TESTING FOR A JUMP IN SPECTRA

AND CROSS-SPECTRA

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Various physical problems in meteorology, oceanography and radar involve the analysis of mixed spectra, i.e. spectra containing both a continuous and a discrete component. After a brief outline of the basic concepts of spectral theory Chapter I is concerned with an analytical derivation of Fisher's classical periodogram test (for the presence of a harmonic (or discrete) component in a mixed spectrum) for the case of independent observations. In Chapter 2 this test is generalized to the case when the observations are from a stationary Gaussian process. Two particular tests, one due to Whittle and the other due to Bartlett (the grouped periodogram test) are described in the latter half of this chapter and their advantages, disadvantages and limitations discussed.

Chapter 3 is based on two papers by Priestley (see [18], I, II). Following Priestley a test based on the correlogram is derived along with the asymptotic power of Whittle's test, the grouped periodogram test and the P(\lambda) test. These asymptotic powers are compared. A critical discussion of Priestley's P(\lambda) test and the asymptotic power comparisons concludes the chapter.

In Chapter 4 Hannan's smoothed periodogram test, where the spectral density function is estimated by a smoothing of the periodogram, is adjusted to give a new spectral estimate. This new estimate, while being far from ideal, represents a marked improvement over previous spectral estimates. This is verified by means of a
numerical example.

The final chapter of this thesis is an extension of these ideas to the case of two independent series of observations. Two independent tests, one based on the phases and the other on the amplitudes, are derived and combined to give a test for the presence of a harmonic component. A practical application of this test was examined for the case of two independent sets of rainfall data. This data was analysed (using an I.B.M. 1620 computer) and tested for periodicities: a summary of the results is given at the end of Chapter 5.
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CHAPTER 1

1.1 Basic concepts of spectral theory.

The sequence of real valued random variables

\[ \{ x_t \} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \]

is called a stochastic sequence if every finite collection

\[ (x_{t_1}, x_{t_2}, \ldots x_{t_n}) \]  \hspace{1cm} (1.1.1)

has a prescribed joint probability distribution and these distributions are compatible. A time series could be defined as a chronologically ordered sequence of values assumed by a time dependent variable \( x_t \) at equal intervals of time.

In the modern theory of time series analysis it is assumed that such a sequence is a realization of a stochastic sequence \( \{ x_t \} \). This implies that infinitely many other realizations are possible. In practice we are usually given a finite portion \( (x_1, x_2, \ldots x_n) \) of one realization only. Even if it seems unrealistic to think of more than one realization, e.g. daily sunspot numbers, it is still worthwhile to postulate a mathematical model (namely a stochastic sequence) in which infinitely many realizations are conceptually possible.
2.

A process is called strictly stationary if the distribution of \((l.l.l)\) is the same as the distribution of

\[
(x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_n+h})
\]

for every \(n, t_1, t_2, \ldots, t_n, h\). In other words the probability distributions are invariant under translation of the origin of time.

A process is called weakly stationary or stationary in the wide sense if, for every \(t\),

\[
\varepsilon(x_t) = m = \text{constant}
\]

\[
\varepsilon(x_t^2) < \infty
\]

\[
\varepsilon(x_s x_{s+t}) \text{ is independent of } s.
\]

In what follows a process which is stationary in the wide sense will be referred to simply as a stationary process.

\[
\gamma_t = \varepsilon(x_s x_{s+t})
\]

is called the \(t\)th serial covariance and it is clear that \(\gamma_t = \gamma_{-t}\) i.e. the sequence of serial covariances is symmetric about zero. This sequence, known as the correlogram, in the past formed the basis of the analysis of time series. Modern analysis however is based on the Fourier cosine transform of the \(\gamma_t\).

\[
\lim_{n \to \infty} \sum_{t=1}^{n-1} \gamma_t
\]

If \(\lim_{n \to \infty} \Sigma \gamma_t\) exists then
3.

\[ f(\lambda) = \lim_{n \to \infty} \frac{1}{2\pi} \sum_{t=-n+1}^{n-1} \gamma_t \cos \lambda t \quad (1.1.2) \]

\[ = \lim_{n \to \infty} \frac{1}{2\pi} \sum_{t=-n+1}^{n-1} \gamma_t e^{it\lambda} \quad (1.1.3) \]

The function \( f(\lambda) \) is called the spectral density function of the process \([x_t]\). Furthermore \( f(\lambda) \) is an even function of \( \lambda \) and is considered only in the range \([-\pi, \pi]\) since it is periodic with period \(2\pi\) outside this range. It is noted that

\[ \gamma_t = \int_{-\pi}^{\pi} f(\lambda) e^{it\lambda} d\lambda \quad (1.1.4) \]

\[ = 2\int_{0}^{\pi} f(\lambda) e^{it\lambda} d\lambda \quad (1.1.5) \]

If \( F(\lambda) \) has a density function \( f(\lambda) \) then \( f(\lambda) \, d\lambda \) may be replaced by \( dF(\lambda) \) in (1.1.4) and (1.1.5) to give a Reimann-Stieltjes Integral. \( F(\lambda) \) is called the spectral distribution function: instead of calling \( f(\lambda) \) the spectral density we sometimes call it the spectrum although this term properly refers to the set of values over which \( F(\lambda) \) is increasing.

\( F(\lambda) \) may be written in the form

\[ F(\lambda) = F_1(\lambda) + F_2(\lambda) + F_3(\lambda) \quad (1.1.6) \]
4.

where

(i) $F_1(\lambda)$ is absolutely continuous,

\[ F_1(\lambda) = \int_{-\pi}^{\lambda} f(\lambda) \, d\lambda \quad (1.1.7) \]

(ii) $F_2(\lambda)$ is a jump function, being constant save for jumps at a finite or denumerable set of points.

(iii) $F_3(\lambda)$ is continuous with zero derivative almost everywhere. $F_3(\lambda)$ is a singular function and is usually ignored in spectral analysis.

When only $F_1(\lambda)$ is present in (1.1.6) we say the process has a continuous spectrum with spectral density $f(\lambda)$. In the case of $F_2(\lambda)$ being the only component present in (1.1.6) we say the process has a discrete spectrum.

If

\[ F(\lambda) = F_1(\lambda) + F_2(\lambda) \quad (1.1.8) \]

the process is said to have a mixed spectrum. Due to the form of $F_2(\lambda)$ there will be distinctive jumps in $F(\lambda)$ at the frequencies at which the discrete (harmonic) components are present.

In an attempt to find a suitable estimate of $f(\lambda)$ for the sample $\{x_t, t = 1, ..., n\}$ we are prompted by the form of (1.1.2) to define the statistic called the sample periodogram as
\[ I_n(\lambda, x) = \frac{2}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} x_s x_t \cos((s-t)\lambda), \quad |\lambda| \leq \pi \]

\[ = 2 \sum_{s=-n+1}^{n-1} \left(1 - \frac{|s|}{n}\right) c_s e^{is\lambda}, \quad |\lambda| \leq \pi \]

(1.1.9)

where the sample covariance \( c_s \) is given by

\[ c_s = \frac{1}{n-s} \sum_{t=1}^{n-s} x_t x_{t+s}, \quad c_{-s} = c_s; \quad s=0, 1, \ldots, (n-1). \quad (1.1.10) \]

Now

\[ \lim_{n \to \infty} \varepsilon(I_n(\lambda, x)) = \lim_{n \to \infty} 2 \sum_{s=-n+1}^{n-1} \left(1 - \frac{|s|}{n}\right) \gamma_s e^{is\lambda} \]

\[ = \frac{4\pi}{h} f(\lambda) \]

at all points of continuity of \( f(\lambda) \) so that the periodogram is an asymptotically unbiased estimate of \( \frac{4\pi}{h} f(\lambda) \). The variance of the periodogram however does not approach zero (see Hannan [13] pp.52-54) so that it is not a consistent estimator.

A consistent estimator of the average value of the spectrum over a small band of frequencies can be obtained by taking a weighted average of periodogram values in the neighbourhood of the band. If it is required to estimate the average value of the spectrum over a band then an averaging of the periodogram over the band is quite satisfactory. The trouble comes however when one
6.

wishes to obtain detailed information so that bands have
to be made very narrow.

* * * * * * * *
1.2 The distribution of the periodogram.

Tests in harmonic analysis are, in general, based on a statistic called the periodogram which is defined by (1.1.9) or alternatively by

\[ I_n(\lambda_j, x) = \frac{2}{n} \left| \sum_{t=1}^{n} x_t e^{\frac{it\lambda_j}{n}} \right|^2, \quad \lambda_j = \frac{2\pi j}{n} \]  

(1.2.1)

where \( j = 1, 2, \ldots \lfloor \frac{1}{2}n \rfloor \), the squared brackets representing the integral part of \( \frac{1}{2}n \) (i.e. \( \frac{1}{2}n \) if \( n \) is even, \( \frac{1}{2}(n-1) \) if \( n \) is odd).

Assuming \( x_t \) to be i.i.d. \( \mathcal{N}(0,1) \) and writing

\[ a_t = \sqrt{\frac{2}{n}} \cos \lambda_j \quad \text{and} \quad b_t = \sqrt{\frac{2}{n}} \sin \lambda_j \]  

(1.2.2)

with \( \lambda_j \) defined as before so that \( \sum a_t b_t = 0 \) for each \( j \), it follows that the linear functions

\[ A = \sum_{t=1}^{n} a_t x_t \quad \text{and} \quad B = \sum_{t=1}^{n} b_t x_t \]  

(1.2.3)

will also be i.i.d. \( \mathcal{N}(0,1) \) except for \( j=0, \frac{1}{2}n \).

From (1.2.2) and (1.2.3)

\[ A^2 + B^2 = \frac{2}{n} \left[ \left( \sum_{t=1}^{n} x_t \cos \lambda_j \right)^2 + \left( \sum_{t=1}^{n} x_t \sin \lambda_j \right)^2 \right] \]  

(1.2.4)

\[ = \frac{2}{n} \left| \sum_{t=1}^{n} x_t e^{\frac{it\lambda_j}{n}} \right|^2 = I_n(\lambda_j, x) \]  

(1.2.5)
Since A and B are both normally distributed, $I_n(\lambda, x)$ will be distributed as a $\chi^2$ variate with two degrees of freedom (with the exception of $I_n(\lambda_0, x)$ and, for $n$ even, $I_n(\lambda_{2n}, x)$ whose distributions have only one degree of freedom) i.e. putting

$y_j = I_n(\lambda_j, x)$ we have

$$f(y_j) = \frac{1}{2} e^{-\frac{1}{2}y_j} \quad 0 \leq y_j < \infty$$

(1.2.6)

The omission of the two exceptional cases will not in general affect the test which is about to be derived (see Hannan [13] p.77).

From (1.2.6) it is seen that the probability of a periodogram ordinate being equal to, or exceeding any particular value, say $k$, is given by

$$\Pr \{ y_j \geq k \} = \int_k^{\infty} \frac{1}{2} e^{-\frac{1}{2}y_j} dy_j = e^{-\frac{1}{2}k}$$

(1.2.7)

For the case we are considering $\text{Cov}(y_j, y_k) = 0(n^{-2}), j \neq k$ (Hannan [13] p53) so that the $y_j$ are, asymptotically, independent. Hence if $y_j$ is the largest of $n$ independent values, the probability that all $n$ values are less than $k$ is given by

$$P(k) = \Pr(y_1 < k, y_2 < k, \ldots, y_n < k) = (1-e^{-\frac{1}{2}k})^n$$

(1.2.8)

This is usually referred to as Walker's criterion (see Walker [22]).
1.3 A periodogram test for independent observations.

Knowing the distribution of the periodogram a test (first derived geometrically by Fisher [8] and analytically by Whittle[25]) will be developed to test for the presence of a harmonic component on the null hypothesis that the observations \( \{x_t, t = 1, \ldots, n\} \) are independent, stationary and Gaussian. Placing

\[
g_j = \frac{I_n(\lambda_j, x)}{\sum_{k=1}^{\frac{1}{2}n} I_n(\lambda_k, x)} = \frac{y_j}{\sum_k y_k} \quad (1.3.1)
\]

a suitable test statistic is the greatest of the \( g_j \), (see Fisher [8]). It may be desired, however, once one or more jumps has been established, to test for further jumps. We will therefore consider the more general analytical case of the distribution of the \( r^{th} \) greatest of the \( g_j \). On deriving this distribution the distribution of \( \max_j g_j \) will be obtained simply by putting \( r = 1 \).

The method used, remembering that each of the \( y_j = I_n(\lambda_j, x) \) is distributed as a \( X^2 \) variate, is to find the characteristic function and hence the density function of the reciprocal of \( g_r \), the \( r^{th} \) greatest of the \( g_j \). A simple transformation gives the density function and hence the distribution of \( g_r \) (see Whittle [25]). Ordering the \( g_j \) so that
where \( N = [\frac{1}{2} n] \), the characteristic function of \( g_r^{-1} \) is given by

\[
\phi(\theta) = e^{\left( \frac{i\theta}{g_r} \right)}
\]

\[
= N \binom{N-1}{r-1} \int \cdots \int f(y_1, \ldots, y_N) e^{\frac{i\theta}{g_r}} \, dy_1 \ldots dy_N
\]

The first term arises from the fact that any one of the \( g_j \) may be the \( r^{th} \) greatest and there are \( \binom{N-1}{r-1} \) ways of choosing the \( (r-1) \) of the \( g_j \) which will be greater than \( g_r \).

Now

\[
f(y_j) = \frac{1}{2} e^{-\frac{1}{2} y_j}
\]

and the \( y \)'s are all independent so that

\[
f(y_1, \ldots, y_N) = (\frac{1}{2})^N \exp \left\{ -\frac{1}{2} \sum_{j=1}^{N} y_j \right\}
\]

\[
\therefore \phi(\theta) = N \binom{N-1}{r-1} \int_{y_r=0}^{\infty} \cdots \int_{y_r+1=0}^{\infty} \exp \left\{ \frac{i\theta}{y_r} \sum_{j=1}^{N} y_j - \frac{1}{2} \sum_{j=1}^{N} y_j \right\} dy_1 \ldots dy_N
\]

(1.3.4)
where the limits of integration are easily seen from (1.3.2).

Performing the first \((N-1)\) integrations and putting \(\frac{1}{2} y_r = y\)

\[
\phi(\theta) = N \binom{N-1}{r-1} \int_0^{\infty} e^{r(10-y)} \frac{(1 - e^{i\theta-y})^{N-r}}{(1 - \frac{i\theta}{y})^{N-1}} dy \quad (1.3.5)
\]

and as the probability density function of \(g_r^{-1}\) is given by

\[
f_r(g_r^{-1}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\theta) e^{r(1\theta-y)} d\theta
\]

then

\[
f_r(g_r^{-1}) = \frac{1}{2\pi} \frac{N!}{(N-r)!} \left( \frac{N-r}{r} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{g_r} \frac{e^{r(1\theta-y)} (1 - e^{i\theta-y})^{N-r}}{(1 - \frac{i\theta}{y})^{N-1}} dy \, d\theta
\]

\[
= \frac{r}{2\pi} \left( \frac{N}{r} \right) \sum_{j=0}^{N-r} (-1)^j \binom{N-r}{j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{g_r \sum_{r+j}^{N-r} (-1)^j \binom{N-r}{j} (i\theta-y) dy \, d\theta
\]

\[
\text{since} \quad (1 - e^{i\theta-y})^{N-r} = \sum_{j=0}^{N-r} (-1)^j \binom{N-r}{j} e^{i\theta-y)j}
\]

The integrand of (1.3.6) is absolutely integrable so that the order of integration may be reversed to evaluate
\[ \int_{-\infty}^{\infty} \frac{e^{i\theta(j+r-g_r^{-1})}}{(1 - \frac{i\theta}{y})^{N-1}} \, d\theta \]  

(1.3.7)

The contour around which we are integrating here is composed of the real axis and the infinite circle above or below it, as the case may be, depending on whether \( e^{i\theta} \) occurs having a positive or negative power. The pole occurs at \( \theta = -iy \) so that if \( j + r > \frac{1}{g_r} > 1 \) the integral vanishes. If \( j + r < \frac{1}{g_r} \) by applying the generalized Cauchy result

\[ f_n(z_0) = \frac{n!}{2\pi i} \int_{c} \frac{F(z)}{(z-z_0)^{n+1}} \, dz \]

the integral (1.3.7) is evaluated as

\[ -\frac{2\pi y}{(N-2)!} \left( g_r^{-1} - j - r \right)^{N-2} e^{y(j+r-g_r^{-1})} \]

and so

\[ f_1(g_r^{-1}) = \frac{r}{2\pi} \left( \frac{N}{r} \right)^{N-r} \sum_{j=0}^{N-r} (-1)^j \left( \begin{array}{c} N-r \\ j \end{array} \right) \int_{0}^{\infty} \frac{-2\pi}{(N-2)!} \, y^{N-1}(g_r^{-1} - j - r)^{N-2} e^{-y g_r^{-1}} \, dy \]  

(1.3.8)

Now \( f(g_r) = -g_r^{-2} f_1(g_r^{-1}) \)

\[ \therefore f(g_r) = r \left( \frac{N}{r} \right)^{N-r} \sum_{j=0}^{N-r} (-1)^j \left( \begin{array}{c} N-r \\ j \end{array} \right) \frac{(g_r^{-1} - j - r)^{N-2}}{(N-2)!} g_r^{-2} \int_{0}^{\infty} y^{N-1} e^{-y g_r^{-1}} \, dy \]
This is true however only in the case \( j + r < \frac{1}{g_r} \) so that, putting \( k = j + r \)

\[
f(g_r) = \frac{N!}{(r-1)! (N-r)!} \left[ \frac{g_r^{-1}}{r \sum_{k=r}^{N-r} (1-kg_r)^{r-1}} \right] (-1)^{k-r} \binom{N-r}{k-r} (1-kg_r)^{N-2}
\]

(1.3.9)

From (1.3.2) it is clearly seen that the maximum value \( g_r \) can take is \( \frac{1}{r} \) so that

\[
Pr(g_r \geq x) = \frac{1}{x} \sum_{k=r}^{N-r} \left[ \frac{g_r^{-1}}{r \sum_{k=r}^{N-r} (1-kg_r)^{r-1}} \right] (-1)^{k-r} \binom{N-r}{k-r} (1-kg_r)^{N-2} dg_r
\]

which on reversing the order of summation and integration

\[
= \frac{N!}{(r-1)! (N-r)!} \left[ \frac{x^{-1}}{r \sum_{k=r}^{N-r} (1-kx)^{r-1}} \right] \int_x^1 (-1)^{k-r} \binom{N-r}{k-r} (1-kx)^{N-2} dk
\]

(1.3.10)

This is easily evaluated to give

\[
F(x) = Pr(g_r \geq x)
= \frac{N!}{(r-1)!} \sum_{k=r}^{N-r} \frac{(-1)^{k-r} (1-kx)^{N-1}}{k (N-k)! (k-r)!}
\]

(1.3.11)
The distribution of max \( g_j \) is obtained by placing \( r=1 \) in (1.3.11) to give

\[
F_1(x) = \Pr(g_1 \geq x) = \sum_{k=1}^{[x^{-1}]} \frac{(-1)^{k-1} N! (1-kx)^{N-1}}{k! (N-k)!}
\]

(1.3.12)

This distribution (1.3.12) was obtained geometrically by Fisher who, from it, evaluated a table of significance points [8]. The test is generally referred to as Fisher's \( g \) test.

The successive terms of (1.3.11) decrease so rapidly that quite a satisfactory significance level is obtained by considering only the first term so that

\[
F(x) \approx \binom{N}{r} (1-rx)^{N-1}
\]

(1.3.13)

and similarly (1.3.12) approximates to

\[
F_1(x) \approx N(1-x)^{N-1}
\]

(1.3.14)
1.4 Limitations of Fisher's Test.

The application of Fisher's test is severely limited in two respects. Firstly it is not reasonable, as a null hypothesis, to assume that the observations are independent. Further, the grid points on which the periodogram ordinates are estimated are multiples of $\frac{2\pi}{n}$ and, as a result, when a jump in $F(\lambda)$ occurs between two consecutive grid points the amplitude of the harmonic component is considerably reduced.

The first restriction, namely that the observations must be independent, is easily relaxed as we shall soon see when discussing equivalent tests in the case of observations from a stationary Gaussian population. The second limitation, however, is not easily removed and as to just how much it affects the significance of the test will be seen from an examination of the effect of a jump on the periodogram.

For the independent observations $x_1, x_2, \ldots, x_n$ we have defined the periodogram as

$$I_n(\mu, x) = \frac{2}{n} \left| \sum_{t=1}^{n} x_t e^{i\mu_j} \right|^2$$

$$= \frac{2}{n} x' \mu \mu^* x$$  \hspace{1cm} (1.4.1)

where $\mu' = (e^{-j}, \ldots, e^{-nj})$, $x' = (x_1, \ldots, x_n)$ and $\mu^*$ is the transposed conjugate of $\mu$. The observed stationary process
\[ x_t = y_t + z_t \]  

(1.4.2)

where \( y_t \) is a stationary process with an absolutely continuous spectral density function and \( \varepsilon(y_t) = 0 \). Also, \( z_t \) is assumed to have a discrete spectrum. \( \{z_t\} \) is independent of \( \{y_t\} \).

Substituting (1.4.2) in (1.4.1) gives

\[
\frac{n}{2} I_n (\mu_j, x) = y\mu^* y + y\mu^* \sigma + y\mu^* z + z\mu^* z
\]

and on taking expectations

\[
\varepsilon \left( \frac{n}{2} I_n (\mu_j, x) \right) = y\mu^* y + z\mu^* z
\]

i.e.

\[
\varepsilon \left( I_n (\mu_j, x) \right) = \frac{2}{n} \left| \sum_{t=1}^{n} y_t e^{it\mu_j} \right|^2 + \frac{2}{n} \left| \sum_{t=1}^{n} z_t e^{it\mu_j} \right|^2
\]

(1.4.3)

Hence the increase in \( \varepsilon(I_n (\mu_j, x)) \) due to the presence of a harmonic component

\[
= \frac{2}{n} \left| \sum_{t=1}^{n} z_t e^{it\mu_j} \right|^2
\]

(1.4.4)

Suppose a jump occurs at a frequency \( \lambda_k \), then \( z_t \) can be written in the form
\[ z_t = \alpha_k e^{it\lambda_k} + \bar{\alpha}_k e^{-it\lambda_k} \]

\[\begin{align*}
\therefore \quad \frac{2}{n} \left| \sum_{t=1}^{n} z_t e^{it\mu_j} \right|^2 &= \frac{2}{n} \left| \sum_{t=1}^{n} e^{it\mu_j + \lambda_k} + \bar{\alpha}_k \sum_{t=1}^{n} e^{-it\mu_j + \lambda_k} \right|^2 \\
&= \frac{2}{n} \left| \sum_{t=1}^{n} e^{it\mu_j + \lambda_k} \right|^2 + \frac{2}{n} \left| \sum_{t=1}^{n} e^{-it\mu_j - \lambda_k} \right|^2 \\
&= \frac{2}{n} \left| \alpha_k \right|^2 \left\{ \frac{\sin^2 \frac{n}{2} (\mu_j - \lambda_k)}{\sin^2 \frac{1}{2} (\mu_j - \lambda_k)} + \frac{\sin^2 \frac{n}{2} (\mu_j + \lambda_k)}{\sin^2 \frac{1}{2} (\mu_j + \lambda_k)} \right\} \\
&= \frac{2}{n} \left| \alpha_k \right|^2 \left\{ \frac{\sin^2 \frac{n}{2} (\mu_j - \lambda_k)}{\sin^2 \frac{1}{2} (\mu_j - \lambda_k)} \right\}
\end{align*}\]

(1.4.5)

As \( \lambda_k \) tends towards \( \mu_j \) (i.e. when a jump occurs near to one of the grid points) the first term in the brackets of (1.4.5) is the dominant one and (1.4.5) itself tends towards \( 2n \left| \alpha_k \right|^2 \).

Consider the worst possible case which occurs when the jump lies midway between two consecutive grid points, say \( \frac{2\pi j}{n} \) and \( \frac{2\pi (j+1)}{n} \).

In this case \( \sin \frac{n}{2} (\mu - \lambda_k) = 1 \) and \( \sin \frac{1}{2} (\mu - \lambda_k) = \sin \frac{\pi}{2n} \approx \frac{\pi}{2n} \),

so that (1.4.5) is approximately equal to \( \frac{8n}{\pi^2} \left| \alpha_k \right|^2 \). Hence when a jump occurs at a frequency \( \lambda_k \) midway between two consecutive grid points (\( \frac{2\pi j}{n} \) and \( \frac{2\pi (j+1)}{n} \)), the contribution of the corresponding harmonic to the periodogram at the nearest grid point \( I_n(\lambda_j, x) \) or \( I_n(\lambda_{j+1}, x) \) in this case) is only about \( \frac{4}{\pi^2} \) of its contribution to \( I_n(\lambda_k, x) \). The fact that this is so high may be of importance in testing the significance of a periodogram at a given frequency.
2.1 The asymptotic relationship between the periodogram of a linear process and that of the residual process.

As has already been emphasized, Fisher's $g$ test is applicable only when the observations $\{x_t, t = 1, \ldots, n\}$, besides being normal, are independent of each other.

From (1.3.1) the test statistic is

$$
\max_j \frac{I_n(\lambda_j, x)}{\sum_k I_n(\lambda_k, x)}
$$

which has the distribution given by (1.3.12). In general however, the observations are not independent. By using an asymptotic relationship between the periodogram of a linear process and that of the residual process it will be found that the problem can be reduced to that of classical periodogram analysis.

Consider the situation where we observe $n$ observations $x_1, \ldots, x_n$ of a stationary process (with a mixed spectrum) defined by

$$
x_t = y_t + z_t \quad \quad (2.1.1)
$$

In (2.1.1) $\{x_t\}$ is the observed process and $\{y_t\}$ is a stationary process with an absolutely continuous spectral function, i.e. $\{y_t\}$
may be represented by

\[ y_t = \sum_{n=0}^{\infty} g_n \varepsilon_{t-u} \quad (2.1.2) \]

Here the \( g_n \) are taken to be constants and the \( \varepsilon_t \), the
residuals, are i.i.d. \( N(0,1) \). \( \{ z_t \} \) represents a stationary process
with a discrete spectrum and is independent of \( \{ y_t \} \).

The periodogram of the observed process \( \{ x_t \} \) is defined by

(1.2.1) while \( I_n(\lambda; y) \) and \( I_n(\lambda; \varepsilon) \) are similarly defined for the
linear process \( \{ y_t \} \) and the residual process \( \{ \varepsilon_t \} \) respectively.

Put

\[ J_n(\lambda; y) = \sum_{t=1}^{2} \sum_{j=0}^{n} y_t e^{i t \lambda} \]

\[ = \sqrt{2} \sum_{t=1}^{2} \sum_{j=0}^{\infty} g_j e^{i t \lambda} e^{i t \lambda} \]

\[ = \sum_{j=0}^{\infty} g_j e^{i j \lambda} J_n(\lambda; \varepsilon) + \sqrt{2} \sum_{j=0}^{\infty} h_j(\lambda) \quad (2.1.3) \]

where

\[ h_j(\lambda) = e^{i j \lambda} \left[ \sum_{k=0}^{j-l} e^{i k \lambda} \varepsilon_{-k} - \sum_{k=0}^{j-l} e^{i(n-k)\lambda} \varepsilon_{n-k} \right] , \]

\[ |j| < n \quad (2.1.4) \]

\[ = e^{i j \lambda} \left[ \sum_{k=j-n}^{j-l} e^{-i k \lambda} \varepsilon_{-k} - \sum_{k=0}^{n-l} e^{i(n-k)\lambda} \varepsilon_{n-k} \right] , \]

\[ |j| \geq n \quad (2.1.5) \]
Combining (2.1.4) and (2.1.5) we have

\[ h_j(\lambda) = \sum_{k=\max\{j-n, 0\}}^{j-1} e^{i(j-k)\lambda} e^{-k} - \sum_{k=0}^{\min\{n-1, j-n\}} e^{i(n+j-k)\lambda} e^{n-k} \]

for all \( j \), \hfill (2.1.6)

the limits of summation being of this form since each sum cannot contain more than \( n \) terms (for all \( j \)). Thus for the last term on the right hand side of (2.1.3)

\[ \left\{ \varepsilon \left| \sum_{j} g_j h_j(\lambda) \right|^2 \right\}^{\frac{1}{2}} \leq \sqrt{2} \sum_{j} \left| g_j \right| \left\{ \varepsilon \left| h_j(\lambda) \right|^2 \right\}^{\frac{1}{2}} \]

The residuals \( \varepsilon_t \) are all N.I.D. (0,1) so that, from (2.1.6),

\[ \varepsilon(\left| h_j(\lambda) \right|^2) = 2j \]

and substituting this in (2.1.7) gives

\[ \left\{ \varepsilon \left| \sum_{j} g_j h_j(\lambda) \right|^2 \right\}^{\frac{1}{2}} \leq \sqrt{2} \sum_{j} \left| g_j \right| j^{\frac{3}{2}} \]

By choosing \( g_j = 0(j^{-3/2}) \), the right hand side of (2.1.8) converges and the second moment of \( \sqrt{2} \sum_{j} g_j h_j(\lambda) \) will in this case be \( o(n^{-1}) \).

Put \( H(\lambda) = \sqrt{\frac{2}{n}} \sum_{j} g_j h_j(\lambda) \)

so that
21.

\[ I_n(\lambda, y) = |J_n(\lambda, y)|^2 \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{t=1}^{n} e^{it\lambda} \sum_{j=0}^{\infty} g_j e^{ij\lambda} + H(\lambda) \]

Since \( \sum_{j=0}^{\infty} g_j e^{ij\lambda} \neq 0 \), then

\[ I_n(\lambda, y) = 2\pi f(\lambda) I_n(\lambda, \epsilon) + \left\{ \sqrt{\frac{2}{\pi}} \sum_{t=1}^{n} e^{it\lambda} \left( \sum_{j=0}^{\infty} g_j e^{ij\lambda} \right) \right\} \]

\[ + \sqrt{\frac{2}{\pi}} \sum_{t=1}^{n} e^{it\lambda} \left( \sum_{j=0}^{\infty} g_j e^{ij\lambda} \right) H(\lambda) + |H(\lambda)|^2 \}

\[ = 2\pi f(\lambda) I_n(\lambda, \epsilon) + M_n(\lambda) \quad \text{(say)}, \quad (2.1.9) \]

where \( \bar{A} \) represents the complex conjugate of \( A \).

Hence

\[ \left\{ \epsilon \left| I_n(\lambda, y) - 2\pi f(\lambda) I_n(\lambda, \epsilon) \right|^2 \right\}^{\frac{1}{2}} = \left\{ \epsilon \left| M_n(\lambda) \right|^2 \right\}^{\frac{1}{2}} \]

\[ \leq A \left\{ \epsilon \left| H(\lambda) \right|^2 \right\}^{\frac{1}{2}} + B \left\{ \epsilon \left| H(\lambda) \right|^4 \right\}^{\frac{1}{2}} \]

(\text{where } A \text{ and } B \text{ are independent of } h_j(\lambda) \),

\[ = 0(n^{-\frac{1}{2}}) + 0(n^{-1}) \]

\[ = 0(n^{-\frac{1}{2}}) . \]

This result is easily generalized to the case when the \( 2k^{th} \) moment of \( \epsilon_t \) is finite and \( \sum_{j=0}^{\infty} |g_j|^{j^k} < \infty \). Under these conditions
\[ \varepsilon \left\{ |I_n(\lambda, y) - 2\pi f(\lambda) I_n(\lambda, \varepsilon)|^{2k} \right\} \text{ is of the order } n^{-k}. \]

From (2.1.9)

\[ I_n(\lambda, y) = 2\pi f(\lambda) I_n(\lambda, \varepsilon) + o(n^{\frac{1}{2}}) \]

i.e.

\[ \frac{I_n(\lambda, y)}{2\pi f(\lambda)} \sim \frac{I_n(\lambda, \varepsilon)}{2\pi f(\lambda)} \quad (2.1.10) \]

As the \( I_n(\lambda, \varepsilon) \) are independently distributed it follows, on the hypothesis of no jump being present (i.e. \( z_t = 0 \)), that

\[ K_n(\lambda, x) = \frac{I_n(\lambda, x)}{2\pi f(\lambda)} \quad (2.1.11) \]

may be regarded as the periodogram of an independent process.

From (1.3.1) and (2.1.11) therefore the new test statistic is

\[ e_0 = \frac{\text{max}}{N} \sum_{j=1}^N K_n(\lambda_j, x), \quad N = \left[ \frac{1}{2} n \right] \quad (2.1.12) \]

and in order to construct a test of significance the distribution of this new statistic is required.
2.2 **The asymptotic distribution of the new test statistic.**

Before proceeding to derive the asymptotic distribution of the test statistic (2.1.11) the following lemma is proved.

**Lemma:** For all $j = 1, 2, \ldots \lfloor \frac{1}{2}n \rfloor$, $\max_j |K_n(\lambda_j, x) - I_n(\lambda_j, e)|$

converges in probability to zero.

**Proof.** From (2.1.9) it follows that

$$I_n(\lambda, x) - 2\pi f(\lambda) I_n(\lambda, e) = M_n(\lambda)$$ (2.2.1)

and using a multivariate form of Tchebycheff's inequality it follows, for $\lim_{n} \eta_n = 0$,

$$\Pr \left( \max_j |I_n(\lambda_j, x) - 2\pi f(\lambda) I_n(\lambda_j, e)| \geq \eta_n \right)$$

$$= \Pr \left( \max_j |M_n(\lambda_j)| \geq \eta_n \right)$$

$$= \Pr \left( \max_j |M_n(\lambda_j)|^{2k} \geq \eta_n^{2k} \right)$$

$$\leq \frac{1}{\eta_n^{2k}} \Sigma_j \varepsilon \left( |M_n(\lambda_j)|^{2k} \right)$$

$$= \frac{1}{\eta_n^{2k}} \frac{n}{2} O \left( n^{-k} \right)$$

$$\rightarrow 0 \text{ if } k \geq 2.$$}

From the previous section it has been seen that this holds for $k = 1$ so that the lemma is true for all $k \geq 1$. 
24.

\[ \max_j \left| K_n(\lambda_j, x) - I_n(\lambda_j, \varepsilon) \right| \text{ converges in probability to zero.} \]

Having proved this lemma we now want to show that

\[ \max_j \frac{K_n(\lambda_j, x)}{\Sigma_j K_n(\lambda_j, x)} \]

asymptotically has the same distribution as

\[ \frac{I_n(\lambda_j, \varepsilon)}{\Sigma_j I_n(\lambda_j, \varepsilon)} \]

In other words, if

\[ \Pr \left( \max_j \frac{I_n(\lambda_j, \varepsilon)}{\Sigma_j I_n(\lambda_j, \varepsilon)} > d(n, \alpha) \right) = \alpha \]

(2.2.2)

it is required to show that

\[ \Pr \left( \max_j \frac{K_n(\lambda_j, x)}{\Sigma_j K_n(\lambda_j, x)} > d(n, \alpha) \right) \to \alpha \]

where \( d(n, \alpha) \) is chosen to satisfy (2.2.2).

Consider the relation

\[ \max_j K_n(\lambda_j, x) \]

\[ \frac{d(n, \alpha)}{\Sigma_j K_n(\lambda_j, x)} = \max_j \left\{ \frac{I_n(\lambda_j, \varepsilon)}{d(n, \alpha) \Sigma_j I_n(\lambda_j, \varepsilon)} + \frac{K_n(\lambda_j, \varepsilon) - I_n(\lambda_j, \varepsilon)}{d(n, \alpha) \Sigma_j I_n(\lambda_j, \varepsilon)} \right\} \]

\[ \left\{ 1 + \frac{1}{N} \Sigma_j \frac{K_n(\lambda_j, x) - I_n(\lambda_j, \varepsilon)}{I_n(\lambda_j, \varepsilon)} \right\}^{-1} \]

(2.2.3)

where \( N = \lfloor \frac{1}{2} n \rfloor \).
Now

\[ \max_j \left| K_n(\lambda_j, x) - I_n(\lambda_j, \varepsilon) \right| \geq \frac{1}{\sum_j} |K_n(\lambda_j, x) - I_n(\lambda_j, \varepsilon)| \]

so that, from the lemma,

\[ \frac{1}{N} \sum_j |K_n(\lambda_j, x) - I_n(\lambda_j, \varepsilon)| \to 0 \text{ in probability.} \]

Furthermore, applying Markov's theorem (see Gnedenko [12] p.232), it follows that

\[ \Pr \left\{ \left| \frac{1}{N} \sum_j I_n(\lambda_j, \varepsilon) - \frac{1}{N} \sum_j \varepsilon(I_n(\lambda_j, \varepsilon)) \right| \geq \varepsilon \right\} \to 0. \]

Since

\[ \frac{1}{N} \sum_{j=1}^{N} \varepsilon(I_n(\lambda_j, \varepsilon)) = \frac{1}{N} \sum_{j=1}^{N} \lambda_j f(\lambda_j, \varepsilon) \]

\[ = \text{constant} \ (\varepsilon_t : N(0,1)) \]

it follows that \( \frac{1}{N} \sum_j I_n(\lambda_j, \varepsilon) \) converges in probability to a constant.

Applying these results to the right hand side of (2.2.3) it is found that \( \max_j K_n(\lambda_j, x) \) converges in probability

\[ \frac{1}{d(n, \alpha)} \sum_j K_n(\lambda_j, x) \]

\[ \to \max_j I_n(\lambda_j, \varepsilon) \]

\[ \frac{1}{d(n, \alpha)} \sum_j I_n(\lambda_j, \varepsilon) \]

\[ \Pr \left\{ \max_j \frac{K_n(\lambda_j, x)}{d(n, \alpha) \sum_j K_n(\lambda_j, x)} > d(n, \alpha) \right\} \to \alpha. \quad (2.2.4) \]
26.

By use of the relation (2.2.3) therefore it has been established that the statistic (2.1.11) is asymptotically distributed as Fisher's g distribution.

To test for the presence of harmonic components when the observations are from a stationary Gaussian process form the statistic

\[ K_n(\lambda_j, x) = \frac{I_n(\lambda_j, x)}{2\pi f(\lambda_j)} \]  

(2.2.5)

where the density function has been prescribed a priori. As a consequence of (2.1.10) \( K_n(\lambda_j, x) \) may be used in the same way as would \( I_n(\lambda_j, x) \) if the null hypothesis was that the \( x_t \) were independent. Hence the \( K_n(\lambda_j, x) \) are evaluated at the standard grid points \( \lambda_p = \frac{2\pi p}{n}, \ p = 1, 2, \ldots, [\frac{2n}{3}] \) and the peaks of this function are tested by referring the test statistic (2.1.12) to Fisher's g distribution. However, \( f(\lambda) \) is unlikely to be prescribed a priori and will have to be estimated from the observed process \( \{x_t, t = 1, \ldots, n\} \).

* * * * * * *
2.3 A test based on an autoregressive model.

One of the more common methods of estimating the spectral density function is by means of a process based on an autoregressive model:

\[ a_0 x_t + a_1 x_{t-1} + \cdots = \varepsilon_t \quad (2.3.1) \]

where the \( x_t \) are from a stationary non-deterministic process. \( a_0 \) is assumed to be unity and the residual variates \( \varepsilon_t, t = 0, 1, 2, \ldots \) are taken to be independently distributed with zero mean and variance \( \sigma^2 = \nu \) (say). It is of course impossible to estimate an infinite number of parameters from a finite sample but, by considering sufficiently many, one can obtain an arbitrarily good approximation to the real process. The order of the autoregression may either be fixed a priori or determined from an examination of the earlier serial covariances.

Consider the process

\[ a_0 x_t + a_1 x_{t-1} + \cdots + a_n x_{t-n} = \varepsilon_t, \quad a_0 = 1. \quad (2.3.2) \]

For this process the spectral density function is given by

\[ f(\lambda) = \frac{\nu}{2\pi \left| \sum_{j=0}^{n} a_j e^{ij\lambda} \right|^2} \quad (2.3.3) \]

and the problem is to estimate \( a_1, a_2, \ldots, a_n \) and \( \nu \).
Suppose the series contains a sinusoidal term of the form

\[ \sum_{k=1}^{p} \left( A_k \sin \lambda_k t + B_k \cos \lambda_k t \right) \]  \hspace{1cm} (2.3.4)

Then the least squares estimates of \( A_k, B_k \) and \( \lambda_k \) are found to be (see Whittle [26] pp.49-52)

\[ \hat{A}_k = \frac{2}{n} \sum_{t=1}^{n} x_t \sin \lambda_k t \]  \hspace{1cm} (2.3.5)

\[ \hat{B}_k = \frac{2}{n} \sum_{t=1}^{n} x_t \cos \lambda_k t \]

and \( f(\lambda) \) has a local maximum at \( \lambda_k \).

It has been shown by Whittle ([27] pp.95-99) that in this case the unknown constants in (2.3.3) may be estimated from

\[ \Sigma_{i,j} a_i a_j c'_{i-j} \]  \hspace{1cm} (2.3.6)

\[ \Sigma_{i} a_i c'_{i-j} = 0 \]

where \( c'_{i-j} = c_{i-j} - \frac{p}{\Sigma_{k=1}^{p} I_n(\lambda_k,x)} e^{i\lambda_k (i-j)} \)  \hspace{1cm} (2.3.7)

and \( c_{i-j} \) is defined, as before, by (1.1.10).

In the case of an autoregressive process with a mixed spectrum therefore, the amplitudes and frequencies of the harmonic components are estimated by (2.3.5) and these estimates are used to correct the sample covariances as in (2.3.7). This correction
is such as to remove the effect of the harmonic or discrete components of the spectrum when estimating the spectral density of the continuous component. The systems of equations (2.3.6) give the required estimates of the constants to estimate the spectral density (2.3.3).

Replacing the spectral density by its estimate such that

\[ \hat{K}_n(\lambda, x) = \frac{I_n(\lambda, x)}{2\pi f(\lambda)} \] (2.3.8)

then the test statistic is

\[ \max_j \frac{\hat{K}_n(\lambda_j, x)}{\sum_j \hat{K}_n(\lambda_j, x)} \]

The effect of replacing the spectral function by its estimate is now examined.

From (2.3.3) and (2.3.6)

\[ \hat{f}(\lambda) = \frac{\sum_i \hat{a}_i \hat{d}_j c_{i-j}}{2\pi \sum_j \hat{a}_j e^{ij\lambda}} \] (2.3.9)

Now \( \sum \sum_i \hat{a}_i \hat{d}_j c_{i-j} \) converges in probability to \( \sum \sum_i a_i a_j \gamma'_{i-j} \)

where \( \gamma_t \) is the \( t^{th} \) serial covariance and \( \gamma'_t \) is the \( t^{th} \) serial covariance corrected for the harmonic components. Furthermore,
\[ e \left\{ \max_j \left| \sum_{j=0}^{n} \hat{a}_j e^{ij\lambda} \right|^2 - \left| \sum_{j=0}^{n} a_j e^{ij\lambda} \right|^2 \right\} \]

\[ = e \left\{ \max_j \left| \sum_{j,k} (\hat{a}_j \hat{a}_k - a_j a_k) e^{i(j-k)\lambda} \right|^2 \right\} \]

\[ \leq e \left\{ \sum_{j,k} |\hat{a}_j \hat{a}_k - a_j a_k| \right\} \] \hspace{1cm} (2.3.10)

But

\[ e \left\{ |\hat{a}_j \hat{a}_k - a_j a_k| \right\} \leq e \left\{ |\hat{a}_j - a_j| \cdot |\hat{a}_k - a_k| \right\} + e \left\{ |\hat{a}_j - a_j| \cdot |a_k| \right\} + e \left\{ |\hat{a}_k - a_k| \cdot |a_j| \right\} \] \hspace{1cm} (2.3.11)

Each term on the right hand side of (2.3.11) converges in probability to zero (since \( \hat{a}_k \) converges to \( a_k \) in mean square).

Hence \( |\sum_j a_j e^{ij\lambda}|^2 \) converges in probability to \( |\sum_j a_j e^{ij\lambda}|^2 \)

and so \( \hat{f}(\lambda) \) converges in probability to \( f(\lambda) \). That is to say the test statistic (2.1.11) is not affected by replacing the spectral density function by its estimate. Following exactly the same argument as before (section 2.2) the asymptotic validity of the test based on \( \max_j \hat{K}_n(\lambda_j, x) \) may be established. The test statistic (2.1.12) with \( \hat{K}_n(\lambda_j, x) \) replacing \( K_n(\lambda_j, x) \) is found to have, asymptotically, the same distribution as Fisher's \( g \).
2.4 The grouped periodogram test.

Bartlett's grouped periodogram test has been derived for the \([\frac{1}{2}n]\) periodogram ordinates \(I_n(\lambda_j, x) \quad (\lambda_j = \frac{2\pi j}{n}, j=1, 2, \ldots, [\frac{1}{2}n])\) by Priestley (see [18], I) and has the advantage that, when applying the test, an estimate of the spectral density function is not required. In practical applications, however, restrictions must be placed on the bandwidth of the peaks of \(f(\lambda)\). If there is a peak at \(f(\lambda_0)\) and \(\delta\) is such that \(f(\lambda_0 - \delta) = f(\lambda_0 + \delta) = \frac{1}{2}f(\lambda_0)\) then the width of the peak is defined to be \(2\delta\) and the bandwidth of \(f(\lambda)\) is defined to be the width of the narrowest peak.

Consider a sample of \(n\) observations. It is then obvious that it would be impossible to detect peaks of \(f(\lambda)\) whose bandwidths are less than \(\frac{1}{n}\). It will be assumed in what follows therefore that no peaks of the continuous spectrum \(f(\lambda)\) have widths less than \(k > 0(n)\). Put \(K = [k]\) and divide the \([\frac{1}{2}n]\) periodogram ordinates into \([\frac{1}{2}n]\) sets, each set containing \(K\) ordinates.

Placing

\[
L_{(q)}^K = \frac{I_n(\lambda_q, x)}{2\pi f(\lambda_q)}
\]

(2.4.1)

\[
= \frac{\sum_{p=(\ell-1)K+1}^{\ell K} I_n(\lambda_p, x)}{2\pi f(\lambda_p)}
\]

then \(\max_{(\ell-1)K < q \leq \ell K} L_{(q)}^K\) is asymptotically distributed as
Fisher's $g$. That is to say, the periodogram ordinates of one of the $\left\lceil \frac{3n}{K} \right\rceil$ sets (for which $L_K^{(q)}$ is defined) are being considered. Consequently if a significance level $\alpha$ is observed when applying Fisher's test, then the correct significance level for a test based on $L_K^{(q)}$ is $\frac{\alpha K}{\left\lceil \frac{3n}{K} \right\rceil}$.

Since the bandwidth of $f(\lambda) \geq K$, it is possible, for the region over which $L_K^{(q)}$ is considered, to assume the spectral density to be constant. Then becomes

$$G_K^{(q)} = \frac{I_n(\lambda_p, x)}{\sum_p I_n(\lambda_p, x)}$$

and the test statistic is $\max q G_K^{(q)}$ (where $q$ is over the same range as before) which is asymptotically distributed as Fisher's $g$.

It appears that by using $\max q G_K^{(q)}$ as a test statistic the ideal test has been found. However the question arises as to how good $G_K^{(q)}$ is as an approximation to $L_K^{(q)}$. From (2.4.1) and (2.4.2)

$$G_K^{(q)} = L_K^{(q)} \frac{\sum_p I_n(\lambda_p, x) f(\lambda_q)}{\sum_p I_n(\lambda_p, x) f(\lambda_p)}$$

In order to reduce the right hand side of (2.4.3) it is further assumed that $f(\lambda)$ is differentiable and has at most one zero in the region considered. Also, suppose $M$ and $m$ are the
upper and lower bounds respectively of $f(\lambda)$. Then from the definition of width and the restriction that has been placed on the bandwidth it follows that $\frac{m}{M} \geq \frac{1}{2}$ and so

$$2 - \frac{M}{m} \leq \frac{f(\lambda_q)}{f(\lambda_p)} \leq \frac{M}{m} \quad . \quad (2.4.4)$$

Hence from (2.4.3)

$$\left(2 - \frac{M}{m}\right) L_K^{(q)} \leq G_K^{(q)} \leq L_K^{(q)} \frac{M}{m} \quad . \quad (2.4.5)$$

It is possible therefore for $G_K^{(q)}$ to differ considerably from $L_K^{(q)}$ and consequently one would not use this procedure unless $\frac{M}{m}$ was small.

As Priestley points out, although the problem of estimating the spectral density function does not arise, the power could be greatly reduced by using $G_K^{(q)}$ as an estimate of $L_K^{(q)}$, i.e. assuming $f(\lambda)$ to be constant in the region considered. Even so, as shall be seen in Chapter 3, the asymptotic power of this test compares favourably with other tests.
2.5 Disadvantages of these tests.

Both the periodogram test based on autoregressive model and the grouped periodogram test asymptotically have the same distribution as Fisher's $g$. Since both tests are based on the periodogram ordinates estimated over the set of grid points $\frac{2\pi j}{n}$, $j = 1, 2, \ldots, \lfloor \frac{1}{2}n \rfloor$ they have disadvantages common to Fisher's test. If a jump in the spectrum occurs between two grid points, then, as before, while the test is still consistent its power will be reduced (see section 1.5). Obviously any test based on a set of periodogram ordinates estimated at these grid points will suffer in the same way.

A major fault of Whittle's test (based on the autoregressive model) occurs when examining, say, frequency $\lambda_j$ and the possible periodicity is removed. Then if there are periodicities at other frequencies $\lambda$ these will distort the autoregressive estimate of the continuous spectrum and in particular may distort it at $\lambda_j$. The possibility of a spectral mass at more than one frequency is not remote because of the likelihood of harmonics arising from the fundamental frequency.

For the grouped periodogram test, from (2.4.5) it is seen that the error involved in approximating $L^q_K$ by $G^q_K$ may be quite large and, unless $\frac{M}{m}$ is small, this procedure would not be used. The only way to reduce $\frac{M}{m}$ would be to make $K$
smaller than the bandwidth of \( f(\lambda) \). To do this, however, requires one to know the density function \( f(\lambda) \). Hence, although the need for knowledge of the spectrum is not so explicit as in Whittle's test, there needs to be some familiarity with it in the grouped periodogram test when choosing a value of \( K \).

If the density function has a small bandwidth then \( G_K(q) \), which is asymptotically distributed as Fisher's \( g \), will have a very small number of degrees of freedom and as a result the power will be adversely affected. A value of \( K \) must therefore be chosen which will be a compromise between retaining sufficient degrees of freedom and reducing the factor \( \frac{M}{m} \).
CHAPTER 3

A Test Based on the Correlogram:
A Comparison of Tests.

3.1 Introduction.

In considering tests to determine the presence (or absence) of harmonic components in a set of observations the common statistic used has been the periodogram. Since the periodogram ordinates are evaluated at a discrete set of grid points (at a distance \( \frac{2\pi}{n} \) apart) it is quite likely, particularly if \( n \) is small, that a jump will occur between two of these grid points. As has already been seen the effect of such a situation may drastically affect the significance of the test statistic. A test is now discussed where the test statistic is based not on the periodogram but on the correlogram, i.e. the sequence of serial covariances. This correlogram test, or \( P(\lambda) \) test as it will be called (for reasons which will soon be obvious), was first derived by Priestley (see [18], I). At first it appears that this test will overcome the problem of a jump occurring between two frequency points at which the test statistic is evaluated. When developing the test statistic \( J_q \) to test for a jump at \( \lambda \) Priestley uses \( \sum_{p} P\left(\frac{2\pi p + a}{m}\right) \) and says: "The constant \( a \) is such that \( \frac{2\pi p + a}{m} = \lambda \) for some integral value of \( p \)." (see [18], I p.231). Consequently the frequency
being tested will always occur at one of the grid points over which \( P(\lambda) \) is being summed. In order to choose such a value of "a" however, one must make a priori reference to the data. Priestley has obviously misunderstood the situation here for he has made no allowance for this when deriving the power of the \( P(\lambda) \) test (see section 5.3). This problem will be more fully discussed later and it will be seen that this test does in fact suffer in exactly the same way as have earlier tests for the case when a jump occurs between two grid points.

From (2.1.1)

\[ x_t = y_t + z_t \]

and

\[ \gamma_s = \varepsilon(x_t x_{t+s}) \]

\[ = \varepsilon \left\{ (y_t + z_t)(y_{t+s} + z_{t+s}) \right\} \]

\[ = \gamma_s^{(1)} + \gamma_s^{(2)} \]

(3.1.1)

since \( \varepsilon(y_t z_{t+s}) = \varepsilon(z_t y_{t+s}) = 0 \).

Now the serial covariance \( \gamma_s^{(1)} \) is the Fourier transform of an absolutely continuous spectral density function so that

\[ \lim_{s \to \infty} \gamma_s^{(1)} = \lim_{s \to \infty} \left[ \int_{-\pi}^{\pi} e^{i \lambda s} f(\lambda) \, d\lambda \right] \]

\[ = 0 \]

(by the Reimann-Lebesgue Lemma).
Hence, for large $s$, the serial covariance of the observed process $x_t$, i.e. $\gamma_s = \gamma_s^{(2)}$, will settle down to a steady oscillation due to the presence of the harmonic components. It is this phenomenon which enables us to construct a test of significance to test the null hypothesis

$$H_0 : x_t \text{ is a stationary Gaussian process with an absolutely continuous spectral density function}$$

against the alternative hypothesis

$$H_1 : x_t \text{ is the sum of a stationary Gaussian process with an absolutely continuous spectrum and a process with a discrete spectrum.}$$

We now derive the test statistic which is in fact a convolution of weight functions and the sample serial covariance.

** ** ** ** ** ** **
3.2 The $P(\lambda)$ statistic and the test procedure.

As we have already seen, for large $s \geq m$ (say), where $m < N$, $\gamma_s^{(1)} \sim 0$. For the $N$ observations $x_t$, $t = 1, \ldots, N$ denote the sample serial covariance function of $x_t$, as before, by

$$c_s = \frac{1}{N-s} \sum_{t=1}^{N-s} x_t x_{t+s}$$

(Note: Previously we have chosen $n$ observations from a sample population. In this chapter however, so as to acquire results in the same form as those derived by Priestley ([18], I, II) we use his notation and consider a sample of $N$ observations. The reason for this standardization will become clear throughout this chapter and, in particular, in section 5.4).

We now define the $P(\lambda)$ statistic as

$$P(\lambda) = \frac{1}{2\pi} \sum_{s=-N+1}^{N-1} \left( w_{n,s,\lambda}^{(1)} - w_{m,s,\lambda}^{(2)} \right) c_s$$

where $w_{n,s,\lambda}^{(1)}$ and $w_{m,s,\lambda}^{(2)}$ are two sequences of weights depending on $m$, $n$, $s$, $\lambda$ (see Priestley [18], I, p.221) with $m = o(n)$ and $n < N$.

Also

$$w_n^{(1)}(\lambda_1) = \frac{1}{2\pi} \sum_s w_{n,s,\lambda}^{(1)} e^{-is\lambda_1}$$

with $w_m^{(2)}(\lambda_1)$ being similarly related to $w_{m,s,\lambda}^{(2)}$. Alternatively, since
\[
\frac{1}{2\pi} \sum_{s= -N+1}^{N-1} w_{n,s,\lambda}^{(1)} c_s = \frac{1}{4\pi} \int_{-\pi}^{\pi} I_N(\lambda_1,x) w_n^{(1)}(\lambda_1) d\lambda_1,
\]

\[
= \hat{f}_1(\lambda)
\]

(3.2.3)

and similarly for \(w_{m,s,\lambda}^{(2)}\) (where \(I_N(\lambda_1,x)\) is the periodogram of the observed process \(x_t\), then \(P(\lambda)\) can be written in the form

\[
P(\lambda) = \frac{1}{4\pi} \int_{-\pi}^{\pi} I_N(\lambda_1,x) (w_n^{(1)}(\lambda_1) - w_m^{(2)}(\lambda_1)) d\lambda_1
\]

\[
= \hat{f}_1(\lambda) - \hat{f}_2(\lambda)
\]

(3.2.4)

If a harmonic component is present in the observed process \(x_t\), then it follows from (3.1.1) that there will be a sharp increase in the serial covariance \(\gamma_s\). Furthermore, from (3.2.2) it is seen that a sharp increase in the sample covariance will result in a peak occurring in \(P(\lambda)\): the peak is present, of course, at the same frequency at which the harmonic component occurs. Thus if \(x_t\) has a mixed spectrum \(P(\lambda)\) will contain several well defined peaks, each peak corresponding to a harmonic component which is present in \(x_t\).

Using \(P(\lambda)\) defined by (3.2.2) or (3.2.4), following Priestley we compose types of standardized cumulative sums of the form

\[
J_q = \Lambda^q \sum_{p=0}^{[\frac{2mq}{m}]} \frac{1}{q} P\left(\frac{2mp}{m}\right), q = 0, 1, 2\ldots [\frac{2m}{m}]
\]

(3.2.5)
where

\[ A = \frac{N}{m} V_{n,m} \]

and

\[ V_{n,m} = 2 \sum_{s} \left( w_{n,s,\lambda}^{(1)} - 2 w_{n,s,\lambda}^{(1)} w_{m,s,\lambda}^{(2)} + w_{m,s,\lambda}^{(2)} \right) \]

\[ = 4\pi \int_{-\pi}^{\pi} \left( w_{n,1}^{(1)}(\lambda) - w_{m,1}^{(2)}(\lambda) \right)^2 d\lambda \]

(3.2.6)

and it is found that under general conditions regarding the derivations of \( f(\lambda) \)

\[ \lim_{N \to \infty} \left\{ \frac{\max \frac{1}{\phi} J_q}{\frac{1}{2\pi} \hat{G}(\pi)^{1/2}} \leq \alpha_0 \right\} = 2 \phi(\alpha_0) - 1 \]  

(3.2.7)

where, writing \( G(t) = \int_{0}^{t} f^2(\lambda) d\lambda \),

then

\[ \hat{G}(\pi) = \int_{0}^{\pi} f^2(\lambda) d\lambda \]

\[ = \frac{1}{4\pi} \sum_{s=-\infty}^{\infty} \gamma_s^{(1)} \]

and it may be shown that

\[ \hat{G}(\pi) = \frac{1}{4\pi} \left( 2 \sum_{s=-m+1}^{m-1} c_s^{2} - \sum_{s=-2m+1}^{2m-1} c_s^{2} \right) \]  

(3.2.8)

Also \( \phi(x) \) is the distribution function of the standardized normal distribution (see [18], I pp. 223-4). From (3.2.7) it is possible to determine a level of significance above which a value of \( J_q \) will indicate the presence of a harmonic component in \( x_t \).
In practice, once a value for $m$ has been obtained (such that $\gamma_s^{(2)} \sim 0$ for all $s \geq m$), two weight functions must be chosen. Two suitable weight functions are the Féjer and the Dirichlet kernels, defined respectively by the sequences

$$w_{n,s,\lambda}^{(1)} = \begin{cases} (1 - \frac{|s|}{n}) \cos s\lambda & |s| < n \\ 0 & \text{otherwise} \end{cases} \quad (3.2.9)$$

and

$$w_{m,s,\lambda}^{(2)} = \cos s\lambda \quad |s| < m \quad (3.2.10)$$

Having chosen suitable weight functions it is now possible to evaluate $P(\lambda)$ over the set of grid points $\lambda_j = \frac{2\pi j}{N}$, $j = 1, 2, \ldots, [\frac{1}{2}N]$ and select the first peak (in order of frequency). Priestley says: "If $x_t$ has a mixed spectrum, then $P(\lambda)$ will, in general, contain several well defined peaks". ([18], I p.231). He fails however, to give a precise definition of the term "peak", such a definition being required so as to keep the number of them small. (Even so, in the practical application of his test, there seems to be no ambiguity (see [18], II pp.526-7)). Suppose the first peak occurs at $\lambda_0$. The frequency range $[0, \pi]$ is then subdivided at intervals $\frac{2\pi}{m}$ on both sides of $\lambda_0$ and these points used to form the cumulative sums $\Sigma F(\lambda)$. The statistic $J_q$ defined by

$$J_q = \Lambda^2 \sum_{p=0}^{q} \frac{\left(\frac{2\pi p + a}{m}\right)}{p = 0}, \quad q = 0, 1, 2, \ldots, [\frac{1}{2}m] \quad (3.2.11)$$
is derived and used in (3.2.7) to determine whether

$$\max_{q} J_{q} \leq \alpha_0 \left( \frac{1}{2\pi} \hat{G}(\pi) \right)^{\frac{1}{2}}$$

where $\hat{G}(\pi)$ is defined by (3.2.8). As mentioned earlier, Priestley chooses "a" such that the peak at $\lambda_0$ occurs at a frequency which is a multiple of $\frac{2\pi}{m}$.

If the first peak at frequency $\lambda_0$ is found to be significant, before proceeding to the second peak, we must remove the effect of the harmonic component (with frequency $\lambda_0$) by correcting the sample covariance $c_s$. For the weight functions defined by (3.2.9) and (3.2.10)

$$P(\lambda) = \frac{1}{\pi} \sum_{s=m}^{n-1} \left( 1 - \frac{|s|}{n} \right) c_s \cos s\lambda \quad (3.2.12)$$

and the amplitude of the harmonic component may be estimated from

$$\mathbb{E} \left( P(\lambda_0) \right) \sim \hat{A}_0^2 \frac{(n-2m)}{\pi}$$

$$\therefore \quad \hat{A}_0^2 \sim \frac{\pi}{(n-2m)} P(\lambda_0)$$

Correcting the sample covariance to give

$$c_s^{(1)} = c_s - \frac{1}{2} \hat{A}_0^2 \cos s\lambda_0$$

we may write

$$P^{(1)}(\lambda) = \frac{1}{\pi} \sum_{s=m}^{n-1} \left( 1 - \frac{|s|}{n} \right) c_s^{(1)} \cos s\lambda$$
and proceed to test $P^{(1)}(\lambda)$ for the presence of further harmonic components. If another component is detected the procedure is repeated until no further components are found significant.
3.3 Asymptotic power comparisons.

Before proceeding to establish the asymptotic powers of the $P(\lambda)$ test, the grouped periodogram test and a modified version of the test derived by Whittle (section 2.3) we prove two lemmas.

Lemma 1:
If an observed process $x_t$ contains a harmonic component at frequency $\lambda_k$, i.e.

$$x_t = y_t + \alpha_1 \cos \lambda_k t + \alpha_2 \sin \lambda_k t$$

where $y_t$ is an independent process with zero mean and variance $\sigma_y^2$, then the ratio $\frac{I_N(\lambda_k, x)}{\sigma_y^2}$ has a non central $\chi^2$ distribution with non centrality factor

$$\frac{\frac{1}{2} N (\alpha_1^2 + \alpha_2^2)}{\sigma_y^2}$$

Proof. We may write $x_t$ in the form

$$x_t = y_t + A \cos (\lambda_k t + \delta)$$

(3.3.1)

where $A^2 = \alpha_1^2 + \alpha_2^2$. The contribution of the harmonic component $A \cos (\lambda_k t + \delta)$ to $\frac{I_N(\lambda_k, x)}{\sigma_y^2}$ may be found by substituting
\[ z = A \cos(\lambda_k t + \delta) \] (where \( A \) and \( \lambda_k \) are constants and \( \delta \) is rectangularly distributed in \([0, 2\pi]\) in (1.4)) and following an identical argument as before (section 1.4).

Doing this we find the increase in
\[ \mathcal{E}( \frac{I_N(\lambda_k', x)}{\sigma_y^2} ) \]
due to the harmonic component is \( \frac{1}{2} \frac{N A^2}{\sigma_y^2} \).

Now the \( \frac{I_N(\lambda_p, x)}{\sigma_y^2} \), \( p = 1, 2, \ldots \lfloor \frac{1}{2} N \rfloor \), \( p \neq k \) are unaffected by the harmonic components and are independently distributed as \( \chi^2 \) variables with two degrees of freedom. It follows immediately that \( \frac{I_N(\lambda_k', x)}{\sigma_y^2} \) has a noncentral \( \chi^2 \) distribution

with non centrality parameter
\[ \Theta = \frac{1}{2} \frac{N (\alpha_1^2 + \alpha_2^2)}{\sigma_y^2} \]

Lemma 2:

If \( x_t \) is of the form (3.3.1) but \( y_t \) is now of the form
\[ y_t = \sum_{0}^{\infty} g_u \varepsilon_{t-u} \], where \( \varepsilon_t \) are independent and normal with zero mean and unit variance, then \( \frac{I_N(\lambda_k', x)}{\sigma_y^2} \) is distributed

as a non central \( \chi^2 \) variate with non centrality parameter
\[ \Theta_1 = \frac{1}{2} \frac{N (\alpha_1^2 + \alpha_2^2)}{2\pi^2(\lambda_k')} \]
where $f(\lambda)$ is the density function of the continuous process $y_t$.

**Proof.** Write

$$J_N(\lambda_k, x) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{t=1}^{N} x_t e^{it\lambda_k}$$

Now

$$x_t = y_t + \alpha_1 \cos \lambda_k t + \alpha_2 \sin \lambda_k t$$

$$= y_t + \alpha e^{-it\lambda_k} + \alpha^* e^{it\lambda_k}$$

where

$$\alpha = \frac{1}{2}(\alpha_1 - i\alpha_2) \quad \text{and} \quad \alpha^* = \frac{1}{2}(\alpha_1 + i\alpha_2)$$

$$\therefore \quad J_N(\lambda_k, x) = J_N(\lambda_k, y) + (2N)^{\frac{1}{2}} \alpha^* \quad (3.3.3)$$

Following Bartlett ([3] p.279)

$$J_N(\lambda_k, y) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{t=1}^{N} \sum_{u=0}^{\infty} e^{it\lambda_k} \varepsilon_{t-u}$$

$$= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{u=0}^{\infty} \sum_{t=1}^{N} e^{i(t-u)\lambda_k} \varepsilon_{t-u}$$

$$= h^*(\lambda_k) J_N(\lambda_k, \varepsilon) \left[1 + O\left(\frac{1}{N}\right)\right]$$

where

$$h^*(\lambda_k) = \sum_u e^{iu\lambda_k}$$

Hence

$$J_N(\lambda_k, x) = h^*(\lambda_k) J_N(\lambda_k, \varepsilon) \left[1 + O\left(\frac{1}{N}\right)\right] + (2N)^{\frac{1}{2}} \alpha^*$$
and similarly

\[ J_N^*(\lambda_k, x) = h(\lambda_k) J_N^*(\lambda_k, x) \left[ 1 + O\left(\frac{1}{N}\right) \right] + \left(2N\right)^{\frac{1}{2}} \alpha . \]

Now

\[ J_N(\lambda_k, x) J_N^*(\lambda_k, x) = I_N(\lambda_k, x) \]

and

\[ h(\lambda_k)h^*(\lambda_k) = 2\pi f(\lambda_k) \]

so that

\[ \frac{I_N(\lambda_k, x)}{2\pi f(\lambda_k)} = \left\{ J_N(\lambda_k, \varepsilon) \left(1 + O\left(\frac{1}{N}\right)\right) + \frac{(2N)^{\frac{1}{2}} \alpha^*}{h(\lambda_k)} \right\} \]

\[ \left\{ J_N^*(\lambda_k, \varepsilon) \left(1 + O\left(\frac{1}{N}\right)\right) + \frac{(2N)^{\frac{1}{2}} \alpha}{h(\lambda_k)} \right\} \]

\[ \sim \left\{ J_N(\lambda_k, \varepsilon) + \frac{(2N)^{\frac{1}{2}} \alpha^*}{h(\lambda_k)} \right\} \left\{ J_N^*(\lambda_k, \varepsilon) + \frac{(2N)^{\frac{1}{2}} \alpha}{h(\lambda_k)} \right\} \]

(3.3.5)

Defining a process \( u_t \) by

\[ u_t = \left\{ \frac{\alpha}{h(\lambda_k)} e^{it\lambda_k} + \frac{\alpha^*}{h^*(\lambda_k)} e^{-it\lambda_k} \right\} + \varepsilon_t \]

(3.3.6)

(\text{where, as before, } \varepsilon(\varepsilon_t) = 0 \text{ and } \varepsilon(\varepsilon^2_t) = 1), \text{ from Lemma 1, (3.3.3) and (3.3.4), since the } \varepsilon_t \text{ are independent it follows that} \]

\( I_N(\lambda_k, u) \) is distributed as a non central \( \chi^2 \) distribution with

non centrality parameter

\[ \theta_1 = \frac{\left[ (2N)^{\frac{1}{2}} \right]^2 \sum_{t=1}^{N} u_t e^{it\lambda_k}}{2\pi f(\lambda_k)} \]

Now

\[ J_N(\lambda_k, u) = \left(2N\right)^{\frac{1}{2}} \sum_{t=1}^{N} u_t e^{it\lambda_k} \]
and similarly
\[ J^*_N(\lambda_k, u) = J^*_N(\lambda_k, \varepsilon) + (2N)^{\frac{1}{2}} \frac{\alpha^*}{h^*(\lambda_k)} \]

so that
\[ J^*_N(\lambda_k, u) = J^*_N(\lambda_k, \varepsilon) + (2N)^{\frac{1}{2}} \frac{\alpha^*}{h^*(\lambda_k)} \]

From (3.3.5) and (3.3.7) it is clearly seen that
\[ \frac{I_N(\lambda_k, x)}{2\pi f(\lambda_k)} \sim I_N(\lambda_k, u) \quad (3.3.8) \]

But we have seen that $I_N(\lambda_k, u)$ has a non central $\chi^2$ distribution with non centrality parameter $\theta_1$. Hence
\[ \frac{I_N(\lambda_k, x)}{2\pi f(\lambda_k)} \]

is asymptotically distributed as a non central $\chi^2$ distribution with non centrality parameter
\[ \theta_1 = \frac{1}{2N}(\alpha_1^2 + \alpha_2^2) \quad (\frac{2\pi f(\lambda_k)}{2\pi f(\lambda_k)}) \]

Thus the lemma is proved.
Following Hartley and Patnaik ([16], [19]) Priestley, in deriving the asymptotic power of a test based on the periodogram, approximates the non central $\chi^2$ distribution (with non centrality parameter $\theta_1$) to $\rho \chi^2_\nu$ where $\rho$ is a scale factor and $\chi^2_\nu$ is a central $\chi^2$ distribution with $\nu$ degrees of freedom. Hence

$$F(\lambda_k, x) = \text{Pr} \left\{ \frac{I_N(\lambda_k, x)}{\sigma^2_y} \leq x \right\}$$

$$= \frac{2^{\frac{1}{2}\nu}}{\Gamma\left(\frac{1}{2}\nu\right)} \int_0^{x/\rho} e^{-\frac{1}{2}\tau} \tau^{\frac{1}{2}\nu - 1} d\tau \quad (3.3.9)$$

Patnaik (see [19] p.206) finds values of $\rho$ and $\nu$ in terms of the non centrality parameter $\theta_1$ to be

$$\rho = \frac{2+2\theta_1}{2+\theta_1} \quad \text{and} \quad \nu = \frac{(2+\theta_1)^2}{2+2\theta_1} \quad (3.3.10)$$

In forming the asymptotic power for each of the various tests it will be sufficient for us to approximate $F(\lambda_k, x)$ by a normal distribution with mean $\rho \nu = (2+\theta_1)$ and variance $\rho^2 2\nu = 2(2+2\theta_1)$ so that

$$F(\lambda_k, x) \sim \phi \left( \frac{x-\theta_1}{2\sqrt{\theta_1}} \right) \quad (3.3.11)$$

where $\phi$ is the standard normal distribution function.
Whittle's Test

Whittle's test, it will be recalled, is based on the test statistic

$$g = \frac{\max_p I_N(\lambda_p, x)}{\frac{1}{2\pi} f(\lambda_p)} , \quad N' = \left\lceil \frac{1}{2}N \right\rceil$$  (3.3.12)

where $f(\lambda_p)$ is estimated from an autoregressive model. In deriving the asymptotic power of the test Priestley considers a truncated periodogram estimate of the spectral density, i.e. $f(\lambda)$ is of the form

$$f_1(\lambda) = \frac{1}{2\pi} \sum_{s=-m+1}^{m-1} c_s e^{is\lambda}$$

Substituting this in (3.3.12) gives a test statistic $g_1$ which is asymptotically distributed in the same way as Fisher's $g$.

In deriving the asymptotic power of this modified form of Whittle's test the statistic

$$g_1^* = \frac{\max_p I_N(\lambda_p, x)}{\frac{1}{2\pi} f_1(\lambda_p)} , \quad N' = \left\lceil \frac{1}{2}N \right\rceil$$  (3.3.13)

is considered. By using $g_1^*$ rather than $g_1$ we avoid having to
consider the non null distribution of the denominator when the observed process contains a harmonic component. Suppose \( x_t \) is of the form (3.3.1). From our definition of \( f_1(\lambda) \) it is found that

\[
\mathbb{E} \left\{ f_1(\lambda_k) \right\} \sim f(\lambda_k) + \frac{1}{4\pi} \lambda^2
\]

where \( f(\lambda) \) is the density function of \( y_t \).

Define \( g^*_o \) in the same way as \( g^*_1 \) except that \( f(\lambda_k) \) replaces \( f_1(\lambda_k) \). Then

\[
g^*_1 \sim \frac{f(\lambda_k)}{f_1(\lambda_k)} g^*_o = \left( \frac{1}{1+\Delta} \right) g^*_o \quad \text{where} \quad \Delta = \frac{\frac{1}{4\pi} \lambda^2}{f(\lambda_k)}
\]

so that

\[
\Pr (g^*_1 \geq x_o) \sim \Pr (g^*_o \geq \Delta x_o) \sim \Pr (g^* \geq \Delta x_o) \quad (3.3.14)
\]

where

\[
g^* = \frac{\max \sum_{p=1}^{N'} I_N(\lambda_p, x)}{\frac{1}{2N'-2} \left[ \sum_{p=1}^{N'} I_N(\lambda_p, x) - \max \sum_{p=1}^{N'} I_N(\lambda_p, x) \right] + \frac{1}{2} \max \sum_{p=1}^{N'} I_N(\lambda_p, x)}
\]

which is an equivalent form of Fisher's \( g \) (1.3.1).

To determine \( x_o \) for significance level \( \alpha \), using Walker's criterion (1.2.8), it follows that (see Priestley [18], II, p.515)

\[
\Pr (g^* \geq x_o) = 1 - \left( e^{-\frac{1}{2} x_o} \right)^{N'} = \alpha
\]


\[
\therefore (1 - \alpha) \approx 1 - N' e^{-\frac{1}{2}x_0}
\]

and so

\[
x_0 \approx 2 \log \left( \frac{N'}{\alpha} \right)
\]

Knowing \(x_0\) we can now use (3.9) and (3.14) to give

\[
Pr (g_1^* \geq x_0) = 1 - (1 - e^{-\frac{1}{2}x_0})^{N' - 1}
\]

and from (3.11)

\[
\sim 1 - (1 - e^{-\frac{1}{2}x_0})^{N' - 1} \phi \left( \frac{x_1 - \theta_1}{2\sqrt{\theta_1}} \right)
\]

where \(\theta_1 = \frac{1}{2} N A^2 \mu_{\lambda_k} \), \(x_1 = \Delta x_0\) and \(\phi\) is the distribution function of the standardized normal distribution.

Hence the asymptotic power of a modified form of Whittle's test is given by

\[
P_1 = Pr (g_1^* \geq x_0) \sim 1 - (1 - e^{-\frac{1}{2}x_0})^{N' - 1} \phi \left( \frac{x_1 - \theta_1}{2\sqrt{\theta_1}} \right)
\]

(3.16)

or, for \(\alpha\) small

\[
P_1 \sim 1 - (1 - \alpha) \phi \left( \frac{x_1 - \theta_1}{2\sqrt{\theta_1}} \right)
\]

(3.17)

The Grouped periodogram test

As before, so as to avoid having to consider the non null distribution of the denominator when the observed process contains
a harmonic component, we consider an equivalent form of \( L^*_K (q) \)
(2.4.1), the test statistic for the grouped periodogram test.

Put

\[
L^*_K = \max_q \frac{I_N(\lambda_q, x)}{2\pi f(\lambda_q)} \quad \text{(3.3.18)}
\]

\[
\frac{1}{(2K-2)} \left[ \sum_q \frac{I_N(\lambda_q, x)}{2\pi f(\lambda_q)} - \max_q \frac{I_N(\lambda_q, x)}{2\pi f(\lambda_q)} \right]
\]

where, as before, \((\ell-1)K + 1 \leq q \leq \ell K\).

As we have already seen (Chapter 2) the appropriate significance level is now \( \frac{\alpha K}{[\frac{1}{2}N]} \) and so the critical value for \( L^*_K \), from (3.3.15), is given by

\[
x^*_o = 2 \log \left( \frac{K}{\alpha K / N'} \right) \quad , \quad N' = \left[ \frac{1}{2}N \right]
\]

\[
= 2 \log \left( \frac{N'}{\alpha} \right) = x_o.
\]

Writing

\[
G^*_K = \max_q I_N(\lambda_q, x)
\]

\[
\frac{1}{(2K-2)} \left( \sum_q I_N(\lambda_q, x) - \max_q I_N(\lambda_q, x) \right)
\]

as an equivalent form of \( \max_q G^*_K \) (where \((\ell-1)K + 1 \leq q \leq \ell K\),
from (2.4.5) it follows that the critical value of \( G^*_K \) should be taken as \( \frac{M}{m} \) times the critical value of \( L^*_K \) to ensure that the error of the first kind is less than or equal to \( \frac{\alpha K}{[\frac{1}{2}N]} \).
Using Walker's criterion the asymptotic power of the test using (3.3.11) and (3.3.18) is given (for large n and K) by

\[
Pr(G^* > x) = 1 - \left(1 - e^{\frac{Mx_0}{2m}}\right)^{K-1} F\left(\frac{M}{m}x_0\right)
\]

\[
\sim 1 - \left(1 - e^{\frac{Mx_0}{2m}}\right) \varphi\left(\frac{x_2 - \theta_1}{a_1}\right)
\]

where \(\varphi\) and \(\theta_1\) are as before and \(x_2 = \frac{M}{m}x_0\). For small \(\alpha\) the asymptotic power of the grouped periodogram test may be written as

\[
P_2 \sim 1 - (1-\alpha) \varphi\left(\frac{x_2 - \theta_1}{a_1}\right)
\]

(3.3.19)

The \(P(\lambda)\) test.

If a harmonic component occurs at \(\lambda_k\) and \(P(\lambda)\) is considered only at points given by \(|\lambda + \lambda_k| = \frac{2\pi q}{m}\) (\(q = 1, 2, \ldots, \lfloor \frac{4m}{\lambda_k} \rfloor\)) then it can be shown (see Priestley [18], II pp.518-520) that the distribution of \(P(\lambda)\) for \(\lambda \neq \lambda_k\) is asymptotically unaffected by the harmonic term and, for large \(N\), \(P(\lambda_k)\) may be regarded as being asymptotically normally distributed with mean \(\mu_k\) and variance \(\sigma_k^2\) where

\[
\mu_k = \frac{1}{8\pi^2} A^2 (n-2m)
\]

and

\[
\sigma_k^2 = r^2(\lambda_k) \left( \frac{V_{n,m}}{N} + \frac{[A(n-2m)]^2}{4\pi^2N^2 f(\lambda_k)} \right)
\]
Furthermore, Priestley has shown ([18], II p.521) that, assuming \( w_{n,s,\lambda}^{(1)} \) and \( w_{n,s,\lambda}^{(2)} \) are defined, as before, by (3.2.9) and (3.2.10),

\[
P = \lim_{N \to \infty} \Pr \left( \max_{q \leq k} \frac{J_q}{\left( \frac{1}{2\pi} G(\pi) \right)^{1/2}} \leq \alpha_0 \right)
\]

\[
= \int_{-\infty}^{\alpha_0 \sqrt{\gamma(\pi)}} p_2(x,\lambda_k) \phi \left( \frac{(\alpha_0 \sqrt{\gamma(\pi)} - x) \Lambda^{1/2} - \mu_k}{\sigma_k} \right) \, dx
\]

where

\[
\gamma(\pi) \sim \frac{1}{2\pi} \hat{G}(\pi),
\]

\[
p_2(x,t) = \frac{1}{\sqrt{2\pi\gamma(t)}} \left[ e^{-\frac{x^2}{2\gamma(t)}} - e^{-\frac{(x-2\alpha_0)^2}{2\gamma(t)}} \right]
\]

and \( \phi \) is the standard normal distribution function.

Putting

\[
a = -\left( \Lambda^{1/2} \sigma_k \right)^{-1} \quad \text{and} \quad b = \alpha_0 \sqrt{\gamma(\pi)} \Lambda^{1/2} - \mu_k
\]

then

\[
P = \int_{-\infty}^{\alpha_0 \sqrt{\gamma(\pi)}} p_2(x,\lambda_k) \phi(ax+b) \, dx \quad \text{(3.3.22)}
\]

Expanding \( \phi(ax+b) \) about \( \phi(b) \) in a Taylor series

\[
\phi(ax+b) = \phi(b) + \frac{ax}{\sqrt{2\pi}} e^{-\frac{b^2}{2}} + \ldots
\]
and since $e^{-\frac{1}{2}b^2} \sim 0(e^{-N})$

$$\phi(ax+b) \sim \phi(b).$$

Hence

$$P \sim \phi(b) \int_{-\infty}^{\alpha\sqrt{\gamma(\pi)}} p_2(x, \lambda_k) \, dx$$

and the power of the $P(\lambda)$ test is given by

$$P_3 = 1 - P$$

$$\sim 1 - \phi(b) \int_{-\infty}^{\alpha\sqrt{\gamma(\pi)}} p_2(x, \lambda_k) \, dx \quad (3.3.23)$$

For small $\alpha$

$$P_3 \sim 1 - (1-\alpha) \phi(b) \quad (3.3.24)$$

or alternatively

$$P_3 \sim 1 - (1-\alpha) \phi\left(\frac{x_3^{\theta_1}}{\alpha\sqrt{\theta_1}}\right) \quad (3.3.25)$$

where

$$x_3 = \frac{2\alpha_o \gamma_{n,m} \sqrt{\gamma(\pi)}}{A(n-2m) f(\lambda_k^{\frac{1}{2}})} \quad (3.3.26)$$

**Power comparisons**

From equations (3.3.17), (3.3.20) and (3.3.25) we have the asymptotic powers of a modified form of Whittle's test, the grouped periodogram test and Priestley's $P(\lambda)$ test respectively as
\[
P_1 \sim 1 - (1-\alpha) \phi \left( \frac{x_1 - \theta_1}{2\sqrt{\theta_1}} \right) = 1 - (1-\alpha) \phi \left( y_1 - \frac{1}{2\sqrt{\theta_1}} \right) \]
\[
P_2 \sim 1 - (1-\alpha) \phi \left( \frac{x_2 - \theta_1}{2\sqrt{\theta_1}} \right) = 1 - (1-\alpha) \phi \left( y_2 - \frac{1}{2\sqrt{\theta_1}} \right) \quad \text{(3.3.27)}
\]
\[
P_3 \sim 1 - (1-\alpha) \phi \left( \frac{x_3 - \theta_1}{2\sqrt{\theta_1}} \right) = 1 - (1-\alpha) \phi \left( y_3 - \frac{1}{2\theta_1} \right)
\]

Now \( x_0 = O(\log N) \), \( \Delta = O(m) \) and \( V_{n,m} = O(n) \) so that
\[
x_1 = O(m \log N), \quad x_2 = O(\log N), \quad x_3 = O \left( \frac{1}{n} \right)
\]

The non centrality parameter \( \theta_1 \) is of order \( N \) so that
\[
\begin{align*}
y_1 &= O \left( \frac{m \log N}{N^2} \right) \\
y_2 &= O \left( \frac{\log N}{N^2} \right) \\
y_3 &= O \left( \frac{1}{n} \right)
\end{align*}
\]

(3.3.28)

Remembering that \( m = O(n) \) and \( n \leq N \) then \( y_2 = o(y_3) \) so that the grouped periodogram test is the more powerful.

In drawing a conclusion from this comparison of powers, it may be said that in general the grouped periodogram test is the more powerful while the modified form of Whittle’s test (where we have used a truncated periodogram estimate rather than an autoregressive estimate of the spectral density) is the least powerful.

* * * * * * * *
A critical discussion of the $P(\lambda)$ test.

In setting up the $P(\lambda)$ test based on the statistic $J_q$, the test frequency $\lambda_0$ say (the frequency of the peak to be tested), which is not necessarily of the form $\frac{2\pi p}{m}$, is chosen and values of $P(\lambda)$ at points separated from $\lambda_0$ by multiples of $\frac{2\pi}{m}$ are used to form $\sum P(\lambda_p)$. By constructing the statistic $\sum P(\lambda_p)$ in this way, the probability of a harmonic component occurring between two grid points is asymptotically zero. It appears, therefore that the $P(\lambda)$ test is superior in that the frequency of the jump cannot occur between two grid points. Previously it will be recalled, in the case of tests based on the periodogram, in the worst possible case when the jump occurred midway between two grid points, there was a reduction of $\frac{h}{\pi^2}$ in the height of the maximum periodogram ordinate.

Priestley states ([18], II p.523) that by putting $m = \log N$ and $n = N$ then $y_2 = o(y_2)$ and the $P(\lambda)$ test is superior to the other two and in particular to the grouped periodogram test. The explanation of the greater power of Priestleys' test appears however to lie in the sequence of alternatives he has chosen when evaluating the power of the $P(\lambda)$ test. These appear to involve the assumption that a periodicity, if it occurs, has a frequency which is a multiple of $\frac{2\pi}{m}$ (see [18], II pp.520-521). By prior reference to the data Priestley is able to find a value of the commencing point $\lambda_0$ (i.e. the frequency at which the maximum $P(\lambda)$ occurs) about which the
grid points \( \frac{2\pi p}{m} \), \( p = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor \) are constructed, i.e. since "a" is not known Priestley chose it so that \( \frac{2\pi p+a}{m} = \frac{2\pi j}{m} \) for some \( p, j \). The effect of doing this on the significance level would be hard to determine but could be large. In setting up the power of his test Priestley has not taken into account this a priori reference to the data. Consequently the significance level used for the \( P(\lambda) \) test is not appropriate and the power of the \( P(\lambda) \) test will be overstated.

Suppose that a jump occurs at a point \( \lambda_0 \) separated from the point \( \frac{2\pi p}{m} \) (=\( \lambda \)) by a distance which is \( O(m^{-1}) \), say \( \frac{\pi}{m} \), and \( m = o(n) \). For the sequence (3.2.9) it follows that

\[
W_n^{(1)}(\lambda_0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{n^{-1}}{s} W_n^{(1)} e^{-is\lambda_0}
\]

\[
= \frac{1}{4\pi n} \left[ \frac{\sin^2 \frac{n}{2} (\lambda_0 + \lambda_0)}{\sin^2 \frac{1}{2} (\lambda_0 + \lambda_0)} + \frac{\sin^2 \frac{n}{2} (\lambda - \lambda_0)}{\sin^2 \frac{1}{2} (\lambda - \lambda_0)} \right].
\]

Hence as \( n \to \infty \) \( W_n^{(1)}(\lambda_0) \) tends to zero, and consequently

\[
P(\lambda) = -\frac{1}{2\pi} \sum_{m=1}^{m-1} W_{m,s}^{(2)} \cos s \lambda_0.
\]

From (3.1.1) and (3.3.1)

\[
\gamma^{(2)}_s = \varepsilon(z_t z_{t+s}) = \frac{1}{2} A^2 \cos s \lambda_0
\]

and using the weight function defined by the sequence (3.2.10)
Thus, from (3.3.21) and (3.3.24), it follows that

\[ P_3 \sim 1 - (1-\alpha) = \alpha \]

i.e. the power of the \( P(\lambda) \) test will approach \( \alpha \), the level of significance. As \( m, n \to \infty \) therefore the set where a jump can be without the power falling to the significance level is of measure zero.

When \( m = \log N \) and \( n = N \) Priestley claims his \( P(\lambda) \) test to be the more powerful. In this particular case however he is considering only \( \log N \) ordinates over which \( P(\lambda) \) is evaluated. If it was known a priori that it was one of these \( \log N \) ordinates at which the jump occurred it would then be necessary to consider only \( \log N \) ordinates when considering periodogram tests. As a result the power of both Whittle's test and the grouped periodogram test would be greatly increased, e.g. in the case of the grouped periodogram test the expression \( \frac{\log N}{\sqrt{N}} \) for \( y_2 \) in (3.3.28) would be replaced by \( \frac{\log m}{\sqrt{N}} \). Thus the grouped
periodogram test will have maximum power for all choices of m and n.

To avoid having to refer to the data to set the grid over which \( P(\lambda) \) is summed one could use the fixed grid points 
\[
\frac{2\pi p}{n}, \quad p = 1, 2, \ldots, \left[ \frac{n}{2} \right]
\]
and hence replace the statistic
\[
\sum_{p} P\left( \frac{2\pi p + \alpha}{m} \right) \quad \text{by} \quad \sum_{p} P\left( \frac{2\pi p}{n} \right)
\]
so that now \( \alpha = 0 \). This new statistic is now free from parameters which can only be determined after reference to the data. By summing the \( P(\lambda) \)'s over this new (fixed) grid it is possible, as with previous tests, that a jump may occur at a frequency between two consecutive grid points.

Priestley's \( P(\lambda) \) test, as it has been presented, depends on the selection of a grid over the frequency axis so that one of the points of subdivision coincides with the frequency of the maximum \( P(\lambda) \). This grid can only be achieved by prior reference to the data. Removing this need to refer to the data by fixing the grid points it is found that the superiority of the test is lost in that it is now possible for the frequency of the harmonic component to occur between two grid points. Furthermore, for all values of \( m \) and \( n \), the asymptotic power of the \( P(\lambda) \) test is always lower than that of the grouped periodogram test.
4.1 Preliminary.

Periodogram tests which have been considered to date may be divided into two classes, namely the class in which the \( n \) observations (of a stationary time series) are independent and normal and the class in which the \( n \) observations are from a stationary Gaussian process. When the observations are independent, the appropriate test statistic is

\[
g = \max_j \frac{I_n(\lambda_j, x)}{\sum_j I_n(\lambda_j, x)}
\]  

which, as we have seen, has the distribution (1.3.13) first derived by Fisher. In the more general case when there is no independence, the statistic

\[
K_n(\lambda_j, x) = \frac{I_n(\lambda_j, x)}{2\pi f(\lambda_j)}
\]  

is used to form the test statistic

\[
\max_j \frac{K_n(\lambda_j, x)}{\sum_j K_n(\lambda_j, x)}
\]
which is asymptotically distributed as Fisher's $g$. In this second case if the spectral density function is not known it has to be estimated.

There are numerous methods of estimation of the spectral density function, one (which has already been discussed) being based on an autoregressive model. In using such a model to estimate $f(\lambda)$ however, we introduce the problem of determining whether a certain peak of the periodogram is in fact evidence of a genuine harmonic term in the series or whether it corresponds to a relatively broad band of frequencies. An estimate in which this does not arise is that based on a smoothing or averaging of the periodogram.

In this case, the spectral density function is defined by

$$\hat{F}(\lambda) = \int_{-\pi}^{\pi} W_n(\lambda - \theta) I_n(\theta, x) \, d\theta \quad (4.1.4)$$

or equivalently, in the discrete case, using (1.1.9) it follows that

$$\hat{f}(\lambda) = \sum_{-n+1}^{n-1} \left(1 - \frac{|t|}{n}\right) w_{n,t} c_t e^{it\lambda} \quad (4.1.5)$$

where $w_{n,t}$ are Fourier coefficients of $4\pi W_n(\lambda)$ and prior assumptions about the smoothness of the $f(\lambda)$ are expressed in the form chosen by the smoothing function $W_n(\lambda)$. Putting
\[ \hat{K}_n(\lambda_j, x) = \frac{I_n(\lambda_j, x)}{2\pi \hat{f}(\lambda_j)} \]

where \( \hat{f}(\lambda_j) \) is given by (4.1.4) or (4.1.5), the test statistic is then

\[
\max_j \frac{\hat{K}_n(\lambda_j, x)}{\sum_j \hat{K}_n(\lambda_j, x)} \tag{4.1.6}
\]

which is asymptotically distributed as Fisher's g. We now establish the asymptotic validity of this test.
4.2 The asymptotic validity of the test.

In section 2.2 it was found that \( K_n(\lambda_j, x) \) (where \( f(\lambda) \) has been prescribed a priori) could be used in the same way as \( I_n(\lambda_j, x) \) when the null hypothesis was that the \( x_t \) were independent. Consequently

\[
\max_j \frac{K_n(\lambda_j, x)}{\sum_j K_n(\lambda_j, x)}
\]

is asymptotically distributed as Fisher's \( G \). To show that

\[
\max_j \frac{\hat{K}_n(\lambda_j, x)}{\sum_j \hat{K}_n(\lambda_j, x)}
\]

is asymptotically distributed as Fisher's \( G \) all that will be required will be to show that \( \max_j |\hat{f}(\lambda_j) - f(\lambda_j)| \to 0 \) in probability.

The Fourier coefficients \( w_{n,t} \) of \( W_n(\lambda) \) are assumed to be of the form \( w_{n,t} = k(b_{n,t}) \) where the function \( k(x) \) is bounded and \( |x|^{-\delta} k(x) \) is absolutely integrable for all \( \delta > 0 \). This places little restriction on our choice of weight function as all those in common use satisfy the condition (see Hannan [13] p.59). Expressing \( x_t \) in the form of an infinite autoregression i.e. \( x_t = \sum_{-\infty}^{\infty} \alpha_j \varepsilon_{t-j} \) and \( \varepsilon_t : N,1.D (0,1), \) we assume that

\( |\alpha_j| = o(j^{-3/2}) \). Then
\[ f(\lambda) = \frac{1}{2\pi} \left| \sum_j \alpha_j e^{ij\lambda} \right|^2 \]

and using (4.1.5) it follows that

\[ \varepsilon \left\{ \max_j \left| \hat{f}_n(\lambda_j) - f(\lambda_j) \right| \right\} = \varepsilon \left\{ \max_j \left| \frac{1}{2\pi} \sum_{-n+1}^{n-1} (c_t \gamma_t) (1 - \frac{|t|}{n}) w_{n,t} e^{it\lambda_j} \right| \right\} \]

\[ \leq \varepsilon \left\{ \frac{1}{2\pi} \sum_{-n+1}^{n-1} |c_t \gamma_t| (1 - \frac{|t|}{n}) \left| w_{n,t} \right| \right\} \]

Now \((n-t) \varepsilon(c_t \gamma_t)^2 = (n-t) \text{var } c_t \) and Hannan ([13] p.39) has shown that \((n-t) \text{var } c_t \) converges boundedly to a finite limit, say \(A\).

\[ \therefore \varepsilon \left\{ \max_j \left| \hat{f}_n(\lambda_j) - f(\lambda_j) \right| \right\} \leq A \cdot \frac{1}{\sqrt{n}} \sum_{-n+1}^{n-1} (1 - \frac{|t|}{n}) \left| w_{n,t} \right| \]

\[ \leq A \left\{ \frac{n^\delta}{n^{\frac{3}{2}} b^{1-\delta}} \sum_{-n+1}^{n-1} \frac{|k(b_n t)|}{(b_n t)^\delta} b_n \right\} \]

so, if \(n^{\frac{3}{2}} b^{1-\delta} \to \infty\) this last expression converges and consequently

\[ \max_j \left| \hat{f}_n(\lambda_j) - f(\lambda_j) \right| \to 0 \text{ in probability.} \]

We may therefore use the

\[ \hat{K}_n(\lambda_j, x) \ (j = 1, 2, \ldots, \lfloor \frac{1}{2}n \rfloor) \]

as we would the \(I_n(\lambda_j, x)\)

\(j = 1, 2, \ldots, \lfloor \frac{1}{2}n \rfloor\) for a series of independent normal observations.

* * * * * * *
4.3 The test procedure.

As in the case of the autoregressive estimate, unless the spectral estimate (4.1.4) is designed to omit the effect of a harmonic component which may be present at a given frequency, the power of the test based on the statistic (4.1.6) will be adversely affected. Before estimating $f(\lambda_j)$ we take out the regression, for each $j$, on the harmonic with this frequency. The term \( \frac{2\pi}{n} W_n(0) I_n(\lambda_j, x) \) is seen, from (4.1.4), to be an approximation (to order $n^{-1}$) of the modification which would result if $f(\lambda_j)$ were estimated from the observations after taking out regression on the harmonic with frequency $\lambda_j$. Hence

\[
    f'(\lambda_j) = \hat{f}(\lambda_j) - \frac{2\pi}{n} W_n(0) I_n(\lambda_j, x) \quad (4.3.1)
\]

and on taking expectations

\[
    \mathbb{E}\left\{f'(\lambda_j)\right\} = f(\lambda_j) - \frac{2\pi}{n} W_n(0) \mathbb{E}\left(I_n(\lambda_j, x)\right) = f(\lambda_j)\left(1 - \frac{8\pi^2}{n} W_n(0)\right) \quad (4.3.2)
\]

since $\mathbb{E}(I_n(\lambda_j, x)) = 4\pi f(\lambda_j)$.

In order to minimize the bias arising from the term subtracted from $\hat{f}(\lambda_j)$, as a consequence of (4.3.2) divide (4.3.1) by \(1 - \frac{8\pi^2}{n} W_n(0)\) and denoting the first estimate
by $f^*(\lambda_j)$ it follows that

$$f^*(\lambda_j) = \frac{\hat{f}(\lambda_j) - \frac{2\pi}{n} W_n(0) I_n(\lambda_j, x)}{1 - \frac{8\pi^2}{n} W_n(0)} \quad (4.3.3)$$

Using this estimate we have

$$K_n^*(\lambda_j, x) = I_n(\lambda_j, x) \left(1 - \frac{8\pi^2}{n} W_n(0)\right)$$

and the test statistic referred to Fisher's $g$ distribution is

$$g^* = \max_j \frac{K_n^*(\lambda_j, x)}{\sum_j K_n^*(\lambda_j, x)}$$

The smoothed periodogram test has similar disadvantages to the test based on the autoregressive model. The most serious of these is the reduction in power due to the occurrence of a jump in the spectrum between two of the grid points over which the periodogram ordinates are being estimated. Without prior information however this difficulty cannot be avoided.
4.4 Adjusting the spectral estimate.

Periodogram tests which we have considered have all been based on the periodogram ordinates being estimated over grid points which are $\frac{2\pi}{n}$ apart. We now consider a modified form of the smoothed periodogram estimate (4.1.5) in which the grid points are taken at a distance of $\frac{\pi}{n}$ apart. It will now be shown that (4.1.5) may be written in the form

$$\hat{f}(\lambda_k) = \frac{1}{2\pi} \sum_{j=-n+1}^{n} a_j(k) I_n\left(\frac{\pi j}{n}, x\right), \quad \lambda_k = \frac{2\pi k}{n} \quad (4.4.1)$$

To evaluate the coefficients $a_j(k)$ we have

$$\hat{f}(\lambda_k) = \frac{1}{2\pi} \sum_{j=-n+1}^{n} a_j(k) I_n\left(\frac{\pi j}{n}, x\right)$$

$$= \frac{1}{2\pi} \sum_{j=-n+1}^{n} a_j(k) \left\{ 2 \sum_{t=-n+1}^{n} c_t \left(1-\frac{|t|}{n}\right) e^{it\pi j/n} \right\}$$

$$= \frac{1}{2\pi} \sum_{t=-n+1}^{n-1} c_t \left(1-\frac{|t|}{n}\right) \left\{ 2 \sum_{j=-n+1}^{n} a_j(k) e^{it\pi j/n} \right\} \quad (4.4.2)$$

From (4.1.5)

$$\hat{f}(\lambda_k) = \frac{1}{2\pi} \sum_{t=-n+1}^{n-1} c_t \left(1-\frac{|t|}{n}\right) w_{n,t} e^{it\lambda_k} \quad (4.4.3)$$

A solution is obtained by equating coefficients of $c_t(1-\frac{|t|}{n})$ in
(4.4.2) and (4.4.3) to give

\[ 2 \sum_{-n+1}^{n} a(k) \left( \frac{\pi j}{n} \right) e^{it\frac{\pi j}{n}} e^{it\lambda_k}, \quad t = -n+1, \ldots, n \]

i.e.

\[ 2 \sum_{-n+1}^{n} a(k) \left( \frac{\pi j}{n} \right) e^{it\frac{\pi s}{n}} e^{it\lambda_k} = \sum_{t=-n+1}^{n} \sum_{t=-n+1}^{n} w_n, t e^{it\lambda_k - \frac{\pi s}{n}} \]

Now

\[ \sum_{t=-n+1}^{n} \left( \frac{\pi j}{n} \right) e^{it\lambda_k - \frac{\pi s}{n}} = 2n \quad j = s \]

\[ = 0 \quad j \neq s \]

\[ \therefore a(k) = \frac{1}{4n} \sum_{t=-n+1}^{n} \sum_{t=-n+1}^{n} w_n, t e^{it\lambda_k - \frac{\pi s}{n}} \]

We have

\[ a(k) = \frac{1}{4n} \sum_{t=-n+1}^{n} w_n, t e^{it\lambda_k - \frac{\pi j}{n}} \]

\[ = \frac{\pi^2}{n} w_n \left( \lambda_k - \frac{\pi j}{n} \right) \]

Thus

\[ \hat{f}(\lambda_k) = \frac{\pi}{n} \sum_{t=-n+1}^{n} w_n \left( \lambda_k - \frac{\pi j}{n} \right) I_n \left( \frac{\pi j}{n}, x \right) \]

\[ = \int_{-\pi}^{\pi} w_n \left( \lambda_k - \theta \right) I_n \left( \theta, x \right) d\theta \quad \text{as before.} \]
In estimating the spectral density function we must take into account the presence and effect of the harmonic component at frequency $\lambda_k$ (say). For the earlier tests, when the periodogram ordinates were estimated at a distance $2\pi/n$ apart and a jump occurred midway between two such ordinates, the contribution of the harmonic component to the nearest periodogram ordinate was only about $41\% \left(= \frac{4}{\pi^2} \right)$ the value at its actual point of occurrence. The estimate of $f(\lambda)$ being considered here is estimated over grid points at a distance $\frac{\pi}{n}$ apart. The harmonic component present at $\lambda_k$ will affect the spectral estimate at frequencies $\lambda_{k+\frac{1}{2}}$ and $\lambda_{k+\frac{1}{2}}$ also so that in order to remove this effect we estimate $f(\lambda)$ by

$$
\tilde{f}(\lambda_k) = \frac{\pi}{n} \sum_{-n+1}^{n} W_n(\lambda_k - \frac{\pi}{n}) I_n(\frac{\pi}{n}, x) - \frac{\pi}{n} \left( W_n(0) I_n(\lambda_k, x) + W_n(\frac{\pi}{n}) I_n(\lambda_k + \frac{\pi}{n}, x) + W_n(\frac{\pi}{n}) I_n(\lambda_k - \frac{\pi}{n}, x) \right) \quad (4.4.6)
$$

By subtracting these three terms a bias has been introduced since

$$
\mathcal{E}(\tilde{f}(\lambda_k)) = f(\lambda_k) - \frac{1}{n} \left( f(\lambda_k) \left( W_n(0) + 2 W_n(\frac{\pi}{n}) \right) \right)
$$

$$
= f(\lambda_k) \left( 1 - \frac{1}{n} \left( W_n(0) + 2 W_n(\frac{\pi}{n}) \right) \right) \quad (4.4.7)
$$
Hence the final estimate of the spectral density function is

$$f^*(\lambda_k) = \hat{f}(\lambda_k) - \frac{\pi}{n} \left[ W_n(0) I_n(\lambda_k, x) + W_n\left(\frac{\pi}{n}\right) I_n(\lambda_k + \frac{\pi}{n}, x) + W_n\left(\frac{2\pi}{n}\right) I_n(\lambda_k - \frac{\pi}{n}, x) \right] \left\{ 1 - \frac{\frac{\pi^2}{n}}{n} \left( W_n(0) + 2 \left( W_n\left(\frac{\pi}{n}\right) \right) \right) \right\}$$

(4.4.8)

where the term in the denominator has been included to minimize the bias arising as a result of the three terms subtracted in the numerator.

By using (4.4.5) as an estimate of the spectral function the probability of a jump occurring between two grid points is less and it will be seen that in the worst possible case (using (4.4.6)) nearly 73% of the effect of the harmonic component is removed (by subtracting these three terms) compared with 41% previously. It would appear that subtraction of more terms in (4.4.8) would provide an even better estimate. However one must be careful for usually $n$ is not very large and subtracting too many terms leaves $f(\lambda)$ to be estimated over a small number of grid points. Consequently the power of the test using such an estimate of the spectral density function will be adversely affected. Furthermore, the practical application of this new estimate (4.4.8) has the disadvantage that there is an increase in the amount of computational work involved.
The superiority of the estimate (4.4.5) over (4.1.5) is now illustrated by considering the weight function whose Fourier coefficients are given by

\[ w_{n,t} = \frac{1}{2}(1 + \cos \frac{\pi t}{m}) \quad |t| \leq m \]  

\[ = 0 \quad |t| > m \quad \text{(4.4.9)} \]

Then from (4.4.4)

\[ a_j = \frac{1}{4n} \sum_{n=-m}^{m} e^{-it \lambda_k \frac{\pi j}{n}} \]

\[ = \frac{1}{6n} \sum_{m} (1 + \cos \frac{\pi t}{m}) e^{-it \lambda_k \frac{\pi j}{n}} \]

\[ = \frac{1}{6n} \left( \sin \frac{n \lambda_k}{n} + \frac{1}{2} \sin \frac{n \lambda_k}{2m} + \sin \frac{n \lambda_k}{2m} \right) \quad \text{(4.4.10)} \]

where \( S_n(\mu) = \sin \frac{n \mu}{2} / \sin \frac{\mu}{2} \).

Further, from (1.4.5)

\[ \text{bias} \left\{ I_n \left( \frac{\pi j}{n}, x \right) \right\} = \frac{2}{n} \left| \lambda_k \right|^2 \left\{ \sin^2 \left( \frac{\pi j}{n} + \lambda_k \right) + \sin^2 \left( \frac{\pi j}{n} - \lambda_k \right) \right\} \quad \text{(4.4.11)} \]

where the jump is at frequency \( \lambda_k \).
It is easily seen from (4.4.1) that the effect of a jump at \( \lambda_k \) on the estimate \( \hat{f}(\lambda_k) \) is given by

\[
\frac{1}{2\pi} \sum_{n=1}^{\infty} a_j(k) \cdot \text{bias} \left\{ I_n \left( \frac{\pi j}{n}, x \right) \right\}
\]

\[
= \frac{|\alpha_k|^2}{16\pi n^2} \sum_{n=1}^{\infty} \left( 2S_{2n+1}(\lambda_k - \frac{\pi j}{n}) + S_{2n+1}(\lambda_k - \frac{\pi j}{n} + \frac{\pi}{m}) + S_{2n+1}(\lambda_k - \frac{\pi j}{n} - \frac{\pi}{m}) \right)
\]

\[
\left( S^2_n \left( \frac{\pi j}{n} + \lambda_k \right) + S^2_n \left( \frac{\pi j}{n} - \lambda_k \right) \right)
\]

(4.4.12)

In order to see the effect of a jump on the estimate \( f(\lambda_k) \) choose typical values for \( n, m \) and \( k \), say \( n = 500, m = 50 \) and \( k = 125 \).

Assuming the jump to occur at frequencies \( \lambda_k, \lambda_k + \frac{1}{4}, \) and \( \lambda_k - \frac{1}{2} \) respectively find the ratio of the sum of the effects at points \( \lambda_k - \frac{1}{2}, \lambda_k \) and \( \lambda_k + \frac{1}{2} \) to the total effect due to the harmonic components.

These results are summarized in Table 1. Hence by removing these three points from the range on which \( f(\lambda) \) is estimated, it is seen that even in the worst possible case when a jump occurs at a distance \( \frac{\pi}{n} \) from \( \lambda_k \), approximately 73% of the effect due to the presence of the harmonic component is removed.
Frequency at which harmonic occurs | Ratio of the effect due to the harmonic at frequency $\lambda_j$ to the total effect $j = k - \frac{1}{2}$ | $k$ | $k + \frac{1}{2}$ | Total
---|---|---|---|---
$\lambda_k$ | 0.201 | 0.527 | 0.201 | 0.929
$\lambda_k - \frac{1}{4}$ | 0.419 | 0.429 | 0.043 | 0.891
$\lambda_k - \frac{1}{2}$ | 0.521 | 0.208 | 0 | 0.729

**TABLE I**

Previously we had $\hat{f}(\lambda_k) = \frac{2\pi}{n} \sum_{n=-\infty}^{n=\infty} W_n(\lambda-x) I_n(\theta,x) - \frac{2\pi}{n} W_n(0) I_n(\theta,x)$.

In the example being considered, taking the frequency of the harmonic component to be $\lambda_k$, $\lambda_k - \frac{1}{4}$ and $\lambda_k - \frac{1}{2}$ respectively as before, on calculation it is found that the ratio of the effect at the point $\lambda_k$ to the total effect due to the harmonic component to be 0.968, 0.858 and 0.416 respectively. Comparing these values with the totals in Table I, in the worst possible case when a jump occurs at a distance $\frac{\pi}{2}$ from a grid point, 73% of the bias is removed using (4.4.6) as against 41% using (4.3.1). In the comparison of the three values obtained (for the bias removed) in each of the two cases, while there is a sharp increase when the jump occurs at $\lambda_k - \frac{1}{2}$, there is little difference at the other two frequencies. Nothing will be lost therefore by using (4.4.8) as an estimate of the
spectral density in place of (4.3.3). (When deriving these figures, due to the fact that a hand computer was used, $\lambda_j$ was considered only over the range $k - 10 \leq j \leq k + 10$).

We can conclude therefore by saying that this new estimate (4.4.6) based on grid points at a distance $\frac{\pi}{n}$ apart is superior to previous spectral estimates. This superiority is demonstrated by comparing this estimate with Hannan's smoothed periodogram estimate (4.3.1). In the worst possible case when a harmonic component occurs midway between two grid points ($\frac{2\pi j}{n}$ and $\frac{2\pi(j+1)}{n}$ say) then 73% of the effect of this harmonic component is removed using (4.4.6) as against 41% using (4.3.1).
CHAPTER 5

Testing for a Jump in Cross-Spectra

5.1 The distribution of the product of two periodograms.

Our ideas are now extended to the problem of testing for the presence of a harmonic component in two sets of observations \( \{x_t\} \) and \( \{y_t\} \). We make two distinct assumptions here, first the \( x_t \) are i.i.d. and so are the \( y_t \), second that \( \{x_t\} \) is independent of \( \{y_t\} \). For example if \( x_t \) and \( y_t \) represent two independent sets of rainfall data, we would like to set up a test to determine whether or not there is any tendency for rainfall readings to oscillate periodically.

Consider two independent series \( \{x_t, t = 1, \ldots, n\} \) and \( \{y_t, t = 1, \ldots, n\} \) both of which are normally distributed with zero mean and unit variance. For \( x_t \) and \( y_t \) in this form the corresponding periodogram ordinates \( I_n(\lambda_j, x) \) and \( I_n(\lambda_j, y) \) are each distributed as \( \chi^2 \) variables with two degrees of freedom.

If \( R^2(\lambda_j) = I_n(\lambda_j, x) I_n(\lambda_j, y) \), \( \lambda_j = \frac{2m_j}{n} \) \hspace{1cm} (5.1.1)

then in order to form a test of significance it is required to know the distribution of \( \max_j R^2(\lambda_j) \).

Put

\[ l = I_n(\lambda_j, x) \sim \chi^2_2 \]
\[ m = I_n(\lambda_j, y) \sim \chi^2_2 \]
Then
\[ \phi(\ell,m) = \frac{1}{4} e^{-\frac{1}{2}(\ell+m)} \quad 0 \leq \ell,m < \infty. \quad (5.1.2) \]

Define
\[ z = \ell m \quad w = \ell \quad 0 \leq z,w < \infty. \]

Then
\[ f(z,w) = \phi(\ell,m) |J| \quad \text{where} \quad |J| = \left| \frac{\partial (m, \ell)}{\partial (z,w)} \right| = \frac{1}{w} \]
\[ = \frac{1}{4w} e^{-\frac{1}{2}(w+z)w} \]
\[ \therefore f(z) = \int_0^\infty \frac{1}{4w} e^{-\frac{1}{2}(w+z)w} dw \]

and
\[ F_\perp(c) = \Pr(z \leq c) = \int_0^c \int_0^\infty \frac{1}{4w} e^{-\frac{1}{2}(w+z)w} dw \, dz \]
\[ = 1 - \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}(w+c)w} \, dw \quad (5.1.3) \]

If we have a change of variable such that
\[ t = \frac{1}{2}w, \quad dt = \frac{1}{2}dw \]

then
\[ F_\perp(c) = 1 - \int_0^\infty e^{-t} e^{-\frac{c}{4t}} \, dt \]
\[ = 1 - \sqrt{c} K_\perp(\sqrt{c}) \quad (5.1.4) \]

where \( K_\perp(\sqrt{c}) \) is a modified Bessel function of the second kind, of order unity.
Thus the distribution of $R^2(\lambda_j)$, $j = 1, 2, \ldots, \lfloor \frac{1}{2} n \rfloor$ is given by

$$F_1(c) = \Pr\left(R^2(\lambda_j) \leq c\right)$$

$$= 1 - \sqrt{c} K_1(\sqrt{c})$$

and hence the distribution of $\max_j R^2(\lambda_j)$ is given by

$$F(c) = \Pr\left(\max_j R^2(\lambda_j) \leq c\right)$$

$$= \left[1 - \sqrt{c} K_1(\sqrt{c})\right]^{n/2}$$

(5.1.5)

where, for $x$ large,

$$K_1(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[1 + \frac{3}{8x} - \frac{15}{128x^2} + \frac{105}{1024x^3}\right]$$

(5.1.6)

With the distribution of $\max_j R^2(\lambda_j)$ known it is now possible to form a test of significance for the presence of harmonic components. This test, however, as shall soon be seen, depends only on the amplitudes. A test is now derived for the phases of the two independent series $x_t$ and $y_t$. These two tests, which will be found to be independent of each other, are then combined to give a test to determine the presence of harmonic components in spectra.
5.2 The Distribution of the Phase Differences

Let $\hat{\theta}_x(\lambda_j)$ and $\hat{\theta}_y(\lambda_j)$ be the phase angles of the components of $x_t$ and $y_t$ (with frequency $\lambda_j$) respectively, i.e.

$$\hat{\theta}_x(\lambda_j) = \arctan \left( \frac{\sum_{t=1}^{n} x_t \sin t\lambda_j}{\sum_{t=1}^{n} x_t \cos t\lambda_j} \right), \quad -\pi/2 \leq \hat{\theta}_x(\lambda_j) \leq \pi/2 \quad (5.2.1)$$

and similarly for $\hat{\theta}_y(\lambda_j)$.

Suppose

$$a_1 = \sum_{t=1}^{n} x_t \cos t\lambda_j, \quad a_2 = \sum_{t=1}^{n} y_t \cos t\lambda_j$$

$$b_1 = \sum_{t=1}^{n} x_t \sin t\lambda_j, \quad b_2 = \sum_{t=1}^{n} y_t \sin t\lambda_j$$

then since

$$\sum_{t=1}^{n} \cos^2 t\lambda_j = \frac{n}{2} = \sum_{t=1}^{n} \sin^2 t\lambda_j$$

and

$$x_t : N(0,1) , \quad y_t : N(0,1)$$

we have

$$a_1 : N(0, \frac{n}{2}) \quad a_2 : N(0, \frac{n}{2})$$

$$b_1 : N(0, \frac{n}{2}) \quad b_2 : N(0, \frac{n}{2})$$

therefore

$$f(a_1, a_2, b_1, b_2) = \frac{1}{\pi^{2} n^2} e^{-\frac{1}{n} \left( \frac{a_1^2}{2} + \frac{b_1^2}{2} + \frac{a_2^2}{2} + \frac{b_2^2}{2} \right)} \quad (5.2.2)$$
Transforming to polar co-ordinates

\[
\begin{align*}
    a_1 &= r_1 \cos \theta_x \\
    a_2 &= r_2 \cos \theta_y \\
    b_1 &= r_1 \sin \theta_x \\
    b_2 &= r_2 \sin \theta_y
\end{align*}
\]

so

\[
\begin{align*}
    r_1^2 &= a_1^2 + b_1^2 \\
    \theta_x &= \arctan \left( \frac{b_1}{a_1} \right) \\
    r_2^2 &= a_2^2 + b_2^2 \\
    \theta_y &= \arctan \left( \frac{b_2}{a_2} \right)
\end{align*}
\]

\[
\therefore f(r_1, \theta_x, r_2, \theta_y) \; dr_1 \; dr_2 \; d\theta_x \; d\theta_y = \frac{1}{n \pi^2} \; e^{-\frac{1}{n}(r_1^2 + r_2^2)} \; r_1 \; r_2 \; dr_1 \; dr_2 \; d\theta_x \; d\theta_y
\]

\[
\therefore f(\theta_x, \theta_y) = \int_0^\infty \int_0^{\frac{2\pi}{n}} \frac{1}{n \pi^2} r_1^2 \; e^{-\frac{1}{n}(r_1^2 + r_2^2)} \; dr_1 \; dr_2
\]

\[
\int_0^\infty \int_0^{\frac{2\pi}{n}} \frac{1}{n \pi} r_1 \; e^{-\frac{1}{n}r_1^2} \; dr_1 \int_0^\infty \int_0^{\frac{2\pi}{n}} \frac{1}{n \pi} r_2 \; e^{-\frac{1}{n}r_2^2} \; dr_2
\]

\[
= \frac{1}{4\pi^2} \quad 0 \leq \theta_x \leq 2\pi \\
0 \leq \theta_y \leq 2\pi
\]

Put

\[
\ell = \theta_x - \theta_y \quad 0 \leq \ell \leq 2\pi
\]

\[
m = \theta_x \quad 0 \leq m \leq 2\pi
\]

Then

\[
f(\ell, m) = f(\theta_x, \theta_y) \mid |J| = \frac{1}{4\pi^2}
\]

\[
\therefore f(\ell) = f(\theta_x - \theta_y) = \int_0^{2\pi} \frac{1}{4\pi^2} \; dm = \frac{1}{2\pi}
\]

\[
(5.2.4)
\]
Hence \[ \hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j) \] is uniformly distributed on the unit circle \((\text{mod } 2\pi)\) and is independent of \(R^2(\lambda_j)\).

Converting \(R^2(\lambda_j)\) to polar co-ordinates it follows from (5.2.3) that \(R^2(\lambda_j) = I_n(\lambda_j,x) I_n(\lambda_j,y)\)

\[
\frac{2}{n} \left[ \left( \sum_{t=1}^{n} x_t \cos t\lambda_j \right)^2 + \left( \sum_{t=1}^{n} x_t \sin t\lambda_j \right)^2 \right]
\]

\[
\times \frac{2}{n} \left[ \left( \sum_{t=1}^{n} y_t \cos t\lambda_j \right)^2 + \left( \sum_{t=1}^{n} y_t \sin t\lambda_j \right)^2 \right]
\]

\[
= \frac{4}{n^2} \left( a_1^2 + b_1^2 \right) \left( a_2^2 + b_2^2 \right)
\]

\[
= \frac{4}{n^2} r_1^2 r_2^2
\]

which is of course dependent only on the amplitudes \(r_1, r_2\) and not on the phases \(\hat{\theta}_x(\lambda_j), \hat{\theta}_y(\lambda_j)\).

Using (5.2.4) a test of significance can be derived; this test being based on the phase differences for the two independent series. Usually, however, the observations \(x_t\) and \(y_t\) are not independent. These results are now generalized to the case of stationary Gaussian processes.

* * * * * * *
5.3 The application of the results to stationary Gaussian processes.

It is now assumed that each of the processes $x_t$ and $y_t$ are of a stationary Gaussian nature with spectral densities $f(\lambda_j, x)$ and $f(\lambda_j, y)$ respectively. Thus from (2.1.9)

$$I_n(\lambda_j, x) = 2\pi f(\lambda_j, x) I_n(\lambda_j, \varepsilon) + I_n(\lambda_j)$$

(5.3.1)

where $I_n(\lambda_j)$ is of order $n^{-1/2}$, and similarly for $I_n(\lambda_j, y)$ so that

$$I_n(\lambda_j, x) I_n(\lambda_j, y) = 2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y) I_n(\lambda_j, \varepsilon) I_n(\lambda_j, \eta)$$

$$+ M_n(\lambda_j)$$

(5.3.2)

where $M_n(\lambda_j)$ is of order $n^{-1/2}$ and $\varepsilon_t, \eta_t$ are both independent random variables with zero means and unit variances.

Let $K_n(\lambda_j, x) = \frac{I_n(\lambda_j, x)}{2\pi f(\lambda_j, x)}$ and $K_n(\lambda_j, y) = \frac{I_n(\lambda_j, y)}{2\pi f(\lambda_j, y)}$

so that

$$K_n(\lambda_j, x) K_n(\lambda_j, y) = \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)} = z \text{ (say)}$$

(5.3.3)

From (5.3.2) it follows that
\[ Pr \left\{ \max_j I_n(\lambda_j, x) I_n(\lambda_j, y) - 2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y) \right\} \]

\[ = Pr \left\{ \max_j M_n(\lambda_j) \mid \geq a \right\} \]

and applying Tchebycheff's inequality as before,

\[ Pr \left\{ \max_j M_n(\lambda_j) \mid \geq a \right\} \]

\[ = Pr \left\{ \max_j M_n(\lambda_j) \mid 2^k \geq a^{2k} \right\} \]

\[ \leq \frac{1}{a^{2k}} \sum_j \epsilon \left( |M_n(\lambda_j)|^{2k} \right) \]

\[ = \frac{1}{a^{2k}} \cdot \frac{n}{2} o(n^{-k}) \]

\[ \rightarrow 0 \text{ if } k \geq 2. \]

Hence

\[ \max_j \left| \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)} - R^2(\lambda_j, \varepsilon, \eta) \right| \]

converges in probability to zero.

Putting

\[ R^2_M(\lambda) = \max_j \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)} \]

\[ g_M(\varepsilon, \eta) = \max_j I_n(\lambda_j, \varepsilon) I_n(\lambda_j, \eta) \]
and \( \Pr \left( g_n(\varepsilon, \eta) > d(n, \alpha) \right) = \alpha . \) (5.3.6)

Then

\[
\frac{K_n(\lambda_j, x)K_n(\lambda_j, y)}{d(n, \alpha)} = \frac{I_n(\lambda_j, \varepsilon)I_n(\lambda_j, \eta)}{d(n, \alpha)} + \frac{K_n(\lambda_j, x)K_n(\lambda_j, y) - I_n(\lambda_j, \varepsilon)I_n(\lambda_j, \eta)}{d(n, \alpha)}
\]

so that

\[
\max_j \frac{K_n(\lambda_j, x)K_n(\lambda_j, y)}{d(n, \alpha)} = \max_j \frac{I_n(\lambda_j, \varepsilon)I_n(\lambda_j, \eta)}{d(n, \alpha)}
\]

\[
+ \max_j \left( \frac{K_n(\lambda_j, x)K_n(\lambda_j, y) - I_n(\lambda_j, \varepsilon)I_n(\lambda_j, \eta)}{d(n, \alpha)} \right).
\]

(5.3.5)

But it has already been seen that

\[
\max_j \left| K_n(\lambda_j, x)K_n(\lambda_j, y) - I_n(\lambda_j, \varepsilon)I_n(\lambda_j, \eta) \right|
\]

converges in probability to zero. Hence from (5.3.4) and (5.3.5) it follows that

\[
\Pr \left( \frac{R_n^2(\lambda)}{d(n, \alpha)} > 1 \right) \rightarrow \alpha
\]

This is so only if \( \lim_{n \to \infty} \Pr \left( \frac{g_n(\varepsilon, \eta)}{d(n, \alpha)} > x \right) \)

is continuous at \( x = 1. \]
i.e.

\[
\Pr \left( \max_j \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)} > d(n, \alpha) \right) \rightarrow \alpha
\]

Asymptotically, therefore, the statistic \((5.3.3)\) will have the same distribution as \((5.1.1)\) and will thus have the distribution \((5.1.5)\). Also, as we have already seen in the last chapter, replacing the density function by its smoothed periodogram estimate asymptotically does not affect the statistic \((5.3.3)\). In estimating \(f(\lambda_j, x), f(\lambda_j, y)\) however, the same kind of difficulty occurs as before, i.e.

\[
\frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)}
\]

will be reduced, when the null hypothesis (i.e. no periodicity) is false, by inflation of the denominator. To minimize this effect the spectral densities are estimated using \((4.4.8)\). Hence

\[
\max_j \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x) 2\pi f(\lambda_j, y)}
\]

asymptotically has the same distribution as if \(x_t\) and \(y_t\) were N.I.D. \((0,1)\) i.e. it is asymptotically distributed the same as \(\max_j R^2(\lambda_j)\) (see \(5.1.5)\).

We now consider the distribution of the phase differences

\[
\left[ \hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j) \right]
\]

when \(x_t\) and \(y_t\) are each stationary Gaussian processes i.e. \(x_t : N(0, \sigma_x^2)\) and \(y_t : N(0, \sigma_y^2)\).
As before \( a_1 = \sum_{t=1}^{n} x_t \cos t\lambda_j \)

so

\[
\text{var} (a_1) = \varepsilon(a_1^2) - \left[ \varepsilon(a_1) \right]^2
\]

\[
= \varepsilon \left( \sum_{t=1}^{n} x_t \cos t\lambda_j \right)^2
\]

\[
= \sum_{s=1}^{n} \sum_{t=1}^{n} \gamma_{s-t} \cos t\lambda_j \cos s\lambda_j
\]

\[
= 2 \int_{\pi}^{\pi} \sum_{s=1}^{n} \sum_{t=1}^{n} \cos (s-t) \lambda \cos s\lambda_j \cos t\lambda_j \, dF(\lambda_j)
\]

\[
= \int_{\pi}^{\pi} \sum_{s=1}^{n} \sum_{t=1}^{n} \cos (s-t) \lambda \left( \cos (s+t)\lambda_j + \cos (s-t)\lambda_j \right) \, dF(\lambda)
\]

\[
= \int_{\pi}^{\pi} \sum_{s=1}^{n} \sum_{t=1}^{n} \cos (s-t)\lambda \cos (s+t)\lambda_j \, dF(\lambda)
\]

\[
+ \int_{\pi}^{\pi} \sum_{s=1}^{n} \sum_{t=1}^{n} \cos (s-t)\lambda \cos (s+t)\lambda_j \, dF(\lambda)
\]

Now the first term on the right hand side

\[
= \frac{1}{2} \sum_{s=1}^{n} \sum_{t=1}^{n} \gamma_{s-t} \cos (s-t) \lambda_j
\]

\[
= \frac{n+1}{2} \sum_{u=-n}^{n-1} \left( 1 - \frac{|u|}{n} \right) \gamma_u e^{iu\lambda_j}
\]

while the second term

\[
= \frac{1}{2} \int_{\pi}^{\pi} \sum_{s=1}^{n} \sum_{t=1}^{n} e^{i(s-t)\lambda} \left( e^{i(s+t)\lambda_j} + e^{-i(s+t)\lambda_j} \right) \, dF(\lambda)
\]
\[
\frac{1}{2} \int_{-\pi}^{\pi} \sin \left( \frac{n}{2} (\lambda_j + \lambda) \right) \frac{\sin \left( \frac{n}{2} (\lambda - \lambda_j) \right)}{\sin \left( \frac{1}{2} (\lambda - \lambda_j) \right)} \cos (n+1) \lambda_j \, dF(\lambda)
\]

and this is negligible unless \( \lambda_j \) is near to \( \lambda \)

\[
\therefore \frac{2}{n} \text{var} (a_1) = \sum_{u=-n+1}^{n-1} \left( 1 - \frac{|u|}{n} \right) \gamma_u e^{iu\lambda_j}
\]

\[
\approx 2\pi f(\lambda_j, x)
\]

\[
\therefore \text{var} (a_1) \approx \pi n f(\lambda_j, x)
\]

So \( a_1 : N(0, \pi f(\lambda_j, x)) \) \quad \quad a_2 : N(0, \pi f(\lambda_j, y)) \)

\( b_1 : N(0, \pi f(\lambda_j, x)) \) \quad \quad b_2 : N(0, \pi f(\lambda_j, y)) \)

Putting \( J_n(\lambda_j, x) = \sum_{t=1}^{n} x_t e^{it\lambda_j} \)

\[
= \sum_{t=1}^{n} x_t \cos t\lambda_j + i \sum_{t=1}^{n} x_t \sin t\lambda_j
\]

\[
= a_1 + ib_1
\]

Then from (2.1.3) \( J_n(\lambda_j, x) \) converges in probability to \( k(\lambda_j) J_n(\lambda_j, \epsilon) \) so that arg \([J_n(\lambda_j, x)]\) converges in probability to arg \([k(\lambda_j) J_n(\lambda_j, \epsilon)]\) where, as before, \( \epsilon_t \) are independent random variables mean zero and unit variance. Then, for a fixed \( \lambda_j \), we can put
\[ k(\lambda_j) J_n(\lambda_j, \varepsilon) = [c(\lambda_j) + i\delta(\lambda_j)] [u(\lambda_j) + iv(\lambda_j)] \]

\[ = [c(\lambda_j) u(\lambda_j) - d(\lambda_j) v(\lambda_j)] + i [c(\lambda_j) v(\lambda_j) + d(\lambda_j) u(\lambda_j)] \]

and, for \( \lambda_j \) of the form \( \frac{2\pi j}{n} \), \( j = 0, 1, 2, \ldots, \lfloor \frac{1}{2}\pi \rfloor \), \( u(\lambda_j) \) and \( v(\lambda_j) \) are independent so that

\[ \begin{bmatrix} c(\lambda_j) u(\lambda_j) - d(\lambda_j) v(\lambda_j) \end{bmatrix} \text{ and } \begin{bmatrix} c(\lambda_j) v(\lambda_j) + d(\lambda_j) u(\lambda_j) \end{bmatrix} \]

are independent.

Hence \( a_1 \) and \( b_1 \) are asymptotically independent and so

\[ f(a_1, b_1, a_2, b_2) = \frac{1}{2\pi} e^{-\frac{(a_2^2 + b_2^2)}{\pi f(a_1, x)}} \cdot \frac{1}{2\pi} e^{-\frac{(a_2^2 + b_2^2)}{\pi f(a_1, y)}} \]

\[ \text{(5.3.4)} \]

Transforming this to polar co-ordinates we find the marginal density of \( \hat{\theta}_x(\lambda_j) \), \( \hat{\theta}_y(\lambda_j) \) and hence the density of \( \begin{bmatrix} \hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j) \end{bmatrix} \) to be the same as before

i.e.

\[ f(\hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j)) = \frac{1}{2\pi} \quad \bullet \leq \theta_x, \theta_y \leq 2\pi \]

Also transforming (5.3.3) to polar co-ordinates gives
\[ z = K_n(\lambda_j, x) K_n(\lambda_j, y) \]

\[ = \frac{r_1^2 r_2^2}{\pi^2 n^2 f(\lambda_j, x) f(\lambda_j, y)} \]

which is dependent only on the amplitudes \( r_1, r_2 \) and not on the phases \( \hat{\theta}_x(\lambda_j), \hat{\theta}_y(\lambda_j) \). As before we may form two independent tests, one being based on the amplitudes and the other on the phases, of the observations \( x_t \) and \( y_t \).

* * * * * * * *
5.4 A test for the presence of a harmonic component.

The test about to be derived will, as has been suggested, be a combination of two independent tests, one based on the amplitudes and the other on the phases of the two series of observations \( x_t \) and \( y_t \). For the statistic

\[
\max_j \frac{R^2(\lambda_j)}{2\pi \hat{f}(\lambda_j, x) 2\pi \hat{f}(\lambda_j, y)}
\]

the distribution function is given by

\[
F_1(c_1) = \left[ 1 - \frac{\sqrt{c_1}}{K_1(\sqrt{c_1})} \right]^{n/2}
\]

(5.4.1)

where \( K_1(x) \) is a modified Bessel function of the second kind, of order unity. The phase difference \( [\hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j)] \) is uniformly distributed about the unit circle, i.e. \( [\hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j)] \) has the distribution function

\[
F_2(c_2) = \frac{c_2}{2\pi}
\]

(5.4.2)

As it has been seen these two tests are independent so that they may be combined; the method of combination used is that given by Anderson and Bancroft [1]. For

\[
P_1 = \int_{-\infty}^{x} f(t) \, dt
\]
it can be shown that \(-2 \log e p_1\) is distributed as \(\chi^2_2\) and in general, since the \(p_i\) are independent, the sum of \(k\) such \(\chi^2\) values is distributed as \(\chi^2\) with \(2k\) degrees of freedom. Therefore

\[
-2 \left( \log e p_1 + \log e p_2 \right)
\]

(5.4.3)

is distributed as a \(\chi^2\) distribution with four degrees of freedom.

For \(F_1(c_1)\) we require \(-2 \log e p_1\) to be large when \(F_1(c_1)\) is near to 1. Thus we choose \(p_1 = 1 - F_1(c_1)\) and the first term in (5.4.3) is \(-2 \log e (1 - F_1(c_1))\), where \(c_1\) is the observed value.

If we reduce \([\hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j)]\) modulo \(2\pi\) then values near zero and values near \(2\pi\) are significant of a small difference in phase angle. The simplest procedure is to plot the two phases \(\hat{\theta}_x(\lambda_j)\) and \(\hat{\theta}_y(\lambda_j)\) on the unit circle and take the smaller of the two angles between them. If \(c_2\) is this value then \(\frac{2c_2}{2\pi}\) is an appropriate choice of \(p_2\). (The 2 in the numerator comes from the fact that we have no prior reason for believing that the phase difference is in one direction or the other). It might be noted here that if \(\hat{\theta}_x(\lambda_j)\) and \(\hat{\theta}_y(\lambda_j)\) are diametrically opposed in some circumstances one might wish to treat this as significant since such a phase difference might correspond to a mere difference in sign between two components. Whether or not one will want to allow for this possibility will depend upon prior knowledge.
Substituting these values for \( p_1 \) and \( p_2 \) in (5.4.3) gives

\[
-2 \left( \log_e (1 - F_1(c_1)) + \log \left( \frac{2c_2}{2\pi} \right) \right)
\]

which is distributed as a \( \chi^2 \) distribution with four degrees of freedom.

* * * * * * * *
5.5 A problem on rainfalls.

The theoretical results obtained to date in this chapter are now applied to two independent sets of rainfall data to determine whether there is a tendency for rainfall readings to oscillate periodically. The two sets of rainfall readings used are U.S. Rainfall 1952-57 (which will be denoted by $x_t : x_t$ is a three day running total of rainfall in points on the $t^{th}$ day of the year) and World Rainfall 1880-1950 (denoted by $y_t : y_t$ being taken, on the $t^{th}$ day of the year, as a percentage of the mean rainfall). The two series are independent (see Brier [5]).

For each of these series the periodogram was evaluated (using an I.B.M. 1620 computer) from (1.2.4) i.e.

$$ I_n(\lambda_j, x) = \frac{2}{n} \left[ \left( \sum_{t=1}^{n} x_t \cos \lambda_j t \right)^2 + \left( \sum_{t=1}^{n} x_t \sin \lambda_j t \right)^2 \right] $$ (5.5.1)

and similarly for $I_n(\lambda_j, y)$ where $\lambda_j = \frac{2\pi j}{n}$, $j = 1, ..., 182$ and $n = 365$. These periodogram values are given in Table 2. Using weights

$$ W_n(\lambda) = \frac{n}{16\pi^2} \left( \frac{1}{16} \right), \frac{\pi j}{n} - \frac{\pi}{2n} \leq \lambda < \frac{\pi j}{n} + \frac{\pi}{2n} ; \ j = -4, 4 $$

$$ = \frac{n}{16\pi^2} \left( \frac{2}{16} \right), \frac{\pi j}{n} - \frac{\pi}{2n} \leq \lambda < \frac{\pi j}{n} + \frac{\pi}{2n} ; \ j = -3, ..., 3 $$

$$ = 0 \quad \text{otherwise} \quad (5.5.2) $$
both \( \hat{f}(\lambda_j,x) \) and \( \hat{f}(\lambda_j,y) \) were estimated, for \( \lambda_j = \frac{2\pi j}{365} \), \( j = 1, 2, \ldots, 182 \), using (4.4.8), i.e.

\[
\hat{f}(\lambda_j) = \frac{\pi}{n} \left( \frac{4}{k=1} \sum W_n(\lambda_j - \frac{\pi k}{n}) I_n(\frac{\pi k}{n}) - \left\{ W_n(0) I_n(\lambda_j) + W_n(\frac{\pi}{n}) I_n(\lambda_j + \frac{\pi}{n}) + W_n(\frac{\pi}{n}) I_n(\lambda_j - \frac{\pi}{n}) \right\} \right)
\]

\[
\left\{ 1 - \frac{4n^2}{n} \left( W_n(0) + 2W_n(\frac{\pi}{n}) \right) \right\}^{\frac{1}{2}}
\]

(5.5.3)

Putting \( R^2(\lambda_j) = I_n(\lambda_j,x) I_n(\lambda_j,y) \) and using the spectral estimates \( \hat{f}(\lambda_j,x) \) and \( \hat{f}(\lambda_j,y) \) the statistic

\[
\frac{R^2(\lambda_j)}{2\pi \hat{f}(\lambda_j,x) 2\pi \hat{f}(\lambda_j,y)}
\]

(5.5.4)

may be formed and tabulated.

An examination of Table 2 shows that the maximum of the test statistic (based on coherence) seems certain to occur at a multiple of \( j = 9 \) so that we will examine these to find which is the greatest. Table 3 gives values of \( \hat{f}(\lambda_j,x) \) and \( \hat{f}(\lambda_j,y) \) for multiples of \( j = 9 \) while Table 4 gives values of the statistic (5.5.4) for the same values of \( j \). It is seen from Table 4 that the maximum of this statistic is \( 0.10934 \times 10^3 \) occurring at \( j = 45 \). Using (5.2.1) the phase differences \( \hat{\theta}_x(\lambda_j) \) and \( \hat{\theta}_y(\lambda_j) \) (for the series \( x_t \) and \( y_t \) respectively) are evaluated (Table 5) and at \( j = 45, [\hat{\theta}_x(\lambda_j) - \hat{\theta}_y(\lambda_j)] \) is found to be 1.355247.
Letting \( c_1 = 0.10934 \times 10^3 \), from (5.1.6)

\[
K_1(\sqrt{c_1}) = 0.11493064 \times 10^{-4}
\]

and so, from (5.4.1)

\[
F_1(c_1) = \Pr \left( \max_j \frac{R^2(\lambda_j)}{2\pi f(\lambda_j, x)2\pi f(\lambda_j, y)} \leq c_1 \right) = \left[ 1 - \sqrt{c_1} K_1(\sqrt{c_1}) \right]^{182} = 0.97812
\]

and

\[
(1 - F_1(c_1)) = 0.02188 \quad (5.5.5)
\]

Furthermore,

\[
\frac{2c_2}{2\pi} = \frac{1.355247}{\pi} = 0.43139 \quad (5.5.6)
\]

Placing (5.5.5) and (5.5.6) in (5.4.4), on calculation

\[
-2 \left( \log_e (1 - F_1(c_1)) + \log \left( \frac{2c_2}{2\pi} \right) \right) = 9.33 \quad (5.5.7)
\]

Now \( \chi^2_4 = 9.49 \) at the 5% level so that (5.5.7) is not significant.

Summarising these results it may be said that the test derived from an examination of the cross-spectra of two independent series of rainfall readings does not indicate any tendency for rainfall values to oscillate periodically.
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<th>j</th>
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<th>$I_n(\lambda_j^y)$</th>
<th>j</th>
<th>$I_n(\lambda_j^x)$</th>
<th>$I_n(\lambda_j^y)$</th>
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**TABLE 2**
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\lambda_j = \frac{\pi j}{365} & \hat{\theta}_x (\lambda_j) & \hat{\theta}_y (\lambda_j) \\
\hline
9 & 0.80172798 \times 10^0 & -0.13805136 \times 10^1 \\
18 & -0.14458133 \times 10^1 & 0.57387043 \times 10^0 \\
27 & 0.25706527 \times 10^0 & -0.91503821 \times 10^0 \\
36 & -0.9434158 \times 10^0 & 0.3613001 \times 10^0 \\
45 & 0.13633439 \times 10^1 & 0.80967830 \times 10^{-2} \\
54 & 0.1294778 \times 10^1 & -0.14816456 \times 10^1 \\
63 & 0.25673062 \times 10^0 & 0.28415624 \times 10^0 \\
72 & -0.14712662 \times 10^1 & -0.94535071 \times 10^0 \\
81 & -0.19076877 \times 10^0 & 0.97192661 \times 10^0 \\
90 & 0.31864434 \times 10^0 & 0.25112989 \times 10^{-1} \\
99 & -0.5079245 \times 10^0 & 0.52244326 \times 10^0 \\
108 & -0.20537733 \times 10^0 & -0.41529418 \times 10^0 \\
117 & 0.71850879 \times 10^0 & 0.14848382 \times 10^1 \\
126 & -0.13856376 \times 10^1 & 0.14532585 \times 10^0 \\
135 & 0.18688698 \times 10^0 & 0.28836720 \times 10^0 \\
144 & 0.13110796 \times 10^1 & 0.97920810 \times 10^0 \\
153 & -0.70018909 \times 10^0 & -0.29743366 \times 10^0 \\
162 & 0.21544606 \times 10^0 & -0.14929045 \times 10^1 \\
171 & 0.15498822 \times 10^1 & 0.57682377 \times 10^0 \\
180 & -0.31506106 \times 10^0 & -0.19203252 \times 10^0 \\
\hline
\end{array}
\]

**TABLE 5**
# REFERENCES

<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s)</th>
<th>Publication Details</th>
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