}(r)$, which are independent of λ. Hence for the atoms,

$$(3.99)\quad \phi_{ea,a}(-\infty,0) = \int g\sigma \sin\chi\, d\chi = \int \frac{g(\cdot 14\times 10^{-3})\, 2^8\pi^2 a_0^2 m_r^2 k_f \kappa e^4 \sin\chi\, d\chi d\kappa}{\hbar^4 k_0 K^4 (a_0^2K^2+1)^4}$$

where $\chi = \cos^{-1}\left((-\underline{k}_0)\cdot(-\underline{k}_f)/k_0k_f\right) = \chi_1$.

Again replacing $(a_0^2K^2+1)^4$ by $(a_0^2k_0^2+1)^4$, since on differentiating (3.63),

$$(3.100)\quad \sin\chi_1\, d\chi_1 = \frac{K\, dK}{k_0 k_f},$$

$$(3.101)\quad \phi_{ea}(-\infty,0) = C\int \frac{\kappa\, d\kappa\, dK}{K^3}$$

where

$$(3.102)\quad C = \frac{g(\cdot 14\times 10^{-3}) m_r^2 e^4 2^8\pi^2 a_0^2}{\hbar^4 k_0^2 (a_0^2k_0^2+1)^4}.$$

The upper and lower limits on K are $k_0 \pm k_f$. Using (3.64) and (3.68),

$$(3.103)\quad k_f^2 = \alpha^2 - \kappa^2,$$

and therefore, on integrating over K ,

$$(3.104)\quad \phi_{ea}(-\infty,0) = \int_0^{\alpha} \frac{\kappa\, d\kappa\, 2 k_0 k_f}{(k_0^2 - k_f^2)^2} \sim 2\int_0^{\alpha} \frac{\kappa\, d\kappa\, (\alpha^2-\kappa^2)^{\frac{1}{2}} k_0}{(2m_r I/\hbar^2)^2}.$$

The remaining integration is straightforward also:

$$(3.105)\quad \phi_{ea}(-\infty,0) = \frac{2}{3}\frac{C k_0 \alpha^3}{(2m_r I/\hbar^2)^2} .$$

It is apparent on comparing (3.105) with (3.70), for example, that the $\phi_{ea}(-\infty,0)$ obtained for the different cases are equal; this suggests that the different approximations made are consistent.

(iv) The protons

As discussed in Appendix B, the Ω^1s for the protons produced in the ionising collisions are readily obtained from those for the atoms. In particular, when terms of order $\frac{m_e}{m_a}$ are neglected, the Ω^1s are the same for the protons as for the atoms. There is, therefore, no need to calculate the particular $\phi_{ea_{jp}}(\mu,\nu)$ for these protons.

(c) Charge exchange collisions

As discussed in Appendix A, only resonant charge exchange collisions are considered. These charge exchange collisions are elastic in the sense that there is no kinetic energy absorbed in the collision, although in the strict sense of a change in the nature of the colliding particles the collisions are inelastic. Nevertheless, the most convenient way of considering this case is to calculate the $\Delta_{jk}(\psi_i)$ and hence the ϕ^1s for the ψ - functions of the atom (or proton) present after the collision minus the ψ - functions of the

atom (or proton), involving the other nucleus, before the collision. From this point of view, the collision is elastic, with differential cross section (3.10).

Since $\delta(1) = \delta(2) = 0$ for elastic collisions, there is no need to consider $\phi(-\infty, 0)$, while $\phi(2,0) = 0$ and $\phi(3,1) = \phi(1,1), = \phi(1)$ say. Also let $\phi(2,2) = \phi(2)$; then

$$(3.106)\quad \phi(\nu) = D \int_0^{\pi} \frac{(1 - \cos^{\nu}\chi)\sin\chi \, d\chi}{(1 + K^2 a_0^2)^6}$$

where

$$(3.107)\quad D = \frac{(2.1 \times 10^{-2})\, g\, m_r^2\, a_0^4\, 2^8\, e^4}{\hbar^4} .$$

Substituting for K^2 (defined for equation (3.10))

$$(3.108)\quad \phi_{ap}(\nu) = D \int_0^{\pi} \frac{(1 - \cos^{\nu}\chi)\sin\chi \, d\chi}{(1 + 2k_0^2 a_0^2 (1 + \cos\chi))^6} ,$$

from which is obtained

$$(3.109)\quad \phi_{ap}(1) = D\left(\frac{1}{10 k_0^2 a_0^2} - \frac{1}{10 k_0^2 a_0^2} + \frac{1}{16 k_0^4 a_0^4} \right)$$

$$= D / 16 k_0^4 a_0^4$$

$$(3.110)\quad \phi_{ap}(2) = D\left(\frac{1}{10 k_0^2 a_0^2} + \frac{1}{10 k_0^2 a_0^2} \right)$$

$$= D / 5 k_0^2 a_0^2 .$$

For these calculations, it has been assumed that $k_o^2 a_o^2 >> 1$.

3.4 Evaluation of the Ω^{l}s

The Ω^{l}s, defined in Chapter 2, are essentially the result of integrating the ϕ^{l}s over the impact velocity. Restating the definition,

$$(3.111)\quad \Omega^{\mu,\nu}_{jk,i}(r) = \pi^{\frac{1}{2}} \int_{Y_o}^{\infty} e^{-Y^2} Y^{2r+2} \phi_{jk,i}(\mu,\nu)\, dY$$

where $\underline{Y} = \gamma \underline{g}$, or since $\hbar \underline{k}_o = m_r \underline{g}$, $\underline{Y} = \frac{\gamma \hbar \underline{k}_o}{m_r}$. Y_o is the threshold value of Y for a particular collision. Thus for elastic collisions Y_o is zero, while for a collision with excitation energy $\Delta \mathcal{E}_o$,

$$(3.112)\quad Y_o^2 = \frac{2\gamma^2}{m_r} \Delta \mathcal{E}_o .$$

This is taken account of, in a sense, in the ϕ^{l}s, in that being functions of the differential cross section, below the threshold energy they take the value zero. However, the approximate cross sections used are not zero for all energies below the collision threshold, and it is necessary to specify the particular lower limit as in (3.112).

3.4.1 Elastic Collisions

For these collisions, with the Ω^{l}s as with the ϕ^{l}s, the definitions are formally the same as those given by Chapman and Cowling[1]. The results for Coulomb collisions given below are in fact the same as in reference 1; however, as already mentioned, the use of "adjusted" Born approximations for the differential cross sections for other elastic collisions leads to different results from those of reference 1,

based on inverse power force laws. Since $\phi(\mu,\nu) \equiv \phi(\nu)$ for elastic collisions, it is evident that

$$(3.113)\quad \Omega^{\mu,\nu}_{jk,i}(r) \equiv \Omega^{\nu}_{jk}(r)$$

(using this equation to define $\Omega^{\nu}_{jk}(r)$).

(a) Electron-electron collisions

Equation (3.12) for $\phi(1)$, with (3.111) gives

$$(3.114)\quad \Omega^{1}_{ee}(r) = \pi^{\frac{1}{2}} \int_0^\infty \frac{e^{-Y^2}\, Y^{2r+2}\, e^4\, \gamma^3\, \log(1+\Lambda^2_{ee})\, dY}{m_r^2\, Y^3} .$$

Although Λ_{ee} is a function of Y , owing to its occurrence only in the logarithmic term, it is generally adequate to replace it by an "averaged" value (Liley[3]), regarding it as constant for the purposes of integration. In this case the integral is easily reduced to an expression involving the factorial function:

$$(3.115)\quad \Omega^{1}_{ee}(r) = \frac{\pi^{\frac{1}{2}}\, \gamma^3 e^4 \log(1+\Lambda^2_{ee})\, \Gamma(r)}{2\, m_r^2} .$$

Similarly, since $\phi_{ee}(2) = 2\, \phi_{ee}(1)$,

$$(3.116)\quad \Omega^{2}_{ee}(r) = 2\, \Omega^{1}_{ee}(r) .$$

The "averaged" value of Λ_{ee} is

$$(3.117)\quad \Lambda_{ee} = \frac{3kT_e \lambda_D}{e^2} = \frac{3(kT_e)^{\frac{3}{2}}}{e^2 (4\pi n_e e^2)^{\frac{1}{2}}}$$

where λ_D is the Debye length.

(b) Electron-hydrogen atom collisions

Using equation (3.17) for $\phi(1)$ gives

$$(3.118)\quad \Omega^{1}_{ea}(r) = \pi^{\frac{1}{2}} \int_0^\infty \frac{e^{-Y^2} Y^{2r+2} (\cdot 45)\, 4 m_r^2 e^4 a_0^4 g}{\hbar^4 (2 k_0^2 a_0^2)^2}\, dY \times \left(\log(a_0^2 k_0^2 + 1) + \frac{(2a_0^2 k_0^2)^3 - 6(2a_0^2 k_0^2)^2 - 24.\, 2a_0^2 k_0^2}{6(2a_0^2 k_0^2 + 2)^3} \right).$$

Retaining only the logarithm term, this equation becomes

$$(3.119)\quad \Omega^{1}_{ea}(r) = \pi^{\frac{1}{2}} (\cdot 45) \frac{e^4 \gamma^3}{m_r^2} \int_0^\infty e^{-Y^2} Y^{2r-1} \log\left(1 + \frac{a_0^2 m_r^2}{\gamma^2 \hbar^2} Y^2\right) dY.$$

For the various r, the following expressions for the $\Omega^{1}(r)$ are obtained:

$$(3.120)\quad \Omega^{1}_{ea}(1) = \frac{\pi^{\frac{1}{2}} (\cdot 45)\, e^4 \gamma^3}{m_r^2} \frac{1}{2} \left(-e^{\frac{1}{b}} Ei\left(-\tfrac{1}{b}\right)\right),$$

$$(3.121)\quad \Omega^{1}_{ea}(2) = \frac{\pi^{\frac{1}{2}} (\cdot 45)\, e^4 \gamma^3}{m_r^2} \frac{1}{2} \left(1 + \frac{1-b}{b} e^{\frac{1}{b}} Ei\left(-\tfrac{1}{b}\right)\right),$$

$$(3.122)\quad \Omega^{1}_{ea}(3) = \frac{\pi^{\frac{1}{2}} (\cdot 45)\, e^4 \gamma^3}{m_r^2} \frac{1}{2} \left(3 - \frac{1}{b} + \frac{(-2b^2 + 2b - 1)}{b^2} e^{\frac{1}{b}} Ei\left(-\tfrac{1}{b}\right)\right).$$

$Ei\left(-\frac{1}{b}\right)$ is the exponential-integral function (Gradshteyn and Ryzhik[4]), and the argument b is given by

$$(3.123)\quad b = \frac{a_0^2 m_r^2}{\gamma^2 \hbar^2}.$$

The similar approximation of discarding terms other than the logarithm term in $\phi_{ea}(2)$ (equation (3.18)) gives

$$\phi_{ea}(2) = 2\,\phi_{ea}(1) \tag{3.124}$$

and therefore

$$\Omega^{2}_{ea}(r) = 2\,\Omega^{1}_{ea}(r)\,. \tag{3.125}$$

(c) Electron-proton collisions

The calculation of the Ω's is formally the same as for electron-electron collisions. The results are

$$\Omega^{1}_{ep}(r) = \frac{\pi^{\frac{1}{2}}\,\gamma^{3}e^{4}\,\log(1+\Lambda^{2}_{ep})\,\Gamma(r)}{2\,m_r^{2}}\,; \tag{3.126}$$

$$\Omega^{2}_{ep}(r) = 2\,\Omega^{1}_{ep}(r)\,,$$

where in this case Λ_{ep} is given by

$$\Lambda_{ep} = \frac{3}{2}\,\lambda_D\left(\frac{\alpha_e+\alpha_p}{\alpha_e\alpha_p}\right)\frac{m_r}{e^2} \sim \frac{3\,(k\,T_e)^{\frac{3}{2}}}{e^2\,(4\pi\,n_e e^2)^{\frac{1}{2}}}\,. \tag{3.127}$$

(d) Hydrogen atom-hydrogen atom collisions

The relevant $\phi(v)$ are given in (3.25), (3.26). Using (3.25) for $\phi(1)$, $\Omega^{1}_{aa}(r)$ is the integral

$$\Omega^{1}_{aa}(r) = \frac{(2\times10^{-2})\,\gamma^{3}e^{4}\pi^{\frac{1}{2}}}{m_r^{2}}\int_0^{\infty} e^{-Y^{2}}Y^{2r-1}\log(1+bY^{2})\,dY, \tag{3.128}$$

where $b = m_r^2 a_o^2 / \hbar^2 \gamma^2$ as before.

From the results involving similar integrals for the electron-hydrogen atom collisions, in the same notation, the $\Omega^1(r)$ become

$$(3.129)\quad \Omega^1_{aa}(1) = \frac{(2\times 10^{-2})\,\gamma^3 e^4 \pi^{\frac{1}{2}}}{m_r^2} \frac{1}{2}\left(-e^{\frac{1}{b}}\,Ei(-\tfrac{1}{b})\right)$$

$$(3.130)\quad \Omega^1_{aa}(2) = \frac{(2\times 10^{-2})\,\gamma^3 e^4 \pi^{\frac{1}{2}}}{m_r^2} \frac{1}{2}\left(1+\frac{1-b}{b}\,e^{\frac{1}{b}}\,Ei(-\tfrac{1}{b})\right)$$

$$(3.131)\quad \Omega^1_{aa}(3) = \frac{(2\times 10^{-2})\,\gamma^3 e^4 \pi^{\frac{1}{2}}}{m_r^2} \frac{1}{2}\left(3-\frac{1}{b}+\frac{(-2b^2+2b-1)}{b^2}\,e^{\frac{1}{b}}\,Ei(-\tfrac{1}{b})\right)$$

Again, since $\phi_{aa}(2) = 2\,\phi_{aa}(1)$ it follows that

$$(3.132)\quad \Omega^2_{aa}(r) = 2\,\Omega^1_{aa}(r) \quad .$$

(e) Hydrogen atom-proton collisions

From the equation (3.28) for $\phi_{ap}(1)$,

$$(3.133)\quad \Omega^1_{ap}(r) = \frac{(2\times 10^{-2})\,e^4 \gamma^3 \pi^{\frac{1}{2}}}{m_r^2} \int_0^\infty e^{-Y^2} Y^{2r-1} \log(1+bY^2)\,dY$$

with b as in (3.123). The integrals have the same form as for the previous cases involving atoms, being expressible in terms of the exponential-integral functions. The results are

$$(3.134)\quad \Omega^1_{ap}(1) = \frac{(2\times 10^{-2})\,e^4 \gamma^3 \pi^{\frac{1}{2}}}{m_r^2} \frac{1}{2}\left(-e^{\frac{1}{b}}\,Ei(-\tfrac{1}{b})\right),$$

$$(3.135)\quad \Omega^1_{ap}(2) = \frac{(2\times 10^{-2})\,e^4 \gamma^3 \pi^{\frac{1}{2}}}{m_r^2} \frac{1}{2}\left(1+\frac{1-b}{b}\,Ei(-\tfrac{1}{b})\right),$$

$$(3.136)\quad \Omega^{1}_{ap}(3) = \frac{(2\times10^{-2})e^4\gamma^3\pi^{\frac{1}{2}}}{m_r^2}\frac{1}{2}\left(3-\frac{1}{b}+\frac{(-2b^2+2b-1)}{b^2}e^{\frac{1}{b}}E_i(-\tfrac{1}{b})\right).$$

Again, since $\phi_{ap}(2) = 2\phi_{ap}(1)$, it follows that

$$(3.137)\quad \Omega^{2}_{ap}(r) = 2\,\Omega^{1}_{ap}(r).$$

(f) Proton-proton collisions

Formally the same as for electron-electron collisions and electron-proton collisions, the results are

$$(3.138)\quad \Omega^{1}_{pp}(r) = \frac{\pi^{\frac{1}{2}}\gamma^3e^4}{2m_r^2}\log(1+\Lambda^2_{pp})\,\Gamma(r),$$

$$(3.139)\quad \Omega^{2}_{pp}(r) = 2\,\Omega^{1}_{pp}(r).$$

For these collisions, however,

$$(3.140)\quad \Lambda_{pp} = \frac{3\lambda_D\, m_r}{\alpha_p e^2} = \frac{3kT_p}{e^2}\left(\frac{kT_e}{4\pi n_e e^2}\right)^{\frac{1}{2}}.$$

3.4.2 Electron-Hydrogen Atom 1s - 2p Excitation Collisions

The $\Omega^{\mu,\nu}_{ea}(r)$, given by (3.111) are functions of, amongst other variables, the lower limit of integration $Y_0 = (2\gamma^2\Delta\mathcal{E}_0/m_r)^{\frac{1}{2}}$. Since γ is a function of the temperature, the $\Omega^{\mu,\nu}(r)$ will essentially be functions of the electron and atom temperatures. As will be seen, in general the $\Omega^{\mu,\nu}(r)$ can only be reduced to expressions in terms of incomplete Gamma functions, which in this case are dimensionless functions of the temperatures. The incomplete Gamma functions are, however, tabulated functions, and in certain extreme

cases they can be written in terms of analytical functions. The following Ω's are required: $\Omega^{-\infty,0}(r)$, $\Omega^{1,1}(r)$, $\Omega^{2,2}(r)$, $\Omega^{2,0}(r)$, with $r = 0, 1, 2,$ and 3 ; but since relations (3.60) express $\Omega^{2,2}(r)$ and $\Omega^{2,0}(r)$ in terms of $\Omega^{1,1}(r)$, it is only necessary to evaluate $\Omega^{-\infty,0}(r)$ and $\Omega^{1,1}(r)$, for these values of r.

The calculation of $\Omega_{eq}^{-\infty,0}(r)$ involves using (3.56) for $\phi_{eq}(-\infty,0)$. This gives

$$(3.141)\quad \Omega^{-\infty,0}(r) = \frac{(\cdot 45)\, e^4\, 2^{15} a_0^2\, \gamma\, \pi^{\frac{1}{2}}}{3^{10}\, \hbar^2} \int_{Y_0}^{\infty} e^{-Y^2} Y^{2r+1} \log(bY)\, dY$$

where b has the value $3\hbar / 2a_0 \gamma \Delta\varepsilon_0$ (bearing no relation to the term b, defined in (3.123)).

Consider the function $I(r, Y_0)$ defined by

$$(3.142)\quad I(r, Y_0) = \int_0^{\infty} e^{-Y^2} Y^{2r+1} \log(bY)\, dY .$$

A change of variable yields

$$(3.143)\quad I(r, Y_0) = \frac{1}{4} \int_{Y_0^2}^{\infty} e^{-x} x^r \log(b^2 x)\, dx ,$$

and this can be integrated by parts for the various values of r. Let $\Gamma(r,z)$ be an incomplete Gamma function, defined by

$$(3.144)\quad \Gamma(r,z) = \int_z^{\infty} e^{-x} x^{r-1}\, dx .$$

(Some authors define the incomplete Gamma function in the inverse way, that is, by ($\Gamma(r) - \Gamma(r,z)$) in the above notation, but the

definition (3.144) is more suitable for the purposes of this Chapter). Provided $z > 0$, the $\Gamma(r,z)$ are defined for all r.

In terms of these functions (3.144), integration of (3.143) gives

$$(3.145)\quad I(0,Y_0) = \frac{1}{4}\left(e^{-Y_0^2}\log b^2 Y_0^2 + \Gamma(0,Y_0^2)\right),$$

$$(3.146)\quad I(1,Y_0) = \frac{1}{4}\left(e^{-Y_0^2}(Y_0^2+1)\log b^2 Y_0^2 + \Gamma(1,Y_0^2) + \Gamma(0,Y_0^2)\right),$$

$$(3.147)\quad I(2,Y_0) = \frac{1}{4}\left(e^{-Y_0^2}(Y_0^4+2Y_0^2+2)\log b^2 Y_0^2 + \Gamma(2,Y_0^2) + 2\Gamma(1,Y_0^2) + 2\Gamma(0,Y_0^2)\right),$$

$$(3.148)\quad I(3,Y_0) = \frac{1}{4}\left(e^{-Y_0^2}(Y_0^6+3Y_0^4+6Y_0^2+6)\log b^2 Y_0^2 + \Gamma(3,Y_0^2) + 3\Gamma(2,Y_0^2) + 6\Gamma(1,Y_0^2) + 6\Gamma(0,Y_0^2)\right).$$

It is evident that the $\Omega_{(r)}^{-\infty,0}$ are obtained using these results for $I(r,Y_0)$ in equation (3.141), that is,

$$(3.149)\quad \Omega_{ea}^{-\infty,0}(r) = \frac{(\cdot 45)\, e^4 2^{15} a_0^2 \gamma \pi^{\frac{1}{2}} I(r,Y_0)}{3^{10}\hbar^2}.$$

The $\Omega_{ea}^{1,1}(r)$ are calculated using equation (3.57) for $\phi_{ea}(1,1)$. This gives

$$(3.150)\quad \Omega_{ea}^{1,1}(r) = \frac{(\cdot 45)\pi^{\frac{1}{2}} e^4 2^{15} a_0^2 m_r \Delta\varepsilon_0 \gamma^3}{3^{10} m_r^2 \hbar^2}\int_{Y_0}^{\infty} e^{-Y^2} Y^{2r-1}\log(bY)\, dY$$

where again $b = 3\gamma\hbar / 2a_0\Delta\mathcal{E}_0$. It follows from this equation that the $\Omega^{1,1}(r)$ can also be expressed in terms of the $I(r, Y_0)$, defined in (3.142), and hence in terms of the incomplete Gamma functions. In particular,

$$(3.151)\quad \Omega^{1,1}_{ea}(r) = \frac{(\cdot 45)\pi^{\frac{1}{2}} e^4 2^{15} a_0^2 m_r \gamma^3 \Delta\mathcal{E}_0}{3^{10} m_r^2 \hbar^2} I(r-1, Y_0) .$$

The integral $I(-1, Y_0)$, not previously given, has the value

$$(3.152)\quad I(-1, Y_0) = \frac{1}{4}\left(\Gamma(0, Y_0^2) \log b^2 + \int_{Y_0^2}^{\infty} \frac{e^{-x} \log x \, dx}{x} \right).$$

The last integral on the right hand side of (3.152) exists for $Y_0^2 > 0$; being of a similar nature to the incomplete Gamma functions, it may also readily be evaluated numerically.

Finally, the remaining $\Omega^{l,s}$ are related to (3.149) and (3.151) by

$$(3.153)\quad \Omega^{2,2}_{ea}(r) = 2\,\Omega^{1,1}_{ea}(r)$$

$$(3.154)\quad \Omega^{2,0}_{ea}(r) = 2\,\Omega^{1,1}_{ea}(r) .$$

3.4.3 Electron-Hydrogen Atom Ionising Collisions

For these collisions the $\Omega^{l,s}$ are defined by (3.111). The lower limit of integration is given by $Y_0 = (2\gamma^2 I / m_r)^{\frac{1}{2}}$ where I is the ionisation energy. As with excitation collisions, since γ is a function of the electron and atom temperatures, the $\Omega^{l,s}$ are also functions of these temperatures. When $m_e T_a << m_a T_e$,

then $\gamma^2 \sim \alpha_e = \frac{m_e}{2kT_e}$, in which case the Ω's are simply functions of the electron temperature. This also applies to the excitation collisions of course. For each of the particles (incident electron, atom, atomic electron, and proton produced), the following Ω's are required: $\Omega^{-\infty,0}(r)$, $\Omega^{1,1}(r)$, $\Omega^{2,2}(r)$, $\Omega^{2,0}(r)$, $\Omega^{3,1}(r)$ with r taking the values 0, 1, 2, and 3. Fortunately, it is not necessary to calculate all 80 Ω's . As shown in Appendix B, the Ω's for the protons are essentially equal to the Ω's for the atoms. Furthermore, for the individual particles, relations have been given between various $\phi(\mu,\nu)$, enabling many Ω's to be expressed in terms of others.

In particular it has been shown that for the incident electron, all $\phi_{ea,e_1}(\mu,\nu)$ can be expressed in terms of $\phi_{ea,e_1}(-\infty,0)$ or $\phi_{ea,e_1}(2,2)$; for the atomic electron, all $\phi_{ea,e_2}(\mu,\nu)$ can be expressed in terms of $\phi_{ea,e_2}(-\infty,0)$; and for the protons and atoms, the only $\phi(\mu,\nu)$ involved is $\phi_{ea}(-\infty,0)$; and finally that the $\phi_{ea}(-\infty,0)$ are equal for all particles involved in a particular collision. The general form of $\Omega^{\mu,\nu}(r)$ being given in equation (3.111), it is evident that it is a function of $\phi(\mu,\nu)$, Y_0 and certain other constants of the collision. Therefore all particles in the collision have the same $\Omega^{-\infty,0}(r)$. It follows that for the ionisation collisions, only 8 of the Ω's need be calculated, namely $\Omega_{ea}^{-\infty,0}(r)$ and $\Omega_{ea,e_1}^{2,2}(r)$ with $r = 0, 1, 2$ and 3.

Since

$$(3.155)\quad Y_0^2 = \frac{2\gamma^2}{m_r} I \quad ,$$

using this relation and also those after (3.111),

$$(3.156)\quad k_o^2 - \frac{2m_r I}{\hbar^2} = \frac{m_r^2}{\gamma^2\hbar^2}\left(Y^2 - Y_o^2\right)$$

$$a_o^2 k_o^2 + 1 = \frac{a_o^2 m_r^2}{\gamma^2\hbar^2}\left(Y^2 + Y_1^2\right)$$

where $Y_1^2 = \gamma^2\hbar^2 / m_r^2 a_o^2$, and, like Y_o , is a constant of the collision. With $\phi_{ea}(-\infty, o)$, given in equation (3.70) for example, from (3.111),

$$(3.157)\quad \Omega_{ea}^{-\infty,o}(r) = \frac{(.14\times10^{-3})\pi^{\frac{1}{2}}\hbar^2 2^9\pi^2\gamma^5 e^4\, I(r; Y_o, Y_1)}{3 a_o^6 m_r^4 (2m_r I/\hbar^2)^2}$$

where $I(r; Y_o, Y_1)$ is defined by

$$(3.158)\quad I(r; Y_o, Y_1) = \int_{Y_o}^{\infty} \frac{e^{-Y^2} Y^{2r+2} (Y^2 - Y_o^2)^{\frac{3}{2}}}{(Y^2 + Y_1^2)^4}\, dY \,.$$

$I(r; Y_o, Y_1)$ may be reduced on making appropriate approximations, or it may be numerically evaluated. As discussed previously, $I(r; Y_o, Y_1)$ is essentially a function of the electron temperature.

Similarly, using equation (3.72) for $\phi_{ea,e_1}(2,2)$, it may be verified that

$$(3.159)\quad \Omega_{ea,e_1}^{2,2}(r) = \frac{(.14\times10^{-3})\pi^{\frac{1}{2}}\hbar^2 e^4 2^{10}\pi^2\gamma^7 m_r I\, I(r-1; Y_o, Y_1)}{3 m_r^6 a_o^6 (2m_r I/\hbar^2)^2} \,,$$

$I(r; Y_o, Y_1)$ being again given by (3.158).

Finally, using equations (3.70), to (3.78), (3.95) to (3.98), and (3.105), the set of 48 Ω^1s for the collision (as already mentioned, 32 of the Ω^1s associated with the atom and proton particle types need not be considered), in terms of the 8 independent Ω^1s are as follows. For convenience, $\Omega_{ea}^{-\infty,0}(r)$ and $\Omega_{ea,e_1}^{2,2}(r)$ are also repeated.

$$(3.160)\quad \Omega_{ea}^{-\infty,0}(r) = \frac{(.14\times10^{-3})\,\hbar^2 e^4 \pi^{\frac{5}{2}} 2^9 \gamma^5}{3 a_0^6 m_r^4 (2m_r I/\hbar^2)^2}\, I(r;Y_0,Y_1),$$

with $I(r;Y_0,Y_1)$ defined in equation (3.158);

$$(3.161)\quad \Omega_{ea,e_1}^{1,1}(r) = \frac{2}{3}\Omega_{ea}^{-\infty,0}(r),$$

$$(3.162)\quad \Omega_{ea,e_1}^{2,0}(r) = \frac{2}{5}\Omega_{ea}^{-\infty,0}(r),$$

$$(3.163)\quad \Omega_{ea,e_1}^{3,1}(r) = \frac{2}{5}\Omega_{ea}^{-\infty,0}(r),$$

$$(3.164)\quad \Omega_{ea,e_1}^{2,2}(r) = \frac{(.14\times10^{-3})\,\pi^{\frac{5}{2}}\hbar^2 e^4 2^{10}\gamma^7 m_r I}{3 m_r^6 a_0^6 (2m_r I/\hbar^2)^2}\, I(r-1;Y_0,Y_1),$$

$$(3.165)\quad \Omega_{ea,e_2}^{1,1}(r) = \Omega_{ea}^{-\infty,0}(r),$$

$$(3.166)\quad \Omega_{ea,e_2}^{2,2}(r) = \frac{13}{15}\Omega_{ea}^{-\infty,0}(r),$$

$$(3.167)\quad \Omega_{ea,e_2}^{2,0}(r) = \Omega_{ea}^{-\infty,0}(r),$$

$$(3.168)\quad \Omega_{ea,e_2}^{3,1}(r) = \Omega_{ea}^{-\infty,0}(r),$$

$$(3.169)\quad \Omega^{-\infty,0}_{ea,a}(r) = \Omega^{-\infty,0}_{ea}(r),$$

$$(3.170)\quad \Omega^{-\infty,0}_{ea,a}(r) = \Omega^{-\infty,0}_{ea}(r).$$

3.4.4 Hydrogen Atom-Proton Charge Exchange Collisions

The nature of the charge exchange collisions has already been discussed. The $\phi_{ap}(\mu,\nu)$ are given by (3.109) and (3.110), being independent of μ, while the Ω^ls required are $\Omega^1(r), \Omega^2(r)$ with $r = 1, 2$ and 3. Using (3.109) and since the collisions are essentially elastic, from (3.111),

$$(3.171)\quad \Omega^1_{ap}(r) = \frac{(2.1\times10^{-2})\, 2^4 e^4 \gamma^3 \pi^{\frac{1}{2}}}{2 m_r^2}\, \Gamma(r).$$

Similarly with $\phi_{ap}(2)$ given by (3.110), it follows that

$$(3.172)\quad \Omega^2_{ap}(r) = \frac{(2.1\times10^{-2})\, 2^8 e^4 a_o^2 \gamma}{10\, \hbar^2}\, \Gamma(r+1).$$

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Chapter 4

Characteristic Forms and Boundary Conditions

4.1 Introduction

Appreciating that the 13-moment equations are in tensor form, and independent of a particular coordinate system, when these equations are applied to a physical situation they must be expressed in terms of physical components relevant to the coordinate system used. For an infinite gas or plasma, where boundaries are not important, a cartesian frame is often most suitable. However, for laboratory experiments, plasmas are usually generated in cylindrical or toroidal apparatus. To take account of the resulting symmetry, it is preferable to use a coordinate system whose coordinate surfaces are the surfaces of symmetry of the plasma. As a particular example, in Appendix C the 13-moment equations are referred to cylindrical coordinates, this system having the advantage of a comparatively simple affine connexion.

Since the 13-moment equations, together with Maxwell's equations, essentially form a set of first order quasilinear hyperbolic partial differential equations in space and time, it is important in the application of these equations to obtain the characteristics and eigenfunctions of the equations. The eigenvalues or characteristic speeds are the speeds at which discontinuities travel. Furthermore, this is necessary to find what boundary conditions are required, as well as permitting the stability of various methods of numerical integration to be determined.

Again, knowing the eigenvectors and characteristic speeds, the equations can be expressed in normal form. This leads to a very simple method for explicit numerical integration. However, in cases

involving Maxwell's equations certain of the characteristic speeds are the velocity of light, and straightforward explicit methods require extremely short time steps making numerical integration somewhat impractical. Recently, however, Keller[1] has devised an essentially implicit method for the numerical integration of such equations which is unconditionally stable, irrespective of step length. Although at first sight the direct simplicity of the explicit method is lost, it turns out that Keller's method, although implicit in principle, is effectively explicit in practice. Unfortunately the proof of stability and convergence of the method depends on the equations being completely reducible to characteristic normal form. In general, however, the 13-moment equations can only be reduced to Jordan normal form. Nevertheless, extensions of the proof should be possible for this case, while as will be seen in subsequent sections, the associated difference method can still be readily applied.

In the subsequent sections, the equations are expressed in characteristic normal or equivalent form, and the imposition of the appropriate boundary conditions examined in detail. Although, as stated in the previous Chapter, the main object of this Part of the thesis is to discuss the application of the equations of Part I to (for example) the formation of a hydrogen-like plasma, the discussion in the present Chapter is, in actual fact, more general than this.

4.2 Characteristics and the Normal Form of the Moment Equations

The techniques which have been developed for studying hyperbolic partial differential equations (without necessarily solving them) are quite lucid and elegant for the case of two independent variables, but relatively complicated and much less elegant for more than two

independent variables. For this reason, only two differential independent variables, r and t, are assumed for this Chapter (the moment equations are still considered for three spatial variables, but spatial derivatives in other than the r-direction are ignored). With a set of first order partial differential equations for a vector variable $\underline{U}$ written in the form

$$(4.1) \quad A \frac{\partial \underline{U}}{\partial t} + B \frac{\partial \underline{U}}{\partial r} + C \underline{U} = 0 ,$$

the equations are hyperbolic provided that all k eigenvalues $c_1, \ldots, c_k$ of the matrix $A^{-1}B$ are real, and in addition that there exist k linearly independent eigenvectors $\underline{\ell}_1, \ldots, \underline{\ell}_k$. In particular, this is the case if all of the roots c_i are distinct (Courant and Hilbert[2]). More general cases are discussed later.

An important property of such hyperbolic equations is that they can be linearly transformed to an equivalent set of equations where the derivative of the vector at a given point is only in one (a "characteristic") direction. The eigenvalues of the matrix $A^{-1}B$, having the dimensions of velocity, are called the characteristic speeds of the system of equations, while the corresponding eigenvectors form a basis for the matrix. Provided that this basis spans the vector space, the operator $A^{-1}B \frac{\partial}{\partial r}$ may be diagonalised, using the eigenvectors as a basis. This diagonalisation has several advantages. Writing $c_i, \underline{\ell}_i$ for the ith characteristic speed and corresponding left eigenvector,

$$(4.2) \quad \underline{\ell}_i A^{-1}B = c_i \underline{\ell}_i$$

so that (assuming for the present that the $\underline{\ell}_i$ span the space of the matrix representation) the equations (4.1) are linearly equivalent to the set, over i, of equations

$$(4.3)\quad \underline{\ell}_i\left(\frac{\partial \underline{U}}{\partial t} + c_i \frac{\partial \underline{U}}{\partial r}\right) + \underline{\ell}_i A^{-1} C \underline{U} = 0 .$$

This is called the characteristic normal form of the equations (4.1).

The importance of this form arises from the fact that a directional derivative,

$$(4.4)\quad D_i \equiv \frac{\partial}{\partial t} + c_i \frac{\partial}{\partial r}$$

can be defined, effectively yielding a set of ordinary differential equations,

$$(4.5)\quad \underline{\ell}_i D_i \underline{U} + \underline{\ell}_i A^{-1} C \underline{U} = 0 .$$

Again, the normal form is most suitable for the application of numerical methods, including the analysis of the stability of difference schemes. Finally, it is at the characteristic speeds that disturbances propagate, and the characteristics --- the curves of the equation

$$(4.6)\quad \frac{dr}{dt} = c_i$$

--- are the curves on which all discontinuities and branch lines of the solutions lie. A knowledge of the characteristics of a set of hyperbolic differential equations is necessary to adequately specify the boundary conditions, while the relevant initial conditions are within the arc that

is bounded by the characteristics passing through the most distant points of interest.

Supposing, however, that the eigenvectors $\underline{\ell}_i$ do not span the vector space, it is necessary to complete a basis for $A^{-1}B$. The matrix can no longer be completely diagonalised. Nevertheless, by a suitable choice of basis, a relatively simple representation may be made, in that the matrix can be transformed to the Jordan cannonical form. Let the eigenvalues c_i have multiplicity m_i , and suppose that the equation

$$\underline{\ell}_i(A^{-1}B - c_i I) = 0 \tag{4.7}$$

has q_i linearly independent solutions for $\underline{\ell}_i$. Then there exist generalised (left) eigenvectors of rank k , $\underline{\ell}_i^{(k)}$ say, such that

$$\underline{\ell}_i^{(k)}(A^{-1}B - c_i I)^k = 0 \tag{4.8}$$

while

$$\underline{\ell}_i^{(k)}(A^{-1}B - c_i I)^{k-1} \neq 0 \tag{4.9}$$

and it can be shown that eigenvectors of differing rank are independent (Friedman[3]). It may also be shown (reference 3) that there are $m_i - q_i$ such generalised eigenvectors, and that the largest value of the integer k is not greater than the dimension of the space. The inclusion of the generalised eigenvectors provides a basis which spans the space. However, while a basis of first order eigenvectors

provides a diagonal matrix representation, the basis of generalised eigenvectors (in the appropriate linear combinations) can only give a representation in the Jordan form. Thus

$$(4.10)\qquad \begin{pmatrix} \underline{\ell}_i^{(\alpha)} \\ \underline{\ell}_i^{(\beta)} \\ \underline{\ell}_i^{(\gamma)} \\ \cdot \end{pmatrix} M = \begin{pmatrix} \underline{\ell}_i^{(\alpha)} \\ \underline{\ell}_i^{(\beta)} \\ \underline{\ell}_i^{(\gamma)} \\ \cdot \end{pmatrix} J_i$$

where

$$(4.11)\qquad J_i = \begin{pmatrix} c_i & 1 & 0 & \cdot \\ 0 & c_i & 1 & \cdot \\ 0 & 0 & c_i & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

is the Jordan form for the generalised eigenvectors $\underline{\ell}_i^{(\alpha)}, \underline{\ell}_i^{(\beta)}, \ldots,$ of the root c_i , for the matrix M .

In considering the 13-moment - Maxwell equation system, there are, in general (and after minor simplifications of the nature of taking $J = \sum_j n_j e_j \underline{u}_j$, $p_j = n_j k T_j$ etc.), some $13N + 6$ simultaneous

equations to be solved, where N is the number of components in the gas mixture. The coupling between the equations for each gas component is of course through the collision terms, the field vectors, and the reference velocity for the moments; some simplification is obtained by referring the moments to a rest frame, that is, taking $\underline{V} = 0$ in equations (1.17) to (1.21) of Chapter 1. From a mathematical point of view, however, it is the differential terms which are relevant to the propagation of disturbances, the remaining terms being relevant to such phenomena as resistivity, viscosity, and relaxation. Since the characteristics are associated with the coefficients of the differential terms, when $\underline{V}=0$ the eigenvalues and eigenvectors for each component of the gas mixture will be independent of the other components. Obtaining the characteristic speeds is therefore straightforward for this case, and it is evident that the physics of the situation is unaffected by setting $\underline{V} = 0$.

The 13-moment equations for the i-th gas component are readily written in the following form:

$$\text{(4.12)} \quad \mathbf{I}\frac{\partial \underline{U}_i}{\partial t} + Q_i \frac{\partial \underline{U}_i}{\partial r} + S_i \underline{U}_i + \sum_j w_j \underline{U}_j = 0.$$

In this equation, $\mathbf{I}$ is the identity matrix, and $\underline{U}_i$ is a 13 component column vector, with transposed form:

$$\text{(4.13)} \quad \underline{U}_i' = \left(\rho_i, u_i^1, u_i^2, u_i^3, p_i, P_i^{11}, P_i^{12}, P_i^{13}, P_i^{22}, P_i^{23}, R_i^1, R_i^2, R_i^3 \right).$$

Taking r as having the dimensions of length, the coefficient Q_i is essentially independent of the coordinate system used, although it should be remembered that $\underline{U}_i$ is assumed independent of spatial coordinates other than r. With a coordinate system (r, s, η), the r, s, η components are designated $1, 2, 3$.

Ignoring subscripts, Q_i is given below. In this matrix,

$$(4.14)\quad A = -\frac{7}{2}\frac{p\,p''}{(\rho)^2} - \frac{5}{2}\left(\frac{p}{\rho}\right)^2 + \frac{5}{2}\frac{p\,u'u'}{\rho},$$

$$B = -\frac{7}{2}\frac{p\,p'^2}{(\rho)^2} + \frac{5}{2}\frac{p\,u'u^2}{\rho},$$

$$C = -\frac{7}{2}\frac{p\,p'^3}{(\rho)^2} + \frac{5}{2}\frac{p\,u'u^3}{\rho},$$

$$F = \frac{7}{2}\frac{p''}{\rho} + \frac{5}{2}\frac{p}{\rho} - \frac{25}{6}u'u',$$

$$G = \frac{7}{2}\frac{p'^2}{\rho} - \frac{25}{6}u'u^2,$$

$$H = \frac{7}{2}\frac{p'^3}{\rho} - \frac{25}{6}u'u^3.$$

$$
(4.15)\quad Q = \begin{pmatrix}
u^1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{u^1u^1}{\rho} & -u^1 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{u^1u^2}{\rho} & -u^2 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{u^1u^3}{\rho} & -u^3 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{3}p & 0 & 0 & \frac{5}{3}u^1 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & \frac{4}{3}p & 0 & 0 & \frac{4}{3}u^1 & 0 & 0 & 0 & 0 & 0 & \frac{8}{15} & 0 & 0 \\
0 & 0 & p & 0 & u^2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\
0 & 0 & 0 & p & u^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5} \\
0 & -\frac{2}{3}p & 0 & 0 & -\frac{2}{3}u^1 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A & -\frac{5}{3}pu^1 & 0 & 0 & F & \frac{p}{\rho} & 0 & 0 & 0 & 0 & -\frac{5}{3}u^1 & 0 & 0 \\
B & -\frac{5}{3}pu^2 & 0 & 0 & G & 0 & \frac{p}{\rho} & 0 & 0 & 0 & -\frac{5}{3}u^2 & 0 & 0 \\
C & -\frac{5}{3}pu^3 & 0 & 0 & H & 0 & 0 & \frac{p}{\rho} & 0 & 0 & -\frac{5}{3}u^3 & 0 & 0
\end{pmatrix}.
$$

The coefficient matrices S_i and W_j in equation (4.12) are functions of the particular coordinate system chosen. For a cylindrical coordinate system, S_i is given in Appendix C (section C3). W_j can also be written out, in terms of the $\delta_{jk,i}$. Since, however, only the $\underline{u}$, $\underset{\sim}{P}$ and $\underline{R}$ are explicit in $W_j \underline{U}_j$, the $\delta_{jk,i}$ being functions of ρ_j, p_j etc., the coefficients of ρ_j, p_j in W_j are not clear without considering particular cases. For cylindrical coordinates, and with the particular values of the $\delta_{jk,i}$ from Chapter 3, the form of W_j can be obtained from the moment equations for this coordinate system, given in section C2 of Appendix C.

The eigenvalues of Q_i, that is, the characteristic speeds for the i-th gas component, are the solutions of the determinant equation

$$|Q - \lambda I| = 0. \tag{4.16}$$

This characteristic equation, on expanding the determinant, is found to be

$$\lambda^5 \left(\frac{7}{5} \frac{kT}{m} - \lambda^2 \right)^2 \left[\lambda^4 - \lambda^2 \left(\frac{26}{5} \frac{kT}{m} + \frac{7}{3} \frac{P''}{\rho} \right) + 3 \left(\frac{kT}{m} \right)^2 + \frac{21}{5} \frac{kT}{m} \frac{P''}{\rho} \right] = 0. \tag{4.17}$$

The 13 roots are

$$\lambda = 0 \qquad \text{(five times)}, \tag{4.18}$$

$$\lambda = \pm \left(\frac{7}{5} \frac{kT}{m} \right)^{\frac{1}{2}} \qquad \text{(twice)}, \tag{4.19}$$

$$(4.20)\quad \lambda = \pm\left[\frac{13}{5}\frac{kT}{m} + \frac{7}{6}\frac{P''}{\rho} + \left(\frac{376}{100}\left(\frac{kT}{m}\right)^2 + \frac{49}{36}\left(\frac{P''}{\rho}\right)^2 + \frac{28}{15}\frac{kTP''}{m\rho}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

and

$$(4.21)\quad \lambda = \pm\left[\frac{13}{5}\frac{kT}{m} + \frac{7}{6}\frac{P''}{\rho} - \left(\frac{376}{100}\left(\frac{kT}{m}\right)^2 + \frac{49}{36}\left(\frac{P''}{\rho}\right)^2 + \frac{28}{15}\frac{kTP''}{m\rho}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

The left eigenvectors are the row vectors which are the solutions of the equation

$$(4.22)\quad \underline{\ell}_\mu(\lambda)\, Q_i = \lambda\, \underline{\ell}_\mu(\lambda)\,.$$

There are 12 linearly independent left eigenvectors of rank 1; ignoring the particle-type subscripts, they are as follows.

(1) Roots $\lambda = \pm\left(\frac{7}{5}\frac{kT}{m}\right)^{\frac{1}{2}} \equiv \pm\lambda_1$

$$(4.23)\quad \left.\begin{matrix}\underline{\ell}_1\ [\lambda=\lambda_1] \\ \underline{\ell}_3\ [\lambda=-\lambda_1]\end{matrix}\right\} \equiv \left(0,\ 0,\ P,\ -pP^{12}/P^{13},\ u^2-u^3P^{12}/P^{13},\right.$$
$$\left.0,\ \lambda,\ -\lambda P^{12}/P^{13},\ 0,\ 0,\ 0,\ \frac{2}{5},\ -\frac{2}{5}P^{12}/P^{13}\right)$$

$$(4.24)\quad \left.\begin{matrix}\underline{\ell}_2\ [\lambda=\lambda_1] \\ \underline{\ell}_4\ [\lambda=-\lambda_1]\end{matrix}\right\} \equiv \left(-8(\lambda+u')P/\rho,\ -\frac{11}{2}P,\ 0,\right.$$

(ctd)

$$\left(\tfrac{87}{14}p-\tfrac{5}{2}P''\right)p/P^{13},\ \left(\tfrac{5}{2}u'+\tfrac{57}{14}\lambda+\tfrac{5}{2}u^3\left[\tfrac{87}{35}p/P^{13}-P''/P^{13}\right]\right),-\tfrac{45}{14}\lambda,$$

$$0,\ \tfrac{5}{2}\lambda\left(\tfrac{87}{35}p/P^{13}-P''/P^{13}\right),\ 0,\ 0,\ 1,\ 0,\ \left(\tfrac{87}{35}p/P^{13}-P''/P^{13}\right)\Big).$$

(2) Roots $\lambda = \pm\lambda_2, \pm\lambda_3$. $\left[\lambda^4-\lambda^2\left(\tfrac{26}{5}\tfrac{kT}{m}+\tfrac{7}{3}\tfrac{P''}{\rho}\right)+\tfrac{21}{5}\left(\tfrac{P''}{\rho}\right)^2+3\left(\tfrac{kT}{m}\right)^2=0\right]$.

(4.25) $\left.\begin{matrix}\underset{\sim}{\ell}_5\ [\lambda=\lambda_2],\ \underset{\sim}{\ell}_6[\lambda=\lambda_3],\\ \underset{\sim}{\ell}_7\ [\lambda=-\lambda_2],\ \underset{\sim}{\ell}_8\ [\lambda=-\lambda_3]\end{matrix}\right\} \equiv \Big((u'+\lambda)\left(\tfrac{5}{6}\lambda^2-\tfrac{35}{18}P''/\rho-\tfrac{13}{3}p/\rho\right),$

$$\left(\tfrac{5}{6}\rho\lambda^2-\tfrac{11}{6}p-\tfrac{35}{18}P''\right),\ 0,\ 0,\ \left(\tfrac{5}{6}u'+\tfrac{5}{6}\lambda+\tfrac{2}{3}p/\rho\lambda\right.$$

$$\left.-\tfrac{14}{9}P''/\rho\lambda\right),\ \left(\tfrac{5}{6}\lambda-\tfrac{5}{6}p/\rho\lambda-\tfrac{35}{18}P''/\rho\lambda\right),\ 0,\ 0,$$

$$0,\ 0,\ 1,\ 0,\ 0\Big).$$

(3) Roots $\lambda = 0$.

(4.26) $\underset{\sim}{\ell}_9 \equiv \left(0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\right.$

$$\left.\tfrac{5}{2},\ 0,\ 0,\ 0,\ 0\right)$$

(4.27) $\underset{\sim}{\ell}_{10} \equiv \left(0,\ 0,\ 0,\ 0,\ 1,\ -\tfrac{5}{4},\ 0,\ 0,\right.$

$$\left.0,\ 0,\ 0,\ 0,\ 0\right)$$

(4.28) $\underset{\sim}{\ell}_{11} \equiv \left(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\right.$

$$\left.0,\ 1,\ 0,\ 0,\ 0\right)$$

$$(4.29)\quad \underline{\ell}_{12} \equiv \Big(\tfrac{7}{2}(u^2 P^{13} - u^3 P^{12})p/\rho,\ 0,\ pP^{13},\ -pP^{12},$$

$$0,\ 0,\ 0,\ 0,\ -\tfrac{25}{4}(u^3 P^{12} - u^2 P^{13}),\ 0,\ 0,\ -P^{13},\ P^{12}\Big).$$

The remaining eigenvector necessary to complete the basis is a generalised eigenvector of rank 2 corresponding to the root $\lambda = 0$. This generalised eigenvector is the solution of the equation

$$(4.30)\quad \underline{\ell}^{(2)}(Q)^2 = 0,$$

being the vector

$$(4.31)\quad \underline{\ell}_{13} \equiv \Big((P^{12}u^3 - P^{13}u^2)(\tfrac{29}{45}p - \tfrac{28}{9}P^{11})pu'/\rho,$$

$$-\tfrac{28}{9}p(P^{12}u^3 - P^{13}u^2)(P^{11} - \tfrac{41}{35}P),\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,$$

$$-\tfrac{15}{2}pu'(P^{12}u^3 - P^{13}u^2),\ 0,\ -7(P^{12}u^3 - P^{13}u^2)(P^{11} - \tfrac{41}{35}P),\ ((5p+7P^{11})\times$$

$$(P^{11} - \tfrac{41}{35}P)u^3 + 7P^{13}u'(P - P^{11})),\ -((5p+7P^{11})(P^{11} - \tfrac{41}{35}P)u^2 + 7P^{12}u'(p - P^{11}))\Big).$$

The five eigenvectors $\underline{\ell}_9$ to $\underline{\ell}_{13}$, corresponding to the zero roots, are associated with the Jordan rather than diagonal form of the matrix respresentation.

Maxwell's equations, as already mentioned, are hyperbolic with characteristic speeds $\pm c$ (twice). Written in the form of equation (4.12) they become

$$(4.32)\qquad \frac{\partial \underline{\xi}}{\partial t} + \begin{pmatrix} 0 & 0 & 0 & (c)^2 \\ 0 & 0 & -(c)^2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{\partial \underline{\xi}}{\partial r} + \Lambda \underline{\xi} = 0$$

where $\underline{\xi}$ is the column vector (E^2, E^3, B^2, B^3) and Λ is a function of the coordinate system, being given in Appendix C for cylindrical coordinates. The four left eigenvectors are

$$(4.33)\qquad \underline{l}_\alpha \equiv (1, 0, 0, c), \quad \underline{l}_\beta \equiv (0, 1, -c, 0),$$

$$\underline{l}_\gamma \equiv (1, 0, 0, -c), \quad \underline{l}_\delta \equiv (0, 1, c, 0),$$

the first two being associated with the root $+c$. The Maxwell equations are of course put into characteristic normal form by multiplying the equation (4.32) by these eigenvectors, and since the components of the vector $\underline{\xi}$ do not enter the differential side of the 13-moment equations, this is the relevant characteristic normal form for use in conjunction with these equations. The convenient nature of this form is shown by expressing the Maxwell equations in normal form:

$$(4.34)\qquad \frac{\partial}{\partial t}(E^2 + cB^3) + c\frac{\partial}{\partial r}(E^2 + cB^3) + (1, 0, 0, c)\Lambda\underline{\xi} = 0$$

$$\frac{\partial}{\partial t}(E^3 - cB^2) + c\frac{\partial}{\partial r}(E^3 - cB^2) + (1, 0, -c, 0)\Lambda\underline{\xi} = 0$$

$$\frac{\partial}{\partial t}(E^2 - c B^3) - c\frac{\partial}{\partial r}(E^2 - c B^3) + (1,0,0,-c)\wedge\underline{\xi} = 0$$

$$\frac{\partial}{\partial t}(E^3 + c B^2) - c\frac{\partial}{\partial r}(E^3 + c B^2) + (0,1,c,0)\wedge\underline{\xi} = 0.$$

Using the eigenvectors $\underline{\ell}_\alpha$ to $\underline{\ell}_\delta$ as a basis, it is evident that, in terms of this basis, the four vector components in equation (4.34) are $\mathcal{F}^1 = \underline{\ell}_\alpha \cdot \underline{\xi}$, $\mathcal{F}^2 = \underline{\ell}_\beta \cdot \underline{\xi}$, $\mathcal{F}^3 = \underline{\ell}_\gamma \cdot \underline{\xi}$ and $\mathcal{F}^4 = \underline{\ell}_\delta \cdot \underline{\xi}$; in terms of these it is obvious that the equations (4.34) become

$$(4.35)\quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r}\right)\mathcal{F}^1 + \underline{\ell}_\alpha \wedge \underline{\xi} = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r}\right)\mathcal{F}^2 + \underline{\ell}_\beta \wedge \underline{\xi} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial r}\right)\mathcal{F}^3 + \underline{\ell}_\gamma \wedge \underline{\xi} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial r}\right)\mathcal{F}^4 + \underline{\ell}_\delta \wedge \underline{\xi} = 0 .$$

Equations (4.35) demonstrate the conciseness of the characteristic normal form.

In the case of a quasilinear set of equations, the eigenvectors and the differential operators do not necessarily commute, and it is not always possible to change the basis to obtain the form (4.35). However they can still be expressed in the form (4.34), and there is of course no fundamental difference between (4.34) and (4.35). The moment equations are largely reducible to this form by multiplying equation (4.12) by the eigenvectors $\underline{\ell}_1$ to $\underline{\ell}_{13}$. The result is, in general notation:

$$(4.36)\qquad \underline{\ell}_\mu\left(\frac{\partial \underline{U}}{\partial t} + c_\mu \frac{\partial \underline{U}}{\partial r}\right) + \underline{\ell}_\mu S\underline{U} + \underline{\ell}_\mu \sum_j W_j \underline{U}_j = 0$$

where $\mu = 1, 2, \ldots, 12$, the c_μ being the roots associated with the respective $\underline{\ell}_\mu$, together with

$$(4.37)\qquad \underline{\ell}_{13}\left(\frac{\partial \underline{U}}{\partial t} + J \frac{\partial \underline{U}}{\partial r}\right) + \underline{\ell}_{13} S\underline{U} + \underline{\ell}_{13} \sum_j W_j \underline{U}_j = 0.$$

Equations (4.36) are in normal form, while since J is the matrix,

$$(4.38)\qquad J =$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{63}{91}\rho/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

equation (4, 37) is in a form pertaining to Jordan normal form. (In (4. 38) the row of zeros along the principal diagonal represent the relevant zero eigenvalues). The matrix J is not in Jordan form of course. The conversion to this form is simply achieved by using an appropriate linear combination of the zero root eigenvectors, $\underline{\ell}_9$ to $\underline{\ell}_{13}$. Let

$$(4.39)\quad \underline{\ell}'_{13} = \underline{\ell}_{13} - \frac{28}{9}\left(\frac{P^{12}u^3 - P^{13}u^2}{P^{12}}\right)\left(\frac{41}{35}P - P^{11}\right)\underline{\ell}_{12} + \frac{91}{9}P\left(\frac{P^{12}u^3 - P^{13}u^2}{\rho u^3}\right)\left(\frac{41}{35}P - P^{11}\right)\underline{\ell}_9 ,$$

then

$$(4.40)\quad \underline{\ell}'_{13} Q = \ell'_{13} J$$

where J is (essentially) a Jordan matrix, with non-zero elements $J_{45} = P/\rho$, $J_{56} = u^3$. (These elements are P/ρ , u^3 instead of the customary 1's due to the dimensionality of $\underline{U}$, the non-commuting of ℓ'_{13} , $\frac{\partial}{\partial r}$ preventing a change of basis from being made as in (3. 35).) Using the eigenvector ℓ'_{13} with equation (4. 37) gives the Jordan form:

$$(4.41)\quad \ell'_{13}\left(\frac{\partial \underline{U}}{\partial t} + J\frac{\partial \underline{U}}{\partial r}\right) + \ell'_{13} S \underline{U} + \ell'_{13}\sum_j W_j \underline{U}_j = 0.$$

(For the particular case of one dimensional flow involving only five dynamical variables, Grad[4] obtained the five characteristic speeds, four of which were non-zero. Similarly for a two-dimensional

flow, the "13-moment" equations (there being actually nine for this case) have nine such speeds, three of which are zero. In each case, on using an analysis similar to that presented above, all the eigenvectors are of first rank, and therefore reduction to true characteristic normal form is possible.).

4.2.1 The Moment Equations in Normal Form

The column vector $\underline{U}$ is defined in terms of its components in equation (4.13). Substituting for these components in the general equations (4.36), (4.37), the characteristic normal form of the moment equations is as follows.

(a) For characteristic speeds $c = \pm \lambda_1$, let

$$\frac{d_1(\pm)}{dt} \equiv \frac{\partial}{\partial t} \pm \lambda_1 \frac{\partial}{\partial r} . \tag{4.42}$$

In writing the corresponding equations, to avoid confusion, $\frac{d_1}{dt}$ will be used, with the understanding that the upper sign in (4.42) is to be taken with the upper sign in equations (4.43) and (4.44).

$$\begin{aligned} & p \frac{d_1 u^2}{dt} - p \frac{P^{12}}{P^{13}} \frac{d_1 u^3}{dt} + \left(u^2 - u^3 \frac{P^{12}}{P^{13}}\right) \frac{d_1 P}{dt} \pm \lambda_1 \frac{d_1}{dt} P^{12} \\ & \mp \lambda_1 \frac{P^{12}}{P^{13}} \frac{d_1}{dt} P^{13} + \frac{2}{5} \frac{d_1 R^2}{dt} - \frac{2}{5} \frac{P^{12}}{P^{13}} \frac{d_1 R^3}{dt} \\ & + \begin{matrix} \underline{\ell}_1 \,[c = \lambda_1] \\ \underline{\ell}_2 \,[c = -\lambda_1] \end{matrix} \left[S \underline{U} + \sum_j W_j \underline{U}_j \right] = 0 . \end{aligned} \tag{4.43}$$

$$(4.44)\quad -8\frac{p}{\rho}\left(\pm\lambda_1+u'\right)\frac{d_1\rho}{dt}-\frac{11}{2}p\frac{d_1u'}{dt}+\frac{5}{2}p\left(\frac{87}{35}\frac{p}{p^{13}}-\frac{p^{11}}{p^{13}}\right)\frac{d_1u^3}{dt}+\left[\frac{5}{2}u^3\left(\frac{87}{35}\frac{p}{p^{13}}-\frac{p^{11}}{p^{13}}\right)+\frac{5}{2}u'\pm\frac{57}{14}\lambda_1\right]\frac{d_1p}{dt}$$
$$\mp\frac{45}{14}\lambda_1\frac{d_1p^{11}}{dt}\pm\frac{5}{2}\lambda_1\left(\frac{87}{35}\frac{p}{p^{13}}-\frac{p^{11}}{p^{13}}\right)\frac{d_1p^{13}}{dt}+\frac{d_1R'}{dt}$$
$$+\left(\frac{87}{13}\frac{p}{p^{13}}-\frac{p^{11}}{p^{13}}\right)\frac{d_1R^3}{dt}$$
$$+\begin{matrix}\underline{\ell}_2\,[c=\lambda_1]\\ \underline{\ell}_4\,[c=-\lambda_1]\end{matrix}\left[S\underline{U}+\sum_j W_j\underline{U}_j\right]=0.$$

(b) Characteristic speeds $c=\pm\lambda_2\,,\ \pm\lambda_3$. Let

$$(4.45)\quad \frac{d_\mu(\pm)}{dt}\equiv\frac{\partial}{\partial t}\pm\lambda_\mu\frac{\partial}{\partial r}\,,$$

μ taking the values 2 and 3 . Using $\frac{d_\mu}{dt}$ for (4.45) with the $(\pm)$ understood as in the previous case, the four equations are:

$$(4.46)\quad \left(u'\pm\lambda_\mu\right)\left(\frac{5}{6}(\lambda_\mu)^2-\frac{35}{18}\frac{p^{11}}{\rho}-\frac{13}{3}\frac{p}{\rho}\right)\frac{d_\mu\rho}{dt}$$
$$+\left(\frac{5}{6}\rho(\lambda_\mu)^2-\frac{11}{6}p-\frac{35}{18}p^{11}\right)\frac{d_\mu u'}{dt}$$
$$+\left(\frac{5}{2}u'\pm\frac{5}{6}\lambda_\mu\pm\frac{2}{3}\frac{p}{\rho\lambda_\mu}\mp\frac{14}{9}\frac{p^{11}}{\rho\lambda_\mu}\right)\frac{d_\mu p}{dt}$$
$$+\left(\pm\frac{5}{6}\lambda_\mu\mp\frac{5}{6}\frac{p}{\rho\lambda_\mu}\mp\frac{35}{18}\frac{p^{11}}{\rho\lambda_\mu}\right)\frac{d_\mu p^{11}}{dt}+\frac{d_\mu R'}{dt}$$

(ctd)

$$+ \begin{matrix} \underline{\ell}_5 \ [c = \lambda_2] \\ \underline{\ell}_6 \ [c = \lambda_3] \\ \underline{\ell}_7 \ [c = -\lambda_2] \\ \underline{\ell}_8 \ [c = -\lambda_3] \end{matrix} \left[S\underline{U} + \sum W_j \underline{U}_j \right] = 0.$$

(c) Characteristic speeds $c = 0$. The corresponding equations are

$$\frac{\partial}{\partial t} p + \frac{5}{2} \frac{\partial P^{22}}{\partial t} + \underline{\ell}_9 \left(S\underline{U} + \sum_j W_j \underline{U}_j \right) = 0 \tag{4.47}$$

$$\frac{\partial}{\partial t} p - \frac{5}{4} \frac{\partial P^{11}}{\partial t} + \underline{\ell}_{10} \left(S\underline{U} + \sum W_j \underline{U}_j \right) = 0 \tag{4.48}$$

$$\frac{\partial}{\partial t} P^{23} + \underline{\ell}_{11} \left(S\underline{U} + \sum_j W_j \underline{U}_j \right) = 0 \tag{4.49}$$

$$\begin{aligned} &\frac{7}{2} \frac{p}{\rho} (u^2 P^{13} - u^3 P^{12}) \frac{\partial \rho}{\partial t} + p P^{13} \frac{\partial u^2}{\partial t} - p P^{12} \frac{\partial u^3}{\partial t} \\ &+ \frac{25}{4} (u^2 P^{13} - u^3 P^{12}) \frac{\partial P^{22}}{\partial t} - P^{13} \frac{\partial R^2}{\partial t} + P^{12} \frac{\partial R^3}{\partial t} \\ &+ \underline{\ell}_{12} \left(S\underline{U} + \sum_j W_j \underline{U}_j \right) = 0 . \end{aligned} \tag{4.50}$$

$$\begin{aligned} &(P^{12} u^3 - P^{13} u^2)\left(\frac{29}{45} p - \frac{28}{9} P^{11}\right) \frac{p u^1}{\rho} \frac{\partial \rho}{\partial t} \\ &- (P^{12} u^3 - P^{13} u^2)\left(P^{11} - \frac{41}{35} p\right) \frac{28}{9} p \frac{\partial u^1}{\partial t} \\ &- \frac{15}{2} (P^{12} u^3 - P^{13} u^2) p u^1 \frac{\partial P^{22}}{\partial t} - 7 (P^{12} u^3 - P^{13} u^2)\left(P^{11} \right. \\ &\left. - \frac{41}{35} p\right) \frac{\partial R^1}{\partial t} + \Big[(5p + 7P^{11})\left(P^{11} - \frac{41}{35} p\right) u^3 + 7 P^{13} u^1 \left(p \right. \end{aligned} \tag{4.51}$$

(ctd)

$$- P^{11}) \Big] \frac{\partial R^2}{\partial t} - \Big[(5P + 7P^{11})(P^{11} - \tfrac{41}{35}P)u^2 + 7P^{12}u^1(p$$

$$- P^{11})\Big] \frac{\partial R^3}{\partial t} + \tfrac{28}{9}(P^{12}u^3 - P^{13}u^2)(\tfrac{41}{35}\tfrac{p}{\rho} - \tfrac{P^{11}}{\rho})p\frac{\partial p}{\partial r}$$

$$- \tfrac{91}{9}(P^{12}u^3 - P^{13}u^2)(P^{11} - \tfrac{41}{35}P)\tfrac{p}{\rho}\frac{\partial P^{11}}{\partial r}$$

$$+ \underline{\ell}_{13}(S\underline{u} + \textstyle\sum_j W_j \underline{u}_j) = 0.$$

As already discussed, the last equation (4.51) is not in characteristic normal form, and neither has it been written in the Jordan normal form. Should this equation be preferred in the Jordan form, this is obtained by using the eigenvector of equation (4.39), $\underline{\ell}'_{13}$. The result is:

(4.52)
$$\Big((P^{12}u^3 - P^{13}u^2)(\tfrac{29}{45}p - \tfrac{28}{9}P^{11})\tfrac{p\,u^1}{\rho} + \tfrac{98}{9}(P^{12}u^3 - P^{13}u^2)^2(\tfrac{41}{35}p$$

$$- P^{11})(P/\rho P^{12})\Big)\frac{\partial \rho}{\partial t} - (P^{12}u^3 - P^{13}u^2)(P^{11} - \tfrac{41}{35}P)\tfrac{28}{9}p\frac{\partial u^1}{\partial t}$$

$$- \tfrac{28}{9}(P^{12}u^3 - P^{13}u^2)(\tfrac{41}{35}p - P^{11})(pP^{13}/P^{12})\frac{\partial u^2}{\partial t} + \tfrac{28}{9}(P^{12}u^3$$

$$- P^{13}u^2)(\tfrac{41}{35}p - P^{11})(p)\frac{\partial u^3}{\partial t} - \tfrac{91}{9}(P^{12}u^3 - P^{13}u^2)(P^{11}$$

$$- \tfrac{41}{35}P)(P/\rho u^3)\frac{\partial p}{\partial t} + \Big(-\tfrac{15}{2}u^1 p(P^{12}u^3 - P^{13}u^2) - \tfrac{455}{18}(P^{12}u^3$$

$$- P^{13}u^2)(P^{11} - \tfrac{41}{35}P)(P/\rho u^3) + \tfrac{175}{9}(P^{12}u^3 - P^{13}u^2)^2(\tfrac{41}{35}p$$

$$- P^{11})(1/P^{12})\Big)\frac{\partial P^{22}}{\partial t} - 7(P^{12}u^3 - P^{13}u^2)(P^{11} - \tfrac{41}{35}P)\frac{\partial R^1}{\partial t}$$

$$+\Big((5p + 7P^{11})(P^{11} - \tfrac{41}{35}P)u^3 + 7P^{13}u^1(p - P^{11}) + \tfrac{28}{9}(P^{12}u^3 - P^{13}u^2)(\tfrac{41}{35}p$$

(ctd)

$$-P^{11})\left(P^{13}/P^{12}\right)\Big)\frac{\partial R^2}{\partial t}+\Big(-(5p+7P^{11})(P^{11}-\tfrac{41}{35}p)u^2$$

$$-7P^{12}(p-P^{11})u^1-\tfrac{28}{9}(P^{12}u^3-P^{13}u^2)(\tfrac{41}{35}p-P^{11})\Big)\frac{\partial R^3}{\partial t}$$

$$+\tfrac{28}{9}(P^{12}u^3-P^{13}u^2)(\tfrac{41}{35}p-P^{11})p/\rho\frac{\partial p}{\partial r}-\tfrac{91}{9}(P^{12}u^3-P^{13}u^2)(P^{11}$$

$$-\tfrac{41}{35}p)p/\rho\frac{\partial P^{11}}{\partial r}+\underline{\ell}'_{13}\left(S\underline{U}+\sum_j W_j\underline{U}_j\right)=0\,.$$

There is no great advantage of having equation (4.51) in this form, and in fact the resulting coefficients are, if anything, more complicated.

4.3 Boundary Conditions

4.3.1 Necessary Boundary Conditions

In the previous subsection it has been shown that for the i-th gas component moment equations, five of the 13 characteristic speeds are zero. From the theory of characteristics it follows that a total of eight boundary conditions must be specified for these equations. Furthermore, since four of the non-zero roots are negative and four positive, four boundary conditions must be specified on each boundary of the variable r (e.g. for a cylindrical system the boundaries are the "surface" $r=0$, and an outer surface, say $r=a$). Similarly, for the Maxwell equations (4.32) there are two positive and two negative characteristic speeds, showing that two boundary conditions are required on each boundary. It is shown later in the Chapter that the moment equation boundary conditions must specify the variables u^1, P^{12}, P^{13}, and R^1 on each boundary. The Maxwell equations are not considered again in this section, since for any particular physical system, the relevant boundary conditions are

readily obtained (in particular, for cylindrical coordinates, on the "boundary" $r = 0$, $E^2 = B^2 = 0$ while E^3, B^3 must be specified on the boundary $r = a$).

4.3.2 Physical Boundary Conditions on a Wall

Although a cylindrical coordinate system is not being considered in this Chapter, except as a specific example in the associated Appendix C, the boundary conditions are given in such a way that they are directly relevant, for instance, to a cylindrical wall. In view of this, a brief mention may be made of the boundary conditions that must be taken at the centre of a cylindrical system.

It is evident from Appendix C that the form of the moment equations in cylindrical coordinates involves terms of the nature s/r where s is a dynamical variable, and in particular the terms u'/r, P^{12}/r, P^{13}/r and R'/r. Since all such terms must remain finite (although certain discontinuities are possible), the relevant boundary conditions at $r = 0$ are obviously

$$u' = 0; \quad P^{12} = 0; \quad P^{13} = 0; \quad R' = 0. \tag{4.53}$$

There are in fact an excessive number of such boundary conditions at $r = 0$; presumably those other than the specified conditions (4.53) follow in any case from initial conditions. Another simple case that may be mentioned is that of an infinite plasma. Obviously at the extremes of this plasma, the variables u', P^{12}, P^{13}, R' are zero as in (4.53).

Boundary conditions for the moment equations at a wall have been discussed by Grad[4]. In the present section, however, more general

conditions, applicable to physical phenomena, are considered. Also, reduction of the boundary conditions is taken to a stage where the particular dynamical variables (u^1, P^{12}, P^{13}, R^1) can be directly expressed in terms of the boundary coefficients.

Consider a boundary wall that is a coordinate surface $\tau =$ constant, in a coordinate system (τ, ξ, η). Let the velocity distribution near the wall be represented as follows:

$$(4.54) \quad f(w^1, w^2, w^3) = f^+(w^1+V^1, w^2, w^3) + f^-(w^1+V^1, w^2, w^3).$$

(The τ, ξ, η components are designated 1, 2, 3 respectively). V^1 is the τth component of the reference velocity of the moments, $\underline{V}$. The terms on the right hand side are defined by this equation together with the specifications

$$(4.55) \quad f^+(c^1, w^2, w^3) = 0 \quad , \quad c^1 < 0 \; ;$$

$$f^-(c^1, w^2, w^3) = 0 \quad , \quad c^1 > 0 \; ;$$

where the direction of c^1 is such that $c^1 > 0$ is away from the wall. The correct boundary conditions are those which specify the "reflected half" of the distribution function of the particles adjacent to the wall, f^+ , when the "incident half", f^- , is given.

This break-up of the distribution function may be applied to determine which variables must be specified for the four boundary conditions on the wall. Since the question of which are the relevant variables is independent of the physical nature of the boundary

conditions, consider a perfectly reflecting wall. Then on this wall,

$$(4.56)\quad f^{+}(c^{1}, w^{2}, w^{3}) = f^{-}(-c^{1}, w^{2}, w^{3}),$$

that is, f is even in c^{1}. Therefore all moments odd in c^{1} must vanish. These are the moments

$$(4.57)\quad \int f\, m(w^{1}+V^{1})\, d\underline{w} = \rho(u^{1}+V^{1})$$

$$\int f\, m(w^{1}+V^{1})\, w^{2}\, d\underline{w} = P^{12} + \rho V^{1} u^{2}$$

$$\int f\, m(w^{1}+V^{1})\, w^{3}\, d\underline{w} = P^{13} + \rho V^{1} u^{3}$$

$$\int f\, \tfrac{1}{2} m(w^{1}+V^{1})(w)^{2}\, d\underline{w} = R^{1} + \frac{5}{4\alpha_{i}}\rho u^{1} + \frac{3}{2} p V^{1}.$$

Since $\underline{V}$ can be specified arbitrarily, it follows that in all cases the boundary conditions must be impressed on the four variables, u^{1}, P^{12}, P^{13}, R^{1}, and of course in general they need not be zero. Again, the particular case of $r = 0$ in a cylindrical system may be considered as a perfectly reflecting line, since in passing through $r = 0$ the r-component of the velocity, c^{1}, is effectively reversed.

Properties of a boundary wall that affect the contained dilute gas or plasma can be described as follows. For each particle of type i which collides with the wall:

(1) a fraction, α, of the incident particles may be specularly reflected, $-c^{1}$ being converted to $+c^{1}$;

(2) a fraction, β, may be adsorbed by the wall and re-emitted with a Maxwellian velocity dsitribution about some temperature, T_{1} (possibly after a time interval τ_{1});

(3) a fraction, γ , may be permanently adsorbed by the wall or re-emitted as particles of another type (possibly after time interval τ_2);

(4) there may be δ particles of type j produced with temperature T_2 (for example from secondary emission or process (3));

(5) finally, the wall may be an independent source of N particles per unit area per unit time, at a temperature T_3 , irrespective of the state of the gas in the vessel.

Although in certain cases some or all of these coefficients are functions of w^1, w^2 and w^3, any such dependence will be ignored for the present section. This allows the boundary values of u^1, P^{12}, P^{13} and R^1 to be readily expressed in terms of the above coefficients. As elsewhere in this Chapter particle-type subscripts are neglected for the i-th gas component.

The "incident half" of the distribution function, $f^-(-c^1, w^2, w^3)$, may be regarded as known. Situation (1) contributes to $f^+(c^1, w^2, w^3)$ a term

$$\propto f^-(-c^1, w^2, w^3) . \tag{4.58}$$

Situations (2), (4), and (5) respectively contribute the terms

$$k_1(t-\tau_1)\, e^{-\frac{1}{2}\frac{m_i(c_i)^2}{kT_1}} \tag{4.59}$$

$$k_2(t-\tau_2)\, e^{-\frac{1}{2}\frac{m_i(c_i)^2}{kT_2}} \tag{4.60}$$

(4.61) $k_3 e^{-\frac{1}{2}\frac{m_i (c_i)^2}{kT_3}}$

where k_1, k_2 and k_3 can be expressed in terms of α, β, γ, δ, N, T_1, T_2 and T_3.

Defining the "half moments" by

(4.62) $$n\overline{\psi}^{\pm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^{\pm}(c', w^2, w^3)\,\psi(\underline{w})\,d\underline{w},$$

the gas mass current incident on the wall is

(4.63) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^{-}(c', w^2, w^3)\, mc'\, d\underline{w} = (\rho u')^{-} + \rho^{-} V'.$$

Then $k_1(t)$ is such that the Maxwellian gas current leaving the wall owing to situation (2) is

(4.64) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{c'=0}^{c'=\infty} k_1(t)\, e^{-\frac{m(c)^2}{2kT_1}}\, mc'\, d\underline{w}.$$

It is also given by

(4.65) $$\beta\left[(\rho u')^{-} + \rho^{-} V'\right]$$

taken at time t. On integrating (4.64),

(4.66) $$k_1(t) = \beta\left[(\rho u')^{-} + \rho^{-} V'\right] \Big/ \left(\frac{2\pi (kT_1)^2}{(m)^2}\right).$$

The calculation for k_2 is very similar. The mass flux of particles of type i due to impact of particles of type j is

$$(4.67)\quad \delta\left[(\rho_j u_j')^- + \rho_j^- V'\right].$$

Therefore

$$(4.68)\quad k_2(t) = \delta\left[(\rho_j u_j')^- + \rho_j V'\right] \Big/ \left(2\pi\left(\frac{kT}{m}\right)^2\right).$$

For the particles introduced from the wall as an independent (Maxwellian) source, it is usual to specify the particle current density, that is, the number of particles per unit time per unit area, say N. The mass current is mN. Since this mass current is also given by (case (5))

$$(4.69)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{c'=0}^{c'=\infty} k_3\, e^{-\frac{m(c)^2}{2kT}}\, mc'\, d\underline{w} = k_3\,\frac{2\pi}{m}\left(kT_3\right)^2$$

it follows that

$$(4.70)\quad k_3 = \frac{N}{2\pi}\left(\frac{m}{kT_3}\right)^2.$$

With the k_1, k_2 and k_3 in terms of known constants or functions, the boundary relations can be completed by obtaining u', P^{12}, P^{13}, and R' as functions of these k's. From equations (4.58) to (4.61),

$$(4.71)\quad f^+(c', w^2, w^3) = \alpha f^-(-c', w^2, w^3) + k_1(t-\tau_1)\, e^{-\frac{m(c)^2}{2kT_1}} + k_2(t-\tau_2)\, e^{-\frac{m(c)^2}{2kT_2}} + k_3\, e^{-\frac{m(c)^2}{2kT_3}}.$$

Multiplying this equation in turn by

(4.72) mw' , $m\{w'w^2\}$, $m\{w'w^3\}$, $\frac{1}{2}mw'((w)^2 - \frac{5}{2\alpha_i})$,

integration over $\underline{w}$ (where the Maxwellian functions on the right hand side exist only for $c' > 0$) gives, in terms of the definition (4.62):

(4.73) $$(\rho u')^+ = \alpha(\rho u')^- + \frac{2\pi k^2}{m}(k_1T_1^2 + k_2T_2^2 + k_3T_3^2) - \frac{mV'}{2}\left(\frac{2\pi k}{m}\right)^{\frac{3}{2}}(k_1T_1^{\frac{3}{2}} + k_2T_2^{\frac{3}{2}} + k_3T_3^{\frac{3}{2}})$$

(4.74) $$(P^{12})^+ = \alpha(P^{12})^- + \frac{\pi^{\frac{1}{2}}m}{2}\left(\frac{2k}{m}\right)^{\frac{5}{2}}(k_1T_1^{\frac{5}{2}} + k_2T_2^{\frac{5}{2}} + k_3T_3^{\frac{5}{2}}) - \frac{2\pi k^2}{m}V'(k_1T_1^2 + k_2T_2^2 + k_3T_3^2)$$

(4.75) $$(P^{13})^+ = \alpha(P^{13})^- + \frac{\pi^{\frac{1}{2}}m}{2}\left(\frac{2k}{m}\right)^{\frac{5}{2}}(k_1T_1^{\frac{5}{2}} + k_2T_2^{\frac{5}{2}} + k_3T_3^{\frac{5}{2}}) - \frac{2\pi k^2}{m}V'(k_1T_1^2 + k_2T_2^2 + k_3T_3^2)$$

(4.76) $$(R')^+ = \alpha(R')^- + \frac{m}{2\pi^2}\left(\frac{2\pi k}{m}\right)^3(k_1T_1^3 + k_2T_2^3 + k_3T_3^3) - \frac{5}{4\alpha_i}\frac{2\pi k^2}{m}(k_1T_1^2 + k_2T_2^2 + k_3T_3^2) + \frac{3m(V')^2}{4\pi}\left(\frac{2\pi k}{m}\right)^2(k_1T_1^2 + k_2T_2^2 + k_3T_3^2) - \frac{5mV'}{8\pi}\left(\frac{2\pi k}{m}\right)^{\frac{5}{2}}(k_1T_1^{\frac{5}{2}} + k_2T_2^{\frac{5}{2}} + k_3T_3^{\frac{5}{2}})$$

(ctd)

$$- \frac{m}{4}(V')^3\left(\frac{2\pi k}{m}\right)^{\frac{3}{2}}\left(k_1 T_1^{\frac{3}{2}} + k_2 T_2^{\frac{3}{2}} + k_3 T_3^{\frac{3}{2}}\right)$$

$$+ \frac{5}{4\alpha_i}\,\frac{mV'}{2}\left(\frac{2\pi k}{m}\right)^{\frac{3}{2}}\left(k_1 T_1^{\frac{3}{2}} + k_2 T_2^{\frac{3}{2}} + k_3 T^{\frac{3}{2}}\right).$$

The "half moments" can be expressed in terms of the moments by the use of the 13-moment approximation for the distribution function. This distribution function is given in the Introduction, and from (4.54) it follows that the $f^{\pm}$ for this function are

(4.77) $$f^{\pm}(\pm c', w_j^2 w^3) = n\left(\frac{\alpha_i}{\pi}\right)^{\frac{3}{2}} e^{-\alpha_i\left((c')^2+(w^2)^2+(w^3)^2\right)}\Big(1$$

$$+ 2\alpha_i(\pm u'c' + u^2w^2 + u^3w^3) + \frac{\alpha_i}{p}\Big[(c')^2P^{11} + (w^2)^2P^{22}$$

$$+ (w^3)^2P^{33} \pm 2c'w^2P^{12} \pm 2c'w^3P^{13} + 2w^2w^3P^{23}\Big]$$

$$+ \frac{2\alpha_i}{5p}\Big(2\alpha_i\left[(c')^2+(w^2)^2+(w^3)^2\right] - 5\Big)(\pm c'R' + w^2R^2$$

$$+ w^3R^3)\Big),$$

for $c' \geqslant 0$;

$$f^{\pm}(\pm c', w_j^2 w^3) = 0,$$

for $c' < 0$.

Taking moments of the variables m , mw^1 , mw^1w^2 , mw^1w^3 , $\frac{1}{2} mw^1((\underline{w})^2 - 5/2\alpha_i)$, with respect to distribution functions (4.77):

$$(4.78)\quad \rho^{\pm} = \rho\left(\tfrac{1}{2} \pm \left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} u^1 \mp \tfrac{1}{5}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} \frac{R^1}{p}\right)$$

$$(4.79)\quad (\rho u^1)^{\pm} = \frac{\rho}{4\alpha_i p}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}}\left(2p(1 \pm (\pi\alpha_i)^{\frac{1}{2}} u^1) + P^{11}\right) - \rho V^1\left(\tfrac{1}{2} \pm \left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} u^1 \mp \tfrac{1}{5}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} \frac{R^1}{p}\right)$$

$$(4.80)\quad (P^{12})^{\pm} = \frac{\rho}{2\alpha_i}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}}\left(u^2 \pm \frac{P^{12}}{2p}\left(\frac{\pi}{\alpha_i}\right)^{\frac{1}{2}} + \frac{R^2}{5p}\right) - \frac{\rho V^1}{2}\left(u^2 \pm \frac{P^{12}}{(\pi\alpha_i)^{\frac{1}{2}} p}\right)$$

$$(4.81)\quad (P^{13})^{\pm} = \frac{\rho}{2\alpha_i}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}}\left(u^3 \pm \frac{P^{13}}{2p}\left(\frac{\pi}{\alpha_i}\right)^{\frac{1}{2}} + \frac{R^3}{5p}\right) - \frac{\rho V^1}{2}\left(u^3 \pm \frac{P^{13}}{(\pi\alpha_i)^{\frac{1}{2}} p}\right)$$

$$(4.82)\quad (R^1)^{\pm} = \frac{2\rho}{\pi^2}\left(\frac{\pi}{\alpha_i}\right)^{\frac{3}{2}}\left(-\frac{1}{16} + \frac{P^{11}}{3p} \pm (\pi\alpha_i)^{\frac{1}{2}} \frac{R^1}{8p}\right) - \frac{\rho(V^1)^3}{2}\left(\tfrac{1}{2} \pm \left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} u^1 \mp \tfrac{1}{5}\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}} \frac{R^1}{p}\right) + \frac{3\rho}{8\alpha_i}(V^1)^2\left(\frac{\alpha_i}{\pi}\right)^{\frac{1}{2}}\left(2p(1 \pm (\pi\alpha_i)^{\frac{1}{2}} u^1) + P^{11}\right) - \rho V^1\left(-\frac{5}{8\alpha_i} \pm \frac{5}{4}\frac{u^1}{(\pi\alpha_i)^{\frac{1}{2}}} + \frac{P^{11}}{4p\alpha_i} \pm \frac{13}{20}\frac{R^1}{(\pi\alpha_i)^{\frac{1}{2}} p}\right) .$$

Finally substitution of these values for the "half moments" into equations (4.73) to (4.76) gives four algebraic equations relating $\rho u'$, P^{12}, P^{13}, and R' to the boundary coefficients. For various reasons, such a simplicity of numerical integration or particular properties of the gas, special values of $\underline{V}$ may be chosen. Therefore the elementary but tedious algebra of substituting the "half moments" into these equations and solving the resulting algebraic equations is not carried out for the general form of (4.73) to (4.76).

4.3.3 Boundary Conditions for Hydrogen Atoms, Protons, and Electrons

As discussed previously, molecular hydrogen and its compounds are not considered in the idealised hydrogenic plasma of this part of the thesis. However it should be mentioned that a well known property of metallic surfaces at normal temperatures when impacted by hydrogen atoms is to adsorb a proportion of these atoms, re-emitting them some time later as molecular hydrogen. The phenomena associated with the impact of electrons, protons and atoms with surfaces are not all properly understood, and experimental results in the lower energy range ($<$ 100ev) are often difficult to interpret. Nevertheless there are sufficient conclusions available on which to base approximate boundary conditions for these particles.

Concerning electron impact with surfaces, incident electrons may undergo inelastic collisions with the lattice or with atoms on the surface, or may be reflected with little energy loss. For incident energies greater than 10 to 20 ev, considerable secondary emission occurs, the secondary electrons being in the energy range 2 to 3 ev, largely independent of the primary electron energy.

Protons striking a surface can also produce secondary electrons. However, most studies appear to show that the secondary emission coefficient is small for incident energies less than 100 ev. Many protons incident on the surface will be scattered with some loss of energy. Such "reflected" protons are observed at all angles of reflection although the specular angle is most probable. Another process that may occur is the capture of an electron from the surface. The neutral atom so formed can be either adsorbed by the solid or reflected as a fast neutral atom. The production of negative ions is also possible by this phenomena, but this process and that of sputtering will be ignored. An additional, practical, effect of positive ion bombardment is the liberation of adsorbed gas layers from the surface.

The possibility of excited atoms colliding with the wall will be ignored. Atoms impinging on a surface with low energies (less than a few ev) may be captured or "condensed", forming an adsorbed layer, or may undergo diffraction. An energy transfer to or from the lattice vibrational energy is also possible. Specular and diffuse reflection occur in all ranges of energy. Outgassing of a wall, either by heating or by fast particles colliding with it, can provide a steady flow of atoms at the wall temperature into the gas.

The expression of these surface effects in a form relevant to the boundary equations is as follows.

(a) Electrons

The fraction of incident electrons specularly reflected is $\alpha(e)$, the fraction re-emitted with a Maxwellian distribution at temperature $T_1(e)$ is $\beta(e)$, and for each electron impacting with the wall, $\delta(e,e)$ secondary electrons at $T_2(e)$ are emitted. For each proton impact, $\delta(e,p)$ secondary

electrons are produced at $T_2(e)$. $\alpha(e)$, $\beta(e)$, $\delta(e,e)$, $\delta(e,p)$, $T_1(e)$, $T_2(e)$ and also $\gamma(e)$, the fraction of electrons absorbed by the surface, are all dependent on the nature of the surface, while $\delta(e,e)$ in particular is a rapidly varying function of the electron temperature. Approximate values of these parameters for several surfaces are given in Chapter V of Massey and Burhop[5].

(b) Protons

The coefficients correspond to a fraction $\alpha(p)$ of incident protons specularly reflected, a fraction $\beta(p)$ re-emitted with temperatue $T_1(p)$, and a fraction $\gamma(p)$ adsorbed by the wall, being emitted as neutral atoms.

(c) Atoms

A fraction $\alpha(a)$ of the impacting atoms are specularly reflected, a fraction $\beta(a)$ re-emitted with temperature $T_1(a)$, and a fraction $\gamma(a)$ adsorbed by the surface. Each ion impact produces $\delta(a,p)$ atoms at $T_2(a)$, where the protons which have captured electrons are reasonably included. Finally there is a gas current of atoms (N atoms per unit area per unit time) at temperature $T_3(a)$.

Again, for the atoms and protons, all the parameters are functions of the surface material and temperature as well as of other circumstances. Approximate values for these parameters are given in Massey and Burhop[5] (Chapter IX), and also Kaminsky[6]. The substitution of these parameters in the boundary equations is basically straightforward (although temperature dependence, for example, must be averaged), and need not be re-iterated.

4.4 Unconditionally Stable Finite Difference Methods

It is well known that an explicit finite difference scheme for the numerical solution of a set of hyperbolic partial differential equations converges only when the fixed mesh ratio, $s = \Delta t / \Delta r$, is less than the maximum inverse characteristic slope. On the other hand certain implicit schemes are convergent for arbitrary mesh ratios, the difficulty being an enormous increase in the amount of computation required to obtain a numerical solution. Recently, however, Keller[1] proposed a scheme for semi-linear hyperbolic systems, which is explicit for all characteristic speeds less than the inverse of the mesh ratio, $1/s$, and implicit by decision for all characteristic speeds greater than $1/s$, and which is unconditionally stable. Subsequently, Keller and Thomée[7] published a modified form of the scheme which is applicable to quasilinear hyperbolic systems. As mentioned in the introduction to this Chapter, the form of the scheme and proof of unconditional stability as given in reference 7 assume the equations are in characteristic normal form.

Consider a system of equations in this normal form:

$$\frac{\partial \underline{U}}{\partial t} - D(r,t,\underline{U}) \frac{\partial \underline{U}}{\partial r} = F(r,t,\underline{U}); \tag{4.83}$$

with initial conditions

$$\underline{U} = \underline{h}(r) , \quad (t = 0); \tag{4.84}$$

and with boundary conditions as determined by the non-zero characteristic speeds, c say:

$$(4.85)\qquad \sigma^- \underline{U} = \underline{h}^-(t,(I-\sigma^-)\underline{U}) \quad , \quad r = r_0 \quad (c<0),$$

$$\sigma^+ \underline{U} = \underline{h}^+(t,(I-\sigma^+)\underline{U}) \quad , \quad r = r_1 \quad (c>0),$$

where

$$(4.86)\qquad \sigma^{\pm} = 0 \quad , \quad \text{if} \quad c \lessgtr 0 \ ;$$

$$= 1 \quad , \quad \text{if} \quad c \gtrless 0 \ .$$

Then the finite difference scheme, for mesh ratio $s = \Delta t/\Delta r$, with $\underline{v}$ the finite difference analogue of $\underline{U}$, is:

$$(4.87a)\qquad \left(v(i)_{j+1}^{n+1} - v(i)_{j+1}^{n}\right) - s\,c(i)_j^n \left(v(i)_{j+1}^{n+1} - v(i)_j^{n+1}\right)$$

$$(4.87b)\qquad \left(v(i)_j^{n+1} - v(i)_j^{n}\right) - s\,c(i)_j^n \left(v(i)_{j+1}^{n} - v(i)_j^{n}\right)$$

$$(4.87c)\qquad \left(v(i)_j^{n+1} - v(i)_j^{n}\right) - s\,c(i)_j^n \left(v(i)_j^{n} - v(i)_{j-1}^{n}\right)$$

$$(4.87d)\qquad \left(v(i)_{j-1}^{n+1} - v(i)_{j-1}^{n}\right) - s\,c(i)_j^n \left(v(i)_j^{n+1} - v(i)_{j-1}^{n+1}\right)$$

$$= \Delta t\, F(i)_j^n \begin{cases} \text{if} \quad 1 < s\,c(i)_j^n \ ; \\ \text{if} \quad 0 \leqslant s\,c(i)_j^n \leqslant 1 \ ; \\ \text{if} \quad -1 \leqslant s\,c(i)_j^n < 0 \ ; \\ \text{if} \quad s\,c(i)_j^n < -1 \ , \end{cases}$$

$V(i)_j^n$ being the i-th component of $\underline{V}$ at mesh space and time points j, n, and similarly for the characteristic speeds $C(i)_j^n$.

When $|c| < 1/S$ the decision scheme chooses the appropriate explicit finite difference form, (4.87b) or (4.87c). However, when $|c| > 1/S$ the finite difference representation chosen effectively reverses the space and time variables, the integration being "forward" in space, and by definition, the characteristic "speed" $1/C$ satisfies the inequality $|1/C| < S$. That is, in this form, the difference scheme is explicit. Nevertheless, it can be seen that more computation is required per step than for a purely explicit case, but, depending on the particular problem, the additional labour is generally slight, while the number of steps may be enormously reduced.

The equations as given in section 4.2 are in characteristic normal form except for one (equation (4.51)). The finite difference scheme (4.87) may, however, still be applied without difficulty, terms involving the spatial derivatives in (4.51) being readily approximated to at any step of the integration, being regarded in fact as no different from the S and W terms. However, because such terms do exist, the questions of stability and convergence are still open to doubt.

Finally, it should be noted that the discussion given in this Part of the thesis is, in one sense, far from complete. The tremendous task of programming, and of giving realistic values to the boundary conditions associated with any exact experimental arrangement, has not been discussed. These aspects are such that even the case of the formation of a simple hydrogenic-like plasma would probably constitute a thesis in itself. For this reason the actual numerical integration of a realistic example has not been pursued.

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PART III

High Frequency Wave Motion in Plasma

Chapter 5

High Frequency Waves in Homogeneous Plasma

5.1 Introduction

In this Part of the thesis the propagation of high frequency waves in a plasma is considered. By using the 13-moment equations, collision phenomena such as resistivity, viscosity, thermal conduction, and thermo-electric effects are automatically taken into account. Although such effects are probably included in direct approaches which employ the Boltzmann equation, the distinction between them is lost. However, in the 13-moment equations, the relationships involving collision phenomena, such as mentioned above, with the wave motion may be independently recognised in the associated dispersion equations.

For this Chapter, a homogeneous plasma is considered, while in the next, wave motion in inhomogeneous plasma is discussed. The prime object is to obtain the dispersion relationships for various types of wave motion and for various orientations of magnetic field. Particular properties of the waves are briefly discussed, and in Appendix D, explicit determination of Faraday rotation is given for several cases.

The emphasis being on propagating waves rather than plasma oscillations, for a given frequency ω, approximate solutions of the dispersion equations are found that give the phase velocities, damping, and other propagation properties. Non-propagating plasma oscillations are ignored. Only continuous wave solutions are considered in which case, as is well known, the Fourier transform of the time derivative of a function is simply the Fourier transform of the function multiplied by $i\omega$ with no initial conditions. (It has already been mentioned that

Landau damping does not occur with a finite set of moment equations. In any case this is not a continuous wave phenomena. Nevertheless, for a transient wave, consideration of the Fourier transforms of initial conditions shows that there are no growing or decaying solutions when collisions are neglected. Again it would appear that the Krammers-Kronnig relations (Kittel[1]) predict a decay term in the transient wave solution. However, on using these relations to calculate the decay term for solutions of the 13-moment equations for wave motion, it is found to be zero.)

In general, in most cases of practical interest, the results for the usual electromagnetic waves are not very different from those obtained by other authors (e.g. Ginzburg[2]). However, contained in the dispersion equations are both transverse and longitudinal "stress" waves. Although perhaps inherent in a Boltzmann approach, these waves are not readily recognised from this source, whereas in the present approach, the physical association with anisotropic stress components and the heat flux is obvious.

As shown in Chapter 2, the 13-moment equations for the electron component of a plasma or gas mixture "decouple", provided that

$$(5.1) \quad \alpha_e \ll \alpha_p, \alpha_q, \ldots$$

$$T_e \ll \frac{m_p}{m_e} T_p, \frac{m_q}{m_e} T_q, \ldots$$

where $p, q, \ldots$ are "heavy" particle types. Also for sufficiently high frequencies it is apparent that the ions and neutrals, owing to their large inertia, will be effectively stationary, with electrons determining the wave motion. This is so when

(5.2) $$|\omega| >> |\omega_{bi}|,$$

$$|\omega| >> \frac{5}{3}\left(\frac{p_{eo}}{\rho_o}\right)^{\frac{1}{2}}|\kappa|,$$

where ω_{bi} is the ion cyclotron frequency, p_{eo} the electron pressure, ρ_o the plasma density, and κ is the wave number for the wave oscillations. The last restriction of (5.2) is equivalent to requiring that the wave speed is much greater than the "heavy" particle mean thermal speeds, and (Liley[3]) is the condition that

(5.3) $$|\underline{V}_o| << |\underline{u}_e|$$

where $\underline{V}_o$ is the plasma mean mass velocity.

In the following sections the analysis applies explicitly to a completely ionised plasma, and implicitly to the partially ionised case. For convenience in treating a fully ionised plasma in particular, the equations are written in a form in which the collision terms are related to certain "characteristic" parameters. Using the current density, $\underline{J} = ne\underline{u}$, these become the "equations of plasma physics" (Herdan and Liley[4]); neglecting the particle-type subscripts for electrons, the equations for this gas component are as follows.

5.1.1 Continuity Equation*

(5.4) $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u} + \rho\underline{V}) = 0 \quad .$$

*Ionisation collisions have been neglected for this equation. In a partially ionised gas such effects could be important. However, the continuity equation is only associated with longitudinal (compressible) modes, and while these are considered in subsequent sections, the emphasis is on transverse modes. Hence, for simplicity, ionisation processes in the continuity and thermal energy equations are ignored.

5.1.2 Generalised Ohm's Law Equation

$$(5.5)\quad \frac{m}{e}\frac{d\underline{J}}{dt} + \frac{m}{e}\underline{J}\nabla\cdot\underline{V} + \nabla\cdot\underset{\sim}{P} + \nabla p - \frac{\rho e}{m}(\underline{E} + \underline{V}\times\underline{B}) - \underline{J}\times\underline{B} + \frac{m}{e}\underline{J}\cdot\nabla\underline{V} + ne\eta\underline{J} + ne\alpha\underline{R} = 0.$$

5.1.3 Thermal Energy Equation

$$(5.6)\quad \frac{d\frac{3}{2}p}{dt} + \frac{3}{2}p\nabla\cdot\underline{V} + \nabla\cdot\underline{R} + \frac{5}{2}\frac{m}{e}\nabla\cdot\left(\frac{p\underline{J}}{\rho}\right) + \frac{m}{e}\underline{J}\cdot\frac{d\underline{V}}{dt} - \underline{J}\cdot(\underline{E} + \underline{V}\times\underline{B}) + \underset{\sim}{P}:\nabla\underline{V} + p\nabla\cdot\underline{V} = \frac{1}{2}\sum_j \delta^{(o)}_{ej,e}.$$

5.1.4 Stress Tensor Equation

$$(5.7)\quad \frac{d}{dt}\underset{\sim}{P} + \underset{\sim}{P}\nabla\cdot\underline{V} + \frac{4}{5}\{\nabla\underline{R}\} + \frac{2m}{e}\left\{\nabla\frac{p\underline{J}}{\rho}\right\} - 2\{(\underline{E} + \underline{V}\times\underline{B})\underline{J}\} + 2\frac{m}{e}\left\{\underline{J}\frac{d\underline{V}}{dt}\right\} + 2\{\underset{\sim}{P}\cdot\nabla\underline{V}\} + 2p\{\nabla\underline{V}\} - \frac{2e}{m}\{\underset{\sim}{P}\times\underline{B}\} + \frac{\Phi}{\tau}\underset{\sim}{P} = 0.$$

5.1.5 Heat Flux Equation

$$(5.8)\quad \frac{d\underline{R}}{dt} + \frac{7}{2}\underset{\sim}{P}\cdot\nabla\left(\frac{p}{\rho}\right) + \frac{5}{2}p\nabla\left(\frac{p}{\rho}\right) + \frac{p}{\rho}\nabla\cdot\underset{\sim}{P} + \frac{5}{2}\frac{m}{e}\underline{J}\frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{e}{m}\underset{\sim}{P}\cdot\left(\underline{E} + \underline{V}\times\underline{B} - \frac{m}{e}\frac{d\underline{V}}{dt}\right) - \frac{e}{m}\underline{R}\times\underline{B} + \frac{7}{5}\underline{R}\nabla\cdot\underline{V}$$

$$+ \frac{7}{5}\underline{R}\cdot\nabla\underline{V} + \frac{2}{5}\nabla\underline{V}\cdot\underline{R} + \frac{mp}{e\rho}\underline{J}\nabla\cdot\underline{V} + \frac{mp}{e\rho}\nabla\underline{V}\cdot\underline{J}$$

$$+ \frac{mp}{e\rho}\underline{J}\cdot\nabla\underline{V} + \gamma\underline{J} + \frac{1}{\tau}\underline{R} = 0 .$$

These "equations of plasma physics" for the electrons are completed by the inclusion of Maxwell's equations.

On comparing equations (5.4) to (5.8) with the general 13-moment equations (1.17) to (1.21) of Chapter 1, it can be seen that

(5.9)
$$\eta = -\frac{m}{\rho n e^2}\,\frac{1}{2}\sum_j \delta^{(1)}_{ej,e} \qquad \text{(momentum eqn.)},$$

$$\alpha = -\frac{1}{ne}\,\frac{1}{2}\sum_j \delta^{(5)}_{ej,e} \qquad \text{(momentum eqn.)},$$

$$\frac{\Phi}{\tau} = -\tfrac{1}{2}\left(\delta^{(3)}_{ee,e} + \delta^{(4)}_{ee,e} + \sum_j \delta^{(3)}_{ej,e}\right) \qquad \text{(stress eqn.)},$$

$$\frac{1}{\tau} = -\tfrac{1}{2}\left(\delta^{(5)}_{ee,e} + \delta^{(6)}_{ee,e} + \sum_j \delta^{(5)}_{ej,e}\right) \qquad \text{(heat flux eqn.)},$$

$$\gamma = -\frac{1}{ne}\,\frac{1}{2}\sum_j \delta^{(1)}_{ej,e} \qquad \text{(heat flux eqn.)}.$$

The relevant δ's are given in section 2.6 of Chapter 2 (used in conjunction with equation (1.16)), in terms of the associated Ω's, while for certain particular cases the Ω's are calculated explicitly in Chapter 3. For a completely ionised gas consisting of electrons and one species (j) of ions with charge $e_j = -Z_j e_e$, the following relations apply:

(5.10)
$$\tfrac{1}{2}\delta^{(1)}_{ej,e} \quad \text{(moment.)} \quad = \quad -\tfrac{16}{3}\frac{mn^2}{Z_j}\Omega'_{ej}(1)$$

$$\tfrac{1}{2}\delta^{(5)}_{ej,e} \quad \text{(moment.)} \quad = \quad \tfrac{32}{5}\frac{\alpha_e n}{Z_j}\Omega'_{ej}(1)$$

$$\tfrac{1}{2}\left(\delta^{(3)}_{ee,e} + \delta^{(4)}_{ee,e} + \delta^{(3)}_{ej,e}\right)_{\text{(stress)}} = \quad -\tfrac{16}{5} n\Omega'_{ee}(1) - \tfrac{32}{5}\frac{n}{Z_j}\Omega'_{ej}(1)$$

$$\tfrac{1}{2}\delta^{(1)}_{ej,e} \quad \text{(heat)} \quad = \quad \frac{4mn^2}{\alpha_e Z_j}\Omega'_{ej}(1)$$

$$\tfrac{1}{2}\left(\delta^{(5)}_{ee,e} + \delta^{(4)}_{ee,e} + \delta^{(5)}_{ej,e}\right)_{\text{(heat)}} = \quad -\tfrac{32}{15} n\Omega'_{ee}(1) - \tfrac{104}{15}\frac{n}{Z_j}\Omega'_{ej}(1)$$

$$\tfrac{1}{2}\delta^{(0)}_{ej,e} \quad \text{(energy)} \quad = \quad 8\,\frac{mn^2\vartheta_{ej}}{\alpha_e Z_j}\Omega'_{ej}(1)\,.$$

Considering for a moment the momentum equation (1.18) of Chapter 1 (being essentially the Ohm's law equation (5.5)), the collisional relaxation term associated with $\frac{\partial \underline{u}_e}{\partial t}$ is $-\frac{1}{2}(u_e/\rho_e)\sum_j \delta^{(1)}_{ej,e}$. Therefore, apart from the term in $\underline{R}_e$ the collision time for momentum relaxation with the ions etc. is τ_D, say, where

(5.11)
$$\tau_D = -\frac{2\rho_e}{\sum_j \delta^{(1)}_{ej,e}}\,.$$

Hence a collision frequency, ν, may be defined by

(5.12)
$$\nu = \frac{1}{\tau_D}\,, \quad = \quad \tfrac{16}{3}\frac{n}{Z_j}\Omega'_{ej}(1)\,,$$

the last value applying only to the particular fully ionised two component plasma. This equation shows, incidentally, that apart from the factor n, the Ω's are generalised collision "frequencies". Using the results of Chapter 3 for Coulomb collisions,

$$(5.13)\quad \Omega^{1}_{ee}(1) = \left(\frac{m_{je}}{m_{ee}}\right)^2 \frac{1}{2^{3/2} Z_j^2} \frac{\log \Lambda^2_{ee}}{\log \Lambda^2_{ej}} \Omega^{1}_{ej}(1)$$
$$\equiv \psi \Omega^{1}_{ej}(1)$$

where

$$(5.14)\quad \psi = \frac{2^{\frac{1}{2}} \log \Lambda_{ee}}{Z_j^2 \log \Lambda_{ej}} .$$

Then

$$(5.15)\quad \eta = \frac{m}{ne^2} \nu$$
$$\alpha = -\frac{6}{5} \frac{\alpha_e}{ne} \epsilon \nu = -\frac{3}{5} \frac{\rho}{pne} \epsilon \nu$$
$$\frac{\Phi}{\tau} = \beta \nu$$
$$\frac{1}{\tau} = \vartheta \nu$$
$$\gamma = -\frac{3}{4} \frac{m}{e\alpha_e} \theta \nu = -\frac{3}{2} \frac{mp}{\rho e} \theta \nu .$$

For the two component fully ionised plasma, the dimensionless parameters β , ϑ , ϵ , θ take the values

$$(5.16)\quad \beta = \frac{3}{5} \psi Z_j + \frac{6}{5}$$
$$\vartheta = \frac{2}{5} \psi Z_j + \frac{13}{10}$$
$$\epsilon = 1$$
$$\theta = 1 ,$$

and also

$$(5.17)\quad \nu = \frac{n Z_j}{T^{3/2}} \log \Lambda_{ep} \times 3\cdot 676\times 10^{-6} \quad (\text{sec})^{-1},$$

Λ_{ep} being given in Chapter 3.

However, since these parameters are retained throughout the dispersion equations, the solutions obtained are in fact more general, being applicable to all cases of partially ionised plasmas when the appropriate values of β, $\mathcal{S}$, ϵ, Θ and ν are included. A further point to note is that by retaining such coefficients, the effects associated with the thermo-electric processes (related to ϵ, Θ), thermal conductivity ($\mathcal{S}$), viscosity (β), and resistivity processes (ν) are easily recognised.

The remaining collision term to be considered is that of the thermal energy equation.* The self-collision term being zero, for elastic collisions,

$$(5.18)\quad \tfrac{1}{2}\delta_{ej}^{(0)} = \frac{16\, n_e n_j k m_e}{m_j}(T_j - T_e)\,\Omega_{ej}^{1}(1)$$

$$= \frac{3\, n_e m_e}{m_j}\nu\,(kT_j - kT_e)$$

$$\sim (\pm)\left|\frac{3 m_e}{m_j}\nu p_e\right|,$$

*Since this equation is only associated with the longitudinal modes, inelastic processes have been excluded for simplicity. Without considering in detail particular collisions, it is not possible to estimate the magnitude of the inelastic term.

where k is Boltzmann's constant. This pressure relaxation frequency, $3m_e \nu / m_j$, is essentially negligible compared with the momentum relaxation frequency, and it may therefore be ignored.

5.2 The Dispersion Equations

The steady state solution of the equations (5.4) to (5.8) for a homogeneous plasma is obviously $\underline{J} = 0$, $\underset{\sim}{P} = 0$, $\underline{R} = 0$, and $p = p_0$, $\rho = \rho_0$, where the subscript 0 is used in this Chapter to designate steady state values of the parameters. Also, from Maxwell's equations, $\underline{E} = 0$ and $\underline{B} = \underline{B}_0$. From equation (5.3), it follows that, for practical purposes, the mean mass velocity of the plasma, $\underline{V}_0$, can be taken as zero in considering high frequency electron motion, and therefore it is satisfactory for the moments to be referred to this velocity, putting $\underline{V} = \underline{V}_0 = 0$ in the moment equations (5.4) to (5.8). Linearising these equations, and using unsubscripted symbols for the dynamical variables which represent the departures (assumed small) from the steady state, gives the equations for wave motion. Using equations (5.15) to express the collision terms in terms of the collision frequency and associated coefficients, the wave motion equations are:

$$(5.19)\quad \frac{\partial \rho}{\partial t} + \frac{m}{e} \nabla \cdot \underline{J} = 0$$

$$(5.20)\quad \frac{\partial \underline{J}}{\partial t} + \frac{e}{m} \nabla \cdot \underset{\sim}{P} + \frac{e}{m} \nabla p - \frac{\rho_0 e^2}{m^2} \underline{E} - \frac{e}{m} \underline{J} \times \underline{B}_0 + \nu \underline{J} - \frac{3}{5} \frac{e}{m} \frac{\rho_0}{p_0} \epsilon \nu \underline{R} = 0$$

$$(5.21)\quad \frac{\partial}{\partial t} p + \frac{2}{3} \nabla \cdot \underline{R} + \frac{5}{3} \frac{m}{e} \frac{p_o}{\rho_o} \nabla \cdot \underline{J} = 0$$

$$(5.22)\quad \frac{\partial}{\partial t} \underset{\sim}{P} + \frac{4}{5} \left\{ \nabla \underline{R} \right\} + \frac{2m}{e} \frac{p_o}{\rho_o} \left\{ \nabla \underline{J} \right\} - \frac{2e}{m} \left\{ \underset{\sim}{P} \times \underline{B}_o \right\} + \beta \nu \underset{\sim}{P} = 0$$

$$(5.23)\quad \frac{\partial}{\partial t} \underline{R} + \frac{5}{2} \frac{p_o}{\rho_o} \nabla p - \frac{5}{2} \frac{p_o^2}{\rho_o^2} \nabla \rho + \frac{p_o}{\rho_o} \nabla \cdot \underset{\sim}{P} - \frac{e}{m} \underline{R} \times \underline{B}_o - \frac{3}{2} \frac{m}{e} \frac{p_o}{\rho_o} \Theta \nu \underline{J} + \mathcal{S} \nu \underline{R} = 0 .$$

Maxwell's equations are already linear:

$$(5.24)\quad \nabla \times \underline{B} - \mu_o \underline{J} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = 0$$

$$(5.25)\quad \nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 .$$

These equations can represent various types of wave motion, including very complicated plasma oscillations. As will be seen, for plane waves they admit of 12 solutions, with 6 modes propagating in each ($\pm$) direction. Depending on the orientation of the magnetic field there may be a smaller number of independent modes. Three principal cases will be considered, (1) no applied magnetic field, (2) an applied magnetic field parallel to the direction of propagation of the wave, and (3) an applied magnetic field perpendicular to the direction of propagation. A fourth possible case, that of the applied magnetic field at an arbitrary angle to the direction of propagation, although important, is exceedingly complex and will not be discussed.

Apart from the usual electromagnetic waves, other transverse waves, associated with the anisotropic components of the stress tensor, are possible. For obvious reasons they are called stress waves, although it should be noted that similar waves also occur as longitudinal modes (these being essentially the normal acoustic mode, and unlike the transverse stress waves, are compressible). For this latter case, "heat" waves are also possible. The stress waves are characterised by a wave speed $\sim f = \left(\frac{p_0}{\rho_0}\right)^{\frac{1}{2}} = \left(\frac{kT_0}{m}\right)^{\frac{1}{2}}$. Throughout this Chapter, it is convenient to distinguish between the stress waves, with speed $\sim f$, and the Maxwell's equations type of wave modes (which also propagate in vacuum) with speed $\sim c$. Since, except for relativistic temperatures, $f^2 << c^2$, the Maxwell-type waves will be called fast waves.

5.2.1 No Applied Magnetic Field

Throughout the rest of this Chapter, an orthogonal Cartesian coordinate system (x, y, z) is used. Continuous plane waves are assumed, with the direction of propagation being the z-axis. As is customary, the direction of the oscillating electric field vector component is used to designate the wave.

For this case the equations separate into three distinct groups, two corresponding to transverse oscillations and one to longitudinal oscillations, with respect to the direction of propagation. The groups of equations are:

(a) Transverse (E_x) wave motion

(5.26)
$$-\frac{\partial B_y}{\partial z} - \mu_0 J_x - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial z} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial J_x}{\partial t} + \frac{e}{m}\frac{\partial P_{xz}}{\partial z} - \frac{e^2}{m^2}\rho_0 E_x + \nu J_x - \frac{3}{5}\frac{e}{m}\frac{\rho_0}{p_0}\epsilon\nu R_x = 0$$

$$\frac{\partial P_{xz}}{\partial t} + \frac{2}{5}\frac{\partial R_x}{\partial z} + \frac{m}{e}\frac{p_0}{\rho_0}\frac{\partial J_x}{\partial z} + \beta\nu P_{xz} = 0$$

$$\frac{\partial R_x}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial P_{xz}}{\partial z} - \frac{3}{2}\frac{m}{e}\frac{p_0}{\rho_0}\theta\nu J_x + \vartheta\nu R_x = 0.$$

(b) Transverse (E_y) wave motion

(5.27)
$$\frac{\partial B_x}{\partial z} - \mu_0 J_y - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$-\frac{\partial E_y}{\partial z} + \frac{\partial}{\partial t}B_x = 0$$

$$\frac{\partial J_y}{\partial t} + \frac{e}{m}\frac{\partial P_{yz}}{\partial z} - \frac{e^2}{m^2}\rho_0 E_y + \nu J_y - \frac{3}{5}\frac{e}{m}\frac{\rho_0}{p_0}\epsilon\nu R_y = 0$$

$$\frac{\partial P_{yz}}{\partial t} + \frac{2}{5}\frac{\partial R_y}{\partial z} + \frac{m}{e}\frac{p_0}{\rho_0}\frac{\partial J_y}{\partial z} + \beta\nu P_{yz} = 0$$

$$\frac{\partial R_y}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial P_{yz}}{\partial z} - \frac{3}{2}\frac{m}{e}\frac{p_0}{\rho_0}\theta\nu J_y + \vartheta\nu R_y = 0.$$

(c) Longitudinal (E_z) wave motion

$$(5.28)\quad \mu_0 J_z + \frac{1}{c^2}\frac{\partial E_z}{\partial t} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{m}{e}\frac{\partial J_z}{\partial z} = 0$$

$$\frac{\partial J_z}{\partial t} - \frac{e}{m}\frac{\partial}{\partial z}(P_{xx}+P_{yy}) + \frac{e}{m}\frac{\partial P}{\partial z} - \frac{e^2}{m^2}\rho_0 E_z + \nu J_z - \frac{3}{5}\frac{e}{m}\frac{\rho_0}{p_0}\epsilon\nu R_z = 0$$

$$\frac{\partial P}{\partial t} + \frac{2}{5}\frac{\partial R_z}{\partial z} + \frac{5}{3}\frac{m}{e}\frac{p_0}{\rho_0}\frac{\partial J_z}{\partial z} = 0$$

$$\frac{\partial P_{xx}}{\partial t} - \frac{4}{15}\frac{\partial R_z}{\partial z} - \frac{2}{3}\frac{m}{e}\frac{p_0}{\rho_0}\frac{\partial J_z}{\partial z} + \beta\nu P_{xx} = 0$$

$$\frac{\partial P_{yy}}{\partial t} - \frac{4}{15}\frac{\partial R_z}{\partial z} - \frac{2}{3}\frac{m}{e}\frac{p_0}{\rho_0}\frac{\partial J_z}{\partial z} + \beta\nu P_{yy} = 0$$

$$\frac{\partial R_z}{\partial t} + \frac{5}{2}\frac{p_0}{\rho_0}\frac{\partial P}{\partial z} - \frac{5}{2}\frac{p_0^2}{\rho_0^2}\frac{\partial \rho}{\partial z} - \frac{p_0}{\rho_0}\frac{\partial}{\partial z}(P_{xx}+P_{yy}) - \frac{3}{2}\frac{m}{e}\frac{p_0}{\rho_0}\Theta\nu J_z + \vartheta\nu R_z = 0 .$$

Since $P_{xx} + P_{yy} + P_{zz} = 0$, no equation for P_{zz} need be given, while the remaining equation, involving P_{xy} (from equations (5.22)), cannot produce a propagating solution, and so is ignored.

Since these sets of equations represent plane waves in a homogeneous plasma, the Fourier wave representation may be taken as $e^{i(\omega t - \kappa z)}$ for the oscillatory part of the dynamical variables. With the restriction to continuous waves, as mentioned

previously, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ may be replaced by $i\omega$ and $-i\varkappa$ in each equation. The dispersion equation is the condition that these homogeneous equations are consistent, that is, it is the condition that the determinant of the coefficients for each independent set of equations is zero. Without giving the details of the determinants or their evaluation, the respective dispersion relations relating ω , $\varkappa$, and the plasma parameters are as follows.

(a) Transverse Waves involving E_x , B_y , J_x , P_{xz} and R_x

$$
\begin{aligned}
(5.29)\quad & \varkappa^4 f^2 c^2 \left(\tfrac{7}{5} i\omega + \tfrac{2}{5}\nu + \vartheta\nu + \tfrac{3}{5}\Theta\nu + \tfrac{3}{5}\epsilon\nu \right) \\
& + \varkappa^2 c^2 \left(i\omega\beta\vartheta\nu^2 - i\omega^3 - \beta\omega^2\nu - \vartheta\omega^2\nu + \beta\vartheta\nu^3 \right. \\
& \qquad \left. - \omega^2\nu + i\omega\beta\nu^2 + i\omega\vartheta\nu^2 - \tfrac{9}{10}\epsilon\Theta\beta\nu^3 - \tfrac{9}{10} i\omega\epsilon\Theta\nu^2 \right) \\
& + \varkappa^2 f^2 \left(-\tfrac{7}{5} i\omega^3 - \tfrac{2}{5}\omega^2\nu - \omega^2\vartheta\nu - \tfrac{3}{5}\omega^2\Theta\nu - \tfrac{3}{5}\omega^2\epsilon\nu \right. \\
& \qquad \left. + \tfrac{2}{5} i\omega\omega_p^2 \right) \\
& + \left(-i\omega^3\beta\vartheta\nu^2 + i\omega^5 + \omega^4\beta\nu + \omega^4\vartheta\nu - \omega^2\beta\vartheta\nu^3 + \omega^4\nu \right. \\
& \qquad - i\omega^3\beta\nu^2 - i\omega^3\vartheta\nu^2 + \tfrac{9}{10}\omega^2\epsilon\Theta\beta\nu^3 + \tfrac{9}{10} i\omega^3\epsilon\Theta\nu^2 \\
& \qquad \left. + i\omega\omega_p^2\beta\vartheta\nu^2 - \omega^2\omega_p^2\beta\nu - \omega^2\omega_p^2\vartheta\nu - i\omega^3\omega_p^2 \right) \\
& = 0 .
\end{aligned}
$$

In this equation, ω_p is the electron plasma frequency, given by

$$\omega_p^2 = \frac{ne^2}{m\epsilon_o} \quad , \tag{5.30}$$

and

$$f = \left(\frac{p_o}{\rho_o}\right)^{\frac{1}{2}} , \tag{5.31}$$

having the dimensions of velocity. A mean free path, λ_p, may be defined by $\lambda_p = f/\nu$.

(b) Transverse waves involving E_y, B_x, J_y, P_{yz} and R_y

The dispersion equation for this case is exactly the same as that for the previous case, equation (5.29).

(c) Longitudinal waves involving E_z, ρ, J_z, P, P_{xx}, P_{yy}, and R_z

This dispersion equation is

$$\begin{aligned}
&\kappa^4 f^4 \left(-\frac{3i}{\omega} - \frac{5}{3}\frac{\beta\nu}{\omega^2} \right) \\
&+ \kappa^2 f^2 \Big(\frac{11}{5}\nu - \frac{5}{3}\frac{i\beta\nu^2}{\omega} + \frac{26}{5} i\omega + \frac{10}{3}\beta\nu - \frac{5}{3}\frac{i\beta\vartheta\nu^2}{\omega} \\
&\quad - \frac{11}{5}\frac{i\omega_p^2}{\omega} - \frac{5}{3}\frac{\beta\nu\omega_p^2}{\omega^2} + 3\vartheta\nu + \frac{9}{5}\theta\nu - \frac{i\theta\beta\nu^2}{\omega} \\
&\quad + \frac{9}{5}\epsilon\nu - \frac{i\beta\nu^2\epsilon}{\omega} \Big) \\
&+ \Big(\beta\vartheta\nu^3 - \omega^2\nu + i\omega\beta\nu^2 + i\omega\vartheta\nu^2 + i\omega\beta\vartheta\nu^2 \\
&\quad - i\omega^3 - \omega^2\beta\nu - \omega^2\vartheta\nu - \frac{i\omega_p^2\beta\vartheta\nu^2}{\omega} + i\omega\omega_p^2 \\
&\quad + \omega_p^2\beta\nu + \omega_p^2\vartheta\nu - \frac{9}{10} i\omega\theta\epsilon\nu^2 - \frac{9}{10}\epsilon\theta\beta\nu^3 \Big) = 0.
\end{aligned} \tag{5.32}$$

5.2.2 Applied Magnetic Field Parallel to the Direction of Propagation: $\underline{B}_o = (0, 0, B_o)$

This case is considerably more complicated than the previous one, since the independent transverse modes of the previous section are coupled by the applied magnetic field, simultaneously involving E_x, E_y, J_x, J_y, B_x, B_y, P_{xz}, P_{yz}, R_x and R_y. The longitudinal oscillations are affected by the field in that they involve the stress component P_{xy}, in addition to E_z, J_z, ρ, P_{xx}, P_{yy}, and R_z. However the longitudinal dispersion equation remains independent of the magnetic field, and therefore for this case also, the longitudinal dispersion equation is (5.32).

The equations involving the transverse components become:

$$(5.33)\quad -\frac{\partial B_y}{\partial z} - \mu_o J_x - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial z} - \mu_o J_y - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial z} + \frac{\partial B_y}{\partial t} = 0$$

$$-\frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial J_x}{\partial t} + \frac{e}{m}\frac{\partial P_{xz}}{\partial z} - \frac{\rho_o e^2}{m^2} E_x - \frac{e}{m} J_y B_o + \nu J_x - \frac{3}{5}\frac{e}{m}\frac{\rho_o}{p_o}\epsilon\nu R_x = 0$$

$$\frac{\partial}{\partial t} J_y + \frac{e}{m} \frac{\partial}{\partial z} P_{yz} - \frac{\rho_0 e^2}{m^2} E_y + \frac{e}{m} J_x B_0 + \nu J_y$$

$$- \frac{3}{5} \frac{e}{m} \frac{\rho_0}{p_0} \epsilon \nu R_y = 0$$

$$\frac{\partial}{\partial t} P_{xz} + \frac{2}{5} \frac{\partial}{\partial z} R_x + \frac{m}{e} \frac{p_0}{\rho_0} \frac{\partial}{\partial z} J_x - \frac{e}{m} B_0 P_{yz}$$

$$+ \beta \nu P_{xz} = 0$$

$$\frac{\partial}{\partial t} P_{yz} + \frac{2}{5} \frac{\partial}{\partial z} R_y + \frac{m}{e} \frac{p_0}{\rho_0} \frac{\partial}{\partial z} J_y + \frac{e}{m} B_0 P_{xz}$$

$$+ \beta \nu P_{yz} = 0$$

$$\frac{\partial}{\partial t} R_x + \frac{p_0}{\rho_0} \frac{\partial}{\partial z} P_{xz} - \frac{e}{m} B_0 R_y - \frac{3}{2} \frac{m}{e} \frac{p_0}{\rho_0} \Theta \nu J_x$$

$$+ \Im \nu R_x = 0$$

$$\frac{\partial}{\partial t} R_y + \frac{p_0}{\rho_0} \frac{\partial}{\partial z} P_{yz} + \frac{e}{m} B_0 R_x - \frac{3}{2} \frac{m}{e} \frac{p_0}{\rho_0} \Theta \nu J_y$$

$$+ \Im \nu R_y = 0 .$$

By taking the same form of plane wave representation as in the non-magnetic case, again with continuous waves, these equations may be combined to yield an eighth order algebraic equation for $\mathcal{K}$. This is the condition that the 10 x 10 determinant of the coefficients, after replacement of the derivatives, is zero. However, and more conveniently for subsequent use, this dispersion equation may be factorised. The two factors are, when set equal to zero, the dispersion equations for the ordinary and extraordinary wave modes. This is the form in which they are given.

In addition to parameters already defined, these equations involve the term ω_b . Since the applied field vector $\underline{B}_o$ has components $(0, 0, B_o)$, ω_b is defined by

$$\omega_b = \frac{e}{m} B_o . \tag{5.34}$$

As the electron charge e is negative, ω_b will be positive or negative according to whether B_o is negative or positive. In dealing with the ordinary and extraordinary modes, throughout this Chapter the discussion and conclusions are given as they apply to the case $\omega_b > 0$. However, the other case is also completely covered, the designated ordinary and extraordinary modes becoming the extraordinary and ordinary modes respectively for $\omega_b < 0$.

With the upper sign representing the ordinary wave, the dispersion equations are:

$$
\begin{aligned}
(5.35)\quad & \mathcal{K}^4 f^2 c^2 \Big[\Big(\tfrac{7}{5}\omega_b^2 - \tfrac{7}{5}\omega^2 + \tfrac{12}{5} i\omega\vartheta\nu + \tfrac{2}{5} i\omega\nu + \tfrac{2}{5}\vartheta\nu^2 \\
& + \tfrac{3}{5}\Theta\vartheta\nu^2 + \tfrac{3}{5} i\omega\Theta\nu + \tfrac{3}{5}\vartheta\epsilon\nu^2 + \tfrac{3}{5} i\omega\epsilon\nu + \vartheta^2\nu^2 \Big) \\
& \pm \Big(\tfrac{2}{5} i\omega_b\vartheta\nu - \tfrac{2}{5} i\omega_b\nu - \tfrac{3}{5} i\omega_b\epsilon\nu - \tfrac{3}{5} i\omega_b\Theta\nu \Big) \Big] \\
+ & \mathcal{K}^2 f^2 \Big[\Big(-\tfrac{2}{5}\omega^2\omega_p^2 + \tfrac{2}{5} i\omega\omega_p^2\vartheta\nu - \tfrac{7}{5}\omega^2\omega_b^2 + \tfrac{7}{5}\omega^4 \\
& - \tfrac{12}{5}\omega^3\vartheta\nu - \tfrac{2}{5} i\omega^3\nu - \tfrac{2}{5}\omega^2\vartheta\nu^2 - \tfrac{3}{5}\omega^2\Theta\vartheta\nu^2 \\
& - \tfrac{3}{5} i\omega^3\Theta\nu - \tfrac{3}{5}\omega^2\epsilon\vartheta\nu^2 - \tfrac{3}{5} i\omega^3\epsilon\nu - \omega^2\vartheta^2\nu^2 \Big) \\
& \pm \Big(-\tfrac{2}{5} i\omega_b\omega^2\vartheta\nu + \tfrac{2}{5} i\omega_b\omega^2\nu + \tfrac{3}{5} i\omega_b\omega^2\epsilon\nu \\
& + \tfrac{3}{5} i\omega_b\omega^2\Theta\nu + \tfrac{2}{5}\omega_b\omega\omega_p^2 \Big) \Big] \\
+ & \mathcal{K}^2 c^2 \Big[\Big(-\tfrac{9}{10}\omega_b^2\Theta\epsilon\nu^2 - \tfrac{9}{10}\Theta\epsilon\beta\vartheta\nu^4 - \tfrac{9}{10} i\omega\epsilon\Theta\beta\nu^3 \\
& - \tfrac{9}{10} i\omega\epsilon\Theta\vartheta\nu^3 + \tfrac{9}{10}\omega^2\epsilon\Theta\nu^2 - \omega_b^2\vartheta^2\nu^2 - 2i\omega\omega_b^2\vartheta\nu \\
& - \omega_b^4 + i\omega\beta\vartheta^2\nu^3 - i\omega^3\beta\nu - 2\omega^2\beta\vartheta\nu^2 + i\omega\omega_b^2\beta\nu \\
& - \omega^2\vartheta^2\nu^2 + \omega^4 - 2i\omega^3\vartheta\nu + \beta\vartheta^2\nu^4 - \omega^2\beta\nu^2 + 2i\omega\beta\vartheta\nu^3 \\
& + \omega_b^2\beta\nu^2 + i\omega\vartheta^2\nu^3 - 2\omega^2\vartheta\nu^2 + i\omega\omega_b^2\nu - i\omega^3\nu \Big) \\
& \pm \Big(\tfrac{9}{10} i\omega_b\beta\Theta\epsilon\nu^3 - \tfrac{9}{10} i\omega_b\Theta\epsilon\vartheta\nu^3 + i\omega_b\beta\vartheta^2\nu^3
\end{aligned}
$$

$$- i\omega_b\omega^2\beta\nu - 2\omega_b\omega\beta\vartheta\nu^2 + i\omega_b^3\beta\nu - 2\omega_b\omega\vartheta^2\nu^2$$

$$+ 2\omega_b\omega^3 - 4i\omega_b\omega^2\vartheta\nu - 2\omega_b^3\omega + i\omega_b\vartheta^2\nu^3$$

$$- i\omega_b\omega^2\nu - 2\omega_b\omega\vartheta\nu^2 + i\omega_b^3\nu)\Big]$$

$$+\Big[\Big(\frac{9}{10}\omega^2\omega_b^2\Theta\epsilon\nu^2 + \frac{9}{10}i\omega^3\beta\Theta\epsilon\nu^3 + \frac{9}{10}\omega^2\beta\vartheta\Theta\epsilon\nu^4$$

$$+ \frac{9}{10}i\omega^3\epsilon\Theta\vartheta\nu^3 - \frac{9}{10}\omega^4\epsilon\Theta\nu^2 + i\omega\omega_p^2\beta\vartheta^2\nu^3$$

$$- i\omega^3\omega_p^2\beta\nu - 2\omega^2\omega_p^2\beta\vartheta\nu^2 + i\omega\omega_p^2\omega_b^2\beta\nu - \omega^2\omega_p^2\vartheta^2\nu^2$$

$$+ \omega^4\omega_p^2 - 2i\omega^3\omega_p^2\vartheta\nu - \omega^2\omega_b^2\omega_p^2 + \omega^2\omega_b^2\vartheta^2\nu^2$$

$$+ 2i\omega^3\omega_b^2\vartheta\nu + \omega^2\omega_b^4 - i\omega^3\beta\vartheta^2\nu^3 + i\omega^5\beta\nu$$

$$+ 2\omega^4\beta\vartheta\nu^2 - i\omega^3\omega_b^2\beta\nu + \omega^4\vartheta^2\nu^2 - \omega^6 + 2i\omega^5\vartheta\nu$$

$$- \omega^2\beta\vartheta^2\nu^4 + \omega^4\beta\nu^2 - 2i\omega^3\beta\vartheta\nu^3 - \omega^2\omega_b^2\beta\nu^2$$

$$- i\omega^3\vartheta^2\nu^3 + i\omega^5\nu + 2\omega^4\vartheta\nu^2 - i\omega_b^2\omega^3\nu\Big)$$

$$\pm\Big(-\frac{9}{10}i\omega_b\omega^2\beta\Theta\epsilon\nu^3 + \frac{9}{10}i\omega_b\omega^2\Theta\epsilon\vartheta\nu^3$$

$$- \omega_b\omega\omega_p^2\vartheta^2\nu^2 + \omega_b\omega^3\omega_p^2 - 2i\omega_b\omega^2\omega_p^2\vartheta\nu$$

$$- \omega_b^3\omega_p^2\omega - i\omega_b\omega^2\beta\vartheta^2\nu^3 + i\omega_b\omega^4\beta\nu + 2\omega_b\omega^3\beta\vartheta\nu^2$$

$$- i\omega_b^3\omega^2\beta\nu + 2\omega_b\omega^3\beta^2\nu^2 - 2\omega_b\omega^5 + 4i\omega_b\omega^4\beta\nu$$

$$+ 2\omega_b^3\omega^3 - i\omega_b\omega^2\beta^2\nu^3 + i\omega_b\omega^4\nu + 2\omega_b\omega^3\beta\nu^2$$

$$- i\omega_b^3\omega^2\nu)\Big] = 0 .$$

As with the previous cases, the detailed solution of this equation is considered later in the Chapter. The coupling of the fast transverse waves by the parallel magnetic field and the resultant "splitting" into the ordinary and extraordinary modes is well known. It is evident from this dispersion relation that the stress waves also form ordinary and extraordinary modes. Their propagation conditions of course remain different from those of the fast waves.

5.2.3 Applied Magnetic Field Perpendicular to the Direction of Propagation: $\underline{B}_0 = (B_0, 0, 0)$

This differs from the previous case in that the mode unaffected by the magnetic field is a transverse mode, being the mode with electric vector parallel to the magnetic field. Despite the fact that certain terms in the relevant equations contain ω_b, the dispersion equation for this mode is identical to (5.29), the equation for the zero field situation. Since the transverse equations with electric vector perpendicular to the direction of the applied magnetic field and the longitudinal component equations are coupled by the magnetic field, propagation of arbitrarily polarised waves is relatively complicated. As the propagating longitudinal modes are essentially stress waves (although in this case associated with the diagonal components of the stress tensor), the coupled equations have a dispersion relation of eighth order in the wave number $\mathcal{K}$, with three stress modes and one fast mode as solutions.

Taking the applied magnetic field direction as the x-axis, the coupled longitudinal-transverse equations are:

(5.36)
$$\frac{\partial B_x}{\partial z} - \mu_o J_y - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$\mu_o J_z + \frac{1}{c^2}\frac{\partial E_z}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial t} - \frac{\partial E_y}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{m}{e}\frac{\partial}{\partial z} J_z = 0$$

$$\frac{\partial}{\partial t} J_y + \frac{e}{m}\frac{\partial}{\partial z} P_{yz} - \frac{e^2}{m^2}\rho_o E_y - \frac{e}{m} B_o J_z + \nu J_y - \frac{3}{5}\frac{e}{m}\frac{\rho_o}{p_o}\epsilon\nu R_y = 0$$

$$\frac{\partial}{\partial t} J_z - \frac{e}{m}\frac{\partial}{\partial z}(P_{xx} + P_{yy}) + \frac{e}{m}\frac{\partial}{\partial z} p - \frac{e^2}{m^2}\rho_o E_z + \frac{e}{m} B_o J_y + \nu J_z - \frac{3}{5}\frac{e}{m}\frac{\rho_o}{p_o}\epsilon\nu R_z = 0$$

$$\frac{\partial}{\partial t} P + \frac{2}{3}\frac{\partial}{\partial z} R_z + \frac{5}{3}\frac{m}{e}\frac{p_o}{\rho_o}\frac{\partial}{\partial z} J_z = 0$$

$$\frac{\partial}{\partial t} P_{yz} + \frac{2}{5}\frac{\partial}{\partial z} R_y + \frac{m}{e}\frac{p_o}{\rho_o}\frac{\partial}{\partial z} J_y + \omega_b (P_{xx} + 2P_{yy}) + \beta\nu P_{yz} = 0$$

$$\frac{\partial}{\partial t} P_{xx} - \frac{4}{15}\frac{\partial}{\partial z} R_z - \frac{2}{3}\frac{m}{e}\frac{p_o}{\rho_o}\frac{\partial}{\partial z} J_z + \beta\nu P_{xx} = 0$$

$$\frac{\partial}{\partial t} P_{yy} - \frac{4}{15}\frac{\partial}{\partial z} R_z - \frac{2}{3}\frac{m}{e}\frac{p_o}{\rho_o}\frac{\partial}{\partial z} J_z - 2\frac{e}{m} B_o P_{yz} + \beta\nu P_{yy} = 0$$

$$\frac{\partial}{\partial t} R_y + \frac{p_o}{\rho_o} \frac{\partial}{\partial z} P_{yz} - \frac{e}{m} B_o R_z - \frac{3}{2} \frac{m}{e} \frac{p_o}{\rho_o} \Theta \nu J_y + \vartheta \nu R_y = 0$$

$$\frac{\partial}{\partial t} R_z + \frac{5}{2} \frac{p_o}{\rho_o} \frac{\partial}{\partial z} P - \frac{5}{2} \frac{p_o^2}{\rho_o^2} \frac{\partial}{\partial z} \rho - \frac{p_o}{\rho_o} \frac{\partial}{\partial z} (P_{xx} + P_{yy}) + \frac{e}{m} B_o R_y - \frac{3}{2} \frac{m}{e} \frac{p_o}{\rho_o} \Theta \nu J_z + \vartheta \nu R_z = 0 .$$

On assuming continuous plane waves of the form $e^{i(\omega t - \mathcal{K} z)}$, the dispersion equation is obtained by taking the 12 x 12 determinant of the coefficients (after replacement of the derivatives) equal to zero. However, the evaluation of this determinant is exceedingly complicated, and therefore to simplify the problem, collisions are ignored, the determinant only being evaluated for the particular case $\nu = 0$. It is interesting to note that the longitudinal-transverse fast mode is a compressible wave.

The dispersion relationship is:

(5.37) $$\mathcal{K}^8 \left(-\frac{21}{5} f^6 c^2 \right) + \mathcal{K}^6 \left(-\frac{6}{5} \omega_p^2 f^6 + \frac{21}{5} \omega^2 f^6 - \frac{257}{25} \omega_b^2 f^4 c^2 + \frac{257}{25} \omega^2 f^4 c^2 - \frac{77}{25} \omega_p^2 f^4 c^2 \right)$$

(ctd)

$$+ \mathcal{K}^4 \left(\frac{454}{25} \omega^2 \omega_p^2 f^4 - \frac{272}{25} \omega_p^4 f^4 - 18 \omega_b^2 \omega_p^2 f^4 \right.$$

$$+ \frac{257}{25} \omega^2 \omega_b^2 f^4 - \frac{257}{25} \omega^4 f^4 - \frac{48}{5} \omega_b^4 f^2 c^2 + \frac{81}{5} \omega^2 \omega_b^2 f^2 c^2$$

$$\left. - \frac{33}{5} \omega_p^2 \omega_b^2 f^2 c^2 - \frac{33}{5} \omega^4 f^2 c^2 + \frac{18}{5} \omega^2 \omega_p^2 f^2 c^2 \right)$$

$$+ \mathcal{K}^2 \left(\frac{116}{5} \omega^2 \omega_b^2 \omega_p^2 f^2 - \frac{28}{5} \omega_p^4 \omega_b^2 f^2 + \frac{13}{5} \omega^2 \omega_p^4 f^2 \right.$$

$$- 8 \omega_b^4 \omega_p^2 f^2 - \frac{46}{5} \omega^4 \omega_p^2 f^2 + \frac{48}{5} \omega^2 \omega_b^4 f^2 - \frac{81}{5} \omega^4 \omega_b^2 f^2$$

$$- 6 \omega_b^2 \omega^4 c^2 + \omega^6 c^2 - \omega^4 \omega_p^2 c^2 + 9 \omega_b^4 \omega^2 c^2$$

$$\left. + 5 \omega^2 \omega_b^2 \omega_p^2 c^2 - 4 \omega_b^6 c^2 - 4 \omega_p^2 \omega_b^4 c^2 \right)$$

$$+ \left(6 \omega^6 \omega_b^2 - \omega^8 + 2 \omega^6 \omega_p^2 - 9 \omega^4 \omega_b^4 - 10 \omega^4 \omega_b^2 \omega_p^2 \right.$$

$$+ 4 \omega_b^6 \omega^2 + 8 \omega_p^2 \omega^2 \omega_b^4 - \omega_p^4 \omega^4 + 5 \omega^2 \omega_p^4 \omega_b^2$$

$$\left. - 4 \omega_b^4 \omega_p^4 \right) \qquad = 0 .$$

5.3 Solutions of the Wave Dispersion Equations

All but one of the dispersion relationships given in the preceding section are quadratic equations in $\mathcal{K}^2$, and for these, exact solutions are possible. However, due to the general complexity of the solutions, various methods of successive approximation are adopted in order to emphasise dominant terms and effects. For instance, when collisions

are "dominant" the wave number $\mathcal{K}$ can be expressed as a rapidly convergent series expansion:

$$(5.38)\quad \mathcal{K} = a_0 + a_1/\nu + a_2/\nu^2 + \cdots .$$

In general, the first few terms of this expansion provide a good approximation for $\mathcal{K}$ when $a_1/a_0\nu << 1$. On the other hand, when collisions are "small", the expansion

$$(5.39)\quad \mathcal{K} = a_0 + a_1\nu + a_2\nu^2 + \cdots$$

applies, and $\mathcal{K}$ is adequately approximated by the first few terms provided $a_1\nu/a_0 << 1$. When a magnetic field is present, similar expansions used are in terms of the electron cyclotron frequency, ω_b.

In the case of the dispersion equations associated with purely transverse wave motion, (5.29) and (5.35), these quadratic equations in $\mathcal{K}^2$ can be written in the form

$$(5.40)\quad \alpha f^2 c^2 \mathcal{K}^4 + (\beta f^2 + \gamma c^2)\mathcal{K}^2 + \delta = 0$$

where the coefficients α, β, γ and δ are obtained directly on comparing the relevant equations with (5.40). Since $f^2 = kT/m$, even for $T \sim 10^8 K$, $f^2 \sim 10^{-2} c^2$ and therefore assuming an upper limit on T of this order (i.e. non-relativistic plasma), the following relationship holds:

$$(5.41)\quad \frac{f^2}{c^2} << 1 .$$

Applying the quadratic formula to equation (5.40), and neglecting terms of order f^2/c^2 compared with unity, gives

$$(5.42)\quad \mathcal{K}^2 = -\frac{\delta}{\gamma c^2}$$

and

$$(5.43)\quad \mathcal{K}^2 = -\frac{\gamma}{\alpha f^2} \quad .$$

The first solution, (5.42), is a fast wave mode, while the second solution, (5.43) is a stress mode. Physically this means that waves of kinetic origin and of electromagnetic origin are essentially distinct. However, when $f^2/c^2 \to 1$ there is obviously no clear-cut distinction between these two basic wave types.

5.3.1 Transverse Modes with No Applied Magnetic Field

The dispersion equation (5.29) applies to both of the uncoupled independent E_x and E_y transverse modes.

(a) Collisions "small"

On expanding $\mathcal{K}$ as in (5.39), i.e.

$$(5.44)\quad \mathcal{K} = a_0 + a_1\nu + a_2\nu^2 + \ldots$$

and using the approximation (5.41), the zero order (collisionless) solutions of the dispersion equation (5.29) are

$$(5.45)\quad a_0^2 = \frac{\omega^2 - \omega_p^2}{c^2}$$

and

$$(5.46)\qquad a_0^2 = \frac{5}{7}\,\frac{\omega^2}{f^2} \quad .$$

The fast wave solution, (5.45), is the well known collisionless result for microwave propagation in a homogeneous plasma. With a cut-off at $\omega = \omega_p$, this wave does not propagate for $\omega < \omega_p$. The phase velocity is clearly $c \Big/ \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$.

The second result, (5.46), is the solution for a collisionless stress wave (as mentioned previously, this solution occurs because of the anisotropic stress components included in the 13-moment equations). Since a_0^2 is always positive for this wave, there is no cut-off, the wave always propagating with phase and group velocities $(7/5)^{\frac{1}{2}} f$. Obviously it is a much slower mode than the fast wave.

It is evident from (5.46) that the stress wave is independent of the plasma frequency, ω_p. This suggests that a fast wave with frequency approaching cut-off at the plasma frequency may, with appropriate boundary conditions, excite stress waves at this frequency, and since the latter are more strongly damped by collisions (as will be shown), the fast wave would appear to be anomalously damped. Some experimental results favour this conclusion (Sharp and Hooker[5]).

Continuing the solutions of (5.29) to higher order terms in (5.44), the fast wave solution to first order in ν is

$$(5.47)\qquad \mathcal{K} = \pm \left(\left(\frac{\omega^2 - \omega_p^2}{c^2} \right)^{\frac{1}{2}} - \frac{i\,\omega_p^2\,\nu}{2\omega c\,(\omega^2 - \omega_p^2)^{\frac{1}{2}}} \right) .$$

This solution, valid for

$$\nu \ll \frac{2\omega(\omega^2 - \omega_p^2)}{\omega_p^2}, \tag{5.48}$$

shows that the effect of the collisions (to this order) is purely damping. As $\omega \to \omega_p$ the damping becomes severe.

For the stress wave, the solution to first order is

$$\kappa = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{\beta}{2} + \frac{\vartheta}{7} - \frac{3}{14}(\theta+\epsilon) + \frac{5}{14}\right)\right). \tag{5.49}$$

As before, the effect of the collisions is simply to damp the wave. This solution for the stress wave is valid when

$$\nu \ll \omega. \tag{5.50}$$

Again, retention of the coefficients β, ϑ, θ and ϵ indicates the relative effects of the various collision terms in the moment equations on the damping.

Replacing the $a_1\nu$ term (i.e. the second terms in equations (5.47) and (5.49)) by $-i\alpha$, the damping term is evidently

$$e^{-\alpha\nu z} \tag{5.51}$$

for the forward-going waves. The damping length, z_d, is conveniently defined as the distance in which the wave amplitude is reduced, owing to collisions, by a factor e. That is,

$$(5.52)\quad z_d = \frac{1}{\alpha \nu} \quad .$$

For the fast wave mode,

$$(5.53)\quad z_d = \frac{2\omega c\,(\omega^2 - \omega_p^2)^{\frac{1}{2}}}{\nu\,\omega_p^2}$$

and for the stress mode,

$$(5.54)\quad z_d = \left(\frac{7}{5}\right)^{\frac{1}{2}} \frac{28 f}{\nu(14\beta + 4\delta - 6(\theta+\epsilon) + 10)} \quad .$$

Since the mean free path is $\lambda_p = f/\nu$, for the stress wave $z_d \sim \lambda_p$ and therefore this mode is damped out in a few mean free path lengths. Except for wave frequencies ω near to ω_p , the ratio of the fast to stress wave damping lengths is

$$(5.55)\quad \sim \frac{\omega^2}{\omega_p^2} \cdot \frac{c}{f} >> 1 \ ,$$

verifying that the stress wave is much more strongly damped by collisions. By comparison, it is clear that when the inequality (5.48) is satisfied, the effect of collisions on the fast wave is slight except for frequencies near to the cut-off at $\omega = \omega_p$.

(b) Collisions "dominant"

Expanding $\mathcal{K}$ as in (5.38),

$$(5.56)\quad \mathcal{K} = a_0 + a_1/\nu + a_2/\nu^2 + \ldots \quad .$$

Substituting this into the dispersion equation (5.29) and grouping terms of the same order in ν^{-1} provides solutions as in the previous case, although of a different nature. As may be inferred from the form of the collision damping in the "small" collisions case, there is only one zero order solution,

$$(5.57) \quad a_o^2 = \frac{\omega^2}{c^2} ,$$

corresponding to the fast wave mode. There is no cut-off while assumption (b) applies. Comparing (5.57) with (5.45), the zero order solutions become identical when $\omega >> \omega_p$.

The solution to first order in ν^{-1} is

$$(5.58) \quad \varkappa = \pm \left(\frac{\omega}{c} - \frac{i \omega_p^2 \vartheta}{\nu c (2\vartheta - 9/5 \epsilon\theta)} \right) .$$

The effect of the collisions to this order, for the fast wave, is again pure damping, and the condition for validity of this solution is

$$(5.59) \quad \nu >> \omega_p^2 / \omega .$$

The damping length is

$$(5.60) \quad z_d = \frac{\nu c (2\vartheta - 9/5 \epsilon\theta)}{\omega_p^2 \vartheta} .$$

5.3.2 Longitudinal Waves with No Applied Magnetic Field

The dispersion equation is (5.32). As indicated previously, these waves are also called stress waves, partly for convenience. Various forms of the two modes that arise are well known; according to the level of the treatment, the particular frequency range, and the inclination of various authors, they may be called electron waves, electrostatic waves, pseudosonic waves, and possibly other terms as well.

Although a quadratic equation in $\mathcal{K}^2$, (3.32) does not take the form (5.40) and so the solutions (5.42) and (5.43) are not applicable. However, relatively simple solutions are obtained for two cases, $\omega >> \omega_p$ and $\omega << \omega_p$.

(a) Collisions "small"

$\mathcal{K}$ may be expanded as the power series in ν:

$$\mathcal{K} = a_0 + a_1\nu + a_2\nu^2 + \cdots \quad . \tag{5.61}$$

Substituting this expansion for $\mathcal{K}$ in (5.32) and grouping terms of the same order in ν, the zero order solutions are

$$a_0^2 = \frac{1}{30f^2}\left[26\omega^2 - 11\omega_p^2 \pm \left(376\omega^4 + 121\omega_p^4 - 272\omega^2\omega_p^2\right)^{\frac{1}{2}}\right] . \tag{5.62}$$

It is easily verified that the two solutions for a_0^2 are always distinct. It also follows that the solution

$$(5.63)\quad a_o^2 = \frac{1}{30f^2}\left[26\omega^2 - 11\omega_p^2 + (376\omega^4 + 121\omega_p^4 - 272\omega^2\omega_p^2)^{\frac{1}{2}}\right]$$

is always positive and hence represents a propagating wave with no cut-off. This is the slower of the two waves. The remaining solution,

$$(5.64)\quad a_o^2 = \frac{1}{30f^2}\left[26\omega^2 - 11\omega_p^2 - (376\omega^4 + 121\omega_p^4 - 272\omega^2\omega_p^2)^{\frac{1}{2}}\right],$$

has a cut-off at $\omega = \omega_p$ and represents a propagating wave for $\omega > \omega_p$. Direct expressions for a_o are obtained for the extreme cases:

(i) $\omega >> \omega_p$

In this case, the solutions for a_o^2 are

$$(5.65)\quad a_o^2 = \frac{1.513\,\omega^2 - .601\,\omega_p^2}{f^2}$$

and

$$(5.66)\quad a_o^2 = \frac{.220\,\omega^2 - .133\,\omega_p^2}{f^2}.$$

As already mentioned, both solutions are propagating modes. Again using the series expansion (5.61), to first order in ν the approximate solutions for $\mathcal{K}$ are

$$(5.67)\quad \mathcal{K} = \pm\left(\frac{1\cdot230\,\omega - \cdot244\,\omega_p^2/\omega}{f} - \frac{i\nu}{f}\left(\cdot244 + \cdot024\beta + \cdot371\vartheta + \cdot285(\theta+\epsilon)\right)\right)$$

$$(5.68)\quad \mathcal{K} = \pm\left(\frac{\cdot469\,\omega - \cdot142\,\omega_p^2/\omega}{f} - \frac{i\nu}{f}\left(\cdot096\beta + \cdot142 + \cdot093\vartheta - \cdot109(\theta+\epsilon)\right)\right).$$

In addition to condition (i), these solutions are valid provided that

$$(5.69)\quad \nu \ll \omega .$$

The first order part of these solutions gives pure collisional damping. Although the precise damping length is only obtained on evaluating β, ϑ, θ and ϵ, it is obvious that for both solutions the damping length is of order

$$(5.70)\quad z_d \sim \frac{f}{\nu} = \lambda_p ,$$

that is, the mean free path. This is of the same order as for the transverse stress waves. Also of interest is the fact that, since the respective longitudinal solutions (5.66) and (5.67) give speeds $\sim \cdot808f$ and $\sim 2\cdot132f$, one of these waves is faster and the other slower than the transverse stress waves, which have speed $(7/5)^{\frac{1}{2}} f$.

(ii) $\omega \ll \omega_p$

When this condition occurs, only one wave propagates, for which

$$(5.71)\quad a_0^2 = \frac{\cdot 454\ \omega^2}{f^2} \quad .$$

The propagating solution for $\mathcal{K}$ to first order in ν is

$$(5.72)\quad \mathcal{K} = \pm\left(\frac{\cdot 674\,\omega}{f} - \frac{i\nu}{f}(\cdot 354\,\mathcal{S} + \cdot 111\beta)\right).$$

The expansion (5.61) is valid for this solution when

$$(5.73)\quad \nu << \omega ,$$

the same as (5.69). Again, the solution (5.72) shows that the effect of collisions, to this order, is pure damping. The properties of this wave do not differ greatly from those of the same wave (i.e. (5.67)) in the extreme case (i), although the wave speed has become $1\cdot 557\, f$ and the damping length $f/\nu(\cdot 354\,\mathcal{S} + \cdot 111\beta)$.

(b) Collisions "dominant"

An inspection of the terms of highest order in ν in the dispersion equation (5.32) suggests that the relevant expansion for $\mathcal{K}$ is

$$(5.74)\quad \mathcal{K} = a_0\nu^{\frac{1}{2}} + a_1 + a_2\nu^{-\frac{1}{2}} + \cdots \quad .$$

With the usual method of seeking a solution of (5.32) in this form from consideration of the dominant terms in ν, the solutions for a_0^2 are given by

$$(5.75)\quad a_0^2 = -\frac{i\Phi^2}{f^2} ,$$

where

$$(5.76)\quad \phi^2 = \frac{\omega}{2}\left[\left(1+\vartheta+\frac{3}{5}(\Theta+\epsilon)\right) \pm \left(1+\vartheta^2+\frac{9}{25}\Theta^2+\frac{9}{25}\epsilon^2 + \frac{6}{5}\Theta-\frac{2}{5}\vartheta+\frac{6}{5}\Theta\vartheta+\frac{6}{5}\epsilon+\frac{6}{5}\epsilon\vartheta+\frac{72}{25}\epsilon\Theta\right)^{\frac{1}{2}}\right].$$

These do not truly represent propagating solutions, for the effective phase angle is $\pi/4$, i.e.

$$(5.77)\quad a_0 = \pm \frac{\Phi}{2^{\frac{1}{2}}f}(1-i),$$

the wave amplitude being reduced by a factor e for each wavelength travelled. For this reason there is no point in deriving the terms of the next order in $\nu^{-\frac{1}{2}}$.

5.3.3 Transverse Waves with an Applied Magnetic Field Parallel to the Direction of Propagation

Since a magnetic field parallel to the direction of propagation does not alter the dispersion equation for the longitudinal waves, the results of subsection 5.3.2 apply equally well in the present case for these waves. Therefore only the transverse waves need to be reconsidered for the parallel magnetic field, as indicated by the title of this subsection.

The coupled transverse wave dispersion equation was obtained in the subsection 5.2.2, where it was written essentially as two factors (equation (5.34)). It is apparent from these factors that the waves are coupled and "split" into two modes which are usually called the ordinary and extraordinary modes. In the collisionless case, these modes represent circularly polarised waves, that is, the electric vector

rotates with constant amplitude as the wave progresses. In particular, the mode with electric vector rotating in the same sense as the positive charge gyromagnetic motion is the ordinary or left hand wave, while that with electric vector rotating in the electron gyromagnetic sense is the extraordinary or right hand wave. For the ordinary wave, then,

$$\frac{E_y}{E_x} = i \,, \tag{5.78}$$

while for the extraordinary wave,

$$\frac{E_y}{E_x} = -i \,. \tag{5.79}$$

In the derivation of the dispersion equation (5.34), the equations (5.33) were reduced to the form

$$\begin{aligned} \phi E_x + \psi E_y &= 0 \\ \psi E_x - \phi E_y &= 0 \end{aligned} \tag{5.80}$$

for which the dispersion equation is

$$\phi^2 + \psi^2 = 0 \tag{5.81}$$

or

$$(\phi \pm i\psi) = 0 \,. \tag{5.82}$$

As may be verified, the upper sign, corresponding to the upper sign in equation (5.34), represents the ordinary wave, and the lower sign the extraordinary wave.

5.3.3 (A). The Ordinary Wave Modes

From the above discussion it follows that the ordinary dispersion equation is (5.34) with the upper (positive) sign being taken. It is a quadratic equation in $\mathcal{K}^2$ with the form (5.40) and therefore the solutions (5.42) and (5.43) apply. Nevertheless, it is still necessary to use expansions to obtain the dominant terms and principal properties of the wave number solutions for the particular cases.

(a) Collisions small

The appropriate expansion for $\mathcal{K}$ is the power series in ν :

$$\mathcal{K} = a_0 + a_1\nu + a_2\nu^2 + \cdots \quad . \tag{5.83}$$

The equations obtained for a_0 by grouping terms of lowest order in ν are easily solved. For the fast wave mode,

$$a_0^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2 + \omega\omega_b}\right). \tag{5.84}$$

This is the same solution as given by Spitzer[6], for example, for the ordinary wave, provided that the ion cyclotron frequency is taken to be small (as in this Chapter). It follows that the wave propagates for

$$\omega^2 + \omega\omega_b > \omega_p^2 \ , \tag{5.85}$$

with a cut-off at

$$\omega = \frac{-\omega_b + (\omega_b^2 + 4\omega_p^2)^{\frac{1}{2}}}{2} \quad . \tag{5.86}$$

When $\omega_b^2 >> \omega_p^2$, the cut-off is at

$$\omega \sim \omega_p^2 / \omega_b \, , \tag{5.87}$$

while for $\omega_p^2 >> \omega_b^2$, the cut-off is near the plasma frequency, at

$$\omega \sim \omega_p - \frac{\omega_b}{2} \quad . \tag{5.88}$$

The wave can propagate at all frequencies above the cut-off frequency, which is always less than ω_p , but it cannot propagate for any frequency below the cut-off. There are no resonances (i.e. when $\mathcal{K} \to \infty$) for this wave. The wave speed, $c/(1 - \frac{\omega_p^2}{\omega^2 + \omega\omega_b})^{\frac{1}{2}}$ is a function of the magnetic field strength, which "slows down" the wave as ω_b increases.

The zero order solution of the ordinary stress wave is

$$a_o^2 = \frac{5}{7} \frac{(\omega + \omega_b)^2}{f^2} \quad . \tag{5.89}$$

Evidently this mode always propagates with no cut-offs or resonances. The velocity, $\left(\frac{7}{5}\right)^{\frac{1}{2}} \frac{\omega f}{\omega + \omega_b}$, is a function of the magnetic field strength, which "slows down" this wave also.

The formal application of perturbation methods to obtain successively the coefficients in the expansion (5.83) is straightforward, although

the intermediate results are somewhat lengthy. Again with the neglect of terms of order f^2/c^2 compared with unity, the fast wave solution to first order in ν is

$$(5.90)\quad \mathcal{K} = \pm\left(\frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2+\omega\omega_b}\right)^{\frac{1}{2}} - \frac{i\nu\omega^{\frac{1}{2}}\omega_p^2}{2c(\omega+\omega_b)^{3/2}(\omega^2+\omega\omega_b-\omega_p^2)^{\frac{1}{2}}}\right).$$

As $\omega_b \to 0$, this result becomes identical with the non-magnetic transverse fast wave solution for the corresponding case of collisions "small", equation (5.47). There are no resonances in the pure damping part of (5.90), and in general, damping is not large except near the cut-off frequency, (5.86). The fast wave damping length is

$$(5.91)\quad z_d = \frac{2c(\omega+\omega_b)^{3/2}(\omega^2+\omega\omega_b-\omega_p^2)^{\frac{1}{2}}}{\omega^{\frac{1}{2}}\omega_p^2\nu},$$

and it is apparent that the magnetic field terms reduce the collisional damping as compared with the case $\omega_b \to 0$. When ω_b is large,

$$(5.92)\quad z_d \sim 2c\omega_b^2 / \omega_p^2\nu$$

and for ω_b small,

$$(5.93)\quad z_d \sim \frac{2c\omega(\omega^2-\omega_p^2)^{\frac{1}{2}}}{\omega_p^2\nu}\left(1 + \frac{\omega_b(4\omega^2-3\omega_p^2)}{2\omega(\omega^2-\omega_p^2)}\right).$$

The solution (5.90) is an adequate approximation for $\mathcal{K}$ provided that

$$(5.94)\quad \nu << \frac{2(\omega+\omega_b)(\omega^2+\omega\omega_b-\omega_p^2)}{\omega_p^2}.$$

For the stress wave the complete solution to first order in ν is found to be

$$(5.95)\quad \kappa = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega+\omega_b}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{5}{14}+\frac{\beta}{2}+\frac{\gamma}{7}-\frac{3}{14}(\theta+\epsilon)\right)\right).$$

Expansion (5.83) is valid for this solution when

$$(5.96)\quad \nu \ll \omega + \omega_b \quad .$$

The pure damping part of this solution being independent of ω_b, it would be expected that this term is the same as for the corresponding non-magnetic case; inspection of equation (5.49) shows that this is so.

Returning for a moment to the fast wave case, when terms of order f^2/c^2 are not neglected in obtaining the a_0 or a_1 of the expansion (5.83), it is found that

$$(5.97)\quad a_o^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega(\omega+\omega_b)}\right) - \frac{\omega^2 f^2}{c^4}\cdot\frac{\left(7\omega(\omega+\omega_b)-2\omega_p^2\right)^2}{140(\omega+\omega_b)^4} \quad .$$

Writing

$$(5.98)\quad a_1 = a_{1,0} + a_{1,1} \quad ,$$

where $a_{1,0}$ is the damping term involving ν in equation (5.90),

$$(5.99)\quad a_{1,1} = -\frac{i\omega^{\frac{1}{2}} f^2}{2c^3(\omega+\omega_b)^{5/2}(\omega^2+\omega\omega_b-\omega_p^2)^{1/2}}\left[\left(\frac{2}{5}\vartheta\omega\omega_p^2 - \left(\frac{2}{5}+\frac{3}{5}\Theta+\frac{3}{5}\epsilon\right)\omega\omega_p^2\right)\frac{\omega(\omega^2+\omega\omega_b-\omega_p^2)}{(\omega+\omega_b)^2}\right.$$

$$+\frac{\omega^2(7\omega(\omega+\omega_b)-2\omega_p^2)^2(1+\beta)}{140(\omega+\omega_b)^2}$$

$$\left.+\frac{\omega^2\vartheta}{70(\omega+\omega_b)^2(\omega-\omega_b)}\left(49\omega^5+49\omega^3\omega_b^2-196\omega^3\omega_p^2+98\omega^4\omega_b+172\omega\omega_p^4+28\omega\omega_b^2\omega_p^2-28\omega_p^4\omega_b\right)\right].$$

The interesting point is that within these terms associated with f^2/c^2 there is a resonance at $\omega = \omega_b$, namely in the terms underlined. Unimportant for all other frequencies because $f^2/c^2 << 1$, it follows from the inclusion of these terms that there is resonant damping of the ordinary fast wave at ω_b. Of course the resonant damping term for this wave is $\sim f^2/c^2$ of the resonant damping of the extraordinary fast wave near $\omega = \omega_b$.

(b) Collisions "dominant"

The relevant expansion for $\mathcal{K}$ becomes

$$(5.100)\quad \mathcal{K} = a_0 + a_1/\nu + a_2/\nu^2 + \ldots ,$$

as in the non-magnetic case. There is only one highest order solution in ν (that is, for a_0), from equation (5.34) (with upper sign taken), and this corresponds to a fast wave:

$$(5.101)\quad a_0^2 = \frac{\omega^2}{c^2} .$$

It appears, at least to this order, that effects of the magnetic field and the plasma frequency are "swamped out" by the collisions. The ordered kinetic motion involved in the propagation of stress waves is destroyed by the randomising effect of the high frequency of collisions.

The complete solution to first order in ν^{-1} is

$$(5.102)\quad \kappa = \pm \left(\frac{\omega}{c} - \frac{i \vartheta \omega_p^2}{2\nu c \left(\vartheta - \frac{9}{10}\Theta\epsilon\right)} \right)$$

and this is valid for

$$(5.103)\quad \nu >> \omega_p^2 / \omega$$

$$\nu >> \omega_b .$$

Again, being independent of ω_b, (5.102) is identical with the corresponding result for the non-magnetic transverse waves. Provided $\vartheta > \frac{9}{10}\Theta\epsilon$ (which, physically must be true when this approximation applies), the second term in (5.102), involving ν^{-1}, represents pure damping. The second of the conditions (5.103) for validity of the solution (5.102) is not a deduction from this equation or similar solutions; however it is evident on grouping the terms with respect to powers of ν that such a condition must apply.

(c) Magnetic field "dominant"

In considering the case of a strong applied magnetic field, the appropriate expansion for κ analogous to (5.100) is

$$(5.104)\quad \kappa = a_0 + a_1 / \omega_b + a_2 / \omega_b^2 + \cdots .$$

Successive approximations are obtained by grouping terms of the same order in the dispersion equation (5.34) after using (5.104) for $\mathcal{K}$.

For this situation also, there is only a fast wave solution, namely

$$\mathcal{K} = \pm \left(\frac{\omega}{c} - \frac{\omega_p^2}{2c\omega_b} \right) . \tag{5.105}$$

This wave has velocity c to lowest order in ω_b^{-1} ; the first order term affects only the wave speed (increasing it) and the wavelength (increasing it), collisions being "swamped out" by the field. The disappearance of the stress waves is presumably because the particle motion is dominated by the magnetic field, which largely "blocks" thermal motion. The solution (5.105) is applicable for

$$\omega_b >> \omega_p^2 / 2\omega , \tag{5.106}$$

$$\omega_b >> \nu$$

(as in (b) above, the last conditions of (5.106) is necessary for the appropriate ordering of the terms in using (5.104) to obtain a solution of (5.34)).

(d) Magnetic field "weak"

Using the expansion

$$\mathcal{K} = a_0 + a_1 \omega_b + a_2 \omega_b^2 + \cdots , \tag{5.107}$$

it follows that the solutions for a_0 are the complete solutions of the non-magnetic transverse wave dispersion equation, (5.29). In

considering the solutions for the non-magnetic case, it is necessary to consider separately the two approximations, ν "large", and ν "small", and this applies to the present case also.

(i) Collisions "small"

There is a fast wave and also a stress wave solution for a_0 in this case. Beginning with the fast wave for which

$$(5.108)\quad a_0 = \pm\left(\left(\frac{\omega^2-\omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{i\nu\omega_p^2}{2\omega c(\omega^2-\omega_p^2)^{\frac{1}{2}}}\right),$$

on neglecting terms of order ν^2, ω_b^2, $\nu\omega_b$ and also f^2/c^2, the complete fast wave solution of the dispersion equation to first order in ν and ω_b is

$$(5.109)\quad \kappa = \pm\left(\left(\frac{\omega^2-\omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{i\nu\omega_p^2}{2\omega c(\omega^2-\omega_p^2)^{1/2}} + \frac{\omega_b\omega_p^2}{2\omega c(\omega^2-\omega_p^2)^{1/2}}\right).$$

This is a valid solution provided that

$$(5.110)\quad \nu << \frac{2\omega(\omega^2-\omega_p^2)}{\omega_p^2},$$

$$\omega_b << \frac{2\omega(\omega^2-\omega_p^2)}{\omega_p^2}.$$

The effect of collisions is the same as in the non-magnetic transverse fast wave solution (5.47). There are no resonances, while the cut-off is depressed by the magnetic field to

$$(5.111)\quad \omega = \omega_p - \frac{\omega_b}{4\omega_p^2}.$$

The solution corresponding to (5.49) is:

$$(5.112)\quad a_0 = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{\vartheta}{7}+\frac{\beta}{2}+\frac{5}{14}-\frac{3}{14}(\theta+\epsilon)\right)\right).$$

With the same approximations as for (5.109), to first order in ω_b and ν the stress wave solution is

$$(5.113)\quad \mathcal{K} = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{\vartheta}{7}+\frac{\beta}{2}+\frac{5}{14}-\frac{3}{14}(\theta+\epsilon)\right) + \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega_b}{f}\right).$$

This solution is identical to (5.95), showing that the restriction to "small" magnetic field is unnecessary, and therefore this is a valid solution provided that

$$(5.114)\quad \nu << \omega \quad .$$

(This is not the case, however, with the fast wave solution, (5.109)).

(ii) Collisions "large"

When ν is "large" there is no stress wave, while the fast wave solution follows from (5.107), (5.58), and (5.34), with

$$(5.115)\quad a_0 = \pm\left(\frac{\omega}{c} - \frac{i\omega_p^2\vartheta}{2\nu c(\vartheta-\frac{9}{10}\theta\epsilon)}\right).$$

The solution to first order in ν^{-1} and ω_b is

$$(5.116)\quad \mathcal{K} = \pm\left(\frac{\omega}{c} - \frac{i\omega_p^2\vartheta}{2\nu c(\vartheta-\frac{9}{10}\theta\epsilon)} + \frac{\omega_b\omega_p^2\vartheta}{2\beta c\nu^2(\vartheta-\frac{9}{10}\theta\epsilon)}\right).$$

Collisional damping in this solution corresponds to that for (5.58), having already been discussed. The effect of the magnetic field is to reduce the wave velocity, but there is no cut-off, and the presence of a factor ν^{-2} in the magnetic field term indicates that this term will be small. (5.115) is a valid solution when

$$\nu \gg \omega_p^2/\omega \qquad (5.117)$$

$$\omega_b \ll \omega\nu^2/\omega_p^2 .$$

5.3.3 (B) The Extraordinary Wave Modes

The dispersion equation for the extraordinary modes is equation (5.34) with the lower (negative) sign being taken. Similarly to the ordinary wave dispersion equation, this can be written in the form (5.40) and therefore has solutions (5.42) and (5.43), for $\mathcal{K}^2$. The α, β, γ and δ are of course different from those for the ordinary wave equations. In general the methods of obtaining approximate solutions for this case parallel those used for the ordinary waves.

(a) Collisions "small"

Using

$$\mathcal{K} = a_0 + a_1\nu + a_2\nu^2 + \cdots , \qquad (5.118)$$

the zero order solution of the extraordinary wave equation (5.34), for the fast wave, is

$$a_0^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega(\omega-\omega_b)}\right) , \qquad (5.119)$$

where terms of order f^2/c^2 have been neglected. This solution (5.118) is the well known extraordinary fast wave for the high frequency collisionless case (for example, Spitzer[6]). A fundamental difference between this solution and the corresponding ordinary solution (5.84) is apparent, namely the resonance at $\omega = \omega_b$. This is the frequency at which the electron gyromagnetic motion is rotation in phase with the rotating electric vector. Furthermore, there is a cut-off of the wave at

$$\omega = \frac{\omega_b + (\omega_b^2 + 4\omega_p^2)^{1/2}}{2}, \tag{5.120}$$

below which the wave can again propagate. For ω_b large, this cut-off is at

$$\omega \sim \omega_b + \frac{2\omega_p^2}{\omega_b}, \tag{5.121}$$

while for ω_b small, the cut-off is elevated above the plasma frequency to

$$\omega \sim \omega_p + \frac{\omega_b}{2}. \tag{5.122}$$

Whenever

$$1 - \frac{\omega_p^2}{\omega(\omega - \omega_b)} > 0 \tag{5.123}$$

the wave can propagate, and therefore for $\omega > \omega_b$ this propagation condition becomes

$$\omega > \frac{\omega_p^2}{|\omega - \omega_b|}, \tag{5.124}$$

while for $\omega < \omega_b$ the propagation condition is

$$(5.125) \qquad \omega < \frac{\omega_p^2}{|\omega_b - \omega|} ,$$

showing that the wave propagates at frequencies down to zero. At any given frequency above the extraordinary wave cut-off, the extraordinary fast wave phase velocity is greater than that of the ordinary wave.

On reaching a cut-off (the phase velocity is "infinite") the wave is normally reflected, although strong collisional damping also occurs (equation (5.47) for example). At a resonance (where the wave speed is "zero"), either reflection or absorption may occur. For waves in a region of inhomogeneous magnetic field tending to the resonance condition, Stix[7] has briefly discussed certain particular cases.

Completing the solution for $\mathcal{K}$ to first order in ν gives:

$$(5.126) \qquad \mathcal{K} = \pm \left(\frac{\omega}{c} \left(1 - \frac{\omega_p^2}{\omega(\omega - \omega_b)} \right)^{\frac{1}{2}} - \frac{i \nu \omega^{\frac{1}{2}} \omega_p^2}{2c(\omega - \omega_b)^{3/2} (\omega^2 - \omega\omega_b - \omega_p^2)^{1/2}} \right) .$$

Evidently the collision term also resonates at $\omega = \omega_b$. As already mentioned, this resonance is when the electrons rotate at the same angular speed as the electric vector, and therefore they experience a "constant" electric field from which they absorb considerable energy, subsequently losing it in collisions. (It may be mentioned that a similar type of mechanism is thought to produce Landau damping).

A collisional resonance effect was also found for the ordinary wave mode by retaining terms of order f^2/c^2 ; this gave a resonance

damping term $\sim f^2/c^2$ that of the collision term in (5.126). For this case, the damping length is

$$(5.127)\quad z_d = \frac{2c(\omega-\omega_b)^{3/2}(\omega^2-\omega\omega_b-\omega_p^2)^{1/2}}{\nu\,\omega_p^2\,\omega^{1/2}}.$$

Away from the resonance, and away from the cut-off (which also produces strong damping), the collision damping of the extraordinary fast waves is slight.

The zero order term in (5.118) for the extraordinary stress wave is given by

$$(5.128)\quad a_o^2 = \frac{5}{7}\frac{(\omega-\omega_b)^2}{f^2}.$$

In contrast to the ordinary stress wave for the corresponding case, this mode has a cut-off at $\omega = \omega_b$, coinciding with the resonance frequency of the fast wave. The stress mode, however, has no resonances and propagates on both sides of the cut-off.

With the first order terms in ν included, the stress wave solution for κ is

$$(5.129)\quad \kappa = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{|\omega-\omega_b|}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{\beta}{2}+\frac{8}{7}+\frac{5}{14}-\frac{3}{14}(\theta+\epsilon)\right)\right).$$

The pure damping part of this solution, being independent of ω_b is the same as that for the ordinary stress wave. Also, as with the fast waves, the extraordinary stress waves have a phase velocity greater than the corresponding ordinary modes, in this case by a factor $(\omega+\omega_b)/(\omega-\omega_b)$. The solution (5.129) is valid when

$$(5.130)\quad \nu << |\omega-\omega_b|.$$

(b) Collisions "large"

The expansion for $\mathcal{K}$ and the results for this case are identical with those for the same assumption with the ordinary wave modes, equations (5. 100) to (5. 103). Only the fast waves exist, for which the solution is

$$\mathcal{K} = \pm \left(\frac{\omega}{c} - \frac{i \vartheta \omega_p^2}{2\nu c (\vartheta - \frac{9}{10}\Theta\epsilon)} \right) . \tag{5.131}$$

(c) Magnetic field "large"

Using the expansion

$$\mathcal{K} = a_0 + a_1/\omega_b + a_2/\omega_b^2 + \ldots \tag{5.132}$$

in equation (5. 34)(with lower sign), a_0 is given by

$$a_0^2 = \frac{\omega^2}{c^2} . \tag{5.133}$$

This zero order solution is identical to the corresponding ordinary wave solution in (5. 105), while again there is no stress wave. On the other hand, the first order term differs in sign from that in (5. 105); to this order, the extraordinary solution is

$$\mathcal{K} = \pm \left(\frac{\omega}{c} + \frac{\omega_p^2}{2c\omega_b} \right) . \tag{5.134}$$

By definition of this case (c), (5. 134) is a wave propagation below (the extraordinary) cut-off, while solution (5. 105) of course represents an ordinary wave above its cut-off. Therefore, unlike the situation

where both waves are above respective cut-offs, the ordinary wave is faster than the extraordinary wave, by a factor $(\omega_p^2 + 2\omega\omega_b)/(2\omega\omega_b - \omega_p^2)$.

(d) Magnetic field "small"

This is the final case to be considered in this subsection. The appropriate expansion for $\mathcal{K}$ is

$$\mathcal{K} = a_0 + a_1\omega_b + a_2\omega_b^2 + \cdots \quad . \tag{5.135}$$

As with the ordinary wave, the zero order solution is the complete non-magnetic transverse wave solution. This takes into account collisions.

(i) Collisions "small"

The zero order fast wave solution is clearly equation (5.47), which may be rewritten as

$$a_0 = \pm\left(\left(\frac{\omega^2 - \omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{i\nu\omega_p^2}{2\omega c(\omega^2 - \omega_p^2)^{1/2}}\right). \tag{5.136}$$

Using the expansion (5.135) in equation (5.34) (with the lower sign) and neglecting terms involving ω_b^2, $\omega_b\nu$, ν^2 and f^2/c^2 gives the fast wave solution:

$$\mathcal{K} = \pm\left(\left(\frac{\omega^2 - \omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{i\nu\omega_p^2}{2\omega c(\omega^2 - \omega_p^2)^{1/2}} - \frac{\omega_p^2\omega_b}{2\omega c(\omega^2 - \omega_p^2)^{1/2}}\right). \tag{5.137}$$

This expression differs from the ordinary wave solution for the corresponding situation only in the sign of the term involving ω_b. It is valid provided

(5.138) $$\nu << \frac{2\omega(\omega^2 - \omega_p^2)}{\omega_p^2}$$

$$\omega_b << \frac{2\omega(\omega^2 - \omega_p^2)}{\omega_p^2},$$

the same condition as for the validity of the ordinary wave solution (5.109). Apart from mentioning that this mode propagates with a higher speed than the corresponding ordinary wave, the discussion for (5.109) applies.

Similarly, the stress wave solution to the same order is

(5.139) $$K = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f} - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{i\nu}{f}\left(\frac{9}{7} + \frac{\beta}{2} + \frac{5}{14} - \frac{3}{14}(\theta+\epsilon)\right) - \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega_b}{f}\right),$$

valid for

(5.140) $$\nu << \omega$$

$$\omega_b << \omega .$$

Again, this expression differs from the ordinary solution (5.113) only in the sign of the term containing ω_b. The discussion following equation (5.113) applies also to (5.139) (including the fact that this solution is valid for the relaxed restriction, as compared with (5.140), $\nu << \omega$).

(ii) Collisions "large"

There is no stress wave for this case, and in fact the solution for the fast wave is the same as the corresponding ordinary fast mode, (5.116), except in the sign of the term involving ω_b. This solution is

$$(5.141)\quad \mathcal{K} = \pm\left(\frac{\omega}{c} - \frac{i\omega_p^2 \mathcal{S}}{2\nu c(\mathcal{S} - \frac{9}{10}\Theta\epsilon)} - \frac{\omega_p^2 \omega_b \mathcal{S}}{2\beta c\nu^2(\mathcal{S} - \frac{9}{10}\Theta\epsilon)}\right),$$

representing a faster wave than the solution (5.116). The criterion for the validity of (5.141) are identical to (5.117):

$$(5.142)\quad \nu \gg \omega_p^2 / \omega$$

$$\omega_b \ll \omega\nu^2 / \omega_p^2 \ .$$

5.3.4 Longitudinal-Transverse Waves with Applied Magnetic Field Perpendicular to the Direction of Propagation

The nature of the dispersion equation for these waves has been discussed in subsection 5.2.3, and the equation, of fourth order in $\mathcal{K}^2$, derived for the particular case of collisions negligible, is (5.36). This equation is essentially a coupling between modes of different types, and it does not represent phenomena similar to those associated with a parallel magnetic field. In general, solutions for extreme cases of weak or strong magnetic fields can be obtained by successive approximations, despite the non-quadratic nature of the equation, by making use of the corresponding solutions obtained for the uncoupled (non-magnetic) modes.

(a) Magnetic field "weak"

The relevant expansion for $\mathcal{K}$ in this case is

$$(5.143)\quad \mathcal{K} = a_0 + a_1 \omega_b^2 + a_2 \omega_b^4 + \ldots \quad .$$

On substituting this into the dispersion equation (5.36) and grouping terms of the same order in ω_b, successive approximations for a_0, a_1, ... are obtained. Physically it is evident that for the zero order term, a_0, the dispersion equation is effectively the product of the non-magnetic transverse and longitudinal dispersion equations. The four solutions for a_0^2 are therefore those of equations (5.45), (5.46), (5.63), and (5.64), and may be rewritten as:

$$(5.144)\quad a_0^2 = \frac{\omega^2 - \omega_p^2}{c^2}$$

$$(5.145)\quad a_0^2 = \frac{5}{7}\frac{\omega^2}{f^2}$$

$$(5.146)\quad a_0^2 = \frac{\omega^2}{30f^2}\left[26 - 11\frac{\omega_p^2}{\omega^2} + \left(376 + 121\frac{\omega_p^4}{\omega^4} - 272\frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}\right]$$

$$(5.147)\quad a_0^2 = \frac{\omega^2}{30f^2}\left[26 - 11\frac{\omega_p^2}{\omega^2} - \left(376 + 121\frac{\omega_p^4}{\omega^4} - 272\frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}\right].$$

These equations give the basic propagation properties for the modes; obviously in the present case the properties are much the same as for the respective non-magnetic collisionless modes discussed previously. (5.144) is a fast wave, while (5.145) to (5.147) are stress modes.

(i) The fast mode

After a lengthy calculation using (5.143) in (5.36), with the neglect of terms in f^2/c^2, the fast wave solution corresponding to (5.144)

as zero order solution is found to be

$$(5.148)\quad \mathcal{K} = \pm\left(\left(\frac{\omega^2-\omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{\omega_p^2\,\omega_b^2}{2c\,(\omega^2-\omega_p^2)^{3/2}}\right).$$

It follows from this result that the effect of the magnetic field is to "speed up" the wave. The solution (5.148) is valid when

$$(5.149)\quad \omega_b^2 \ll \frac{2\,(\omega^2-\omega_p^2)^2}{\omega_p^2}\,.$$

In view of this restriction, the resonance which occurs in the last term of (5.148), at the same frequency as the cut-off in the first term, is probably of no significance. However, the effect of the magnetic field, through the last term in (5.148), is to shift the cut-off of the wave from $\omega = \omega_p$ to

$$(5.150)\quad \omega^2 = \omega_p^2 \pm \frac{\omega_p\omega_b}{2^{1/2}}\,.$$

This cut-off occurs for each of the signs in (5.150), and while the wave obviously cannot propagate below $\omega^2 = \omega_p^2 - \omega_p\omega_b/2^{1/2}$, the behaviour of the wave in the region between the cut-offs is not entirely clear.

(ii) The transverse stress mode

With a_o given by (5.145), the solution for this stress mode is

$$(5.151)\quad \mathcal{K} = \pm\left(\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f} + \left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega_b^2}{\omega f}\left(\frac{8\omega^2+7\omega_p^2}{58\omega^2+28\omega_p^2}\right)\right).$$

This solution is valid when

(5.152) $\omega_b << \dfrac{\omega(58\omega^2 + 28\omega_p^2)}{8\omega^2 + 7\omega_p^2}$.

There are no resonances or cut-offs of this wave. It is easily seen that the effect of the magnetic field is speed up this wave. Also, unlike the non-magnetic transverse stress waves, this solution is dispersive, with the group velocity no longer equal to the phase velocity.

(iii) The longitudinal stress modes

The remaining cases, with a_o given by (5.146) and (5.147) are grouped together, since they have direct zero order solutions only when $\omega >> \omega_p$ or $\omega << \omega_p$. Taking the former conditions first, the solutions are obtained after considerable algebra. In particular, with $\omega >> \omega_p$, to first order in ω_b , the two solutions are

(5.153) $$\mathcal{K} = \pm\left(\frac{1{\cdot}23\,\omega}{f} - \frac{{\cdot}244\,\omega_p^2}{\omega f} + \frac{\omega_b^2}{\omega f}\left[\frac{22{\cdot}77\,\omega^2 - 6{\cdot}13\,\omega_p^2}{13{\cdot}11\,\omega^2 + 22{\cdot}51\,\omega_p^2}\right]\right)$$

$$\simeq \pm\left(\frac{1{\cdot}23\,\omega}{f} - \frac{{\cdot}244\,\omega_p^2}{\omega f} + \frac{1{\cdot}736\,\omega_b^2}{\omega f}\right)$$

and

(5.154) $$\mathcal{K} \simeq \pm\left(\frac{{\cdot}469\,\omega}{f} - \frac{{\cdot}142\,\omega_p^2}{\omega f} - \frac{1{\cdot}247\,\omega_b^2}{\omega f}\right).$$

These solutions are valid for

(5.155) $\omega_b << \omega$

$\omega_p << \omega$.

While little more can be said about these solutions, it may be noted that (5.153), the slower of the two waves, has its speed further decreased by the magnetic field, while the faster wave, (5.154) is "speeded up" by the field.

The second approximation, $\omega << \omega_p$, gives only one propagating solution, and again to first order in ω_b, this is found to be

$$(5.156)\quad \mathcal{K} = \pm \left(\frac{.674\,\omega}{f} - \frac{1.848\,\omega_b^2}{\omega f} \right).$$

This solution is valid for

$$(5.157)\quad \omega_b << \omega$$

$$\omega << \omega_p .$$

It has already been mentioned that this solution is essentially the counterpart of (5.154), and like the latter wave, is "speeded up" by the magnetic field.

To a certain extent, it is possible to take account of collisions indirectly for these cases, provided the assumption 'collisions "small"' applies. This is by using the first order 'collisions "small"' solutions for the respective non-magnetic wave solutions for the a_o instead of the zero order (in ν) solutions used. That is, instead of using (5.45), (5.46), (5.63) and (5.64) in equations (5.144) to (5.147), use (5.47), (5.49), and (5.67), (5.68), or (5.72). The collision effects in this case are of course identical with the corresponding non-magnetic solutions.

(b) Magnetic field "large"

This is the final case to be considered in this Chapter. The relevant expansion for $\mathcal{K}$ is

$$(5.158)\quad \mathcal{K} = a_0 + a_1/\omega_b^2 + a_2/\omega_b^4 + \cdots \quad .$$

Using this expansion with (5.36), the highest order terms give only one solution,

$$(5.159)\quad a_0^2 = \frac{\omega^2}{c^2} \quad .$$

Since this fast wave solution is the only zero order solution of (5.36) consistent with (5.158), it is apparent that a strong magnetic field "swamps out" longitudinal as well as transverse stress waves in confining the motion of the particles. Neglecting terms in f^2/c^2, to first order in ω_b^{-2} the solution is

$$(5.160)\quad \mathcal{K} = \pm\left(\frac{\omega}{c} + \frac{\omega_p^2(\omega^2-\omega_p^2)}{2\omega\omega_b^2 c}\right) .$$

This solution is valid under the condition

$$(5.161)\quad \omega_b^2 >> \omega_p^2\left(1 - \frac{\omega_p^2}{\omega^2}\right) .$$

There are of course no resonances, and the cut-off at $\omega = \omega_p$ has also been lost. The speed of this wave also is decreased by the magnetic fields, provided that $\omega > \omega_p$.

5.4 Discussion

In concluding this Chapter, several points may be noted. The results for the usual electromagnetic waves are similar to those obtained in other treatments (although few authors give the collision terms in the direct form used in this Chapter for the particular cases). Of other methods, the most exact is to obtain solutions of the linearised Boltzmann equation, using these to determine the equivalent dielectic tensor in Maxwell's equations. However, although such an approach is mathematically correct, it is extremely difficult to account for most types of collisions. Of all possible types of elastic and inelastic collisions, only inverse fifth power law interactions are easily included. On the other hand, the present approach is such that the collision effects for all possible binary encounters are automatically included. Furthermore, the various phenomena associated with thermo-electric effects, viscosity, thermal conduction, etc. are easily identified.

Again, although all stress-type waves are probably included in a direct Boltzmann approach, they are not readily recognised. In particular, the transverse stress waves which are associated with the 13-moment equations are essentially unique to this form of treatment (the case of low frequency stress waves has been discussed by Liley[3]). One of the most important aspects of the approach adopted in this Chapter, however, is the coupling between the electromagnetic and kinetic properties of a plasma. This is easily apprehended, and the strength of the coupling is of order f^2 / c^2 . In most practical cases, such coupling is unimportant, and for this reason it has only been considered in detail for one or two special cases (e. g. resonance damping). Nevertheless, at temperatures of order 10^9 degrees Kelvin, which are of interest in CTR work, the coupling factor is $f^2/c^2 \sim 10^{-1}$

and significant perturbations of the "pure" electromagnetic modes should be observable. Although solutions for this near-relativistic case have not been considered explicitly, the dispersion equations given in the Chapter are applicable.

Finally, as an application of the results of this Chapter, Faraday rotation of the polarisation of both fast and stress waves is discussed in Appendix D. Explicit calculations are given for several of the wave solutions obtained in the Chapter, both with magnetic fields parallel and perpendicular to the direction of propagation.

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Chapter 6

Wave Motion in Inhomogeneous Plasma

6.1 Introduction

As already noted, the 13-moment equations are adequate to describe the behaviour of a plasma even at moderate departures from equilibrium. Therefore it is interesting to consider the application of these equations to wave motion in inhomogeneous plasma. In this Chapter, the propagation of several types of waves in a steady state inhomogeneous plasma is examined.

In particular, this Chapter may in part be regarded as an investigation of the effects of the higher moment equations on the usual electromagnetic wave results for an inhomogeneous plasma, while analogous solutions for the stress waves are also derived. To the degree of approximation used, reflections or coupling between independent modes are not necessarily expected, although this problem is briefly discussed in the final section. Again, as found in the preceding Chapter, it is to be expected that the "kinetic" terms due to the higher order moment equations influence the Maxwell-type waves to order f^2/c^2, and are, therefore, again only likely to be of importance in high temperature plasma.

Solutions are obtained to the approximation of geometrical optics. A relatively simple form of inhomogeneity, where the density and pressure gradients are non-zero, is considered, while collision terms are neglected. In the case of an applied magnetic field, solutions are only obtained for the field parallel to the direction of propagation.

6.2 The Steady State Inhomogeneous Plasma

Any possible state of the plasma is a solution of the moment-Maxwell equation system to the extent that these equations are valid. For the present Chapter, since collisions are neglected, the relevant form of the 13-moment equations (similar to Chapter 5), is:

$$(6.1)\quad \frac{\partial \rho}{\partial t} + \frac{m}{e}\nabla\cdot\underline{J} = 0$$

$$\frac{m^2}{e^2}\frac{\partial \underline{J}}{\partial t} + \frac{m}{e}\nabla\cdot\underset{\sim}{P} + \frac{m}{e}\nabla p - \frac{m}{e}\underline{J}\times\underline{B} - \rho\underline{E} = 0$$

$$\frac{\partial}{\partial t}\overset{3}{\underset{2}{}}p + \nabla\cdot\underline{R} + \frac{5}{2}\frac{m}{e}\frac{p}{\rho}\nabla\cdot\underline{J} + \frac{5}{2}\frac{m}{e\rho}\underline{J}\cdot\nabla p$$

$$- \frac{5}{2}\frac{m}{e}\frac{p}{\rho^2}\underline{J}\cdot\nabla\rho - \underline{J}\cdot\underline{E} = 0$$

$$\frac{\partial}{\partial t}\underset{\sim}{P} + \frac{4}{5}\left\{\nabla\underline{R}\right\} + \frac{2m}{e}\frac{p}{\rho}\left\{\nabla\underline{J}\right\} + \frac{2m}{e\rho}\left\{\underline{J}\nabla p\right\}$$

$$- \frac{2mp}{e\rho^2}\left\{\underline{J}\nabla\rho\right\} - 2\left\{\underline{E}\,\underline{J}\right\} - \frac{2e}{m}\left\{\underset{\sim}{P}\times\underline{B}\right\} = 0$$

$$\frac{\partial \underline{R}}{\partial t} + \frac{7}{2}\frac{\underset{\sim}{P}\cdot\nabla p}{\rho} - \frac{7}{2}\frac{p}{\rho^2}\underset{\sim}{P}\cdot\nabla\rho + \frac{5}{2}\frac{p}{\rho}\nabla p - \frac{5}{2}\frac{p^2}{\rho^2}\nabla\rho$$

$$+ \frac{p}{\rho}\nabla\cdot\underset{\sim}{P} + \frac{5}{2}\frac{m}{e}\frac{\underline{J}}{\rho}\frac{\partial p}{\partial t} - \frac{5}{2}\frac{m}{e}\frac{p}{\rho^2}\underline{J}\frac{\partial \rho}{\partial t}$$

$$- \frac{e}{m}\underset{\sim}{P}\cdot\underline{E} - \frac{e}{m}\underline{R}\times\underline{B} = 0\ .$$

The notation used in these equations, and in this Chapter, is identical to that of Chapter 5, including the neglect of subscripts for the electron plasma component.

The simplest solution of these equations and the associated Maxwell's equations is an equilibrium homogeneous plasma, with

$$\text{(6.2)} \quad \underline{J} = 0 \; ; \quad \underset{\sim}{P} = 0 \; ; \quad \underline{R} = 0 \; ;$$

$$p = \text{const.} \; ; \quad \rho = \text{const.}$$

Consider instead an inhomogeneous plasma with a density gradient maintained (for example) by an applied electric field. A solution of (6.1) is easily obtained. Let the density gradient have characteristic length L, i.e.,

$$\text{(6.3)} \quad \left| \frac{\nabla \rho}{\rho} \right| = \frac{1}{L} \quad .$$

Then, in terms of L, the dynamical parameters of the steady state inhomogeneous plasma are:

$$\text{(6.4)} \quad \underline{E}_0 = \frac{m p_0}{e \rho_0} \frac{\hat{\underline{r}}}{L} \equiv \underline{F} \; ,$$

$$\nabla \rho_0 = \frac{\rho_0}{L} \hat{\underline{r}} = \frac{e}{m} \frac{\rho_0^2}{p_0} \underline{F} \; ,$$

$$\nabla p_0 = \frac{p_0}{L} \hat{\underline{r}} = \frac{e}{m} \rho_0 \underline{F} \; ,$$

$$\underline{J}_0 = 0 \; , \quad \underset{\sim}{P}_0 = 0 \; , \quad \underline{R}_0 = 0 \; ,$$

where $\hat{\underline{r}}$ is a unit vector in the direction of the applied electric field. The subscript o is used to signify steady state parameters, and the vector $\underline{F}$ denotes the steady state electric field. Although the temperature is not used explicitly in these equations, it should be noted that

$$(6.5) \qquad \nabla T_o = \frac{m}{k} \nabla\left(\frac{p_o}{\rho_o}\right) = \frac{m}{k}\left(\frac{1}{\rho_o}\nabla p_o - \frac{p_o}{\rho_o^2}\nabla\rho_o\right) = 0 ,$$

the last relation in (6.5) following of course from (6.4). As usual, k is Boltzmann's constant.

6.3 The Wave Equations for Inhomogeneous Plasma

As with the equations for wave motion in homogeneous plasma, the moment equations for the wave variables are linearised. Again, unsubscripted symbols are used for the variables which represent departures from the steady state (6.4). In particular, linearising equations (6.1) and subtracting the steady state (zero order) equations gives the wave dynamical equations:

$$(6.6) \qquad \frac{\partial}{\partial t}\rho + \frac{m}{e}\nabla\cdot\underline{J} = 0$$

$$\frac{m^2}{e^2}\frac{\partial \underline{J}}{\partial t} + \frac{m}{e}\nabla\cdot\underset{\sim}{P} + \frac{m}{e}\nabla p - \frac{m}{e}\underline{J}\times\underline{B}_o - \rho_o\underline{E} - \rho\underline{F} = 0$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla\cdot\underline{R} + \frac{5}{2}\frac{m}{e}\frac{p_o}{\rho_o}\nabla\cdot\underline{J} - \underline{F}\cdot\underline{J} = 0$$

$$\frac{\partial}{\partial t}\underset{\sim}{P} + \frac{4}{5}\{\nabla\underline{R}\} + \frac{2m}{e}\frac{p_o}{\rho_o}\{\nabla\underline{J}\} - 2\{\underline{F}\,\underline{J}\} - \frac{2e}{m}\{\underset{\sim}{P}\times\underline{B}_o\} = 0$$

$$\frac{\partial \underline{R}}{\partial t} + \frac{p_0}{\rho_0} \nabla \cdot \underset{\sim}{P} - \frac{e}{m} \underset{\sim}{P} \cdot \underline{F} - \frac{e}{m} \underline{R} \times \underline{B}_0 + \frac{5}{2} \frac{p_0}{\rho_0} \nabla p - \frac{5}{2} \frac{p_0^2}{\rho_0^2} \nabla \rho = 0 .$$

Maxwell's equations are also to be included:

$$\nabla \times \underline{B} - \mu_0 \underline{J} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = 0 \tag{6.7}$$

$$\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 .$$

These equations are similar to the homogeneous plasma wave dynamical equations of Chapter 5, in general giving various independent wave dispersion equations (although in this case differential instead of algebraic). The main complications arise from the fact that not only an applied magnetic field but also plasma gradients can couple modes which are independent in a simple non-magnetic homogeneous plasma. Also, of course, the linearised equations are quasilinear, the coefficients being functions of plasma parameters. Consequently the differential operators and coefficients do not commute; the wave equations have to be derived by careful sequential elimination of the variables.

The simplifications of these equations to obtain meaningful solutions are twofold. Firstly, only continuous plane waves are considered, propagating in the z direction, at a given frequency ω, independent of the plasma parameters. Secondly, it is assumed that the characteristic lengths of the gradients are much greater than the wavelengths

(λ) of the propagating waves. This means that terms of order λ^2/L^2 are to be neglected compared with unity, and is known (Ginzburg[1]) as the approximation of geometric optics. Finally, this Chapter is restricted to the particular cases of $\underline{F}$ parallel and perpendicular to the direction of propagation, the former case being also considered in the presence of an applied magnetic field.

6.3.1 Density Gradient Perpendicular to the Direction of Propagation

This case is chosen first as it may relatively easily be dismissed. It is clear that in the presence of a transverse density gradient, when the wave velocity is a function of density, for example, different points on the "plane" wave front at a particular z coordinate have different speeds. This is inconsistent with the assumption of plane wave propagation. Ignoring this consideration for the present, assume a transverse gradient such that $\underline{F} = (0, F, 0)$, in a rectangular cartesian coordinate system. Let the time dependence be of the form $e^{i\omega t}$, and consider the case of no applied magnetic field. From equations (6.6) it may be shown that there is a pure transverse mode, and also a coupled longitudinal-transverse mode, the coupling being due to the transverse gradients. Substituting for the time derivative, the associated set of equations for the transverse mode, involving E_x, B_y, B_z, J_x, P_{xy}, P_{xz} and R_x , is:

$$(6.8)\quad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0 J_x - \frac{i\omega}{c^2} E_x = 0$$

$$\frac{\partial E_x}{\partial z} + i\omega B_y = 0$$

$$-\frac{\partial E_x}{\partial y} + i\omega B_z = 0$$

$$\frac{i\omega m^2}{\rho_0 e^2} J_x + \frac{m}{e\rho_0}\frac{\partial}{\partial y} P_{xy} + \frac{m}{e\rho_0}\frac{\partial}{\partial z} P_{xz} - E_x = 0$$

$$i\omega P_{xy} + \frac{2}{5}\frac{\partial}{\partial y} R_x + \frac{m}{e} f^2 \frac{\partial}{\partial y} J_x - F J_x = 0$$

$$i\omega P_{xz} + \frac{2}{5}\frac{\partial}{\partial z} R_x + \frac{m}{e} f^2 \frac{\partial}{\partial z} J_x = 0$$

$$i\omega R_x + f^2 \frac{\partial}{\partial y} P_{xy} + f^2 \frac{\partial}{\partial z} P_{xz} - \frac{e}{m} F P_{xy} = 0.$$

For this particular case, the continuous "plane" wave assumption amounts to the wave variables being of the form

$$\phi = \phi(y)\, e^{i(\omega t - \kappa z)} \tag{6.9}$$

with κ independent of y and z, and $\phi(y)$ as yet undetermined. Hence, replacing $\frac{\partial}{\partial z}$ by $-i\kappa$, with terms of higher than first order in F neglected, equations (6.8) may be reduced to the following wave equation.

$$\begin{aligned} 0 = {} & \left(\frac{2}{5}\frac{m^2\kappa^2 f^2 \omega_p^2}{e^2 c^2} - \frac{m^2\omega^2\omega_p^2}{e^2 c^2} + \frac{7}{5}\frac{m^2 f^2 \kappa^4}{e^2} - \frac{m^2\kappa^2\omega^2}{e^2} \right. \\ & \left. - \frac{7}{5}\frac{m^2\omega^2\kappa^2 f^2}{e^2 c^2} + \frac{m^2\omega^4}{e^2 c^2} \right) E_x + \left(-\frac{4}{5}\frac{m f^4 \omega_p^2}{e c^2} \right. \\ & + \frac{4}{25}\frac{m\kappa^2 f^2 \omega_p^2}{e c^2 \omega^2} + \frac{14}{25}\frac{m f^2 \kappa^4}{e \omega^2} + \frac{m}{e}\kappa^2 - \frac{14}{25}\frac{m\kappa^2 f^2}{e c^2} \\ & \left. - \frac{m\omega^2}{e c^2} \right) F \frac{\partial E_x}{\partial y} + \left(-\frac{2}{5}\frac{m^2 f^2 \omega_p^2}{e^2 c^2} - \frac{9}{5}\frac{m^2\kappa^2 f^2}{e^2} \right. \end{aligned} \tag{6.10}$$

$$+ \frac{m^2\omega^2}{e^2} - \frac{2}{5}\frac{m^2}{e^2}\frac{f^4\kappa^4}{\omega^2} + \frac{2}{5}\frac{m^2 f^2\omega^2}{e^2 c^2} + \frac{2}{5}\frac{m^2}{e^2}\frac{f^4\kappa^2}{c^2}\Bigg)\frac{\partial^2 E_x}{\partial y^2}$$

$$+ \text{ high order terms in } \frac{\partial^n E_x}{\partial y^n} \ .$$

However, since the wave motion is along the z-axis, spatial derivatives of the wave variables, including E_x, in the y direction will be of the same order in characteristic length as the density gradient, i.e.,

$$(6.11)\qquad \frac{\partial}{\partial y} \sim \frac{1}{L} \ .$$

Therefore it is consistent to neglect, in equation (6.10), along with terms in F^2, all higher than first derivatives with respect to y and all first derivatives with respect to y when multiplied by F. With these considerations, it is apparent that to first order in F, the only term retained in (6.10) is the term involving E_x without derivatives with respect to y. Since this coefficient of E_x is in fact the collisionless transverse wave homogeneous plasma dispersion relationship, it follows that the effect of transverse gradients is negligible to first order in F for this case.

Similarly, the equations for the gradient coupled longitudinal-transverse equations can be written out by taking the appropriate component equations in (6.6). These longitudinal-transverse component equations involve B_x, E_y, E_z, J_y, J_z, P, ρ, P_{xx}, P_{yy}, P_{yz}, R_y and R_z. Reduction of the associated set of equations

is excessively complicated, but by systematic elimination of variables, and neglect of higher than first derivatives with respect to y, it is possible to reduce these equations to two equations of the form:

$$(6.12)\quad i\alpha E_y + \left(\beta\frac{\partial}{\partial y} + \gamma F\right) E_z = 0$$

$$i\Theta E_z + \left(\phi\frac{\partial}{\partial y} + \psi F\right) E_y = 0,$$

where α, β, γ, Θ, ϕ and ψ are complicated algebraic functions that need not be given.

Combining equations (6.12) to eliminate E_y,

$$(6.13)\quad \Theta E_z + \left(\phi\frac{\partial}{\partial y} + \psi F\right)\left(\frac{\beta}{\alpha}\frac{\partial}{\partial y} + \frac{\gamma}{\alpha}F\right) E_z = 0.$$

Since

$$(6.14)\quad \frac{\partial}{\partial y}(\beta/\alpha) \sim \frac{1}{L}(\beta/\alpha)$$

$$\frac{\partial}{\partial y}(\gamma/\alpha) \sim \frac{1}{L}(\gamma/\alpha),$$

put

$$(6.15)\quad \frac{\partial}{\partial y}(\beta/\alpha) = \eta F$$

$$\frac{\partial}{\partial y}(\gamma/\alpha) = \mu F.$$

Therefore equation (6.13) becomes

$$(6.16)\quad \left(\Theta + \phi\eta F\frac{\partial}{\partial y} + \phi\mu F^2 + \phi\frac{\beta}{\alpha}\frac{\partial^2}{\partial y^2} + \phi\frac{\gamma}{\alpha}F\frac{\partial}{\partial y} + \psi\frac{\beta}{\alpha}F\frac{\partial}{\partial y} + \psi\frac{\gamma}{\alpha}F^2\right)E_z = 0 .$$

Again to first order in F or L^{-1}, this equation is simply

$$(6.17)\quad \Theta E_z = 0 ,$$

where $\Theta = 0$ is the homogeneous plasma dispersion equation given by the product of the non-magnetic transverse and longitudinal dispersion equations obtained in Chapter 5 (with ν put equal to zero).

It may be concluded that the effect of a transverse density gradient on the propagation of waves is negligible to the present order of approximation, when there is no magnetic field. It does not necessarily follow that the same situation will apply when a magnetic field is present. However, in view of the results obtained above, for the remainder of this Chapter only the case of a density gradient parallel to the direction of the wave propagation will be considered.

6.3.2 Density Gradient Parallel to the Direction of Propagation

The components of the applied electric field vector $\underline{F}$ for longitudinal gradients are $\underline{F} = (0, 0, F)$. In this subsection two principal cases are considered, (1) when there is no magnetic field, and (2) an applied magnetic field parallel to the direction of propagation. Because the reduction of the magnetically coupled equations for wave motion in the inhomogeneous plasma is extremely complex, the case of a transverse applied magnetic field has been excluded.

(a) No applied magnetic field

In a non-magnetic plasma, even in the presence of longitudinal density gradients, the equations for transverse wave motion are degenerate, there being two essentially identical and independent sets of equations for transverse waves involving E_x, B_y, J_x, P_{xz}, R_x; and E_y, B_x, J_y, P_{yz}, R_y, respectively. Considering only one of these sets of equations, with the operator D being used for $\frac{\partial}{\partial z}$, from equations (6.6) it may be shown that the relevant equations are

$$(6.18)\quad -DB_y - \mu_0 J_x - \frac{i\omega}{c^2} E_x = 0$$

$$DE_x + i\omega B_y = 0$$

$$\frac{i\omega m^2}{e^2} J_x + \frac{m}{e} DP_{xz} - \rho_0 E_x = 0$$

$$i\omega P_{xz} + \frac{2}{5} DR_x + \frac{m}{e} f^2 DJ_x - FJ_x = 0$$

$$i\omega R_x + f^2 DP_{xz} - \frac{e}{m} FP_{xz} = 0,$$

where $f^2 = P_0/\rho_0$ as before.

Combining these equations, and discarding terms of higher than first order in F, the transverse wave equation obtained is

$$(6.19)\quad D^4 E_x - \frac{e}{mf^2} FD^3 E_x + \left(\frac{5}{7}\frac{\omega^2}{f^2} - \frac{2}{7}\frac{\omega_p^2}{c^2} + \frac{\omega^2}{c^2}\right) D^2 E_x$$

$$+ \left(-\frac{e}{m}\frac{\omega^2}{f^2 c^2} - \frac{2}{7}\frac{e}{m}\frac{\omega_p^2}{f^2 c^2}\right) FDE_x + \left(-\frac{5}{7}\frac{\omega^2 \omega_p^2}{f^2 c^2}\right.$$

$$+ \frac{5}{7}\frac{\omega^4}{f^2c^2}\Big) E_x = 0 \ .$$

Solutions of this equation are discussed in a subsequent section.

The longitudinal wave components involve E_z, J_z, ρ, P, P_{xx}, P_{yy}, R_z. Again using (6.6), the component equations for longitudinal wave motion are

$$\mu_0 J_z + \frac{i\omega E_z}{c^2} = 0 \tag{6.20}$$

$$i\omega\rho + \frac{m}{e} D J_z = 0$$

$$\frac{i\omega m^2}{e^2} J_z - \frac{m}{e} D(P_{xx} + P_{yy}) + \frac{m}{e} Dp - \rho F - \rho_0 E_z = 0$$

$$\frac{3}{2} i\omega p + D R_z + \frac{5}{2}\frac{m}{e} f^2 D J_z - F J_z = 0$$

$$i\omega P_{xx} - \frac{4}{15} D R_z - \frac{2}{3}\frac{m}{e} f^2 D J_z + \frac{2}{3} F J_z = 0$$

$$i\omega P_{yy} - \frac{4}{15} D R_z - \frac{2}{3}\frac{m}{e} f^2 D J_z + \frac{2}{3} F J_z = 0$$

$$i\omega R_z + \frac{5}{2} f^2 Dp - \frac{5}{2} f^4 D\rho - f^2 D(P_{xx} + P_{yy})$$

$$- \frac{e}{m} F (P_{xx} + P_{yy}) = 0 \ .$$

With the neglect of terms in F of higher than first order, equations (6.20) yield the longitudinal wave equation

$$(6.21)\quad f^4D^4E_z - \frac{e}{m}f^2FD^3E_z + f^2\left(\frac{26}{15}\omega^2 - \frac{11}{15}\omega_p^2\right)D^2E_z + \frac{e}{m}\left(-\frac{37}{45}\omega^2 - \frac{74}{45}\omega_p^2\right)FDE_z + \left(\frac{\omega^4}{3} - \frac{\omega^2\omega_p^2}{3}\right)E_z = 0.$$

This equation and (6.19) are identical to the algebraic dispersion equations for the corresponding cases when only "zero order" terms are considered, collisions are neglected, and the appropriate wave representation in terms of a wave vector is used.

(b) Applied magnetic field parallel to the direction of propagation

As in the case of a homogeneous plasma with a magnetic field parallel to the direction of propagation, the previously independent sets of transverse wave variables are coupled by the magnetic field. In addition there is an independent set of equations representing longitudinal wave motion, some terms of which involve the magnetic field. It is found, however, that the longitudinal wave equation is unaffected by the applied field, and is identical to (6.21). The main difference is that the set of longitudinal wave variables is extended to include an anisotropic stress component, P_{xy} .

Writing $\underline{B}_o = (0, 0, B_o)$, ω_b is defined, as in Chapter 5, by

$$(6.22)\quad \omega_b = \frac{eB_o}{m} \quad .$$

Since for electrons, e is negative, it is evident that ω_b is negative for B_o positive, that is, for the direction of $\underline{B}_o$ along the positive $z-$ axis. In the following equations and results, for convenience, ω_b is regarded as positive. As with the homogeneous plasma of the previous Chapter, when $\omega_b < 0$, the results for the ordinary and extraordinary modes are simply interchanged.

In terms of ω_b, and with the time dependence $e^{i\omega t}$, $\omega \neq \omega(z)$, from (6.6) it may be seen that the coupled transverse component equations are:

(6.23)
$$DB_y + \mu_o J_x + \frac{i\omega}{c^2} E_x = 0$$

$$-DB_x + \mu_o J_y + \frac{i\omega}{c^2} E_y = 0$$

$$DE_x + i\omega B_y = 0$$

$$-DE_y + i\omega B_x = 0$$

$$\frac{i\omega m^2}{e^2} J_x + \frac{m}{e} D P_{xz} - \rho_o E_x - \omega_b \frac{m^2}{e^2} J_y = 0$$

$$\frac{i\omega m^2}{e^2} J_y + \frac{m}{e} D P_{yz} - \rho_o E_y + \omega_b \frac{m^2}{e^2} J_x = 0$$

$$i\omega P_{xz} + \frac{2}{5} DR_x + \frac{m}{e} f^2 DJ_x - FJ_x - \omega_b P_{yz} = 0$$

$$i\omega P_{yz} + \frac{2}{5} DR_y + \frac{m}{e} f^2 DJ_y - FJ_y + \omega_b P_{xz} = 0$$

$$i\omega R_x + f^2 DP_{xz} - \frac{e}{m} FP_{xz} - \omega_b R_y = 0$$

(ctd)

$$i\omega R_y + f^2 D P_{yz} - \frac{e}{m} F P_{yz} + \omega_b R_x = 0 \ .$$

While it is difficult to combine these equations to yield a single eighth order differential equation, systematic elimination of variables to reduce (6.23) to a set of two fourth order equations is both less arduous and more rewarding. With this reduction the equations become

$$(6.24)\quad f(D) E_x + g(D) E_y = 0$$

$$g(D) E_x + f(D) E_y = 0 ,$$

with the operators $f(D)$, $g(D)$ given by

$$(6.25)\quad f(D) = i\Big[\Big(\frac{7}{5} \frac{\omega_b^2}{\omega} f^2 - \frac{7}{5} \omega f^2 \Big) D^4 + \Big(\frac{7}{5} \omega - \frac{7}{5} \frac{\omega_b^2}{\omega} \Big) \frac{e}{m} F D^3$$
$$+ \Big(-\omega^3 + \frac{2}{5} \frac{f^2}{c^2} \omega \omega_p^2 + \frac{\omega_b^4}{\omega} - \frac{7}{5} \omega^3 \frac{f^2}{c^2}$$
$$+ \frac{7}{5} \omega \omega_b^2 \frac{f^2}{c^2} \Big) D^2 + \Big(\frac{7}{5} \frac{\omega^3}{c^2} - \frac{7}{5} \frac{\omega \omega_b^2}{c^2} + \frac{2}{5} \frac{\omega \omega_p^2}{c^2} \Big) \frac{e}{m} F D$$
$$+ \Big(\frac{\omega^3 \omega_p^2}{c^2} - \frac{\omega \omega_b^2 \omega_p^2}{c^2} - \frac{\omega^5}{c^2} + \frac{\omega \omega_b^4}{c^2} \Big) \Big]$$

and

$$(6.26)\quad g(D) = \Big[\Big(-2\omega^2 \omega_b + 2\omega_b^3 - \frac{2}{5} \frac{f^2}{c^2} \omega_p^2 \omega_b \Big) D^2 + \Big(-\frac{2\omega^4 \omega_b}{c^2}$$
$$+ \frac{2\omega^2 \omega_b^3}{c^2} + \frac{\omega_b \omega^2 \omega_p^2}{c^2} - \frac{\omega_b^3 \omega_p^2}{c^2} \Big) \Big] .$$

Should $f(D)$, $g(D)$ commute, the wave equation would be simply

$$(6.27)\quad \left(f^2(D) - g^2(D) \right) E_x = 0 .$$

In general, however, $f(D)$ and $g(D)$ do not commute. Nevertheless, on neglecting terms of order f^2/c^2 when multiplied by F , it is found that

$$(6.28)\quad \left(f^2(D) - g^2(D) \right) E_x = \propto FDE_y$$

$$\left(f^2(D) - g^2(D) \right) E_y = \propto FDE_x ,$$

where

$$(6.29)\quad \propto FD = g(D) f(D) - f(D) g(D)$$

$$= \frac{2e\omega_b \omega_p^2}{m f^2} \left(\omega^2 - \omega_b^2 \right)^3 FD .$$

From (6.28), writing

$$(6.30)\quad E_y = \left(f^2(D) - g^2(D) \right)^{-1} \propto FDE_x ,$$

it follows that

$$(6.31)\quad \left(f^2(D) - g^2(D) \right) E_x = \propto F^2 D \left(f^2(D) - g^2(D) \right)^{-1} \propto DE_x .$$

To the approximation of retaining first order terms only in F, the right hand side can be put equal to zero, and therefore the wave equation is (6.27). The magnitude of α is apparent from (6.29), being of "zero order", and therefore it appears that the above procedure is valid. It should be noted, however, that this result is not equivalent to being able to write

$$(6.32)\quad \left(f(D)-g(D)\right)\left(f(D)+g(D)\right)E_x = 0 .$$

In detail, the wave equation (6.27) becomes

$$(6.33)\quad \left(\frac{7}{5}\right)^2\left(\frac{\omega^2-\omega_b^2}{\omega}\right)^2 f^4\Bigg[D^8-\frac{2e}{mf^2}FD^7+\Bigg(-\frac{4}{7}\frac{\omega^2\omega_p^2}{c^2(\omega^2-\omega_b^2)}+\frac{2\omega^2}{c^2}+\frac{10}{7}\frac{(\omega^2+\omega_b^2)}{f^2}\Bigg)D^6+\Bigg(-4\frac{\omega^2}{c^2}-\frac{10}{7}\frac{(\omega^2+\omega_b^2)}{f^2}-\frac{8}{7}\frac{\omega^2\omega_p^2}{c^2(\omega^2-\omega_b^2)}\Bigg)\frac{e}{mf^2}FD^5+\Bigg(\frac{4}{49}\frac{\omega^4\omega_p^4}{c^4(\omega^2-\omega_b^2)^2}+\frac{\omega^4}{c^4}+\frac{25}{49}\frac{(\omega^2-\omega_b^2)^2}{f^4}-\frac{4}{7}\frac{\omega^4\omega_p^2}{c^4(\omega^2-\omega_b^2)}-\frac{20}{49}\frac{\omega^2\omega_p^4\omega_b^4}{c^4(\omega^2-\omega_b^2)^2}+\frac{20}{7}\frac{\omega^2(\omega^2+\omega_b^2)}{f^2c^2}-\frac{10}{7}\frac{\omega^2\omega_p^2}{f^2c^2}-\frac{4}{49}\frac{\omega^2\omega_p^4\omega_b^4}{c^4(\omega^2-\omega_b^2)^2}-\frac{60}{49}\frac{\omega^2\omega_b^2\omega_p^2}{f^2c^2(\omega^2-\omega_b^2)}\Bigg)D^4+\Bigg(-\frac{2\omega^4}{c^4}-\frac{20}{7}\frac{\omega^2(\omega^2+\omega_b^2)}{f^2c^2}+\frac{20}{49}\frac{\omega^2\omega_p^2(\omega^2+\omega_b^2)}{f^2c^2(\omega_b^2-\omega^2)}+\frac{4}{7}\frac{\omega^4\omega_p^2}{c^4(\omega_b^2-\omega^2)}+\frac{16}{49}\frac{\omega^4\omega_p^4}{c^4(\omega_b^2-\omega^2)^2}-\frac{10}{7}\frac{\omega^2\omega_p^2}{f^2c^2}-\frac{20}{49}\frac{\omega^2\omega_p^2(\omega^2+\omega_b^2)}{f^2c^2(\omega^2-\omega_b^2)}\Bigg)\frac{e}{mf^2}FD^3$$

(ctd)

$$+\left(\frac{50}{49}\frac{\omega^2\omega_b^2\omega_p^2}{f^4c^2}-\frac{100}{49}\frac{\omega^4\omega_b^2}{f^4c^2}+\frac{20}{49}\frac{\omega^2\omega_b^2\omega_p^4}{f^2c^4(\omega^2-\omega_b^2)}\right.$$

$$-\frac{60}{49}\frac{\omega^4\omega_b^2\omega_p^2}{f^2c^4(\omega^2-\omega_b^2)}-\frac{10}{7}\frac{\omega^4\omega_p^2}{f^2c^4}+\frac{10}{7}\frac{\omega^6}{f^2c^4}+\frac{10}{7}\frac{\omega^4\omega_b^2}{f^2c^4}$$

$$-\frac{50}{49}\frac{\omega^4\omega_p^2}{f^4c^2}+\frac{50}{49}\frac{\omega^6}{f^4c^2}+\frac{50}{49}\frac{\omega^2\omega_b^4}{f^4c^2}+\frac{20}{49}\frac{\omega^4\omega_p^4}{f^2c^4(\omega^2-\omega_b^2)}$$

$$\left.-\frac{20}{49}\frac{\omega^6\omega_p^2}{f^2c^4(\omega^2-\omega_b^2)}\right)D^2+\left(\frac{20}{49}\frac{\omega^2\omega_b^2\omega_p^4}{f^2c^4(\omega^2-\omega_b^2)}\right.$$

$$+\frac{20}{49}\frac{\omega^4\omega_p^2(\omega^2+\omega_b^2)}{f^2c^4(\omega_b^2-\omega^2)}-\frac{20}{49}\frac{\omega^4\omega_p^4}{f^2c^4(\omega_b^2-\omega^2)}$$

$$+\frac{10}{7}\frac{\omega^4(\omega_p^2-\omega^2-\omega_b^2)}{f^2c^4}+\frac{20}{49}\frac{\omega^4\omega_p^4}{f^2c^4(\omega^2-\omega_b^2)}$$

$$\left.-\frac{10}{7}\frac{\omega^4\omega_p^2}{f^2c^4}-\frac{50}{49}\frac{\omega^2\omega_p^2(\omega^2+\omega_b^2)}{f^4c^2}+\frac{100}{49}\frac{\omega^2\omega_b^2\omega_p^2}{f^4c^2}\right)\frac{e}{mf^2}FD$$

$$\left.+\left(-\frac{25}{49}\frac{\omega^2\omega_b^2(\omega_p^2-2\omega^2)^2}{f^4c^4}+\frac{25}{49}\frac{\omega^4}{f^4c^4}(\omega_p^2-\omega^2-\omega_b^2)^2\right)\right]E_x$$

$$=0.$$

6.4 Solutions of the Wave Dispersion Equations

The wave dispersion equations derived in section 6.3 for plane wave propagation in inhomogeneous plasma are fourth or eighth order quasi-linear ordinary differential equations. The quasilinearity arises from the terms involving $\omega_p^2 = \rho_0 e^2/m\epsilon_0$ and F in the coefficients. Quasilinear equations, like non-linear equations, in most cases do not have exact solutions, and approximation methods must be used. Several general methods are available, but all (including the WKBJ and Van der Pol methods) are essentially related to an expansion in powers of a parameter that is small. While for certain special forms of equations these methods are very concise (e.g. for Ricci equations), their particular forms have no special advantages for the above fourth or eighth order equations. For the latter equations, an appropriate method of solution is successive approximation by means of an expansion in inverse powers of L.

As only first order solutions in L^{-1} are sought, it is convenient to express the waves in terms of a propagation vector. For this reason the propagating variables are considered in the form:

$$E(z) = A\, e^{i\omega t - \int \phi\, dz} \quad . \tag{6.34}$$

On substituting (6.34) into the quasilinear dispersion equations, non-linear equations for ϕ are obtained. Since F is related to L by (6.4), the general form of the equations derived in the previous section is

$$\left(D^8 + \frac{\alpha}{L} D^7 + \beta D^6 + \frac{\gamma}{L} D^5 + \delta D^4 + \frac{\epsilon}{L} D^3 + \vartheta D^2 + \frac{\rho}{L} D + \theta \right) E = 0 \quad . \tag{6.35}$$

It is assumed that L includes a sign according to the direction of F. As $D\phi \sim \phi/L$, $D^2\phi \sim \phi/L^2$, etc., using (6.34) in (6.35), the equation for ϕ is

$$
\begin{aligned}
(6.36)\quad & \Big[\left(\phi^8 + \beta\phi^6 + \delta\phi^4 + \varepsilon\phi^2 + \theta \right) \\
& + \left(-28\phi^6 D\phi - \frac{\alpha}{L}\phi^7 - 15\beta\phi^4 D\phi - \frac{\gamma}{L}\phi^5 \right. \\
& \left. - 6\delta\phi^2 D\phi - \frac{\epsilon}{L}\phi^3 - 3D\phi - \frac{\eta}{L}\phi \right) \\
& + \left(56\phi^5 D^2\phi + 200\phi^4 (D\phi)^2 + 21\frac{\alpha}{L}\phi^5 D\phi \right. \\
& + \beta\left[45\phi^2 (D\phi)^2 + 20\phi^3 D^2\phi \right] + 10\frac{\gamma}{L}\phi^3 D\phi \\
& \left. + \delta\left[4\phi D^2\phi + 3(D\phi)^2 \right] + 3\frac{\epsilon}{L}\phi D\phi \right) + O\left(\frac{\phi}{L^3}\right) \Big] = 0.
\end{aligned}
$$

Consider the following expansion for ϕ:

$$
(6.37)\quad \phi = \phi_0 + \phi_1/L + \phi_2/L^2 + \cdots .
$$

Substituting this in equation (6.36) and grouping terms of the same order in L^{-1} provides successive approximations for ϕ. The equation for ϕ_0 is obviously

$$
(6.38)\quad \phi_0^8 + \beta\phi_0^6 + \delta\phi_0^4 + \varepsilon\phi_0^2 + \theta = 0.
$$

Terms of first order in L^{-1} give

$$(6.39)\quad \phi_1 = \Big[(28\phi_0^6 D\phi_0 + 15\beta\phi_0^4 D\phi_0 + 6\delta\phi_0^2 D\phi_0 + 9D\phi_0)L + (\alpha\phi_0^7 + \gamma\phi_0^5 + \epsilon\phi_0^3 + \eta\phi_0)\Big] \Big/ \Big[8\phi_0^7 + 6\beta\phi_0^5 + 4\delta\phi_0^3 + 2\vartheta\phi_0 \Big] .$$

Subsequent successive approximations are similarly obtained.

6.4.1 Solutions with No Applied Magnetic Field

This category includes the longitudinal wave equation (6.21) and the (degenerate) transverse wave equation (6.19). The solutions for ϕ_0 are of course those corresponding to a homogeneous collisionless plasma.

In the case of the longitudinal wave equation (6.21), for which

$$(6.40)\quad \phi_0^4 + \left(\frac{26}{15}\frac{\omega^2}{f^2} - \frac{11}{15}\frac{\omega_p^2}{f^2}\right)\phi_0^2 + \left(\frac{\omega^4}{3f^4} - \frac{\omega^2\omega_p^2}{3f^4}\right) = 0,$$

the solutions for ϕ_0 are direct in the extreme frequency regions $\omega >> \omega_p$ and $\omega << \omega_p$. In particular, with $\omega >> \omega_p$, the solutions for ϕ_0^2 are

$$(6.41)\quad \phi_0^2 = \frac{-\cdot 220\,\omega^2 + \cdot 133\,\omega_p^2}{f^2}$$

and

$$(6.42)\quad \phi_0^2 = \frac{-1\cdot 513\,\omega^2 + \cdot 601\,\omega_p^2}{f^2} .$$

When $\omega << \omega_p$ there is only one propagating solution for ϕ_o^2, i.e.,

$$(6.43) \quad \phi_o^2 = -\cdot 454 \frac{\omega^2}{f^2} \quad .$$

Neglecting the derivatives of ω_p^2 in (6.41) and (6.42) (since $\omega_p << \omega$ for these results), direct substitution of equations (6.41) to (6.43) in (6.39) gives the first order solutions of the expansion (6.37). The zero order solution (6.41) in (6.39) yields

$$(6.44) \quad \phi_1 = -\cdot 403 \quad .$$

Similarly, (6.42) has first order term

$$(6.45) \quad \phi_1 = +\cdot 903$$

and finally, in the other extreme frequency region, (6.43) yields

$$(6.46) \quad \phi_1 = +1\cdot 126 \quad .$$

Summarising, the complete propagating solutions for the longitudinal wave equation (6.21), to first order in L^{-1}, and subject to the above approximations concerning the relative magnitudes of ω and ω_p, are:

(a) $\omega >> \omega_p$

$$(6.47)\quad E_z = A_z e^{i\omega t \mp i\int \cdot 469 \frac{\omega}{f}(1 - \cdot 303 \frac{\omega_p^2}{\omega^2})dz + \cdot 403 \int \frac{dz}{L}}$$

and

$$(6.48)\quad E_z = A_z e^{i\omega t \mp i\int 1 \cdot 229 \frac{\omega}{f}(1 - \cdot 198 \frac{\omega_p^2}{\omega^2})dz - \cdot 903 \int \frac{dz}{L}} .$$

(b) $\omega << \omega_p$

$$(6.49)\quad E_z = A_z e^{i\omega t \mp i\int \cdot 674 \frac{\omega}{f} dz - 1 \cdot 126 \int \frac{dz}{L}} .$$

These solutions are discussed in section 6.5.

The transverse wave equation (6.19) may be solved in a similar way. The equation for ϕ_o is

$$(6.50)\quad \phi_o^4 + \left(\frac{5}{7}\frac{\omega^2}{f^2} - \frac{2}{7}\frac{\omega_p^2}{c^2} + \frac{\omega^2}{c^2}\right)\phi_o^2 + \left(\frac{5}{7}\frac{\omega^4}{f^2 c^2} - \frac{5}{7}\frac{\omega^2 \omega_p^2}{f^2 c^2}\right) = 0 .$$

For a slightly more general case, this equation has been discussed in Chapter 5. To the lowest order in f^2/c^2, the solutions are a fast wave,

$$(6.51)\quad \phi_o^2 = -\frac{\omega^2 - \omega_p^2}{c^2}$$

and a stress wave,

$$(6.52)\quad \phi_0^2 = -\frac{5}{7}\frac{\omega^2}{f^2} \quad .$$

The derivative of the fast wave solution, (6.51), is

$$(6.53)\quad D\phi_0 = \frac{\omega_p^2}{2c^2\phi_0 L} \quad .$$

Therefore, using (6.51) and (6.53) in equation (6.39), the coefficients being apparent from (6.19), the first order solution, ϕ_1 , is obtained. With terms in f^2/c^2 again neglected compared with unity, the result is

$$(6.54)\quad \phi_1 = -\frac{\omega_p^2}{4(\omega^2-\omega_p^2)} \quad .$$

The stress solution, (6.52), has no derivative, and in the same manner as for the fast wave, the first order term may be derived:

$$(6.55)\quad \phi_1 = -\frac{1}{2} \quad .$$

In summary, the non-magnetic transverse wave solutions are, to first order in L^{-1} :

(a) Fast wave

$$(6.56)\quad E_x = A_x\, e^{i\omega t \mp i\int\left(\frac{\omega^2-\omega_p^2}{c^2}\right)^{\frac{1}{2}}dz + \int \frac{\omega_p^2\, dz}{4L(\omega^2-\omega_p^2)}} \quad .$$

(b) Stress wave

$$(6.57)\quad E_x = A_x\, e^{i\omega t \;\mp\; i\int\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f}\,dz \;+\; \int \frac{dz}{2L}} .$$

A discussion of these solutions is given in section 6.5.

6.4.2 Solutions with the Applied Magnetic Field Parallel to the Direction of Propagation

The effect of a parallel magnetic field on the wave variables has already been noted, and the resulting transverse wave equation derived. This equation, (6.33), is an eighth order quasilinear differential equation. The longitudinal wave equation, being unaffected by the field, need not be reconsidered.

The homogeneous plasma transverse wave equation for the parallel magnetic field case corresponds to

$$(6.58)\quad \left(f(D) - g(D)^2 \right) E_x = 0 ,$$

but with $f(D)$, $g(D)$ commuting (see equations (6.25), (6.26), and (6.27)). Hence the equations for ϕ_0 are

$$(6.59)\quad \phi_0^4 \left(\frac{7}{5}\,\frac{\omega_b^2}{\omega} f^2 - \frac{7}{5}\,\omega f^2 \right) + \phi_0^2 \left[\left(-\omega^3 + \frac{2}{5}\,\frac{f^2}{c^2}\,\omega\omega_p^2 + \frac{\omega_b^4}{\omega} - \frac{7}{5}\,\omega^3\frac{f^2}{c^2} + \frac{7}{5}\,\omega\omega_b^2\frac{f^2}{c^2} \right) \pm \left(2\omega_b^3 - 2\omega^2\omega_b - \frac{2}{5}\,\omega_b\omega_p^2\frac{f^2}{c^2} \right) \right] + \left[\left(\frac{\omega^3\omega_p^2}{c^2} - \frac{\omega\omega_b^2\omega_p^2}{c^2} - \frac{\omega^5}{c^2} + \frac{\omega\omega_b^4}{c^2} \right) \pm \left(\frac{2\omega^2\omega_b^3}{c^2} + \frac{\omega_b\omega^2\omega_p^2}{c^2} - \frac{2\omega^4\omega_b}{c^2} - \frac{\omega_b^3\omega_p^2}{c^2} \right) \right] = 0 .$$

The upper (positive) sign gives the ordinary wave modes. Again, a slightly more general equation of this form has been discussed in Chapter 5.

The fast wave solutions for ϕ_o^2 are

$$(6.60) \quad \phi_o^2 = -\frac{\omega}{c^2}\left(\frac{\omega^2 \pm \omega\omega_b - \omega_p^2}{\omega \pm \omega_b}\right),$$

where terms of order f^2/c^2 have been neglected. In these equations also, and for the following results, the upper sign denotes the ordinary wave. To the same approximation, the stress wave solutions for ϕ_o are

$$(6.61) \quad \phi_o^2 = -\frac{5}{7}\frac{(\omega \pm \omega_b)^2}{f^2}.$$

The appropriate coefficients for use in equation (6.39) are obtained from (6.33), and after some elementary algebra the solutions appropriate to (6.60) and (6.61) can be obtained. The fast wave ϕ_o, given by (6.60), have derivatives

$$(6.62) \quad D\phi_o = \frac{\omega\,\omega_p^2}{2c^2(\omega \pm \omega_b)\,L\,\phi_o}.$$

Using this result, it is found that

$$(6.63) \quad \phi_1 = \frac{(2\omega^2 \pm 2\omega\omega_b - 3\omega_p^2)}{4(\omega^2 \pm \omega\omega_b - \omega_p^2)}.$$

Taking the stress solution (6.61), there are no derivatives to be considered, and the first order term is

(6.64) $\phi_1 = -\frac{1}{2}$.

In obtaining (6.63) and (6.64), terms of order f^2/c^2 have again been neglected. It should also be mentioned that the fast wave result does not tend to the corresponding non-magnetic result as $\omega_b \to 0$. This is because the ordering in terms of f^2/c^2 used in deriving the first order solution (6.62) becomes invalid for $\omega_b = 0$ (this situation representing division of numerator and denominator by zero). More exact results for this case involve retaining terms of order f^2/c^2 and are very complicated.

Summarising, the solutions to first order in L^{-1} for a parallel magnetic field are:

(a) Fast waves

The ordinary wave solution is

(6.65) $$E_x = A_x e^{i\omega t \mp i\int \left(\frac{\omega(\omega^2+\omega_b\omega-\omega_p^2)}{c^2(\omega+\omega_b)}\right)^{\frac{1}{2}} dz - \int \frac{(2\omega^2+2\omega\omega_b-3\omega_p^2)dz}{4L(\omega^2+\omega\omega_b-\omega_p^2)}}$$.

Similarly, the extraordinary wave solution is

(6.66) $$E_x = A_x e^{i\omega t \mp i\int \left(\frac{\omega(\omega^2-\omega\omega_b-\omega_p^2)}{c^2(\omega-\omega_b)}\right)^{\frac{1}{2}} dz - \int \frac{(2\omega^2-2\omega\omega_b-3\omega_p^2)dz}{4L(\omega^2-\omega\omega_b-\omega_p^2)}}$$.

(b) Stress waves

The ordinary wave solution is

(6.67) $$E_x = A_x e^{i\omega t \mp i\int \left(\frac{\rho}{T}\right)^{\frac{1}{2}} \frac{(\omega+\omega_b)}{f} dz + \int \frac{dz}{2L}}$$

and the extraordinary solution is

$$(6.68)\quad E_x = A_x e^{i\omega t \mp i\int (\frac{5}{7})^{\frac{1}{2}} \frac{(\omega-\omega_b)dz}{f} + \int \frac{dz}{2L}} .$$

The results are discussed in the following section.

6.5 Discussion

Apart from the stress waves and effects of the order of the kinetic coupling term, f^2/c^2, it may be demonstrated that the results derived in the previous section, corresponding to the approximation of geometrical optics, are identical with those given by other authors, e.g. Ginzburg[1]. The stress waves also have an analogous form.

The non-magnetic transverse fast wave solution is given in equation (6.56). With the time dependent terms in the wave solutions ignored throughout this section, consider the non-oscillatory part of (6.56), i.e.

$$(6.69)\quad A_x\, e^{\int \frac{\omega_p^2\, dz}{4L(\omega^2-\omega_p^2)}} .$$

Using equation (6.53), the exponent becomes:

$$(6.70)\quad \int \frac{\omega_p^2\, dz}{4L(\omega^2-\omega_p^2)} = -\int \frac{2LC^2\, D\phi_0\, dz}{4LC^2\phi_0}$$

$$= -\frac{1}{2}\log \phi_0 .$$

Therefore the fast wave solution (6.56) may alternatively be written

$$(6.71)\quad E_x = \frac{B_x}{(\omega^2-\omega_p^2)^{\frac{1}{4}}}\, e^{\mp i\int \frac{(\omega^2-\omega_p^2)^{1/2}}{c} dz} ,$$

and this is essentially the form used by Ginzburg[1]. The constant B_x, related to the constant A_x in (6.56), depends on boundary conditions.

This result has of course been obtained by a different and possibly more general method than that of the equivalent dielectric tensor, in that the kinetic coupling terms, f^2/c^2, are readily included. For example, apart from algebraic complexity, the solutions for a near-relativistic plasma are obtained in a straightforward manner by retaining terms of this order.

The remaining first order solutions obtained in the previous section may also be given in a form similar to (6.71), as follows. The transverse stress wave solution (6.57) gives

$$
\begin{aligned}
(6.72)\quad E_x &= A_x\, e^{\mp i\int \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{\omega}{f}\, dz \;+\; \int \frac{dz}{2L}} \\
&= A_x\, e^{\mp i\int \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{\omega}{f}\, dz \;+\; \int \frac{dz}{2} \frac{d\rho_0}{dz} \frac{1}{\rho_0}} \\
&= A_x\, e^{\mp i\int \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{\omega}{f}\, dz \;+\; \frac{1}{2} \log \rho_0} \\
&= A_x\, \rho_0^{\frac{1}{2}}\, e^{\mp i\int \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{\omega}{f}\, dz} .
\end{aligned}
$$

Similarly the longitudinal stress solutions may be rewritten. Solution (6.47) becomes

$$
(6.73)\quad E_z \sim A_z\, \rho_0^{.403}\, e^{\mp i\int .469\, \frac{\omega}{f}\, dz} .
$$

Solution (6.48) gives

$$(6.74)\quad E_z \sim A_z \rho_0^{-.903} e^{\mp i \int 1.229 \frac{\omega}{f} dz}$$

and (6.49) is equivalent to

$$(6.75)\quad E_z \sim A_z \rho_0^{-1.126} e^{\mp i \int .674 \frac{\omega}{f} dz} .$$

Again, in the case of a magnetic field parallel to the direction of propagation, the ordinary and extraordinary fast wave solutions are (considering only forward-going waves to avoid confusion with $\pm$ signs):

$$(6.76)\quad E_x = A_x e^{-i\int \left(\frac{\omega(\omega^2 \pm \omega\omega_b - \omega_p^2)}{\omega \pm \omega_b}\right)^{\frac{1}{2}} dz \; - \int \frac{(2\omega^2 \pm 2\omega\omega_b - 3\omega_p^2) dz}{4L(\omega^2 \mp \omega\omega_b - \omega_p^2)}} .$$

Using equation (6.62) for the derivative of the ϕ_0 for these waves (equation (6.60)), the non-oscillating part of (6.76) is

$$(6.77)\quad A_x e^{-\int \frac{-\omega_p^2}{4(\omega^2 \pm \omega\omega_b - \omega_p^2)} \frac{dz}{L} - \int \frac{2(\omega^2 \pm \omega\omega_b - \omega_p^2)}{4(\omega^2 \pm \omega\omega_b - \omega_p^2)} \frac{dz}{L}}$$

$$= A_x e^{-\int \frac{D\phi_0}{2\phi_0} dz - \int \frac{dz}{2L}}$$

$$= A_x e^{-\frac{1}{2}\log \phi_0 - \frac{1}{2}\log \rho_0}$$

$$= A_x / \phi_0^{\frac{1}{2}} \rho_0^{\frac{1}{2}}$$

Therefore a solution for the ordinary and extraordinary fast waves is

$$(6.78)\quad E_x = \frac{B_x}{\rho_0^{\frac{1}{2}}} \left(\frac{\omega \pm \omega_b}{\omega(\omega^2 \pm \omega\omega_b - \omega_p^2)}\right)^{\frac{1}{4}} e^{-i\int \left(\frac{\omega(\omega^2 \pm \omega\omega_b - \omega_p^2)}{\omega \pm \omega_b}\right)^{\frac{1}{2}} \frac{dz}{c}} .$$

This result is, in essence, similar to results given in reference 1. Finally, the ordinary and extraordinary stress waves in this form give

$$(6.79)\quad E_x = A_x \rho_0^{\frac{1}{2}} e^{-i\int \left(\frac{S}{T}\right)^{\frac{1}{2}} \frac{(\omega \pm \omega_b)}{f} dz} .$$

Comparing these results with those of Ginzburg[1], it therefore follows that to zero order in f^2/c^2, the higher moment equations included in the present treatment do not affect the fast waves. (The stress waves, of course, are found as a result of including the higher moment terms). In addition, to first order in L^{-1}, different wave solutions remain independent. For example, waves of a particular type propagating in opposite directions obviously correspond to distinct uncoupled equations.

The second point to be considered concerns coupling and reflections. The form in which the wave solutions were expressed in section 6.4 consists of an oscillatory propagating function and an exponentially growing or decaying term, with respect to distance. At first sight it may appear that this is due to reflection of the wave by the inhomogeneous media (for decaying solutions) or augmentation by reflection from another wave (growing solutions), or thirdly, due to coupling with other wave modes. However, the only really satisfactory way to determine whether reflection or coupling between different modes is occurring is to consider the wave energy at various points along the propagation path. Brekhovskikh[2] has calculated the time average of the Poynting vector in the case of the usual (fast) electromagnetic waves (e.g. solutions (6.71)), and has shown to first order in L^{-1} (the approximation of geometric optics) that it is constant along the propagation axis.

It follows for this case, then, that there is no reflection, and solutions to order L^{-2} must therefore be sought.

As indicated above, the correct approach to the problem of reflection and coupling of this nature is to consider the time average of the complete energy equation for the wave variables. This equation may be written (cf. the energy equation in (6.1)):

$$(6.80)\qquad \frac{\partial}{\partial t}\left(\frac{3}{2}p + \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}\right) + \nabla\cdot\left(\underline{R} + \frac{5}{2}\frac{m}{e}\frac{p}{\rho}\underline{J} + \frac{\underline{E}\times\underline{B}}{\mu_0}\right) = 0,$$

collisions being neglected. This equation refers to the complete (rather than perturbation) dynamical variables.

Defining

$$(6.8)\qquad \underline{S} = \underline{R} + \frac{5}{2}\frac{m}{e}\frac{p}{\rho}\underline{J} + \frac{\underline{E}\times\underline{B}}{\mu_0},$$

with time average $\overline{\underline{S}}$, for a given region of plasma through which the wave is propagating,

$$(6.82)\qquad \int_{\text{Volume}} \nabla\cdot\overline{\underline{S}}\, d\underline{\tau} = 0$$

or

$$(6.83)\qquad \int_{\text{Surface}} \overline{S}_n\, dT = 0,$$

where $d\Gamma$ is the surface element, and $\bar{S}_n$ is the component of $\bar{\underline{S}}$ normal to the surface. In particular, for plane waves in the z direction, the energy loss, $\Delta \bar{S}$, in the rectangular region bounded by the planes $z = z_1$, $z = z_2$ ($z_2 > z_1$) is obviously

$$(6.84) \quad \Delta \bar{S} = \bar{S}_z(z_1) - \bar{S}_z(z_2)$$

for forward propagation. Since collisions are ignored, if and only if $\Delta \bar{S} \neq 0$, the energy transported to the region by the wave under consideration is removed from the region by another wave, either by reflection or by coupling with another mode (e. g. a fast wave coupling with a stress mode). From (6.81),

$$(6.85) \quad S_z = R_z + \frac{5}{2} \frac{m}{e} \frac{p_0}{\rho_0} J_z + \frac{(\underline{E} \times \underline{B})_z}{\mu_0}$$

and therefore for purely transverse waves,

$$(6.86) \quad S_z = \frac{(\underline{E} \times \underline{B})_z}{\mu_0} ,$$

while for purely longitudinal waves,

$$(6.87) \quad S_z = R_z + \frac{5}{2} \frac{m}{e} \frac{p_0}{\rho_0} J_z .$$

It follows from these equations that the energy in both the fast and the stress transverse waves is carried by the electromagnetic fields, and not by the motion of the particles. For longitudinal waves it is of course obvious that the energy is transferred by the particle motion.

Consider the transverse stress wave solution, (6.72). Maxwell's equations give

$$(6.88)\quad B_y = \frac{i}{\omega}\frac{\partial}{\partial z}E_x = \frac{i}{\omega}A_x\rho_0^{\frac{1}{2}}\left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} \mp i\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f}\right)e^{\mp i\int\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\omega}{f}dz}.$$

Therefore the (time averaged) Poynting vector gives

$$(6.89)\quad \bar{S}_z = \frac{A_x^2}{2\mu_0}\left(\frac{5}{7}\right)^{\frac{1}{2}}\frac{\rho_0}{f}.$$

This dependence of $\bar{S}_z$ on ρ_0 indicates that energy is transferred from the wave by reflection (it is evident from the fact that $\bar{S}_z$ is constant for fast waves that energy cannot be transferred to these by coupling). The mechanism of the reflection is not fully understood and will not be further discussed.

The case of longitudinal waves is more complicated, requiring J_z and R_z to be expressed in terms of E_z. This process is similar to that involved in deriving the dispersion equations. Brekhovskikh[2] has shown that to first order in L^{-1}, the energy flux vector for sound waves is constant. Since, however, the heat flux term was excluded, this result is possibly not relevant to the longitudinal waves considered above.

The question of boundary conditions for the waves has not been discussed in this Chapter. However, it is obvious that reflection and coupling between modes are closely associated with boundary conditions. And while the fast wave geometrical optics solutions do not correspond

to reflection in the plasma, reflection at a sharp boundary is easily accounted for. This is a property of the boundary, and applies also for waves in a bounded homogeneous medium. Since this is discussed in many text books, it is adequate to mention that in most cases the reflection of waves at a cut-off is satisfactorily described by a "sharp boundary" approximation.

Finally, the usefulness of the solutions obtained in this Chapter may be mentioned. The approximation of geometrical optics has, in fact, a wide range of applicability. In many laboratory plasmas, in the ionosphere, and in space, for example, provided the wave frequency is not close to cut-off, this approximation is usually adequate. Furthermore, the effect of ("small") collisions, although specifically excluded from this Chapter, may be approximately accounted for in these results by including the collisional damping terms obtained in Chapter 5.

References

1. Ginzburg, V. L., "The Propagation of Electromagnetic Waves in Plasmas", (transl. from the Russian), Pergamon (1964).
2. Brekhovskikh, L. M., "Waves in Layered Media", (transl. from the Russian), Academic (1960).

Appendix A

The Differential Cross Sections

A1.1 The Born Approximation

In this Appendix, quantum-mechanical expressions are obtained for the differential cross sections for certain processes involving electrons, hydrogen atoms and protons, which are required to calculate the ϕ's and Ω's. A Born type of approximation has been chosen to give analytical expressions for the differential cross section, and a numerical adjustment factor is included which is selected to obtain a "reasonable" approximation to the known total cross sections in the energy range 0 - 100 ev. The notation is the same as used in Chapter 3.

Proceeding formally, consider a collision involving atomic particles j and k referred to a frame moving with the centre of mass. Suppose that $\underline{r}_j$, $\underline{r}_k$ represent the electron position vectors of the particles relative to their respective nuclei (these being zero of course if the particles are electrons), and $\underline{r}$ the inter-nuclear distance, with $V(\underline{r}, \underline{r}_j, \underline{r}_k)$ the interaction potential. $\mathcal{H}_j(\underline{r}_j)$, $\mathcal{H}_k(\underline{r}_k)$ are the internal (unperturbed) Hamiltonians of the two particles, referred to their respective nuclei, $\underline{g}_{jk}$ is the initial asymptotic relative velocity, and $\Psi(\underline{r}, \underline{r}_j, \underline{r}_k)$ is the complete wave function for both particles. The the Schrodinger equation for the system is

$$\text{(A1)} \qquad \left[\frac{\hbar^2}{2m_r}\frac{\partial^2}{\partial \underline{r}^2} - \mathcal{H}_j - \mathcal{H}_k + \tfrac{1}{2} m_r g_{jk}^2 + E_o - V\right]\Psi = 0$$

where $m_r = m_j m_k/(m_j+m_k)$ is the reduced mass, and E_o the unperturbed ground state energy of the system associated with $\mathcal{H}_j$ and $\mathcal{H}_k$. The term $\frac{1}{2} m_r g_{jk}^2$ is the asymptotic kinetic energy of the system.

Neglecting the possibility of electron exchange, Ψ can be expanded as

$$(A2)\quad \Psi(\underline{r},\underline{r}_j,\underline{r}_k) = \left(\sum_q + \int\right) \psi_q(\underline{r}_j,\underline{r}_k)\, F_q(\underline{r})$$

where $\psi_q(\underline{r}_j,\underline{r}_k)$ is the set of all possible $\phi_q(\underline{r}_j)\,\phi_q(\underline{r}_k)$ which are eigenfunctions for the particles j and k. $F_q(\underline{r})$ is essentially a scattering function, to be determined by solving the Schrodinger equation. $E_q = E_{q,j} + E_{q,k}$, is the sum of the eigenvalues of the two particles corresponding to the eigenfunctions $\phi_q(\underline{r}_j)$ and $\phi_q(\underline{r}_k)$. The summation is over discrete states and the integration over continuum states. Substituting this expansion in the Schrodinger equation (A1) gives

$$(A3)\quad \left(\sum_q + \int\right) \psi_q(\underline{r}_j,\underline{r}_k) \left[\frac{\hbar^2}{2m_r}\frac{\partial^2}{\partial \underline{r}^2} + \frac{1}{2} m_r g_{jk}^2 + E_o - E_q\right] F_q(\underline{r})$$
$$= V(\underline{r},\underline{r}_j,\underline{r}_k)\, \Psi(\underline{r},\underline{r}_j,\underline{r}_k)\ .$$

Multiplying (A1) by $\psi^*(\underline{r}_j,r_k)\,d\underline{r}_j\,d\underline{r}_k$ and integrating, this becomes

$$(A4)\quad \left[\frac{\hbar^2}{2m_r}\frac{\partial^2}{\partial \underline{r}^2} + \frac{1}{2} m_r g_{jk}^2 + E_o - E_q\right] F_q(\underline{r})$$
$$= \int V(\underline{r},\underline{r}_j,\underline{r}_k)\, \Psi(\underline{r},\underline{r}_j,\underline{r}_k)\, \psi_q^*(\underline{r}_j,\underline{r}_k)\, d\underline{r}_j\, d\underline{r}_k .$$

$F_o(\underline{r})$ represents the sum of an incident and a scattered wave, while $F_q(\underline{r})$ $(q \neq o)$ must represent scattered waves only. Hence the respective asymptotic forms can be shown to be (Mott and Massey[1]):

(A5) $$F_o \sim e^{i\underline{k}_o\cdot\underline{r}} + \frac{1}{r} e^{i\underline{k}_f(o)\cdot\underline{r}} f_o(\chi,\epsilon)$$

(A6) $$F_q \sim \frac{1}{r} e^{i\underline{k}_f(q)\cdot\underline{r}} f_q(\chi,\epsilon)$$

where $\underline{k}_o$ is the (reduced) incident wave vector such that the reduced momentum, $\underline{p}$, is given by $\hbar\underline{k}_o$. Similarly $\underline{k}_f(q)$ is the reduced wave vector of the scattered particle for the state q. Then the scattered current density for the qth state of excitation is

(A7) $$\frac{\hbar}{m_r}\underline{k}_f(q)\left|\frac{1}{r} e^{i\underline{k}_f(q)\cdot\underline{r}} f_q(\chi,\epsilon)\right|^2 = \frac{\hbar}{m_r}\underline{k}_f(q)\left|\frac{f_q(\chi,\epsilon)}{r^2}\right|^2.$$

The differential cross section for the deflection of the final wave vector, $\underline{k}_f(q)$, into a solid angle $(\Omega, \Omega+d\Omega)$ with excitation state q is the ratio of the scattered current per unit solid angle about Ω to the incident current density, giving

(A8) $$\sigma_q d\Omega = \frac{k_f}{k_o}\left|f_q(\chi,\epsilon)\right|^2 d\Omega .$$

Subject to the plane wave Born approximation,

(A9) $$\Psi = e^{i\underline{k}_o\cdot\underline{r}}\,\psi_o(\underline{r}_j,\underline{r}_k).$$

Substituting this expression for Ψ in the right hand side of (A4), this equation becomes

(A10) $$\left[\frac{\hbar^2}{2m_r}\frac{\partial^2}{\partial\underline{r}^2} + \frac{\hbar^2 k_f^2(q)}{2m_r}\right] F_q(\underline{r}) = \int V(\underline{r},\underline{r}_j,\underline{r}_k)\, e^{i\underline{k}_o\cdot\underline{r}} \times \psi_o(\underline{r}_j,\underline{r}_k)\,\psi_q^*(\underline{r}_j,\underline{r}_k)\, d\underline{r}_j\, d\underline{r}_k$$

with asymptotic solution

$$(A11)\quad F_q(\underline{r}) = \frac{1}{r}\, e^{i\underline{k}_f(q)\cdot\underline{r}}\, \frac{m_r}{2\pi\hbar^2} \int V e^{i\underline{K}\cdot\underline{r}}\, \psi_o\, \psi_q^*\, d\underline{r}_j\, d\underline{r}_k$$

or

$$(A12)\quad f_q(\chi,\epsilon) = \frac{m_r}{2\pi\hbar^2} \int V e^{i\underline{K}\cdot\underline{r}}\, \psi_o\, \psi_q\, d\underline{r}_j\, d\underline{r}_k\, d\underline{r}.$$

In these equations, $\underline{K} = \underline{k}_o - \underline{k}_f(q)$ and $k_f^2 = \frac{m_r^2}{\hbar^2} g_{jk}^2 + \frac{2m_r}{\hbar^2}(E_o - E_q)$.

In the case of excitation to the continuum level, ψ_q is the wave function whose matrix element, taken with the ground state wave function and interaction potential, gives the probability of a particular direction and energy of the atomic electron (as well as the direction and energy of the scattered particle). With the Hamiltonians $H_j(\underline{r}_j)$, $H_k(\underline{r}_k)$ of the electrons of the particles j and k referred to their respective centres of mass, the eigenfunctions $\phi_q(r_j)$, $\phi_q(r_k)$, grouped under the term ψ_q, therefore relate the continuum state electron parameters to the centre of mass of the particle, say k, from which it was displaced. Consequently the angle and energy of the ejected electron is not referred to the centre of mass of the system j and k, but to particle k. In order to use the differential cross sections in the centre of mass system, it becomes necessary to convert these parameters to this system. Bearing this in mind, the above results give, formally, for ionisation (with $\underline{\chi}$ the atomic electron wave vector):

$$\text{(A13)}\quad \sigma\, d\Omega_1\, d\Omega_2\, d\mathcal{K} = \frac{k_f}{k_0}\left| f(\chi_1, \epsilon_1, \chi_2, \epsilon_2, \mathcal{K})\right|^2 d\Omega_1\, d\Omega_2 d\mathcal{K}$$

where the subscripts $1, 2$ refer to the scattered (incident) particle and the atomic (ejected) electron, respectively.

Regarding the frames of reference, the differential cross section as obtained above has to be transformed to the frame moving with the centre of mass of the colliding particles (which will be called the "c.m. frame"). It is also necessary to obtain an expression for the internal energy of the "composite" particle produced in the collision, as discussed in Chapter 2. The types of collisions being considered are those in which one of the colliding particles, say j, is scattered, while the other particle, say k, on collision breaks up into two "sub-particles", r and s. The form (A13) of the differential cross section refers the angle and velocity of j to the c.m. frame, but (owing to the reference frame of the Hamiltonians) it refers the velocities and solid angles of r and s to a frame moving with their centre of mass (the "k" frame). Using $\underline{v}$, χ and Ω to denote velocities, angles and solid angles in the c.m. frame, and $\underline{u}$, θ and ω velocities, angles, and solid angles in the "k" frame, the problem is to express $\underline{u}_r$, $\underline{u}_s$, ω_r and ω_s in terms of Ω and $\underline{v}$. Clearly,

$$\text{(A14)}\quad m_j \underline{v}_j + m_k \underline{v}_k = 0 \quad \text{or} \quad \underline{v}_k = -\frac{m_j}{m_k}\underline{v}_j .$$

Then

$$\text{(A15)}\quad \underline{v}_r = \underline{u}_r + \underline{v}_k = \underline{u}_r - \frac{m_j}{m_k}\underline{v}_j ,$$

$$\underline{v}_s = \underline{u}_s - \frac{m_j}{m_k}\underline{v}_j .$$

The azimuthal angle, that is the angle with the plane of the collision, is the same for both frames. Hence the elements of solid angle in the two frames are respectively $d\Omega = \sin\chi \, d\chi \, d\epsilon$ and $d\omega = \sin\theta \, d\theta \, d\epsilon$. Taking components of velocity of (A15) in a cartesian frame gives

$$(A16) \quad v_{rx} = v_r \cos\chi_r = u_r \cos\theta_r - \frac{m_j}{m_k} v_j \cos\chi_j$$

$$v_{ry} = v_r \sin\chi_r = u_r \sin\theta_r - \frac{m_j}{m_k} v_j \sin\chi_j .$$

From these relations it follows that

$$(A17) \quad u_r^2 = v_r^2 + \left(\frac{m_j}{m_k}\right)^2 v_j^2 + 2 v_j v_r \frac{m_j}{m_k} \cos(\chi_j - \chi_r)$$

$$(A18) \quad u_s^2 = v_s^2 + \left(\frac{m_j}{m_k}\right)^2 v_j^2 + 2 v_j v_s \frac{m_j}{m_k} \cos(\chi_j - \chi_s)$$

$$(A19) \quad \sin\theta_r = \frac{v_r \sin\chi_r + (m_j/m_k) v_j \sin\chi_j}{\left(v_r^2 + (m_j/m_k)^2 v_j^2 + 2 v_r v_j (m_j/m_k) \cos(\chi_j - \chi_r)\right)^{\frac{1}{2}}}$$

$$(A20) \quad \sin\theta_s = \frac{v_s \sin\chi_s + (m_j/m_k) v_j \sin\chi_j}{\left(v_s^2 + (m_j/m_k)^2 v_j^2 + 2 v_s v_j (m_j/m_k) \cos(\chi_j - \chi_r)\right)^{\frac{1}{2}}} .$$

When j is an electron and k an atom, on neglecting $\frac{m_e}{m_a}$ compared with unity, the c.m. and " k " frames become identical, as would be expected.

The $\Delta\mathcal{E}$ term is discussed in Chapter 2. Essentially it is the sum of the ionisation energy and the internal kinetic energy of the composite "particle". Suppose of three particles, q, s, t, q and s form a composite "particle", then for this case

$$(A21) \quad \Delta\mathcal{E} = I + \frac{1}{2}\frac{m_q m_s}{m_q + m_s}\left(\underline{v}_q - \underline{v}_s\right)^2 .$$

This is given later (in Appendix B) for the particular case of electron-hydrogen atom ionising collisions.

A1.2 The Born Differential Cross Sections

For convenience of writing in this and subsequent sections, as in Chapter 3, define $e^2 \equiv q^2/(4\pi\epsilon_0)$ where q is the elementary unit of charge. The notation in this section is the same as that of the previous section, and in particular, of Chapter 3, where many of the symbols are defined.

A1.2.1 Elastic Collisions

(a) Electron-electron collisions

With the neglect of the electron spin, the classical method, the Born approximation, and the exact partial wave calculation all yield the well known Coulomb collision result for the differential cross section:

$$(A22) \quad \sigma\, d\Omega = \frac{m_r^2 e^4 \operatorname{Cosec}^4 \chi/2}{4\hbar^4 k_0^4}\, d\Omega .$$

Since the nature of the collision terms in the moment equations involves treating the colliding particles as distinguishable (when they are identical), it is doubtful whether it would be adequate to take account of exchange effects purely in the differential cross section. Although the energy range 0 - 100 ev is that in which exchange effects make their greatest contribution to the cross sections, they are not considered in this thesis.

(b) Electron-hydrogen atom collisions

Using (A8), (A12) with the ground state wave functions for hydrogen, since the interaction potential is

$$(A23)\quad V = e^2\left(\frac{1}{r} - \frac{1}{|\underline{r}-\underline{r}_1|}\right)$$

where $\underline{r}_1$ is the atomic electron position vector with respect to the nucleus, the Born differential cross section is

$$(A24)\quad \sigma\, d\Omega = \frac{4\, a_0^4\, m_r^2\, e^4\, (8 + K^2 a_0^2)^2}{\hbar^4\, (4 + K^2 a_0^2)^4}\, d\Omega \,.$$

In this equation, (as defined in Chapter 3), $K^2 = (\underline{k}_0 - \underline{k}_f)^2 = 2k_0^2(1-\cos\chi)$.

(c) Electron-proton collisions

The Coulomb result applies for this case, the differential cross section being

$$(A25)\quad \sigma\, d\Omega = \frac{m_r^2\, e^4\, \mathrm{Cosec}^4\, \chi/2}{4\, \hbar^4\, k_0^4}\, d\Omega \,.$$

(d) Hydrogen atom-hydrogen atom collisions

Using $\underline{r}_1$, $\underline{r}_2$ for the position vectors of the atomic electrons referred to their respective nuclei, and $\underline{r}$ for the internuclear distance, the interaction potential is

$$(A26)\quad V = e^2\left(-\frac{1}{r} + \frac{1}{|\underline{r}-\underline{r}_1|} + \frac{1}{|\underline{r}+\underline{r}_2|} - \frac{1}{|\underline{r}+\underline{r}_2-\underline{r}_1|}\right).$$

The colliding atoms are regarded as distinguishable, in which case, using the ground state wave functions, from (A8) and (A12), the differential cross section for these collisions is

$$\text{(A27)}\quad \sigma\, d\Omega = \frac{4\, m_r^2 e^4 a_o^8 K^4 (a_o^2 K^2 + 8)^4}{\hbar^4 (a_o^2 K^2 + 4)^8}\, d\Omega .$$

(e) Hydrogen atom-proton collisions

Apart from the difference in the reduced masses, these collisions are identical with the electron-hydrogen atom collisions, of case (b). The differential cross section is

$$\text{(A28)}\quad \sigma\, d\Omega = \frac{4\, m_r^2 e^4 a_o^4 (a_o^2 K^2 + 8)^2}{\hbar^4 (a_o^2 K^2 + 4)^4}\, d\Omega .$$

(f) Proton-proton collisions

These Coulomb collisions are formally the same as case (a). The differential cross section is therefore

$$\text{(A29)}\quad \sigma\, d\Omega = \frac{m_r^2 e^4 \operatorname{Cosec}^4 \chi/2}{4 \hbar^4 k_o^4}\, d\Omega .$$

A1.2.2 Inelastic Collisions

It is reasonable to suppose that in an inelastic collision with an atom, owing to the screening effect of the atomic electron, atoms will have a smaller cross section than protons. Again, using the Born approximation, an estimate of the relative magnitudes of the cross sections of protons and electrons in inelastic collisions with atoms can be obtained as follows (Bates and Griffing[2]). Let $Q_e(E)$, $Q_p(E)$

be total cross sections for a given electron or proton energy E respectively. Then if $\Delta\mathcal{E}$ is the kinetic energy defect in the collision the Born approximation gives

$$(A30)\quad Q_e(\gamma E) = \frac{1}{\gamma}\left[Q_p\left(\frac{m_p}{m_e}E\right) - \left(\frac{\Delta\mathcal{E}}{4E}\right)^2 Q_p\left(\frac{m_p\,\Delta\mathcal{E}}{16\,m_e E}\right)\right]$$

where

$$(A31)\quad \gamma = (1 + \Delta\mathcal{E}/4E)^2 \sim 1 + \Delta\mathcal{E}/2E$$

and m_e, m_p are the electron and proton masses. Very approximately, this gives

$$(A32)\quad Q_p(E) = \gamma\, Q_e\left(\frac{\gamma m_e E}{m_p}\right).$$

According to (A32), then, the proton-atom cross section for a given incident energy is of the same order as the electron-atom cross section for an incident energy m_e/m_p of the proton energy. Since for low energies the electron cross section is small, the proton-atom cross section (and the atom-atom cross section also) will be very small in the energy range 0 - 100ev. It is reasonable, therefore, to neglect protons and atoms as inelastic collision projectiles for the calculations of Chapter 3.

(a) Electron-hydrogen atom 1s-2p excitation collisions

The only excitation process that will be considered is 1s-2p excitation, this process having a much larger total cross section than that to any other level from the ground state. The interaction potential

is given in (A23). Using the ground state hydrogen atom wave function and the $2p\ (m=0)$ wave function in (A11) to calculate $f(1s-2p)$, from (A8) the differential cross section is

$$\text{(A33)}\qquad \sigma\, d\Omega = \frac{k_f\, 3^2\, 2^{15} m_r^2 a_0^2\ d\Omega}{k_0\, \hbar^4\, K^2 (9 + 4K^2 a_0^2)^6} .$$

(b) Electron-hydrogen atom ionising collisions

For these collisions, using the exact continuum wave function (Mott and Massey[1], p. 233) in the calculation of the Born approximation for the differential cross section leads to an exceedingly complicated expression. Let $d\Omega_1$ be the solid angle into which the incoming electron is scattered; $d\Omega_2$ the solid angle into which the atomic electron is ejected with wave vector $\underline{\kappa}$; and $\underline{K} = \underline{k}_0 - \underline{k}_f$; $\underline{k}_0$, $\underline{k}_f$ being initial and final wave vectors of the incident electron, as before. Then the differential cross section for this collision is

$$\text{(A34)}\qquad \sigma\, d\Omega_1\, d\Omega_2\, d\kappa = \frac{2^8 m_r^2 a_0^4 e^4 k_f\, \kappa\, e^{-\frac{2}{\kappa a_0}\tan^{-1}\left(\frac{2\kappa a_0}{1 + a_0^2(K^2-\kappa^2)}\right)}}{4\pi^2 \hbar^4 \pi\, k_0 K^2 (a_0^2(K^2+\kappa^2-2\kappa K\cos\delta)+1)^4}$$

$$\cdot\ \frac{(a_0^2(K-\kappa\cos\delta)^2 + \cos^2\delta)\ \ d\Omega_1\, d\Omega_2\, d\kappa}{(1 - e^{-\frac{2\pi}{\kappa a_0}})((1-\kappa^2 a_0^2 + K^2 a_0^2)^2 + 4\kappa^2 a_0^2)} .$$

In this equation, δ is the angle between $\underline{K}$ and $\underline{\kappa}$. $d\Omega_2$ and $\underline{\kappa}$ are in fact referred to a frame moving with the centre of mass of the atom after the collision, but it is apparent from equations (A17) to (A20) that the error involved in taking these variables as referred to the c. m. for the collision is only of order m_e / m_a .

However, it is obvious that this expression (A34) is unsuitable for calculation of the ϕ's and Ω's in Chapter 3, where successive integrations over functions involving the differential cross section are necessary. Even when this differential cross section is averaged over the solid angle of ejection of the atomic electron, the resulting expression is too complicated to be used in evaluating the Ω's .

The problem may, however, be approached in a simpler way. The main reason for the complicated form of (A34) is the complex nature of the continuum wave function for the hydrogen atomic electron. It is therefore reasonable to consider replacing it by a plane wave function, in a similar manner to that in which the relevant Born approximation was obtained by using a plane wave (A9) in equation (A10). Such a plane wave is

$$\phi_1 = e^{i \underline{\kappa} \cdot \underline{r}_1} , \tag{A35}$$

$\underline{r}_1$ being the position vector of the atomic electron relative to the c.m. (But, similarly to the Born approximation being valid for large energies of the incident electron, so this approximation is actually valid only for high energies of the ejected electron.) The appropriate normalising factor is not entirely clear, but to obtain the correct form of the differential cross section, $\phi_1 \phi_1^* \, d\kappa$ should represent a probability density of particles with wave numbers κ in the range $(\kappa, \kappa + d\kappa)$, and so have dimensions (volume)$^{-1}$. A comparison of the final form obtained, using this wave function, with (A34), obtained by using the correct wave function, suggests that the correct form for ϕ_1 is

$$\text{(A36)} \quad \phi_1 = \left(\frac{\kappa}{a_0}\right)^{\frac{1}{2}} e^{i \underline{\kappa} \cdot \underline{r}} .$$

The use of an adjusting factor with the total cross section obtained using this wave function will mean that the neglect of numerical factors is of no consequence. With (A36) for the final wave function and the normal ground state hydrogen wave function as the initial wave function, the Born approximation, equation (A13), may be used to calculate $f(\kappa)$. From (A8) the differential cross section is found to be

$$\text{(A37)} \quad \sigma \, d\Omega_1 \, d\Omega_2 \, d\kappa = \frac{2^6 \pi a_0^2 m_r^2 e^4 k_f \kappa \; d\Omega_1 \, d\Omega_2 \, d\kappa}{\hbar^4 k_0 K^4 \left(a_0^2 (K^2 - \kappa^2) + 1\right)^4} .$$

In general, $|\underline{K}| >> |\underline{\kappa}|$ and therefore it is convenient to simplify (A37) to

$$\text{(A38)} \quad \sigma \, d\Omega_1 \, d\Omega_2 \, d\kappa = \frac{2^6 \pi a_0^2 m_r^2 e^4 k_f \kappa}{\hbar^4 k_0 K^4 (a_0^2 K^2 + 1)^4} \, d\Omega_1 \, d\Omega_2 \, d\kappa .$$

Again, since $\hbar^2 \kappa^2 = 2 m_r E_\kappa$, (A37) is easily rearranged to give the probability of ejection of the atomic electron in the energy range $(E_\kappa, E_\kappa + dE_\kappa)$.

(c) Proton-hydrogen atom charge exchange collisions

Resonant charge exchange, where the electron energy relative to the nucleus is the same for the final "atoms" as for the initial "atoms", is in general the only type of charge exchange collision with an appreciable cross section. This can occur between hydrogen atoms and protons. It has been shown (Bates[3]) that the correct interaction potential for such collisions is not the simple "post" or "prior" interaction (identical for the symmetrical proton hydrogen atom system), but a much more

complicated function. Furthermore, in calculating the collision cross section, account should be taken of the change in translational motion of the electron that accompanies the jump between the nuclei. However, in order to achieve reasonable analytical expressions, both of these considerations will be ignored, it being assumed that the inclusion of a numerical adjustment factor will bring the Born calculation under these simplifications to an acceptable approximation for this process.

The "prior" interaction potential is

$$(A39)\quad V = e^2\left(\frac{1}{|\underline{r}+\underline{r}_1|} - \frac{1}{r}\right)$$

where $\underline{r}$ is the internuclear separation, and $\underline{r}_1$ the position vector of the electron relative to the nucleus that it has before collision. The Born approximation for this case gives

$$(A40)\quad f = \frac{m_r}{2\pi\hbar^2}\int V e^{i\underline{K}\cdot\underline{r}}\,\psi_o(r_1)\,\psi_f^*(\underline{r}_1)\,d\underline{r}_1\,d\underline{r}\ ,$$

but, somewhat differently from the previous cases, $\underline{k}_o$ is the (reduced) wave vector of the incident nucleus, $\underline{k}_f$ the wave vector of the "scattered" nucleus; ψ_o, ψ_f are the wave functions of the electron in the initial and final atoms; and also

$$(A41)\quad \underline{K} = \underline{k}_o + \underline{k}_f\ ;\quad K^2 = 2k_o^2(1+\cos\chi)\ .$$

Using ground state hydrogen atom wave functions for ψ_o, ψ_f to obtain f, the differential cross section follows from (A8):

$$(A42)\quad \sigma\,d\Omega = \frac{m_r^2 e^4 2^8 a_o^4}{\hbar^4(a_o^2K^2+1)^6}\,d\Omega\ .$$

A1.3 The Total Cross Sections

In this section, the expressions obtained for the differential cross sections in the previous section are integrated over the scattering angles to obtain the total collision cross sections as a function of incident energy and other parameters. These total cross sections are compared with experimental results or exact numerical calculations to give a numerical adjustment factor. This factor is used to bring the "Born" approximations into reasonable agreement with the probable cross sections in a region for which the plane wave assumption is entirely inadequate. However since the energy range of interest is limited, the approximations are considered sufficient to be used in the calculations of the $\Omega^{1,s}$ to obtain an estimate of the collision terms in the moment equations for the formation of a plasma. As discussed previously, it is likely that certain classical approximations better describe the non-Coulomb elastic collisions involved.

A1.3.1 Elastic Collisions

(a) Coulomb collisions (electron-electron, electron-proton, proton-proton)

With Coulomb collisions the integral of the differential cross section diverges. For this reason the well known Debye cut-off is introduced, the Debye length representing a distance from a particle beyond which the electrostatic shielding of the other particles is "complete". Usually put forward as an upper limit on an impact parameter, it may equally well be expressed as a minimum angle of deflection in a collision.

Since the details of Coulomb collisions are discussed in many books, only the bare facts need be given. In terms of an impact

parameter b , the scattering angle χ is

$$\chi = 2 \sin^{-1}\left(\frac{1}{(1+\Lambda^2)^{\frac{1}{2}}}\right) \tag{A43}$$

where

$$\Lambda = \frac{b\, m_r\, g^2}{e^2} . \tag{A44}$$

The Debye length, λ_D , being

$$\lambda_D = \left(\frac{k\, T_e}{4\pi n_e e^2}\right)^{\frac{1}{2}}, \tag{A45}$$

it follows at once that the minimum deflection χ_o is given by (A43) on taking $\Lambda = \Lambda_{jk}$, where

$$\Lambda_{jk} = \frac{\lambda_D\, m_r\, g_{jk}^2}{e^2} = \left(\frac{k T_e}{4\pi n_e e^2}\right)^{\frac{1}{2}} \frac{m_r\, g_{jk}^2}{e^2} . \tag{A46}$$

The total cross section, in terms of χ_o, is

$$\sigma_T = \frac{2\pi m_r^2 e^4}{4\hbar^4 k_o^4}\left(\frac{8}{\chi_o^2}\right) . \tag{A47}$$

No correction factors are required for this well known result.

(b) Electron-hydrogen atom collisions

Since $K^2 = 2k_o^2(1-\cos\chi)$, on differentiating,

$$K\, dK = k_o^2 \sin\chi\, d\chi . \tag{A48}$$

The upper and lower limits of integration over K become $2k_o$ and 0. Using the differential cross section (A24), with the substitution (A48) the total cross section is

$$(A49)\quad \sigma_T = \int_0^{2k_o}\int_0^{2\pi} \frac{4a_o^4 m_r^2 e^4 (K^2a_o^2+8)^2 K\,dK\,d\epsilon}{\hbar^4 k_o^2 (K^2a_o^2+4)^4}$$

$$= \frac{\pi m_r^2 e^4 a_o^2 (7k_o^6a_o^6 + 18k_o^4a_o^4 + 12k_o^2a_o^2)}{3\hbar^4 k_o^2 (1+k_o^2a_o^2)^3} .$$

(c) Hydrogen atom-hydrogen atom collisions

Using (A48) in the differential cross section (A27), the application of standard integrals gives the total cross section:

$$(A50)\quad \sigma_T = \frac{8\pi m_r^2 e^4 a_o^2}{\hbar^4 k_o^2 (1+k_o^2a_o^2)^7}\left(\frac{33}{140}k_o^{14}a_o^{14} + \frac{7}{5}k_o^{12}a_o^{12} + \frac{16}{5}k_o^{10}a_o^{10} + \frac{10}{3}k^8a_o^8 + \frac{4}{3}k_o^6a_o^6\right) .$$

(d) Hydrogen atom-proton collisions

This is algebraically the same result as for electron-hydrogen atom collisions:

$$(A51)\quad \sigma_T = \frac{\pi m_r^2 e^4 a_o^2 (7k_o^6a_o^6 + 18k_o^4a_o^4 + 12k_o^2a_o^2)}{3\hbar^4 k_o^2 (1+k_o^2a_o^2)^3} .$$

A1.3.2 Inelastic Collisions

Since

$$(A52)\quad K^2 = k_o^2 + k_f^2 - 2k_ok_f\cos\chi ,$$

differentiation yields

$$(A53)\quad K\,dK = k_o k_f \sin\chi\, d\chi \ .$$

The limits of integration with respect to K are $k_o \pm k_f$; from the energy equation for the type of collision being considered,

$$(A54)\quad k_o^2 = k_f^2 + \frac{2 m_r \Delta\mathcal{E}}{\hbar^2} \ ,$$

giving

$$(A55)\quad k_f = \left(k_o^2 - \frac{2 m_r \Delta\mathcal{E}}{\hbar^2} \right)^{\frac{1}{2}} \sim k_o \left(1 - \frac{m_r \Delta\mathcal{E}}{\hbar^2 k_o^2} \right)$$

where $\Delta\mathcal{E}$ is the energy transferred inelastically to the atom. The upper and lower limits of K are taken as $2k_o$ and $m_r \Delta\mathcal{E}/\hbar^2 k_o$, respectively. The approximate limits cannot be expected to be reasonable near the threshold energy, and they are therefore "over-ruled" in this region by taking $\sigma_T = 0$ at threshold.

(a) Electron-hydrogen atom 1s-2p excitation

For these collisions $\Delta\mathcal{E}$ is the (discrete) excitation energy (10.2 ev). From (A33) and using (A53) the total cross section is

$$(A56)\quad \sigma_T = \int_{m_r \Delta\mathcal{E}/\hbar^2 k_o}^{2k_o} \int_0^{2\pi} \frac{k_f m_r^2 e^4 3^2 2^{15} a_o^2 \, K\,dK}{k_f k_o^2 \hbar^4 K^2 (9 + 4K^2 a_o^2)^6}$$

$$\sim \frac{2^{17} \pi m_r^2 e^4 a_o^2}{9^5 \hbar^4 k_o^2} \log \frac{3 k_o \hbar^2}{a_o m_r \Delta\mathcal{E}}$$

for sufficiently large k_o .

(b) Electron-hydrogen atom ionisation collisions

With ionisation collisions, in equation (A54), $\Delta\mathcal{E}$ becomes

$$\Delta\mathcal{E} = I + \frac{\hbar^2 \mathcal{K}^2}{2m_e} \tag{A57}$$

where m_e is the mass of the ejected electron, and I is the ionisation energy. For this case the upper and lower limits on K are more conveniently written as $2k_0$ and $(m_r I/\hbar^2 k_0 + \mathcal{K}^2/2k_0)$. The limits of integration over $\mathcal{K}$ are 0 and $(k_0^2 - 2m_r I/\hbar^2)^{\frac{1}{2}} \equiv \alpha$. From (A38) and using (A53) the total cross section is given by

$$\sigma_T = \int_0^{\alpha} \int_{k_0-k_f}^{k_0+k_f} \int_0^{2\pi} \frac{k_f K m_r^2 e^4 2^6 \pi a_0^2 \mathcal{K} d\mathcal{K}\, dK\, d\epsilon\, d\Omega_2}{k_f \hbar^4 k_0^2 K^4 (a_0^2 K^2 + 1)^4} \tag{A58}$$

where $d\Omega_2$ is the solid angle of the ejected electron. With k_0 sufficiently large,

$$(k_0 + k_f) a_0 \sim 2k_0 a_0 - \frac{a_0}{k_0}\left(\frac{m_e I}{\hbar^2} + \frac{\mathcal{K}^2}{2}\right) >> 1 ;$$

$$(k_0 - k_f) a_0 \sim \frac{a_0}{k_0}\left(\frac{m_e I}{\hbar^2} + \frac{\mathcal{K}^2}{2}\right) << 1 . \tag{A59}$$

Using these approximations (invalid near threshold), the result is

$$\sigma_T = \frac{m_e^2 e^4 2^9 \pi^3 a_0^2}{\hbar^4}\left(\frac{\hbar^2}{2m_e I} - \frac{1}{k_0^2}\right) \tag{A60}$$

(since, on neglecting the term $\frac{m_e}{m_a}$ compared with unity, the reduced mass, m_r, is equal to the electron mass, m_e).

(c) Charge exchange collisions

As discussed earlier for this case, K^2 is given by $2k_o^2(1+\cos\chi)$ and therefore

$$\text{(A61)} \quad \sin\chi \, d\chi = - K dK / k_o^2 \, .$$

This collision being effectively elastic, using this relation in (A42) and integrating over K with limits 0, $2k_o$ gives the total cross section. For k_o large, (owing to the large reduced mass in this collision), the result is

$$\text{(A62)} \quad \sigma_T = \frac{\pi m_r^2 e^4 \, 2^9 a_o^2}{5 \hbar^4 k_o^2} \, .$$

A1. 3. 3 Comparison with Known Values of Cross Sections

Since the Coulomb calculations are exact (apart from neglect of exchange) these are not discussed. For all of the other cases considered except elastic hydrogen atom-hydrogen atom collisions (for which no data appears to be available, perhaps owing to the problem of recombination to form molecular hydrogen), curves showing a comparison of numerically adjusted Born-type total cross sections with known values are given. Most of the "known" data is taken from Barnett, Ray and Thompson[4]. The charge exchange cross sections are from Bates[5].

The numerical factors which the calculated Born-type total cross sections have been multiplied by to give the results in the figures are as follows:

(a) Electron-hydrogen atom elastic collisions.

Born calculation multiplied by 0.45.

(b) Hydrogen atom-proton elastic collisions.

Born calculation multiplied by 2.0×10^{-2}.

(c) Hydrogen atom-hydrogen atom elastic collisions.

No comparison with "known" results being possible, the same factor as with (b) above is taken: 2.0×10^{-2}.

(d) Electron-hydrogen atom 1s-2p excitation.

Born calculation multiplied by 0.45.

(e) Electron-hydrogen atom ionisation.

Born calculation (simplified) multiplied by 0.14×10^{-3}.

(f) Hydrogen atom-proton charge exchange.

Born calculation (simplified) multiplied by 2.1×10^{-2}.

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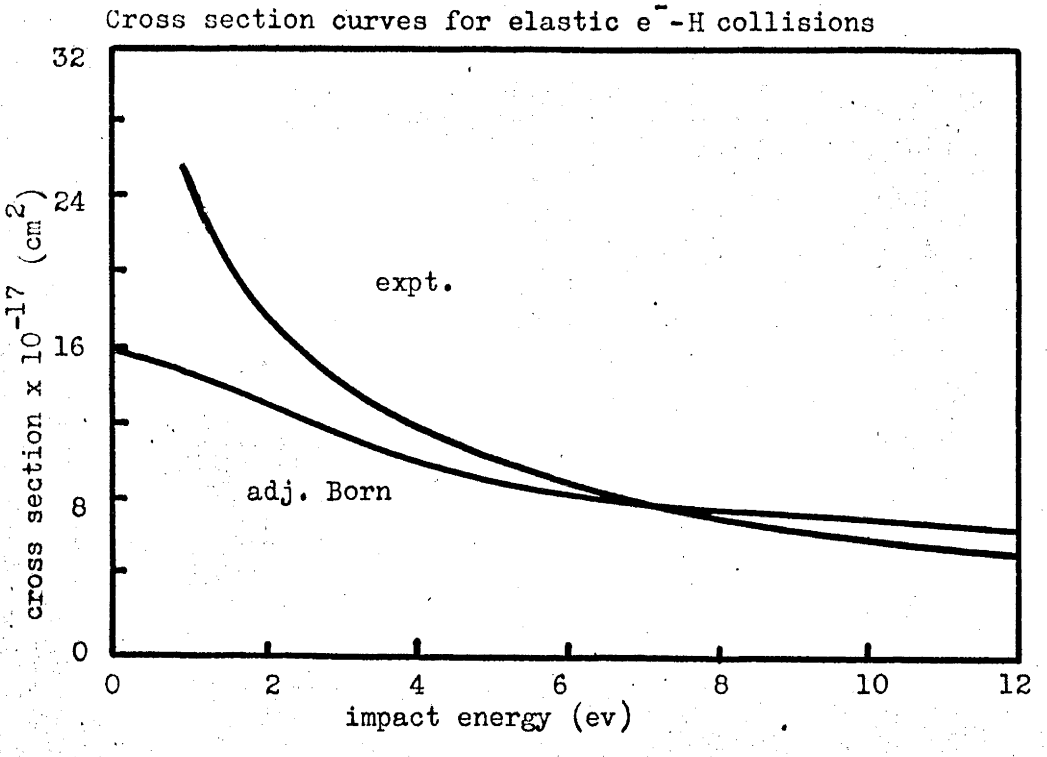
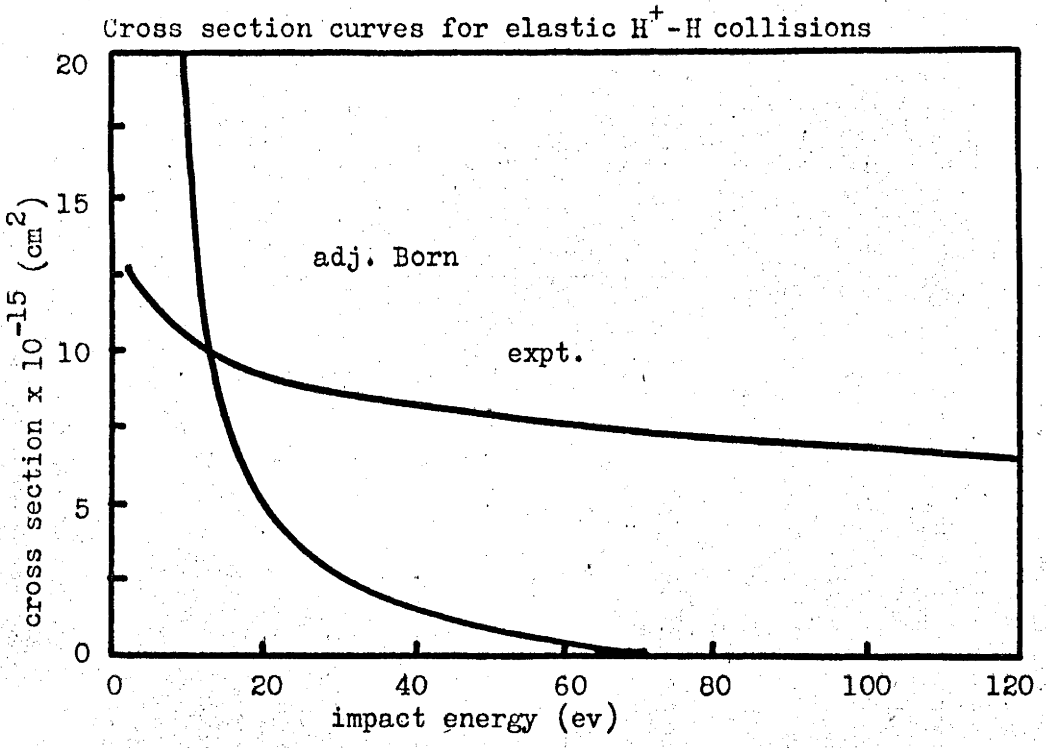
3. Bates, D. R., Proc. Roy. Soc., A247, 294 (1958).

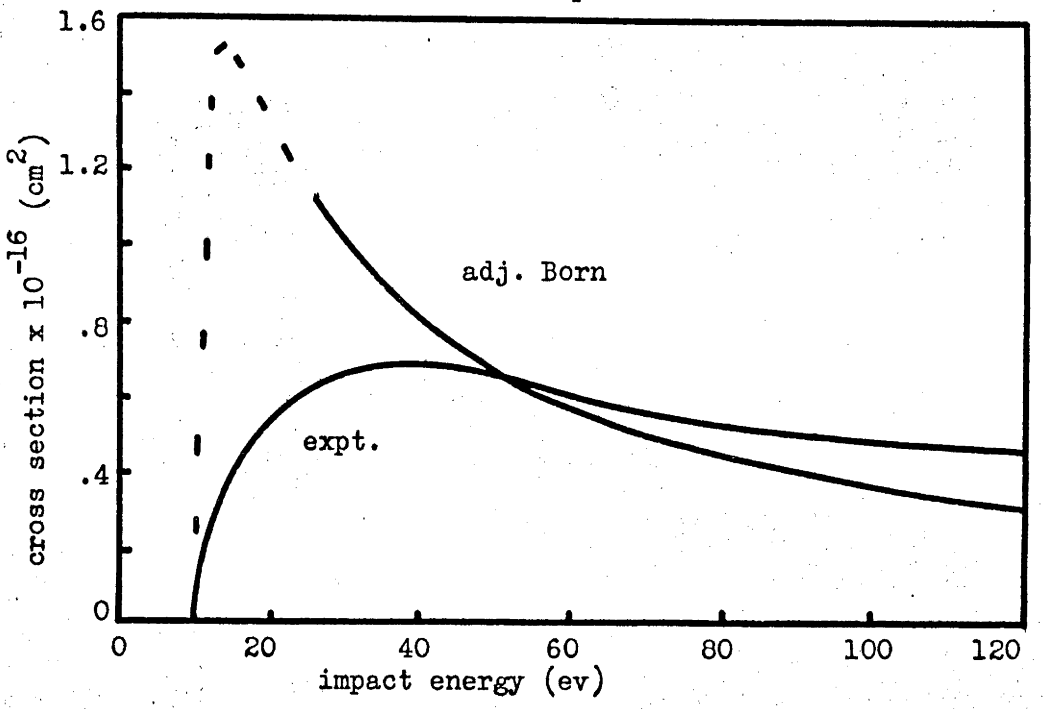
4. Barnett, C. F., Ray, J. A., and Thompson, J. C., "Atomic and Molecular Cross Sections of Interest in Controlled Thermonuclear Research", Oak Ridge National Laboratory, Report ORNL-3113, Revised (August 1964).

5. Bates, D. R., "Theoretical Treatment of Collisions Between Atomic Systems", in Bates, D. R. (ed), "Atomic and Molecular Processes", Academic (1962).

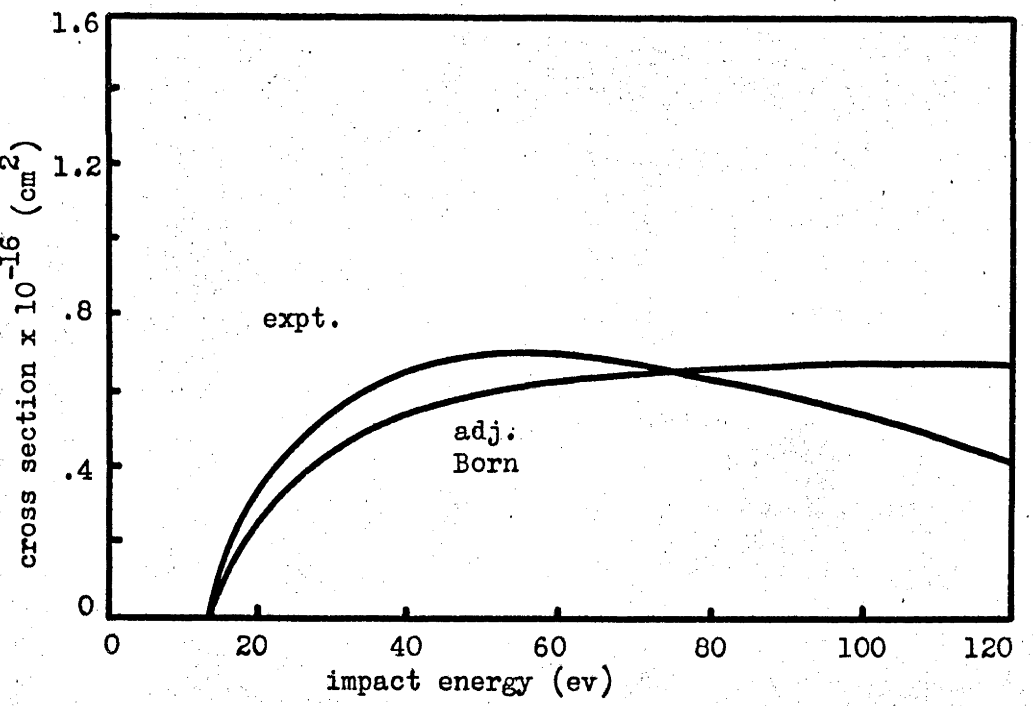

Cross section curves for elastic e⁻-H collisions
32
24
16
8
0
cross section x 10⁻¹⁷ (cm²)
expt.
adj. Born
0
2
4
6
8
10
12
impact energy (ev)

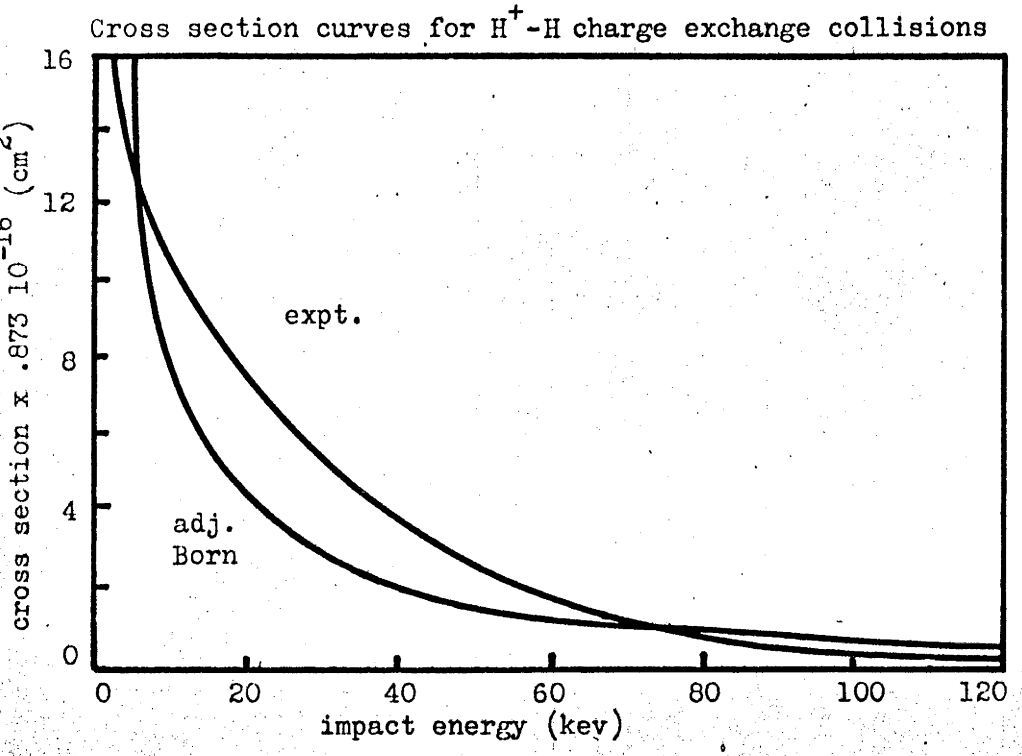

Cross section curves for elastic H⁺-H collisions
20
15
10
5
0
cross section x 10⁻¹⁵ (cm²)
adj. Born
expt.
0
20
40
60
80
100
120
impact energy (ev)

Cross section curves for 1s-2p e^--H excitation collisions



Cross section curves for e^--H ionisation collisions




Cross section curves for H^+-H charge exchange collisions
cross section x .873 10^{-16} (cm^2)
16
12
8
4
0
expt.
adj.
Born
0
20
40
60
80
100
120
impact energy (kev)

Appendix B

Subsidiary Relations for Electron-Hydrogen Atom Ionisation Collisions

B1. The λ's and $\Delta\mathcal{E}$'s

In considering the dynamics of inelastic binary collisions, in Chapter 2, the λ's were defined by

$$\text{(B1)} \quad \lambda^2 = \frac{m_j m_k}{m_{j'} m_{k'}} - \frac{2\Delta\mathcal{E}\, m_0}{g^2 m_{j'} m_{k'}}$$

where $\Delta\mathcal{E}$ is the apparent kinetic energy loss in the collision. For an ionisation collision for example, $\Delta\mathcal{E}$ is the sum of the ionisation energy and the change in internal energy of the particles j', k' compared with j, k. With the particular ionisation collisions being considered, considerable simplification is achieved by neglecting terms of order m_e/m_a compared with unity; it has been shown in Appendix A that this permits the frame of reference for all differential cross section parameters to be taken as the c.m. frame. This is not necessarily so for the "composite" particle internal energies, however.

(a) The incident electron, e_1

For the collision dynamics of this particle, the two "particles" after the collision are the incident (and scattered) electron, and the proton + atomic electron system. The apparent energy loss in the collision, $\Delta\mathcal{E}$ is, quite simply,

$$\text{(B2)} \quad \Delta\mathcal{E} = I + \frac{\hbar^2 K^2}{2 m_r} ,$$

the last term in (B2) being the internal kinetic energy of the proton + electron system; as before $\underline{K}$ is the wave vector of the atomic

electron, and m_e/m_a is neglected compared with unity. I is the ionisation energy. Using (B1), λ is given by

$$\text{(B3)} \quad \lambda^2 = 1 - 2\left(I + \frac{\hbar^2 K^2}{2m_r}\right)/m_r g^2 \quad .$$

(b) The Atomic Electron, e_2

Corresponding to this particle being one of the particles after the collision, the remaining "particle" is the system proton + incident electron. In the c.m. system, this latter "particle" has internal energy $\hbar^2 k_0^2/2m_r$ (neglecting terms of order m_e/m_a). The apparent energy loss is therefore

$$\text{(B4)} \quad \Delta\mathcal{E} = I + \frac{\hbar^2 k_f^2}{2m_r} \quad .$$

From this it follows that

$$\text{(B5)} \quad \lambda^2 = 1 - 2\left(I + \frac{\hbar^2 k_f^2}{2m_r}\right)/m_r g^2 \quad .$$

(c) The Proton, p

The composite "particle" after this collision is the system of the two electrons. In the c.m. system, the respective velocities of the scattered and atomic electrons are $\hbar \underline{k}_f/m_e$ and $\hbar \underline{K}/m_e$. The internal kinetic energy relative to their centre of mass is therefore $\frac{m_e}{4}(\hbar/m_e)^2(\underline{k}_f - \underline{K})^2$, and so

$$\text{(B6)} \quad \Delta\mathcal{E} = I + \frac{\hbar^2}{4m_e}(\underline{k}_f - \underline{K})^2 \quad .$$

Therefore

$$\text{(B7)} \quad \lambda^2 = 1 - 2\left(I + \frac{\hbar^2}{4m_e}\left(k_f^2 + K^2 - 2\underline{k}_f \cdot \underline{K}\right)\right)/m_r g^2 .$$

In general, away from threshold $|\underline{K}| << |\underline{k}_f|$, and so in this region it is possible to write

$$\text{(B8)} \quad \lambda^2 = 1 - 2\left(I + \frac{\hbar^2 k_f^2}{4m_e}\right)/m_r g^2 .$$

(d) The atom, a

This particle is not present after the collision, and therefore, using the δ-functions defined in Chapter 2, it may be verified that the collision integrals for the atoms only involve $\Omega^{-\infty,0}(r)$. Since by definition these Ω's are independent of λ, they are also independent of $\Delta\mathcal{E}$. The $\Delta\mathcal{E}$, however, are implicit in the integration of the differential cross section over all possible energies of the ejected electron in the calculation of the Ω's for this case.

It is worthwhile noting that, with the neglect of terms of order m_e / m_a, there is a simple relationship between the Ω's for the atoms and those for the protons in an ionisation collision. This may be seen by considering the δ-functions of Chapter 2 and neglecting all terms of order m_e / m_a. It can also be seen in another simple way. Neglect of terms involving the electron mass is equivalent to regarding the collision as an event which strips the electron from the atom, forming a proton with the same mass and velocity as that of the atom before the collision. In the notation of Chapter 2, this means that

(B9) $\Delta_{ea}(\psi_p) = \psi_p' = -\psi_a .$

Since only the $\Omega^{-\infty,o}(r)$ are involved, it follows that (the δ-functions containing the sign relevant to (B9)):

(B10) $\Omega^{-\infty,o}_{ea,p}(r) = \Omega^{-\infty,o}_{ea,a}(r)$

B2. Integrals Occurring in the Calculation of the ϕ's and Ω's

(a) The Integral $J_1(\mu,\nu)$

Consider

(B11) $$J_1(\mu,\nu) = \int_0^{\pi}\int_0^{(k_o^2 - \frac{2m_r I}{\hbar^2})^{\frac{1}{2}}} \frac{(1-\lambda^{\mu}\cos^{\nu}\chi)\, k_f\, \kappa \sin\chi \, d\chi \, d\kappa}{(k_o^2 + k_f^2 - 2k_o k_f \cos\chi)^2}$$

with notation as in Chapter 3;

(B12) $\lambda^2 = 1 - 2\left(I + \frac{\hbar^2\kappa^2}{2m_e}\right) m_r / \hbar^2 k_o^2 .$

This integral (B11) occurs for the case of an ionisation collision where the particle being considered is the incident (and scattered) electron. Conservation of energy in the collision gives

(B13) $\frac{\hbar^2 k_o^2}{2m_r} = I + \frac{\hbar^2 k_f^2}{2m_r} + \frac{\hbar^2 \kappa^2}{2m_r}$

and since $m_e \sim m_r$, the reduced mass for the collision, it follows from (B11) that

(B14) $\lambda^2 = k_f^2 / k_o^2 .$

The following values of $J_1(\mu,\nu)$ are required: $(-\infty, 0)$, $(1,1)$, $(2,2)$, $(2,0)$, $(3,1)$.

(i) Integration over χ

In equation (B11), integration over χ is readily effected with the use of a substitution of the form $x = \cos\chi$. This gives

$$(B15)\quad J_1(\mu,\nu) = \int_0^{(k_o^2 - 2m_r I/\hbar^2)^{\frac{1}{2}}} \left(\frac{2}{(k_o^2 - k_f^2)^2} - \lambda^\mu I(\nu) \right) k_f \kappa \, d\kappa$$

where

$$(B16)\quad I(0) = \frac{2}{(k_o^2 - k_f^2)^2}$$

$$(B17)\quad I(1) = \frac{1}{4 k_o^2 k_f^2} \left(2 \log \frac{k_o^2 - k_f^2}{k_o^2 + k_f^2} + \frac{4 k_o k_f (k_o^2 + k_f^2)}{(k_o^2 - k_f^2)^2} \right)$$

$$(B18)\quad I(2) = \frac{2}{4 k_o^2 k_f^2} \left(1 + \left(\frac{k_o^2 + k_f^2}{k_o^2 - k_f^2} \right)^2 + \frac{k_o^2 + k_f^2}{k_o k_f} \log \frac{k_o^2 - k_f^2}{k_o^2 + k_f^2} \right).$$

Ignoring the logarithm term and, in addition, retaining only the terms which are largest away from the threshold, these formulae are represented by

$$(B19)\quad I(\nu) = \frac{2(k_o^2 + k_f^2)^\nu}{(k_o^2 - k_f^2)^2 (2 k_o k_f)^\nu}.$$

Accordingly, $J_1(\mu,\nu)$ becomes

$$(B20)\quad J_1(\mu,\nu) = \int_0^{(k_o^2 - 2m_r I/\hbar^2)^{\frac{1}{2}}} \frac{2}{(k_o^2 - k_f^2)^2} \left(1 - \lambda^\mu \frac{(k_o^2 + k_f^2)^\nu}{(2 k_o k_f)^\nu} \right) \kappa \, k_f \, d\kappa .$$

(ii) Integration over κ

Using α as defined in Chapter 3:

$$\alpha^2 = k_o^2 - 2m_r I/\hbar^2 \quad , \tag{B21}$$

k_f and λ are

$$k_f^2 = \alpha^2 - \kappa^2 \tag{B22}$$

$$\lambda^2 = (\alpha^2 - \kappa^2)/k_o^2 \quad .$$

From equation (B20) $J_1(\mu,\nu)$ can be written as

$$J_1(\mu,\nu) = P(\mu,\nu) - Q(\mu,\nu) \tag{B23}$$

where

$$P(\mu,\nu) = \int_0^\alpha \frac{2k_f \kappa\, d\kappa}{(k_o^2 - k_f^2)^2} \tag{B24}$$

$$Q(\mu,\nu) = \int_0^\alpha \frac{2k_f \kappa\, d\kappa}{(k_o^2 - k_f^2)^2}\, \lambda^\mu\, \frac{(k_o^2 + k_f^2)^\nu}{(2k_o k_f)^\nu} \quad . \tag{B25}$$

In general, for ionisation collisions of this type, the atomic electrons are ejected with low energy. In view of this, the error introduced by replacing the terms

$$(k_o^2 - k_f^2)^2 = (\kappa^2 + \frac{2m_r I}{\hbar^2})^2 \tag{B26}$$

in the denominator by $(2m_r I/\hbar^2)^2$ is not unduly large. This simplification gives

(B27) $$P(\mu,\nu) = \int_0^\alpha \frac{2(\alpha^2-\kappa^2)^{\frac{1}{2}}\,\kappa\,d\kappa}{(2m_rI/\hbar^2)^2} = \frac{2\alpha^3}{3(2m_rI/\hbar^2)^2}.$$

Again, in calculating $Q(\mu,\nu)$, in the term involving $(k_o^2 + k_f^2)$, κ^2 is neglected so that

(B28) $$k_o^2 + k_f^2 = 2k_o^2 - \frac{2m_rI}{\hbar^2} - \kappa^2$$

$$\sim 2k_o^2 - \frac{2m_rI}{\hbar^2}$$

$$= k_o^2 + \alpha^2 .$$

In this case,

(B29) $$Q(\mu,\nu) = \frac{2(k_o^2+\alpha^2)^\nu}{(2m_rI/\hbar^2)^2\,2^\nu\,k_o^{\nu+\mu}} \int_0^\alpha (\alpha^2-\kappa^2)^{\frac{\mu-\nu+1}{2}}\,\kappa\,d\kappa$$

$$= \frac{2(k_o^2+\alpha^2)^\nu}{(2m_rI/\hbar^2)^2\,2^\nu\,k_o^{\mu+\nu}}\, S(\mu,\nu)$$

where the $S(\mu,\nu)$ are given by

(B30) $$S(-\infty,0) = 0$$

$$S(1,1) = S(2,2) = \frac{\alpha^3}{3}$$

$$S(3,1) = S(2,0) = \frac{\alpha^5}{5} .$$

Finally, therefore,

(B31) $$J_1(-\infty,0) = \frac{2\alpha^3}{3(2m_rI/\hbar^2)}$$

$$(B32)\quad J_1(1,1) = \frac{2\alpha^3}{3(2m_rI/\hbar^2)^2}\left(1 - \frac{1}{2}\frac{(k_o^2+\alpha^2)}{k_o^2}\right)$$

$$(B33)\quad J_1(2,2) = \frac{2\alpha^3}{3(2m_rI/\hbar^2)^2}\left(1 - \frac{1}{4}\frac{(k_o^2+\alpha^2)^2}{k_o^4}\right)$$

$$(B34)\quad J_1(2,0) = \frac{2\alpha^3}{3(2m_rI/\hbar^2)^2}\left(1 - \frac{3}{5}\frac{\alpha^2}{k_o^2}\right)$$

$$(B35)\quad J_1(3,1) = \frac{2\alpha^3}{3(2m_rI/\hbar^2)^2}\left(1 - \frac{3}{10}\frac{\alpha^2(k_o^2+\alpha^2)}{k_o^4}\right).$$

(b) The Integral $L(\mu,\nu)$

Consider

$$(B36)\quad L(\mu,\nu) = \int_0^{(k_o^2 - 2m_rI/\hbar^2)^{\frac{1}{2}}} \frac{k_f\, K\, dK}{(2m_rI/\hbar^2 + K^2)^2}\left(2 - \lambda^\mu \frac{(1+(-1)^\nu)}{\nu+1}\right).$$

This integral arises when the particle being considered is the ejected atomic electron. For this case,

$$(B37)\quad \lambda^2 = 1 - \frac{(k_o^2 - K^2)}{k_o^2},$$

the notation being as previously used in this section, and in Chapter 3. (B36) is conveniently separated into two parts, the first being independent of (μ,ν). Taking

$$(B38)\quad L(\mu,\nu) = W + X(\mu,\nu),$$

$$(B39)\quad W = 2\int_0^{\alpha} \frac{K\,(\alpha^2 - K^2)^{\frac{1}{2}}\, dK}{(2m_rI/\hbar^2 + K^2)^2},$$

with α as in (B21). Although it is possible to evaluate the integral in this form (B39), usable expressions for calculating the Ω's are only obtained if the denominator is replaced by $(2m_r I/\hbar^2)^2$ as in earlier integrals of this section. With this substitution, W is easily obtained:

$$(B40)\quad W = \frac{2\alpha^3}{3(2m_r I/\hbar^2)^2}.$$

The second part of integral (B36), with (B38) and using (B37), is

$$(B41)\quad X(\mu,\nu) = -\int_0^\alpha \frac{\lambda^\mu (1+(-1)^\nu)\ k_f\ K\,dK}{(\nu+1)\quad (2m_r I/\hbar^2)^2}$$
$$\sim -\int_0^\alpha \left(1-\frac{k_o^2-K^2}{k_o^2}\right)^{\frac{\mu}{2}} \frac{(1+(-1)^\nu)\,k_f\,K\,dK}{(\nu+1)\,(2m_r I/\hbar^2)^2}.$$

For the individual cases,

$$(B42).\quad X(-\infty,o) = X(1,1) = X(3,1) = 0$$

while

$$(B43)\quad X(2,2) = -\int_0^\alpha \frac{2\,k_f\,K\,dK}{3(2m_r I/\hbar^2)^2}\left(1-\frac{k_o^2-K^2}{k_o^2}\right)$$

and this becomes

$$(B44)\quad X(2,2) = -\frac{2}{3(2m_r I/\hbar^2)^2}\left(\frac{\alpha^3}{3}-\frac{\alpha^5}{5k_o^2}\right)$$

while lastly,

(B45) $$X(2,0) = -\int_0^{\alpha} \frac{2k_f K\,dK}{(2m_r I/\hbar^2)^2}\left(1-\frac{k_0^2-K^2}{k_0^2}\right) = 3X(2,2).$$

Using (B38), $L(\mu,\nu)$ has the values

(B46) $$L(3,1) = L(1,1) = L(-\infty,0) = \frac{2\alpha^3}{3(2m_r I/\hbar^2)^2}$$

(B47) $$L(2,2) = \frac{2\alpha^3}{3(2m_r I/\hbar^2)^2}\left(\frac{2}{3}+\frac{\alpha^2}{5k_0^2}\right)$$

(B48) $$L(2,0) = \frac{2\alpha^3}{3(2m_r I/\hbar^2)^2}\cdot\frac{3\alpha^2}{5k_0^2}\,.$$

Appendix C

The Moment Equations in Cylindrical Geometry

C.1 The Tensor Relations for Cylindrical Geometry

In order to simplify the moment equations, spatial derivatives with respect to Θ and Z will be neglected. This excludes waves and instabilities in these directions, but nevertheless should not detract from the gross phenomena involved by the application of uniform fields in these directions. In this case, for the cylindrical coordinate system (r, Θ, Z),

$$\text{(C1)} \qquad \nabla \equiv \left(\frac{\partial}{\partial r} , 0 , 0 \right) .$$

The following tensor relations are required for the expression of the 13-moment - Maxwell equations in cylindrical coordinates, where a is a scalar, $\underline{A}$, $\underline{B}$ are vectors, and $\underset{\sim}{H}$ is a second rank symmetrical tensor.

$$\text{(C2)} \qquad \{\nabla \underline{A}\} \; ; \quad \underline{A} \cdot \nabla \underline{B} \; ; \quad \nabla \underline{A} \cdot \underline{B} \; ;$$

$$\{\underset{\sim}{H} \cdot \nabla \underline{A}\} \; ; \quad \{\underline{A} \cdot \nabla \underset{\sim}{H}\} \; ; \quad \nabla \cdot \underset{\sim}{H} \; ;$$

$$\underset{\sim}{H} : \nabla \underline{A} \; ; \quad \underset{\sim}{H} \cdot \nabla a \; ; \quad \nabla \times \underline{A} \; ; \quad \nabla \cdot \underline{A} \; .$$

In a cylindrical coordinate system (r, Θ, Z) , the metric tensor, $\underset{\sim}{g}$, is given by

(C3) $$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (r)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or

(C4) $$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (r)^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The only non-zero components of the affine connexion, $\Gamma^{\alpha}_{\beta\gamma}$, are

(C5) $$\Gamma^{1}_{22} = -r \; ; \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = 1/r \; ,$$

where the r, θ, z components are designated $1, 2, 3$. Also

(C6) $$\nabla \equiv \underline{e}^{\nu} D_{\nu} = \underline{e}^{1} D_{1}$$

where $D_1 \equiv \frac{\partial}{\partial r}$ and $\underline{e}^{\nu}$ is a contravariant base vector. The remaining relation required is that

(C7) $$D_{\nu} \underline{e}_{\mu} = \underline{e}_{\eta} \Gamma^{\eta}_{\nu\mu} \; ; \quad D_{\nu} \underline{e}^{\mu} = \underline{e}^{\eta} \Gamma^{\mu}_{\nu\eta} \; .$$

The physical components are obtained by using

$$(C8) \qquad |\underline{e}_\mu|^2 = \underline{e}_\mu \cdot \underline{e}_\mu = g_{\mu\mu} \; ;$$

$$|\underline{e}^\mu|^2 = \underline{e}^\mu \cdot \underline{e}^\mu = g^{\mu\mu} \qquad \text{(no summation).}$$

Finally, since the components of the dynamical variables are measured in physical components (i. e. referred to unit vectors), these must be divided by the relevant $(g_{\mu\mu})^{\frac{1}{2}}$ for the tensor operations. Hence the following relations, where the tensors are expressed in their physical components, and derivatives with respect to Θ and z neglected, are obtained:

$$(C9) \qquad \nabla \underline{A} = \begin{pmatrix} D_1 A^1 & D_1 A^2 & D_1 A^3 \\ -A^2/r & A^1/r & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

$$(C10) \qquad \{\nabla A\} = \frac{1}{2} \begin{pmatrix} \frac{4}{3} D_1 A^1 - \frac{2}{3} A^1/r & D_1 A^2 - A^2/r & D_1 A^3 \\ D_1 A^2 - A^2/r & \frac{4}{3} A^1/r - \frac{2}{3} D_1 A^1 & 0 \\ D_1 A^3 & 0 & -\frac{2}{3}(D_1 A^1 + A^1/r) \end{pmatrix} .$$

$$(C11) \qquad \nabla \cdot \underline{A} = D_1 A^1 + A^1/r \; .$$

(C12) $$\nabla \underline{A} \cdot \underline{B} = \left(B^1 D_1 A^1 + B^2 D_1 A^2 + B^3 D_1 A^3 , 0 , 0 \right).$$

(C13) $$\underset{\sim}{H} \cdot \nabla \underline{A} =$$

$$\begin{pmatrix} H^{11} D_1 A^1 - H^{12} A^2/r & H^{11} D_1 A^2 + H^{12} A^1/r & H^{11} D_1 A^3 \\ H^{21} D_1 A^1 - H^{22} A^2/r & H^{21} D_1 A^2 + H^{22} A^1/r & H^{21} D_1 A^3 \\ H^{31} D_1 A^1 - H^{32} A^2/r & H^{31} D_1 A^2 + H^{32} A^1/r & H^{31} D_1 A^3 \end{pmatrix}.$$

(C14) $$\left\{ \underset{\sim}{H} \cdot \nabla \underline{A} \right\} =$$

$$\begin{pmatrix} \frac{2}{3}(H^{11} D_1 A^1 - H^{12} A^2/r) - \frac{1}{3}(H^{12} D_1 A^2 + H^{22} A^1/r + H^{13} D_1 A^3), & & \\ \frac{1}{2}(H^{11} D_1 A^2 + H^{12} A^1/r + H^{12} D_1 A^1 - H^{22} A^2/r), & & \\ \frac{1}{2}(H^{11} D_1 A^3 + H^{13} D_1 A^1 - H^{23} A^2/r) & & \\ \frac{1}{2}(H^{11} D_1 A^2 + H^{12} A^1/r + H^{12} D_1 A^1 - H^{22} A^2/r), & & \\ \frac{2}{3}(H^{12} D_1 A^2 + H^{22} A^1/r) - \frac{1}{3}(H^{11} D_1 A^1 - H^{12} A^2/r + H^{13} D_1 A^3), & & \\ \frac{1}{2}(H^{12} D_1 A^3 + H^{13} D_1 A^2 + H^{23} A^1/r) & & \\ \frac{1}{2}(H^{11} D_1 A^3 + H^{13} D_1 A^1 - H^{23} A^2/r) & & \\ \frac{1}{2}(H^{12} D_1 A^3 + H^{13} D_1 A^2 + H^{23} A^1/r) & & \\ \frac{2}{3}(H^{13} D_1 A^3) - \frac{1}{3}(H^{11} D_1 A^1 - H^{12} A^2/r + H^{12} D_1 A^2 + H^{22} A^1/r) & & \end{pmatrix}.$$

(C15) $\underline{A} \cdot \nabla \underset{\sim}{H} =$

$$\begin{pmatrix} A' D_1 H^{11} - \frac{2A^2 H^{12}}{r} & A' D_1 H^{12} - \frac{A^2 H^{22}}{r} + \frac{A^2 H^{11}}{r} & A' D_1 H^{13} - \frac{A^2 H^{23}}{r} \\ A' D_1 H^{12} - \frac{A^2 H^{22}}{r} + \frac{A^2 H^{11}}{r} & A' D_1 H^{22} + \frac{2A^2 H^{12}}{r} & A' D_1 H^{23} + \frac{A^2 H^{13}}{r} \\ A' D_1 H^{31} - \frac{A^2 H^{23}}{r} & A' D_1 H^{23} + \frac{A^2 H^{13}}{r} & A' D_1 H^{33} \end{pmatrix}.$$

Let

(C16) $$S = H^{11} + H^{22} + H^{33},$$

and therefore if $\underset{\sim}{H}$ is a traceless tensor, $S = 0$. In terms of S,

(C17) $\left\{ \underline{A} \cdot \nabla \underset{\sim}{H} \right\} =$

$$\begin{pmatrix} A' D_1 H^{11} - \frac{2A^2 H^{12}}{r} - \frac{A' D_1 S}{3} & A' D_1 H^{12} - \frac{A^2 H^{22}}{r} + \frac{A^2 H^{11}}{r} & A' D_1 H^{13} - \frac{A^2 H^{23}}{r} \\ A' D_1 H^{12} + \frac{A^2 H^{11}}{r} - \frac{A^2 H^{22}}{r} & A' D_1 H^{22} + \frac{2A^2 H^{12}}{r} - \frac{A' D_1 S}{3} & A' D_1 H^{23} + \frac{A^2 H^{13}}{r} \\ A' D_1 H^{13} - \frac{A^2 H^{23}}{r} & A' D_1 H^{23} + \frac{A^2 H^{13}}{r} & A' D_1 H^{33} - \frac{A' D_1 S}{3} \end{pmatrix}.$$

(C18) $\nabla \cdot \underset{\sim}{H} =$

$$\left(D_1 H^{11} + \frac{H^{11}}{r} - \frac{H^{22}}{r}, D_1 H^{12} + \frac{2 H^{12}}{r}, D_1 H^{13} + \frac{H^{13}}{r} \right).$$

(C19) $\underline{A} \cdot \nabla \underline{B} =$

$$\left(A' D_1 B^1 - \frac{A^2 B^2}{r}, A' D_1 B^2 + \frac{A^2 B^1}{r}, A' D_1 B^3 \right).$$

$$(C20)\quad \underset{\sim}{H} : \nabla \underline{A} = H^{11} D_1 A^1 + H^{12} D_1 A^2 + H^{13} D_1 A^3 + \frac{H^{22} A^1}{r} .$$

$$(C21)\quad \underset{\sim}{H} \cdot \nabla a = \left(H^{11} D_1 a , H^{12} D_1 a , H^{13} D_1 a \right) .$$

$$(C22)\quad \nabla \times A = \left(0 , - D_1 A^3 , D_1 A^2 + \frac{A^2}{r} \right) .$$

C2. The Moment Equations in Cylindrical Coordinates

In this section the tensor relationships derived above are used to express the 13-moment equations and Maxwell's equations in the cylindrical coordinate system (r, θ, z) . This procedure being purely a matter of substitution in the equations of Chapter 1, only the results are given. All moments are referred to a frame with velocity $\underline{V}$. $\left(\frac{d}{dt} \right)$ is replaced by its definitive relation, $\frac{\partial}{\partial t} + \underline{V} \cdot \nabla$, and for reasons of convenience, the particle type subscripts i , for the left hand side, are not written in.

C2.1 Continuity Equation

$$(C23)\quad \frac{\partial}{\partial t} \rho + D_1 (\rho u^1 + \rho V^1) + \frac{\rho u^1 + \rho V^1}{r} = \frac{1}{2} \sum_{j,k} \delta^{(0)}_{jk,i} .$$

C2.2 Momentum Equations

$$(C24)\quad \frac{\partial \rho \underline{u}}{\partial t} + \begin{pmatrix} V^1 D_1 (\rho u^1) - \frac{\rho u^2 V^2}{r} \\ V^1 D_1 (\rho u^2) - \frac{\rho u^1 V^2}{r} \\ V^1 D_1 (\rho u^3) \end{pmatrix} + \begin{pmatrix} D_1 P^{11} + \frac{P^{11}}{r} - \frac{P^{22}}{r} \\ D_1 P^{12} + \frac{2 P^{12}}{r} \\ D_1 P^{13} + \frac{P^{13}}{r} \end{pmatrix}$$

$$+ \begin{pmatrix} D_1 P \\ 0 \\ 0 \end{pmatrix} + \rho \underline{u}(D_1 V^1 + \frac{V^1}{r}) - \rho \underline{F} - \rho \underline{u} \times \underline{b} + \rho \begin{pmatrix} u^1 D_1 V^1 - \frac{u^2 V^2}{r} \\ u^1 D_1 V^2 - \frac{u^2 V^1}{r} \\ u^1 D_1 V^3 \end{pmatrix}$$

$$= \frac{1}{2} \sum_{j,k} \left(\delta^{(1)}_{jk,i} \underline{u}_j + \delta^{(2)}_{jk,i} \underline{u}_k + \delta^{(5)}_{jk,i} \underline{R}_j + \delta^{(6)}_{jk,i} \underline{R}_k \right) .$$

C2.3 Thermal Energy Equation

$$\text{(C25)} \quad \frac{\partial}{\partial t}\left(\frac{3}{2}P\right) + \frac{3}{2} V^1 D_1 P + D_1 R^1 + \frac{R^1}{r} + \frac{5}{2} D_1 (p u^1) + \frac{5}{2} \frac{p u^1}{r} - \rho \underline{F} \cdot \underline{u}$$

$$+ P^{11} D_1 V^1 + P^{12} D_1 V^2 + P^{13} D_1 V^3 + \frac{P^{22} V^1}{r} + \frac{5}{2} P \left(D_1 V^1 + \frac{V^1}{r} \right)$$

$$= \frac{1}{2} \sum_{j,k} \delta^{(0)}_{jk,i} .$$

C2.4 Stress Tensor Equations

$$\text{(C26)} \quad \frac{\partial}{\partial t} \underset{\sim}{P} + \underset{\sim}{P} D_1 V^1 + \underset{\sim}{P} \frac{V^1}{r}$$

$$+ \begin{pmatrix} V^1 D_1 P^{11} - \frac{2 V^2 P^{12}}{r} & , V^1 D_1 P^{12} + \frac{V^2 P^{11}}{r} - \frac{V^2 P^{22}}{r} , & V^1 D_1 P^{13} - \frac{V^2 P^{23}}{r} \\ V^1 D_1 P^{12} + \frac{V^2 P^{11}}{r} - \frac{V^2 P^{22}}{r} , & V^1 D_1 P^{22} + \frac{2 V^2 P^{12}}{r} & , V^1 D_1 P^{23} + \frac{V^2 P^{13}}{r} \\ V^1 D_1 P^{13} - \frac{V^2 P^{23}}{r} , & V^1 D_1 P^{23} - \frac{V^2 P^{13}}{r} & , V^1 D_1 P^{33} \end{pmatrix}$$

$$
+\frac{2}{5}\begin{pmatrix}
\frac{4}{3}D_1R^1-\frac{2}{3}\frac{R^1}{r}, & D_1R^2-\frac{R^2}{r}, & D_1R^3,\\
D_1R^2-\frac{R^2}{r}, & \frac{4}{3}\frac{R^1}{r}-\frac{2}{3}D_1R^1, & 0,\\
D_1R^3, & 0, & -\frac{2}{3}D_1R^1-\frac{2}{3}\frac{R^1}{r}.
\end{pmatrix}
$$

$$
+\begin{pmatrix}
\frac{4}{3}D_1(pu^1)-\frac{2}{3}\frac{pu^1}{r}, & D_1(pu^2)-\frac{pu^2}{r}, & D_1(pu^3),\\
D_1(pu^2)-\frac{pu^2}{r}, & \frac{4}{3}\frac{pu^1}{r}-\frac{2}{3}D_1(pu^1), & 0,\\
D_1(pu^3), & 0, & -\frac{2}{3}D_1(pu^1)-\frac{2}{3}\frac{pu^1}{r}
\end{pmatrix}
$$

$$
-\rho\begin{pmatrix}
\frac{4}{3}F^1u^1-\frac{2}{3}(F^2u^2+F^3u^3), & F^1u^2+F^2u^1, & F^1u^3+F^3u^1,\\
F^1u^2+F^2u^1, & \frac{4}{3}F^2u^2-\frac{2}{3}(F^1u^1+F^3u^3), & F^2u^3+F^3u^2,\\
F^1u^3+F^3u^1, & F^2u^3+F^3u^2, & \frac{4}{3}F^3u^3-\frac{2}{3}(F^1u^1+F^2u^2)
\end{pmatrix}
$$

$$
-\frac{e}{m}\begin{pmatrix}
2(P^{12}B^3-P^{13}B^2), & P^{13}B^1-P^{23}B^2+B^3(P^{22}-P^{11}), & P^{23}B^3-P^{12}B^1+B^2(P^{11}-P^{33}),\\
P^{13}B^1-P^{23}B^2+B^3(P^{22}-P^{11}), & 2(P^{23}B^1-P^{12}B^3), & P^{12}B^2-P^{13}B^3+B^1(P^{33}-P^{22}),\\
P^{23}B^3-P^{12}B^1+B^2(P^{11}-P^{33}), & P^{12}B^2-P^{13}B^3+B^1(P^{33}-P^{22}), & 2(P^{13}B^2-P^{23}B^1)
\end{pmatrix}
$$

$$
+\left(\begin{array}{ccc}
\left[\frac{4}{3}(P^{11}D_1V^1-\frac{P^{12}V^2}{r})-\frac{2}{3}(P^{12}D_1V^2\right. & \left[P^{11}D_1V^2+\frac{P^{12}V^1}{r}+P^{12}D_1V^1\right. & \left[P^{11}D_1V^3+P^{13}D_1V^1\right. \\
\left.+\frac{P^{22}V^1}{r}+P^{13}D_1V^3\right] & \left.-\frac{P^{22}V^2}{r}\right] & \left.-\frac{P^{23}V^2}{r}\right] \\
\left[P^{11}D_1V^2+\frac{P^{12}V^1}{r}+P^{12}D_1V^1\right. & \left[\frac{4}{3}(P^{12}D_1V^2+\frac{P^{22}V^1}{r})-\frac{2}{3}(P^{11}D_1V^1\right. & \left[P^{12}D_1V^3+P^{13}D_1V^2\right. \\
\left.-\frac{P^{22}V^2}{r}\right] & \left.-\frac{P^{12}V^2}{r}+P^{13}D_1V^3)\right] & \left.+\frac{P^{23}V^1}{r}\right] \\
\left[P^{11}D_1V^3+P^{13}D_1V^1\right. & \left[P^{12}D_1V^3+P^{13}D_1V^2\right. & \left[\frac{4}{3}(P^{13}D_1V^3)-\frac{2}{3}(P^{11}D_1V^1\right. \\
\left.-\frac{P^{23}V^2}{r}\right] & \left.+\frac{P^{23}V^1}{r}\right] & \left.-\frac{P^{12}V^2}{r}+P^{12}D_1V^2+\frac{P^{22}V^1}{r}\right]
\end{array}\right)
$$

$$
=\frac{1}{2}\sum_{j,k}\left(\delta^{(3)}_{jk,i}\underset{\sim}{P}_j+\delta^{(4)}_{jk,i}\underset{\sim}{P}_k\right)\ .
$$

C2.5 Heat Flux Equations

(C27)

$$
\frac{\partial \underline{R}}{\partial t}+\frac{7}{5}\underline{R}(D_1V^1+\frac{V^1}{r})+\frac{5}{2}nk\underset{\sim}{\mu}\left(\frac{\partial T}{\partial t}+V^1D_1T\right)-\frac{e}{m}\underline{R}\times\underline{B}
$$

$$
+\begin{pmatrix} V^1D_1R^1-\frac{V^2R^2}{r} \\ V^1D_1R^2-\frac{V^2R^1}{r} \\ V^1D_1R^3 \end{pmatrix}+\frac{5}{2}\frac{kP}{m}\begin{pmatrix} D_1T \\ 0 \\ 0 \end{pmatrix}+\frac{kT}{m}\begin{pmatrix} D_1P^{11}+\frac{P^{11}-P^{22}}{r} \\ D_1P^{12}+\frac{2P^{12}}{r} \\ D_1P^{13}+\frac{P^{13}}{r} \end{pmatrix}
$$

$$
-\begin{pmatrix} P^{11}F^1+P^{12}F^2+P^{13}F^3 \\ P^{12}F^1+P^{22}F^2+P^{23}F^3 \\ P^{13}F^1+P^{23}F^2+P^{33}F^3 \end{pmatrix}+\frac{7}{5}\begin{pmatrix} R^1D_1V^1-\frac{R^2V^2}{r} \\ R^1D_1V^2-\frac{R^2V^1}{r} \\ R^1D_1V^3 \end{pmatrix}
$$

$$+\frac{2}{5}\begin{pmatrix} R^1 D_1 V^1 + R^2 D_1 V^2 + R^3 D_1 V^3 \\ 0 \\ 0 \end{pmatrix} + p\,\underline{u}\left(D_1 V^1 + \frac{V^1}{r}\right)$$

$$+ p\begin{pmatrix} u^1 D_1 V^1 + u^2 D_1 V^2 + u^3 D_1 V^3 \\ 0 \\ 0 \end{pmatrix} + p\begin{pmatrix} u^1 D_1 V^1 - \frac{u^2 V^2}{r} \\ u^1 D_1 V^2 - \frac{u^2 V^1}{r} \\ u^1 D_1 V^3 \end{pmatrix}$$

$$= \frac{1}{2}\sum_{j,k}\left(\delta^{(1)}_{jk,i}\,\underline{u}_j + \delta^{(2)}_{jk,i}\,\underline{u}_k + \delta^{(5)}_{jk,i}\,\underline{R}_j + \delta^{(6)}_{jk,i}\,\underline{R}_k\right).$$

C2.6 Maxwell's Equations

(C28)
$$\begin{pmatrix} 0 \\ -D_1 E^3 \\ D_1 E^2 + \frac{E^2}{r} \end{pmatrix} + \frac{\partial \underline{B}}{\partial t} = 0.$$

(C29) $$\begin{pmatrix} 0 \\ -D_1 B^3 \\ D_1 B^2 + \frac{B^2}{r} \end{pmatrix} - \mu_0 \underline{J} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} - \mu_0 \underline{V} \sum_i n_i e_i = 0,$$

where $\underline{J} = \sum_i n_i e_i \underline{u}_i$, there being no subscripts intended for Maxwell's equations.

C3. The Coefficient Matrix S in the 13-Moment Equations in Cylindrical Coordinates

It has been mentioned in Chapter 4 that the coefficient occurring in the form of the 13-moment equations in equation (4. 12), where the equations are in a diagonal form with respect to the time derivatives, is a function of the coordinate system. In the present section, this matrix coefficient is given for the equations in a cylindrical coordinate system (r , θ , z) where derivatives with respect to θ and z have been neglected. The particle-type subscripts, i , have again been ignored for ease of writing.

(C30)

$$
S = \begin{pmatrix}
0 & \frac{\rho}{r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\rho u^1}{r}-\frac{F^1}{u^1} & -\frac{eB^3}{m} & \frac{eB^2}{m} & 0 & \frac{1}{\rho r} & 0 & 0 & -\frac{1}{\rho r} & 0 & 0 & 0 & 0 \\
0 & \frac{eB^3}{m} & -\frac{\rho u^1}{r}-\frac{F^2}{u^2} & -\frac{eB^1}{m} & 0 & 0 & \frac{2}{\rho r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{eB^2}{m} & \frac{eB^1}{m} & -\frac{\rho u^1}{r}-\frac{F^3}{u^3} & 0 & 0 & 0 & \frac{1}{\rho r} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3}\rho F^1 & -\frac{2}{3}\rho F^2 & -\frac{2}{3}\rho F^3 & \frac{5}{3}u^1 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3r} & 0 & 0 \\
0 & \frac{2}{3}\frac{p}{r}-\frac{4}{3}\rho F^1 & \frac{2}{3}\rho F^2 & \frac{2}{3}\rho F^3 & 0 & 0 & -\frac{2eB^3}{m} & \frac{2eB^2}{m} & 0 & 0 & -\frac{4}{15r} & 0 & 0 \\
0 & -\rho F^2 & -\frac{p}{r}-\rho F^1 & 0 & 0 & \frac{eB^3}{m} & 0 & -\frac{eB^1}{m} & -\frac{eB^3}{m} & \frac{eB^2}{m} & 0 & -\frac{2}{5r} & 0 \\
0 & -\rho F^3 & 0 & -\rho F^1 & 0 & -\frac{2eB^2}{m} & \frac{eB^1}{m} & 0 & \frac{eB^2}{m} & -\frac{eB^3}{m} & 0 & 0 & 0 \\
0 & \frac{4}{3}\frac{p}{r}+\frac{2}{3}\rho F^1 & -\frac{4}{3}\rho F^2 & \frac{2}{3}\rho F^3 & 0 & 0 & \frac{2eB^3}{m} & 0 & 0 & -\frac{2eB^1}{m} & \frac{8}{15r} & 0 & 0 \\
0 & 0 & -\rho F^3 & -\rho F^2 & 0 & \frac{eB^1}{m} & -\frac{eB^2}{m} & \frac{eB^3}{m} & \frac{2eB^1}{m} & 0 & 0 & 0 & 0 \\
0 & \frac{5}{3}\rho \underline{F}\cdot\underline{u} & 0 & 0 & -\frac{5}{3}\frac{u^1u^1}{r} & \frac{p}{\rho r}-F^1 & -F^2 & -F^3 & -\frac{p}{\rho r} & 0 & -\frac{5}{3}\frac{u^1}{r} & -\frac{eB^3}{m} & \frac{eB^2}{m} \\
0 & 0 & \frac{5}{3}\rho \underline{F}\cdot\underline{u} & 0 & -\frac{5}{3}\frac{u^1u^2}{r} & 0 & \frac{2p}{\rho r}-F^1 & 0 & -F^2 & -F^3 & \frac{eB^3}{m}-\frac{5}{3}\frac{u^2}{r} & 0 & -\frac{eB^1}{m} \\
0 & 0 & 0 & \frac{5}{3}\rho \underline{F}\cdot\underline{u} & -\frac{5}{3}\frac{u^1u^3}{r} & F^3 & 0 & \frac{p}{\rho r}-F^1 & F^3 & -F^2 & -\frac{eB^2}{m}-\frac{5}{3}\frac{u^3}{r} & \frac{eB^1}{m} & 0
\end{pmatrix}.
$$

C4. The Non-Derivative Coefficient in Maxwell's Equations in Cylindrical Coordinates, Λ

This term is relevant to Maxwell's equations in the form (4.32) of Chapter 4. For the particular case of cylindrical coordinates, with derivatives in other than the r direction neglected, and using the notation of this Chapter, Λ is given by:

$$(C31) \quad \Lambda = \begin{pmatrix} \frac{\mu_0 (c)^2 J^2}{E^2} & 0 & 0 & 0 \\ 0 & \frac{\mu_0 (c)^2 J^3}{E^3} & \frac{-(c)^2}{r} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 \end{pmatrix}.$$

Appendix D

Polarisation and Faraday Rotation

D1. Introduction

The polarisation, S, of a transverse electromagnetic wave propagating in the z direction is usually defined by the ratio of the electric 'vectors':

$$S = \frac{E_y}{E_x} \quad . \tag{D1}$$

Since for transverse waves in a non-magnetic plasma it has been shown that E_x and E_y are independent, the polarisation is indeterminate. More explicitly, the plasma has no bearing on the polarisation, which is controlled by the manner of launching of the wave, and remains constant, since the propagation properties of both transverse waves are identical in all respects.

As evident from section 5.3 of Chapter 5, an applied magnetic field has a profound effect on one or both of the transverse waves. The propagation properties, including waves speeds and damping, become different for the two transverse modes and their polarisation becomes a function of position and of plasma parameters. As in Chapter 5, only two orientations of magnetic field are considered, that of a magnetic field parallel to the direction of propagation, and secondly, a magnetic field perpendicular to the direction of propagation.

D2. The Applied Magnetic Field Parallel to the Direction of Propagation

As shown in Chapter 5, a field parallel to the direction of propagation couples the E_x and E_y transverse wave modes so that all such propagating waves involve simultaneous oscillations of both transverse

electric fields, amongst other variables. In particular it was found that for both fast waves and stress waves, there are two distinct solutions for the wave vector $\mathcal{K}$; the ordinary wave solution having polarisation

$$\text{(D2)} \qquad \frac{E_y}{E_x} = i$$

and the extraordinary wave with polarisation

$$\text{(D3)} \qquad \frac{E_y}{E_x} = -i \ .$$

The wave modes with polarisation (D2) or (D3), independent of position or of plasma parameters, are commonly called characteristic waves. Irrespective of collisions, at any given point equations (D2) and (D3) respectively represent constant electric vectors rotating in the sense of positive and negative charge gyromagnetic rotation. Nevertheless, owing to the damping effect of collisions for example, the amplitude of the rotating electric vector (i. e. the radius of the "circle" described by this resultant vector) is a function of position along the z-axis.

In general any transverse wave propagating parallel to a magnetic field can be represented as the sum of two characteristic waves (either fast or stress waves according to the nature of the wave; provided $f^2/c^2 << 1$ it is reasonable to consider the stress waves and fast waves as separate types. Should the plasma temperature be sufficiently high, it may be necessary to regard a propagating wave as the sum of four characteristic waves, two fast and two stress). Thus a wave of given polarisation at any point can be represented as the sum of two contra-rotating circularly polarised characteristic waves. As

these waves travel at different speeds, the resultant polarisation at different points will be altered in angle. This property is called Faraday rotation. Furthermore, owing to collisional damping, the shape of the ellipse described by the resultant electric vector may be different at differing points along the z-axis.

Let κ_1, κ_2 be the wave numbers of the ordinary and extra-ordinary waves respectively. Continuing the use of the subscripts 1 and 2 for the ordinary and extraordinary cases, let

$$E_{x1} = A_1(z)\, e^{i(\omega t - \kappa_1 z)}$$

$$E_{x2} = A_2(z)\, e^{i(\omega t - \kappa_2 z)} \,. \tag{D4}$$

Then

$$E_{y1} = iE_{x1} = A_1 e^{i(\omega t - \kappa_1 z + \frac{\pi}{2})}$$

$$E_{y2} = -iE_{x2} = A_2 e^{i(\omega t - \kappa_2 z - \frac{\pi}{2})} \,. \tag{D5}$$

Since for any wave,

$$E_x = E_{x1} + E_{x2}$$

$$E_y = E_{y1} + E_{y2} \,, \tag{D6}$$

the polarisation S is given by

$$(D7) \qquad S = \frac{E_y}{E_x} = \frac{A_1 e^{-i\kappa_1 z + i\frac{\pi}{2}} + A_2 e^{-i\kappa_2 z - i\frac{\pi}{2}}}{A_1 e^{-i\kappa_1 z} + A_2 e^{-i\kappa_2 z}} .$$

Writing

$$(D8) \qquad A = \frac{A_1}{A_2} ,$$

$$(D9) \qquad S = \frac{i(A e^{-i(\kappa_1 - \kappa_2)z} - 1)}{(A e^{-i(\kappa_1 - \kappa_2)z} + 1)} \equiv a(z) e^{i\phi(z)} .$$

For any given z , equation (D9) corresponds to the resultant vector $\underline{E} = (E_x, E_y, 0)$ describing an ellipse. Arbitrarily taking $z = 0$, for example,

$$(D10) \qquad S = \frac{i(A-1)}{A+1} ,$$

this ellipse becoming a circle for $A = 0$ or $A = \infty$, while in general, the ratio of major to minor diameters of the ellipse, a_1 / a_2 , is

$$(D11) \qquad \frac{a_1}{a_2} = \frac{A+1}{A-1} .$$

The angle of polarisation for elliptical polarisation (including linear polarisation but obviously excluding circular polarisation) is the angle that the major axis of the polarisation ellipse makes with the x-axis (apart from an arbitrary constant). It is the angle ψ in figure D5.1.

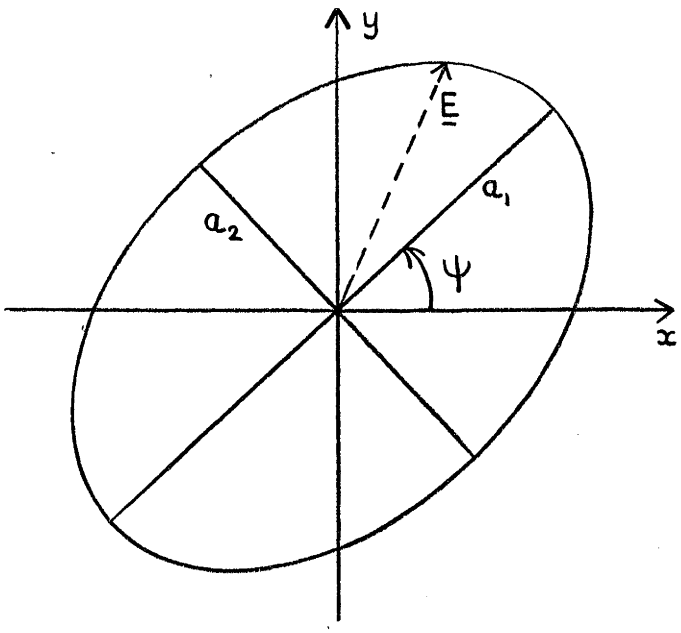


Fig. D5.1 Polarisation ellipse

From equation (D9), it is convenient to define

(D12) $$a(z) \equiv \tan(\theta/2)$$

and also to take

(D13) $$\frac{a_2}{a_1} \equiv \tan\lambda \ ,$$

where as mentioned above, a_1, a_2 are the major and minor diameters of the polarisation ellipse. Positive λ, that is $A > 1$, corresponds to right handed rotation (positive gyromotion) and negative λ ($A < 1$) to left handed rotation of the electric vector around the ellipse. A certain amount of algebraic manipulation yields the results (e.g., see Holt and Haskell[1]):

(a) The orientation of the ellipse, i.e. the angle of polarisation, ψ , is given by

(D14) $$\tan 2\psi = \tan\theta \cos\phi .$$

(b) λ , related to a_2/a_1 by (D13), is given by

(D15) $$\sin 2\lambda = \sin\theta \sin\phi .$$

(c) As mentioned, the direction of rotation of the electric vector is in the electron sense for $\lambda<0$ and in the opposite sense for $\lambda>0$.

The Faraday rotation being the change in ψ with position along the axis, from (D9), (D12), and (D14), elementary trignometric identities yield

(D16) $$\tan 2\psi = \tan \alpha z$$

or

(D17) $$\psi = \alpha z/2 ,$$

where

(D18) $$\alpha = (\kappa_1 - \kappa_2)$$

is the difference between the propagation vectors for the two characteristic waves. Hence the Faraday rotation is independent of collisions, and of the relative proportion of the two characteristic modes in the resultant wave.

Considering specific cases, when collisions are "small", equations (5.90) and (5.126) give, for fast waves,

$$(D19) \quad \mathcal{K}_1 = \left(\frac{\omega\,(\omega^2 + \omega\omega_b - \omega_p^2)}{c^2\,(\omega + \omega_b)} \right)^{\frac{1}{2}}$$

$$(D20) \quad \mathcal{K}_2 = \left(\frac{\omega(\omega^2 - \omega\omega_b - \omega_p^2)}{c^2\,(\omega - \omega_b)} \right)^{\frac{1}{2}}.$$

Therefore, by (D17),

$$(D21) \quad \psi = \frac{\omega^{\frac{1}{2}} z}{2c} \left[\left(\frac{\omega^2 + \omega\omega_b - \omega_p^2}{\omega + \omega_b} \right)^{\frac{1}{2}} - \left(\frac{\omega^2 - \omega\omega_b - \omega_p^2}{\omega - \omega_b} \right)^{\frac{1}{2}} \right].$$

When collisions are "large", the fast waves have (from (5.101) and (5.128))

$$(D22) \quad \mathcal{K}_1 = \mathcal{K}_2 = \frac{\omega}{c}$$

and so there is no Faraday rotation. On the other hand, from (5.105) and (5.134), when the magnetic field is large,

$$(D23) \quad \mathcal{K}_1 = \frac{\omega}{c} - \frac{\omega_p^2}{2c\omega_b}$$

$$\mathcal{K}_2 = \frac{\omega}{c} + \frac{\omega_p^2}{2c\omega_b}.$$

Therefore $\alpha = -\omega_p^2 / \omega_b c$, giving a Faraday rotation angle

$$(D24) \quad \psi = -\frac{\omega_p^2 z}{2\omega_b c}.$$

This angle is obviously proportion to $1/\omega_b$, and in fact this is apparent by considering the corresponding extreme case of (D20).

In the case of stress waves, from (5.113) and (5.129),

$$(D25) \quad \kappa_1 = \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{(\omega + \omega_b)}{f} \; ; \quad \kappa_2 = \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{(\omega - \omega_b)}{f} .$$

From (D17), these give the value

$$(D26) \quad \psi = \left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{\omega_b z}{f} .$$

This result is for the case of collisions "small"; it may be recalled that for collisions "large", the stress waves no longer exist.

D3. The Applied Magnetic Field Perpendicular to the Direction of Propagation

Finally in this Appendix, mention may be made of the polarisation in the case of a magnetic field perpendicular to the direction of propagation, say in the x direction. Although for this case the polarisation is indeterminate in the sense of E_x, E_y being independent, once specified at a given point the polarisation becomes a definite function of position along the z- axis. The transverse mode involving E_x, and the coupled longitudinal-transverse modes, associated with E_y, take the role of the characteristic waves.

The polarisation is again elliptical, and equations (D12) to (D17) still apply. Considering the fast waves first, with collisions "small", the equation (5.47) gives the uncoupled transverse wave solution for E_x:

$$(D27) \quad E_x = A_x(z)\, e^{i(\omega t - \kappa_1 z)},$$

with

$$(D28) \quad \mathcal{K}_1 = \left(\frac{\omega^2 - \omega_p^2}{c^2}\right)^{\frac{1}{2}} .$$

The coupled longitudinal-transverse mode, involving E_y, from equation (5.148) (magnetic field "small"), has solution

$$(D29) \quad E_y = A_y(z)\, e^{i(\omega t - \mathcal{K}_2 z)}$$

where

$$(D30) \quad \mathcal{K}_2 = \left(\frac{\omega^2 - \omega_p^2}{c^2}\right)^{\frac{1}{2}} - \frac{\omega_p^2 \omega_b^2}{2c(\omega^2 - \omega_p^2)^{3/2}} .$$

Then the polarisation is

$$(D31) \quad S = \frac{A_y}{A_x}\, e^{\frac{i\,\omega_p^2 \omega_b^2 z}{2c(\omega^2 - \omega_p^2)^{3/2}}} .$$

This is, of course, elliptical polarisation with S a function of position, plasma frequency, and magnetic field. The Faraday rotation, ψ, is given by

$$(D32) \quad \tan 2\psi = \tan 2\theta \cos\left(\frac{\omega_p^2 \omega_b^2 z}{2c(\omega^2 - \omega_p^2)^{3/2}}\right) ,$$

where however

$$(D33) \quad \tan \theta/2 = A_y / A_x .$$

Similarly for the stress waves, it is easily verified that

$$(D34) \qquad S = \frac{A_y(z)}{A_x(z)} = e^{\frac{i}{f}\left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{(8\omega^2 + 7\omega_p^2)\ \omega_b^2 z}{(58\omega^2 + 28\omega_p^2)\ \omega}} ,$$

and the Faraday rotation is given by

$$(D35) \qquad \tan 2\psi = \tan\theta \cos\left(\left(\frac{5}{7}\right)^{\frac{1}{2}} \frac{1}{f} \frac{(8\omega^2 + 7\omega_p^2)\ \omega_b^2 z}{(58\omega^2 + 28\omega_p^2)\ \omega}\right) .$$

It is clear that other cases such as that of a "large" magnetic field are worked out in a similar way, using the results of Chapter 5.

Reference

1. Holt, E. H., and Haskell, R. E., "Foundations of Plasma Dynamics", MacMillan (1965).

Principal Symbols and Notation Used in the Thesis

The symbols and notation are listed below together with the page number (or numbers) on which they are defined. Symbols used only locally are not included, and while in some cases the same symbol is used for different parameters, it is evident from the relevant text what is intended. However, it may be mentioned again that in certain parts of the thesis, the particle-type subscripts have been neglected. Whenever this is the case, it is indicated in the associated text.

(1) Lower case Roman

$\underline{a}_i$	44
a_0	71
b	48
$\underline{b}_i$	4
c_i	109
$\underline{c}_i$	4
$c_1^{uu}(\alpha)$	53, 54
$c_2^{uu}(\alpha)$	53, 54
$c_1^{uq}(\alpha)$	53, 54
$c_2^{uq}(\alpha)$	53, 54
$c_1^{p}(\alpha)$	55, 56
$c_2^{p}(\alpha)$	55, 56
$c_1^{qu}(\alpha)$	56, 57
$c_2^{qu}(\alpha)$	56, 57

$c_1^{qq}(\alpha)$	56, 57
$c_2^{qq}(\alpha)$	56, 57
$\frac{d}{dt}$	4
$d\sigma_{jk,i}$	8
$d\Omega$	8
$d\Omega_1$	74
$d\Omega_2$	74
e_i	4
e	71
$\underline{e}^{\mu}$, $\underline{e}_{\mu}$	5; a-33
f	155, 159
f_i	5
f_i (13-moment)	10
$f(D)$	219
$f_q(x,\epsilon)$	a-4
$\underline{g}_{jk}$, g_{jk}	8
$\underline{g}$, g	43
$\underline{g}'$, g'	43
$\underset{\sim}{g}$ ($g^{\mu\nu}$ etc.)	5
$g(D)$	219

$\hbar$	71
k	7
$\underline{k}$	71
$\underline{k}_o$, k_o	72, a-3
$\underline{k}_f$, k_f	72, a-3
$\underline{\ell}_i$	109
$\underline{\ell}_i^{(k)}$	111
m_i	4
m_{ir}	45
m_o	43
m_r	71, a-1
n_i	4
p_i	7
p_{ji}	23
$\underline{P}$	71, a-3
q	71
r	109
s	a-45
s_j	42
$\underline{u}_i$	6

$\underline{w}_i$	4
$\underline{w}_i'$	42
z	a-32
z_d	172

(2) Upper case Roman

B	4
D	215
D_1	a-33
$D(\underline{x}, \underline{Y})$	51
$\underline{E}$	4
$\underline{F}_i$	4
$\underline{F}$	207
(components F	210, 214)
$\underline{g}_o$	43
$\underline{g}$	44
$\mathcal{H}_j(\underline{r}_j)$	a-1
I	86, 113
$I_{jk}(\Psi_i)$	7
$\underline{J}$	27
$\underline{K}$	72

L	207
M_j	43
$\underset{\sim}{P}_i$	7
$\underset{\sim}{P}_{ji}$	22
$\underline{Q}_i$	6
Q_i	113, 115
$\underline{R}_i$	6
R_{ji}	22
S_i	113
$\underset{\sim}{S}_i$	6
$\underline{S}$, $\bar{\underline{S}}$	236
T_i	6
T_{ji}	22
$\underline{U}_i$	113
$\underset{\sim}{U}$	7
$\underline{V}$	4
$\underline{V}_o$	5
$\underline{V}_i$	5
$V(\underline{r}, \underline{r}_j, \underline{r}_k)$	a-1
W_j	113

$\underline{X}$	50
$\underline{Y}$, Y	50
Y_0	94
Z_j	149

(3) Lower case Greek

α_i	7
α_{ji}	23
α	31, 149
α	a-27
β_{jk}	51
β	151
γ_{jk}	51
γ	29, 149
$\delta_{jk,i}$	16
$\delta(1)$	45
$\delta(2)$	45
$\delta(3)$	45
$\delta(4)$	45
ϵ	45
ϵ_0	71

ϵ	151
ξ_{jk}	51
ς	151
η	31, 149
θ	151
θ	a-32
$\underline{\kappa}, \kappa$	74
κ	147
λ	44
λ	116
λ_p	159
λ_D	a-16
ν	150
ρ_i	6
σ	47
σ_q	a-3
σ_T	a-16 to a-20
τ	29, 149
τ_D	150
$\phi_{jk,i}(\mu,\nu)$	47

$\phi(\nu)$	74
ϕ, ϕ/τ	149
ϕ	223
$\phi_0, \phi_1, \phi_2, \ldots$	224
χ	45
χ_0	72
$\psi_i(\underline{r}, \underline{w}_i, t)$	4
ψ	151
$\psi_q(\underline{r}_j, \underline{r}_k)$	a-2
ω	145
ω_p	159
ω_b	162, 217

(4) Upper case Greek

$\Gamma(r,z)$	100
$\Delta_{jk}(\psi_i)$	8, 42
$\Delta\xi$	8, a-7
Λ_{ee}	75, 95
Λ_{ep}	76, 97
Λ_{pp}	78, 99
Λ	120

$\Phi(\underline{\varsigma}, \underline{q})$	52
$\Phi(\underline{X}, \underline{Y})$	52
$\Psi(\underline{r}, \underline{r}_j, \underline{r}_k)$	a-1
$\Omega^{\mu,\nu}_{jk,i}(r)$	53
$\Omega^{\nu}_{jk}(r)$	95

(5) Notation

(In this group, the Greek letters represent arbitrary parameters used to illustrate the notation).

$\frac{d}{dt}$	4
D	215
$\underset{\sim}{\alpha}$	5
$\underset{\sim}{\alpha} \times \underline{\beta}$	5
$\underset{\sim}{\alpha} : \underset{\sim}{\beta}$	5
$\{\underset{\sim}{\alpha}\}$	7
α_o	153
$\alpha_i, \beta_j, \ldots$	7
$\overline{\alpha_i}$	5