ON BRAZILIAN PARACONSISTENT LOGICS

Igor Urbas

A thesis submitted for
the degree of Doctor of Philosophy
of the Australian National University

Department of Philosophy
Research School of Social Sciences
August 1987

Except where otherwise acknowledged,

this thesis is my own work.

ACKNOWLEDGEMENTS

I would like to thank Dr. Richard Sylvan, Dr. Robert K. Meyer and Dr. Michael A. McRobbie for their supervision, and Frances Redrup for her typing. I would also like to acknowledge the assistance afforded by the computer program TESTER developed by Nuel D. Belnap, Jr. and Dale Isner, which reduced the labour of constructing the matrices in the thesis, and which verifies the matrix claims made therein.

Synopsis

This thesis embodies a detailed investigation of two families of paraconsistent logics developed by the Brazilian logician Newton C.A. da Costa and his collaborators. These two families - the J-systems and the C-systems - are among the most well-known contributions of Brazilian logicians to the paraconsistent logic programme, and they also exhibit most clearly what has come to be regarded as the distinguishing feature of the Brazilian approach to paraconsistency.

This approach is enshrined in da Costa's two conditions for paraconsistent logics. The first uncontroversially requires that arbitrary conclusions not be deducible from inconsistent premisses, and the second requires that such logics approximate classical logic as far as is compatible with the satisfaction of the first condition. It is this second condition which distinguishes the Brazilian approach, and it is the major task of this thesis both to assess whether the J-systems and C-systems satisfy these conditions and also to critically examine the conditions, particularly the second, in the process.

In Chapter One, it is shown that the positive theorems of J_1, the weakest of the J-systems, are exactly those of positive intuitionistic logic, while the positive theorems of the remaining J-systems are exactly those of positive classical logic.

In Chapter Two, it is noted that the stronger J-systems either explicitly or substantively fail to satisfy da Costa's first condition, leaving only J_1 for further consideration. However, the fact that J_1 does not enjoy SE, the property of intersubstitutivity of provable equivalents, indicates that this system does not adequately meet the second condition. Attempts to extend J_1 so as to secure SE are unsuccessful, in that its satisfaction of the first condition is thereby forfeited.

In Chapter Three, attention is turned to subsystems of J_1, in an attempt to find J-systems which both enjoy SE and satisfy the paraconsistency conditions. Axiomatic and Gentzen-style equivalents of most of the subsystems investigated are provided.

In Chapter Four, it is shown that the subsystems of J_1 defined in Chapter Three do not enjoy SE. However, the attempts to extend them so as to secure this property do not compromise their satisfaction of the first condition. Their satisfaction of the second condition, however, is slightly tarnished by the existence of axiomatic counterparts which

approximate classical logic more closely. These latter systems form the bridge between the J-systems studied in the first four chapters and the C-systems studied in the last two.

In Chapter Five, it is shown that the C-systems also fail to enjoy SE, and that the extensions necessary to secure this property collapse all but the base system C_ω into classical logic.

In Chapter Six, two variations to the C-systems are considered. The first, which involves the least drastic revision, is only partially successful in producing C-systems which both enjoy SE and satisfy the paraconsistency conditions. The second variation, which involves replacing C_ω by alternative base systems from among the axiomatic systems of Chapter Four, produces only a few systems with the desired properties, the remainder collapsing into either classical logic or other systems which do not satisfy the first condition. The final conclusion is that still weaker base systems should be investigated, indicating that a significant departure from da Costa's second condition is warranted.

TABLE OF CONTENTS

Chapter One: The Positive Parts of the J-Systems

1. THE J-SYSTEMS

The J-systems of Arruda and da Costa (introduced and investigated in [4]) have as primitive connectives $\rightarrow$, $\supset$, $\wedge$, &, $\vee$ and $\neg$. Formulas involving only propositional variables p,q,r, ... ,p_1,p_2, ... and the connectives $\supset$, &, $\vee$ and $\neg$ will be denoted by the letters A,B,C, ... , A_1, A_2, ... , and the set of such formulas will be denoted by $\mathcal{L}$. The letters Δ, Γ, ... will be reserved for formulas of the form $A_1 \wedge ... \wedge A_n$, where $\{A_1, ... , A_n\}$ is a (possibly empty) subset of $\mathcal{L}$. The connective $\rightarrow$ can occur only in formulas of the form $\Gamma \rightarrow$ A; and if the set of $\mathcal{L}$-formulas occurring in Γ is empty, this is abbreviated as $\rightarrow$A. The set of formulas of the form $\Gamma \rightarrow$ A will be denoted by $\mathcal{K}$.

Evidently, a natural analogy holds between the connectives $\wedge$ and $\rightarrow$ of the J-systems and, respectively, the comma and arrow (or turnstile) of Gentzen-style formulations of more familiar logics. Following this analogy, we henceforth express Δ, Γ, etc. simply as *sequences* A_1, A_2, ... , A_n (in conformity with the notation of [4] and [7]), and refer to the members of $\mathcal{K}$ as *sequents*. Further, if $\Gamma \rightarrow$ A is a derivable sequent in a J-system, we will say that A is *derivable* from Γ; and if both A $\rightarrow$ B and B $\rightarrow$ A are derivable sequents, we will say that A and B are *interderivable* or *provably equivalent*. Following [4], we adopt the shorthand A $\leftrightarrow$ B to represent the (metalogical) conjunction of sequents A $\rightarrow$ B and B $\rightarrow$ A, and say that A $\leftrightarrow$ B is derivable when both of these sequents are. Finally, any formula A of $\mathcal{L}$ such that $\rightarrow$A is a derivable sequent of a J-system will be called a *theorem* of that system.

The postulates (axiom schemata and rules of inference) of J_1 are as follow:

$\to_1$) $A \to A$ $\qquad$ $\to_2$) $\dfrac{\Gamma\to A}{C,\Gamma\to A}$ $\qquad$ $\to_3$) $\dfrac{C,C,\Gamma\to A}{C,\Gamma\to A}$

$\to_4$) $\dfrac{\Delta,C,D,\Gamma\to A}{\Delta,D,C,\Gamma\to A}$ $\qquad$ $\to_5$) $\dfrac{\Delta\to C \quad C,\Gamma\to A}{\Delta,\Gamma\to A}$

$\supset_1$) $\dfrac{\Gamma,A\to B}{\Gamma\to A\supset B}$ $\qquad$ $\supset_2$) $(A \supset B) \,\&\, (A \supset (B \supset C)) \to A \supset C$

$\&_1$) $A, B \to A \,\&\, B$ $\qquad$ $\&_2$) $A \,\&\, B \to A$ $\qquad$ $\&_3$) $A \,\&\, B \to B$

$\&_4$) $(A \supset B) \,\&\, (A \supset C) \to A \supset (B \,\&\, C)$ $\qquad$ $\vee_1$) $A \to A \vee B$

$\vee_2$) $B \to A \vee B$ $\qquad$ $\vee_3$) $(A \supset C) \,\&\, (B \supset C) \to (A \vee B) \supset C$

$\vee_4$) $\dfrac{\Gamma,A\to C \quad \Gamma,B\to C}{\Gamma,A\vee B\to C}$ $\qquad$ $\neg_1$) $A \to \neg\neg A$

$\neg_2$) $\neg\neg A \to A$ $\qquad$ $\neg_3$) $\to \neg(A \,\&\, \neg A)$ $\qquad$ $\neg_4$) $\to A \vee \neg A$

The postulates of J_2 are those of J_1 together with the following:

$\neg_5$) $\to (A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ $\qquad$ $\neg_6$) $\neg A, \neg B \to \neg(A \vee B)$.

The postulates of J_3 are those of J_2 together with the following:

$\neg_7$) $A, \neg B \to \neg(A \supset B)$ $\qquad$ $\neg_8$) $A, \neg B \to \neg(A \,\&\, B)$

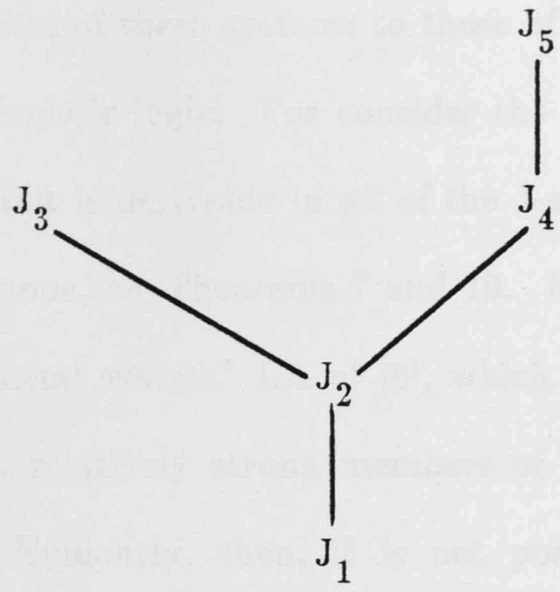
$\neg_9$) $\neg A, B \to \neg(A \,\&\, B)$ $\qquad$ $\neg_{10}$) $\neg A, \neg B \to \neg(A \,\&\, B)$.

The postulates of J_5 are those of J_2 together with the following:

$\to_7$) $\dfrac{\Delta,A\to B \quad \Delta\to\neg B}{\Delta\to\neg A}$.

Finally J_4 has all of the postulates of J_2, together with $\neg_{11}$), which is just $\to_7$) with Δ empty.

The relations of containment holding between these systems are as set out below, with weaker systems placed below stronger ones (see Theorems 11 and 21 of [4]).



2. THEOREMS AND SEQUENTS

Notably absent from the postulates of the J-systems is the rule of *modus ponens* for ⊃. Consequently, there are sequents $\Gamma \rightarrow A \supset B$ derivable in these systems such that Γ, $A \rightarrow B$ is not derivable. In particular, this holds for empty Γ: there are theorems $A \supset B$ of the J-systems such that $A \rightarrow B$ is not derivable.

The absence of *modus ponens* is meant to ensure that the systems are *paraconsistent*, i.e. that they can support inconsistent theories without collapse into triviality. Specifically, Arruda and da Costa construct on the basis of each J_i a set theory ZF_i in which the postulate of separation occurs without the restrictions which are placed on it in Zermelo-Fraenkel set theory in order to guard against paradox. The guard against paradox -- or more accurately, the guard against triviality in the presence of paradox -- is built into the theories ZF_i not at the properly set-theoretical but already at the propositional level, i.e. in the postulates of the J-systems themselves.

This strategy is at least partially successful, for even though the sequent $\rightarrow (A \,\&\, \neg A) \supset B$ is derivable in each $J_i (2 \leqslant i \leqslant 5)$, by Theorem 4 of [4], its paraconsistency-defeating correlate $A, \neg A \rightarrow B$ is derivable only in J_5, by Theorems 3, 5, 7, 10 and 18. Of course, it is precisely the absence of *modus ponens* which is responsible for the impossibility of deriving the latter from the former in the systems $J_i (2 \leqslant i \leqslant 4)$.

However, the absence of *modus ponens* from the J-systems also complicates the task of relating the positive parts of these systems to those of more familiar propositional logics such as classical and intuitionistic logic. For consider the sequents $\rightarrow (A \;\&\; (A \supset B)) \supset B$ and $A \;\&(A \supset B) \rightarrow B$. The first is derivable in all of the J-systems, by Theorem 2 of [1], while the second is derivable in none, by Theorems 7 and 19. However, both sequents are derivable in the "absolute propositional system" LA of [8], which is the weakest of Curry's hierarchy of sequent-based systems, relatively strong members of which are intuitionistic logic (LJ) and classical logic (LK). Evidently, then, it is not possible to demonstrate any coincidence between the positive sequents derivable in the J-systems and those derivable in the more familiar sequent-based systems; it is only with respect to positive theorems that such coincidence can be established.

3. POSITIVE PARTS

Definition 1. A formula of $\mathcal{L}$ is *positive* just in case it does not contain the connective $\neg$. The set of positive $\mathcal{L}$-formulas is denoted by $\mathcal{L}^+$.

Definition 2. A sequent of $\mathcal{K}$ is *positive* just in case all of the $\mathcal{L}$-formulas which occur in it are positive. The set of positive sequents is denoted by $\mathcal{K}^+$.

Definition 3. For each J_i, the *positive subsystem* of J_i, denoted J_i^+, is that system generated by precisely those instances of the postulates of J_i which do not involve the connective $\neg$.

Definition 4. J_i is a *conservative extension* of its positive subsystem J_i^+ just in case every positive sequent derivable in J_i is also derivable in J_i^+.

We note that any conservative extension result for a system J_i will provide useful information about the positive theorems of J_i. For if J_i is a conservative extension of J_i^+, then in particular, every positive sequent of the form $\rightarrow A$ which is derivable in J_i is also derivable in J_i^+, i.e. the positive theorems of J_i are precisely the theorems of J_i^+.

We begin our investigation of the positive parts of the J-systems by considering their positive subsystems.

4. POSITIVE SUBSYSTEMS

Theorem 1. All of the J-systems share a common positive subsystem, i.e. $J_1^+ = J_i^+$ ($1 \leqslant i \leqslant 5$).

Proof: This is evident from the fact that all of the postulates which are added to those of J_1 in the construction of the remaining J-systems involve negation explicitly; hence there are no positive instances of these postulates to be added to those of J_1^+ in the elaboration of J_2^+ to J_5^+.

Thus, our work in this section is reduced by the fact that we have only one positive subsystem to investigate. It is minimised by the fact that it is established in [7] precisely what the theorems of that subsystem are. We quote this result without proof.

Theorem 2. The theorems of J_1^+ are precisely those of positive intuitionistic logic.

5. THE POSITIVE THEOREMS OF J_1

Theorem 3. J_1 is a conservative extension of J_1^+

Proof: We need to show that every positive sequent derivable in J_1 has a derivation in J_1^+. To this end, we define the function p: $\mathcal{K} \cup \mathcal{L} \rightarrow \mathcal{K}^+ \cup \mathcal{L}^+$ as follows.

$p(A_1, \dots, A_n \rightarrow A_{n+1}) = p(A_1), \dots, p(A_n) \rightarrow p(A_{n+1})$ (where $n \geq 1$);

$p(\rightarrow A) \quad = \quad \rightarrow p(A)$;

$p(A \vee B) \quad = \quad p(A) \vee p(B)$;

$p(A \& B) \quad = \quad p(A) \& p(B)$;

$p(A \supset B) \quad = \quad p(A) \supset p(B)$;

$p(\neg\neg A) \quad = \quad p(A)$;

$p(A) \quad = \quad$ A, if $A \in \mathcal{L}^+$;
q ⊃ q (where q is a fixed propositional variable of $\mathcal{L}$) if A is of the form ¬B and B is not of the form ¬C.

Let $\Gamma \rightarrow$ A be any sequent derivable in J_1. We show by induction on the length n of derivation of $\Gamma \rightarrow$ A that p($\Gamma \rightarrow$ A) is derivable in J_1^+.

Base case ($n = 1$). In this case $\Gamma \rightarrow$ A is an axiom of J_1. For all such axioms except the $\neg$-postulates, it is trivial to verify that their p-translates are derivable in J_1^+ (indeed, they are J_1^+-postulates). For example, the p-translate of $\supset_2$) is just (p(A) $\supset$ p(B)) & (p(A) $\supset$ (p(B) $\supset$ p(C))) $\rightarrow$ p(A) $\supset$ p(C), which is a positive instance of $\supset_2$). The negation postulates are also fairly straightforward. Postulates $\neg_1$) and $\neg_2$) both translate as p(A) $\rightarrow$ p(A), which is just a positive instance of $\rightarrow_1$). $\neg_3$) becomes $\rightarrow$ q $\supset$ q whatever A is, which is derivable from the J_1^+-postulate q $\rightarrow$ q by one step of $\supset_1$). Finally, postulate $\neg_4$) translates as $\rightarrow$ p(A) $\vee$ p($\neg$A), but evidently one of the disjuncts must be q $\supset$ q, and so the whole disjunction has an easy derivation in J_1^+. This concludes the base case.

Inductive step ($n = k$ for some $k > 1$; inductive hypothesis: if $\Gamma \rightarrow$ A has a J_1-derivation of length less than k, p($\Gamma \rightarrow$ A) has a derivation in J_1^+). Again, this is straightforward: the sequent in question must have been derived by application of a rule. We consider the case of $\rightarrow_5$); the other rules can be treated similarly. In this case, the sequent with derivation of length k is Δ, $\Gamma \rightarrow$ A, and is derived from $\Delta \rightarrow$ C and C, $\Gamma \rightarrow$ A by $\rightarrow_5$). Let Δ and Γ be the sequences D_1, ..., D_m and B_1, ..., B_n respectively. Then p($\Delta \rightarrow$ C) and p(C, $\Gamma \rightarrow$ A) are the sequents p(D_1), ..., p(D_m) $\rightarrow$ p(C) and p(C), p(B_1), ..., p(B_n) $\rightarrow$ p(A) respectively. But on inductive hypothesis, these latter have derivations in J_1^+, and hence, by one step of $\rightarrow_5$), so does p(D_1), ..., p(D_m), p(B_1), ..., p(B_n) $\rightarrow$ p(A). But this is just p(Δ, $\Gamma \rightarrow$ A).

We have established that the p-translate of any sequent derivable in J_1 is derivable in J_1^+. To complete the proof of Theorem 3, it suffices simply to note that if $\Gamma \rightarrow$ A is already positive (i.e. a member of $\mathcal{K}^+$), then p($\Gamma \rightarrow$ A) is just $\Gamma \rightarrow$ A itself, and hence $\Gamma \rightarrow$ A is derivable in J_1^+ if derivable in J_1.

Corollary. The positive theorems of J_1 are precisely the theorems of J_1^+, which are precisely the theorems of positive intuitionistic logic.

6. THE POSITIVE THEOREMS OF J_2 TO J_5

Unfortunately, the conservative extension method used to determine the positive theorems of J_1 cannot be applied to the remaining J-systems.

Theorem 4. None of the systems $J_i (2 \leqslant i \leqslant 5)$ is a conservative extension of its positive subsystem J_i^+.

Proof: In theorem 4 of [4], it is noted that the intuitionistically underivable sequent $\rightarrow ((A \supset B) \supset A) \supset A$ is derivable in J_2 (and hence, also in J_3, J_4 and J_5). However, this cannot be derivable in $J_i^+ (2 \leqslant i \leqslant 5)$ by Theorems 1 and 2 of Section 4.

To determine the positive theorems of J_2 to J_5, then, other methods will have to be employed. We note that there is a result in [4] which implicitly determines the positive theorems of J_3. It is Theorem 6, which states that all theorems of classical (propositional) logic are theorems of J_3. It is easy to verify that all of the postulates of all of the J-systems are classically derivable in a sequent-based formulation such as LK of [8]; hence, the converse also holds. The theorems of J_3, then, are precisely the theorems of classical logic; and in particular, the positive theorems of J_3 are precisely the positive theorems of classical logic. These in turn are precisely the theorems of positive classical logic.

Again, the above argument cannot be employed to determine the positive theorems of J_2, J_4 and J_5, for unlike J_3, these systems do not contain all of the theorems of classical logic, by Theorem 19 of [4]. Nonetheless, it can be established by means of a general but fairly complicated proof that the positive theorems of these systems are, like those of J_3, precisely the theorems of positive classical logic. Because this proof depends on numerous technical preliminaries, we preface its presentation with an informal sketch of the strategy to be deployed.

The object of the proof is to show that the positive theorems of $J_i (2 \leqslant i \leqslant 5)$ are precisely those of positive classical logic. This is achieved not directly, but through the use of a transformation procedure which has two component functions. The first component, the u-function, simply reorders and/or reassociates disjunctions, thereby transforming formulas of

$\mathcal{L}^+$ into so-called *u-normal form*. Some initial lemmas are used to establish that every formula of $\mathcal{L}^+$ is provably equivalent, both in the J-systems and classically, to its u-normal form. The job of the u-function is simply to ensure that application of the second component, the t-function, terminates in the desired way. The joint application of the two functions transforms formulas of $\mathcal{L}^+$ into so-called *t-normal form*. Again, several lemmas are used to establish that every formula of $\mathcal{L}^+$ is provably equivalent, both in $J_i(2 \leqslant i \leqslant 5)$ and classically, to its t-normal form. The final stage of the proof consists in showing that the t-normal positive theorems of $J_i(2 \leqslant i \leqslant 5)$ correspond precisely to the t-normal theorems of positive classical logic.

We now turn to the proof in detail, interjecting explanatory remarks where appropriate. We begin by adducing some of the more general properties of the J-systems.

As observed earlier, all of the postulates of the J-systems are classically derivable. We therefore have the following.

Lemma 1. Every sequent drivable in $J_i(1 \leqslant i \leqslant 5)$ is also classically derivable. In particular, every theorem of $J_i(1 \leqslant i \leqslant 5)$ is also a classical theorem, and any formulas provably equivalent in $J_i(1 \leqslant i \leqslant 5)$ are also provably equivalent classically.

Lemma 2. The systems $J_i(1 \leqslant i \leqslant 5)$ enjoy SE^+, the property of intersubstitutivity of provable equivalents in negation-free contexts, i.e. where $\Gamma \rightarrow A$ is a derivable sequent and B is a subformula of some members(s) of Γ and/or of A not occuring within the scope of any $\neg$, then the sequent obtained by substituting any formula C which is provably equivalent to B for any or all occurences of B in $\Gamma \rightarrow A$ is also derivable.

Proof: It is easy to verify that in $J_i(1 \leqslant i \leqslant 5)$, the following are derivable rules:

$$\frac{B \leftrightarrow C}{B^*D \leftrightarrow C^*D} \qquad \frac{B \leftrightarrow C}{D^*B \leftrightarrow D^*C},$$

where * stands for any of the connectives &, $\vee$ or $\supset$. Given these rules, it is straightforward to construct an inductive argument showing that the J-systems enjoy SE^+.

Lemma 3. In $J_i(1 \leqslant i \leqslant 5)$, the following are derivable:

$(B \vee C) \leftrightarrow (C \vee B)$ $\qquad$ $(B \vee (C \vee D)) \leftrightarrow ((B \vee C) \vee D)$.

Proof: By Theorem 2 of [4].

Definition 5. A formula of $\mathcal{L}^+$ is *disjunctive* if it is of the form $B \vee C$, *conjunctive* if it is of the form $B \& C$, and *implicative* if it is of the form $B \supset C$.

Lemma 4. Any finite disjunction of non-disjunctive formulas of $\mathcal{L}^+$ can, without loss of equivalence in $J_i (1 \leqslant i \leqslant 5)$, be reordered and/or reassociated.

Proof: This follows by a straightforward inductive argument from Lemmas 2 and 3.

We now introduce the first component of the transformation procedure.

Definition 6. The function u: $\mathcal{L}^+ \rightarrow \mathcal{L}^+$ reorders and/or reassociates (as necessary) any finite disjunction A of non-disjunctive formulas $A_1,...,A_n$ so that $u(A) = A_1 \vee (A_2 \vee ... \vee (A_{n-1} \vee A_n)...)$ where, for distinct A_i and A_j, A_i occurs to the left of A_j in u(A) just in case A_i occurs before A_j in some given total ordering of the non-disjunctive formulas of $\mathcal{L}^+$ satisfying the following conditions:

(i) every implicative formula occurs before every conjunctive formula in the ordering; and

(ii) every conjunctive formula occurs before every propositional variable in the ordering.

For completeness, where A is a non-disjunctive formula of $\mathcal{L}^+$, u(A) is defined to be A. For any formula A of $\mathcal{L}^+$, u(A) is referred to as the *u-normal* form of A.

Lemma 5. Let A be a formula of $\mathcal{L}^+$. Then both in $J_i (1 \leqslant i \leqslant 5)$ and classically, A is provably equivalent to its u-normal form u(A).

Proof: If A is disjunctive, then we have $A \leftrightarrow u(A)$ in $J_i (1 \leqslant i \leqslant 5)$ by Lemma 4. If A is non-disjunctive, then $u(A) = A$, and we have $A \leftrightarrow u(A)$ by postulate $\rightarrow_1$). In general, then, A is provably equivalent to u(A) in $J_i (1 \leqslant i \leqslant 5)$, and hence also classically by Lemma 1.

The following technical observation will facilitate later proofs.

Lemma 6. Let A be a formula of $\mathcal{L}^+$. Then A and u(A) have the same number of connectives.

Proof: This is obvious from the definition of u(A), since the number of connectives in a formula is preserved under the operations of reordering and reassociation of disjuncts.

We now introduce the second component of the transformation procedure.

Definition 7. The function t: $\mathcal{L}^+ \rightarrow \mathcal{L}^+$ is defined as follows:

t(A) = A if A is not of the form B & C or (B ⊃ C) ∨ D or (B & C) ∨ D;

t(B & C) = t(u(B)) & t(u(C));

t((B ⊃ C) ∨ D) = B ⊃ (C ∨ D);

t((B & C) ∨ D) = t(u(B ∨ D)) & t(u(C ∨ D)).

For any formula of $\mathcal{L}^+$, the formula t(u(A)) is referred to as the *t-normal* form of A.

We can describe in less formal terms the way in which a positive formula A is transformed by the joint application of the t- and u-functions. In the interesting case, in which A is a positive formula containing all of the connectives &, ∨ and ⊃, the transformation "drives inwards" any occurrences of ∨ not within the scope of any occurrence of ⊃, through occurrences of & if necessary, and eventually through some occurrence of ⊃ if possible. The contribution of the u-function is to locate implicative disjuncts, or failing this, conjunctive disjuncts, and push them to the left so that the t-function does in fact drive occurrences of ∨ inwards in the desired direction. In general, the endproduct t(u(A)) has no occurrences of & within the scope of any occurrence of ∨ but not within the scope of some occurrence of ⊃. In the case in which A is a theorem of positive classical logic, it will emerge that t(u(A)) is either implicative or a finite conjunction of implicative formulas.

To establish the relationship between positive formulas and their t-normal forms, several lemmas are again required.

Lemma 7. In $J_i (1 \leqslant i \leqslant 5)$, the following is derivable:

(B & C) ∨ D ↔ (B ∨ D) & (C ∨ D).

Proof: By Theorem 2 of [4].

The following is the first lemma which requires more substantial proof.

Lemma 8. In $J_i (2 \leqslant i \leqslant 5)$, the following is derivable:

$$(B \supset C) \vee D \leftrightarrow B \supset (C \vee D).$$

Proof: The left-to-right half is in fact derivable in J_1 by fairly straightforward moves. Only the right-to-left half requires the extra postulates of J_2. We sketch a derivation below, borrowing from results in [4] where convenient.

An instance of one of the sequents derivable in J_1 by Theorem 2 of [4] is

$$(B \supset (C \vee D)) \,\&\, ((C \vee D) \supset C) \rightarrow B \supset C,$$

which by $\&_1)$ and $\rightarrow_5)$, yields

$$B \supset (C \vee D), (C \vee D) \supset C \rightarrow B \supset C \ldots (1)$$

As an instance of $\vee_1)$ we have

$$B \supset C \rightarrow (B \supset C) \vee D,$$

and this together with (1) yields by $\rightarrow_5)$

$$B \supset (C \vee D), (C \vee D) \supset C \rightarrow (B \supset C) \vee D \ldots (2).$$

Easily derivable is

$$B \supset (C \vee D), D \rightarrow (B \supset C) \vee D,$$

which together with (2) gives by $\vee_4)$

$$B \supset (C \vee D), ((C \vee D) \supset C) \vee D \rightarrow (B \supset C) \vee D.$$

Permuting antecedents by $\rightarrow_4)$ yields

$$((C \vee D) \supset C) \vee D, B \supset (C \vee D) \rightarrow (B \supset C) \vee D \ldots \ (3),$$

which will evidently give us the desired right-to-left half if we can derive

$$\rightarrow ((C \vee D) \supset C) \vee D \ldots (4).$$

This is the part of the derivation which requires the extra J_2-postulates.

By Theorem 4 of [4], $\neg A \rightarrow A \supset B$ is derivable in J_2, though not in J_1. We use the instance

$$\neg (C \vee D) \rightarrow (C \vee D) \supset C,$$

which by $\vee_1)$ and $\rightarrow_5)$ quickly gives

$$\neg(C \vee D) \rightarrow ((C \vee D) \supset C) \vee D \ldots (5).$$

Easily derivable is

$$C \rightarrow (C \vee D) \supset C.$$

which by $\vee_1)$ and $\rightarrow_5)$ gives

$$C \rightarrow ((C \vee D) \supset C) \vee D \ldots (6).$$

An instance of $\vee_2)$ is

$$D \rightarrow ((C \vee D) \supset C) \vee D,$$

which can be combined with (6) to give by $\vee_4)$

$$(C \vee D) \rightarrow ((C \vee D) \supset C) \vee D.$$

Again using $\vee_4)$, this latter sequent together with (5) gives

$$(C \vee D) \vee \neg(C \vee D) \rightarrow ((C \vee D) \supset C) \vee D.$$

But the antecedent is just an instance of $\neg_4)$, so by $\rightarrow_5)$ we get the desired (4), yielding the right-to-left half of the equivalence of Lemma 8 as indicated.

We turn now to the relationship between positive formulas and their t-normal forms.

Lemma 9. Let A be a formula of $\mathcal{L}^+$. Then both in $J_i (2 \leqslant i \leqslant 5)$ and classically, A is provably equivalent to its t-normal form t(u(A)).

Proof: It suffices to show that A is provably equivalent to t(u(A)) in $J_i (2 \leqslant i \leqslant 5)$; from this it follows that they are also provably equivalent classically by Lemma 1. We proceed by induction on the number n of connectives in A.

Base case ($n = 0$). In this case, A is a propositional variable, so t(u(A)) = t(A) = A.

Inductive step ($n = k$ for some $k > 0$; inductive hypothesis: if B is a formula of $\mathcal{L}^+$ with fewer than k connectives, then B $\leftrightarrow$ t(u(B)) in J_i ($2 \leqslant i \leqslant 5$)). Since $n > 0$, A must be implicative, conjunctive or disjunctive. We consider each of these possible forms in turn.

(i) Suppose A = B ⊃ C. Then t(u(A)) = t(u(B ⊃ C)) = t(B ⊃ C) = B ⊃ C = A.

(ii) Suppose A = B & C. Then t(u(A)) = t(u(B & C)) = t(B & C) = t(u(B)) & t(u(C)). But the numbers of connectives in B and in C are smaller than the number of connectives in B & C; hence, on inductive hypothesis, we have B $\leftrightarrow$ t(u(B)) and C $\leftrightarrow$ t(u(C)). By virtue of the rules exhibited in the proof of Lemma 2, we then also have B & C $\leftrightarrow$ t(u(B)) & t(u(C)), i.e. A $\leftrightarrow$ t(u(A)).

(iii) In this case, A is disjunctive, and hence so is u(A). If u(A) is of the form (B ⊃ C) ∨ D, then we have t(u(A)) = t((B ⊃ C) ∨ D) = B ⊃ (C ∨ D). But then we have t(u(A)) $\leftrightarrow$ u(A) by Lemma 8, and u(A) $\leftrightarrow$ A by Lemma 5, whence t(u(A)) $\leftrightarrow$ A. If u(A) is of the form (B & C) ∨ D, then t(u(A)) = t((B & C) ∨ D) = t(u(B ∨ D)) & t(u(C ∨ D)). But the number of connectives in B ∨ D is less than the number of connectives in (B & C) ∨ D which is u(A) in the case we are considering and which therefore has the same number as A by Lemma 6. Similarly for C ∨ D. Hence, on inductive hypothesis, we have t(u(B ∨ D)) $\leftrightarrow$ B ∨ D and t(u(C ∨ D)) $\leftrightarrow$ C ∨ D, which together yield t(u(B ∨ D)) & t(u(C ∨ D)) $\leftrightarrow$ (B ∨ D) & (C ∨ D) by virtue of the rules in the proof of Lemma 2. But by Lemma 7, we have (B ∨ D) & (C ∨ D) $\leftrightarrow$ (B & C) ∨ D, i.e. t(u(A)) $\leftrightarrow$ u(A). Again by Lemma 5, we have u(A) $\leftrightarrow$ A, and thus t(u(A)) $\leftrightarrow$ A. Finally, if u(A) is disjunctive but not of the form (B ⊃ C) ∨ D or of the form (B & C) ∨ D, then t(u(A)) = u(A), and hence t(u(A)) $\leftrightarrow$ A again by Lemma 5.

This concludes the proof of Lemma 9.

To establish the main result of this section, it remains to prove that t(u(A)) is a theorem of positive classical logic if and only if also a positive theorem of J_i($2 \leqslant i \leqslant 5$); and it is precisely the form of t(u(A)) which will facilitate such proof.

<u>Lemma 10</u>. Let A be a theorem of positive classical logic. Then t(u(A)) is either implicative or a finite conjunction of implicative formulas.

<u>Proof</u>: We begin with the observation that no single propositional variable, nor any disjunction of propositional variables, is a theorem of positive classical logic. Thus, if A is

such a theorem, it must be conjunctive, implicative or a finite disjunction of formulas, at least one of which is conjunctive or implicative. It follows by Definition 6 that u(A) must itself be conjunctive, implicative or of the form $A_1 \vee (A_2 \vee \ldots \vee (A_{m-1} \vee A_m)\ldots)$ $(m \geqslant 2)$, where A_1 is either conjunctive or implicative.

We proceed by induction on the number n of connectives in A.

Base case $(n = 1)$. The smallest number of connectives that a theorem A of positive classical logic can have is 1, in which case $A = p \supset p$ for some propositional variable p, and $t(u(A)) = t(u(p \supset p)) = t(p \supset p) = p \supset p$, which is implicative.

Inductive step $(n = k > 1)$. We consider in turn the three possible forms of u(A) mentioned in the preceding observation.

(i) u(A) is of the form $B \supset C$. In this case, $t(u(A)) = t(B \supset C) = B \supset C$, which is implicative.

(ii) u(A) is of the form B & C. In this case $u(A) = A = B \,\&\, C$, and since A is a theorem of positive classical logic, so are B and C, but with fewer connectives than A. Hence, on inductive hypothesis, each of t(u(B)) and t(u(C)) is either implicative or a finite conjunction of implicative formulas. It follows that t(u(B)) & t(u(C)) is also such a conjunction, but $t(u(A)) = t(B \,\&\, C) = t(u(B)) \,\&\, t(u(C))$.

(iii) u(A) is of the form $A_1 \vee (A_2 \vee \ldots \vee (A_{m-1} \vee A_m)\ldots)$ $(m \geqslant 2)$, where A_1 is of the form B & C or $B \supset C$. In the first case, $t(u(A)) = t((B \,\&\, C) \vee D) = t(u(B \vee D)) \,\&\, t(u(C \vee D))$, where D stands for $A_2 \vee \ldots \vee A_m$. Again, each of $B \vee D$ and $C \vee D$ has fewer connectives than u(A) which has the same number as A by Lemma 6; hence, on inductive hypothesis, each of $t(u(B \vee D))$ and $t(u(C \vee D))$ is either implicative or a finite conjunction of implicative formulas, and therefore $t(u(B \vee D)) \,\&\, t(u(C \vee D))$ is also such a conjunction. In the second case, $t(u(A)) = t((B \supset C) \vee D) = B \supset (C \vee D)$, which is implicative.

So ends the proof of Lemma 10; its significance will now become apparent.

Lemma 11. If A is a theorem of positive classical logic, then $D \supset A$ is a theorem of J_i $(2 \leqslant i \leqslant 5)$ for any $D \in \mathcal{L}$.

Proof: (The proof that follows is analogous to the proof of the Lemma of [7], which relates positive intuitionistic theorems and the theorems of J_1 in a similar way).

Positive classical logic can be axiomatised by adding Peirce's law, $((A \supset B) \supset A) \supset A$, to a suitable axiomatisation of positive intuitionistic logic. For present purposes, we follow [7] in taking positive intuitionistic logic to be formulated with *modus ponens* as the sole rule of inference.

The proof proceeds by induction on the length n of the derivation of A in positive classical logic.

Base case ($n = 1$). In this case, A is an axiom of positive classical logic. By Theorem 2, all of the axioms of positive intuitionistic logic are theorems of $J_1{}^+$, and hence also theorems of J_i ($2 \leqslant i \leqslant 5$). And Peirce's Law is a theorem of J_2 and hence of J_i ($2 \leqslant i \leqslant 5$) by Theorem 4 of [4]. Thus, if A is an axiom of positive classical logic, then A is a theorem of J_i ($2 \leqslant i \leqslant 5$). It follows straightforwardly that $D \supset A$ is also a theorem of J_i ($2 \leqslant i \leqslant 5$) for any $D \in \mathcal{L}$, by $\rightarrow_2$) and $\supset_1$).

Inductive step ($n = k > 1$). Since positive classical logic is here taken to be formulated with *modus ponens* as sole rule of inference, A must be derived from premisses B and $B \supset A$ by an application of this rule. On inductive hypothesis, we have $\rightarrow D \supset B$ and $\rightarrow D \supset (B \supset A)$ in J_i ($2 \leqslant i \leqslant 5$). By $\&_1$) and $\rightarrow_5$), we hence have $\rightarrow (D \supset B) \ \& \ (D \supset (B \supset A))$, but as an instance of $\supset_2$) we have $(D \supset B) \ \& \ (D \supset (B \supset A)) \rightarrow D \supset A$, so by $\rightarrow_5$) we get $\rightarrow D \supset A$.

Lemma 11 leads quickly to the following powerful result.

Lemma 12. $B \supset C$ is a theorem of positive classical logic iff $B \supset C$ is a positive theorem of J_i ($2 \leqslant i \leqslant 5$).

Proof. Assume that $B \supset C$ is a theorem of positive classical logic. Then by Lemma 11, we have $\rightarrow D \supset (B \supset C)$ in J_i ($2 \leqslant i \leqslant 5$) for any $D \in \mathcal{L}$. In particular, we have $\rightarrow B \supset (B \supset C)$. Since we also have $\rightarrow B \supset B$ by $\rightarrow_1$) and $\supset_1$), we get $\rightarrow (B \supset B) \ \& \ (B \supset (B \supset C))$ by $\&_1$), which by $\supset_2$) and $\rightarrow_5$) quickly yields $\rightarrow B \supset C$.

The converse follows by Lemma 1.

Finally, we present the main result of this section.

Theorem 4. A is a theorem of positive classical logic iff A is a positive theorem of J_i

$(2 \leqslant i \leqslant 5)$.

Proof: Since it has been established that, for $A \in \mathcal{L}^+$, A and t(u(A)) are provably equivalent both classically and in J_i $(2 \leqslant i \leqslant 5)$, we need only prove that t(u(A)) is a theorem of positive classical logic iff t(u(A)) is a theorem of J_i $(2 \leqslant i \leqslant 5)$. This we show by induction on the number n of connectives in t(u(A)).

Base case $(n = 1)$. In this case, t(u(A)) is implicative as in Lemma 10. By Lemma 12, it follows that t(u(A)) is a theorem of positive classical logic iff t(u(A)) is a positive theorem of J_i $(2 \leqslant i \leqslant 5)$.

Inductive step $(n = k > 1)$. Assume that t(u(A)) is a theorem of positive classical logic. By Lemma 10, t(u(A)) is either implicative or a conjunction of implicative formulas.

If t(u(A)) is implicative, then it is a theorem of J_i $(2 \leqslant i \leqslant 5)$ by Lemma 12. If t(u(A)) is a conjunction of implicative formulas, then each conjunct is a theorem of positive classical logic. Moreover, inspection of Definition 7 reveals that (i) each conjunct must itself be of the form $t(u(A_i))$ for some $A_i \in \mathcal{L}^+$, and (ii) each conjunct must contain fewer connectives than t(u(A)). Hence on inductive hypothesis, each conjunct is a positive theorem of J_i $(2 \leqslant i \leqslant 5)$, and hence so is their conjunction, i.e. t(u(A)). Thus, if t(u(A)) is a theorem of positive classical logic, it is also a positive theorem of J_i $(2 \leqslant i \leqslant 5)$. The converse follows by Lemma 1.

We have shown, then, that the positive theorems of each J_i $(2 \leqslant i \leqslant 5)$ correspond precisely to the theorems of positive classical logic. This concludes our investigation of the positive parts of the J-systems of Arruda and da Costa.

Chapter Two: Paraconsistency and the J-systems

1. CONDITIONS FOR PARACONSISTENCY

Da Costa and Alves state in [10] that, in general, systems of paraconsistent logic must satisfy the following conditions:

(I) from two contradictory formulas A and ¬A, it must not be possible in general to deduce an arbitrary formula B; and

(II) such systems should contain most of the schemata and deduction rules of classical logic that do not interfere with (I).

Although further requirements have occasionally been added for particular paraconsistent systems (see, for example, [9]), these two conditions have consistently operated as the primary guiding principles for da Costa and his collaborators in the consutruction of their paraconsistent logics.

However, this approach is not beyond controversy, for while (I) is universally accepted as a necessary condition for paraconsistent systems, (II) is less generally endorsed (see, for example, [6], [7] and [15]. Indeed, dissent over (II) is sufficiently widespread that adherence to this condition has come to be regarded as a distinguishing feature of the Brazilian approach to paraconsistency (see [15]). Without rehearsing the arguments of others in detail, a number of grounds for dissatisfaction with (II) are worth mentioning.

Firstly, da Costa himself notes that this condition is "vague" ([9], p.498). Certainly, it is not clear exactly what measure of containment of classical schemata and rules constitutes satisfaction of the condition. But more significant is the fact that it is not *determinative*. For it is conceivable - indeed, this will emerge in later sections - that two different schemata or rules could be singly but not jointly incorporated into a paraconsistent system without compromising condition (I). In such a case, (II) suggests that one of the pair ought to be incorporated, or at least considered for incorporation, but no means of deciding between them is suggested. This indicates that, for (II) to be coherently applied in the construction of paraconsistent systems, it must be coupled with some account of the relative merits of competing candidate schemata and rules. One suggestion for such an account will be advanced in Section 5.

A second objection to condition (II) is that it needlessly places on paraconsistent logics the burden of ensuring that inconsistent theories based on these logics sufficiently resemble their classical competitors to be considered as serious rival theories. For example, it is plausible that inconsistent set theories (such as those constructed on the basis of the J-systems in [4]) should sufficiently approximate classically based set theories for them to be considered to be genuine rival formal models of the same informal intuitions concerning sets. But it does not follow that this overall similarity should extend also to the components of the respective theories, and in particular, to their logical bases.

Finally, even if the general tenor of (II) is accepted, a case still exists for slightly modifying it. For it is noteworthy that, as a matter of practice, da Costa and his collaborators consider worth mentioning not only substantial containment of classical logic, but also containment of intuitionistic logic. (For example, Theorem 1 of [2] states that C_ω, the weakest of the C-systems, contains the theorems of the intuitionistic positive calculus). This practice has led some authors to read into (II) the proviso that substantial containment of intuitionistic logic is a next-best alternative, or even a no worse alternative, to substantial containment of classical logic (see, for example, [6] and [19]). This suggests that condition (II) be modified as follows: (II′) paraconsistent systems should contain most of the schemata and deduction rules of classical or intuitionistic logic that do not interfere with (I).

2. THE J-SYSTEMS AND THE PARACONSISTENCY CONDITIONS

Contrary to their intended purpose as logical bases for inconsistent set theories, it turned out that the J-systems do not all satisfy condition (I). J_5 explicitly fails in that the sequent A, ¬A → B is derivable in this system (by Theorem 18 of [4]). Of the remainder, J_2 to J_4 satisfy (I) in letter but not in spirit, for while A, ¬A → B is not derivable in these systems (by Theorems 5, 7 and 10 of [4]), the sequent A, ¬A → B ⊃ C is (by Theorem 1 of [5]). As a result, although the inconsistent theories based on J_2 to J_4 do not collapse totally (as does the theory based on J_5), the collapse is near enough to complete for these systems to be effectively disqualified from further consideration as paraconsistent logics (see Theorem 4 of [5]).

This leaves only J_1, the weakest of the J-systems. As far as condition (I) is concerned, J_1 fares better than its stronger siblings: A, ¬A → B is not derivable in J_1 (by Theorem 3 of [4]), nor is A, ¬A → B ⊃ C (by Theorem 2 of [5]). It is rather because of its apparent failure to satisfy (II) that Arruda and da Costa take their leave not only of the stronger J-systems, but also of J_1 (see [5], p.186).

Certainly, J_1 fails to contain some classically derivable sequents which would not obviously interfere with its satisfaction of (I) if they were incorporated. For example, it does not contain even all of the *theorems* of *positive* classical logic, since →((A ⊃ B) ⊃ A) ⊃ A is not derivable in J_1 (by Theorem 3 of [4]). But this constitutes only *prima facie* evidence that (II) is not satisfied; the case is not settled until it is demonstrated that significant portions of classical logic not contained by J_1 can be added without compromising (I). And in any case, (II) is itself open to question, as noted in Section 1.

Of particular interest in this regard is the suggested replacement of (II) by the weaker (II′). For when it comes to containing substantial parts of intuitionistic rather than classical logic, J_1 fares somewhat better. Theorem 1 of [7] shows that the theorems of J_1^+, the subsystem generated by only the negation-free postulates of J_1, are precisely the theorems of positive intuitionistic logic; and the conservative extension result (Theorem 3) of Chapter 1 shows that these are precisely the negation-free theorems of J_1.

It is where negation is involved that J_1 diverges more spectacularly from intuitionistic logic, possibly to a greater extent than condition (II′) envisages. Some of the intuitionistically derivable sequents involving negation that are not derivable in J_1 are listed in Theorem 3 of [4], but one which is particularly striking, and which would least obviously interfere with the satisfaction of (I) were it incorporated into J_1, is cited in Theorem 5 of that paper. The (infinite) matrices supporting Theorem 5 are needlessly complex for present purposes, so we restate the result in question in greater generality, and supply simpler matrices for its proof.

Theorem 1. In J_1, the following sequents are not derivable:

→ ¬((A & ¬A) & B);

→ ¬X, where X is any reassociation and/or permutation of (A & ¬A) & B.

Proof: The following matrices validate the postulates of J_1, but invalidate these sequents when A is assigned the value 0 and B is assigned the value 1.

→/⊃	0	1	2	¬
*0	0	1	2	0
*1	0	0	2	2
2	0	0	0	1

∧/&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

∨	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(The values 0 and 1 are designated; and the value of →A is taken in this and all subsequent proofs involving matrices for the J-systems to be the same as that of A).

Even more striking than Theorem 1 is the following.

Theorem 2. In J_1, the following sequents are not derivable:

→ ¬(¬A & A);

→ ¬((A & ¬A) & (A & ¬A));

→ ¬((A & ¬A) ∨ (A & ¬A)).

Proof: The following matrices validate the postulates of J_1, but invalidate these sequents when A is assigned the value 0.

→/⊃	0	1	2	3	4	¬
*0	0	0	0	0	4	1
*1	0	0	0	0	4	2
*2	0	0	0	0	4	1
*3	0	0	0	0	4	4
4	0	0	0	0	0	3

∧/&	0	1	2	3	4
0	3	1	3	3	4
1	3	3	2	3	4
2	3	2	3	2	4
3	3	3	3	3	4
4	4	4	4	4	4

∨	0	1	2	3	4
0	3	3	3	3	3
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	4

(Only the value 4 is not designated).

The above results provide a number of reasons for dissatisfaction with J_1. Firstly, it is counterintuitive that the sequents of Theorems 1 and 2 are not derivable in J_1. This is not because of their underivability *per se*, but because the sequent of which they would ordinarily be taken to be merely syntactic variants, → ¬(A & ¬A), is explicitly incorporated as a postulate. It may be possible to provide a plausible motivation for such fine discrimination, but no such motivation is to be discerned in the discussion accompanying the construction of the J-systems in [4], nor in the sequel [5]. In the absence of any illuminating motivation, such fine discrimination is simply anomalous.

Secondly, it may be that the absence of the (intuitionistically derivable) sequents of Theorems 1 and 2 from the stock of derivable sequents of J_1 constitutes an infringement of (II′). For the following result indicates that these sequents can be added to J_1 without endangering the satisfaction of (I).

<u>Theorem 3</u>. In the system formed by adding the sequents of Theorems 1 and 2 to the postulates of J_1, the sequents A, ¬A → B and A, ¬A → B ⊃ C are not derivable.

<u>Proof</u>: The following matrices validate the postulates of J_1 and the sequents of Theorems 1 and 2, but invalidate A, ¬A → B when A assigned the value 1 and B is assigned the value 3, and A, ¬A → B ⊃ C when A is assigned the value 1, B is assigned the value 0 and C is assigned the value 3.

→/⊃	0	1	2	3	¬
*0	0	0	2	3	3
*1	0	0	2	3	2
2	0	0	0	3	1
3	0	0	0	0	0

∧/&	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

∨	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

(The values 0 and 1 are designated).

Even if the sequents of Theorems 1 and 2 were added to J_1, however, there is no guarantee that further deficiencies could not be exhibited. In particular, it is likely that other syntactic variants of → ¬(A & ¬A) would still prove to be underivable. Obviously, a more systematic strategy is needed; and this in turn requires that we establish which more general property is shown by Theorems 1 and 2 to be lacking from J_1.

3. J_1 AND THE PROPERTY OF INTERSUBSTITUTIVITY OF PROVABLE EQUIVALENTS

The underivability of the sequents of Theorem 2 in J_1 is symptomatic of a more general deficiency, namely, that this system lacks SE, the property of intersubstitutivity of provable equivalents. The most natural way of defining provable equivalence in the J-systems is as follows: two formulas C and D are *provably equivalent* just in case the pair of sequents C → D and D → C (abbreviated C ↔ D) is derivable. SE, then, is the property that, if Γ → A is a derivable sequent, B a subformula of some member(s) of Γ and/or of A, and C a formula which is provably equivalent to B, then the sequent obtained by substituting C for some or all occurrence(s) of B in Γ → A is also derivable. Equivalent in this context is the property that, if B is a subformula of A, and B and C are provable equivalent, then so are A and the formula obtained by substituting C for some or all occurrence(s) of B in A.

Theorem 4. J_1 does not enjoy SE.

Proof: Easily derived in J_1 are the sequents A & ¬A → ¬A & A and ¬A & A → A & ¬A. If J_1 enjoyed SE, then → ¬(¬A & A), the result of substituting ¬A & A for A & ¬A in postulate $¬_3$), would also be derivable. But this sequent is not derivable, by Theorem 2; hence, J_1 does not enjoy SE.

The proof of Theorem 4 illustrates how unsystematically the connective $\neg$ behaves in J_1; two formulas are provably equivalent and yet their putative negations are not (indeed, one is a postulate while the other is underivable). Of course, it may be possible to interpret $\neg$ in such a way that this result is acceptable, but there is nothing in [4] and [5] to indicate that $\neg$ is to be interpreted even as a special kind of negation, let alone as something other than negation. Again, in the absence of any such illumination, the behaviour of this connective in J_1 is simply anomalous. (In [15], a similar view is expressed about the behaviour of $\neg$ in the C-systems of da Costa, which similarly fail to enjoy SE; and in [13], it is argued that no reasonable conditional or biconditional can be expressed in the C-systems, again because of their failure to enjoy SE).

In general, the absence of SE makes it difficult to provide a natural and uniform interpretation of the connectives of a logic and the relations thereby definable. Technically, this tends to be reflected in the complexity of formal semantical and algebraic perspectives (again, see [13] and [15], and also [14] and [19]).

The desired general strategy for removing the deficiencies of J_1 exhibited in (at least) Theorem 2, then, is to attempt to secure the property SE. In [19], the parallel problem of securing SE for the C-systems is addressed, and the respective addition of two rules is proposed. The appropriate versions of these rules in the present context are:

RC $\dfrac{C \to D}{\neg D \to \neg C}$ and EC $\dfrac{C \leftrightarrow D}{\neg D \to \neg C}$.

By Lemma 2 of Chapter 1, J_1 enjoyes SE^+, the property of intersubstitutivity of provable equivalents in negation-free contexts. It follows that, for any extension of J_1 (in the same vocabulary), the admissibility of RC or of EC is sufficient to guarantee SE in full, and the admissibility of EC is also evidently necessary. (A rule is *admissible* in a sequent-based system just in case the system formed by adding that rule is a conservative extension of, i.e. has the same stock of derivable sequents as, the original system. Thus, every derivable rule is admissible, but the converse does not generally hold).

We now investigate the result of respectively adding RC and EC to J_1. That neither rule is admissible in J_1 follows from Theorem 4; it is to be expected, therefore, that their addition will strengthen J_1 in an interesting way.

The strength imparted by the addition of RC proves to be somewhat excessive.

<u>Theorem 5</u>. In J_1 + RC, the sequent A, ¬A → B is derivable.

<u>Proof</u>: Application of $\rightarrow_2$) to postulate $\neg_3$) of J_1 yields ¬B → ¬(A & ¬A), from which ¬¬(A & ¬A) → ¬¬B follows by RC. Together with A & ¬A → ¬¬ (A & ¬A), which is an instance of $\neg_1$), this yields A & ¬A → ¬¬B by $\rightarrow_5$). From this, A & ¬A → B follows by $\neg_2$) and $\rightarrow_5$), and from this A, ¬A → B is easily derived using $\&_1$) and $\rightarrow_5$).

Unfortunately, the addition of the ostensibly weaker EC has precisely the same result.

<u>Theorem 6</u>. In J_1 + EC, the sequent A, ¬A → B is derivable.

<u>Proof</u>: Application of $\rightarrow_2$) to postulate $\neg_3$) yields ¬(B & ¬B) → ¬(A & ¬A). A parallel derivation yields the converse, ¬(A & ¬A) → ¬(B & ¬B). From these, EC delivers ¬¬(A & ¬A) → ¬¬(B & ¬B), which quickly reduces to A & ¬A → B & ¬B using $\neg_1$), $\neg_2$) and $\rightarrow_5$). An instance of $\&_2$) is B & ¬B → B, whence $\rightarrow_5$) yields A & ¬A → B. As in the proof of Theorem 5, this suffices to deliver A, ¬A → B.

As with J_5, the derivability of A, ¬A → B in J_1 + RC and J_1 + EC constitutes an explicit violation of condition (I), effectively disqualifying these systems from contention as paraconsistent logics. In fact, these three systems are equivalent.

<u>Theorem 7</u>. J_1 + EC = J_1 + RC = J_5.

<u>Note</u>: In this and subsequent proofs, we will make use of the following rules and sequent,

which are easily shown to be derivable in J_1:

Transitivity (of $\supset$): $$\frac{\Gamma \rightarrow C \supset D \qquad \Delta \rightarrow D \supset E}{\Gamma, \Delta \rightarrow C \supset E}$$

Permutation of antecedents: $$\frac{\rightarrow C \supset (D \supset E)}{\rightarrow D \supset (C \supset E)}.$$

Restricted *modus ponens* (sequent): $C, C \supset (D \supset E) \rightarrow D \supset E$

Restricted *modus ponens* (rule): $$\frac{\Gamma \rightarrow C \supset (D \supset E)}{\Gamma, C \rightarrow D \supset E}$$

The sequent form of restricted *modus ponens* is shown to be derivable in J_1 in [7] (p.45); from this, $\rightarrow_4$) and $\rightarrow_5$) deliver the rule form.

Proof: That J_1 + EC is a subsystem of J_1 + RC is evident, since the derivability of RC ensures the derivability of the weaker EC. Moreover, J_1 + RC is a subsystem of J_5, since J_1 is a subsystem of J_5 to begin with, and RC is easily derived in J_5 as follows. Assume C $\rightarrow$ D. From this, $\neg$D, C $\rightarrow$ D follows by $\rightarrow_2$). An instance of $\rightarrow_1$) is $\neg$D $\rightarrow$ $\neg$D. But the last two sequents yield $\neg$D $\rightarrow$ $\neg$C by postulate $\rightarrow_7$) of J_5.

To complete the proof of Theorem 7, it suffices to show that J_5 is a subsystem of J_1 + EC, i.e. that those postulates which are added to J_1 in the construction of J_5 are derivable in J_1 + EC. These are $\rightarrow_7$), $\neg_6$) and $\neg_5$).

Postulate $\rightarrow_7$) is the rule $$\frac{\Delta, A \rightarrow B \qquad \Delta \rightarrow \neg B}{\Delta \rightarrow \neg A}.$$

This is derived in J_1 + EC as follows. Assume Δ, A $\rightarrow$ B and Δ $\rightarrow$ $\neg$B. Applications of $\rightarrow_2$) and $\rightarrow_4$) to the second sequent give Δ, A $\rightarrow$ $\neg$B, which can be combined with the first to yield Δ, A $\rightarrow$ B & $\neg$B (see Theorem 2 of [4]). But B & $\neg$B $\rightarrow$ $\neg$A is derivable in J_1 + EC by Theorem 6; hence by $\rightarrow_5$), we get Δ, A $\rightarrow$ $\neg$A. Easily derived by $\rightarrow_1$) and $\rightarrow_2$) is Δ, $\neg$A $\rightarrow$ $\neg$A; hence $\vee_4$) yields Δ, A $\vee$ $\neg$A $\rightarrow$ $\neg$A. Applications of $\rightarrow_4$) transform this into A $\vee$ $\neg$A,

$\Delta \rightarrow \neg A$, from which $\Delta \rightarrow \neg A$ follows by $\neg_4$) and $\rightarrow_5$). Thus, $\rightarrow_7$) is derivable in J_1 + EC.

Postulate $\neg_6$) is the sequent $\neg A, \neg B \rightarrow \neg(A \vee B)$. This is derived in J_1 + EC as follows. Firstly, we have $\neg A, \neg B, A \vee B \rightarrow (\neg A \& \neg B) \& (A \vee B)$ by $\&_1$) and $\rightarrow_5$). By distribution (see Theorem 2 of [4]), we have $(\neg A \& \neg B) \& (A \vee B) \rightarrow ((\neg A \& \neg B) \& A) \vee ((\neg A \& \neg B) \& B)$, whence $\neg A, \neg B, A \vee B \rightarrow ((\neg A \& \neg B) \& A) \vee ((\neg A \& \neg B) \& B)$ by $\rightarrow_5$). But $(\neg A \& \neg B) \& A \rightarrow A \& \neg A$ is easily derived, and $(\neg A \& \neg B) \& B \rightarrow A \& \neg A$ is derivable in J_1 + EC by virtue of Theorem 6; hence, we have $((\neg A \& \neg B) \& A) \vee ((\neg A \& \neg B) \& B) \rightarrow A \& \neg A$ by $\vee_4$), and therefore $\neg A, \neg B, A \vee B \rightarrow A \& \neg A$ by $\rightarrow_5$). From $\neg_3$), $\neg A, \neg B \rightarrow \neg(A \& \neg A)$ follows by $\rightarrow_2$). From the last two sequents, the above-derived $\rightarrow_7$) delivers the desired $\neg A, \neg B \rightarrow \neg(A \vee B)$.

Postulate $\neg_5$) is the sequent $\rightarrow (A \supset B) \supset ((A \supset \neg B) \supset \neg A)$. This is derived in J_1 + EC as follows. Firstly, we have $A \supset B, A \supset \neg B \rightarrow A \supset (B \& \neg B)$ by $\&_1$) and $\&_4$). But by Theorem 6, $B \& \neg B \rightarrow \neg A$ is derivable in J_1 + EC, and hence so is $\rightarrow (B \& \neg B) \supset \neg A$ by $\supset_1$). By transitivity, we therefore have $A \supset B, A \supset \neg B \rightarrow A \supset \neg A$. Easily derived using $\vee_3$) is $A \supset \neg A \rightarrow (A \vee \neg A) \supset \neg A$, so by $\rightarrow_5$), we get $A \supset B, A \supset \neg B \rightarrow (A \vee \neg A) \supset \neg A$. Applying $\supset_1$) and permuting antecedents gives $\rightarrow (A \vee \neg A) \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$, from which $A \vee \rightarrow (A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ follows by the restricted *modus ponens* rule. The desired $\neg_5$) then follows by $\neg_4$) and $\rightarrow_5$).

This concludes the proof of Theorem 7.

We note that the proofs of Theorems 6 and 7 do not rely upon the actual derivability of EC in J_1 + EC; it suffices merely that this rule is admissible. But the admissibility of EC in any extension of J_1 is a necessary condition for SE. We can therefore state the following more general result.

<u>Theorem 8</u>. There is no extension of J_1 which enjoys SE but which is weaker than J_5.

We conclude that the deficiencies of J_1 exhibited in Theorems 1 and 2 cannot be remedied by extending this system so as to secure SE without thereby rendering zt too strong to satisfy paraconsistency condition (I). Two alternative strategies present themselves: (i) to extend J_1 so as to secure not SE but some weaker version of this property which would nonetheless at least mitigate these deficiencies; and (ii) to explore methods of variation other than extension. These strategies will be investigated in the next two sections.

4. J_1 AND WEAKER VERSIONS OF THE PROPERTY OF INTERSUBSTITUTIVITY OF PROVABLE EQUIVALENTS

Two weaker versions of SE suggest themselves, each a restricted variant of one of the equivalent statements of SE presented in Section 3. Firstly, let SE′ be the property that, if A is a theorem (i.e. →A is derivable), B is a subformula of A, and C is provably equivalent to B, then the formula obtained by substituting C for some or all occurrence(s) of B in A is also a theorem. SE′ is the property of intersubstitutivity of provable equivalents in theorems, and its obtaining would at least avoid the deficiencies exhibited in Theorem 2. A second variant is SE″, which is the property that, if B is a subformula of A, and A(C) is the result of substituting a formula C which is provably equivalent to B for some or all occurrence(s) of B in A, then $\rightarrow A \equiv A(C)$ is a derivable sequent. ($C \equiv D$ abbreviates the formula $(C \supset D)$ & $(D \supset C)$).

<u>Theorem 9</u>. J_1 does not enjoy SE′.

<u>Proof</u>: As for Theorem 4.

<u>Theorem 10</u>. J_1 does not enjoy SE″.

<u>Proof</u>: As in the proof of Theorem 4, A & $\neg$A and $\neg$A & A are provably equivalent in J_1. If J_1 enjoyed SE″, then $\rightarrow \neg(A \,\&\, \neg A) \equiv \neg(\neg A \,\&\, A)$ would be derivable. But the matrices of Theorem 2, which validate the postulates of J_1, invalidate this sequent when A is assigned the value 0; hence, J_1 does not enjoy SE″.

<u>Theorem 11</u>. Every extension of J_1 which enjoys SE′ also enjoys SE″, but not conversely.

<u>Proof</u>: To establish the first part, it suffices to note that $A \equiv A$ is a theorem of J_1 for any formula A. If SE′ holds, then any occurrence of a subformula B of A can be replaced by a provably equivalent formula C; hence, $A \equiv A(C)$ is also a theorem, and SE″ holds.

To establish the second part, it suffices to exhibit an extension of J_1 which enjoys SE″ but not SE′. Consider the system $J_1{}'$ constructed by adding the postulate $\rightarrow A \equiv B$ to J_1. Trivially, $J_1{}'$ enjoys SE″. However, the matrices in the proof of Theorem 2, with the modification that $A \supset B$ is assigned the value 0 always, validate the postulates of $J_1{}'$ but continue to invalidate the sequents of Theorem 2; hence $J_1{}'$ does not enjoy SE′.

It follows from Theorem 11 that any addition to J_1 which is necessary to secure SE″ is necessary also for SE′. We therefore proceed to establish which additions are necessary for the weaker property.

It is easy to verify that the admissibility of the following variant of EC in any extension of J_1 is a necessary condition for SE″:

$$\text{EC}\supset \qquad \frac{C \leftrightarrow D}{\rightarrow \neg D \supset \neg C}.$$

Moreover, a straightforward inductive argument can be used to establish that the admissibility of EC ⊃ is also sufficient for SE″, since the following rules are already derivable in J_1:

$$\frac{C \leftrightarrow D}{\rightarrow B^* C \equiv B^* D} \qquad \frac{C \leftrightarrow D}{\rightarrow C^* B \equiv C^* D},$$

where * represents any of the connectives &, ∨ or ⊃.

The admissibility of any of the following rules is evidently also sufficient:

$$\text{RC}\supset \qquad \frac{C \rightarrow D}{\rightarrow \neg D \supset \neg C}$$

$$\supset\text{EC}\supset \qquad \frac{\rightarrow C \equiv D}{\rightarrow \neg D \supset \neg C}$$

$$\supset\text{RC}\supset \qquad \frac{\rightarrow C \supset D}{\rightarrow \neg D \supset \neg C}.$$

We now consider the systems formed by respectively adding these rules to J_1. In fact, these systems are all equivalent.

Theorem 12. $J_1 + EC \supset = J_1 + RC \supset = J_1 + \supset EC \supset = J_1 + \supset RC \supset$.

Proof: Evidently, $J_1 + EC \supset$ is a subsystem of both $J_1 + RC \supset$ and $J_1 + \supset EC \supset$, and both of these are subsystems of $J_1 + \supset RC \supset$. We need only show, therefore, that $J_1 + \supset RC \supset$ is a subsystem of $J_1 + EC \supset$, i.e. that $\supset RC \supset$ is derivable in $J_1 + EC \supset$.

By $\neg_3$) and $\rightarrow_2$), we have $\neg(C \,\&\, \neg C) \leftrightarrow \neg(D \,\&\, \neg D)$ in J_1. Applying $EC \supset$ yields $\rightarrow \neg\neg(D \,\&\, \neg D) \supset \neg\neg(C \,\&\, \neg C)$, which, using $\neg_1$), $\neg_2$), $\supset_1$) and transitivity, is easily reduced to $\rightarrow (D \,\&\, \neg D) \supset (C \,\&\, \neg C)$. By $\&_3$), $\supset_1$) and transitivity, this further reduces to $\rightarrow (D \,\&\, \neg D) \supset \neg C$. Applying $\supset_1$) to an instance of $\&_1$) gives $D \rightarrow \neg D \supset (D \,\&\, \neg D)$, which together with the preceding sequent, yields by transitivity, $D \rightarrow \neg D \supset \neg C$. Applying $\supset_1$) gives $\rightarrow D \supset (\neg D \supset \neg C)$.

An instance of $\vee_3$) is $(C \supset \neg C) \,\&\, (\neg C \supset \neg C) \rightarrow (C \vee \neg C) \supset \neg C$, which is transformed into $C \supset \neg C, \neg C \supset \neg C \rightarrow (C \vee \neg C) \supset \neg C$ by $\&_1$) and $\rightarrow_5$), and further into $\neg C \supset \neg C \rightarrow (C \supset \neg C) \supset ((C \vee \neg C) \supset \neg C)$ by $\rightarrow_4$) and $\supset_1$). But $\rightarrow \neg C \supset \neg C$ is derivable by $\rightarrow_1$) and $\supset_1$), so $\rightarrow_5$) delivers $\rightarrow (C \supset \neg C) \supset ((C \vee \neg C) \supset \neg C)$. Permutation of antecedents transforms this into $\rightarrow (C \vee \neg C) \supset ((C \supset \neg C) \supset \neg C)$, from which the restricted *modus ponens* rule gives $C \vee \neg C \rightarrow (C \supset \neg C) \supset \neg C$. From this, $\rightarrow (C \supset \neg C) \supset \neg C$ follows by $\neg_4$) and $\rightarrow_5$).

We now consider $\supset RC \supset$. Assume $\rightarrow C \supset D$. Together with the sequent $\rightarrow D \supset (\neg D \supset \neg C)$ derived above, this yields by transitivity, $\rightarrow C \supset (\neg D \supset \neg C)$. Permuting antecedents gives $\rightarrow \neg D \supset (C \supset \neg C)$, which, together with the last sequent of the preceding paragraph, yields by transitivity, $\rightarrow \neg D \supset \neg C$. Thus, $\supset RC \supset$ is derivable in $J_1 + EC \supset$.

This concludes the proof of Theorem 12.

Interestingly, $J_1 + EC \supset$ also has an equivalent formulation purely in terms of the postulates of the J-systems.

Theorem 13. Let $J_{1.5}$ be the system formed by adding to J_1 postulate $\neg_5$) of J_2. Then J_1 +

EC $\supset$ = $J_{1.5}$.

<u>Proof</u>: To show that J_1 + EC $\supset$ is a subsystem of $J_{1.5}$, it suffices to derive EC $\supset$ in the latter system. This is derived as follows. An instance of postulate $\neg_5$) is $\rightarrow$ (C $\supset$ D) $\supset$ ((C $\supset$ $\neg$D) $\supset$ $\neg$C). Using the restricted *modus ponens* rule, this is transformed into C $\supset$ D $\rightarrow$ (C $\supset$ $\neg$D) $\supset$ $\neg$C. Easily derived using $\rightarrow_1$), $\neg_2$) and $\supset_1$) is $\rightarrow$ $\neg$D $\supset$ (C $\supset$ $\neg$D), whence transitivity yields C $\supset$ D $\rightarrow$ $\neg$D $\supset$ $\neg$C. But $\rightarrow$ C $\supset$ D follows from the premise of EC $\supset$ (or of any of the other three rules); whence $\rightarrow_5$) yields the conclusion, $\rightarrow$ $\neg$D $\supset$ $\neg$C. Thus, EC $\supset$ is derivable in $J_{1.5}$.

To show conversely that $J_{1.5}$ is a subsystem of J_1 + EC $\supset$, it suffices to derive $\neg_5$) in J_1 + EC $\supset$. An instance of &$_4$) is (A $\supset$ B) & (A $\supset$ $\neg$B) $\rightarrow$ A $\supset$ (B & $\neg$B), which by &$_1$) and $\rightarrow_5$) is transformed into A $\supset$ B, A $\supset$ $\neg$B $\rightarrow$ A $\supset$ (B & $\neg$B). As in the proof of Theorem 12, the sequent $\rightarrow$ (B & $\neg$B) $\supset$ $\neg$A is derivable in J_1 + EC $\supset$; hence transitivity yields A $\supset$ B, A $\supset$ $\neg$B $\rightarrow$ A $\supset$ $\neg$A. Applying $\supset_1$) gives A $\supset$ B $\rightarrow$ (A $\supset$ $\neg$B) $\supset$ (A $\supset$ $\neg$A). Again as in the proof of Theorem 12, $\rightarrow$ (A $\supset$ $\neg$A) $\supset$ $\neg$A is derivable in J_1 + EC $\supset$, so transitivity delivers A $\supset$ B $\rightarrow$ (A $\supset$ $\neg$B) $\supset$ $\neg$A, from which the desired $\neg_5$) follows by $\supset_1$). Thus, $\neg_5$) is derivable in J_1 + EC.

This concludes the proof of Theorem 13.

$J_{1.5}$ lies between J_1 and J_2, but it is equivalent to neither.

<u>Theorem 14</u>. $J_1 \neq J_{1.5}$.

<u>Proof</u>: That postulate $\neg_5$) of $J_{1.5}$ is not derivable in J_1 is shown in Theorem 3 of [4]. However, the result also follows from the fact that $J_{1.5}$ enjoys SE″, while J_1 does not, by Theorem 10 above.

<u>Theorem 15</u>. $J_{1.5} \neq J_2$.

<u>Proof</u>: The matrices in the proof of Theorem 2, as modified in the proof of Theorem 11, validate the postulates of $J_{1.5}$ but invalidate postulate $\neg_6$) of J_2 when A and B are both assigned the value 0.

Unfortunately, even though J_1 + EC ⊃ (= $J_{1.5}$) is weaker than J_2, it similarly fails to substantively satisfy paraconsistency condition (I).

Theorem 16. In J_1 + EC ⊃, the sequent A, ¬A → B ⊃ C is derivable.

Proof: As in the proof of Theorem 12, →(A & ¬A) ⊃ (C & ¬C) is derivable in J_1 + EC ⊃. By $\&_2$), $\supset_1$) and transitivity, this yields →(A & ¬A) ⊃ C. But → C ⊃(B ⊃ C) is easily derived using $\rightarrow_1$), $\rightarrow_2$) and $\supset_1$); whence transitivity again yields →(A & ¬A) ⊃ (B ⊃ C). Using the restricted version of *modus ponens* cited in the proof of Theorem 7, this is transformed into A & ¬A → B ⊃ C, from which A, ¬A → B ⊃ C follows by $\&_1$) and $\rightarrow_5$).

We note that the proofs of Theorems 13 and 16 (and those parts of the proof of Theorem 12 which they presuppose) do not rely upon the actual derivability of EC ⊃ in J_1 + EC ⊃; it suffices merely that this rule is admissible. But the admissibility of EC ⊃ in any extension of J_1 is a necessary condition for SE″. We can therefore state the following more general result.

Theorem 17. There is no extension of J_1 which enjoys SE″ but which is weaker than $J_{1.5}$.

From Theorems 11 and 17, the following is also immediate.

Theorem 18. There is no extension of J_1 which enjoys SE′ but which is weaker than $J_{1.5}$.

We conclude that even the weaker versions of SE considered in this section cannot be secured by extending J_1 without substantially compromising its satisfaction of (I). We turn instead to methods of variation other than extension.

5. OTHER METHODS OF VARIATION

The obvious alternative to extension is subtraction. In particular, it may be that systems obtained by deleting some of the postulates of J_1, i.e. subsystems of J_1, could be shown to either enjoy SE naturally or be amenable to extension so as to secure this property without infringing condition (I).

A likely candidate for deletion is postulate $\neg_3$). The motivation for extending J_1 in the first place was to remove the anomalies exhibited in Theorems 1 and 2; but these results are anomalous only because J_1 incorporates $\neg_3$) - the anomalies might just as well be removed by deleting this postulate as by adding its variants. The subsystem so obtained would still enjoy SE^+, since the rules required to guarantee this property would not be affected by the deletion of a negation postulate, and it would also evidently satisfy (I), since the sequents A, $\neg$A $\rightarrow$ B and A, $\neg$A $\rightarrow$ B $\supset$ C would still not be derivable. If, in addition, the admissibility of RC or EC in this subsystem could be established, then it would enjoy SE naturally; and even if not, it may be that these rules could be added without compromising its satisfaction of (I).

Similar considerations apply also to the removal of any of the other negation postulates of J_1. Of course, an obvious constraint on this strategy of subtraction is condition (II). At first glance, it would appear that the deletion of any of the postulates of J_1 would increase the degree to which (II) is not satisfied. However, there are several considerations which indicate that the matter is not so straightforward

Firstly, there is the suggested replacement of (II) by (II′). It may be that the deletion from J_1 of the intuitionistically underivable postulates $\neg_2$) and $\neg_4$) would leave unaffected its stock of intuitionistically derivable rules and sequents, in which case the satisfaction of (II′) would not be diminished. Of course, the subsystem so obtained would fail to enjoy SE, since it would still incorporate $\neg_3$) but not its variants listed in Theorem 2; but again it may be that the addition of RC or EC in order to secure SE would not entail the undesirable consequences of the corresponding additions to J_1.

Even without resorting to the intuitionistic escape clause of (II′), however, there are good reasons for not being too concerned by the deletion of some of the negation postulates of J_1, especially if this allows the addition of such a rule as RC without harm to the satisfaction of (I). This is the case foreshadowed in Section 1; we are assuming that, for some subsystem of J_1, the addition of a missing negation postulate results in J_1, which satisfies (I), and the addition of RC results in some other system which also satisfies (I), but the addition of both the postulate and RC results in J_5, which does not satisfy (I). Condition (II) suggests that one or the other ought to be added, but it does not suggest which.

An argument which weighs heavily in favour of adding RC in this case is that this addition would guarantee the systematic behaviour of $\neg$, while the addition of the competing postulate would only result in the unsystematic behaviour documented in Theorems 1 and 2. This suggests that RC is more general, or expresses a more fundamental property of negation, than any of the negation postulates of J_1. Certainly, this rule is incorporated in a very broad range of logics, including all of the negation systems of [8] (among them, classical and intuitionistic logic, and Johansson's "minimal calculus"), and also all of the main relevant logics (see [18]). On the other hand, postulates $\neg_2$) and $\neg_4$) are not so universally incorporated, which suggests that they express properties of a somewhat special (and strong) type of negation; and, perhaps more interestingly, (the appropriate versions of) $\neg_1$) and $\neg_3$) are notably absent from the C-systems of da Costa.

The widespread inclusion of RC is hardly surprising, for this rule expresses little more than that negation reverses the order of strength among propositions: the weaker a proposition, the stronger is its rejection or denial; and the stronger the proposition, the weaker its denial. Indeed, it is difficult to see how a connective not conforming to this rule can be interpreted as negation at all, rather than as some more enigmatic functor. These considerations apply also to the weaker EC; indeed, this rule expresses the even less arresting precept that a logic which identifies two propositions should not distinguish between their denials. Again, it is difficult to see how a connective which does not conform to this rule can be interpreted as anything other than a very selective type of negation, if as negation at all.

The considerations expressed in this section indicate that an investigation into the subsystems of J_1, augmented by RC or EC if required, is well warranted. Accordingly, a detailed investigation is undertaken in the next two chapters.

Chapter Three: Subsystems of J_1

1.THE SYSTEMS $J_i (0 \leqslant i \leqslant 1)$

The system J_1 is the weakest of the J-systems introduced by Arruda and da Costa in [4]. As shown in Chapter 2, however, J_1 is still too strong to be amenable to extension so as to secure the property of intersubstitutivity of provable equivalents without forfeiting its satisfaction of the main condition for paraconsistent logics. It is of interest, therefore, to define and investigate still weaker J-systems, i.e. subsystems of J_1.

A natural collection of subsystems is obtained by varying only the negation postulates of J_1. Specifically, we define the base system J_0 to be that subsystem obtained by omitting postulates $\neg_1$) to $\neg_4$) from J_1. (Although J_0 has no postulates specifically governing negation, it is understood to be defined over the full vocabulary $\mathcal{L}$ (including $\neg$) of J_1; thus, it differs from the positive subsystem J_1^+ of J_1, which has the same postulates as J_0, but which is defined over just the positive vocabulary $\mathcal{L}^+$ (see Chapter 1)). The collection $J_i (0 \leqslant i \leqslant 1)$ is defined to include any system formed by adding to J_0 some subset of the postulates $\neg_1$) to $\neg_4$).

The systems $J_i (0 \leqslant i \leqslant 1)$ can be grouped into three distinct branches. Firstly, the *intuitionistic* branch comprises J_0 and those systems formed by adding to J_0 either or both of the intuitionistically derivable negation postulates $\neg_1$) and $\neg_3$). Since all of the postulates of J_0 are themselves intuitionistically derivable, these systems have the property that every sequent derivable in them is also intuitionistically derivable. Opposite to the intuitionistic branch is the *dualintuitionistic* branch, which comprises those systems formed by adding to J_0 either or both of the negation postulates $\neg_2$) and $\neg_4$), which are "dual" to the intuitionistic postulates $\neg_1$) and $\neg_3$). Finally, the *intermediate* branch comprises those systems which contain both intuitionistic and dualintuitionistic negation postulates.

The following systems will prove to be particularly interesting:

$J_0 \quad = \quad J_1 - \neg_1) - \neg_2) - \neg_3 - \neg_4);$

$J_{0\cdot 1} \quad = \quad J_0 + \neg_1);$

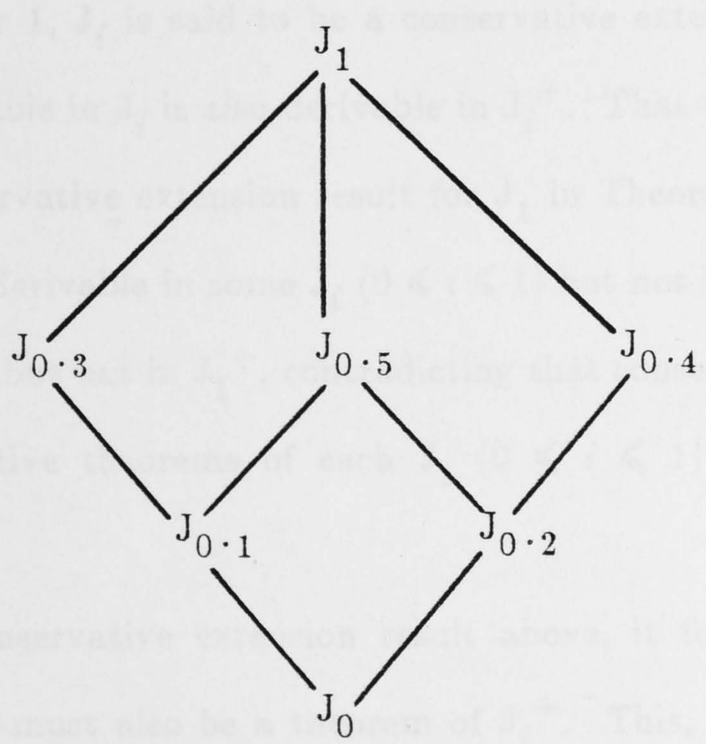
$J_{0\cdot 2} \quad = \quad J_0 + \neg_2);$

$J_{0\cdot 3} \quad = \quad J_{0\cdot 1} + \neg_3);$

$J_{0\cdot 4} \quad = \quad J_{0\cdot 2} + \neg_4);$

$J_{0\cdot 5} \quad = \quad J_0 + \neg_1) + \neg_2).$

J_0, $J_{0\cdot 1}$ and $J_{0\cdot 3}$ are intuitionistic; $J_{0\cdot 2}$ and $J_{0\cdot 4}$ are dualintuitionistic; and $J_{0\cdot 5}$ and, of course, J_1 are intermediate. The relations of containment holding between these systems are set out below, with weaker systems placed below stronger ones. (It will be shown in Section 8 that these systems are in fact distinct).



2. POSITIVE PARTS

We begin our investigation of the deductive strength of the systems $J_i (0 \leqslant i \leqslant 1)$ by considering their *positive parts*. Terminology is as in Chapter 1.

Theorem 1. For each J_i $(0 \leqslant i \leqslant 1)$, $J_i^+ = J_0^+$.

Proof: As in [7] and Chapter 1, J_i^+ is the *positive subsystem* of J_i, i.e. the system generated by only the negation-free instances of the postulates of J_i. Evidently, J_0 is a subsystem of each J_i $(0 \leqslant i \leqslant 1)$, hence J_0^+ is a subsystem of each J_i^+ $(0 \leqslant i \leqslant 1)$. But there can be no negation-free instances of postulates of any J_i $(0 \leqslant i \leqslant 1)$ in addition to those of J_0^+, for each system in this family is constructed by adding to J_0 only postulates which involve negation explicitly. Therefore, for each J_i $(0 \leqslant i \leqslant 1)$, $J_i^+ = J_0^+$.

Corollary. The theorems of each J_i^+ $(0 \leqslant i \leqslant 1)$ are exactly those of positive intuitionistic logic. (As in Chapter 1, A is a theorem of a J-system just in case the sequent $\rightarrow$A is derivable in that system).

Proof: This follows directly from the above result and Theorem 1 of [7], in which it is shown that the theorems of J_1^+ are exactly those of positive intuitionistic logic.

Theorem 2. Each J_i $(0 \leqslant i \leqslant 1)$ is a *conservative extension* of its positive subsystem, J_i^+.

Proof: As in Chapter 1, J_i is said to be a conservative extension of J_i^+ just in case every positive sequent derivable in J_i is also derivable in J_i^+. That this is so for each J_i $(0 \leqslant i \leqslant 1)$ follows from the conservative extension result for J_1 in Theorem 3 of Chapter 1. For if any positive sequent were derivable in some J_i $(0 \leqslant i \leqslant 1)$ but not in J_i^+, then that sequent would also be derivable in J_1 but not in J_1^+, contradicting that conservative extension result.

Corollary. The positive theorems of each J_i $(0 \leqslant i \leqslant 1)$ are exactly those of positive intuitionistic logic.

Proof: From the conservative extension result above, it follows in particular that every positive theorem of J_i must also be a theorem of J_i^+. This, together with the Corollary to Theorem 1, suffices to establish the result.

The above results go some way towards illuminating the deductive strength of the systems J_i $(0 \leqslant i \leqslant 1)$. However, it would be much more informative if *all* of the theorems of each of these systems, not just the negation-free theorems, could be shown to coincide with those of some axiomatic system. Indeed, this can be achieved for most of the J-systems under consideration. The remainder of the chapter will be devoted mainly to this task.

3. AXIOMATIC SYSTEMS

We provide a list of postulates from which various axiomatic systems can be assembled:

(1) $A \supset (B \supset A)$

(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

(3) $\dfrac{A \qquad A \supset B}{B}$

(4) $(A \,\&\, B) \supset A$

(5) $(A \,\&\, B) \supset B$

(6) $A \supset (B \supset (A \,\&\, B))$

(7) $A \supset (A \vee B)$

(8) $B \supset (A \vee B)$

(9) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

(10) $A \vee \neg A$

(11) $\neg\neg A \supset A$

(12) $A \supset \neg\neg A$

(13) $\neg(A \,\&\, \neg A)$.

(The inclusion of (3) as a postulate is in conformity with the axiomatics for the C-systems provided in [9]).

Postulates (1) to (11) form C_ω, the weakest of the C-systems. If (10) and (11) are replaced by their "duals" (12) and (13), the resulting system is an intuitionistic variant of C_ω, which is henceforth referred to as IC_ω. (This system is related to the intuitionistic variants of the C-systems constructed in [3] and [6]). Similarly, NC_ω is defined to be the system constituted by (1) and (9) plus the double-negation postulates (11) and (12). Finally, OC_ω is defined to be the system constituted by just (1) to (9). (Of course, (1) to (9) also axiomatise positive intuitionistic logic, but like J_0, OC_ω is understood to be defined over the full vocabulary $\mathcal{L}$ rather than just $\mathcal{L}^+$).

4. GENTZEN-STYLE SYSTEMS

For each of the axiomatic systems defined in Section 3, we define a corresponding Gentzen-style system. Terminology is as in Gentzen's [12].

LOC_ω is a (singular in the succedent) system which has the following components:

(i) initial sequents of the form A→A (where A is any formula of $\mathcal{L}$);

(ii) structural inference figures: Thinning, Contraction, Interchange (all in the antecedent), and Cut;

(iii) operational inference figures: &-IA, &-IS, ∨-IA, ∨-IS, ⊃-IA and ⊃-IS (all singular in the succedent).

We extend LOC_ω by adding one or both of the following operational inference figures:

$$\neg\neg\text{ - IA: } \frac{A, \Gamma \rightarrow B}{\neg\neg A, \Gamma \rightarrow B} \qquad \neg\neg\text{ - IS: } \frac{\Gamma \rightarrow B}{\Gamma \rightarrow \neg\neg B}.$$

The systems LOC_ω + ¬¬ - IA and LOC_ω + ¬¬ - IS will not be given special names, but LOC_ω + ¬¬ - IA + ¬¬ - IS will be called LNC_ω.

These systems may be further extended by adding either or both of the following as initial sequents:

(i) those of the form → A ∨ ¬A (EM);

(ii) those of the form → ¬(A & ¬A) (NC).

The system $LOC_\omega + \neg\neg$ - IA + EM will be called LC_ω, while the system $LOC_\omega + \neg\neg$ - IS + NC will be called LIC_ω. (We note that the addition of EM in this context is equivalent to the addition of Curry's rule Nx (see [8]):

$$\frac{A, \Gamma \to B \quad \neg A, \Gamma \to B}{\Gamma \to B}\ ).$$

5. EQUIVALENCE OF AXIOMATIC AND GENTZEN-STYLE SYSTEMS

Theorem 3. The systems LOC_ω and OC_ω are equivalent in the following sense: if $A_1, \ldots, A_n \to B$ (respectively, $\to B$) is a derivable sequent in LOC_ω, then $(A_1 \& \ldots \& A_n) \supset B$ (respectively, B) is a theorem of OC_ω, and (ii) if B is a theorem of OC_ω then $\to B$ is a derivable sequent in LOC_ω. Similarly equivalent are the following pairs of systems:

$LOC_\omega + \neg\neg$ - IA	and	OC_ω + (11);
$LOC_\omega + \neg\neg$ - IS	and	OC_ω + (12);
LNC_ω	and	NC_ω ;
LIC_ω	and	IC_ω ;
LC_ω	and	C_ω ;
LNC_ω + NC + EM	and	NC_ω + (10) + (13).

Proof: The proof is by induction on length of derivation. We illustrate with LC_ω and C_ω.

To prove (i), we establish initially that the formulas $A \supset A$ and $A \vee \neg A$ corresponding to the initial sequents of LC_ω are theorems of C_ω. Then we consider the inference figures. If, for example, the sequent $\neg\neg A, C_1, \ldots, C_n \to B$ is derived in LC_ω by $\neg\neg$ - IA from $A, C_1, \ldots, C_n \to B$, then on inductive hypothesis $(A \& C_1 \& \ldots \& C_n) \supset B$ is a theorem of C_ω, from which $(\neg\neg A \& C_1 \& \ldots \& C_n) \supset B$ is derivable with the assistance of (11).

To prove (ii), we establish initially that the axioms of C_ω are all derivable in LC_ω. For example, (1) corresponds to $\to A \supset (B \supset A)$, which is derived from the initial sequent $A \to A$ by Thinning and two applications of $\supset$ - IS. Then we consider the case in which a formula B is derived by (3) in C_ω from formulas A and A ⊃ B. On inductive hypothesis, → A and → A ⊃ B are derivable in LC_ω, and hence so is → A & (A ⊃ B). But A &(A ⊃ B) → B is easily derived in LC_ω, whence → B follows by Cut.

Such inductive proofs can be constructed for all of the pairs of systems listed in Theorem 3.

6. CUT-ELIMINABILITY FOR THE GENTZEN-STYLE SYSTEMS

We now prove that the inference figure Cut is eliminable from most of the Gentzen-style systems mentioned in Theorem 3. By this is meant that any sequent which is derivable in such a system also has a derivation in which Cut does not figure. The value of such proofs lies in the fact that they enable straightforward demonstrations of specific deductive properties which might otherwise be difficult to establish. For the systems presently under consideration, it will be shown that each has properties from among those listed below.

CP (Conjunction Property). A system is said to have this property just in case B & C is a theorem only if both B and C are also theorems.

DP (Disjunction property). A system is said to have this property just in case B ∨ C is a theorem only if at least one of B and C is also a theorem.

NP1 (Negation Property 1). A system is said to have this property just in case no formula of the form ¬B is a theorem.

NP2 (Negation Property 2). A system is said to have this property just in case a formula ¬B is a theorem only if B is itself of the form ¬C where C is a theorem.

NP3 (Negation Property 3). A system is said to have this property just in case a formula ¬B is a theorem only if B is either itself of the form ¬C where C is a theorem, or of the form C & ¬C.

<u>General comment</u>: In what follows, we will not present entire Cut-elimination proofs for the systems under consideration, but rather indicate only how Gentzen's Cut-elimination proof for intuitionistic logic LJ in [12] needs to be modified in each case. The sections of Gentzen's proof which are of particular interest are the following.

3.111. In this section, it is supposed that the left-hand upper sequent of the mix at the end of a derivation is an initial sequent.

3.112. In this section, it is supposed that the right-hand upper sequent of the mix is an initial sequent.

3.113. In this section, it is supposed that neither the left-hand nor the right-hand upper sequent of the mix is an initial sequent; both are therefore lower sequents of inference figures.

Gentzen observes in **3.113** that the following holds generally of inference figures of LJ: if a formula occurs in the antecedent (succedent) of the lower sequent of an inference figure, it is either a principal formula or the $\mathcal{D}$ of a Thinning, or else it also occurs in the antecedent (succedent) in at least one upper sequent of the inference figure. The special case in which the mix formula $\mathcal{M}$ occurs in at least one upper sequent of the inference figures immediately above the mix is dealt with by Gentzen in section **3.12**; the case in which $\mathcal{M}$ is introduced by Thinning is dealt with in section **3.113.2**; and the sole remaining case, in which $\mathcal{M}$ occurs both in the succedent of the left-hand upper sequent and in the antecedent of the right-hand upper sequent of the mix as the principal formula of one of the operational inference figures, is dealt with in section **3.113.3**.

Turning to the Gentzen-style systems listed in Theorem 3, we note that Gentzen's observation in section **3.113** holds also of the inference figures ¬¬-IA and ¬¬-IS. There are therefore no cases except those considered in sections **3.12**, **3.113.2** and **3.113.3** which can arise as a result of the presence of these additional figures. Moreover, Gentzen's treatment of the operational inference figures with one upper sequent in section **3.12** (in particular, in **3.121.22** and **3.122**) applies without modification to both of the figures ¬¬-IA and ¬¬-IS; and the treatment of Thinning in section **3.113.2** also applies without modification to the systems under consideration. So the only section in which the presence of ¬¬-IA and ¬¬-IS can make an interesting difference is **3.113.3** - in particular, **3.13.35**, in which it is supposed that the terminal symbol of the mix formula $\mathcal{M}$ is ¬.

For those systems incorporating NC or EM as additional initial sequents, sections **3.111** and **3.112** also need to be reconsidered.

With these considerations expressed, we proceed to the Cut-elimination proofs.

Theorem 4. The inference figure Cut is eliminable from LOC_ω.

Proof: Only the following section of Gentzen's proof needs to be reconsidered.

3.113.3. In this section, the mix formula $\mathcal{M}$ is assumed to occur both in the succedent of the left-hand upper sequent and in the antecedent of the right-hand upper sequent solely as the principal formula of one of the operational inference figures. But this precludes the possibility that the terminal symbol of $\mathcal{M}$ is $\neg$, since LOC_ω has no operational inference figures dealing specifically with negation. Hence, paragraph **3.113.35** is simply omitted.

Corollary. OC_ω has **CP**, **DP** and **NP1**.

Proof: By Theorem 3, a formula A is a theorem of OC_ω if and only if $\rightarrow$ A is a derivable sequent of LOC_ω. By Theorem 4, if $\rightarrow$ A is a derivable sequent of LOC_ω, then there is a Cut-free derivation of $\rightarrow$ A in LOC_ω. These facts enable a straightforward proof of the Corollary as follows.

CP: If A is a theorem of OC_ω and of the form B & C, then in LOC_ω there is a Cut-free derivation of $\rightarrow$ B & C. There are no initial sequents of this form, so the only possible Cut-free derivation is one in which $\rightarrow$ B & C is derived by & - IS from $\rightarrow$ B and $\rightarrow$ C. But then B and C are both theorems of OC_ω. Hence, OC_ω has **CP**.

DP: If A is a theorem of OC_ω and of the form B $\vee$ C, then in LOC_ω there is a Cut-free derivation of $\rightarrow$ B $\vee$ C. There are no initial sequents of this form, so the only possible Cut-free derivation is one in which $\rightarrow$ B $\vee$ C is derived by $\vee$ - IS from $\rightarrow$ B or from $\rightarrow$ C. But then at least one of B and C is a theorem of OC_ω. Hence, OC_ω has **DP**.

NP1: If there were a theorem of OC_ω of the form $\neg$ B, then in LOC_ω there would be a Cut-free derivation of $\rightarrow \neg$ B. But there are no initial sequents of this form in LOC_ω, nor any operational inference figures with a lower sequent of this form; thus, there is no possible Cut-free derivation in LOC_ω terminating in a sequent of the form $\rightarrow \neg$B. Hence, OC_ω has **NP1**.

Theorem 5. The inference figure Cut is eliminable from $LOC_\omega + \neg\neg$ - IA.

Proof: Only the following section of Gentzen's proof needs to be reconsidered.

3.113.3. As in Theorem 4, the possibility that the terminal symbol of $\mathcal{M}$ is $\neg$ is precluded, since in LOC_ω + ¬¬ - IA, a negated formula cannot occur in the succedent of a sequent as the principal formula of one of the operational inference figures. Hence, paragraph **3.113.35** is simply omitted.

Corollary. OC_ω + (11) has **CP, DP** and **NP1**.

Proof: This proof is analogous to the proof of the Corollary to Theorem 4, except that Theorem 5 is invoked instead of Theorem 4.

Theorem 6. The inference figure Cut is eliminable from LOC_ω + ¬¬ - IS.

Proof: Only the following section of Gentzen's proof needs to be reconsidered.

3.113.3. As in Theorem 4, the possibility that the terminal symbol of $\mathcal{M}$ is ¬ is precluded, since in LOC_ω + ¬¬ - IS, a negated formula cannot occur in the antecedent of a sequent as the principal formula of one of the operational inference figures. Hence, paragraph **3.113.35** is simply omitted.

Corollary. OC_ω + (12) has **CP, DP** and **NP2**.

Proof: This proof is analogous to the proof of the Corollary to Theorem 4, except that Theorem 6 is invoked instead of Theorem 4, and instead of the section on **NP1**, we have the following.

NP2: If A is a theorem of OC_ω + (12) and of the form ¬ B, then in LOC_ω + ¬¬ - IS there is a Cut-free derivation of → ¬ B. There are no initial sequents of this form, so the only possible Cut-free derivation is one in which B is of the form ¬ C and → ¬ B (i.e. → ¬¬ C) is derived by ¬¬ - IS from → C. But then C is also a theorem of OC_ω + (12). Hence, OC_ω + (12) has **NP2**.

Theorem 7. The inference figure Cut is eliminable from LNC_ω (= LOC_ω + ¬¬ - IA + ¬¬ - IS).

Proof: Only the following section of Gentzen's proof needs to be reconsidered.

3.113.35. Suppose the terminal symbol of $\mathcal{M}$ is ¬. Then the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow A}{\Gamma_1 \rightarrow \neg\neg A}\ \neg\neg\text{-}IS \qquad \dfrac{A,\ \Gamma_2 \rightarrow B}{\neg\neg A,\ \Gamma_2 \rightarrow B}\ \neg\neg\text{-}IA}{\Gamma_1,\ \Gamma_2 \rightarrow B}\ mix.$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_1 \rightarrow A \qquad A, \Gamma_2 \rightarrow B}{\Gamma_1, \Gamma_2{}^* \rightarrow B}\ mix}{\Gamma_1, \Gamma_2 \rightarrow B}\ \textit{possibly several interchanges and thinnings.}$$

The new mix may be elimated by virtue of the induction hypothesis.

Corollary. NC_ω has **CP**, **DP** and **NP2**.

Proof: This proof is analogous to the proof of the Corollary to Theorem 6, except that Theorem 7 is invoked instead of Theorem 6.

Theorem 8. The inference figure Cut is eliminable from LIC_ω (= LOC_ω + ¬¬ - IS + NC).

Proof: Only the following sections of Gentzen's proof need to be reconsidered.

3.111. Suppose that the left-hand upper sequent of the mix is an initial sequent. Here the extra possibility arises that this initial sequent is of the form NC, so that the mix reads:

$$\frac{\rightarrow \mathcal{M} \qquad \Delta \rightarrow B}{\Delta^* \rightarrow B}$$

where $\mathcal{M}$ is of the form ¬(A & ¬A). Since the rank of the derivation in this case is supposed to be 2, the right-hand upper sequent $\Delta \rightarrow$ B must be either an initial sequent or the lower sequent of a Thinning. (It cannot be the lower sequent of an operational inference figure, since ¬(A & ¬A) cannot be the principal formula of any operational figure of LIC_ω).

In the first case, $\Delta \rightarrow$ B cannot also be of the form NC, since the mix formula must occur in the antecedent of this sequent; therefore, $\Delta \rightarrow$ B must be the initial sequent $\mathcal{M} \rightarrow \mathcal{M}$. But then the lower sequent of the mix, $\Delta^* \rightarrow$ B, is $\rightarrow \mathcal{M}$, which already has a derivation without a mix.

In the second case, $\Delta \rightarrow$ B is the lower sequent of a Thinning with upper sequent $\Delta' \rightarrow$ B. Since the rank of the mix is 2, Δ' cannot contain $\mathcal{M}$; therefore, $\Delta^* = \Delta'$ and there is already a derivation of $\Delta^* \rightarrow$ B without a mix.

3.112. Suppose the right-hand upper sequent of the mix is an initial sequent. The extra possibility that it is of the form NC does not arise, since the mix formula $\mathcal{M}$ must occur in the antecedent of this sequent.

3.113.35. This section is dealt with as in Theorem 6.

Corollary. IC_ω has **CP**, **DP** and **NP3**.

Proof: This proof is analogous to the proof of the Corollary to Theorem 6, except that Theorem 8 is invoked instead of Theorem 6, and instead of the section on **NP2**, we have the following.

NP3: If A is a theorem of IC_ω and of the form $\neg$ B, then in LIC_ω there is a Cut-free derivation of $\rightarrow \neg$ B. The only possible Cut-free derivations of $\rightarrow \neg$ B are (i) a derivation in which $\rightarrow \neg$ B is an instance of NC, in which case B is of the form C & $\neg$ C; and (ii) a derivation in which B is of the form $\neg$ C and $\rightarrow \neg$ B (i.e. $\rightarrow \neg\neg$ C) is derived by $\neg\neg$ - IS from $\rightarrow$ C, in which case C is also a theorem of IC_ω. Hence IC_ω has **NP3**.

7. THEOREMS OF THE SUBSYSTEMS OF J_1

The results of the preceding section enable us to determine more precisely the deductive strength of all but one of the subsystems of J_1 listed in Section 1.

Theorem 9. The theorems of J_0 are precisely those of OC_ω. A similar relationship holds also between the following pairs of systems:

$J_{0\cdot 1}$ and $OC_\omega + (12)$;

$J_{0\cdot 2}$ and $OC_\omega + (11)$;

$J_{0\cdot 3}$ and IC_ω ;

$J_{0\cdot 5}$ and NC_ω .

Proof: The proof that follows is essentially the same as the proof of Theorem 1 of [7], the only substantial additions being those which deal with the negation properties of the axiomatic systems. As in [7], we interject a useful lemma.

Lemma 1. If A is a theorem of OC_ω, then D $\supset$ A is a theorem of J_0, for any D ε $\mathcal{L}$. A similar relationship holds between all of the pairs of systems listed in Theorem 9.

Proof: The proof proceeds by induction on the length n of derivation of A in OC_ω.

Base case ($n = 1$). In this case, A is an axiom of OC_ω. It is straightforward to verify that each axiom of OC_ω is a theorem of J_0. For each such A, it follows by $\rightarrow_2$) and $\supset_1$) that $D \supset A$ is also a theorem of J_0, for any $D \varepsilon \mathcal{L}$.

Inductive step ($n = k$ for some $k > 1$: inductive hypothesis: if B is a theorem of OC_ω with derivation of length less than k, then $D \supset B$ is a theorem of J_0 for any $D \varepsilon \mathcal{L}$). In this case, A is derived in OC_ω from premisses B and $B \supset A$ by an application of (3). On inductive hypothesis, $D \supset B$ and $D \supset (B \supset A)$ are theorems of J_0 for any $D \varepsilon \mathcal{L}$. By $\&_1$) and two applications of $\rightarrow_5$), $(D \supset B) \& (D \supset (B \supset A))$ is also a theorem, and hence by $\supset_2$) and $\rightarrow_5$), so is $D \supset A$.

A similar proof can be given for each of the remaining pairs of systems listed in Theorem 9.

Corollary. If $B \supset C$ is a theorem of OC_ω, then $B \supset C$ is a theorem of J_0. A similar relationship holds between all of the pairs of systems listed in Theorem 9.

Proof: Assume that $B \supset C$ is a theorem of OC_ω. Then by Lemma 1, $D \supset (B \supset C)$ is a theorem of J_0 for any $D \varepsilon \mathcal{L}$. In particular, $B \supset (B \supset C)$ is a theorem of J_0. Since $B \supset B$ is also a theorem, by $\rightarrow_1$) and $\supset_1$), it follows that $(B \supset B) \& (B \supset (B \supset C))$ is also a theorem, by $\&_1$) and $\rightarrow_5$), whence so is $B \supset C$, by $\supset_2$) and $\rightarrow_5$).

A similar proof can be given for each of the remaining pairs of systems listed in Theorem 9.

Proof of Theorem 9 (continued): To show that every theorem of J_0 is a theorem of OC_ω, it suffices to verify that all of the postulates of J_0 are derivable in LOC_ω, from which the desired result follows by Theorem 3. This is easily verified, and a similar verification can be given for all of the remaining pairs of systems listed.

To prove the converse, we assume that A is a theorem of OC_ω and proceed by induction on the number n of connectives in A.

Base case ($n = 1$). Evidently, the smallest number of connectives that a theorem A of OC_ω can have is one, in which case $A = p \supset p$, for some propositional variable p. It follows that A is a theorem of J_0 by the Corollary to Lemma 1.

Inductive step ($n = k$ for some $k > 1$; inductive hypothesis: if B is a theorem of OC_ω with fewer than k connectives, then B is a theorem of J_0). We distinguish between the possible forms of A.

(i) Suppose A is of the form B ⊃ C. In this case, A is a theorem of J_0 by the Corollary to Lemma 1.

(ii) Suppose A is of the form B & C. In this case, both B and C are also theorems of OC_ω, since OC_ω has **CP** by the Corollary to Theorem 4. But each of B and C has fewer connectives than B & C; hence, on inductive hypothesis, B and C are both theorems of J_0, from which it follows that B & C (i.e. A) is also a theorem of J_0 by $\&_1$) and two applications of $\rightarrow_5$).

(iii) Suppose A is of the form B ∨ C. In this case, at least one of B and C is also a theorem of OC_ω, since OC_ω has **DP** by the Corollary to Theorem 4. But each of B and C has fewer connectives than B ∨ C; hence, on inductive hypothesis, at least one of B and C is also a theorem of J_0, from which it follows that B ∨ C (i.e. A) is also a theorem of J_0 by either $\vee_1$) or $\vee_2$) together with $\rightarrow_5$).

(iv) A cannot be of the form ¬B, since OC_ω has **NP1** by the Corollary to Theorem 4.

Paragraphs (i) to (iv) exhaust the possible forms of A; hence, it has been shown that if A is a theorem of OC_ω then A is also a theorem of J_0.

To deal with the remaining pairs of systems listed in Theorem 9, it suffices to modify the above proof in the appropriate way for each pair. The only modification which requires explication is the alteration of paragraph (iv).

For the pair $J_{0\cdot 1}$ and $OC_\omega + (12)$, paragraph (iv) is replaced by the following:

(iv)′ Suppose A is of the form ¬B. In this case, B is itself of the form ¬C where C is a theorem of $OC_\omega + (12)$, since $OC_\omega + (12)$ has **NP2** by the Corollary to Theorem 6. But C has fewer connectives than ¬¬C, hence on inductive hypothesis, C is also a theorem of $J_{0\cdot 1}$, from which it follows that ¬¬C (i.e. A) is also a theorem of $J_{0\cdot 1}$ by $\neg_1$) and $\rightarrow_5$).

For the pair $J_{0\cdot 2}$ and $OC_\omega + (11)$, paragraph (iv) is not substantially altered, since $OC_\omega + (11)$, like OC_ω, has **NP1** by the Corollary to Theorem 5.

For the pair $J_{0\cdot 3}$ and IC_ω, paragraph (iv) is replaced by the following:

(iv)″ Suppose A is of the form ¬B. In this case, B is either itself of the form ¬C where C is a theorem of IC_ω, or of the form C & ¬ C, since IC_ω has **NP3** by the Corollary to Theorem 8. The first subcase is dealt with as in paragraph (iv)′, while in the second, A is of the form ¬(C & ¬ C) and hence a theorem of $J_{0\cdot 3}$ by $\neg_3$).

For the pair $J_{0\cdot 5}$ and NC_ω, paragraph (iv) is replaced by a paragraph substantially similar to paragraph (iv)′, since NC_ω has **NP2** by the Corollary to Theorem 7.

Corollary. J_0 and $J_{0\cdot 2}$ have properties **CP**, **DP** and $\mathbf{NP}_1$; $J_{0\cdot 1}$ and $J_{0\cdot 5}$ have **CP**, **DP** and $\mathbf{NP}_2$; and $J_{0\cdot 3}$ has **CP**, **DP** and $\mathbf{NP}_3$.

Proof: This follows from Theorem 9 and the Corollaries to Theorems 4 through to 8.

Notably absent from the list of subsystems of J_1 considered in Theorem 9 is $J_{0\cdot 4}$, which is formed by adding postulate $\neg_4$) to $J_{0\cdot 2}$. It might be thought that the theorems of $J_{0\cdot 4}$ could likewise be shown to coincide with those of its apparently corresponding axiomatic system, C_ω. Indeed, it might also be thought that the theorems of J_1 itself could similarly be shown to coincide with those of its apparently corresponding axiomatic system, NC_ω +(10)+(13). Both of the systems C_ω and NC_ω +(10)+(13) were shown in Theorem 3 to be equivalent to their respective Gentzen formulations, LC_ω and LNC_ω +NC+EM. However, no proof of the eliminability of Cut from either of these Gentzen-style systems was given in Section 6. The following theorem shows why.

Theorem 10. The inference figure Cut is not eliminable from either of the systems LC_ω and LNC_ω +NC+EM.

Proof: It is straightforward to show that the sequent → ¬A ∨ A is derivable in both of these systems, even if neither → ¬A nor → A is derivable. If Cut were eliminable, then there would be a derivation of this sequent in which Cut does not figure. But then → ¬A ∨ A would either be an initial sequent or be derived from either → ¬A or → A by an application of ∨-IS. But there are no initial sequents of this form in either system, and it has been assumed that neither → ¬ A nor → A is derivable. Hence, Cut is not eliminable from either system.

It was noted in Section 4 that systems incorporating EM could equivalently be formulated with Curry's rule Nx instead of EM. In fact, it can be shown that Cut is eliminable from such formulations. However, Nx is itself just a special form of Cut, and it is by no means obvious that such Cut-eliminability proofs would enable demonstrations that C_ω and NC_ω +(10)+(13) enjoy properties of any usefulness in proving the coincidence of their theorems with those of $J_{0\cdot 4}$ and J_1 respectively. As the final result of this section, we show that such coincidence does not in fact obtain.

Theorem 11. The theorems of $J_{0\cdot4}$ (respectively, J_1) constitute a proper subset of the theorems of C_ω (respectively, $NC_\omega +(10)+(13)$).

Proof: That every theorem of $J_{0\cdot4}$ (J_1) is a theorem of C_ω ($NC_\omega +(10)+(13)$) can be straightforwardly established as in Theorem 9.

To prove that the converse does not hold, it suffices to exhibit a formula which is a theorem of C_ω (and therefore also of $NC_\omega +(10)+(13)$), but which is not a theorem of J_1 (and therefore also not of $J_{0\cdot4}$). Such is the following: $B \vee \neg(A \& (A \supset B))$. That this is a theorem of C_ω follows from the fact that, in this system, B is derivable from $A \& (A \supset B)$, and so the theorem in question is derivable from $(A \& (A \supset B)) \vee \neg(A \& (A \supset B))$, which is an instance of (10). That it is not a theorem of J_1 is established by the following argument.

If $B \vee \neg(A \& (A \supset B))$ were a theorem of J_1, then it would also be a theorem of J_5 (see [4]). But the rule from $\rightarrow D \vee \neg C$ to $C \rightarrow D$ is derivable in J_5, as follows. Assume $\rightarrow D \vee \neg C$. An instance of $\&_1)$ is $D \vee \neg C, C \rightarrow (D \vee \neg C) \& C$, so by $\rightarrow_5)$, we have $C \rightarrow (D \vee \neg C) \& C$. An instance of one of the distribution laws derivable in J_1 (and therefore also in J_5) by Theorem 2 of [4] is $(D \vee \neg C) \& C \rightarrow (D \& C) \vee (\neg C \& C)$. Hence, by $\rightarrow_5)$, we have $C \rightarrow (D \& C) \vee (\neg C \& C)$. But $C \& \neg C \rightarrow D$ is derivable in J_5 by Theorem 18 of [4], and hence so is $\neg C \& C \rightarrow D$; and an instance of $\&_2)$ is $D \& C \rightarrow D$, so by $\vee_4)$, we have $(D \& C) \vee (\neg C \& C) \rightarrow D$. This, together with $C \rightarrow (D \& C) \vee (\neg C \& C)$, suffices to deliver $C \rightarrow D$ by $\rightarrow_5)$. Thus, if $B \vee \neg(A \& (A \supset B))$ were a theorem of J_5, then $A \& (A \supset B) \rightarrow B$ would be derivable. But this sequent is not derivable in J_5, by Theorem 19 of [4].

8. DISTINCTNESS OF THE SUBSYSTEMS OF J_1

Theorem 12. The systems J_0, $J_{0\cdot1}$, $J_{0\cdot2}$, $J_{0\cdot3}$, $J_{0\cdot4}$, $J_{0\cdot5}$ and J_1 are distinct.

Proof: We begin by noting that none of the intuitionistic systems J_0, $J_{0\cdot1}$ and $J_{0\cdot3}$ can be equivalent to any of the remaining systems, since these latter all incorporate intuitionistically invalid sequents as postulates. Moreover, these three systems are distinct: $J_{0\cdot1}$ contains theorems of the form $\neg\neg A$ (for any theorem A), but J_0 has property **NP1** by the Corollary to Theorem 9; and $J_{0\cdot3}$ contains theorems of the form $\neg(A \& \neg A)$, but $J_{0\cdot1}$ has **NP2** by the

Corollary to Theorem 9.

Similarly, the dualintuitionistic systems are distinct: $J_{0\cdot4}$ contains theorems of the form $A \vee \neg A$ (where neither disjunct is a theorem), but $J_{0\cdot2}$ has **DP** by the Corollary to Theorem 9.

A similar argument shows that the intermediate systems $J_{0\cdot5}$ and J_1 are also distinct. It remains to show that neither of the dualintuitionistic systems is equivalent to either of the intermediate systems. That $J_{0\cdot2}$ is distinct from both $J_{0\cdot5}$ and J_1 follows from the fact that both of the latter systems have theorems of the form $\neg A$, but $J_{0\cdot2}$ has **NP1** by the Corollary to Theorem 9. That $J_{0\cdot4}$ is distinct from $J_{0\cdot5}$ follows again from the fact that the former has theorems of the form $A \vee \neg A$ (where neither disjunct is a theorem), while the latter has **DP** by the Corollary to Theorem 9. Finally, that $J_{0\cdot4}$ is distinct from J_1 follows from the fact that the latter contains theorems of the form $\neg(A \,\&\, \neg A)$, while the former does not, since by Theorem 11, its theorems form a subset of those of C_ω, which lacks such theorems (see Theorem 3 of [17]).

This concludes our investigation into the deductive strength of the subsystems of J_1 introduced in this paper. An investigation into the suitability of these systems for paraconsistent purposes is undertaken in Chapter 4.

Chapter Four: Paraconsistency and the Subsystems of J_1

1. $J_i(0 \leqslant i \leqslant 1)$ AND THE PARACONSISTENCY CONDITIONS

The systems $J_i(0 \leqslant i \leqslant 1)$, defined in Chapter 3, are all subsystems of J_1, the weakest of the J-systems originally constructed by Arruda and da Costa. As noted in Chapter 2, J_1 satisfies the first of da Costa's conditions for paraconsistent systems: (I) from contradictory formulas A and $\neg$A, it must not be possible to deduce an arbitrary formula B. Because they are subsystems of J_1, it follows that the systems $J_i(0 \leqslant i \leqslant 1)$ also satisfy this condition.

However, it is also noted in Chapter 2 that J_1 does not satisfy the second of da Costa's conditions: (II) paraconsistent systems should contain most of the schemata and deduction rules of classical logic that do not interfere with (I). And even if, as suggested in Chapter 2, (II) is weakened to condition (II′), which allows substantial containment of intuitionistic schemata and rules as an alternative, it is still apparent that J_1 does not adequately meet this requirement. Again, because they are subsystems of J_1, it follows that the systems $J_i(0 \leqslant i \leqslant 1)$ also fail, to at least the same degree, to satisfy this second condition.

The main ground for dissatisfaction with J_1 advanced in Chapter 2, both in terms of condition (II′) and for independent reasons, is its failure to enjoy SE, the property of intersubstitutivity of provable equivalents. However, the attempts in Chapter 2 to extend J_1 so as to secure SE proved to be unsatisfactory, in that the necessary additions collapse J_1 into J_5, which explicitly violates condition (I). Hence the suggestion that subsystems of J_1 be investigated, in the hope that such subsystems might either enjoy SE naturally or be amenable to extension so as to secure this property without similarly compromising (I). We turn to the systems $J_i(0 \leqslant i \leqslant 1)$ to see whether this hope can be realised.

2. $J_i(0 \leqslant i \leqslant 1)$ AND THE PROPERTY OF INTERSUBSTITUTIVITY OF PROVABLE EQUIVALENTS

Theorem 1. The systems $J_i(0 \leqslant i \leqslant 1)$ do not enjoy the property SE.

Proof: Easily derived in each $J_i(0 \leqslant i \leqslant 1)$ are the sequents $A \rightarrow A \,\&\, A$ and $A \,\&\, A \rightarrow A$ (abbreviated $A \leftrightarrow A \,\&\, A$), and $\neg A \rightarrow \neg A$. If these systems enjoyed SE, then $\neg(A \,\&\, A) \rightarrow \neg A$, the result of substituting A & A for (one occurrence of) A in $\neg A \rightarrow \neg A$, would also be derivable. However, the following matrices validate the postulates of each $J_i(0 \leqslant i \leqslant 1)$, but invalidate $\neg(A \,\&\, A) \rightarrow \neg A$ when A is assigned the value 0; hence, these systems do not enjoy SE.

→/⊃	0	1	2	3	¬
*0	0	1	2	3	3
*1	0	0	2	3	2
2	0	0	0	3	1
3	0	0	0	0	0

∧/&	0	1	2	3
0	1	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

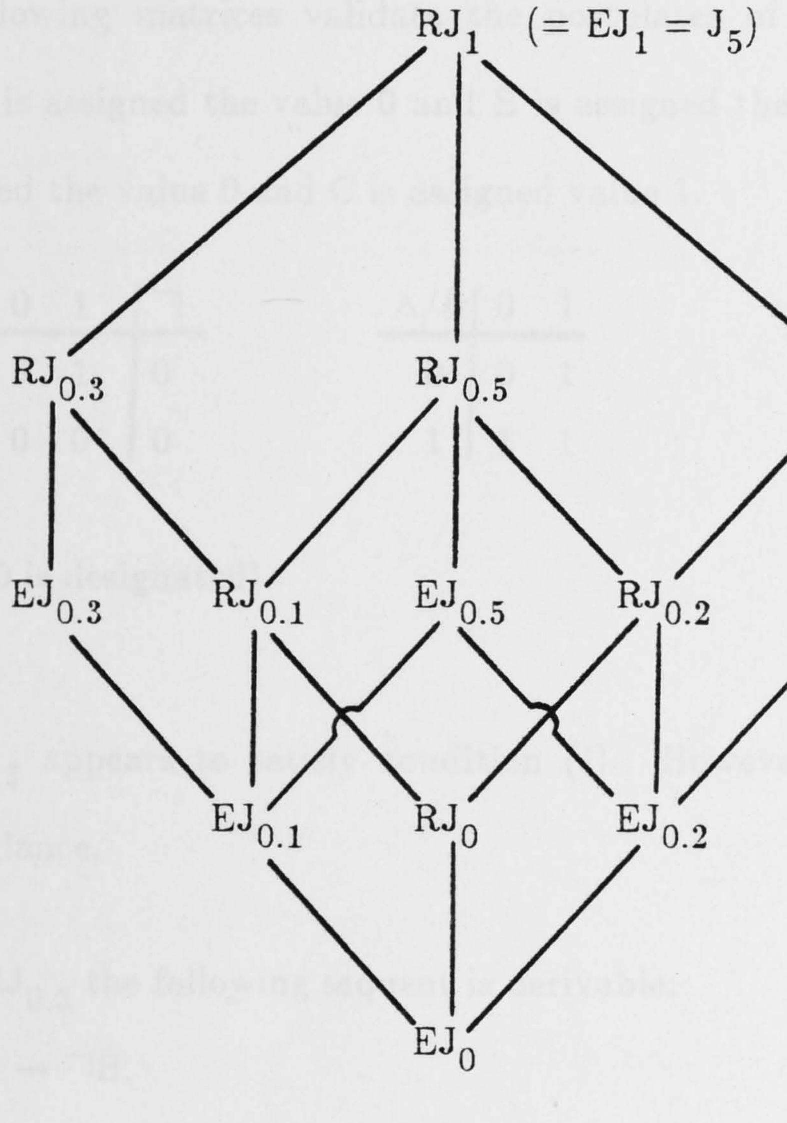
∨	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

(The values 0 and 1 are designated; the value of $\rightarrow A$ is the same as the value of A).

As in Chapter 2, the obvious strategies for securing SE are by adding the rules RC and EC respectively. Like the original J-systems, the systems $J_i(0 \leqslant i \leqslant 1)$ enjoy SE^+, the property of intersubstitutivity of provable equivalents in negation-free contexts (see Lemma 2 of Chapter 1). It follows that, for any extension of these systems (in the same vocabulary), the admissibility of either RC or EC is sufficient to guarantee SE in full, and the admissibility of EC is also necessary. For each J_i, the system formed by adding RC will be called RJ_i, and the system formed by adding EC will be called EJ_i.

In Chapter 3, the systems $J_i(0 \leqslant i \leqslant 1)$ were divided into three separate branches. It is convenient to divide the extended systems $RJ_i(0 \leqslant i \leqslant 1)$ and $EJ_i(0 \leqslant i \leqslant 1)$ similarly: for each J_i, the systems RJ_i and EJ_i will be deemed to belong to the same branch as J_i. In particular, RJ_0, EJ_0, $RJ_{0.1}$, $EJ_{0.1}$, $RJ_{0.3}$ and $EJ_{0.3}$ all belong to the *intuitionistic* branch (and since the rules RC and EC are intuitionistically derivable, these systems, like the intuitionistic members of $J_i (0 \leqslant i \leqslant 1)$, have the property that any sequent derivable in them is intuitionistically derivable); $RJ_{0.2}$, $EJ_{0.2}$, $RJ_{0.4}$ and $EJ_{0.4}$ are *dualintuitionistic*; and $RJ_{0.5}$, $EJ_{0.5}$ and RJ_1 ($=EJ_1$) are *intermediate*. (That $RJ_1 = EJ_1 = J_5$ is proved in Theorem 7 of Chapter 2).

The relations of containment holding between these systems are as set out below, with weaker systems placed below stronger ones. (That these systems are distinct will be shown in Section 7).



For each of the three branches, we will determine the strongest member or members which satisfy condition (I), since these are also the most likely to satisfy (II′). For the J-systems, explicit satisfaction of (I) amounts to the underivability of the sequent A, ¬A → B. (Thus, RJ_1 and EJ_1 fail to satisfy (I), since they are equivalent to J_5, in which this sequent is derivable). We note, however, that a J-system can fail to satisfy (I) substantively, even if not explicitly. For example, the sequent A, ¬A → B ⊃ C is derivable in the systems J_2, J_3

and J_4 even though A, ¬A → B is not (see Chapter 2). We shall therefore need to consider not only the latter sequent, but variants of it as well.

3. THE INTUITIONISTIC BRANCH

Theorem 2. In $RJ_{0.3}$ (and therefore also in the remaining intuitionistic systems), neither of the following sequents is derivable:

A, ¬A → B

A, ¬A → B ⊃ C.

Proof: The following matrices validate the postulates of $RJ_{0.3}$, but invalidate the first sequent when A is assigned the value 0 and B is assigned the value 1, and the second when A and B are assigned the value 0 and C is assigned value 1.

→/⊃	0	1	¬
*0	0	1	0
1	0	0	0

∧/&	0	1
0	0	1
1	1	1

∨	0	1
0	0	0
1	0	1

(Only the value 0 is designated).

Thus, $RJ_{0.3}$ appears to satisfy condition (I). However, not all is as promising as it appears at first glance.

Theorem 3. In $RJ_{0.3}$ the following sequent is derivable:

A, ¬A → ¬B.

Proof: Applying $\rightarrow_2$) to postulate $\neg_3$) yields B → ¬(A & ¬A), from which ¬¬(A & ¬A) → ¬B follows by RC. An instance of $\neg_1$) is A & ¬A → ¬¬(A & ¬A), which, together with the above sequent, yields A & ¬A → ¬B by $\rightarrow_5$). From this, A, ¬A → ¬B follows by $\&_1$) and $\rightarrow_5$).

The derivability of A, ¬A → ¬B can be seen to be (at least) as substantive an infringement of (I) as is the derivability of A, ¬A → B ⊃ C in some of the other J-systems. For an obvious consequence is that, for any inconsistent theory based on $RJ_{0.3}$, the negation of every formula in the theory would also be in the theory. Thus, such a theory would be thoroughly inconsistent; no part of it could reasonably be said to be "locally consistent".

The system $RJ_{0.3}$, then, must be rejected as too strong for paraconsistent purposes. We turn instead to the weaker members of the intuitionistic branch.

Theorem 4. In $RJ_{0.1}$, the sequent A, ¬A → ¬B is not derivable. More generally, the sequent A, ¬A → $\neg^n$B is not derivable, where $\neg^n$ represents any finite string of occurrences of ¬.

Proof: The following matrices validate the postulates of $RJ_{0.1}$, but invalidate A, ¬A → $\neg^n$B when A is assigned the value 1 and B is assigned whichever of 0 and 2 results in $\neg^n$B taking the value 2.

→/⊃	0	1	2	¬
*0	0	1	2	2
1	0	0	2	1
2	0	0	0	0

∧/&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

∨	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 0 is designated).

Thus, $RJ_{0.1}$ appears to substantively satisfy (I), and therefore so do the weaker intuitionistic systems $EJ_{0.1}$, RJ_0 and EJ_0. However, $EJ_{0.3}$ is not weaker than $RJ_{0.1}$; this system must therefore be considered separately.

Theorem 5. In $EJ_{0.3}$, the sequent A, ¬A → $\neg^n$B is not derivable.

Proof: The following matrices validate the postulates of $EJ_{0.3}$, but invalidate the above sequent when A is assigned the value 0 and B is assigned the value 1.

→	0	1	2	3	¬
*0	0	1	2	3	0
1	0	0	2	3	2
2	0	1	0	3	1
3	0	0	0	0	0

⊃	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

∧/&	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

∨	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

(Only the value 0 is designated).

Thus, $EJ_{0.3}$, like $RJ_{0.1}$, appears to substantively satisfy (I).

Whether these two systems satisfy (II′), however, is less clear. Certainly, they enjoy SE, and are therefore an improvement on most of the original J-systems and the subsystems $J_i(0 \leqslant i \leqslant 1)$, at least insofar as the enjoyment of this property contributes to the satisfaction of (II′) (see Chapter 2). However, they are in other respects fairly weak relative to classical and intuitionistic logic; in fact, they are subsystems not only of (a sequent-based formulation of) intuitionistic logic, but also of the "minimal calculus" of Johansson (see [8]). But then, (II′) does not require that paraconsistent systems approximate classical or intuitionistic logic unconditionally, but only insofar as satisfaction of (I) allows. And when it comes to approximating intuitionistic logic, at least, $RJ_{0.1}$ and $EJ_{0.3}$ are the strongest members of the intuitionistic branches of $RJ_i(0 \leqslant i \leqslant 1)$ and $EJ_i(0 \leqslant i \leqslant 1)$ which substantively satisfy (I).

A final remark about $EJ_{0.3}$ is in order. It follows from Theorem 5 that the rule RC is not admissible in $EJ_{0.3}$, for if it were, then A, ¬A → ¬B would be derivable in this system, as in the proof of Theorem 3. While this has the desired consequence that $EJ_{0.3}$ substantively satisfies (I), it also means that this system fails to contain certain variants of the sequent → ¬(A & ¬A), even though this sequent is explicitly incorporated as a postulate.

<u>Theorem 6.</u> In $EJ_{0.3}$, the following sequents are not derivable:

→ ¬((A & ¬A) & B)

→ ¬X, where X is any permutation and/or reassociation of (A & ¬A) & B.

<u>Proof</u>: The matrices in the proof of Theorem 5 invalidate these sequents when A is assigned the value 0 and B is assigned the value 1.

As argued after presenting the same result for J_1 in Theorem 1 of Chapter 2, the absence of these sequents is not only anomalous in the presence of postulate $¬_3$), but also contrary to at least the spirit of (II′). Of the systems $EJ_{0.3}$ and $RJ_{0.1}$, then, the latter presents itself as somewhat more attractive.

4. THE DUALINTUITIONISTIC BRANCH

Theorem 7. In $RJ_{0.4}$ (and therefore also in the remaining dualintuitionistic systems), the following sequents are not derivable:

$$A, \neg A \rightarrow B$$
$$A, \neg A \rightarrow B \supset C$$
$$A, \neg A \rightarrow \neg^n B.$$

Proof: The following matrices (adapted from those provided for CC_ω in [19]) validate the postulates of $RJ_{0.4}$, but invalidate these sequents. The first is invalidated when A is assigned the value 1 and B is assigned the value 2; the second when A is assigned the value 1, B is assigned the value 0 and C is assigned the value 2; and the third when A is assigned the value 1 and B is assigned whichever of 0 and 1 results in $\neg^n B$ taking the value 2.

→/⊃	0	1	2	¬
*0	0	1	2	2
1	0	0	2	0
2	0	0	0	0

∧/&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

∨	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 0 is designated).

Thus, $RJ_{0.4}$ appears to substantively satisfy (I). In investigating whether this system also satisfies (II′), we note firstly that $J_{0.4}$ lacks the intuitionistic negation postulates $\neg_1$) and $\neg_3$), but instead includes their "duals", $\neg_2$) and $\neg_4$). In this sense, it can be said that what $J_{0.4}$ lacks from the intuitionistic point of view, it makes up for from the broader classical perspective. And with respect to the variants of $\rightarrow \neg(A \mathbin{\&} \neg A)$, it is at least more uniform than some of the other J-systems investigated (see, for example, Theorems 1 and 2 of Chapter 2 and Theorem 6 above).

Theorem 8. In $J_{0.4}$, the following sequents are not derivable:

$$\rightarrow \neg(A \mathbin{\&} \neg A)$$

$\rightarrow \neg(\neg A \,\&\, A)$

$\rightarrow \neg((A \,\&\, \neg A) \,\&\, (A \,\&\, \neg A))$

$\rightarrow \neg((A \,\&\, \neg A) \,\&\, B)$

$\rightarrow \neg X$, where X is any permutation and/or reassociation of $(A \,\&\, \neg A) \,\&\, B$.

Proof: The following matrices validate the postulates of $J_{0.4}$, but invalidate these sequents when A and B are both assigned the value 0.

$\rightarrow/\supset$	0	1	2	$\neg$
*0	0	0	2	1
*1	0	0	2	2
2	0	0	0	1

$\wedge/\&$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\vee$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 2 is not designated).

Thus, $J_{0.4}$ is itself reasonably attractive but for the fact that it does not enjoy SE, by Theorem 1. Addition of the rule RC, resulting in $RJ_{0.4}$, suffices to secure this property without compromising condition (I), as shown in Theorem 7. It also has an interesting consequence concerning the sequents listed in Theorem 8.

Theorem 9. In $RJ_{0.4}$, all of the sequents listed in Theorem 8 are derivable.

Proof: We provide the derivation of only $\rightarrow \neg(A \,\&\, \neg A)$; from this, the remaining sequents can easily be derived with the assistance of RC.

An instance of postulate $\&_2)$ is $A \,\&\, \neg A \rightarrow A$, and an instance of $\&_3)$ is $A \,\&\, \neg A \rightarrow \neg A$. Applying RC to these yields $\neg A \rightarrow \neg(A \,\&\, \neg A)$ and $\neg\neg A \rightarrow \neg(A \,\&\, \neg A)$, whence $\neg A \vee \neg\neg A \rightarrow \neg(A \,\&\, \neg A)$ follows by $\vee_4)$. But the antecedent is an instance of $\neg_4)$, so by $\rightarrow_5)$, we get the desired $\rightarrow \neg(A \,\&\, \neg A)$.

Thus, $RJ_{0.4}$ is as uniform in including the sequents of Theorem 8 as $J_{0.4}$ is in excluding them. Moreover, Theorem 9 shows that $RJ_{0.4}$ differs from RJ_1 $(= J_5)$ only in not containing postulate $\neg_1)$; yet the former system substantively satisfies condition (I), while the latter explicitly violates it. It follows that $RJ_{0.4}$ also satisfies (II′), at least insofar as there is no system in $RJ_i(0 \leqslant i \leqslant 1)$ or $EJ_i(0 \leqslant i \leqslant 1)$ which is stronger and which also satisfies (I).

As a final remark on $RJ_{0.4}$, we recall that the reason for dissatisfaction with its intuitionistic "dual", $RJ_{0.3}$, was that this system was shown in Theorem 3 to contain the sequent A, ¬A → ¬B. In particular, the instance A, ¬A → ¬¬B is derivable in $RJ_{0.3}$, though in the absence of the negation postulate $\neg_2$), this is not quite sufficient to yield the unwanted A, ¬A → B. It might be that a "dual" result holds for $RJ_{0.4}$, similarly compromising its satisfaction of (I). The following presents itself as such.

Theorem 10. In $RJ_{0.4}$, the sequent ¬¬(A & ¬A) → B is derivable.

Proof: From → ¬(A & ¬A), shown to be derivable in $RJ_{0.4}$ in Theorem 9, ¬B → ¬(A & ¬A) follows by $\rightarrow_2$). Applying RC yields ¬¬(A & ¬A) → ¬¬B, from which the sequent follows by $\neg_2$) and $\rightarrow_5$).

Again, in the absence of the negation postulate $\neg_1$), this is not sufficient to yield the unwanted A, ¬A → B. However, there is no reason to consider the sequent of Theorem 10 to be as undesirable as its "dual", A & ¬A → ¬¬B. For the presence of the latter in an inconsistent theory ensures virtual collapse; but the presence of the former need have no such consequence, unless of course ¬¬(A & ¬A) is derivable from A & ¬A in the theory. Thus, Theorem 10 does not indicate that the satisfaction of (I) by $RJ_{0.4}$ is significantly compromised.

5. THE INTERMEDIATE BRANCH

As noted earlier, RJ_1 (= EJ_1) violates condition (I) explicitly; however, $RJ_{0.5}$ is more satisfactory.

Theorem 11. In $RJ_{0.5}$ (and therefore also in $EJ_{0.5}$), the sequents listed in Theorem 7 are not derivable.

Proof: The matrices in the proof of Theorem 4 validate the postulates of $RJ_{0.5}$, but invalidate these sequents. The first is invalidated when A is assigned the value 1 and B is assigned the value 2; the second when A is assigned the value 1, B is assigned the value 0 and C is assigned the value 2; and the third as in the proof of Theorem 4.

Thus, $RJ_{0.5}$, appears to substantively satisfy (I). Moreover, like $RJ_{0.4}$, it offers a uniform treatment of the sequents listed in Theorem 8.

Theorem 12. In $RJ_{0.5}$, none of the sequents listed in Theorem 8 is derivable.

Proof: The matrices in the proof of Theorem 4 validate the postulates of $RJ_{0.5}$, but invalidate these sequents when A is assigned the value 1 and B is assigned the value 0.

Finally, $RJ_{0.5}$ also satisfies (II′), at least insofar as there is no system in $RJ_i(0 \leqslant i \leqslant 1)$ or $EJ_i(0 \leqslant i \leqslant 1)$ which is stronger and which also satisfies (I), for the addition of either $\neg_3)$ or $\neg_4)$ to $RJ_{0.5}$ collapses this system into RJ_1.

Theorem 13. $RJ_{0.5} + \neg_3) = RJ_{0.5} + \neg_4) = RJ_1$.

Proof: That $RJ_{0.5} + \neg_4) = RJ_1$ follows from the fact that the former system contains $RJ_{0.4}$, in which $\rightarrow \neg(A \,\&\, \neg A)$, i.e. $\neg_3)$, is derivable by Theorem 9. But $RJ_{0.5} + \neg_4) + \neg_3) = RJ_1$.

To show that $RJ_{0.5} + \neg_3) = RJ_1$, it suffices to derive $\neg_4)$ in the former system. The derivation is as follows. An instance of $\vee_1)$ is $A \rightarrow A \vee \neg A$, which by RC yields $\neg(A \vee \neg A) \rightarrow \neg A$. Similarly, $\neg(A \vee \neg A) \rightarrow \neg\neg A$ results by $\vee_2)$ and RC. These two sequents yield $\neg(A \vee \neg A) \rightarrow \neg A \,\&\, \neg\neg A$ by $\&_1)$ and two applications of $\rightarrow_5)$. Applying RC again yields $\neg(\neg A \,\&\, \neg\neg A) \rightarrow \neg\neg(A \vee \neg A)$. But the antecedent is an instance of $\neg_3)$; hence $\rightarrow_5)$ yields $\rightarrow \neg\neg(A \vee \neg A)$. From this, $\rightarrow A \vee \neg A$, i.e. $\neg_4)$, follows by $\neg_2)$ and $\rightarrow_5)$.

We note that the results of this section have a bearing on whether the intuitionistic system $RJ_{0.1}$ really satisfies (II′); for it is now apparent that there is a stronger system, namely $RJ_{0.5}$, which also substantively satisfies (I). However, $RJ_{0.1}$ retains its interest as the strongest intuitionistic system in $RJ_i(0 \leqslant i \leqslant 1)$ which satisfies (I).

6. AXIOMATIC FORMULATIONS

In Theorem 9 of Chapter 3, it is shown for most of the systems $J_i(0 \leqslant i \leqslant 1)$ that the theorems of each coincide with those of an axiomatic counterpart. In this section, we investigate whether similar results can be obtained for $RJ_i(0 \leqslant i \leqslant 1)$ and $EJ_i(0 \leqslant i \leqslant 1)$.

The appropriate forms of the rules RC and EC for axiomatic systems are the following:

$$\text{Ax.RC} \quad \frac{C \supset D}{\neg D \supset \neg C} \qquad\qquad \text{Ax.EC} \quad \frac{C \equiv D}{\neg D \supset \neg C},$$

where C ≡ D abbreviates (C ⊃ D) & (D ⊃ C).

The names ROC_ω and EOC_ω will be used to denote the systems respectively formed by adding Ax.RC and Ax.EC to OC_ω; and similarly for the remaining axiomatic systems studied in Chapter 3. (We note that in [19], ROC_ω and RC_ω are called, respectively, CC and CC_ω).

As in Chapter 3, we can provide Gentzen-style formulations of these axiomatic systems. For present purposes, it is simplest just to add the rules RC and EC, respectively, to the Gentzen-style systems introduced in Chapter 3. The resulting systems will be denoted $LROC_\omega$, $LEOC_\omega$, etc.

An initial equivalence between the axiomatic and the Gentzen-style systems is easily established.

<u>Theorem 14</u>. The systems $LROC_\omega$ and ROC_ω are equivalent in the following sense: (i) if $A_1,\ldots,A_n \to B$ (respectively, $\to B$) is a derivable sequent in $LROC_\omega$, then $(A_1 \,\&\, \ldots \,\&\, A_n) \supset B$ (respectively, B) is a theorem of ROC_ω; and (ii) if B is a theorem of ROC_ω, then $\to B$ is a derivable sequent in $LROC_\omega$. Similarly equivalent are the following pairs of systems:

$LEOC_\omega$	and EOC_ω;
$LROC_\omega + \neg\neg\text{-IA}$	and $ROC_\omega + (11)$;
$LEOC_\omega + \neg\neg\text{-IA}$	and $EOC_\omega + (11)$;
$LROC_\omega + \neg\neg\text{-IS}$	and $ROC_\omega + (12)$;
$LEOC_\omega + \neg\neg\text{-IS}$	and $EOC_\omega + (12)$;

$LRIC_\omega$ and RIC_ω;

$LEIC_\omega$ and EIC_ω;

$LRNC_\omega$ and RNC_ω;

$LENC_\omega$ and ENC_ω;

LRC_ω and RC_ω;

LEC_ω and EC_ω; and

$LRNC_\omega$ + NC + EM (= $LENC_\omega$ + NC + EM)
and RNC_ω + (10) + (13) (= ENC_ω + (10) + (13)).

(We note that the members of this last pair are respectively, a sequent-based and an axiomatic formulation of classical logic).

Proof: The proof is by induction on length of derivation, as in the proof of Theorem 3 of Chapter 3. We need to consider only the additional cases in which RC and Ax.RC, or EC and Ax.EC, are involved

In proving (i), we consider the case in which a sequent $\neg B \rightarrow \neg A$ is derived in $LROC_\omega$ from $A \rightarrow B$ by RC. On inductive hypothesis, $A \supset B$ is a theorem of ROC_ω, to which Ax.RC can be applied to yield $\neg B \supset \neg A$.

In proving (ii), we consider the case in which $\neg B \supset \neg A$ is a theorem of ROC_ω and is derived by Ax.RC from $A \supset B$. On inductive hypothesis, $\rightarrow A \supset B$ is derivable in $LROC_\omega$. But $A \supset B, A \rightarrow B$ is also derivable in $LROC_\omega$; hence, Cut can be applied to yield $A \rightarrow B$. Application of RC yields $\neg B \rightarrow \neg A$, from which the desired $\rightarrow \neg B \supset \neg A$ follows by $\supset$ -IS.

A similar proof can be constructed for the remaining pairs involving RC and Ax.RC; and analogously for those involving EC and Ax.EC.

We now consider the relationship between the J-systems and the axiomatic systems.

Theorem 15. The theorems of RJ_0 constitute a proper subset of the theorems of ROC_ω. A similar relationship holds also between the following pairs of systems:

EJ_0	and EOC_ω;
$RJ_{0.1}$	and ROC_ω + (12);
$EJ_{0.1}$	and EOC_ω + (12);
$RJ_{0.2}$	and ROC_ω + (11);
$EJ_{0.2}$	and EOC_ω + (11);
$RJ_{0.3}$	and RIC_ω;
$EJ_{0.3}$	and EIC_ω;
$RJ_{0.4}$	and RC_ω;
$EJ_{0.4}$	and EC_ω;
$RJ_{0.5}$	and RNC_ω;
$EJ_{0.5}$	and ENC_ω; and
RJ_1 (= EJ_1 = J_5)	and RNC_ω + (10) + (13) (= classical logic).

Proof: To show that every theorem of RJ_0 is a theorem of ROC_ω, it suffices to verify that all of the postulates of RJ_0 are derivable in $LROC_\omega$, from which the desired result follows by Theorem 14. This is easily verified, and a similar verification can be given for all of the remaining pairs of systems listed.

To show that the converse does not hold, it suffices to exhibit, for each pair of systems, a theorem of the axiomatic system which is not a theorem of the corresponding J-system.

For the following pairs, a single argument will suffice:

RJ_0	and ROC_ω;
EJ_0	and EOC_ω;

$RJ_{0.1}$ and ROC_ω + (12);

$EJ_{0.1}$ and EOC_ω + (12);

$RJ_{0.2}$ and ROC_ω + (11);

$EJ_{0.2}$ and EOC_ω + (11);

$RJ_{0.5}$ and RNC_ω; and

$EJ_{0.5}$ and ENC_ω.

For all of the above axiomatic systems -- indeed, for all of those listed in Theorem 15 -- the following are theorems: $((A \supset A) \supset A) \supset A$ and $A \supset ((A \supset A) \supset A)$. Hence, by Ax.RC or Ax.EC as appropriate, so is $\neg A \supset \neg((A \supset A) \supset A)$. The following matrices, however, show that this is not a theorem of any of the eight J-systems listed above, for they validate the postulates of these systems, but invalidate $\rightarrow \neg A \supset \neg((A \supset A) \supset A)$ when A is assigned the value 1.

→	0	1	2	3	¬
*0	0	1	2	3	3
1	0	0	2	3	2
2	0	0	0	3	1
3	0	0	0	0	0

⊃	0	1	2	3
0	0	0	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

∧/&	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

∨	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

(Only the value 0 is designated).

For the pairs $RJ_{0.4}$ and RC_ω, and $EJ_{0.4}$ and EC_ω, exactly the same argument holds, except that in the matrices presented, the negation table is replaced by the following:

	¬
0	3
1	0
2	0
3	0 .

(Alternatively, the proof of Theorem 11 of Chapter 3 can be straightforwardly adapted to cover these pairs, as well as the pair RJ_1 and RNC_ω + (10) + (13). There it is shown that $B \vee \neg(A \,\&\, (A \supset B))$ is a theorem of C_ω; hence, it is also a theorem of RC_ω, EC_ω and RNC_ω + (10) + (13). But it is also shown to not be a theorem of J_5 (= RJ_1); hence, it is also not a theorem of $RJ_{0.4}$ or $EJ_{0.4}$, which are subsystems of J_5).

A final argument applies to those pairs that remain, as well as to some of those already considered:

$RJ_{0.1}$	and	ROC_ω + (12);
$EJ_{0.1}$	and	EOC_ω + (12);
$RJ_{0.3}$	and	RIC_ω;
$EJ_{0.3}$	and	EIC_ω;
$RJ_{0.5}$	and	RNC_ω;
$EJ_{0.5}$	and	ENC_ω; and
RJ_1	and	RNC_ω + (10) + (13).

For all of the above axiomatic systems -- indeed, for all of those listed in Theorem 15 -- the following are theorems: $((A \supset A) \supset \urcorner(A \supset A)) \supset \urcorner(A \supset A)$ and $\urcorner(A \supset A) \supset ((A \supset A) \supset \urcorner(A \supset A))$. Hence, by Ax.RC or Ax.EC as appropriate, so is $\urcorner\urcorner(A \supset A) \supset \urcorner((A \supset A) \supset \urcorner(A \supset A))$. But all of the above axiomatic systems incorporate (12); hence, $\urcorner\urcorner(A \supset A)$ is a theorem of these systems and, therefore, so is $\urcorner((A \supset A) \supset \urcorner(A \supset A))$. The following matrices, however, show that this is not a theorem of any of the J-systems listed in Theorem 15, for they validate the postulates of all of these systems, but invalidate $\rightarrow \urcorner((A \supset A) \supset \urcorner(A \supset A))$ for all assignments to A.

$\rightarrow$	0	1	$\urcorner$
*0	0	1	1
1	0	0	0

$\supset$	0	1
0	0	0
1	0	0

$\wedge$/&	0	1
0	0	1
1	1	1

$\vee$	0	1
0	0	0
1	0	1

(Only the value 0 is designated).

This concludes the proof of Theorem 15.

There is, then, no coincidence between the theorems of any of the J-systems under investigation and those of their axiomatic counterparts. The question arises how the J-systems can, and whether they should, be further augmented so that such coincidence obtains.

An initial step towards securing such coincidence might be to add to each J- system as a new postulate that axiomatic theorem shown to be missing from its stock of theorems. However, there is no guarantee that this would suffice; for it is likely that further formulas provable in the axiomatic system but not in the corresponding J-system would be discovered. A more attractive strategy is to ascertain which more general principles, particularly those governing negation, are shown by Theorem 15 to be still lacking from the J-systems.

Consider, for example, those systems shown in Theorem 15 to lack $\rightarrow \neg A \supset \neg((A \supset A) \supset A)$. This is despite the fact that the sequents $\rightarrow((A \supset A) \supset A) \supset A$ and $\rightarrow A \supset ((A \supset A) \supset A)$ can easily be derived in these systems. What is needed to derive $\rightarrow \neg A \supset \neg((A \supset A) \supset A)$ from the first of this pair is not RC, but the rule called ⊃ RC ⊃ in Chapter 2:

$$\supset \text{RC} \supset \qquad \frac{\rightarrow C \supset D}{\rightarrow \neg D \supset \neg C}.$$

Similarly, what is needed to derive $\rightarrow \neg A \supset \neg((A \supset A) \supset A)$ from both sequents is not EC, but the following:

$$\supset \text{EC} \supset \qquad \frac{\rightarrow C \equiv D}{\rightarrow \neg D \supset \neg C}.$$

It is immediate from Theorem 15 that neither of these rules is admissible in those J-systems lacking $\rightarrow \neg A \supset \neg((A \supset A) \supset A)$.

Of course, ⊃ RC ⊃ and ⊃ EC ⊃ are just the respective correlates of Ax.RC and Ax.EC as they might appear in a sequent-based system like the J-systems or the Gentzen-style systems presented earlier. In fact, because these Gentzen-style systems have *modus ponens* as a derivable rule (in the form: from $\rightarrow A$ and $\rightarrow A \supset B$, infer $\rightarrow B$), the presence of RC is sufficient also for the derivability of ⊃ RC ⊃, and the presence of EC is sufficient also for the derivability of ⊃ EC ⊃. The J-systems, on the other hand, lack *modus ponens*, so the presence of RC or EC is not in general sufficient for the derivability, or even the admissibility, of $\supset$ RC $\supset$ or $\supset$ EC $\supset$. (We note that in the systems J_2 to J_5, $\supset$ RC $\supset$ and $\supset$ EC $\supset$ are derivable, but this is due to the presence of postulate $\neg_5$) rather than RC or EC, which are derivable only in J_5).

Thus, although RC and EC guarantee the desired intersubstitutivity property SE, they do not suffice in the context of the J-systems to properly reflect the incorporation of Ax.RC and Ax.EC into the axiomatic systems. To achieve this, it seems that $\supset$ RC $\supset$ would need to be further added to those J-systems incorporating RC, and $\supset$ EC $\supset$ to those incorporating EC.

Even if such additions were made, however, there is still no guarantee that coincidence of theorems would obtain. For consider those J-systems shown to lack the sequent $\rightarrow \neg((A \supset A) \supset \neg(A \supset A))$ in Theorem 15; among them, some of those in which $\supset$ RC $\supset$ and $\supset$ EC $\supset$ have just been noted to be inadmissible. Even if these rules were added to the J-systems as suggested, there is no reason to believe that the above sequent would then be derivable. What is needed to derive this sequent in the J-systems seem to be something more like postulate $\neg_7$) of J_3: $A, \neg B \rightarrow \neg(A \supset B)$. And to secure $\rightarrow B \vee \neg(A \ \& \ (A \supset B))$ for the systems $RJ_{0.4}$, $EJ_{0.4}$ and RJ_1, a different addition again would seem to be required.

But it has not yet been established that augmenting the J-systems so as to achieve a coincidence between their theorems and those of their axiomatic counterparts is at all desirable. Among the systems for which such augmentation has been considered are $RJ_{0.1}$, $RJ_{0.4}$ and $RJ_{0.5}$, the most likely candidates among all the J-systems to simultaneously enjoy SE and satisfy both of da Costa's conditions for paraconsistency. But it may be that augmenting these systems in order to secure such coincidence would also compromise their satisfaction of condition (I), in which case such augmentation would be clearly undesirable. On the other hand, if satisfaction of (I) is not thereby compromised, then condition (II′) virtually enjoins us to so augment them.

The following result indicates which way this question will be resolved.

Theorem 16. If the rule ⊃ RC ⊃ is added to each of the systems $RJ_{0.1}$, $RJ_{0.4}$ and $RJ_{0.5}$, the following sequents are still not derivable:

$A, \neg A \rightarrow B$

$A, \neg A \rightarrow B \supset C$

$A, \neg A \rightarrow \neg^n B.$

Proof: The matrices in the proof of Theorem 4, which validate the postulates of $RJ_{0.1}$ and of $RJ_{0.5}$ but invalidate the above sequents, and those in the proof of Theorem 7, which validate the postulates of $RJ_{0.4}$ but similarly invalidate these sequents, also validate ⊃ RC ⊃.

The rule ⊃ RC ⊃, then, can be added to the systems $RJ_{0.1}$, $RJ_{0.4}$ and $RJ_{0.5}$ without compromising their satisfaction of (I). Similarly, it can be verified that these matrices validate also $\neg_7$); this too can be added. Further, the matrices of Theorem 7 validate $\rightarrow B \vee \neg(A \,\&\, (A \supset A))$; this sequent also can be added to $RJ_{0.4}$ without compromising condition (I).

The logical conclusion of this line of argument can quickly be established, for another sequent which is validated by the matrices referred to in the proof of Theorem 16 is $A, A \supset B \rightarrow B$. The effect of adding this sequent to the J-systems is to restore *modus ponens*, thereby collapsing these systems into the Gentzen-style equivalents of their axiomatic counterparts. While this neatly solves the problem of augmenting the J-systems so that their theorems coincide with those of their axiomatic counterparts, it does so at the expense of continued interest in the former, for we might as well dispense with the J-systems so augmented in favour of their more simply formulated axiomatic and Gentzen-style equivalents. Accordingly, a more detailed investigation of the latter systems (and their extensions) is undertaken in the next two chapters.

However, it would be premature to discard the J-systems outright, for the underivability of the sequents of Theorem 16 is only an indication, not a guarantee, of the satisfaction of (I);

it might be that the restoration of *modus ponens* to $RJ_{0.1}$, $RJ_{0.4}$ and $RJ_{0.5}$ would precipitate in some as yet unforseen way the substantive collapse of certain otherwise stable inconsistent theories based on these three systems.

7. DISTINCTNESS OF THE SYSTEMS

We conclude by showing that the systems occurring in the diagram in Section 2 are in fact distinct. We show firstly that the three branches are disjoint.

Theorem 17. No system occurring in the diagram in Section 2 is both a member of the intuitionistic branch and also a member of either the dualintuitionistic or the intermediate branch.

Proof: This is immediate from the fact that every system occurring in the diagram which is a member of either of the latter two branches incorporates the intuitionistically underivable postulate $\neg_2$).

Theorem 18. No system occurring in the diagram in Section 2 is both a member of the dualintuitionistic and the intermediate branch.

Proof: Every system in the diagram which is a member of the intermediate branch incorporates $\neg_1$) as a postulate. However, the matrices in the proof of Theorem 7 validate the postulates of $RJ_{0.4}$ (and therefore also of every other member of the dualintuitionistic branch), but invalidate $\neg_1$) when A is assigned the value 1.

Theorems 17 and 18 suffice to establish that the three branches of systems occurring in the diagram in Section 2 are disjoint. We now show that the systems belonging to each branch are distinct.

Theorem 19. The intuitionistic systems RJ_0, EJ_0, $RJ_{0.1}$, $EJ_{0.1}$, $RJ_{0.3}$ and $EJ_{0.3}$ are distinct.

Proof: That RJ_0 is distinct from $RJ_{0.1}$ and that EJ_0 is distinct from $EJ_{0.1}$ are established by the following matrices, which validate the postulates of RJ_0 (and therefore also of EJ_0), but invalidate postulate $\neg_1$) of $RJ_{0.1}$ and $EJ_{0.1}$ when A is assigned the value 0.

$\to/\supset$	0	1	$\neg$
*0	0	1	1
1	0	0	1

$\wedge/\&$	0	1
0	0	1
1	1	1

$\vee$	0	1
0	0	0
1	0	1

(Only the value 0 is designated).

That $RJ_{0.1}$ is distinct from $RJ_{0.3}$ and that $EJ_{0.1}$ is distinct from $EJ_{0.3}$ are established by the matrices in the proof of Theorem 4, which validate the postulates of $RJ_{0.1}$ (and therefore also of $EJ_{0.1}$), but invalidate postulate $\neg_3$) of $RJ_{0.3}$ and $EJ_{0.3}$ when A is assigned the value 1.

That EJ_0 is distinct from RJ_0 and that $EJ_{0.1}$ is distinct from $RJ_{0.1}$ are established as follows. The sequent $\neg A \to \neg (A \& B)$ is derivable in RJ_0 and $RJ_{0.1}$ by one application of RC to postulate $\&_2$). However, that it is not derivable in EJ_0 or $EJ_{0.1}$ is established by the first set of matrices occurring in this proof, with the modification that the negation-table is replaced by the following:

	$\neg$
0	0
1	1 .

The matrices thus modified validate the postulates of $EJ_{0.1}$ (and therefore also of EJ_0), but invalidate $\neg A \to \neg (A \& B)$ when A is assigned the value 0 and B is assigned the value 1.

Finally, that $EJ_{0.3}$ is distinct from $RJ_{0.3}$ follows from the fact that the sequent A, $\neg A \to \neg B$ is derivable in $RJ_{0.3}$ by Theorem 3, but is not derivable in $EJ_{0.3}$ by Theorem 5.

These observations suffice to establish that the systems listed in Theorem 19 are distinct.

<u>Theorem 20</u>. The dualintuitionistic systems $RJ_{0.2}$, $EJ_{0.2}$, $RJ_{0.4}$ and $EJ_{0.4}$ are distinct.

<u>Proof</u>: That $RJ_{0.2}$ is distinct from $RJ_{0.4}$ and that $EJ_{0.2}$ is distinct from $EJ_{0.4}$ are established by the matrices in the proof of Theorem 4, which validate the postulates of $RJ_{0.2}$ (and therefore also of $EJ_{0.2}$), but invalidate postulate $\neg_4$) of $RJ_{0.4}$ and $EJ_{0.4}$ when A is assigned the value 1.

That $EJ_{0.2}$ is distinct from $RJ_{0.2}$ is established by the fact that the sequent $\neg A \to \neg$ (A

& B) is derivable in $RJ_{0.2}$ by one application of RC to postulate $\&_2$), but is not derivable in $EJ_{0.2}$, since the modified matrices in the proof of Theorem 19 validate the postulates of $EJ_{0.2}$, but invalidate ¬A → ¬ (A & B) as in that proof.

Finally, that $EJ_{0.4}$ is distinct from $RJ_{0.4}$ is established by the following matrices, which validate the postulates of $EJ_{0.4}$, but invalidate the sequent → ¬ (A & ¬A), shown to be derivable in $RJ_{0.4}$ in Theorem 9, when A is assigned the value 1.

→/⊃	0	1	2	3	4	5	6	7	¬
*0	0	1	2	3	4	5	6	7	7
1	0	0	2	3	2	3	6	6	2
2	0	1	0	3	1	5	3	5	5
3	0	1	2	0	4	1	2	4	4
4	0	0	0	3	0	3	3	3	3
5	0	0	2	0	2	0	2	2	2
6	0	1	0	0	1	1	0	1	0
7	0	0	0	0	0	0	0	0	0

∧/&	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	4	5	4	5	7	7
2	2	4	2	6	4	7	6	7
3	3	5	6	3	7	5	6	7
4	4	4	4	7	4	7	7	7
5	5	5	7	5	7	5	7	7
6	6	7	6	6	7	7	6	7
7	7	7	7	7	7	7	7	7

∨	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	0	1	1	0	1
2	0	0	2	0	2	0	2	2
3	0	0	0	3	0	3	3	3
4	0	1	2	0	4	1	2	4
5	0	1	0	3	1	5	3	5
6	0	0	2	3	2	3	6	6
7	0	1	2	3	4	5	6	7

(Only the value 0 is designated).

These observations suffice to establish that the systems listed in Theorem 20 are distinct.

Theorem 21. The intermediate systems $RJ_{0.5}$, and $EJ_{0.5}$ and RJ_1 (= EJ_1) are distinct.

Proof: That $RJ_{0.5}$ is distinct from RJ_1 is established by the matrices in the proof of Theorem 4, which validate the postulates of $RJ_{0.5}$, but invalidate postulate $\neg_3$) of RJ_1 when A is assigned the value 1.

That $EJ_{0.5}$ is distinct from $RJ_{0.5}$ is established by the fact that the sequent $\neg A \rightarrow \neg (A \,\&\, B)$ is again derivable in $RJ_{0.5}$ but not in $EJ_{0.5}$, since the modified matrices in the proof of Theorem 19 validate the postulates of $EJ_{0.5}$, but invalidate $\neg A \rightarrow \neg (A \,\&\, B)$ as in that proof.

These observations suffice to establish that the systems listed in Theorem 21 are distinct.

Theorems 17 through to 21 together suffice to establish that all of the systems occurring in the diagram in Section 2 are distinct.

Chapter Five: Paraconsistency and the C-Systems

1. THE C-SYSTEMS AND THE PARACONSISTENCY CONDITIONS

In discussing the C-systems in [9], da Costa states that "it seems natural that they satisfy" not only (I) and (II), but also the following:

(III) in these systems, the schema $\neg(A \,\&\, \neg A)$ must not be derivable; and

(IV) it must be simple to extend the systems to predicate calculi (with or without equality) of first order.

As noted in Chapter 2, (I) is unanimously accepted as a necessary condition for paraconsistent systems; and (IV) is also uncontroversial, if only because paraconsistency researchers have standardly sought to guarantee the stability of their systems under the burden of inconsistency at the propositional level, rather than by tampering with the usual (classical) rules governing the quantifiers. Like (II), however, condition (III) is not so generally endorsed. Certainly, the construction of paraconsistent logics incorporating $\neg(A \,\&\, \neg A)$ is not unusual, even in Brazilian circles, as the J-systems of Arruda and da Costa testify.

Even if all four conditions are accepted, however, it is not absolutely clear that they are all satisfied by the C-systems. Conditions (I) and (III) present no problem, for $\neg(A \,\&\, \neg A)$ is indeed not derivable in $C_n (1 \leqslant n \leqslant \omega)$, nor is an arbitrary proposition B derivable from contradictory formulas A and $\neg A$. And the first-order extension of the systems, as described in [9], is evidently simple enough to satisfy (IV). Rather, it is with respect to (II) that some room for doubt emerges.

As with the J-systems, the reason for this doubt is that the C-systems similarly fail to enjoy SE, the property of intersubstitutivity of provable equivalents. (This is noted in [11], Corollary to Theorem 1). This failure is investigated more precisely in the following section.

2. THE C-SYSTEMS AND THE PROPERTY OF INTERSUBSTITUTIVITY OF PROVABLE EQUIVALENTS

For convenience, we restate the postulates of C_ω:

(1) $A \supset (B \supset A)$

(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

(3) $\dfrac{A \quad A \supset B}{B}$

(4) $(A \,\&\, B) \supset A$

(5) $(A \,\&\, B) \supset B$

(6) $A \supset (B \supset (A \,\&\, B)$

(7) $A \supset (A \vee B)$

(8) $B \supset (A \vee B)$

(9) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$.

(10) $A \vee \neg A$

(11) $\neg\neg A \supset A$.

We note that, by Theorem 1 of [11], sufficient to collapse C_ω into C_0 (classical logic) is addition of the *reductio* schema, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$. The remaining systems $C_n (1 \leqslant n < \omega)$ extend C_ω by adding this schema, though in a qualified fashion rather than *simpliciter*. For each system, the schema is qualified by a formula which can be interpreted as expressing the proposition that B is not paradoxical, or "behaves classically". For C_1, the qualification is the formula B°, which is defined as $\neg(B \,\&\, \neg B)$. In addition, compounding principles ensure that compounds of "classical" formulas are themselves "classical".

The postulates of C_1 are those of C_ω together with the following:

$(12)^\circ$ $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

$(13)^\circ$ $(A^\circ \,\&\, B^\circ) \supset (A \,\&\, B)^\circ$

$(14)^\circ$ $(A^\circ \,\&\, B^\circ) \supset (A \vee B)^\circ$

$(15)^\circ$ $(A^\circ \,\&\, B^\circ) \supset (A \supset B)^\circ$

$(16)^\circ$ $B^\circ \supset (\neg B)^\circ$.

For each remaining $C_n (1 \leqslant n < \omega)$, B° is replaced by $B^{(n)}$, which is the conjunction $B^n \,\&\, B^{n-1} \,\&\, \ldots \,\&\, B^1$, where $B^1 = B^\circ$ and $B^i = B^{\overset{0\ldots0}{i\ times}}$. (For completeness, $B^{(1)}$ is defined to be B^1).

The postulates of $C_n (1 < n < \omega)$ are those of C_1, except that $(12)^\circ$ to $(16)^\circ$ are replaced by the following:

$(12)^{(n)}$ $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

$(13)^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \,\&\, B)^{(n)}$

$(14)^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \vee B)^{(n)}$

$(15)^{(n)}$ $(A^{(n)} \,\&\, B^{(n)}) \supset (A \supset B)^{(n)}$

$(16)^{(n)}$ $B^{(n)} \supset (\neg B)^{(n)}$.

By Theorem 9 of [9], the systems $C_0, C_1, C_2, \ldots, C_\omega$ are all distinct and form a linear hierarchy with strongest member C_0 and weakest member C_ω.

We now turn to SE, the property of intersubstitutivity of provable equivalents. For axiomatic systems like the C-systems, it is most natural to define two formulas B and C to be *provably equivalent* just in case $(B \supset C) \,\&\, (C \supset B)$, abbreviated $B \equiv C$, is derivable. (We note that this is not the only sort of equivalence definable in these systems; other definitions are given in [14]). SE, then, is the property that, where B is a subformula of a theorem A, and C is a formula which is provably equivalent to B, the result of substituting C for some or all occurrences of B in A is also a theorem.

Because the C-systems incorporate positive intuitionistic logic, they at least enjoy the property of intersubstitutivity of provable equivalents in negation-free contexts (SE^+). It is where negation is involved that intersubstitutivity fails, as the following result shows.

<u>Theorem 1</u>. The systems $C_n (1 \leqslant n \leqslant \omega)$ do not enjoy SE.

<u>Proof</u>: Easily derived in C_ω, and therefore in each $C_n (1 \leqslant n \leqslant \omega)$, are the schemata $A \equiv (A \,\&\, A)$ and $\neg A \supset \neg A$. If $C_n (1 \leqslant n \leqslant \omega)$ enjoyed SE, then $\neg A \supset \neg(A \,\&\, A)$, the result of substituting A & A for (one occurrence of) A in $\neg A \supset \neg A$, would also be derivable. However, the following matrices show that $\neg A \supset \neg(A \,\&\, A)$ is not derivable in $C_n (1 \leqslant n \leqslant \omega)$, for they validate the postulates of these systems, but invalidate this schema when A is assigned the value 1.

⊃	0	1	2	3	4	¬
*0	0	1	2	3	4	4
*1	0	0	2	3	4	3
2	0	0	0	3	3	3
3	0	0	2	0	2	2
4	0	0	0	0	0	0

&	0	1	2	3	4
0	0	1	2	3	4
1	1	0	2	3	4
2	2	2	2	4	4
3	3	3	4	3	4
4	4	4	4	4	4

∨	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	0	2
3	0	1	0	3	3
4	0	1	2	3	4

(Only the values 0 and 1 are designated).

A much sharper result can be obtained for C_ω.

<u>Theorem 2</u>. In C_ω, the schema $\neg B \equiv \neg C$ is derivable if and only if B and C are the same formula.

<u>Proof</u>: The proof proceeds by considering the Gentzen-style system WG_ω of [16], which is stronger than C_ω in that, while for every formula $A \supset B$ derivable in C_ω the sequent $A \rightarrow B$ is derivable in WG_ω, the converse does not hold. (The terminology used in this proof is that of

Gentzen's [12]).

We note firstly that the inference figure Cut is proved in [16] to be eliminable from WG_ω, and that the system is shown to be not finitely trivialisable, i.e. there is no formula B such that B → C is derivable for an arbitrary formula C. From this, it follows that WG_ω has no derivable sequent of the form → ¬B. For WG_ω has no initial sequents of this form; hence, such a sequent could only be derived by the inference figure ¬-IS from the sequent B → . But from the latter, B → C follows by Thinning in the succedent for an arbitrary formula C, contradicting the fact that WG_ω is not finitely trivialisable. Hence, WG_ω has no derivable sequent of the form → ¬B.

We now show by induction on the length of derivation in WG_ω that, if Γ → ¬B is a derivable sequent, then ¬B is a (possibly improper) subformula of some member of Γ.

Base case. In this case, Γ → ¬B is an initial sequent of the form ¬B → ¬B. Of course, ¬B is a subformula of itself.

Inductive step. In this case, Γ → ¬B is derived by application of some inference figure. The only candidates are the structural figures (Thinning, Contraction and Interchange), the -IA figures (⊃-IA, &-IA and ∨-IA), and ¬-IS.

In fact, ¬-IS is not a possibility. For if Γ → ¬B were derived from B, Γ → by ¬-IS, then the sequent B & (&Γ) → would also be derivable (where (&Γ) represents the conjunction of all of the members of Γ), and hence so would be B & (&Γ) → C for arbitrary C by Thinning, contradicting the fact that WG_ω is not finitely trivialisable. A similar argument shows that Γ → ¬B cannot be the result of Thinning in the succedent.

This leaves only those figures in which the principal formula occurs in the antecedent. We consider only Thinning and ⊃-IA; the remaining figures can be dealt with similarly.

If Γ → ¬B is derived by Thinning in the antecedent, then Γ is a sequence of the form C, Γ', and the upper sequent of the figure is Γ' → ¬B. The observation that → ¬B cannot be derived in WG_ω ensures that Γ' is not empty. On inductive hypothesis, then, ¬B is a subformula of some member of Γ', and hence also of some member of Γ.

If Γ → ¬B is derived by ⊃-IA, then Γ is a sequence of the form C ⊃ D, Γ_1, Γ_2, and the upper sequents of the figure are Γ_1 → C and D, Γ_2 → ¬B. On inductive hypothesis, ¬B is a subformula of D or of some member of Γ_2. But then it must also be a subformula of some member of Γ.

We have shown, then, that for every sequent of the form $\Gamma \rightarrow \neg B$ derivable in WG_ω, $\neg B$ must be a subformula of some member of Γ. To complete the proof of Theorem 2, it suffices to note that, if $\neg B \equiv \neg C$ is a theorem of C_ω, then $\neg B \rightarrow \neg C$ and $\neg C \rightarrow \neg B$ are both derivable sequents in WG_ω, and therefore $\neg B$ and $\neg C$ must each be a subformula of the other, from which it follows that they are in fact the same formula.

Theorem 2 shows that, despite the incorporation of postulates (10) and (11), C_ω is a very weak system with respect to negation: no two (different) negated formulas are provably equivalent in this system. We turn now to the stronger C-systems. For simplicity, we restrict our attention initially to C_1.

Particular interest attaches to the question which formulas are provably equivalent in C_1 to the schema B°, because of the special role which this formula plays. It might be expected that such trivial variants as $\neg(\neg B \ \& \ B)$ or $\neg((B \ \& \ \neg B) \ \& \ (B \ \& \ \neg B))$ could be proved equivalent in C_1 to B°, and hence equally capable of expressing the proposition that B "behaves classically". But in the absence of SE, there is no guarantee of a uniform argument to this effect, for although $(B \ \& \ \neg B) \equiv (\neg B \ \& \ B)$ and $(B \ \& \ \neg B) \equiv ((B \ \& \ \neg B) \ \& \ (B \ \& \ \neg B))$ are easily derived in C_1, it does not follow that $\neg(B \ \& \ \neg B) \equiv \neg(\neg B \ \& \ B)$ and $\neg(B \ \& \ \neg B) \equiv \neg((B \ \& \ \neg B) \ \& \ (B \ \& \ \neg B))$ are also derivable. These cases must therefore be considered individually.

A first result is promising.

<u>Theorem 3</u>. In C_1, the schema $\neg(B \ \& \ \neg B) \equiv \neg((B \ \& \ \neg B) \ \& \ (B \ \& \ \neg B))$ is derivable.

<u>Note</u>: In this and subsequent proofs, we will make use of the following rules and schemata, easily shown to be derivable in positive intuitionistic logic and therefore in all of the C-systems.

Transitivity: $\dfrac{C \supset D \quad D \supset E}{C \supset E}$

Permutation of antecedents: $\dfrac{C \supset (D \supset E)}{D \supset (C \supset E)}$

Importation: $\dfrac{C \supset (D \supset E)}{(C \& D) \supset E}$

Exportation: $\dfrac{(C \& D) \supset E}{C \supset (D \supset E)}$

Prefixing: $(D \supset E) \supset ((C \supset D) \supset (C \supset E))$

Suffixing: $(C \supset D) \supset ((D \supset E) \supset (C \supset E))$

Distribution: $(C \&(D \vee E)) \equiv ((C \& D) \vee (C \& E))$
$(C \vee (D \& E)) \equiv ((C \vee D) \& (C \vee E))$.

In addition, we will use the following schemata, easily shown to be derivable in C_ω with the assistance of (9) and (10).

C_ω-*reductio*: $(C \supset \neg C) \supset \neg C$
$(\neg C \supset C) \supset C$.

<u>Proof of Theorem 3</u>: The derivation of $\neg(B \& \neg B) \supset \neg((B \& \neg B) \& (B \& \neg B))$ is as follows. By postulate (4), we have both

$((B \& \neg B) \& (B \& \neg B)) \supset (B \& \neg B)$ and $(B \& \neg B) \supset B$,

from which follows, by transitivity,

$((B \& \neg B) \& (B \& \neg B)) \supset B$.

A similar argument, using also (5), yields

$((B \& \neg B) \& (B \& \neg B)) \supset \neg B$.

Substituting $((B \& \neg B) \& (B \& \neg B)$ for A in $(12)^{\circ}$ and permuting antecedents yields

$(((B \& \neg B) \& (B \& \neg B)) \supset B) \supset ((((B \& \neg B) \& (B \& \neg B)) \supset \neg B) \supset$

$(B^{\circ} \supset \neg ((B \& \neg B) \& (B \& \neg B)))).$

But the first two antecedents have been shown to be derivable, hence by two applications of (3), we obtain the desired

$B^{\circ} \supset \neg((B \& \neg B) \& (B \& \neg B)).$

The converse derivation is as follows. We show firstly that $(B \& \neg B)^{\circ}$ is derivable in C_1. Substituting $(B \& \neg B) \& \neg(B \& \neg B)$ for A in $(12)^{\circ}$ and permuting antecedents yields

$(((B \& \neg B) \& \neg(B \& \neg B)) \supset B) \supset (((B \& \neg B) \& \neg(B \& \neg B)) \supset \neg B) \supset$

$(B^{\circ} \supset \neg((B \& \neg B) \& \neg(B \& \neg B)))).$

The first two antecedents are easily derived, leaving

$B^{\circ} \supset \neg((B \& \neg B) \& \neg(B \& \neg B))).$

But an instance of (5) is

$((B \& \neg B) \& \neg(B \& \neg B)) \supset B^{\circ},$

hence by transitivity, we obtain

$((B \& \neg B) \& \neg(B \& \neg B)) \supset \neg((B \& \neg B) \& \neg(B \& \neg B)).$

From this follows, by C_{ω}-*reductio* and (3),

$\neg((B \& \neg B) \& \neg(B \& \neg B)),$

which is by definition $(B \& \neg B)^{\circ}$.

To continue, it follows straightforwardly from the above result and $(13)^{\circ}$ that $((B \& \neg B) \& (B \& \neg B))^{\circ}$ is also derivable in C_1. We now consider the result of substituting $B \& \neg B$ and $(B \& \neg B) \& (B \& \neg B)$ for, respectively, A and B in $(12)^{\circ}$. As just observed, the first antecedent is derivable in C_1, leaving

$((B \& \neg B) \supset ((B \& \neg B) \& (B \& \neg B))) \supset$

$(((B \& \neg B) \supset \neg((B \& \neg B) \& (B \& \neg B))) \supset \neg(B \& \neg B)).$

The antecedent of the above schema is also easily derived, using (6) and (2), leaving

$((B \& \neg B) \supset \neg((B \& \neg B) \& (B \& \neg B))) \supset \neg(B \& \neg B).$

But an instance of (1) is

$\neg((B \& \neg B) \& (B \& \neg B)) \supset ((B \& \neg B) \supset \neg((B \& \neg B) \& (B \& \neg B))),$

whence by transitivity, we obtain the desired

$\neg((B \& \neg B) \& (B \& \neg B)) \supset \neg(B \& \neg B).$

This concludes the proof of Theorem 3.

We have shown, then, that one of the formulas put forward as a trivial variant of B° is in fact provably equivalent to it in C_1. Unfortunately, the same cannot be said for the second formula, $\neg(\neg B \,\&\, B)$.

<u>Theorem 4</u>. In C_1, the schema $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ is not derivable.

<u>Proof</u>: One half on this schema, $\neg(B \,\&\, \neg B) \supset \neg(\neg B \,\&\, B)$ is in fact derivable in C_1 as follows. Instances of postulates (5) and (4) respectively are $(\neg B \,\&\, B) \supset B$ and $(\neg B \,\&\, B) \supset \neg B$. Substituting $\neg B \,\&\, B$ for A in $(12)^{\circ}$ and permuting antecedents yields $((\neg B \,\&\, B) \supset B) \supset (((\neg B \,\&\, B) \supset \neg B) \supset (B^{\circ} \supset \neg(\neg B \,\&\, B)))$. But the first two antecedents have been shown to be derivable; hence by two applications of (3), we obtain $B^{\circ} \supset \neg(\neg B \,\&\, B)$, which is just the desired $\neg(B \,\&\, \neg B) \supset \neg(\neg B \,\&\, B)$.

The converse, however, is not derivable. This is shown by the following matrices, which validate the postulates of C_1 but invalidate this schema when B is assigned the value 1.

$\supset$	0	1	2	3	4	5	$\neg$
*0	0	0	0	0	0	5	5
*1	0	0	0	0	0	5	2
*2	0	0	0	0	0	5	3
*3	0	0	0	0	0	5	4
*4	0	0	0	0	0	5	5
5	0	0	0	0	0	0	4

&	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	4	3	4	5
2	2	3	2	4	4	5
3	3	3	3	3	4	5
4	4	4	4	4	4	5
5	5	5	5	5	5	5

$\vee$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	1	1
2	0	0	2	0	2	2
3	0	1	2	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

(Only the value 5 is not designated).

That $\neg(B \,\&\, \neg B)$ and $\neg(\neg B \,\&\, B)$ are not provably equivalent in C_1 is certainly curious, if not anomalous. For, as is argued with respect to the similarly deficient system J_1 in Chapter 2, such fine discrimination between what would ordinarily be regarded as mere syntactic variants demands some sort of justification. But the motivating considerations for C_1, namely, conditions (I) to (IV), not only do not support such discrimination but militate against it.

For although $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ is not derivable in C_1, the matrices in the proof of Theorem 1 show that this schema can be added to C_1 without compromising its satisfaction of condition (I). For the C-systems, satisfaction of (I) is equivalent to the underivability of the schema $(A \,\&\, \neg A) \supset B$. But the matrices in question invalidate this schema (when A is assigned the value 1 and B is assigned the value 2, for example) while at the same time validating not only the postulates of C_1 but also $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$.

The absence of the above schema from C_1, then, presents itself as not only anomalous in its own right, but in contravention of at least the spirit of condition (II). These considerations univocally suggest that C_1 should be extended to include this schema.

Of course, it is not to be expected that the mere addition of $\neg(B \,\&\, \neg B) \equiv \neg(\neg B \,\&\, B)$ will remedy any other deficiencies of C_1. For, as Theorems 3 and 4 show, the presence of a schema stating the equivalence of B° to one syntactic variant is not sufficient to guarantee its equivalence to any other. And even if sufficiently many such schemata could be added to secure the equivalence of B° to all such variants, there is no reason to believe that the deficiencies exhibited in Theorem 1 would not remain, for it is shown there that, in general, $\neg A \equiv \neg(A \,\&\, A)$ is not derivable in C_1, notwithstanding that the particular instance of this schema obtained by substituting $B \,\&\, \neg B$ for A is shown to be derivable in Theorem 3.

Rather than attempting to treat individually the symptoms of the failure of the C-systems to enjoy SE, a more obviously systematic strategy is to attempt to secure this property directly.

In [19], two methods of extending the C-systems in order to secure SE are proposed:

(i) addition of the rule RC: $\dfrac{C \supset D}{\neg D \supset \neg C}$; and

(ii) addition of the (weaker) rule EC: $\dfrac{C \equiv D}{\neg D \supset \neg C}$.

(We note that the above rules were given the names Ax.RC and Ax.EC in Chapter 4; however, since we do not need to distinguish between the axiomatic and sequent-based versions here, we revert to the original and simpler nomenclature).

Because the C-systems enjoy SE^+, and lack only the property of intersubstitutivity of provable equivalents in negated contexts, it is evident that the addition of either RC or EC is sufficient to guarantee SE in full. Moreover, the admissibility of EC (in any extension of a C-system) is also a necessary condition for SE.

3. THE RC-SYSTEMS

We investigate firstly the result of adding RC to the C-systems. For each $C_n (1 \leqslant n \leqslant \omega)$, the result of adding RC will be called RC_n. (We note that this diverges from the nomenclature of [19], in which the resulting systems are called $CC_n (1 \leqslant n \leqslant \omega)$).

The following initial result and its proof are taken directly from [19].

Theorem 5. $RC_\omega \neq C_0$ (classical logic). In particular, the schema $(A \,\&\, \neg A) \supset B$ is not derivable in RC_ω.

Proof: The following matrices validate the postulates of RC_ω but invalidate the above schema when A is assigned the value 1 and B is assigned the value 2.

⊃	0	1	2	¬
*0	0	1	2	2
1	0	0	2	0
2	0	0	0	0

&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

∨	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2.

(Only the value 0 is designated).

Thus, the addition of RC to C_ω does not result in any compromise of condition (I), and the system so obtained certainly does not suffer from deficiencies exhibited in Theorem 2.

Unfortunately, the same is not true of the stronger C-systems. The following result is proved in [19], but we employ a rather simpler proof below.

Theorem 6. For $1 \leqslant n < \omega$, $RC_n = C_0$.

Proof: It suffices to show that, for any $1 \leqslant n < \omega$, the formula $B^{(n)}$ is derivable in RC_ω. Since this is a subsystem of each $RC_n (1 \leqslant n < \omega)$, it follows that, in each such RC_n, the formula qualifying the *reductio* schema in $(12)^{(n)}$ is derivable, and hence so is unqualified *reductio.* As noted earlier, this suffices to collapse C_ω, and therefore every $RC_n (1 \leqslant n < \omega)$, into classical logic.

We show firstly that B° is derivable in RC_ω for any formula B. An instance of postulate (4) is $(B \ \& \ \neg B) \supset B$. Applying RC yields $\neg B \supset \neg(B \ \& \ \neg B)$. Similarly, (5) and RC yield $\neg\neg B \supset \neg(B \ \& \ \neg B)$. These, together with (9), yield $(\neg B \vee \neg\neg B) \supset \neg(B \ \& \ \neg B)$. But the antecedent is an instance of (10), so (3) yields $\neg(B \ \& \ \neg B)$, which is B° by definition.

From this, it follows straightforwardly that B^n is derivable in RC_ω for $1 < n < \omega$, since each such B^n is itself of the form $(B^{n-1})^\circ$. A simple inductive argument then shows that the conjunction $B^n \ \& \ B^{n-1} \ \& \ ... \ \& \ B^\circ$, which is by definition $B^{(n)}$, is also derivable.

Thus, the addition of RC to the C-systems collapses all but the weakest system, C_ω, into classical logic. We turn instead to EC, in the hope that the addition of this rule will not have such drastic consequences.

4. THE EC-SYSTEMS

For each $C_n (1 \leqslant n \leqslant \omega)$, the result of adding EC is called EC_n. (This is as in [19]).

Theorem 7. $EC_\omega \neq C_0$.

Proof: This follows from Theorem 5 and the fact the EC is derivable from RC in any extension of C_ω; hence, EC_ω is a subsystem of RC_ω.

In fact, EC_ω is a proper subsystem of RC_ω.

Theorem 8. $EC_\omega \neq RC_\omega$.

Proof: The following matrices validate the postulates of EC_ω, but invalidate the schema $\neg(B \ \& \ \neg B)$, shown to be derivable in RC_ω in the proof of Theorem 6, when B is assigned the value 1.

⊃	0	1	2	3	4	5	6	7	¬
*0	0	1	2	3	4	5	6	7	7
1	0	0	2	3	2	3	6	6	2
2	0	1	0	3	1	5	3	5	5
3	0	1	2	0	4	1	2	4	4
4	0	0	0	3	0	3	3	3	3
5	0	0	2	0	2	0	2	2	2
6	0	1	0	0	1	1	0	1	0
7	0	0	0	0	0	0	0	0	0

&	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	4	5	4	5	7	7
2	2	4	2	6	4	7	6	7
3	3	5	6	3	7	5	6	7
4	4	4	4	7	4	7	7	7
5	5	5	7	5	7	5	7	7
6	6	7	6	6	7	7	6	7
7	7	7	7	7	7	7	7	7

∨	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	0	1	1	0	1
2	0	0	2	0	2	0	2	2
3	0	0	0	3	0	3	3	3
4	0	1	2	0	4	1	2	4
5	0	1	0	3	1	5	3	5
6	0	0	2	3	2	3	6	6
7	0	1	2	3	4	5	6	7

(Only the value 0 is designated).

Thus, EC_ω is distinct from RC_ω. We now determine whether $EC_n(1 \leqslant n < \omega)$ are distinct from $RC_n(1 \leqslant n < \omega)$. Several lemmas are required.

In the following lemmas, F_n is defined for each system $C_n(1 \leqslant n < \omega)$ to be the schema $B^{(n)}$ & (B & ¬B).

Lemma 1. In each $C_n(1 \leqslant n < \omega)$, the schema $F_n \supset A$ is derivable.

Proof: An instance of postulate $(12)^{(n)}$ of each $C_n(1 \leqslant n < \omega)$ is $B^{(n)} \supset ((\neg A \supset B) \supset ((\neg A \supset \neg B) \supset \neg\neg A))$. Permuting antecedents yields $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset (B^{(n)} \supset \neg\neg A))$. An instance of (1) is $B \supset (\neg A \supset B)$, so transitivity delivers $B \supset ((\neg A \supset \neg B) \supset (B^{(n)} \supset \neg\neg A))$. Permutation yields $(\neg A \supset \neg B) \supset (B \supset (B^{(n)} \supset \neg\neg A))$, which by (1) and transitivity is further reduced to $\neg B \supset (B \supset (B^{(n)} \supset \neg\neg A))$. Permutation and importation transform this into $(B^{(n)} \& (B \& \neg B)) \supset \neg\neg A$. By (11) and transitivity, this yields $(B^{(n)} \& (B \& \neg B)) \supset A$, which is, by definition, $F_n \supset A$.

Lemma 2. In C_ω, the schema $A \vee A^{(i)}$ is derivable, for $1 \leqslant i < \omega$.

Proof: We consider firstly the schema $\neg A^i$. For $1 < i < \omega$, $\neg A^i$ is, by definition, $\neg\neg(A^{i-1} \& \neg A^{i-1})$. By (11), we have $\neg\neg(A^{i-1} \& \neg A^{i-1}) \supset (A^{i-1} \& \neg A^{i-1})$, and by (5), $(A^{i-1} \& \neg A^{i-1}) \supset \neg A^{i-1}$; hence, by transitivity, we obtain $\neg A^i \supset \neg A^{i-1}$. A straightforward inductive argument shows that we therefore have $\neg A^i \supset \neg A^\circ$ for all $1 < i < \omega$, and thus for all $1 \leqslant i < \omega$, since $\neg A^1$ is just $\neg A^\circ$. But $\neg A^\circ$ is, by definition, $\neg\neg(A \& \neg A)$. By (11), we have $\neg\neg(A \& \neg A) \supset (A \& \neg A)$, and by (4), $(A \& \neg A) \supset A$; whence by transitivity again, we obtain $\neg A^\circ \supset A$. This, together with the above-derived $\neg A^i \supset \neg A^\circ$, yields $\neg A^i \supset A$, for $1 \leqslant i < \omega$.

To continue, an instance of postulate (10) of C_ω is $A^i \vee \neg A^i$, which, together with the schema $\neg A^i \supset A$ of the preceding paragraph, quickly leads to $A \vee A^i$, for all $1 \leqslant i < \omega$. Explicitly, we have $A \vee A^i$, $A \vee A^{i-1}$, ..., $A \vee A^\circ$, which can be conjoined to yield $(A \vee A^i) \& (A \vee A^{i-1}) \& \ldots \& (A \vee A^\circ)$. An appropriate number of applications of distribution and (3) transform this schema into $A \vee (A^i \& A^{i-1} \& \ldots A^\circ)$, which is, by definition, the desired $A \vee A^{(i)}$.

Lemma 3. In each $C_n (1 \leqslant n < \omega)$, the schema $F_n{}^{(n)}$ is derivable.
Proof: The following is an instance of (9):

$$(F_n \supset F_n{}^{(n)}) \supset ((F_n{}^{(n)} \supset F_n{}^{(n)}) \supset ((F_n \vee F_n{}^{(n)}) \supset F_n{}^{(n)})).$$

In each $C_n (1 \leqslant n < \omega)$, the first antecedent is derivable by Lemma 1. The second is easily derived in C_ω, and the third is also derivable in C_ω by Lemma 2. This leaves the desired $F_n{}^{(n)}$ as a schema derivable in each $C_n (1 \leqslant n < \omega)$.

Lemma 4. In each $C_n (1 \leqslant n < \omega)$, the schema $(A \supset F_n) \supset A^{(n)}$ is derivable.

Proof: The following is an instance of prefixing:

$$(F_n \supset A^{(n)}) \supset ((A \supset F_n) \supset (A \supset A^{(n)})).$$

The antecedent is derivable in each $C_n (1 \leqslant n < \omega)$ by Lemma 1, leaving

$$(A \supset F_n) \supset (A \supset A^{(n)}).$$

Permuting antecedents yields

$$A \supset ((A \supset F_n) \supset A^{(n)}).$$

An instance of postulate (1) is

$$A^{(n)} \supset ((A \supset F_n) \supset A^{(n)}),$$

which, together with the preceding schema and an instance of (9), yields

$$(A \vee A^{(n)}) \supset ((A \supset F_n) \supset A^{(n)}).$$

But the antecedent is derivable in C_ω by Lemma 2, leaving the desired

$$(A \supset F_n) \supset A^{(n)}.$$

Lemma 5. In each $C_n (1 \leqslant n < \omega)$, the schema $(A \supset F_n) \supset (A \supset F_n)^{(n)}$ is derivable.

Proof: An instance of postulate (6), with antecedents permuted, is

$$F_n^{(n)} \supset (A^{(n)} \supset (A^{(n)} \& F_n^{(n)}))$$

The antecedent is derivable in each $C_n (1 \leqslant n < \omega)$ by Lemma 3, leaving

$$A^{(n)} \supset (A^{(n)} \& F_n^{(n)}).$$

An instance of postulate $(15)^{(n)}$ of each $C_n (1 \leqslant n < \omega)$ is

$$(A^{(n)} \& F_n^{(n)}) \supset (A \supset F_n)^{(n)}.$$

By transitivity, the two preceding schemata yield

$$A^{(n)} \supset (A \supset F_n)^{(n)}.$$

But by Lemma 4, in each $C_n (1 \leqslant n < \omega)$, we have

$$(A \supset F_n) \supset A^{(n)},$$

whence, by transitivity again, we obtain the desired

$$(A \supset F_n) \supset (A \supset F_n)^{(n)}.$$

Lemma 6. In each $C_n (1 \leqslant n < \omega)$, the schemata and deduction rules of positive classical logic are derivable.

Proof: This is stated in various places, e.g. for C_1 in Theorem 3 of [9]. But it also follows fairly easily from Lemma 2 above. For to obtain an axiomatics for positive classical logic, it suffices to add the schema $A \vee (A \supset B)$ to positive intuitionistic logic as axiomatised by postulates (1) to (9) of C_ω. By Lemma 2, we have $A \vee A^{(n)}$ in C_ω. Conjoined with postulate (10), this yields $(A \vee A^{(n)}) \& (A \vee \neg A)$. By distribution, this is equivalent to $A \vee (A^{(n)} \&$

$\neg$A). In each $C_n(1 \leqslant n < \omega)$, the schema $((A \ \& \ \neg A) \ \& \ A^{(n)}) \supset B$ follows straightforwardly from an instance of postulate $(12)^{(n)}$, and this reduces easily to $(A^{(n)} \ \& \ \neg A) \supset (A \supset B)$. Together with the preceding schema, and with the assistance of (9), this yields the desired $A \vee (A \supset B)$.

Lemma 7. In each $C_n(1 \leqslant n < \omega)$, the schema $B \equiv ((B \supset F_n) \supset F_n)$ is derivable.

Proof: The following is an instance of postulate (9):

$$(F_n \supset A) \supset ((A \supset A) \supset ((F_n \vee A) \supset A)).$$

The first antecedent is derivable in each $C_n(1 \leqslant n < \omega)$by Lemma 1, and the second is easily derived in C_ω, leaving

$$(F_n \vee A) \supset A.$$

Substituting $B \equiv ((B \supset F_n) \supset F_n)$ for A yields

$$(F_n \vee (B \equiv ((B \supset F_n) \supset F_n))) \supset (B \equiv ((B \supset F_n) \supset F_n)).$$

But the antecedent is an instance of the positive classical tautology

$$A \vee (B \equiv ((B \supset A) \supset A)),$$

and is therefore derivable in $C_n(1 \leqslant n < \omega)$ by Lemma 6, leaving the desired

$$B \equiv ((B \supset F_n) \supset F_n).$$

Lemma 8. In each $C_n(1 \leqslant n < \omega)$, the schema $((B \supset F_n) \supset F_n)^{(n)}$ is derivable.

Proof: Substituting $B \supset F_n$ for A in the schema of Lemma 5 yields

$$((B \supset F_n) \supset F_n) \supset ((B \supset F_n) \supset F_n)^{(n)}.$$

Substituting $(B \supset F_n) \supset F_n$ for A in the schema of Lemma 4 yields

$$((B \supset F_n) \supset F_n) \supset F_n) \supset ((B \supset F_n) \supset F_n)^{(n)}.$$

These two schemata, with the assistance of (9), yield

$$(((B \supset F_n) \supset F_n) \vee (((B \supset F_n) \supset F_n) \supset F_n)) \supset$$
$$((B \supset F_n) \supset F_n)^{(n)}.$$

But the antecedent is an instance of the positive classical tautology $A \vee (A \supset B)$, and is therefore derivable in each $C_n(1 \leqslant n < \omega)$ by Lemma 6, leaving the desired

$$((B \supset F_n) \supset F_n)^{(n)}.$$

Finally, we are in a position to determine the result of adding the rule EC to the systems $C_n (1 \leqslant n < \omega)$.

Theorem 9. For $1 \leqslant n < \omega$, $EC_n = C_0$.

Proof: As noted at the end of Section 2, the addition of EC to $C_n (1 \leqslant n < \omega)$ suffices to guarantee the property SE. This permits the following very simple proof.

In each $EC_n (1 \leqslant n < \omega)$, B and $(B \supset F_n) \supset F_n$ are provably equivalent by Lemma 7, and the schema $((B \supset F_n) \supset F_n)^{(n)}$ is derivable by Lemma 8. Because they enjoy SE, it follows that $B^{(n)}$, which is the result of substituting B for $(B \supset F_n) \supset F_n$ in this schema, is also derivable in each $EC_n (1 \leqslant n < \omega)$. As in the proof of Theorem 6, this yields unqualified *reductio*, which suffices to collapse each $EC_n (1 \leqslant n < \omega)$ into classical logic.

It will be noted that the proof of Theorem 9 does not rely upon the actual derivability of the rule EC in $EC_n (1 \leqslant n < \omega)$; it is sufficient merely that this rule is admissible. But, as noted at the end of Section 2, the admissibility of EC in any extension of $C_n (1 \leqslant n \leqslant \omega)$ is a necessary condition for SE. We can therefore state the following more general result.

Theorem 10. There is no extension of any $C_n (1 \leqslant n < \omega)$ which enjoys SE but which is weaker than classical logic.

(We note that an alternative proof of Theorem 9 can be constructed using the schema $\neg\top$ in place of F_n, where $\top$ is defined to be an arbitrary theorem of C_ω. While this has the slight advantage that, unlike F_n, $\neg\top$ is not relative to each $C_n (1 \leqslant n < \omega)$, it has the disadvantage that Lemma 1 and those subsequent lemmas that rely upon it must be restated to apply to $EC_n (1 \leqslant n < \omega)$ rather than $C_n (1 \leqslant n < \omega)$. This is because the schema $\neg\top \supset A$ is not derivable in the latter systems, but only in the former. This alternative proof, therefore, does not demonstrate as clearly that it is precisely because the C-systems fail to enjoy SE that they do not collapse into classical logic).

5. CONCLUSION

Obtaining analogues of the C-systems which both enjoy SE and satisfy the paraconsistency conditions, then, is not to be achieved by extension but perhaps by some other method of variation. One possibility, which involves the least revision of the C-systems as they stand, is to retain all of the postulates of these systems but to redefine the schema B°. For there is nothing sacrosanct about the original definition of this schema as $\neg(B \,\&\, \neg B)$; in different contexts other candidates may well prove more adequate in expressing the proposition that B "behaves classically". A second and more radical possibility is to retain the method of constructing the higher C-systems but to change the base system. Among the possible alternatives to C_{ω} which suggest themselves are the systems NC_{ω} and OC_{ω} defined in Chapter 3. Both of these possibilities are considered in greater detail in Chapter 6.

Chapter Six: Variations on the C-Systems

1. METHODS OF VARIATION

In Chapter 5, it was shown that the systems C_n $(1 \leqslant n < \omega)$ cannot be extended so as to secure the property SE without collapse into classical logic. In order to obtain paraconsistent analogues of the C-systems which enjoy SE, then, other methods of variation must be considered. In this chapter, we investigate two such methods:

(i) retaining the postulates of $C_n (1 \leqslant n \leqslant \omega)$, but redefining the schema B° (and derivatively, $B^{(n)}$; and

(ii) retaining the method of constructing the stronger systems C_n $(1 \leqslant n < \omega)$ on the basis of C_{ω}, but replacing this system with some other base system.

2. REDEFINING B°

In postulate $(12)^{(n)}$ of each C_n $(1 \leqslant n < \omega)$, the schema $B^{(n)}$ is used to qualify the *reductio* schema $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$, the unqualified addition of which suffices to collapse these systems into classical logic. The reason for such collapse when RC or EC is added to the systems C_n $(1 \leqslant n < \omega)$ is that in each resulting RC_n $(1 \leqslant n < \omega)$ and EC_n $(1 \leqslant n < \omega)$ the schema $B^{(n)}$ is derivable, and hence so is unqualified *reductio*. An initial constraint on the redefinition of B°, then, is that $B^{(n)}$ must not be derivable in the resulting systems.

A second but less formal constraint pertains to the interpretation of the schema B°, originally defined as $\neg(B \ \& \ \neg B)$. This schema lends itself to the interpretation that B is not paradoxical, or "behaves classically". Accordingly, C_1 is constructed so that, for any classically derivable schema A with subformulas $A_1,...,A_m$, while A may not be derivable in C_1, the schema $(A_1{}^{\circ} \,\&\, ... \,\&\, A_m{}^{\circ}) \supset A$ is. An analogous property holds also of the remaining C_n $(1 \leqslant n < \omega)$. While this construction does not depend on the precise definition of B°, it is desirable that any candidate replacement for $\neg(B \,\&\, \neg B)$ should similarly lend itself to this informal interpretation.

3. THE C′-SYSTEMS

A candidate for such redefinition is suggested by the construction of C_ω. This system is neatly constructed by adding to positive intuitionistic logic not the intuitionistic negation postulates $\neg(A \,\&\, \neg A)$ and $A \supset \neg\neg A$, but rather their "duals" $A \vee \neg A$ and $\neg\neg A \supset A$. Given that it is untenable to define B° as $\neg(B \,\&\, \neg B)$ in any extension of $C_n(1 \leqslant n < \omega)$ which enjoys SE, it is natural to turn to $B \supset \neg\neg B$ as an alternative. Accordingly, we define $C'_n(1 \leqslant n \leqslant \omega)$ to be those systems which have, respectively, exactly the same postulates as $C_n(1 \leqslant n \leqslant \omega)$ except that B° is defined as $B \supset \neg\neg B$. ($B^{(n)}$ is defined recursively in terms of B° as before. We note also that C_ω is unaltered by the redefinition of B°; hence, $C'_\omega = C_\omega$).

This redefinition arguably satisfies at least the second constraint outlined above. For while $\neg(B \,\&\, \neg B)$ can be read informally as "It is not the case that both B and not-B", $B \supset \neg\neg B$ can be read as "If it is the case that B, then it is not the case that not-B". Each of these formulas presents itself as well-suited to represent the proposition that B is not paradoxical or "behaves classically".

Moreover, the first constraint is also satisfied.

Theorem 1. In each of the systems $C'_n(1 \leqslant n < \omega)$, neither the schema $B^{(n)}$ nor the *reductio* schema $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ is derivable.

Proof: The matrices in the proof of Theorem 5 of Chapter 5 validate the postulates of each $C'_n(1 \leqslant n \leqslant \omega)$, but invalidate B°, and therefore $B^{(n)}$ for all $1 \leqslant n < \omega$, as well as *reductio*, when A is assigned the value 0 and B is assigned the value 1.

It is also easy to establish that the systems $C'_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy in analogy with the original $C_n(1 \leqslant n \leqslant \omega)$. A lemma is required.

Lemma 1. In each $C'_n(1 \leqslant n < \omega)$, the schema $B^{(n)} \supset B^{(n+1)}$ is derivable.

Note: The rules and schemata listed before the proof of Theorem 3 of Chapter 5 will be used also in the proofs which follow.

Proof of Lemma 1: An instance of postulate (6) of each $C'_n(1 \leqslant n < \omega)$ is $B^{(n)} \supset ((\neg\neg B)^{(n)} \supset (B^{(n)} \& (\neg\neg B)^{(n)}))$. But $B^{(n)} \supset (\neg\neg B)^{(n)}$ follows by transitivity from $B^{(n)} \supset (\neg B)^{(n)}$ and $(\neg B)^{(n)} \supset (\neg\neg B)^{(n)}$, both of which are instances of postulate $(16)^{(n)}$. By (2), therefore, we have $B^{(n)} \supset (B^{(n)} \& (\neg\neg B)^{(n)})$. An instance of postulate $(15)^{(n)}$ is $(B^{(n)} \& (\neg\neg B)^{(n)}) \supset (B \supset \neg\neg B)^{(n)}$, so by transitivity, we have $B^{(n)} \supset (B \supset \neg\neg B)^{(n)}$. But $B \supset \neg\neg B$ is B°, by definition, and $(B \supset \neg\neg B)^{(n)}$ is $(B^{\circ})^{(n)}$. And $(B^{\circ})^{(n)} \supset (B^{\circ})^{n}$ is an instance of (4) if $n > 1$, or of the easily derived $A \supset A$ if $n = 1$, so by transitivity again, we have $B^{(n)} \supset (B^{\circ})^{n}$. Again by definition, $(B^{\circ})^{n}$ is just B^{n+1}, so we have $B^{(n)} \supset B^{n+1}$, from which $B^{(n)} \supset (B^{n+1} \& B^{(n)})$ follows by (6) and (2). But this is the desired $B^{(n)} \supset B^{(n+1)}$, by definition.

Theorem 2. For $1 \leqslant n < \omega$, each C'_{n+1} is a subsystem of C'_n, and C'_ω $(=C_\omega)$ is a subsystem of every $C'_n(1 \leqslant n \leqslant \omega)$.

Proof: That C'_ω is a subsystem of every C'_n $(1 \leqslant n \leqslant \omega)$ is immediate from the construction of these systems. To show that, for $1 \leqslant n < \omega$, each C'_{n+1} is a subsystem of C'_n, it suffices to show that postulates $(12)^{(n+1)}$ to $(16)^{(n+1)}$ of C'_{n+1} are derivable in C'_n.

From postulate $(12)^{(n)}$ of C'_n and $B^{(n+1)} \supset B^{(n)}$, which is an instance of (5), $(12)^{(n+1)}$ follows by transitivity.

From postulate $(13)^{(n)}$, $(A^{(n+1)} \& B^{(n+1)}) \supset (A \& B)^{(n)}$ follows as in the preceding paragraph. But $(A \& B)^{(n)} \supset (A \& B)^{(n+1)}$ is derivable in C'_n by Lemma 1, whence transitivity yields the desired $(13)^{(n+1)}$.

A similar argument shows that $(14)^{(n+1)}$ to $(16)^{(n+1)}$ are also derivable in C'_n.

Thus $C'_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy of (not necessarily distinct) systems, with strongest member C'_1 and weakest member C'_ω.

4. THE C′-SYSTEMS AND THE PROPERTY OF INTERSUBSTITUTIVITY OF PROVABLE EQUIVALENTS

As with the original C-systems, the schemata and rules of $C'_n (1 \leqslant n \leqslant \omega)$ do not suffice to guarantee SE.

Theorem 3. The systems $C'_n (1 \leqslant n \leqslant \omega)$ do not enjoy SE.

Proof: Easily derived in C'_ω, and therefore in each $C'_n (1 \leqslant n \leqslant \omega)$, are the schemata $A \equiv (A \& A)$ and $\neg A \supset \neg A$. If $C'_n (1 \leqslant n \leqslant \omega)$ enjoyed SE, then $\neg A \supset \neg(A \& A)$, the result of substituting $A \& A$ for (one occurrence of) A in $\neg A \supset \neg A$, would also be derivable in these systems. But the matrices in the proof of Theorem 1 of Chapter 5 validate the postulates of $C'_n (1 \leqslant n \leqslant \omega)$, but invalidate $\neg A \supset \neg(A \& A)$ when A is assigned the value 1. Hence, these systems do not enjoy SE.

To secure SE, the obvious strategies are again to add RC or EC. The systems so formed are, respectively, $RC'_n (1 \leqslant n \leqslant \omega)$ and $EC'_n (1 \leqslant n \leqslant \omega)$. We will investigate in detail only the former family of systems. As with the C-systems, the admissibility of RC in any extension of a C′-system (in the same vocabulary) is sufficient for SE, and the admissibility of EC is both necessary and sufficient. Unlike the C-systems, however, the addition of these rules does not collapse the C′-systems into classical logic.

Theorem 4. For $1 \leqslant n \leqslant \omega$, $RC'_n \neq C_0$ (classical logic). Similarly, $EC'_n \neq C_0$.

Proof: The matrices referred to in the proof of Theorem 1, which invalidate the classically derivable *reductio* schema, validate not only the postulates of $C'_n (1 \leqslant n \leqslant \omega)$, but also the rules RC and EC.

5. THE RC′-SYSTEMS AND THE PARACONSISTENCY CONDITIONS

Theorem 4 indicates that, not only do the systems $RC'_n (1 \leqslant n \leqslant \omega)$ enjoy SE, but they also promise to satisfy da Costa's conditions for paraconsistent logics.

Condition (I) is explicitly satisfied by axiomatic systems such as those under investigation just in case the schema (A & $\neg$A) $\supset$ B is not derivable. And indeed, this schema is not derivable in $RC'_n(1 \leqslant n \leqslant \omega)$, for it is invalidated by the matrices referred to in the proof of Theorem 4 when A is assigned the value 1 and B is assigned the value 2. It is possible, however, that (I) may be substantively, though not explicitly, compromised by the derivability of some variant of the above schema (as is the case with some of the J-systems of Chapter 4). A judgement on whether the systems $RC'_n(1 \leqslant n \leqslant \omega)$ substantively satisfy (I) is, therefore, better reserved until the deductive strength of these systems is more fully explored in the following section.

Similarly, whether condition (II), or even the weaker (II′), is satisfied by the the systems $RC'_n(1 \leqslant n \leqslant \omega)$ will emerge more clearly as their deductive strength is explored.

In addition to (I) and (II), the two further conditions listed by da Costa when dealing specifically with the C-systems should perhaps also be considered. Of these, condition (IV) is unproblematic. It requires that it must be simple to extend these systems to predicate calculi of first order. There is no reason to think that the systems currently under investigation do not satisfy this condition as well as the original C-systems do. Condition (III), however, is certainly not satisfied by the systems $RC'_n(1 \leqslant n \leqslant \omega)$. This condition requires that the schema $\neg$(B & $\neg$B) not be derivable. But it is shown to be derivable in RC_ω in the proof of Theorem 6 of Chapter 5; hence, it is derivable in each $RC'_n(1 \leqslant n \leqslant \omega)$. There is no compelling reason, however, to insist that (III) should be satisfied by these systems; for it is important that $\neg$(B & $\neg$B) not be derivable in the original C-systems only because this is the schema in terms of which $B^{(n)}$ is defined. Where $B^{(n)}$ is defined otherwise, as in the C′-systems, the derivablility of $\neg$(B & $\neg$B) is less significant. Rather, what is important is that $B^{(n)}$ not be derivable in such systems; and it follows from Theorem 4 that this schema is indeed not derivable in the systems $RC'_n(1 \leqslant n \leqslant \omega)$.

6. THE RC′-SYSTEMS

That the systems $RC'_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy, with strongest member RC'_1 and weakest member RC'_ω, follows immediately from Theorem 2. However, it remains to be established whether these systems are actually distinct.

The following result will prove very powerful.

Lemma 2. In RC'_ω, the schema $(\neg B)^{(n)}$ is derivable, for $1 \leqslant n < \omega$.

Proof: The proof proceeds by induction on the value of n.

Base case ($n = 1$). By definition, $(\neg B)^{(1)} = (\neg B)^\circ = \neg B \supset \neg\neg\neg B$. But this follows from $\neg\neg B \supset B$, which is an instance of postulate (11) of RC'_ω, by RC.

Inductive step ($n = k > 1$). On inductive hypothesis, $(\neg B)^{(k-1)}$ is derivable in RC'_ω, whence so is $(\neg B)^{k-1}$, from which follows $\neg(B \mathbin{\&} \neg B) \supset (\neg B)^{k-1}$ by (1). Applying RC twice yields $\neg\neg\neg(B \mathbin{\&} \neg B) \supset \neg\neg(\neg B)^{k-1}$. On inductive hypothesis, $(\neg(B \mathbin{\&} \neg B))^\circ$, i.e. $\neg(B \mathbin{\&} \neg B) \supset \neg\neg\neg(B \mathbin{\&} \neg B)$, is derivable in RC'_ω, whence by transitivity, we get $\neg(B \mathbin{\&} \neg B) \supset \neg\neg(\neg B)^{k-1}$. But $\neg(B \mathbin{\&} \neg B)$ is shown in the proof of Theorem 6 of Chapter 5 to be derivable in RC_ω ($= RC'_\omega$), whence by (3) we get $\neg\neg(\neg B)^{k-1}$. By (1), this yields $(\neg B)^{k-1} \supset \neg\neg(\neg B)^{k-1}$, which is, by definition, $((\neg B)^{k-1})^\circ$, i.e. $(\neg B)^k$. Again on inductive hypothesis, we have $(\neg B)^{(k-1)}$, which conjoined with $(\neg B)^k$ yields $(\neg B)^k \mathbin{\&} (\neg B)^{(k-1)}$, or by definition, $(\neg B)^{(k)}$.

Lemma 2 will be used to establish that there is in fact a great deal of collapse in the RC′-hierarchy. A further lemma is required.

Lemma 3. In RC'_ω, the schema $B^{(n+1)} \supset B^{(n+2)}$ is derivable, for $1 \leqslant n < \omega$.

Proof: An instance of postulate (1) of RC'_ω is $\neg\neg B^n \supset (B^n \supset \neg\neg B^n)$, or, by definition, $\neg\neg B^n \supset B^{n+1}$. Applying RC twice yields $\neg\neg\neg\neg B^n \supset \neg\neg B^{n+1}$, from which follows $(\neg\neg B^n \supset \neg\neg\neg\neg B^n) \supset (\neg\neg B^n \supset \neg\neg B^{n+1})$ by prefixing. But the antecedent is just $(\neg\neg B^n)^\circ$, which is derivable in RC'_ω by Lemma 2, leaving $\neg\neg B^n \supset \neg\neg B^{n+1}$. Easily derived in RC'_ω is $(B^n \mathbin{\&} (B^n \supset \neg\neg B^n)) \supset \neg\neg B^n$, or, by definition, $(B^n \mathbin{\&} B^{n+1}) \supset \neg\neg B^n$. Together with the previous schema, this yields, by transitivity, $(B^n \mathbin{\&} B^{n+1}) \supset \neg\neg B^{n+1}$. Another instance of

(1) is $\neg\neg B^{n+1} \supset (B^{n+1} \supset \neg\neg B^{n+1})$, or, by definition, $\neg\neg B^{n+1} \supset B^{n+2}$. Again, transitivity yields $(B^n \,\&\, B^{n+1}) \supset B^{n+2}$. From this, it is straightforward to derive $B^{(n+1)} \supset (B^{n+2} \,\&\, B^{(n+1)})$, i.e. $B^{(n+1)} \supset B^{(n+2)}$.

Theorem 5. For $1 \leqslant n < \omega$, $RC'_{n+1} = RC'_{n+2}$.

Proof: Since the RC′-systems form a linear hierarchy, it suffices to show that each RC'_{n+1} is a subsystem of RC'_{n+2}. We need only show that postulates $(12)^{(n+1)}$ to $(16)^{(n+1)}$ of RC'_{n+1} are derivable in RC'_{n+2}.

Postulate $(12)^{(n+1)}$ follows by transitivity from $(12)^{(n+2)}$ and $B^{(n+1)} \supset B^{(n+2)}$, which is derivable in RC'_{ω}, and hence in each RC'_{n+2}, by Lemma 3.

From $(13)^{(n+2)}$, $(A^{(n+2)} \,\&\, B^{(n+2)}) \supset (A \,\&\, B)^{(n+1)}$ is easily derived, whence follows $(13)^{(n+1)}$ as in the preceding paragraph.

A similar argument shows that $(14)^{(n+1)}$ to $(16)^{(n+1)}$ are also derivable in RC'_{n+2}.

Thus, there are at most three distinct systems in the RC′-hierarchy: RC'_1, RC'_2 and RC'_{ω}. We now show that these three systems are indeed distinct.

Theorem 6. $RC'_{\omega} \neq RC'_2$.

Proof: The following matrices validate the postulates of RC'_{ω}, but invalidate postulate $(12)^{(2)}$ of RC'_2 when A is assigned the value 0 and B is assigned the value 1.

$\supset$	0	1	2	3	4	$\neg$
*0	0	1	2	3	4	4
1	0	0	2	2	4	2
2	0	1	0	1	4	1
3	0	0	0	0	4	0
4	0	0	0	0	0	0

&	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

$\vee$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

(Only the value 0 is designated).

Theorem 7. $RC'_2 \neq RC'_1$.

Proof: The following matrices validate the postulates of RC'_2, but invalidate postulate $(15)^{\circ}$ of RC'_1 when A is assigned the value 1 and B is assigned the value 3.

$\supset$	0	1	2	3	$\neg$
*0	0	1	2	3	3
1	0	0	2	2	0
2	0	1	0	1	0
3	0	0	0	0	0

&	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

$\vee$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

(Only the value 0 is designated).

The derivability of $(\neg B)^{(n)}$ in RC'_ω, shown in Lemma 2, thus proves very powerful in collapsing the RC'-hierarchy. It also provides a way of greatly simplifying the formulations of RC'_1 and RC'_2.

We define RC'_* to be the system formed by adding to RC'_ω the following schema, called $\neg$-*reductio*: $(A \supset \neg B) \supset ((A \supset \neg\neg B) \supset \neg A)$.

Lemma 4. RC'_* is a subsystem of RC'_2 (and therefore also of RC'_1).

Proof: It suffices to show that $\neg$-*reductio* is derivable in RC'_2. An instance of postulate $(12)^{(2)}$ of RC'_2 is $(\neg B)^{(2)} \supset ((A \supset \neg B) \supset ((A \supset \neg\neg B) \supset \neg A))$. But $(\neg B)^{(2)}$ is derivable in RC'_ω, and hence also in RC'_2, by Lemma 2, leaving the desired $\neg$-*reductio*.

Thus, RC'_* ostensibly falls between RC'_ω and RC'_2 in the RC'-hierarchy. In fact, it will be shown that postulates $(12)^{(2)}$ to $(16)^{(2)}$ of RC'_2 are derivable in RC'_*, from which it follows that $RC'_2 = RC'_*$, and that postulates $(12)^\circ$ to $(14)^\circ$ and $(16)^\circ$ of RC'_1 are also derivable in RC'_*, yielding a simpler formulation also of RC'_1.

Lemma 5. Postulate $(12)^\circ$ of RC'_1 is derivable in RC'_*.

Proof: Permuting antecedents in $\neg$-*reductio* yields

$$(A \supset \neg\neg B) \supset ((A \supset \neg B) \supset \neg A).$$

Prefixing yields

$$((A \supset B) \supset (A \supset \neg\neg B)) \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)).$$

Another instance of prefixing is

$$(B \supset \neg\neg B) \supset ((A \supset B) \supset (A \supset \neg\neg B)),$$

whence by transitivity, we have

$$(B \supset \neg\neg B) \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A)),$$

which is, by definition, $(12)^\circ$.

Lemma 6. Postulate $(12)^{(2)}$ of RC'_2 is derivable in RC'_*.

Proof: From $(12)^\circ$, which is derivable in RC'_* by Lemma 5, it is straightforward to derive $(12)^{(2)}$ as in the proof of Theorem 2.

Lemma 7. Postulate $(13)^\circ$ of RC'_1 is derivable in RC'_*.

Proof: We begin by showing that the following de Morgan law is derivable in RC'_ω: $\neg(A \& B) \supset (\neg A \vee \neg B)$. Instances of postulates (7) and (8) of RC'_ω are $\neg A \supset (\neg A \vee \neg B)$ and $\neg B \supset (\neg A \vee \neg B)$, respectively. Applying RC to both yields $\neg(\neg A \vee \neg B) \supset \neg\neg A$ and $\neg(\neg A \vee \neg B) \supset \neg\neg B$, which, by (11) and transitivity, reduce to $\neg(\neg A \vee \neg B) \supset A$ and $\neg(\neg A \vee \neg B) \supset B$. From these, $\neg(\neg A \vee \neg B) \supset (A \& B)$ follows with the assistance of (6). Applying RC again yields $\neg(A \& B) \supset \neg\neg(\neg A \vee \neg B)$, whence (11) and transitivity deliver the desired $\neg(A \& B) \supset (\neg A \vee \neg B)$.

From this de Morgan law,

$$((A \& B) \& \neg(A \& B)) \supset ((A \& B) \& (\neg A \vee \neg B))$$

is easily derived, whence by distribution and transitivity we obtain

$$((A \& B) \& \neg(A \& B)) \supset (((A \& B) \& \neg A) \vee ((A \& B) \& \neg B)).$$

With the assistance (4) and (5), this is reduced to

$$((A \& B) \& \neg(A \& B)) \supset ((A \& \neg A) \vee (B \& \neg B)).$$

An instance of $(12)^\circ$, which is derivable in RC'_* by Lemma 5, is

$$A^\circ \supset ((\neg(A \& B) \supset A) \supset ((\neg(A \& B) \supset \neg A) \supset \neg\neg(A \& B))),$$

from which $A^\circ \supset ((A \& \neg A) \supset \neg\neg(A \& B))$ is easily derived. This yields

$$(A^\circ \& B^\circ) \supset ((A \& \neg A) \supset \neg\neg(A \& B))$$

by (4) and transitivity. Permuting antecedents yields

$$(A \& \neg A) \supset ((A^\circ \& B^\circ) \supset \neg\neg(A \& B)).$$

A similar derivation gives

$$(B \& \neg B) \supset ((A^\circ \& B^\circ) \supset \neg\neg(A \& B)),$$

whence by (9), we have

$$((A \mathbin{\&} \neg A) \vee (B \mathbin{\&} \neg B)) \supset ((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)).$$

Together with the last schema in the preceding paragraph, this yields

$$((A \mathbin{\&} B) \mathbin{\&} \neg(A \mathbin{\&} B)) \supset ((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)),$$

by transitivity. Permuting antecedents yields

$$(A^\circ \mathbin{\&} B^\circ) \supset ((A \mathbin{\&} B) \mathbin{\&} \neg(A \mathbin{\&} B)) \supset \neg\neg(A \mathbin{\&} B)),$$

which, using exportation, is transformed into

$$(A^\circ \mathbin{\&} B^\circ) \supset ((A \mathbin{\&} B) \supset (\neg(A \mathbin{\&} B) \supset \neg\neg(A \mathbin{\&} B))).$$

Finally, an instance of C_ω-*reductio* is

$$(\neg(A \mathbin{\&} B) \supset \neg\neg(A \mathbin{\&} B)) \supset \neg\neg(A \mathbin{\&} B),$$

which, together with the preceding sequent and prefixing, quickly delivers

$$(A^\circ \mathbin{\&} B^\circ) \supset ((A \mathbin{\&} B) \supset \neg\neg(A \mathbin{\&} B)), \text{ i.e. } (13)^\circ.$$

Lemma 8. Postulate $(13)^{(2)}$ of RC'_2 is derivable in RC'_*.

Proof: By Lemma 7, $(13)^\circ$ is derivable in RC'_*. Applying RC twice yields

$$\neg\neg(A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)^\circ.$$

Prefixing yields

$$((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg((A^\circ \mathbin{\&} B^\circ)) \supset ((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)^\circ),$$

which is, by definition,

$$(A^\circ \mathbin{\&} B^\circ)^\circ \supset ((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)^\circ).$$

As an instance of $(13)^\circ$, we have

$$((A^\circ)^\circ \mathbin{\&} (B^\circ)^\circ) \supset (A^\circ \mathbin{\&} B^\circ)^\circ,$$

or, by definition,

$$(A^2 \mathbin{\&} B^2) \supset (A^\circ \mathbin{\&} B^\circ)^\circ,$$

which, together with the preceding schema, yields by transitivity,

$$(A^2 \mathbin{\&} B^2) \supset ((A^\circ \mathbin{\&} B^\circ) \supset \neg\neg(A \mathbin{\&} B)^\circ).$$

From this,

$$((A^2 \mathbin{\&} A^\circ) \mathbin{\&} (B^2 \mathbin{\&} B^\circ)) \supset \neg\neg(A \mathbin{\&} B)^\circ$$

is easily derived using importation; but this is, by definition,

$$(A^{(2)} \mathbin{\&} B^{(2)}) \supset \neg\neg(A \mathbin{\&} B)^\circ.$$

As an instance of (1), we have

$$\neg\neg(A \,\&\, B)^\circ \supset ((A \,\&\, B)^\circ \supset \neg\neg(A \,\&\, B)^\circ),$$

which, by definition, is just

$$\neg\neg(A \,\&\, B)^\circ \supset (A \,\&\, B)^2.$$

Also, we have

$$\neg\neg(A \,\&\, B)^\circ \supset (A \,\&\, B)^\circ$$

by (11), whence we have

$$\neg\neg(A \,\&\, B)^\circ \supset ((A \,\&\, B)^2 \,\&\, (A \,\&\, B)^\circ),$$

which is, by definition,

$$\neg\neg(A \,\&\, B)^\circ \supset (A \,\&\, B)^{(2)}.$$

Together with the last schema in the preceding paragraph, this delivers the desired $(13)^{(2)}$ by transitivity.

<u>Lemma 9</u>. Postulate $(14)^\circ$ of RC'_1 is derivable in RC'_ω (and therefore also in RC'_*).

<u>Proof</u>: An instance of prefixing is

$$(\neg\neg A \supset \neg\neg(A \vee B)) \supset ((A \supset \neg\neg A) \supset (A \supset \neg\neg(A \vee B))).$$

But the antecedent follows from postulate (7) of RC'_ω by two applications of RC, leaving

$$(A \supset \neg\neg A) \supset (A \supset \neg\neg(A \vee B)),$$

which is, by definition,

$$A^\circ \supset (A \supset \neg\neg(A \vee B)).$$

From this,

$$(A^\circ \,\&\, B^\circ) \supset (A \supset \neg\neg(A \vee B))$$

is easily derived, whence permuting antecedents gives

$$A \supset ((A^\circ \,\&\, B^\circ) \supset \neg\neg(A \vee B)).$$

A similar derivation yields

$$B \supset ((A^\circ \,\&\, B^\circ) \supset \neg\neg(A \vee B)).$$

These two schemata yield, by (9),

$$(A \vee B) \supset ((A^\circ \,\&\, B^\circ) \supset \neg\neg(A \vee B)).$$

Permuting antecedents delivers

$$(A^\circ \,\&\, B^\circ) \supset ((A \vee B) \supset \neg\neg(A \vee B)),$$

which is $(14)^\circ$ by definition.

Lemma 10. Postulate $(14)^{(2)}$ of RC'_2 is derivable in RC'_*.

Proof: Starting with $(14)^\circ$, which is derivable in RC'_* by Lemma 9, the schema $(A^{(2)} \& B^{(2)}) \supset \neg\neg(A \vee B)^\circ$ can be derived in analogy with the derivation of $(A^{(2)} \& B^{(2)}) \supset \neg\neg(A \& B)^\circ$ from $(13)^\circ$ in the first paragraph of the proof of Lemma 8. Also, as in the second paragraph of that proof, $\neg\neg(A \vee B)^\circ \supset (A \vee B)^{(2)}$ can be derived. Transitivity then delivers the desired $(14)^{(2)}$.

Although postulate $(15)^\circ$ of RC'_1 is not derivable in RC'_*, by the proof of Theorem 7, the following variant of $(15)^\circ$ is.

Lemma 11. In RC'_*, the schema $(A^{(2)} \& B^\circ) \supset (A \supset B)^\circ$ is derivable.

Proof: An instance of $(12)^\circ$, which is derivable in RC'_* by Lemma 5, is

$$A^\circ \supset((\neg B \supset A) \supset ((\neg B \supset \neg A) \supset \neg\neg B)),$$

from which

$$A^\circ \supset(A \supset(\neg A \supset B))$$

is easily derived using permutation, (1) and transitivity. Permuting antecedents yields

$$\neg A \supset(A^\circ \supset (A \supset B)).$$

Also easily derived, with the assistance of (1), is

$$B \supset(A^\circ \supset(A \supset B)).$$

These two schemata yield, by (9),

$$(\neg A \vee B) \supset (A^\circ \supset (A \supset B)),$$

from which

$$(A^\circ \& (\neg A \vee B)) \supset (A \supset B)$$

follows straightforwardly. Applying RC twice gives

$$\neg\neg(A^\circ \& (\neg A \vee B)) \supset \neg\neg(A \supset B).$$

Prefixing yields

$$((A^\circ \& (\neg A \vee B)) \supset \neg\neg(A^\circ \& (\neg A \vee B)) \supset ((A^\circ \& (\neg A \vee B)) \supset \neg\neg(A \supset B)),$$

which is, by definition,

$$(A^\circ \& (\neg A \vee B))^\circ \supset ((A^\circ \& (\neg A \vee B)) \supset \neg\neg(A \supset B)).$$

An instance of $(13)^\circ$, which is derivable in RC'_* by Lemma 7, is

$$((A^\circ)^\circ \mathbin{\&} (\neg A \vee B)^\circ) \supset (A^\circ \mathbin{\&} (\neg A \vee B))^\circ,$$

which, together with the preceding schema, yields by transitivity,

$$((A^\circ)^\circ \mathbin{\&} (\neg A \vee B)^\circ) \supset ((A^\circ \mathbin{\&} (\neg A \vee B)) \supset \neg\neg(A \supset B)).$$

This is straightforwardly reduced to

$$(((A^\circ)^\circ \mathbin{\&} A^\circ) \mathbin{\&} (\neg A \vee B)^\circ) \supset ((\neg A \vee B) \supset \neg\neg(A \supset B)),$$

which is, by definition,

$$(A^{(2)} \mathbin{\&} (\neg A \vee B)^\circ) \supset ((\neg A \vee B) \supset \neg\neg(A \supset B)).$$

An instance of $(14)^\circ$, which is derivable in RC'_* by Lemma 9, is

$$((\neg A)^\circ \mathbin{\&} B^\circ) \supset (\neg A \vee B)^\circ.$$

But $(\neg A)^\circ$ is derivable in RC'_ω by Lemma 2, so this reduces to

$$B^\circ \supset (\neg A \vee B)^\circ,$$

which quickly leads to

$$(A^{(2)} \mathbin{\&} B^\circ) \supset (A^{(2)} \mathbin{\&} (\neg A \vee B)^\circ).$$

Together with the last schema in the preceding paragraph, this yields, by transitivity,

$$(A^{(2)} \mathbin{\&} B^\circ) \supset ((\neg A \vee B) \supset \neg\neg(A \supset B)).$$

Finally, it is easy to derive

$$(A \supset B) \supset (\neg A \vee B)$$

in RC'_ω, using (7), (8), (9) and (11), from which suffixing yields

$$((\neg A \vee B) \supset \neg\neg(A \supset B)) \supset ((A \supset B) \supset \neg\neg(A \supset B)),$$

which is, by definition,

$$((\neg A \vee B) \supset \neg\neg(A \supset B)) \supset (A \supset B)^\circ.$$

Together with the last schema in the preceding paragraph, this delivers, by transitivity, the desired

$$(A^{(2)} \mathbin{\&} B^\circ) \supset (A \supset B)^\circ.$$

<u>Lemma 12</u>. Postulate $(15)^{(2)}$ of RC'_2 is derivable in RC'_*.

<u>Proof</u>: Applying RC twice to the schema of Lemma 11 yields

$$\neg\neg(A^{(2)} \mathbin{\&} B^\circ) \supset \neg\neg(A \supset B)^\circ.$$

Prefixing yields

$$((A^{(2)} \mathbin{\&} B^\circ) \supset \neg\neg(A^{(2)} \mathbin{\&} B^\circ)) \supset ((A^{(2)} \mathbin{\&} B^\circ) \supset \neg\neg(A \supset B)^\circ),$$

which is, by definition,

$$(A^{(2)} \& B^{\circ})^{\circ} \supset ((A^{(2)} \& B^{\circ}) \supset \neg\neg(A \supset B)^{\circ}).$$

An instance of postulate $(13)^{\circ}$, which is derivable in RC'_{*} by Lemma 7, is

$$((A^{(2)})^{\circ} \& (B^{\circ})^{\circ}) \supset ((A^{(2)} \& B^{\circ})^{\circ};$$

hence, by transitivity, we have

$$((A^{(2)})^{\circ} \& (B^{\circ})^{\circ}) \supset ((A^{(2)} \& B^{\circ}) \supset \neg\neg(A \supset B)^{\circ}),$$

from which it is straightforward to derive

$$(((A^{(2)})^{\circ} \& A^{(2)}) \& ((B^{\circ})^{\circ} \& B^{\circ})) \supset \neg\neg(A \supset B)^{\circ}.$$

But $(A^{(2)})^{\circ}$ is, by definition, $(A^{2} \& A^{\circ})^{\circ}$. Again by $(13)^{\circ}$, we have

$$((A^{2})^{\circ} \& (A^{\circ})^{\circ}) \supset (A^{2} \& A^{\circ})^{\circ},$$

which is, by definition,

$$(A^{3} \& B^{2}) \supset (A^{(2)})^{\circ}.$$

From this follows

$$((A^{3} \& A^{2}) \& A^{(2)}) \supset ((A^{(2)})^{\circ} \& A^{(2)}),$$

or more simply,

$$A^{(3)} \supset ((A^{(2)})^{\circ} \& A^{(2)}).$$

But by Lemma 3, $A^{(2)} \supset A^{(3)}$ is derivable in RC'_{ω}; hence, transitivity yields

$$A^{(2)} \supset ((A^{(2)})^{\circ} \& A^{(2)}).$$

Also, $((B^{\circ})^{\circ} \& B^{\circ})$ is just $B^{(2)}$, so the last schema in the preceding paragraph reduces to

$$(A^{(2)} \& B^{(2)}) \supset \neg\neg(A \supset B)^{\circ}.$$

An instance of (1) is

$$\neg\neg(A \supset B)^{\circ} \supset ((A \supset B)^{\circ} \supset \neg\neg(A \supset B)^{\circ}),$$

which is, by definition,

$$\neg\neg(A \supset B)^{\circ} \supset (A \supset B)^{2}.$$

Also, we have

$$\neg\neg(A \supset B)^{\circ} \supset (A \supset B)^{\circ}$$

as an instance of (11), whence we have

$$\neg\neg(A \supset B)^{\circ} \supset (A \supset B)^{(2)}.$$

Together with the last schema in the preceding paragraph, this yields, by transitivity,

$$(A^{(2)} \& B^{(2)}) \supset (A \supset B)^{(2)},$$

which is the desired $(15)^{(2)}$.

Lemma 13. Postulate $(16)^{\circ}$ of RC'_1 is derivable in RC'_{ω} (and hence also in RC'_*).

Proof: An instance of postulate (1) of RC'_{ω} is $(\neg B)^{\circ} \supset (B^{\circ} \supset (\neg B)^{\circ})$. By Lemma 2, $(\neg B)^{\circ}$ is derivable in RC'_{ω}, leaving $B^{\circ} \supset (\neg B)^{\circ}$, i.e. $(16)^{\circ}$.

Lemma 14. Postulate $(16)^{(2)}$ of RC'_2 is derivable in RC'_{ω} (and hence also in RC'_*).

Proof: An instance of postulate (1) of RC'_{ω} is $(\neg B)^{(2)} \supset (B^{(2)} \supset (\neg B)^{(2)})$. By Lemma 2, $(\neg B)^{(2)}$ is derivable in RC'_{ω}, leaving $B^{(2)} \supset (\neg B)^{(2)}$, i.e. $(16)^{(2)}$.

Theorem 8. $RC'_2 = RC'_*$.

Proof: By Lemma 4, RC'_2 contains RC'_*. Conversely, by Lemmas 6, 8, 10, 12 and 14, RC'_* contains not only RC'_{ω}, but also all of the postulates which are added to RC'_{ω} in the construction of RC'_2.

Theorem 9. $RC'_1 = RC'_* + (15)^{\circ}$.

Proof: By Lemma 4, RC'_1 contains RC'_*. Conversely, by Lemmas 5, 7, 9 and 13, RC'_* contains not only RC'_{ω}, but also all of the postulates which are added to RC'_{ω} in the construction of RC'_1, with the exception of $(15)^{\circ}$.

We are now in a position to comment further on the paraconsistency conditions. It was noted in Section 5 that the schema $B^{(n)}$ should not be derivable in the systems $RC'_n(1 \leqslant n < \omega)$, for if it were, then so would be unqualified *reductio*, collapsing these systems into classical logic. And indeed, Theorem 4 establishes that neither $B^{(n)}$ nor *reductio* is derivable in $RC'_n(1 \leqslant n \leqslant \omega)$. But by Lemma 2, the schema $(\neg B)^{(n)}$ is derivable in RC'_{ω}, and hence also in each $RC'_n(1 \leqslant n < \omega)$; consequently, $\neg$-*reductio* is also derivable in these latter systems, as in the proof of Lemma 4. It quickly follows that a variant of the paraconsistency- defeating $(A \& \neg A) \supset B$ is also derivable in these systems.

Theorem 10. In $RC'_n(1 \leqslant n < \omega)$, the schema $(\neg A \& \neg\neg A) \supset B$ is derivable.

Proof: An instance of $\neg$-*reductio* is $(\neg B \supset \neg A) \supset ((\neg B \supset \neg\neg A) \supset \neg\neg B)$, which is easily reduced to $(\neg A \& \neg\neg A) \supset \neg\neg B$ with the assistance of (1), requiring only (11) and transitivity to deliver $(\neg A \ \& \ \neg\neg A) \supset B$.

As noted in Section 5, the derivability of $(A \ \& \ \neg A) \supset B$ would explicitly violate condition (I). Does the derivability of this variant substantively violate (I)?

The answer is arguably that it does not, for there is no reason to expect that inconsistent theories based on any $RC'_n(1 \leqslant n < \omega)$ would substantially collapse. Certainly, those containing a contradiction of the form $\neg A \ \& \ \neg\neg A$ would collapse totally, but there is nothing to suggest that this would be true of inconsistent theories in general. There is therefore nothing in the results of this section which indicates that condition (I) is not substantively satisfied by the RC′ -systems. However, it remains that, even though the derivability of $(\neg B)^{(n)}$ may not be in this sense damaging, neither is it well-motivated, for there is no particular reason why negated formulas should in general be expected to "behave classically".

As far as condition (II) is concerned, there is also nothing in this section which suggests that the RC′ -systems fail to contain significant parts of classical logic which would not interfere with the satisfaction of (I) if they were incorporated. On the contrary, the derivability of $(\neg A \ \& \ \neg\neg A) \supset B$ suggests that, if anything, they approximate classical logic a little too closely. And in enjoying SE, the systems $RC'_n(1 \leqslant n \leqslant \omega)$ come closer to satisfying (II) than do either $C'_n(1 \leqslant n \leqslant \omega)$ or the original $C_n(1 \leqslant n \leqslant \omega)$.

Finally, the fact that there are only three distinct systems in $RC'_n(1 \leqslant n \leqslant \omega)$, though perhaps a little surprising, need not diminish their philosophical interest. For even with the original infinite hierarchy of C-systems, the interesting ones are those at the extremes, the remainder, as noted in [19], being somewhat less obviously motivated.

Of course, these considerations are in no way conclusive for the method of variation which has been dealt with in this half of the chapter, for we have investigated only one way of redefining B°, resulting in the systems $C'_n(1 \leqslant n \leqslant \omega)$, and only one way of extending these systems so as to secure SE. It may be, for example, that the hierarchy $EC'_n(1 \leqslant n \leqslant \omega)$ would not exhibit a similar degree of collapse, or would prove more satisfactory in terms of the paraconsistency conditions. However, as the final result of this section, we show that at least EC'_1 shares some of the curious features of the RC′ -systems. For the derivability of $(\neg B)^\circ$ in RC'_1 does not in fact depend on the presence of RC.

Theorem 11. In C'_1 (and therefore in EC'_1), the schemata $(\neg B)^\circ$, $\neg$-*reductio* and $(\neg A \,\&\, \neg\neg A) \supset B$ are all derivable.

Proof: An instance of postulate (1) of C'_1 is $\neg\neg B \supset (B \supset \neg\neg B)$, i.e. $\neg\neg B \supset B^\circ$. Together with postulate $(16)^\circ$, this yields, by transitivity, $\neg\neg B \supset (\neg B)^\circ$. Another instance of (1) is $\neg\neg\neg B \supset (\neg B \supset \neg\neg\neg B)$, i.e. $\neg\neg\neg B \supset (\neg B)^\circ$. With the assistance of (9), these two schemata yield $(\neg\neg B \vee \neg\neg\neg B) \supset (\neg B)^\circ$. But the antecedent is an instance of postulate (10), leaving $(\neg B)^\circ$. Hence, from $(\neg B)^\circ \supset ((A \supset \neg B) \supset ((A \supset \neg\neg B) \supset \neg A))$, which is an instance of postulate $(12)^\circ$ of C'_1, $\neg$-*reductio* follows by (3). Using $\neg$-*reductio*, $(\neg A \,\&\, \neg\neg A) \supset B$ is then further derived as in the proof of Theorem 10.

7. ALTERNATIVE BASES

We now turn to the second method of variation described in Section 1. Several alternative bases are to be found among the axiomatic systems constructed in Chapter 3. We will consider three of these: NC_ω, OC_ω + (12) and OC_ω. Again, it will turn out that the hierarchies $NC_n(1 \leqslant n \leqslant \omega)$, $OC_n(1 \leqslant n \leqslant \omega)$ + (12) and $OC_n(1 \leqslant n \leqslant \omega)$ based on these systems do not enjoy SE; and we will consider the effect of adding RC in order to secure this property. The respective bases of these extended hierarchies, therefore, will be RNC_ω, ROC_ω + (12) and ROC_ω. We note that the first two of these are the respective axiomatic counterparts of $RJ_{0.5}$ and $RJ_{0.1}$, which were found in Chapter 4 to be the strongest members of, respectively, the intermediate and intuitionistic branches of subsystems of J_1 extended by RC which satisfy the paraconsistency conditions. (The strongest member of the dualintuitionistic branch is $RJ_{0.4}$, the axiomatic counterpart of which is RC_ω, already investigated as a base system in Chapter 5 and the first half of this chapter). The results obtained for the hierarchies based on RNC_ω and ROC_ω + (12) will lead us to consider the still weaker ROC_ω (the axiomatic counterpart of RJ_0), which is the weakest of the axiomatic systems incorporating RC defined in Chapter 4.

8. THE NC-SYSTEMS

The base system NC_ω is constituted by postulates (1) to (9) of C_ω, which axiomatise positive intuitionistic logic, together with the double-negation postulates (11) and (12), only the first of which is incorporated in C_ω. The systems $NC_n(1 \leqslant n < \omega)$ are constructed on the basis of NC_ω exactly as the original $C_n(1 \leqslant n < \omega)$ are constructed on the basis of C_ω: each NC_n is formed by adding to NC_ω postulates $(12)^{(n)}$ to $(16)^{(n)}$. The schema $B^{(n)}$ is defined in terms of B°, and B° is defined as $\neg(B \;\&\; \neg B)$, exactly as in the original C-systems.

We establish initially that the systems $NC_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy. A lemma is required.

<u>Lemma 15</u>. In each $NC_n(1 \leqslant n < \omega)$, the schema $B^{(n)} \supset B^{(n+1)}$ is derivable.

<u>Proof</u>: An instance of postulate (6) of each $NC_n(1 \leqslant n < \omega)$ is $B^{(n)} \supset ((\neg B)^{(n)} \supset (B^{(n)} \;\&\; (\neg B)^{(n)}))$. But $B^{(n)} \supset (\neg B)^{(n)}$ is an instance of postulate $(16)^{(n)}$; by (2), therefore, we have $B^{(n)} \supset (B^{(n)} \;\&\; (\neg B)^{(n)})$. An instance of postulate $(13)^{(n)}$ is $(B^{(n)} \;\&\; (\neg B)^{(n)}) \supset (B \;\&\; \neg B)^{(n)}$; by transitivity, therefore, we have $B^{(n)} \supset (B \;\&\; \neg B)^{(n)}$. Another instance of $(16)^{(n)}$ is $(B \;\&\; \neg B)^{(n)} \supset (\neg(B \;\&\; \neg B))^{(n)}$; transitivity again yields $B^{(n)} \supset (\neg(B \;\&\; \neg B))^{(n)}$. But $(\neg(B \;\&\; \neg B))^{(n)}$ is, by definition, $(B^\circ)^{(n)}$, and we have $(B^\circ)^{(n)} \supset (B^\circ)^n$; transitivity yields $B^{(n)} \supset (B^\circ)^n$. But $(B^\circ)^n$ is just B^{n+1}, so we have $B^{(n)} \supset B^{n+1}$, from which $B^{(n)} \supset (B^{n+1} \;\&\; B^{(n)})$ follows straightforwardly. But, by definition, this is exactly $B^{(n)} \supset B^{(n+1)}$.

<u>Theorem 12</u>. For $1 \leqslant n < \omega$, each NC_{n+1} is a subsystem of NC_n, and NC_ω is a subsystem of every $NC_n(1 \leqslant n \leqslant \omega)$.

<u>Proof</u>: That NC_ω is a subsystem of every $NC_n(1 \leqslant n \leqslant \omega)$ is immediate from the construction of these systems. To show that, for $1 \leqslant n < \omega$, each NC_{n+1} is a subsystem of NC_n, it suffices to establish that postulates $(12)^{(n+1)}$ to $(16)^{(n+1)}$ of NC_{n+1} are derivable in NC_n.

From postulate $(12)^{(n)}$ of NC_n and $B^{(n+1)} \supset B^{(n)}$, which is an instance of (5), $(12)^{(n+1)}$ follows by transitivity.

From postulate $(13)^{(n)}$, $(A^{(n+1)} \,\&\, B^{(n+1)}) \supset (A \,\&\, B)^{(n)}$ follows as in the preceding paragraph. But $(A \,\&\, B)^{(n)} \supset (A \,\&\, B)^{(n+1)}$ is derivable in NC_n by Lemma 15, whence transitivity yields the desired $(13)^{(n+1)}$.

A similar argument shows that $(14)^{(n+1)}$ to $(16)^{(n+1)}$ are also derivable in NC_n.

Thus, $NC_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy of (not necessarily distinct) systems, with strongest member NC_1 and weakest member NC_ω.

Moreover, none of these systems is equivalent to C_0 (classical logic), or to any $C_n(1 \leqslant n \leqslant \omega)$.

Theorem 13. For $1 \leqslant n \leqslant \omega$, $NC_n \neq C_0$. In particular, the schema $(A \,\&\, \neg A) \supset B$ is not derivable in $C_n(1 \leqslant n \leqslant \omega)$.

Proof: The following matrices validate the postulates of each $NC_n(1 \leqslant n \leqslant \omega)$, but invalidate $(A \,\&\, \neg A) \supset B$ when A is assigned the value 1 and B is assigned the value 4.

$\supset$	0	1	2	3	4	$\neg$
*0	0	0	0	0	4	4
*1	0	0	0	0	4	2
*2	0	0	0	0	4	1
*3	0	0	0	0	4	4
4	0	0	0	0	0	3

&	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

$\vee$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

(Only the value 4 is not designated).

Theorem 14. No $NC_n(1 \leqslant n \leqslant \omega)$ is equivalent to any $C_n(1 \leqslant n \leqslant \omega)$.

Proof: The following matrices validate the postulates of $NC_n(1 \leqslant n \leqslant \omega)$, but invalidate $A \vee \neg A$, which is postulate (10) of each $C_n(1 \leqslant n \leqslant \omega)$, when A is assigned the value 2.

$\supset$	0	1	2	$\neg$
*0	0	1	2	1
1	0	0	0	0
2	0	0	0	2

&	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$\vee$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 0 is designated).

As with the original C-systems, the schemata and rules of $C_n(1 \leqslant n \leqslant \omega)$ do not suffice to guarantee SE.

Theorem 15. The systems $NC_n(1 \leqslant n \leqslant \omega)$ do not enjoy SE.

Proof: Easily derived in NC_ω, and therefore in each $NC_n(1 \leqslant n \leqslant \omega)$, are the schemata $A \equiv (A \mathbin{\&} (A \supset A))$ and $\neg A \supset \neg A$. If $NC_n(1 \leqslant n \leqslant \omega)$ enjoyed SE, then $\neg A \supset \neg(A \mathbin{\&} (A \supset A))$, the result of substituting $A \mathbin{\&} (A \supset A)$ for (one occurence of) A in $\neg A \supset \neg A$, would also be derivable in these systems. But the matrices in the proof of Theorem 14, which validate the postulates of $NC_n(1 \leqslant n \leqslant \omega)$, invalidate this schema when A is assigned the value 1. Hence, these systems do not enjoy SE.

To secure SE, the obvious strategies are again to add RC or EC. The systems so formed are, respectively, $RNC_n(1 \leqslant n \leqslant \omega)$ and $ENC_n(1 \leqslant n \leqslant \omega)$. We will investigate only the former family of systems.

9. THE RNC-SYSTEMS

Theorem 16. $RNC_\omega \neq C_0$. In particular, the schema $(A \mathbin{\&} \neg A) \supset B$ is not derivable in RNC_ω.

Proof: The following matrices (from the proof of Theorem 4 of Chapter 4) validate the postulates of RNC_ω, but invalidate $(A \mathbin{\&} \neg A) \supset B$ when A is assigned the value 1 and B is assigned the value 2.

$\supset$	0	1	2	$\neg$
*0	0	1	2	2
1	0	0	2	1
2	0	0	0	0

&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\vee$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(Only the value 0 is designated).

Thus, the addition of RC to NC_ω does not collapse this system into classical logic; and in particular, the paraconsistency-defeating $(A \& \neg A) \supset B$ remains underivable.

Unfortunately, the same is not true of the remaining $NC_n (1 \leqslant n \leqslant \omega)$. Several lemmas are required.

Lemma 16. In RNC_ω, the schema $A^{(i)} \supset A^{(i+1)}$ is derivable, for $1 \leqslant i < \omega$.

Proof: An instance of postulate (5) of RNC_ω is $(A^i \& \neg A^i) \supset \neg A^i$. Applying RC yields $\neg\neg A^i \supset \neg(A^i \& \neg A^i)$, which is by definition $\neg\neg A^i \supset A^{i+1}$. By (12), this reduces to $A^i \supset A^{i+1}$; but we have $A^{(i)} \supset A^i$, so by transitivity, we get $A^{(i)} \supset A^{i+1}$, which quickly leads to $A^{(i)} \supset (A^{i+1} \& A^{(i)})$, i.e. $A^{(i)} \supset A^{(i+1)}$.

Corollary. In RNC_ω, the schema $A^\circ \supset A^{(i)}$ is derivable, for $1 \leqslant i < \omega$.

Proof: This follows from Lemma 16 by a straightforward inductive argument.

Lemma 17. In RNC_ω, the schema $A^\circ \supset (A \vee \neg A)$ is derivable.

Proof: Instances of postulates (7) and (8) of RNC_ω are $\neg\neg A \supset (\neg\neg A \vee \neg A)$ and $\neg A \supset (\neg\neg A \vee \neg A)$, respectively. Applying RC to each yields $\neg(\neg\neg A \vee \neg A) \supset \neg\neg\neg A$ and $\neg(\neg\neg A \vee \neg A) \supset \neg\neg A$, which, with the assistance of (11), reduce to $\neg(\neg\neg A \vee \neg A) \supset \neg A$ and $\neg(\neg\neg A \vee \neg A) \supset A$. From these, $\neg(\neg\neg A \vee \neg A) \supset (A \& \neg A)$ quickly follows through (6). Applying RC again yields $\neg(A \& \neg A) \supset \neg\neg(\neg\neg A \vee \neg A)$, from which $\neg(A \& \neg A) \supset (A \vee \neg A)$ quickly follows with the assistance of (11). But this is just $A^\circ \supset (A \vee \neg A)$.

Lemma 18. In each $NC_n (1 \leqslant n < \omega)$, the schema A° is derivable.

Proof: An instance of postulate (4) of each $RNC_n (1 \leqslant n < \omega)$ is $(A \& \neg A) \supset A$. Applying RC yields $\neg A \supset \neg(A \& \neg A)$, which is, by definition, $\neg A \supset A^{\circ}$. An instance of (5) is $(A \& \neg A) \supset \neg A$, which, together with the preceding schema, yields by transitivity, $(A \& \neg A) \supset A^{\circ}$. By the Corollary to Lemma 16, $A^{\circ} \supset A^{(n)}$ is derivable in RNC_{ω} and hence in each $RNC_n(1 \leqslant n < \omega)$; by transitivity, therefore, we have $(A \& \neg A) \supset A^{(n)}$. Let T be a theorem of RNC_{ω}. An instance of postulate $(12)^{(n)}$ of each $RNC_n(1 \leqslant n < \omega)$ is $A^{(n)} \supset ((T \supset A) \supset ((T \supset \neg A) \supset \neg T))$, which is easily reduced to $A^{(n)} \supset ((A \& \neg A) \supset \neg T)$. Together with the preceding schema, this yields by transitivity, $(A \& \neg A) \supset ((A \& \neg A) \supset \neg T)$, which reduces to $(A \& \neg A) \supset \neg T$ with the assistance of (2). Applying RC yields $\neg\neg T \supset \neg(A \& \neg A)$, i.e. $\neg\neg T \supset A^{\circ}$. But $\neg\neg T$ follows by (12) from T, which has been assumed to be a theorem of RNC_{ω}; this leaves the desired A°.

<u>Theorem 17</u>. For $1 \leqslant n < \omega$, $RNC_n = C_0$.
<u>Proof</u>: By Lemma 17, the schema $A^{\circ} \supset (A \vee \neg A)$ is derivable in RNC_{ω}; and by Lemma 18, A° is derivable in each $RNC_n(1 \leqslant n < \omega)$. It follows that $A \vee \neg A$ is also derivable in each $RNC_n(1 \leqslant n < \omega)$. But this is postulate (10) of the C-systems. It follows that, for $1 \leqslant n < \omega$, each RC_n is a subsystem of RNC_n. But by Theorem 6 of Chapter 5, each such $RC_n = C_0$; hence, C_0 is a subsystem of each $RNC_n(1 \leqslant n < \omega)$, and since each RNC_n is also a subsystem of C_0, it follows that each $RNC_n = C_0$, for $1 \leqslant n < \omega$.

10. THE OC + (12) -SYSTEMS

The base system $OC_{\omega} + (12)$ is obtained from NC_{ω} simply by omitting postulate (11). Thus, $OC_{\omega} + (12)$ is constituted by postulates (1) to (9), which axiomatise positive intuitionistic logic (or OC_{ω}, when the vocabulary is taken to include $\neg$), together with the intuitionistic double-negation postulate (12). This system is therefore intuitionistic: every schema derivable in $OC_{\omega} + (12)$ is also intuitionistically derivable. The systems $OC_n(1 \leqslant n < \omega) + (12)$ are constructed on the basis of $OC_{\omega} + (12)$ exactly as the original $C_n(1 \leqslant n < \omega)$ are constructed on the basis of C_{ω}: each $OC_n + (12)$ is formed by adding to $OC_{\omega} + (12)$ postulates $(12)^{(n)}$ to $(16)^{(n)}$, with $B^{(n)}$ and B° defined as in the C-systems.

Since $(12)^{(n)}$ to $(16)^{(n)}$ are intuitionistically derivable, the systems $OC_n(1 \leqslant n \leqslant \omega)$ + (12) are all intuitionistic. In fact, they are subsystems of the still weaker HM (Johansson's "minimal calculus"), which is constituted by postulates (1) to (9) together with the *reductio* schema $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ (see [8]).

As with the NC-systems, it is straightforward to establish that $OC_n(1 \leqslant n \leqslant \omega) + (12)$ form a linear hierarchy of (not necessarily distinct) systems, with strongest member OC_1 + (12) and weakest member OC_ω + (12).

Lemma 19. In each $OC_n(1 \leqslant n < \omega) + (12)$, the schema $B^{(n)} \supset B^{(n+1)}$ is derivable.

Proof: As for Lemma 15.

Theorem 18. For $1 \leqslant n < \omega$, each OC_{n+1} + (12) is a subsystem of OC_n + (12), and OC_ω + (12) is a subsystem of every $OC_n(1 \leqslant n \leqslant \omega) + (12)$.

Proof: As for Theorem 12, invoking Lemma 19 where appropriate.

Moreover, none of the systems $OC_n(1 \leqslant n \leqslant \omega) + (12)$ is equivalent to HM (or, therefore, to intuitionistic or classical logic), or to any $C_n(1 \leqslant n \leqslant \omega)$ or $NC_n(1 \leqslant n \leqslant \omega)$.

Theorem 19. For $1 \leqslant n \leqslant \omega$, $OC_n + (12) \neq$ HM. In particular, the *reductio* schema is not derivable in $OC_n(1 \leqslant n \leqslant \omega) + (12)$.

Proof: The matrices in the proof of Theorem 13, which validate the postulates of $NC_n(1 \leqslant n \leqslant \omega)$ and therefore also those of $OC_n(1 \leqslant n \leqslant \omega) + (12)$, invalidate *reductio* when A is assigned the value 0 and B is assigned the value 1.

Theorem 20. No $OC_n(1 \leqslant n \leqslant \omega) + (12)$ is equivalent to any $C_n(1 \leqslant n \leqslant \omega)$ or to any $NC_n(1 \leqslant n \leqslant \omega)$.

Proof: It suffices to note that all of the systems in the latter hierarchies incorporate the intuitionistically underivable postulate (11).

As with the C-systems and NC-systems, the postulates of $OC_n(1 \leqslant n \leqslant \omega) + (12)$ do not suffice to guarantee SE.

Theorem 21. The systems $OC_n(1 \leqslant n \leqslant \omega) + (12)$ do not enjoy SE.

Proof: As for Theorem 15.

To secure SE, the obvious strategies are again to add RC or EC. We will investigate only the addition of RC, resulting in the systems $ROC_n(1 \leqslant n \leqslant \omega) + (12)$.

11. THE ROC + (12) -SYSTEMS

Since the rule RC is derivable in HM, it follows that $ROC_n(1 \leqslant n \leqslant \omega) + (12)$ are also subsystems of HM. Moreover, $ROC_\omega + (12)$ at least is a proper subsystem of HM.

Theorem 22. $ROC_\omega + (12) \neq HM$. In particular, the *reductio* schema is not derivable in $ROC_\omega + (12)$.

Proof: The matrices in the proof of Theorem 16, which validate the postulates of RNC_ω and therefore also those of $ROC_\omega + (12)$, invalidate *reductio* when A is assigned the value 0 and B is assigned the value 1.

Thus, the addition of RC to $OC_\omega + (12)$ does not collapse this system into HM. Unfortunately, the same is not true of the remaining $OC_n(1 \leqslant n \leqslant \omega) + (12)$. Several lemmas are required.

Lemma 20. In $ROC_\omega + (12)$, the schema $A^{(i)} \supset A^{(i+1)}$ is derivable, for $1 \leqslant i < \omega$.

Proof: As for Lemma 16 (since postulate (11) of RNC_ω is not involved in the proof of Lemma 16).

Corollary. In $ROC_\omega + (12)$, the schema $A^\circ \supset A^{(i)}$ is derivable, for $1 \leqslant i < \omega$.

Proof: As for the Corollary to Lemma 16.

Lemma 21. In each $ROC_n(1 \leqslant n < \omega) + (12)$, the schema A° is derivable.

Proof: As for Lemma 18.

Lemma 22. In each $ROC_n(1 \leqslant n < \omega) + (12)$, the schema $A^{(n)}$ is derivable.

Proof: This follows from Lemma 21 and the Corollary to Lemma 20.

Theorem 23. For $1 \leqslant n < \omega$, $ROC_n + (12) = HM$.

Proof: By Lemma 22, the antecedent of postulate $(12)^{(n)}$ of each $ROC_n(1 \leqslant n < \omega) + (12)$ is derivable in that system, leaving the *reductio* schema. This suffices to collapse each $ROC_n(1 \leqslant n < \omega) + (12)$ into HM.

Thus, the addition of RC to $OC_n(1 \leqslant n < \omega) + (12)$ results in HM. However, HM is too strong for paraconsistent purposes, for easily derived from *reductio* in HM is the schema $(A \; \& \; \neg A) \supset \neg B$. This schema substantively (though not explicitly) violates condition (I), for its presence ensures that an inconsistent theory is thoroughly inconsistent: for every formula in the theory, its negation is also in the theory. (This is also the ground on which the system $RJ_{0.3}$ was rejected in Chapter 4).

We turn instead to OC_ω, the weakest of the axiomatic systems defined in Chapter 3, as an alternative basis.

12. THE OC-SYSTEMS

The base system OC_ω is constituted by postulates (1) to (9), and differs from positive intuitionistic logic only in that its vocabulary includes the connective $\neg$. The systems $OC_n(1 \leqslant n < \omega)$ are constructed on the basis of OC_ω exactly as the original $C_n(1 \leqslant n < \omega)$ are constructed on the basis of C_ω: each OC_n is formed by adding to OC_ω postulates $(12)^{(n)}$ to $(16)^{(n)}$, with $B^{(n)}$ and B° defined as in the C-systems.

As with the NC-systems and OC + (12)-systems, it is easy to establish that $OC_n(1 \leqslant n \leqslant \omega)$ form a linear hierarchy of (not necessarily disntinct) systems, with strongest member OC_1 and weakest member OC_ω.

Lemma 23. In each $OC_n(1 \leqslant n < \omega)$, the schema $B^{(n)} \supset B^{(n+1)}$ is derivable.

Proof: As for Lemma 15.

Theorem 24. For $1 \leqslant n < \omega$, each OC_{n+1} is a subsystem of OC_n, and OC_ω is a subsystem of every $OC_n(1 \leqslant n \leqslant \omega)$.

Proof: As for Theorem 12, invoking Lemma 23 where appropriate.

Moreover, none of the systems $OC_n(1 \leqslant n \leqslant \omega)$ is equivalent to HM (or, therefore, to inuitionistic or classical logic), or to any $OC_n(1 \leqslant n \leqslant \omega)$ + (12) (or, therefore, to any $NC_n(1 \leqslant n \leqslant \omega)$), or to any $C_n(1 \leqslant n \leqslant \omega)$.

Theorem 25. For $1 \leqslant n \leqslant \omega$, $OC_n \neq$ HM. In particular, the *reductio* schema is not derivable in $OC_n(1 \leqslant n \leqslant \omega)$.

Proof: This follows from Theorem 19, since, for $1 \leqslant n \leqslant \omega$, each OC_n is a subsystem of OC_n + (12).

Theorem 26. No $OC_n(1 \leqslant n \leqslant \omega)$ is equivalent to any $OC_n(1 \leqslant n \leqslant \omega)$ + (12).

Proof: The following matrices (from the proof of Theorem 19 of Chapter 4) validate the postulates of $OC_n(1 \leqslant n \leqslant \omega)$, but invalidate postulate (12) of $OC_n(1 \leqslant n \leqslant \omega)$ + (12) when A is assigned the value 0.

$\supset$	0	1	$\neg$
*0	0	1	1
1	0	0	1

&	0	1
0	0	1
1	1	1

$\vee$	0	1
0	0	0
1	0	1

(Only the value 0 is designated).

Theorem 27. No $OC_n(1 \leqslant n \leqslant \omega)$ is equivalent to any $C_n(1 \leqslant n \leqslant \omega)$.

Proof: It suffices to note that all of the systems in the latter hierarchy incorporate the intuitionistically underivable postulates (10) and (11).

As with the C-systems, NC-systems and OC + (12)-systems, the postulates of $OC_n(1 \leqslant n \leqslant \omega)$ do not suffice to guarantee SE.

Theorem 28. The systems $OC_n(1 \leqslant n \leqslant \omega)$ do not enjoy SE.

Proof: As for Theorem 15.

To secure SE, the obvious strategies are again to add RC or EC. We will investigate only the addition of RC, resulting in the systems $ROC_n(1 \leqslant n \leqslant \omega)$.

13. THE ROC-SYSTEMS

For $1 \leqslant n \leqslant \omega$, each ROC_n is a subsystem of $ROC_n + (12)$. It follows $ROC_n(1 \leqslant n \leqslant \omega)$ are all subsystems of HM. Unlike the ROC + (12)-systems, however, none of the ROC-systems is equivalent to HM.

Theorem 29. For $1 \leqslant n \leqslant \omega$, $ROC_n \neq HM$. In particular, the *reductio* schema is not derivable in $ROC_n(1 \leqslant n \leqslant \omega)$.

Proof: The matrices in the proof of Theorem 26 validate not only the postulates of $OC_n(1 \leqslant n \leqslant \omega)$, but also the rule RC; however, they invalidate *reductio* when A is assigned the value 1 and B is assigned the value 0.

Despite the fact that *reductio* is not derivable in $ROC_n(1 \leqslant n \leqslant \omega)$, the undesirable $(A \& \neg A) \supset \neg B$, is still derivable in at least ROC_1.

Theorem 30. In ROC_1, the schema $(A \& \neg A) \supset \neg B$ is derivable.

Proof: An instance of postulate $(12)^\circ$ of ROC_1 is $A^\circ \supset ((B \supset A) \supset ((B \supset \neg A) \supset \neg B))$, which is easily reduced to $A^\circ \supset ((A \& \neg A) \supset \neg B)$. But $(A \& \neg A) \supset A^\circ$ is also easily derived in ROC_1, as in the proof of Lemma 18; hence, by transitivity, we have $(A \& \neg A) \supset ((A \& \neg A) \supset \neg B)$, which, with the assistance of (2), reduces to $(A \& \neg A) \supset \neg B$.

However, this result does not extend to the remaining ROC-systems.

Theorem 31. For $2 \leqslant n \leqslant \omega$, $ROC_n \neq ROC_1$. In particular, the schema $(A \& \neg A) \supset \neg B$ is not derivable in $ROC_n(2 \leqslant n \leqslant \omega)$.

Proof: The following matrices validate the postulates of $ROC_n(2 \leqslant n \leqslant \omega)$, but invalidate $(A \& \neg A) \supset \neg B$ when A is assigned the value 2 and B is assigned the value 0.

$\supset$	0	1	2	3	4	$\neg$
*0	0	1	2	3	4	3
1	0	0	2	3	4	3
2	0	0	0	3	3	1
3	0	0	2	0	2	3
4	0	0	0	0	0	1

&	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	4	4
3	3	3	4	3	4
4	4	4	4	4	4

$\vee$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	1	2
3	0	1	1	3	3
4	0	1	2	3	4

(Only the value 0 is designated).

Despite the fact that Theorem 31 shows ROC_1 and ROC_2 to be distinct, there is still a great deal of collapse in the ROC-hierarchy. Again, several lemmas are required.

Lemma 24. In ROC_ω, the following schemata are derivable, for $1 \leqslant i < \omega$:

(i) $\neg B \supset B^\circ$

(ii) $\neg\neg B \supset B^\circ$

(iii) $\neg B^i \supset B^{i+1}$

(iv) $\neg\neg B^i \supset B^{i+1}$.

Proof: Schema (i) follows by RC from $(B \,\&\, \neg B) \supset B$, which is an instance of postulate (4) of ROC_ω.

Schema (ii) follows by RC from $(B \,\&\, \neg B) \supset \neg B$, which is an instance of postulate (5).

Schema (iii) follows by RC from $(B^i \,\&\, \neg B^i) \supset B^i$, which is an instance of (4).

Schema (iv) follows by RC from $(B^i \,\&\, \neg B^i) \supset \neg B^i$, which is an instance of (5)

Lemma 25. In ROC_ω, the schema $\neg B^i \supset B^i$ is derivable, for $1 \leqslant i < \omega$.

Proof: For $i = 1$, we have $\neg B \supset B^\circ$ by (i) of Lemma 24, from which $(B \,\&\, \neg B) \supset B^\circ$ follows by (5) and transitivity. Applying RC yields the desired $\neg B^\circ \supset B^\circ$.

For $i > 1$, we have $\neg B^{i-1} \supset B^i$ by (iii) of Lemma 24, from which $(B^{i-1} \,\&\, \neg B^{i-1}) \supset B^i$ follows by (5) and transitivity. Applying RC yields the desired $\neg B^i \supset B^i$.

In the following lemmas, $\top$ represents an arbitrary theorem of ROC_ω, and $(\neg\neg)^i\top$ represents $\top$ preceded by $2i$ occurences of $\neg$, for $1 \leqslant i < \omega$.

Lemma 26. In ROC_ω, the schema $(\neg\neg)^i\top \equiv \top^i$ is derivable, for $1 \leqslant i < \omega$.

Proof: The proof proceeds by induction on the value of i.

Base case ($i = 1$). Evidently, we have in ROC_ω both $\neg\top \supset \top$ and $\neg\top \supset \neg\top$; together, these yield $\neg\top \supset (\top \ \& \ \neg\top)$ with the assistance of (6). Applying RC yields $\top^\circ \supset \neg\neg\top$. Conversely, we have $\neg\neg\top \supset \top^\circ$ by (ii) of Lemma 24. Hence, we have $\neg\neg\top \equiv \top^\circ$.

Inductive step ($i > 1$). On inductive hypothesis, we have $(\neg\neg)^{i\text{-}1}\ \top \equiv \top^{i\text{-}1}$, and in particular, we have $(\neg\neg)^{i\text{-}1}\ \top \supset \top^{i\text{-}1}$. Applying RC twice yields $\neg\neg\ (\neg\neg)^{i\text{-}1}\ \top \supset \neg\neg\top^{i\text{-}1}$, i.e. $(\neg\neg^i)\ \top \supset \neg\neg\top^{i\text{-}1}$. But by (iv) of Lemma 24, we have $\neg\neg\top^{i\text{-}1} \supset \top^i$; transitivity, therefore, yields $(\neg\neg^i)\ \top \supset \top^i$. Conversely, again on inductive hypothesis, we have $(\neg\neg)^{i\text{-}1}\ \top \equiv \top^{i\text{-}1}$, and in particular, $\top^{i\text{-}1} \supset (\neg\neg)^{i\text{-}1}\ \top$. Applying RC yields $\neg(\neg\neg)^{i\text{-}1}\ \top \supset \neg\top^{i\text{-}1}$. By Lemma 25, we have $\neg\top^{i\text{-}1} \supset \top^{i\text{-}1}$, and therefore $\neg\top^{i\text{-}1} \supset (\top^{i\text{-}1} \ \& \ \neg\top^{i\text{-}1})$. By transitivity, then, we have $\neg(\neg\neg^{i\text{-}1})\ \top \supset (\top^{i\text{-}1} \ \& \ \neg\top^{i\text{-}1})$. Applying RC again yields $\top^i \supset \neg\neg(\neg\neg)^{i\text{-}1}\ \top$, i.e. $\top^i \supset (\neg\neg)^i\ \top$. Together with $(\neg\neg)^i\ \top \supset \top^i$, derived above, this gives the desired $(\neg\neg)^i\ \top \equiv \top^i$.

Lemma 27. In ROC_ω, the schema $\top^{i+1} \supset \top^i$ is derivable, for $1 \leqslant i < \omega$.

Proof: The proof proceeds by induction on the value of i.

Base case ($i = 1$). We have $\neg\neg\top \supset \top$ in ROC_ω; applying RC twice yields $\neg\neg\neg\neg\top \supset \neg\neg\top$. But by Lemma 26, we have $\neg\neg\top \equiv \top^\circ$ and $\neg\neg\neg\neg\top \equiv \top^2$. Hence, we have $\top^2 \supset \top^\circ$.

Inductive step ($i > 1$). On inductive hypothesis, we have $\top^{i\text{-}1+1} \supset \top^{i\text{-}1}$, i.e. $\top^i \supset \top^{i\text{-}1}$. Applying RC twice yields $\neg\neg\top^i \supset \neg\neg\top^{i\text{-}1}$. But by Lemma 26, we have $\neg\neg\top^i \equiv (\top^i)^\circ$, i.e. $\neg\neg\top^i \equiv \top^{i+1}$, and $\neg\neg\top^{i\text{-}1} \equiv (\top^{i\text{-}1})^\circ$, i.e. $\neg\neg\top^{i\text{-}1} \equiv \top^i$. Hence we have $\top^{i+1} \supset \top^i$.

Corollary. In ROC_ω, the schema $\top^{(i)} \equiv \top^i$ is derivable, for $1 \leqslant i < \omega$.

Proof: This follows by a straightforward inductive argument from Lemma 27.

Lemma 28. In each $ROC_{n+1}(1 \leqslant n < \omega)$, the schema $B^{(n)} \supset (\neg B^n \supset \neg\top)$ is derivable.

Proof: We note firstly that, by Lemma 23, we have $B^{(n+1)} \supset B^{(n+2)}$, $(B^\circ)^{(n+1)} \supset (B^\circ)^{(n+2)}$, and in general, for $1 \leqslant k < \omega$, $(B^k)^{(n+1)} \supset (B^k)^{(n+2)}$ in each $OC_{n+1}(1 \leqslant n < \omega)$ and therefore in each $ROC_{n+1}(1 \leqslant n < \omega)$. But we also have $B^{(n+2)} \supset (B^\circ)^{(n+1)}$, $(B^\circ)^{(n+2)} \supset (B^2)^{(n+1)}$, and in general, $(B^k)^{(n+2)} \supset (B^{k+1})^{(n+1)}$; so by transitivity we obtain $B^{(n+1)} \supset$

$(\mathrm{B}^{\circ})^{(n+1)}$, $(\mathrm{B}^{\circ})^{(n+1)} \supset (\mathrm{B}^{2})^{(n+1)}$, and in general, $(\mathrm{B}^{k})^{(n+1)} \supset (\mathrm{B}^{k+1})^{(n+1)}$. These latter schemata deliver, with the assistance of transitivity, the result $\mathrm{B}^{(n+1)} \supset (\mathrm{B}^{k})^{(n+1)}$, and in particular, $\mathrm{B}^{(n+1)} \supset (\mathrm{B}^{n})^{(n+1)}$. An instance of postulate $(12)^{(n+1)}$ of each $\mathrm{ROC}_{n+1}(1 \leqslant n < \omega)$ is $(\mathrm{B}^{n})^{(n+1)} \supset ((\top \supset \mathrm{B}^{n}) \supset ((\top \supset \neg\mathrm{B}^{n}) \supset \neg\top)$, which is easily reduced, using permutation, (1) and transitivity, to $(\mathrm{B}^{n})^{(n+1)} \supset (\mathrm{B}^{n} \supset (\neg\mathrm{B}^{n} \supset \neg\top))$. But we have just shown $\mathrm{B}^{(n+1)} \supset (\mathrm{B}^{n})^{(n+1)}$ to be derivable; hence by transitivity, we obtain $\mathrm{B}^{(n+1)} \supset (\mathrm{B}^{n} \supset (\neg\mathrm{B}^{n} \supset \neg\top))$. But we have $\mathrm{B}^{(n+1)} \supset \mathrm{B}^{n}$, so this reduces to $\mathrm{B}^{(n+1)} \supset (\neg\mathrm{B}^{n} \supset \neg\top)$, from which $\mathrm{B}^{(n)} \supset (\mathrm{B}^{n+1} \supset (\neg\mathrm{B}^{n} \supset \neg\top))$ quickly follows. But by (iii) of Lemma 24, we have $\neg\mathrm{B}^{n} \supset \mathrm{B}^{n+1}$; hence, this schema reduces further to the desired $\mathrm{B}^{(n)} \supset (\neg\mathrm{B}^{n} \supset \neg\top)$.

<u>Lemma 29</u>. In each $\mathrm{ROC}_{n+1}(1 \leqslant n < \omega)$, the schema $\top^{\circ} \supset \top^{(n+1)}$ is derivable.

<u>Proof</u>: By Lemma 28, in each $\mathrm{ROC}_{n+1}(1 \leqslant n < \omega)$, we have $\top^{(n)} \supset (\neg\top^{n} \supset \neg\top)$. But by the Corollary to Lemma 27, we have $\top^{(n)} \equiv \top^{n}$, so this yields $\top^{n} \supset (\neg\top^{n} \supset \neg\top)$. By Lemma 25, we have $\neg\top^{n} \supset \top^{n}$, so this schema reduces to $\neg\top^{n} \supset \neg\top$. Applying RC yields $\neg\neg\top \supset \neg\neg\top^{n}$. But by Lemma 26, we have $\neg\neg\top \equiv \top^{\circ}$ and $\neg\neg\top^{n} \equiv \top^{n+1}$; hence we have $\top^{\circ} \supset \top^{n+1}$. By the Corollary to Lemma 27 again, we have $\top^{(n+1)} \equiv \top^{n+1}$; so this delivers the desired $\top^{\circ} \supset \top^{(n+1)}$.

<u>Lemma 30</u>. In each $\mathrm{ROC}_{n+1}(1 \leqslant n < \omega)$, the schema $\top^{\circ} \supset (\mathrm{B}^{(n)} \supset \mathrm{B}^{(n+1)})$ is derivable.

<u>Proof</u>: An instance of postulate $(12)^{(n+1)}$ of each $\mathrm{ROC}_{n+1}(1 \leqslant n < \omega)$ is $\top^{(n+1)} \supset ((\neg\mathrm{B}^{n} \supset \top) \supset ((\neg\mathrm{B}^{n} \supset \neg\top) \supset \neg\neg\mathrm{B}^{n}))$. By Lemma 29, we have $\top^{\circ} \supset \top^{(n+1)}$, so by transitivity, we get $\top^{\circ} \supset ((\neg\mathrm{B}^{n} \supset \top) \supset ((\neg\mathrm{B}^{n} \supset \neg\top) \supset \neg\neg\mathrm{B}^{n}))$. But $\neg\mathrm{B}^{n} \supset \top$ is obviously derivable, and by Lemma 28, we have $\mathrm{B}^{(n)} \supset (\neg\mathrm{B}^{n} \supset \neg\top)$, so this schema simplifies to $\top^{\circ} \supset (\mathrm{B}^{(n)} \supset \neg\neg\mathrm{B}^{n})$. But by (iv) of Lemma 24, we have $\neg\neg\mathrm{B}^{n} \supset \mathrm{B}^{n+1}$, from which $(\mathrm{B}^{(n)} \supset \neg\neg\mathrm{B}^{n}) \supset (\mathrm{B}^{(n)} \supset \mathrm{B}^{(n+1)})$ quickly follows. By transitivity, therefore, we have $\top^{\circ} \supset (\mathrm{B}^{(n)} \supset \mathrm{B}^{(n+1)})$.

<u>Lemma 31</u>. In ROC_{ω}, the schema $\mathrm{B}^{2} \supset \top^{\circ}$ is derivable.

<u>Proof</u>: Evidently, we have both $(\mathrm{B} \mathbin{\&} \neg\mathrm{B}) \supset \top$ and $\mathrm{B}^{\circ} \supset \top$ in ROC_{ω}. Applying RC to each yields $\neg\top \supset \mathrm{B}^{\circ}$ and $\neg\top \supset \neg\mathrm{B}^{\circ}$, which can be combined to give $\neg\top \supset (\mathrm{B}^{\circ} \mathbin{\&} \neg\mathrm{B}^{\circ})$. Applying RC again yields $\neg(\mathrm{B}^{\circ} \mathbin{\&} \neg\mathrm{B}^{\circ}) \supset \neg\neg\top$, which is, by definition $\mathrm{B}^{2} \supset \neg\neg\top$. By Lemma 26, we

have $\neg\neg\top \equiv \top^\circ$; hence, we have $B^2 \supset \top^\circ$.

Lemma 32. In each $ROC_{n+1}(2 \leqslant n < \omega)$, the schema $B^{(n)} \supset B^{(n+1)}$ is derivable.

Proof: By Lemma 30, in each $ROC_{n+1}(1 \leqslant n < \omega)$ we have $\top^\circ \supset (B^{(n)} \supset B^{(n+1)})$. But for $2 \leqslant n < \omega$, we have $B^{(n)} \supset B^2$, and therefore $B^{(n)} \supset \top^\circ$ by Lemma 31. Hence, in each $ROC_{n+1}(2 \leqslant n < \omega)$, the above schema simplifies to the desired $B^{(n)} \supset B^{(n+1)}$.

Theorem 32. For $2 \leqslant n < \omega$, $ROC_n = ROC_2$.

Proof: Using Lemma 32, it is straightforward to show that postulates $(12)^{(n)}$ to $(16)^{(n)}$ of ROC_n are derivable in ROC_{n+1}, for $2 \leqslant n < \omega$; from this it follows that each such ROC_n is a subsystem of ROC_{n+1}. But conversely, it follows from Theorem 24 that each ROC_{n+1} is a subsystem of ROC_n; hence, each $ROC_n = ROC_{n+1}$, for $2 \leqslant n < \omega$. Equivalently, each $ROC_n = ROC_2$, for $2 \leqslant n < \omega$.

Thus, there are at most three distinct systems in the ROC-hierarchy: ROC_1, ROC_2 and ROC_ω. Our final result establishes that there are exactly three.

Theorem 33. $ROC_\omega \neq ROC_2$.

Proof: The matrices in the proof of Theorem 16, which validate the postulates of RNC_ω and therefore also those of ROC_ω, invalidate postulate $(12)^{(2)}$ of ROC_2 when A is assigned the value 0 and B is assigned the value 1.

14. CONCLUSION

The conclusion of our investigations in this second half of the chapter is mainly negative. All of the systems in the hierarchies obtained by replacing the original base system C_ω by, respectively, NC_ω, OC_ω + (12) and OC_ω fail to enjoy SE. Yet the consequences of adding RC in order to secure SE are that: (i) each of the hierarchies of systems incorporating RC collapses into just two systems, with the exception of the ROC-hierarchy, which comprises three distinct systems; and (ii) the strongest member of each such hierarchy either explicitly or substantively fails to satisfy condition (I), leaving only the base systems RNC_ω, ROC_ω + (12) and ROC_ω, together with ROC_2, as potentially acceptable paraconsistent logics. The second method of variation of the C-systems, then, has so far failed to produce the desired hierarchies of paraconsistent systems analogous to the original $C_n (1 \leqslant n \leqslant \omega)$ but enjoying SE.

Of course, these results are not conclusive for the second method of variation, for the addition of RC is ostensibly more than is required to secure SE; it may be that the hierarchies obtained by adding the weaker rule EC would prove to be more satisfactory. On the other hand, Theorem 9 of Chapter 5 and Theorem 11 of this chapter warn against too much optimism on this score, for they show that results obtained for at least the stronger C-style systems incorporating RC can often be extended to those incorporating EC, even though the proofs are perhaps more complicated.

A more radical alternative, perhaps the only course compatible with the retention of RC, is to weaken the basis of the C-systems still further. It may be desirable to explore bases even weaker than positive intuitionistic logic, such as the (positive) relevant logics developed in [1] and in subsequent works, including [18]. (This is the course advocated in [19]). It must be recognised, however, that such a course represents a significant departure from the approach taken by da Costa and embodied in his paraconsistency condition (II); but then, it might also be exactly what is required for the construction of paraconsistent systems which enjoy SE, and which, therefore, emulate what is desirable in classical logic rather than merely what is classical.

REFERENCES

[1] Anderson, A.R. and Belnap, N.D.Jnr. *Entailment: The Logic of Relevance and Necessity, vol. 1*, Princeton University Press (1975), Princeton, New Jersey.

[2] Arruda, A.I. "A survey of paraconsistent logic", *Mathematical Logic in Latin America* (ed. A.I. Arruda, R. Chuaqui and N.C.A. da Costa), North-Holland (1980), Amsterdam.

[3] Arruda, A.I. and Alves, E.H. "Some remarks on the logic of vagueness", *Bulletin of the Section of Logic*, v.8, no.3 (1979), pp.133-138.

[4] Arruda, A.I. and da Costa, N.C.A. "Sur le schéma de la séparation", *Nagoya Mathematical Journal*, v.38, (1970), pp.71-84.

[5] Arruda, A.I. and da Costa, N.C.A. "Le schéma de la séparation et les calculs J_n", *Mathematica Japonicae*, v.19, (1974), pp.183-186.

[6] Bunder, M.W. "A new hierarchy of paraconsistent logics", *Proceedings of the Third Brazilian Conference on Mathematical Logic* (ed. A.I. Arruda, N.C.A. da Costa and A.M. Sette), *Sociedade Brasileira de Lôgica* (1980), São Paulo, pp.13-22.

[7] Bunder, M.W. "On Arruda and da Costa's Logics J_1 to J_5", *The Journal of Non-classical logic*, v.2, no.1 (1983), pp.43-48.

[8] Curry, H.B. *Foundations of Mathematical Logic*, Dover Publications (1976), New York.

[9] da Costa, N.C.A. "On the theory of inconsistent formal systems", *Notre Dame Journal of Formal Logic*, v.15, no.4 (1974), pp.497-510.

[10] da Costa, N.C.A. and Alves, E.H. "Relations between paraconsistent and many-valued logics", *Bulletin of the Section of Logic*, v.10, no.4 (1981), pp.185-191.

[11] da Costa, N.C.A. and Guillaume, M. "Sur les calculs C_n", *Anais da Academia Brasileira de Ciências*, v.36, (1964), pp.379-82.

[12] Gentzen, G. "Investigations into logical deduction", *The Collected papers of*

Gerhard Gentzen, (ed. M.E. Szabo), North-Holland (1969), Amsterdam.

[13] Mortensen, C. "Every quotient algebra for C_1 is trivial", *Notre Dame Journal of Formal Logic*, v.21, no.4 (1980), pp.694-700.

[14] Mortensen, C. "Paraconsistency and C_1", *Logic Research Paper no.10*, (1981), Australian National University (Philosophy, RSSS), Canberra.

[15] Priest, G. and Routley, R. "On paraconsistency", *Research Series in Logic and Metaphysics*, (1983), Australian National University (Philosophy, RSSS), Canberra.

[16] Raggio, A.R. "Propositional sequence-calculi for inconsistent systems", *Notre Dame Journal of Formal Logic*, v.9, (1968), pp.359-366.

[17] Raggio, A.R. "A proof-theoretic analysis of da Costa's C_ω^* ", *Proceedings of the First Brazilian Conference on Mathematical Logic* (ed. A.I. Arruda, N.C.A. da Costa and R. Chuaqui), Marcel Dekker (1977), New York, pp.233-240.

[18] Routley, R., Meyer, R.K., Plumwood, V. and Brady, R.T. *Relevant logics and their rivals 1*, Ridgeview (1982), Atascadero, California.

[19] Sylvan, R. "Variations on da Costa C-systems and dual-intuitionistic logics", (forthcoming).