SOME TECHNIQUES OF HARMONIC ANALYSIS
ON COMPACT LIE GROUPS
WITH APPLICATIONS TO LACUNARITY

by

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STATEMENT

All results in this thesis, except those specifically attributed to others, are the product of my own research.

A. H. Dooley

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I should like to thank the following individuals and organizations for their encouragement and assistance throughout my course:

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The study of lacunary subsets of the duals of compact abelian groups can be traced back to Sidon and Hadamard, and it is still an area of activity today. During the last decade, this study has been extended to the compact non-abelian case, which forms the central theme of this thesis.

A notion of lacunary set due to Bozejko and Pytlik is generalized from the abelian to the non-abelian setting, and a study is made of the class of sets thus obtained for compact Lie groups. This class includes most types of lacunary sets previously considered. In the process, it is necessary to develop some new techniques of harmonic analysis on compact Lie groups; the results obtained generalize and improve a number of known results on previously considered notions of lacunarity.

Chapter 1 is a collection of the definitions, notation and elementary results of harmonic analysis which will be used throughout the thesis, followed by a brief introduction to some aspects of the theory of lacunarity for compact nonabelian groups.

In Chapter 2, I begin the study of sets of type \( V(p, q) \) and \( A(p, q) \) \((p, q \in [1, \infty])\) and their central and local central analogues. An important subclass of these sets consists of the \( p \)-Sidon and central \( p \)-Sidon sets \((p \in [1, 2])\). Most of the theory of this chapter is valid for arbitrary compact groups.

Chapter 3 deals with sets of type local central \( V(p, q) \) and local central \( A(p, q) \), specializing to the case of a compact Lie group. The theorems are proved by extensive use of the theory of semisimple Lie algebras to obtain estimates for the \( p \)-norms of the irreducible characters in terms of their dimensions. It is shown that the dual of a compact Lie group is of type local central \( A(2+\epsilon) \), but that no infinite subset of the dual is of type local central \( A(4-\delta) \). Here \( \epsilon > 0 \) and \( \delta > 0 \) are
constants which depend on the group. It is also a result of this chapter that the dual of a compact Lie group contains no infinite $p$-Sidon sets ($1 \leq p < 2$).

The results of Chapter 4 concern the convolution centres of the classical function and measure algebras of a compact connected Lie group (i.e. the space of trigonometric polynomials, the space of continuous functions, the $L^p$ spaces ($1 \leq p \leq \infty$) and the space of measures). It is shown that the convolution centre of each of these spaces is linearly isometrically isomorphic to a subspace of the same space of functions or measures on the maximal torus, and a formula is derived which makes it possible to calculate the Fourier transform of a central function or measure in terms of the Fourier transform of its image under this isometric isomorphism. One is thus able to reduce many questions concerning convolution centres to questions about abelian groups. Again, important use is made of the representation theory of Lie groups and Lie algebras.

In Chapter 5, the theory of Chapter 4 is applied to sets of type central $V(p, q)$ and central $A(p, q)$ for compact connected Lie groups. In particular, it is shown that every infinite subset of the dual contains an infinite set of type $A(2+\varepsilon)$ (where $\varepsilon$ is as in Chapter 3), and it is shown that a set of representations is of type central $A(2)$ if and only if the corresponding set of characters of the maximal torus is of type $A(2)$. Finally, central $p$-Sidon sets are considered; central $p$-Sidonicity is related to a simple arithmetic condition - $r$-boundedness - on sets of characters, a new proof of the Ragozin-Rider result that the dual contains no infinite central Sidon sets is given, and it is shown that the dual of $SU(2)$ contains no infinite central $p$-Sidon sets ($1 \leq p < 2$).

I make much wider use of the theory of Lie groups, Lie algebras and their representations than previous authors in this field: the requisite facts from this theory are outlined in two appendices, appendix A on Lie
algebras, and appendix B on Lie groups. A list of symbols and definitions is provided for the reader's convenience, and a list of references relates the results in this thesis to the literature.

1. Set theory

\[ A \cup B \] union of sets
\[ A \cap B \] intersection of sets
\[ A \subseteq B \] containment
\[ A^c \] complement of the set

\[ A \setminus B \] \( A \) minus \( B \)

\[ \{ x \mid P(x) \} \] the set of elements of \( A \) satisfying property \( P \)

\[ \text{card } A \] cardinal number of \( A \)

\[ f \mid A \] restriction of function \( f \) to a subset \( A \) of the domain

\[ f : A \rightarrow B \] function \( f \) has domain \( A \) and range \( B \)

\[ f(a) = b \] \( f \) evaluated at \( a \)

\[ f \circ g \] composition of functions

Definitions and elementary properties of partially ordered sets.

\( \mathbb{N} \) set of natural numbers \( \{0, 1, 2, \ldots \} \)

\( \mathbb{Z} \) set of integers (positive integers \( \{1, 2, 3, \ldots \} \))

\( \mathbb{Q} \) set of rational numbers

\( \mathbb{R} \) set of real numbers

\( \mathbb{C} \) set of complex numbers

2. Algebra

Definitions and elementary properties of group, subgroups, and rings.

Subgroup, cosets of a group, products, quotients of groups, action of a group on a set.
DEFINITIONS AND NOTATION

The following definitions and notation will be used without comment in the remainder of the text.

1. Set theory

\( \in \) \hspace{1cm} \text{elementhood}

\( \cup \) \hspace{1cm} \text{union of sets}

\( \cap \) \hspace{1cm} \text{intersection of sets}

\( \subseteq \) \hspace{1cm} \text{containment}

\( A^c \) \hspace{1cm} \text{complement of the set}

\( A \setminus B \) \hspace{1cm} \( A \cup B^c \)

\( \{x \in A \mid P(x)\} \) \hspace{1cm} \text{the set of elements of } A \text{ satisfying property } P

\( \text{card } A \) \hspace{1cm} \text{cardinal number of } A

\( f \mid_A \) \hspace{1cm} \text{restriction of function } f \text{ to a subset } A \text{ of its domain}

\( f : A \to B \) \hspace{1cm} \text{function } f \text{ has domain } A \text{ and range } B

\( f : a \mapsto b \) \hspace{1cm} f(a) = b

\( f \circ g \) \hspace{1cm} \text{composition of functions}

Definitions and elementary properties of partially ordered sets.

\( \mathbb{N} \) \hspace{1cm} \text{set of natural numbers } \{0, 1, 2, \ldots\}

\( \mathbb{Z} \) \hspace{1cm} \text{set of integers (positive integers) } \{0, \pm1, \pm2, \ldots\}

\( \mathbb{Z}^+ \) \hspace{1cm} \text{set of positive integers} \{1, 2, 3, \ldots\}

\( \mathbb{Q} \) \hspace{1cm} \text{set of rational numbers}

\( \mathbb{R} \) \hspace{1cm} \text{set of real numbers}

\( \mathbb{C} \) \hspace{1cm} \text{set of complex numbers}

2. Algebra

Definitions and elementary properties of group, subgroup, normal subgroup, centre of a group, products, quotients of groups, action of a group on a set.
Definitions and elementary properties of vector space, matrices.

$\dim V$ dimension of the vector space $V$

$\text{GL}(V)$ set of invertible endomorphisms of $V$

(See Herstein [1] for these concepts.)

3. Topology

Definitions and elementary properties of open sets, closed sets, compactness, connectedness, arcwise connectedness, simple connectedness, fundamental group.

If $a, b \in \mathbb{R} \cup \{\infty\}$, $a < b$, let

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

$$]a, b] = \{x \in \mathbb{R} \mid a < x < b\}.$$

Similarly define $[a, b]$ and $]a, b]$. 

4. Functional analysis

Definitions and elementary properties of Hilbert space, Banach space, Banach algebra, Banach $\ast$-algebra, ideals, homomorphisms of Banach algebras.

$V^*$ set of continuous complex-valued linear functionals on the normed vector space $V$

$\mathcal{B}(V)$ set of norm continuous linear endomorphisms of $V$

$\|\phi\|$ the operator norm of $\phi \in \mathcal{B}(V)$

(see Edwards [1] for these terms).
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CHAPTER 1
PRELIMINARIES

(1.1) Introduction

This chapter has two main aims. Firstly, for completeness, it sets forth the definitions and fundamental concepts from the theory of harmonic analysis on compact groups which will be used later in the thesis. These have been well expounded previously - notably by the encyclopaedic Hewitt and Ross [1]; also Edwards [3] and Dunkl and Ramirez [1] provide well-written introductions to this subject. My treatment is accordingly brief. In addition to material from these books, I have included a statement of the structure theorem for compact connected groups (section 2), and a somewhat expanded original treatment of central function and measure algebras which was hinted at in lemma (1.1) of Parker [1] (section 5).

The second aim of this chapter is to provide an outline of the subject of lacunarity for compact groups, since this is the main motivating force behind this thesis. Hewitt and Ross [1], Volume II, was published in 1970, and gave a very good summary of what was known up to that time. In section 6, I give a brief outline of the material presented there, followed by a sketch of some of the more recent developments. My treatment cannot pretend to be exhaustive or impartial: I have stressed those lines of research which lead to the research contained in this thesis, but I hope that it will help to motivate and set in context that which follows.

(1.2) Compact groups

(1.2.1) A set $G$ which is endowed with a group structure (an identity $e$, and operations $x \mapsto x^{-1}$, $(x, y) \mapsto xy$) and which is also a
topological space is said to be a topological group if the map

\((x, y) \mapsto xy^{-1} : G \times G \to G\) is continuous. The morphisms for the category of topological groups are the continuous homomorphisms. If there exists a homeomorphism which is also an isomorphism \(G_1 \to G_2\), I shall write \(G_1 \cong G_2\), and say that \(G_1\) and \(G_2\) are isomorphic (as topological groups).

If the underlying topological space of a topological group \(G\) is a locally compact Hausdorff space, \(G\) is called a locally compact group; if the underlying space is a compact Hausdorff space, \(G\) is a compact group. (I shall never consider non-Hausdorff groups.)

(1.2.2) The class of abelian groups (those for which \(\forall x, y \in G, xy = yx\)) is an important subclass of the class of topological groups. Perhaps the most important compact abelian group is \(\mathbb{T}\), the circle group. \(\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}\), has the topology induced from \(\mathbb{C}\) and the multiplication induced from the multiplication of \(\mathbb{C}\). Sometimes, it will be convenient to identify the interval \([-\pi, \pi]\) with \(\mathbb{T}\), via the map \(\theta \mapsto e^{i\theta}\).

(1.2.3) Let \(G\) be a topological group. A subgroup \(H\) of \(G\) which is a closed set is called a closed subgroup of \(G\). It may happen that the number of \(H\) cosets \(xH\) \((x \in G)\) is finite. In this case, we say that \(H\) has finite index in \(G\), and denote the number of cosets by \([G : H]\).

If \(N\) is a closed normal subgroup of \(G\), we can form the quotient group, \(G/N\). This is also a topological group, under the strongest topology for which the quotient map \(G \to G/N\) is continuous (see Hewitt and Ross [1], §5).

The direct product, \(\prod_{i \in I} G_i\), of any family of topological groups, \((G_i)_{i \in I}\), endowed with the product topology, is again a topological group. If the \(G_i\) are all compact, so is the product group (see Hewitt and Ross
If (the underlying topological space of) \( G \) is a connected topological space, \( G \) is called a connected group. For any topological group \( G \), the connected component of the identity, \( G_0 \), is a closed normal subgroup of \( G \). We say \( G \) is totally disconnected if \( G_0 = \{e\} \). Note that \( G/G_0 \) is always totally disconnected. (For further details, see Hewitt and Ross [1], §7.)

(1.2.4) A group \( G \) which has the structure of an analytic manifold such that the operation \((x, y) \mapsto xy^{-1}\) is analytic is called a Lie group. Lie groups are locally compact groups. Some further information on Lie groups is presented in appendix B.

It is perhaps worth mentioning here that if \( G \) is a compact Lie group, then the connected component of the identity, \( G_0 \), has finite index in \( G \) (i.e. \( G/G_0 \) is finite). This can be seen by noting that, if \( G \) is a Lie group, \( G_0 \) is an open subset of \( G \) (Price [4], Lemma 2.1.4 (b)). Hence, since \( G \) is compact a finite number of the cosets \( xG_0 \) (\( x \in G \)) covers \( G \), i.e. \( G/G_0 \) is finite.

A Lie group is called simple if it contains no non-trivial connected normal subgroups. A Lie group is semi-simple if it is the direct product of finitely many simple Lie groups (cf. (B1)).

A Lie group is simply connected if its underlying topological space is arcwise connected and has trivial fundamental group. In some senses, the compact simply connected simple Lie groups are the fundamental building blocks for non-abelian compact connected groups, as is shown in the next paragraph.

(1.2.5) THEOREM (The Structure Theorem for Compact Connected Groups).

Let \( G \) be a compact connected group. Then there exists a family
(\(G_i\))_{i \in I}$ of compact simply connected simple Lie groups, a compact connected abelian group $\Lambda$, and a totally disconnected subgroup $\mathbb{Z}$ of the centre of \(A \times \bigsqcup_{i \in I} G_i\) such that

\[ G \cong \left( A \times \bigsqcup_{i \in I} G_i \right) / \mathbb{Z}. \]

To the best of my knowledge, this theorem first appeared in Weil [1]. A more recent exposition of it is given in Price [4], (6.5.6).

Note that a structure theorem for connected compact abelian groups is given in Hewitt and Ross [1], (25.23). This theorem is rather technical and I shall not discuss it here.

If $G$ is a compact Lie group, then we may assume that $A$ is \(\prod^n\) (i.e. \(\prod \times \ldots \times \prod\) (n times)) for some \(n \in \mathbb{N}\), and that the set $I$ is finite. This special case is discussed in (B2).

(1.2.6) Let $G$ be a compact group. I shall denote by \(C(G)\) the set of complex-valued continuous functions on $G$. Then \(C(G)\) is a vector space, which is a Banach *-algebra under the norm \(\|f\|_\infty = \sup_{x \in G} |f(x)|\), pointwise multiplication, and involution \(f^*(x) = f(x^{-1})\). The set \(M(G)\) of (Radon) measures on $G$, is defined to be the set of continuous functionals on \(C(G)\). If \(\nu \in M(G)\), I shall write \(\nu(f)\) and \(\int f d\nu\) interchangeably to mean the value of \(\nu\) at \(f \in C(G)\). The space \(M(G)\) is a Banach space with norm \(\|\nu\|_1 = \sup\{|\nu(f)| \mid \|f\|_\infty = 1\}\).

If \(f \in C(G)\), define \(f_x\) and \(x^f \in C(G)\) by \((x^f)(y) = f(xy)\) and \(f_x(y) = f(yx)\). There exists a unique regular measure on $G$, \(\lambda_G\), called Haar measure such that: for all \(f \in C(G)\), \(\lambda_G(f_x) = \lambda_G(f)\); if \(f(x) \geq 0\) for all \(x \in G\) and \(f \neq 0\) then \(\lambda_G(f) > 0\); and \(\lambda_G(1) = 1\).

(1 : \(x \to 1 \in C(G)\)) (Hewitt and Ross [1], (15.8)). Note that compact groups are unimodular; so we also have, for \(f \in C(G)\), \(\lambda_G(f^*) = \overline{\lambda_G(f)}\),
and \( \lambda_G(f^*_w) = \lambda_G(f) \). For \( \mu \in M(G) \), I shall denote by \( \mu^X \) (respectively \( \mu^X \)) the measure \( f \mapsto \mu\left(f^*_w\right) \) (respectively \( f \mapsto \mu(f^*_w) \)).

Let \( L^p(G) \) \((1 \leq p \leq \infty)\) denote the usual \( L^p \)-spaces with respect to the measure \( \lambda_G \); \( L^p(G) \) is the Banach space of equivalence classes of functions with integrable \( p \)th powers (or, in the case \( p = \infty \), of \( \lambda_G \)-essentially bounded functions), two functions being declared equivalent if they are equal almost everywhere. The norm for \( L^p(G) \) is 
\[
\|f\|_p = \left( \int |f|^p \, d\lambda_G \right)^{1/p} \quad \text{if } 1 \leq p < \infty \quad \text{and} \quad \|f\|_\infty = \lambda_G - \text{ess sup} \{|f(x)| \}.
\]

The space \( C(G) \) is a dense subspace of \( L^p(G) \) for \( 1 \leq p < \infty \).

As usual, 
\[
C(G) \subseteq L^\infty(G) \subseteq L^p(G) \subseteq L^q(G) \subseteq L^1(G) \subseteq M(G)
\]
for \( 1 < q < p < \infty \). Notice that the embedding \( L^1(G) \to M(G) \) is given by 
\[
f \mapsto f.\lambda_G, \quad \text{where} \quad (f.\lambda_G)(g) = \int g \cdot f \, d\lambda_G.
\]

If \( \nu, \mu \in M(G) \), define \( \nu * \mu \) by, for \( g \in C(G) \),
\[
(\nu * \mu)(g) = \nu(x \mapsto \mu(\cdot^{-1} x)) , \quad \text{and} \quad (\mu^*)(f) = \mu(f^*)\).
\]
These operations make \( M(G) \) into a Banach \*\-algebra; further \( L^1(G) \) is a closed ideal. Note that for \( f, g \in L^1(G) \), 
\[
(f * g)(y) = \int f(x)g(x^{-1}y) \, d\lambda_G(x)
\]

Notice also that \( L^p(G) \) is an \( L^1(G) \)-module - for \( f \in L^p(G) \), \( g \in L^1(G) \), 
\[
f * g \in L^p(G) \quad \text{and} \quad \|f * g\|_p \leq \|f\|_1 \|g\|_p . \]

\( L^p(G) \) is a Banach \*\-algebra with convolution and \( f^*(x) = f(\cdot^{-1} x) \). Further, for \( p \in [1, \infty[ \), let, \( p' \in [1, \infty[ \) be such that \( (1/p) + (1/p') = 1 \) (where we set \((1/\infty) = 0\)). Then for \( p \in [1, \infty[ \), \( L^{p'}(G) \) is the dual space of \( L^p(G) \). The duality is given by \( (f, g) = (f.\lambda_G)(g) = \int f \cdot g \, d\lambda_G \quad (f \in L^{p'}(G), \ g \in L^p(G)) \).
These matters have been dealt with extensively elsewhere, and I do not wish to dwell on them here. A very full account is given in Hewitt and Ross [1], Chapters 3, 4 and 5.

(1.3) Representation theory of compact groups

(1.3.1) If $H$ is a Hilbert space over $\mathbb{C}$, denote by $U(H)$ the set of all unitary operators $H \to H$ (the reader will recall that the linear operator $U : H \to H$ is unitary if for all $\xi, \zeta \in H$, $(U\xi, U\zeta) = (\xi, \zeta)$). $U(H)$ is a topological group (with composition as operation, and the strong operator topology). If $H$ is a finite dimensional Hilbert space, $U(H)$ is a compact group. If $H$ is the simplest of all Hilbert spaces, $\mathbb{C}$, then $U(H) = \Pi$.

(1.3.2) If $G$ is a locally compact group and $H$ is a Hilbert space, then a representation of $G$ on $H$ is a continuous homomorphism $\sigma : G \to U(H)$. The dimension of the Hilbert space $H$ (possibly infinite) is called the dimension of $\sigma$. I shall be concerned almost exclusively with finite dimensional representations.

Let $\sigma$ be a representation of $G$ on $H$. A closed subspace $V$ of $H$ is called invariant if for all $x \in G$, $\sigma(x)V \subseteq V$. The representation $\sigma$ is called irreducible if the only invariant subspaces of $H$ are $H$ and $\{0\}$.

Two representations of $G$, $\sigma_1$ on $H_1$ and $\sigma_2$ on $H_2$ are called equivalent (written $\sigma_1 \cong \sigma_2$) if there is a linear isometry $\phi : H_1 \to H_2$ such that for all $x \in G$, $\phi \circ \sigma_1(x) = \sigma_2(x) \circ \phi$. This is the same as "unitary equivalence" (see Hewitt and Ross [1], (13.43)). Without worrying too much about foundational problems, I denote by $\Sigma(G)$ the dual of $G$, a maximal set of pairwise inequivalent representations of $G$. (An argument dealing with the foundational problems raised here is given by
The Gelfand-Raikov theorem (Hewitt and Ross [1], (22.12)) assures us that there are sufficiently many elements of $\Sigma(G)$ to separate points of $G$; i.e. for every $x \neq y \in G$, there is $\sigma \in \Sigma(G)$ such that $\sigma(x) \neq \sigma(y)$.

Some authors have considered non-unitary representations of locally compact groups. Actually, it turns out that provided $G$ is a compact group this gives us no more generality: if $H$ is a Hilbert space, and $\sigma : G \to \mathcal{B}(H)$ is a weakly continuous irreducible homomorphism, then a new inner product can be introduced into $H$ so that $\sigma$ is a representation in the above sense (see Weil [1], §18).

(1.3.3) Suppose $G$ is a compact group. Then it is known (see Hewitt and Ross [1], (22.13)) that every irreducible representation of $G$ has finite dimension. Henceforth, if $\sigma \in \Sigma(G)$, I shall denote by $H_\sigma$ the finite dimensional space such that $\sigma : G \to \cup(H_\sigma)$ and by $d_\sigma$ the dimension of $H_\sigma$. I shall always denote by $I_{d_\sigma} : H_\sigma \to H_\sigma$ the identity operator with domain $H_\sigma$.

If $\sigma \in \Sigma(G)$, let $\overline{\sigma}$ denote the unique element of $\Sigma(G)$ equivalent to its conjugate representation (which acts in the conjugate Hilbert space $H_\sigma^*$ — see Hewitt and Ross [1], (27.27)).

If \((\sigma_i)_{i \in I}\) is a family of representations of $G$, we may form their direct sum \(\bigoplus_{i \in I} \sigma_i\), which acts in the space \(\bigoplus_{i \in I} H_i\). Every representation of $G$ may be uniquely decomposed as a direct sum of irreducible representations (Hewitt and Ross [1], (27.29) and (27.44)).

If $\sigma_1$, $\sigma_2$ are two representations of $G$, their tensor product, $\sigma_1 \otimes \sigma_2$, may be defined in a more or less natural fashion (see Hewitt and Ross [1], (27.33)), acting in $H_{\sigma_1} \otimes H_{\sigma_2}$. 

Hewitt and Ross [1], footnote to (27.3).
Using the unique decomposition of a representation into a direct sum of irreducibles, I may, for \( \sigma, \eta \in \Sigma(G) \), write \( \sigma \otimes \eta = \bigoplus_{\beta \in \Sigma(G)} n_{\sigma \eta}(\beta)\beta \), where \( n_{\sigma \eta} : \Sigma(G) \to \mathbb{N} \) and has finite support. The support of \( n_{\sigma \eta} \) is denoted \( \sigma \times \eta \).

We may think of "\( \times \)" and "\( - \)" as arithmetic operations on \( \Sigma(G) \). These make \( \Sigma(G) \) into a hypergroup (in the sense of Helgason [1]). A subset of \( \Sigma(G) \) closed under \( \times \) and \( - \) is called a subhypergroup.

Hypergroups have been considered by Helgason [1], Dunkl and Ramirez [4], McMullen [1], McMullen and Price [1], [2] (see also Hewitt and Ross [1], (27.36)-(27.38)).

(1.3.4) If \( \{G_i\}_{i \in I} \) is a family of compact groups, then \( \Sigma\left(\bigoplus_{i \in I} G_i\right) \) consists of those representations of the form \( \sigma_{i_1} \otimes \ldots \otimes \sigma_{i_n} \), \( \{i_1, \ldots, i_n\} \) a finite subset of \( I \), and \( \sigma_{i_j} \in \Sigma(G_{i_j}) \) for \( j \in \{1, \ldots, n\} \).

If \( G \) is a compact group and \( N \) a normal subgroup, let \( A(\Sigma(G), N) = \{ \sigma \in \Sigma(G) \mid \sigma|_N = I_{d_\sigma} \} \). Then \( A(\Sigma(G), N) \) may be identified with \( \Sigma(G/N) \) (Hewitt and Ross [1], (28.10)).

It is possible to get conditions on the hypergroup structure of \( \Sigma(G) \) which are equivalent to conditions on the structure of \( G \). The most important of these are undoubtedly:

(i) \( G \) is a Lie group if and only if \( \Sigma(G) \) is finitely generated (i.e. there exist \( \{\sigma_1, \ldots, \sigma_n\} \subseteq \Sigma(G) \) such that the smallest subhypergroup containing \( \{\sigma_1, \ldots, \sigma_n\} \) is \( \Sigma(G) \)); for a proof of this, see Price [4], (6.1.1);

(ii) \( G \) is totally disconnected if and only if every element of \( \Sigma(G) \) has finite order (i.e. each \( \sigma \in \Sigma(G) \) is contained in
a finite subhypergroup); this is proved by Hewitt and Ross [1], (28.19).

In this connection, I would like to record the following

**DEFINITION.** A compact group $G$ is **tall** if, for every $n \in \mathbb{N}$, 
\[
\{ \sigma \in \Sigma(G) \mid d_{\sigma} = n \}
\]
is a finite set. It will be seen in (B4) that compact connected semi-simple Lie groups are tall.

(1.3.5) Let $G$ be a compact group. If $\sigma \in \Sigma(G)$ and $\xi, \zeta \in H_{\sigma}$, then 
\[
t_{\xi, \zeta}^{(\sigma)} : x \mapsto \langle \sigma(x)\xi, \zeta \rangle \in \mathcal{C}(G).
\]
The linear span of all such functions is called the set of **trigonometric polynomials** on $G$, and is denoted $T(G)$. By the Gelfand-Raikov and Stone-Weierstrasse theorems, $T(G)$ is a dense subalgebra of $\mathcal{C}(G)$. It is worth noting that, for $\sigma, \eta \in \Sigma(G)$, $\xi, \zeta \in H_{\sigma}$, $\xi_0, \zeta_0 \in H_{\eta}$,
\[
\int t_{\xi, \zeta}^{(\sigma)} \cdot \overline{t_{\xi_0, \zeta_0}^{(\eta)}} \, d\lambda_{\sigma} = \begin{cases} 
\frac{1}{d_{\sigma}} \langle \xi, \xi_0 \rangle \langle \zeta, \zeta_0 \rangle & \text{if } \sigma = \eta, \\
0 & \text{otherwise},
\end{cases}
\]
(see Hewitt and Ross [1], (27.18)). These equations are called the **orthogonality relations**.

Suppose $\sigma$ is any finite dimensional representation of $G$. Then $\chi_{\sigma} : x \mapsto \text{tr}(\sigma(x))$ is well-defined and an element of $T(G)$. This function is called the **character** of $\sigma$. Characters of irreducible representations are called **irreducible characters**. The following statements are then true.

(i) If $\sigma$ and $\sigma_1$ are irreducible representations of $G$, then $\chi_{\sigma} = \chi_{\sigma_1}$ if and only if $\sigma$ is equivalent to $\sigma_1$.

(ii) If $\sigma$ is a finite dimensional representation of $G$, then $\chi_{\sigma} = \overline{\chi_{\sigma}}$. 

(iii) If $\sigma, \eta$ are two finite-dimensional representations of $G$, then $\chi_{\sigma \oplus \eta} = \chi_\sigma + \chi_\eta$, and $\chi_{\sigma \otimes \eta} = \chi_\sigma \cdot \chi_\eta$.

(iv) If $\sigma, \eta, \beta \in \Sigma(G)$, then

$$\int \chi_\sigma \overline{\chi_\eta} d\lambda_G = \begin{cases} 1 & \text{if } \sigma = \eta, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int \chi_\sigma \chi_\eta \overline{\chi_\beta} d\lambda_G = n_{\sigma \eta}(\beta).$$

(1.3.6) If $T$ is a compact abelian group, every irreducible representation is, by Schur's lemma, one-dimensional; that is, is a homomorphism $T \to \mathbb{T}$. Thus, the set of irreducible representations and the set of irreducible characters coincide. If $\sigma, \eta \in \Sigma(T)$, then $\sigma \times \eta$ contains precisely one element, and the above arithmetical operations make $\Sigma(T)$ into a group, the character group of $T$. The theory of character groups of abelian groups has been extensively studied (e.g. Hewitt and Ross [1], Rudin [2], Edwards [2]).

(1.3.7) I conclude this section on the representation theory of compact groups with a brief introduction to the theory of induced representations. This powerful tool is classical for the study of representations of finite groups (which form, of course, a subclass of the class of compact groups). It was extended to the class of locally compact groups by Mackey [1], [2], [3]. I shall, however, have no need of the theory at the level of generality proposed by Mackey - I shall use it as a tool to extend results on compact connected Lie groups to arbitrary compact Lie groups, by inducing from the component of the identity. Thus the groups I shall consider are compact, and I shall always induce from a subgroup of finite index. At this level of simplicity, most theorems can be proved by simple extensions of the method for finite groups, which is well outlined in Curtis and Reiner [1], or Dornhoff [1].

**Definition.** Let $G$ be a compact group, $H$ a closed subgroup of
finite index in $G$. Suppose $\xi$ is a finite dimensional representation of $H$ on $H_{\xi}$. Define a representation, $\xi^G$ of $G$ as follows:

$$H_{\xi}^G = \{ f : G \to H_{\xi} \mid \forall h \in H, \forall x \in G, f(h \cdot x) = \xi(h)(f(x)) \}.$$ 

This is clearly a finite-dimensional space. (Its dimension is $d_{\xi} \cdot [G : H]$.) For $x \in G$ and $f \in H_{\xi}^G$, define $(\xi^G(x) \cdot f) \in H_{\xi}^G$ by

$$(\xi^G(x) \cdot f) : y \mapsto f(xy).$$

It is not hard to check that $\xi^G$ is a representation of $G$ on $H_{\xi}^G$; it is called the representation of $G$ induced from $\xi$.

**Lemma.** Let $x_1, \ldots, x_{[G:H]}$ be a set of coset representatives for $G/H$. Define, for $x \in G$,

$$\chi_{\xi}(x) = \begin{cases} \chi_{\xi}(x) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

Then

$$\chi_{\xi^G}(y) = \frac{[G:H]}{\sum_{t=1} \chi_{\xi}(x^{-1} y x_t)}$$

This lemma is easy to prove from the above definition. I shall not include a proof here.

**Theorem (The Frobenius Reciprocity Theorem).** Let $G$ be a compact group, $H$ a closed subgroup of finite index in $G$. Let $\xi \in \Sigma(H)$ and $\sigma \in \Sigma(G)$. (Then $\xi^G$ is a representation of $G$ and $\sigma|_H$ is a representation of $H$.) The number of times $\sigma$ occurs in the decomposition of $\xi^G$ as a direct sum of irreducible representations is the same as the number of times $\xi$ occurs in the decomposition of $\sigma|_H$ as a direct sum of irreducible representations.

A proof of this theorem appears in Mackey [2] (Theorem 5.1).
(1.3.9) THEOREM (Clifford). Suppose $G$ is a compact group, $H$ a normal closed subgroup of finite index in $G$.

Let $\sigma \in \Sigma(G)$. Then there exists $\xi \in \Sigma(H)$, and there exists $n_{\sigma} \in \mathbb{N}$ such that

$$\sigma|_H = n_{\sigma} \cdot \left( \xi(g_1) \oplus \cdots \oplus \xi(g_k) \right),$$

where for $g \in G$ and $h \in H$, $\xi(g)(h) = \xi(g^{-1}h)$, and $\{g_1, \ldots, g_k\}$ is a complete set of coset representatives for $H \xi = \{g \in G \mid \xi(g) \approx \xi\}$.

This theorem is a very special case of theorem 12.1 of Mackey [1]. Perhaps a more amenable presentation is that of Curtis and Reiner [1], (49.7).

(1.4) The spaces $L^p(\Sigma(G))$. Fourier transforms

Throughout this section, $G$ will be a compact group, and $\Sigma(G)$ will be as defined in (1.3.2).

(1.4.1) Suppose $H$ is a finite-dimensional Hilbert space. For $A \in \mathcal{B}(H)$ and $p \in [1, \infty]$, define $|A|$ to be the unique positive-definite square root of the Hermitian operator $A^*A$, and let $a_1, \ldots, a_k$ be the (necessarily non-negative) eigenvalues of $|A|$. Then set

$$\|A\|_{\varphi_p} = \left( \sum_{i=1}^{k} a_i^p \right)^{1/p}(1 \leq p < \infty),$$

$$\|A\|_{\varphi_{\infty}} = \max\{a_1, \ldots, a_k\}.$$

These define norms on the space $\mathcal{B}(H)$; $\|A\|_{\varphi_{\infty}}$ co-incides with the usual operator norm. For further details, see Hewitt and Ross [1], appendix D.

(1.4.2) Define $\pi : \bigcup_{\sigma \in \Sigma(G)} \mathcal{B}(H^G) \to \Sigma(G)$ by $\pi(A) = \sigma$ if $A \in \mathcal{B}(H^G)$. 

This defines a (rather trivial) vector bundle structure over $\Sigma(G)$. Let $E(\Sigma(G))$ denote the set of all sections of this bundle - i.e. maps $\sigma \mapsto A_\sigma : \Sigma(G) \to \bigcup_{\sigma \in \Sigma(G)} B(H_\sigma)$ so that $\pi(A_\sigma) = \sigma$. If $R$ is any subset of $\Sigma(G)$, let $E(R)$ be the set of all restrictions of elements of $E(\Sigma(G))$ to $R$; that is, $E(R)$ is the set of all sections of the restricted bundle map $\pi : \bigcup_{\sigma \in R} B(H_\sigma) \to R$.

Now for any subset $R$ of $\Sigma(G)$, let

$$E^p(R) = \left\{ A \in E(R) \mid \left( \sum_{\sigma \in R} d_\sigma \| A_\sigma \|_{\mathcal{P}}^p \right)^{1/p} = \| A \|_p < \infty \right\}$$

for $p \in [1, \infty]$, and let

$$E^\infty(R) = \left\{ A \in E(R) \mid \sup_{\sigma \in R} \| A_\sigma \|_{\mathcal{P}} = : \| A \|_\infty < \infty \right\}.$$ 

It is shown by Hewitt and Ross [1], §28, that for each $p$, $E^p(R)$ is a Banach $\ast$-algebra with pointwise composition as multiplication and with involution given by $(A^*)_\sigma = (A_\sigma)^*$. Further, if $p \in [1, \infty]$, $E^p(R)$ is the dual of $E^p(R)$, under $\langle A, B \rangle = \sum_{\sigma \in \Sigma(G)} d_\sigma \text{tr}(A_\sigma B^*)$.

Finally, let

$$E^0_0(R) = \left\{ A \in E(R) \mid \| A_\sigma \|_{\mathcal{P}} \to 0 \text{ as } \sigma \to \infty \right\},$$

$$E^0_0(R) = \left\{ A \in E(R) \mid A_\sigma = 0 \text{ outside a finite set} \right\}.$$ 

Note that, if $1 \leq p < q < \infty$, then

$$E^0_0(R) \subset E^p(R) \subset E^q(R) \subset E^0_0(R) \subset E^\infty(R);$$

provided $R$ is infinite, all these containments are strict.

(1.4.3) Let $\mu \in M(G)$ and $\sigma \in \Sigma(G)$. Define $\hat{\mu}(\sigma)$ to be the Bochner integral $\int_{\sigma} \overline{\sigma d\mu}$ (for the theory of Bochner integrals, see Edwards [1], §8.14.1). Then $\hat{\mu} \in E(\Sigma(G))$. In fact, it can be shown that $\hat{\mu} \in E^\infty(\Sigma(G))$, and that $\| \hat{\mu} \|_\infty \leq \| \mu \|_1$. The section $\hat{\mu}$ is called the Fourier transform of
It is then true that \( \mu \mapsto \hat{\mu} \) is a non norm-increasing \( \ast \)-isomorphism of the Banach \( \ast \)-algebra \( M(G) \) into the Banach \( \ast \)-algebra \( E^\infty_0(\Sigma(G)) \). It can be further shown that if \( f \in L^1(G) \), then \( \hat{f} \in E^0_0(\Sigma(G)) \). (These results are to be found in Hewitt and Ross [1], (28.36) and (28.40) respectively - the latter is known as the Riemann-Lebesgue lemma.) A further, less difficult, result in this direction is that

\[
\hat{\cdot} : T(G) \to E^0_0(\Sigma(G))
\]

is a bijective linear map.

(1.4.4) The following theorems are true:

(i) \( \hat{\cdot} : L^2(G) \to E^2(\Sigma(G)) \) is an isometric isomorphism of Banach \( \ast \)-algebras (Parseval's theorem - Hewitt and Ross [1], (28.43)).

(ii) If \( f \in L^p(G) \) for some \( p \in [1, 2] \), then \( \hat{f} \in E^p(\Sigma(G)) \) and \( \|\hat{f}\|_p \leq \|f\|_p \). (The Hausdorff-Young inequality - Hewitt and Ross [1], (31.22).)

(iii) If \( g \in L^2(G) \) and \( p \in [1, 2] \) then \( \|g\|_p \leq \|\hat{g}\|_p \) (Hewitt and Ross [1], (31.25).)

(1.5) Central functions and measures: the central projection

In this section, \( G \) will denote an arbitrary compact group. Other notation is taken from previous sections.

(1.5.1) LEMMA. Let \( \nu \in M(G) \). The following propositions are equivalent:

(i) for all \( f \in T(G) \), \( \nu \ast f = f \ast \nu \);

(ii) for all \( \mu \in M(G) \), \( \nu \ast \mu = \mu \ast \nu \);

(iii) for all \( \sigma \in \Sigma(G) \), \( \hat{\nu}(\sigma) \) is a multiple of \( I_{d_{\sigma}} \).
(iv) for all \( x \in G \), \( x^y = v_x \).

The proof of this lemma is omitted; it is based on standard techniques of harmonic analysis (see Hewitt and Ross [1], (28.48) and (28.49)).

If \( A \) is any subalgebra of \( M(G) \), define the centre of \( A \), \( ZA \), by
\[
ZA = \{ v \in A \mid \text{for all } \mu \in A, v \ast \mu = \mu \ast v \}.
\]
The above lemma shows that, provided that \( A \supseteq T(G) \), \( ZA = A \cap ZM(G) \).

(1.5.2) Let \( f \in C(G) \). Define, for \( y \in G \),
\[
\Xi(f)(y) = \int f(x^{-1}yx) d\lambda_G(x) .
\]

**Lemma.** \( \Xi : C(G) \to ZC(G) \) is an idempotent bounded linear map of norm 1.

Further, \( \Xi \big|_{ZC(G)} \) is the identity map.

**Proof.** Now \( \Xi \) is clearly linear; that \( \Xi \big|_{ZC(G)} \) is the identity follows from (1.5.1) (iv).

Notice that
\[
\|\Xi(f)\|_\infty = \sup_{y \in G} \left| \int f(x^{-1}yx) d\lambda_G(x) \right| \leq \sup_{x \in G, y \in G} |f(x^{-1}yx)|
\]
\[
= \sup_{x \in G} |f(x)| = \|f\|_\infty .
\]

This completes the proof. \( \square \)

(1.5.3) One may extend \( \Xi \) to a map \( \Xi_0 : M(G) \to ZM(G) \) in the following manner; for \( v \in M(G) \) and \( f \in C(G) \), let
\[
\Xi_0(v)(f) = v(\Xi(f)) .
\]

If we consider the continuous functions \( g \) and \( \Xi(g) \) as measures, then for all \( f \in C(G) \),
\[ (E(g) . \lambda_G)(f) = \int f(y) \Xi(g)(y) d\lambda_G(y) \]
\[ = \int \int f(y) g(x^{-1}yx) d\lambda_G(y) d\lambda_G(x) \]
\[ = \int \int f(x^{-1}yx) g(y) d\lambda_G(y) d\lambda_G(x) \]
\[ = (g \lambda_G)(\Xi(f)) = (\Xi_0(g \lambda_G))(f) . \]

Thus \( \Xi_0 \) extends \( \Xi \). I shall henceforth denote both maps by "\( \Xi \)".

It is now not hard to show:

**PROPOSITION** (i) \( \Xi : M(G) \to ZM(G) \) is an idempotent bounded linear map of norm 1.

(ii) \( \Xi|_{ZM(G)} \) is the identity map.

(iii) Suppose \( A \) is a Banach algebra of measures under convolution, with norm \( \| \cdot \|_A \), such that \( T(G) \subseteq A \), and such that \( \nu \mapsto x^{-1}x \nu \) is an isometry of \( A \). Then \( \Xi|_A : A \to ZA \) is an idempotent bounded linear map of norm 1, and \( \Xi|_{ZA} \) is the identity map.

**Proof.** (i) and (ii) may be proved by appealing to (1.5.2). The details are left to the reader.

To prove (iii), notice that, for fixed \( \nu \in A \), \( x \mapsto x^{-1}x \nu : G \to A \)

{the reader will recall that, for \( x \in G \), \( \nu \in M(G) \), and \( f \in C(G) \),

\[ x^{-1}f(x) = \nu \left( f(x) \right) \] - see (1.2.6)} is an integrable vector valued function, and \( \Xi(\nu) = \int_G x^{-1}x \nu d\lambda_G(x) \). But, using the theory of Bochner integrals (see Edwards [1], 8.14), this shows that

\[ \| \Xi(\nu) \|_A \leq \int_G \| x^{-1}x \nu \|_A d\lambda_G(x) = \| \nu \|_A . \]

It is a result of this theorem that, if \( A \) satisfies the hypotheses of (iii), \( ZA \) is a closed subalgebra of \( A \).
It is worthwhile noticing the effect of $\mathcal{E}$ on the trigonometric polynomials $t_{\xi, \zeta}^{(\sigma)}$ defined in (1.3.5). In fact, for $\sigma \in \Sigma(G)$, $x \in G$ and $\xi, \zeta \in H_\sigma$, $t_{\xi, \zeta}^{(\sigma)}(x) = (\sigma(x)\xi, \zeta)$, and

$$\mathcal{E}\left[t_{\xi, \zeta}^{(\sigma)}(x)\right] = \frac{1}{d_\sigma}(\xi, \zeta)v_\sigma(x).$$

(This can be proved by use of the orthogonality relations - see (1.3.5).)

Thus, it is easy to see directly that $\mathcal{E}(T(G)) = ZT(G)$. From this, the density of $ZT(G)$ in $ZC(G)$ and $ZL^p(G)$ ($1 \leq p < \infty$) follows; one uses (1.5.3) and the density of $T(G)$ in $C(G)$ and $L^p(G)$ ($1 \leq p < \infty$).

(1.5.5) If $A$ is some subalgebra of $M(G)$, containing $T(G)$, $\mathcal{E}|_A$ is called the central projection of $A$ on $ZA$.

The following proposition is now immediate from (1.5.3). A proof of this proposition in the case $A = C(G)$, $L^p(G)$ ($1 \leq p \leq \infty$) is given by Parker [1], lemma (1.1).

**PROPOSITION.** Let $G$ be a compact group, and let $A$ be a Banach algebra of measures on $G$ under convolution, with norm $\|\cdot\|_A$, such that for all $x \in G$, $\nu \mapsto x^n \lambda_{x^{-1}}$ is an isometry of $A$. Suppose further that $C(G)$ is densely contained in $A$ (i.e. $C(G) \subseteq A$) and there exists $\kappa \in \mathbb{R}$ so that for all $f \in C(G)$, $\|f\|_A \leq \kappa \|f\|_\infty$).

Then $(ZA)^*$ is isometrically isomorphic to $Z(A^*)$. In particular, this conclusion is valid if $A$ is any one of the algebras $M(G)$, $L^p(G)$ ($1 \leq p \leq \infty$) or $C(G)$.

**Proof.** Denote by $i_A : ZA \to A$ the canonical injection. By lemma (1.5.3), $\mathcal{E}|_A \circ i_A = 1_{ZA}$.

Thus, taking adjoints, I obtain
Now consider the map \((ZA)^* \rightarrow Z(A^*)\) given by \(\mathbb{E}|_{A^*} \circ (\mathbb{E}|_A)^*\). Since \(C(G)\) is continuously contained in \(A\), \(A^*\) is a subalgebra of \(M(G)\), hence this is a bounded map; it has an inverse, viz. \(i_1^* \circ i_2\), where \(i_2: Z(A^*) \rightarrow A^*\) is the canonical injection. Since each of the maps \(\mathbb{E}|_{A^*}, \mathbb{E}|_{A^*} \circ i_1^* \circ i_2\) has norm 1, both \(\mathbb{E}|_{A^*} \circ (\mathbb{E}|_A)^*\) and its inverse have norm 1 - thus this map is an isometry. \(\Box\)

(1.5.6) **Proposition.** Let \(v \in M(G), \sigma \in \Sigma(G)\). Then

\[
(\mathbb{E}(v))^{\sim}(\sigma) = \frac{1}{d_\sigma} v(\mathcal{X}_\sigma).I_{d_\sigma}
\]

The proof of this proposition is not difficult given (1.5.4), and hence is omitted. \(\Box\)

(1.5.7) **Lemma.** Suppose \(A \in E(E(G))\). The following statements are equivalent:

(i) for all \(B \in E_{\infty}(\Sigma(G)), AB = BA\);

(ii) for all \(B \in E(\Sigma(G)), AB = BA\);

(iii) for all \(\sigma \in \Sigma(G)\), exists \(a_\sigma \in \mathbb{C}\) such that

\[
A_\sigma = a_\sigma \cdot I_{d_\sigma}.
\]

If \(E\) is a subset of \(E(\Sigma(G))\), let \(Z_E\) denote the set of \(A \in E\) satisfying any one of the above equivalent statements. Note that (1.5.6) says that

\[
\sim: ZM(G) \rightarrow Z_{E_{\infty}}(\Sigma(G)).
\]

(1.5.8) It is clear that for each \(p \in [1, \infty]\), \(Z_{E^p}(\Sigma(G))\) is a closed subalgebra of \(E^p(\Sigma(G))\) - a similar statement is true for \(Z_{E_{\infty}^0}(\Sigma(G))\).

**Lemma.** Let \(p \in [1, \infty]\). The map which associates with

\[
\sigma \mapsto a_\sigma \cdot I_{d_\sigma} \in Z_{E^p}(\Sigma(G)) \quad (or \quad Z_{E_{\infty}^0}(\Sigma(G)))
\]
the map \( \sigma \mapsto d_\sigma^{2/p} \cdot a_\sigma \in l^p(\Sigma(G)) \) (or \( \sigma \mapsto a_\sigma \in c_0(\Sigma(G)) \)) is a linear isometric isomorphism. \( \square \)

Note that this map is not a Banach *-algebra isomorphism \((p \neq \infty)\).

(1.6) Lacunarity

(1.6.1) Since a large part of this thesis is devoted to a study of lacunary or "thin" subsets of the duals of compact groups, it seems appropriate to give some background upon this subject here. The study of lacunarity for abelian groups is today a vast study - several books have been devoted to it alone: Lindahl and Poulsen [1], Lopez and Ross [1], Kahane and Salem [1], to name but a few. I shall make no attempt to summarise developments in this area - indeed, I shall only touch upon those which are of importance in the study of non-abelian lacunarity.

(1.6.2) Suppose \( G \) is a compact group, \( R \subseteq \Sigma(G) \) and \( A \subseteq M(G) \). I shall denote by \( A_R \) the set \( \{ \mu \in A \mid \text{for all } \sigma \notin R, \hat{\mu}(\sigma) = 0 \} \). I shall often write \( L^p_R(G) \), \( C_R(G) \), etc. in place of \( (L^p(G))_R \), \( (C(G))_R \), etc.

An element of \( M_R(G) \) is called \( R \)-spectral.

A subset \( R \) of \( \Sigma(G) \) is said to be a Sidon set if \( (C_R(G))^\sim \subseteq E^1(\Sigma(G)) \).

This condition is equivalent to several other conditions; the existence of \( \kappa \in \mathbb{R} \) so that for all \( f \in T_R(G) \), \( \| f \|_1 \leq \kappa \| f \|_\infty \); \( (L^\infty_R(G))^\sim \subseteq E^1(\Sigma(G)) \); \( (M(G))^\sim |_R \supseteq E^\infty(R) \). These, and several other equivalences, are proved in (37.2) of Hewitt and Ross. They are also a special case of (2.2.3) below. In \( \Sigma(\mathbb{Z}) = \mathbb{Z} \), every Hadamard sequence of integers (i.e. each sequence \( \{ n_k | k \in \mathbb{N} \} \) with \( (n_{k+1})/(n_k) > \lambda > 1 \) is a Sidon set; in fact, if \( T \) is any compact abelian group, every dissociate set is a Sidon set (for a definition of dissociate (which is an algebraic
condition) and a proof of these facts, using Riesz polynomials, see 
(37.12)-(37.15) of Hewitt and Ross [1]). From this, it follows that every
infinite subset of $\Sigma(T)$ contains an infinite Sidon set. This is
definitely not true for non-abelian groups, as the following material will
show. An excellent treatment of recent developments in the theory of Sidon
sets for abelian groups is given by Lopez and Ross [1].

(1.6.3) Let $p \in ]1, \infty[$. A subset $R$ of $\Sigma(G)$ is said to be of
type $A(p)$ if for some $q \in [1, p[$, $L^p_R(G) \supseteq L^q_R(G)$. This, again, is
equivalent (see Hewitt and Ross [1], (37.7) and (37.9)) to several other
conditions: in particular, to $L^p_R(G) \supseteq L^q_R(G)$ for all $q \in [1, p[$, and to
the existence of $\kappa \in R$ such that, for all $f \in T^*_R(G)$, $\|f\|_p \leq \kappa \|f\|_1$.

It can be shown that for every Sidon set $R$ there exists $\kappa \in R$ such
that for $p \in ]2, \infty[$ and for $f \in T^*_R(G)$,

$$\|f\|_p \leq \kappa p^{\frac{1}{2}} \|f\|_2.$$ 

Hence a Sidon set is of type $A(p)$ set for all $p$ (Hewitt and Ross [1],
(37.10)).

(1.6.4) The preceding two paragraphs have presented a summary of some
of the results pertinent to lacunarity for non-abelian groups which are
presented in §37 of Hewitt and Ross [1]. Very little information was
available at that time on the existence of lacunary sets in the duals of
compact non-abelian groups. However, three results in this direction are
given which I should like to mention here, since they have been important
for subsequent developments in the area.

(i) By estimating the $L^4$-norm of the irreducible characters of $SU(2)$,
the authors are able to show that $\Sigma(SU(2))$ contains no infinite $A(4)$
sets and hence no infinite Sidon sets. This example can actually be
strengthened to show that $\Sigma(SU(2))$ contains no infinite $A(3)$ sets.
(ii) Let \( G = \bigcap_{n=2}^{\infty} U(n) \), where \( U(n) \) is the group of \( n \times n \) unitary matrices. Then \( G \) is a compact group. For each \( n \in \mathbb{N} \), the projection \( \pi_n : G \to U(n) \in \Sigma(G) \), and \( \{ \pi_n \mid n \in \mathbb{N} \} \) is a Sidon set.

(iii) For each \( n \in \mathbb{N} \), let \( G \) be the set of \( n \times n \) unitary matrices with precisely one \(+1\) or \(-1\) in each row and each column. Then \( G = \bigcap_{n=3}^{\infty} G_n \) is a compact group; for each \( n \), the projection \( \pi_n : G \to U(n) \) is an irreducible representation of \( G \), but no infinite subset of \( \{ \pi_n \mid n \in \mathbb{N} \} \) is of type \( \Lambda(p) \) for any \( p > 1 \). (This example is due to Figà-Talamanca [2].)

Since the publication of Hewitt and Ross [1], Volume II, research into this area has developed considerably. It is my aim in the rest of this section to give an outline of some of the subsequent developments, with particular emphasis on those which directly motivate the remainder of this thesis. Some developments not discussed here (notably those concerned with uniform approximability and the union problem for Sidon sets) appear in the book of Dunkl and Ramirez [1].

(1.6.5) One offshoot of the existence question has been the proliferation of types of lacunary sets considered. I wish to mention here three additional types: central sets, local sets and local central sets. Parker [1] defines a set \( R \) to be central Sidon if \( (ZC_R(G))^\perp \subseteq ZL^1(G) \), and central \( \Lambda(p) \) if \( (ZP_R(G))^\perp \supseteq ZL^q_R(G) \) for some (and hence for all) \( q \in [1, p[ \). Equivalent statements such as those mentioned in (1.6.2) and (1.6.3) for Sidon sets and \( \Lambda(p) \) sets are available (with appropriate changes - simply replace each function algebra, measure algebra, or space \( E \) by its centre - see Parker [1], Theorems 2.1 and 2.2, for details).

Clearly Sidon sets are central Sidon sets, and \( \Lambda(p) \) sets are central \( \Lambda(p) \) sets.
sets. For abelian groups, the two notions co-incide. Dunkl and Ramirez [3] show that $\Sigma(G)$ cannot be central Sidon unless $G$ is a finite group.

Rider [2] defines a set $R$ to be a **local Sidon set** if there exists a constant $\kappa \in \mathbb{R}$ such that for every $f \in T(G)$ such that $\hat{f}$ is supported on a single element of $R$,

$$\|\hat{f}\|_1 \leq \kappa \|f\|_\infty,$$  

and to be of type **local $\Lambda(p)$** if there is $\kappa \in \mathbb{R}$ so that for every such $f$,

$$\|f\|_p \leq \kappa \|f\|_1. \tag{2}$$

A set $R$ is of type **local central $\Lambda(p)$** if (2) holds as $f$ runs through the set of all irreducible characters of elements of $\Sigma(G)$. (It can quickly be seen that every subset of $\Sigma(G)$ is "local central Sidon".)

If $\sup\{d_\sigma \mid \sigma \in R\} < \infty$ (in particular, this condition is satisfied for any subset $R$ of the dual of an abelian group), $R$ is local Sidon, and local $\Lambda(p)$ for all $p$.

It is clear that the following implications hold:

$$\Lambda(p) \Rightarrow \text{local } \Lambda(p) \Rightarrow \text{local central } \Lambda(p),$$

$$\Lambda(p) \Rightarrow \text{central } \Lambda(p) \Rightarrow \text{local central } \Lambda(p),$$

$$\text{Sidon} \Rightarrow \text{local Sidon}.$$  

Price [3] proves that a local Sidon set is local $\Lambda(p)$ for all $p$.

However (see (1.6.8) (iii)), a central Sidon set need not even be local central $\Lambda(p)$ for any $p$. It seems an open question whether $\Sigma(G)$ can be local Sidon with $\sup\{d_\sigma \mid \sigma \in \Sigma(G)\} = \infty$.

(1.6.6) Hutchinson [1] has recently shown that if $R$ is an infinite set with $\sup\{d_\sigma \mid \sigma \in R\} < \infty$, then $R$ contains an infinite Sidon set.

His method uses the technique of Riesz polynomials on the entry functions of the representations (Parker [1] was only able to show that such a set contains an infinite central Sidon set). It follows from (1.6.7) below that a central Sidon set $R$ with $\sup\{d_\sigma \mid \sigma \in R\} < \infty$ is actually of type $\Lambda(p)$.
for all $p$. It seems to be an open question whether such a set must be a Sidon set.

(1.6.7) Rider [2] proves the following relations between the sets introduced in (1.6.5):

Suppose $R$ is a central Sidon set, and $p \in [2, \infty]$. Then

(i) if $R$ is a local central $\Lambda(p)$ set, $R$ is a central $\Lambda(p)$ set;

(ii) if $R$ is a local $\Lambda(p)$ set, $R$ is a $\Lambda(p)$ set.

Secondly, he proves that if $s \in \mathbb{N}$, $s > 1$, and $R$ is any subset of $\Sigma(G)$, then $R$ is of type $\Lambda(2s)$ if and only if $R$ is both local $\Lambda(2s)$ and central $\Lambda(2s)$.

Thus central sets, local sets and local central sets can be used to give examples of "ordinary" lacunary sets.

Functional analytic conditions for local sets have been discussed by Armstrong [1], Price [2], and Picardello [1].

(1.6.8) Motivated by (1.6.4) (ii) and (iii), one has the following results.

(i) Let $\{G_i\}_{i \in I}$ be a family of compact groups, and for each $i \in I$, let $\sigma_i$ be an element of $\Sigma(G_i)$. Let $G = \prod_{i \in I} G_i$. Then

$\{\sigma_i \circ \pi_i \mid i \in I\} \subseteq \Sigma(G)$ (where $\pi_i : G \to G_i$ is the projection) is a central Sidon set. (This is proved by using a notion of independence; see Parker [1], theorem 4.2.)

(ii) Rider [2] shows that the set $R = \{\pi_n \mid n \in \mathbb{N}\}$ of (1.6.4) (iii) is local central $\Lambda(p)$ for every $p$. The results of (1.6.7) now show that this set is central Sidon and central $\Lambda(p)$ for every $p$, but not local $\Lambda(p)$ for any $p$. This example seems to indicate that "central" properties and "local" properties are somehow independent. I have been able to tighten Rider's analysis to show that $R$ satisfies
Thus there is no analogue of the result of (1.6.3) for central Sidon sets which are central $A(p)$ for all $p$. This proves incorrect a rather wild claim of Benke [1], p. 22.

(iii) Another example of Rider [2], in the same genre, gives a set which is a central Sidon set which is not of type central $A(p)$ (and hence not of type local central $A(p)$) for any $p$. Thus, one can expect no analogue of the theorem in the last paragraph of (1.6.3). In both this example and the previous one, the group $G$ is an infinite product of finite groups, and hence is totally disconnected.

(iv) Hutchinson [1] has given the following construction:

Let $G$ be any compact connected tall group which is not a semi-simple Lie group. Then it is not too hard to see, by use of (1.2.5), that

$$G \cong \left( \bigoplus_{i \in I} G_i \right) / Z,$$

with $I$ infinite, where the $G_i$ are simple simply connected Lie groups, and at most finitely many of the $G_i$ belong to any one isomorphism class of simple simply connected Lie groups. Thus, we must have an infinite sequence $\{i_j | j \in \mathbb{N}\} \subseteq I$ such that each $G_{i_j}$ belongs to one of the families of the classical simple simply connected Lie groups (see appendix B). Suppose first that $Z = \{e\}$, and for each $j \in \mathbb{N}$, let $\sigma_{i_j}$ be the irreducible representation afforded by the "natural" injection $G_{i_j} \rightarrow U(\sigma_{i_j})$ (see (B3)). Then $\{\pi_{i_j} \circ \sigma_{i_j} | j \in \mathbb{N}\}$ is a Sidon set in $\Sigma(G)$. (The details are in Hutchinson [1], §4.) It seems likely that a similar construction can be made even if $Z \neq \{e\}$. (Certainly, each set defined in this way is central Sidon.)

By (1.6.4), the groups for which the existence of infinite lacunary sets is uncertain are the compact tall groups. It is apparent from the discussion in the preceding section that there are many interesting
questions concerning lacunary sets for compact tall totally disconnected groups. I have, however, made no attempt to deal with this class of groups in this thesis. The thesis of Hutchinson [2] contains, I believe, some information on this subject, but I have had no opportunity to peruse it.

The other extreme is represented by the tall compact connected groups; the discussion of (1.6.8) (iv) indicates that it is likely that each compact connected tall group which is not a semi-simple Lie group has an infinite Sidon set (by (1.6.8) (i), it certainly has an infinite central Sidon set). Thus, there remains the class of compact connected semi-simple Lie groups. It is to this class that this thesis is devoted.

The remainder of this section relates some of the known results for this class of groups.

(i) Cecchini [1], motivated by (1.6.4) (i), estimates the $4$ norms of the irreducible characters of compact Lie groups, and is able to show that if $G$ is a compact Lie group, the only local central $\Lambda(4)$ sets in $\Sigma(G)$ are those with $\sup\{d_\sigma | \sigma \in \mathbb{R}\} < \infty$.

(ii) Price [3] shows that if $G$ is a compact Lie group, then $\Sigma(G)$ is of type local central $\Lambda(2)$.

(iii) Rider [3] discusses local central $\Lambda(p)$ sets for $G = \mathbb{U}(n)$ and $\mathbb{SU}(n)$. In particular he shows that $\Sigma(G)$ is of type local central $\Lambda(p)$ for all $p < 2 + 2/n$, but that there exist sets which contain no infinite subset of type local central $\Lambda(2 + (2/n))$.

I shall give theorems which generalize and improve (i)-(iii) in chapter 3.

(iv) Rider [5] has shown that neither $\Sigma(\mathbb{SU}(n))$ nor $\Sigma(\mathbb{U}(n))$ contains an infinite set of type local $\Lambda(p)$ for any $p > 1$. (The case $n = 2$ was done by Price [1].)

(v) Let $G$ be a simple Lie group, $\mu \in ZM(G)$ a continuous measure. Then Ragozin [1] shows that, for $n$ large enough (independent of $\mu$)
\( \mu * \ldots * \mu \in ZL^1(G) \). In particular, \( \| \hat{\mu}(\sigma) \|_{Q_0} \to 0 \) as \( \sigma \to \infty \). It follows that \( \Sigma(G) \) contains no infinite central Sidon sets.

(vi) Rider [2] has improved Ragozin's argument to show that, for \( G \) a connected group \( \Sigma(G) \) contains an infinite central Sidon set if and only if \( G \) is not a semi-simple Lie group.

It seems plausible from (1.6.8) (iv) that these conditions are also equivalent to the condition that \( \Sigma(G) \) contains an infinite Sidon set.

(1.6.10) In this paragraph, I shall give some details of the argument of Rider mentioned in (1.6.9) (vi), since I shall, later in the thesis be interested in trying to improve this result.

(i) First recall that one of the conditions equivalent to \( R \) being central Sidon is \( ZM(G) \supseteq ZE^0(R) \).

(ii) Recall from Hewitt and Ross [1], §19.20 that for any locally compact group \( G \), \( M(G) = M^0(G) \oplus M^d(G) \) - where \( M^d(G) \) consists of the "discrete" measures (those of the form \( \sum_{i=1}^{\infty} a_i \delta_{x_i} \), where \( a_i \in \mathbb{C} \),

\[
\sum_{i=1}^{\infty} |a_i| < \infty \), and \( \delta_{x_i} : f \mapsto f(x_i) \in M(G) \) - and \( M^0(G) \) consists of the "continuous" measures (those with no point masses).

(iii) It is an easy argument to see that for any locally compact group \( G \) the only measures \( \delta_x \) (\( x \in G \)) which are central are those with \( x \) belonging to the centre of \( G \) (see Ragozin [1], 2.1). Hence, for \( \mu \in ZM(G) \), \( \mu \) is continuous if and only if \( \mu(\{x\}) = 0 \) for every \( x \) belonging to the centre \( ZG \) of \( G \).

(iv) Now suppose that \( G \) is a compact connected simple Lie group and that \( x \notin ZG \). Then \( E(\delta_x) \) is a central measure, and it is not hard, using (iii), to see that \( E(\delta_x) \) is continuous. Now, by Ragozin's result
(1.6.9) (vi), \( \Xi(\delta_x) \ast \ldots \ast \Xi(\delta_x) \in Z\Lambda^1(G) \). Thus, by the Riemann-Lebesgue lemma (1.4.3),

\[
\|\Xi(\delta_x)^*(\sigma)\|_{\Phi^\infty} \to 0 \text{ as } \sigma \to \infty
\]

(where \( n = \text{dim } G \)). But, using (1.5.6), it is easy to see that

\[
\Xi(\delta_x)^*(\sigma) = \frac{1}{d_\sigma} \cdot \chi_\sigma(x) \cdot I_{d_\sigma} ;
\]

hence for \( x \notin ZG \), \( \frac{1}{d_\sigma} \chi_\sigma(x) \to 0 \) as \( \sigma \to \infty \).

This result is now immediate for every compact connected semi-simple Lie group, since each such group is a direct product of finitely many simple Lie groups.

(v) Now suppose \( \mu \) is a continuous central measure on a compact connected semi-simple Lie group \( G \). Then

\[
\|\hat{\mu}(\sigma)\|_{\Phi^\infty} = \left| \mu\left(\frac{1}{d_\sigma} \chi_\sigma\right) \right| .
\]

But \( \left| \frac{1}{d_\sigma} \chi_\sigma \right| \) is a uniformly bounded set of functions, and by (iv), as \( \sigma \to \infty \), this approaches (pointwise) the characteristic function of \( ZG \).

Hence (since \( \mu(ZG) = 0 \)), by the bounded convergence theorem,

\[
\|\hat{\mu}(\sigma)\|_{\Phi^\infty} \to 0 \text{ as } \sigma \to \infty .
\]

That is \( \hat{\mu} \in ZE_0(\Sigma(G)) \).

(vi) Since, for \( R \) infinite, \( ZE_0(R) \) has infinite codimension in \( ZE_\infty(R) \), this shows that (1.6.10) (i) cannot hold.
CHAPTER 2

SOME TYPES OF LACUNARY SETS

(2.1) Introduction

In view of the material presented in §1.6 and particularly in view of (1.6.9), the question arises as to whether there is some notion of lacunary set (or perhaps, central lacunary set) which is admitted by the dual of every compact group, and in particular by the dual of every compact Lie group.

Working entirely within the framework of abelian groups, Bozejko and Pytlik [1] introduced lacunary sets of type $S_{p,q}$ and $S^*_{p,q}$. These notions provide a unified treatment of most types of lacunary sets previously considered. (For example, Sidon sets are those of type $S_{\infty,1}$, and $\Lambda(p)$ sets ($p > 2$) are those of type $S^*_{p',2}$.) At the same time, it appears from their work at least a priori likely that some of these sets are not lacunary in any previous sense. A result in this direction is provided by Edwards and Ross [1] (again for abelian groups only). These authors show that the dual of any abelian group contains $3/2$-Sidon sets (these are Bozejko and Pytlik sets of type $S_{\infty,3/2}$) which are not Sidon. Johnson and Woodward [1] have subsequently improved this result.

It is my aim in this chapter to extend Bozejko and Pytlik's definitions and principal theorems to arbitrary (i.e. not necessarily abelian) compact groups. Of course, this leads to consideration of central and local central analogues of these sets (cf. (1.6.7)). In future chapters, I will analyse these notions more closely, specializing to the case of compact Lie groups. Specifically, in Chapter 3 I will deal with local central sets, and in Chapter 5 with central sets.
Closer examination of Bozejko and Pytlik [1] shows that many of the results therein (and particularly theorems 1 and 2) have straightforward generalizations to the non-abelian case. I summarize these results in §2.2. It should be noted that I have not conserved Bozejko and Pytlik's notation. Rather than sets of type $S_{p,q}$ and $S^*_{p,q}$, I shall speak of sets of type $V(p,q)$ and $A(p,q)$. §2.3 is devoted to central and local central analogues of these sets. In §2.4, I consider $p$-Sidon and central $p$-Sidon sets, giving a generalization of the theorem that a Sidon set is $A(q)$ for all $q > 1$ (see (1.6.3)). §2.5 contains some necessary conditions for a set to be of type central $A(p,q)$ in terms of the hypergroup structure on the dual; I extend the notion (due to Edwards, Hewitt and Ross [2]) of test family from abelian to non-abelian groups. Using this notion, I am able to obtain an improvement to a theorem of Benke [1] (which is, in turn, based on a theorem of Rudin concerning intersections of arithmetic progressions with lacunary sets). There are two applications of these techniques. The first is to show that the dual of $SU(2)$ contains no infinite $p$-Sidon sets for any $p \in [1, 2]$; the second is to lacunary sets of projections in a class of totally disconnected groups (cf. (1.6.4) (iii)).

This chapter is a somewhat expanded version of Dooley [1].

(2.2) Definitions and Elementary Properties: sets of type $V(p,q)$ and $A(p,q)$

(2.2.1) All notation is taken from Chapter 1.

(2.2.2) Definition. Let $G$ be a compact group, $R \subseteq \Sigma(G)$, and suppose $p, q \in [1, \infty]$. Then $R$ is of type $V(p,q)$ if

(i) there exists $\kappa \in \mathbb{R}$ such that for all $f \in T_R(G)$,

$$\|\hat{f}\|_q \leq \kappa \|f\|_p;$$

and $R$ is of type $A(p,q)$ if
(ii) there exists $\kappa \in \mathbb{R}$ such that for all $f \in T_R(G)$,
\[\|f\|_p \leq \kappa \|\hat{f}\|_q .\]

For typographical and historical reasons, I have preferred to alter the notation of Boşejo and Pytlik [1]. The correspondence is given by:
\[
R \text{ is of type } S_{\frac{p}{q}} \iff R \text{ is of type } V(p', q') ;
\]
\[
R \text{ is of type } S^A_{\frac{p}{q}} \iff R \text{ is of type } A(p', q') .
\]

Notice that for $p > 2$, $R$ is of type $A(p)$ if and only if $R$ is of type $A(p, 2)$, and also that $R$ is of type $A(2)$ if and only if there exists $p > 2$ such that $R$ is of type $V(p, 2)$. Every set of type $A(p)$ ($p < 2$) is also of type $V(\infty, p)$.

The proofs of the following theorems may safely be left to the reader familiar with §37 of Hewitt and Ross [1], Edwards and Ross [1], and Boşejo and Pytlik [1].

(2.2.3) THEOREM. Let $G$ be a compact group, $p, q \in [1, \infty]$, $q \neq 1$, and suppose $R \subseteq \Sigma(G)$. The following propositions are pairwise equivalent:

(i) $R$ is of type $V(p, q)$;

(ii) there is $\kappa \in \mathbb{R}$ such that for all $f \in C_R(G)$ (or $L^p_R(G)$), $\hat{f} \in \mathbb{E}^{q'}(\Sigma(G))$ and $\|\hat{f}\|_q \leq \kappa \|f\|_p$;

(iii) $\left( L^p_R(G) \right)^{\ast} \subseteq \mathbb{E}^{q'}(\Sigma(G))$;

(iv) $\left( L^p(G) \right)^{\ast} |_{R} \supseteq \mathbb{E}^{q}(R) \left( \mathbb{E}_0^{q}(R) \right.$ if $p = 1$ and $q = \infty \}$;

(v) there is $\kappa \in \mathbb{R}$ such that for all $\phi \in \mathbb{E}^{q}(R) \left( \mathbb{E}_0^{q}(R) \right.$ if $p = 1$ and $q = \infty \}$ there exists $h \in L^p(G)$ such that $\phi = \hat{h}|_R$ and $\|h\|_p \leq \kappa \|\phi\|_q$.

If $p = 1$, (i)-(v) are also equivalent to each of the following
conditions:

(vi) \( C_R(G)^\cap \subseteq E^q(R) \);

(vii) \( (M(G))^\cap |_R \supseteq E^q(R) \).

If \( p = \infty \), (i)-(v) are equivalent to each of the following conditions:

(viii) \( M_R(G)^\cap \subseteq E^q(|\Sigma(G)|) \);

(ix) \( (C(G))^\cap |_R \supseteq E^q(R) \) (if \( q = \infty \)).

(2.2.4) THEOREM. Let \( G \) be a compact group, \( p, q \in [1, \infty] \), \( p \neq \infty \) and suppose \( R \subseteq \Sigma(G) \). The following statements are pairwise equivalent:

(i) \( R \) is of type \( \Lambda(p, q) \);

(ii) there is \( \kappa \in R \) such that for all \( f \in C_R(G) \) (or

\[ L^\infty_R(G) \]}, \hat{f} \in E^q(\Sigma(G)) \Rightarrow \|f\|_p \leq \kappa \|\hat{f}\|_q ; \]

(iii) there is \( \kappa \in R \) such that for all \( \forall \in M_R(G) \) (\( L^1_R(G) \) if \( p = 1 \), \( \hat{\nu} \in E^q(\Sigma(G)) \) (\( E^q_\infty(\Sigma(G)) \) if \( q = \infty \)) \( \Rightarrow \forall \in L^p(G) \) and

\[ \|\forall\|_p \leq \kappa \|\hat{\forall}\|_q . \]

(iv) \( \left( L^p(R) \right)^\cap |_R \supseteq E^q(R) \)

(v) \( \left( L^p'(G) \right)^\cap |_R \subseteq E^q'(R) \) (if \( p = 1 \), \( C(G)^\cap |_R \subseteq E^q'(R) \) );

(vi) there is \( \kappa \in R \) such that for all \( g \in L^p'(G) \) (\( C(G) \) if \( p = 1 \), \( \hat{g}\) \( |_R \in E^q'(R) \) and \( \|\hat{g}\|_R \leq \kappa \|g\|_p \),

If \( p = \infty \), one has (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv) \( \iff \) (v) \( \iff \) (vi).

(2.2.5) REMARKS. The above theorems show that every set of type \( \Lambda(p, q) \) is also of type \( V(p, q) \). It is clear also that each type is hereditary - a subset of a set of type \( V(p, q) \) or \( \Lambda(p, q) \) is of the same
type. Furthermore, it is easily seen that if \( p_1 \leq p \), \( q_1 \leq q \) then every set of type \( V(p, q) \) (respectively \( A(p, q) \)) is also of type \( V(p_1, q_1) \) (respectively \( A(p_1, q_1) \)).

Of course, for many values of \( (p, q) \) the class of sets of type \( A(p, q) \) is trivial; either it includes all subsets of \( \Sigma(G) \) or it includes only finite sets (which are clearly of type \( A(p, q) \) for all \( (p, q) \)). A consequence of the Hausdorff-Young inequality ((1.4.4) (ii)) is that every subset of \( \Sigma(G) \) is of type \( A(p, p') \) for all \( 2 \leq p < \infty \). Let me also note here that the preceding theorems show \( \Sigma(G) \) to be of type \( V(p, q) \) if and only if \( \Sigma(G) \) is of type \( A(p, q) \).

(2.2.6) It is easy to show (using (2.2.4) (v)) that the union of two sets of type \( A(p, q) \) is again of type \( A(p, q) \). The problem for sets of type \( V(p, q) \), however, is not at all trivial, even for the abelian case. Most seems to be known about Sidon sets (those of type \( V(1, \infty) \)); Drury's lemma [1] settled this for the abelian case. Rider [2] gives an example to show that the union of two central Sidon sets is not necessarily central Sidon. In Rider [6], a new proof of Drury's lemma is given for the abelian case, and it is asserted (without proof) that this will enable one to extend the result to the non abelian case. I do not know if this has been done. A partial result was obtained independently by Dunkl and Ramirez [2], and Bożejko [2]; they showed that the union of two Sidon sets of uniformly bounded dimension is again a Sidon set.

(2.3) Central and Local Central Sets

(2.3.1) DEFINITION. Let \( G \) be a compact group. In analogy with (1.6.7), I shall say that \( R \subseteq \Sigma(G) \) is of type central \( V(p, q) \) if (2.2.2) (i) holds for all \( f \in Z_T(R) \).

Further, \( R \) is of type local central \( V(p, q) \) if (2.2.2) (i) holds as
f runs through all irreducible characters of G.

In a similar way, one defines sets of type central $\Lambda(p, q)$ and local central $\Lambda(p, q)$.

(2.3.2) REMARKS. The following implications are clear for $R \subseteq \Sigma(G)$: $R$ is of type $V(p, q)$ $\Rightarrow$ $R$ is of type central $V(p, q)$ $\Rightarrow$ $R$ is of type local central $V(p, q)$. Similar implications hold for type $\Lambda(p, q)$.

Analogues of theorems (2.2.3) and (2.2.4) hold for sets of type central $V(p, q)$ and central $\Lambda(p, q)$. One replaces, in the statements, $T(G)$, $C(G)$, $L^p(G)$, $M(G)$ and $\mathbb{T}^p(R)$ by $\mathbb{Z}T(G)$, $\mathbb{Z}C(G)$, $\mathbb{Z}L^p(G)$, $\mathbb{Z}M(G)$ and $\mathbb{Z}\mathbb{E}^p(R)$. Thus, for example, (2.2.3) (iii) becomes

\[
\left[\mathbb{Z}L^p_H(G)\right]^* \subseteq \mathbb{Z}E(q')^{-1}(\Sigma(G)).
\]

The task of writing out and proving the central analogues of (2.2.3) and (2.2.4) is left to the interested reader. The proofs are again trivial modifications of the proofs in Bożejko and Pytlik [1] - although one needs to use (1.5.5) in showing the implications (ii) $\Rightarrow$ (iii) in (2.2.3) and (iv) $\Rightarrow$ (v) in (2.2.4).

(2.3.3) LEMMA. Let $G$ be a compact group, and let $\sigma \in \Sigma(G)$.

(i) If $q \in [1, \infty]$, $\|\hat{\chi}_\sigma\|_q = d_\sigma^{1-(2/q')}$.  

(ii) If $p \leq 2$, $d_\sigma^{1-(2/p)} \leq \|\chi_\sigma\|_p$, and if $p \geq 2$, 

$\|\chi_\sigma\|_p \leq d_\sigma^{1-(2/p)}$.

Proof. Let $\tau \in \Sigma(G)$. Then

$$
\hat{\chi}_\sigma(\tau) = \begin{cases} 
\frac{1}{d_\sigma} I & \text{if } \sigma = \tau, \\
0 & \text{if } \sigma \neq \tau.
\end{cases}
$$

Hence $\|\hat{\chi}\|_q = d_\sigma^{(2-q)/q} = d_\sigma^{1-(2/q')}$.  

(ii) follows from (i) by the Hausdorff-Young inequality and its converse (see (1.4.4)).
(2.3.4) PROPOSITION. Let $G$ be a compact group. Then

(i) $\Sigma(G)$ is of type local central $V_l(q)$ for all $q \in [1, \infty]$. 

(ii) Suppose $p \in [1, 2]$, $q \in [2, \infty]$ and $p > q'$. Then if $R \subseteq \Sigma(G)$ is of type local central $V_l(p, q)$, $\sup\{d_\sigma | \sigma \in R\} < \infty$.

Further, if $R$ is of type central $V_l(p, q)$, $R$ is finite.

(iii) Suppose $q \in [1, \infty]$. Then if $R \subseteq \Sigma(G)$ is of type local central $\Lambda(\infty, q)$, $\sup\{d_\sigma | \sigma \in R\} < \infty$. If $R$ is of type central $\Lambda(\infty, q)$, $R$ is finite.

Proof. By (2.3.3) (i),

$$\sup\{\|x_\sigma\|_q / \|x_\sigma\|_\infty | \sigma \in \Sigma(G)\} \leq \sup\{d_\sigma^{-(2/q')} | \sigma \in \Sigma(G)\} \leq 1.$$ 

This proves (i).

To show (ii), first notice that if $R$ is of type local central $V_l(p, q)$ for $p \leq 2$, then $\sup\{\|x_\sigma\|_q / \|x_\sigma\|_p' | \sigma \in R\} < \infty$. By (2.3.3) (ii) this implies that $\sup\{d_\sigma^{(2/p')-(2/q')} | \sigma \in R\} < \infty$, so either $p \leq q'$ or $\sup\{d_\sigma | \sigma \in R\} < \infty$.

Suppose now that $R$ is an infinite set of type central $V_l(p, q)$ ($p \leq 2$, $p > q'$). Then $\sup\{d_\sigma | \sigma \in R\} < \infty$, so by (1.6.6) we may assume that $R$ is of type $\Lambda(p')$. Then $Z_{L_{p_1}}^l(R) \supseteq \left(Z_{L_{p_1}}^l(G)\right)^\sigma | \sigma = \left(Z_{L_{p_1}}^r(G)\right)^\sigma | \sigma = Z_{E_2}(R)$. This implies that $R$ is finite or that $q \leq 2$. The latter contradicts the stated constraint on the values of $q$.

The proof of (iii) is similar, except that one uses the fact that any infinite set of uniformly bounded degree contains an infinite central Sidon set (1.6.6).

(2.3.5) REMARK. Suppose that $q \in [2, \infty]$, $p \in [1, 2]$, and suppose that $R$ is an infinite set of type central $V_l(p, q)$. Then the argument of (ii) shows that $R$ contains no infinite set of type central $\Lambda(p')$. If $G$ is abelian, every infinite set contains such a subset and so there are no
infinite sets of type $V(p, q)$ for any $p, q$ with $p > 1$ and $q > 2$.

However, in the nonabelian case, not every infinite subset need contain an
infinite central $A(p')$ set (cf. (1.6.8) (iii), (1.6.9) (i)); the
situation is unclear.

(2.3.6) It may clarify the above theorems to give a graphical
representation of the information which they contain.

The conventions for the following graphs are as follows:

Identify the set $[1, \infty] \times [1, \infty]$ with the unit square
$[0, 1] \times [0, 1]$ via the map $(p, q) \mapsto (1/p, 1/q)$. If a boundary is
contained in the region below it and to the left, it is shown as a dotted
line; if it is contained in the region above it and to the right, it is
shown as a solid line.

A region is marked "all" if for every $(1/p, 1/q)$ in that region,
$E(G)$ (and hence every subset of $E(G)$) is of type local central $V(p, q)$
(respectively $A(p, q)$). A region is marked "b.d." if for $(1/p, 1/q)$ in
that region the only sets of type local central $V(p, q)$ (respectively
$A(p, q)$) are those whose representations have uniformly
bounded dimension (i.e. are sets $R$ such that $\sup \{d_\sigma | \sigma \in R\} < \infty$).

Finally, a region marked "?" is one for which the above theorems give
no information concerning which sets can occur there.

Sets occurring on the boundary labelled "A" (Figure 1) are precisely
local central $A(2)$ sets. Sets occurring on the boundary labelled "B"
(Figure 2) are precisely local central $A(p)$ sets $(p > 2)$.
The reader should note that if a set is of type local central \( V(p, q) \) (respectively \( A(p, q) \)) for \((1/p, 1/q)\) in the diagram, then it is also of type \( V(p_1, q_1) \) (respectively \( A(p_1, q_1) \)) for each \((1/p_1, 1/q_1)\) belonging to the region of all points above and to the right of \((1/p, 1/q)\).

Also, each set of type local central \( A(p, q) \) is also of type local central \( V(p, q) \).

Similar diagrams can be drawn for sets of type \( V(p, q) \), \( A(p, q) \), central \( V(p, q) \) and central \( A(p, q) \). Diagrams for these sets are nearly identical to those in Figure 1 and Figure 2. One simply replaces "b.d." by "finite". However, notice that the boundary in Figure 1 labelled "all" \( \{(1, l/q) \mid 0 \leq l/q < 1/2\} \) now consists of the \( q' \)-Sidon sets (or central \( q' \)-Sidon sets).

I shall return to these diagrams in (3.6.3), in (5.3.1) and in (5.5.1), for the special case where \( G \) is a connected Lie group; in this case much fuller information will be available.

### (2.4) Some results on \( p \)-Sidon sets

Sets of type (central) \( V(1, p') \) are said to be (central) \( p \)-Sidon sets. These sets have been discussed for \( G \) abelian by Bozejko and Pytlik [1], L.-S. Hahn [1], Edwards and Ross [1] and Johnson and Woodward [1], and Woodward [1]. Edwards and Ross showed the existence of \((3/2)\)-Sidon sets, in the dual of any abelian group, which are not \( p \)-Sidon for any \( p < 3/2 \), and Johnson and Woodward [1] (again for abelian groups), the existence of \((2n/(n+1))\)-Sidon sets which are not \( p \)-Sidon for any \( p < (2n/n+1) \). None of these authors was able to decide whether a \( p \)-Sidon set need be of type \( A(q) \) for all, or for any \( q > 1 \).

By the Hausdorff-Young inequality ((1.4.4) (ii)), \( \Sigma(G) \) is \( p \)-Sidon for all \( p \geq 2 \). Thus I shall consider only \( 1 \leq p < 2 \). The (central) \( 1 \)-Sidon sets are simply the (central) Sidon sets of §1.6.
Let $G$ be a compact group, $p \in [1, 2]$. 

(2.4.1) **Theorem.** Let $R \subset \Sigma(G)$ be $p$-Sidon. Then
(i) \( R \) is of type \( V(\infty, 2p/(3p-2)) \);
(ii) \( R \) is of type \( \Lambda(r, 2p/(3p-2)) \) for all \( r \in [1, \infty] \); in fact there is \( \kappa \in \mathbb{R} \) such that for all \( f \in T^*_R(G) \) and

\[
\text{for every } r \in [1, \infty], \quad \|f\|_r \leq \kappa r^{\frac{1}{r}} \|f\|_{2p/(3p-2)}. \]

**Proof.** (i) is proved by an adaptation of the argument of Edwards and Ross [1], theorem 2.1:

Let \( g \in L^1_R(G) \). Then for all \( f \in C(G) \),

\[
(g \ast f)^\wedge = \hat{g} \cdot \hat{f} \in \mathcal{E}^p(G). \]

Thus \( \hat{g} \) is a \( C(G, \mathcal{E}^p(G)) \) multiplier (see Hewitt and Ross [1], (35.1)). Hence, by table (36.20) of Hewitt and Ross [1],

\[
\hat{g} \in \mathcal{E}(2p)/(2-p)(\Sigma(G)). \quad \text{I have shown that } \left(L^1_R(G)\right)^\wedge \subseteq \mathcal{E}(2p)/(2-p)(\Sigma(G)). \]

Now apply (2.2.3) (iii), noting that \( (2p)/(2-p)' = (2p)/(3p-2) \).

A version of (ii) for abelian groups is proved by Bożejko and Pytlik [1]. Using methods of Figà-Talamanca and Rider [1], I shall indicate how to extend their proof to the non-abelian case.

Let \( G = \bigcap_{\sigma \in R} U(d_\sigma) \); let \( f \in T^*_R(G) \), and \( 0 < \alpha < 1 \). Recall from (D.26) of Hewitt and Ross [1], the definition of \( |A|^\alpha \) for \( A \in \mathcal{B}(H_\sigma) \).

Define

\[
f_\alpha : G \times G \rightarrow \mathbb{C} : (x, W) \mapsto \sum_{\sigma \in R} d_\sigma \text{Tr} \left( \hat{f}(\sigma) \cdot |\hat{f}(\sigma)|^{-\alpha} \cdot W_\sigma \sigma(x) \right). \]

Note that for \( W \in G \), \( \sigma \mapsto W_\sigma^* \cdot |\hat{f}(\sigma)|^\alpha \in \mathcal{E}_{00}(\Sigma(G)) \). Since \( R \) is P-Sidon, there exists \( \kappa \in \mathbb{R} \) (independent of \( f \) ) and there exists \( g^\wedge_W \in L^1(G) \) (depending on \( f \) ) such that

\[
\hat{g}_W^\wedge(\sigma) = W_\sigma^* \cdot |\hat{f}(\sigma)|^\alpha, \quad \text{and} \quad \|g^\wedge_W\|_1 \leq \kappa \|W^* \cdot |\hat{f}|^\alpha\|_p. \tag{1} \]

It is easy to see that \( \{(f_\alpha^\wedge(\cdot, W)) \cdot g^\wedge_W\}(\xi) = \hat{f}(\xi) \) for all \( \xi \in \Sigma(G) \), so that
\( f = f^\alpha_\omega(\cdot, W) \ast g^\omega \).

Hence, for \( r \in [1, \infty[ \) ,

\[
\|f\|_r^p \leq \|f^\alpha_\omega(\cdot, W)\|_r^p \|g^\omega\|_1^p . \tag{2}
\]

If \( \alpha \geq 1/p \), (D.41) and (D.48) of Hewitt and Ross [1], show that for \( \sigma \in R \),

\[
\|\hat{\nu}_G(\sigma)\|_\hat{\nu}_G^p = \|f(\sigma)\|_{\hat{\phi}_p}^p , \tag{3}
\]

so (1) implies that

\[
\|g^\omega\|_1 \leq \kappa \|\hat{\nu}_G\|_{\hat{\phi}_p}^\alpha .
\]

Combining (2) and (3) and integrating over \( G \), one obtains

\[
\|f\|_r^p \leq \kappa \|\hat{\nu}_G\|_{\hat{\phi}_p}^{\alpha p} , \tag{4}
\]

Since \( R \) considered as a subset of \( \Sigma(G) \) is a Sidon set (1.6.4) (ii), there exists \( B \in R \), independent of \( f \), such that for all \( r \in [2, \infty[ \) and for each \( x \in G \),

\[
\int |f^\alpha_\omega(x, W)|^p d\lambda_\omega(W) = \|f^\alpha_\omega(x, \cdot)\|_2^p \leq (B\sqrt{r})^p \|f^\alpha_\omega(x, \cdot)\|_2^p \leq \left( B\sqrt{r} \|\hat{f}\|_{\hat{\phi}_p}^{1-\alpha} \right)^p \tag{5}
\]

provided that \( 1 - \alpha \geq \frac{1}{2} \). Now (4) and (5) give

\[
\|f\|_r^p \leq B\kappa \sqrt{r} \|\hat{\nu}_G\|_{\hat{\phi}_p}^{\alpha} \|f\|_{\hat{\phi}_p}^{\alpha} ,
\]

provided that \( (1-\alpha) \geq \frac{1}{2} \) and \( \alpha p' \geq 1 \). Choosing \( \alpha \) such that \( \alpha p' = 2(1-\alpha) \), it is easily established that \( \alpha p' = 2p/(3p-2) \), which completes the proof. \( \square \)

(2.4.2) REMARK. The reader will recognize (2.4.1) (ii) as an analogue for \( p \)-Sidon sets of the result mentioned in the last paragraph of (1.6.3) (that a Sidon set is \( \Lambda(r) \) for all \( r > 1 \)). I note (with Edwards and Ross [1], 2.5 (ii)) that in the case \( p = 1 \), (2.4.1) (i) and (2.4.2) (ii) combine to show the existence of \( \kappa \in R \) such that for all \( f \in T_R(G) \), and for all \( r \in [1, \infty[ \),

\[
\|f\|_r^p \leq \kappa r^{\frac{p}{p-2}} \|f\|_1^p ,
\]

but that for \( p > 1 \), these two
results no longer combine in this way.

The example of Rider given in (1.6.8) (iii) leads one to suspect that there will be, in general, no analogue of (2.4.1) (ii) for central \( p \)-Sidon sets (for any \( p \in [1, 2[ \)). I can show, however, that a central \( p \)-Sidon set \( R \) such that \( \sup \{ d_0 | \sigma \in R \} < \infty \) must satisfy (2.4.1) (ii) with "\( f \in T_R(G) \)" replaced by "\( f \in ZT_R(G) \)" (cf. (1.6.7)). The proof is a straightforward adaptation of the method of Parker [1]; however, in the case \( p = 1 \), the method yields (2.4.1) (ii) unchanged.

The problem of extending (2.4.1) (i) to central \( p \)-Sidon sets is even more interesting. Again, Rider's example ((1.6.4) (iii)) shows that this is not always possible, at least for \( p = 1 \). However, it seems reasonable to ask whether the set of \( (ZC(G), Z_{\mathcal{E}}^p(\Sigma(G))) \) multipliers can ever be equal to \( Z_{\mathcal{E}}^{2p/(3-p)}(\Sigma(G)) \), for some values of \( p \in [1, 2[ \) and for some class of non-abelian groups - perhaps for the class of compact connected Lie groups.

(2.5) Test families for non-abelian groups

Let \( G \) be a compact group.

(2.5.1) DEFINITION. For \( R \subseteq \Sigma(G) \), let \( \nu(R) = \sum_{\sigma \in R} d_0^2 \). If \( G \) is abelian, \( \nu(R) \) is simply the cardinality of \( R \). The measure \( \nu \) is the Plancherel measure on \( \Sigma(G) \).

(2.5.2) The reader is reminded of the operations \( \times \) and \( - \) and the integers \( n_\eta(\xi) \) introduced in (1.3.3). If \( P \) and \( Q \) are subsets of \( \Sigma(G) \), let \( P \times Q = \bigcup_{\sigma \in P, \eta \in Q} \sigma \times \eta \) and \( \overline{P} = \{ \sigma | \sigma \in P \} \). 

LEMMA. Let \( G \) be a compact group, and let \( P, Q \) be non-empty finite subsets of \( \Sigma(G) \). There exists \( k \in ZT(G) \) such that

\[(i) \hat{k}(\sigma) = I_{d_0^2} ; \; \sigma \in P,\]

\( (ii) \quad \hat{k}(\sigma) = 0 \), \( \sigma \in \Sigma(G) \setminus (P \times Q \times \overline{Q}) \),

\( (iii) \quad 0 \leq \hat{k}(\sigma) \leq I_{d_\sigma} \) for all \( \sigma \in \Sigma(G) \),

\( (iv) \quad \|k\|_1 \leq \left( \frac{v(P \times Q)}{v(Q)} \right)^{1/2} \cdot \frac{v(Q)}{v(P \times Q)} \).

Proof. Let \( f = \sum_{\eta \in Q} d_\eta \chi_\eta \), \( g = \sum_{\zeta \in P \times Q} d_\zeta \chi_\zeta \), and

\[
\hat{k} = \frac{1}{v(Q)} f \cdot g = \frac{1}{v(Q)} \sum_{\eta \in Q} \sum_{\zeta \in P \times Q} d_\eta d_\zeta \left( \sum_{\xi \in \eta \times \zeta} n_{\eta}(\xi) \chi_\xi \right).
\]

It is easily seen that, for all \( \sigma \in \Sigma(G) \),

\[
\hat{k}(\sigma) = \left( \frac{1}{v(Q)} \sum_{\eta \in Q} \sum_{\zeta \in P \times Q} d_\eta d_\zeta n_{\eta}(\sigma) \right) I_{d_\sigma}.
\]

Notice that \( n_{\eta}(\sigma) = n_{\sigma}(\eta) \), and for \( \sigma \in P \), \( \eta \in Q \),

\[
\sum_{\zeta \in P \times Q} d_\eta n_{\sigma}(\zeta) = d_\sigma d_\eta.
\]

Hence, \( (i) \) follows from \( (1) \); \( (iii) \) also follows from \( (1) \) because, for any \( \sigma \in \Sigma(G) \),

\[
\sum_{\zeta \in P \times Q} d_\eta n_{\sigma}(\zeta) \leq \sum_{\zeta \in \Sigma(G)} d_\eta n_{\sigma}(\zeta) = d_\sigma d_\eta.
\]

The equality \( (ii) \) results from \( (1) \) together with the fact that for \( \sigma \in P \times Q \times \overline{Q} \), and for any \( \eta \in Q \), \( \zeta \in P \times \overline{Q} \), \( n_{\eta}(\sigma) = 0 \).

Finally, by Hölder's inequality,

\[
\|k\|_1 \leq \frac{1}{v(Q)} \|f\|_2 \|g\|_2 = \left( \frac{v(Q)}{v(P \times Q)} \right)^{1/2} \left( \frac{v(P \times Q)}{v(Q)} \right)^{1/2}.
\]

Let \( G \) be a compact group.

\( (2.5.3) \) DEFINITION. Let \( M \in \mathbb{R} \), \( M \geq 1 \). A family \( \Omega \) of finite non-empty subsets of \( \Sigma(G) \) is called a test family of order \( M \) if for all \( P \in \Omega \), \( \frac{v(P \times \overline{P})}{v(P)} \leq M \) and \( \frac{v(P \times P \times \overline{P})}{v(P \times \overline{P})} \leq M \).

This is a generalization of the definition given in Edwards, Hewitt and Ross [1], (3.1). It should be noted that in the abelian case, the first condition follows from the second, but that in the non-abelian case such an implication need not hold.
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(2.5.4) THEOREM. Let $G$ be a compact group, and let $\Omega$ be a test family of order $M$ in $\Sigma(G)$. Suppose that $R \subseteq \Sigma(G)$ is of type central $\Lambda(r, s)$ for some $r, s \in [1, \infty]$, with constant $\kappa$. Then for any $P \in \Omega$, $v(R \cap P) \leq \kappa^s M^{s'/2} (v(P))^{s'/r}$.

Proof. Choose a function $k \in \mathcal{Z}(G)$ satisfying (2.5.2) (i)-(iv), with $P = Q$.

Let $g = \sum_{\sigma \in \mathcal{R} P} d_{\sigma} \chi_{\sigma}$. It is clear that $v(R \cap P) = g \ast k(e) \leq \|k\|_p, \|g\|_p \leq \kappa \|k\|_p, \|\hat{g}\|_g$.

Now

$$\|\hat{g}\|_g = \left( \sum_{\sigma \in \mathcal{R} P} d_{\sigma}^2 \right)^{1/s} = v(R \cap P)^{1/s}.$$

Thus $v(R \cap P)^{1/s} \leq \kappa \|k\|_p$.

But by Hölder's inequality,

$$\|k\|_p \leq \|k\|_1^{1-(2/r)} \|k\|_2^{2/r} \leq (v(P \times \overline{P})/v(P))^{(1/2) - (1/r)} (v(P \times P \times \overline{P}))^{1/r} \leq M^{1/2} (v(P))^{1/r}.$$

This completes the proof. □

(2.5.4) combines with (2.4.1) (ii) to prove:

(2.5.5) COROLLARY. Let $\Omega$ be a test family of order $M$. If $R$ is a $p$-Sidon set then there is $\kappa \in \mathbb{R}$ such that for every $P \in \Omega$ with $v(P) \geq 3$, $v(R \cap P) \leq \kappa M^{p/(2-p)} (\log v(P))^{p/(2-p)}$.

Let $G$ be a compact group.

(2.5.6) EXAMPLE. Recall ((1.3.3)) that a subset $P$ of $\Sigma(G)$ is called a subhypergroup if $\overline{P} \subseteq P$ and $P \times P \subseteq P$. Let $\Omega$ be the family of all finite subhypergroups of $\Sigma(G)$. Then $\Omega$ is a test family of order 1.

(2.5.7) NOTATION. Let $G$ be a compact group, and let $\sigma \in \Sigma(G)$. 

Denote by $\sigma^n$ the subset $\sigma \times \ldots \times \sigma$ ($n$ times) of $\Sigma(G)$. Let

$$S_n(\sigma) = \bigcup_{j+k=n} \sigma^j \times \sigma^{-k}, \quad S_0(\sigma) = \{1\},$$

the trivial representation, and let $P_n(\sigma) = \bigcup_{r=0}^n S_n$. Notice that $P_n(\sigma)$ contains $\bigcup_{r=0}^n \sigma^r$. (The latter set is denoted $A(\sigma, n)$ by Benke [1]. The two sets are equal if $\sigma = \overline{\sigma}$. ) If $G$ is abelian, $P_n(\sigma)$ is the arithmetic progression of length $n$ generated by $\sigma$.

(2.5.8) **LEMMA.** Let $G$ be a compact group. For each $\sigma \in \Sigma(G)$, there is a polynomial $p$ with degree at most $2d_\sigma^2$ such that, for $n$ sufficiently large, $\nu(P_n(\sigma)) = p(n)$. If $\sigma = \overline{\sigma}$, the degree of $p$ is at most $d_\sigma^2$.

First note that the case $\sigma = 1$ is trivial; henceforth, suppose $\sigma \neq 1$.

**Proof.** Let $J_0 \subseteq L^2(G)$ be the two-sided convolution ideal generated by $\{1 + \chi_\sigma + \overline{\chi_\sigma}\}$. It is not hard to see that $J_\sigma$ is the linear span of the function 1 together with the coordinate functions $\xi_{\xi_0, \eta_0}$, $\xi_{\xi_0, \eta_0}$, $\xi_0, \eta_0 \in H_\sigma$ (introduced in (1.3.5)). Thus $\dim J_\sigma \leq 2d_\sigma^2 + 1$.

(If $\sigma = \overline{\sigma}$, $\dim J_\sigma = d_\sigma^2 + 1$.)

Let $J_\sigma^n$ be the subspace of $L^2(G)$ spanned by functions of the form

$\phi_1 \cdot \ldots \cdot \phi_n$, $\phi_j \in J_\sigma$. Notice that $J_\sigma^n$ is spanned by co-ordinate functions of the representations 1 and $\sigma^j \otimes \sigma^{-k}$, $1 \leq j+k \leq n$; hence

$$\dim J_\sigma^n = \sum_{\xi \in \xi P_n(\sigma)} d_\xi^2 = \nu(P_n(\sigma)).$$

Now $J_\sigma^n$, considered as a vector space, is isomorphic to a quotient of
the vector space of polynomials of degree $n$ in $\dim J_\sigma$ variables. Thus by lemma 8.9 of Mayer [1], there exists a polynomial $p$ of degree at most $\dim J_\sigma-1$ such that for $n$ sufficiently large, $\dim J_\sigma^n = p(n)$. \hfill \Box

(2.5.9) PROPOSITION. \{P_n(\sigma) \mid n \in \mathbb{N}\} is a test family.

Proof. Now $\overline{P_n(\sigma)} = P_n(\sigma)$, and one has $P_n(\sigma) \times P_n(\sigma) = P_{2n}(\sigma)$, and $P_n(\sigma) \times P_n(\sigma) \times P_n(\sigma) = P_{3n}(\sigma)$. By (2.5.8), for $n$ sufficiently large,

$$\frac{\nu(P_{2n}(\sigma))}{\nu(P_n(\sigma))} \leq 2d_{\sigma}^2 + \frac{1}{3}$$

and

$$\frac{\nu(P_{3n}(\sigma))}{\nu(P_n(\sigma))} \leq 3d_{\sigma}^2 + \frac{1}{3}. \hfill \Box$$

(2.5.10) REMARKS. Applying (2.5.4) to (2.5.9) gives (for $G$ any compact group):

Let $\sigma \in \Sigma(G)$ and suppose that $R \subseteq \Sigma(G)$ is of type central $\Lambda(p, s)$, $n, s \in [1, \infty]$. Then there exists $k \in \mathbb{R}$ such that

$$\nu(R \cap P_n(\sigma)) \leq kn \text{ for all } n; \text{ if } \sigma = \overline{\sigma},$$

$$\nu(R \cap P_n(\sigma)) \leq kn \text{ for all } n.$$

In particular, if $R$ is a central $\Lambda(p)$ set $(p > 2)$,

$$\nu(R \cap P_n(\sigma)) \leq kn \text{ for all } n.$$ Benke [1] shows that if $R$ is a central $\Lambda(p)$ set, $\text{card}(R \cap A(\sigma, n)) \leq O(n^{(\nu d_{\sigma}^2)/p})$. The result of the preceding paragraph is an improvement of his result, since $P_n(\sigma) \supseteq A(\sigma, n)$ and also since, for any set $R$, $\nu(R) \geq \text{card } R$, equality prevailing if and only if $d_{\sigma} = 1$ for all $\sigma \in R$. 

(2.5.11) PROPOSITION. Let $G$ be an infinite compact group. Then $\Sigma(G)$ has an infinite test family.

Proof. For $\sigma \in \Sigma(G)$, let $[\sigma]$ denote the smallest hypergroup containing $\sigma$ (see (1.3.3)). If there is a $\sigma \in \Sigma(G)$ such that $[\sigma]$ is infinite, then (2.5.9) provides such a test family. If not, for every $\sigma \in \Sigma(G)$, $[\sigma]$ is finite. In this case, the family of finite subhypergroups of $\Sigma(G)$ is an infinite test family. □

Applying (2.5.11) to (2.5.4) gives:

(2.5.12) COROLLARY. Let $G$ be an infinite compact group. Then $\Sigma(G)$ is never of type central $\Lambda(p, q)$ for $p > q'$.

(2.5.13) PROPOSITION. $\Sigma(SU(2))$ contains no infinite sets of type central $\Lambda(p, q)$ for $p > \frac{3}{2}q'$, $p, q \in [1, \infty[.$

Proof. Let (see Hewitt and Ross [1], (29.13)) $\sigma_{\frac{j}{l}}$ be the (unique) irreducible representation of $SU(2)$ of dimension $2l + 1$, $l \in \{0, \frac{1}{2}, 1, \ldots \}$. Then for $\sigma \in \mathbb{N}$, $P_{\sigma_{\frac{j}{l}}} = \{\sigma_0, \sigma_{\frac{j}{l}}, \ldots, \sigma_{n/2}\}$.

Suppose that $R = \{\sigma_{\frac{j}{l}} | 2n \in \mathbb{N}\}$ is an infinite set of type $\Lambda(p, q)$.

One has

$$\nu(P_{\sigma_{\frac{j}{l}}} \cap R) = \sum_{j=1}^{k} (2n_{\frac{j}{l}} + 1)^2, \quad \nu(P_{\sigma_{\frac{j}{l}}}) = \sum_{j=0}^{n} (2j+1)^2 = \frac{4n^3}{3}.$$ 

Thus by (2.5.10) there is a constant $\kappa$ such that for all $i$,

$$(2n_{\frac{j}{l}} + 1)^2 \leq \nu(P_{\sigma_{\frac{j}{l}}} \cap R) \leq \kappa \left(\frac{4n^3}{3}\right)^{q'/p},$$

Since $R$ is infinite, this implies that $p \leq \frac{3}{2}q'$. □

Hence, by (2.4.1) (ii), $\Sigma(SU(2))$ contains no infinite $p$-Sidon sets for any $p \in [1, 2[.$ In Chapter 3, I will be able to extend this result to the case of an arbitrary compact connected semi-simple Lie group.
(2.5.14) As a second application of the theory of this chapter, I shall show how it can be applied to a class of totally disconnected groups, one of which is given in (1.6.4) (iii).

Let \((G_n)_{n \in \mathbb{N}}\) be a sequence of finite groups. Let \(G = \prod_{n=1}^{\infty} G_n\), and, for each \(n\) let \(\pi_n : G \to G_n\) be the projection. Let \(\sigma_n\) be an irreducible representation of \(G_n\), for each \(n\). Then \(R = \{\sigma_n \circ \pi_n | n \in \mathbb{N}\} \subseteq \Sigma(G)\) is a central Sidon set ((1.6.8) (i)) and hence a central \(p\)-Sidon set for all \(p \in [1, 2]\).

**Proposition.** In order that \(R\) be of type central \(\Lambda(r, s)\), it is necessary that there should exist \(\kappa \in \mathbb{R}\) such that for all \(n \in \mathbb{N}\),

\[
\frac{d_{\sigma_n}^2}{\kappa (\text{card } G_n)^{s'/r}}.
\]

In order that \(R\) should be a \(p\)-Sidon set \((p \in [1, 2])\), it is necessary that there exist \(\kappa \in \mathbb{R}\) with, for all \(n \in \mathbb{N}\),

\[
\frac{d_{\sigma_n}^2}{\kappa \left(\log(\text{card } G_n)\right)^{p/(2-p)}}.
\]

**Proof.** Let \(\Omega\) be the test family of finite subhypergroups \(\{\Sigma(G_n) | n \in \mathbb{N}\}\). Then \(\Omega\) is a test family of order 1. By (27.57) (i) of Hewitt and Ross [1], \(\cup \{\Sigma(G_n)\} = \text{card } G_n\). Now apply (2.5.4) and (2.5.5).

(2.5.15) Let \(k\) be a prime number. Then it is known (see Herstein [1]) that there exists a nonabelian group of order \(k^3\). Choose such a group for each \(k\), and denote it by \(G_k\). Then \(G_k\) has an irreducible representation \(\sigma_k\) of dimension \(k\). (To see this, note that since \(G_k\) is nonabelian, it must have a representation \(\sigma\) with \(d_{\sigma} > 1\). By Hewitt and Ross (27.57) (i) and (ii), \(d_{\sigma}\) is a divisor for \(k^3\), and \(d_{\sigma}^2 \leq k^3\). It follows that \(d_{\sigma} = k\).
Now let $G = \prod_{k \text{ prime}} G_k$. The set $R = \{s_k^p \mid k \text{ prime} \}$ is, by (1.6.8) (i), a central Sidon set in $\Sigma(G)$. But, applying (2.5.14), one sees that $R$ is of type central $\Lambda(p, s)$ only if there exists a constant $K$ such that for all primes $k$,

$$\frac{2^2}{K} \leq \left(3s'/r\right)^{3s'}/r ; \quad \text{i.e. } r \leq \frac{3}{2}s'.$$

In particular, $R$ is not a $p$-Sidon set for any $p \in [1, 2]$. This method can be applied to other examples of this sort.
CHAPTER 3

NORMS OF CHARACTERS AND LOCAL CENTRAL SETS

(3.1) Introduction

In this chapter, I shall take up the analysis of sets of type local central \( V(p, q) \) and local central \( \Lambda(p, q) \), which were introduced in (2.3.1), specializing to the case where \( G \) is a compact Lie group. The theorems of chapter 2 will enable me to deduce from these some results about sets of type \( V(p, q) \) and \( \Lambda(p, q) \).

It is quickly seen that analysis of these local central sets involves finding estimates for the \( p \)-norm of an irreducible character of \( G \) in terms of the dimension of its associated irreducible representation, as was done for arbitrary compact \( G \) in (2.3.3). The bulk of this chapter is thus devoted to finding estimates which complement and improve those of (2.3.3), for the case where \( G \) is a compact Lie group.

The main results are as follows (\( G \) being a compact Lie group).

(i) There exists \( \varepsilon_G > 0 \) such that for all \( s \in [0, \varepsilon_G] \), there exists \( \kappa_s \in \mathbb{R} \) with, for all \( p \in [2-s, \infty[ \), and for all \( \sigma \in \Sigma(G) \),

\[
\|\chi_\sigma\|_p \leq \kappa_s^{1/p} \cdot d_\sigma^{1-(2+s)/p}.
\]

(ii) There exists \( M_G \in \mathbb{R}^+ \) with the following property:
For all \( p \in [M_G, \infty[ \), there is a constant \( \kappa'_p \) such that for all \( \sigma \in \Sigma(G) \),

\[
\|\chi_\sigma\|_p \geq \kappa'_p d_\sigma^{1-(M_G/p)}.
\]

(iii) There exist \( \delta_G \in \mathbb{R}^+ \) and \( \kappa \in \mathbb{R}^+ \) with the following properties:
If $p \in [1, \infty]$, choose $r \in \mathbb{N}$ with $p < 2r$. Then for all $\sigma \in \Sigma(G)$,

$$\|\chi_\sigma\|_p \geq \kappa^{1/p} \frac{1-(2r-\delta_G)}{d_\sigma}.$$ 

These estimates are applied to yield some results concerning local central lacunary sets for compact Lie groups. It requires a certain amount of technicality to state the theorems on sets of type local central $V(p, q)$ and local central $\Lambda(p, q)$ in full generality, but I would like to mention here some of the consequences for "ordinary" lacunary sets (i.e. those discussed in §1.6). Firstly ($G$ being a compact Lie group), $\Sigma(G)$ is of type local central $\Lambda(2+s)$ for every $s < \varepsilon_G$. This was proved by Rider for the special case $G = U(n)$ or $SU(n)$ (cf. (1.6.9) (iii)); it improves the result of Price given in (1.6.9) (ii). Secondly, every subset $R$ of $\Sigma(G)$ which is of type local central $\Lambda(4-\delta_G)$ has $\sup\{d_\sigma \mid \sigma \in R\} < \infty$. This result improves the well-known result of Cecchini (cf. (1.6.9) (i)). Finally, combining estimate (ii) with (2.4.1) (ii), one sees that every $p$-Sidon subset $R$ of $\Sigma(G)$ ($1 \leq p < 2$) also satisfies $\sup\{d_\sigma \mid \sigma \in R\} < \infty$.

The chapter is organized as follows. The special case of compact simply connected semi-simple Lie groups is dealt with first, the estimate (i) being given in §3.2 and the estimates (ii) and (iii) in §3.3. In §3.4 the constants $\varepsilon_G$, $\delta_G$ and $M_G$ are tabulated for the simple simply connected Lie groups. In §3.5 I show how to extend these results to arbitrary compact Lie groups. Finally, §3.6 contains the results on lacunarity referred to above. This material has already been accepted for publication (Dooley [2]).
(3.2) An upper estimate for $\|x_\alpha\|_p$

(3.2.1) NOTATION. Throughout §3.2, $G$ will denote a simply connected simple Lie group of rank $l$, $T$ a maximal torus for $G$ (see (B3) for definitions). Let $\Phi$ denote the set of roots of $\mathfrak{G}(G)$ (the complexified Lie algebra of $G$) with respect to $\mathfrak{G}(T)$ (a CSA of $\mathfrak{G}(G)$), and $\Phi^+$ the set of positive roots with respect to some base $\Delta$. The root $\alpha \in \Phi$ defines a unique character $\chi_\alpha'$ of $T$ via $(\chi_\alpha')_e = \alpha$ (see (B5)). The Weyl integration formula (B7) states that for any continuous class function $f : G \to \mathbb{C}$,

$$
\int f d\lambda_G = \frac{1}{\text{card}(\mathcal{W})} \int f|_T \cdot |q|^2 d\lambda_T.
$$

Here, $\mathcal{W}$ is the Weyl group, $\lambda_G$ and $\lambda_T$ denote the Haar measures on $G$ and $T$ respectively, and $q$ is a trigonometric polynomial on $T$ given by

$$
q = \prod_{\alpha \in \Phi^+} (1 - \chi_\alpha').
$$

(3.2.2) Before giving the next theorem, it seems in order to make some remarks on its history. Rider [3] quoted this theorem without proof, for the special case of $G = U(n)$. It was subsequently proved by Clerc [1] for arbitrary simply connected simple Lie groups, though his proof is somewhat incomplete. Clerc's proof is improved by Stanton and Tomas [1]. I had obtained an independent proof of this theorem before I became aware of the work of Clerc or of Stanton and Tomas, so it seemed worthwhile including a proof here.

THEOREM. Notation is as in (3.2.1). Let $0 \leq s < \frac{l}{\text{card}(\Phi^+)}$. Then

$$
|q|^{-s} \in L^1(T).
$$

Proof. Note first that for $X \in L(T)$,

$$
|q \circ \exp_T(X)| = 2^{\text{card}(\Phi^+)} \prod_{\alpha \in \Phi^+} |\sin \pi a(X)|.
$$
Identify $L(T)$ with $\mathbb{R}^L$. Since Haar measure on $T$ is, to within a constant the image under $\exp_T$ of Lebesgue measure on $L(T)$, one sees that $|q|^{-s} \in L^1(T)$ if and only if $|q \circ \exp_T|^{-s}$ is integrable over some neighbourhood $U$ of 0 in $L(T)$. In fact, choosing a base $\Delta$ for $\phi$, I may assume $U = \{X \in L(T) \mid |\gamma(X)| \leq 1 \text{ for all } \gamma \in \Delta\}$. Hence, using the fact that $|q(t)| = |q(t^{-1})|$, it will suffice to show that

$$X \mapsto \prod_{\alpha \in \phi^+} |\sin \alpha(X)|^{-s}$$

is integrable over

$$\{X \in L(T) \mid 0 \leq \gamma(X) \leq 1 \text{ for all } \gamma \in \Delta\}.$$ 

Notice that for $\alpha \in \phi^+$, we may write $\alpha(X) = \sum_{\gamma \in \Delta} n^\alpha_\gamma \gamma(X)$ where the $n^\alpha_\gamma$ are non-negative integers. It hence suffices to show that for some $\varepsilon > 0$,

$$X \mapsto \prod_{\alpha \in \phi^+} |\sin \alpha(X)|^{-s}$$

is integrable over

$$L_\varepsilon = \{X \in L(T) \mid 0 < \gamma(X) < \varepsilon \text{ for all } \gamma \in \Delta\}.$$ 

(The function $X \mapsto \prod_{\alpha \in \phi^+} |\sin \alpha(X)|^{-s}$ has singularities bounded away from zero, but since these all have lower order than the singularity at zero, they can be dealt with by a slight modification of the argument which follows.) By choosing $\varepsilon$ sufficiently small, one may approximate $\sin(\alpha(X))$ by $\alpha(X)$ uniformly on $L_\varepsilon$. Thus, finally, I am reduced to proving, for

$$0 < s < \frac{L}{\text{card}_\phi^+}, \quad h^s : X \mapsto \prod_{\alpha \in \phi^+} |\alpha(X)|^{-s}$$

is integrable over $L_\varepsilon$.

For any ordering "<" of $\Delta$ (there are $L!$ different orderings), let

$$L^<_\varepsilon = \{X \in L_\varepsilon \mid \forall \gamma, \beta \in \Delta, \gamma < \beta \Rightarrow \gamma(X) \leq \beta(X)\}.$$ 

It is clear that $L_\varepsilon = \bigcup L^<_\varepsilon$. I will show that for each order <, $h^s$ is integrable over $L^<_\varepsilon$. Suppose $\Delta = \{\gamma_1, \ldots, \gamma_L\}$ with

$$\gamma_1 < \gamma_2 < \ldots < \gamma_L.$$
Then

\[ \prod_{\alpha \in \Phi^+} a(X) = \prod_{\alpha \in \Phi^+} \left( \sum_{i=1}^{L} n_{Y_i}^{\alpha} Y_i(X) \right), \]

where \( n_{Y_i}^{\alpha} \in \mathbb{N} \). Hence I may write

\[ \prod_{\alpha \in \Phi^+} a(X) = \sum_{(k_1, \ldots, k_L) \in \mathbb{N}^L} D(k_1, \ldots, k_L) \prod_{j=1}^{L} (Y_j(X))^{k_j}, \]

where the \( D(k_1, \ldots, k_L) \) are non-negative integers. It is a non-trivial fact that for some choice of \( (N_1, \ldots, N_L) \in \mathbb{N}^L \) with

\[ N_1 + \ldots + N_L = \text{card } \Phi^+ \] and \( N_1 \leq N_2 \leq \ldots \leq N_L \), \( D(N_1, \ldots, N_L) \neq 0 \). This fact is proved by Cahn [1], lemma 3, lemma 4 and lemma 5, for \( G \) belonging to each of the classical families of simply connected simple Lie groups (using a case-by-case argument); the proof for the exceptional simply connected simple Lie groups is (Stanton and Tomas [1]) "an exercise in tedium". However, given the explicit description of the irreducible root systems of \( (A_1) \), such a program can be carried out, although I have not done so here.

Then

\[ \prod_{\alpha \in \Phi^+} |a(X)| = \prod_{i=1}^{L} (Y_i(X))^{N_i}. \]

Since \( \Delta \) is a base, \( d_{Y_1} \wedge \ldots \wedge d_{Y_L} \) is Lebesgue measure on \( L(T) \) (to within a constant). Thus, it suffices to show that

\[ \int \ldots \int_{\{x | 0 < x_1 < x_2 < \ldots < x_L < \varepsilon\}} \prod_{i=1}^{L} X_i^{-sN_i} dX_1 dX_2 \ldots dX_L < \infty. \]

Now, the above conditions on \( N_1, \ldots, N_L \) ensure that for each \( i \),

\[ s \cdot (N_1 + \ldots + N_L) < 1. \]
\[ \int_0^1 \int_0^1 \ldots \int_0^1 x_1^{2+s} \ldots x_L^{2+s} d\lambda_1 \ldots d\lambda_L = \frac{1}{(1-s N_1) \ldots (1-s N_L)} \epsilon \]

This completes the proof. \( \square \)

(3.2.3) COROLLARY. Notation is taken from (3.2.1). For \( 0 \leq s < \frac{L}{\text{card} \Phi^+} \), there exists a constant \( \kappa_s \in \mathbb{R}^+ \) such that for all \( \sigma \in \Sigma(G) \),

\( \|x_\sigma\|_{2+s} \leq \kappa_s \).

Proof. By the Weyl integration formula (B7),

\[ \|x_\sigma\|_{2+s}^2 = \frac{1}{\text{card} \mathcal{W}} \int |q . x_\sigma|_{T}^{2+s} |q|^{-s} d\lambda_T. \]

By the Weyl character formula (B9),

\[ q . x_\sigma|_{T} = \sum_{w \in \mathcal{W}} \text{sgn } w . w(\chi^{(s)}) \chi^I_{\sigma-w \delta} \]

where \( \chi^{(s)} \) and \( \chi^I_{\sigma-w \delta} \) are as defined in (B9).

Thus \( \|q . x_\sigma|_{T}\|_{\infty} \leq \text{card } \mathcal{W} \) and it follows that

\[ \|x_\sigma\|_{2+s}^2 \leq \int_{T} |q|^{-s} d\lambda_T = \kappa_s . \]

(3.2.4) COROLLARY. Notation is from (3.2.1). Let \( 0 \leq s < \frac{L}{\text{card} \Phi^+} \).

Then for all \( p \in [2+s, \infty[ \) and for all \( \sigma \in \Sigma(G) \),

\[ \|x_\sigma\|_p \leq (\kappa_s)^{1/p} d_\sigma^{1-((2+s)/p)}. \]

Proof. Recall that \( \|x_\sigma\|_\infty \leq d_\sigma \) and use (3.2.3) together with Hölder's inequality. \( \square \)
(3.3) Lower estimates for $\|x_\sigma\|_p$

In this next lemma, I shall use notation from (2.5.1) and (2.5.7).

(3.3.1) LEMMA. Let $G$ be an arbitrary compact group; let $n \in \mathbb{N}$. Then for $\sigma \in \Sigma(G)$,

(i) $\|\chi_\sigma\|_{2n} \geq (\text{card } \sigma^n)^{1/2}$;

(ii) $\|\chi_\sigma\|_{2n} \geq d_\sigma \cdot (\nu(\sigma^n))^{-1/2}$.

Proof. Let $\eta = \sigma \otimes \ldots \otimes \sigma$ ($n$ times), and write $\eta = \oplus n(\zeta) \cdot \zeta$, $\zeta \in \sigma^n$

where the $\zeta$ are irreducible representations and $n(\zeta) \geq 1$ is the multiplicity of $\zeta$ in the decomposition. Now

$$\|x_\sigma\|^2 = \int \chi_\sigma^{n(\zeta)} \cdot \lambda_\sigma = \sum_{\zeta \in \sigma^n} (n(\zeta))^2 ;$$

(i) is now clear. For (ii), use Hölder's inequality:

$$d_\sigma^{2n} = \sum_{\zeta \in \sigma^n} n(\zeta) \cdot d_\eta \leq \sum_{\zeta \in \sigma^n} (n(\zeta))^2 \cdot \sum_{\zeta \in \sigma^n} d_\zeta^2 = \|x_\sigma\|_{2n} \cdot \nu(\sigma^n).$$

Neither of these estimates involves calculating multiplicities directly.

(3.3.2) NOTATION. In this paragraph, I fix some notation which will be used throughout the remainder of this section. Fuller explanation of these matters appears in appendix A.

Let $L$ be a semi-simple Lie algebra of rank $r$ over $\mathbb{C}$, and let $\Lambda$ be the set of weights of $L$. The Weyl group $W$ of $L$ has a natural action on $\Lambda$. One defines a partial order $<$ on $\Lambda$ by

$$\lambda_1 < \lambda_2 \iff \lambda_2 - \lambda_1 = \sum_{\alpha \in \Phi^+} k_\alpha \cdot \alpha, \quad k_\alpha \geq 0.$$ 

Let $W(\lambda) = \{ \mu \in \Lambda \mid \text{for all } \\ \nu \in W, \nu \mu < \lambda \}$. By $\Lambda^+$, I denote the set of dominant weights of $L$ with respect to some choice of base.
\[ \Delta = \{ \alpha_1, \ldots, \alpha_l \} \]. Choose a basis \( \{ \lambda_1, \ldots, \lambda_l \} \) for \( \Lambda^+ \), dual (under the form \( \langle , \rangle \)) to the basis \( \Delta \) such that any element of \( \Lambda^+ \) may be uniquely written \( \sum_{i=1}^l n_i \lambda_i \), \( n_i \geq 0 \). These concepts are all explained in (A7). Recall from (A9) that there is a bijective correspondence between \( \Lambda^+ \) and the set of equivalence classes of finite-dimensional irreducible \( L \)-modules. For \( \lambda \in \Lambda^+ \), let \( V(\lambda) \) denote the corresponding \( L \)-module (of highest weight \( \lambda \)). The Weyl dimension formula (A10) asserts that

\[ \dim V(\lambda) = \prod_{\alpha \in \Phi^+} (\lambda + \delta, \alpha), \text{ where } \delta = \sum_{i=1}^l \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \]

Thus \( \dim V(\sum_{i=1}^l n_i \lambda_i) \) is a polynomial with rational coefficients in \( (n_{l+1}, \ldots, n_{l+1}) \), homogeneous of degree \( \text{card} \Phi^+ \). For each \( i \in \{1, \ldots, l\} \) let \( N_i \) be the degree of this polynomial in \( n_i \) \( n_j \), \( j \neq i \) being considered fixed. Let \( N_L = \min\{N_i \mid i = 1, \ldots, l\} \). Clearly \( N_L \) depends only on \( L \). Note that \( N_i = \text{card}\{\alpha \in \Phi^+ \mid (\lambda, \alpha) \neq 0\} \). Thus \( 1 \leq N_L \leq \text{card} \Phi^+ \). Values for \( N_L \) when \( L \) is a simple Lie algebra are given in §3.4.

(3.3.3) For \( \lambda \in \Lambda^+ \), let \( \lambda^n \) be the set of dominant weights \( \mu \) such that \( V(\mu) \) is a direct summand of \( V(\lambda) \otimes \ldots \otimes V(\lambda) \) \( (r \) times). If \( F \) is a set of dominant weights, let \( v(F) = \sum_{\mu \in F} (\dim(V(\mu)))^2 \).

**Lemma.** Notation from (3.3.2). Let \( r \in \mathbb{N} \). There is a constant \( \kappa'_r \in \mathbb{R} \) such that for all \( \lambda \in \Lambda^+ \),

\[ v(\lambda^n) \leq \kappa'_r (\dim V(\lambda))^{(\dim L / N_L)}. \]

Proof. Let \( \lambda = \sum_{i=1}^l n_i \lambda_i \). It is an easy result of Steinberg's
formula (A10), that if \( \mu \in \lambda^p \) then \( \mu < r\lambda \). Thus

\( \lambda^p \subset \{ \mu \in \Lambda^+ \mid \mu < r\lambda \} \). As is shown in (A7), one may write

\[
\lambda_i = \sum_{j=1}^I q_{i,j} \alpha_j \quad \text{where} \quad q_{i,j} \in \mathbb{Q}^+. \]

Suppose that \( \mu = \sum_{i=1}^I m_i \lambda_i \in \Lambda^+ \) and

\( \mu < r\lambda \). Then

\[
r\lambda - \mu = \sum_{j=1}^I \left( \sum_{i=1}^I q_{i,j} (r \cdot n_i - m_i) \right) \alpha_j \]

is a sum of positive roots, so for each \( i \),

\[
\sum_{i=1}^I q_{i,j} (r \cdot n_i - m_i) \in \mathbb{N}. \]

It follows that

\[
\sum_{\mu \in \lambda^p} (\dim V(\mu))^2 \leq \sum_{\mu \in \lambda^p} \left( \dim V(\sum_{i=1}^I m_i \lambda_i) \right)^2 \]

\[
= \sum_{\{m_i \mid i=1 \leq \mathbb{N} \} \text{for } j=1 \ldots I, \sum_{i=1}^I q_{i,j} (r \cdot n_i + 1 - m_i) \in \mathbb{N} \}
\]

where \( \mathcal{S} \) is a polynomial in \( n \) variables, homogeneous of degree \( 2 \cdot \text{card } \Phi^+ \).

Note that if \( \left( \sum_{i=1}^I q_{i,j} (r \cdot n_i + 1 - m_i) \right) \in \mathbb{N}, \) then for all \( t \in \{1, \ldots, l\}, \)

\[
Q = \inf_{j=1 \ldots I} \frac{1}{\sup_{k=1 \ldots l} q_{k,j} \sum_{i=1}^I q_{i,j} (r \cdot n_i + 1)} \geq m_i. \]

Thus

\[
\sum_{\mu \in \lambda^p} (\dim V(\mu))^2 \leq \sum_{m_1=1}^Q \sum_{m_2=1}^Q \ldots \sum_{m_l=1}^Q \mathcal{S}(m_1, \ldots, m_l). \]

Using the fact that \( \sum_{m=1}^Q m^t \) is a polynomial in \( Q \) of degree \( t + 1 \), one sees that there exists \( \kappa \in \mathbb{R} \), independent of \( \lambda \) such that

\[
\nu(\lambda^p) \leq \kappa Q^{2 \cdot \text{card } \Phi^+ + l}. \]

But by the definition of \( Q \), \( Q^{2 \cdot \text{card } \Phi^+ + l} \) is dominated by a polynomial
in \((n_1, \ldots, n_l)\) in which each \(n_i\) occurs with degree

\[2 \text{card } \Phi^+ + l = \dim L.\]

But \(\dim V\left(\sum n_i \lambda_i\right)\) is a polynomial in \(n_i\) in which each \(n_i\) occurs with degree at least \(N_L\). Therefore there exists a constant \(\kappa'_n\) such that for all \(\lambda \in \Lambda^+\),

\[v(\lambda^n) \leq \kappa'_n(\dim V(\lambda))^{(\dim L/N_L)}.\]

\((3.3.4)\) COROLLARY. For the sequence \(\lambda_n = n.\delta\) one has for all \(n \in \mathbb{N}\),

\[v(\lambda^n) \leq \kappa'_n(\dim V(\lambda))^{2+(l/\text{card } \Phi^+)}.
\]

Proof. By the argument of (3.3.3), \(v(\lambda^n)\) is dominated by

\[
\inf_{j=1 \ldots l} \left( \frac{1}{\sup_{k=1 \ldots l} q_k^{(j)}} \sum_{i=1}^j q_{i,j} (n+1) \right)^{2 \text{card } \Phi^+ + l},
\]

a polynomial in \(n\) of degree \(2 \text{card } \Phi^+ + l\). But, by (3.3.2),

\(\dim(V(n\delta)) = \kappa_n \text{card } \Phi^+\) where \(\kappa \in \mathbb{R}^+\). The corollary follows. \(\square\)

\((3.3.5)\) Let \(G\) be a compact simply connected semi-simple Lie group. Then, by (B4) the finite-dimensional representations of \(G\) are in bijective correspondence with the finite-dimensional representations of \(\mathbb{C}L(G)\), the complexification of the Lie algebra of \(G\); this correspondence preserves irreducibility, dimension, direct sum and tensor product of representations. The following theorem now results from (3.3.1) and (3.3.3). (Note that \(\dim G = 2 \text{card } \Phi^+ + l\), where \(l\) and \(\Phi^+\) are respectively the rank and set of positive roots of \(\mathbb{C}L(G)\) - see (3.3.2).)

THEOREM. Let \(G\) be a compact simply connected semi-simple Lie group. Let \(N_G = N\mathbb{C}L(G)\) and let \(\Phi^+\) denote the set of positive roots of \(\mathbb{C}L(G)\) (see (3.3.2)). Let \(r \in \mathbb{N}\). There exists \(\kappa'_n \in \mathbb{R}\) such that for all \(\sigma \in \Sigma(G)\),
Further, there exists an infinite sequence \( \{\sigma_n \mid n \in \mathbb{N}\} \subseteq \Sigma(G) \) such that

\[
\|X_{\sigma_n}\|_{2r} \geq \kappa'_r d_{\sigma_n}^{1-(1/2r)}((\dim G/\text{card} \Phi^+)).
\]

(3.3.6) COROLLARY. Let \( G \) be as in (3.3.5), \( p \in \left[\frac{\dim G}{N_G}, \infty\right] \). There exists \( \kappa'_p \in \mathbb{R} \) such that for all \( \sigma \in \Sigma(G) \),

\[
\|X_{\sigma}\|_p \geq \kappa'_p d_{\sigma}^{1-(1/p)}((\dim G/N_G)).
\]

Proof. Choose \( r \in \mathbb{N} \) such that \( p < 2r \). Then, since \( \|X_{\sigma}\|_\infty \leq d_{\sigma} \),

\[
\|X_{\sigma}\|_p = \left\{ \int |x_{\sigma}|^{p-2r} |x_{\sigma}|^{2r} dx_{\sigma}\right\}^{1/p} \geq d_{\sigma}^{1-(2r/p)} \|x_{\sigma}\|^{2r/p}_{2r} \\
\geq \kappa'_p d_{\sigma}^{1-(2r/p)} (2r/p - (1/p))((\dim G/N_G)).
\]

(3.3.7) COROLLARY. Let \( G \) be as in (3.3.5). There exists an infinite sequence \( \{\sigma_n \mid n \in \mathbb{N}\} \subseteq \Sigma(G) \) such that for all \( n \in \mathbb{N} \),

\[
\|X_{\sigma_n}\|_p \geq \kappa'_p d_{\sigma_n}^{1-(1/p)}((\dim G/\text{card} \Phi^+)).
\]

This corollary shows that for \( p > \frac{\dim G}{\text{card} \Phi^+}, \|X_{\sigma_n}\|_p \to \infty \) as \( d_{\sigma} \to \infty \).

Hence we can expect no great improvement of (3.2.3).

(3.3.8) The above method appears to give a good estimate when \( p \) is large. In this paragraph, I shall develop an approach which gives (in general) better results for smaller values of \( p \). In particular, I show how to improve the well-known estimate of Cecchini [1], which states that there is a constant \( \kappa \) such that for all \( \sigma \in \Sigma(G) \), \( \|X_{\sigma}\|_4 \geq \kappa \log d_{\sigma} \).

The proof given here is considerably simpler than that of Cecchini.

LEMMA. Notation is taken from (3.3.2). Let \( \lambda', \lambda'' \in \Lambda^+ \). Suppose
that \( k \in \mathbb{N} \) satisfies \( k \leq \min\{\lambda', \alpha_i\}, (\lambda'', \alpha_i) \) for some \( i \in \{1, \ldots, l\} \). Then \( \mu = \lambda' + \lambda'' - k\alpha_i \in \Lambda^+ \), and \( V(\mu) \) occurs precisely once as a direct summand of \( V(\lambda') \otimes V(\lambda'') \).

**Proof.** To see that \( \mu \in \Lambda^+ \), let \( j \in \{1, \ldots, l\} \) and note that 
\[
\langle \mu, \alpha_j \rangle = \langle \lambda', \alpha_j \rangle + \langle \lambda'', \alpha_j \rangle - k\langle \alpha_i, \alpha_j \rangle.
\]
Since \( \langle \alpha_i, \alpha_j \rangle < 0 \) for \( i \neq j \) and \( \langle \alpha_i, \alpha_i \rangle = 2 \), it follows that \( \langle \mu, \alpha_j \rangle \geq 0 \).

To prove the second claim, I shall use Steinberg's formula (see A10) which states that the number of times \( V(\mu) \) occurs as a direct summand of \( V(\lambda') \otimes V(\lambda'') \) is given by
\[
m_{\lambda', \lambda''}(\mu) = \sum_{w \in W} \sum_{w' \in W} \sgn(w) p\left(\mu + 2\delta - w(\lambda' + \delta) - w'(\lambda'' + \delta)\right)
\]
where for \( \lambda \in \Lambda \), \( p(\lambda) \) is the number of ways of writing \( \lambda = \sum_{\alpha > 0} k_{\alpha} \alpha \) with \( k_{\alpha} \leq 0 \).

Now suppose that
\[
p\left(\lambda' + \lambda'' - k\alpha_i + 2\delta - w(\lambda' + \delta) - w'(\lambda'' + \delta)\right)
\]
\[
= p\left(\lambda' - w\lambda', \lambda'' - w'\lambda'', \delta - w\delta \right) + \delta - w'\delta \text{ is a sum of positive roots (by the lemma of (A7))}, each must be a non-negative multiple of \( \alpha_i \).

Thus in particular, \( w\delta = \delta - m\alpha_i \) for some \( m \in \mathbb{N} \). Now, denoting by \( w_{\alpha_i} \) the reflection determined by \( \alpha_i \),
\[
(w_{\alpha_i}, w)\delta = \delta + (m-1)\alpha_i.
\]
Hence, by lemma (A7), either \( m = 1 \), in which case \( w = w_{\alpha_i} \), or \( m = 0 \), in which case \( w \) is the identity. The same argument applies, of course, to \( w' \).

Thus, precisely four terms occur in (1) corresponding to
(w, w') = (1, 1), (1, w_{a_i}), (w_{a_i}, 1) and (w_{a_i}, w_{a_i}), and

\[ m_{\lambda', \lambda}(u) = p(-ka_i) - p((\lambda' + \delta, a_i) - k) a_i \]

\[ - p((\lambda' + \delta, a_i) - k) a_i + p((\lambda + \lambda + 2\delta, a_i) - k) a_i. \]

But since \( k \leq \langle \lambda', a_i \rangle \) one has \( p((\lambda' + \delta, a_i) - k) a_i = 0 \); similarly all terms vanish except the first, which is 1. Thus \( m_{\lambda', \lambda}(u) = 1. \)

**PROPOSITION.** If \( r \geq 2 \), then \( \lambda'' \geq \{ r, \lambda - ka_i \mid 0 \leq k \leq \langle \lambda, a_i \rangle \} \).

This proposition is an easy induction, based on the previous lemma.

**THEOREM.** Let \( G \) be a compact simply connected semi-simple Lie group, and let \( \Phi^+ \) denote the set of positive roots of \( \mathfrak{L}(G) \). There is a constant \( \kappa \in \mathbb{R} \) such that for all \( r \in \mathbb{N} \) and for all \( \sigma \in \Sigma(G) \),

\[ \| \chi_\sigma \|_{2^r} \geq \kappa \cdot \frac{1}{\operatorname{card} \Phi^+}. \]

**Proof.** The preceding proposition together with (3.3.2) (i) shows that

\[ \| \chi_\sigma \|_{2^r} \geq \text{card} \lambda'' \geq \sum_{i=1}^I \langle \lambda_{\sigma}, a_i \rangle \text{ where } \lambda_{\sigma} \text{ is the dominant weight associated with } \sigma. \]

On the other hand, \( d_\sigma \) is a polynomial in \((\lambda_{\sigma}, a_i+1)\), \(\ldots\), \((\lambda_{\sigma}, a_i+1)\), homogeneous of degree \( \text{card} \Phi^+ \). Thus there is a constant \( \kappa \) such that for all \( \sigma \in \Sigma(G) \),

\[ \kappa \cdot \frac{1}{\operatorname{card} \Phi^+} \leq \max_{i=1, \ldots, I} \langle \lambda_{\sigma}, a_i \rangle. \]

This completes the proof. \( \Box \)

(3.3.9) By a proof similar to that of (3.3.6) one may now show

**COROLLARY.** Let \( G \) and \( \Phi^+ \) be as in Theorem (3.3.8). There exists \( \kappa \in \mathbb{R} \) with the following property: Let \( p \in [1, \infty[ \) and choose \( r \in \mathbb{N} \) such that \( p \leq 2r \). Then for all \( \sigma \in \Sigma(G) \),

\[ \| \chi_\sigma \|_p \geq \kappa \cdot \frac{1}{\operatorname{card} \Phi^+} \left( \frac{1}{2^r} \right) \left( \frac{1}{\operatorname{card} \Phi^+} \right). \]

In particular, if \( p > 4 - \frac{1}{\operatorname{card} \Phi^+} \), \( \| \chi_\sigma \|_p \geq \kappa d_\sigma \) where
\[ \varepsilon = 1 - \frac{1}{p} \left( 4 - \frac{1}{\text{card} \Phi^+} \right) > 0 . \] The estimate given in (3.3.9) improves that of (3.3.6) if and only if for some \( r \in \mathbb{N} \),

\[ p \leq 2r \leq \frac{2 \text{card} \Phi^+ + l}{N_G} + \frac{1}{\text{card} \Phi^+} . \]

(3.4) The simple Lie groups

(3.4.1) The following table gives the various constants in \( 3.2 \) and \( 3.3 \) for the compact simply connected simple Lie groups. Its verification, which is left to the reader, is a simple mechanical task, given (All).

It should be noted that the upper and lower estimates of (3.2.4) and (3.3.6)-(3.3.9) coincide if (and only if) \( G = \text{SU}(2) \).

<table>
<thead>
<tr>
<th>Group</th>
<th>Lie Algebra</th>
<th>( \frac{L}{\text{card} \Phi^+} )</th>
<th>( \frac{\text{dim} L}{N_L} )</th>
<th>( \frac{1}{\text{card} \Phi^+} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SU}(l+1) )</td>
<td>( A_l ) (( l \geq 1 ))</td>
<td>( \frac{2}{l+1} )</td>
<td>( l+2 )</td>
<td>( \frac{2}{(l+1)l} )</td>
</tr>
<tr>
<td>( \text{SO}(2l+1) )</td>
<td>( B_l ) (( l \geq 2 ))</td>
<td>( \frac{1}{l} )</td>
<td>( \frac{l^2+l}{2l-1} )</td>
<td>( \frac{1}{l^2} )</td>
</tr>
<tr>
<td>( \text{Sp}(l) )</td>
<td>( C_l ) (( l \geq 3 ))</td>
<td>( \frac{1}{l} )</td>
<td>( \frac{l^2+l}{2l-1} )</td>
<td>( \frac{1}{l^2} )</td>
</tr>
<tr>
<td>( \text{SO}(2l) )</td>
<td>( D_l ) (( l \geq 4 ))</td>
<td>( \frac{1}{l-1} )</td>
<td>( \frac{l^2}{2(l-1)} )</td>
<td>( \frac{1}{l(l-1)} )</td>
</tr>
<tr>
<td>( \mathfrak{e}_8 )</td>
<td>( E_8 )</td>
<td>( \frac{1}{15} )</td>
<td>( 128 )</td>
<td>( \frac{1}{120} )</td>
</tr>
<tr>
<td>( \mathfrak{e}_7 )</td>
<td>( E_7 )</td>
<td>( \frac{1}{9} )</td>
<td>( 70 )</td>
<td>( \frac{1}{63} )</td>
</tr>
<tr>
<td>( \mathfrak{e}_6 )</td>
<td>( E_6^- )</td>
<td>( \frac{1}{6} )</td>
<td>( 42 )</td>
<td>( \frac{1}{36} )</td>
</tr>
<tr>
<td>( \mathfrak{f}_4 )</td>
<td>( F )</td>
<td>( \frac{1}{6} )</td>
<td>( 28 )</td>
<td>( \frac{1}{24} )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( G_2 )</td>
<td>( \frac{1}{3} )</td>
<td>( 2 )</td>
<td>( \frac{1}{6} )</td>
</tr>
</tbody>
</table>
(3.5) Extensions to arbitrary compact Lie groups

(3.5.1) I remind the reader of the structure theorem for compact connected Lie groups (see B2). If $G$ is a compact connected Lie group, one may write

$$G = \left( A \times \prod_{i=1}^{m} G_i \right)/Z$$

where $A$ is compact abelian, $G_1, \ldots, G_m$ are compact simply connected simple Lie groups, and $Z$ is a finite subgroup of the centre of $A \times \prod_{i=1}^{m} G_i$.

(3.5.2) Let $\chi_0$ be an irreducible character of $A \times \prod_{i=1}^{m} G_i$. Then, appealing to (1.3.4), one sees that $\chi_0 = \chi^A \cdot \prod_{i=1}^{m} \chi_i$, where $\chi^A$ is a character of $A$, and for each $i \in \{1, \ldots, m\}$, $\chi_i$ is an irreducible character of $G_i$. Clearly

$$d_\sigma = \prod_{i=1}^{m} d_{\chi_i} \quad \text{and} \quad \|\chi_0\|_p = \prod_{i=1}^{m} \|\chi_i\|_p .$$

Further, suppose $G$ is any compact group with normal subgroup $N$. Then by (1.3.4) there is a bijection $\phi$ from

$$A(\Sigma(G), N) = \{ \sigma \in \Sigma(G) \mid \sigma(x) = I_{d_{\sigma}} \quad \text{for all} \quad x \in N \}$$

to $\Sigma(G/N)$. For $\sigma \in A(\Sigma(G), N)$, one has $d_\sigma = d_{\phi(\sigma)}$, and it is banal to show that $\|\chi_0\|_p = \|\chi_{\phi(\sigma)}\|_p$. These remarks show that the results of §§3.2 and 3.3 are valid for arbitrary connected Lie groups.

(3.5.3) Now let $G$ be a compact not necessarily connected Lie group. Then $G_0$, the connected component of the identity of $G$ is a compact connected Lie group and $[G : G_0]$ is finite. Suppose $[G : G_0] = n$. 
Let $\sigma$ be an irreducible representation of $G$. Then, by Clifford's theorem (1.3.9) the restriction $\sigma|_{G_0}$ decomposes into a direct sum of the form $n_\sigma \cdot \xi_1 \oplus \ldots \oplus \xi_k$, where $n_\sigma$ is a positive integer, $\xi$ an irreducible representation of $G_0$, for $g \in G$, $h \in G_0$, $\xi(g)(h) = \xi(g^{-1}hg)$, and $\{ e = g_1, g_2, \ldots, g_k \}$ is a complete set of coset representatives for $H_\xi = \{ g \in G | \xi(g) \cong \xi \}$.

**PROPOSITION.**

(i) $d_\xi \leq d_\sigma \leq n \cdot d_\xi$.

(ii) Let $r \in \mathbb{N}$. Then

$$n^{(-1/2r)} \|X_\xi\|_{2^r} \leq \|X_\sigma\|_{2^r} \leq n^{1-(1/2r)} \|X_\xi\|_{2^r}.$$

**Proof.** Note that $G_0 \leq H_\xi \leq G$, so that $k \leq n$. Now by Hewitt and Ross [1], (28.54),

$$\|X_\sigma\|_{2^r} = \left\{ \frac{1}{n} \sum_{i=1}^{n} \left| X_\sigma(g_i,h) \right|^{2^r} d\lambda_{G_0}(h) \right\}^{1/2r} \geq n^{(-1/2r)} \left\{ \sum_{i=1}^{k} \left| X_\sigma(g_i,h) \right|^{2^r} d\lambda_{G_0}(h) \right\}^{1/2r} \geq n_\sigma \cdot n^{(-1/2r)} \left\{ \sum_{i=1}^{k} \left| \chi_{\xi}(g_i)(h) \right|^{2^r} d\lambda_{G_0}(h) \right\}^{1/2r}.$$

(The last inequality is valid because all the mixed terms in the sum which immediately precedes it are multiplicities of a certain representation in a direct decomposition and hence are non-negative integers.) Thus, since $h \mapsto g_i^{-1}hg_i$ leaves the Haar measure of $G_0$ invariant,

$$\|X_\sigma\|_{2^r} \geq n_\sigma \cdot n^{(-1/2r)} \cdot k^{(1/2r)} \|X_\xi\|_{2^r} \geq n^{(-1/2r)} \|X_\xi\|_{2^r}. \quad (1)$$
This proves the left hand inequality of (ii). Putting \( r = 1 \) in (1) gives
\[ n_\sigma \cdot k^\delta \leq n_\sigma^k. \]
Now \( d_\sigma = \chi_\sigma(e) = n_\sigma \cdot k \cdot d_\xi. \) Thus (i) follows. It remains to prove the right hand inequality of (ii).

By the Frobenius reciprocity theorem (1.3.8), \( \sigma \) is contained in the decomposition of the induced representation \( \xi^G \) precisely \( n_\sigma \) times. Now, for \( x \in G, \)
\[ \chi_{\xi^G}(x) = \begin{cases} \frac{n}{k} \sum_{i=1}^k \chi_{\zeta^i}(x) & \text{if } x \in G_0, \\ 0 & \text{otherwise.} \end{cases} \]
(This follows from lemma (1.3.7).)

Hence by arguments similar to those used in the proof of (ii),
\[
n_\sigma \|\chi_\sigma\|_{2r}^2 \leq \left\{ \int |\chi_{\xi^G}|^{2r} d\lambda_G \right\}^{1/2r} = n^{(1/2r)} \left\{ \int \left| \frac{n}{k} \sum_{i=1}^k \chi_{\zeta^i}(x) \right|^{2r} d\lambda_G(x) \right\}^{1/2r} \leq n^{1-(1/2r)} \cdot \frac{1}{k} \sum_{i=1}^k \left\{ \int \left| \chi_{\xi^G} g_i^{-1} h_\zeta \right|^{2r} d\lambda_G(x) \right\}^{1/2r} = n^{1-(1/2r)} \|\chi_\xi\|_{2r}^2.
\]

(3.5.4) THEOREM. Let \( G \) be a compact Lie group. There exist constants \( \epsilon_G, M_G, \delta_G > 0 \) such that:

(i) for all \( s \in [0, \epsilon_G] \) and for all \( p \in [2+s, \infty[ \) there exists \( \kappa_s \in \mathbb{R} \) such that for all \( \sigma \in \Sigma(G), \)
\[ \|\chi_\sigma\|_p \leq \kappa_s^{1/p} d_\sigma^{-1/(2+s)/p}; \]
(ii) for all \( p \in [M_G, \infty[ \) there exists \( \kappa_p \in \mathbb{R} \) such that,
for all \( \sigma \in \Sigma(G), \)
(iii) there exists \( \kappa \in \mathbb{R} \) with the following property:

Let \( p \in [1, \infty[ \) and choose \( r \in \mathbb{N} \) such that \( p \leq 2r \).

Then for all \( \sigma \in \Sigma(G) \),

\[
\|x_\sigma\|_p \geq \kappa^{1/(2P + 2)} \left( \frac{2 + s}{2P + 2} \right) .
\]

Proof. Let \( G_0 \) be the connected component of the identity of \( G \). In view of (3.5.2), together with (3.2.4), (3.3.6) and (3.3.9), the theorem holds for \( G_0 \), and applying (3.5.3) shows that the theorem is true if \( p \) is constrained to belong to the set of even integers. The technique of (3.3.6) now enables one to deduce that (ii) and (iii) are valid for all values of \( p \). To extend the validity of (i) to all values of \( p \), proceed as follows; let \( s \in [0, e_{G_0}] \), suppose that \( p \in [2+s, \infty[ \), and choose \( r \) with \( 2r < p \leq 2r+2 \). The above arguments ensure the existence of \( \kappa_s \in \mathbb{R} \) with

\[
\|x_\sigma\|_2^r \leq \kappa_s^{rac{1}{2r}} \left( \frac{(2+s)/2r}{2P} \right) \quad \text{and} \quad \|x_\sigma\|_{2r+2} \leq \kappa_s^{rac{1}{2r+2}} \left( \frac{(2+s)/(2r+2)}{2P} \right) .
\]

A simple application of Holder's inequality gives

\[
\|x_\sigma\|_p \leq \|x_\sigma\|_2^{r+1}(p-2r)/p \|x_\sigma\|_{2r+2}^{(2r+2-p)/p} ,
\]

which combines with the above to yield

\[
\|x_\sigma\|_p \leq \kappa_s^{1/p} \left( \frac{2+s}{2P} \right) .
\]

as required. \( \square \)

(3.5.5) Note that once the structure of \( G \) is known, the various constants in (3.5.4) can be determined. Let \([G : G_0] = n\) and suppose \( G_1, \ldots, G_m \) are the simply connected simple groups associated with \( G_0 \) as in (3.5.1). Then
\[ \varepsilon_G = \min_{i=1}^{\dim G_i} \left( \frac{2 \rank G_i}{\dim G_i - \rank G_i} \right), \quad M_G = \max_{i=1}^{\dim G_i} \left( \frac{\dim G_i}{N G_i} \right); \]

\[ \delta_G = \min_{i=1}^{\dim G_i} \left( \frac{2}{\dim G_i - \rank G_i} \right); \quad \kappa \leq n \cdot \prod_{i=1}^{\dim G_i} \kappa(i), \]

\[ \kappa_p' \geq n^{-1} \prod_{i=1}^{\dim G_i} \kappa_p(i), \quad \text{and} \quad \kappa \geq n^{-1} \prod_{i=1}^{\dim G_i} \kappa(i), \]

where \( \kappa(i), \kappa_p(i) \) and \( \kappa(i) \) are the relevant constants for \( G_i \).

### (3.6) Applications to lacunary sets

#### (3.6.1) In this section, I return to the sets of type local central \( \Sigma(p, q) \) and local central \( \Lambda(p, q) \) which were defined in (2.3.1). It is clear, via (2.3.3), how (3.5.4) provides information on these sets. Furthermore, by use of (2.4.1) (ii), I can obtain information concerning \( p \)-Sidon sets. We now have

**Proposition.** Let \( G \) be a compact Lie group.

(i) Suppose \( p \in \left[ 1, 2 + \varepsilon_G \right], \quad q' \in \left[ \frac{2p}{2 + \varepsilon_G}, \infty \right]. \) Then every subset of \( \Sigma(G) \) is of type local central \( \Lambda(p, q) \). In particular, every subset of \( \Sigma(G) \) is of type local central \( \Lambda(p) \) for all \( p < 2 + \varepsilon_G \).

(ii) Suppose \( p \in \left[ M_G, \infty \right], \quad q' \in \left[ \frac{2p}{M_G}, \infty \right]. \) Then if \( R \subseteq \Sigma(G) \) is of type local central \( \Lambda(p, q) \), \( \sup \{ d_\sigma \mid \sigma \in R \} < \infty \). In particular, if \( R \) is a \( p \)-Sidon set then \( \sup \{ d_\sigma \mid \sigma \in R \} < \infty \).

(iii) Suppose that for some \( r \in \mathbb{N}, \quad p \in \left[ 2r - \delta_G, \infty \right], \) and \( q' \in \left[ 1, \frac{2p}{2r - \delta_G} \right]. \) Then if \( R \subseteq \Sigma(G) \) is of type local central \( \Lambda(p, q) \), \( \sup \{ d_\sigma \mid \sigma \in R \} < \infty \). In particular if \( R \) is of type (local central) \( \Lambda(4 - \delta_G) \), then \( \sup \{ d_\sigma \mid \sigma \in R \} < \infty \).
(Notice that the second statement of (ii) follows from theorem (2.4.1) (ii).)

**3.6.2** PROPOSITION. Let $G$ be a compact Lie group.

(i) Suppose $p' \in [2+\varepsilon_G, \infty[$, $q \in [\frac{2p'}{2+\varepsilon_G}, \infty[$. Then $\Sigma(G)$ is of type local central $V(p, q)$.

(ii) Suppose that for some $r \in \mathbb{N}$, $p' \in [2r-\delta_G, \infty[$ and $q' \in [1, \frac{2}{2r-\delta_G}p']$. Then $\Sigma(G)$ is of type local central $V(p, q)$.

(iii) Suppose $p' \in [1, 2+\varepsilon_G[$, $q \in [\frac{2p'}{2+\varepsilon_G}, \infty[$. Then if $R \subseteq \Sigma(G)$ is of type local central $V(p, q)$, $\sup \{d_G | \sigma \in R \} < \infty$.

**3.6.3** One may diagrammatically represent the sets of type local central $\Lambda(p, q)$ and $V(p, q)$ as was done in (2.3.6). Adopting the conventions explained in (2.3.6), theorems (3.6.1) and (3.6.2) can be expressed by the diagrams of Figure 1 and Figure 2. The letters $a$, $b$ and $c$ represent the numbers $2 + \varepsilon_G$, $4 - \delta_G$ and $M_G$ respectively, and the letters $a'$, $b'$, $c'$ their conjugate indices (see (1.2.6)).
FIGURE 1. Sets of type local central $V(p, q)$ ($G$ compact Lie)

FIGURE 2. Sets of type local central $A(p, q)$ ($G$ compact Lie)
(3.6.4) It seems worthwhile to mention here the situation for sets of type \( A(p, q) \) and \( V(p, q) \). Figures 3 and 4 represent knowledge of these sets which can be gleaned from (2.2.5), (2.3.5), (3.6.1), (3.6.2), (3.6.3). The sets occurring on the boundary labelled "C" (Figure 3) are precisely the \( A(2) \) sets. If \( G = U(n) \) or \( SU(n) \), Rider has shown that these sets are finite.

Notice that if \( G \) is semi-simple, then \( G \) is tall, so "b.d." is equivalent to "finite".

FIGURE 3. Sets of type \( V(p, q) \) (\( G \) compact Lie)
CHAPTER 4

CENTRAL FUNCTIONS AND MEASURES

1.4.1 Introduction

In this chapter we shall develop the theory for analyzing the one-dimensional Fourier transforms of the standard functions and measures associated with compact Lie groups. The theory is derived in a general study of the mapping \( \mathbb{R} \) of the standard functions and measures associated with compact Lie groups. It leads to the special case where \( \mathbb{R} \) is a compact connected Lie group. In an important example, we show that the Fourier transform of the compact Lie group \( G \) is isometrically isomorphic to the algebra of the group algebra of \( G \), which is the center of the group algebra. This example is an element of the group algebra \( \mathbb{C} \) consisting of those elements invariant under the action of the Heisenberg group.

In §4.3 it is shown that the Fourier transform of an element of \( L^1(G) \) may be calculated in terms of the Fourier transform of its image under the isometric isomorphism mentioned in the previous paragraph. A formula applicable to central kernels is also developed. These techniques are applied in §4.6 to get convolution formulas for central functions and measures.

In §4.7, we work through in some detail two related examples \( G = SU(3) \) and \( G = U(3) \). These examples are simple enough to be easily understandable and yet exhibit many of the characteristics of the general case. The reader is invited to consult §4.7 or to read the other sections of this chapter, as this will perhaps aid his understanding of the general case. Finally, in §4.9, I have included a discussion of the possibility of...
CHAPTER 4

CENTRAL FUNCTIONS AND MEASURES

(4.1) Introduction

In this chapter, I shall develop methods for analysing the convolution centres (introduced in §1.5) of the standard function and measure algebras: \( Z^1(G), Z^L(G), Z^P(G) \ (1 \leq p \leq \infty) \), \( Z^M(G) \). §4.2 is devoted to a closer study of the mapping \( Z \) of §1.5; its action on functions (rather than on equivalence classes of functions) is studied for arbitrary compact \( G \), and this leads, in the special case where \( G \) is a compact connected Lie group, to an extension of the Weyl integration formula (see (B7)) to elements of \( L^1(G) \). Using this extended formula, I am able, in §§4.2 and 4.3, to show that the centre of any of the standard function or measure algebras on \( G \) is isometrically isomorphic to the subalgebra of the same function or measure algebra on \( T \) (a maximal torus for \( G \)) consisting of those elements invariant under the action of the Weyl group.

In §4.5 it is shown that the Fourier transform of an element of \( Z^\infty(G) \) may be calculated in terms of the Fourier transform of its image under the isometric isomorphism mentioned in the previous paragraph. A formula applicable to central measures is also developed. These techniques are applied in §4.6 to get convolution formulae for central functions and measures.

In §4.7, I work through in some detail the two related examples \( G = SU(2) \) and \( G = U(2) \). These examples are simple enough to be easily understandable and yet exhibit many of the characteristics of the general case. The reader is invited to consult §4.7 as he reads the other sections of this chapter: this will perhaps aid his understanding of the general case. Finally, in §4.8, I have included a discussion of the possibility of
extending the Weyl integration formula to some class of compact connected groups more general than the class of connected Lie groups.

The main motivation for the theory presented in this chapter is its applicability to central lacunary sets, which will be dealt with in the next chapter. However, the theorems presented here have interest in their own right, because they elucidate the structure of the convolution centres of the standard function and measure algebras. I should mention that the technique of reducing questions about central functions on $G$ to questions about functions on $T$ is not new - it has been applied successfully by Rider [3], Stanton [1], Clerc [2]. However, the theorems of this chapter have apparently not previously been proved. It seems likely that this work will have wider applicability - for example, to the problem of determining multipliers of central function and measure algebras; see the second paragraph of (2.4.2). (Indeed, I have already obtained some partial results in this direction - although they are not included as part of this thesis.)

(4.2) An extension of the Weyl integration formula

(4.2.1) In this section, I shall show how to extend the Weyl integration formula of (B7) to elements of $L^1(G)$. This will involve first a closer examination of the mapping $\mathcal{E}$ introduced in §1.5.

(4.2.2) LEMMA. Let $G$ be a compact group, and let $f$ be a $\lambda_G$-integrable complex-valued function on $G$. Then for almost all $y \in G$, the integral $\int f(xy^{-1})d\lambda_G(x)$ exists.

The function $\mathcal{E}(f)$ defined by

$$\mathcal{E}(f)(y) = \begin{cases} \int f(xy^{-1})d\lambda_G(x) & \text{if this integral exists,} \\ 0 & \text{otherwise.} \end{cases}$$
Proof. The map \( \theta : G \times G \to G \) is continuous; thus \( f \circ \theta \) is a \( \lambda_G \)-measurable function. Furthermore,

\[
\int |f \circ \theta| d\lambda_G \times d\lambda_G = \int \int |f(xy x^{-1})| d\lambda_G(y) d\lambda_G(x) = \int |f| d\lambda_G < \infty.
\]

Thus the lemma follows from Tonelli's theorem (4.17.8 of Edwards [1]).

(4.2.3) Lemma. Let \( G \) be a compact group. If \( f \) is \( \lambda_G \)-integrable, then for all \( x \in G \) and for all \( y \in G \),

\[ \Xi(f)(x y^{-1} x) = \Xi(f)(y). \]

Proof. Since \( \lambda_G \) is left invariant, \( \int f(xy x^{-1} x^{-1}) d\lambda_G(z) \) exists if and only if \( \int f(x y x^{-1}) d\lambda_G(z) \) exists; when these integrals exist they are equal.

(4.2.4) Lemma. Let \( G \) be a compact group. Suppose \( f \) is a \( \lambda_G \)-integrable function such that \( \frac{f}{x} \) is \( \lambda_G \)-integrable function such that \( \frac{f}{x} = f \) a.e. for all \( x \in G \). Then

\[ \Xi(f) = f \] a.e.

Proof.

\[
\int |\Xi(f) - f| d\lambda_G = \int \int |f(xy x^{-1}) - f(y)| d\lambda_G(x) d\lambda_G(y) \leq \int \int |f(xy x^{-1}) - f(y)| d\lambda_G(x) d\lambda_G(y) = \int \int |f(xy x^{-1}) - f(y)| d\lambda_G(y) d\lambda_G(x) = 0.
\]

by Fubini's theorem (Edwards [1], 4.17.7)

(4.2.5) The preceding lemmas show that the mapping \( \Xi \) of (4.2.2) has the property that if \( f = f_1 \) a.e. then \( \Xi(f) = \Xi(f_1) \) a.e. The map obtained on equivalence classes of functions thus co-incides with the map \( \Xi \) of section (1.5).
(4.2.6) Theorem. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Let $f$ be a $\lambda_G$-integrable complex valued function. Then
\[ \int_T fd\lambda_G = \frac{1}{\dim G} \left[ \mathbb{E}(f) \circ \iota \right] |q|^2 d\lambda_T. \]
Here $\iota$ denotes the canonical injection $T \to G$, and $q$ is the function defined in (B7).

Proof. By (B7), the formula holds for all $f \in C(G)$. Now let $f$ be a $\lambda_G$-integrable function. By (4.6.7) and (4.6.9) of Edwards [1] there exists a $\|\cdot\|_1$-Cauchy sequence $(g_n)_{n \in \mathbb{N}} \subseteq C(G)$ such that $\lim g_n = f$ a.e., and such that $\|g_n - f\|_1 \to 0$ as $n \to \infty$.

By (4.2.2), it follows that $\mathbb{E}(g_n)$ is a $\|\cdot\|_1$-Cauchy sequence, that $\lim \mathbb{E}g_n = \mathbb{E}(f)$ a.e., and that $\|\mathbb{E}g_n - \mathbb{E}f\|_1 \to 0$ as $n \to \infty$.

By the Weyl integration formula of (B7), $(\mathbb{E}g_n \circ \iota)$ is an $L^1 \left[ T, |q|^2 d\lambda_T \right]$-Cauchy sequence. By Edwards [1], (4.6.7), it now suffices to show that $\lim \mathbb{E}g_n(t) = \mathbb{E}f(t)$ for $\lambda_T$-almost all $t \in T$.

Suppose this not so, and let $S = \{ t \in T \mid \mathbb{E}g_n(t) \not\to \mathbb{E}f(t) \}$. By (4.2.3), for all $x \in G$ and for all $t \in T$, $\mathbb{E}g_n(xtx^{-1}) = \mathbb{E}g_n(t)$ and $\mathbb{E}(xtx^{-1}) = \mathbb{E}f(t)$. Thus, for $y \in \{ xtx^{-1} \mid x \in G, t \in S \} = Y$, $\mathbb{E}g_n(y) \not\to \mathbb{E}f(y)$. But $\lambda_G(Y) = \lambda_G(G/T \times S)$, where $\phi$ is as defined in (B6). The formula of (B7) shows that
\[ \lambda_G(\phi(G/T \times S)) = (\lambda_G(G/T) \times \lambda_T)(G/T \times S) = \lambda_T(S). \]

Now we have $\lambda_G(Y) = \lambda_T(S) > 0$, and this contradicts the fact that $\lim \mathbb{E}g_n = \mathbb{E}f$ a.e. \qed

(4.3) The centre of $L^p(G)$ ($1 \leq p \leq \infty$)

(4.3.1) Notation. Throughout this section, $G$ will denote a compact...
connected Lie group, \( T \) a maximal torus for \( G \) (see B3), \( 1 : T \to G \) the canonical injection. The Weyl group \( W \) (isomorphic to \( N_G(T)/T \) - see also (B3)) has an action on \( T \) defined by \( w \cdot t = gtg^{-1} \) for \( w = gT \in W \).

Suppose \( f : T \to \mathbb{C} \). Define \( w.f : T \to \mathbb{C} \) by \( (w.f)(t) = f(w^{-1}t) \).

Clearly \( f \mapsto w.f : C(T) \to C(T) \) is an isometric isomorphism. Furthermore, by Hewitt and Ross [1], (15.26), the transformation \( t \mapsto w.t \) of \( T \) preserves the \( \lambda_T \)-measure of any set - thus \( f \mapsto w.f \) also defines an isometric isomorphism \( L^p(T) \to L^p(T) \) \((1 \leq p \leq \infty)\).

Finally, for \( v \in M(T) \) and \( f \in C(T) \), let \((w.v)(f) = v(w^{-1}f)\).

Then \( v \mapsto w.v : M(T) \to M(T) \) is an isometric isomorphism coinciding with the map \( f \mapsto w.f \) of the previous paragraph on the subspaces \( C(G) \), and \( L^p(G) \) \((1 \leq p \leq \infty)\).

If \( A \) is a subset of \( M(T) \), let \( A_W = \{ v \in A \mid \forall w \in W, w.v = v \} \).

Note that I shall usually write \( C_W(G) \), \( L^p_W(G) \), etc. instead of \( (C(G))_W \), \( (L^p(G))_W \) etc.

(4.3.2) LEMMA. If \( A \) is a normed vector subspace of \( M(T) \) such that for all \( w \in W \), \( v \mapsto w.v \) is an isometry of \( A \), then
\[
\Omega : v \mapsto \frac{1}{\text{card}W} \sum_{w \in W} w.v : A \to A_W
\]
is a bounded linear map of norm 1 whose restriction to \( A_W \) is the identity map. Thus, under these hypotheses \( A_W \neq \{0\} \).

Proof. It is clear that for any \( v \in A \), \( \Omega(v) \in A_W \), and that for \( v \in A_W \), \( \Omega(v) = v \). Let \( \| \cdot \|_A \) denote the norm of \( A \). Now, for \( v \in A \),
\[
\| \Omega(v) \|_A \leq \frac{1}{\text{card}W} \sum_{w \in W} \| w.v \|_A = \| v \|_A .
\]

(4.3.3) COROLLARY. If \( A \) is a complete normed vector subspace of
This corollary may be proved by the method of proof of (1.5.5).

(4.3.4) DEFINITION. Suppose \( f \) is a \( \lambda_G \)-integrable function \( G \to \mathbb{C} \).

Define, for \( p \in [1, \infty[ \), a measurable function \( \psi_p(f) : T \to \mathbb{C} \) by

\[
\psi_p(f) = \frac{|q|^{2/p}}{(\text{card } W)^{1/p}} \cdot \varphi(f) \circ \iota
\]

and define

\[
\psi_\infty(f) = \varphi(f) \circ \iota.
\]

(4.3.5) THEOREM. Let \( G \) be a compact connected Lie group, \( T \) a maximal torus for \( G \). For each \( p \in [1, \infty[ \), the map \( \psi_p \) of (4.3.4) is a linear isometric isomorphism \( ZL^p(G) \to L^p_W(T) \).

Proof. I shall first check that the maps \( \psi_p \) are well-defined.

Suppose that \( f = g \) a.e., and \( f, g \in ZL^p(G) \). Then by (4.2.4),

\[
\varphi(f) = \varphi(g) \lambda_G \text{ a.e.}
\]

By the argument of (4.2.6), this implies that

\[
\varphi(f) \circ \iota = \varphi(g) \circ \iota \lambda_T \text{ a.e.}
\]

Thus for every \( p \in [1, \infty[ \), the measurable functions \( \psi_p(f) \) and \( \psi_p(g) \) are equal \( \lambda_T \)-almost everywhere.

Suppose that \( f \in ZL^p(G) \) for \( p \in [1, \infty[ \). Then by (4.2.6),

\[
\infty > \int |f|^p d\lambda_G = \int \varphi(f)^p d\lambda_G = \frac{1}{\text{card } W} \int \varphi(\varphi(f)^p) |q|^{2} d\lambda_T
\]

\[
= \int \frac{|q|^2}{\text{card } W} \cdot |\varphi(f)|^p \circ \iota d\lambda_T
\]

\[
= \int |\psi_p(f)|^p d\lambda_T.
\]

(The reader may easily verify the third equality by use of (4.2.3).)

Thus \( \psi_p(f) \in L^p(T) \). This argument also establishes that \( \psi_p \) is an
isometry for $p \in [1, \infty[$. 

To see that $\psi_\infty$ is an isometry into $L^\infty(T)$, note that, for $f \in ZL^\infty(G)$,

$$\|f\|_\infty = \|\varpi(f)\|_\infty = \lambda_G - \text{ess sup}_{x \in G} |\varpi(f)(x)|$$

$$= \lambda_T - \text{ess sup}_{t \in T} |\varpi(f)(t)|.$$ 

(To see this, note that, by (4.2.6), for any $M \in \mathbb{R}$,

$$\lambda_G\{x \in G \mid \varpi(f)(x) > M\} = 0 \text{ if and only if } \lambda_T\{t \in T \mid \varpi(f)(t) > M\} = 0.$$ 

Next, I show that if $f \in ZL^p(G)$, then $\psi_p(f)$ is invariant under the action of $W$ defined in (4.3.1). To see this, first recall that

$$|q|^2 = \prod_{\alpha \in \Phi} (1-\chi_\alpha')$$

(see (B9)). Since each element of $W$ maps $\Phi$ into itself (A5, B9), $|q|^2$ is invariant under this action. Secondly, note that for any $\lambda_G$-integrable function $f$, and for $w = xt \in W$ (cf. (4.3.1)),

$$(\varpi(f)(xt^{-1})) = \varpi(f)(xt^{-1}) = \varpi(f)(t) \text{ (by 4.2.3).}$$

It follows that for any $p \in [1, \infty]$, $\psi_p : ZL^p(G) \to L^p_W(T)$ is an isometry.

To complete the proof, it is necessary to show that $\psi_p$ is a surjective map for each $p$. I shall treat only the case $p \in [1, \infty[$. The case

$p = \infty$ is entirely similar but slightly simpler, and so I have omitted its proof. I shall use information from (B6), whence all unexplained notation

is taken.

Let $g$ be a function $T \to \mathbb{C}$, invariant under the action of $W$, and such that $|g|^p$ is $\lambda_T$-integrable. Then $f_1$, defined by

$$f_1(t) = \begin{cases} 
\frac{(\text{card } W)^{1/p}}{|q(t)|^{2/p}} \cdot g(t) & \text{if } q(t) \neq 0, \\
0 & \text{if } q(t) = 0,
\end{cases}$$

and this shows that $f_1 \in ZL^p(G)$, by (4.2.1).
is a $\lambda_T$-measurable function on $T$, invariant under the action of $W$.

Thus \( f_2 : (G/T) \times T \to \mathbb{C}, \) defined by \( f_2(gT, t) = f_1(t) \) is a $\lambda_{G/T} \times \lambda_T$-measurable function, invariant under the action of $W$ on $(G/T) \times T$. It follows that there exists a measurable function \( f_3 : ((G/T) \times T)/W \to \mathbb{C} \) such that \( f_2 = f_3 \circ \varphi \).

Now $\varphi_1 : ((G/T) \times T)/W \to G$ is injective, and $((G/T) \times T)/W$ and $G_R$ are two manifolds of the same dimension. Thus, by the inverse function theorem (Matsushima [1], p. 19), $\varphi_1$ is a $C^\infty$ diffeomorphism; hence

\[ f_3 \circ \varphi_1^{-1} : G \to \mathbb{C} \]

is a well-defined $\lambda_G$-measurable function. (I have used here, of course, the results of (B6) and (B7) that the two measures $\lambda_G$ and $\lambda_{G/T} \times \lambda_T \circ \varphi$ are mutually absolutely continuous.) Thus

\[
\begin{align*}
    f(x) &= \begin{cases} 
        f_3 \circ \varphi_1^{-1}(x) & \text{if } x \in G_R, \\
        0 & \text{if } x \notin G_R,
    \end{cases}
\end{align*}
\]

is a $\lambda_G$-measurable function on $G$.

Furthermore, for $x \in G$, $y \in G_R$,

\[
    f(xy^{-1}) = f_3 \left( \varphi_1^{-1}(xy^{-1}) \right)
    = f_3(W(xzT, t)) \quad \text{where } y = ztx^{-1}
    = f_2((xzT, t)) = f_2(zT, t) = f(y).
\]

Now applying (4.2.6) yields

\[
\int_{G} |g|^P d\lambda_T = \frac{1}{\text{card}W} \int |f_1|^P \cdot |q|^2 d\lambda_T
= \frac{1}{\text{card}W} \int |f \circ \varphi|^P |q|^2 d\lambda_G = \int |f|^P d\lambda_G,
\]

and this shows that $f \in L^P(G)$. 
Finally, for $t \in \mathcal{T}_R$, $|q(t)| > 0$. Thus

$$
\psi_p(f)(t) = \frac{|q(t)|^{2/p} (E(f))(t)}{(\text{card}\mathcal{W})^{1/p}} = \frac{|q(t)|^{2/p} f(t)}{(\text{card}\mathcal{W})^{1/p}}
$$

$$
= \frac{|q(t)|^{2/p} f_2(eT, t)}{(\text{card}\mathcal{W})^{1/p}} = \frac{|q(t)|^{2/p} f_1(t)}{(\text{card}\mathcal{W})^{1/p}} = g(t).
$$

Thus $\psi_p(f) = g \lambda_T$ - a.e. □

It should be noted that the maps $\psi_p$ are not (even for $p = 2$) Banach *-algebra isomorphisms. This point will be further discussed in section 4.6.

(4.4) The centres of $T(G)$, $C(G)$ and $M(G)$

In this section, I show how to extend the results of the previous section to $T(G)$, $C(G)$ and $M(G)$. Notation is as in (4.3.1) and (4.3.4).

(4.4.1) LEMMA. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Suppose $f \in Z(T(G))$. Then $\psi_\infty f \in T^w(T)$, and $\|f\|_\infty = \|\psi_\infty f\|_\infty$.

Proof. Recall that if $f \in Z(T(G)$, then $f$ is a linear combination of irreducible characters of $G$. Now, by (B8) (or, indeed, by more elementary considerations), the restriction of an irreducible character of $G$ to $T$ is an (integral) linear combination of characters of $T$. Thus $\psi_\infty(f) \in T(T)$. The other claims now follow from (4.3.5). □

(4.4.2) PROPOSITION. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. $\psi_\infty : Z(T(G) \rightarrow T^w(T)$ is a linear isometric isomorphism.

Proof. In view of (4.4.1), I need only show surjectivity.

Let $g \in T^w(T)$. Then $g = \Omega(g)$ ($\Omega$ is defined in (4.3.2)). But $\Omega(g)$ is a linear combination of elements of $T^w(T)$ of the form
Thus it suffices to show that every \( S(\chi) \) is in the image of \( \psi_\infty|_{\Sigma(T)} \),
and since, for all \( \omega \in \mathcal{W} \), \( S(\omega \cdot \chi) = S(\chi) \), it suffices to show that
\( S(\chi(\sigma)) \) is in the image of \( \psi_\infty|_{\Sigma(T)} \), for each \( \sigma \in \Sigma(G) \). (By lemma A of
B9, every character is carried onto a \( \chi(\sigma) \) by some element of \( \mathcal{W} \).)

Moreover, also by lemma A of (B9), the number of elements \( \chi \in \Sigma^+(T) \)
such that \( \chi < \chi(\sigma) \) is finite. The proposition now follows by induction
on the number of elements \( < \chi(\sigma) \).

In fact, this method proves that each \( S(\chi) \) can be written as a
finite linear combination with integer co-efficients of elements of
\( \{ \psi_\infty(\chi(\sigma)) \mid \sigma \in \Sigma(G) \} \).

(4.4.3) COROLLARY. Let \( G \) be a compact connected Lie group, \( T \) a
maximal torus for \( G \). Then \( \psi_\infty|_{\Sigma(T)} : \Sigma(T) \rightarrow C_w(T) \) is a linear
isometric isomorphism.

Proof. That the map is into \( C(T) \) is clear (since the injection 1 is
continuous). That it is into \( C_w(T) \), linear and isometric follows from
(4.3.5) (the case \( p = \infty \)). Since the map is an isometry, its range is a
closed linear subspace of \( C_w(T) \). But by (4.4.2), the range contains
\( T_w(T) \), and it is a consequence of (4.3.2) that \( T_w(T) \) is dense in \( C_w(T) \).

(4.4.4) DEFINITION. Let \( G \) be a compact connected Lie group, \( T \) a
maximal torus for \( G \). Define, for \( \nu \in \Sigma(M) \), \( \psi_\nu(\nu) \in \Sigma_w(T) \) by
\[ \psi_1(v)(f) = v(\psi_\infty^{-1} f) \text{, for all } f \in C_\omega(T). \] (Note that, by (4.3.3), \( M_\omega(T) \) is the dual space of \( C_\omega(T) \).)

Now \( C_\omega(T) \) is isometrically isomorphic to \( ZC(G) \). The dual spaces of these two spaces, \( M_\omega(T) \) and \( ZM(G) \), are hence isometrically isomorphic.

It is not hard to see that \( \psi_1 \) is an isometric isomorphism. Thus we have

**(4.4.5) Proposition.** Let \( G \) be a compact connected Lie group, \( T \) a maximal torus for \( G \). Then \( \psi_1 : ZM(G) \rightarrow M_\omega(T) \) is a linear isometric isomorphism, and \( \psi_1 \mid_{ZL^1(G)} \) co-incides with the map \( \psi_1 \) of (4.3.5).

(4.5) Formulae for Fourier transforms of central functions and measures

**(4.5.1)** In this section, I shall show how it is possible to calculate the Fourier transform of elements of \( ZL^\infty(G) \) and \( ZM(G) \) in terms of the transforms of their images under the maps \( \psi_\infty \) and \( \psi_1 \) respectively. The formulae developed in this section are essential for the treatment of central lacunary sets which appears in Chapter 5.

Let \( G \) be a compact connected Lie group, \( T \) a maximal torus for \( G \).

In this section, I will freely use notation from (B9). In particular, the reader is reminded of the characters \( \chi^{\prime}_\alpha \) (\( \alpha \in \Phi^+ \), \( \Phi \) a root system for \( \mathbb{C}L(G) \) with respect to \( \mathbb{C}L(T) \)), and \( \chi^{\prime}_w \), \( \omega \in \Omega \). My first lemma concerns the function \( q = \prod_{\alpha \in \Phi^+} (1-\chi^{\prime}_\alpha) \).

**(4.5.2) Lemma.** (i) \( q = \sum_{\omega \in \Omega} \text{sgn } \omega \cdot \bar{\chi}^{\prime}_{\omega \Omega} \).

(ii) For all \( \omega \in \Omega \), \( \omega \cdot q = \text{sgn } \omega \cdot \chi^{\prime}_{\omega \Omega} \cdot q \).

(iii) For all \( \chi \in \Sigma(T) \).
\[ \hat{q}(\chi) = \begin{cases} 
\text{sgn } w & \text{if } \chi = \chi_{\delta-w_0}^T \text{ for some } w \in \mathcal{W}, \\
0 & \text{otherwise}. 
\end{cases} \]

Proof. Statement (i) comes from (B9); (iii) is a trivial consequence of (i).

To prove (ii), let \( w_0 \in \mathcal{W} \). Then

\[
w_0 \cdot q = \sum_{w \in \mathcal{W}} \text{sgn } w \chi_{\delta-w_0}^T \delta \hat{w}_0 \delta
\]

\[
= \sum_{w \in \mathcal{W}} \text{sgn } w \chi_{\delta-w_0}^T \delta \chi_{\delta-w_0}^T \delta
\]

\[
= \text{sgn } w_0 \chi_{\delta-w_0}^T \delta \sum_{w \in \mathcal{W}} \text{sgn } w_0 \chi_{\delta-w_0}^T \delta
\]

\[
= \text{sgn } w_0 \cdot \chi_{\delta-w_0}^T \delta \cdot q. 
\]

\[ \boxed{\text{(4.5.3) THEOREM. Suppose } G \text{ is a compact connected Lie group with maximal torus } T, \text{ and let } \psi_\infty \text{ be the map defined in (4.3.4). Then for all } f \in L^\infty(G) \text{ and for all } \sigma \in \Sigma(G),}
\]

\[ \hat{f}(\sigma) = \frac{1}{d_\sigma} \left( q \cdot \psi_\infty(f) \right)^* (\chi^{(\sigma)}) \cdot I_{d_\sigma}. \]

(Notice that the symbol "\(^*\)" on the left-hand side of this equation denotes the Fourier transform of \( G \), whereas on the right-hand side it denotes the Fourier transform of \( T \).)

Proof. By (1.5.6),

\[ \hat{f}(\sigma) = \frac{1}{d_\sigma} \left( \int f \cdot \chi_G \, d\lambda_G \right) I_{d_\sigma}. \]

Now, by (4.2.6),

\[ \int f \cdot \chi_G \, d\lambda_G = \frac{1}{\text{card } \mathcal{W}} \int \psi_\infty(f) \cdot \chi_G \, |q|^2 \, d\lambda_T \]

\[ = \frac{1}{\text{card } \mathcal{W}} \int (q \cdot \psi_\infty(f))^* \cdot \chi_G \cdot |q|^2 \, d\lambda_T. \]

Hence, using the Weyl character formula (B9),

\[ \int f \cdot \chi_G \, d\lambda_G = \frac{1}{\text{card } \mathcal{W}} \sum_{w \in \mathcal{W}} \text{sgn } w \psi_\infty(f) \omega(\sigma) \cdot \chi_{\delta-w_0}^T \delta \, d\lambda_T. \]

Remark that for all \( w \in \mathcal{W} \), \( w \cdot \psi_\infty(f) = \psi_\infty(f) \) (since by (4.3.5),
Thus, by (4.5.2),
\[ w \cdot (q \cdot \psi_\infty(f)) = \chi_{\delta - \omega \delta}^t \cdot \text{sgn} \; w \cdot q \cdot (\psi_\infty f) . \]

Hence,
\[ \int f \cdot \chi_0 d\lambda_T = \frac{1}{\text{card}W} \sum_{\omega \in W} \int w(q \cdot \psi_\infty(f) \cdot \chi^{(\sigma)}) d\lambda_T \]
\[ = (q \cdot \psi_\infty(f))^{\wedge}([\chi^{(\sigma)}]) \]
(since (4.3.1) \( \lambda_T \) is invariant under the action of \( W \)). \( \square \)

(4.5.4) COROLLARY. Notation is as in (4.5.3). Let \( f \in Z^\infty\ell(G) \), \( \sigma \in \Sigma(G) \). Then
\[ \hat{f}(\sigma) = \frac{1}{\delta_\sigma} \left( \sum_{\omega \in W} \text{sgn} \; w \left( \psi_\infty(f) \right)^{\wedge} \left( \chi^{(\sigma)} \cdot \chi_{\delta - \omega \delta}^t \right) \right) \cdot I_{\delta_\sigma} . \]

Proof. By (4.5.3), \( \hat{f}(\sigma) = \frac{1}{\delta_\sigma} \left( \hat{q} \ast (\psi_\infty(f))^{\wedge} \left( \chi^{(\sigma)} \chi \right) \right) \cdot I_{\delta_\sigma} \), where "\( \ast \)" denotes convolution in \( \Sigma(T) \). Now use (4.5.2) (iii) to see that
\[ \hat{q} \ast (\psi_\infty(f))^{\wedge} \left( \chi^{(\sigma)} \chi \right) = \sum_{\chi \in \Sigma(T)} \hat{q}(\chi) (\psi_\infty(f))^{\wedge} \left( \chi^{(\sigma)} \chi \right) \]
\[ = \sum_{\omega \in W} \text{sgn} \; w \left( \psi_\infty(f) \right)^{\wedge} \left( \chi^{(\sigma)} \chi_{\delta - \omega \delta}^t \right) . \] \( \square \)

(4.5.5) I shall need the following lemma in the proof of (4.5.7).

LEMMA. Let \( G \) be a compact connected Lie group, \( T \) a maximal torus for \( G \). Let \( \chi \in \Sigma(T) \). Then either

(i) there exists \( \alpha \in \Phi \) such that \( w_\alpha(\chi) = \chi \cdot \chi_{\delta - \omega \delta}^t \) (recall the definition of \( w_\alpha \) from (A5)), or

(ii) there exists \( \sigma \in \Sigma(G) \) and \( w \in W \) such that
\[ \chi = w(\chi^{(\sigma)}) \cdot \chi_{\delta - \omega \delta}^t . \]

These two possibilities are mutually exclusive.

Proof. Let \( G_1, A, Z \) and \( \rho \) be as in (B9). Then \( \chi \circ \rho = \chi_A \cdot \chi_\mu \) where \( \chi_A \) is a character of \( A \), \( \mu \in A = \Lambda(\mathbb{L}(G_1)) \), and \( \chi_A \cdot \chi_\mu \) is trivial.
on \( Z \).

Consider \( \mu + \delta \in \Lambda \). By the lemma of (A7), there exists \( w \in \mathcal{W} \) such that \( w(\mu + \delta) \in \Lambda^+ \).

If \( w(\mu + \delta) \) is strongly dominant, then \( \lambda = w(\mu + \delta) - \delta \in \Lambda^+ \), and

\[
\chi_\lambda = w(\chi_{\mu + \delta}) \cdot \chi_\delta = w(\chi_\mu) \cdot \chi_{\delta - \omega_\delta}.
\]

Thus \( \chi_A \cdot \chi_\lambda \) is trivial on \( Z \); hence there exists \( \sigma \in \Sigma(G) \) with

\[
\chi(\sigma) \circ \rho = \chi_A \cdot \chi_\lambda = w(\chi_A \cdot \chi_\mu) \chi_{\delta - \omega_\delta} = w\chi \circ \rho \cdot \chi_{\delta - \omega_\delta} \circ \rho,
\]

and (ii) obtains.

If, on the other hand, \( w(\mu + \delta) \) is not strongly dominant, then

\[
w(\mu + \delta) = \sum_{i=1}^l n_i \lambda_i \quad (\lambda_1, \ldots, \lambda_l \text{ are the fundamental dominant weights}) \text{ with } n_j = 0 \text{ for some } j \in \{1, \ldots, l\}.
\]

Then for some \( \beta \in \Delta \),

\[
w_\beta \cdot w(\mu + \delta) = \sum_{i=1}^l n_i \lambda_i - \sum_{i=1}^l n_i \langle \lambda_i, \beta \rangle = w(\mu + \delta).
\]

Thus \( \left( w^{-1}w_\beta w\right)(\mu + \delta) = \mu + \delta \). But it can easily be proved (see Humphreys [1], lemma 9.2) that \( w^{-1}w_\beta w = w_\omega(\beta) \). Let \( \alpha = w(\beta) \in \Phi \). Then

\[
w_\alpha(\chi_\mu \chi_\delta) = \chi_\mu \chi_\delta \text{, so } w_\alpha \chi_\mu = \chi_\mu \chi_{\delta - \omega_\delta} \text{. It follows that } w_\alpha(\chi) = \chi \cdot \chi_{\delta - \omega_\delta} \text{,}
\]

and so in this case (i) holds. \( \square \)

(4.5.6) NOTATION. As usual, \( G \) is a compact connected Lie group, \( T \) a maximal torus for \( G \). For the purposes of the next lemma, I shall denote \( \Sigma(T) \) by "\( T \)" and \( \Sigma^+(T, G) \) by \( X^+ \) (this is in line with the notation of Hewitt and Ross [1]). If \( R \) is a subset of \( \Sigma(G) \), let

\[
X^+(R) = \{ \chi(\sigma) \mid \sigma \in R \}.
\]

Then \( X^+(R) \subseteq X^+ \). Further, let

\[
X(R) = \bigcup_{w \in \mathcal{W}} \{ w \cdot \chi^+(R) \cdot \chi_{\delta - \omega_\delta} \}.
\]
**(4.5.7) Proposition.** Notation is conserved from (4.5.6). Let \( f \in \mathbb{ZL}(G) \), and \( R \subseteq \Sigma(G) \). The following propositions are equivalent:

(i) for every \( \sigma \notin R \), \( \hat{f}(\sigma) = 0 \);

(ii) for every \( \chi \in \mathbb{X}^+ \setminus \mathbb{X}^+(R) \), \( (q \cdot \psi_{\omega_\sigma})^* (\chi) = 0 \);

(iii) for every \( \chi \notin \mathbb{X}(R) \), \( (q \cdot \psi_{\omega_\sigma})^* (\chi) = 0 \);

(iv) for every \( \chi \notin \mathbb{X}(R) \), \( \sum_{w \in \mathbb{W}} \text{sgn} (q \cdot \psi_{\omega_\sigma})^* (\chi \cdot \overline{X_{\omega_\delta-w_\delta}}) = 0 \).

**Proof.** The equivalence of (i) and (ii) results from (4.5.3), and the equivalence of (iii) and (iv) from (4.5.2).

(iii) \( \Rightarrow \) (ii). It suffices to show that for \( \sigma \in \Sigma(G) \),

\[ \chi(\sigma) \in \mathbb{X}(R) \Rightarrow \chi(\sigma) \in \mathbb{X}^+(R) . \]

Suppose thus that \( \eta \in \mathbb{X}^+(R) \) and

\[ \chi(\sigma) = \omega(\chi(\eta)) \cdot \overline{X_{\omega_\delta-w_\delta}} . \tag{1} \]

Let \( A, G_1, Z \) and \( \rho \) be as in (B9), and write \( \chi(\sigma) \circ \rho = \chi_A \cdot \chi_{\lambda_{\eta_1}} \),

\[ \chi(\eta) \circ \rho = \chi_{A'} \cdot \chi_{\lambda_{\eta_1}} , \]

where \( \chi_A, \chi_{A'} \) are characters of \( A \) and

\[ \lambda_{\sigma_1}, \lambda_{\eta_1} \in \mathbb{A}^+ \mathbb{L}(G_1) \]. Then (1) assures us that \( \chi_A = \chi_{A'} \), and

\[ \chi_{\lambda_{\sigma_1}} = \chi_w(\lambda_{\eta_1} + \delta) - \delta . \]

Thus \( \lambda_{\sigma_1} + \delta = \omega(\lambda_{\eta_1} + \delta) \). It follows by the lemma of (A7) that \( w = 1 \) and \( \sigma_1 = \eta_1 \). Thus \( \eta = \sigma \).

(ii) \( \Rightarrow \) (iii). Suppose (ii) is true, and let \( \chi \notin \mathbb{X}(R) \). By (4.5.5),

either \( \chi = \omega(\chi(\sigma)) \cdot \overline{X_{\omega_\delta-w_\delta}} \) (for some \( w \in \mathbb{W} \) and \( \sigma \in \Sigma(G) \)), or for some \( \alpha \in \Phi \), \( \omega(\chi) = \chi \cdot \overline{X_{\omega_\delta-w_\delta}} \). In the former case, \( \chi(\sigma) \notin \mathbb{X}^+(R) \) (for otherwise \( \chi \in \mathbb{X}(R) \)), and hence

\[ (q \cdot \psi_{\omega_\sigma})^* (\chi) = (q \cdot \psi_{\omega_\sigma})^* (w(\chi(\sigma)) \cdot \overline{X_{\omega_\delta-w_\delta}}) = ((\chi_{\omega_\delta-w_\delta} \cdot q) \cdot \psi_{\omega_\sigma})^* (\omega(\chi(\sigma))). \]
But (4.5.2) shows that $w . q = \text{sgn} \, w \cdot x_{\delta - \omega \delta} \cdot q$ and (4.3.5) that $w(\psi_{\alpha}f) = \psi_{\alpha}f$. Thus, invariance of $\lambda_{T}$ under the action of $w$ proves that

$$(q \cdot \psi_{\alpha}f)^{\wedge}(\chi) = \text{sgn} \, w(q \cdot \psi_{\alpha}f)^{\wedge}(\chi^{(\alpha)}) = 0.$$ 

In the latter case, by (4.5.3),

$$(q \cdot \psi_{\alpha}f)^{\wedge}(\chi) = \sum_{w \in \mathcal{W}} \text{sgn} \, w(\psi_{\alpha}f)^{\wedge}(x \cdot x_{\delta - \omega \delta}^{\alpha}).$$ 

Notice that, since $w_{\alpha}$ is a reflection, $\text{sgn} \, w_{\alpha} = -1$. Thus

$$(q \cdot \psi_{\alpha}f)^{\wedge}(\chi) = \sum_{\{w \in \mathcal{W} \mid \text{sgn} \, w = 1\}} \{(\psi_{\alpha}f)^{\wedge}(x \cdot x_{\delta - \omega \delta}^{\alpha}) - (\psi_{\alpha}f)^{\wedge}(x \cdot x_{\delta - \omega \delta}^{\alpha})\}.$$ 

But

$$x \cdot x_{\delta - \omega \delta}^{\alpha} W_{\alpha} = x \cdot x_{\delta - \omega \delta}^{\alpha} \delta_{\alpha}(x_{\delta - \omega \delta}^{\alpha}) = w_{\alpha}(x \cdot x_{\delta - \omega \delta}^{\alpha}).$$

Thus, since $(\psi_{\alpha}f)(w_{\alpha}x) = \psi_{\alpha}f(x)$, $(q \cdot \psi_{\alpha}f)^{\wedge}(\chi) = 0$. \hfill \Box

(4.5.8) Recall from (B9), lemma B, the integers $n_{\chi}(x_{1})$, $\chi \in \Sigma^{+}(T)$, $x_{1} \in \Sigma(T)$.

**PROPOSITION.** Let $\nu \in \mathcal{ZM}(G)$, $\sigma \in \Sigma(G)$. Then

$$\hat{\nu}(\sigma) = \frac{1}{d_{\sigma}} \left( \sum_{\chi \in \Sigma(T)} n_{\chi}(\sigma) \cdot (\psi_{\nu}^{(\sigma)})^{\wedge}(\chi) \right)_{d_{\sigma}}.$$ 

$$= \frac{1}{d_{\sigma}} \left( \sum_{\chi \in \Sigma^{+}(T)} n_{\chi}(\sigma) \cdot \text{card}(w, \chi) \cdot (\psi_{\nu}^{(\sigma)})^{\wedge}(\chi) \right)_{d_{\sigma}}.$$ 

**Proof.** By (4.2.6), $\hat{\nu}(\sigma) = \frac{1}{d_{\sigma}} \nu(\overline{\chi_{\sigma}})_{d_{\sigma}}$. But

$$\nu(\overline{\chi_{\sigma}}) = \nu\left(\psi_{\sigma}^{-1} \psi_{\nu} \chi_{\sigma}\right) = \left(\psi_{\nu}^{(\sigma)} \chi_{\sigma}\right)$$

$$= \sum_{\chi \in \Sigma(T)} n_{\chi}(\sigma) \cdot \left(\psi_{\nu}^{(\sigma)}\right)^{\wedge}(\chi).$$ 

(by lemma B of (B9)). To see the equality of the two expressions, recall that $\psi_{\nu} \in M_{w}(T)$. Thus for any $w \in \mathcal{W}$, $(\psi_{\nu}^{(\sigma)})^{\wedge}(\chi) = (\psi_{\nu}^{(\sigma)})^{\wedge}(w \chi)$. \hfill \Box
(4.6) A convolution formula

(4.6.1) The isometric isomorphisms $\psi_p (1 \leq p \leq \infty)$ of (4.3.4) and (4.4.4) are not, unfortunately, Banach *-algebra isomorphisms, since they fail to conserve convolutions. In this section, I shall derive a formula which shows how to calculate the convolution of two elements of $ZL^\infty(G)$ ($G$ a compact connected Lie group) in terms of their images under the map $\psi_\infty$ of (4.3.4).

My first result, however, concerns the involution * which was introduced in (1.2.6).

**LEMMA.** Let $G$ be a compact connected Lie group, and let $p \in [1, \infty]$. Then for $f \in ZL^p(G)$, $\psi_p(f^*) = (\psi_p(f))^*$. For $\nu \in ZM(G)$,

$$\psi_1(\nu^*) = (\psi_1(\nu))^*.$$ 

(4.6.2) **NOTATION.** Define $d : \Sigma(T) \to \mathbb{R}^+ \cup \{0\}$ as follows: for $\chi \in \Sigma(T)$,

$$d(\chi) = \begin{cases} 
\text{sgn } w \cdot d_\sigma \text{ if } \sigma \in \Sigma(G) \text{ and } w \in \mathcal{W} \text{ are such that} \\
\chi = w \cdot \chi_{\sigma}^{\mathcal{T}} \cdot \chi_{\alpha}^{-}^{\mathcal{W}} \\
0 \text{ if there exists } \alpha \in \Phi \text{ such that } w_\alpha \chi = \chi \cdot \chi_{\alpha}^{\mathcal{T}} \cdot \chi_{\alpha}^{-}^{\mathcal{W}}.
\end{cases}$$

By (4.5.5), this defines $d(\chi)$ for all $\chi \in \Sigma(T)$.

Define $\hat{\nu} \in \mathbb{E}^\infty_{\Sigma}(\Sigma(T))$ as follows: for $\chi \in \Sigma(T)$,

$$\hat{\nu}(\chi) = \begin{cases} 
\frac{1}{d(\chi)} \text{ if } d(\chi) \neq 0, \\
0 \text{ if } d(\chi) = 0.
\end{cases}$$

The reader is reminded of the definition of $A(T)$ (Hewitt and Ross [1], (34.4)); $A(T) = \{ f \in C(T) \mid \hat{f} \in \mathbb{E}^1(\Sigma(T)) \}$. By (34.7) of Hewitt and Ross [1], $(A(T))^\perp = \mathbb{E}^1_{\Sigma}(\Sigma(T))$. 


For $f \in A(T)$, one defines $\nabla * f \in A(T)$ by $(\nabla * f)^\wedge(\chi) = \hat{\nabla}(\chi) \cdot \hat{f}(\chi)$.

(I am here thinking of $\nabla$ as a pseudomeasure (see Hewitt and Ross [1], (34.46)).)

(4.6.3) PROPOSITION. Let $G$ be a compact connected Lie group. Let $f, g \in L^\infty(G)$. Then

$$f * g = \psi_{\infty}^{-1} \left( \frac{1}{q} \cdot \nabla * (q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g) \right).$$

(REMARK. I should note here that, since $|q \cdot \psi_{\infty} f| = |\psi_{2} f|$, both $q \cdot \psi_{\infty} f$ and $q \cdot \psi_{\infty} g$ are elements of $L^2(T)$. It follows that $(q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g) \in A(T)$ (see Hewitt and Ross [1], (34.16)).)

Proof. Suppose first that $\sigma \in \Sigma(G)$. Then by (4.5.2),

$$(q \cdot \psi_{\infty}(f * g))^\wedge(\chi(\sigma))_{L_\sigma} = d_{\sigma}(f * g)^\wedge(\sigma)$$

$$= d_{\sigma}(\hat{f}(\sigma) \hat{g}(\sigma) = \frac{1}{d_{\sigma}} (q \cdot \psi_{\infty} f)^\wedge(\chi(\sigma))(q \cdot \psi_{\infty} g)^\wedge(\chi(\sigma)).$$

Now suppose $\chi \in \Sigma(T)$. Then if $\chi$ satisfies (4.5.5) (i), the argument of (4.5.6) shows that

$$(q \cdot \psi_{\infty}(f * g))^\wedge(\chi) = 0 = (q \cdot \psi_{\infty} f)^\wedge(\chi) = (q \cdot \psi_{\infty} g)^\wedge(\chi) .$$

Hence, in this case,

$$(q \cdot \psi_{\infty}(f * g))^\wedge(\chi) = (\nabla * (q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g))^\wedge(\chi) .$$

If, on the other hand, $\chi$ satisfies (4.5.5) (ii), for $\sigma \in \Sigma(G)$ and $w \in W$, then for any $h \in ZL^\infty(G)$, the argument of (4.5.6) shows that

$$(q \cdot \psi_{\infty} h)^\wedge(\chi) = \text{sgn } w(q \cdot \psi_{\infty} h)^\wedge(\chi(\sigma)).$$

It follows that

$$(q \cdot \psi_{\infty}(f * g))^\wedge(\chi) = \hat{\nabla}(\chi) \cdot (q \cdot \psi_{\infty} f)^\wedge(\chi)(q \cdot \psi_{\infty} g)^\wedge(\chi)$$

$$= (\nabla * (q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g))^\wedge(\chi) .$$

Thus $q \cdot \psi_{\infty}(f * g) = \nabla * (q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g)$.

Since $q$ is non-zero almost everywhere, this implies that

$$\psi_{\infty}(f * g) = \frac{1}{q} \left( \nabla * (q \cdot \psi_{\infty} f) * (q \cdot \psi_{\infty} g) \right) \in L^\infty(T),$$

and this gives the formula. \qed
(4.6.4) PROPOSITION. Suppose $G$ is a compact connected semi-simple Lie group. Let $l$ denote the rank of $G$ and $\Phi^+$ the set of positive roots of $\mathfrak{g}(G)$. Then for $p > (\text{card } \Phi^+) - l + 1$,

$$\hat{\nu} \in \mathcal{D}^P(\Sigma(T)) = \mathcal{D}^P(\Sigma(T)).$$

Proof. $\|\hat{\nu}\|_P^p \leq \text{card } \mathcal{W} \sum_{n=1}^{\infty} \frac{1}{n^p} \times \text{card}\{\sigma \mid d_{\sigma} \leq n\}$. By a recent result of Cahn and Rothaus (see Rothaus [1], Theorem 1) $\text{card}\{\sigma \mid d_{\sigma} \leq n\}$ approaches asymptotically $n^{\text{card } \Phi^+ - l}$. The result is now obvious. \[\Box\]

(4.6.5) I now give a convolution formula for central measures. This can be deduced from (4.5.8).

PROPOSITION. Let $\mu, \nu \in ZM(G)$, and suppose $\sigma \in \Sigma(G)$. Then

$$\sum_{\chi \in \Sigma(T)} \frac{n(\sigma)(\chi)}{n(\sigma_0)(\chi_0)} \hat{\psi}_1(\mu \ast \nu) \chi(\chi) = \sum_{\chi \in \Sigma(T)} \frac{1}{n(\sigma_0)(\chi_0)} \sum_{\chi \in \Sigma(T)} \frac{n(\sigma)(\chi)}{n(\sigma_0)(\chi_0)} \hat{\psi}_1(\mu) \chi(\chi) \cdot \hat{\psi}_1(\nu) \chi(\chi_0).$$

(4.7) An example

(4.7.1) In this section, I shall work through in some detail the examples $G = SU(2)$ and $G = U(2)$. (These two groups are closely related.) I hope that this will provide some motivation for the theory of the preceding sections. It should be noted that $ZM(SU(2))$ has been discussed by Mayer [1], [2], [3], [4], and also by Coifman and Weiss [1]. The reader may be familiar with the discussion of the representation theory of $U(2)$ given in (29.48) of Hewitt and Ross [1].

(4.7.2) I shall first discuss $SU(2)$. The reader will recall that $SU(2)$ is the group of $2 \times 2$ complex unitary matrices of determinant +1. A maximal torus for $SU(2)$ is $T$, embedded in $SU(2)$ by the mapping

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}.$$
The complexified Lie algebra \( \mathfrak{g} \mathfrak{l}(\text{SU}(2)) \) is precisely \( \mathfrak{sl}(2, \mathbb{C}) \), which is of type \( A_1 \). (See (All) for further details of \( \mathfrak{sl}(2, \mathbb{C}) \).) Now \( \mathfrak{sl}(2, \mathbb{C}) \) consists of the \( 2 \times 2 \) complex matrices of trace zero, and the above embedding of \( \Pi \) in \( \text{SU}(2) \) lifts to the map

\[
\mathbb{R} \to \mathfrak{g} \mathfrak{l}(\text{SU}(2)) : r \mapsto \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}.
\]

Thus a Cartan subalgebra for \( \mathfrak{sl}(2, \mathbb{C}) \) is

\[
H = \left\{ \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \mid c \in \mathbb{C} \right\}.
\]

There is only one positive root, \( \alpha \) (of \( \mathfrak{sl}(2, \mathbb{C}) \) with respect to \( H \)), i.e. \( \Delta = \Phi^+ = \{ \alpha \} \). The action of \( \alpha \) on \( H \) is \( \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \mapsto 2c \). Now \( \chi_\alpha \) is a character of \( \Pi \), and it follows that \( \chi_\alpha(z) = z^2 \) (\( z \in \Pi \)). We note that \( \chi_\alpha \) is trivial on \( \{1, -1\} \) which is precisely the centre of \( \text{SU}(2) \).

It is now clear that \( q(z) = 1 - z^2 \), \( z \in \Pi \). (Hence \( |q(e^{i\pi})| = 4 |\sin \frac{\pi}{2}| \).

Note that the Weyl group, isomorphic to \( S_2 = \{1, w\} \) has as action on \( \Pi \), \( l : z \mapsto z \) and \( w : z \mapsto \bar{z} \). Thus its action on characters of \( \Pi \) is given by \( w(\chi) = \bar{\chi} \).

There is just one fundamental dominant weight of \( \mathfrak{sl}(2, \mathbb{C}) \) - denote it by \( \lambda \). Then one has, for \( z \in \Pi \), \( \chi_\lambda(z) = \chi_\delta(z) = z \) (and for \( \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \in H, \lambda\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} = c \)). It follows that \( \mathfrak{z}^+(\Pi, \text{SU}(2)) = \left\{ \chi_\lambda^n \mid n \in \mathbb{N} \right\} \).

Note that \( w.\chi_\lambda^n = \bar{\chi}_\lambda^n \), and that \( \chi_\lambda^n > \chi_\delta^n \) if and only if \( n - m \) is an even integer.

The integers \( n_{\lambda}(\chi) (n \in \mathbb{N}) \) are given by \( \chi_\lambda^n \).
\( n_{\lambda}^{H}(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_{\lambda}^{m} (m \in \mathbb{Z}) \text{ with } n - |m| \in 2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \)

Further \( d\left(\chi_{\lambda}^{n}\right) = n + 1 \).

The results of this chapter now become:

(i) \( \mathcal{ZT}(SU(2)) \) is isometrically isomorphic to

\[ T_{w}(\Pi) = \{ f \in T(\Pi) \mid \text{for all } z \in \Pi, f(z) = f(w) \} . \]

Similarly for \( \mathcal{ZC}(SU(2)) \).

(ii) The maps \( \psi_{p} : \mathcal{ZL}(SU(2)) \to T_{w}(\Pi) \) are isometric isomorphisms.

(iii) For any \( f \in \mathcal{ZL}(SU(2)) \), and for \( \sigma_{n} \in \Sigma(SU(2)) \) of dimension \( n + 1 \),

\[ \hat{f}(\sigma_{n}) = (q \cdot \psi_{n})^\wedge(n) \]

\[ = (\psi_{n}^\wedge)^{(n-1)} - (\psi_{n}^\wedge)^{(n+1)} . \]

(iv) If \( R = \{ \sigma_{n} \}_{n_{K} \in \mathbb{N}} \subseteq \Sigma(SU(2)) \), then

\[ X^{+}(R) = \{ n_{K} \mid n \in \mathbb{N} \} \subseteq \mathbb{Z} = \Sigma(\Pi) , \]

and

\[ \chi(R) = \{ \pm n_{K} \mid k \in \mathbb{N} \} . \]

The function \( f \in \mathcal{ZL}(SU(2)) \) is \( R \) spectral if and only if \( (q \cdot \psi_{n}) \)

is \( \chi(R) \) spectral.

(v) \( \mathcal{ZM}(SU(2)) \cong M_{w}(\Pi) \), and for \( \nu \in \mathcal{ZM}(SU(2)) \),

\[ \hat{\nu}(\sigma_{n}) = \frac{1}{n+1} \left[ \sum_{m=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (\psi_{m})^\wedge \left( x_{\lambda}^{2(m+(n/2) - \lfloor n/2 \rfloor)} \right) \right] . \]

(4.7.3) I shall now carry out the analogous discussion for the group \( U(2) \).

Let \( Z \) denote the normal subgroup \( \{(1, e), (-1, -e)\} \) of \( \Pi \times SU(2) \)

(where \( e \) denotes the identity of \( SU(2) \)). Then the decomposition of \( B_{2} \) is:
Let \( \rho : \Pi \times SU(2) \to U(2) \) be the quotient map. The discussion of (B5) shows that a maximal torus for \( U(2) \) (i.e. \( (\Pi \times SU(2))/Z \)) is \( T = (\Pi \times \Pi)/Z \). Let \( \eta : \Pi \times \Pi \to \Pi \times \Pi : (s_1, s_2) \mapsto (s_1s_2, s_2) \), we see that \( \ker \eta = Z \), so that the quotient map \( \eta \) identifies \( T \) with \( \Pi \times \Pi \).

I shall henceforth consider the maximal torus of \( U(2) \) to be \( \Pi \times \Pi \), embedded in \( U(2) \) by \( \eta^{-1} \).

By the discussion of (4.7.2) and (B5), we can see that \( \chi'_{\alpha} \), which is a character of \( \Pi \times \Pi \) is \( (s_1, s_2) \mapsto s_2 \). (To see this, notice that \( (\chi'_{\alpha} \circ \rho)(s_1, s_2) = s_2^2 = \chi_{\alpha}(s_2) \).) It follows that \( q(s_1, s_2) = 1 - s_2 \).

Note that the action of \( \omega = \{1, \omega\} \) on \( \Pi \times \Pi \) is given by \( 1 : (s_1, s_2) \mapsto (s_1, s_2) \) and \( \omega : (s_1, s_2) \mapsto (s_1\bar{s_2}, \bar{s_2}) \).

Denote by \( \chi(k_1, k_2) \) (\( k_1, k_2 \in \mathbb{Z} \)) the character \( (s_1, s_2) \mapsto k_1 s_1 s_2 k_2 \) of \( \Pi \times \Pi \). We have \( \omega(\chi(k_1, k_2)) = \chi(1, -(k_1 + k_2)) \). Furthermore, \( \chi'_{\alpha} = \chi_{\delta' \delta} = \chi(0, 1) \). Finally, \( \chi(k_1, k_2) \geq \chi(m_1, m_2) \) if and only if \( m_1 = k_1 \) and \( k_2 \geq m_2 \).

Which characters \( \chi(k_1, k_2) \) are elements of \( \Sigma^+(\Pi \times \Pi, U(2)) \)? To answer this, notice that

\[
(\chi(k_1, k_2) \circ \eta_1) \in \mathfrak{sl}(SU(2))^* = \{ (2k_1 + k_2) \lambda \},
\]

where \( \lambda \) is as defined in (4.7.2) (cf. also (B5)). Hence,

\[
\Sigma^+(\Pi \times \Pi, U(2)) = \{ \chi(k_1, k_2) \mid k_1 + 2k_2 \in \mathbb{N} \}.
\]

Denoting by \( \sigma(k_1, k_2) \) the representation of \( U(2) \) corresponding to \( \chi(k_1, k_2) \in \Sigma^+(\Pi \times \Pi, U(2)) \), one has, for \( (z, u) \in \Pi \times SU(2) \),
\[(\sigma(k_1,k_2) \circ \rho)(z, U) = z^k \cdot \sigma(k_1+2k_2)(U).\]

Hence, in particular \(d(\sigma(k_1,k_2)) = k_1 + 2k_2 + 1\). It can now be seen that, if \(\chi(k_1,k_2)\) and \(\chi(m_1,m_2)\) are elements of \(\Sigma^+(\Pi \times \Pi)\),

\[\begin{cases} 1 & \text{if } k_1 = m_1 \text{ and } k_2 \geq m_2, \\ 0 & \text{otherwise.} \end{cases}\]

(i), (ii). We now have analogues of (4.7.2) (i) and (ii); notice that \(T_w(\Pi \times \Pi) = \{f \in T(\Pi \times \Pi) \mid f(z_1, z_2) = f(z_1 \overline{z_2}, \overline{z_2})\} \).

(iii) The analogue of (4.7.2) (iii) is, for \(f \in \mathcal{ZL}^\infty(U(2))\),

\[\hat{f}(\sigma(k_1,k_2)) = (q \cdot \psi_{\infty} f) \chi(k_1,k_2) \]
\[= (\psi_{\infty} f) \chi(k_1,k_2) - (\psi_{\infty} f) \chi(k_1,k_2+1)\] .

(iv) If \(R = \{\sigma(n_j,m_j) \mid (n_j, m_j) \in \mathbb{Z} \times \mathbb{Z}\} \subset \Pi \times \Pi\) and \(2m_j + n_j \in \mathbb{N}\) \(\subset \Sigma(U(2))\), then

\[X^+(R) = \{\chi(n_j,m_j) \mid j \in \mathbb{N}\}\]

and

\[X(R) = \{\chi(n_j,m_j) \mid j \in \mathbb{N}\} \cup \{\chi(n_j,-(n_j+m_j+1)) \mid j \in \mathbb{N}\}\] .

Again, \(f \in \mathcal{ZL}^\infty(U(2))\) is \(R\) spectral if and only if \(q \cdot \psi_{\infty} f\) is \(X(R)\) spectral.

(v) Finally, by (4.4.5), \(\mathcal{ZM}(U(2))\) is isometrically isomorphic to \(M_w(\Pi \times \Pi)\), and one has, for \(v \in \mathcal{ZM}(U(2))\),
(4.8) A counter example

(4.8.1) In this section, I examine the possibility of extending the Weyl integration formula to some more general class of compact connected groups than compact connected Lie groups. The results of this chapter show what a powerful tool this formula is, so this question is an interesting one. I am indebted to Professor Edwards for posing it.

(4.8.2) More specifically, suppose \( G \) is a compact connected group, \( T \) a connected abelian subgroup of \( G \). Can one write, for \( f \in C(G) \),

\[
\int f \, d\lambda_G = \int_{G/T \times T} f(\varphi(gT, t)) \cdot h(gT, t) \, d\lambda_{G/T \times \lambda_T},
\]

where \( \varphi : (G/T) \times T \to G : (gT, t) \mapsto (gtg^{-1}) \), and

\( h \in L^1((G/T) \times T, \lambda_{G/T \times \lambda_T}) \).

It is not hard to show that, if such a formula as (1) holds for a connected abelian subgroup \( T \) of \( G \), a similar formula holds for a maximal connected abelian subgroup containing \( T \). Thus, I may assume \( T \) is a maximal connected abelian subgroup of \( G \).

By the structure theorem for compact connected groups (1.2.5), \( G \) is a quotient of a group of the form \( A \times \prod_{i \in I} G_i \) (where \( A \) is a compact abelian group, and for \( i \in I \), \( G_i \) is a compact connected simple Lie group), by a totally disconnected subgroup \( Z \) of its centre. A maximal abelian subgroup of \( G \) is then the quotient by \( Z \) of \( A \times \prod_{i \in I} T_i \), where, for each \( i \), \( T_i \) is a maximal torus of \( G_i \). If a formula such as (1) holds for such a group,
it holds for \( G = A \times \prod_{i \in I} G_i \), \( T = A \times \prod_{i \in I} T_i \).

To ask whether (1) holds is equivalent to asking whether \( \lambda_G \) is absolutely continuous with respect to the measure \( (\lambda_G/T \times \lambda_T) \circ \varphi^{-1} \). The following example shows that this is not so for the very simple case where \( I = \mathbb{N} \) and \( G_i = \text{SU}(2) \) for all \( i \in \mathbb{N} \). It is not hard to extend this to cover the case where \( \{G_i \mid i \in I\} \) is an arbitrary infinite family of simple simply connected Lie groups, and hence to conclude that \( I \) must be a finite set, and \( G \) a connected Lie group.

Such an extension would, however, obscure the argument by introducing technical difficulties, so I shall not include it here.

(4.8.3) Let \( G = \prod_{n \in \mathbb{N}} \text{SU}(2) \), \( T = \prod_{n \in \mathbb{N}} T \). I will show that \( \lambda_G \) is not absolutely continuous with respect to \( \lambda_{G/T} \times \lambda_T \circ \varphi^{-1} \), by giving an example of a set \( E \subseteq G \) for which \( (\lambda_{G/T} \times \lambda_T)(\varphi^{-1}(E)) = 0 \) but \( \lambda_G(E) > 0 \).

It is known (see Hewitt and Ross [1], §13) that \( \lambda_G = \prod_{n \in \mathbb{N}} \lambda_{\text{SU}(2)} \), and it is not hard to see that \( \lambda_{G/T} \times \lambda_T = \prod_{n \in \mathbb{N}} (\lambda_{\text{SU}(2)/\Pi} \times \lambda_\Pi) \).

I shall identify \( \Pi \) with \([-\pi, \pi] \). Let \( \alpha_i \to 0 \) be some sequence, \( 1 > \alpha > 0 \), which will be chosen later. Let \( \Pi \supseteq F_i = [-\pi, -\pi \alpha_i] \cup [\pi \alpha_i, \pi] \), and let

\[
E = \prod_{n \in \mathbb{N}} E_n = \prod_{n \in \mathbb{N}} \varphi((\text{SU}(2)/\Pi) \times F_n) \subseteq G.
\]

Then \( E \) is a measurable set. Now

\[
(\lambda_{G/T} \times \lambda_T)(\varphi^{-1}(E)) = \prod_{n \in \mathbb{N}} (\lambda_{\text{SU}(2)/\Pi} \times \lambda_\Pi) \varphi^{-1}(E_n)
= \prod_{n \in \mathbb{N}} (\lambda_{\text{SU}(2)/\Pi} \times \lambda_\Pi)(\varphi((\text{SU}(2)/\Pi) \times F_n))
= \prod_{n \in \mathbb{N}} \lambda_\Pi(F_n) = \prod_{n \in \mathbb{N}} (1-\alpha_n).
\]

Also, letting \( \psi_E \) denote the characteristic function of the set \( E \), one
has

\[ \lambda_G(E) = \prod_{n \in \mathbb{N}} \lambda_{SU(2)}(E^n) \]

\[ = \prod_{n \in \mathbb{N}} \int_{SU(2)/\Pi} \psi_{SU(2)/\Pi}(g\Pi)\psi_{F_t} \left( \frac{|q(t)|^2}{2} \right) \frac{d\lambda_{SU(2)/\Pi}(g\Pi)}{\Pi(t)} \]

\[ = \prod_{n \in \mathbb{N}} \frac{1}{2} \int_{F_t} |q(t)|^2 \frac{d\lambda_{\Pi}(t)}{\Pi câr} = \prod_{n \in \mathbb{N}} \frac{2}{2\pi} \int_{\pi n} (1 - \cos t) dt \]

\[ = \prod_{n \in \mathbb{N}} \left( 1 - \frac{n}{n+1} \sin \pi n \right) \]

Now choose \( a_n = \frac{1}{n+1}, \quad n = 1, 2, 3, \ldots \). The above formulae yield:

\[ \left( \lambda_{G/T} \times \lambda_{\Pi} \right)(\phi^{-1}E) = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{n+1} \right) = \lim_{n \to \infty} \frac{1}{n+1} = 0, \]

but

\[ \lambda(E) = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{n!} \left( \frac{\pi n}{\pi n} - \sin \pi n \right) \right) \]

\[ = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{n!} \left( \frac{\pi n}{\pi n} \right)^3 \right) = \prod_{n=1}^{\infty} \left( 1 - \frac{\pi^2}{6(n+1)^3} \right) \]

\[ = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{(n+1)^3} \right) = \frac{1}{2}. \]
CHAPTER 5

CENTRAL LACUNARY SETS

(5.1) Introduction

In this chapter, I shall return to the sets of type central \( \Lambda(p, q) \) and central \( V(p, q) \) introduced in (2.4.1), specializing here to the case where \( G \) is a compact connected Lie group. The results of §3.6 give some information on these sets, but it turns out that, using the ideas developed in chapter 4, considerably more information can be obtained. The general strategem of this chapter is to reduce questions about the lacunarity of a set \( R \subseteq \Sigma(G) \) (\( G \) a compact connected Lie group) to questions which only involve some sort of lacunary condition (to be specified) on the set \( \hat{X}^+(R) \subseteq \Sigma(T) \) (see (4.5.6)), where \( T \) is a maximal torus for \( G \).

The chapter is organized as follows. Section (5.2) introduces the set of antisymmetric trigonometric polynomials and gives a lemma which will be useful in the sequel. In §5.3, I shall recall from chapters 2 and 3 the known facts about sets of type central \( \Lambda(p, r) \), and generally provide background for the discussion of these sets which appear in §5.4. Perhaps the most important result of §5.4 is that, if \( G \) is a compact connected Lie group, every infinite subset of \( \Sigma(G) \) contains an infinite set of type central \( \Lambda(2 + \epsilon_G) \) (where \( \epsilon_G \) is the number introduced in (3.5.5)). This result was proved by Rider [3] for the special case where \( G = U(n) \). The following section, §5.5, moves on to a discussion of the background to sets of type central \( V(p, r) \). Here there are two main areas of interest; those for which \( 2 < p' < p < \infty \) (which include sets of type central \( \Lambda(2) \)); and central \( p \)-Sidon sets \( (1 \leq p < 2) \). The former are discussed in §5.6. It is shown, in particular, that \( R \subseteq \Sigma(G) \) is of type central \( \Lambda(2) \) if and only if \( \hat{X}^+(R) \) is of type \( \Lambda(2) \). Using this, I am able to...
show the existence of infinite sets which are of type central $V(p, r)$ for every $p, r$ satisfying $2 < r' < p < \infty$, but which are not of type central $\Lambda(2)$ (the reader will recall that a set is of type central $\Lambda(2)$ if and only if it is of type central $V(p, 2)$ for some $p > 2$). Finally, §5.7 is devoted to a discussion of central $p$-Sidon sets. The results are disappointingly fragmentary. Nevertheless, I am able to give an independent proof of the result of Ragozin-Rider that if $G$ is a compact connected semi-simple Lie group, $\Sigma(G)$ contains no infinite central Sidon sets, and I show that $\Sigma(SU(2))$ contains no infinite central $p$-Sidon sets for any $p \in [1, 2]$. For certain other simply connected groups $G$ (those of low rank), I can show that $\Sigma(G)$ contains no infinite $p$-Sidon sets for $p \in [1, K_G]$, where $K_G \in [1, 2]$ is a constant depending on the group.

(5.2) The set of antisymmetric trigonometric polynomials: a lemma

In this section, I introduce the set of antisymmetric trigonometric polynomials and prove a lemma which will be useful in later sections.

(5.2.1) NOTATION. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Recall from (B9) the set $\Sigma^+(T) \subseteq \Sigma(T)$. Let $\Phi^+$ denote the set of positive roots of $\mathfrak{u}L(G)$ with respect to the CSA $\mathfrak{u}L(T)$. Recall (A5) that the Weyl group of $(G, T)$ is generated by the reflections $\{w_\alpha \mid \alpha \in \Phi^+\}$.

Let $\chi \in \Sigma^+(T)$ and $\chi_1 \in \Sigma(T)$. The reader will recall from (B9) the nonnegative integer $n_\chi(\chi_1)$.

(5.2.2) DEFINITION. In this paragraph, I shall use (4.5.5) to extend the definition of the integers $n_\chi(\chi_1)$ to the case where $\chi \in \Sigma(T)$. This extension will enable me to state (5.2.4) in a rather simpler form.

Define, for $\chi, \chi_1 \in \Sigma(T)$,
\[
\begin{cases}
\text{sgn } w \cdot n_{\chi}(\sigma)(x_1) & \text{if } \sigma \in \Sigma(G) \text{ and } w \in \mathcal{W} \\
0 & \text{if there exists } \alpha \in \Phi \text{ such that } \omega_\alpha \chi = \chi \cdot \chi_{\delta - w \delta}.
\end{cases}
\]

are such that \( \chi = \omega \chi(\sigma) \cdot \chi_{\delta - w \delta} \).

By (4.5.5), this defines the integers \( n_{\chi}(x_1) \) for all \( \chi, x_1 \in \Sigma(T) \).

Let \( d \) be the function defined in (4.6.2); notice that for all \( \chi \in \Sigma(T) \),
\[
d(\chi) = \sum_{x_1 \in A(T)} n_{\chi}(x_1).
\]

(5.2.3) DEFINITION. Notation is taken from (5.2.1). Recall from (B9) the characters \( \chi'_{\delta - w \delta} \) \((w \in \mathcal{W})\). A function \( f : T \to \mathbb{C} \) is said to be antisymmetric if for all \( w \in \mathcal{W} \),
\[
wf = \text{sgn } w \cdot \chi'_{\delta - w \delta} f.
\]

Denote by \( AT(T) \) the set of antisymmetric trigonometric polynomials on \( T \).

It was shown in (4.5.2) that the function \( q = \prod_{\alpha \in \Phi} (1 - \chi_{\alpha}^T) \) (see (B7)) is antisymmetric.

(5.2.4) LEMMA. Notation is taken from (5.2.1), (5.2.2), (5.2.3) and (4.3.1). Suppose \( f \in AT(T) \). Then there exists \( g \in T_\mathcal{W}(T) \) such that \( q \cdot g = f \). In fact, if \( f = \sum_{\chi \in F} a(\chi) \cdot \chi \) (where \( F \) is a finite subset of \( \Sigma(T) \) and the \( a(\chi) \) are complex numbers),
\[
g = \frac{1}{q} \cdot f = \frac{1}{\text{card } \mathcal{W}} \sum_{x_1 \in A(T)} \left( \sum_{\chi \in F} n_{\chi}(x_1) \cdot a(\chi) \right) \cdot x_1.
\]

Proof. Consider \( a \) as a map \( \Sigma(T) \to \mathbb{C} \), vanishing outside \( F \).

Since \( f = \sum_{\chi \in \Sigma(T)} a(\chi) \cdot \chi \) and \( w \cdot f = \text{sgn } w \chi'_{\delta - w \delta} f \) one sees that, for any \( w \in \mathcal{W} \),
\[
\sum_{\chi \in \Sigma(T)} a(w^{-1} \chi) \cdot \chi = \text{sgn } w \cdot \chi_{\delta - \omega_\delta} \sum_{\chi \in \Sigma(T)} a(\chi) \cdot \chi = \sum_{\chi \in \Sigma(T)} a(\chi) \cdot \text{sgn } w \cdot \chi \cdot \chi_{\delta - \omega_\delta} = \sum_{\chi \in \Sigma(T)} \text{sgn } w z(\chi \chi_{\delta - \omega_\delta}) \cdot \chi.
\]

Therefore, for all \( w \in \mathcal{W} \) and for all \( \chi \in \Sigma(T) \),

\[
a(w^{-1} \chi) = \text{sgn } w \cdot a(\chi \chi_{\delta - \omega_\delta}).
\]

Hence, if \( \chi \) satisfies (4.5.5) (i), \( a(\chi) = 0 \). It follows that

\[
f = \sum_{\chi \in \Sigma(T)} a(\chi) \sum_{w \in \mathcal{W}} \text{sgn } w \cdot \chi_{\delta - \omega_\delta} \cdot w \chi.
\]

However, the Weyl character formula (B9) shows that

\[
\frac{1}{q} \sum_{w \in \mathcal{W}} \text{sgn } w \chi_{\delta - \omega_\delta} w(\chi) = \sum_{\chi_1 \in \chi} n_{\chi}(\chi_1) \cdot \chi_1.
\]

Thus,

\[
\frac{1}{q} \cdot f = \sum_{\chi \in \Sigma(T)} a(\chi) \sum_{\chi_1 \in \Sigma(T)} n_{\chi}(\chi_1) \cdot \chi_1 = \frac{1}{\text{card } \mathcal{W}} \sum_{\chi_1 \in \Sigma(T)} \left( \sum_{\chi \in \Sigma(T)} n_{\chi}(\chi_1) \cdot a(\chi) \right) \chi_1.
\]

(5.2.5) In the light of some later work (particularly (5.7.2)) it will be interesting to ask whether such a factorization is possible for elements of \( AC(T) \), i.e. whether for every \( \chi \), there exists \( g \in C_\omega(T) \) with \( q \cdot g = f \). I shall show that it is not even possible to find \( g \in L_\omega^1(T) \) satisfying this equation. I treat here only the simplest case of \( G = SU(2) \), but the proof is easily susceptible of generalization. Refer to (4.7.2) for the notation in the following. One has \( T = \Pi \), \( q(z) = 1 - \frac{z^2}{2} \), \( |q(e^{it})| = 2|\sin t| \). Consider the function \( f \in C(\Pi) \) defined by
\[ f(z) = \begin{cases} \frac{\text{sgn}(z-\overline{z})}{\log|z-\overline{z}|} & \text{if } 0 < |z-\overline{z}| < \frac{1}{\varepsilon}, \\ 0 & \text{if } |z-\overline{z}| = 0, \\ z \text{ sgn}(z-\overline{z}) & \text{if } |z-\overline{z}| \geq \frac{1}{\varepsilon}. \end{cases} \]

It is clear that \(1 \cdot f = f\) and that

\[(w \cdot f)(z) = f(\overline{z}) = -\overline{z}^2 f(z) = \text{sgn} w \left( \chi_{\delta, \omega}^* \cdot f \right)(z).\]

It is banal to check that \(f \in C(\mathbb{T})\). Thus \(f \in AC(\mathbb{T})\). But consider the measurable function \(\frac{f}{q}\). It is clear that for \(t\) sufficiently small,

\[\left(\frac{f}{q}\right)(e^{it}) \sim \frac{1}{t \log t}; \text{ hence this function is not even in } L^1(\mathbb{T}).\]

(5.3) Sets of type central \(\Lambda(p, q)\); motivating discussion

(5.3.1) In this section, \(G\) is a compact Lie group. (For the moment, I shall not need the hypothesis of connectedness.) By (2.3.6), (2.4.4) and (3.6.3), I may depict the present state of our knowledge of sets of type central \(\Lambda(p, r)\) as follows:
The notational conventions for this diagram are explained in (2.3.6) and (3.6.3).

In the region labelled "A" there occur only sets $R$ with $\sup \{d_\sigma \mid \sigma \in R\} < \infty$. Any infinite set with this property contains an infinite Sidon set (1.6.5) which is, of course, of type central $\Lambda(p, q)$ for all $(p, q)$.

The sets of type central $\Lambda(p, 2)$ for $p > 2$ are precisely the central $\Lambda(p)$ sets.

(5.3.2) In the light of (5.3.1), it is natural to ask whether there are any infinite sets $R$ in region B or C for which $\sup \{d_\sigma \mid \sigma \in R\} = \infty$.

In particular, if $G$ is a connected semi-simple Lie group, are there any infinite sets in regions "B" or "C"? (It is to be recalled that connected semi-simple Lie groups are tall.) It should be noted that by (2.6.12), $\Sigma(G)$ can never be of type central $\Lambda(p, q)$ for $[\frac{1}{p}, \frac{1}{q}]$ lying in any of the regions A, B or C, but that $\Sigma(G)$ is local central $\Lambda(p, q)$ for $[\frac{1}{p}, \frac{1}{q}]$ lying in region C.

(5.4) Existence of sets of type central $\Lambda(p, q)$

(5.4.1) DEFINITION. Suppose $T$ is a compact abelian group, and suppose $w : \Sigma(T) \to \mathbb{R}^+ \cup \{0\}$. Further suppose $r, s \in [1, \infty]$, $s \neq \infty$. Then $Y \subseteq \Sigma(T)$ is said to be of type $w - \Lambda(r, s)$ if there exists $\kappa \in \mathbb{R}$ such that for all $f \in T_y(T)$,

$$\|f\|_r \leq \kappa \left( \sum_{\chi \in Y} w(\chi) |\hat{f}(\chi)|^s \right)^{1/s}.$$  

These sets are related to the weighted lacunary sets discussed by Sanders [1], [2]. I shall only be interested in the case where $w$ is $d^r$ for some $r \in \mathbb{R}$, where $d \cdot$ is defined in (4.6.2).
(5.4.2) PROPOSITION. Suppose $G$ is a compact connected Lie group, $T$ a maximal torus for $G$. Let $r, s \in [1, \infty]$, $s \neq \infty$, and suppose that $R \subseteq \Sigma(G)$. The following conditions are equivalent:

(i) $R$ is of type central $\Lambda(r, s)$;

(ii) there exists $\kappa \in \mathbb{R}$ such that for all $f \in \left(\mathbb{AT}(T)\right)_{X(R)}$,

$$\|q^{(2/r)-1} \cdot f\|_r \leq \kappa \left[ \sum_{\chi \in X(R)} d(\chi)^{2-s} |\hat{f}(\chi)|^s \right]^{1/s}.$$

(Remarks: Central $\Lambda(p, q)$ is defined in (2.3.1), $\mathbb{AT}(T)$, in (5.2.3), $X(R)$ in (4.5.6). As in (1.6.2), $\left(\mathbb{AT}(T)\right)_{X(R)}$ denotes the set of $X(R)$-spectral elements of $\mathbb{AT}(T)$.

Note that the left hand side of the inequality of (ii) is well-defined for any $r \in [1, \infty]$ by (5.2.4).

Proof. Now $R$ is of type central $\Lambda(r, s)$ if and only if there exists $\kappa \in \mathbb{R}$ such that for all $f \in ZT_R(G)$, $\|f\|_r \leq \kappa \|f\|_s$.

Appealing to (4.3.5) we see that

$$\|f\|_r = \|\psi_{\omega_f} f\|_r = \left\| \frac{q^{2/r}}{(\text{card}(W))^{1/r}} \psi_{\omega_f} f \right\|_r = \left\| \frac{q^{(2/r)-1}}{(\text{card}(W))^{1/r}} (q \cdot \psi_{\omega_f} f) \right\|_r,$$

and to (4.5.3), that

$$\|\hat{f}\|_s = \left( \sum_{\phi \in \mathcal{R}} d_\phi \|\hat{f}\|_{\phi}^s \right)^{1/s} = \left[ \sum_{\lambda \in X(R)} d(\chi)^{2-s} |(q \cdot \psi_{\omega_f}) \gamma(\chi)|^s \right]^{1/s}.$$

But by (4.4.2) and (5.1.2), $f \mapsto q \cdot \psi_{\omega_f} : ZT(G) \rightarrow \mathbb{AT}(T)$ is a bijective map, and by (4.5.7), this map restricts to a bijective map

$$ZT_R(G) \rightarrow \left(\mathbb{AT}(T)\right)_{X(R)}.$$

These facts now combine to conclude the proof. □

(5.4.3) PROPOSITION. Suppose $G$ is a compact connected Lie group, $T$ a maximal torus for $G$. Let $\varepsilon_G$ be the number defined in (3.2.2). Let
$R \subseteq \Sigma(G)$, $r \in ]2, 2+\varepsilon_G[,$ $s \in [1, \infty]$. In order that $R$ should be of type central $\Lambda(r, s)$, it is sufficient that $\chi^t(R) \subseteq \Sigma(T)$ (see (4.5.6)) should be of type $d^{2-s} - \Lambda(p, s)$ for some $p \in \left[ \frac{-\varepsilon_G}{2+\varepsilon_G-r}, \infty \right]$. (For the definition of $d^{2-s} - \Lambda(p, s)$, refer to (4.6.2) and (5.4.1).)

Proof. Choose $p \in \left[ \frac{-\varepsilon_G}{2+\varepsilon_G-r}, \infty \right]$. Note that, since $2 < r < 2+\varepsilon_G$ we have $1 < \frac{-\varepsilon_G}{2+\varepsilon_G-r} < \frac{p}{r} < \infty$. Thus $\left( \frac{p}{r} \right)' < \left( \frac{-\varepsilon_G}{2+\varepsilon_G-r} \right)' = \frac{-\varepsilon_G}{r-2}$, and so $-\varepsilon_G < (2-r) \left( \frac{p}{r} \right)' < 0$. Note that, in fact, $\left( \frac{p}{r} \right) = \frac{p}{p-r}$.

Now, by Hölder's inequality, for $f \in T(T)$,

$$\|q^{(2/r)-1}f\|_p = \left( \int |q|^{2-r}|f|^p d\mu T \right)^{1/r} \leq \left( \left( \int |q|^{(2-r)(p/r)}' d\mu T \right)^{(p-r)/p} \left( \int |f|^p d\mu T \right)^{r/p} \right)^{1/r} = \kappa(p, r)\|f\|_p.$$  

I have denoted by $\kappa(p, r)$ the number $\left( \int |q|^{(2-r)(p/r)}' d\mu T \right)^{(1/r)-(1/p)}$.

Since $-\varepsilon_G < (2-r) \left( \frac{p}{r} \right)' < 0$, (3.2.2) assures us that

$$\int |q| (2-r)(p/r)' d\mu T < \infty.$$  

Now $X(R)$, a finite union of translates of a set of type $d^{2-s} - \Lambda(p, s)$, is also of type $d^{2-s} - \Lambda(p, s)$ (cf. (2.2.6)). Thus there is a constant $\kappa \in \mathbb{R}$ such that for all $f \in (AT(T))_{X(R)}$, 

$$\|f\|_p \leq \kappa \left( \sum_{\chi \in X(R)} d(\chi)^{2-s} |\hat{f}(\chi)|^s \right)^{1/s}.$$  

Now, combining (1) and (2), I obtain, for all $f \in (AT(T))_{X(R)}$, the inequality
By (5.2.2), this implies that $R$ is of type central $A(r, s)$. □

(5.4.4) Of course, for $s = 2$, sets of type $d^{2-s}$ - $A(r, s)$ are simply $A(r)$ sets. Thus, we obtain

THEOREM. Suppose $G$ is a compact connected Lie group. Let $r \in \mathbb{R}$, $2 + \varepsilon_G].$ Then in order for $R \subseteq \Sigma(G)$ to be of type central $A(r)$, it is sufficient that $x^+(R)$ should be of type $A(p)$ for some $p \in [r, r + \varepsilon_G]$. □

In particular, every infinite subset of $\Sigma(G)$ contains an infinite set which is of type central $A(r)$ for all $r < 2 + \varepsilon_G$.

Proof. The first statement results directly from (5.4.3), and the second from the fact that every infinite subset of $\Sigma(T)$ contains an infinite set which is of type $A(p)$ for all $p$ (see (1.6.2), (1.6.3)). □

(5.4.5) COROLLARY. Suppose $G$ is an infinite compact connected group. Then for some $p > 2$, $\Sigma(G)$ contains an infinite central $A(p)$ set.

Proof. By the structure theorem (1.2.6), we may write $G = G_1 \times G_2$ where $G_1$ is an infinite connected Lie group. Now $\Sigma(G_1)$ certainly contains an infinite central $A(2 + \varepsilon_{G_1})$ set, and it is not hard to check that, considered as a subset of $\Sigma(G)$, $\Sigma(G_1)$ is a central $A(2 + \varepsilon_{G_1})$ set of $G$. □

(5.4.6) Suppose $G$ is a simply connected Lie group. Example (3.3.4) gives an infinite set which contains no infinite subset of type (local) central $A(2 + \varepsilon_{G_1})$. Hence there exist infinite subsets of $\Sigma(G)$ which are of type central $A(p)$ for all $p < 2 + \varepsilon_G$ but which are not of type central.
This is perhaps rather surprising, in view of the fact that for abelian groups there are no known sets of type \( \Lambda(2) \) which are not of type \( \Lambda(3) \).

Price [1] showed that \( SU(2) \) has no infinite local \( \Lambda(p) \) sets and hence no infinite \( \Lambda(p) \) sets for any \( p > 1 \), and Rider [5] has extended this result to \( SU(n) \). It seems likely that this result is true for any connected semisimple Lie group.

The result of (5.4.4) is in sharp contrast to these results. The special case of (5.4.4) when \( G = U(n) \) was proved by Rider [3], Theorem 5. Rider ([3], Corollary 7) also showed that every compact connected group has an infinite central \( \Lambda(2) \) set.

(5.4.7) It would be interesting to know if there are any sets of type central \( \Lambda(p, s) \) for \( s < 2 \) which are not of type central \( \Lambda(p) \) \((p > 2)\). For \( p > 4-\delta_{G} \), this question reduces to the problem of existence of infinite sets of type central \( \Lambda(p, s) \). Of course, by (3.6.1), we can only hope to find such sets provided that

\[
s' \geq \frac{2p}{2n-\delta_{G}} \quad (2n - \delta_{G} < p \text{ for some } n \in \mathbb{N})
\]

and \( s' \geq \frac{2p}{N_{G}} \) (see (3.6.1)).

Another interesting question which I have left unanswered is the problem of finding a necessary condition on \( X^{+}(R) \) for \( R \) to be of type central \( \Lambda(p) \), which is "close" to the sufficient condition (5.4.3). In particular, is it possible to have \( R \) of type central \( \Lambda(p) \) without \( X^{+}(R) \) being of type \( \Lambda(p) \)? In this connection, see (5.6.6).

Finally, I have not attacked the problem of whether connectedness is necessary for the preceding - in particular, are (5.4.4) and (5.4.5) valid without the hypothesis that \( G \) is connected? It may be possible to prove this extension of (5.4.4) by recourse to the theory of induced representations.
(5.5) Sets of type central $V(p, q)$: motivating discussion

(5.5.1) Suppose $G$ is a compact Lie group. As in (5.3.1), the knowledge of sets of type central $V(p, r)$ which can be gleaned from Chapter 2 and Chapter 3 may be depicted thus:

![Diagram]

Again, notation is taken from (2.3.6) and (3.6.3).

Recall from (2.3.1) that a set is of type central $V(r, 2)$ for some (and hence for all) $r > 2$ if and only if it is of type central $\Lambda(2)$, and that every central $\Lambda(r)$ set ($r < 2$) is of type $V(\infty, r)$. We are further assured, by (2.6.12), that we can never have "all" in the region labelled "A".

The sets of type $V(1, r)$ ($r > 2$) are just the $r'$-Sidon sets.

Note that we can never have "all" in region B or C, for then by (2.3.4), $\Sigma(G)$ is of type central $\Lambda(p, q)$ for $(1/p, 1/q)$ in region B or region C and this contradicts (5.3.1).
It is worthwhile recalling here, from (2.3.5), that if $R$ is a set of type central $V(p, q)$ ($p > 1$, $q < 2$) then $R$ contains no infinite subset of type central $A(p')$. Thus there are no sets in regions B or C which are not actually central $p$-Sidon sets for some $p < 2$, and which contain infinite subsets of uniformly bounded degree.

(5.5.2) I list some questions about sets of type central $V(p, r)$.

(i) Are there any infinite sets in region "A" which are not of type central $A(2)$?

(ii) For $p \in ]1, 2[\,$, are there any infinite central $p$-Sidon sets?

(iii) Are there any points $\left(\frac{1}{p'}, \frac{1}{r'}\right)$ in region "C" for which there are infinite sets of type central $V(p, r)$? (This question is answered in the negative for abelian groups by the last paragraph of (5.5.1).)

(5.6) Existence of sets of type central $V(p, q)$

(5.6.1) In this section, I shall consider the sets lying in region "A" of the diagram of (5.5.1).

(5.6.2) DEFINITION. Let $T$ be a compact abelian group, $w : \Sigma(T) \rightarrow \mathbb{R}^+$. Suppose $r, s \in [1, \infty]$. Then $Y \subseteq \Sigma(T)$ is of type $w - V(r, s)$ if there exists $\kappa \in \mathbb{R}$ such that for all $f \in T'(T)$,

$$\left(\sum_{\chi \in Y} w(\chi) |\hat{f}(\chi)|^{s'/s}\right)^{(1/s')} \leq \kappa \|f\|_{p'}.$$

(5.6.3) I have, of course, an analogue of (5.4.2) for sets of type central $V(p, r)$. Its proof, which is almost word-for-word the same - provided several of the inequalities are reversed - is omitted.

PROPOSITION. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Suppose $p, r \in [1, \infty]$, and $R \subseteq \Sigma(G)$. The following conditions are equivalent:

(i) $R$ is of type central $V(p, r)$;
(ii) There exists $\kappa \in \mathbb{R}$ such that for all $f \in (\text{AT}(T))_{X(R)}$, 
\[
\left( \sum_{\chi \in X(R)} |d(\chi)|^{2/p'} \right)^{1/p'} \left( \sum_{\chi \in X(R)} |\hat{f}(\chi)|^{p'} \right)^{1/p'} \leq \kappa \|q\|_{2/p'}^{-1} \|f\|_{p'}.
\]

(Refer to the remarks following the statement of (5.4.2).)

(5.6.4) COROLLARY. Notation is taken from (5.6.2) and (5.6.3). Let $p \in [2, \infty]$ and $r \in [1, \infty]$. If $R \subseteq \Sigma(G)$ is of type central $V(p, r)$ then

\[X^+(R) \subseteq \Sigma(T) \text{ is of type } d^{2-n'} - V(p, r).\]

Proof. Suppose $R$ is of type central $V(p, r)$. Then since $p \geq 2$, 
\[
\frac{2}{p'} - 1 \geq 0 .
\]
Thus by (5.6.3) (ii) and Hölder's inequality (recall that $\|q\|_{\infty} = \text{card } \mathcal{W}$), for all $f \in (\text{AT}(T))_{X(R)}$, 
\[
\left( \sum_{\chi \in X(R)} |d(\chi)|^{2/p'} |\hat{f}(\chi)|^{p'} \right)^{1/p'} \leq \kappa (\text{card } \mathcal{W})^{(2/p')-1} \|f\|_{p'}.
\]

Now let $g \in T$. Then $f = \sum_{w \in \mathcal{W}} w \cdot g \cdot \chi_{\delta^k \cdot \omega^k} \in (\text{AT}(T))_{X(R)}$, and so 
\[
\frac{1}{\|\mathcal{W}\|^{1/p'}} \left( \sum_{\chi \in X^+(R)} |d(\chi)|^{2/p'} |\hat{f}(\chi)|^{p'} \right)^{1/p'} 
\]
\[
= \left( \sum_{\chi \in X(R)} |d(\chi)|^{2/p'} |\hat{f}(\chi)|^{p'} \right)^{1/p'} 
\]
\[
\leq \kappa (\text{card } \mathcal{W})^{2/p'} \|g\|_{p'} \cdot \kappa (\text{card } \mathcal{W})^{2/p'} \|g\|_{p'} . \quad \square
\]

(5.6.5) PROPOSITION. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Suppose $p \in [2, \infty]$ and $r \in [1, \infty]$, and let $R \subseteq \Sigma(G)$. Then if $X(R)$ is of type $d^{2-n'} - V(p, r)$, $R$ is of type central $V(s, r)$ for every $s < \left[\frac{2+\varepsilon_G}{p+\varepsilon_G}\right] \cdot p$. (Recall that $\varepsilon_G$ is defined in (3.2.2).)
Proof. Suppose $X(R)$ is of type $d^{2-p'} - V(p, r)$. Then in particular, there exists $\kappa \in \mathbb{R}$ such that for every $f \in (\mathcal{M}(T))_X(R)$,

$$\left( \sum_{\chi \in X(R)} d(\chi)^{2-p'} |\hat{f}(\chi)|^{r'} \right)^{1/r'} \leq \kappa \|f\|_{p'}.$$  \hspace{1cm} (1)

Notice that $s < \left( \frac{2+\varepsilon_G}{p+\varepsilon_G} \right)$. $p < p'$; it follows that $p' < s'$. Now, for $f \in \mathcal{T}(T)$,

$$\|f\|_{p'} = \left( \int |q^{(2/s')-1} f|^{p'} |q|^{p' \left( 1 - (2/s') \right)} d\lambda_T \right)^{1/p'},$$
and applying Hölder's inequality, I obtain

$$\|f\|_{p'} \leq \left( \int |q^{(2/s')-1} f|^{s'} d\lambda_T \right)^{1/s'} \left( \int |q|^{p' \left( 1 - (2/s') \right) (s'/p')} d\lambda_T \right)^{1/(p'(s'/p')')} \hspace{1cm} (2),$$

where

$$\kappa \left( p(2-s)/(p-s) \right) = \int |q|^{p \left( (2-s)/(p-s) \right)} d\lambda_T \quad (cf. \ (3.2.3)).$$

(Remark that $\left( \frac{s'}{p'} \right)' = \frac{s'}{s'-p'}$. Thus

$$p' \left( 1 - \frac{2}{s'} \right) \left( \frac{s'}{p'} \right)' = \frac{p's'}{s'-p'} \left( \frac{s'-2}{s'} \right) = -p \left( \frac{s-2}{p-s} \right).$$

Since $s < \frac{2+\varepsilon_G}{p+\varepsilon_G}$, $p(2-s) < -\varepsilon_G (p-s)$, and so $\frac{p(2-s)}{p-s} < \varepsilon_G$. Thus by

$(3.2.2), \ \kappa \left( p(2-s)/(p-s) \right) < \infty.$)

Combining (1) and (2) gives the desired result. \quad \square

$(5.6.6)$ COROLLARY. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. Then $R \subseteq \Sigma(G)$ is a central $\Lambda(2)$ set if and only if $X^+(R) \subseteq \Sigma(T)$ is a $\Lambda(2)$ set.

Proof. Recall that $R$ is a central $\Lambda(2)$ set if and only if $R$ is of type central $V(p, 2)$ for some $p > 2$. Notice also that $X(R)$ is a finite union of translates of $X^+(R)$. Thus $X^+(R)$ is a $\Lambda(2)$ set if and
only if \( X(R) \) is a \( \Lambda(2) \) set. The result thus follows from (5.6.4) and (5.6.5).

\[ \square \]

(5.6.7) The results of the previous two paragraphs have reduced the problem of finding sets in region "A" of diagram (5.5.1) which are not central \( \Lambda(2) \) sets to one in abelian lacunary theory.

Are there any infinite subsets of \( \Sigma(T) \) of type \( d^{2-n'} - V(p, r) \) for some \( (p, r) \) such that \( 2 < r' < p \frac{(2+\varepsilon_G)}{(p+\varepsilon_G)} \) which are not of type \( \Lambda(2) \)?

One way of answering this question is given in the following lemma.

**Lemma.** Let \( T \) be the maximal torus of the compact connected Lie group \( G \). Let \( \varepsilon_G \) be the number of (3.2.2) and \( d \) the function of (4.6.2).

Suppose \( \Sigma \subseteq \Sigma(T) \) is such that \( \sum_{\chi \in \Sigma} \frac{d(\chi)}{(2+r')t^{1/r'}} < \infty \). Then, for every \( \Sigma \subseteq \Sigma(T) \) such that \( \sum_{\chi \in \Sigma} \frac{d(\chi)}{(2+r')t^{1/r'}} < \infty \), \( \Sigma \) is of type \( d^{2-n'} - V(p, r) \).

**Proof.** Let \( f \in T_y(T) \). Then, using Hölder's inequality, for any \( t \in ]1, \infty[ \),

\[
\left( \sum_{\chi \in \Sigma} d(\chi)^{2-n'} |N(\chi)|^{r' t'} \right)^{1/r'} \leq \left( \sum_{\chi \in \Sigma} d(\chi)^{(2-n')t} \right)^{1/2} \cdot \left( \sum_{\chi \in \Sigma} |N(\chi)|^{(r' t')^{1/r'} t'} \right)^{1/r'}
\]

Choosing \( t = \frac{\varepsilon_G}{r'-2} \) (since \( \frac{2+\varepsilon_G}{1+\varepsilon_G} < r' < 2 \), one sees that \( 2 < r' < 2+\varepsilon_G \));

thus \( t \in ]1, \infty[ \), I obtain

\[
\left( \sum_{\chi \in \Sigma} \frac{d(\chi)}{(2+r')t^{1/r'}} \left| \frac{N(\chi)}{r' t'} \right| \right)^{1/r'} \leq \left( \sum_{\chi \in \Sigma} d(\chi)^{-\varepsilon_G} \right)^{(2+\varepsilon_G)^{r'-2}/(2+\varepsilon_G) t'} \cdot \left( \sum_{\chi \in \Sigma} |N(\chi)|^{(2+\varepsilon_G)^{r'-2}/(2+\varepsilon_G) t'} \right)^{1/r'}
\]

\[
\leq \left( \sum_{\chi \in \Sigma} \frac{d(\chi)}{(p+\varepsilon_G)} \right)^{(2+\varepsilon_G)^{r'-2}/(2+\varepsilon_G) t'} \cdot \left( \sum_{\chi \in \Sigma} |N(\chi)|^{(2+\varepsilon_G)^{r'-2}/(2+\varepsilon_G) t'} \right)^{1/r'}
\]
It is not hard to see that the condition $r' < \left(\frac{2 + \varepsilon_G}{p + \varepsilon_G}\right)^p$ is equivalent to

$$\frac{r' \in G}{2 + \varepsilon_G - p'} < p.$$ 

Thus, letting $\kappa = \left\{ \sum_{\chi \in Y} d(\chi)^{-(r'-2)/r'} \varepsilon_G \right\}^{(r'-2)/r'} \in G$, I have

$$\left( \sum_{\chi \in Y} \left( d(\chi)^{2-r'} \hat{f}(\chi) \right)^{p'} \right)^{1/p'} \leq \kappa \left( \sum_{\chi \in \mathcal{E}(T)} \left( \hat{f}(\chi) \right)^p \right)^{1/p} \leq \kappa \| f \|_{p'}^p.$$

(The last step follows from the Hausdorff-Young inequality - (1.4.4) (ii)).

(5.6.8) Combining (5.6.6) and (5.6.7), I obtain

**PROPOSITION.** Let $G$ be a compact connected Lie group, and let $\varepsilon_G$ be the constant of (3.2.2). Suppose $R \subseteq \Sigma(G)$. Then if $\sum_{\sigma \in R} d^{-\varepsilon_G} < \infty$, $R$ is a set of type central $V(s, r)$ for all $(s, r)$ satisfying $2 < r' < s < \infty$. □

This proposition can be used to produce a wealth of examples of sets of type central $V(s, r)$ ($2 < r' < s < \infty$) of unbounded degree in the dual object of any compact connected Lie group.

(5.6.9) **EXAMPLE.** I will give an example of an infinite set in $\Sigma(SU(2))$ which is of type central $\Lambda(s, r)$ for all $(s, r)$ satisfying $2 < r' < s < \infty$ but which is not of type central $\Lambda(2)$. Note that $T = \Pi$, and $\varepsilon_{SU(2)} = 1$. In view of (5.6.8), it will suffice to display a set $E$ of positive integers such that $E$ is not $\Lambda(2)$, but such that $\sum_{n \in E} \frac{1}{n} < \infty$.

Consider the set of integers $E = \{2m!+(2m-1)! \cdot k \mid 0 \leq k \leq m, m \in \mathbb{N}\}$. It is shown by Rudin [1], (4.3), that $E$ is not of type $\Lambda(2)$ (in fact, $E$ is not even of type $\Lambda(1)$). On the other hand $\sum_{n \in E} \frac{1}{n} \leq \sum_{m=1}^{\infty} \frac{2m}{(2m)!} < \infty$. 
It seems likely that examples of sets with these properties can be found for any compact connected Lie group.

(5.7) Existence of sets of type central $V(p, q)$ II

(5.7.1) The sets in regions "B" and "C" of diagram (5.5.1) are the subject of this section. I shall, however, limit my discussion to central $p$-Sidon sets ($1 \leq p < 2$): I have made no attempt to answer the question posed in (5.5.2) (iii).

Let $G$ be a compact Lie group, and let $p \in [1, 2]$. Then I have shown in (2.4.4) (i) that every subset of $\Sigma(G)$ is a local central $p$-Sidon set. On the other hand, it is a result of (3.6.2) (ii) that the only $p$-Sidon subsets of $\Sigma(G)$ are sets whose representations have uniformly bounded dimensions. (So that if $G$ is semi-simple, there are no infinite $p$-Sidon subsets of $\Sigma(G)$.) Central $p$-Sidon sets "lie between" local central $p$-Sidon and $p$-Sidon sets, so the problem of their existence is interesting.

In the case where $G$ is a compact abelian group Edwards and Ross [1] showed that there are $3/2$-Sidon sets which are not $p$-Sidon for any $p < 3/2$. Their result has been subsequently improved by Johnson and Woodward [1], who show that, for each $n \in \mathbb{N}$, there are $\frac{2n}{n+1}$-Sidon sets which are not $p$-Sidon for any $p < \frac{2n}{n+1}$.

This leads one to hope that the dual of a compact connected semi-simple Lie group $G$ might contain infinite central $p$-Sidon sets for some $p > 1$, despite the result of Ragozin-Rider that it cannot contain infinite central Sidon sets (cf. (1.6.10)). Furthermore, close examination of the proof given in (1.6.10), shows that it uses the equivalence of (2.2.3) (viii); that $R \subseteq \Sigma(G)$ is central $p$-Sidon if and only if

$$\left(\mathbb{Z}_p^\times(G)\right)^\wedge \subseteq \mathbb{Z}_p^\times(R).$$

(1)
Now by (1.6.10) (v), \((ZM^c(G))^c \subseteq ZE_0(\Sigma(G))\), and if \(p = 1\), this shows that (1) cannot hold for \(R\) infinite. However, if \(p > 1\), \(p' < \infty\) and one knows that \(ZE_0(\Sigma(G)) \subseteq ZE_0(\Sigma(G))\). Thus the method of (1.6.10) gives no information concerning \(p\)-Sidon sets.

In this and the following sections, it is my aim to carry out a discussion of central \(p\)-Sidon sets, continuing to use the methods of the previous sections; i.e. reducing to questions concerning sets of characters on the maximal torus. In this section, I shall introduce a simple arithmetic condition \((r\)-boundedness, \(r \in [0, 1]\)) on the set of characters \(\chi^+(R)\) of the maximal torus associated (via (4.5.6)) with a set \(R \subseteq \Sigma(G)\), and show that if \(\chi^+(R)\) is \(r\)-bounded, then \(R\) contains no infinite central \(\frac{2}{1+r}\)-Sidon sets. The \(r\)-bounded sets are discussed in §5.8.

\textbf{(5.7.2) PROPOSITION. Let} \(G\) be a compact connected Lie group, \(T\) a maximal torus for \(G\). Let \(R \subseteq \Sigma(G)\) and recall from (4.5.6) the set \(\chi^+(R)\) of characters of \(T\) associated with \(R\). Let \(p \in [1, 2]\). Then \(R\) is a central \(p\)-Sidon set if and only if there exists \(\kappa \in \mathbb{R}\) such that for all \(f \in T \chi^+(R)\),

\[ \|\hat{f}\|_p \leq \kappa \left\| \sum_{\varphi \in \Sigma(T)} \left( \sum_{\chi \in \chi^+(R)} d(\chi)^{1-2/p} n(\chi) \hat{\varphi}(\chi) \right) \varphi \right\|_{\infty}. \] (1)

\textbf{Proof.} Recall from (2.5.1) that \(R\) is central \(p\)-Sidon if and only if \(R\) is of type central \(V(1, p')\). By (5.6.3), this is equivalent to the existence of \(\kappa \in \mathbb{R}\) such that for all \(f \in \left(\mathcal{A}(T)\right)_{\chi^+(R)}\),

\[ \left( \sum_{\chi \in \chi^+(R)} |d(\chi)^{(2/p) - 1}\hat{f}(\chi)|^p \right)^{1/p} \leq \kappa \|f\|_{\infty}^{-1} \|f\|_{\infty}. \] (2)

Using (5.2.4), for any \(f \in \left(\mathcal{A}(T)\right)_{\chi^+(R)}\),
Now $A : f \mapsto \sum_{w \in \mathcal{W}} \text{sgn } w \cdot w \cdot f \cdot \chi_0 - \omega_0$ is a bijection from $T_{X^+(R)}(T)$ onto $\mathcal{A}_{X^+(R)}(T)$, and

$$\| (A^f)^* \|_p = (\text{card } \mathcal{W})^{-(1/p)} \| f \|_p.$$ 

Thus (2) holds precisely when, for all $f \in T_{X^+(R)}(T)$,

$$\left\{ \sum_{\chi \in X^+(R)} |d(\chi)^{(2/p') - 1} \hat{f}(\chi)|^p \right\}^{1/p} \leq \kappa (\text{card } \mathcal{W})^{(1/p) - (1/r)} \left\| \sum_{\chi \in \Sigma(T)} \left( \sum_{\chi_1 \in X^+(R)} n_{\chi_1}^r \hat{f}(\chi_1) \right) \chi \right\|_{\infty}. \quad (3)$$

The equivalence of (1) and (3) is easily seen.

(5.7.3) The next lemma will considerably facilitate my consideration of the existence of $p$-Sidon sets. All unexplained notation is taken from (B9).

**Lemma.** Suppose $G$ is a connected semi-simple Lie group, $T$ a maximal torus for $G$. Every infinite subset of $\Sigma^+(T)$ contains an infinite subset which is totally ordered under $\preceq$.

**Proof.** In view of (B9), it will suffice to prove that if $L$ is a semi-simple Lie algebra over $\mathbb{C}$, every infinite subset of $\Lambda^+$ where $\Lambda^+$ is the set of dominant weights of $L$ contains an infinite subset which is totally ordered under the order $\preceq$ of $(A7)$.

Suppose first that $\lambda = \sum_{i=1}^{\ell} n_i \lambda_i$, $n_i \geq 0$ is a minimal element of $\Lambda^+$. Then for each $\alpha_j \in \Delta = \{ \alpha_1, \ldots, \alpha_\ell \}$, 

$$q^{-1} \cdot f = \frac{1}{\text{card } \mathcal{W}} \sum_{\chi \in \Sigma(T)} \left( \sum_{\chi_1 \in X^+(R)} n_{\chi_1} \hat{f}(\chi_1) \chi \right)^{1/p} \leq \kappa (\text{card } \mathcal{W})^{(1/p) - (1/r)} \| f \|_{\infty}. \quad (3)$$

The equivalence of (1) and (3) is easily seen.
\[ \lambda - \alpha_j = \sum_{i=1}^{\ell} \left( n_i - \langle \alpha_j, \alpha_i \rangle \right) \lambda_i \not\in \Lambda^+. \]

Since for \( j \neq i \), \( \langle \alpha_j, \alpha_i \rangle \leq 0 \), it follows that for each \( i \in \{1, \ldots, \ell\} \), \( n_i \leq \langle \alpha_i, \alpha_i \rangle = 2 \). Thus there are only finitely many minimal elements in \( \Lambda^+ \).

Since \( \Lambda^+ = \bigcup \{ \lambda \in \Lambda^+ \mid \mu > \lambda \} \), every infinite subset of \( \Lambda^+ \) must have infinite intersection with a set of the form \( \{ \mu \in \Lambda^+ \mid \mu > \lambda_0 \} \) for some \( \lambda_0 \in \Lambda^+ \) — that is, every infinite subset of \( \Lambda^+ \) contains an infinite set of the form \( \{ \lambda_0 + \sum_{i=1}^{\ell} k_i^{(j)} \alpha_i \mid j \in \mathbb{N} \} \), where \( \{ k_1^{(j)}, \ldots, k_{\ell}^{(j)} \} \) is an infinite subset of \( \mathbb{N}^{\ell} \).

Recall that \( \lambda_0 + \sum_{i=1}^{\ell} k_i^{(j)} \alpha_i \succ \lambda_0 + \sum_{i=1}^{\ell} k_i^{(n)} \alpha_i \) if and only if for each \( i \in \{1, \ldots, \ell\} \), \( k_i^{(j)} \geq k_i^{(n)} \). Thus, it suffices to show that any infinite subset of \( \mathbb{N}^{\ell} \) contains a totally ordered subset (where \( \mathbb{N}^{\ell} \) is given the order \( (n_1, \ldots, n_{\ell}) \succeq (m_1, \ldots, m_{\ell}) \) if \( n_1 \geq m_1, \ldots, n_{\ell} \geq m_{\ell} \)).

This, I shall do by induction.

The case \( \ell = 1 \) being clear, suppose \( \ell > 1 \) and that the statement is true for subsets of \( \mathbb{N}^{\ell-1} \). For \( j \in \{1, \ldots, \ell\} \) let \( \pi_j : \mathbb{N}^{\ell} \to \mathbb{N} : (n_1, \ldots, n_{\ell}) \mapsto n_j \). Let \( R \) be an infinite subset of \( \mathbb{N}^{\ell} \).

If \( \pi_j^{-1}(n_0) \cap R \) is an infinite set for any \( j \in \{1, \ldots, \ell\} \) and any \( n_0 \in \mathbb{N} \), then I may use the inductive hypothesis to find a totally ordered subset of \( R \). Otherwise, for each \( j \in \{1, \ldots, \ell\} \) and for each \( n_0 \in \mathbb{N} \), \( \pi_j^{-1}(n_0) \cap R \) is finite. But in this case, it is easy to inductively choose a totally ordered subset of \( R \). The first element is chosen at random.
Suppose $n^j = \{n_1^{(j)}, \ldots, n_I^{(j)}\}$ is chosen. Since \[ \bigcup_{i=1}^I \left( \bigcup_{k=0}^{\infty} \pi_i^{-1}(k) \cap R \right) \]
is a finite subset of $R$ and $R$ is infinite, I may choose $n_I^{j+1}$ to be any
element of
\[
R \setminus \left( \bigcup_{i=1}^I \left( \bigcup_{k=0}^{\infty} \pi_i^{-1}(k) \cap R \right) \right) = R \cap \left( \bigcap_{i=1}^I \bigcap_{k=0}^{\infty} \pi_i^{-1}(k) \right).
\]
Such an element is clearly greater than $n_I^{(j)}$. \[ \Box \]

(5.7.4) DEFINITION. Let $G$ be a compact Lie group, $T$ its maximal
torus. Other notation is explained in (B9). A subset $Y$ of $\Sigma^+(T)$ is
said to be $r$-bounded $(0 \leq r \leq 1)$ if it satisfies the following two
conditions:

(i) $\sup_{\chi \in Y} \sup_{\chi_0 \in \Sigma^+(T)} \frac{n_{X_\chi}(\chi_0)}{d(\chi)^n} < \infty$;

(ii) for all $\chi_0 \in \Sigma^+(T)$, $\lim_{\chi \in Y \to \infty} \frac{n_{X_\chi}(\chi_0)}{d(\chi)^n} = 0$.

(By (ii) I understand the following statement: for every $\varepsilon > 0$,
there exists a finite subset $F$ of $Y$ such that
\[ \chi \in Y \setminus F \Rightarrow \frac{n_{X_\chi}(\chi_0)}{d(\chi)^n} < \varepsilon. \]

I shall discuss the $r$-bounded sets in more detail in section 5.8; in
particular, it will be shown that, provided $G$ is semi-simple, every
subset of $\Sigma^+(T)$ is 1-bounded. Further, it will be shown that every
subset of $\Sigma^+(T, SU(2))$ is $r$-bounded for each $r \in [0, 1]$. The
motivation for introducing $r$-bounded sets is given by the following
theorem.

(5.7.5) THEOREM. Let $G$ be a compact connected semi-simple Lie group,
$T$ a maximal torus for $G$. Suppose that $R \subseteq \Sigma(G)$ is such that $X^+_R$
is \( r \)-bounded for some \( r \in ]0, 1] \). Then \( R \) contains no infinite central \( \frac{2}{1+r} \)-Sidon sets.

**Proof.** The general stratagem of this proof is to use (5.7.2) to reduce the problem to one concerning characters on \( T \), the maximal torus of \( G \), and then to use "randomness" arguments for these characters.

Suppose, therefore that \( R \) is an infinite set such that \( X^+(R) \) is \( r \)-bounded. Taking (if necessary) a subset of \( R \), I may, by (5.7.4), assume also that \( X^+(R) \) is totally ordered under \( \prec \). Write \( X^+(R) = \{ x_m \mid m \in \mathbb{N} \} \), with \( x_i \prec x_j \) for \( i \leq j \). Let \( F(R) \) denote the filter generated by \( X^+(R) \) in \( \Sigma^+(T) \); i.e.

\[
F(R) = \{ \eta \in \Sigma^+(T) \mid \eta \prec \chi \text{ for some } \chi \in X^+(R) \}.
\]

For \( \eta \in F(R) \), let \( m(\eta) = \inf \{ m \mid x_m \prec \eta \} \); notice that, for \( \eta \in \Sigma^+(T) \), \( x_m(\eta) = 0 \) unless \( \eta \in F(R) \), and \( m \geq m(\eta) \).

Let \( \{ \alpha_1, \ldots, \alpha_\ell \} \) be a base for \( \Phi \). Then for \( m \in \mathbb{N} \),

\[
\chi_m = x_1 \cdot \prod_{i=1}^{\ell} \left( x_{\alpha_i} \right)^{k_i(m)} , \quad \text{where } \left\{ \left( k_1^{(m)}, \ldots, k_\ell^{(m)} \right) \mid m \in \mathbb{N} \right\} \text{ is an infinite subset of } \mathbb{N}^\ell , \text{ and each element } \eta \text{ of } F(R) \text{ has the form}
\]

\[
\eta = x_1 \cdot \prod_{i=1}^{\ell} \left( x_{\alpha_i} \right)^{k_i(\eta)} , \quad \text{where } \left\{ k_1^{(\eta)}, \ldots, k_\ell^{(\eta)} \right\} \in \mathbb{N}^\ell . \quad \text{In fact, by (5.7.3), } \Sigma^+(T) \text{ (and hence } F(R) \text{) has finitely many minimal elements; this implies that there exists a constant } M \geq 0 \text{ such that for all } \eta \in F(R) \text{, and for all } i \in \{1, \ldots, \ell \} , \ k_i^{(\eta)} \geq -M .
\]

Since \( R \) is infinite, I may (again taking an infinite subset if necessary) suppose that, for \( (k_1, \ldots, k_\ell) \in \mathbb{N}^\ell \),

\[
m \left( x_1 \cdot \prod_{i=1}^{\ell} \left( x_{\alpha_i} \right)^{k_i} \right) \geq \prod_{i=1}^{\ell} k_i . \quad (1)
\]

Finally, I may assume that, for all \( \eta \in F(R) \),
\( m > m(\eta) \Rightarrow n_{\chi_m}(\eta) \cdot d(\chi_m)^{-p} < m^{-\frac{3}{2}} \). \hspace{1cm} (2)

To see this, choose a subset \( X^+_{R_0} = \{ \chi_{j}^{(0)} \mid j \in \mathbb{N} \} \) of \( X^+(R) \) according to the following inductive rule:

(i) \( \chi_0^{(0)} = \chi_0 \).

(ii) Suppose \( \chi_j^{(0)} \) is chosen \( (j \geq 0) \). Notice that, by (5.7.4) (ii),

\[
F_j = \bigcup_{n \leq \chi_j^{(0)}} \left\{ \chi \in X^+(R) \mid n_{\chi}^{(\eta)} d(\chi)^{-p} > (j+1)^{-\frac{3}{2}} \right\}
\tag{1}
\eta \in \Sigma_j^+(T)
\]

is a finite set. (That the union is finite follows from lemma A (iii) of (B9).) Let \( \chi_{j+1}^{(0)} \) be any element of \( X^+(R) \setminus F_j \). It is clear that \( X^+_j(R_0) \) has property (2), and also that \( F(R_0) = F(R) \). Further, since \( R_0 \) is a subset of \( R \), all the properties previously attributed to \( R \) also hold for \( R_0 \).

Recall from Edwards [2], (14.1.1), the Rademacher functions

\[ r_m : [0, 1] \rightarrow \{-1, 0, 1\} \quad (m \in \mathbb{N}) \]. If \( \chi = \chi_1 \cdot \prod_{i=1}^{L} k_i \in F(R) \), and

if \( t \in [0, 1] \), let \( r_{\chi}(t) = \prod_{i=1}^{L} r_{k_i}(t) \).

For \( t \in [0, 1] \), \( m \in \mathbb{N} \), let \( a(t)(\chi_m) = r_{\chi_m}(t) \cdot m^{-\frac{1}{2}} \). Then

for almost all \( t \in [0, 1] \),

\[
\sum_{\chi \in X^+(R)} |a(t)(\chi)|^{2/(1+r)} = \sum_{m=1}^{\infty} m^{-\frac{1}{2}} = \infty.
\]

I will show that, for almost all \( t \in [0, 1] \), the sum

\[
\sum_{\chi \in \Sigma_j^+(T)} \left\{ \sum_{m=1}^{\infty} n_{\chi_m}^{(\eta)} d(\chi_m)^{-p} \cdot a(t)(\chi_m) \right\} \eta
\tag{3}
\]

is uniformly convergent, and hence has uniformly bounded partial sums; by
(5.7.2), this will show that \( R \) cannot be a central \( \frac{2}{1+r} \)-Sidon set.

In fact, it will suffice to show that for almost all \( t \in [0, 1] \), the series

\[
\sum_{(k_1, \ldots, k_l) \in \mathbb{N}^l} \left[ \sum_{m=1}^{\infty} n_{\chi_m} \left( \sum_{i=1}^{l} (\chi_{\alpha_i}^{(t)})^k_i \right) \cdot d(\chi_m)^{-\rho} \cdot \frac{1}{\alpha_i^{(t)}} \right] \frac{1}{l} \sum_{i=1}^{l} (\chi_{\alpha_i}^{(t)})^k_i
\]

is uniformly convergent.

Let

\[
\eta = \eta(k_1, \ldots, k_l) = \sum_{i=1}^{l} \chi_{\alpha_i}^{(t)} \cdot \chi_{\alpha_i}^{(t)} \cdot \eta_{\chi_m} \cdot d(\chi_m)^{-\rho} \cdot m^{-(1+r)/2} \leq m^{-(3/2) - (1+r)/2} = m^{-2-(r/2)}. \tag{5}
\]

Now, for \( (k_1, \ldots, k_l) \in \mathbb{N}^l \), let \( \{\varepsilon_m(k_1, \ldots, k_l)\}_{m=1}^{\infty} \) be any sequence consisting entirely of elements of \( \{+1, -1\} \). The inequality (5) shows that the series

\[
\sum_{m=1}^{\infty} \varepsilon_m(k_1, \ldots, k_l) n_{\chi_m} \cdot d(\chi_m)^{-\rho} \cdot m^{-(1+r)/2}
\]

is convergent to a number whose modulus is at most

\[
n_{\chi_m}(\eta) \cdot d(\chi_m(\eta))^{-\rho} \cdot m(\eta)^{-(1+r)/2} + \sum_{m=m(\eta)+1}^{\infty} m^{-2-(r/2)} = \kappa \cdot m(\eta)^{-(1+r)/2}, \text{ where } \kappa = \sup_{m \in \mathbb{N}} \sup_{\chi \in \mathcal{F}(R)} n_{\chi_m}(\chi) \cdot d(\chi_m) + 1,
\]

\[
\leq \kappa \cdot \sum_{k=1}^{l} k^{-(1+r)/2}.
\]
(Notice that $K < 00$ by (5.7.4) (i), and that the last inequality comes from (1).)

It is trivial to remark that

$$\sum_{i=1}^{l} \log^{1+r}(k_{i}) . \prod_{i=1}^{l} k_{i}^{-(1+r)} = \prod_{i=1}^{l} \sum_{k=1}^{\infty} \log^{1+r}(k_{i}) k_{i}^{-(1+r)} < \infty .$$

Thus the result of (14.3.6) of Edwards [2] entails that for each choice of \( \{ \varepsilon_{m}(k_{1}, \ldots, k_{l}) \} \) \( m \in \mathbb{N}, (k_{1}, \ldots, k_{l}) \in \mathbb{N}^{l} \), for almost all \( t_{0} \in [0, 1] \), the series

$$\sum_{(k_{1}, \ldots, k_{l}) \in \mathbb{N}^{l}} r^{n(k_{1}, \ldots, k_{l})} \left( \varepsilon_{0} \left( \sum_{m=1}^{\infty} \varepsilon_{m}(k_{1}, \ldots, k_{l}) n_{m}^{\lambda_{m}}(n(k_{1}, \ldots, k_{l})) \right)^{m-(1+r)/2} \right) \prod_{i=1}^{l} \left( \chi_{\alpha_{i}}^{k_{i}} \right)^{k_{i}}$$

converges uniformly.

It follows since there are only countably many possible ways of choosing the values \( \varepsilon_{m}(k_{1}, \ldots, k_{l}) \), \( m \in \mathbb{N}, (k_{1}, \ldots, k_{l}) \in \mathbb{N}^{l} \), that for almost all \( t_{0} \in [0, 1] \), the series (6) converges uniformly for every choice of \( \varepsilon_{m}(k_{1}, \ldots, k_{l}) \).

In particular, for any choice of \( t_{0} \) with this property, choosing

$$\varepsilon_{m}(k_{1}, \ldots, k_{l}) = r_{n(k_{1}, \ldots, k_{l})} \cdot r_{\chi_{m}^{\lambda_{m}}(t_{0})} ,$$

we see that

$$\sum_{(k_{1}, \ldots, k_{l}) \in \mathbb{N}^{l}} \left( \prod_{m=1}^{\infty} n_{m}^{\lambda_{m}}(n(k_{1}, \ldots, k_{l})) d(\chi_{m})^{-r} \alpha^{(t_{0})}(\chi_{m}) \right) \prod_{i=1}^{l} \left( \chi_{\alpha_{i}}^{k_{i}} \right)^{k_{i}}$$

converges uniformly. As indicated above, this is sufficient to conclude the proof. \( \Box \)
(5.8) The \( r \)-bounded sets

(5.8.1) In this section, I shall discuss the \( r \)-bounded sets which were introduced in (5.7.4). This discussion will be combined with theorem (5.7.5) to get some results concerning central \( p \)-Sidon sets. My first result concerns central Sidon sets - I show that \( \Sigma(G) \) is 1-bounded and hence contains no infinite central Sidon sets. It should be noted that this constitutes a new proof of the Ragozin-Rider result (see (1.6.10)). Secondly, I show that \( \Sigma(SU(2)) \) contains no infinite \( p \)-Sidon sets for any \( p < 2 \). Finally, the existence of \( r \)-bounded sets \( (r < 1) \) in the dual of any semi-simple Lie group is shown, and it is proved for the compact simply connected simple Lie groups of low rank that their duals are \( r \)-bounded for some \( r < 1 \).

Throughout §5.8, \( G \) will denote a compact connected semi-simple Lie group, \( T \) a maximal torus for \( G \). The notations \( \Sigma^+(T) \) and \( \chi^+(R) \) (for \( R \subseteq \Sigma(G) \)) are explained in (B9) and (4.5.6).

(5.8.1) LEMMA. \( \Sigma^+(T) \) is 1-bounded.

Proof. We have to verify (5.7.4) (i) and (5.7.4) (ii). Now (5.7.4) (i) is easy, for all \( \chi \in \Sigma^+(T) \), \( \sum_{\chi_0 \in \Sigma(T)} n_{\chi}(\chi_0) = d(\chi) \); thus

\[
\sup_{\chi \in \Sigma^+(T)} \sup_{\chi_0 \in \Sigma(T)} n_{\chi}(\chi_0) \cdot d(\chi)^{-1} \leq 1.
\]

To see that (5.7.4) (ii) is satisfied, let \( \chi_0 \in \Sigma(T) \). Then

\[
\Omega(\chi_0) = \frac{1}{\text{card}|W|} \sum_{w \in W} \omega(x_0) \in T^W(T). \text{ Thus, by (4.3.5),}
\]

\[
\psi_1^{-1}(\Omega(x_0)) \in ZL^1(G). \text{ By the Riemann-Lebesgue lemma (see (1.4.3)), this implies that } \|\psi_1^{-1}(\Omega(x_0))\|_{1} \to 0 \text{ as } \sigma \to \infty.
\]

But, by (4.5.8),
\[ \| \psi_n^{-1}(\Omega(x_0)) \| = \left| \int_{x_0} \sum_{\chi \in \Sigma(T)} n(\chi)^{\alpha(\chi)} \cdot (\Omega(x_0))^{\alpha(\chi)} \right| \]
\[ = \frac{1}{d_{\alpha}} n(x_0). \]

This completes the proof. \( \Box \)

(5.8.2) PROPOSITION. Let \( G \) be a compact connected semi-simple Lie group. Then \( \Sigma(G) \) contains no infinite central Sidon sets.

Proof. This proposition is immediate from (5.7.5) and (5.8.1). \( \Box \)

(5.8.3) PROPOSITION. Let \( r \in ]0, 1] \). Then \( \Sigma^+(\Pi, \text{SU}(2)) \) is \( r \)-bounded. Hence, \( \Sigma(\text{SU}(2)) \) contains no infinite central \( p \)-Sidon sets for any \( p \in [1, 2[. \)

Proof. Recall from (4.7.2) that \( \Sigma^+(\Pi, \text{SU}(2)) = \{ x_m \mid m \in \mathbb{N} \} \) and that, for \( m, n \in \mathbb{N} \), \( d(x_m) = (m+1) \), and

\[ n_{x_m}(x_n) = \begin{cases} 1 & \text{if } n-m \in 2\mathbb{N}, \\ 0 & \text{otherwise}. \end{cases} \]

Given these explicit expression, it is easy to verify (5.7.4) (i) and (ii). \( \Box \)

(5.8.4) The remainder of this section is motivated by (5.8.3). In order to find \( r \)-bounded subsets of \( \Sigma(G) \) for \( G \) an arbitrary compact connected semi-simple Lie group, I shall need a bound on the integers \( n_{x_0}(x) \) \( (x, x_0 \in \Sigma(T) - \text{see (B9), (5.2.2)}) \). It is my intention to seek for such a bound by use of Kostant's formula (see (A10)). To this end, I shall first need to discuss the function \( p \), defined in (A10). Denote by \( \Phi \) the set of roots for \( \mathfrak{L}(G) \) with respect to \( \mathfrak{L}(T) \); choose a base \( \Delta \) for \( \Phi \), and let \( \Phi^+ \) be the set of positive roots (cf. (A5)). Finally, let \( \Lambda \) (respectively \( \Lambda^+ \)) denote the set of weights (respectively dominant weights) of \( \Phi \). The reader will remember that, for \( \lambda \in \Lambda \), \( p(\lambda) \) is defined to be the number of card \( \Phi^+ \)-tuples of positive integers
\{k^\alpha \mid \alpha \in \Phi^+\}$, such that

$$\lambda = - \sum_{\alpha \in \Phi^+} k^\alpha \cdot \alpha.$$  

Recall that each $\alpha \in \Phi^+$ may be written $\alpha = \sum_{i=1}^{l} m_i^{(\alpha)} \cdot \alpha_i$ $(m_i^{(\alpha)} \in \mathbb{N})$. Thus, if $\lambda = - \sum r_i \cdot \alpha_i$ $(r_i \in \mathbb{N})$, then $p(\lambda)$ is the number of solutions $\{k^\alpha \mid \alpha \in \Phi^+\} \subseteq \mathbb{N}^{\text{card} \Phi^+}$ of the system of equations:

$$\sum_{\alpha \in \Phi^+} m_i^{(\alpha)} k^\alpha = r_i \quad (1 \leq i \leq l).$$

If $\lambda$ has any other form, $p(\lambda) = 0$. My estimate for $p$ will thus follow from this lemma.

**Lemma.** Let $l, N \in \mathbb{N}$, $N \geq l$. Suppose $\{m_i^j \mid 1 \leq i \leq l, 1 \leq j \leq N\}$ is a set of non-negative integers with the properties that for $i, j \in \{1, \ldots, l\}$, $m_i^j = \delta_{i,j}$, and for fixed $j \in \{1, \ldots, N\}$, at least one $m_i^j$ is non-zero. Let $r = (r_1, \ldots, r_l) \in \mathbb{N}^l$.

Then the number of solutions $(k_1, \ldots, k_N)$, $k_j \in \mathbb{N}$, to the system of equations

$$\sum_{j=1}^{N} m_i^j k_j = r_j \quad (1 \leq i \leq l) \quad (1)$$

is at most

$$\inf_{j=l+1}^{N} \left\{ \left\lfloor \frac{k_j}{m_i^j} \right\rfloor \mid 1 \leq i \leq l \text{ and } m_i^j \neq 0 \right\} + 1.$$

([ ] denotes "the integer part of").

**Proof.** The proof is by induction on $N$. For $r \in \mathbb{N}^l$, denote by $R_N(r)$ the number of solution to (1).

Note that, by the restrictions on $\{m_i^j \mid j \leq l\}$, $R_N(r) = 1$ for each
Thus, the theorem is verified in this case. Now suppose it to be verified for \( N \). Then \( (k_1, \ldots, k_{N+1}) \) is a solution to

\[
\sum_{j=1}^{N+1} m_i^j k_j = r_i \quad \text{if and only if} \quad (k_1, \ldots, k_N) \text{ is a solution to}
\]

\[
\sum_{j=1}^{N} m_i^j k_j = r_i - m_i^{N+1} k_{N+1} .
\]

It follows that the only possible values of \( k_{N+1} \) are 0, 1, \ldots, \( \left\lfloor \frac{r_i}{m_i^{N+1}} \right\rfloor \), and hence that

\[
R_{N+1}^{N+1} = \min \left\{ \left\lfloor \frac{r_i}{m_i^{N+1}} \right\rfloor | m_i^{N+1} \neq 0 \right\}.
\]

(I have denoted by \( \frac{N+1}{m} \) the vector \( \left( \frac{N+1}{m_1}, \ldots, \frac{N+1}{m_N} \right) \in \mathbb{N}^L \).) Each of the \( \min \left\{ \left\lfloor \frac{r_i}{m_i^{N+1}} \right\rfloor | m_i^{N+1} \neq 0 \right\} + 1 \) terms in this sum is dominated by \( R_{N}^{N} \). Thus the lemma follows by induction. \( \square \)

The method of this lemma can be used to prove that

\[
R_{N}^{N} \geq \frac{1}{(N-L-1)!} \inf_{j=L+1}^{N} \inf \left\{ \left[ \frac{k_i}{m_i^j} \right] \mid 1 \leq i \leq L \text{ and } m_i^j \neq 0 \right\}.
\]

I shall not need this result, so I have not given its proof here.

(5.8.5) COROLLARY. Let \( (k_1, \ldots, k_L) \in \mathbb{N}^L \). Then

\[
p \left( - \sum_{i=1}^{L} k_i \alpha_i \right) \leq \prod_{\alpha \in \Phi^+ \setminus \Delta} \left( \inf \left\{ \left[ k_i / m_i^{(\alpha)} \right] \mid 1 \leq i \leq L \text{ and } m_i^{(\alpha)} \neq 0 \right\} + 1 \right). \quad \square
\]

It is interesting to note that, for \( G = SU(3) \),

\[
p(-k_1 \alpha_1 - k_2 \alpha_2) = \inf \left( k_{1}^{+1}, k_{2}^{+1} \right) ,
\]

so the inequality is an equality in this case.

(5.8.6) PROPOSITION. Notation is taken from (5.8.4). There exists a polynomial \( Q \), in \( l \) variables of degree \( \text{card} \Phi^+ - l \) such that for every \( \lambda, u \in \Lambda^+ \) with \( u < \lambda \),
\[ n_\lambda(\mu) \leq q(\langle \lambda, \alpha_i \rangle - \langle \mu, \alpha_i \rangle, \ldots, \langle \lambda, \alpha_i \rangle - \langle \mu, \alpha_i \rangle) , \]
\[ q(X_1, \ldots, X_l) \text{ may be taken to be} \]
\[ \text{card } W \sum_{\alpha \in \Phi^+ \setminus \Delta} \left( \sum_{i=1}^l q_i^{(\alpha)} X_i + 1 \right) , \]
where \( q_i^{(\alpha)} \in \mathbb{Q}^+ \).

**Proof.** By Kostant's formula (A10) for all \( \lambda, \mu \in \Lambda^+ \),
\[ n_\lambda(\mu) \leq \sum_{\omega \in W} |p((\mu+\omega) - w(\lambda+\omega))| . \]

Let \( \omega \in W \). Then \( w(\lambda+\omega) = \lambda + \omega - \sum_{i=1}^l n_i^{(\omega)} \alpha_i \), with \( n_i^{(\omega)} \in \mathbb{N} \), and so
\[ p(\mu+\omega - w(\lambda+\omega)) = p \left( \sum_{i=1}^l \langle \mu, \alpha_i \rangle - \langle \lambda, \alpha_i \rangle \right) \lambda_i + \sum_{i=1}^l n_i^{(\omega)} \alpha_i \]
\[ = p \left( \sum_{j=1}^l \sum_{i=1}^l q_i^{(\alpha)} \left( \langle \mu, \alpha_i \rangle - \langle \lambda, \alpha_i \rangle \right) \alpha_j + \sum_{i=1}^l n_i^{(\omega)} \alpha_i \right) . \]

Recall from (A6) that the \( q_i^{\alpha} \in \mathbb{Q}^+ \) are such that \( \sum_{j=1}^l q_i^{\alpha} \alpha_j = \lambda_i \)
\( (i = 1, \ldots, l) \).

Using corollary (5.8.3), I obtain, for all \( \omega \in W \),
\[ p(\mu+\omega - w(\lambda+\omega)) \]
\[ \leq \inf_{\alpha \in \Phi^+ \setminus \Delta} \left( \frac{1}{m_j^{(\alpha)}} \sum_{i=1}^l q_i^{(\alpha)} \left( \langle \lambda, \alpha_i \rangle - \langle \mu, \alpha_i \rangle \right) \sum_{1 \leq j \leq l} \text{ and } m_j^{(\alpha)} \neq 0 \right) + 1 \).

Of course, since each \( q_i^{\alpha} \) is non-zero, the right hand side of this inequality is dominated by a polynomial of the desired form. The multiplicity \( n_\lambda(\mu) \) is then dominated by \( \text{card } W \) times this polynomial. This completes the proof of the proposition. \( \square \)

**(5.8.7) Corollary.** Suppose that \( G \) is a compact connected semi-simple Lie group. Let \( l \) and \( \text{card } \Phi^+ \) be the rank and number of positive
roots of $\mathfrak{L}(G)$. Then $\Sigma(G)$ contains an infinite set which is $r$-bounded for every $r > 1 - \frac{l}{\text{card}^{+}}$, and which hence contains no infinite central $p$-Sidon sets for any $p < \frac{2\text{card}^{+}}{2\text{card}^{+}-l}$.

**Proof.** Let $\delta$ be the weight of $(A7)$, for $\mathfrak{L}(G)$. Since $2\delta = \sum_{\alpha \in \Phi^{+}} \alpha$, it is clear that the character $\chi_{2\delta}' = \prod_{\alpha \in \Phi^{+}} \chi_{\alpha}'$ has the property that $(\chi_{2\delta}')^* = 2\delta$ (see B9). Let $R \subseteq \Sigma(G)$ be such that $X(R) = \left\{ \chi_{2\delta}' | n \in \mathbb{N} \right\}$. Then, applying (5.8.6), we see that there exists $\kappa \in \mathbb{R}$ such that for all $\chi_{\mu}' \in \Sigma^{+}(T)$ (where $\mu \in \Lambda^{+}(\mathfrak{L}(G))$) and for all $m \in \mathbb{N}$, $n \left( \chi_{2\delta}' \right)^{m} = n_{2m\delta}(\mu) \leq \kappa(2m+1)\text{card}^{+}-l$. Further, by the Weyl dimension formula (see A10),

$$\left( \chi_{2\delta}' \right)^{m} = \prod_{\alpha \in \Phi^{+}} \left( \frac{(2m+1)\delta, \alpha}{\delta, \alpha} \right) = (2m+1)\text{card}^{+}.$$ 

Thus, for $r \in [0, 1]$, 

$$n \left( \chi_{2\delta}' \right)^{m} \cdot d \left( \left( \chi_{2\delta}' \right)^{m} \right)^{-n} < \kappa(2m+1)\text{card}^{+}-l-r\text{card}^{+}.$$ 

So if $r > 1 - \frac{l}{\text{card}^{+}}$, the set $R = \left\{ \left( \chi_{2\delta}' \right)^{m} | m \in \mathbb{N} \right\}$ satisfies (5.7.1) (i) and (5.7.4) (ii). □

**Corollary.** Suppose $G$ is a compact simply connected Lie group. Let $l$ and $\text{card}^{+}$ denote the rank and the number of positive roots of $\mathfrak{L}(G)$, and let $N_{G}$ be as in (3.3.2). Then if $\text{card}^{+}-l < N_{G}$, $\Sigma(G)$ is $r$-bounded for all $r > \frac{\text{card}^{+}-l}{N_{G}}$, and hence contains no infinite central $p$-Sidon sets for any $p < \frac{2}{1+\left((\text{card}^{+}-l)/N_{G}\right)}$.

**Proof.** Suppose that $\chi_{\lambda}', \chi_{\mu}' \in \Sigma^{+}(T)$, $\lambda, \mu \in \Lambda^{+}$. Then by (5.8.4)
and the Weyl dimension formula (A10),
\[ n^{\chi_{\lambda}}(\chi_{\lambda})^{-n} = n^\lambda(\mu) d(V(\lambda))^{-n} \]

\[ \leq K \prod_{\alpha \in \Phi^+} \left( \sum_{\ell=1}^{L} q_{\ell}^{(\alpha)}(\lambda, \alpha_\ell) \cdot \langle \mu, \alpha_\ell \rangle + 1 \right) \cdot \prod_{\alpha \in \Phi^+} (\langle \lambda, \alpha \rangle + 1)^{-n} \]

\[ \leq K \prod_{\alpha \in \Phi^+ \setminus \Delta} \left( \sum_{\ell=1}^{L} q_{\ell}^{(\alpha)}(\lambda, \alpha_\ell) + 1 \right) \cdot \prod_{\alpha \in \Phi^+} \left( \sum_{\ell=1}^{L} n_{\ell}^{(\alpha)}(\lambda, \alpha_\ell) + 1 \right)^{-n} \]

\[ \leq K \left( \max_{i=1 \ldots L} \langle \lambda, \alpha_i \rangle + 1 \right)^{\text{card} \Phi^+ - L} \cdot \prod_{\alpha \in \Phi^+} \left( \sum_{\ell=1}^{L} n_{\ell}^{(\alpha)}(\lambda, \alpha_\ell) + 1 \right)^{-n} . \]

Thus (5.7.4) (i) and (ii) hold provided that \( \text{card} \Phi^+ - L < N_G \), and

\[ r < \frac{\text{card} \Phi^+ - L}{N_G} \].

(5.8.9) REMARKS. Unfortunately, it is only for the simply connected groups of low rank that one has \( \text{card} \Phi^+ - L < N_G \); this holds for those of type \( A_1, A_2, A_3, B_2, D_4 \) and \( G_2 \) (to prove this, refer to table (3.4.1)).

It seems likely that the estimate of (5.8.6) can be improved; after all, one forfeits a substantial amount of information when one replaces Kostant's alternating sum by the sum of the absolute values of its terms. Perhaps it is possible to use Freudenthal's formula (see (A10)) to get a better estimate. Such an improvement would hopefully yield an extension of (5.8.8) to arbitrary compact connected semi-simple Lie groups.

I have no information concerning sets which are not \( r \)-bounded (for some \( r \in [0, 1) \)). To prove that a set is not \( r \)-bounded would require a lower estimate for \( n^\lambda(\mu) \). It seems conceivable, nevertheless, at least in the case \( G = SU(3) \) that the set of (5.8.7) is not \( r \)-bounded for

\[ r < 1 - \frac{L}{\text{card} \Phi^+} = \frac{2}{3} . \]
APPENDIX A

LIE ALGEBRAS

In this appendix, I shall summarize some information about semi-simple Lie algebras and their representation theory which I shall require elsewhere. It is beyond the scope of the present work to pretend to give an exhaustive treatment. I have had to be selective, often in an arbitrary manner. For further details, the reader is referred to the excellent treatment of Humphreys [1]. Other readable treatments are those of Varadarajan [1], Hochschild [1], and Serre [1]. There is also a certain amount of directly relevant information in Wallach [1].

(A1) Lie Algebras

Let $\mathbb{F}$ be a field. (For my purposes, $\mathbb{F}$ will always be $\mathbb{R}$ or $\mathbb{C}$, but much of the theory presented in this appendix is true for arbitrary algebraically closed fields of characteristic zero. I shall assume for the moment only that the characteristic of $\mathbb{F}$ is zero.) A Lie algebra over $\mathbb{F}$ is a finite dimensional vector space $L$ with an operation $(x, y) \mapsto [xy] : L \times L \to L$ satisfying:

(i) $[\ ]$ is bilinear;

(ii) for all $x \in L$, $[xx] = 0$;

(iii) for all $x, y, z \in L$, $[x[yz]] + [y[xz]] + [z[xy]] = 0$.

One defines Lie subalgebras, Lie homomorphisms and isomorphisms, direct sums and products of Lie algebras as usual. $L$ is said to be abelian if $[LL] = \{0\}$. If $V$ is any vector space over a field $\mathbb{F}$, let $\mathfrak{gl}(V)$ denote the set of vector space endomorphisms of $V$, equipped with the product $[AB] = AB - BA$. Then $\mathfrak{gl}(V)$ is a Lie algebra over $\mathbb{F}$. Further examples will be given in (A.11), but in fact, Ado's theorem assures us that every Lie algebra is a subalgebra of $\mathfrak{gl}(V)$ for suitable $V$. 
(A2) Ideals, Simple Lie Algebras

Let $L$ be a Lie algebra over $\mathbb{F}$. An ideal of $L$ is a subset $I$ such that whenever $x \in L$ and $y \in I$, $[xy] \in I$. $L$ is simple if $[LL] \neq \{0\}$ and $L$ has no ideals other than $\{0\}$ and $L$. $L$ is semisimple if there exist ideals $L_1, \ldots, L_t$ of $L$, which are simple as Lie algebras and such that $L = L_1 \oplus \cdots \oplus L_t$. Note that if $L$ is semisimple so are all homomorphic images and ideals; further $[LL] = L$.

(A3) Representations, Modules

A representation of a Lie algebra $L$ over $\mathbb{F}$ is a Lie homomorphism of $L$ into $\mathfrak{gl}(V)$, where $V$ is a finite dimensional vector space over $\mathbb{F}$. (I shall not here be interested in the case where $V$ is an infinite-dimensional space.) It will often be more convenient (and it is more conventional in the context of Lie algebras) to use the language of module theory (which is completely equivalent to that of representation theory). Thus an $L$-module is a vector space $V$ over $\mathbb{F}$, together with a map $(x, v) \mapsto xv : L \times V \to V$ satisfying

(i) $(ax+by)v = a(xv) + b(yv)$ ($v \in V$, $x, y \in L$, $a, b \in \mathbb{F}$),

(ii) $x(av+bw) = a(xv) + b(xw)$ ($v, w \in V$, $x \in L$, $a, b \in \mathbb{F}$),

(iii) $[xy]v = x(yv) - y(xv)$ ($x, y \in L$, $v \in V$).

It is clear how this corresponds to a representation of $L$ on $V$. Again, I may define $L$-homomorphisms and isomorphisms as usual. Isomorphic $L$-modules are said to be equivalent. $L$-modules with no non-trivial submodules are said to be irreducible. Direct sums and tensor products of $L$-modules are defined as one would expect. (Notice that all these notions carry over to the language of representation theory; thus one speaks of equivalent representations, direct sums of representations, etc). The theorem of Weyl guarantees that, if $\mathbb{F}$ is algebraically closed, any
representation of a semi-simple Lie algebra over $\mathbb{F}$ may be decomposed into a direct sum of irreducible representations.

The adjoint representation, a representation of $L$ on itself, is defined by $\text{ad}_L(x)y = [xy]$. When there is no possibility of confusion, I shall write "ad" for $\text{ad}_L$. For $x, y \in L$, $\kappa(x, y) = \text{tr}(\text{ad} x \circ \text{ad} y)$, the Killing form, is a symmetric bilinear form on $L$. It is well known that $\kappa$ is non-degenerate if and only if $L$ is semisimple.

(A4) Cartan Subalgebra - Root Space Decomposition

Let $L$ be a semisimple Lie algebra over $\mathbb{F}$ (algebraically closed of characteristic zero). An endomorphism $\phi$ of $L$ is called semisimple if there exists a basis for $L$ so that the matrix of $\phi$ is diagonal. The reader will recall that every normal operator is semisimple (see Herstein [1], Theorem 6.2). A Cartan subalgebra (CSA) is a maximal abelian subalgebra $H$ of $L$ such that for every $x \in H$, $\text{ad}_L x$ is a semisimple transformation of $L$. Such a subalgebra always exists but is not unique in general. It is not hard to see that if $H$ is a CSA of $L$, then $H = \{x \in L \mid \text{ for all } h \in H, [xh] = 0\}$.

Now $\text{ad}_L(H)$ is a commuting set of semisimple endomorphisms of $L$, and hence, by a standard theorem in linear algebra (see Herstein [1], p. 262), one may choose a basis for $L$ in such a way that all matrices $\text{ad}_L(h)$ ($h \in H$) are diagonal; equivalently, $L$ is the direct sum of the subspaces $L_\alpha = \{x \in L \mid [hx] = \alpha(h).x \text{ for all } h \in H\}$, where $\alpha$ runs through $H^*$ (the vector space dual of $H$). By the above, $L_0 = H$. Let $\Phi$ be the set of non-zero $\alpha \in H^*$ such that $L_\alpha \neq \{0\}$. Then $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$. This is called the Cartan decomposition of $L$, and $\Phi$ is called the set of roots of $L$ (with respect to $H$). Actually, $\Phi$ can be shown, in a certain
sense, to be independent of the choice of $H$.

The Killing form $K$ has non-degenerate restriction to $H$. One may, by the usual argument, use its restriction to $H$ to obtain a Euclidean structure on $H$, and thence, identifying $H$ and $H^*$, a non-degenerate bilinear form $(\ , \ )$ on $H^*$. This makes $H^*$ into a Euclidean space, so that $\Phi$ is a root system in the Euclidean space $E = H^*$, in the sense of the following paragraph. $\Phi \subseteq H^*$ is called the root system of $L$ with respect to $H$.

(A5) Root Systems - Bases - Weyl Group

Let $E$ be a Euclidean space with inner product $(\ , \ )$. For $\alpha, \beta \in E, \beta \neq 0$, let $\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$. Note that this number is not affected by multiplication of the inner product by a scalar. The reflection in $E$ determined by $\alpha \in E, \psi_\alpha$, is the transformation of $E$ defined by:

$$w_\alpha : \beta \mapsto w_\alpha \beta = \beta - \langle \beta, \alpha \rangle \alpha.$$

It is not hard to check that this transformation is an involution, sends $\alpha$ to $-\alpha$, and leaves fixed all vectors perpendicular to $\alpha$.

A subset $\Phi$ of $E$ is said to be a root system in $E$ if

(i) $\Phi$ is finite, spans $E$, and $0 \notin \Phi$;

(ii) if $\alpha \in \Phi$, the only multiples of $\alpha$ which are elements of $\Phi$ are $\alpha$ and $-\alpha$;

(iii) if $\alpha \in \Phi$, $w_\alpha(\Phi) = \Phi$;

(iv) if $\alpha, \beta \in \Phi$ then $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

The integer $l = \dim E$ is called the rank of $\Phi$. If $\Phi$ is the root system associated with a Lie algebra $L$ with respect to a CSA, $H$ (see (A4)), one refers, rather, to rank $L$, which is equal to the dimension of $H$ as a vector space. (Rank $L$ is independent of the choice of $H$.)
The group of transformations of $E$ generated by the set $\{\omega_\alpha | \alpha \in \Phi\}$ is called the Weyl group $W$. This is a finite group (by (iii), it is a subgroup of the symmetric group on $\Phi$). Further, for $\alpha, \beta \in \Phi$, $w \in W$, $\langle w.\alpha, w.\beta \rangle = \langle \alpha, \beta \rangle$.

One may choose a subset $\Delta$ of $\Phi$ with the following properties:

(i) $\Delta$ is a basis for $E$;

(ii) each element of $\Phi$ may be written $\sum_{\alpha \in \Delta} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}$ are either all positive or all negative.

Such a set is called a base for $\Phi$.

An element of $\Phi$ is said to be positive (with respect to the base $\Delta$) if it can be written $\sum_{\alpha \in \Delta} k_\alpha \alpha$ with all $k_\alpha \geq 0$. The set of positive roots is denoted by $\Phi^+$.

If $\Delta, \Delta'$ are two different bases for $\Phi$, then there exists just one $w \in W$ such that $w(\Delta) = \Delta'$. Choose any base $\Delta$. $W$ is in fact generated by $\{\omega_\alpha | \alpha \in \Delta\}$.

Let $w \in W$. Then it can be shown (Humphreys [1], §10.3) that the number $n(w)$ of elements in $\Phi^+$ whose image under the action of $w$ is not in $\Phi^+$ is precisely the same as the smallest integer $t$ such that $w$ can be written $\omega_{\alpha_1} \ldots \omega_{\alpha_t}$ ($\alpha_i \in \Delta$). For $w \in W$, I shall denote by $\text{sgn } w$ the quantity $(-1)^n(w)$.

(A6) Irreducible Root Systems. Cartan Classification

A root system $\Phi$ is called irreducible if it cannot be written as a disjoint union of two proper subsets $\Phi_1, \Phi_2$ with each element of $\Phi_1$ orthogonal to each element of $\Phi_2$. Every root system can be written as a
disjoint union of irreducible root systems. Irreducible root systems correspond to simple Lie algebras. The well-known Cartan classification of the simple Lie algebras is achieved by classifying the root systems. I shall not give details of the process here; suffice it to say that the simple root systems are as follows: there are four infinite families (the so-called classical root systems), $A_l$ ($l \geq 1$), $B_l$ ($l \geq 2$), $C_l$ ($l \geq 3$), $D_l$ ($l \geq 4$) - here $l$ is the rank of the root system and the restrictions on $l$ are imposed to ensure that there is no overlap. Then, there are the five "exceptional" root systems: $E_8$, $E_7$, $E_6$, $F_4$, $G_2$. A detailed discussion of the simple root systems is given below ((A.11)).

(A7) The Set of Weights

Let $\Lambda = \{ \lambda \in E \mid \text{for all } \alpha \in \Phi, (\lambda, \alpha) \in \mathbb{Z} \}$. $\Lambda$ is called the set of weights of $\Phi$; $\lambda \in \Lambda$ is said to be dominant with respect to the base $\Phi$ if all the integers $(\lambda, \alpha)$ are non-negative; $\lambda$ is strongly dominant if these integers are all positive. Let $\Lambda^+$ denote the set of dominant weights.

Let $\Lambda = \{ \alpha_1, \ldots, \alpha_l \}$ be a base for $\Phi$, and choose a basis $\lambda_1, \ldots, \lambda_l$ for $E$, dual to the basis $\{ 2\alpha_i/(\alpha_i, \alpha_i) \mid i = 1, \ldots, l \}$. Then $(\lambda_i, \alpha_j) = \delta_{i,j}$, and any $\lambda \in \Lambda$ may be written $\sum_{i=1}^l m_i \lambda_i$, where $m_i = (\lambda, \alpha_i) \in \mathbb{N}$. Note that $\lambda \in \Lambda^+$ if and only if $m_i \geq 0$ for $i = 1, \ldots, l$. By property (iv) of root systems, $\Phi \subset \Lambda$ - in fact, $\alpha_i = \sum_{i=1}^l (\alpha_i, \alpha_j) \alpha_j$ - however, beware, for it is not in general true that $\Phi^+ \subset \Lambda^+$. Since $\{ \lambda_i \mid i = 1, \ldots, l \}$ and $\{ \alpha_i \mid i = 1, \ldots, l \}$ are both bases for $E$, one may write $\lambda_i = \sum_{j=1}^l q_{ij} \alpha_j$. It is not too hard to show
that $q_{i,j} \in \Phi^*$ for all $i, j$. Values of $q_{i,j}$ for the irreducible root systems are tabulated by Humphreys [1], §13.2, Table 1.

It is easy to see that the action of $W$ carries $\Lambda$ into $\Lambda$. Now define a partial order "\(\prec\)" on $\Lambda$ by $\mu \prec \lambda$ if $\lambda - \mu = \sum k_\alpha \alpha$ with $k_\alpha \in \mathbb{N}$. The following lemma is easy to prove (see Humphreys [1], Lemma 13.2A):

**LEMMA.** Each weight is conjugate under $W$ to one and only one dominant weight. If $\lambda \in \Lambda^+$, then for all $w \in W$, $w\lambda \prec \lambda$, and if $\lambda$ is strongly dominant, then $w\lambda = \lambda$ only when $w = 1$.

These considerations show that if $\lambda \in \Lambda^+$, then the number of dominant weights $\mu \prec \eta$ is finite. Hence $\Pi(\lambda) = \{\lambda \in \Lambda \mid \text{for all } w \in W, w\mu \prec \lambda\}$ is a finite set.

Before leaving this subject, I introduce some notation which will be useful later. Set $\delta = \sum_{i=1}^{L} \lambda_i$. Then $\delta$ is a strongly dominant weight, and it can be shown that $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

(A8) Root Systems and Lie Algebras over an Algebraically Closed Field

Suppose, for the purposes of this paragraph, that $\mathbb{F}$ is an algebraically closed field of characteristic zero. In (A4), I showed how a semisimple Lie algebra over $\mathbb{F}$ gives rise to a system of roots $\Phi$ in a Euclidean space $E$. If $L = L_1 \oplus \ldots \oplus L_t$ with the $L_i$ simple Lie algebras, then one may write $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$ (disjoint union with the $\Phi_i$'s pairwise orthogonal, irreducible root systems) where $\Phi_i$ is the root system of $L_i$ ($i = 1, \ldots, t$). Two semisimple Lie algebras are isomorphic if and only if their root systems are isomorphic. (Two root systems $\Phi \subseteq E$, $\Phi' \subseteq F$ ($E, F$
Euclidean spaces) are isomorphic if there exists an isomorphism (not necessarily an isometry) $\phi : E \to F$ carrying $\phi$ onto $\phi'$, such that for $\alpha, \beta \in \Phi$, $(\phi\alpha, \phi\beta) = (\alpha, \beta)$.

Finally, given any one of the irreducible root systems $\Phi$ of (A6), there exists a simple Lie algebra having $\Phi$ as root system. As indicated in (A6), this gives a complete classification of all simple Lie algebras over $\mathbb{F}$. We thus refer to Lie algebras of type $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4$ and $G_2$.

(A9) Representation Theory

Let $L$ be a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. In this paragraph, I give a description of the finite dimensional irreducible $L$-modules.

Let $V$ be a finite-dimensional $L$-module. It follows from a theorem in linear algebra that, considered as an $H$-module, $V = \bigoplus V_\lambda$, where $\lambda$ runs through $H^*$, and $V_\lambda = \{v \in V \mid \text{for all } h \in H, h.v = \lambda(h)v\}$. (In (A4), I carried out this process in the case of the representation ad to obtain the roots.) If $V_\lambda \neq \{0\}$, $\lambda$ is called a weight of $H$ on $V$, and $V_\lambda$ a weight space of $V$. The weights of $H$ on $V$ can be seen to be weights of the root system $\Phi$ of $L$ with respect to $H$, in the sense of (A7). (For the details of this, see Humphreys [1], §20.2).

Let $\lambda \in \Lambda^+$. Then there exists a finite-dimensional irreducible $L$-module, $V(\lambda)$, (called the module of highest weight $\lambda$) with the property that each weight $\mu$ of $H$ on $V(\lambda)$ satisfies $\mu \prec \lambda$, and $\dim(V(\lambda))_\lambda = 1$. Any two irreducible $L$-modules with this property are isomorphic. Furthermore, any finite dimensional irreducible $L$-module is isomorphic to $V(\lambda)$ for some $\lambda \in \Lambda^+$. Thus
PROPOSITION. \( \lambda \mapsto V(\lambda) \) induces a bijective correspondence between \( \Lambda^+ \) and the set of equivalence classes of finite dimensional irreducible \( L \)-modules (Humphreys [1], §21.2).

In fact, the set of weights of \( H \) on \( V(\lambda) \) is precisely \( \Pi(\lambda) \). Each \( \omega \in \Omega \) permutes this set, and one has, for all \( \omega \in \Omega \),
\[
dim(V(\lambda))_\mu = \dim(V(\lambda))_{\omega\mu}.
\]

(A10) Some Useful Formulae

Notation is taken from the previous paragraph.

For \( \lambda \in \Lambda^+ \) and \( \mu \in \Lambda \), let \( n_\lambda(\mu) \) be the number of times that \( (V(\lambda))_\mu \) occurs as a direct summand of \( V(\lambda) \), where \( V(\lambda) \) is considered as an \( H \)-module. Freudenthal's formula gives a recursive method for calculating \( n_\lambda(\mu) \). One has \( n_\lambda(\lambda) = 1 \), and
\[
((\lambda+\delta, \lambda+\delta)-(\mu+\delta, \mu+\delta)) \cdot n_\lambda(\mu) = 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} n_\lambda(\mu+ia)(\mu+ia, \alpha).
\]

(In order to apply this formula, remark that for \( \mu \in \Pi(\lambda) \), \( (\mu+\delta, \mu+\delta) \leq (\lambda+\delta, \lambda+\delta) \), equality prevailing only if \( \mu = \lambda \).)

If \( \eta \in H^* \), let \( p(\eta) \) be the number of ways of expressing
\[
\eta = \sum_{\alpha \in \Phi^+} -k_\alpha \cdot \alpha, \text{ with } k_\alpha \in \mathbb{N}.
\]
Note that \( p(\eta) = 0 \) unless \( -\eta \) is in the non-negative integral span of \( \Phi^+ \). In general, \( p \) is a difficult function to calculate; nevertheless, it is clear that for \( \alpha \in \Delta, k \in \mathbb{Z} \),
\[
p(k_\alpha) = \begin{cases} 0 & \text{if } k > 0, \\ 1 & \text{if } k \leq 0. \end{cases}
\]

Kostant's formula gives an alternative expression for the multiplicities \( n_\lambda(\mu) \); viz.
\[ n_{\lambda}(\mu) = \sum_{\omega \in W} \text{sgn} \omega \cdot p(\mu + \delta - \omega(\lambda + \delta)) . \]

Weyl's dimension formula states that for \( \lambda \in \Lambda^+ \),
\[ \dim V(\lambda) = \prod_{\alpha \in \Phi^+}^\lambda (\lambda + \delta, \alpha) (\delta \text{ is defined in (A7)}) . \]

Let \( \lambda', \lambda'' \in \Lambda^+ \). Then \( V(\lambda') \otimes V(\lambda'') \) is a finite dimensional \( L \)-module - thus it decomposes as a direct sum of modules \( V(\mu), \mu \in \Lambda^+ \). Steinberg's formula asserts that, for \( \mu \in \Lambda^+ \), \( V(\mu) \) occurs in the decomposition with multiplicity
\[ \sum_{\omega, \nu} \text{sgn}(\omega \nu) p(\mu + 2\delta - \omega(\lambda' + \delta) - \nu(\lambda'' + \delta)) . \]

Alternative approaches to the decomposition of tensor products are provided by Klimyk [1] and Straumann [1]. None of these approaches is as transparent as one could wish - nevertheless, they are programmable on a computer.

(A11) The Simple Lie Algebras

So far, my presentation of the theory of Lie algebras has been virtually unblemished by examples. In this section, I shall give a fairly detailed discussion of some features of the root systems of the simple Lie algebras, partly to remedy this defect, but also because I shall want to use (particularly in Chapter 3) some details of their structure which I have been unable to find in the published literature. The basis for this analysis is the discussion on pp. 64-65 of Humphreys [1]. I shall discuss each type in turn.

Type \( A_L \). The Lie algebra \( \mathfrak{sl}(L+1, \mathbb{F}) \) - this is the subalgebra of \( \mathfrak{gl}(L+1, \mathbb{F}) \) of matrices of trace zero. Let \( E \) be the \( L \)-dimensional subspace of \( \mathbb{R}^{L+1} \) orthogonal to \( \varepsilon_1 + \ldots + \varepsilon_{L+1} \) (where \( \varepsilon_1, \ldots, \varepsilon_{L+1} \) is the
canonical basis for \( \mathbb{R}^{l+1} \). \( \Phi^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq l+1 \} \). There are

\( l+1 \) positive roots. Setting \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) \((i = 1, \ldots, l)\),

\( \{ \alpha_1, \ldots, \alpha_l \} \) is a base; one has

\[ \varepsilon_i - \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1} \]

\((1 \leq i < j \leq l+1)\). \( W \) is the symmetric group acting on \( \varepsilon_1, \ldots, \varepsilon_{l+1} \).

Values for \( \langle \alpha_i, \alpha_j \rangle \) are the entries of the Cartan matrices given in

Humphreys [1], §11, Table 1. For type \( A_l \),

\[ \langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j + 1 \text{ or } j - 1, \\ 0 & \text{otherwise.} \end{cases} \]

The irreducible representations of \( \mathfrak{sl}(l+1, \mathbb{F}) \) are indexed by

\( n_1 \lambda_1 + \ldots + n_l \lambda_l \), where \( \lambda_1, \ldots, \lambda_l \) are the fundamental weights and

\( n_1, \ldots, n_l \in \mathbb{N} \). Weyl's dimension formula allows one to deduce that

\[ \dim(V(n_1 \lambda_1 + \ldots + n_l \lambda_l)) = \prod_{i=1}^{l} \prod_{j=1}^{l} \frac{1}{n_i + j + 1} \left( \begin{array}{c} n_i + n_j + 2 \\ n_i + n_j + 1 \end{array} \right) \]

\[ \ldots \left( \begin{array}{c} n_i + \ldots + n_{i+j} + j \\ n_i + \ldots + n_{i+j} \end{array} \right). \]

Steinberg's formula gives a method of decomposition any tensor product, but it is difficult to write down an explicit formula, even for \( l = 2 \).

(For \( l = 1 \), it reduces to the well-known Clebsch-Gordon series.)

Type \( B_l \). The orthogonal Lie algebra \( \mathfrak{o}(2l+1, \mathbb{F}) \). Let \( s \) be the

\( (2l+1) \times (2l+1) \) matrix

\[ \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & I_l & \ldots & I_l \\ \vdots \\ 0 & I_l & \ldots & I_l \end{pmatrix}, \]

where \( I_l \) is the \( l \times l \) identity matrix. \( \mathfrak{o}(2l+1, \mathbb{F}) \) is the set of

\( (2l+1) \times (2l+1) \) matrices \( x \) over \( \mathbb{F} \) such that \( sx = -x^t s \). Let \( E \) be
\( \mathbb{R}^l \), \( \Phi^+ = \{ \varepsilon_i \mid i = 1, \ldots, l \} \cup \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l \} \). There are \( l^2 \) positive roots. A base is \( \{ \alpha_1, \ldots, \alpha_l \} \) where
\[
\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \; 1 \leq i \leq l-1, \\
\alpha_i = \varepsilon_i ;
\]
one has
\[
\varepsilon_i = \alpha_i + \ldots + \alpha_l, \; 1 \leq i \leq l, \\
\varepsilon_i - \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1}, \; 1 \leq i < j \leq l, \\
\varepsilon_i + \varepsilon_j = \alpha_i + \ldots + \alpha_j - 1 + 2\alpha_j + \ldots + 2\alpha_l, \; 1 \leq i < j \leq l.
\]
One may, in a manner similar to the previous case, write down from this an explicit formula for the degree of the representation
\[
\nu(n_1\lambda_1 + \ldots + n_l\lambda_l).
\]
Type \( C_l \). The symplectic Lie algebra \( \mathfrak{sp}(2l, \mathbb{F}) \). Let \( s \) be the matrix
\[
\begin{pmatrix}
0 & I_l \\
-I_l & 0
\end{pmatrix}
\]
\( \mathfrak{sp}(2l, \mathbb{F}) \) is the set of \( 2l \times 2l \) matrices \( x \) over \( \mathbb{F} \) such that \( sx = -x^t s \).

Let \( E \) be \( \mathbb{R}^l \); \( \Phi^+ = \{ 2\varepsilon_i \mid 1 \leq i \leq l \} \cup \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l \} \).

There are again \( l^2 \) positive roots. A base is \( \{ \alpha_1, \ldots, \alpha_l \} \), where
\[
\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \; 1 \leq i \leq l-1, \\
\alpha_i = 2\varepsilon_i ;
\]
\[
2\varepsilon_i = 2\alpha_i + \ldots + 2\alpha_{l-1} + \alpha_l, \; 1 \leq i \leq l, \\
\varepsilon_i - \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1}, \; 1 \leq i < j \leq l, \\
\varepsilon_i + \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{l-1} + \alpha, \; 1 \leq i < j < l, \\
\varepsilon_i + \varepsilon_i = \alpha_i + \ldots + \alpha_l, \; i < l.
\]
Type $D_l$. The orthogonal Lie algebra $\mathfrak{o}(2l, \mathbb{F})$. Let $s = \begin{bmatrix} 0 & I \vspace{2mm} \hline I & 0 \end{bmatrix}$. 

$\mathfrak{o}(2l, \mathbb{F})$ is the set of $2l \times 2l$ matrices $x$ over $\mathbb{F}$ such that $sx = -x^t s$. Let $E = \mathbb{R}^l$. $\Phi^+ = \{ \varepsilon_i \pm \varepsilon_j \ | \ 1 \leq i < j \leq l \}$. There are $l(l-1)$ positive roots. A base is $\{ \alpha_1, \ldots, \alpha_l \}$ where

$$\begin{align*}
\alpha_i &= \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \leq l-1, \\
\alpha_l &= \varepsilon_{l-1} + \varepsilon_l.
\end{align*}$$

One has

$$\begin{align*}
\varepsilon_i - \varepsilon_j &= \alpha_i + \ldots + \alpha_{j-1}, \quad 1 \leq i < j \leq l, \\
\varepsilon_i + \varepsilon_j &= \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l, \\
& \quad l \leq i < j \leq l-1, \\
\varepsilon_i + \varepsilon_j &= \alpha_i + \ldots + \alpha_{l-2} + \alpha_l, \quad 1 \leq i \leq l-2, \\
\varepsilon_{l-1} + \varepsilon_l &= \alpha_l.
\end{align*}$$

Type $G_2$. The Cayley octonian algebra.

Let $E$ be the subspace of $\mathbb{R}^3$ orthogonal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$,

$$\Phi^+ = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3, -\varepsilon_1 - 2\varepsilon_2 + 2\varepsilon_3 \}.$$

A base is $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. The six positive roots are then $\alpha_1$, $\alpha_2$, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$.

Type $F_4$. $E = \mathbb{R}^4$,

$$\Phi^+ = \{ \varepsilon_i \ | \ 1 \leq i \leq 4 \} \cup \{ \varepsilon_i \pm \varepsilon_j \ | \ 1 \leq i < j \leq 4 \} \cup \{ \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \}.$$

Choosing as base

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4, \quad \alpha_4 = \frac{1}{2}(\varepsilon_1 - 2\varepsilon_2 - \varepsilon_3 - \varepsilon_4),$$

the twenty-four positive roots are...
Type $E_8$. Let $E = \mathbb{R}^8$. The one hundred and twenty positive roots are

$$\{e_j \pm e_i \mid 1 \leq i < j \leq 8\} \cup \left\{\frac{1}{8} \sum_{i=1}^{8} (-1)^k(i) e_i \mid k(i) \in \{0, 1\}, \sum k(i) \text{ is even and } k(8) = 0\right\}.$$

For a base, choose

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 + e_8 - (e_2 + \ldots + e_7)), \\
\alpha_2 &= e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, \\
\alpha_4 &= e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \\
\alpha_6 &= e_5 - e_4, \\
\alpha_7 &= e_6 - e_7, \\
\alpha_8 &= e_7 - e_8.
\end{align*}$$

The positive roots are then
\[ \varepsilon_j - \varepsilon_i = \alpha_i^{i+2} + \ldots + \alpha_{j+1}^i, \quad 1 \leq i < j \leq 7, \]

\[ \varepsilon_8 - \varepsilon_i = 3\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - (\alpha_2 + \alpha_4 + \ldots + \alpha_{i+1}^i), \quad i < 8, \]

\[ \varepsilon_1 + \varepsilon_j = \alpha_2 + \alpha_4 + \ldots + \alpha_{j+1}^i, \quad 2 \leq j \leq 7, \]

\[ \varepsilon_i + \varepsilon_j = \alpha_2 + \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{i-1}^i + \alpha_{i+2}^i + \ldots + \alpha_{j+1}^i, \quad 2 \leq i < j \leq 7, \]

\[ \varepsilon_8 + \varepsilon_j = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 + (\alpha_3 + \alpha_4 + \ldots + \alpha_{j+1}^j), \quad 1 \leq j \leq 7, \]

\[ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_8) = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \]

\[ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_8) - \varepsilon_i - \varepsilon_j = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 2(\alpha_4 + \alpha_5 + \ldots + \alpha_{i+1}^i) - (\alpha_{i+2}^i + \ldots + \alpha_{j+1}^j), \quad 2 \leq i < j \leq 7, \]

\[ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_8) - \varepsilon_1 - \varepsilon_j = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - (\alpha_2 + \alpha_4 + \alpha_5 + \ldots + \alpha_{j+1}^j), \quad 2 \leq j \leq 7, \]

\[ \varepsilon_8 + \varepsilon_j - \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_8) = \alpha_1 + \alpha_3 + \ldots + \alpha_{j+1}^j, \quad 1 \leq j \leq 7, \]
\[ \varepsilon_0 + \varepsilon_i + \varepsilon_j + \varepsilon_k - \tfrac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_6) = \]
\[ = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \ldots + 3\alpha_{i+1} + 2\alpha_{i+2} + \ldots \]
\[ \ldots + 2\alpha_{j+1} + \alpha_{j+2} + \ldots + \alpha_{k+1}, \quad 1 \leq i < j < k \leq 7. \]

Type \( E_7 \). The positive roots are those positive roots of \( E_7 \) in which \( \alpha_8 \) does not occur; a base is \( \{\alpha_1, \ldots, \alpha_7\} \). There are 63 positive roots.

Type \( E_6 \). The positive roots are those positive roots of \( E_6 \) in which neither \( \alpha_7 \) nor \( \alpha_8 \) occurs; a base is \( \{\alpha_1, \ldots, \alpha_6\} \). There are 36 positive roots.

\textit{(A12) Complexification of a Real Lie Algebra}

The rather beautiful and complete representation theory of \( \text{A}(10) \) requires that the field \( \mathbb{F} \) be algebraically closed. However, the Lie algebra of a Lie group is a Lie algebra over \( \mathbb{R} \), so in this paragraph I consider real Lie algebras.

Let \( L \) be a Lie algebra over \( \mathbb{R} \). Define a Lie algebra, \( CL \) over \( \mathbb{C} \) as follows: the elements of \( CL \) are \( \{a+ib \mid a, b \in L\} \), with Lie product given by

\[ [(a+ib)(c+id)] = [ac] - [bd] + i([ad]+[bc]). \]

It is not hard to verify that \( CL \) is, in fact a Lie algebra over \( \mathbb{C} \); it is called the \textit{complexification} of \( L \).

It can be shown that \( CL \) is (semi) simple if and only if \( L \) is (semi) simple. If \( K \) is any Lie algebra over \( \mathbb{C} \), then any real Lie homomorphism \( L \to K \) may be uniquely extended to a complex Lie homomorphism \( CL \to K \).

Thus any homomorphism of the real Lie algebra \( L \) into \( \mathfrak{gl}(n, \mathbb{C}) \) (considered as a Lie algebra over \( \mathbb{R} \)) can be uniquely extended to a representation of \( CL \) on \( \mathfrak{gl}(n, \mathbb{C}) \).
Here I summarize some of the theory of compact Lie groups and their representation theory which I shall require in the body of the text. Since this is a vast subject, I have made a very arbitrary selection of material, often giving more weight to points which have less importance in the theory simply because I have not found a treatment in the literature which is satisfactory from my point of view. Again, proofs are not generally given - an exception is the Weyl integration formula. The treatment of this formula given by Adams [1] contains several mistakes; the proof I give here is perhaps simpler than Wallach's [1], which is the only other published proof I know of (apart from Weyl's original proof for the special case of $U(n)$ [1]). Moreover, I use parts of this proof in Chapter 4. Much of the material in this appendix is traceable to one of the following sources: Adams [1], Price [4], Wallach [1]. Other expositions dealing with this subject are those of Varadarajan [1], Hochschild [1] and Serre [1]. See also the summary given by Clerc [1].

(B1) Basics

The reader familiar with Price [4] could omit this section.

Suppose $G$ is a Lie group (note - at this stage I do not assume $G$ compact) - a topological group equipped with the structure of an analytic manifold in such a way that the group operations are analytic. Recall that associated with $G$ there is a Lie algebra $L(G)$ over $\mathbb{R}$, called the Lie algebra of $G$. (This can be variously defined as the set of one-parameter subgroups of $G$, or the tangent space at the identity, $T_e(G)$, equipped with a certain canonical Lie product, $[,]$.) The exponential map
\[
\exp = \exp_G : L(G) \to G
\]
is a local homeomorphism from a neighbourhood of 0 in \( L(G) \) onto a neighbourhood of \( e \) in \( G \). The Campbell-Baker-Hausdorff (CBH) formula says further that for any \( X, Y \) in some neighbourhood of 0 in \( L(G) \),

\[
\exp X \cdot \exp Y = \exp(X+Y+\frac{1}{2}[X,Y] + \text{higher order terms}).
\]

Suppose \( \phi : G \to H \) is an analytic map defined near \( p \in G \), and let \( \psi_p : L(G) \to G : X \mapsto (\exp_G X) \cdot p \). Then \( \psi_p \) is a local homeomorphism from a neighbourhood of 0 in \( L(G) \) onto a neighbourhood of \( p \) in \( G \). (Its local inverse is a chart at \( p \).) Similarly,

\[
\phi(p) : L(H) \to H : Y \mapsto (\exp_H Y) \cdot \phi(p)
\]
is a homeomorphism from a neighbourhood of 0 in \( L(H) \) onto a neighbourhood of \( \phi(p) \) in \( H \). Denote by \( F \) the unique map defined near 0, such that the diagram

\[
\begin{array}{ccc}
L(G) & \xrightarrow{F} & L(H) \\
\downarrow{\phi_p} & & \downarrow{\phi(p)} \\
G & \xrightarrow{\phi} & H
\end{array}
\]

commutes, and define \( \phi_{*,p} \), a linear map, \( L(G) \to L(H) \) by

\[
\phi_{*,p} = F'(0).
\]

Here \( ' \) denotes the Fréchet differential of the map \( F \). This is the special case of the manifold derivative (formula (1.2.4) of Price [4]) when \( M = G \) and \( N = H \) are each given the canonical chart (see Price [4], (2.3.4)).

If \( \phi \) is an analytic homomorphism, then \( \phi_{*,e} \) is a Lie algebra homomorphism. The CBH formula shows a partial converse; if there is a Lie homomorphism \( F : L(G) \to L(H) \), then there exists a local analytic homomorphism \( f : U \to H \), where \( U \) is a neighbourhood of \( e \) in \( G \), such that \( f_{*,e} = F \). If \( G \) is a simply connected Lie group, then more is true: for any Lie homomorphism \( F : L(G) \to L(H) \) there exists a unique analytic
homomorphism $f : G \to H$ such that $f^* e = F$.

Every Lie group is locally isomorphic to a unique simply connected Lie group, called its universal covering group. Two Lie groups have the same Lie algebra if and only if their universal covering groups are isomorphic (see Price [4]).

Recall further that the connected Lie group $H$ is a closed subgroup of $G$ if and only if $L(H)$ is a subalgebra of $L(G)$ - further $H$ is normal in $G$ if and only if $L(H)$ is an ideal of $L(G)$. A Lie group $G$ is said to be simple if it contains no nontrivial connected normal subgroups - the above considerations show that this is equivalent to the statement that $L(G)$ is simple. The Cartan classification of simple Lie algebras now enables one to write down all the compact simply connected simple Lie groups - see table 1. Combined with the theory of central subgroups of simply connected groups, this yields comprehensive information about simple Lie groups (Tits [1]).

$G$ is semisimple if $L(G)$ is semisimple. This can be shown to be equivalent to the statement that $G$ has no connected abelian normal subgroups.

A compact connected abelian Lie group is called a torus. It can be shown that every torus is isomorphic to $\mathbb{T}^k$ for some $k \in \mathbb{N}$.

To conclude this miscellany of information, I remind the reader of the adjoint representation. For $x \in G$, let $A_x : G \to G : y \mapsto xyx^{-1}$. $A_x$ is an analytic homomorphism. For $x \in G$, let $\text{Ad}(x) = (A_x)^* e$. Then $\text{Ad} : G \to \text{GL}(L(G))$ is a representation of $G$, and $(\text{Ad})^* e : L(G) \to \frak{gl}(L(G))$, and in fact $(\text{Ad})^* e = \text{ad}$ (where ad is defined in (A3)). One has (see Price [4], (5.2.4) the following commutative diagram ($x$ being an element of $G$):
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathfrak{g}(G)$</th>
<th>description of $G$</th>
<th>dimension of $G$</th>
<th>rank of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(z+1)$ ($z \geq 1$)</td>
<td>$A_z$</td>
<td>$(z+1) \times (z+1)$ complex unitary matrices of determinant $+1$</td>
<td>$z(z+2)$</td>
<td>$z$</td>
</tr>
<tr>
<td>special unitary group</td>
<td>[G \setminus {G}]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$SO(2z+1)$ ($z \geq 2$)</td>
<td>$B_z$</td>
<td>$(2z+1) \times (2z+1)$ real orthogonal matrices of determinant $+1$</td>
<td>$z(z+2)$</td>
<td>$z$</td>
</tr>
<tr>
<td>special orthogonal group</td>
<td>[G \setminus {G}]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Sp(z)$ ($z \geq 3$)</td>
<td>$C_z$</td>
<td>$z \times z$ quaternionic matrices $g$ such that $g^{-1}g = I_z$</td>
<td>$z(z+2)$</td>
<td>$z$</td>
</tr>
<tr>
<td>symplectic group</td>
<td>[G \setminus {G}]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$SO(2z)$ ($z \geq 4$)</td>
<td>$D_z$</td>
<td>$2z \times 2z$ real orthogonal matrices of determinant $+1$</td>
<td>$z(2z-1)$</td>
<td>$z$</td>
</tr>
<tr>
<td>special orthogonal group</td>
<td>[G \setminus {G}]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathfrak{g}(G)$</th>
<th>[G \setminus {G}]</th>
<th>[G \setminus {G}]</th>
<th>[G \setminus {G}]</th>
<th>[G \setminus {G}]</th>
<th>[G \setminus {G}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_8$</td>
<td>$E_8$</td>
<td>128</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_7$</td>
<td>$E_7$</td>
<td>63</td>
<td>7</td>
<td></td>
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</tr>
<tr>
<td>$E_6$</td>
<td>$E_6$</td>
<td>36</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_4$</td>
<td>$F_4$</td>
<td>24</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_2$</td>
<td>$G_2$</td>
<td>6</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 1.** Table of the compact simply connected simple Lie groups

(This table can be derived from Helgason [1], p. 346, Tits [1], or Price [4].)
Here, $e^{(\cdot)} : \mathfrak{gl}(L(G)) \rightarrow \mathfrak{gl}(L(G))$ is the exponential map

$$\forall X \in \mathfrak{g} : e^X = I + X + \frac{X^2}{2!} + \ldots .$$

**B2) Structure of Compact Connected Lie Groups**

Suppose $G$ is a compact connected Lie group. Then $L_1 = [L(G), L(G)]$ is semisimple, and $L(G) = Z_L(G) \oplus L_1$, where $Z_L(G)$ is the centre of $L(G)$, $\{ h \in L(G) \mid \forall x \in L(G), [hx] = 0 \}$. Thus (since $Z_L(G)$ is an abelian ideal), $G$ is semisimple precisely when $Z_L(G) = \{0\}$. These remarks can be used to prove the structure theorem; a compact connected Lie group $G$ is a quotient of a group of the form $A \times G_1$ (where $A$ is a torus (in fact $A$ may be taken to be the identity component of the centre of $G$), and $G_1$ is simply connected semisimple) by a finite subgroup $Z$ of its centre. (This is a special case of the structure theorem for compact connected groups given in (1.2.5).)

Notice that if $G$ is a compact simply connected Lie group then $Z$ must be trivial - but then also $A$ must be trivial (since it is a torus). Thus compact simply connected groups are semisimple. For further details, see Wallach [1] (§3.6).

**B3) Maximal Tori in Compact Connected Lie Groups**

In this paragraph, $G$ is a compact connected Lie group. A subgroup $T$ of $G$ is a *maximal torus* if it is a torus which is maximal with respect to
inclusion (such a subgroup is clearly closed in \( G \)). A connected subgroup \( T \) is a maximal torus if and only if \( L(T) \) is a maximal abelian subalgebra of \( L(G) \) and, \( G \) being compact, this is equivalent to \( L(T) \) being a Cartan subalgebra of \( L(G) \) (see Wallach [1], (3.6)). Fix a maximal torus \( T \) in \( G \). It can be shown (see Wallach [1], (3.9)) that every element of \( G \) is contained in \( gTg^{-1} \) for some \( g \in G \); thus two maximal tori are conjugate; there is a Lie algebra formulation of this result; if \( H_1 \) and \( H_2 \) are maximal abelian subalgebras of \( L(G) \), then there exists \( g \in G \) such that \( \text{Ad}(g)H_1 = H_2 \).

The normalizer of \( T \) in \( G \) is defined to be

\[
N_G(T) = \{ g \in G \mid gTg^{-1} = T \},
\]

and the centralizer of \( T \) in \( G \),

\[
Z_G(T) = \{ g \in G \mid \forall t \in T, gtg^{-1} = t \}.
\]

Then it can be shown that \( Z_G(T) = T \); furthermore, \( Z_G(T) \) is a normal subgroup of \( N_G(T) \), which coincides with the connected component of the identity. It follows \( N_G(T) \) is a closed subgroup of \( G \), hence a compact Lie group) that \( W = N_G(T)/T \) is a finite group. This group is called the Weyl group of \((G, T)\). How does this relate to the Weyl group defined in (A5)? Let \( L_1 \) be as in (B2). Referring to (A10), we see that \( L \) is a complex semisimple Lie algebra; as in the preceding paragraph, \( H = \mathfrak{C}(L(T) \cap L_1) \) is a Cartan sub-algebra. \( W \) is then isomorphic to the Weyl group of \( (\mathfrak{C}L_1, H) \) defined in (A5). Note that \( t \mapsto wt = ntn^{-1} \), where \( w = nT \in W \) defines an action of \( W \) on \( T \). We also have an action of \( W \) on \( H \), dual to the action on \( H^* \) of (A5) (in fact for \( \phi \in H^* \), \( X \in H \) and \( w \in W \), \( (w\phi)(X) = \phi(w^{-1}X) \)). These two actions are related by, for \( X \in L(T) \cap L_1 \), \( w \cdot \exp_T(X) = \exp_T(w.X) \).
(B4) Correspondence of Representation Theories of $G$ and $\mathfrak{g} L(G)$

Let $G$ be a simply connected Lie group (I shall not, for the purposes of this paragraph, assume $G$ compact). Suppose $\sigma : G \to U(H_0)$ is a finite dimensional unitary representation of $G$. Then by (A12), $\sigma^* e : L(G) \to \mathfrak{gl}(H_0)$ extends to a unique representation of $\mathfrak{g} L(G)$ on $\mathfrak{g} l(H_0)$. Conversely, given a (finite dimensional) representation $\phi$ of $\mathfrak{g} L(G)$ on $\mathfrak{g} l(V)$, where $V$ is some complex vector space, the restriction of $\phi$ to $L(G)$ is a Lie algebra homomorphism from $L(G)$ to $L(U(V)) = \mathfrak{g} l(V)$. Thus, since $G$ is simply connected, there exists a unique analytic homomorphism $G \to U(V)$ so that $\sigma^* e = \phi|_{L(G)}$. Hence one obtains a bijective correspondence between the set of finite dimensional representations of $G$ and the set of representations of $\mathfrak{g} L(G)$. It is not hard to show that this correspondence preserves equivalence, and thus irreducibility, direct sums and tensor products of representations.

If $G$ is a connected Lie group, it is a quotient of its universal covering group $\tilde{G}$ by a totally disconnected subgroup $Z$ of the centre of $\tilde{G}$. The finite-dimensional representations of $G$ are precisely the finite-dimensional representations of $\tilde{G}$ which are trivial on $Z$. Hence, in this case, the finite-dimensional representations of $G$ are in bijective correspondence with a subset of the set of finite-dimensional representations of $\mathfrak{g} L(G) = \mathfrak{g} L(\tilde{G})$.

(B5) Weights and Characters

If $T$ is a torus and $\chi$ is a character of $T$ (i.e. $\chi : T \to \mathbb{T}$ is a continuous homomorphism), then $\chi^* e$ has a unique extension as an element of $(\mathfrak{g} L(T))^\ast$. The elements of $(\mathfrak{g} L(T))^\ast$ which can arise in this way can be characterized as follows: let $\Gamma(T) = \{ \chi \in L(T) \mid \exp_T \chi = e \}$. $\Gamma(T)$
may, of course, be considered as a subset of $L(T)$. Then $\lambda \in (L(T))^*$ is the extension of $\chi^*_e$, for some character $\chi$ of $T$ if and only if $\lambda(\Gamma(T)) \subseteq \mathbb{Z}$.

Now suppose that $G$ is a compact simply connected Lie group with maximal torus $T$. Then it can easily be shown (Wallach [1], (4.6.7)) that for $\lambda \in (L(T))^*$, the condition $\lambda(\Gamma(T)) \subseteq \mathbb{Z}$ is equivalent to the statement that $\lambda$ is a weight for the root system of $L(G)$ with respect to $L(T)$ in the sense of (A7). Thus the weights of $(L(G), L(T))$ are in bijective correspondence with the characters of $T$; if $\chi_\lambda$ is the character corresponding to the weight $\lambda$, then the following diagram commutes:

$$
\begin{array}{ccc}
L(T) & \xrightarrow{\lambda} & \mathbb{R} \\
\exp_T & \downarrow & \text{Exp} 2\pi i \\
T & \xrightarrow{\chi_\lambda} & \pi \\
\end{array}
$$

in fact, $\chi_\lambda \circ \exp_T = \text{Exp}(2\pi i \lambda|_{L(T)})$. Recall that the Weyl group $\mathcal{W}$ acts both on $T$ and on $\Lambda$ (the set of weights). One may define an action of $\mathcal{W}$ on the set of characters of $T$ by, for each character $\chi$,

$$(w\chi)(t) = \chi(w^{-1}t).$$

The following lemma is now easily proved.

**Lemma.** (i) $\chi_0 = 1$ (the identity character).

(ii) For all $\lambda \in \Lambda$, $\chi_{-\lambda} = \overline{\chi_\lambda}$.

(iii) For $\lambda_1, \lambda_2 \in \Lambda$, $\chi_{\lambda_1} \chi_{\lambda_2} = \chi_{\lambda_1 + \lambda_2}$.

(iv) For $\lambda \in \Lambda$, $w \in \mathcal{W}$, $w\chi_\lambda = \chi_{w\lambda}$.

By (B3), the centre of $G$ is contained in $T$; one can now identify it explicitly; let $\Delta$ be a base for $\Phi$, the root system associated with $(L(G), L(T))$. Then

$$E_G \triangleq \{ t \in T | \forall \alpha \in \Delta, \chi_\alpha(t) = 1 \}. $$
Finally, suppose $G$ is a compact connected Lie group. Choose a maximal torus $T$ for $G$. Recall from (B2) that $G \cong (A \times G_1)/Z$ where $A$ is a torus, $G_1$ simply connected and $Z$ a finite subgroup of the centre of $A \times G_1$. We then have $T \cong (A \times T_1)/Z$, where $T_1$ is a maximal torus for $G_1$. Suppose $\alpha$ is a root of $L(G_1)$. Then $\chi_{\alpha}(z) = 1$ for all $z \in Z$. Thus there exists a character $\chi'_{\alpha}$ of $T$ such that $\chi'_{\alpha}(a, t)Z = \chi_{\alpha}(t)$ for $a \in A$, $t \in T_1$. However, not every weight of $L(G_1)$ need come from a character of $T$ in this way. Some further details of the relationship between weights and characters is given in (B9).

(B6) Preliminaries to the Weyl integration formula

Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$, and consider the map $\varphi : (G/T) \times T \to G : (gT, t) \mapsto g^t g^{-1}$. We are assured by (B3) that $\varphi$ is onto. Furthermore, $\varphi$ is clearly analytic. Recall that $W$ can be identified with $N_G(T)/T$. If $w = nt \in W$, define $w(gT, t) = (g n^{-1} T, n t n^{-1})$. This defines an action of $W$ on $(G/T) \times T$ such that for all $w \in W$, $\varphi(w(gT, t)) = \varphi(gT, t)$. Hence, there exists $\varphi_1 : \left((G/T) \times T\right)/W \to G$ (by $\left((G/T) \times T\right)/W$, I denote the quotient manifold; let $\upsilon$ be the quotient map) so that $\varphi_1 \circ \upsilon = \varphi$.

Let $T_R$ denote the set of generators of $T$. It is clear that $\left((G/T) \times T_R\right)/W$ is injective. Recall (Hewitt and Ross [1], (25.27)) that the $\lambda_T$ measure of $T_R^c$ is zero. Let $G_R = \varphi_1 \left(\left((G/T) \times T_R\right)/W\right)$. Then $G_R^c$, the set of singular points of the analytic map $\varphi_1$, has $\lambda_G$-measure zero. This follows from Sard's theorem (see Matsushima [1], (II, §8)) -
recall that $\lambda_G$ arises from a differential form (Matsushima [1], (V, §3)).

Let $\lambda_{G/T}$ denote the usual unique $G$-invariant measure on $G/T$ (see Hewitt and Ross [1], (15.22) and (15.24)), and let $\lambda_T$ be the Haar measure of $T$. Then the product measure $\lambda_{G/T} \times \lambda_T$ is a measure on $(G/T) \times T$, invariant under the action of $W$.

(B7) The Weyl integration formula

Notation is conserved from (B5), (B6).

THEOREM. Let $G$ be a compact connected Lie group, $T$ a maximal torus for $G$. For $f \in C(G)$,

$$\int_G f d\lambda_G = \frac{1}{\text{card } W} \int_T \left( \int_G f(gt^{-1}) d\lambda_G(g) \right) |q(t)|^2 d\lambda_T(t),$$

where $q = \prod_{\alpha \in \Phi^+} (1-x_\alpha')$ is a trigonometric polynomial on $T$.

Proof. In this proof I shall use the relationship between differential forms and measures - if $\eta$ is a differential form on a manifold $M$, then $f \mapsto \int_M f \eta$ defines a measure on $C(M)$. This material is treated by Chevalley [1], but I have preferred the more recent treatment of Matsushima [1].

Let $\eta$ be the left invariant positive differential form on $G$ which defines $\lambda_G$ (Matsushima, V, §3). The information in (B6), together with Matsushima's theorem 1, V, §6, shows that, for $f \in C(G)$,

$$\text{card } W \int_M f d\lambda_G = \int_{(G/T) \times T} \phi^*(f \eta) = \int_{(G/T) \times T} (f \circ \phi) \phi^*(\eta) \ldots . \quad (1)$$

Let $\xi, \zeta$ be differential forms defining the measures $\lambda_{G/T}$ and $\lambda_T$ respectively. The positive differential $n$-form $\xi \wedge \zeta$ defines a measure on $(G/T) \times T$ which is invariant under the action (by multiplication) of $G \times T$. Thus $\phi^*(\eta) = h(gT, t).\xi \wedge \zeta$, for some positive $h \in C^\infty((G/T) \times T)$
(Matsushima, V, §1 (1)). Using the invariance properties of both measures, one sees that for all $g \in G$ and for all $t \in T$,
\[ h(gT, t) = h(eT, t) = h(t) . \]

From (1), it now follows that
\[ \int_{T} f d\lambda_{G} = \frac{1}{\text{card} \Omega} \int_{T} \left( \int_{G} f(stg^{-1}) d\lambda_{G}(g) \right) h(t) d\lambda_{T}(t) . \]

Thus, I must calculate $h(t) = \det \varphi^{*}(eT, t)$ (see Matsushima [1], III, §5).

I may identify the tangent space of $G/T$ at $eT$ with $L(G)/L(T)$, and hence (since $L(G)$ is finite dimensional) with a subspace of $L(G)$. (Choose a basis $X_1, \ldots, X_{n-l}, X_{n-l+1}, \ldots, X_n$ for $L(G)$ so that $X_{n-l+1}, \ldots, X_n$ is a basis for $L(T)$. Then $L(G)/L(T)$ is identified with the subspace spanned by $X_1, \ldots, X_{n-l}$.) Denote by $\exp_{G/T} : L(G)/L(T) \to G/T$ the map induced by $\exp_G$. This map will depend on the particular basis chosen for $L(G)/L(T)$. (Further details of technicalities involved here may be found in Matsushima [1], (IV, §14).)

By the definition given in (Bl), $\varphi^{*}(eT, t)$ is the Fréchet differential of the map
\[ F : (L(G)/L(T)) \times L(T) \to L(G) : (X, Y) \mapsto \log_{G}\left( \varphi(\exp_{G/T}X, \exp_{T}Y, t) t^{-1} \right) \]
evaluated at the point $(0, 0)$.

Notice that
\[ F'(0, 0)(X, Y) = F'(0, 0)(X, 0) + F'(0, 0)(0, Y) . \]

Now
\[ F'(0, 0)(0, Y) = \lim_{h \to 0} \frac{1}{h} \left( F(0, hY) - F(0, 0) \right) \]
\[ = \lim_{h \to 0} \frac{1}{h} \left( \log_{G}\left( \varphi(eT, \exp_{T}hY, t) t^{-1} \right) \right) \]
\[ = Y \quad \text{(since } \exp_{T} = \exp_{G}|_{L(T)} \text{)} . \]

Further,
\[
F'(0,0)(X,0) = \lim_{h \to 0} \frac{1}{h} \left[ F(hX,0) - F(0,0) \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \log_G \left( \exp_{G/T} hX.t \cdot \exp_{G/T}^{-1} hX.t \right) \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \log_G \left( \exp_{G} hX \cdot A_t \left( \exp_{G} hX \right) \right) \right].
\]

\(A_t\) is defined in (Bl). By the commutative diagram in the last paragraph of (Bl), and the CBH formula,
\[
\exp_{G} hX \cdot A_t \left( \exp_{G} hX \right) = \exp_{G} hX \cdot \exp_{G} - h(Ad(t)X)
\]
\[
= \exp_G \left( h(X - Ad(t)X) + O(h^2) \right).
\]
Thus
\[
F'(0,0)(X,0) = \lim_{h \to 0} \frac{1}{h} \left[ \log_G \left( h(X - Ad(t)X) + O(h^2) \right) \right]
\]
\[
= (I - Ad(t)) \cdot X.
\]

But \(Ad(t)X = e \cdot X\), and by (A4), \(ad(\log_T(t))|_{L(G)/L(T)}\)
is the card \(\Phi \times \text{card } \Phi\) diagonal matrix whose entries are \(\alpha(\log_T(t))\),
\[
\alpha \in \Phi.\text{ By (B5), } e^{\alpha(\log_T(t))} = \chi_{\alpha}(t).\text{ Hence, finally,}
\]
\[
h(t) = \det \Phi_{\chi}(eT,t) = \prod_{\alpha \in \Phi} \left( 1 - \chi_{\alpha}'(t) \right) = \prod_{\alpha \in \Phi^+} |1 - \chi_{-\alpha}'(t)|^2.
\]

(B8) The restriction of characters to \(T\)

Let \(G\) be a compact connected Lie group, \(T\) a maximal torus for \(G\).
If \(\chi\) is a character \(T\), let \(A(\chi) = \sum_{\omega \in \Omega} \text{sgn } \omega \cdot \omega(\chi)\). It is not hard to show that, if \(A(\chi) \neq 0\) then \(\omega \chi = \chi\) only if \(\omega = 1\). Furthermore,
Suppose further that $G$ is simply connected. Let $\delta$ be the weight
defined in (A8). Then it can be shown that (Wallach [1], 4.9.5)
\[ A(\delta) = q \cdot x_\delta, \]
where $q = \prod_{\alpha \in \Phi^+} (1-x_{-\alpha})$ is the function of (B7). Note that
\[ A(\delta)(w t) = \text{sgn } w A(\delta)(t) \text{ for all } t \in T, \ w \in W. \]

**THEOREM** (The Weyl Character Formula). Let $\sigma \in \Sigma(G)$, and let
$\lambda_\sigma \in \Lambda^+$ be the highest weight associated with $\sigma$ by (A9) and (B3); let
$\chi_\sigma = \text{tr}(\sigma)$ be the character of $\sigma$. Then
\[ A(\delta) \cdot \chi_\sigma|_T = A(\lambda_{\sigma + \delta}). \]

This result can be proved entirely in the context of Lie algebras
(Humphreys [1], (24.3)), but it would have taken us too far afield even to
state it in that context. The Weyl dimension formula (see (A10)) is a
corollary of this result.

There is another, more direct expression for $\chi_\sigma|_T$. Recall Freudenthal's
formula (A10), which gives a recursive expression for the integers $n_\lambda(\mu)$
where $V(\lambda)|_{E_L(T)} = \sum_{\mu \in \Pi(\lambda)} n_\lambda(\mu) \cdot V(\mu)$. It is clear that
\[ \chi_\sigma|_T = \sum_{\mu \in \Pi(\lambda_\sigma)} n_{\lambda_\sigma}(\mu) \cdot \chi_\mu. \]
Note that $n_{\lambda_\sigma}(\lambda_\sigma) = 1$, and that
\[ n_{\lambda_\sigma}(w \mu) = n_{\lambda_\sigma}(\mu) \text{ for all } w \in W, \text{ and for all } \mu \in \Lambda. \]
For notational convenience, write $n_{\lambda}(\mu) = 0$ for all $\mu \in \Lambda$ for which $\exists w \in W$ with
$\mu \not\leq \lambda$. Thus, abbreviating $\sum_{\chi \in W \cdot \chi} \chi_\perp$ by $S(\chi)$, one has
Compact connected Lie groups

It is of some interest to generalize the formulae of the last section, and those of (B5) to the setting of compact connected Lie groups, since I make essential use of these formulae in Chapter 4. The generalizations I give here will enable me to state and prove results without the restrictive hypothesis of simple connectedness.

If \( G \) is a compact connected Lie group then by (B2), \( G \cong (A \times G_1)/Z \) where \( A \) is a torus, \( G_1 \) is simply connected and \( Z \) is a finite subgroup of the centre of \( A \times G_1 \). Fix a maximal torus \( T \cong (A \times T_1)/Z \) for \( G \), where \( T_1 \) is a maximal torus for \( G_1 \). Let \( \rho : A \times G_1 \rightarrow (A \times G_1)/Z \) be the quotient map.

A character of \( T \) is called a weight of \( G \) on \( T \). For each weight \( \chi \) of \( h \) on \( T \), denote by \( \tilde{\chi}_* \), e the unique extension of \( \chi \) to an element of \( (CL(T))^* \). Remark that \( L(T) = L(A) \oplus L(T_1) \); thus \( \chi \mapsto (\tilde{\chi}_*, e) \mid _{CL(T_1)} \) defines a mapping from the set of weights of \( G \) on \( T \) to the set of weights of \( CL(G_1) \) with respect to \( CL(T_1) \). (If \( G \) is simply connected, this map is a bijection (see (B5)).) If \( \chi \) and \( \chi_1 \in \Sigma(T) \) have the same image under this map then \( \chi \circ \rho \cdot \chi_1 \circ \rho \) is trivial on \( T_1 \), and hence is a character of \( A \).

Define a partial order "\( \preceq \)" on \( \Sigma(T) \) by \( \chi_1 \preceq \chi_2 \) if there exists \( \{n_\alpha \mid \alpha \in \Phi^-\} \subseteq \mathbb{N} \) such that \( \chi_2 \chi_{-1} = \prod_{\alpha \in \Phi^+} (\chi_{\alpha}^*)^{n_\alpha} \) (the \( \chi_{\alpha}^* \) are defined in (B5)). Notice that \( \chi_1 \preceq \chi_2 \) if and only if \( \chi_1 \circ \rho \mid _A = \chi_2 \circ \rho \mid _A \) and
(\hat{\chi}_{1\ast}^\ast,e)\mid_{\mathbb{L}(T)} \prec (\hat{\chi}_{2\ast}^\ast,e)\mid_{\mathbb{L}(T)}$ in the sense of (A7). Recall that $W$, the Weyl group of $(G, T)$ acts on $T$ (B3). As in (B5), I may extend this to an action of $W$ on the weights of $G$ on $T$ by the formula

$$w(\chi)(t) = \chi(w^{-1}t),$$

and this action is related to the action of $\hat{\gamma}$ on $\mathbb{L}$ given in (A7) by

$$w \cdot \left((\hat{\chi}_{\ast}^\ast,e)\mid_{\mathbb{L}(T)}\right) = (w\hat{\chi})_{\ast}^\ast,e\mid_{\mathbb{L}(T)}.$$

Let $\sigma$ be an irreducible representation of $G$. Then

$$\sigma \circ \rho = \chi_A \otimes \sigma_\lambda,$$

where $\chi_A$ is a character of $A$, and $\sigma_\lambda$ is an irreducible representation of $G_\lambda$ such that $\chi_A \otimes \sigma_\lambda$ is trivial on $Z$.

Now $\chi_A \otimes \sigma_\lambda$ is trivial on $Z$ if and only if $\chi_A \cdot \chi_{\lambda_{\sigma_\lambda}}$ is trivial on $Z$.

(To see this, use the formula for $\chi_{\sigma_\lambda}^\ast\mid_{\mathbb{T}}$ of (B8), the fact that $Z \subseteq T$ (B5), and the fact that for all $\alpha \in \Phi^+$, $\chi_\alpha$ is trivial on $Z$ (B5).) Thus there exists a unique character $\chi^{(\sigma)}$ of $T$ such that

$$\chi^{(\sigma)} \circ \rho = \chi_A \cdot \chi_{\lambda_{\sigma_\lambda}}.$$  

(Of course, if $G$ is simply connected, $\chi^{(\sigma)}$ is the character which I have been denoting by $\chi_{\lambda_{\sigma}}$.)

Hence the irreducible representations of $G$ are in bijective correspondence with a certain subset of the set of weights of $G$ on $T$. Call the characters belonging to this subset dominant weights of $G$ on $T$, and denote the set of all dominant weights by $\Sigma^+(T, G)$ or $\Sigma^+(T)$ when no confusion will arise.

One now has (from (A7))

**LEMMA A.** (i) Every weight is congruent under the action of $W$ to a dominant weight.

(ii) If $\chi$ is dominant, for all $w \in W$, $w\chi \prec \chi$.

(iii) If $\chi$ is dominant $\{\chi_1 \in \Sigma^+(T) \mid \chi_1 \prec \chi\}$ is a finite set.
I now come to the problem of generalizing the Weyl character formula. The problem is, of course, that \( \delta \) need not be a weight of \( G \) on \( T \). To be more precise, this means that \( \delta \) need not be in the image of the map \( \chi \mapsto (\chi_\delta, e) \mid L(T_1) \). For if \( G \) is not simply connected, this map may not be a surjection. Of course, if \( \delta \) is a weight of \( G \) on \( T \) (this is equivalent to Adams' condition "ad lifts to Spin", see Adams [1], (5.56) and (5.57)), then the Weyl character formula as stated in B8 holds. (One replaces \( \chi_{\lambda+\delta} \) by \( \chi^{(\sigma)} \cdot \chi_\delta' \) where \( \chi_\delta' \) is such that \( (\chi_\delta')^*, e \mid L(T_1) = \delta \) and \( \chi_\delta' \circ \rho \mid A \) is trivial.)

However, we do know from (B5) that each \( \alpha \in \Phi \) is in the image of this map. Thus every integral linear combination \( \sum_{\alpha \in \Phi} k_\alpha \cdot \alpha \), \( k_\alpha \in \mathbb{Z} \) is in the image. Now \( \delta \) is a strongly dominant weight, so for all \( \omega \in W \), \( \delta \succ \omega \delta \). This means that \( \delta - \omega \delta = \sum_{\alpha \in \Delta} k_\alpha^{(\omega)} \cdot \alpha \), \( k_\alpha^{(\omega)} \in \mathbb{N} \). Thus, the character
\[
\chi_{\delta - \omega \delta} = \prod_{\alpha \in \Delta} (\chi_\alpha')^{k_\alpha^{(\omega)}}
\]
has the properties that \( (\chi_{\delta - \omega \delta})^*, e \mid L(T_1) = \delta - \omega \delta \), and that, \( \chi_{\delta - \omega \delta} \circ \rho \) is trivial on \( A \).

Now it is not hard to see that
\[
q = \prod_{\alpha \in \Phi^+} (1 - \chi_\alpha') = \sum_{\omega \in W} \text{sgn } \omega \chi_{\delta - \omega \delta}^T.
\]
Thus the Weyl character formula can be written
\[
q \cdot \chi_\sigma \mid T = \sum_{\omega \in W} \text{sgn } \omega \cdot w^{(\sigma)} \cdot \chi_{\delta - \omega \delta}^T.
\]
This form is perhaps a little clumsier than that of B8, but it avoids all dependence on hypotheses such as "\( \delta \) is a weight of \( G \) on \( T \)."

Finally, define, for \( \chi \in \Sigma^+(T) \), \( \chi_1 \in \Sigma(T) \),
\[ n_\chi(x_1) = \begin{cases} 0 & \text{if } x_1 \leq x, \\ n_\lambda(w) & \text{if } x_1 \leq x, \quad (x_1)_{*,e}^* \mu(T_1) = \mu \text{ and } (x)_{*,e}^* \mu(T_1) = \lambda. \end{cases} \]

Clearly \( n_\chi(x) = 1 \), and \( n_\chi(x_1) = 0 \) unless for all \( w \in \mathcal{W} \), \( w x_1 \leq x \). Also \( n_\chi(w x_1) = n_\chi(x_1) \) for all \( w \in \mathcal{W} \), and \( n_\chi(x_1) \) can be calculated as before from Freudenthal's or Kostant's formula.

**Lemma B.** With this notation,

\[ \chi_S|_T = \sum_{\chi \in \Sigma(T)} n_{(\sigma)}(\chi) \cdot \chi \]

\[ = \sum_{\chi \in \Sigma^+(T)} n_{(\sigma)}(\chi) \cdot \mathcal{S}(\chi) \]

where

\[ \mathcal{S}(\chi) = \sum_{x_1 \in \mathcal{W} \cdot x} \chi_1. \]

It is probably worth while remarking here as a corollary that, for \( \sigma \in \Sigma(G) \),

\[ d_\sigma = \sum_{\chi \in \Sigma(T)} n_{(\sigma)}(\chi) = \sum_{\chi \in \Sigma^+(T)} n_{(\sigma)}(\chi) \cdot \text{card}(\mathcal{W}_x). \]
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[5] "SU(n) has no infinite local A_p sets", preprint.

O.S. ROTHAUS


W. RUDIN


J.W. SANDERS


J.P. SERRE


R.J. STANTON


R.J. STANTON and P.A. TOMAS

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R.S. Strichartz

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\[ E_6, E_7, E_8 \]
types of root system/Lie algebra

\[ \exp_{\mathcal{G}} \]
exponential map

\[ F_4 \]
type of root system/Lie algebra

\[ f \ast g \]
convolution of functions

\[ f^* \]
adjoint of a function

\[ \mathfrak{x}^f, \mathfrak{x}_x \]
translates of a function

\[ G_2 \]
type of root system/Lie algebra

\[ G \]
compact group

\[ \tilde{G} \]
universal covering group

\[ G_0 \]
component of identity

\[ G_1 \]
semi-simple part of covering group

\[ \mathfrak{g}_R \]
set of regular elements of \( G \)

\[ [G : H] \]
index of \( H \) in \( G \)

\[ \mathfrak{gl}(V) \]
general linear Lie algebra

\[ \mathfrak{GL}(V) \]
general linear group

\[ \mathcal{H}_\sigma \]
Hilbert space in which \( \sigma \) acts

\[ I_{d_\sigma} \]
identity operator with domain \( \mathcal{H}_\sigma \)

\[ \kappa(x, y) \]
Killing form

\[ \kappa_{\mathcal{E}}, \kappa_{\mathcal{R}} \]

\[ \mathcal{L} \]
Lie algebra

\[ \mathcal{L}_1 \quad [L, L] \]

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\( p' \) conjugate index of \( p \)
\( p(\lambda) \) Kostant's function
\( \pi \) projection
\( \prod \) product of numbers, groups
\( \Pi(\lambda) \) tensor product of representations
\( [xy] \) Lie product
\( \langle f, g \rangle \) scalar product
\( \langle A, B \rangle \) scalar product
\( \langle \alpha, \beta \rangle \)
\( \frac{2(\alpha,\beta)}{(\beta,\beta)} \)
\( \phi^* \) differential of a map \( \phi \)
\( \psi, \varphi \) set of roots, positive roots
\( \psi \) \((1 \leq p \leq \infty)\) isometric isomorphisms
\( q \)
\( \sum_{\alpha \in \Phi} (1-x^{-\alpha})^{-1} \)
\( S(\chi) \) elementary symmetric sum
\( sgn \omega \)
\( \text{sl}(l+1, F) \) special linear Lie algebra
\( \text{sp}(l+1, F) \) symplectic Lie algebra
\( \sigma \) irreducible representation
\( \bar{\sigma} \) conjugate representation
\( \sigma^\approx \) \( \sigma \times \ldots \times \sigma \) \((n\ \text{times})\)
\( \sigma \times \eta \) "product" of representations
\( \Sigma(G) \) dual of a group
\( \Sigma(T, G) \), set of dominant weights of \( G \) on \( T \)
\( T \) maximal torus
\( \Pi_R \) set of regular elements of \( T \)
\( \Pi \) circle group
\( t_\xi \) co-ordinate functions
\( T(T) \) set of trigonometric polynomials
\( U(H) \) unitary group

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Group
- abelian
- circle
- connected
- (locally) compact
- Lie (see also Lie group)
- quotient
- (locally) compact
- quotient
- topological
- totally disconnected
- unitary
- universal covering

Hadamard sequence

Hausdorff-Young inequality

Hypergroup

Ideal (in a Lie algebra)

Invariant subspace

Killing form

Kostant's formula

Kostant's function

Lacunary projections

Lie algebra
- abelian
- adjoint representation
- semi-simple
- simple
- of a Lie group

Lie group
- adjoint representation
- Lie algebra of
- simple, semi-simple, simply connected

L-module (L a Lie algebra)
- irreducible, equivalent
- of highest weight \( \lambda \)

Local central lacunary sets

Local lacunary sets

\( L^p \)-spaces

\( \Lambda(p) \)-sets
- central \( \Lambda(p) \)-sets
- local \( \Lambda(p) \)-sets
- local central \( \Lambda(p) \)-sets

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\( A(p, q) \)
- central \( A(p, q) \), local central \( A(p, q) \)

**Measure**
- Haar
- central
- Plancherel (on \( \Sigma(G) \))

**Normalizer**
- unitary

**Orthogonality relations**

**Plancherel's theorem**

**Product**
- of topological groups
- scalar \( \langle f, g \rangle = \langle f \in L^p, g \in L^q \rangle \)
- scalar \( \langle \alpha, \beta \rangle, \alpha \in E, \beta \in E \)
- Lie \( \{ \} \)
- tensor product of representations

**Pseudomeasure**

**Rank** (of a semi-simple Lie algebra)

**Reflection determined by** \( \alpha \in \Phi \)

**Representation**
- adjoint (see adjoint representation)
- conjugate
- correspondence of Lie group and Lie algebra representations
- equivalent
- finite dimensional
- induced
- irreducible
- of a Lie algebra (see also \( L \) module)
- of a topological group

**Riemann-Lebesgue lemma**

**Roots**
- positive

**Root system**
- in Euclidean space
- irreducible
- isomorphic
- of a semi-simple Lie algebra
- and characters of maximal torus

(semi)simple
- endomorphism
- Lie algebra
- Lie group

**Sidon set**
- central Sidon
- local Sidon
- \( p \)-Sidon set

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- central $q$-Sidon set

R-spectral measure (function)
Steinberg's formula
Structure theorem
- compact connected groups
- compact connected Lie groups
Subgroup (closed) of a topological group
Subhypergroup (of $\Sigma(G)$)
Test family
Torus
- maximal (of a connected Lie group)
Trigonometric polynomials
- antisymmetric
- symmetric
Type $\Lambda(p)$, $\Lambda(p, q)$ (see $\Lambda(p)$, $\Lambda(p, q)$)
$V(p, q)$
- central $V(p, q)$
- local central $V(p, q)$
Weight (of a Lie algebra)
- dominant, strongly dominant
- fundamental
- partial order on
- of a representation of $L$
- of $G$ on $T$ (as a character)
Weyl
- character formula
- dimension formula
- integration formula
Weyl group
- of a root system
- of $(G, T)$
- action on $\Lambda$
- action on $T$
- action on $C(T), L^p(T), M(T)$

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