THEORY AND APPLICATION OF RAY TRANSMISSION
COEFFICIENTS IN MULTIMODE OPTICAL FIBRES

CLIVE WINKLER

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PREFACE

This dissertation is an account of work carried out in the Department of Applied Mathematics of the Institute of Advanced Studies in the Australian National University between July, 1975 and June, 1978 under the supervision of Dr. John Love.

While I have benefited greatly from discussions with all members of the department, especially my supervisor, the material presented in this thesis is my own unless specifically stated.

None of the work appearing here has been submitted to any other institution of learning for any degree.

Clive Winkler.
PUBLICATIONS


5. J.D. Love and C. Winkler, "Refracting leaky rays in graded index fibres", *Appl. Optics*, __, 1978, pp


8. J.D. Love and C. Winkler, "Generalised Fresnel power transmission coefficients for curved graded index media", *IEEE Trans.*, MTT-__, 1979, pp


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It is a curious measure of informality, idyllic surroundings and lively debate which makes this department such a rewarding personal experience as well as a stimulating academic one. Add to that some impeccable manipulative diplomacy from our head, Barry Ninham, and some extravagant enthusiasm from Allan Snyder and we have a successful formula for a complex argument.

Their energies and tolerance have been appreciated over the past three years, as have their expressive and occasionally articulate dissertations on a wide range of topics. It has been a privilege to work in their department and I thank Allan for giving me that opportunity. The value of that experience is incalculable.

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In fact all the members of the department have contributed in some way, directly or indirectly, to the completion of these studies. I wish to thank Bernard Pailthorpe from the ceramics group for teaching me to sail, Lee White for his keen political commentary, Kevin Barrell for expanding my knowledge of computer languages and Derek Chan for showing me how to differentiate with respect to four.

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My wife, Merrilee, helped enormously with checking and detailing, and her encouragement over the last three years has made the job far less difficult.

I am grateful to the Australian National University and especially to the Commonwealth Public Service Board for both the opportunity to undertake these studies and for the financial assistance during this period.
ABSTRACT

The complete solution of Maxwell's equations for any practical problem even with the simplest geometry is difficult, if not intractable. For more complicated geometries it is essential to develop simple, accurate methods of describing electromagnetic (e.m.) wave phenomena. The emergence of low loss optical fibres as a powerful challenger to coaxial cables for communication purposes has begged a simple but comprehensive theory of e.m. wave propagation along dielectric waveguides. This thesis presents a substantial part of the basis of such a theory. Based on considerations of geometric optics and restricted to multimode optical waveguides, wave propagation effects are built into a ray optic analysis to provide an elementary straightforward theory based on simple physical concepts. The approach is an analogue of the well-known theory of geometrical diffraction and provides a powerful analytical tool for the description of losses on multimode dielectric structures. It has direct application to fibres, laser cavities and a variety of devices known generally as integrated optics.

The role of ray power transmission coefficients in multimode optical guides is described and its relationship to power attenuation coefficients developed. In Chapter 2 the theory is derived for slab and circular dielectric structures with both step and graded refractive index profiles. The mathematical methods used to calculate the ray transmission coefficients are compared, their regions of validity analysed and the reduction to known solutions for these simple structures is observed. Three regions for the ray solution are
identified. (i) Bound rays confined to the core of a guiding structure but having associated fields extending indefinitely into the cladding medium. (ii) Refracting leaky rays which lose power rapidly by refraction from the core cladding interface. (iii) Tunnelling leaky rays which undergo electromagnetic tunnelling, slowly losing power from the core of the guide at caustics and points of reflection in a manner analogous to quantum mechanical tunnelling. As the loss of fibre materials decreases, the role of the very low loss tunnelling leaky rays becomes significantly more important in describing the propagation characteristics of the fibre over large distances. In Chapter 3 an analytic extension of Fresnel's law is developed to describe reflection from within an arbitrarily curved medium with either a step or a continuous refractive index gradation. These generalisations of Fresnel's law are shown to yield solutions corresponding to those of Chapter 2 for skew rays on straight fibres. Chapters 4 and 5 extend the theory to slightly absorbing and multilayered guides using a straightforward building block approach that allows the solution to be written down by inspection.

The result for ray power attenuation using the transmission coefficient theory are compared with exact numerical results of the e.m. boundary value problem in Chapter 6. Excellent agreement is shown to within a few percent in most cases.

In Chapter 7 the application of ray transmission coefficients to bent step index and graded index slab and fibre guides is detailed. Some numerical results for each case are quoted for a fibre excited by an incoherent source.
Finally, the application of the theory to guides with a small elliptical deformation is illustrated in Chapter 8 where the solution is also compared with the solution derived from the eigenvalue equation.
Notes on the text:

(i) References are numbered consecutively and listed at the end of each chapter.

(ii) Equations are numbered consecutively within each chapter. In the notation \((m,n)\), \(m\) is the chapter number and \(n\) the equation number.

(iii) The figures are numbered consecutively in each chapter.
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CHAPTER 1

INTRODUCTION: OPTICAL FIBRE THEORY

1.1 HISTORICAL PERSPECTIVE

Optical fibre transmission systems developed slowly at first from the period of their initial serious conception. The advantages of low cost wide bandwidth communications were obvious, but their fulfillment required the development of special materials with very low loss. High loss fibres were constructed, but they were unsatisfactory for the long distance applications of greatest potential use. It was not until several years later in 1970, when the achievement of very low glass losses with less than 20db per kilometre attenuation was announced, that the practical implementation of these ideas became a reality.

The interest generated by this activity spawned a rapid sequence of events after 1970. A variety of different fibre structures were being reported, with liquid core fibres and graded-index glass fibres holding the most attention. Successively lower material losses were reported and obtained routinely. Development of devices to be placed at the ends of the fibre proceeded rapidly, as did cabling techniques and connector technology.

The applications for optical fibres have been diverse. Even while optical fibres for long distance communication are still largely in a research and development stage, with lower material losses regularly
being announced, the existing technology has found immediate use in a number of specialised applications. There was a rapid utilisation of the graded-index fibre for its image relay property in medical endoscopy for the purpose of internal examination and diagnosis. Weight reduction in airborne systems along with an absence of crosstalk and interference problems offer significant advantages to the avionics industry. These very factors also make them attractive for shipboard use and this is already being exploited. In harsh environments, at large scale power installations near high voltage power lines, these same advantages are of crucial importance. In the marine environment fibre optics allows underwater hydrophone cabling to be reduced in size and enormously simplified. The dream of underwater communication links over long distances using optical fibres without the need for repeaters may not be too far away in the future.

Nearly all of these applications are insensitive to the cost of the original fibre because of the special considerations. For most terrestrial applications however, the fibres must compete cost-wise with copper structures. Even as now planned, large scale production for low loss fibres should easily produce favourable cost comparisons and the applications to conventional and envisaged communication systems then becomes virtually unlimited.

Within this context, it can easily be seen that there is a major requirement for a timely, simple but comprehensive description of the effects involved in the propagation of electromagnetic waves along these guiding structures.

While spectacular achievements in glass material losses were being proclaimed, considerable effort was being directed towards the development of a theoretical understanding of electromagnetic wave
propagation in dielectric waveguides. Studies of wave guiding phenomena in dielectric structures dates back to at least 1910, and investigations which followed centred mainly on the development of dielectric rod antennae at microwave frequencies. Many significant works in this area, too numerous to mention, were produced and the reader is referred to references 12, 13 for further detail. The analogy with microwave propagation has proved useful, even recently, in the conduct of some experimental studies designed to explore the physical mechanisms involved in bending loss calculations.

A major simplification in optical fibre theory resulted from the adoption of the weakly guiding approximation in 1969, and further refinement of the modal theory followed. Additional simplification arose from the use of WKB and ray optics approximations for the description of some of the observed propagation effects on multimode guides. WKB methods were also applied to determine modal fields. However, most analyses used a mixture of ray and modal descriptions.

In 1975 it was suggested that a tunnelling loss coefficient, which followed directly from a quantum mechanical analogy, be used to describe losses of tunnelling leaky rays in graded index multimode optical fibres and the concept was rapidly adopted and extended. This led to considerable progress in the understanding of the physical loss mechanisms and some particularly simple ray optics descriptions for tunnelling and absorption have resulted, without the need to resort to complex rays. A wide variety of papers dealing with aspects of the geometric optics approach have appeared and, more recently, a comprehensive summary of ray propagation within the core of a fibre using these concepts has been published.
A detailed tracing of the history of the innovations and theory of dielectric optical fibres would be interesting but an almost impossible undertaking to perform adequately. As far as the theory is concerned, a large number of authors have contributed to its development, and the problem of giving proper recognition to their individual efforts is an unenviable exercise because of the sheer volume of literature involved. There are, however, a number of excellent reviews which give details of the historical developments from various points of view and long lists of relevant references. A succinct collection of important reprints up until 1975, has appeared in the I.E.E.E. press, and many books on the subject of fibre optics have been written. With the promise of even lower loss single crystal fibres, rapid progress is sure to continue, especially in the field of materials technology.

From the above remarks it is clear that the development of a comprehensive simplified theoretical base is highly desirable. Several contributions have already been made. Recently a detailed study of the geometric optics approach to the determination of ray paths on multimode structures has been presented, as have studies of the excitation of the fibres by various kinds of light sources. It is a logical progression to now present a simple unified theory to describe propagation losses on multimode dielectric optical fibres, to show that the theory is an accurate representation of the real situation and to demonstrate the application of the theory to some important waveguide problems. This is the aim of this thesis. We begin by discussing the geometric optics model for attenuation, and then following on to introduce the concept of the ray power transmission coefficient.
1.2 THE RAY APPROXIMATIONS

1.2.1 Ray concepts

An electromagnetic wave propagating in free space has a well defined direction of propagation specified by the wave vector \( \mathbf{k} \) normal to the electric and magnetic field vectors, \( \mathbf{E} \) and \( \mathbf{H} \) respectively. The vectors \( \mathbf{E} \), \( \mathbf{H} \) and \( \mathbf{k} \) form a right-handed triad. In a uniform homogeneous medium, no difficulty exists in the specification of the direction of the wave and the concept of a ray is easily managed. A ray indicates the direction of the local plane wave vector \( \mathbf{k} \), which is also the direction of power propagation. Maxwell's equations are exactly satisfied and the direction of the wave is simply the direction of a ray.

In an inhomogeneous medium, where the electromagnetic fields are not in general specified by a single plane wave, the direction of a ray is no longer so simply identified and it is necessary to specify it asymptotically as the short wavelength \((\lambda \to 0)\) approximation for the local plane wave direction. It is used to indicate the direction of \( \mathbf{k} \) and the magnitude, given by the relation

\[
|\mathbf{k}| = k = \frac{2\pi n}{\lambda},
\]

is taken to the limit \( k \to \infty \). \( \lambda \) is the wavelength of light in vacuum. The ray path is then always defined by the eikonal equation,

\[
\frac{d}{ds} \left( n \frac{dR}{ds} \right) = \nabla n
\]

where \( s \) is the distance measured along the path and \( R \) is the position vector for a point on the ray path. \( n \) is the refractive index of the medium at the point under consideration, and \( dR/\text{ds} \) is interpreted physically as a unit vector tangent to the ray path. This is the asymptotic
theory known as geometric optics. The ray direction defined in the short wavelength approximation is specified everywhere including caustics and points of reflection.

Indiscriminate use of geometric optics leads to strange contradictions. It predicts the field intensity at a focus or caustic to be infinite which is obviously ridiculous. The field distribution near caustics are not properly calculated using geometric optics and in order to allow the advantages of the simplicity of geometric optics to be exploited everywhere we aim to extend geometric optics to accommodate these extra effects. A similar approach has already been used by Keller in the Geometric Theory of Diffraction which has proved highly successful.

1.2.2 Ray tube concepts

Consider the situation shown in Fig. 1.1 where two rays are drawn through adjacent points on a wavefront within the core of (a) a graded index and (b) a step index dielectric waveguide forming a characteristic flux tube as shown in each case. The flux or ray tube can be traced as shown through an optical medium by tracing each of the rays individually. In a non-absorbing core, power in the ray tube travels parallel to the ray direction and can only be lost to the cladding at reflections from the core-cladding interface or at caustics where the asymptotic description of geometric optics is inadequate.

1.3 A MODEL FOR ATTENUATION

1.3.1 Leaky mode attenuation

The electromagnetic field which propagates along an ideal non-absorbing optical waveguide has both bound and unbound components. The unbound modes give rise to a continuous spectrum of radiation modes which
Fig. 1.1 Sketch illustrating the formation of flux tubes from rays drawn through adjacent points on a wavefront in (a) a graded index and (b) a step index dielectric waveguide.
may be approximated by a discrete representation of leaky modes. For multimode waveguides, the principal contribution to the fields within the core at virtually all distances along the guide can be accurately approximated using a sum of all the leaky and bound modes of the structure. Each leaky mode is the analytic continuation of a bound mode below its cutoff value.

These modes have a simple physical interpretation. For bound modes the fields outside the core are everywhere evanescent and power remains trapped within the core. For leaky modes, oscillatory field behaviour in the cladding indicates that power is being lost from the core. The fields of leaky modes decay exponentially in the axial direction along the guide.

In order to calculate the modal fields within a guide, we seek solutions of the eigenvalue equation for the axial propagation constant $\beta$ of the electric and magnetic field vectors $E$ and $H$ which have axial $z$ dependence of the form $\exp\{i\beta z\}$. For bound modes $\beta$ is pure real and for leaky modes it is complex and of the form

$$\beta = \beta^r + i\beta^i \quad (1.3)$$

where the superscripts $r$, $i$ denote real and imaginary parts. The sign convention adopted uses an implicit time dependence $\exp\{-i\omega t\}$ which ensures that $\exp\{i\beta z\}$ represents a wave travelling in the $+$ve $z$ direction.

For an ideal non-absorbing waveguide power is lost uniformly from the leaky mode fields within the core and it is well known that the power remaining after some distance $z$ is given by

$$P(z) = P_0 \exp\{-\gamma z\} \quad (1.4)$$
where $P_0$ is the initial power of the mode and the constant $\gamma$ is the modal power attenuation coefficient. $\gamma$ is then given by

$$\gamma = 2\beta^i$$  \hspace{1cm} (1.5)

1.3.2 Leaky ray attenuation

The definition of a ray allows us to associate with a ray power travelling in the same direction. Strictly speaking, the terminology should refer to either a small ray tube or flux tube but it is simpler, physically appealing and unambiguous to merely refer to it as the power of the ray.

In a guiding structure the local plane wave fields of a family of rays, all with the same ray invariants and following similar ray paths, is the geometric optics equivalent of a mode. In order to develop a ray model for attenuation let us consider the situation shown for a ray on a dielectric waveguide in Fig. 1.2. The refractive index of the core is $n_1$ and of the cladding is $n_2$ with $n_2 < n_1$. The ray suffers reflections within the core of the guide losing a fraction of its power at each reflection. An equivalent situation exists for graded media, where a ray loses power each time it touches the outer caustic or turning point along the ray path in the core.

The ray power transmission coefficient $T$ is defined as

$$T = \frac{\text{Power lost from the incident ray}}{\text{Power in the incident ray}}$$  \hspace{1cm} (1.6)

The value of $T$ represents the fraction of power lost at each reflection and for an assymmetrical guide can vary from one reflection to the next. The definition (1.6) is equivalent, as we show in the next chapter, to the definition of the classical ray transmission coefficients of Fresnel.
It allows us to include all power attenuation processes in the cladding in one single expression.

The model for ray attenuation is formulated as follows. Consider the situation where a ray carrying initial power $P_0$ at $z=0$ undergoes a fractional loss $T_1$ at the first reflection, then after this reflection
there will be an amount of power $P_1$ left travelling along the ray, where

$$P_1 = P_0 (1 - T_1) \quad (1.7)$$

After a second reflection we have power $P_2$ left in the ray,

$$P_2 = P_0 (1 - T_1)(1 - T_2) \quad (1.8)$$

where $T_2$ denotes the loss at the second reflection. After $n$ reflections the remaining power is

$$P_n = P_0 \prod_{i=1}^{n} (1 - T_i) \quad (1.9)$$

If the distance to the first reflection is $z_1$ and the distance between the $i-1$ th reflection and the $i$ th reflection is $z_i$, then the distance travelled along the guide by the ray to the $n$ th reflection is

$$z_n = \sum_{i=1}^{n} z_i \quad (1.10)$$

It is convenient to express the power variation along the guide in the form of (1.4) and seek an expression for the ray power attenuation coefficient, $\gamma_R$. This is done by recasting (1.4) as

$$-\gamma_R z = \ln \frac{P(z)}{P_0} \quad (1.11)$$

Substituting $P_n$ for $P(z)$ and $z_n$ for $z$ gives

$$-\gamma_R \sum_{i=1}^{n} z_i P_i = \sum_{i=1}^{n} \ln(1 - T_i) \quad (1.12)$$

or

$$\gamma_R = -\frac{\sum_{i=1}^{n} \ln(1 - T_i)}{\sum_{i=1}^{n} z_i P_i} \quad (1.13)$$
If the fractional power loss at each reflection is the same fraction $T$, then $T_i = T$, $z_{p1} = z_p$ and (1.13) reduces to

$$\gamma_R = -\frac{\ln(1 - T)}{z_p}$$

(1.14)

Only when $T \ll 1$ does (1.14) reduce to the more familiar form $44$

$$\gamma_R = \frac{T}{z_p}$$

(1.15)

Generally, it is the more weakly attenuating rays which contribute significantly to the total power at points distant along the guide. For these rays the expression (1.15) is adequate. For strongly leaky rays (1.15) is deficient and the model (1.14) must be used to describe situations close to the source.

The value of $z_p$ on a straight step or graded index guide as indicated on Fig. 1.3 is easily calculated within the geometric optic approximation from the ray path invariants. $41$ On a bent fibre the calculation of the distance between points of reflection is not so simple even with the geometric optics approximation and the reader is referred to Chapter 7 for a detailed discussion.

We find in this thesis that under certain conditions the value of a small shift, known commonly as the Goos-Hänchen shift, must be added to the geometric optics approximation for the path length in order to give good agreement for attenuation between predictions of geometric optics and an exact modal analysis. In fact a modified form of the Goos-Hänchen shift valid for leaky rays near the critical angle is necessary. Details are presented in Chapter 6.

An examination of (1.13) gives a clue to the potency of this modified geometric optics analysis. For example on a bent fibre with the
Fig. 1.3 Tunnelling ray paths within an optical waveguide core showing distance $z_p$ between reflection points at which power is lost for (a) a graded index profile on a cylindrical waveguide and (b) a step index profile on a slab waveguide.

$z$ axis is lying along the bent axis of the guide, $\gamma$ is a function of $z$. The equivalent modal analysis is as a multimode guide is an intractable mode coupling problem in which power is coupled between the modes whilst radiating from each individual mode. The solution obtained from (1.13) by summing the power in the rays and using ray tracing is rapidly convergent as the number of ray increases.
The assembly of a complete expression for the attenuation coefficient \( \gamma \) to account for both core and cladding losses is a straightforward procedure carried out in Chapter 5. Since the calculation of \( z_P \) is a fairly elementary procedure the determination of the attenuation coefficient \( \gamma_R \) is essentially one of determining the ray power transmission coefficient \( T \). The calculation of \( T \) on dielectric optical waveguides is the prime objective of this thesis.

The author sincerely apologises for the obesity of Chapter 2 which develops the theory of ray transmission coefficients from first principles. A reader familiar with the mathematical methods used in this thesis may skip Chapter 2 and only return to read those parts to which he is referred in the later text.
REFERENCES


In this chapter, methods are developed for calculating the ray power transmission coefficient $T$ accurately for all leaky rays on lossless, multimode dielectric optical waveguides of both slab and circular geometry. For step-index profiles, these coefficients have already been obtained for certain restricted ray sets by several different methods,\textsuperscript{1-3} while for graded index profiles, the coefficients are obtainable in certain cases within the WKB approximation.\textsuperscript{4-6} In this chapter we develop a consistent and comprehensive ray optical theory which covers all these situations and generalises the results for arbitrarily graded fibres to include the effect of any finite number of discontinuities in the refractive index profile. The theory is shown to be uniformly valid for all leaky rays.

We begin with a discussion of elementary plane wave analysis on a simple slab guide, and then examine the WKB technique and its application to dielectric optical waveguides of circular cross section. The use of Airy function solutions for deriving connecting formulae to match WKB expressions on either side of a caustic, where the WKB method fails, is considered in detail. The methods are developed on the slab guide and then applied in a straightforward manner to the circular fibre guide.
A single universally valid expression applicable to both tunnelling and refracting leaky rays is also derived for the power transmission coefficient on step and graded index waveguides using the method of uniform approximation.

A comparison of all these methods for calculating the power transmission coefficient $T$, and hence the attenuation coefficient $\alpha$, with numerical solutions for the exact power attenuation coefficient of leaky modes is carried out in Chapter 6, where excellent agreement between the ray approach and exact electromagnetic theory is demonstrated. The limitations on slope of the refractive index profile inherent in the WKB and uniform approximation methods are examined in detail in section 2.5.

Problems involving fibres constructed with slightly absorbing core and cladding materials are considered in Chapter 4. In chapter 5 we show how to assemble a simple building block approach which encompasses all these effects and allows us to write down expressions for $T$ by inspection for most practical multilayered fibres.

In Chapter 3 we extend the techniques of this chapter to arbitrarily curved interfaces.

2.1 TRANSMISSION COEFFICIENTS IN A SLAB GUIDE

The slab dielectric waveguide is of significant practical importance in diffused optical devices and is expected to become more important as the field of integrated optics develops. There are emerging applications to optical switches and multiplexers. Light emitting diodes and laser diodes are currently fabricated as planar devices and the techniques discussed here are ideally suited to the analysis of losses on these structures.
However, even more significant is the fact that the concepts which we use can be derived on relatively simple slab geometry before being applied to the more complicated circular situation. This provides a convenient analytical tool which we exploit throughout this thesis.

2.1.1 Transmission coefficients on the step-index slab guide

The simplest guiding structure consists of a core of lossless uniform material with refractive index $n_1$ embedded in an infinite lossless cladding medium of index $n_2$ so that $n_2 < n_1$. The situation is depicted in Fig. 2.1 which shows a simple slab guide oriented with its axis along the $z$ axis of a local cartesian coordinate system. The width of the guide is $2\rho$.

![Diagram of a simple slab guide](image)

Fig. 2.1 Simple lossless step-index slab waveguide with uniform core and cladding having refractive indices $n_1$ and $n_2$ respectively such that $n_1 > n_2$. The core width is $2\rho$. The ray loses power by refraction at alternate reflections from the upper and lower core-cladding interfaces.
The waveguide parameter $V$ is defined as

$$V = k \rho (n_1^2 - n_2^2)^{1/2}$$

(2.1)

where $k = 2\pi/\lambda$ and $\lambda$ is the wavelength of the incident radiation in vacuum. For $V < \pi/2$, the guide is monomode and only one mode propagates. For $V >> 1$, many modes propagate and the guide is called multimode. For multimode guides it becomes convenient to use an asymptotic or ray representation.

For a ray being reflected within the core of a multimode guide as illustrated in Fig. 2.1, we define the quantity $\tilde{\beta}$ where

$$\tilde{\beta} = n_1 \cos \theta_1$$

(2.2)

which remains invariant at all points along the guide. The angle $\theta_1$ that the ray makes with the $z$ direction is preserved at each reflection within the core. The same $\tilde{\beta}$ is associated with the ray fields in the cladding and

$$\tilde{\beta} = n_2 \cos \theta_t$$

(2.3)

If $\cos \theta_t > 1$ then no solution exists in the cladding and the ray fields are confined to the core giving, for bound rays, $n_1 < \tilde{\beta} < n_2$. When $\tilde{\beta} < n_2$ then a fraction of the power associated with the ray is transmitted into the cladding at each reflection from the core-cladding boundary. Power therefore leaks from the core of the guide by refraction and these rays are called leaky rays.

We define $\theta_c = \theta_1$ when $\theta_t = 0$ so that $\theta_c = \cos^{-1}(n_2/n_1)$. The usual critical angle $\alpha_c$, measured with respect to the normal to the interface, is defined by $\alpha_c = \pi/2 - \theta_c$. 
If we adopt the convention of assuming an implicit time dependence for the ray fields of the form $\exp\{-i\omega t\}$, then the variation of the fields in the outward or $y$ direction associated with a ray can be simply represented by the form $\exp\{\pm ik_y y\}$ where the $+$ve sign is associated with a wave travelling in the $+y$ direction and the $-$ve sign with a corresponding wave in $-y$ direction. $k_y$ is the outward component of the local plane wave representation for the fields associated with the ray.

Geometric optics predicts\(^9\) that the reflection coefficient $R$ given by the ratio of the reflected and incident ray amplitudes at each point of reflection is

$$R = \frac{k_{1y} - k_{2y}}{k_{1y} + k_{2y}} \quad \tilde{\beta} < n_2$$

for the incident ray polarised with the $y$ component of the electric field vector $E_y = 0$, where $k_{1y}$ and $k_{2y}$ are the outward components of the wave vector in the core and in the cladding on respective sides of the interface. The corresponding ray transmission coefficient, representing the fraction of power lost from the core at each reflection, is defined by eqn (1.6) as

$$T = 1 - |R|^2$$

which reduces to (for $E_y = 0$)

$$T = \frac{4 k_{1y} k_{2y}}{(k_{1y} + k_{2y})^2} \quad \tilde{\beta} < n_2$$

and

$$T = 0 \quad \tilde{\beta} > n_2$$

Eqn (2.6a) represents the classical Fresnel transmission coefficient\(^9\) $T^E_F$. For the incident ray polarised with the $y$ component of the magnetic
field vector $H_y = 0$, then the corresponding expressions for $T$ are

$$T = \frac{4 n^2 n^2 k y_1 k y_2}{(n^2 k_y 1 + n^2 k y_2)^2} \quad \tilde{\beta} < n_2$$

(2.7a)

and

$$T = 0 \quad \tilde{\beta} > n_2$$

(2.7b)

Eqn (2.7a) is the classical Fresnel transmission coefficient $^9 T^H_P$.

In most practical situations the core and cladding refractive indices are closely equal and polarisation becomes unimportant. When $n_1 \approx n_2$ and the incident polarisation is arbitrary then $T$ is given by eqn (2.6).

2.1.2 Transmission coefficients on the graded index slab

Classical geometric optics provides a convenient short wavelength ($\lambda \rightarrow 0$) approximation to the electromagnetic fields propagating in a multimode optical waveguide. When $\lambda = 0$, power flows along the direction of a ray and, in an ideal non absorbing guide, no power can be lost along the ray path. However, when $\lambda \neq 0$, a modified form of geometric optics must be used to describe power losses occurring at caustics where the classical theory is invalid. The caustics are the only points along the ray path in an ideal medium where power can be lost since it is the only point where the local plane wave field representation is inappropriate.

In order to derive expressions for the ray power transmission coefficient, $T$ of eqn (1.6), on a graded index slab waveguide, we begin by first considering the behaviour of the electromagnetic fields in the region of a caustic. The development of formulae to connect the WKB solutions, discussed in Appendix 2A, across the region near a caustic is assisted by using the Airy function solutions of Appendix 2B and by
identifying the asymptotic forms of the Airy functions away from the caustic with the WKB solutions.

Firstly we consider the field behaviour near an incident caustic, then near a radiation caustic, and finally for profiles where both incident and radiation caustics are present and well separated.

2.1.2.1 WKB connecting formulae at an incident caustic

Consider initially the monotone decreasing refractive index profile $n^2(y)$ as shown in Fig. 2.2, which depicts a ray incident from the region $y < y_c$ that reflects at the caustic at $y = y_c$ and returns to

![Graph](image_url)
\( y < y_C \). In the region \( y < y_C \) the electromagnetic fields are wavelike and for \( y > y_C \) only evanescent decaying fields are present and vanish as \( y \to \infty \). In both regions the fields have simple \( z \) dependence and the scalar wave equation which determines the fields in the \( n_1 \approx n_2 \) approximation has solutions of the form

\[
\psi(y, z) = \psi(y) e^{ik\tilde{\beta}z} \tag{2.8}
\]

where \( \tilde{\beta} \) is the ray invariant related to the axial propagation constant \( \beta \) by \( \tilde{\beta} = \beta/k \).

The function \( \psi \) satisfies the equation

\[
(\nabla^2 + k^2 n^2(y))\psi(y, z) = 0 \tag{2.9}
\]

which reduces using (2.8), to

\[
\frac{d^2}{dy^2} + k_y(y)^2 \psi(y) = 0 \tag{2.10}
\]

where \( k_y(y) \) is the outward component of the wave vector in the \( y \) direction and is given by

\[
k_y^2(y) = k^2(n^2(y) - \tilde{\beta}^2) \tag{2.11}
\]

At the caustic \( y = y_C \), \( k_y^2(y) \) vanishes by definition, and the value of \( y_C \) is calculated from

\[
\tilde{\beta}^2 = n^2(y_C) \tag{2.12}
\]

Thus for \( y < y_C \) we have \( k_y^2(y) > 0 \) and for \( y > y_C \), \( k_y^2(y) < 0 \). The appropriate WKB solutions of (2.10) (see Appendix 2A) are

\[
\psi(y) = \frac{A}{|k_y(y)|^{1/2}} \exp \left\{ - \int_y^{y_C} k_y(t) dt \right\} \quad \text{for } y > y_C \tag{2.13}
\]
and

$$\psi(y) = \frac{B,C}{[k_y(y)]^\frac{1}{2}} \exp \left\{ \pm i \int_y^{y_c} k_y(t) dt \right\} \quad \text{for } y < y_c \quad (2.14)$$

where A,B,C are constants to be determined.

The relationship between A,B,C is derived using appropriate connecting formulae across the caustic. This is done via the Airy function solutions for the linearised profile in the region of the caustic where $k_y^2(y)$ is expressed as a Taylor series expansion

$$k_y^2(y) = k_y^2(y_c) + (y-y_c) \frac{dk_y^2(y)}{dy} \bigg|_{y=y_c} + o((y-y_c)^2) \quad (2.15)$$

$$\cong (y-y_c)k^2\delta \quad (2.16)$$

where

$$\delta = \left[ - \frac{dn^2(y)}{dy} \right]_{y=y_c}$$

Substituting (2.16) into (2.10) gives

$$\frac{d^2\psi(y)}{dy^2} - k^2\alpha(y-y_c) \psi(y) = 0 \quad (2.17)$$

which transforms into Airy's equation

$$\frac{d^2\psi(w)}{dw^2} - w\psi(w) = 0 \quad (2.18)$$

under the transformation of eqn. (B.3)

$$w = k^{2/3}\delta^{1/3}(y-y_c) \quad (2.19)$$

The solution of (2.18) is

$$\psi = aAi(w) + bBi(w) \quad (2.20)$$
where \( A_i, B_i \) are Airy functions of the first and second kind respectively and \( a \) and \( b \) are constants. When \( y > y_c, w > 0 \) and when \( y < y_c, w < 0 \).

The asymptotic forms of the Airy functions for large positive values of \( w \) are

\[
A_i(w) \sim \frac{1}{2\sqrt{\pi w}} \exp \left\{ -\frac{2}{3} w^{3/2} \right\}, \quad B_i(w) \sim \frac{1}{\sqrt{\pi w}} \exp \left\{ \frac{2}{3} w^{3/2} \right\}
\]

For large negative

\[
A_i(w) \sim \frac{1}{\sqrt{\pi (-w)^{3/2}}} \sin \left\{ \frac{\pi}{4} + \frac{2}{3} (-w)^{3/2} \right\},
\]

\[
B_i(w) \sim \frac{1}{\sqrt{\pi (-w)^{3/2}}} \cos \left\{ \frac{\pi}{4} + \frac{2}{3} (-w)^{3/2} \right\}
\]

For the solution to be bounded in the evanescent region \( y > y_c \), then in eqn (2.20) \( b = 0 \) and \( \psi(y) \) becomes

\[
\psi(y) \sim \frac{a}{2\sqrt{\pi w^{3/2}}} \exp \left\{ -\frac{2}{3} w^{3/2} \right\} \quad \text{for} \quad y > y_c
\]

\[
\psi(y) \sim \frac{a}{2\sqrt{\pi (-w)^{3/2}}} \left[ \exp \left\{ \frac{2}{3} (-w)^{3/2} - \frac{\pi}{4} \right\} + \exp \left\{ -\frac{2}{3} (-w)^{3/2} - \left( \frac{\pi}{4} - \frac{\pi}{2} \right) \right\} \right]
\]

The phase change of \(-\pi/2\) for a wave touching the caustic can be clearly seen in (2.24). It is now straightforward to relate the WKB solutions (2.13) (2.14) to the asymptotic forms of the Airy functions (2.23), (2.24).

From (2.19) and (2.16)

\[
k_y^2(y) = -(k^2 \delta)^{2/3} w
\]

Substituting (2.25) into (2.13), (2.14) and performing the simple integration in each case
\[ \psi(y) = \frac{A}{(\delta k^2)^{1/6} w^{1/4}} \exp \left\{ -\frac{2}{3} w^{3/2} \right\} \text{ for } y > y_c \] (2.26)

\[ \psi(y) = \frac{B_C}{(\delta k^2)^{1/6} (-w)^{1/4}} \exp \left\{ \pm i \frac{2}{3} (-w)^{3/2} \right\} \text{ for } y < y_c \] (2.27)

Comparing (2.26) and (2.23) shows

\[ a = \frac{2\sqrt{\pi} A}{(\delta k^2)^{1/6}} \] (2.28)

and comparing (2.27) with (2.24) yields the relations

\[ B = A \exp \{-i \frac{\pi}{4}\} \]

\[ C = A \exp \{i \frac{\pi}{4}\} \] (2.29)

The complete connected WKB formulae are

\[ \psi(y,z) = \frac{A}{|k_y(y)|^{1/2}} \exp \left\{ -\int_{y_c}^{y} |k_y(t)| dt \right\} e^{ik\tilde{z}} \]

\[ \text{for } y > y_c \] (2.30)

\[ \psi(y,z) = \frac{2A}{|k_y(y)|^{1/2}} \cos \left\{ \int_{y_c}^{y} k_y(t) dt - \frac{\pi}{4} \right\} e^{ik\tilde{z}} \]

\[ \text{for } y < y_c \] (2.31)

These are equivalent to the WKB formulae of ref. 11.

The ratio of the reflected and incident wave amplitudes \( R \) is the ratio \( B/C \) and in this case we see clearly \( |R| = 1 \). The ray power transmission coefficient (2.5) is \( T = 0 \) for this case, as we would expect, since no power is lost upon reflection.
For a situation where the refractive index profile is no longer monotone decreasing but increases beyond some point in the region $y > y_c$ then it is possible that reradiation may be induced at some point called the radiation caustic. We now examine the WKB connecting formulae in the region of a radiation caustic where the refractive index profile increases.

2.1.2.2 WKB connecting formulae at a radiation caustic

This problem involves the reverse situation of 2.1.2.1 above, and is depicted in Fig. 2.3. The refractive index profile $n^2(y)$ is shown monotone increasing in the vicinity of a radiation caustic at $y = y_d$. In

Fig. 2.3 The region $y < y_d$ contains purely evanescent fields, while the region $y > y_d$ supports a ray originating at $y = y_d$
the region \( y < y_d \) an evanescent wave is assumed to exist, while for \( y > y_d \) a real wave, or ray, emerges from the reradiation caustic and propagates away to \( \infty \). In the vicinity of \( y_d \) a Taylor series expansion for \( k^2(y) \) is valid, provided \( n^2(y) \) is smoothly varying, in a similar way to (2.15) which gives in this case

\[
k_y(y) \approx (y-y_d)k^2\delta
\]  

(2.32)

where

\[
\delta = \left[ \frac{dn^2(y)}{dy} \right]_{y=y_d}
\]

Using a similar transformation as in the previous section gives the solution for this case as

\[
\psi(y) = a \text{Ai}(w) + b \text{Bi}(w)
\]  

(2.33)

where

\[
w = -k^{2/3} \delta^{1/3}(y-y_d)
\]  

(2.34)

As \( y \to \infty \), \( w \to -\infty \) and the appropriate combination of Airy functions is the one which represents an outgoing wave propagating in the \( +y \) direction giving

\[
\psi(y) = a[\text{Ai}(w) - i \text{Bi}(w)]
\]  

(2.35)

Thus in the evanescent region when \( w \to +\infty \) the asymptotic form of (2.35) is

\[
\psi(y) = \frac{a}{\sqrt{\pi w^3}} \left\{ \frac{1}{2} \exp\left( -\frac{2}{3} w^{3/2} \right) - i \exp\left( \frac{2}{3} w^{3/2} \right) \right\}
\]  

(2.36)

which represents physically, incident and reflected evanescent waves at the reradiation caustic \( y = y_d \). In the region \( y > y_d \) the asymptotic form of (2.35) represents a real propagating wave travelling away from \( y = y_d \) and is given by
\[
\psi(y) \equiv -\frac{ai}{\sqrt{\pi(-\omega)^{\frac{1}{4}}}} \exp i\left\{\frac{2}{3} (-\omega)^{3/2}\right\} e^{\frac{i\pi}{4}} \quad (2.37)
\]

The corresponding WKB solutions using

\[
k_y^2(y) = k_y^2(y-y_d) = -(k_y^2)^{2/3} \omega \quad (2.38)
\]
reduce after performing the simple integration as for (2.26) and (2.27) to give in this case

\[
\psi(y) = \frac{A}{(k_y^2)^{1/6}(-\omega)^{\frac{1}{4}}} \exp i\left\{\frac{2}{3} (-\omega)^{3/2}\right\} \quad \text{for } y > y_d \quad (2.39)
\]
and

\[
\psi(y) = \frac{B,C}{(k_y^2)^{1/6}(-\omega)^{\frac{1}{4}}} \exp i\left\{\frac{2}{3} \omega^{3/2}\right\} \quad \text{for } y < y_d \quad (2.40)
\]

Comparing (2.39) with (2.37)

\[
A = -ia(k_y^2)^{1/6} e^{\frac{i\pi}{4}} \quad (2.41)
\]
and (2.40) with (2.36) gives

\[
B = Ae^{-i\frac{\pi}{4}}
\]
\[
C = A e^{+i\frac{\pi}{4}}
\]

The complete analytic expressions connected across the reradiation caustic are then

\[
\psi(y,z) = \frac{A}{|k_y(y)|^{\frac{1}{4}}} \exp i\left\{\int_{y_d}^{y} k_y(t) dt + ikz\right\} \quad \text{for } y > y_d \quad (2.42)
\]
\[
\psi(y,z) = \frac{Ae^{-i\frac{\pi}{4}}}{|k_y(y)|^{\frac{1}{4}}} \exp i\left\{\int_{y_d}^{y} |k_y(t) dt| + ikz\right\} + \frac{Ae^{+i\frac{\pi}{4}}}{2|k_y(y)|^{\frac{1}{4}}} \exp i\left\{\int_{y_d}^{y} |k_y(t) dt| + ikz\right\} \quad \text{for } y < y_d \quad (2.43)
\]
2.1.2.3 Combined WKB connecting formulae

For the refractive index profile containing a dip as shown in Fig. 2.4, and with incident and radiation caustics occurring at $y_c, y_d$ respectively, the connecting formulae derived in the previous two sections must both be applied. An evanescent region exists between $y_c$ and $y_d$, which transports a fraction of the power associated with the incident ray at $y = y_c$, across the evanescent region and reemerges at $y = y_d$, radiating away along the ray path from $y_d$. We connect the WKB formulae across the caustic at $y_c$ and $y_d$ in the following way.

Fig. 2.4 Refractive index profile $n^2(y)$ containing a dip and incident and radiation caustics occurring at $y_c$ and $y_d$ respectively. An evanescent region exists between $y_c$ and $y_d$. 
For the profile shown in Fig. 2.4, consider the amplitude of the wave incident on $y_c$ as $C$ and the reflected wave amplitude $D$. The WKB expressions for the corresponding wave function $\psi_i$, $\psi_r$ in the region $y < y_c$ are

$$\psi_i(y) = \frac{C}{[k_y(y)]^\frac{1}{2}} \exp \left\{ i \left( \int_y^{y_c} k_y(t) \, dt \right) \right\} \quad y < y_c$$

$$\psi_r(y) = \frac{D}{[k_y(y)]^\frac{1}{2}} \exp \left\{ i \left( \int_y^{y_c} k_y(t) \, dt \right) \right\} \quad y < y_c$$

Using the method developed in 2.1.2.1 above, it is straightforward to show that these rays result in evanescent fields in the region $y > y_c$ given by

$$\psi_i(y) = \frac{C}{2[k_y(y)]^\frac{1}{2}} \exp \left\{ \frac{i \pi}{4} - \frac{$$}$$\int_y^{y_c} |k_y(t)| \, dt \right\}$$

$$+ \frac{C e^{-i \frac{\pi}{4}}}{[k_y(y)]^\frac{1}{2}} \exp \left\{ \int_y^{y_c} |k_y(t)| \, dt \right\} \quad y > y_c$$

$$\psi_r(y) = \frac{D}{2} e^{-i \frac{\pi}{4}} \exp \left\{ \int_y^{y_c} |k_y(t)| \, dt \right\}$$

$$+ D e^{i \frac{\pi}{4}} \exp \left\{ \int_y^{y_c} |k_y(t)| \, dt \right\} \quad y > y_c$$

The WKB connected fields for the transmitted ray $y > y_d$ and the evanescent fields $y < y_d$ are already given by (2.42) and (2.43). Between the caustics the fields must remain continuous. Comparing like integrals in (2.43) and the sum of (2.46) and (2.47) gives two equations

$$A e^x = \frac{Ci}{2} - \frac{D}{2}$$

$$\frac{A}{2} e^{-x} = Ci + D$$
where

\[ x = \int_{y_c}^{y_d} k_y(t) \, dt \]

Eliminating \( A \) gives

\[ \frac{e^{-2x}}{4} (C_1 - D) = (C_1 + D) \]  
(2.50)

\[ \frac{D}{C} = i \frac{(e^{-2x} - 4)}{(e^{-2x} + 4)} \]  
(2.51)

The ray power transmission coefficient \( T \) is therefore from eqn (1.6)

\[ T = 1 - \left| \frac{D}{C} \right|^2 \]
\[ = \frac{e^{-2x}}{1 + \frac{e^{-2x}}{4}} \]  
(2.52)

which is of the same form as in ref. 12.

For \( T \ll 1 \) the expression (2.52) reduces simply to

\[ T \approx \exp \left\{ - \frac{1}{2} \int_{y_c}^{y_d} |k_y(t)| \, dt \right\} \]  
(2.53)

This expression (2.53) is valid provided the caustics at \( y_c \) and \( y_d \) are sufficiently well separated. For caustics so close that the asymptotic forms of the Airy functions used to derive the WKB connecting formulae are invalid then more complicated expressions are needed to calculate the power transmission coefficient \( T \). (For example see section 2.4).

2.1.2.4 Transmission coefficients for profiles containing a kink

So far in this section, only smoothly varying refractive index profiles on the slab guide have been considered. In many practical situations a discontinuity in the slope of the profile may occur at the core-cladding boundary, as shown in Fig. 2.5. The profile is continuous
at $y = \rho$ and the classical Fresnel transmission coefficient (2.6) predicts that the power in a ray incident on the interface is totally transmitted. However, a significant amount of power may be reflected back into the core region $y < \rho$ by the discontinuity in the profile slope at the boundary. Two methods are used to illustrate this effect, (i) the WKB and (ii) the Airy function solutions, which are shown to be asymptotically equivalent when the incident ray is not close to grazing incidence.
(i.e. $\theta_t \neq 0$) on the interface.

(i) WKB analysis

Consider the profile $n^2(y)$ shown in Fig. 2.5, which contains a kink or discontinuity in the profile slope at the core cladding interface $y = \rho$. The core has a graded refractive-index profile and the cladding is uniform beyond $y = \rho$. A ray incident on the interface at angle $\alpha_t$, with respect to the normal to the boundary, is partially reflected and partially transmitted. Normalising the incident amplitude to unity and assuming the reflected and transmitted field amplitudes are $R$, $S$ respectively the WKB solutions for the fields on either side of the interface at $y = \rho$ follow from (2.44), (2.45) and (2.42) to give

$$\psi_i + \psi_r = \left[ \frac{1}{[k_{y_1}]} \exp\left\{ i \int_{y}^{\rho} k_{y_1} (t) dt \right\} \right]$$

$$+ \frac{R}{[k_{y_1}]} \exp\left\{ -i \int_{y}^{\rho} k_{y_1} (t) dt \right\} e^{ik_y z}, \quad y < \rho \quad (2.54)$$

and

$$\psi_r = \frac{S}{[k_{y_1}]} \exp\left\{ i \int_{y_1}^{\rho} k_{y_1} (t) dt \right\} e^{ik_y z}, \quad y > \rho \quad (2.55)$$

where $k_{y_1}$ and $k_{y_2}$ are the outward components of the local plane wave vector in the core and the cladding respectively.

At $y = \rho$ the tangential electric and magnetic fields must be continuous across the interface leading to a continuity of $\psi$ and $\psi'$. This gives

$$1 + R = S \quad (2.56)$$
If the slope of the refractive index profile $n^2(y)$ in the core at $y = \rho$ is given by $\delta$, then the linearised form of the profile becomes

$$\frac{dk_y}{dy} \bigg|_{y=\rho} = -\frac{\delta k^2}{2k_y^1}, \quad \frac{dk_y}{dy} \bigg|_{y=\rho} = 0$$ \hspace{1cm} (2.58)

Eliminating $S$ in (2.56) and (2.57) gives

$$\frac{1-R}{1+R} = 1 + \frac{i \delta k^2}{4k_y^1 (\rho)^3}$$ \hspace{1cm} (2.59)

The ray transmission coefficient $T$ is given in terms of the reflection coefficient $R$ by

$$T = 1 - |R|^2$$ \hspace{1cm} (2.60)

$$= 1 - \frac{1}{64} \cdot \frac{\delta k^2}{(n_2^6 \cos^6 \alpha_t)}$$ \hspace{1cm} (2.61)

The WKB solution (2.61) is valid provided that the fields are remote from a turning point or caustic, i.e. the incident ray is not too close to grazing incidence at the interface $y = \rho$.

(ii) Airy function analysis

For the situation shown in Fig. 2.5, linearising the refractive index profile $n^2(y)$ close to $y = \rho$, as in (2.58), allows us to write down solutions for the fields exactly in terms of Airy functions.
Normalising the incident amplitude to unity and assuming the reflected field has amplitude $R$ gives, (from Appendix 2B) and taking the appropriate linear combination of the Airy functions

$$\Psi_i = [\text{Ai}(w) - i \text{Bi}(w)] e^{ik_z z} \quad (2.62)$$

$$\Psi_r = R[\text{Ai}(w) + i \text{Bi}(w)] e^{ik_z z} \quad (2.63)$$

where $w = (\delta k^2)^{1/3} (y - y_c)$ and $y_c$ represents the position of the caustic on the analytic continuation of the core profile into the cladding. The use of this fictitious caustic for refracting rays is a powerful tool and will be discussed further in section 2.4.

In the uniform cladding the solution for the fields is trivially

$$\Psi_{t} = \frac{i}{\gamma} \left( \int_{y_1}^{\rho} k(t) \, dt \right) \quad (2.64)$$

Matching the fields at $y = \rho$ requires continuity of $\Psi$ and $\frac{d \Psi}{dy}$ to give,

using $u = w|_{y=\rho}$ and $u' = w'|_{y=\rho}$

$$[(1+R)\text{Ai}'(u) + i(R-1)\text{Bi}'(u)] u' = -S i k \quad (2.65)$$

Eliminating $S$ and putting $K = u'/k \quad (2.66)$

$$[(1+R)\text{Ai}'(u) + i(R-1)\text{Bi}'(u)] u' = -S i k \quad (2.66)$$

$$\begin{align*}
(1+R)\text{Ai}(u) + i(R-1)\text{Bi}(u) &= S \\
(1+R) \text{i Ai}(u) - (R-1) \text{Bi}(u) &= -[(1+R)\text{Ai}'(u) + i(R-1)\text{Bi}'(u)] K
\end{align*} \quad (2.67)$$

$$R = \frac{(-K\text{Ai}'(u) - \text{Bi}(u)) + i(K\text{Bi}'(u) - \text{Ai}(u))}{(K\text{Ai}'(u) - \text{Bi}(u)) + i(K\text{Bi}'(u) + \text{Ai}(u))} \quad (2.68)$$

This is of the form quoted in Appendix 2D and can be readily simplified using the results of that appendix to give the ray power transmission coefficient $T$, defined as
\[ T = 1 - |R|^2 \] (2.69)

as

\[
T = \frac{4K}{\pi K^2 \left[ A_i'(u)^2 + B_i'(u)^2 \right] + \frac{2K}{\pi} + \left[ A_i^2(u) + B_i^2(u) \right]}
\] (2.70)

This represents the ray transmission coefficient for all angles of ray incidence even for rays which graze the interface \( y = \rho \) provided the incidence angle is not so steep that the linearisation of the profile about \( y = \rho \) is inappropriate. Then the WKB analysis in (i) above is valid.

(iii) Comparison of (i) and (ii) for refracting rays

The asymptotic form of (2.70) for \( u \) large and negative can be evaluated. This represents real propagating waves on each side of \( y = \rho \). Retaining the first three terms in the asymptotic expansion for the Airy functions\(^{10}\) gives

\[
\begin{align*}
A_i(u) &= \frac{1}{\sqrt{\pi}(-u)^{\frac{1}{4}}} \left[ \sin \Omega - \cos \Omega \xi^{-1} c_1 - \sin \Omega \xi^{-2} c_2 \right] \\
A_i'(u) &= \frac{-(-u)^{\frac{1}{4}}}{\sqrt{\pi}} \left[ \cos \Omega - \frac{7}{5} \sin \Omega \xi^{-1} c_1 + \frac{13}{11} \cos \Omega \xi^{-2} c_2 \right] \\
B_i(u) &= \frac{1}{\sqrt{\pi}(-u)^{\frac{1}{4}}} \left[ \cos \Omega + \sin \Omega \xi^{-1} c_1 - \cos \Omega \xi^{-2} c_2 \right] \\
B_i'(u) &= \frac{(-u)^{\frac{1}{4}}}{\sqrt{\pi}} \left[ \sin \Omega + \frac{7}{5} \cos \Omega \xi^{-1} c_1 + \frac{13}{11} \sin \Omega \xi^{-2} c_2 \right]
\end{align*}
\] (2.71)

where \( c_1 = \frac{5}{72}, c_2 = \frac{385}{10368}, \xi = \frac{2}{3} (-u)^{3/2}, \Omega = \xi + \frac{\pi}{4} \). Substituting (2.71) into (2.70) and collecting terms in the denominator in powers of \( \xi \) we see that the coefficient of \( \xi^0 \) is 4, the coefficient of \( \xi^{-1} \) is zero and the coefficient of \( \xi^{-2} \) is \( \frac{4}{25} c_1^2 + \frac{8}{11} c_2 \). Thus the ray power transmission coefficient \( T \) is
\[ T = \frac{4}{4 + \frac{1}{36} \cdot \xi^{-2}} \quad (2.72) \]

which reduces exactly to eqn (2.61).

The two methods (i) and (ii) above are therefore asymptotically equivalent.

(iv) Power law profiles

If the core refractive index profile for the graded slab was of the form

\[ n^2(y) = n_1^2(1 - 2\Delta(y/\rho)^q) \quad (2.73) \]

where \( q \) is an arbitrary positive number and \( \Delta = \frac{\sin^2 \theta_c}{2} \), then

\[ \delta = \frac{2\Delta n_1^2 \rho}{\rho} \quad (2.74) \]

and both (2.72) and (2.61) give the ray power transmission coefficient for angles of incidence not too close to grazing as

\[ T = 1 - \frac{1}{64} \left[ \frac{q^6 \rho^2}{n_2 k \rho \sin^3 \theta_c} \right]^2 \quad (2.75) \]

where \( \theta_c = \frac{\pi}{2} - \alpha_c \). For rays close to grazing the complete expression for \( T \), (2.70), should be used.

(v) Bound ray limit of (ii) above

To recover the bound ray result, the solution (2.64) must represent an evanescent field. Then for \( y_c < \rho \), \( u >> 1 \) and \( T = 0 \).

2.2 TRANSMISSION COEFFICIENTS IN A CIRCULAR GUIDE

The circular dielectric optical fibre is the most convenient practical guiding structure and is expected to be deployed widely in
communication systems. In this section we investigate the extension of the methods used to analyse slab guides in the preceding sections, to the problem of calculating the ray transmission coefficients on circular optical fibres. The effects of radiation due to both tunnelling and refraction, which cause loss of power from the core of a fibre, are analysed using the WKB and Airy function solutions for the ray transmission coefficients, since these methods are simple and are closely identifiable with the physics. The more powerful method of uniform approximation which results in more complicated expressions for $T$ is discussed in section 2.4.

Fig. 2.6 Sketch of a refractive index profile $n^2(r)$ for a simple step index circular fibre of radius $\rho$. For $0 \leq r < \rho$, $n(r) = n_1$ and for $r > \rho$, $n(r) = n_2$. 


2.2.1 Transmission coefficients on the step-index fibre

The simple step index fibre consists of a core of uniform loss-less material with refractive index $n_1$ surrounded by a cladding of similar material of refractive index $n_2$ but with $n_2 < n_1$. For $n_1 \equiv n_2$ the polarisation effects are small and ignored here (see section 2.3).

For a multimode fibre in circular cylindrical coordinates $(r, \phi, z)$ with the z axis oriented along the axis of the guide and with radial refractive index profile as shown in Fig. 2.6, the incident and reflected fields at the interface $r = \rho$ resemble local plane waves. The divisions between bound, tunnelling and refracting rays in ray angle space is represented in Fig. 2.7. If the angle the incident ray makes with the axial z direction is $\theta_z$, then rays with $\theta_z < \frac{\pi}{2} - \alpha_c$, where $\alpha_c = \sin^{-1}(n_2/n_1)$ is the critical angle, are trapped. All other rays lose a fraction of the incident power at each reflection and are therefore leaky. The border line between tunnelling and refracting leaky rays is the angle $\alpha_1 = \alpha_c$.

We now proceed to construct expressions for the ray transmission coefficient $T$ of eqn (1.6) and show how these expressions reduce to the results for refracting and bound rays in the appropriate limits.

2.2.1.1 Derivation of $T$ on step index fibres

For the simple step index profile depicted in Fig. 2.6 solutions for the wave equation

$$\nabla^2 \psi + k^2 \psi = 0 \quad (2.76)$$

are sought in cylindrical polar coordinates $(r, \phi, z)$. If the azimuthal variation around the fibre cross section takes the form $\exp\{i \tilde{k} \phi \}$ then the wave equation becomes
Fig. 2.7 Visual representation of the three regions of incident ray angle space for a ray that arrives at P from the core. Trapped ray angles lie in the half cone $\theta_z \leq \pi/2 - \alpha_c$, and refracting rays lie in the half cone $\alpha_d \leq \alpha_c$. The two half-cones touch along a common generator in the meridional plane. Tunnelling rays are incident in the two symmetric regions on either side of the meridional plane through P and exterior to the half cones.

\[
\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + k^2 \left(n^2 - \tilde{\beta}^2 - \left(\frac{\rho\tilde{\ell}}{r}\right)^2\right)\psi = 0
\]  

(2.77)

where $\Psi(\rho, \phi, z) = \psi(\rho) \exp\{i k (\tilde{\ell}\rho \phi + \tilde{\beta} z)\}$ and we have used the ray invariants $\tilde{\beta}, \tilde{\ell}$ related to $\beta, \ell$ by

\[
\beta = k\tilde{\beta}, \quad \ell = \rho k\tilde{\ell}.
\]

(2.78)
The convention of the suppressed time variation of section 2.1.1 has been adopted. The transformation $\Phi = \psi / \sqrt{r}$ reduces (2.77) to

$$\frac{d^2 \Phi}{dr^2} + k^2 \left[ n^2 - \beta^2 - \left( \frac{\rho}{r} \right)^2 \left( \bar{\chi}^2 - \frac{1}{4(k_0)^2} \right) \right] \Phi = 0 \quad (2.79)$$

Comparing (2.79) with (2.10) we see immediately that, ignoring the factor $\frac{1}{4k_0}$ with respect to $\bar{\chi}^2$ since we are considering multimode fibres with $V \gg 1$ and hence $\beta_0$ is negligible, the equations are of the same form but with $y$ replaced by $r$ and

$$k_r^2(r) = k^2 \left[ n^2 - \beta^2 - \left( \frac{\beta_0}{r} \right)^2 \right] \quad (2.80)$$

which is the radial component of the wave vector in cylindrical geometry.

The factor $\sqrt{r}$ used in the transformation of (2.77) to (2.79) is necessary to conserve energy within a given flux tube in cylindrical geometry. For tunnelling leaky rays, evanescent fields exist in the cladding from the core-cladding interface at $r = \rho$ out to the point $r = r_{\text{rad}}$, called the radiation caustic, and real propagating waves exist for $r > r_{\text{rad}}$. The position of $r_{\text{rad}}$ is a root of (2.80). A fraction of the power associated with the incident ray is transmitted across the interface to appear at $r_{\text{rad}}$ and radiate away in the cladding.

Taking unit normalised incident ray amplitude in the core and reflected ray amplitude $R$, the local plane wave fields near the core cladding interface have the form

$$\Phi(r) = \exp{i k_r (r)} + R \exp{-i k_r r} \quad r < \rho \quad (2.81)$$

The solutions of (2.79) is a linear combination of Airy functions

$$\Phi(r) = a \text{Ai}(w) + b \text{Bi}(w) \quad (2.82)$$
where
\[ w = \left( \frac{2k_2^2}{\rho} \right)^{1/3} (r_{rad} - r) \] (2.83)

and \( a, b \) are constants.

The appropriate linear combination of Airy functions so that \( \Phi \) represents an outgoing wave as \( r \to \infty \) is \( b = -ia \) so that the fields in the cladding are represented as
\[ \Phi(r) = a[Ai(w) - iBi(w)] \] (2.84)

Continuity of the fields at \( r = \rho \) is equivalent to a continuity of \( \Phi \) and \( \frac{d\Phi}{dr} \) leading to
\[ \exp{ik_1 \rho} + R \exp{-ik_1 \rho} = a[Ai(u) - iBi(u)] \] (2.85)
\[ ik_1 \exp{ik_1 \rho} - R \exp{-ik_1 \rho} = u'a[Ai'(u) - Bi'(u)] \] (2.86)

where
\[ u = w \bigg|_{r=\rho}, \quad u' = w' \bigg|_{r=\rho} \]

and the prime denotes differentiation with respect to argument.

Putting \( K = \frac{-u'}{k_1} \) gives
\[ ik_1 [1 + R \exp{-2ik_1 \rho}] [Ai'(u) - iBi'(u)] \]
\[ = [1 - R \exp{-2ik_1 \rho}] [Ai(u) - iBi(u)] \]
\[ R \exp{-2ik_1 \rho} = \frac{[Ai(u) - KBi'(u)] - i[Bi(u) + KAi'(u)]}{[Ai(u) + KBi'(u)] - i[Bi(u) - KAi'(u)]} \] (2.87)

The ray power transmission coefficient \( T \) defined by
\[ T = 1 - |R|^2 \] (2.88)
becomes after a little algebra and using (D.1) and (D.3)

\[ T = \frac{4k}{\pi} \frac{1}{[\text{Ai}(u)^2 + \text{Bi}(u)^2] + \frac{2k}{\pi} + K^2[\text{Bi}'(u)^2 + \text{Ai}'(u)^2]} \]  

(2.89)

and K and u are given as above

\[ K = \left( \frac{2k^2p^2}{\rho} \right)^{1/3} / k, \quad u = 2(k^2p^2)^{1/3} \left( \frac{\text{rad}}{\rho} - 1 \right) \]  

(2.90)

The asymptotic forms of T, eqn (2.89), can be related to earlier expressions.

(i) Tunnelling rays

The asymptotic forms of the Airy functions for \( u \gg 1 \) are

\[ \text{Ai}(u) = \frac{1}{2\sqrt{\pi}u^{3/2}} \exp\left\{ -\frac{2}{3} u^{3/2} \right\} \]

\[ \text{Bi}(u) = \frac{1}{\sqrt{\pi}(u)^{1/4}} \exp\left\{ \frac{2}{3} u^{3/2} \right\} \]

\[ \text{Ai}'(u) = -\frac{(u)^{1/4}}{2\sqrt{\pi}} \exp\left\{ -\frac{2}{3} u^{3/2} \right\} \]

\[ \text{Bi}'(u) = \frac{(u)^{1/4}}{\sqrt{\pi}} \exp\left\{ \frac{2}{3} u^{3/2} \right\} \]  

(2.91)

Noting that for \( u \gg 1 \) then \( \text{Ai}(u) \ll \text{Bi}(u) \), \( \text{Ai}'(u) \ll \text{Bi}'(u) \) and substituting (2.91) in (2.89) gives T as

\[ T = |T_F| \exp\left\{ -\frac{4}{3} u^{3/2} \right\} \]  

(2.92)

where

\[ T_F = \frac{4k_{r_1}(p)k_{r_2}(p)}{k_{r_1}^2(p) + k_{r_2}^2(p)} \]  

(2.93)

The factor \( T_F \) is the analytic continuation of the classical Fresnel power transmission coefficient to complex wave numbers. In this case \( k_{r_1}(p) \) is associated with a real propagating wave in the core and is therefore
real. In the cladding $k_2'(\rho)$ is associated with an evanescent region and is pure imaginary. $T_F$ is therefore pure imaginary and has a simple physical interpretation. It allows for the power reflected due to the jump discontinuity in the refractive index profile whereas the exponent in (2.92) represents a "reflection" of the power over the continuous part of the profile in the cladding.

The WKB calculation for the power transmission coefficient can be calculated in an exactly analogous way to that for the graded slab guide except that in matching the fields at the discontinuity at the core cladding interface a factor $|T_F|$ also arises. The WKB expression equivalent to (2.92)

$$T_{WKB} = |T_F| \exp \left( -2 \int_{\rho}^{rad} |k_2'(t)| \, dt \right)$$

If the region of applicability of the WKB expression is restricted to $r_{rad}$ close to $\rho$ but not so close that the WKB analysis is invalid then it is appropriate to linearise $k_2'$ about $r = \rho$ and we get eqn (2.92) exactly. For $r_{rad}$ very close to or equal to $\rho$ then the expression (2.89) must be used.

In this tunnelling ray region where $\alpha_1 = \alpha_c$ but not so close that the Airy function form is needed, eqn (2.92) can be written in terms of the incidence angles for the ray on the interface. If the direction cosines of the ray at the point of incidence with respect to local cartesian coordinates as in Fig. 2.8 are $(\cos \theta_x, \cos \alpha_1, \cos \theta_z)$, then we can reexpress in terms of ray angles

$$\bar{\beta} = n_1 \cos \theta_z, \quad \bar{\alpha} = n_1 \cos \theta_x$$

$$k_{y_1} = n_1 k \cos \alpha_1, \quad \alpha_c = \sin^{-1}(n_2/n_1)$$

(2.95)
Fig. 2.8 Direction angles \((\theta_x, \alpha_i, \theta_i)\) of the incident wave vector \(k_i\) relative to the Cartesian axes at the point \(P\) on the core-cladding interface of the step fibre defined in Fig. 2.6.

and the ray transmission coefficient \(T\) of (2.92) can be rewritten as

\[
T = 4 \frac{\theta_i}{\theta_c} \left(1 - \frac{\theta_i}{\theta_c}\right)^{1/2} \exp \left\{- \frac{2 n k \rho}{3 \cos^2 \theta_x} \left(\theta_c^2 - \theta_i^2\right)^{3/2}\right\} \tag{2.96}
\]

where \(\theta_i = \frac{\pi}{2} - \alpha_i\) and \(\theta_c = \frac{\pi}{2} - \alpha_c\) are small. In this form the tunnelling coefficient is equivalent to eqn (13) of ref. 2, and when \(\theta_i \approx \theta_c\) it is equivalent to eqn (10) of ref. 3.
Eqn (2.92) can be thought of as the bridge between the full WKB solution valid for $r_{\text{rad}} \gg \rho$ and the Airy function form (2.89). Following from eqn (1.15) a dimensionless attenuation coefficient $\alpha$ can be defined and is related to the ray power transmission coefficient by

$$\alpha = \frac{\frac{T_0}{\rho}}{z_p} \quad (2.97)$$

If the integral in eqn (2.94) using the expression for $k_{\text{r}_2}(r)$ of eqn (2.80) is performed analytically, then the WKB expression for $T$ becomes

$$T = \left| T_F \right| \exp\left\{2\left(\frac{k^2}{Q^2} - Q^2\right)^{\frac{1}{2}} - 2k\cosh^{-1}\left(\frac{Q}{Q}\right)\right\} \quad (2.98)$$

where

$$Q = \left[ k^2(n^2 - \beta^2) \right]^{\frac{1}{2}}$$

Using this WKB expression (2.98) for $T$ in (2.97) gives the same solution as eqns (40), (46a) in ref. 14. The linearised form for $T$ given by (2.96) in (2.97) gives the same result for $\alpha$ as eqns (40), (46b) in ref. 14.

That (2.98) reduces to (2.96) for $r_{\text{rad}} \cong \rho$ can readily be verified by expanding $\cosh^{-1}$ in terms of a power series and collecting terms to order $(r_{\text{rad}}/\rho - 1)^{3/2}$. In the bound ray limit $Q \to 0$ and eqn (2.98) predicts $T \to 0$.

(ii) Rays at the critical angle

When $u = 0$ then

$$B_{i^2}(o) = 3A_{i^2}(o)$$
$$B_{i^1}(o) = 3A_{i^1}(o) \quad (2.99)$$

and (2.89) reduces to

$$T = \frac{K}{\pi} \left[ A_{i^2}(o) + \frac{K}{2\pi} + A_{i^1}(o) \right] \quad (2.100)$$
This represents the transition point between tunnelling and refracting rays around $\alpha_t = \alpha_c$.

(iii) Refracting rays

The asymptotic form of the Airy functions for $u \ll -1$ are

$$\begin{align*}
\text{Ai}(u) &= \frac{1}{\sqrt{\pi}(-u)^{3/4}} \sin \Omega \\
\text{Bi}(u) &= \frac{1}{\sqrt{\pi}(-u)^{1/4}} \cos \Omega \\
\text{Ai}'(u) &= -\frac{(-u)^{3/4}}{\sqrt{\pi}} \cos \Omega \\
\text{Bi}'(u) &= \frac{(-u)^{1/4}}{\sqrt{\pi}} \sin \Omega
\end{align*}$$

(2.101)

where

$$\Omega = \frac{2}{3} (-u)^{3/2} + \frac{\pi}{4}.$$  

Substituting these asymptotic forms (2.101) into (2.89) gives

after a little algebra

$$T = T_F$$

(2.102)

where $T_F$ is Fresnel's classical ray transmission coefficient. This situation corresponds to refraction where real propagating waves exist on both sides of the interface $r = \rho$. When refraction occurs at the core cladding interface then we allow a fictitious $r_{rad}$ to exist on the analytic continuation of the cladding profile to $r < \rho$. Therefore when $u$ becomes negative, then the position of the reradiation caustic can be interpreted as having receded to $r < \rho$. This case is obviously quite distinct from the case of $r_{rad} = \rho$ which leads to $u = 0$.

2.2.1.2 Comparison with a modal solution for $\gamma$

A derivation of an approximate eigenvalue equation for bound modes using plane wave or WKB considerations has been presented on a
circular step index fibre\(^\text{15}\) but it is not applicable to leaky modes. An equivalent construction of the eigenvalue equation for leaky modes leads to

\[
\cot \phi_1 = \frac{k_r}{|k_2|} \left[ 1 + i \exp\{-2\phi_2 \} \right]
\]  

(2.103)

where

\[
\phi_1 = \begin{cases} 
(U^2 - \xi^2)^{1/2} - \ell \cos^{-1}\left(\frac{\xi}{U}\right) - \frac{\pi}{4} 
\end{cases}
\]

\[
\phi_2 = \begin{cases} 
2(\xi^2 - Q^2)^{1/2} - 2\ell \cosh^{-1}\left(\frac{\xi}{Q}\right)
\end{cases}
\]

(2.104)

and

\[
U = \rho(n_1^2k_2^2 - \beta^2)^{1/2}
\]

\[
Q = \rho(n_2^2k_2^2 - \beta^2)^{1/2}
\]

For tunnelling modes \(\beta\) is complex as given by eqn (1.3). Providing the modes are weakly leaky the imaginary component \(\beta^i\) is small and expanding (2.103) in a Taylor series gives

\[
\beta^i = -\left|T_F\right| \exp\{-2\phi_2 \} \\
\frac{1}{4} \frac{d^2}{d\beta^i} - \frac{k_r}{|k_2|} \left|k_2\right|
\]

(2.105)

Since \(\frac{d^2}{d\beta^i} = \frac{z}{p}\) and \(\gamma = 2\beta^i\) where \(z_p\) is the distance between successive reflections along the guide we get

\[
\gamma = \frac{1}{z_p} \left|T_F\right| \exp\{-2\phi_2 \}
\]

(2.106)

which is exactly the form derived from a ray analysis using eqns (2.98) and (1.15).

This proves the equivalence of calculating the attenuation coefficient \(\gamma\) within the WKB approximation on a circular step index fibre from a modal analysis and from a ray optical analysis.
2.2.2 Transmission coefficients on the graded index fibre

The derivation of ray transmission coefficients on the graded-index slab waveguide has been demonstrated in section 2.1.2 of this thesis. Here the extension of these methods to the round graded-fibre is described.

2.2.2.1 Derivation of T on graded fibres

A fibre with a continuously graded power law refractive index profile in the core of the form

\[ n_2^2(r) = n_2^2(o) \left[ 1 - 2A \left( \frac{r}{\rho} \right)^q \right] \quad 0 > r > \rho \]  

and in the cladding

\[ n_2^2(r) = n_2^2 = n_2^2(o) \left[ 1 - 2A \right] \quad r \geq \rho \]

has continuity of the profile at the core-cladding interface as shown in Fig. 2.9. If the total variation of \( n \) over the whole fibre cross-section is small, then polarisation effects can be ignored. For rays having their incident caustic in the core close to the core-cladding interface, the core profile can be linearised close to \( r = \rho \) as shown on Fig. 2.9 to give

\[ n_2^2(r) = n_2^2 + \delta(r-\rho), \quad r \leq \rho \]

\[ = n_2^2 \quad r \geq \rho \]  

(2.108)

where \( \delta = -\frac{dn_2^2(r)}{dr} \bigg|_{r=\rho} = \tan \chi. \)

The solutions for the fields are related to solutions of the scalar wave equation in circular cylindrical geometry \((r,\phi,z)\) where \(\phi\) is the azimuthal angle and the \(z\) axis is the axis of the cylinder. The wave
Fig. 2.9 The continuous core-cladding profile \( n(r)^2 \), where \( n(\rho)^2 = n^2 \), and \( r = \rho \) is the interface. If \( \chi \) is the angle the tangent to the core profile at \( r = \rho \) makes with the \( r \) axis, then \( \delta = -\tan \chi \).

Equation (2.76) has solutions of the form \( \Psi(r, \phi, z) = \psi(r) \exp ik[\bar{\beta}z + \rho \phi] \)

where \( \psi(r) \) satisfies

\[
\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d \psi}{dr} + k^2 \left[ n^2(r) - \bar{\beta}^2 - \frac{\chi^2 \rho^2}{r^2} \right] \psi = 0
\]  (2.109)

Via the transformation \( \Phi = \psi/\sqrt{r} \) eqn (2.109) reduces to, in the core,

\[
\frac{d^2 \phi}{dr^2} + k^2 \left[ n^2(r) - \bar{\beta}^2 - \left( \frac{\rho^2}{r^2} \right) \right] \phi = 0 \quad r < \rho
\]  (2.110)
and in the cladding

\[ \frac{d^2 \Phi}{dr^2} + k^2 \left[ n_2^2 - \beta^2 - \left( \frac{\beta}{r} \right)^2 \right] \Phi = 0 \quad r > \rho \]

where we have ignored the factor $1/(4k^2\rho^2)$ compared with $\chi^2$ for multimode fibres. This can be rewritten in the general form

\[ \frac{d^2 \Phi}{dr^2} + k^2(r) \Phi = 0 \quad (2.111) \]

where $k^2(r)$ takes the value of the radial component of the wave vector in the core and cladding respectively. The transition between real and evanescent waves occurs at the incident caustic at $r_{\text{inc}}$ in the core and at the reradiation caustic at $r_{\text{rad}}$ in the cladding as indicated in Fig. 2.10.

In each case, in addition to linearising the profile, we also linearise the term $\left( \frac{\beta}{r} \right)^2$ about $r = \rho$ and by comparison of (2.111) with (2.110) we see

\[ k^2_{r}(r) = k^2 \left[ n^2 - \beta^2 - \chi^2 + (1 - \frac{\chi}{\rho}) \delta \right] \]

\[ r < \rho \]

\[ k^2_{r}(r) = k^2 \left[ n^2 - \beta^2 - \chi^2 - (1 - \frac{\chi}{\rho}) \chi^2 \right] \]

\[ r > \rho \quad (2.112) \]

A change of variables from $r$ to

\[ \xi_1 = (k\rho)^{2/3} \left[ \left( \frac{\chi}{\rho} - 1 \right) (\delta \rho - 2\chi^2)^{1/3} - \frac{(n_2^2 - \beta^2 - \chi^2)}{(\delta \rho - 2\chi^2)^{2/3}} \right] \quad (2.113) \]

\[ \xi_2 = (k\rho)^{2/3} \left[ (1 - \frac{\chi}{\rho}) (2\chi^2)^{1/3} - (n_2^2 - \beta^2 - \chi^2)(2\chi^2)^{2/3} \right] \]

converts eqns (2.110) to

\[ \frac{d^2 \Phi}{d\xi^2_1} - \xi_1^{2} \Phi = 0, \quad r < \rho \quad (2.114) \]

\[ \frac{d^2 \Phi}{d\xi^2_2} - \xi_2^{2} \Phi = 0, \quad r > \rho \]
Fig. 2.10  Qualitative description of tunnelling rays. The core ray path has a turning point or caustic at $r_p$ and reappears at $r_{rad}$ in the cladding. Between $r_p$ and $r_{rad}$ the fields of the rays are evanescent.

which have solutions $\text{Ai}(\xi_1)$, $\text{Bi}(\xi_1)$ and $\text{Ai}(\xi_2)$, $\text{Bi}(\xi_2)$ respectively.

These solutions are related through the asymptotic forms of the Airy function to the fields of the incident and reflected rays when $\xi_1 \to -\infty$ and to the fields of the transmitted ray when $\xi_2 \to -\infty$. The linear combination of Airy functions which asymptotically represent these waves are, for the incident and reflected waves within the core,

\[
\phi^i = \text{Ai}(\xi_1) - i \text{Bi}(\xi_1) \\
\phi^r = R[\text{Ai}(\xi_1) + i \text{Bi}(\xi_1)]
\]  

(2.115)
and for transmitted wave in the cladding

\[ \phi^t = S[\text{Ai}(\xi_2) - i \text{Bi}(\xi_2)] \]  

(2.116)

where the incident ray normalisation is arbitrary and R, S are constants. The fields of the rays are continuous at the interface giving

\[ \phi_i + \phi_r = \phi^t, \]  

(2.117)

\[ \frac{d\phi_i}{dr} + \frac{d\phi_r}{dr} = \frac{d\phi^t}{dr} \quad \text{at} \quad r = \rho \]

Solving for R and eliminating S in eqns (2.115) and (2.116) gives

\[ R = [\text{Ai}(v)\text{Ai}'(u) - \text{Bi}(v)\text{Bi}'(u) - K(\text{Ai}(u)\text{Ai}'(v) - \text{Bi}(u)\text{Bi}'(v))] \]

\[ + i[-\text{Bi}(v)\text{Ai}'(u) - \text{Bi}'(u)\text{Ai}(v) + K(\text{Bi}'(v)\text{Ai}(u) + \text{Bi}(u)\text{Ai}'(v))] \]

\[ [K(\text{Ai}(u)\text{Ai}'(v) + \text{Bi}(u)\text{Bi}'(v)) - (\text{Ai}(v)\text{Ai}'(u) + \text{Bi}(v)\text{Bi}'(u))] \]

\[ + i[K\text{Bi}(u)\text{Ai}'(v) - \text{Bi}(v)\text{Ai}(u) + (\text{Bi}(v)\text{Ai}'(u) - \text{Bi}'(u)\text{Ai}(v)) \]  

(2.118)

where \( K = -v'/u' \) and

\[ u = \xi_1 \bigg|_{r=\rho}, \quad v = \xi_2 \bigg|_{r=\rho} \]  

(2.119)

\[ u' = \xi_1' \bigg|_{r=\rho}, \quad v' = \xi_2' \bigg|_{r=\rho} \]

Then using the results of Appendix 2D we calculate the ray transmission coefficient \( T \), defined by

\[ T = 1 - |R|^2, \]  

(2.120)

which simplifies after some algebra to

\[ T = \frac{4K}{\pi^2 [C_0 + 2C_1 K + C_2 K^2]} \]  

(2.121)
where

\[ C_0 = [\text{Ai}(v)^2 + \text{Bi}(v)^2][\text{Ai}'(u)^2 + \text{Bi}'(u)^2] \]

\[ C_1 = \frac{1}{\pi^2} + [\text{Ai}(u)\text{Ai}'(u) + \text{Bi}(u)\text{Bi}'(u)]. \]

\[ [\text{Ai}(v)\text{Ai}'(v) + \text{Bi}(v)\text{Bi}'(v)] \]  

\[ C_2 = [\text{Ai}(u)^2 + \text{Bi}(u)^2][\text{Ai}'(v)^2 + \text{Bi}'(v)^2] \]

and

\[ K = -\frac{v'}{u'} = \left(\frac{2k^2}{\rho^3}\right)^{1/3} \left[\delta - \left(\frac{2k^2}{\rho_c^2}\right)^{1/3}\right] \]

The asymptotic forms of (2.121) can be related to previously established results.

Asymptotic forms for \( T \)

(i) Tunnelling rays

If the asymptotic forms for the Airy functions (2.91) for \( u, v \gg 1 \) are substituted into (2.121), \( T \) reduces simply to the expression

\[ T = e^{-4/3} u^{3/2} - \frac{4}{3} v^{3/2} \]  

(2.123)

If a discontinuity occurs in the profile, an extra factor \( |T_p| \) to account for the presence of the jump appears in the right hand side of (2.123). This is discussed further in section 2.3.

For the tunnelling rays, the core ray trajectory does not reach the core cladding interface and only evanescent fields associated with the ray reach \( r = \rho \). The position of the incident caustic in the core at \( r = r_{tp} \) is a root of the equation

\[ n^2(r) - \tilde{\beta}^2 - \left(\frac{\rho \tilde{\gamma}}{r}\right)^2 = 0 \]

(2.124)
The ray reappears at the reradiation caustic in the cladding \( r = r_{\text{rad}} \), which is also a root of (2.124). If no root exists in the cladding of (2.124), then the ray is a bound ray and the evanescent fields associated with the ray in the cladding extend to infinity. Only when \( \rho < r_{\text{rad}} < \infty \) can tunnelling take place and a fraction of the incident ray power is lost to the cladding.

Provided the caustics are well separated from each other and from the core-cladding interface, the ray power transmission coefficient can be calculated by applying the WKB approximation developed earlier in this thesis on the slab guide. This gives

\[
T = \exp \left\{ -2k \int_{r_{\text{tp}}}^{r_{\text{rad}}} \left[ \tilde{\beta}^2 + \left( \frac{\partial \tilde{\beta}}{\partial r} \right)^2 - n^2(r) \right]^{1/2} \, dr \right\} \tag{2.125}
\]

When \( r_{\text{tp}} \approx \rho \) and \( r_{\text{rad}} \approx \rho \) but not so close that the complete Airy function expression (2.121) must be used, then (2.125) reduces to (2.123). The limitation for the position of the caustics \( r_{\text{tp}}, r_{\text{rad}} \approx \rho \) is brought about by the linearisation of the profile about the interface in order to simply solve the equations (2.110) in terms of Airy functions. The WKB method does not require this linearisation and is valid for \( r_{\text{tp}}, r_{\text{rad}} \) removed from \( \rho \). The Airy function solution (2.121) provides the connection formulae for WKB method across the region of the caustic where WKB techniques are invalid.

(ii) Rays at the critical angle

For \( u, v = 0 \), both the incident and reradiation caustics, \( r_{\text{tp}} \) and \( r_{\text{rad}} \), are just located at the interface \( r = \rho \). This represents the border line between tunnelling and refracting rays. Eliminating Bi using (2.99) gives for (2.121)
\[ T = \frac{2K}{A_1^2(0)A_1''(0)[8K^2 + 8 + 4K] + \frac{K}{\pi^2}} \] (2.126)

and since

\[ (A_1(0)A_1'(0))^x = \frac{1}{12\pi^2} \]

we have

\[ T = \frac{3K}{(K+1)^2} \] (2.127)

The function (2.127) has a maximum value of 3/4 and only applies for profiles with no discontinuity or jump at the core cladding interface at \( r = \rho \).

(iii) Refracting rays

In order to recover the complete WKB expression for the ray transmission coefficient for refracting rays we follow the method used in Section 2.1.2.4 for the graded slab and retain the first three terms in the asymptotic expansions of the Airy functions for \( u, v < -1 \) as in eqn (2.71).

For \( u, v \) negative, real propagating waves exist on both sides of the core cladding interface. The position of the caustic \( r_{cp} \) only exists ficticiously on the analytic extension of the core profile into the cladding. Similar comments apply in this case also for the ficticious caustic \( r_{rad} \). This is a mathematical convenience which allows us to envisage the situation for refracting rays on each side of \( r = \rho \) physically.

The appropriate asymptotic forms of the Airy functions for \( A_i(u), B_i(u), A_i'(u), B_i'(u) \) are given in (2.71) in descending powers of \( \xi \) where \( \xi = 2/3(-u)^{3/2} \). The same expansions (2.71) are valid for
Ai(v), Bi(v), Ai'(v), Bi'(v) in descending powers of $\zeta$ where $\zeta = 2/3(-v)^{3/2}$. Substituting for Ai(u), Ai(v) etc. in (2.121) and collecting terms in descending powers of $\xi, \zeta$ in the denominator we find the non zero coefficients of the denominator up to order $\xi^{-2}\zeta^{-2}$ are

- the coefficient of $\xi^0\zeta^0 = 4$
- the coefficient of $\xi^1\zeta^1 = \frac{K}{18}$
- the coefficient of $\xi^2\zeta^2 = -\frac{70}{(72)^2}$
- the coefficient of $\xi^{-2}\zeta^0 = \frac{1}{36}$
- the coefficient of $\xi^0\zeta^{-2} = \frac{1}{36}$

These terms to order $\xi^{-2}\zeta^{-2}$ give rise to a ray transmission coefficient $T$ given by

$$T = \frac{1}{1 + \frac{1}{64} \frac{(\delta k^2)^2}{k^2 (n_2^2 - \beta^2 - k^2)^3}}$$

(2.128)

provided the ray is incident at angles different from grazing, or, in terms of ray angles,

$$T = 1 - \left(\frac{\delta}{8n^3k}\right)^2 \frac{1}{\sin^6 \theta_t} \quad \text{for} \quad \theta_t \gg 0$$

(2.129)

where $\theta_t$ is the angle the incident ray makes with the tangent to the interface in the plane of incidence as shown in Fig. 2.11. Since the profile is continuous at $r = \rho$ the angle of incidence of the ray on the interface is equal to the angle the transmitted ray makes with the tangent to the interface. The form (2.129) is only valid when $\theta_t$ is large.

The same result for the ray transmission coefficient (2.129) may be derived directly from the scalar wave equation when the appropriate
WKB representations for the fields of the incident, reflected and transmitted rays are substituted into the boundary value problem at the interface instead of the Airy function analysis.

2.2.2.2 Examples

To illustrate the behaviour of the full transmission coefficient and the accuracy of the two asymptotic forms for tunnelling and refracting rays, all three expressions are plotted against the angle $\theta_z$ the rays make with the axial direction at the interface of the parabolic fibre. The values of the subsidiary variables are $n_1 = 1.508$, $\Delta = .0056$,

![Diagram of ray paths](image)

**Fig. 2.11** The incident, reflected and transmitted rays lie in a plane and make angle $\alpha_{\perp} = \pi/2 - \theta_{\perp}$ with the normal at $P$ where $\theta_{\perp}$ is the angle between each ray and the tangent to the interface in the plane of the rays.
Fig. 2.12 ($\vec{z} \neq 0$). Plots of the full transmission coefficient eqn.(2.121) and the two WKB forms eqns (2.125) and (2.129) against $\theta_z$ for skew rays. The dashed line is the division between tunnelling rays to the left and refracting rays to the right, with $\theta_t \geq 0$ for the latter. ($\theta_z$ is in radians, $\theta_t$ is in degrees).

$\rho = 5 \times 10^{-5}$ m, $V = 50$ and $\lambda = 10^{-6}$ m. Figure 2.12 shows plots for skew rays with constant $\vec{z} = .032$ corresponding to $\lambda = 10$. The division between tunnelling and refracting rays occurs at $\theta_z = .1078$ when $n_z^2 = \beta^2 + \vec{z}^2$. For $\theta_z > .1078$ or $\theta_t > 0$ all rays refract and the asymptotic WKB form
given by 2.129 is not accurate until $T$ is close to unity. On the other hand for $\theta_z < 0.1708$ the value of $T$ for tunnelling rays decreases rapidly with decreasing $\theta_z$ and the asymptotic WKB form (2.129) becomes accurate away from the point $\theta_t = 0$.

The qualitative behaviour of $T$ for meridional rays is quite different as Fig. 2.13 verifies. In this case $\chi = 0$ and the division at $\theta_z = 0.1057$ when $n_z^2 = \beta^2$ is between refracting rays, for large $\theta_z$, and bound rays, for small $\theta_z$. Here $T \to 0$ as $\theta_t \to 0$ and again the WKB form of (2.125) is only accurate for $T \approx 1$.

![Fig. 2.13 ($\chi = 0$). Plots of the full transmission coefficient eqn (2.121) and the WKB form eqn (2.129) against $\theta_z$ for meridional rays. The dashed line is the division between bound rays to the left and refracting rays to the right, with $\theta_t \geq 0$ for the latter. ($\theta_z$ is in radians, $\theta_t$ is in degrees).](image)
2.2.2.3 Comparison with modal solutions

We can compare the ray power attenuation coefficient based on \( T \) of (2.121) with the corresponding modal coefficient \(^{16}\) \( 2\text{Im}(\beta) \) where \( \text{Im} \) denotes the magnitude of the imaginary part if we relate the modal parameters \( \beta \) and \( \xi \) with the ray parameters \( \tilde{\beta}, \tilde{\xi} \) by eqn (2.78) The mode is then viewed as the family of all ray paths with common \( \tilde{\beta} \) and \( \tilde{\xi} \) values. Since each member of this family can be transformed into any other member by a combination of rotation and translation the attenuation of the modal approach is equivalent to the attenuation of any individual ray of the family. An expression for the modal attenuation coefficient is available as the solution to an eigenvalue equation derived within the WKB approximation for refracting rays.\(^{16}\) However we do not find good agreement with the analysis performed here using either (1.14) or (1.15) as the model for ray attenuation for refracting rays.

It seems that the accuracy of the modal eigenvalue equation is based on a WKB representation which is no more accurate than the corresponding curves in Figs (2.12), (2.13). If this is so the values of \( \beta^1 \) for which a comparison is valid must correspond to \( T \to 1 \). The perturbation solution of the eigenvalue equation may be inappropriate for such large values of \( \beta^1 \).

2.2.2.4 Discussion

For tunnelling rays the WKB solution for \( T \) (2.125) has been derived previously.\(^{17}\) Performing the integral in (2.125) analytically by separating the integral into two parts, one from \( r_{tp} \) to \( \rho \), the other from \( \rho \) to \( r_{rad} \), for a parabolic fibre gives

\[
\ln T = (\xi^2 - Q^2)^{1/2} - \left[ \frac{U^2}{4V} - \frac{\xi}{2} \right] \ln(U^2 - 4\xi^2V^2) + \frac{U^2}{2V^2} \ln(2V^2 - U^2 + 2V(\xi^2 - Q^2)^{1/2})
\]

\[+ 2\xi \ln Q - 2 \ln[\xi + (\xi^2 - Q^2)^{1/2}] + \xi \ln[\xi - 2(\xi^2 - Q^2)^{1/2}] \]

(2.130)
where $U$, $Q$ and $V$ were defined previously. This expression is exactly
the expression given by previous authors\textsuperscript{17} apart from an error in the
sign of the factor $(2\gamma x + 2\gamma^2 - u^2)$ in their notation.

Now we compare the forms of the various solution for $T$ for
tunnelling rays that have already been established. Fig. 2.14 shows this
comparison for one particular value of $\tilde{\beta}$ but the curves show essentially
similar behaviour for all $\tilde{\beta}$. Curve 1 shows a plot of the complete
Airy function expression (2.123) for $T$ for varying $\tilde{\lambda}$. Curve 2 is the
WKB expression obtained using Stewart's corrected integral (2.130).

![Fig. 2.14 Comparison of five methods of calculating the ray power transmission coefficient $T$. The curves 1 - 5 are discussed in the text. $\tilde{\beta} = 1.5166$, $n_1 = 1.527$, $n_2 = 1.517$.](image-url)
Curve 3 is obtained using a simplified formula ignoring the integral within the core. The fourth curve 4 shows a linearised form of the WKB expression, (2.125) given in ref. 4. If the solution (2.125) is denoted $T_{WKB}$ then the solution corresponding to (2.52) which is plotted as curve 5 is

$$T = \frac{T_{WKB}}{(1 + T_{WKB}/4)^2} \quad (2.131)$$

From the above discussion and the divergence of the curves on Fig. 2.14 we deduce that no single expression is valid over the entire range of $T$. The Airy function expression (curve 1) is valid only for $r_{rad} \approx \rho$ while the WKB expressions of either curve 5 or 2 is valid for $r_{rad} \to \infty$.

In section 2.4 we describe a more powerful mathematical method for calculating the ray transmission coefficient which is uniformly valid across the entire range of ray angle space.

### 2.3 A COMBINED GENERAL THEORY FOR TRANSMISSION COEFFICIENTS

In the preceding section of this chapter, the derivation of ray transmission coefficients on optical waveguides of both slab and cylindrical geometry has been demonstrated making extensive use of WKB theory. It was noted that the WKB method was restricted to profiles which vary slowly over a wavelength of the incident radiation, a condition which is readily satisfied in almost all situations of practical interest provided that the profile is continuous.

In this section the WKB analysis for graded profiles is extended and generalised to include jump discontinuities by building in the boundary matching process and using the Airy function solutions to provide
the WKB connecting formulae near caustics. In so doing we develop the first step of a building block technique which enables the ray transmission coefficient to be written down immediately for waveguides with an arbitrary number of layers of graded or uniform index materials separated by jump discontinuities. These results are valid for both slab and cylindrical waveguides.

The ray power attenuation along these multilayered waveguides can be written down from a knowledge of the ray transmission coefficient using the simple attenuation model (1.15). Compared with this method the evaluation of attenuation by an electromagnetic mode analysis presents a boundary value problem that is virtually intractable for more than two or three layers.

2.3.1 Derivation for slab geometry

When a tunnelling leaky ray is incident on an arbitrarily graded refractive index medium of either slab or circular geometry the situation is as depicted in Fig. 2.15. The incident ray is partially reflected at the incident caustic \( y_{tp} \) and the evanescent or tunnelling region associated with the ray extends from \( y_{tp} \) to \( y_{rad} \). At the reradiation caustic \( y_{rad} \) the ray reappears and propagates away from the guide. Between \( y_{tp} \) and \( y_{rad} \) there are assumed to be \( N \) finite jumps in the profile \( n(y) \), which is otherwise continuous and arbitrary.

To calculate the ray transmission coefficient, we begin by solving the electromagnetic boundary value problem for an arbitrary graded profile containing a single jump discontinuity as shown in Fig. 2.15, in terms of Airy functions using a linearisation of the square of the refractive index profile at each end of the jump.
Fig. 2.15 The refractive index profile $n(y)$ showing arbitrary variation in the continuous sections between discontinuities or jumps. Note that the continuous sections may contain kinks, i.e. points at which there is an abrupt change in the slope or derivative of $n(y)$. The positions of the caustics at $y_{tp}$ and $y_{rad}$ depend upon both the geometry of the structure and the profile. For a slab geometry $n(y_{tp}) = n(y_{rad})$, while for cylindrical geometry, $n(y_{tp}) \neq n(y_{rad})$ because of the azimuthal ray invariant $\lambda$. The evanescent region associated with the tunnelling ray is shown shaded.

From this solution the transmission coefficients can be calculated when either or both $y_{tp}$ and $y_{rad}$ are very close to the jump. When $y_{tp}$ and $y_{rad}$ are well separated from the jump then the WKB connecting formulae can be deduced by examining the asymptotic form of the exact
solution in the linearised region, and matching it to the WKB solution of the scalar wave equation in the same region. It is then straightforward to extend these results to more than one discontinuity in the profile.

This procedure is carried out in two dimensions in Cartesian geometry and the corresponding results for cylindrical geometry follow trivially with a change of variables in the Cartesian solution.
The starting point is the scalar wave equation, (2.9). The solutions are assumed to vary as \(\exp(ik\bar{z})\) in the \(z\) direction along the axis of the guide. Assuming the time dependence is implicit then the \(y\) dependence is given from

\[
\frac{d^2\psi(y)}{dy^2} + k^2(n^2(y) - \beta^2)\psi(y) = 0 \tag{2.132}
\]

where \(n^2(y)\) is the full refractive index profile of Fig. 2.16. In the linearised region close to the top and bottom of the jump

\[
n^2(y) = \begin{cases} 
  n_i^2 + (\rho-y)\delta, & y < \rho \\
  n_f^2 + (\rho-y)\bar{\delta}, & y > \rho 
\end{cases} \tag{2.133}
\]

where \(\delta = \left. \frac{dn^2(y)}{dy} \right|_{y=\rho}\) and \(\bar{\delta} = \left. \frac{dn^2(y)}{dy} \right|_{y=\rho}\)

evaluated for \(y \leq \rho\) and \(y \geq \rho\) respectively. The solution of eqn (2.132) with \(n(y)\) given by (2.133) is a linear combination of Airy functions. For \(y < \rho\) this combination for \(\psi\) must give asymptotically an incident and reflected wave for \(y < y_{tp}\) and for \(y > \rho\) the appropriate combination must represent a +ve travelling transmitted wave for \(y >> y_{rad}\). If the incident wave amplitude is normalised to unity and the reflected wave amplitude is \(R\) then

\[
\psi(y) = \begin{cases} 
  [Ai(s) - i Bi(s)] + R[Ai(s) + i Bi(s)], & y < \rho \\
  S[Ai(t) - i Bi(t)], & y > \rho 
\end{cases} \tag{2.134}
\]

where \(Ai, Bi\) are the Airy functions of the first and second kind, 
\(s = (y-y_{tp})(\delta k^2)^{1/3}\), \(t = (y_{rad}-y)(\delta k^2)^{1/3}\) and \(R\) and \(S\) are constants to be calculated. The boundary conditions satisfied by the fields at \(y = \rho\), when the electric field \(E_y\) is polarised parallel to the interface, i.e. 
\(E_y = 0\), are equivalent to continuity of \(\psi(y)\) and \(\frac{d\psi(y)}{dy}\) at the profile.
discontinuity. If we define

\[ u = s \bigg|_{y=\rho}, \quad u' = s' \bigg|_{y=\rho} \]
\[ v = t \bigg|_{y=\rho}, \quad v' = t' \bigg|_{y=\rho} \]  \hspace{1cm} (2.135)

where the prime denotes differentiation with respect to \( y \), then two linear equations in the unknowns \( R \) and \( S \) are obtained which can be solved for \( R \) giving

\[
R = -\frac{K[A_i'(v)A_i(u) + B_i'(v)B_i(u)] - A_i'(u)A_i(v) - B_i'(u)B_i(v)}{K[A_i(u)A_i'(v) - B_i(u)B_i'(v)] + A_i(v)A_i'(u) - B_i(v)B_i'(u)}
\]
\[ + \frac{i[K[A_i'(v)B_i(u) - B_i'(v)A_i(u)] - A_i'(u)B_i(v) + B_i'(u)A_i(v)]}{K[A_i(u)A_i'(v) - B_i(u)B_i'(v)] + A_i(v)A_i'(u) - B_i(v)B_i'(u)} \]  \hspace{1cm} (2.136)

where \( K = -v'/u' \).

Using the simplifying algebra of Appendix 2D the ray transmission coefficient \( T \), defined by

\[ T = \frac{1 - |R|^2}{2} \]  \hspace{1cm} (2.137)

gives after a little algebra

\[
T = \frac{4K}{\pi^2} \left[ \frac{4K[A_i(u)^2 + B_i(u)^2]A_i'(v)^2 + B_i'(v)^2K^2}{2K[A_i(v)A_i'(v) + B_i(v)B_i'(v)]^2} \right]
\]
\[ + \frac{1}{\pi^2} \left[ \frac{A_i(v)^2 + B_i(v)^2[A_i(u)A_i'(u) + B_i(u)B_i'(u)]}{(A_i(u)^2 + B_i(u)^2)^2} \right] \]  \hspace{1cm} (2.138)

The corresponding expression for \( T \) when the magnetic field vector is polarised parallel to the interface, i.e. \( H_x = 0 \), can be derived by
matching \( n(y)\psi(y) \) and \( [1/n(y)] \frac{d\psi(y)}{dy} \) at the discontinuity at \( y = \rho \) to give the same equation as (2.138) but with \( K = v'n^2/u'n^2 \).

For tunnelling rays with \( y_{\text{tp}} \) and \( y_{\text{rad}} \) sufficiently removed from the interface and each other \( u >> 1 \) and \( v >> 1 \) and substituting the asymptotic forms of the Airy functions (2.91) into (2.138) gives, for \( E_y = 0 \),

\[
T = \left| \frac{4 k_y y_1 y_2}{(k_y + k_y)^2} \right| \exp \left\{ -\frac{4}{3} u^{3/2} - \frac{4}{3} v^{3/2} \right\} \tag{2.139}
\]

The factor before the exponential is the classical Fresnel coefficient, (2.6a).

If \( y_{\text{tp}} \) and \( y_{\text{rad}} \) are not too far removed from the interface so that the linearisation of the profile in Fig. 2.16 still holds then (2.139) is the same result as given by a linearised WKB analysis. In this case the field solutions for the linearised parts of the profile can be readily matched to the WKB forms valid away from the discontinuity to give, for \( E_y = 0 \),

\[
T = \left| \frac{4 k_y y_1 y_2}{(k_y + k_y)^2} \right| \exp \left\{ -2 \int_{y_{\text{tp}}}^{y_{\text{rad}}} |k_y(y)| dy \right\} \tag{2.140}
\]

where \( k_y(y) \) refers to the continuous parts of the profile and \( k_{y_1} \) and \( k_{y_2} \) are the values of \( k_y(y) \) at \( y = \rho \) on respective sides of the discontinuity. An analogous expression follows for \( H_y = 0 \).

Equation (2.138) is the appropriate tunnelling coefficient when both \( y_{\text{tp}} \) and \( y_{\text{rad}} \) are close to the jump, i.e. \( y_{\text{rad}} - y_{\text{tp}} = 0(\lambda) \). When both \( y_{\text{tp}} \) and \( y_{\text{rad}} \) are well distant from the jump the asymptotic form (2.140) is appropriate. There are two other special situations.
Special Cases

(i) $y_{tp} \approx \rho, \ y_{rad} \gg \rho$

In this case $v \gg 1$ and the asymptotic forms for $A_i(v), B_i(v)$ from eqn (2.91) can be used to give

$$T = \frac{(4K/\pi) \frac{1}{2} \exp[-2 \int_{y_{tp}}^{y_{rad}} |k_y(y)| dy]}{K^2[v[A_i^2(u) + B_i^2(u)] + 2Ku^2[A_i(u)A_i'(u) + B_i(u)B_i'(u)]}$$

$$+ (2K/\pi) \frac{1}{2} + [A_i'(u)^2 + B_i'(u)^2]\quad (2.141)$$

with $K = -v'/u'$ for $E_y = 0$ and $K = -v'n^2/u'n^2$ for $H_y = 0$.

(ii) $y_{tp} \ll \rho, \ y_{rad} \approx \rho$

In this case $u \gg 1$ and the asymptotic forms for $A_i(u), B_i(u)$ from eqn (2.91) can be used to give

$$T = \frac{(4K/\pi) \frac{1}{2} \exp[-2 \int_{y_{tp}}^{y_{rad}} |k_y(y)| dy]}{K^2[A_i'(v)^2 + B_i'(v)^2] + 2Ku^2[A_i(v)A_i'(v) + B_i(v)B_i'(v)]}$$

$$+ (2K/\pi) \frac{1}{2} + u[A_i'(u)^2 + B_i'(u)^2]\quad (2.142)$$

with $K$'s defined as for eqn (2.141)

The ray power transmission coefficient $T$ can now be solved where two or more discontinuities exist in the refractive index profile within the evanescent region using the exact solutions for the wave equation in the same way as performed above. However it is simpler and more physical to note that we could have matched the WKB solutions directly at each side of the discontinuity to give the same asymptotic result without using Airy functions with each boundary we associate a factor $|T_F|$, which represents the analytic continuation of Fresnel's transmission formulae to complex wave vectors. We therefore deduce for $E_y = 0$ (TE waves)
The functions $T^i_{FE}$ and $T^i_{FH}$ are defined to be the analytic continuation of the classical Fresnel power transmission coefficients to complex wave vectors for $E_y = 0$ (TE waves) and $H_y = 0$ (TM waves) respectively and are evaluated at the $i$-th discontinuity.

If $n_1$ and $n_2$ are the values of $n(y)$ on either side of the jump and if $k_{y1}$ and $k_{y2}$ are corresponding values of $k_y(y)$, then

$$|T^i_{FE}| = \left| 4k_{y1} k_{y2} / (k_{y1} + k_{y2})^2 \right|$$

and

$$|T^i_{FH}| = \left| 4n_1^2 n_2^2 k_{y1} k_{y2} / (n_1^2 k_{y1} + n_2^2 k_{y2})^2 \right|$$

If a jump lies between $y_{tp}$ and $y_{rad}$, then both $k_{y1}$ and $k_{y2}$ are pure imaginary. When a ray is reflected or reradiates from a jump then one of $k_{y1}$ or $k_{y2}$ is real and the other is pure imaginary.

### 2.3.2 Derivation for cylindrical geometry

The same results for cylindrical geometry follow from the solution of the analogous wave equation in cylindrical $(r, \phi, z)$ coordinates where the $z$ axis is aligned with the axis of the cylinder. The wave function $\psi(r)$ in cylindrical coordinates becomes

$$\frac{d^2 \psi(r)}{dr^2} + \frac{1}{r} \frac{d \psi(r)}{dr} + k^2 [n^2(r) - \beta^2 - \left( \frac{\beta}{r} \right)^2] \psi(r) = 0$$

(2.147)
Performing the substitution \( \Phi(r) = \psi(r)/\sqrt{r} \) and rewriting \( r \) as \( y \) yields the equation

\[
\frac{d^2\Phi(y)}{dy^2} + ky^2\left(n^2(y) - \beta^2 - \left(y^2 - \frac{1}{4k^2\rho^2}\right)^2\right)\Phi(y) = 0
\]

(2.148)

which is of the same form as that of (2.132). Ignoring the factor \( 1/2k\rho \) compared to \( \beta \), the solution is found to be of the same form as (2.140) but with

\[
k^2(y) = k^2\{n^2(y) - \beta^2 - \left(\frac{\rho}{r}\right)^2\}
\]

(2.149)

which is the radial component of the wave vector in cylindrical geometry. Therefore eqns (2.143) and (2.144) are valid for both slab and cylindrical geometry with the radial component of the wave vector given by (2.149)

2.3.3 Discussion

The expressions for the ray transmission coefficients (2.143) and (2.144) can be given a simple physical interpretation. The positions of \( y_{\text{rad}} \) and \( y_{\text{tp}} \) are determined from the ray parameters and the refractive index profile. Each of the factors \( |T_F| \) represents the loss due to a jump discontinuity which essentially "reflects" part of the evanescent tunnelling power flow back toward \( y_{\text{tp}} \). The WKB integral represents "reflection" of power over the continuous parts of the profile.

Some previous results for the transmission coefficient using the WKB formalism interprets \( T \) as the area under the positive part of the curve defined by \(-k^2(y)\) between \( y_{\text{tp}} \) and \( y_{\text{rad}} \). This is just the integral of (2.140) and is only valid for continuous profiles with no discontinuity.

Expressions for the attenuation coefficient, \( \gamma \), of modes on circular optical fibres have been derived using the WKB approximations in
the formulation of the eigenvalue equation for the guide. However, if
the resulting expression is compared with (2.140) using (1.15), then it
can be seen that the formulae are only applicable for continuous profiles
since none of the multiplicative factors $|T_r|$ appears. The formula derived
for the attenuation coefficient in ref. 6 for step index fibres is noted
to be deficient by a maximum of a factor of two which is the maximum
value of $|T_F|$ for step fibres.

The case of refracting rays can be recovered from the complete
expression (2.138) when $u << -1$ and $v << -1$. The asymptotic forms for
$A_1, B_1$ eqn (2.101) reduce the expression for $T$ to $T_F$, the classical
Fresnel coefficient for refraction at a plane dielectric interface. The
situation for $u$ negative corresponds to a mathematical trick allowing
the existence of a fictitious incident caustic at some position $y > \rho$
on the analytic continuation of the core profile to $y > \rho$. Then real
propagating waves exist up to the interface $y = \rho$. A similar situation
for $v$ negative considers the reradiation caustic $y_{rad}$ to be on the
analytic continuation of the cladding profile to $y < \rho$. This means that
the same form for $T$, (2.138), can be used to calculate the losses for
refracting rays where real propagating fields extend up to the jump on
both sides of the discontinuity.

2.4 TRANSMISSION COEFFICIENTS DERIVED BY THE METHOD OF UNIFORM
APPROXIMATION

So far in this chapter, the ray transmission coefficient $T$ has
been evaluated using different methods and for different geometries.
Several expressions have been presented for $T$ but all are restricted in
their validity to specific sub domains of the set of all leaky rays,
comprising both tunnelling and refracting rays. For the step index
circular fibre, it requires three different representations to describe
the attenuation of (a) refracting leaky rays, (b) weakly tunnelling rays and (c) leaky rays in the region around the critical angle which overlaps both (a) and (b).\(^1\) Similarly for graded index fibres there are separate expressions for the attenuation of (a) refracting rays,\(^1\) (b) weakly tunnelling rays\(^4\) and (c) rays whose caustics are close to the core cladding interface.\(^19\) A general form for \(T\) using these simple methods has also been assembled and it serves to illustrate clearly the physics involved. A comparable modal derivation using the method of uniform approximation appears in ref. 20.

The motivation of this section is to incorporate all of these results into a single analytical expression valid for all leaky rays on both step and graded index fibres. The resulting expressions are a little more complicated but the wide applicability and universality of the following results substantiate their importance.

Firstly, we derive a general ray transmission coefficient for the general case of an arbitrarily graded refractive index profile containing a step or discontinuity at some point in the profile. Subsequently we show how this can reduce to the situation for a simple step index fibre and a comparison is formulated for our generalised power transmission coefficient through the corresponding power attenuation coefficient with solutions of the eigenvalue equation to verify its accuracy. For a detailed comparison with some exact numerical values the reader is referred to chapter 6.

2.4.1 The general arbitrary profile

The generalised power transmission coefficient is evaluated for the full refractive index profile of Fig. 2.16 where the radial variable \(y\) is replaced by \(r\) and the profile \(n^2(r)\) is continuous in both the core
and the cladding but there is a discontinuity or finite jump (step) at the core cladding interface \( r = \rho \). When a tunnelling ray propagates in the core it is bound by the incident caustic at \( r_{\text{tp}} \), which is the turning point of the ray path and the part of the ray that reappears in the cladding has a caustic at \( r_{\text{rad}} \). Between \( r_{\text{tp}} \) and \( r_{\text{rad}} \) the electromagnetic fields associated with the ray are evanescent. In the case of a refracting ray the path extends to the interface in the core and away from the interface in the cladding so that there is no evanescent region.

The fields of the ray are associated with the solution of the scalar wave equation in cylindrical coordinates \((r, \phi, z)\). In terms of the ray invariants \( \tilde{\beta}, \tilde{\gamma}, \tilde{\xi} \), then

\[
\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + k^2 \psi \left( n^2(r) - \tilde{\beta}^2 - \tilde{\gamma}^2 \left( \frac{\partial}{r} \right)^2 \right) = 0
\]

(2.150)

with solution \( \psi(r) e^{ik(\tilde{\gamma} \rho \phi + \tilde{\beta} z)} \). As previously it is convenient to transform this equation by

\[
\Phi = \frac{\psi}{r^{1/2}}
\]

(2.151)

to

\[
\frac{d^2 \Phi}{dr^2} + k^2 \Phi \left\{ n^2(r) - \tilde{\beta}^2 - \left[ \tilde{\gamma}^2 - \frac{1}{4(kr)^2} \right] \left( \frac{\partial}{r} \right)^2 \right\} = 0
\]

(2.152)

The method of uniform approximation concedes solutions

\[
\Phi = \frac{\xi^{1/3}}{k_r^{1/2}} \left\{ a \text{Ai}(\xi) + b \text{Bi}(\xi) \right\}
\]

(2.153)

where \( a \) and \( b \) are constants and

\[
\xi = \left\{ \frac{3}{2} \int_{\rho}^{R} \left| k_r(r) \right| dr \right\}^{2/3} \quad \text{if } k_r^2 < 0
\]

(2.154)

\[
\xi = -\left\{ \frac{3}{2} \int_{\rho}^{R} k_r(r) dr \right\}^{2/3} \quad \text{if } k_r^2 > 0
\]
The function $k_r(r)$ is the radial component of the wave vector which is given, ignoring the factor of $1/2kp$ compared to $k^2$, by

$$k_r(r)^2 = k^2\{n^2(r) - \beta^2 - \frac{\zeta^2}{r} \}$$ (2.155)

The accuracy of this solution is discussed in Appendix 2C.

When $\zeta >> 1$ the asymptotic forms of the Airy functions enable us to relate $\phi$ to the fields of the ray approaching and leaving the core caustic at $r_{cp}$ and the ray in the cladding which originates at the radiation caustic $r_{rad}$. These rays are sketched on Fig. 2.16. The appropriate forms for the three rays are respectively

$$\phi_i = \frac{A}{|k_r|^\frac{1}{4}} \zeta^\frac{1}{3}(\text{Ai}(\zeta) - i \text{Bi}(\zeta)), \quad r < \rho$$ (2.156)

$$\phi_r = \frac{R}{|k_r|^\frac{1}{4}} \zeta^\frac{1}{3}(\text{Ai}(\zeta) + i \text{Bi}(\zeta)), \quad r < \rho$$ (2.157)

$$\phi_t = \frac{S}{|k_r|^\frac{1}{4}} \zeta^\frac{1}{3}(\text{Ai}(\zeta) - i \text{Bi}(\zeta)), \quad r > \rho$$ (2.158)

Continuity of the fields at $r = \rho$ requires that

$$\phi_i + \phi_r = \phi_t, \quad \frac{d}{dr}(\phi_i + \phi_r) = \frac{d}{dr}(\phi_t)$$

which enables solution of $R, S$ in terms of $A$.

We therefore have

$$\frac{R}{A} = -A'_{12} - B'B_{12} + A\frac{C_{11}}{12} + C\frac{A'}{12} + B\left[\frac{C_{12}}{22} + \frac{C_{22}}{22}\right]$$

$$+ i[-A'B_{12} + B'A_{12} - B\left(\frac{C_{12}}{12} + \frac{C_{12}}{22}\right) + A\left(\frac{C_{22}}{12} + \frac{C_{22}}{22}\right)]$$

$$A'_{12} - B'B_{12} - A\left(\frac{C_{12}}{12} + \frac{C_{12}}{22}\right) + B\left(\frac{C_{22}}{12} + \frac{C_{22}}{22}\right)$$

$$+ i[A'B_{12} + B'A_{12} - B\left(\frac{C_{12}}{12} + \frac{C_{12}}{22}\right) - A\left(\frac{C_{22}}{12} + \frac{C_{22}}{22}\right)]$$ (2.159)

where $A_{12} \equiv \text{Ai}\left(\frac{\zeta}{12}\right), A_{22} \equiv \text{Ai}\left(\frac{\zeta}{22}\right), B_{12} \equiv \text{Bi}\left(\frac{\zeta}{12}\right), B_{22} = \text{Bi}\left(\frac{\zeta}{22}\right)$. The Ai and Bi
are Airy functions of the first and second kinds, and the prime denotes differentiation with respect to the arguments

\[ \xi_1 = \left\{ \frac{3}{2} \right\}^{2/3} \int_{r_{tp}}^{\rho} |k_r(r)| \, dr, \quad \xi_2 = \left\{ \frac{3}{2} \right\}^{2/3} \int_{r_{rad}}^{\rho} |k_r(r)| \, dr \]  

(2.160)

\( r_{tp} \) and \( r_{rad} \) are the roots of \( k_r(r) = 0 \) in the core and in the cladding respectively. The functions \( C_1 \) and \( C_2 \) are defined as

\[ C_1 = \frac{L - L_1}{M_1}, \quad C_2 = \frac{M_2}{M_1} \]

where

\[ L_1 = \frac{k_{r_1}}{4\xi_1^{3/2}} - \frac{k_{r_1}'}{2k_{r_1}}, \quad L_2 = \frac{k_{r_2}}{4\xi_2^{3/2}} - \frac{k_{r_2}'}{2k_{r_2}} \]

\[ M_1 = \frac{4k_{r_1}}{\xi_1^{3/2}}, \quad M_2 = -\frac{k_{r_2}}{\xi_2^{3/2}} \]  

(2.161)

The subscripts 1 and 2 refer to the value of \( k_r(r) \) at the top and bottom of the step at \( r = \rho \) in Fig 2.16.

The ray transmission coefficient \( T \) is the fraction of incident power lost to the cladding.

\[ T = 1 - \left| \frac{R}{A} \right|^2 \]  

(2.162)

which after a little algebra and simplification with the help of Appendix 2D gives

\[ T = \frac{4}{\pi^2} \frac{C_2}{(X^2 + Y^2)} \]  

(2.163)

where

\[ X = A_1^1 A_2^1 + B_1^1 B_2^1 - A_1^1 (C_1 A_2^1 - C_1 A_2^1) - B_1^1 (C_1 B_2^1 - C_1 B_2^1) \]

\[ Y = -A_1^1 B_2^1 + B_1^1 A_2^1 - B_1^1 (C_1 A_2^1 - C_1 A_2^1) + A_1^1 (C_1 B_2^1 - C_1 B_2^1) \]
As defined the transmission coefficient is evaluated in a straightforward manner from a knowledge of the profile $n(r)$, the wave number $k$, and the two ray invariants $\tilde{\beta}, \tilde{\lambda}$. The arguments $\xi_1$ and $\xi_2$ defined above assume $r_{tp} < \rho$ and $r_{rad} > \rho$ as in Fig. 2.16 so that the ray tunnels through evanescent regions on both sides of the interface.

For the same profile in Fig. 2.16 it is possible to have tunnelling rays where the core ray path impinges directly on the interface and reappears in the cladding, with the evanescent region extending from $r = \rho$ to $r_{rad}$. By inverting the discontinuity in the profile we can produce the reverse situation as shown in Fig. 2.17, where a tunnelling ray has its turning point in the core and reappears at the interface. Then the evanescent region extends from $r = r_{tp}$ to $\rho$.

Fig. 2.17 Inversion of the step of Fig. 2.16 enables tunnelling rays to originate at the interface. The ray path is the same as would occur from an imaginary caustic $r_{rad} < \rho$, determined by the analytic continuation of the cladding profile. Both the position of $r_{rad}$ and the profile continuation are shown dotted.
Either profile will support refracting rays at the interface when there is no evanescent region at all. The transmission coefficient of eqn (2.163) covers all these situations if we extend the definitions of $\xi_1$ and $\xi_2$ as follows. When, in the first case, the core ray reflects from the interface we permit a ficticious $r_{tp}$ evaluated from the ray invariants and the analytic continuation of the core profile beyond $r = \rho$. Similarly in the second case, when the ray reappears at the interface we evaluate $r_{rad}$ in terms of the ray invariants and the analytic continuation of the cladding profile to $r < \rho$ as shown in Fig. 2.17. For refracting rays we invoke both procedures. In terms of these extended definitions for refracting rays

$$
\xi_1 = - \left\{ \frac{3}{2} \int_\rho^{r_{tp}} k_r(r) \, dr \right\} \frac{1}{\sqrt{3}}, \quad r_{tp} > \rho,
$$

$$
\xi_2 = - \left\{ \frac{3}{2} \int_{r_{rad}}^{\rho} k_r(r) \, dr \right\} \frac{1}{\sqrt{3}}, \quad r_{rad} < \rho,
$$

so that both $\xi_1, \xi_2 < 0$.

Previously established expressions for the transmission coefficient can be recovered from eqn (2.163) by examining the behaviour in various limits.

(a) Refracting rays

When the arguments of the Airy functions become large and negative the asymptotic forms for $Ai, Bi$ can be used resulting in the reduction of $T$ to the classical Fresnel coefficient $T_F$. If the variation of $n(r)$ across the interface is small then $T$ is independent of polarisation effects.
(b) *Tunnelling rays*

In the opposite extreme, when the arguments $\xi_1, \xi_2 \gg 1$, then $r_{tp}$ and $r_{rad}$ are well separated from the interface, and tunnelling is weak i.e. $T \ll 1$. Then

$$T \rightarrow \left| T_F \right| \exp \left\{ - \int_{r_{tp}}^{r_{rad}} \left| k_r (r) \right| \, dr \right\} \quad (2.165)$$

which is the WKB result produced in section 2.3. When the step discontinuity is absent $\left| T_F \right| = 1$ and the expression has already been obtained as eqn (2.125). The origin of the factor $\left| T_F \right|$ has been discussed above and in ref. 19.

(a) *Rays at the critical angle*

When $\xi_1, \xi_2 \rightarrow 0$, this corresponds to $r_{tp} \approx \rho, r_{rad} \approx \rho$ so that the tunnelling depth ($r_{rad} - r_{tp}$) is very narrow. Under these circumstances, an expression for $T$ has already been derived in terms of Airy functions by linearising the profile at the top and bottom of the step at the interface in Fig. 2.16. If the refractive index profile is linearised in the expressions for $\xi_1, \xi_2$, (2.160), this leads to $C_1 = 0$ and the expression for $T$ (2.163) reduces to the Airy function expression (2.138) of section 2.3.

(d) *Special cases*

$\xi_1 \rightarrow \infty, \xi_2 \rightarrow 0$ and $\xi_1 \rightarrow 0, \xi_2 \rightarrow \infty$.

In these two situations one of $r_{tp}$ or $r_{rad}$ is close to the interface and the other is well away from the interface. Although the tunnelling depth is large for both cases the proximity of one caustic to the profile discontinuity generates expressions for the transmission coefficient which appear as a combination of WKB and Airy functions.
2.4.2 The step-index profile

Performing a similar analysis for the step refractive index profile of Fig. 2.18 where the fields within the core represent local plane waves incident and reflected from the core cladding interface and the cladding fields are represented using the method of uniform approximation, we deduce $T$ as

$$T = \frac{4}{\pi} \frac{K_a}{\left\{X^2 + Y^2\right\}}$$

(2.166)

where

$$X = K_1 A_1(\xi_2) - K_2 A'_1(\xi_2) + B_1(\xi_2)$$

$$Y = A_1(\xi_2) - K_1 B_1(\xi_2) - K_2 B'_1(\xi_2)$$

and $k_{r_1}$ represents the radial component of the incident wave vector in the core at the core cladding interface and $L_2$ and $M_2$ are as defined previously. The value of $\xi_2$ is given by either (2.160) or (2.164).

---

Fig. 2.18 Profile of the step index fibre comprising uniform core index $n_i$ and cladding index $n_2$. All rays in the core are incident on the interface $r = \rho$. Tunnelling rays reappear at $r_{rad} > \rho$ and the ray fields in $\rho < r < r_{rad}$ are evanescent.
depending on $r_{\text{rad}} > \rho$ or $r_{\text{rad}} < \rho$. This same expression can be deduced directly from the generalised transmission coefficient for the arbitrary profile (2.163). In this case when the core profile reduces to its uniform step value, (2.155) dictates $r_{tp} \rightarrow \infty$ in keeping with our interpretation of core rays impinging on the interface. When this limit is applied to (2.163), $T$ reduces to the expression in (2.166) within the accuracy of the method of uniform approximation.

In a similar way as with the graded index transmission coefficient, the step index coefficient can be related to previously established forms by examining the limiting forms of (2.166) as a function of $\xi_2$.

\textit{Asymptotic forms}

(i) Refracting rays

When $\xi_2 \rightarrow -\infty$, then we substitute the asymptotic form of the Airy function (2.101) and the ray power transmission coefficient (2.166) reduces to the classical Fresnel power transmission coefficient $T_F$ for refracting rays in the step index fibre\(^1\).

(ii) Tunnelling rays

When $\xi_2 \rightarrow \infty$, then $r_{\text{rad}} \gg \rho$ and the tunnelling depth becomes large. (2.166) predicts

$$T \rightarrow |T_F| \exp \left\{- \int_\rho^{r_{\text{rad}}} |k_r(r)| \, dr \right\} \quad (2.167)$$

which is the WKB expression for all weakly tunnelling rays (2.94). We note in passing the similarity of (2.167) with (2.165) where $r_{tp}$ is replaced by $\rho$.

(iii) Rays at the critical angle

When $\xi_2 \rightarrow 0$, $r_{tp} \approx \rho$ so that the tunnelling depth becomes very
small. For this transition region between tunnelling and refracting rays, $K_1 \to 0$ and (2.166) reduces to the Airy function expression from ref. 2.

$$T \to \frac{|T_F|}{4\pi \left| \xi_2^{1/2} \left\{ A_i \left( \xi_2 \right)^2 + B_i \left( \xi_2 \right)^2 \right\} }$$

(2.168)

where $T_F$ may be complex and $\xi_2$ can adopt either sign. The expression (2.166) is not unbounded when $\xi_2 = 0$ since $T_F/\xi_2$ remains finite.

2.4.3 Comparison with a modal solution

The normalised power attenuation coefficient for the step index fibre is given by

$$\alpha_T = \frac{\frac{T_0}{z}}{p}$$

(2.169)

where $z_p = 2p \sin \theta$, $\cot \theta$ simply represents the distance between successive outer caustics or reflection points along the ray path and $T$ is the power transmission coefficient (1.6). The ray invariants $\tilde{\beta}, \tilde{\chi}$ are related to the angles used to describe the ray path by

$$\tilde{\beta} = n_1 \cos \theta, \tilde{\chi} = n_1 \sin \theta \cos \phi$$

(2.170)

and the angles are defined in Fig. 2.19. To demonstrate the accuracy of the generalised transmission coefficients, the power attenuation coefficient $\alpha_T$ is compared with the power attenuation coefficient $\alpha_E$ representing the solution of the eigenvalue equation for the step index fibre. It is given from a perturbation expansion as

$$\alpha_E = \frac{2}{\pi} \frac{|H_x^{(1)}(Q)|^{-2}}{V \sin \theta \cos \theta}$$

(2.171)

where $H_x^{(1)}(Q)$ is a Hankel function and the modal parameters are defined in terms of the ray parameters by
Fig. 2.19 A ray DP incident at P on the core-cladding interface of a step index fibre. \( \theta_z \) is the angle between DP and the fibre axis, and \( \alpha_i \) is the angle between DP and the radius OP = \( \rho \). If PR is the projection of DP onto the cross-section, then \( \theta_\phi \) is the angle between PR and the tangent in the cross-section at P. The angles are related by \( \sin \theta_z \sin \theta_\phi = \cos \alpha_i \).

\[
\begin{align*}
\beta &= k \tilde{\beta}, \quad \ell = kp\tilde{\ell}, \quad V = n_1 kp \sin \theta_c, \\
Q &= n_1 kp \left( \cos^2 \theta_c - \cos^2 \theta_z \right)^{V/2}
\end{align*}
\]

and \( \theta_c \) is the complement of the critical angle satisfying

\[
\cos \theta_c = \frac{n_2}{n_1}
\]

A comparison between \( \alpha_T \) and \( \alpha_E \) for fibres with \( V = 50 \) and \( V = 100 \) is shown in Figs. 2.20, 2.21 for tunnelling skew rays with \( \ell = 20 \) in (2.172).

The percentage error relative to the modal attenuation coefficient is plotted as a function of \( \theta_z \). Bound rays occupy \( 0 < \theta_z < \theta_c \) and tunnelling rays fill the zone \( \theta_c < \theta_z < \theta_m \) where \( \sin \theta_m \sin \theta_\phi = \sin \theta_c \).

Larger values of \( \theta_z \) correspond to refracting rays.
Fig. 2.20 Comparison of $\alpha_m$ and $\alpha_F$ of eqns (2.169) and (2.171) for
a $V = 50$ step index fibre with $n_i = 1.52$, $\lambda = 1 \mu m$,
$\rho = 50 \mu m$ and $\lambda = 20$. Cutoff $\theta = 0.1048$ and the division
between tunnelling and refracting rays occurs at $\theta_m = 0.1134$.

Fig. 2.21 The legend is identical with that of Fig. 2.20 except
$V = 100$. 

Fig. 2.21 The legend is identical with that of Fig. 2.20 except
$V = 100$. 

There is excellent agreement for all weakly tunnelling rays in both cases and only as \( \theta_z \to \theta_m \) is there a significant discrepancy. This discrepancy is halved when \( V \) doubles which can be regarded as being due to the much smaller values of \( T \) for the \( V = 100 \) curve at \( \theta_z = \theta_m \). It should also be realised that \( \alpha_E \) is calculated from a perturbation solution of the eigenvalue equation which is increasingly inaccurate as \( \alpha_E \) increases. Fortunately in practical problems it is those tunnelling rays with smaller values of \( T \) that dominate the behaviour of ray power attenuation along the fibre. Both methods agree well in the region \( T << 1 \).

A further illustration of the generalised transmission coefficient appears in Figs. 2.22 and 2.23 which show the relative error between various forms of \( \alpha_E \) and \( \alpha_T \) under the same conditions as Figs. 2.20, 2.21 respectively. Curve (a) uses the Airy function dependence of (2.168) which is valid on either side of \( \theta_z = \theta_m \). This form is only valid for \( r_{tp} \approx \rho \) and loses its accuracy for weakly tunnelling rays as \( \theta_z \to \theta_c \) and also for refracting rays \( \theta_z >> \theta_c \) although the latter is not shown. The WKB representation of (2.167) is used to plot curve (b) which is highly accurate for weakly tunnelling rays but is inappropriate for the more highly attenuated tunnelling rays closer to the refracting ray region. Lastly, curve (c) is based on the classical Fresnel coefficient \( T_F \) and is valid for those refracting rays with \( \theta_z \) somewhat larger than \( \theta_m \).

Further comparisons of ray and modal calculations for the normalised attenuation coefficient \( \alpha \) appear in Chapter 6.
Fig. 2.22 Comparison between the generalized transmission coefficient and (a) the Airy function form, (b) the WKB form and (c) the classical Fresnel form. The parameters and their values correspond to Fig. (2.20)

Fig. 2.23 The legend is identical with that of Fig. 2.22 except $V = 100$. 
APPENDIX 2A: THE WKB METHOD

The WKB method provides a convenient short wavelength approximation for the solution of equations of the form

$$\frac{d^2\psi}{dy^2} + q^2(y)\psi = 0 \quad (A.1)$$

in regions remote from caustics or zeros of $q(y)$ and provided that the function $q(y)$ is slowly varying. In this appendix we demonstrate the method by direct application to the solution of equations of the form

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r} \frac{\partial\psi}{\partial r} + q^2(r)\psi = 0 \quad (A.2)$$

which arise commonly in circular guiding structures. We note that it is already well known that a linear homogeneous equation with constant coefficients of the form

$$\frac{d^2\psi}{dr^2} + P \frac{d\psi}{dr} + Q\psi = 0$$

has solutions of the form

$$\psi \approx e^{u(r)} \quad \left[ -\frac{P}{2} \pm \sqrt{\frac{P^2}{4} - Q} \right] r$$

Therefore if we make $P,Q$ slowly varying then a solution of the form $\psi \approx e^{u(r)}$ will be an approximate solution of the equation.

As a first attempt we try a solution of the form

$$\psi = e^{u(r)}$$

$$\frac{d\psi}{dr} = u'(r)e^{u(r)}$$

$$\frac{d^2\psi}{dr^2} = u''(r)e^{u(r)} + (u'(r))^2e^{u(r)}$$

Substituting in (A.2) and reducing gives

$$u''(r) + (u'(r))^2 + \frac{u'(r)}{r} + q^2 = 0 \quad (A.4)$$
Away from a caustic the second derivative remains relatively small (this is the WKB approximation) and the first estimate $u_0$ satisfies

$$u_0'(r)^2 + \frac{u_0'(r)}{r} + q^2 = 0$$

Thus

$$u_0(r) = \int_{r_1}^{r_2} \left( -\frac{1}{2r} \pm \sqrt{\frac{1}{2} - q^2} \right) dr$$ \hfill (A.5)

A better estimate $u_1(r)$ can now be derived where

$$\left( u_1'(r) \right)^2 + \frac{u_1'(r)}{r} + \left( q^2 + u_0''(r) \right) = 0$$ \hfill (A.6)

which works well where $u_0$ is close to $u_1$ and $|u_0''| \ll |q|^2$.

Now

$$u_0'' = \frac{1}{2r^2} \pm \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4r^2} - q^2}} \cdot \left( -\frac{1}{2r^3} - 2qq' \right)$$ \hfill (A.7)

Ignoring small terms

$$\psi \approx \exp \left\{ -\int_{r_1}^{r_2} \frac{1}{2r} \pm \left( -\frac{q^2 \pm q_1'}{1} \right)^{\frac{1}{2}} dr \right\}$$ \hfill (A.8)

where $q_1'^2 \ll 1$. Using a binomial expansion

$$\psi \sim \exp \left\{ -\int_{r_1}^{r_2} q \pm q_1 dr \right\} \exp \left\{ -\frac{1}{2} \ln q - \frac{1}{2} \ln r \right\}$$ \hfill (A.9)

In the region $r > r_c$ the solutions are

$$\psi = \frac{A_r B_r}{\sqrt{r} |q|^{\frac{1}{2}}} \exp \left\{ \pm \int_r r_c |q(t)| dt \right\}$$ \hfill (A.10)

In the region $r < r_c$ the solutions are

$$\psi = \frac{C_r D_r}{\sqrt{r} q^{\frac{1}{2}}} \exp \left\{ \pm i \int_r r_c q(t) dt \right\}$$ \hfill (A.11)
These are the WKB approximations for the fields on either side of the boundary. The values of the constants $A, B, C, D$ are determined using the relevant connecting formulae across the caustic.
APPENDIX 2B: AIRY FUNCTION SOLUTIONS

The solution of equations of the form

$$\frac{d^2 \phi}{dy^2} + q^2(y) \phi = 0 \quad (B.1)$$

can be obtained easily in terms of Airy functions if the function $q^2(y)$
can be linearised or expressed in the form $F + Gy$. Then the substitution

$$w = -G^{-2/3}(F + Gy) \quad (B.2)$$

gives Airy's equation

$$\frac{d^2 \phi}{dw} - w \phi = 0 \quad (B.3)$$

which has linearly independent solutions $Ai, Bi$.\(^{10}\) The solutions are
exact provided $q^2(y)$ is a linear function of $y$. For examples using
circular geometry so that the wave equation takes the form of (A.2)
we obtain (B.1) using the transformation $\phi = \psi/\sqrt{r}$. 
APPENDIX 2C: THE METHOD OF UNIFORM APPROXIMATION

A method that does not require any linearisation of the function $q^2(y)$ and is uniformly applicable across the entire range is the method of uniform approximation (ref. 10, p. 451). Equations of the form

$$\frac{d^2\phi}{dy^2} + q^2(y)\phi = 0$$

(C.1)

have an asymptotic solution, for any slowly varying function $q^2(y) = k^2 p(y)$, given to order $k^{-1}$ by

$$\phi = \frac{\xi^{1/3}}{|q(y)|^{1/2}} \left( a \text{Ai}(-\xi) + b \text{Bi}(-\xi) \right)$$

(C.2)

where

$$\xi = \left\{ \frac{3}{2} \int y |q(t)| \, dt \right\}^{2/3} \quad \text{if } q^2(y) < 0$$

$$\xi = -\left\{ \frac{3}{2} \int y q(t) \, dt \right\}^{2/3} \quad \text{if } q^2(y) > 0$$

When $q^2(y)$ is a linear function of $y$, the method of uniform approximation is equivalent to the Airy function solutions of Appendix 2B.

An estimate of the errors involved in the solution of the scalar wave equation using the method of uniform approximation can be obtained by substituting the approximate solution (C.2) into (C.1) and examining the remainder term. This gives

$$\frac{d^2\phi}{dy^2} + \phi (q^2(y) + \text{Rem}) = 0$$

where

$$\text{Rem} = \frac{5}{16} \cdot \frac{q^2}{\xi^3} + \frac{q''}{2q} - \frac{3}{4} \cdot \left( \frac{q'}{q} \right)^2$$
and the prime denotes differentiation with respect to $y$. The second and third terms in $\text{Rem}$ represent the error in the WKB solution of the scalar wave equation and are negligible if $r$ is not close to a caustic and $q$ does not vary too rapidly with $y$, accounting for the restriction to slowly varying profiles. The first term in $\text{Rem}$ is also small away from caustics. In the neighbourhood of caustics we may linearise the profile about the position of the caustic and rewriting

$$
\text{Rem} = \frac{5}{16} \left\{ \frac{kr^2}{\xi^3} - \left( \frac{kr^2}{k_r^2} \right)^2 \right\} + \frac{1}{4} \frac{kr^2}{k_r^2}
$$

it may readily be verified that $\text{Rem}$ vanishes at a caustic.
APPENDIX 2D: ALGEBRAIC FORM FOR T

In nearly all calculations involving ray transmission coefficients, the expressions for the reflected ray amplitude, R, normalised with respect to the incident ray can be written in the form

\[ R = \frac{(-p-r) + i(q-s)}{(p-r) + i(q+s)} \]  \hspace{1cm} (D.1)

The corresponding ray transmission coefficient T is then given from (1.6) by

\[ T = 1 - |R|^2 \]  \hspace{1cm} (D.2)

as

\[ T = \left| \frac{4(pr-qs)}{(p-r)^2 + (q+s)^2} \right| \]  \hspace{1cm} (D.3)

by simple algebra. This simple relation considerably simplifies the calculation of T in nearly every problem involving ray transmission coefficients.
REFERENCES


CHAPTER 3

THE GENERALISED FRESNEL'S LAW (G.F.L.)

3.1 INTRODUCTION

The previous chapter considered the application of ray transmission coefficients to determine losses in graded- and step-index dielectric fibres. In this chapter the analysis is extended to consider both reflection from an arbitrarily curved dielectric interface between two uniform media, and reflection from within arbitrarily bent graded-index media. In each case simple formulae for the tray transmission coefficient $T$ are shown to depend only on the radius of curvature in the plane of incidence of the ray path and normal, provided that the radiation caustic in the cladding is not too far removed from either the interface or incident caustic. This covers nearly all practical cases where the radiation is significant, and is required for the calculation of curvature losses on asymmetric structures, such as bent and elliptical waveguides, which are discussed in the later chapters of this thesis.

We begin by discussing the situation for planar dielectric interfaces, and then introducing the effects of curvature as perturbations of the fields of the planar interface.
3.2 PLANAR INTERFACES

We consider the situations of step- and graded-index profiles separately.

3.2.1 Plane interface and step-index profile

The reflection of a ray (or plane wave) at a planar interface between two nonabsorbing dielectric media of refractive indices $n_1$ and $n_2$, where $n_2 < n_1$, occurs such that the angles of incidence and reflection are equal,\(^1\),\(^2\) as shown in Fig. 3.1. When the ray is incident at angle $\alpha_i$ less than the critical angle $\alpha_c$, then a fraction of the power associated with the ray is transmitted into the optically less dense medium at angle $\alpha_t$ so that it obeys Snell's Law,\(^1\),\(^2\)

$$n_1 \sin \alpha_i = n_2 \sin \alpha_t.$$  

(3.1)

For angles of incidence $\alpha_i > \alpha_c$ the ray is totally reflected. In the second medium the electromagnetic field associated with the ray in $n_1$ is evanescent and decays exponentially away from the interface. The critical angle $\alpha_c$ is defined as the angle of incidence for which the angle $\alpha_t = \frac{\pi}{2}$ in Eqn (3.1), or

$$\sin \alpha_c = \frac{n_2}{n_1}.$$  

(3.2)

The power transmission coefficient $T$ represents the fraction of power transmitted across the interface, and is given by the classical Fresnel coefficient $T_F$. When $\alpha_i > \alpha_c$ then $T_F = 0$, and for $\alpha_i < \alpha_c$ then $T_F$ is given by $T_F^E$ or $T_F^H$, depending on whether the electric or magnetic field vectors are polarised parallel to the interface. In terms of the parameters of Fig. 3.1,
Fig. 3.1 The reflection of a ray or plane wave at a planar dielectric interface between two media of refractive indices $n_1$ and $n_2$. For $\alpha_1 < \alpha_c$ the ray is refracted as shown, but for $\alpha_1 > \alpha_c$ only total reflection takes place.

$$T_F^E = \frac{4\left(\cos^2 \alpha_i - \cos^2 \alpha_c\right)^{1/2} \cos \alpha_i}{\left\{\cos \alpha_i + \left(\cos^2 \alpha_i - \cos^2 \alpha_c\right)^{1/2}\right\}^2}, \quad (3.3)$$

$$T_F^H = \frac{4\left(\cos^2 \alpha_i - \cos^2 \alpha_c\right)^{1/2} \cos \alpha_i \sin^2 \alpha_c}{\left\{\cos \alpha_i \sin^2 \alpha_c + \left(\cos^2 \alpha_i - \cos^2 \alpha_c\right)^{1/2}\right\}^2}. \quad (3.4)$$

In the case of nearly equal refractive indices $n_1 \approx n_2$, $\sin \alpha_c \approx 1$ and $T_F^H \approx T_F^E$ for arbitrary polarisation, thus the transmission coefficient is given by (3.3).
3.2.2 Planar interface and graded-index profile

For a lossless, continuously graded medium, an analogous situation exists and is depicted in Fig. 3.2, with the y axis in the direction of the grading. A medium with continuously varying refractive index profile \( n(y) \) adjoins a second medium with uniform refractive index \( n_2 \) so that there is no step in the refractive index profile at the interface, i.e. \( n_2 = n(0) \). A ray arriving at the interface from the optically denser graded medium is partially reflected and transmitted. The angle \( \alpha_t \) the ray makes with the normal to the interface in the second medium for the situation shown in Fig. 3.2 is equal to the angle of incidence \( \alpha_i \). Nearly all the incident power is transmitted because of the continuity of the profile at the interface and \( T = 1 \), provided \( \alpha_i \not\approx \frac{\pi}{2} \), as given by Eqn (2.75). Only a very small amount of power is reflected by the kink or slope discontinuity in the profile at the interface. For rays close to grazing incidence, i.e. \( \alpha_i \approx \frac{\pi}{2} \), the reflected power may become significant. For all other cases the reflected power is small and may be neglected.

The situation for both step- and graded-index profiles is summarised by Fig. 3.3, which shows the four possible combinations. Figures (a) and (b) show rays totally and partially reflected at a step interface. In (a) \( \alpha_i > \alpha_c \) and the ray is totally reflected while for (b) \( \alpha_i < \alpha_c \) and only partial reflection occurs. Figures (c) and (d) show equivalent situations for a plane graded interface where, in this case, if ray reaches the interface it is refracted. For planar interfaces, figures (a) and (c) represent total reflection, i.e. the ray transmission coefficient \( T = 0 \). Case (b) yields a ray transmission coefficient equal to the classical Fresnel coefficient, i.e. \( T = T_F \) of
Fig. 3.2 Reflection of a plane wave or ray at a planar interface between a graded medium of index profile \( n(y) \) and a uniform medium of index \( n_2 = n(0) \). The ray is incident from the optically denser medium and a transmitted ray with \( \alpha_t = \alpha_i \) always exists as long as the incident ray reaches the interface. The discontinuity in profile slope is responsible for partial reflection.

Eqn (3.3). Case (d) is discussed above and \( T \cong 1 \). In cases (a) and (c) only evanescent fields exist in the cladding. For (b) and (d) no evanescent fields exist.

3.2.3 The effect of curvature

Now we wish to investigate the modifications to the results for the planar interface due to curvature.\(^3\)\(^-\)\(^6\) To do this a simple perturbation approach\(^3\) is adopted in which the effect of curvature is
Fig. 3.3 Sketches of four possible conditions for reflection at a planar dielectric interface:
(a) total reflection at a step interface;
(b) partial reflection at a step interface;
(c) reflection within a graded medium;
(d) reflection at an interface in a graded medium.

accounted for only in the fields beyond the interface at \( y = 0 \), i.e. \( y > 0 \) in Figs. 3.1, 3.2. Both the cases for loss from a step and from a graded refractive index profile are treated in the following sections, 3.3 and 3.4 respectively.
In section 3.5 we show how to incorporate both grading and a step in the profile to give a simple expression for the combined case. In each case the solutions for the ray transmission coefficient can be simply expressed in terms of the analytic continuation of the Fresnel transmission coefficient \( T_F \) to pure imaginary values of the wave vector, to account for evanescent fields in the neighbourhood of the interface. In section 3.6 the application of the Generalised Fresnel's Law to some specific examples is discussed. Further detailed examples of the application of these results to losses due to bending and elliptic deformation of a dielectric waveguide are discussed in Chapters 7 and 8.

3.3 THE GENERALISED FRESNEL'S LAW FOR A GENERAL CURVED INTERFACE BETWEEN UNIFORM MEDIA

Fresnel's law for reflection of an electromagnetic wave or ray from a planar dielectric interface can be generalised to describe the phenomenon of ray, or plane wave, tunnelling at a curved interface using straightforward physical concepts. Consider reflection of an incident ray from a general curved interface between two non-absorbing uniform dielectric media, described at the point of reflection by the two principal radii of curvature. The situation is depicted in Figs. 3.4 and 3.5. When the ray is incident from the optically denser medium, then for angles of incidence greater than the critical angle \( \alpha_c \), the ray is only partially reflected, rather than totally reflected as from a planar interface, due to the effects of curvature. The electromagnetic field in the less dense medium decays exponentially out to some distance \( y_{rad} \), where the ray emerges at a tangent to the radiation caustic and propagates away from the interface as illustrated in Fig. 3.4. A fraction of the incident ray power tunnels from the interface.
Fig. 3.4 Reflection of a ray, or local plane wave, from a curved dielectric interface between two uniform non-absorbing dielectric media of refractive index $n_1$ and $n_2$. A ray incident at $\alpha_i > \alpha_c$ is partially reflected and reappears at the radiation caustic $y_{rad}$. Between the interface at $y=0$ and the radiation caustic $y_{rad}$ within the shaded region the fields are evanescent. At $y=y_{rad}$ the transmitted ray is tangential to the caustic.

to the radiation caustic $y_{rad}$ by a mechanism analogous to quantum mechanical tunnelling. The reradiation occurs from the point in the less dense medium where the transverse phase velocity of the electromagnetic fields is equal to the speed of light in that medium. Within
the evanescent region between the interface and the radiation caustic, the transverse phase velocity of the electromagnetic fields is less than this.

As the angle of incidence increases beyond the critical angle, the position of the radiation caustic, $y_{\text{rad}}$, moves further into the cladding away from the interface. Rays which are totally reflected correspond to $y_{\text{rad}} \to \infty$. When $\alpha_i = \alpha_c$, $y_{\text{rad}} = 0$ and the radiation caustic coincides with the interface. For angles of incidence sufficiently less
than the critical angle refraction occurs and Fresnel's and Snell's classical laws are closely obeyed.

In this section we examine the losses due to this tunnelling phenomenon and show how Fresnel's law may be extended to cover this situation. Some special cases of this effect have been noted previously\(^4,5\) and a similar derivation has already been presented.\(^3\) We now present a description of the derivation along with a review of the results. The situation for graded media and media with both gradation and a finite step are discussed subsequently. If \(n_1 \equiv n_2\) all the results are independent of polarisation.

3.3.1 Derivation of the ray transmission coefficient

We consider the curved boundary to be a perturbation of a plane interface, and the analysis is valid provided (a) that the curvature is not too great, (b) that the fields exhibit local plane wave characteristics close to the interface.

In the case of the plane interface, defined by cartesian coordinates \((x,y,z)\) oriented with the y axis normal to the boundary, the magnitudes \(k_1\) and \(k_2\) of the wave vectors \(k_1\) and \(k_2\) in the more dense and less dense medium respectively, can be written as the sum of their components

\[
k_1^2 = k_x^2 + k_y^2 + k_z^2, \quad k_2^2 = k_x^2 + k_y^1 + k_z^2, \tag{3.5}
\]

where \(k_x = k_1 \cos \theta_x\), \(k_y = k_1 \cos \alpha_y\), \(k_z = k_1 \cos \theta_z\)

\[
k_y^2 = (k_2^2 - k_1^2 + k_y^1)^2 = k_1 (\sin^2 \alpha_c - \sin^2 \alpha_y) \tag{3.6}
\]

\[
k_1 = \frac{2\pi n_1}{\lambda}, \quad k_2 = \frac{2\pi n_2}{\lambda}.
\]
The angles $\theta_x$, $\alpha_z$, $\theta_z$ are the angles of inclination to the cartesian $x$, $y$ and $z$ axes respectively. In the less dense medium, we seek a solution for the vector fields $\Psi$ to satisfy the scalar wave equation such that

$$\Psi = \psi(y) \exp(i \kappa_x x + i \kappa_z z)$$

(3.7)

where

$$\left( \frac{d^2}{dy^2} + k_{y_2}^2 \right) \psi(y) = 0.$$  

(3.8)

For the curved interface an exact solution for the fields must have the functional forms $\exp\{i \ell_x \phi_x\}$ and $\exp\{i \ell_z \phi_z\}$ where $\phi_x$, $\phi_z$ are the azimuthal angles in the $O_{xy}$ and $O_{yz}$ planes referred to the centres of curvature $O_x$, $O_z$ as on Fig. 3.5. Curvature is taken into account by assuming $k_x$, $k_z$ are functions of $y$ satisfying

$$\exp\{i k_x(y)x\} \approx \exp\{i \ell_x \phi_x\}$$

$$\exp\{i k_z(y)z\} \approx \exp\{i \ell_z \phi_z\}$$

(3.9)

near the interface. Then noting

$$x \approx (\rho_x + y) \phi_x, \quad z \approx (\rho_z + y) \phi_z$$

(3.10)

expressions for $k_x(y)$, $k_z(y)$ can be derived.

Since $k_x(0) \approx k_x$ and $k_z(0) \approx k_z$ at the interface and restricting the analysis to $y << \rho_x, \rho_z$

$$k_x^2(y) + k_z^2(y) \approx \left( k_x^2 + k_z^2 \right) \left( 1 - \frac{2y}{\rho_c} \right)$$

(3.11)

where

$$\rho_c = \frac{\rho_x \rho_z \sin^2 \alpha_1}{\rho_x \cos^2 \theta_x + \rho_z \cos^2 \theta_z}.$$  

(3.12)
This parameter $\rho_c$ is exactly equal to the radius of curvature of the interface in the plane of incidence at 0 defined by the normal to the interface and the incident ray direction. The function $k_{y_2}^2(y)$ can then be shown to be

$$k_{y_2}^2(y) = k_1^2 \left( \sin^2 \alpha_c - \sin^2 \alpha_1 + 2y \frac{\sin^2 \alpha_1}{\rho_c} \right)$$  \hspace{1cm} (3.13)$$

which can simply be substituted into Eqn (3.8) and the wave equation solved using a change of variables to give Airy's equation

$$\left( \frac{d^2}{d\xi^2} - \xi \right) \psi(\xi) = 0$$  \hspace{1cm} (3.14)$$

where

$$\xi = \left( \frac{k_1 \rho_c}{2 \sin^2 \alpha_1} \right)^{2/3} \left( \sin^2 \alpha_1 - \sin^2 \alpha_c - 2y \frac{\sin^2 \alpha_1}{\rho_c} \right).$$  \hspace{1cm} (3.15)$$

The zero in $\xi$ occurs at the radiation caustic $y_{rad}$ so that

$$y_{rad} = \frac{\rho_c}{2} \left( 1 - \frac{\sin^2 \alpha_c}{\sin^2 \alpha_1} \right).$$  \hspace{1cm} (3.16)$$

For $y < y_{rad}$ then $\xi > 0$ and the solutions of Eqn (3.14) represent evanescent or exponential behaviour while for $y > y_{rad}$ they represent oscillatory behaviour. We therefore select the appropriate linear combination of the Airy functions $\text{Ai}, \text{Bi}$ to represent an outward travelling wave as $y \to \infty$. The coefficients for the field amplitudes are found by solving the scalar wave equation (3.8) subject to the appropriate boundary conditions at the interface. The analysis is also subject to the approximations $k_1 \rho_c \gg 1$, $y \ll \rho_x, \rho_z$.

Within the first medium with index $n_1$ the incident ray is represented using arbitrary normalisation by
\[ \psi^i = \exp\{i k y_1 y\} \]  \hspace{1cm} (3.17)

and the reflected ray by

\[ \psi^r = R \exp\{-i k y_1 y\}. \]  \hspace{1cm} (3.18)

In the second medium with index \( n_2 \) the transmitted wave has the form

\[ \psi^t = S[A_i(\xi) - i B_i(\xi)]. \]  \hspace{1cm} (3.19)

The constants \( R, S \) are to be determined. At the boundary \( y = 0 \) we require continuity of fields depending on the incident polarisation. For incident fields with the electric field vector \( \vec{E} \) parallel to the interface at the point of reflection \( (E_y = 0) \), \( \psi \) and \( \psi' \) must be continuous and we put

\[ \psi^i + \psi^r = \psi^t, \quad \frac{d\psi^i}{dy} + \frac{d\psi^r}{dy} = \frac{d\psi^t}{dy} \text{ at } y = 0. \]  \hspace{1cm} (3.20)

The equations (3.17) - (3.20) can therefore be solved for \( R \) and the ray power transmission coefficient \( T \) becomes

\[ T = 1 - |R|^2. \]  \hspace{1cm} (3.21)

Defining

\[ v = \xi \bigg|_{y=0}, \quad v' = \bigg[ \frac{d\xi}{dy} \bigg]_{y=0}, \]

and after some algebra

\[ T = \frac{4K/\pi}{[K A_i(\nu) - B_i(\nu)]^2 + [A_i(\nu) + K B_i(\nu)]^2}, \]  \hspace{1cm} (3.22)

where the prime denotes differentiation with respect to the argument and

\[ K = -\frac{v'}{k y_1}, \]

\[ \nu = \gamma_{\text{rad}} \cdot \left( \frac{2 \sin^2 \alpha_i k_1^2}{\rho_c} \right)^{1/3}, \]  \hspace{1cm} (3.23)
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\[ v' = -\left(\frac{2 \sin^2 \alpha \rho_i}{\rho_c}\right)^{1/3} \]

and \( y_{rad} \) is given by (3.16) above.

3.3.2 Asymptotic forms of \( T \)

The asymptotic forms of Eqn (3.22) enables us to derive simplified expressions for \( T \) which are needed in later chapters.

(i) Tunnelling rays

When \( v \gg 1 \) then \( A_i(v) \) becomes small and \( B_i(v) \) is the dominant term. This gives

\[ T \approx \frac{4K/T}{[B_i(v)^2 + K^2 B_i'(v)^2]} \quad (3.24) \]

and using \( |k| = -\sqrt{v} v' \) we have substituting the asymptotic forms for \( B_i \) (Eqn 2.91),

\[ T = |T_F| \exp\left\{ -\frac{4}{3} v^{3/2} \right\}, \quad (3.25) \]

where \( T_F \) is the analytic continuation of the Fresnel coefficient for \( E_y = 0 \) and is represented in this situation by

\[ T_F^E = \frac{4 k_{y_1} k_{y_2}}{(k_{y_1} + k_{y_2})^2}, \quad (3.26) \]

where \( k_{y_1} \) is real and \( k_{y_2} \) is pure imaginary. For those fields polarised with \( H_y = 0 \), matching the fields at the boundary corresponds to a continuity of \( n \psi \) and \( \frac{1}{n} \frac{d\psi}{dy} \) in place of Eqn (3.20). A corresponding analysis for this situation gives the Fresnel coefficient \( T_F^H \) as

\[ T_F^H = \frac{4 n_1^2 n_2^2 k_{y_1} k_{y_2}}{(n_2^2 k_{y_1} + n_1^2 k_{y_2})^2}. \quad (3.27) \]
In most cases of interest \( n_1 \approx n_2 \) and \( T_F^H \approx T_F^E \approx T_F^T \).

(ii) Refracting rays

For \( v \ll -1 \) then the asymptotic forms of the Airy functions (2.100) can be used and \( T \) reduces to

\[
T = T_F, \quad (3.28)
\]

where \( T_F \) is the classical Fresnel coefficient discussed above.

(iii) Rays close to the critical angle

When \( v \approx 0 \) the ray transmission coefficient \( T \) reduces to

\[
T = \frac{4K/\pi}{4Ai'(0)^2k^2 + \frac{2K}{\pi} + 4Ai(0)^2}. \quad (3.29)
\]

This is different from the form of Eqn (3.25) but provided \( T \ll 1 \) at \( v = 0 \) the difference between \( T \) given by (3.29) and (3.25) is small. It was found previously \(^3\) that \( T \) could be re-expressed in the simple form

\[
T = |T_F| C, \quad (3.30)
\]

where \( C \) was the factor introduced by curvature representing the exponential of (3.25) for \( v > 0 \) and \( C = 1 \) for \( v < 0 \). The restriction \( T \ll 1 \) is equivalent to \( k_1 \rho_C \gg \cos^{-3} \alpha_C \) necessary \(^3\) for the fields to be represented by a local plane wave decomposition at the boundary.

3.3.3 Application to round guides

The problem of calculating the ray transmission coefficient \( T \) has been reduced to a two-dimensional one in plane of incidence, defined by the normal to the interface and the incident ray direction, within the restrictions \( y \ll \rho_x, \rho_z, k_1 \rho \gg 1 \) and for the radiation caustic...
\( y_{\text{rad}} \) not too far removed from the interface. This covers most practical situations of interest which arise in multimode fibre optic waveguides. For larger values of \( y_{\text{rad}} \) the value of \( T \) is extremely small and for most situations the generalisation of the Fresnel Law is an excellent approximation.

The validity of the G.F.L. can be demonstrated by considering the example of skew rays on a straight cylindrical step index guide. In this case \( \rho_z \to \infty \) and from Eqn (3.12)

\[
\rho_c = \frac{\rho_x \sin^2 \alpha_i}{\cos^2 \theta_x},
\]

where \( \rho_x \) is the radius of the cylinder and \( \theta_x \) and \( \alpha_i \) are the angles of the incident ray to the \( x \) and \( y \) axes respectively according to (3.6). Then using \( \ell = k_1 \rho_x \cos \theta_x \), the G.F.L. gives the same result for \( T \) as a WKB analysis\(^8\) for tunnelling rays. The attenuation coefficient given by \( T \rho_x / z_p \), where \( z_p \) is the distance between successive reflections along the axis of the round guide and \( T \) is given by the G.F.L. for tunnelling rays, is exactly the same as the attenuation coefficient for leaky modes derived from a solution of the eigenvalue equation for a step index fibre\(^{14}\) when the radiation caustic is close to the interface.

On the straight circular step index guide\(^6\) it has been shown that the curvature causes a slight rotation of the plane of the transmitted ray relative to the plane of incidence.

Some special cases appeared previously\(^4,5,15\) for losses of rays on bent step index slabs and for whispering gallery rays on a variety of structures.\(^4,16\) A full detailed discussion of the application of the G.F.L. to determine losses on bent guides appears in Chapter 7.
It now remains to complete the picture by deriving an appropriate generalisation of Fresnel's law for graded dielectric media.

3.4 THE GENERALISED FRESNEL'S LAW FOR A GENERAL CURVED INTERFACE IN A GRADED-INDEX MEDIUM

In the previous section the reflection of a ray, or local plane wave, at a general curved interface between two media of uniform, but slightly different, refractive indices was examined. For a concave interface where the ray is incident from the more dense medium, we saw that a generalisation of Snell's and Fresnel's classical laws for planar interfaces allows a description of the tunnelling phenomenon for rays incident at angles greater than the critical angle defined by Eqn (3.2).

In this section we consider the more general situation of incidence from a continuously varying refractive index profile \( n(y) \) arranged so that it is stratified in curved layers orthogonal to the change in \( n \). The situation is depicted in Fig. 3.6. A ray is shown approaching \( y = 0 \) from the denser graded medium, and being reflected at the caustic at \( y = 0 \) by the refractive index grading \( n(y) \). There are evanescent fields associated with the ray path extending beyond the caustic into the region \( y > 0 \). Beyond the point \( y = d \), the second medium has constant refractive index \( n_2 \). For the curved graded medium the physical description is similar to the curved step index case since there is an analogous point in the second medium where a fraction of the incident ray power dissociates itself from the interface and radiates away. This point of radiation again defines the point at which the nature of the fields changes from evanescent, or exponentially decaying behaviour, to oscillatory and occurs where the transverse phase velocity of the electromagnetic fields is equal to the speed of light in the
Fig. 3.6 A ray propagating in a medium with refractive index profile $n(y)$ is reflected at $0$ before reaching the interface with the uniform medium of index $n_2$ at $y=d$. The evanescent field shown shaded extends to the reradiation caustic at $y_d$ where a fraction of the incident ray power reappears and propagates away from the interface.

second medium, and is due to the curvature of the interface. A fraction of the incident power is, therefore, lost by the mechanism of tunnelling upon reflection of the ray at $0$. In non-absorbing media those rays which fail to reach the curved interface can only lose power by tunnelling.
3.4.1 Derivation of the ray transmission coefficient

The analysis follows from a planar graded interface in a similar manner to the previous development for the curved step interface. The curved contours of the constant refractive index profile are treated as a perturbation of a planar interface so that the analysis is again a modification of that used to derive Fresnel’s classical laws.

Consider a ray incident at the extremity of its ray path 0 in a graded medium as sketched on Fig. 3.7. The local cartesian coordinates Oxyz are orientated so that Oy is in the direction of the refractive index grading \( n(y) \) and the planes Oxy, Oyz contain the principal radii of curvature \( \rho_x, \rho_z \) respectively as sketched on Fig. 3.8. The ray makes an angle \( \alpha \) with the Oz axis at 0 and the plane of incidence is defined by this angle \( \alpha \) and the Oy axis. The ray path does not necessarily lie in this incidence plane at points other than \( y = 0 \). For a skew ray in a graded index fibre, the incidence plane is only tangential to the ray path at \( y = 0 \), while a meridional ray on a circular waveguide by definition remains in the plane of incidence.

A characteristic refractive index profile is shown in Fig. 3.9 where \( n(y) \) decreases monotonically until, beyond the interface at \( y = d \), it assumes a constant value \( n_2 \). At \( y = 0 \) classical geometric optics is invalid for light of finite wavelength, and close to \( y = 0 \) we must work with the electromagnetic fields. The asymptotic forms of these fields can be identified with the curved ray path approaching and leaving \( y = 0 \).

The local plane wave vector \( \mathbf{k} \) is assumed to have components \( k_x(y), k_y(y), k_z(y) \) parallel to the \( x,y,z \) axes defined in Fig. 3.7, so that in the graded medium \( n(y) \),
Fig. 3.7 Sketch of a ray approaching the caustic at 0 from below and reflecting from 0 back to the region $y < 0$. The plane of incidence of the ray at 0 defined by the tangent is the ray path at 0 and the normal, makes angle $\alpha$ to the Oz axis.

\[ n^2(y) k^2 = k_x^2(y) + k_y^2(y) + k_z^2(y). \]  \hspace{1cm} (3.32)

At $y = 0$ they assume the values

\[ k_x(0) = n(0) k \sin \alpha \]
\[ k_y(0) = 0 \]  \hspace{1cm} (3.33)
\[ k_z(0) = n(0) k \cos \alpha. \]

We define the complement of the local critical angle at 0 as
Fig. 3.8 The contour \( n(y) = \text{constant} \) in a graded refractive index medium with radii of curvature \( \rho_x, \rho_z \) and centres of curvature \( 0x, 0z \) respectively. The \( y \) axis is normal to the contour and the \( xz \) plane is tangential to it.

\[
\theta_c(0) = \cos^{-1} \frac{n_z}{n(0)},
\]

(3.34)

and seek solutions of the form

\[
\Psi = \psi(y) \exp\{i k_x(y) x + i k_z(y) z\},
\]

(3.35)

where \( \psi(y) \) satisfies the scalar wave equation

\[
\left\{ \frac{d^2}{dy^2} + k_y(y)^2 \right\} \psi(y) = 0.
\]

(3.36)

To account for curvature we see that in the neighbourhood of the turning point the fields must have azimuthal dependence
Fig. 3.9 A characteristic refractive index profile $n^2(y)$, monotonically decreasing until $y = d$ and uniform for $y > d$. Caustics are shown at $y = 0$ and $y = d$. The linearisation of the profile between $y = 0$ and $y = d$ is shown dashed.

$$\exp\{i \ell_x \phi_x + i \ell_z \phi_z\}, \quad (3.37)$$

where $\ell_x, \ell_z$ are constants and $\phi_x, \phi_z$ are azimuthal angles in the $0$ and $0$ planes. Close to the interface

$$x \approx (\rho_x + y) \phi_x, \quad z \approx (\rho_z + y) \phi_z, \quad (3.38)$$

where $\rho_x$ and $\rho_z$ are the principal radii of curvature. The representations of (3.35) and (3.37) must be locally equivalent and hence

$$k_x(y) x \approx \ell_x \phi_x, \quad k_z(y) z \approx \ell_z \phi_z \quad (3.39)$$

which on substituting Eqn (3.38) gives
Comparing Eqn (3.40) with Eqn (3.32) gives $\frac{k_x}{(\rho_x + y)}$, $\frac{k_y}{(\rho_y + y)}$ and then

$$k_x(y) = \frac{n(0) k \rho_x \sin \alpha}{(\rho_x + y)}, \quad k_z(y) = \frac{n(0) k \rho_z \cos \alpha}{(\rho_z + y)}.$$  

(3.41)

If $|y| \ll \rho_x, \rho_z$, then a binomial expansion, discarding terms in $\frac{y}{\rho_x}$, gives

$$k_x(y)^2 + k_z(y)^2 \cong n(0)^2 k^2 \left(1 - \frac{2y}{\rho_c}\right).$$  

(3.42)

where

$$\rho_c = \frac{\rho_x \rho_z}{\rho_x \cos^2 \alpha + \rho_z \sin^2 \alpha}.$$  

(3.43)

This is exactly the radius of curvature of the graded medium in the plane of the ray path tangent and the normal at 0, and is again given by Eqn (3.12), noting that in this case for continuous graded media $\sin \alpha_1 = \frac{\pi}{2}$ at $y=0$. This establishes the analogous situation for graded media with that for the step index interface in that the reflection losses only depend on the radius of curvature in the plane of incidence.

Finally substituting into Eqn (3.32) gives

$$k_y(y)^2 = n(0)^2 k^2 \left[\frac{2y}{\rho_c} - \left(1 - \frac{n(y)^2}{n(0)^2}\right)\right].$$  

(3.44)

When the incident caustic is close to d it is convenient to linearise the refractive index profile as shown by the dashed line on Fig. 3.9. Expressions for $T$ in terms of the exact profile using the method of uniform approximation are possible but extremely cumbersome. For those situations where $T$ is significant the linearisation gives a highly accurate result. The refractive index profile then becomes
\( n^2(y) \) \[
\begin{cases}
  n^2(0) \left\{ 1 - \frac{y}{d} \right\} + n_2^2 \frac{y}{d}, & y \leq d \\
  n_2^2, & y \geq d.
\end{cases}
\] (3.45)

Therefore Eqn (3.44) becomes, using (3.34),
\[
\begin{align*}
  k_y^2(y) & = n(0)^2k^2 \left\{ \frac{2d}{\rho_c} - \theta_c(0)^2 \right\} \frac{y}{d}, & y \leq d \\
  & = n(0)^2k^2 \left\{ \frac{2y}{\rho_c} - \theta_c(0)^2 \right\}, & y \geq d.
\end{align*}
\] (3.46)

Substituting Eqn (3.47) into the scalar wave equation leads to two Airy equations for \( \psi(y) \)
\[
\begin{align*}
  \left\{ \frac{d^2}{d\xi_1^2} - \xi_1 \right\} \psi(\xi_1) & = 0, & y \leq d \\
  \left\{ \frac{d^2}{d\xi_2^2} - \xi_2 \right\} \psi(\xi_2) & = 0, & y \geq d,
\end{align*}
\] (3.47)
where
\[
\begin{align*}
  \xi_1 & = \frac{y}{d} \left\{ \left( n(0)^2kd \right)^2 \left\{ \theta_c(0)^2 - \frac{2d}{\rho_c} \right\} \right\}^{1/3} \\
  \xi_2 & = \left( \theta_c(0)^2 - \frac{2y}{\rho_c} \right) \left\{ \frac{n(0)^2\rho_c}{2} \right\}^{2/3}.
\end{align*}
\] (3.48)

The solutions for \( \psi \) are in terms of the Airy functions \( \text{Ai} \) and \( \text{Bi} \). The combination of these functions for \( y < 0 \) must asymptotically represent a ray approaching and leaving the turning point. For large \( y \) they must also represent the ray transmitted across the evanescent region and propagating away from the outer caustic as in Fig. 3.6.

The appropriate solution representing the incident ray is
\[
\psi^i(y) = \text{Ai}(\xi_1) - i \text{Bi}(\xi_1), \quad y \leq d
\] (3.49)
and representing the reflected ray is
\[
\psi^r(y) = R[\text{Ai}(\xi_1) + i \text{Bi}(\xi_1)], \quad y \leq d.
\] (3.50)
For the transmitted ray

\[ \psi^S(y) = S[Ai(\xi_2) - i Bi(\xi_2)], \quad y \geq d. \] (3.51)

Matching the fields at the interface

\[ \psi^i + \psi^r = \psi^S, \quad \frac{d\psi^i}{dy} + \frac{d\psi^r}{dy} = \frac{d\psi^S}{dy} \quad \text{at} \quad y = d. \] (3.52)

The two equations (3.52) can then be solved for the ray power transmission coefficient \( T \) which represents the fraction of power lost when the ray passes through 0. Then

\[ T = 1 - |R|^2 \] (3.53)

giving

\[ T = \frac{4}{\pi} \frac{\kappa}{\{a + b\kappa + c\kappa^2\}}, \] (3.54)

where

\[ a = \{Ai(v)^2 + Bi(v)^2\} \{Ai'(u)^2 + Bi'(u)^2\} \]

\[ b = \frac{2}{\pi^2} + 2\{Ai(u) Ai'(u) + Bi(u) Bi'(u)\} \times \{Ai(v) Ai'(v) + Bi(v) Bi'(v)\} \]

\[ c = \{Ai(u)^2 + Bi(u)^2\} \{Ai'(v)^2 + Bi'(v)^2\} \]

\[ \kappa = -\frac{\xi_2'(d)}{\xi_1'(d)} = \left\{\frac{\rho_c}{\theta_c} \left(\frac{\rho_c(0)^2}{2d} - 1\right)\right\}^{-1/3} \]

\[ u = \xi_1(d) = \kappa^{-1} \left\{\frac{2k^2n(0)^2 d^2}{\rho_c}\right\} \]

\[ v = \xi_2(d) = \kappa^{-2} u \] (3.55)

and the prime denotes differentiation with respect to \( y \).

The position of the reradiation caustic \( y_d \) is defined by the zero of \( k_y(y)^2 \) for \( y \geq d \) and
\[ y_d = \frac{\rho_c \theta_c(0)^2}{2}. \]  

(3.56)

For \( y > y_d \) the fields again represent a propagating real wave and a fraction of the incident power is transmitted from the first medium to the second.

3.4.3 Asymptotic forms of \( T \)

We examine the asymptotic forms of Eqn (3.54) to relate to earlier results.

(i) Tunnelling rays

When \( u, v \gg 1 \) then the asymptotic forms of the Airy functions (Eqn 2.91) are valid.

The expression for \( T \) reduces in a straightforward manner in a continuously graded medium to

\[ T \approx \exp \left\{ -\frac{2}{3} k n(0) \rho_c \theta_c^2(0) \left( \theta_c^2(0) - \frac{2d}{\rho_c} \right)^{1/2} \right\}, \]  

(3.57)

where by definition \( \theta_c^2(0) \rho_c > 2d \). The restriction \( y_d \approx d \) also applies.

(ii) Refracting rays

When \( u, v \ll -1 \) then real propagating fields exist on both sides of the interface at \( y = d \). The ray is refracted by the interface and substituting the appropriate asymptotic forms (Eqn 2.101) for the Airy functions in Eqn (3.54) gives

\[ T \approx 1. \]  

(3.58)

Most the ray power is transmitted into the second medium. Actually a small amount of power may be reflected by the discontinuity in the slope of the refractive index profile but this rapidly becomes
negligible as the angle of incidence of the ray on the interface decreases away from the critical angle. To calculate the small amount of power reflected the complete expression Eqn (3.54) should be used.

(iii) Rays at the critical angle.

As the turning point \( y_d \to d \) then the power transmission coefficient does not approach unity. From Eqn (3.55) we find \( u,v \to 0 \). If we use the appropriate forms of \( \text{Ai}(0),\text{Bi}(0) \) from ref. 7 and Eqn (3.54)

\[
T = \frac{4\kappa/\pi^2}{16[1+\kappa^2] \text{Ai}(0)^2 \text{Ai}'(0)^2 + 2\kappa/\pi^2 + 8\kappa \text{Ai}(0)^2 \text{Ai}'(0)^2}.
\]

Noting

\[
\text{Ai}(0) \text{Ai}'(0))^2 = \frac{3^{-2}}{[\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)]^2}
\]

and

\[
\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}
\]

produces

\[
T = \frac{3\kappa}{(\kappa + 1)^2}
\]

which takes a maximum value of 3/4 when \( \kappa = 1 \). The effect of this limit can clearly be seen from the case of the parabolic fibre in Section 3.6.1. Note that this applies only for graded profiles with no jump discontinuities.

In the next section the extension of these results to media with a discontinuity in the profile is discussed. In section 3.6 some examples of the application of the G.F.L. for graded media are discussed and the reduction to existing results can be seen in some special cases.
3.5 COMBINED STEP- AND GRADED-INDEX PROFILES

The power transmission coefficient of section 3.4 assumes that the refractive index is continuous, and that there are no finite jumps in the profile. Only abrupt changes in slope at the interface \( y=d \) were allowed in Fig. 3.9. If a finite jump is now introduced at the interface \( y=d \), then the situation corresponds to that in Fig. 3.10. For this combination of graded and step profiles the transmission coefficient is also given by the expression (3.54). However, for \( \xi_1, \xi_2 >> 1 \), substituting for \( K \)

\[
T = \left| T_F \right| \cdot \exp \left\{ -\frac{4}{3} \xi_1^{3/2} - \frac{4}{3} \xi_2^{3/2} \right\}, \quad (3.61)
\]

where \( \xi_1 \) and \( \xi_2 \) are given by (3.55) and

\[
T_F = \frac{4 k_{y_B} k_{y_C}}{(k_{y_B} + k_{y_C})^2} \quad (3.62)
\]

is the analytic continuation of the classical Fresnel power transmission coefficient to complex values of \( k_{y_B} \) and \( k_{y_C} \), i.e. either or both may be pure imaginary. These two quantities are defined as the \( y \) component of the local plane wave vector at the bottom B and top C of the jump in Fig. 3.10. They are real where rays, or plane waves propagate, and pure imaginary in regions where rays cannot propagate, or, equivalently, where the local plane wave is evanescent. The exponential term in Eqn (3.61) is evaluated as in the previous section but with the values of the components adjusted to fit the profile AB and CD. For the graded continuous profile of Section 3.4, \( k_{y_B} = k_{y_C} \) and \( |T_F| = 0 \).

When \( u,v << -1 \), no evanescent region exists and refraction occurs at the interface \( y=d \). In this case for the profile of Fig. 3.10, we see \( T \) reduces to the classical Fresnel coefficient.
3.6 EXAMPLES

3.6.1 Skew rays on a straight circular graded-index cylinder

On a straight cylindrical graded index guide of circular cross section of radius $\rho$ with no discontinuity in the refractive index profile at the core-cladding interface, $\rho_z = \infty$, $\rho_x = \rho$ in Eqn (3.43). Using the linearised form for the profile slope within the core we see

$$k_y^2(0)' = k^2 n(0)^2 \left( \frac{2}{\rho_c} - \frac{\theta_c(0)^2}{d} \right)$$

(3.63)

$$k_y^2(y_d)' = k^2 n(0)^2 \cdot \frac{2}{\rho_c}.$$

Substituting into Eqn (3.57) yields...
This is in exact agreement with the triangular approximation of the WKB transmission coefficient on the straight fibre.\textsuperscript{11}

3.6.2 Rays on a bent parabolic index slab waveguide

A WKB analysis for rays on a fibre with circular symmetry\textsuperscript{3,8} gives $T$ as

$$T = \exp \left\{ -\frac{\frac{4}{3} y_d^{3/2}}{\left[ \frac{1}{k_y (y_d)^2} - \frac{1}{k_y (0)^2} \right]^{1/2}} \right\}, \quad (3.64)$$

where $y_{tp}$, $y_{rad}$ represent the position of the incident and reradiation caustics. A bent slab guide of width $2\rho$ and bending radius $R$ mimics the situation of a fibre of radius $R + \rho$. A ray in the cross section of the fibre loses power at each reflection according to Eqn (3.65) in the same way as a ray at the outer caustic of the bent slab as shown in Fig. 3.11.

Linearising the profile around the outer edge of the bent slab, the radial component of the wave vector $k_y (y)$ becomes

$$k_y (y) = \left( n^2(y) k^2 - \frac{\rho^2}{(R_c + y)^2} \right)^{1/2}, \quad (3.66)$$

where

$$R_c = R + \rho - d$$

$$n^2(y) \equiv n(0)^2 - y a$$

$$a = \frac{n(0)^2 - n_z^2}{d}. \quad (3.67)$$

Since the caustic is located at $y = 0$ then $k_y (0) = 0$ and
Fig. 3.11 A ray propagates along a bent graded index slab guide of width $2\rho$ and bending radius $R$ losing the same fraction $T$ of its power at each reflection from the outer caustic.

$$n(0)^2k^2 - \frac{\gamma^2}{(R_c)^2} = 0.$$  

$k_y(y)^2$ is positive in the shaded regions of Fig. 3.12. In these regions real propagating rays exist. Evanescent fields exist elsewhere.

For $y < d$

$$k_y^2(y) \approx y \left( \frac{2\gamma^2}{R_c^3} - a k^2 \right),$$  

(3.68)
Fig. 3.12 The radial component of the wave vector is represented by the difference between the curves $\frac{\ell^2}{R_c^2}$ and $n^2(y) k^2$. The shaded part of the figure indicates $k_y^2(y) > 0$. \( R_c = (R + \rho - d) \).

For \( y \geq d \)

\[
k_y(y)^2 = n_2^2 k^2 - \frac{\ell^2}{(R_c + y)^2}.
\] (3.69)

Also since the outer caustic is at \( y = y_d \), \( k_y(y_d) = 0 \) and

\[
n_2^2 k^2 - \frac{\ell^2}{(R_c + y_d)^2} = 0
\]

then

\[
k_y(y)^2 \approx \frac{2n^2(y - y_d)}{R_c^3} \quad \text{for} \quad y \geq d.
\] (3.70)

Writing the integral of Eqn (3.65) in two parts gives
The integration is straightforward and reduces, noting $\lambda^2 \sim n(0)^2 k^2 R_c^2$, to

$$T = \exp\left\{- 2 \int_0^d \sqrt{y} \left( \frac{c(x_0)^2 n(0)^2 k^2}{d} - \frac{2 \lambda^2}{R_c^3} \right) dy \right\}$$

$$- 2 \int_d^y \frac{2 \lambda^2}{R_c^3} (y_d - y)^{\frac{1}{2}} dy} \right\}.$$  (3.71)

which is the same as Eqn (3.57) above.

We have therefore shown the equivalence of the Generalised Fresnel's Law and a WKB analysis of the bent graded slab waveguide. For step index media this equivalence has already been noted previously. 4

3.6.3 The bent parabolic index fibre

An important application of the Generalised Fresnel's law is to the bent parabolic fibre illustrated in Fig. 3.13. We need to calculate the ray power transmission coefficient $T$ at each point of reflection along the ray path separately because the symmetry of the ray path is destroyed by bending. In general $T$ varies along any ray path. The calculation of the actual ray path and the evaluation of the overall attenuation on the bent parabolic fibre is left until Chapter 7. Here we concentrate on showing how to calculate the losses upon reflection.

If the introduction of the major radius of the bend does not modify the refractive index profile, then $n$ can be expressed in terms of the radial distance $r$ from the bent fibre axis, in local cylindrical coordinates $(r, \phi, z)$ where the $z$ axis lies along the bent axis of the fibre, as
Fig. 3.13 A ray propagating in a bent parabolic fibre of radius $\rho$ and bend radius $R$. The ray has an incident caustic at $r_{tp}$ and makes an angle $\phi$ with the major cross section of the bent fibre. A local cartesian coordinate system is taken with the $z$ axis parallel to the fibre axis at the turning point and with the $y$ axis normal to the ray path.

\[ n(r) = n_x \left\{ 1 - 2\Delta \left( \frac{r}{\rho} \right)^2 \right\}, \quad 0 \leq r \leq \rho \]

\[ n_2^2 = n_1^2 \left( 1 - 2\Delta \right) , \quad r \geq \rho , \quad (3.73) \]

where $\Delta = .0056$ is chosen as a typical practical value. The ray is reflected from the outer turning point along the core ray path at $r = r_{tp}$ corresponding to $y=0$. The fibre core radius is $\rho$.

The radial direction at $r_{tp}$ makes an angle $\phi$ with the plane containing the fibre axis and the centre of the bend of radius $R$ indicated on Fig. 3.13. The principal radii of curvature are therefore
The local cartesian Oz axis at \( r = r_{tp} \) is taken parallel to the fibre axis. The radius of curvature \( \rho_c \) in the plane of the ray at \( r = r_{tp} \) and the normal to the ray path is given by Eqn (3.43).

For the above value of \( \Lambda \), the ray power transmission coefficient \( T \) is shown plotted against the dimensionless parameter \( n_1kd \) where \( d = (\rho - r_{tp}) \) for various values of \( n_1k\rho_c \) in Figs. 3.14 and 3.15. Apart from \( \Lambda \), these curves are universal for any bent parabolic fibre. The first set of curves in Fig. 3.14 show the complete expression for \( T \) given by Eqn (3.54) plotted against \( n_1kd \) for values of \( n_1k\rho_c \). The rapid fall-off of \( T \) as either of the two parameters increases can be seen. As \( n_1kd \) increases the distance between the incident and radiation caustics, \( r_{tp} \) and \( r_{rad} \) respectively, increases and higher decay of the evanescent field results in a smaller value of \( T \). Similarly, increasing the radius of curvature \( \rho_c \), as seen by the ray at its turning point, moves \( y_d \) further into the cladding with a consequent reduction in \( T \).

Fig. 3.15 demonstrates the good agreement between the exact form for \( T \) (Eqn 3.54) shown plotted in solid lines, and its asymptotic form (Eqn 3.57) shown plotted with dashed lines. Only for those rays which lose a large fraction of their power at the turning point, when \( y_d \rightarrow d \), does the agreement become poor. This enables the simpler asymptotic form for \( T \) to be used for calculating the attenuation of rays which lose only a small fraction of their energy at a turning point. Since these are the only rays which contribute significantly to the power remaining in the fibre after travelling some distance along it, the simpler form for \( T \) is used in most practical situations.

\[
\begin{align*}
\rho_x &= r_{tp}, \quad \rho_z = r_{tp} + \frac{R}{\cos \phi}.
\end{align*}
\]
Fig. 3.14 The power transmission coefficient $T$ for a bent parabolic fibre plotted against the dimensionless parameter $k_1d$ for various values of the curvature factor $k_1\rho_c$. $d$ is the distance $\rho - r_{tp}$ and $\rho_c$ is the curvature in the plane of the ray. $k_1 = n_1k$ and $\Delta = .0056$.

On the bent fibre there can be no bound rays$^{12}$ and all rays must be either refracting or tunnelling. The different kinds of rays can be easily distinguished. If the ray path in the bent graded fibre reaches the core-cladding interface at $r = \rho$ then it is a refracting ray, and if it fails to reach the interface it must be a tunnelling ray.

As $\rho_z \to \infty$, $R \to \infty$ then the bent fibre approaches the limiting case of a straight fibre. The position at which the ray path reappears in the cladding, i.e. the reradiation caustic given by $r_{\text{rad}} = y_d + r_{tp}$ determines the division between bound and tunnelling rays on the
Fig. 3.15 A comparison of the ray transmission coefficients $T$ for the exact form given by Eqn (3.54) (shown by solid lines) and the asymptotic form Eqn (3.57) (shown by dashed lines) against the dimensionless parameter $k_1d$ for various values of the curvature factor $k_1\rho_c$. The parameter values are those of Fig. 3.14.

If $y_d \to \infty$ as $R \to \infty$ then the tunnelling ray of the bent fibre becomes the bound ray of the straight fibre. If $y_d$ remains finite then the ray remains as a tunnelling ray. The expressions for $T$ derived above assume $y_d$ is not too far distant from $d$. Therefore Eqn (3.54) and (3.57) are not valid for $y_d \to \infty$ and are only valid on the straight fibre if $y_d \equiv d$ as $R \to \infty$. In this limit we have already shown in Example 3.6.1 above that the expressions for $T$ reduce to the corresponding expressions [Eqn (6) and (7) of Ref. 11] for the straight graded index fibre.

Also on Fig. 3.14 it is worthy of note that for $k_1d \to 0$ and decreasing $k_1\rho_c$ the ray transmission coefficient $T \to 3/4$ as predicted by Eqn (3.60).
REFERENCES


CHAPTER 4
RAY TRANSMISSION COEFFICIENTS IN SLIGHTLY ABSORBING WAVEGUIDES

Thus far in this thesis, our attention has been confined to waveguides made with ideal lossless materials. In this chapter consideration is extended to cover graded- or step-index optical waveguides of slab or cylindrical geometry and composed of slightly lossy core and cladding materials. A building block approach is developed to account for losses due to tunnelling and absorption so that, in general, the total loss can be written down by inspection as a combination of six basic terms. Power loss along the ray path, in the neighbourhood of caustics and in the evanescent regions associated with the ray fields, can all be included in the description.

4.1 INTRODUCTION

All practical materials used in the fabrication of dielectric optical waveguides exhibit the property of bulk absorption, which attenuates ray power travelling along the guide. In general, the core material is made as pure as possible to minimise absorption loss while the cladding materials may have a bulk absorption coefficient which is several orders of magnitude greater. However, some fibres are now being produced in which the cladding absorption is comparable with that of the
core or in which a secondary low loss cladding layer is added between the conventional core and cladding to reduce evanescent field absorption.

For the simple multimode step index fibre a ray analysis has already been performed which includes both absorption and tunnelling losses\(^1\) However, for multimode graded index fibres the analysis is not so complete. Whilst core absorption has been described in terms of loss along the ray path\(^2\), the absorption of the evanescent fields has only been discussed in the cladding\(^3\). In what follows the more general situation of arbitrary absorption in both the core and cladding is examined including losses in the evanescent fields near caustics where classical geometric optics is invalid. In addition the extension of the analysis to general profiles incorporating both grading and step behaviour is considered.

In chapter 2 methods for calculating the ray transmission coefficients for tunnelling leaky rays in non-absorbing waveguides of either slab or cylindrical geometry were presented. Section 2.3 indicated how the WKB methods could be generalised by building in the boundary matching process to take account of discrete jumps in an otherwise arbitrarily graded profile. A similar procedure can be followed to account for material absorption mechanisms since the fields in the slightly absorbing guide are close to those in the non-absorbing situation. In this chapter we show how a perturbation analysis which closely parallel the derivation in chapter 2.3 can be used to develop a highly accurate, simplistic model for absorption losses.

By assembling all the components of loss into a systematic building block approach, we find that only six basic elements need be used to cover most practical situations.
For losses occurring continuously along the core ray path the geometric optics formulation for attenuation presented previously\(^2\) is used. For losses occurring at caustics the ray transmission coefficient \(T\) is calculated. If absorption loss is small then the transmission coefficients due to tunnelling and absorption are additive.\(^1\) The combined ray attenuation coefficient follows immediately from a sum of the components of attenuation along the ray path and at the caustics.

Compared to this procedure, the calculation of power attenuation by an exact electromagnetic boundary value problem rapidly becomes intractable, particularly for problems involving two or more layers which are discussed explicitly in the following chapter.

In this chapter we consider practical situations involving only one layer and where the total variation in the refractive index profile is small, so that polarisation effects can be neglected.

4.2 DERIVATION

Consider the situation shown in Fig. 4.1 which shows a tunnelling leaky ray that is partially reflected within the core of a slab or cylindrical dielectric waveguide at \(y = y_{tp}\) where the \(y\) axis is orthogonal to the waveguide axis. For bound rays the situation is identical except that the radiation caustic \(y_{rad} \to \infty\). Between \(y_{tp}\) and \(y_{rad}\) the fields associated with the ray are evanescent.

The refractive index profile is arbitrarily graded and may contain a step discontinuity. In this derivation we consider specifically just one discontinuity but the extension to multiple layers is straightforward and follows the methods of section 2.3.

To account for absorption, the refractive index profile \(n(y)\) has an additional small imaginary part. If the superscripts \(r, i\) refer to
Fig. 4.1 Arbitrary profile for the square of the refractive index with only one discontinuity at \( y = \rho \) showing the profile linearisation on each side of \( y = \rho \). A ray is reflected at a turning point \( y = y_{tp} \) in the core and reappears at \( y = y_{rad} \) in the cladding. The fields are evanescent between \( y_{tp} \) and \( y_{rad} \).

The real and imaginary parts then

\[
n(y) = n^r(y) + in^i(y). \tag{4.1}
\]

Both \( n^r \) and \( n^i \) may be discontinuous at the interface between each layer but they are otherwise continuous.
We could proceed to analyse profiles for which both $n^r$ and $n^i$ are arbitrary, but the resulting expressions for ray power attenuation are extremely cumbersome and the physical interpretation is unnecessarily obscure. Furthermore, there does not appear to be reliable published information on the variation of material absorption in graded index fibres. It has been assumed that in such fibres $n^i$ follows $n^r$ linearly, but there is no experimental evidence for this hypothesis.

In what follows we shall allow $n^i$ to be either constant or to vary linearly with $y$ within each layer. The first case corresponds to the situation where $n^r$ is constant and the waveguide material is uniform throughout each layer, as in the step index fibre. By invoking the linear dependence, this permits a reasonable approximation to the unknown $n^i$ dependence without invoking an excessive number of building blocks to describe a particular profile. It is believed that these assumptions provide sufficient flexibility to cover all practical waveguide structures while maintaining the essential simplicity of the approach.

The derivation of the losses associated with a ray travelling in the core of a lossy dielectric guide is carried out in two parts in order to avoid the difficulties associated with the calculation of the geometric optics terms. These terms have been calculated elsewhere and estimates for the integral in eqn 4.27 have been given in many cases. In this analysis firstly we examine evanescent field effects remote from the caustic and secondly we estimate core effects at and close to the caustic. We start by considering cladding effects in slab two dimensional cartesian geometry and show how the results in cylindrical geometry follow trivially from a change of variables.
4.2.1 Cladding losses

The first part of the derivation represents a perturbation of the analysis for non absorbing guides carried out in Chapter 2. It relies on the linearisation of square of the refractive index profile at each end of a single jump discontinuity as shown in Fig. 4.1. We solve the electromagnetic boundary value problem for only one discontinuity while the extension to more than one cladding layer is obvious but tedious. We wish to solve the scalar wave equation and assume that the solution has implicit dependences of \( \exp\{\imath k \tilde{\beta} z\} \) and \( \exp\{-\imath \omega t\} \) where \( \tilde{\beta} \) is the ray invariant and \( \omega = kc \) is the angular frequency with \( c \) the speed of light in a vacuum. For the \( y \) dependence

\[
\frac{d^2 \phi(y)}{dy^2} + k^2 (n^2(y) - \tilde{\beta}^2) \phi(y) = 0
\]  
(4.2)

where \( n^2(y) \) is the complex refractive index profile

\[
n^2(y) = (n_r)^2 + (\rho - y)\delta + 2i \frac{n^i}{n_r} \quad y < \rho \tag{4.3}
\]

\[
n^2(y) = (n_r)^2 + (\rho - y)\bar{\delta} + 2i \frac{n^i}{n_r} \quad y > \rho \tag{4.4}
\]

and the respective values of \( n(y) \) on each side of the jump are defined as

\[
n(\rho^-) = n_r + i \frac{n^i}{n_1} \tag{4.5}
\]

\[
n(\rho^+) = n_r + i \frac{n^i}{n_2} \tag{4.6}
\]

with \( \frac{n^i}{n_1} \ll \frac{n_r}{n_1}, \frac{n^i}{n_2} \ll \frac{n_r}{n_2} \) and \( \delta, \bar{\delta} \) are defined by

\[
\delta = \delta^r + i \delta^1, \quad \bar{\delta} = \bar{\delta}^r + i \bar{\delta}^1 \tag{4.7}
\]

\[
\delta^r = -\frac{d}{dy} \text{Re} \ n^2(y) \bigg|_{y=\rho^-}, \quad \bar{\delta}^r = -\frac{d}{dy} \text{Re} \ n^2(y) \bigg|_{y=\rho^+} \tag{4.8}
\]
\[
\delta^1 = 2n_1 \frac{dn^1_i}{dy} \bigg|_{y=\rho^-}, \quad \delta^2 = 2n_2 \frac{dn^2}{dy} \bigg|_{y=\rho^+} \quad (4.9)
\]

The \( \pm \) superscripts represent respective sides of the discontinuity at \( y=\rho \) on Fig. 4.1.

The solution of eqn (4.2) with \( n(y) \) represented by eqns (4.3) and (4.4) is a linear combination of Airy functions. For \( y > \rho \) this combination must be such that \( \phi(y) \) represents a wave travelling in the \(+y\) direction when \( y \gg y_{\text{rad}} \). Similarly for \( y < \rho \) it must represent an incident and reflected wave for \( y \ll y_{\text{tp}} \). We deduce from the asymptotic forms of the Airy functions that

\[
\phi(y) = Ai(s) - i Bi(s) + R(Ai(s) + i Bi(s)), \quad y < \rho \quad (4.10)
\]

\[
\phi(y) = S (Ai(t) - i Bi(t)), \quad y > \rho \quad (4.11)
\]

where \( Ai, Bi \) are Airy functions of the first and second kinds, \( s \) and \( t \) are complex variables given by

\[
s = (\delta r^2)^{1/3} (y-y_{\text{tp}}) \left[ 1 + \frac{\delta^1}{3\delta r} \right] - i \, 2n_1 \frac{i r^2}{2} (\delta r^2)^{-2/3} \)
\]

\[
t = (\delta r^2)^{1/3} (y_{\text{rad}}-y) \left[ 1 + \frac{\delta^1}{3\delta r} \right] - i \left\{ 2n_2 \frac{i r^2}{2} - 2n_1 \frac{i r^2}{2} \right\} (\delta r^2)^{-2/3} \quad (4.12)
\]

and \( R \) and \( S \) are constants to be determined.

The boundary conditions at the jump \( y=\rho \), are equivalent to continuity of \( \phi(y) \) and \( \frac{d\phi(y)}{dy} \). If we define

\[
u = s \bigg|_{y=\rho^-}, \quad \nu' = s' \bigg|_{y=\rho^-}, \quad \nu' = t' \bigg|_{y=\rho^+}
\]

where the prime denotes differentiation with respect to \( y \) then we obtain two linear equations in \( R \) and \( S \) which can be solved for \( R \). The arguments \( u, v \) of the Airy functions have small imaginary components so that the
Airy functions can be expanded in a Taylor series about the real parts of $u, v$ to give

$$\text{Ai}(u) = \text{Ai}(u_r) + iu_r \text{Ai}'(u_r)$$

and corresponding expansions for $\text{Bi}(u), \text{Ai}(v)$ and $\text{Bi}(v)$.

The ray transmission coefficient $T$ describes the fraction of power transmitted across the interface $y = \rho$ and is given as

$$T = 1 - |R|^2$$

leading to a very complicated and unwieldy expression of little practical significance. In general $y_{\text{tp}}$ and $y_{\text{rad}}$ are both sufficiently separated from each other and from $y = \rho$ that the asymptotic forms of the Airy function are valid. In this case the field solutions for the linearised parts of the profile can be matched to the WKB solutions away from the interface following the methods of chapter 2 to give

$$T = |T_F| \exp\left\{ -2 \int_{y_{\text{tp}}}^\rho |k_y(y)| \, dy \right\} .$$

$$+ \exp\left\{ -2 \int_{\rho}^{y_{\text{rad}}} |k_y(y)| \, dy \right\}$$

where $|T_F|$ represents the analytic continuation of the classical Fresnel transmission coefficient to complex wave vectors and

$$T_F = \frac{4k_y k_y}{(k_{y_1}^2 + k_{y_2}^2)}$$
where $k_1$ and $k_2$ are the outward or radial components of the local plane wave vector on respective sides of the step in the profile. We note in passing that (4.18) is the simple sum of both tunnelling and absorption terms.

The results for cylindrical geometry follow from an analogous equation to eqn (4.2) for the radial variation in cylindrical polar coordinates $(r, \phi, z)$ based on the waveguide axis as the $z$ axis,

$$\frac{d^2 \psi(r)}{dr^2} + \frac{1}{r} \frac{d \psi(r)}{dr} + k^2 \left\{ n^2(r) - \beta^2 - \left( \frac{\rho \lambda}{r} \right)^2 \right\} \psi = 0 \quad (4.20)$$

Via the transformation $\psi(r) = r^k \psi(r)$ and writing $r$ as $y$ we see that this equation reduces to

$$\frac{d^2 \psi(y)}{dy^2} + k^2 \left\{ n^2(y) - \beta^2 - \left( \frac{\rho \lambda}{y} \right)^2 \right\} \psi = 0 \quad (4.21)$$

where we have ignored $\frac{1}{y}$ compared to $(k \rho \lambda)^2$. This is the same form as eqn (4.2) but with the $y$ component of the wave vector now given by $k^2 \{ n^2(y) - \beta^2 - \left( \frac{\rho \lambda}{y} \right)^2 \}$ which is the radial component of the wave vector in cylindrical geometry. Again the extra factor $\sqrt{r}$ in the solution of eqn (4.20) compared with eqn (4.2) can be readily accounted for by the variation of the flux tube cross section in cylindrical geometry and the correct coefficients are still given by eqn (4.18) with $y$ as the radial variable.

4.2.2 Core losses

The second part of the derivation estimates the correction to geometric optics due to the presence of the caustic and its evanescent field in a slightly absorbing graded index core. It relies on a linearisation of the square of the refractive index profile $n^2(y)$ across the
narrow Airy region close to the caustic where geometric optics breaks down. Since this region is very narrow the linearisation provides a highly accurate approximation to the amount of this correction provided that there is no discontinuity within this region. This is the case for most situations of interest in a graded fibre.

In slab geometry the complete electromagnetic boundary value problem for the dashed profile in Fig. 4.2 representing a linearisation of the solid profile is readily solvable. For a slightly lossy medium a perturbation analysis similar to that above can be used. Further we can exactly solve for the ray path in this case to obtain the losses according to geometric optics for the linearised profile. The difference between

![Diagram](image)

**Fig. 4.2** Linearisation of the square of the refractive index profile in the region of the reflection or turning point at \( y = y_{tp} \). The original profile is shown as a solid line. The linearisation is shown dashed.
these two quantities is the required correction to geometric optics about the turning point \( y_{tp} \). We calculate a transmission coefficient \( T \) representing this correction as

\[
T_{\text{caus}} = \frac{4kn_n(r(\theta))_n^2}{\delta_c} \left\{ \sin \theta z - \cos^2 \theta z \tanh^{-1}(\sin \theta z) \right\} \tag{4.22}
\]

where \( \delta_c \) is the slope of the real part of \( n^2(y) \) at \( y = y_{tp} \) and \( \theta_z \) is the angle of the ray to the \( z \) axis at \( y = 0 \). For small \( \theta_z \)

\[
T_{\text{caus}} \approx \frac{8}{3} \frac{kn_r(\theta)^r_1}{\delta_c} \sin^3 \theta z \tag{4.23}
\]

The error involved in neglecting this term for the dashed profile of Fig. 4.3 is \( \frac{100 \sin^2 \theta}{3} \% \).

For cylindrical geometry with cylindrical polar coordinates \((r, \phi, z)\) we seek to linearise \( n^2(y) - (\frac{\partial \tilde{y}}{y})^2 \) in eqn (4.21) around each caustic. Replacing \( \delta_c \) by \( \delta_c - 2(\tilde{y}^2/\rho) \) we can reexpress eqn (4.23) to give for cylindrical geometry a correction due to the presence of the outer caustic,

\[
T_{\text{caus}} = \frac{8}{3} \frac{kn^r(\theta)^r_1}{n(y_{ic})} \frac{(n(y_{ic})^2 - \delta^2)^{3/2}}{\left( \delta_c - \frac{2\delta^2}{\rho} \right)} \tag{4.24}
\]

where \( n(y_{ic}) \) is the value of the refractive index at the inner caustic.

Similarly we can solve for the extra loss at the inner caustic for a skew ray in a cylindrical structure. Once again we can solve eqn (4.21) where the coefficient of \( \Psi \) is linearised about the inner caustic. For most practical situations the profile slope \( \delta^r \) at the inner caustic approaches zero and the solution is again given by eqn (4.24) with \( \delta_c - \frac{2\delta^2}{\rho} \) replaced by \( \frac{2(\rho\delta)^2}{(y_{ic})^3} \). It should be noted that as \( y_{ic} \to 0 \), the
correction due to the presence of the inner caustic approaches zero. This correction is often neglected with respect to the correction of the outer caustic.

4.3 THE BUILDING BLOCK APPROACH

We identify four mechanisms for power loss from rays propagating within the core of a multimode waveguide. The overall power attenuation coefficient $\gamma$ is compounded from the sum of the individual loss mechanisms in the following way. If the power at position $z$ along the waveguide axis is denoted $P(z)$ then

$$P(z) = P(0) \exp(-\gamma z) \quad (4.25)$$

The attenuation coefficient $\gamma$ is constructed from the sum

$$\gamma = \gamma_{\text{ray}} + \gamma_{\text{tun}} + \gamma_{\text{evan}} + \gamma_{\text{caus}} \quad (4.26)$$

The first attenuation coefficient $\gamma_{\text{ray}}$ represents the loss of power absorbed along the ray path within the core. Each ray path has the distinctive behaviour shown in Fig. 4.3 regardless of whether the ray path is uniform or graded. We define $z_p$ to be the ray period which is the distance between successive outer caustics measured along the axis of the guide for skew rays, or the distance between successive caustics for meridional rays passing through the waveguide axis. If $\alpha_{\text{co}}(r)$ is the axi-symmetric power absorption coefficient in the core, then

$$\gamma_{\text{ray}} = \frac{1}{z_p} \int_{z_p} \alpha_{\text{co}}(r) dr \quad (4.27)$$

where the integration is along the ray path over a ray period. $\gamma_{\text{ray}}$ represents the geometric optics averaged power loss over a period of the
Fig. 4.3 Tunnelling ray paths within an optical fibre waveguide core showing distance $z_p$ between reflection points at which power is lost for (a) a graded index profile on a cylindrical waveguide and (b) a step index profile on a slab waveguide.

The remaining components of $\gamma$ are all associated with power losses which only occur at the caustics of the ray path. These are all calculated via the ray power transmission coefficient $T$ defined by eqn (1.6).

For tunnelling rays incident at an outer caustic of the ray path within the core, $y = y_{tp}$ on Fig. 4.1, a fraction of the incident ray power is transmitted across the evanescent region and reappears at the radiation caustic $y_{rad}$ as a transmitted ray which is sketched in Fig. 4.4.
Fig. 4.4 Description of the tunnelling mechanism showing two dimensional projection in the plane of incidence of a ray tube reflected at $y = y_{tp}$. A fraction of the energy reflected at $y = y_{tp}$ is transmitted across the evanescent region and the core cladding interface at $y = \rho$ to reappear in the cladding where it is radiated away within another ray tube originating at $y_{rad}$.

Then,

$$Y_{tun} = \frac{T_{tun}}{\rho}$$

(4.28)

is the averaged loss between successive outer caustics and $T_{tun}$ may be readily calculated using the methods of Chapter 2.
The absorption of power from the evanescent fields extending beyond $\rho$ into the cladding medium removes a fraction $T_{\text{evan}}$ of the power from the incident ray. This mechanism, like the tunnelling loss, removes power from the evanescent fields within the cladding and is illustrated in Fig. 4.5. Within the cross hatched area the fraction $T_{\text{evan}}$ of ray power is absorbed by the cladding layers. The power attenuation coefficient $\gamma_{\text{evan}}$ is calculated from the ray power transmission coefficient by

$$\gamma_{\text{evan}} = \frac{T_{\text{evan}}}{z_p} \quad (4.29)$$

Fig. 4.5 Description of the absorption loss showing two dimensional projection in the plane of incidence of a ray tube reflected at $y = y_{tp}$. The evanescent field extending into the lossy cladding, $y > \rho$, is shown cross hatched.
Lastly we can calculate the extra fraction of ray power lost at the inner and outer caustics of the ray path within the core, \(y_{tp}\) and \(y_{ic}\), because of the inadequacy of geometric optics. This extra fraction \(T_{caus}\) accounts for the difference between the ray path attenuation coefficient of eqn (4.27) and more accurate analysis in the neighbourhood of \(y_{tp}\) and \(y_{ic}\). It includes core absorption at and near the caustics out to the core cladding interface at \(y = \rho\). Hence

\[
\gamma_{caus} = \frac{T_{caus}}{z_p}
\]  

(4.30)

averages this effect along the fibre axis.

If we combine eqns (4.25)-(4.30) we can express the overall attenuation coefficient in terms of a total power transmission coefficient \(T\), where

\[
T = T_{tun} + T_{evan} + T_{caus} + \int_{p} \alpha_{co}(s)ds
\]  

(4.31)

and

\[
\gamma = \frac{T}{z_p}
\]  

(4.32)

Equations (4.31) and (4.32) are the essence of the building block approach adopted here. The components of (4.31) are

(A) Tunnelling coefficient

In slightly absorbing guides the fields are approximately those of the non-absorbing guide and again we have the tunneling coefficient given as a product of a WKB integration over the continuous parts of the refractive index profile in the evanescent region between the reflection or turning point \(y_{tp}\) and the position at which the ray reappears \(y_{rad}\), and a factor \(|T_p|\) to account for the jump in the refractive index profile
in the evanescent region where \( T_F \) is the analytic continuation of the Fresnel power transmission coefficient to complex wave vectors.

If \( k_y(y) \) is the \( y \)-component of the wave vector given by the coefficient of \( \Phi \) in eqn (4.2) and (4.20)

\[
T_{\text{tun}} = |T_F| \exp\left\{-2 \int_{y_{\text{tp}}}^{y_{\text{rad}}} |k_y(y)| \, dy \right\} \quad (4.33)
\]

and \(|T_F|\) at that discontinuity is given by eqn (4.19).

The turning points \( y_{\text{tp}}, y_{\text{rad}} \) in a slightly absorbing fibre are approximately those for the non absorbing fibre.

(B) Evanescent field absorption coefficient

The transmission coefficient \( T_{\text{evan}} \) representing the fraction of power lost from the ray at the reflection or turning point by absorption from the evanescent fields in the slightly lossy cladding is given at each layer by the sum of three basic terms. (i) An absorption loss for a uniform absorbing medium of \(-\frac{n_i n_r k^2}{|k_y^2(\rho)|^2}\) where \( k = \frac{2\pi}{\lambda_o} \) and \( \lambda_o \) is the free space wavelength, (ii) a term to account for a backflow of energy to maintain the field distribution outside the guide of \( \frac{n_i n_r k^2}{k_y^2(\rho)} \), (iii) a term to account for the effect of a small linear slope in the profile of \( \frac{\delta^i}{3\delta^r} - \frac{\delta^i}{3\delta^r} \) where \( \delta^i, \delta^i, \delta^r, \delta^r \) are defined by eqns (4.14) and (4.15).

The total evanescent field \( T_{\text{evan}} \) is then

\[
T_{\text{evan}} = |T_F| \cdot \left\{ \frac{n_i n_r k^2}{|k_y^2(\rho^+)|^2} - \frac{n_i n_r k^2}{|k_y^2(\rho^-)|^2} + \frac{\delta^i}{3\delta^r} - \frac{\delta^i}{3\delta^r} \right\} \quad (4.34)
\]

(C) Caustic absorption coefficient

The attenuation due to material absorption along the ray path in a slightly absorbing medium represented by eqn (4.5) is a geometric optics approximation which is accurate everywhere except at a reflection
or turning point. Therefore at each caustic we may account for this
deficiency by assigning a transmission coefficient $T_{caus}$ to allow for the
difference between the losses according to a geometric optics analysis
and an exact electromagnetic theory solution. This term acts as a small
correction to geometric optics and includes absorption of the evanescent
field in the core material adjacent to each caustic. For a slab guide
this correction is given simply by eqn (4.23). For cylindrical geometry
$T_{caus}$ is the sum of corrections to geometric optics at both the inner and
outer caustics. We find

\[ T_{caus} = \frac{8}{3} \frac{\text{kn}}{n(y_{ic})} \left( \frac{(n(y_{ic}^2 - \tilde{\beta}^2)^{3/2}}{\left( \delta_c - \frac{2\tilde{\beta}^2}{\rho} \right)} \right) \mid y = y_{tp} \]

\[ + \frac{8}{3} \frac{\text{kn}}{n(y_{tp})} \left( \frac{n^2(y_{tp}) - \tilde{\beta}^2)^{3/2}}{\left( \delta_c - \frac{2(n\tilde{\beta})^2}{(y_{ic}^3)} \right)} \right) \mid y = y_{ic} \]

(4.35)

where $\delta_c = - \frac{d}{dy} \text{Re } n^2(y)$.

Both corrections are in general a small percentage of the
attenuation along the ray path and can often be neglected.

4.4 EXAMPLES

In the previous section we described the elements of the building
block approach to calculating attenuation for a ray travelling in the core
of a singly clad arbitrarily graded dielectric waveguide. A collection
of simple terms involving elementary physical processes is sufficient to
cover virtually all practical situations. This means that the solution
for ray attenuation can now be written down by inspection from the profile
and the ray invariants $\tilde{\beta}, \tilde{\lambda}$. It remains to demonstrate the significance of the method by application to some specific problems.

4.4.1 Step index slab

For a multimode step index guide rays are reflected from the boundaries of the guide as illustrated in Fig. 4.3(b). In this case the evanescent fields associated with each reflection are totally confined to the cladding and therefore no term $T_{\text{caus}}$ is required. The total transmission loss is then just the sum of the tunnelling and absorption losses in (4.31) with $T_{\text{caus}} = 0$. If the expression (4.34) is used for $T_{\text{evan}}$ then a term to account for backflow of energy at $y = \rho$ is necessarily included. This is the same result as has been previously noted for reflection from a lossy plane dielectric interface.

4.4.2 Step index fibres

An expression for the modal attenuation due to both lossy core and cladding materials in step index fibres has already been given but it only reduces to our results in the approximation of paraxial rays. A ray analysis of the interplay of absorption and tunneling losses in illuminated step fibres has been calculated using the combination $\gamma_{\text{ray}} + \gamma_{\text{tun}} + \gamma_{\text{evan}}$.

4.4.3 Graded index fibres

Both modal and ray expressions for attenuation in graded index fibres have also been given which are equivalent to $\gamma_{\text{ray}}$ of eqn (4.26), but both ignore the effect of the absorption of the evanescent fields beyond the caustics and $\gamma_{\text{caus}}$ of eqn (4.30). We estimate the error involvement in neglecting these terms in a typical fibre is < 0.6%.
Fig. 4.6 A ray is reflected from the turning point at $y = y_{tp}$ within the core of a parabolic index fibre whose profile $n(y)$ is given by $n(y) = n(0)(1 - 2\Delta(y/\rho)^2)^{1/2}$. The cladding (shown shaded) is uniform and slightly lossy. A discontinuity occurs at the core cladding interface at $y = \rho$.

For attenuation in a circularly symmetric guide with a graded lossless core and a lossy uniform cladding with no discontinuity at the core cladding interface, eqn (4.34) reduces to the example of ref. 3. Where there is a step discontinuity in the real part of the refractive index profile, as shown in Fig. 4.6 at the core cladding interface and the cladding is uniform but slightly lossy then a factor $|T_p|$ appears to account for this jump in eqn (4.34). This gives the same results as a modal solution\(^{10}\) which uses a perturbation technique to solve the eigenvalue equation for the imaginary component of the axial propagation constant $\beta$. Our power attenuation coefficient $\gamma$ is just twice the field attenuation coefficient of ref. 10.
REFERENCES


CHAPTER 5
MULTILAYERED WAVEGUIDE STRUCTURES

5.1 INTRODUCTION

In the previous chapters of this thesis the derivation of the ray transmission coefficient, $T$, has been described on slab and fibre guides with continuously graded refractive index profiles containing one step discontinuity. The extension of the derivations has been intimated for profiles with multiple layers of cladding and discontinuities in the profile between the layers. In this chapter we show how tunnelling and absorption losses in multilayered guides can be treated within the building block approach of Chapter 4. Essentially the expressions for the ray transmission coefficients $T_{\text{tun}}$ and $T_{\text{evan}}$, used to account for losses at the outer caustics of ray paths within the core, must be expanded to include a factor $|T_p|$ at each additional jump in the refractive index profile within the evanescent region of the fields associated with the ray path. The absorption introduced by each cladding layer must be summed to give the total contribution to the loss.

In the first section, we extend the building block approach of Chapter 4 to cover the general situation of multilayered waveguide structures. In section 5.3 we discuss a specific example of practical importance concerning the tunnelling loss on a non-absorbing 3 layered symmetrical slab. Finally in order to demonstrate the wide applicability of the concepts developed in this thesis, the situation of a multimode
step-index fibre manufactured with an additional low loss layer included between the conventional core and cladding materials is investigated to find how effective this procedure is in reducing total attenuation along the guide.

5.2 MULTILAYERED WAVEGUIDES

Consider the situation depicted in Fig. 5.1 which shows a sketch of the real part of the refractive index profile of a multilayered waveguide of either slab or cylindrical geometry, where the y axis is perpendicular to the waveguide axis. In the cladding there can be a finite number of layers consisting of arbitrary variation in the continuous sections between discontinuities at their edges. Part of the power of a ray is reflected at $y_{tp}$ and part tunnels through the evanescent region (shown shaded) to reappear at $y_{rad}$. The position of the caustics at $y_{rad}$ and $y_{tp}$ depend on the profile and the ray invariants.

![Fig. 5.1 Sketch of the real part of the refractive index n(y) showing a step in the profile at the core cladding interface at y = ρ. In the cladding there can be a finite number of layers consisting of arbitrary variation in the continuous sections between discontinuities at their edges. Part of the power of a ray is reflected at $y_{tp}$ and part tunnels through the evanescent region (shown shaded) to reappear at $y_{rad}$. The position of the caustics at $y_{rad}$ and $y_{tp}$ depend on the profile and the ray invariants.](image-url)
finite number of layers consisting of an arbitrary variation in the continuous sections of the profile between discontinuities at the layer edges. A tunnelling ray is shown incident upon and partially reflected from the incident caustic at \( y = y_{tp} \). A fraction of the power associated with the incident ray tunnels through the evanescent region, shown shaded, and reappears at the radiation caustic at \( y = y_{rad} \) to be transmitted away from the core.

The general situation for tunnelling rays on a non absorbing guide with two or more discontinuities in the profile within the evanescent region was solved in chapter 2 by building in the boundary matching procedure to the WKB method. The WKB fields in the cladding are therefore diminished by the factor \( |T_F| \) at each step discontinuity in the profile. The same effect occurs in slightly lossy waveguides resulting in the introduction of the factor \( |T_F| \) at each boundary in the expressions for the absorption loss. The case of bound rays is recovered in the tunnelling ray limit when \( y_{rad} \rightarrow \infty \).

In multilayered guides we use the same building block approach represented by eqns (4.30) and (4.31) as we used in the single layered guides of chapter 4. However the expressions for the ray transmission coefficients \( T_{evan} \) and \( T_{tun} \) must take into account the contributions from each of the layers and so they become a little more complicated.

(A) Tunnelling coefficient.

In slightly absorbing guides the fields are approximately those of the non-absorbing case and again we have the tunneling coefficient, \( T_{tun} \) given as a product of two fundamental quantities, (i) a WKB integration over the continuous parts of the refractive index profile in the evanescent region between the reflection or turning point \( y_{tp} \) and the position at
which the ray reappears \( y_{\text{rad}} \) and (ii) a factor \( |T_F| \) at each jump in the refractive index profile between adjacent layers in the evanescent region where \( T_F \) is the analytic continuation of the Fresnel power transmission coefficient to complex wave vectors.

If we define \( k_y(y) \) as the outward \( y \)-component of the wave vector then

\[
T_{\text{run}} = \prod_{m=1}^{N} |T_F^m| \exp \left\{ -2 \int_{y_{\text{tp}}}^{y_{\text{rad}}} |k_y(y)| \, dy \right\} \tag{5.1}
\]

where \( m \) denotes the \( m \)th discontinuity between \( y_{\text{tp}} \) and \( y_{\text{rad}} \) and if \( k_{y_1} \) and \( k_{y_2} \) are the respective values of \( k_y(y) \) on either side of a discontinuity, \( |T_F^m| \) at that discontinuity represents the analytic continuation of the classical Fresnel coefficient given by

\[
T_F = \frac{4 \, k_y y_1 y_2}{(k_y + k_y)^2} \tag{5.2}
\]

The turning points \( y_{\text{tp}} \), \( y_{\text{rad}} \) in a slightly absorbing fibre are approximately those for the non absorbing fibre.

(B) Evanescent field absorption coefficient.

The transmission coefficient, \( T_{\text{evan}} \), representing the fraction of power lost from the ray at the reflection or turning point by absorption from the evanescent fields in the slightly lossy cladding is given at each layer by the sum of three basic terms. If a layer starting at \( y_\lambda \) has a refractive index given by eqn (4.1) then the terms are, (i) an absorption loss \(^1\) for a uniform absorbing medium of \((n_\lambda^i n_\lambda^k k^2)/|k_y(y_\lambda)|^2\) where \( k = 2\pi/\lambda \) and \( \lambda \) is the free space wavelength, (ii) a term to account for a backflow of energy to maintain the field distribution outside the guide of \((n_{\lambda-1}^i n_{\lambda-1}^r k^2)/(k_y(y_\lambda)^2)\), (iii) a term to account for the effect of a small linear slope in the profile of \((\delta^i_\lambda)/(3\delta^r_\lambda) - (\delta^i_\lambda)/(3\delta^r_\lambda)\) where
\[ \bar{\delta}, \bar{\epsilon}, \bar{\rho}, \bar{\rho} \] are defined by eqns (4.18) and (4.19), all multiplied by a factor \(|T_j|\) at each discontinuity between \(y_{tp}\) and \(y\) and a WKB integral between \(y_{tp}\) and \(y\).

The total evanescent field \(T_{\text{evan}}\) is then the sum of the losses in each layer. Formally we have

\[
T_{\text{evan}} = \sum_{\ell=1}^{N} \sum_{m=1}^{N} |T^{m}_\ell| \exp \left\{ -2 \int_{y_{tp}}^{y} |k_\ell(y)| \, dy \right\} .
\]

\[
\left\{ \frac{n_\ell n_\ell^2}{|k_{y_\ell}(y)|^2} - \frac{n_\ell n_{\ell-1}^2 k_\ell^2}{|k_{y_{\ell-1}}(y)|^2} + \frac{\delta_\ell^-}{3 \delta_\ell^+} - \frac{\delta_\ell^+}{3 \delta_\ell^-} \right\} . \quad (5.3)
\]

The same expressions for \(T_{\text{caus}}\) and \(\alpha_{co}\) apply within the core of multilayered guides as for the single layered guides of chapter 4.

### 5.3 THE THREE LAYER SYMMETRIC WAVEGUIDE

Consider the symmetric, non-absorbing step index slab waveguide shown in Fig. 5.2 consisting of a core of index \(n_1\), surrounded by a layer on index \(n_2 < n_1\) and thickness \(d\), beyond which is a third layer of index \(n_3 > n_2\). Under these conditions tunnelling rays can propagate along the core. They are partially reflected at the core boundary, \(y = y_{tp}\), and reappear at the interface between the second and third layers, \(y = y_{\text{rad}}\), i.e. \(k_\ell^2(y) < 0\) for \(y_{tp} < y < y_{\text{rad}}\).

For TE polarization eqn (5.1) predicts

\[
T_{\text{E}} = \frac{16k_{\|} |k_{1y} y_2 y_3^2 |k_{2y} y_3 |k_{3y} y_2 |} {(k_{1y}^2 + |k_{2y} y_3 |^2)(k_{2y}^2 + |k_{3y} y_2 |^2)} . \quad (5.4)
\]

where
\[ k_{y_1} = n_1 k \cos \alpha_1, \quad k_{y_3} = k\{n_1^2 \cos^2 \alpha_1 - (n_3^2 - n_2^2)\}^{1/2} \]
\[ k_{y_2} = ik\{n_1^2 - n_2^2\} - n_1 \cos^2 \alpha_1\}^{1/2} \]

and \( i = \sqrt{-1} \). We can derive the exact tunnelling coefficient by solving the electromagnetic boundary value problem for reflection of plane waves, and obtain

\[
T_E = \frac{4k_{y_1} |k_{y_2}|^2 k_{y_3} \{1 - \tanh^2 (|k_{y_2}| d)\}}{|k_{y_2}|^2 (k_{y_1} + k_{y_3})^2 + \tanh^2 (|k_{y_2}| d)(k_{y_1} k_{y_3} - |k_{y_2}|^2)} .
\]

Fig. 5.2 Symmetrical slab waveguide showing a tunnelling ray. The layer of refractive index \( n_2 \) has thickness \( d \), and \( n_1 > n_2, n_3 > n_2 \). The evanescent region is shown shaded.
Eqn. (5.6) reduces to eqn (5.4) if $|k_y|d \gg 1$. When $k_y \to 0$, $\alpha_1 \to \alpha_c = \sin^{-1}(n_2/n_1)$ where $\alpha_c$ is the critical angle, the WKB analysis becomes invalid and the more exact expression (5.6) must be used.

The power attenuation coefficient is

$$\gamma = \frac{T_E}{2d \tan \alpha_1}, \quad (5.7)$$

which may be obtained from the eigenvalue equation for the waveguide only after an extensive analysis.

The same problem has been tackled using a perturbation analysis\(^2\) which gives the same results as (5.4) and (5.7) provided $|k_y d| \gg 1$.

5.4 STEP INDEX FIBRE WITH DOUBLY LAYERED CORE

It has been suggested that it may be possible to lower the attenuation of a step index optical fibre if during the fabrication process a secondary low loss layer of intermediate refractive index was included between the usual core and cladding materials of a conventional step index fibre as indicated in the cross sectional view of Fig. 5.3. The effect of such a layer would be to reduce the evanescent field absorption in the lossy cladding material for both the bound and leaky rays.

In this analysis we investigate quantitatively the attenuation of a fibre with a double layered core using the building block approach of Section 5.2 to see whether a significant improvement over a conventional step index fibre results from this process.

Previously the effects of absorption and radiation losses on a singly clad step index fibre have been considered\(^3,4\) using a modified form of geometric optics which includes evanescent field absorption in the cladding. The core of the simple step fibre was considered to be excited
Fig. 5.3 Cross sectional view of a model of a fibre with doubly layered core. The core has uniform refractive index \( n_1 \) surrounded by a secondary layer of index \( n_2 \). This structure is imbedded in a lossy cladding medium of infinite extent and index \( n_3 \), where \( n_1 > n_2 > n_3 \).

at one end by (i) an incoherent (or Lambertian) source and (ii) a coherent source like a laser. In each case the overall attenuation of the bound and tunnelling rays were evaluated by summing the total power remaining in each ray over all the ray families. We compare our building block approach with the results of ref. 3.
The refractive index profile for a double layered step index fibre is shown in Fig. 5.4. The core of the fibre contains a low loss material of refractive index $n_1$ and radius $\rho_1$, surrounded by a secondary low loss layer of intermediate index $n_2$ and outer radius $\rho_2$, and the whole is embedded in an infinite, slightly lossy medium of index $n_3$ so that $n_1 > n_2 > n_3$. The cladding in a practical fibre is generally sufficiently
thick that its finiteness is of little consequence and the assumption of an infinite cladding is a good approximation.

For each ray launched within the core the losses at each reflection from the interfaces \( r = \rho_1 \) and \( r = \rho_2 \) can be calculated by invoking elements of the building block approach in section 5.2. Firstly however, it is necessary to classify the rays within the guide and then consider the power launched into the various ray families from a source illuminating the end face of the fibre. We then perform a summation over all rays on the structure to calculate the total ray power at some distance \( z \) along the axis of the guide.

5.4.1 Ray classification

On a multilayered dielectric structure the most convenient way of classifying the rays propagating within the guide is by using the radial component of the plane wave vector \( k_r(r) \) from the local plane wave decomposition at each boundary \( \rho_1, \rho_2 \). We define \( k_r(r) \) in terms of ray invariants \( \tilde{\beta}, \tilde{\gamma} \) within a layer as

\[
k_r(r) = n^2 - \tilde{\beta}^2 - \left( \frac{\tilde{\gamma}}{r} \right)^2
\]

where \( n \) is the refractive index profile of Fig. 5.4 and \( \rho \) is the corresponding radius of the layer. At any point in the guide if \( k_r(r) \) is real then real waves exist but where \( k_r(r) \) is imaginary then evanescent fields exist. Caustics occur where \( k_r(r) = 0 \). Further if the superscripts \( \pm \) are used to describe the left and right hand sides of each interface on Fig. 5.4 then we test the sign of \( k_r^2(r) \) to distinguish the rays. The possibilities are

(a) \( k_r^2(\rho^-_1) > 0 \) while \( k_r^2(\rho^+_1), k_r^2(\rho^-_2), k_r^2(\rho^+_2) \) are all negative

The rays are reflected within the inner core by the interface
$r = r_1$. If some point exists in the cladding $r > r_2$ where $k_r^2(r) = 0$, then the ray is a tunnelling ray and the ray transmission coefficient $T_{\text{tun}}$ of eqn (5.1) consists of a WKB integration over the continuous parts of the refractive index profile and a factor $|T_p|$ at each interface $r_1, r_2$. If no such point in the cladding $r > r_2$ exists then the ray is bound and $T_{\text{tun}} = 0$. In both cases evanescent field absorption can occur in both the secondary layer, $r_1 < r < r_2$, or the cladding $r > r_2$ so that the expression for $T_{\text{evan}}$ (eqn (5.3)) contains two terms.

(b) $k_r^2(r)$, $k_r^2(r_1^+)$, $k_r^2(r_2^-)$ all $> 0$ while $k_r^2(r_2^+) < 0$

These rays are reflected at the interface $r = r_2$. While the rays are within the inner core $r < r_2$, the rays make angle $\theta_z$ with the $z$ axis and while they are within the secondary core $r_1 < r < r_2$ the rays make a corresponding angle $\theta_z$ with the $z$ axis. Refraction occurs at the interface $r_1$ and

$$n_1 \cos \theta_z = n_2 \cos \theta_z = \beta$$ (5.9)

The skew angle $\theta_\phi$ defined by Fig. 5.5 is denoted $\theta_\phi$ at incidence upon $r = r_1$ and is related to the corresponding skew angle $\theta_\phi$ at $r = r_2$ by

$$\cos \theta_\phi_1 = \cos \theta_\phi_2 \cdot \frac{\rho_2}{\rho_1} \cdot \frac{n_2}{n_1} \frac{n_2}{\sqrt{1 - n^2_1 \cos^2 \theta_2}}$$ (5.10)

The rays may be classified as tunnelling rays if a point exists in the cladding $r > r_2$ where $k_r^2(r) = 0$, then $T_{\text{tun}}$ is given by (5.1). Otherwise $T_{\text{tun}} = 0$. Evanescent field absorption occurs in the region $r > r_2$ according to (5.3).

(c) $k_r^2(r_1^-) > 0$, $k_r^2(r_1^+) < 0$, $k_r^2(r_2^-) > 0$ and $k_r^2(r_2^+) < 0$

For these rays a point exists between $r_1$ and $r_2$ where $k_r^2(r) = 0$. Power from the inner core tunnels from $r = r_1$ to the caustic between
Fig. 5.5 Illustration of ray angles defined at incidence on an optical fibre boundary. Only one boundary is shown for simplicity. The rays are characterised by their angles $\theta_Z$, $\theta_{\phi}$. $\alpha_i$ is the angle between the normal to the boundary OP and the incident ray direction $k$. $\sin \alpha_i = \sin \theta_Z \sin \theta_{\phi}$.

$\rho_1$ and $\rho_2$. Under these conditions power is coupled between the inner and outer sections of the core. The dominant sources of power loss are assumed to occur along the ray path and upon reflection of the ray at $r = \rho_2$ because of evanescent field absorption and possible tunnelling loss.

(d) $k^2_r(\rho_1^-), k^2_r(\rho_1^+), k^2_r(\rho_2^-), k^2_r(\rho_2^+)$ all $> 0$

These rays refract at each interface and power is lost rapidly from the core of the guide. They are specifically ignored since they contribute little to the power within the core.\(^8\)
(e) \( k_r^2(\rho^-) \) and \( k_r^2(\rho^+) < 0 \) while \( k_r^2(\rho^-) > 0 \) and \( k_r^2(\rho^+) < 0 \).

In this case the rays are reflected from the interface \( r = \rho_2 \).

Each ray is sufficiently skew that it never enters the central core and remains solely confined to the secondary layer. Absorption loss occurs along the ray path and from the evanescent fields in the cladding \( r > \rho_2 \).

For tunneling rays \( T_{\text{tun}} \) is predicted by (5.1) while for bound rays \( T_{\text{tun}} = 0 \).

(f) \( k_r^2(\rho^-) \) and \( k_r^2(\rho^+) < 0 \) while \( k_r^2(\rho^-) \) and \( k_r^2(\rho^+) > 0 \)

This final case is for refracting rays which, like case (d) above, are ignored.

It now remains to consider how the power is distributed amongst the rays for coherent and incoherent sources.

5.4.2 Illumination of the double layered core

A simple ray treatment which accounts for both bound and tunnelling rays within dielectric optical waveguides has been used successfully to describe the ray power launched along a simple step index optical fibre.\(^4,5\) We extend this theory to consider a fibre having a double layered core by including the illumination of both the annular shaped cross section of the secondary layer as well as the circular shaped cross section of the central core at the fibre face as sketched in Fig. 5.3. The excitation of the bound and tunnelling rays in each of these regions of the core are considered separately. Those rays which become refracting because of the proximity of adjacent sections of the core, are then ignored.

(i) Incoherent source

An incoherent or Lambertian source of uniform intensity over the fibre face is azimuthally symmetric and has angular dependence of the form

\[ I = I_0 \cos \theta_z \]  

(5.11)
where $I_o$ is the source strength constant and $\theta_z$ is the angle to the $z$ axis normal to the end face of the fibre.

If we define the complementary critical angles of the inner and outer sections of the core as

$$\theta_{c1} \approx \sin \theta_{c1} = [1 - (n_3/n_1)^2]^{1/2}$$

$$\theta_{c2} \approx \sin \theta_{c2} = [1 - (n_3/n_2)^2]^{1/2}$$

then the power coupled into the trapped and leaky rays from a given source area $A$ is respectively given from

$$P_{\text{bound}} = \int_{\text{source}} dA I_o \int_0^1 \sin^2 \theta_c R dR \int_0^{r/\rho} \frac{4 \rho}{\sqrt{1 - (\frac{\rho}{r} x)^2}} dx$$ (5.13)

and

$$P_{\text{leaky}} = \int_{\text{source}} dA I_o \int_1^{1/\theta} \sin^2 \theta_c R dR \int_{r/\rho}^{4 \rho/r} \frac{4 \rho}{\sqrt{1 - (\frac{\rho}{r} x)^2}} dx$$ (5.14)

where $x = \cos \theta$, which is defined on Fig. 5.5, $R = \sin \theta_z / \sin \theta_c$ and $A$ is the area of the source coupling to the fibre. ($\rho = \rho_1$ for the inner core, $\rho_2$ for the outer core).

This gives the power coupled into the circular core (radius $\rho_1$) for trapped and leaky rays respectively as

$$P_{\text{TR cyl}}(0) = \pi^2 \rho_1^2 I_o \theta_{c1}^2$$ (5.15)

and

$$P_{\text{LR cyl}}(0) = 8\pi \rho_1^2 I_o \theta_{c1}^2 \left[ \frac{\pi}{8} - \frac{(1 - \theta_{c1}^2)^{1/2}}{4 \theta_{c1}} + \theta_{c1} \left( \frac{1}{4 \theta_{c1}^2} - \frac{1}{2} \right) \right]$$ (5.16)
For annular region of inner radius $\rho_1$ and outer radius $\rho_2$ we have

$$P_{TR \text{ annulus}}(0) = \pi^2 \rho_1^2 \frac{\theta^2}{\omega_c^2} \left[ 1 - \left( \frac{\rho_1}{\rho_2} \right)^2 \right]$$

(5.17)

and

$$P_{LR \text{ annulus}}(0) = 8\pi \rho_1^2 \frac{\theta^2}{\omega_c^2} \left[ \frac{\pi}{8} \frac{1}{\theta_c^2} + \arccos \frac{\theta_0}{\theta_c} \left( 1 - \frac{1}{\theta_c^2} \right) \right]$$

(5.18)

If $\frac{\rho_1}{\rho_2} > \sqrt{1 - \frac{\theta^2}{\omega_c^2}}$, then the integration of (5.14) can be simplified but this situation does not generally hold.

(ii) Coherent source

In this case, following ref. 4, we consider illumination of the endface of the fibre by a single plane wave. For a multimode fibre, diffraction effects can be ignored and all rays make the same angles $\theta_z$ with respect to the z axis normal to the fibre face. This results in a far simpler analysis and the expressions for the bound and leaky ray power launched by the source in the cylinder and the annulus respectively at
the start of the fibre \((z = 0)\) are

\[
P_{\text{bound cyl}} = 2\pi^2 \rho_1^2 I_0 \quad \theta < \theta_c^1 \tag{5.19}
\]

\[
P_{\text{bound annulus}} = 2\pi^2 \rho_1^2 I_0 \left[1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right] \quad \theta < \theta_c^2 \tag{5.20}
\]

\[
P_{\text{leaky cyl}} = 8\pi \rho_1^2 I_0 \left[\frac{\pi}{4} \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right) - \frac{\sqrt{1-R_2^2}}{2R_1} - \frac{1}{2} \arccos \left(\frac{1}{R_2}\right)\right] \quad \theta > \theta_c^1 \tag{5.21}
\]

\[
P_{\text{leaky annulus}} = 8\pi \rho_1^2 I_0 \left[\frac{\pi}{4} \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right) - \frac{\sqrt{1-R_2^2}}{2R_2} - \frac{1}{2} \arccos \left(\frac{1}{R_2}\right)\right]
\]

\[
+ \frac{\sqrt{1-R_2^2}}{2} \left(\frac{\rho_1}{\rho_2}\right)^2 - 1 + R_2^2 + \frac{1}{2} \left(\frac{\rho_1}{\rho_2}\right) \arcsin \sqrt{1-R_2^2} \left(\frac{\rho_1}{\rho_2}\right)\]
\]

when \(\theta_c^1 < \theta < \frac{1}{\sqrt{1 - \left(\frac{\rho_1}{\rho_2}\right)^2}}\)

\[
\left[\frac{\pi}{4} \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right) - \frac{\sqrt{1-R_2^2}}{2R_2} - \frac{1}{2} \arccos \left(\frac{1}{R_2}\right)\right]
\]

when \(\frac{1}{\sqrt{1 - \left(\frac{\rho_1}{\rho_2}\right)^2}} < \theta < \frac{\pi}{2}\)

\[
\sin \theta < \frac{1}{\sin \theta_c^1}, \quad R_1 = \frac{1}{\sin \theta_c^1}, \quad R_2 = \frac{\sin \theta_c^2}{\sin \theta_c^2}
\]
For the combined structure the same technique is adopted as for the case of incoherent light where rays which become refracting as a result of the proximity of the annular and circular regions of the core are neglected.

5.4.3 A comparison of the single and double cored fibres

The distribution of power among the rays on a fibre with double layered core was considered in the previous section. The losses of each of the rays of a particular ray group was described in section 5.4.1. In this section the losses are assembled, as elements of the building block approach for multilayered guides discussed in section 5.2, to give the total attenuation of the power within the guide. We then compare these results with those of reference 3. In particular we choose to carry out the comparison for the more difficult case of an incoherent source. The calculation for a coherent source is trivial since the attenuation coefficient $\gamma$ need be known for only one ray.

For each ray family the attenuation coefficient $\gamma$ is a constant. However $\gamma$ varies from one ray family to the next. We calculate the ray transmission coefficient for a given ray from

$$T = T_{\text{tun}} + T_{\text{evan}} + T_{\text{caus}} + \int_P \alpha_{\text{co}} \, ds$$  \hspace{1cm} (5.24)

as for eqn (4.31) for single layered guides but using (5.1) and (5.3) for $T_{\text{tun}}$ and $T_{\text{evan}}$. In what follows since $T_{\text{caus}}$ is very small, it is ignored, and we put $T_{\text{caus}} = 0$ everywhere. The attenuation coefficient $\gamma$ is related to $T$ by

$$\gamma = \frac{T}{z_p}$$  \hspace{1cm} (5.25)

where the ray paths are calculated using simple geometric optics. $z_p$ is the distance along the axis of the guide of one ray period which may include ray paths in either or both the inner and outer parts of the core.
The distance $z_{p_1}$ is the distance travelled by a ray in the centre part of the core between successive reflections at $r = \rho_1$ and is given simply by

$$z_{p_1} = 2\rho_1 \cot \theta_1 \sin \theta_1$$

(5.26)

If the ray also travels in the secondary core layer then the distance travelled in the medium of index $n_2$ between successive reflections from the interface at $r = \rho_2$, if the ray path crosses $r = \rho_1$, is

$$z_{p_2} = 2 \cot \theta_2 \sqrt{\rho_2^2 + \rho_1^2 - 2\rho_1 \rho_2 \cos \xi}$$

(5.27)

where

$$\cos \xi = \frac{(\rho_1 \sin \theta_2 \cos \theta_2^2 + \sqrt{(\rho_2^2 - \rho_1^2)(\sin^2 \theta_1 - \sin^2 \theta_2) \cos^2 \phi_1 \sin^2 \theta_1}}{\rho_2 (\sin^2 \theta_1 - \sin^2 \theta_2)}$$

(5.28)

where

$$\sin \theta_1 = \sin \theta_1 \sin \phi_1$$

(5.29)

When the ray only travels in the secondary layer and only a very skew ray is possible in this case, then

$$z_{p_1} = 0$$

$$z_{p_2} = 2\rho_2 \cot \theta_2 \sin \theta_2$$

(5.30)

The subscripts 1, 2 are used to denote quantities related to the inner and outer parts of the core and the angles refer to reflection at an interface as defined in Fig. 5.5.
For those rays of case (c) of section 5.4.1 when power is coupled between the inner and outer parts of the core by tunnelling, attenuation due to absorption along each ray path is averaged between the ray paths.

This procedure gives a method of calculating $\gamma$ for any given ray on the structure where the ray power $P$ attenuates according to

$$P(z) = P(0) \exp\{-\gamma z\} \quad (5.31)$$

It now remains to sum over all rays on the structure to find the total attenuation.

Substituting (5.31) into the integrals of eqns (5.13)-(5.14), the expressions then become for the bound and leaky rays respectively in the circular cylinder excited by a Lambertian source,

$$P_{TR_{cyl}}(z) = 8\pi \rho^2 I_o \int_0^1 \sin^2 \theta R dR \int_0^1 (1-x^2)^{\frac{1}{2}} e^{-\gamma z} dx \quad (5.32)$$

$$P_{LR_{cyl}}(z) = 8\pi \rho^2 I_o \int_1^{1/\theta} \sin^2 \theta R dR \int_1^1 (1-x^2)^{\frac{1}{2}} e^{-\gamma z} \frac{1}{\sqrt{1-R^2}} dx \quad (5.33)$$

and with a similar straightforward application to the rays excited in the annular region

$$P_{TR_{annulus}}(z) = 8\pi \rho^2 I_o \int_0^1 \sin^2 \theta R dR \left\{ \int_0^{\rho_1 / \rho_2} \frac{1}{\sqrt{1-x^2}} e^{-\gamma z} dx \right\}$$

$$+ \int_0^{\rho_1 / \rho_2} \left[ \sqrt{1-x^2} - \sqrt{\rho_1 / \rho_2} \right]^{\frac{1}{2}} \sqrt{1-x^2} e^{-\gamma z} dx \right\} \quad (5.34)$$

$$P_{LR_{annulus}}(z) = 8\pi \rho^2 I_o \int_1^{1/\theta} \sin^2 \theta R dR \left\{ \int_0^{\rho_1 / \rho_2} \frac{1}{\sqrt{1-x^2}} e^{-\gamma z} dx \right\}$$
These expressions must then be evaluated numerically because of the complexity of the expressions for \( \gamma \).

For a fibre with refractive indices, \( n_1 = 1.472 \), \( n_2 = 1.46 \), \( n_3 = 1.458 \) and radii \( \rho_1 = 40 \mu m \) and \( \rho_2 = 45 \mu m \) where the material losses in the inner core, secondary layer and the cladding are 6, 6, 100 dB/Km respectively the equations (5.32)-(5.35) were evaluated numerically. The bound ray power remaining within the fibre is shown plotted in solid lines as curve (a) on Fig. 5.6 against

\[
\frac{\rho_1}{\rho_2} \left[ \sqrt{I-x^2} - \sqrt{\left(\frac{\rho_1}{\rho_2}\right)^2-x} \right] \left(e^{-\gamma z}\right) dx
\]

Fig. 5.6 Plots of the fraction \( F \) of power in the total (a), in the bound (b) and in the leaky (c) rays of the double layered fibre against normalised distance along the fibre \( (z/\rho_1) \). Corresponding curves for a conventional single core fibre are shown dotted for comparison.
normalised distance $z/\rho_1$. The dashed lines indicate the corresponding results for the single layered fibre of ref. 3 with core of index $n_1$ and cladding of index $n_3$. Curve (b) gives the same result for the total power within the guide and curve (c) is for leaky rays alone.

Curves plotted in each case using the methods of ref. 4 for a single core step fibre could not be separated from the dashed curves derived using our building block approach.

Fig. 5.7 Fraction $F$ of the total power within the guide for (i) simple step index fibre, (ii) a double layered fibre shown replotted on a logarithmic scale.
Initially the high rate of attenuation from the double cored structure is from the highly attenuated leaky rays in the annular region. For larger $z/\rho_1$ the attenuation is significantly less and the point of cross over of the curves can be readily seen on Fig. 5.7 where the curves for total power in the fibre are replotted on a logarithmic scale. Curve (i) on Fig. 5.7 is for the simple step fibre and curve (ii) is for the double cored fibre.
REFERENCES


CHAPTER 6

NUMERICAL COMPARISON WITH MODAL THEORY

6.1 INTRODUCTION

In the preceding chapters the theory of ray transmission coefficients has been developed and expressions for ray power attenuation based on the simple model of Eqn (1.15) have been compared with approximate solutions of the eigenvalue equation for modal power attenuation on step-index waveguides. Of the methods used to calculate the ray transmission coefficient, the method of uniform approximation has furnished a single universally valid expression for all leaky rays. This expression has been shown to approach the WKB solution when the inner and radiation caustics are well separated from each other, and from any discontinuities, and also to reduce to the Generalised Fresnel's Law solution in the Airy region where the WKB analysis is invalid.

In order to validate the theory it is important to test the accuracy of the ray theory by comparison with some exact leaky mode solutions. The reasons for this are manyfold. It has already been noted that the ray approach is only an asymptotic representation to the exact situation when the mode parameter $V$ is sufficiently large. The actual error for specified $V$ values cannot be determined analytically. Also it is necessary to consider the effect of the inclusion of higher order terms in the asymptotic expansion, such as the Goos-Hänchen shift. The accuracy of both the ray solution and the approximate analytical
eigenvalue equation solutions can also be compared. Evaluation of these expressions for leaky modes and rays for different values of the critical angle $\theta_c$, which measures the difference between the maximum and minimum values of the refractive index profile, enables an estimate of the effect of polarisation to be made. Finally, the validity of the simple (1.15) and more sophisticated (1.14) models for ray attenuation may be established.

We use exact modal solutions for the straight step index slab waveguide and the circularly symmetric step index fibre. This covers cases involving both refracting and tunnelling rays and both refracting and tunnelling leaky modes.

6.1.1 Modal power attenuation coefficient

The variables used in the following analysis are the waveguide parameter $V$ defined by

$$V = kp(n_1^2 - n_2^2)^{1/2}, \tag{6.1}$$

where $k = 2\pi/\lambda$, $\lambda$ is the wavelength of the incident radiation in vacuum, $p$ is the half width of the slab guide or radius of the round guide, $n_1$ is the refractive index of the core, and $n_2$ is the refractive index of the cladding.

Associated parameters are

$$U = \rho(n_2^2k^2 - \beta^2)^{1/2}, \tag{6.2}$$

and

$$Q = \rho(n_2^2k^2 - \beta^2)^{1/2}, \tag{6.3}$$

where $\beta$ is the propagation constant of the mode as used previously.
For bound modes on lossless waveguides no attenuation occurs and
$\beta$ is pure real. However, if the mode attenuates then $\beta$ has both real and
imaginary parts as indicated in Eqn. (1.3). It is convenient to use a
dimensionless attenuation coefficient $\alpha$ given by the relation

$$\gamma = \frac{\alpha}{\rho}$$  (6.4)

In this form the expression for the power along the guide given by (1.4)
becomes

$$P(z) = P(0) \exp\left(-\frac{\alpha z}{\rho}\right).$$  (6.5)

where $P(0)$ is the initial power of the mode. The attenuation coefficient
is related to the imaginary component of $\beta$, denoted $\beta^1$, by

$$\alpha = 2\rho \beta^1$$  (6.6)

To express this in terms of $U$ from (6.2) and using (1.3), we get

$$U^2 = \rho^2(n_1^2k^2 - \beta_r^2 + \beta_1^2 - 2\beta_r \beta^1)$$  (6.7)

Equating real and imaginary parts of each side of (6.7) we obtain a
quadratic equation for $\beta_1^2$. Selecting the appropriate root which
vanishes as $U^1 \to 0$, this gives

$$\alpha = \sqrt{2\left[(\frac{v^2}{\sin^2\theta} - U_r^2 + U_1^2)^2 + 4(U_r U_i^1)^2\right]^{1/2}}$$

- $\left[\frac{v^2}{\sin^2\theta} - U_r^2 + U_1^2\right]^{1/2}$,

where

$$\sin \theta_c = (1 - \frac{n_2^2}{n_1^2})^{1/2},$$

and $U = U_r + iU_i$ is the complex solution of the eigenvalue equation for
the guide. The attenuation coefficient of (6.8) is the modal solution
we wish to compare with the simplified ray analysis. The eigenvalue equation allows the propagation of a number of discrete modes. The notation $\beta_{\ell m}$ is often used to indicate the value of $\beta$ for the particular mode under consideration, where $\ell, m$ are the number of azimuthal and radial phase variations associated with the particular mode. The total attenuation of the power in a multimode guide is therefore taken to be the sum of the attenuations of the individual modes $\alpha_{\ell m}$ weighted with respect to the relative power in that mode, i.e.

$$P_{\text{tot}}(z) = \sum_{\ell, m} P_{\ell, m}(0) \exp(-\alpha_{\ell m} z/\rho),$$

where $P_{\ell, m}(0)$ is the initial power in the $\ell, m$ th mode and $P_{\text{tot}}(z)$ is the total leaky mode power.

6.1.2 Ray power attenuation coefficient

From the definition of a ray in chapter 1, a mode can be considered as being composed of a family of rays all making the same angle $\theta_z$ with the axial $z$ direction, where $\theta_z$ is defined by the relation

$$U^r = n \ell k \rho \sin \theta_z$$

(6.9)

At each point of reflection of a slab guide the angle of incidence relative to the local normal is $\alpha_1 = \pi/2 - \theta_z$. On a circular fibre the angle $\alpha_1$ is given in terms of the ray angles $\theta_z$ and $\theta_\phi$ defined in Fig. 6.1 by

$$\cos \alpha_1 = \sin \theta_z \sin \theta_\phi.$$  

(6.10)

If the field variation in the azimuthal direction of the circular fibre guide is of the form $e^{i\ell \phi}$, then $\ell$ is related to the ray angles by
Fig. 6.1 Illustration of angles defined at incidence on a fibre boundary. P is the point of incidence. O is the centre of the circular cross section. $\alpha_1$ is the angle between the normal to the boundary given by the line OP and the incident ray direction DP. PR is the projection of the incident ray direction onto the cylinder cross section. $\alpha_1$, $\theta_z$ and $\theta_\phi$ are related by $\cos \alpha_1 = \sin \theta_z \sin \theta_\phi$.

\[
\ell = k \rho \sin \theta_z \cos \theta_\phi
\]  

(6.11)

where $\theta_\phi$ is the angle between the projection of the incident ray in the cross section of the fibre and the tangent to the surface shown in Fig. 6.1.

The ray formulation of the power attenuation coefficient $\alpha$ corresponding to (6.5) uses the model of (1.14) as

\[
\alpha = -\frac{\ln(1 - T) \rho}{z_p}
\]  

(6.12)
for significant attenuation rates, when $T$ is no longer small, and of (1.15) as

$$\alpha = \frac{T_p}{z_p}, \quad (6.13)$$

for very weak attenuation, when $T$ is extremely small. In both cases $T$ is the ray transmission coefficient given by one of the methods discussed in chapter 2, and $z_p$ is the distance between successive reflections at which power is lost by radiation.

The attenuation coefficient $\alpha$ has been constructed using both a modal and a ray optics approach. In the following sections these two constructions are compared both analytically and numerically on slab and cylindrical step index waveguides.

6.2 THE DIELECTRIC SLAB

6.2.1 Modal solution

The eigenvalue equation for the fields on a planar dielectric step index slab guide is well known.\(^2,5\) If the structure of the guide is as shown in Fig. 6.2, with uniform core and cladding indices of $n_1$ and $n_2$ respectively, and the slab width is $2p$, then the eigenvalue equation is given by

$$\tan U_p = \eta \frac{W}{U} \quad \text{for even modes}$$

and

$$\tan U_p = -\eta \frac{U}{W} \quad \text{for odd modes}$$

(6.14)

where $\eta = 1$ for TE modes and $\eta = n_2^2/n_1^2$ for TM modes, and $W = iQ$. However, as we are concerned with profiles on which $n_1 \approx n_2$, we can restrict attention to the TE modes. If we define leaky modes as bound modes below cutoff (given by the point $U = V$), then a simple perturbation approach\(^2\)
Fig. 6.2 Sketch of step index dielectric slab waveguide with core and cladding refractive indices of $n_1$ and $n_2$ respectively. The slab width is $2\rho$. A ray description is shown superimposed losing a fraction $T$ of its power by refraction at each reflection.

yields an approximate analytic expression for $U^i$ as

$$U^i \approx \frac{1}{V_c} (U^r + 1 - V_c^2)\frac{1}{2}$$  \hspace{1cm} (6.15)

where $U^r \approx V_c \approx \frac{m\pi}{2}$ and $m$ is a positive integer or zero. $V_c$ represents the cutoff value for the particular mode. Using these values as starting points for the numerical analysis $U^r$ and $U^i$ can be calculated exactly for (6.14) using the Newton-Raphson method. For large $V$ values, numerical calculation is difficult and exact results were not obtainable above about
$V_c \cong 25$. Asymptotic techniques must then be used.

The exact solutions of (6.14) for $U$ as a function of the waveguide parameter $V$, are shown in Fig. 6.3 for the first few TE modes. Points on the diagram to the right of $U = V$ represent bound modes where $U^i = 0$. For leaky modes $U^i$ is non-zero and these values are indicated where $U^i$ is shown dashed. There is a region immediately below cutoff ($U = V = V_c$) and before the leaky mode domain ($U^i$ finite) where the solutions have zero attenuation. Modes in this region are unphysical. The phenomenon has not been satisfactorily explained physically\(^2\) but the resulting inflection can be clearly seen on Fig. 6.3, particularly for the low order modes. This region becomes insignificantly small for high order modes.

![Fig. 6.3 The exact solutions for $U$ of the eigenvalue equation (6.15) for the first few TE modes as a function of $V$. The real components of $U$ are shown as solid lines while the imaginary components are dashed.](image)
6.2.2 Ray solution

The geometric optics model for attenuation on the symmetrical slab guide is particularly simple and is shown superimposed on Fig. 6.2. At each reflection of the ray within the core a constant fraction $T$ of the incident power is lost by refraction. There are no tunnelling leaky rays because of the planar geometry, and thus all leaky rays refract. The ray transmission coefficient is given by Fresnel's classical law for TE waves from Eqn (3.3) as

$$T = T_F^E = \frac{4 k_{y_1} k_{y_2}}{(k_{y_1} + k_{y_2})^2} \quad (6.16)$$

A similar equation for TM waves follows from Eqn. (3.4) but for $n_1 \neq n_2$ then $T_F^E \approx T_F^H$, $k_{y_1}$ and $k_{y_2}$ are the transverse components of the wave vector on the respective sides of the core-cladding interface and are real quantities defined by

$$k_{y_1} = k \sin \frac{\theta_z}{z_1}$$

$$k_{y_2} = k \left( \sin^2 \theta_z - \sin^2 \frac{\theta_c}{c} \right)^{1/2} \quad (6.17)$$

and $k_{y_1} = n_1 k_{y_1}$, $\theta_z$ is the angle of the ray with the $z$ direction defined by Eqn. (6.9) and $\theta_c$ is the complement of the local critical angle and is defined by $\theta_c = \cos^{-1}\left(\frac{n_2}{n_1}\right)$. Since $T$ can become large very rapidly as $\theta_z$ increases we require the general model (6.12) for attenuation where the periodic distance between successive reflections is

$$z_p = 2\rho \cot \frac{\theta_z}{z} \quad (6.18)$$

Thus, the complete expression for the ray power attenuation coefficient is
\[
\alpha = -\ln \left[ 1 - \frac{\sin \theta_z (\sin^2 \theta_z - \sin^2 \theta_c)^{1/2}}{(\sin \theta_z + (\sin^2 \theta_z - \sin^2 \theta_c)^{1/2})^2} \right] \tan \theta_z \quad (6.19)
\]

The attenuation model (6.13) in which \(-\ln(1-T)\) is replaced by \(T\) can only be used for weakly leaky rays when \(T \ll 1\).

6.2.3 A comparison of rays and modes

For weakly leaky modes of high order close to cutoff \(V_c \approx U^r >> 1\). Using this approximation the modal attenuation (6.8) on a step index dielectric slab reduces easily with (6.15) and \(U^i \ll U^r\) to

\[
\alpha = \frac{2(U^r - V^i)^{1/2}}{\beta \rho} \quad (6.20)
\]

For weakly leaky rays on a large \(V\) guide the simple model (6.13) is appropriate and using equation (6.16) for the ray transmission coefficient \(T\) with (6.17), (6.18) and (6.9) for \(U^r \gg 1\) gives the result (6.20) identically.

This establishes analytically the equivalence of the ray and modal approaches for higher order weakly leaky modes on guides with large \(V\).

A comparison of the more general ray attenuation coefficient of Eqn. (6.12) with the exact numerical solution of the TE slab eigenvalue equation for the attenuation coefficient is plotted in Fig. 6.4. The error between these two approaches appears as a function of normalised ray angle \(\theta_z/\theta_c\). Bound rays correspond to \(\theta_z/\theta_c < 1\).

Even for \(V_c = 4.7\), the error is still small provided \(\theta_z \gg \theta_c\). For modes having large \(V_c\) the error between the ray and the exact modal solution is very small. Only in a very narrow region close to \(\theta_z = \theta_c\) is there a significant divergence. For \(V_c > 20.4\) the region in which the
error is greater than 1% resembles a delta function about $\theta_z = \theta_c$. The agreement improves as either the cutoff value increases or as the attenuation increases.

It should be noted that the curves should not be plotted right down to $\theta_z/\theta_c = 1$ since the solutions of the eigenvalue equation are known to be unphysical in a small region immediately below cutoff.
6.3 THE CIRCULAR STEP INDEX DIELECTRIC FIBRE

The exact eigenvalue equation for the fields on a straight circular step index rod is \(^4,6\):

\[
\left( \frac{n_1}{n_2} \right)^2 \frac{J_{\ell}^{(1)}(U)}{UJ_{\ell}^{(1)}(U)} - \frac{H_{\ell}^{(2)}(Q)}{QH_{\ell}^{(2)}(Q)} \left[ \frac{J_{\ell}^{(1)}(U)}{UJ_{\ell}^{(1)}(U)} - \frac{H_{\ell}^{(2)}(Q)}{QH_{\ell}^{(2)}(Q)} \right] = \left( \frac{\omega \beta}{n_2} \right) \cdot \left( \frac{V}{UQ} \right)^4
\]

(6.21)

where \(J_{\ell}\) is a Bessel function of the first kind and \(H_{\ell}^{(2)}\) is a Hankel function of the second kind, and \(U\) and \(Q\) are related to \(V\) by \(^1\)

\[V^2 = U^2 - Q^2\]

The eigenvalue equation reduces in the weakly guiding approximation \((n_1 \approx n_2)\) to

\[
\frac{UJ_{\ell}^{(1)}(U)}{J_{\ell-1}^{(1)}(U)} = \frac{Q}{H_{\ell}^{(2)}(Q)} \cdot \frac{H_{\ell}^{(2)}(Q)}{H_{\ell-1}^{(2)}(Q)}
\]

(6.22)

which is independent of polarisation. This simplified equation has been solved analytically using perturbation techniques \(^1,3\) for the higher order modes on a guide with \(V \gg 1\), and numerically for the lower order modes. \(^3,7\)

In this comparison we use the exact numerical solution of (6.21) for \(U_{\ell m}\), where the subscripts \(\ell, m\) indicate the particular mode involved, to determine the imaginary component \(\beta_{\ell m}^{\dagger}\) of the axial propagation constant. The attenuation coefficient \(\alpha_{\ell m}\) is then given by (6.8).

In chapter 2 the mathematical techniques used to calculate the ray transmission coefficient \(T\) were detailed. The forms for \(T\) used in this comparison were those of the WKB (2.94) and uniform approximation methods (2.166).
6.3.1 Comparison using a simple ray attenuation model

Firstly we used the simple expression given by (6.14) where $z_p$ is the periodic distance between successive reflections

$$z_p = 2\rho \cot \theta_z \sin \theta_\phi$$

(6.23)

The angles $\theta_z$, $\theta_\phi$ have already been defined in Fig. 6.1. The results of this comparison are displayed in Tables 1 to 4 in which all the leaky modes of a circular dielectric cylindrical guide with a cutoff less than 30 (corresponding to $U^1 < 1$) are listed for $V = 20$, $V = 10$ and for $\theta_c = .1$, $\theta_c = .01$ respectively. Also listed in each case are the fractional errors, compared to the exact numerical attenuation coefficients, using

(i) the WKB method to calculate $T$.  
(ii) the method of uniform approximation to calculate $T$.  
(iii) the simplified analytical eigenvalue equation solution for $\alpha$.  

In the fifth column is the exact value of $\alpha$ for each of the modes calculated from (6.21). The sixth column gives the value of the ratio $(\alpha_i - \alpha_c)/\alpha_c$ which is a measure of the closeness of the ray direction to the critical angle $\alpha_c$. When this ratio is negative the ray is a tunnelling leaky ray. Fractional errors are converted to percentages by multiplying by 100. The method of uniform approximation is uniformly valid over the entire range of attenuation. However, using the simple model (6.13) for ray attenuation, gives poor agreement, and this can easily be seen in the fourth column of Table 6.1, where errors between solutions based on the method of uniform approximation and the exact numerical solutions frequently exceed 50%. Better agreement can be seen for the more weakly attenuating rays where $T \ll 1$. 
6.3.2 Comparison using a more sophisticated ray attenuation model

A second comparison was made, in which the improved model for ray attenuation (6.12) was used. This provided a dramatic improvement in the agreement between the ray power attenuation coefficient and the exact numerical solutions of the attenuation coefficient for the corresponding modal quantity. The results of this comparison are displayed in Tables 5 to 8 for the same parameters as for Tables 1 to 4, but using the improved attenuation model. Some cases, for example those modes with \( \ell = 23, 24, 25, 26 \) on Tables 5 and 6, still show poor agreement. The fact that the larger errors occur for higher order modes suggests that a further refinement is required to improve the agreement. In the next section, the incorporation of the lateral, or Goos-Hänchen shift, provides this improvement.

6.3.3 An improved ray model incorporating a lateral shift

(a) Lateral shift

It is well known that for bound modes, there is a lateral shift, \( s \), also called the Goos-Hänchen shift,\(^{10,11,12}\) associated with the evanescent field which extends over the cladding of a guiding structure. This shift gives rise to the displacement along a plane interface between the incident and reflected rays in Fig. 6.5. The extent of the shift for planar\(^{13}\) and curved\(^{14}\) interfaces is related to the phase change \( \Phi \) between the incident and reflected local plane waves associated with the corresponding incident and reflected rays by

\[
 s = - \frac{1}{n_k \cos \alpha_I} \cdot \frac{d\Phi}{d\alpha_I} \tag{6.24}
\]

where \( \alpha_I \) is the angle of incidence relative to the normal. The shift for planar interfaces is explicitly
Fig. 6.5 Sketch illustrating the shift of the origin of a reflected ray along a dielectric interface in the plane of the incident ray and the normal to the interface. The angles of incidence and reflection are $\alpha_1$ measured with respect to the normal to the interface.

For tunnelling leaky rays we would expect a similar shift, $s_L$, associated with the evanescent field that occurs between the core-cladding interface and the radiation caustic in the cladding where the tunnelling ray reappears \(^{15}\) in Fig. 6.6.

$$s = \frac{2 \tan \alpha_1}{n_1 k \left(\cos^2 \alpha_c - \cos^2 \alpha_1\right)^{1/2}}, \quad \alpha_1 > \alpha_c$$  \hspace{1cm} (6.25)
To calculate the leaky ray shift $s_L$ for this situation, we use the phase change $\phi$ for the reflection of leaky rays at the curved core cladding interface as developed in Chapter 3. This is the phase change associated with the reflection coefficient $R$ of Eqn (3.21) for the Generalised Fresnel's Law at a step index interface. To calculate the component of this shift in the $z$ direction, $z_s$, we take
\[ z_s = s \frac{\cos \theta}{L \sin \alpha_i} \]  
(6.26)

where

\[ s_L = -2\text{Im}\left(\frac{\psi'}{(\psi^2 - 1)}\right) \]  
(6.27)

\[ \psi = \frac{-iv \cos \alpha_i}{(\cos^2 \alpha_c - \cos^2 \alpha_i)^{1/2}} \cdot \frac{A(v) - iB(v)}{A'(v) - iB'(v)} \]  
(6.28)

the primes denote differentiation with respect to \( v \), and

\[ v = \left(\frac{n \kappa \rho_c}{2 \sin^2 \alpha_i}\right)^{2/3} \quad (\cos^2 \alpha_c - \cos^2 \alpha_i) \]  
(6.29)

where \( \rho_c \) represents the radius of curvature of the interface in the plane of the incident ray and the normal,\(^{15,16} \) as given in (3.12). The parameter \( v \) is equivalent to \( \Delta \) in ref. 16. We observe the shift (6.26) in the following limits.

(a) For the case of meridional rays which have \( \rho_c = \infty \) and therefore see essentially a planar interface, (6.26) simplifies since \( \theta_z \) and \( \alpha_i \) are complementary. Thus for meridional rays

\[ z_s = s \quad \alpha_i > \alpha_c \]  
(6.30)

\[ z_s = 0 \quad \alpha_i < \alpha_c \]

(b) In the refracting ray limit, we consider the shift \( s_L \) given by (6.27). Performing the limit \( n \kappa \rho_c \to \infty \), on (6.27) then \( z_s \to 0 \) for \( \alpha_i < \alpha_c \) in keeping with the known result of zero shift at a planar interface for refracting rays.

(c) For tunnelling rays \( \alpha_i > \alpha_c \) the shift \( s_L \) follows (6.27). Increasing \( n \kappa \rho_c \) gives a shift in this case which approaches the classical
result, \( z_s + s \frac{\cos \theta}{\sin \alpha_i} \) except for a narrow region near \( \alpha_i = \alpha_c \). This is indicated in Fig. 6.7. The leaky ray shift of (6.27) is shown by solid lines and the classical Goos-Hänchen shift (6.25) is shown dashed with each normalised with respect to \( \rho_c \).

Fig. 6.7 Plot of the ratio \( s_L/\rho_c \) against \( \alpha_i/\alpha_c \) for \( \alpha_c = 85^\circ \) in the region \( \alpha_i \approx \alpha_c \) for various values of \( n_1 k \rho_c \). The ratio using the classical Goos-Hänchen shift of Eqn. 6.25 is shown dashed.
The distance between successive reflections must therefore consist of a small shift at each reflection as well as the optical ray path $z_g$. This is shown schematically in Fig. 6.8 where the total path between successive reflections $z_p$ incorporates the shift $z_s$ and

$$z_p = z_g + z_s$$  \hspace{1cm} (6.31)

Fig. 6.8 Qualitative description of a general ray path in the core of the step-index waveguide. Skewness has been suppressed to highlight the ray period $z_g$ and the lateral shift $s_L$. 
(b) Comparison incorporating a lateral shift

With both the improved attenuation model (6.12) and $z_p$ corrected by the Goos-Hänchen shift for leaky rays, (6.27) for $v < 1.5$ and (6.25) for $v > 1.5$ (since this modified shift is derived from the Generalised Fresnel Law and is only valid for $\alpha_i \approx \alpha_c$), then the comparison between the ray attenuation using the method of uniform approximation and the exact numerical solution shows the agreement is excellent. Using the same values of $V$ and $\theta_c$ as for the previous two comparisons the results appear in Tables 9 to 12. A great improvement in agreement is particularly obvious for modes with $\ell = 23, 24, 25, 26$.

Fig. 6.9 Distribution of the absolute percentage errors $|1 - \gamma_R/\gamma_M| \times 100$ for the step-index round waveguide against attenuation $\rho \gamma_M$ for $V = 20$ and $\sin \theta_c = 0.1$, for the 63 modes with $\text{Im} \rho(n_i^2 k^2 - \beta^2) \gamma < 1$. 
A scatter diagram showing the errors for the 63 HE modes on Table 9 for $V = 20$ and $\theta_c = 0.1$ appears as Fig. 6.9. A general bias in the errors can be observed of about 1%. Comparing the weakest leaky mode $HE_{18,1}$ for $\theta_c = 0.1$ (Table 9) and $\theta_c = 0.01$ (Table 10) shows that reducing $\theta_c$ involves about a 0.8% improvement in the agreement between the approximate ($n_1 \approx n_2$) eigenvalue equation solution and the exact solution (column 5 of the tables). This therefore suggests that part of the bias in the errors may be due to polarisation effects which have been suppressed in this ray analysis.

From these tables we must conclude that the modified ray optical theory for attenuation developed in this thesis gives excellent agreement with modal theory for multimode step index fibres ($V > 20$) but still moderately good agreement for $V$ down to as low as 10. The agreement is a significant improvement over the solution of the approximate eigenvalue equation (Eqn. 6.22) as can be seen from the fifth column of Tables 9-12.
### RESULTS FOR \( K > 20.0000 \) THETA = 0.10

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**Table 6.1** A comparison of the calculation of attenuation on a step index circular fibre with \( V = 20 \) and \( \theta = 0.1 \) by several different methods. All leaky HE modes with \( U_1 < 1 \) are listed. The first two columns label the modes. The fractional errors by methods (i), (ii), (iii) discussed in the text occupy the next 3 columns. The exact attenuation coefficient \( a \) for the mode appears in column 6. Column 7 contains the ratio \( (a^2 - a_c^2) / a_c^2 \). The appearance of a negative quantity in this column means it is a tunnelling mode. For the ray methods in columns 3 and 4, the simple attenuation model \( a = Tp / z \) is used.
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Table 6.2 As for Table 6.1 but with $V = 20$ and $\theta_c = 0.1$. 

204
### Table 6.3
As for Table 6.1 but with $V = 10$ and $\theta_c = .1$.

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Table 6.5 A comparison of the calculation of attenuation using an improved ray attenuation model (6.12) on a step index circular fibre with \( \nu = 20 \) and \( \theta = .1 \) by methods (i), (ii), (iii) from the text with exact numerical values. All leaky HE modes with \( U_i < 1 \) are plotted. The first two columns identify the mode. The fractional errors appear in the next 3 columns. The exact attenuation coefficient \( \alpha \) is in column 6. The ratio \( (\alpha_i - \alpha_c)/\alpha_c \) is in column 7. Column 7 negative indicates a tunnelling leaky ray.
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Table 6.6 As for Table 6.5 but with $V = 20$ and $\theta_c = .01$. 
### Table 6.7 As for Table 6.5 but with $V = 10$ and $\theta_c = .1$

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Table 6.9 A comparison of the calculations of attenuation including the effects of the Goos-Hanchen shift and the model (6.12) for a step index fibre with V = 20 and \( \theta_c = 0.1 \) by methods (i), (ii), (iii) in the text with the exact numerical values. All leaky HE modes with \( U_l < 1 \) are plotted. The first two columns identify the mode. The fractional errors appear in the next 3 columns. The exact attenuation coefficient \( \alpha \) is in column 6. The ratio \( (\alpha_l - \alpha_c) / \alpha_c \) is in column 7. Column 7 negative indicates a tunnelling leaky ray.
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CHAPTER 7

THE APPLICATION OF TRANSMISSION COEFFICIENTS TO THE CALCULATION OF BENDING LOSSES ON DIELECTRIC WAVEGUIDES

An evaluation of the effect of bends on multimode dielectric waveguides is an important consideration in determining the performance of optical fibre systems, since it can critically affect power losses, as well as pulse dispersion, and hence the information carrying capacity of these systems. In this chapter, an example of the use of ray transmission coefficients, developed in the preceding sections of this thesis is presented for the calculation of bending losses. Although the emphasis in this chapter is on the total power attenuation induced by a bend, a similar approach could be applied to the analysis of pulse dispersion. We consider losses on both step and graded index multimode guides, having either slab or cylindrical geometry. In each case, numerical results are included, which, as well as showing absolute attenuation, allow a quantitative comparison of bending losses between the various kinds of structures.

7.1 INTRODUCTION

The problem of calculating the radiation from a bent multimode dielectric waveguide is not a simple calculation, and many approximation methods have been developed for analysing bent slab\textsuperscript{1-3} and bent fibre\textsuperscript{4-20} guides.
7.1.1 Modal analysis

A strict electromagnetic boundary value approach requires that the eigenvalue equation\(^1\) be solved directly for the bent waveguides. Even for the bent step-index slab this is not a simple undertaking, and for bent fibres with circular cross section, an eigenvalue equation for the simple step-index profile is not available. Many approximation techniques have been developed, some involving the adaption of radiation conditions on the bent slab to the bent fibre,\(^4-6\) and some using the fields of the straight guide as approximations of those in the bent guide.\(^7,10,12\) The latter approximations necessarily limit consideration to large bending radii. Certain methods are restricted to uniformly bent fibres,\(^7,10\) while others recognise the existence of a transition region\(^4,8,14,15\) due to the abrupt change in curvature at the beginning of the bend. Experimental investigation\(^11\) of several methods\(^1,5,10\) has been carried out at microwave frequencies, while more recent investigations on multimode guides\(^16-20\) have confirmed a redistribution of energy in the cross-section of the bent guide,\(^13,14\) and the existence of the transition region.

In multimode guides, mode conversion\(^4\) has been shown to be significant in modal calculations of bending loss, and theoretical investigations of mode mixing in the presence of corners\(^21\) and experimental investigations of this effect for uniform bending\(^22\) have been undertaken. The mode coupling coefficients, the radiation loss due to transition regions at the start and finish of a bend, and the total power loss including radiation loss for the uniform bend have been calculated numerically for a step-index fibre with small V values,\(^14,15\) but the methods become extremely difficult for fibres with large V. These methods have been shown to agree well with experimental measurements.
on monomode fibres where the initial rapid attenuation in the transition region was observed.

7.1.2 Ray analysis

For bent multimode guides, there have been some isolated ray analyses, and surprisingly few experimental results, all of which are somewhat inconclusive in determining losses of bent multimode waveguides. In what follows, a straightforward geometric optics approach utilising ray transmission coefficients is applied to the problem of calculating bending losses in multimode dielectric waveguides.

Every ray path in a bent guiding structure necessarily loses some power as it propagates around a section of the bend. An equivalent radiation condition applies to a mode in a bent guide. The mechanism of power loss for the ray is either by refraction or by tunnelling. Depending on curvature, either mechanism can result in a fractional loss of power when a ray is reflected from an outer caustic along the core ray path or from the core-cladding interface. We calculate the loss of power using the ray transmission coefficients of Chapter 3. Within bent guides there are no bound rays, only tunnelling and refracting rays may propagate. Therefore, we must assign an attenuation coefficient to every ray on a bent guide whether or not it is bound on the straight guide.

7.1.3 Bent step-index waveguides

A typical ray suffers successive reflections between two points P and Q on the core cladding interface in a step-index guide as shown in Fig. 7.1. The angular displacement around the bend from P to Q is whether the ray is meridional, i.e. in the plane of the bend, or skew. If
Fig. 7.1 A typical ray in the core of a bent multimode guide is reflected from point P to point Q suffering transmission loss $T$ at Q. A ray makes an angle of incidence $\theta_i$ at Q with the tangent to the surface and moves through an angle $\Delta \xi$ between reflections measured at C the centre of curvature of the bend.

The fraction of the incident ray power lost by reflection at Q from the $i$-th reflection is $T_i$ then the power at angular position $\xi$ is given by

$$P(\xi) = P(o) \exp\left\{ -\xi \sum_{i=1}^{N} T_i / \sum_{i=1}^{N} \Delta \xi_i \right\}$$

(7.1)

using the simplified model (1.15) for power attenuation or

$$P(\xi) = P(o) \exp\left\{ -\xi \prod_{i=1}^{N} \ln(1-T_i) / \sum_{i=1}^{N} \Delta \xi_i \right\}$$

(7.2)

using the more sophisticated model (1.14) for ray attenuation. $P(o)$ is the power associated with the ray at the beginning of the bend and $N$ is
the total number of reflections in the angular displacement $\xi$. The attenuation coefficient $\gamma$ of (1.14) or (1.15) is a function of $\xi$ and is not constant for skew rays. However, the bent slab waveguide is a specific example where $\gamma$ remains constant for a particular ray and thus greatly simplifies the analysis.

The attenuation of the total power around the bend is obtained by summing Eqn 7.1 over all the ray paths originating at $\xi = 0$. The initial power $P(o)$ in each ray depends on the power density distribution in the cross section $X'X$ at the beginning of the bend.

A diffuse Lambertian source illuminates the entire end face of an ideal step-index multimode guide which is straight for distance $d$ until the cross section $X'X$ representing the beginning of a uniform bend of radius $R$. The situation is shown in Fig. 7.2. We assume in each of

Fig. 7.2 A diffuse source $S$ illuminates a multimode step index guide which has uniform refractive index $n_1$ in the core and $n_2$ in the cladding. The width of the core is $2\rho$. At distance $d$ from the source a bend of radius $R$ occurs beginning at the section $XX'$. 
the examples discussed that $d$ is sufficiently large that all the power of
leaky rays on the straight waveguide have left the core by the beginning
of the bend and only bound rays remain. Therefore, the ray power intensity
distribution in the cross section $X'X$ may be considered to be a truncated
Lambertian of the form

$$I = I_o \cos \theta$$  \hspace{1cm} (7.3)

where $I_o$ is a constant, $\theta$ is the angle the ray path makes with the
axial direction of the straight guide and $-\theta_c \leq \theta \leq \theta_c$ for a slab guide
and $0 \leq \theta \leq \theta_c$ for a round guide. $\theta_c$ is the complement of the critical
angle given for a step index guide by $\theta_c = \cos^{-1}(n_2/n_1)$. The refractive
indices of the core and cladding are $n_1$ and $n_2$ respectively. The slab
width and fibre diameter are both $2\rho$. For graded index guides $\theta_c$ varies
with position in the cross section $X'X$ and this is discussed later in
Sections 7.5 and 7.6.

In each of the following examples an incoherent Lambertian
source is used, but it may be replaced by any other source and the same
principles of our analysis employed. For example, a completely coherent
on-axis source would considerably simplify the analysis since only a
single ray direction need be considered at the beginning of the bend.

We begin by seeking the ray invariants on the bent waveguide
$\tilde{\eta}, \tilde{\nu}$, which complement the ray invariants $\tilde{\beta}, \tilde{\kappa}^{33}$ on the straight fibre.
This enables us to exploit the few symmetries available to simplify the
calculations. The analysis is performed for a general arbitrary graded
profile within both slab and cylindrical geometries and this includes
the special case of a step index profile.
All these situations are for an ideal lossless guide where the refractive index profile remains undistorted as the axis of the guide is bent. In practice the radius of curvature $R$ is normally much greater than the bending radius and this assumption is valid. However, if the fibre is assumed to be perfectly elastic then the elastic strain imposed by bending can be shown to cause an effective increase in the radius of curvature. The propagating ray, therefore, behaves as though the fibre was unstrained but had a slightly reduced curvature so that the effect of elastic strain can be readily deduced in the following by appropriately amending the bending radius.

7.2 RAY INVARIANTS

In the analysis of ray paths on straight fibres or slab waveguides the rise of the invariants $\tilde{p}, \tilde{q}$ results in considerable simplification compared with previous analyses. In the same way, analogous invariants are sought for bent guides. Both bent slab and cylindrical geometrics are considered in the general case of an arbitrarily graded medium. The invariants for the step profile then follow merely as a special case of those for the graded profile.

7.2.1 Bent slab geometry

Consider the situation of a bent graded dielectric slab depicted in Fig. 7.3. The slab is bent with uniform radius $R$ about the origin at 0 and the ray is distant $R$ from 0 at an arbitrary angle $\xi$ with respect to a reference plane $\xi = 0$. The ray is travelling at angle $\phi$ to the normal to the position vector $R$. It is convenient to use a local coordinate system to describe the distance of the ray $y$ from the central axis of the bent guide.
Fig. 7.3 Section of a bent graded dielectric slab with uniform bend of radius R beginning at X'X and showing a ray launched from the straight guide propagating around the bend. A point on the ray is distant r from 0 at an angle $\xi$ with respect to the reference plane at $\xi = 0$ containing X'X.
We begin with the eikonal equation

\[ \frac{d}{ds} \left[ n(r) \frac{dR}{ds} \right] = Vn \]  

(7.4)

where \( ds \) is the distance measured along the ray path, \( R \) is the ray position and \( \frac{dR}{ds} \) is therefore a unit vector tangent to the ray path, and

\[ dR = dr \hat{e}_r + rd\xi \hat{e}_\xi, \]  

thus,

\[ \frac{d}{ds} \left[ n(r) \left( \frac{dr}{ds} \hat{e}_r + r \frac{d\xi}{ds} \hat{e}_\xi \right) \right] = \frac{dn}{dr} \hat{e}_r \]  

(7.5)

We also note

\[ \frac{d}{ds} \hat{e}_r = \frac{\xi}{\xi} \hat{e}_\xi \]  

(7.6)

\[ \frac{d}{ds} \hat{e}_\xi = -\frac{\xi}{\xi} \hat{e}_r \]  

where

\[ \cdot = \frac{d}{ds} \]

So

\[ \frac{d}{ds} \left[ n \frac{dr}{ds} \right] \hat{e}_r + n \frac{dr}{ds} \xi \hat{e}_\xi + \frac{d}{ds} \left[ nr \frac{d\xi}{ds} \right] \hat{e}_\xi - nr \frac{d\psi}{ds} \hat{e}_r = \frac{dn}{dr} \hat{e}_r \]  

(7.7)

The radial and angular components give two equations,

\[ \frac{d}{ds} \left[ n \frac{dr}{ds} \right] - nr \xi^2 = \frac{dn}{dr} \]  

(7.8)

The second equation (7.8) simplifies to

\[ \frac{d}{ds} \left( nr^2 \xi \right) = 0 \]  

(7.9)

whence

\[ nr^2 \xi = \text{constant} \]  

(7.10a)

\[ = \tilde{\eta} \]  

(7.10b)
and thus $\tilde{\eta}$ is invariant around the bend for that ray. If the ray is considered to be launched at the reference plane $XX'$, as shown in Fig. 7.3 from a section of straight guide then the invariant $\tilde{\eta}$ is related to the invariant on the straight guide $\check{\eta}$ by

$$\tilde{\eta} = (R + \delta)\check{\eta}.$$  

(7.11)

where $\delta$ is the displacement from the axis of the guide at the start of the bend as shown.

The first equation (7.7) leads to a differential equation for the ray path $r = r(\psi)$ from which we can calculate the angular distance $\Delta \xi_p$ between successive ray turning points. Substituting

$$\frac{d}{ds} = \frac{\tilde{\eta}}{nr^2} \frac{d}{d\xi} ,$$

$$\frac{\tilde{\eta}}{nr^2} \frac{d}{d\xi} \left( \frac{\tilde{\eta}}{r^2} \frac{dr}{d\xi} \right) - \frac{\tilde{\eta}^2}{nr^3} = \frac{dn}{dr}$$  

(7.12)

which is the differential equation for the ray path $r(\xi)$.

7.2.2 Circular Geometry

Consider the situation of a bent circular graded dielectric fibre depicted in Fig. 7.4. The fibre can be considered to be a section of a toroid or ring with radius of curvature $R$ in the major cross section, and radius of curvature $\rho$ in the minor cross section. Starting once again from the eikonal equation we wish to derive the ray invariants for a ray on the bent circular fibre. For convenience local cylindrical coordinates are introduced and defined in 7.4.

Starting again with

$$\frac{d}{ds} \left( n \frac{dp}{ds} \right) = \nabla_n$$  

(7.13)
Fig. 7.4  Section of a bent circular dielectric fibre with centre of the bend at 0 and defining the local cylindrical coordinates \((r,\psi,z)\) for a point on the ray within the core.

and using bent cylindrical coordinates at some point on the ray path we have

\[
\frac{dR}{ds} = \frac{dr}{ds} \mathbf{e}_r + r \frac{d\psi}{ds} \mathbf{e}_\psi + \mathbf{e}_\xi \frac{d\xi}{ds} \tag{7.14}
\]

where

\[
R = R + r \sin \psi, \quad z = R \xi \text{ (on axis)}
\]

as shown in Fig. 7.4. Substituting in 7.13 we have

\[
\frac{d}{ds} \left( n \frac{dr}{ds} \mathbf{e}_r \right) + \frac{d}{ds} \left( nr \frac{d\psi}{ds} \mathbf{e}_\psi \right) + \frac{d}{ds} \left( r^2 \frac{d\xi}{ds} \mathbf{e}_\xi \right)
= \frac{\partial n}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial n}{\partial \psi} \mathbf{e}_\psi + \frac{1}{\xi} \frac{\partial n}{\partial \xi} \mathbf{e}_\xi \tag{7.15}
\]
We are only interested in situations when \( n = n(r) \) so we have, using \( \dot{\gamma} \equiv \frac{d}{ds} \),

\[
\frac{d}{ds} \left[ n \frac{dr}{ds} \right] \hat{e}_r + n \frac{dr}{ds} \hat{e}_r + \frac{d}{ds} \left[ nr \frac{d\psi}{ds} \right] \hat{e}_\psi + nr \frac{d\psi}{ds} \hat{e}_r \\
+ \frac{d}{ds} \left[ \frac{\partial R}{\partial \zeta} \frac{d\zeta}{ds} \right] \hat{e}_\zeta + \frac{\partial R}{\partial \zeta} \frac{d\zeta}{ds} \hat{e}_\zeta = \frac{dn}{dr} \hat{e}_r
\]

(7.16)

We now require expressions for \( \dot{\hat{e}}_r, \dot{\hat{e}}_\psi, \dot{\hat{e}}_\zeta \) for the coordinate system shown below in Fig. 7.5, where \( \hat{i}, \hat{j}, \hat{k} \) are parallel to a fixed coordinate system.

Fig. 7.5 Sketch defining the rotated \( \hat{i}, \hat{j}, \hat{k} \) coordinates parallel to the \((X,Y,Z)\) cartesian coordinates at the start of a bend in circular fibre.
\[ e_r = \cos \psi \hat{i} + \sin \psi \cos \xi \hat{j} + \sin \psi \sin \xi \hat{k} \] (7.17)
\[ e_\psi = -\sin \psi \hat{i} + \cos \psi \cos \xi \hat{j} + \cos \psi \sin \xi \hat{k} \] (7.18)
\[ e_\zeta = -\sin \xi \hat{i} + \cos \xi \hat{k} \] (7.19)
\[ \dot{e}_r = -\sin \psi \hat{i} + (\cos \psi \cos \xi - \sin \psi \sin \xi \hat{j}) + (\cos \psi \sin \xi + \sin \psi \cos \xi \hat{k}) \] (7.20)
\[ \dot{e}_\psi = -\cos \psi \hat{i} - (\sin \psi \cos \xi + \cos \psi \sin \xi \hat{j}) + (\sin \psi \sin \xi + \cos \psi \cos \xi \hat{k}) \] (7.21)
\[ \dot{e}_\zeta = -\cos \xi \hat{i} - \sin \xi \hat{k} \] (7.22)

It immediately follows that
\[ \dot{e}_r = \dot{\psi} e_\psi + \dot{\zeta} \sin \psi e_\zeta \] (7.23)
\[ \dot{e}_\psi = -\dot{\psi} e_r + \dot{\zeta} \cos \psi e_\zeta \] (7.24)
\[ \dot{e}_\zeta = -\dot{\xi}(e_r \sin \psi + e_\psi \cos \psi) \] (7.25)

Substituting in (7.16) for \( \dot{e}'s \) gives 3 equations, one for each of the components in directions \( e_r, e_\psi, e_\zeta \)
\[ \frac{d}{ds}(n \frac{dr}{ds}) - \psi^2nr - n\xi \frac{d\xi}{ds} \sin \psi = \frac{dn}{dr} \] (7.26)
\[ nr \dot{\psi} + \frac{d}{ds} nr \frac{d\psi}{ds} - \xi^2nr \cos \psi = 0 \] (7.27)
\[ nr \sin \psi \dot{\xi} + nr \dot{\psi} \xi \cos \psi + \frac{d}{ds}(n\xi) = 0 \] (7.28)

We use the last two of these equations to derive the invariants in the bent circular fibre. Of course eqs (7.27) and (7.28) simplify for \( \xi \to 0 \), \( R \to \infty \) and reduce to the case for the straight cylinder as in ref. (33).
Eq (7.28) reduces since \( \dot{\mathbf{r}} = \dot{r} \sin \psi + r \cos \psi \hat{\psi} \) to

\[
\mathbf{n} \mathbf{\ddot{\zeta}} + \frac{d}{ds} (\mathbf{n} \mathbf{\dot{\zeta}}) = 0 \tag{7.29}
\]

\[
\mathbf{n} \mathbf{\ddot{\zeta}} + R \frac{d}{ds} (\mathbf{n} \mathbf{\dot{\zeta}}) = 0 \tag{7.30}
\]

\[
\frac{d}{ds} (n R^2 \zeta) = 0 \tag{7.31}
\]

which yields the first invariant \( \tilde{\eta} = n R^2 \zeta \) \( \tag{7.32} \)

Continuing we note \( \frac{d}{ds} \Rightarrow \frac{\tilde{\eta}}{n R^2} \frac{d}{d \zeta} \) and eqn (7.27) reduces to

\[
n \dddot{\psi} + \frac{d}{ds} (n \ddot{\psi}) - \zeta^2 n R \cos \psi = 0 \tag{7.33}
\]

\[
\frac{d}{ds} (n \ddot{\psi}) - \zeta^2 n R \cos \psi = 0 \tag{7.34}
\]

\[
\frac{d}{d \zeta} \left( \frac{r^2}{R^2} \frac{d \psi}{d \zeta} \right) - \frac{r}{\hat{R}} \cos \psi = 0 \tag{7.35}
\]

To obtain a solution to this equation, each side is multiplied by \( \frac{R \hat{\psi}}{\hat{R}} \) where \( \cdot \) now indicates differentiation with respect to \( \zeta \). Then

\[
a \frac{r}{\hat{R}} \psi \frac{d}{d \zeta} \left( \frac{r^2}{R^2} \psi \right) - a \frac{r}{\hat{R}} \cos \psi \hat{\psi} = 0 \tag{7.36}
\]

\[
f \frac{d}{d \zeta} \left( \frac{r^2}{R^2} \psi \right) - f' \left( \frac{r^2}{R^2} \psi \right) = 0 \tag{7.37}
\]

\[
\frac{d}{d \zeta} \left( f \frac{r^2}{R^2} \psi \right) = 0 \tag{7.38}
\]

and the invariant is given by

\[
f \frac{r^2}{R^2} \hat{\psi} = \tilde{\nu} \tag{7.39}
\]

It now remains to determine \( f \). \( f \) and \( f' \) are given by comparison of eq. (7.36) and eq. (7.37).
\[ f = a \frac{r}{\theta} \psi, \quad f' = -a \cos \psi \]

\[ \therefore \frac{a r}{\theta} \psi + \frac{r \psi}{\theta} + r \frac{\psi}{\theta} r \frac{\psi}{\theta} + \cos \psi = 0 \quad (7.40) \]

which gives

\[ a = \frac{\theta}{\psi} \exp \left\{ - \int \frac{\theta}{r \psi} \cos \psi \, d\xi \right\} \quad (7.41) \]

\[ f = \exp \left\{ - \int \frac{\theta}{r \psi} \cos \psi \, d\xi \right\} \quad (7.42) \]

Therefore we have for the ray path two invariants \( \tilde{\eta}, \tilde{\nu} \)

\[ \tilde{\eta} = n \theta^2 \frac{d\xi}{ds}, \quad \tilde{\nu} = \frac{nr^2}{\theta} \frac{d\psi}{ds} \exp \left\{ - \tilde{\eta}^2 \int \frac{\cos \psi \, ds}{r \cdot \frac{d\psi}{ds} \cdot n^2 \theta^3} \right\} \quad (7.43) \]

These two invariants \( \tilde{\eta}, \tilde{\nu} \) on a bent fibre guide are of limited use because of the complexity of the expression for \( \tilde{\nu} \). While \( \tilde{\eta} \) corresponds to the analogous invariant on the slab guide, eqn (7.11), no simple interpretation exists for \( \tilde{\nu} \).

The equation for the ray path on the bent circular guide becomes from eqn (7.26)

\[ \frac{d}{ds} \left[ \eta \frac{dr}{ds} \right] - \left( \frac{d\psi}{ds} \right)^2 n r - \frac{\tilde{\eta}^2}{n \theta^3} \sin \psi = \frac{dn}{dr} \quad (7.44) \]

\[ \frac{d}{d\xi} \left[ \frac{1}{\theta^2} \frac{dr}{d\xi} \right] - \frac{r}{\theta^2} \frac{\psi}{\theta^2} - \sin \psi = \frac{\theta^2}{2\tilde{\eta}^2} \frac{dn^2}{dr} \quad (7.45) \]

Using \( \cdot = \frac{d}{d\xi} \) we get using \( \theta = r \sin \psi + r \cos \psi \)

\[ \frac{\ddot{r}}{\theta} - 2 \frac{r^2 \sin \psi}{\theta^2} - 2 \frac{rr \cos \psi}{\theta^2} - \frac{r \cdot \frac{\psi}{\theta}}{\theta^2} - \sin \psi = \frac{\theta^3}{2\tilde{\eta}^2} \frac{dn^2}{dr} \quad (7.46) \]
We can further substitute for $\psi = \frac{-\sqrt{R^2}}{fr^2}$ but we get a very complicated expression for the differential equation for the ray path and it is difficult to solve. We avoid the need to do this by using ray tracing techniques for the bent fibre in later Sections of this chapter (sections 7.4 and 7.6).

7.3 BENDING LOSSES IN THE BENT STEP INDEX SLAB

Consider the situation in Fig. 7.2, which shows a section of an ideal, lossless, bent, uniform slab guide of width $2\rho$, having refractive index $n_1$ in the core and $n_2$ in the cladding. The radius of the uniform bend is $R$. A Lambertian source illuminates the end face of the long straight section of the guide so that only bound rays are left at the cross section X'X at the beginning of the bend. The distribution of power amongst the rays at X'X is that of a truncated Lambertian source of eqn (7.3).

In a multimode guide with uniform core material, the ray paths are straight lines, so that the angles of incidence of a ray on the core-cladding boundaries are derived by simple geometric arguments.

Consider first those rays which are successively reflected from the inner and outer boundaries of the bent guide. If $\theta_i$ is the angle of incidence on the outer boundary measured to the tangent to the interface, and $\theta'_i$ is the corresponding angle of incidence on the inner boundary, then the property of the invariant $\tilde{n}$ (eqn 7.10) gives

$$(R-\rho)\cos \theta'_i = (R+\rho)\cos \theta_i.$$  \hspace{1cm} (7.47)

If $(R+\rho)/(R-\rho)\cos \theta_i > 1$, then the ray is reflected only between points on the outer boundary, and is a whispering gallery ray. In either case
symmetry considerations maintain the same angle of incidence $\theta_i$ at all subsequent reflection points around the outer boundary, and $\theta_i'$ at each reflection at the inner boundary.

On the bent slab guide, there are no bound rays and only tunnelling and refracting rays propagate. For either kind of leaky ray, at each reflection a small fraction $T_i$ of the incident power may be lost. At the outer boundary, the power transmission coefficient $T_i$ is given for tunnelling rays ($\theta_i < \theta_c$) by the Generalised Fresnel's Law (eqn 3.25) as

$$T_i = \frac{4 \sin \theta_i}{\sin \theta_c} \left( \frac{\sin^2 \theta_i}{\sin^2 \theta_c} \right)^{1/2} \exp \left\{ -\frac{2}{3} n_1 k (R+p)(\theta_c^2 - \theta_i^2) \right\}^{3/2}$$

(7.48)

and for refracting rays $\theta_i > \theta_c$ by eqn. (3.28)

$$T_i = \frac{4 \sin \theta_i}{\sin \theta_c} \left( \frac{\sin^2 \theta_i}{\sin^2 \theta_c} - 1 \right)^{1/2}$$

(7.49)

where

$$\sin \theta_c = \left\{ 1 - \left( \frac{n_2}{n_1} \right)^2 \right\}^{1/2}$$

For $\theta_i \approx \theta_c$ the complete Airy function form (eqn 3.22) should be used but there are very few rays which have incident angles sufficiently close to $\theta_c$ to warrant the use of this much more complicated expression and the error involved in using (7.48) and (7.49) only is small. Therefore we use eqns (7.48) and (7.49) over the whole range of $\theta_i$. At the inner surface for $\theta_i' < \theta_c$

$$T_i = 0$$

(7.50)

and for $\theta_i' > \theta_c$ we use the classical Fresnel coefficient

$$T_i = \frac{4 \sin \theta_i'}{\sin \theta_c} \left( \frac{\sin^2 \theta_i'}{\sin^2 \theta_c} - 1 \right)^{1/2}$$

(7.51)
Eqns (7.48)-(7.51) are the ray transmission coefficients for the bent guide.

Strictly speaking, the Generalised Fresnel's Law of eqn (7.48) is only valid for refracting rays and for tunnelling rays whose outer caustic is close to the outer interface at \( R+p \). For the more weakly leaky tunnelling rays with caustics further from the interface this law is slightly pessimistic, i.e. it predicts a slightly higher attenuation rate compared to the exact value. However, the error produced by this assumption is very small, since it is the leaky rays with the larger attenuation coefficients which dominate the overall attenuation. The error is likely to become significant only after many kilometers of continuous bend, and such physical situations are extremely unlikely.

The angle \( \Delta \xi_i \) between subsequent points of reflection \( P_iQ \) in Fig. 7.1 is easily calculated for rays reflected from both surfaces as

\[
\Delta \xi_i = (\theta_i - \theta'_i)
\]

(7.52)

and for whispering gallery rays which reflect only from outer surfaces by

\[
\Delta \xi_i = 2\theta_i
\]

(7.53)

The ray power at position \( \xi \) around the guide is calculated using eqn (7.1) and then summing by taking a double integration over all rays at each position on the slab cross section and for all angles in the angular distribution of rays at \( X'X \) (eqn. 7.3). This summation leads to the results shown in Fig. 7.6 where the total power loss in db is shown plotted against normalised distance along the axis of the guide \((z/p)\) for a typical multimode waveguide with \( V = 50 \) and for various values of the bending parameter \( \rho/R \).
Fig. 7.6 A plot of power $P$ remaining in a bent slab waveguide versus distance $z$ along the bent slab axis normalised to the half width $\rho$, for various values of $\rho/R$ where $R$ is the radius of the bend. The fibre has parameter $V = 50$. The corresponding curves including the effect of the Goos-Hänchen shift cannot be distinguished from this one on the scale used here.

Each curve shows the same characteristic behaviour. Initially there is a rapid loss of power from rays incident on the core boundaries at angles $\theta_i$ and $\theta_i'$ close to or greater than $\theta_c$. Subsequently the attenuation assumes a lower value for longer $z/\rho$ due to the slower rate of loss from the more weakly leaky rays, which then dominate the overall attenuation rate. If more accurate and complicated expressions like
eqn (3.65) for the transmission coefficients were used, it leads to an indistinguishable change in the results on the scale shown which justifies the choice of the transmission coefficient, eqn. (7.48). Likewise the incorporation of the Goos-Hänchen shift (see refs 24, 39 and Chapter 6) shows no modification of the curves.

7.4 BENDING LOSSES IN THE BENT STEP INDEX FIBRE

Consider the situation shown in Fig. 7.2 where the slab guide of the previous section is replaced by a circular fibre. A dielectric step index fibre of radius $\rho$ is straight for a large distance $d$ until the cross section $X'X$ when a uniform bend of radius $R$ is introduced. The circular fibre has a core of refractive index $n_1$ and is surrounded by a cladding of refractive index $n_2$. The distance $d$ is sufficiently large that only bound rays remain in the core. If the straight section of the fibre is illuminated by a Lambertian source then the distribution of rays at $X'X$ is the truncated Lambertian of eqn (7.3).

Ray paths in the core of a multimode circular guide with uniform refractive index profile are straight lines, but the geometry in a circular guide is more complicated than in a slab guide due to the presence of skew rays and the problem is further compounded on bent guides. In the bent slab, the angle of incidence of a ray at the outer core-cladding interface remains the same along a given ray path, and this property is true of all ray paths.

However, in the bent fibre, only those rays entering the bend in the meridional plane containing the centre of the bend behave in this manner. For rays entering skew to this plane the subsequent reflections
within the core do not follow a simple repeatable pattern because of the
asymmetry introduced by bending. The complicated forms of the ray
invariants (7.43) and the differential equation for the ray path offer
little hope for simplification. We therefore resort to a numerical
technique for tracing each ray separately along the fibre.

If we consider a skew ray entering the bent section of the guide
from the cross section X'X, then the position at which the ray strikes
the core cladding interface is the solution of a quartic equation (A.8).
The quartic arises because in general a straight line intersects a ring
at 4 points. The position of subsequent reflections along the guide
involves the solution of the simpler cubic equation (B.1) in appendix 7B.
Some simple coordinate rotations and translations give the angles of the
incident and reflected ray directions at each point of reflection (see
Appendix 7.C).

The radius of curvature \( \rho_c \) of the core cladding interface in
the plane of the incident and reflected rays can be calculated from the
principal radii of curvature \( \rho_{K1} , \rho_{K2} \) by the relation

\[
\rho_c = \frac{\rho_{K1} \rho_{K2} \sin^2 \alpha}{\rho_{K1} \cos^2 \theta_z + \rho_{K2} \cos^2 \theta_x}
\]  

(7.54)

where \( \rho_{K1} , \rho_{K2} \) are the radii of (3.74) with \( r_{tp} = \rho \).

\( \theta_x , \alpha , \theta_z \) are the direction angles defined in Appendix 7A. The angular
distance \( \Delta \xi_1 \) between bounces around the bend is calculated in Appendix 7A.

Having determined the geometry of the ray path we can then use
the Generalised Fresnel's Law from Chapter 3 to calculate the ray trans­
mission coefficient \( T \) at each point of reflection along a given ray path.
The expressions for the transmission coefficient $T$ are given by eqns (7.48) – (7.51), where the radius of curvature $\rho_c$ for the circular fibre in (7.54) replaces $R+\rho$ on the bent slab guide. The attenuation of each ray varies from reflection to reflection along the guide, and we use the simple attenuation model of eqn (7.1) to determine the power loss of each ray along the bent step index guide.

Finally, it is necessary to sum over all possible rays launched at the beginning of the bend. To do this we carry out a quadruple integration across the cross sectional area of the guide at $X'X$ and the distribution of ray angles $(\theta, \phi)$ as in Appendix 7D.

The total power remaining in the fibre in db with respect to the incident power is shown plotted in Fig. 7.7 against normalised

![Graph](image)

**Fig. 7.7** A plot of power $P$ remaining in a bent circular step index guide of radius $\rho$ versus normalised distance $z/\rho$ with $z$ measured around the axis of the guide for various values of $\rho/R$ where $R$ is the radius of the bend. The fibre has parameter $V = 50$. 
distance \( (z/p) \) along the curved axis of the guide for various values of the bending parameter \( \rho/R \) and for a typical multimode guide with \( V = 50 \). The behaviour of the curves is similar to those of the bent slab, Fig. 7.6, but the initial rapid loss of power is not so great as in the case of the bent slab, since only a few rays close to meridional in the plane of the bend, have the largest bending loss and are equivalent to those of the slab. All other rays are subject to the effects of skewdness and can be reflected into directions of greater attenuation even though they may have very low initial attenuation, e.g. those rays entering the bent guide close to meridional but in a plane normal to the plane of curvature. This results in the larger attenuation for the fibre at greater \( z/\rho \) than was the case for the slab. It is this asymmetry which destroys the initial symmetry of the diffuse source and, after many reflections, results in a concentration of ray paths in the outer half of the bend i.e. \( R < r < R+\rho \) similar to the effect observed in monomode fibres.\(^{20}\)

Attempts to reduce the order of the quadruple integration used to obtain these results have been unsuccessful, as they modify the real spread in the angular distribution caused by the bend. Also an assumption of an equal distribution of rays around the guide in the bend leads to gross errors in the power attenuation, emphasising the importance of the increased density of rays in the outer half of the bend.

Unfortunately because of the amount of computing time involved, it has only proved possible to perform the calculation for a few metres of guide at most. However for those bends for which the losses are significant, a few metres of guide represents perhaps dozens of turns at that bending radius so that most practical situations are well represented.
7.5 BENDING LOSSES IN THE BENT GRADED INDEX SLAB

Let us reconsider the situation shown in Fig. 7.2, where the guide is replaced by a graded step index slab of width $2\rho$. The refractive index in the core is symmetrical about the central axis of the guide and is given by

$$n^2(y) = n_1^2 \left(1 - 2\Delta \left(\frac{y}{\rho}\right)^2\right) \quad |y| \leq \rho$$

$$(7.55)$$

$$n^2(y) = n_2^2 = n_1^2(1 - 2\Delta) \quad |y| \geq \rho$$

where

$$\Delta = \left(\frac{n_2^2 - n_1^2}{2n_1^2}\right)$$

and $y$ is the displacement from the central axis of the bent guide. The profile of eqn. (7.55) is assumed not to be modified by the radius of the bend. The profile is continuous at $y = \rho$ and no step appears at the core cladding interface.

The radius of the uniformly bent section of the guide is again $R$, and the illumination at $X'X$ is also only from rays bound on the straight guide, distributed as a truncated Lambertian source, in the form of eqn (7.3), except that the distribution $I_0 \cos \theta$ is for angles $\theta$ relative to the axial direction and within the local critical angle $\theta_c(y)$ i.e.

$$-\theta_c(y) \leq \theta \leq \theta_c(y)$$

$$\theta_c(y) = \cos^{-1} \frac{n_2}{n(y)}$$

These are the only rays which are bound on the straight guide. We assume all leaky rays have disappeared.
A ray entering the bent guide propagates with ray invariant \( \tilde{n} \), given by eqn (7.10), and evaluated at the cross section X'X in Fig. 7.2. For a ray distant \( \delta \) from the axis of the guide at the beginning of the bend

\[
\tilde{n} = n(\delta) \cos \theta (R+\delta)
\]  
(7.57)

If \( \tilde{n} > n (R+\delta) \), then the ray reaches the core cladding interface at \( y = \rho \) and power is lost from the core by refraction. Otherwise it is a tunnelling ray.

On the bent graded slab guide, the ray path may be determined by differentiating the first term of eqn. (7.12) to give

\[
\frac{\tilde{n}^2}{nr^4} \frac{d^2r}{d\xi^2} - \frac{2\tilde{n}^2}{nr^5} \frac{d^2r}{d\xi^2} + \frac{\tilde{n}^2}{nr^3} = \frac{dn}{dr}
\]  
(7.58)

where \( r \) is the distance of a point on the ray from the centre of the bend so that \( y = r - R \) and \( n = n(y) \). Using \( \frac{d}{dr} \) to denote \( \frac{d}{d\xi} \) gives

\[
\frac{1}{r^4} \frac{dr}{dr} \left( r' - \frac{2}{r^5} r'^2 - \frac{1}{r^3} \right) = \frac{1}{2\tilde{n}^2} \frac{dn^2}{dr}
\]  
(7.59)

Also we note

\[
\frac{1}{2} \frac{d(r')^2}{dr} = \frac{dr'}{dr} \cdot r'
\]  
(7.60)

and

\[
\frac{d}{dr} \left( \frac{(r')^2}{r^4} \right) = \frac{1}{r^4} \frac{d(r')^2}{dr} - \frac{4}{r^5} (r')^2
\]

Substituting (7.60) into (7.59) gives

\[
\frac{d}{dr} \left\{ \frac{1}{r^4} (r')^2 \right\} = \frac{1}{\tilde{n}^2} \frac{dn^2}{dr} + \frac{2}{r^3}
\]  
(7.61)

Integrating with respect to \( r \) the differential equation (7.61) reduces to

\[
\frac{d\xi}{d\tilde{r}} = r' = r^2 \sqrt{\frac{n^2(r)}{\tilde{n}^2} - \frac{1}{r^2}}
\]  
(7.62)
For the purposes of bending loss calculations the ray path need not be known explicitly. We only need the angular displacement between successive outer caustics, $\xi_p$, and the distance of the inner and outer caustics of the ray within the core denoted $y_{tp_1}$, $y_{tp_2}$ respectively. These are relatively simple to calculate. Rearranging (7.62), substituting $y = r-R$ and using (7.55) for $n(y)$ we find the angular displacement $\Delta \xi$ between two points at $y_1$ and $y_2$ along the ray path is given by

$$\Delta \xi = \int_{y_1}^{y_2} \frac{y}{n_1 R^2(1+y/R)} \frac{dy}{\sqrt{1-2\Delta \left(\frac{y}{\rho}\right)^2 + \frac{2y}{n_1} - \frac{n_1^2}{n_2 R^2}}}$$

(7.63)

The zeros of the quadratic under the square root sign in the integrand of (7.63) give the positions of the inner and outer caustics of the ray path on the bent guide, $y_{tp_1}$ and $y_{tp_2}$. The substitution of $t = 1/(y+R)$ then simplifies (7.63) to give the angular distance between successive outer caustics as

$$\xi_p = \frac{\tilde{n}}{\sqrt{a}} \arcsin \left\{ \frac{-2at + 2n_2 R + \frac{4\Delta R^3 n_2}{n_1 \rho^2}}{2n R \sqrt{\frac{2\Delta (\tilde{n} - n_2 R^2)}{n_1 \rho^2} + n_2}} \right\}$$

$$\left| \frac{1}{y_{tp_2} + R} \right| \left| \frac{1}{y_{tp_1} + R} \right|$$

(7.64)

where

$$a = n^2 R^2 + \tilde{n}^2 + \frac{2\Delta n^2 R^4}{\rho^2}$$

Because the ray invariant $\tilde{n}$ is fixed for any one ray, then the ray path is repeated around the bend and the ray transmission coefficient, $T$, and
hence the ray attenuation remains constant along the bent guide.

Rays which do not reach either the inner or outer core cladding interface at \( y = \rho \) on the bent guide are continually reflected back and forth between the inner and outer caustics \( y_{tp_1}, y_{tp_2} \) of the ray path losing a fraction of the power associated with the ray at each outer caustic. For these rays we use the Generalised Fresnel's Law of Chapter 3 to calculate the ray transmission coefficient \( T \), representing the fractional power loss, from the geometry of the ray path. Putting \( \theta_g = \theta_c(y_{tp_2}) \) we find from (3.57)

\[
T = \exp \left\{ -\frac{2}{3} k n(y_{tp_2})(R+y_{tp_2}) \theta_c^2 \left( \frac{\theta_g^2 - \left( \frac{2\rho - y_{tp_2}}{R+y_{tp_2}} \right)^2}{\theta_c^2} \right)^{1/2} \right\}
\]  

(7.65)

For rays which reach either the inner or outer core cladding interface (at \( y = \pm \rho \)) and are refracted from it we put \( T \equiv 1 \).

In this case both the models for attenuation corresponding to eqns (7.1) and (7.2), were used to calculate the ray attenuation coefficient \( \gamma \) for each ray launched at the cross section \( X'X \).

Performing the double summation for the power remaining in the guide over all possible initial ray directions and ray positions at the start of the bend produces the results of Fig. 7.8 which show total power loss in db plotted against normalised distance along the guide axis \( z/\rho \) for various vaues of the bending parameter \( \rho/R \) for a fibre with \( V = 50 \). Two sets of curves are shown plotted. The solid lines are plotted using eqn (7.1) for the power remaining in the guide. The dashed lines are plotted using the eqn (7.2). It can be seen on Fig. 7.8 that both forms (7.1) and (7.2), corresponding to the attenuation models (1.15) and (1.14), agree well for large \( z/\rho \). For small \( z/\rho \) the divergence of the curves is because in this region the losses are caused by the rays
leaving the core by refraction. In these cases $T$ is large and model (1.15) is only valid for $T$ small. The crosses on the curve are points obtained from a numerical ray tracing technique discussed in Section 7.6.

Comparing Fig. 7.8 with Fig. 7.6 then we find that the losses in a bent graded index slab are significantly greater than the loss in a bent step index slab for the same bending radius. It is well known that the pulse dispersion $^{33}$ is much reduced by grading in an optical waveguide but here we find that the opposite is true in the susceptibility to bending losses.

Fig. 7.8 A plot of power $P$ remaining in a bent graded slab waveguide of radius $\rho$ versus normalised distance $z/\rho$ with $z$ measured around the axis of the guide for various values of $\rho/R$ where $R$ is the radius of the bend. The solid lines are for the attenuation model of eqn (7.1). The dashed lines are for the attenuation model of eqn (7.2). The crosses are points obtained by a numerical ray tracing technique relying on a paraxial approximation and discussed in Section 7.6. The fibre has parameter $V = 50$. 

$P$ (db)

$z/\rho$
The enhanced attenuation of graded index slab guides which we have observed can be attributed to the fact that the source excites fewer rays on the graded slab than the step slab and that those rays which are excited are less tightly bound to the core. For example relatively few rays are launched near the core cladding interfaces on a graded slab and they are all weakly bound compared with rays from a similar source near the edge of a step fibre.

7.6 BENDING LOSSES IN THE GRADED CIRCULAR FIBRE

7.6.1 Ray tracing

A skew ray travelling within the core of a straight graded circular fibre follows a helical ray path but the ray invariants \( \tilde{\beta}, \tilde{\lambda} \) are maintained at every point along the path. For the straight guide the geometry at each point of reflection from the outer caustic of a ray travelling within the core remains constant, so that the losses suffered by one member of a ray family determine the losses of every member of that ray family, or mode formed by the ray family.

This is not the case for a bent circular guide. All rays not in the plane formed by the curved axis of the guide and centre of the bend are successively reflected into differing degrees of skewdness as they propagate within the core of the bent guide. Rays in the plane of the bend, however, remain in that plane. Since the differential equation (7.46) for the ray path is complicated, we rely on a numerical ray tracing technique. A convenient solution for the ray path in fibres with power law profile and which can be used to trace a ray around a bend, relies on a paraxial approximation for the ray within the core. The equations used to determine the ray path are given in Appendix 8E. For a fibre
excited by the bound rays from the straight guide, i.e. eqns (7.56) and
provided the radius of the bend is not too small, then the paraxial
approximation is valid, at least initially. We demonstrate the accuracy
of this approximation by using it to trace rays on the bent parabolic slab
of Section 7.5.

7.6.2 Comparison for bent slab guide of exact and paraxial approximations

At each point of reflection of the ray from an outer caustic
along the ray path the Generalised Fresnel's Law for bent graded media
(eqn (3.57)) is applied to determine the fractional power loss $T$ at that
reflection. Then using the model for attenuation of (7.2) for each ray
and summing the power over all rays in the core gives the total power
left at a given distance along the guide. The total power in the guide
using this technique is indicated by the crosses superimposed on the curves
of Fig. 7.8. For large $z/p > 100$ all the crosses lie on the corresponding
curves, and the results are indistinguishable from the analytical solutions
of Section 7.5. For small $z/p < 100$ the crosses lie between the results
of the models of (7.1) and (7.2). This is because the numerical ray
tracing assumes no loss until at least the first point of reflection
whereas the analytical techniques assign a pro-rata loss everywhere.

On the bent graded slab the maximum error in the radial position
of the turning points within the core between the analytical solution of
(7.64) and the position of the turning point obtained from ray tracing for
the parameters plotted in Fig. 7.8 is about 1%. The corresponding errors
in the axial distance between successive reflections at outer caustics
on the bent graded slab can take values as high as 6%. However the rays
with the largest errors in the ray path calculations are the most strongly
attenuating rays and do not contribute significantly to the total power
at some distance along the guide.
7.6.3 Results for the circular parabolic index fibre

If the bent guide depicted in Fig. 7.2 is considered to be a circular graded index fibre with the refractive index profile not modified by the radius of the bend and given, in bent cylindrical coordinates \((r, \phi, z)\), by

\[
n^2(r) = n_1^2 \left(1 - 2\Delta \left(\frac{r}{\rho}\right)^2\right) \quad r \leq \rho
\]

\[
= n_2^2 = n_1^2 \left(1 - 2\Delta\right) \quad r > \rho
\]

where

\[
\Delta = \frac{n_2^2 - n_1^2}{2n_1^2}
\]

then a similar ray tracing analysis applies as for the case of the step index fibre in Section 7.4. The fibre face at the start of the bend in the cross section \(X'X\) is illuminated by a complete set of bound rays which is distributed as a truncated Lambertian, eqn (7.3), where the angle \(\theta\) is for ray angles to the axial \(z\) direction less than the local critical angle

\[
0 \leq \theta \leq \theta_c(r), \quad \theta_c(r) = \cos^{-1}\left(\frac{n_2}{n(r)}\right)
\]

As a ray propagates along the core of the bent graded fibre, a fraction \(T\) of the power associated with the incident ray is lost at each point of reflection from an outer caustic. The Generalised Fresnel's Law of eqn (3.57) is used to calculate \(T\) from the ray path parameters. For rays which strike the core cladding interface at \(r = \rho\) we put \(T = 1\). For tunnelling rays the Generalised Fresnel's law is used to calculate \(T\) from the ray parameters and the curvature of the medium in the plane of incidence at the outer caustic. The radius of curvature in the plane of incidence
defined in Chapter 3.4 is given by eqn (3.43) and (3.74). This radius of curvature at the outer caustic varies along the ray path on the bent circular fibre whereas it remains constant on the bent graded slab of section 7.5.

Summing the powers associated with each ray in the cross section of the guide at a distance \( z \) around the bend gives the results for the total power remaining in the fibre which is shown normalised with respect to the power at the cross section \( X'X \) and plotted against \( (z/\rho) \) in Fig. 7.9 for various values of the bending parameter \( \rho/R \).

![Fig. 7.9 A plot of power \( P \) remaining in a bent circular parabolic index fibre of radius \( \rho \) versus normalised distance \( z/\rho \) with \( z \) measured around the axis of the guide for various values of \( \rho/R \) where \( R \) is the radius of the bend. The fibre has parameter \( V = 50 \).](image)
A comparison with each of the previous sets of results for bent guides, Figs 7.7-7.9 yields some interesting observations.

(i) For large bending radii the power loss on the bent graded fibre is less than the power loss on the bent graded slab. A similar argument as for the bent circular step fibre applies in that only the meridional rays in the plane of the bend suffer the greatest loss and this occurs for all rays on the bent slab.

(ii) As the radius of the bend decreases the losses on the graded fibre become greater than the losses on the graded slab.

(iii) For $\rho/R = .005$ it can be seen that initially the loss on the bent graded fibre is greater than the loss on the bent graded slab.

(iv) The bend graded circular fibre shows much greater loss for the same bend radius than for the bent step index fibre. A similar observation holds between the bent graded slab and the bent step index slab.
APPENDIX 7.A: RAY TRAJECTORIES ON A STEP INDEX TORUS - FIRST REFLECTION

The situation depicted in Fig. 7.10 shows a section through a circular optical fibre radius $\rho$ at the beginning of the bend ($\xi = 0$). This is the cross section $X'X$ in Fig. 7.2. The bent fibre forms a part of a torus by revolution of the circular cross section about the origin 0 with radius of bend $R$. Consider an arbitrary ray incident on the cross section of the fibre at point P. The axis system $OXYZ$ has axis $OX$ pointing directly into the paper. A local coordinate system $oxyz$ with oz pointing into the paper is also indicated at P.

Fig. 7.10 Definition for coordinate systems in toroidal geometry—initial excitation.
An arbitrary ray incident at P has direction cosines \((\cos \theta_x, \cos \alpha, \cos \theta_z)\) relative to Oxyz. The vector \(\vec{OP}\) is given by

\[
\vec{OP} = (0, R-p_1 \cos \phi, p_1 \sin \phi)
\]  

(A.1)

where the distance of the point P from the centre of the fibre is \(p_1\). If we let \(r\) be a point along the ray distant \(\lambda\) from P then

\[
r = [\lambda \cos \theta_z, R-p_1 \cos \phi + \lambda \{\cos \theta_x \sin \phi - \cos \alpha \cos \phi\}, \rho \sin \phi + \lambda \{\cos \theta_x \cos \phi + \cos \alpha \sin \phi\}]
\]  

(A.2)

\[|r|^2 = \lambda^2 + R^2 + p_1^2 - 2R\rho \cos \phi + 2\lambda R(\cos \theta_x \sin \phi - \cos \alpha \cos \phi) + 2\lambda R \cos \alpha
\]  

(A.3)

If the ray meets the torus at Q then we put \(\lambda = PQ\). At Q we have the identity

\[|r - R \sin \xi \hat{X} - R \cos \xi \hat{Y}| = \rho
\]  

(A.4)

and this gives the angle \(\xi\) around the axis of the bent fibre

\[\tan \xi = \frac{r \cdot \hat{X}}{r \cdot \hat{Y}} = \frac{\lambda \cos \theta_x}{R-p_1 \cos \phi + \lambda (\cos \theta_x \sin \phi - \cos \alpha \cos \phi)}
\]  

(A.5)

Squaring (A.4) we have

\[\rho^2 + |r|^2 + R^2 - 2\rho^2 |r|^2 - 2\rho^2 R^2 - 2|r|^2 R^2 = -4R^2 (\xi \hat{Z})^2
\]  

(A.6)

Thus

\[\rho^2 + R^2 - 2(\rho^2 + R^2)\{\lambda^2 + R^2 + p_1^2 - 2R\rho \cos \phi + 2\lambda R(\cos \theta_x \sin \phi - \cos \alpha \cos \phi) + 2\lambda R \cos \alpha\} - 2\rho^2 R^2 + 4R^2 \rho_1 \sin \phi
\]
This multiplies out to give a quartic for the position $\lambda$ of the intersection of the ray with the guide

\[
\lambda^4 + \lambda^3 \{4R(\cos \theta \sin \phi - \cos \alpha \cos \phi) + 4\rho I \cos \alpha\} \\
+ \lambda^2 \{-2\rho^2 + 4R^2(\cos \theta^2 + \cos^2 \alpha) + 4\rho^2 \cos^2 \alpha + 2\rho^2 \cos \phi + 8\rho R \cos \alpha(\cos \theta \sin \phi - \cos \alpha \cos \phi)\} \\
+ \lambda \{-4\rho I \rho^2 \cos \alpha + 4\rho^3 \cos \alpha + 4R(\cos \theta \sin \phi - \cos \alpha \cos \phi)(\rho^2 - \rho_I^2) \\
+ 8R^2 \rho^2 \cos \alpha - 8\rho R \cos \alpha\} \\
+ (\rho^2 - \rho_I^2)^2 - 4R^2(\rho^2 - \rho_I^2) + 4R \rho I \cos \phi \{\rho^2 - \rho_I^2\} = 0
\] (A.8)

The smallest real positive solution for $\lambda$ gives the point Q.
APPENDIX 7.B: RAY TRAJECTORIES ON THE STEP INDEX TORUS - SUBSEQUENT REFLECTIONS

From the point of first reflection on the torus which is calculated as a solution to eqn (A.8), the ray is reflected to another point on the surface of the torus and so on around the bend. After the first reflection all subsequent solutions for the distance \( \lambda \) to the next reflection are given as solutions of a cubic

\[
\lambda^3 + 4\lambda^2[R(\cos \theta \sin \phi - \cos \alpha \cos \phi) + \rho \cos \alpha] \\
+ \lambda[4R^2 \sin^2 \theta_x + 4\rho^2 \cos^2 \alpha - 4\rho \cos \phi] \\
+ 8\rho \cos \alpha(\cos \theta_x \sin \phi - \cos \alpha \cos \phi) \\
+ 8\rho R^2 \cos \alpha - 8\rho \cos \alpha \cos \phi = 0
\]  

(B.1)

where the notation is the same as that in Appendix 7.A.
APPENDIX 7.C: CALCULATION OF ANGLES AT POINTS OF REFLECTION ON STEP INDEX TORUS

Some simple coordinate rotations and translations simplify the calculation of the incident and reflected ray angles at each bounce. They are

(a) The direction cosines at the point P are reexpressed in terms of a local coordinate system rotated by (-\phi) about oz with respect to oxyz on Fig. 7.10 so that the ox axis is parallel to the OZ axis.

(b) a translation of the coordinate system along the ray path from P to Q (no rotation).

(c) a rotation of the coordinate system by -\xi about the ox axis where \xi is the angle around the plane of the bend from P to Q.

(d) a rotation about Q to bring the local x axis tangential to the surface of the torus at Q. This rotation we denote as \zeta.

The direction cosines at Q are then given by multiplying a matrix transformation T by the direction cosines at P where

\[
\begin{bmatrix}
\cos \zeta \cos \phi + \sin \zeta \cos \psi \sin \phi & \sin \phi \cos \zeta - \sin \zeta \cos \psi \cos \phi & \sin \zeta \sin \phi \\
\sin \zeta \cos \phi - \cos \zeta \cos \psi \sin \phi & \sin \zeta \sin \phi + \cos \zeta \cos \psi \cos \phi & -\sin \zeta \cos \psi \\
-\sin \phi \sin \zeta & \sin \zeta \cos \phi & \cos \zeta
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\text{Direction cosines at Q} \\
\cos x \\
\cos \alpha \\
\cos \zeta
\end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix}
\cos \theta \\
\cos \alpha \\
\cos \zeta
\end{bmatrix}
\]

and \(\theta_x, \alpha, \theta_z\) are defined in Appendix 7.A.
APPENDIX 7.D : EXCITATION OF THE SOURCE IN THE $\xi=0$ PLANE

Consider the illumination of a circular guiding structure. An element of the source is considered to be located at position $r, \psi$ in the entrance aperture and the ray direction is given by angles $\theta, \phi$. The angles are defined in Fig. 7.11.

We define a source intensity function $I$ so that the flux radiated by a source element of area $dA$ into a solid angle $d\Omega$, is

$$dF = I(\theta, \phi, r, \psi) d\Omega dA \quad (D.1)$$

where

$$d\Omega = \sin \theta \, d\theta d\phi$$
$$dA = r \, dr d\psi$$

We usually consider Lambertian sources where $I$ takes the form

$$I = I_0 \cos \theta \quad (D.2)$$

The bending of the guide destroys the axial symmetry encountered in the straight guide and the total bound power with the guide is then

$$P_{\text{BOUND}} = 2 \int_0^\theta \int_0^{2\pi} r \, dr d\psi \int_0^\theta \sin \theta \, d\theta I_0 \cos \theta \int_0^{2\pi} e^{-\alpha \xi} d\phi \quad (D.3)$$

where

$$\alpha = \alpha (\theta, r, \psi, \phi, \xi)$$

and the variation with respect to $\xi$ (defined in section 7.3) indicates that the attenuation $\alpha$ varies around the bend. Note that the integration limits of $\phi$ need only be $0$ to $\pi$ since there is a small concession from the symmetry considerations.
Fig. 7.11. Elemental area $dA$ of source at $z = 0$ used to excite a circular fibre.
From ref. 41 and using the coordinate system defined in Fig. 7.12 with the refractive index profile of eqn (7.66), a point \((\zeta_p, \eta_p, z)\) on the ray path within the core of the bent parabolic index circular fibre is defined by the equations

\[
\zeta_p = \frac{1}{Ra^2} (1 - \cos az) + \zeta_1 \cos az + \mu \sin az
\]

\[
\eta_p = \eta_1 \cos az + \nu \sin az
\]

where \(a = \frac{\sqrt{2\Delta}}{\rho}\)

\[
\mu = \frac{1}{a} \frac{d\zeta_p}{dz} \bigg|_{z=0}
\]

\[
\nu = \frac{1}{a} \frac{d\eta_p}{dz} \bigg|_{z=0}
\]

Fig. 7.12 Local cartesian coordinate system \((\zeta, \eta, z)\) used to describe ray paths on a bent circular fibre.
and \( \zeta_1, \eta_1 \) are the initial positions of the ray in the cross section at the start of the bend. To find the turning points of the ray we differentiate the function representing the distance of the ray from the axis of the bent guide.

From

\[
 r = \sqrt{\zeta_1^2 + \eta_1^2}
\]

we find

\[
\frac{dr}{dz} = \frac{\zeta_1}{\sqrt{r}} \cdot \frac{d\zeta_1}{dz} + \frac{\eta_1}{\sqrt{r}} \frac{d\eta_1}{dz}
\]

\[
= \sin \alpha z \left( \frac{1}{R^2 a^3} - \frac{\zeta_1}{Ra} \right) + \cos \alpha z \left( \frac{\mu}{Ra} \right)
\]

\[
+ \sin^2 \alpha z \left( \frac{\mu}{Ra} - \zeta_1 a \mu - \eta_1 a \nu \right) + \cos^2 \alpha z (\mu a \zeta_1 - \frac{\mu}{Ra} + \eta_1 a \nu)
\]

\[
+ \sin \alpha z \cos \alpha z \left( - \frac{1}{R^2 a^3} + \frac{\zeta_1}{Ra} + \frac{\zeta_1^2}{Ra} + \zeta_1^2 a + \mu^2 a - \eta_1^2 a + \nu^2 a \right)
\]

A turning point occurs on the ray path for

\[
\frac{dr}{dz} = 0
\]

The simplest method for selecting the outer turning point of the ray is to select those points where \( \frac{dr}{dz} \) changes from positive to negative while stepping along the ray path numerically.
REFERENCES


CHAPTER 8
THE ELLIPTICAL STEP-INDEX GUIDE

8.1 INTRODUCTION

So far in this thesis our attention has been confined to the study of propagation characteristics of optical waveguides with either circular or slab geometries. In practical situations slight elliptic deformations may be introduced during the fibre drawing process that cause the fibre to be non-circular. In what follows we show how the techniques developed in the preceding chapters can be applied to the calculation of attenuation on slightly elliptic step-index fibres.

Firstly we show that a generalisation of geometric optics can be applied to the classification of rays on a dielectric cylinder of elliptical cross section. We find four groups of rays, (i) bound, (ii) tunnelling, (iii) refracting and (iv) a class which is not found on the circular cylinder\(^1\) and consists of a mixture of tunnelling and refracting type reflections. The last of these cases arises because the circular symmetry apparent in previously analysed cases no longer exists. The classification of rays has a corresponding analogue in the classification of modes which is discussed subsequently.

A ray expression for attenuation is derived by applying the Generalised Fresnel's Law from Chapter 3 at every point in the cross section of the guide to determine the fractional loss of skew ray power.
at each reflection, and then utilising the simple ray attenuation model of (1.15).

In order to demonstrate the validity of the modified ray optics approach, we compare it with an analysis based on a complete modal expansion of the electromagnetic fields. A simplified eigenvalue equation is derived in the weakly guiding approximation \( n_1 \approx n_2 \) and solved to give the modal attenuation for high order tunnelling modes whose radiation caustic in the cladding is close to the surface of the elliptical guide. The modal expansion is derived using Mathieu functions which we expand in terms of small eccentricity and show it to be analytically equivalent to the corresponding perturbation of the ray optics analysis.

8.2 GEOMETRIC OPTICS ANALYSIS

8.2.1 Ray classification

Consider a dielectric cylinder of elliptical cross section with a core of refractive index \( n_1 \) and a cladding with index \( n_2 \) such that \( n_2 < n_1 \). The situation is illustrated in Fig. 8.1. The axis of the cylinder is aligned with the \( z \) axis so that the core cladding boundary is given in cartesian \((x,y,z)\) coordinates as

\[
\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1
\]  \hspace{1cm} (8.1)

where \( A, B \) are the conventional semi-axes of the ellipse and the eccentricity is defined by

\[
\varepsilon_o^2 = 1 - \frac{B^2}{A^2}
\]  \hspace{1cm} (8.2)

The two foci are at the points \( x = \pm \epsilon A, y = 0 \). For the special case where the ellipse becomes circular, then \( \epsilon_o = 0 \).
Fig. 8.1 Sketch showing orientation of Oxyz axis system in the cross section of an elliptical dielectric cylinder with core and cladding refractive indices of $n_1$ and $n_2$ respectively. The major and minor semi-axes have lengths $A$ and $B$ as shown.

At each reflection the incident and reflected rays are in the plane containing the normal $\rho$ to the boundary at the point of reflection as shown in Fig. 8.2. The angle $\theta_z$ is the angle the ray makes with the axial $z$ direction and $\Gamma$ is the angle between the normal $\rho$ and the projection of the ray onto the plane that contains $\rho$ and is perpendicular to the $z$ axis. If we define the complement of the critical angle $\theta_c$ by

$$\cos \theta_c = \frac{n_2}{n_1}$$

(8.3)
Fig. 8.2 Geometry of the reflection of a ray at point P on the core-cladding boundary of a step-index cylindrical waveguide. The ray makes angle $\theta_0$ with the axial z direction and $\Gamma$ is the angle between the normal $\rho$ and the projection of the ray onto the plane that contains $\rho$ and is perpendicular to the z axis.

then the angles of reflection can be used to classify the rays as follows:

bound ray reflection: $\theta_z \leq \theta_c$ \hspace{1cm} (8.4a)

tunnelling ray reflection: $\theta_z > \theta_c$ and $\sin \theta_z \cos \Gamma \leq \sin \theta_c$ \hspace{1cm} (8.4b)

refracting ray reflection: $\theta_z > \theta_c$ and $\sin \theta_z \cos \Gamma > \sin \theta_c$ \hspace{1cm} (8.4c)

As a ray propagates between reflections along uniform cylindrical structures
the angle $\theta_z$ remains unchanged. In the special case of a circular cross section $\Gamma$ remains unchanged as in Fig. 8.3(a) but on the elliptical cylinder it changes at each reflection. However it is easy to see that the envelope of the projection of the ray trajectory onto the fibre cross section is the quantity which is preserved. The theorems of classical geometry show that for an elliptical cross section the envelope, or inner caustic, is a confocal hyperbola when the ray projection cuts the x axis between the foci, and a confocal ellipse otherwise. We can use the eccentricity $\varepsilon_1$ of the inner ellipse to classify the rays.

(i) Hyperbolic caustics

Rays with $\theta_z > \theta_c$ and $\varepsilon_1 > 1$ i.e. those with hyperbolae as inner caustics all have the property that $\Gamma = 0$ for some reflections, as illustrated on Fig. 8.3(b), because confocal ellipses and hyperbolae intersect at right angles. Thus all these rays are classed either as refracting rays (obeying (8.4c) at every reflection) or tunnelling/refracting rays (obeying (8.4c) at some reflections and (8.4b) at others). There are no purely tunnelling rays.

(ii) Elliptical caustics

Rays with $\theta_z > \theta_c$ and $\varepsilon_1 < 1$, i.e. those with elliptical inner caustics as shown in Fig. 8.3(c), make reflections which are purely tunnelling, purely refracting or mixed tunnelling/refracting. We specify these ray classes as follows.

If a point on the ellipse is given in terms of the eccentric angle $v$ by

$$x = A \cos v, \quad y = B \sin v$$

(8.5)

then after some algebra we obtain
Fig. 8.3 Sketch showing the angle $\Gamma$ between the ray projection in the cross section of the guide and the normal to the surface at the point of incidence for (a) a circular guide, (b) an elliptical guide with hyperbolic ray caustics in the core, (c) an elliptical guide with elliptical inner core caustic.
\[ \cos^n = \left( \frac{e_0^{-2} - e_1^{-2}}{e_0^{-2} - \cos^2 \theta} \right) \]  

(8.6)

For rays with given \( \varepsilon_1 \) then

\[ \left( 1 - \frac{e_1^{-2}}{e_0^{-2}} \right)^{\frac{1}{2}} < \cos \Gamma < \left( \frac{e_0^{-2} - e_1^{-2}}{e_0^{-2} - 1} \right)^{\frac{1}{2}} \]  

(8.7)

The rays with elliptical inner caustics are then

**bound rays:** \( \theta z < \theta c \)

**tunnelling rays:** \( \theta z > \theta c \), \((\cos \Gamma)^{\text{max}} < \frac{\sin \theta_c^C}{\sin \theta_z^C}\)  

(8.8a)

**refracting rays:** \( \theta z > \theta c \), \((\cos \Gamma)^{\text{min}} > \frac{\sin \theta_c^C}{\sin \theta_z^C}\)  

(8.8b)

**tunnelling/refracting rays:** \( \theta z > \theta c \), \((\cos \Gamma)^{\text{min}} < \frac{\sin \theta_c^C}{\sin \theta_z^C}\) and \((\cos \Gamma)^{\text{max}} > \frac{\sin \theta_c^C}{\sin \theta_z^C}\)

(8.8c)

Bound rays propagate down the cylinder without attenuation but rays in the other classes suffer a power loss at each reflection according to ref. 6.

For a refracting ray, radiation occurs from the fibre boundary, while for a tunnelling ray radiation occurs from the radiation caustic, distant \( r_{\text{rad}} \) from the centre of curvature in the direction of the radius of curvature as depicted on Fig. 8.4 and given by

\[ r_{\text{rad}} = \rho(v) \frac{\sin \Gamma(v)}{C} \]  

(8.9)

where

\[ C = \left( 1 - \frac{\sin \theta_c^C}{\sin \theta_z^C} \right)^{\frac{1}{2}} \]  

(8.10)

If we let the radiation caustic be the curve \((\xi(v), \eta(v))\) as shown in Fig. 8.3 then some geometry gives

\[ \xi(v) = x(v) + \Delta \xi \]  

(8.11)

\[ \eta(v) = y(v) + \Delta \eta \]
Fig. 8.4 Radiation occurs from the point \((\xi(v), \eta(v))\) in the cladding for a ray reflected at the point \((A\cos v, B\sin v)\) as shown on the diagram. The radius of curvature of surface at the point of reflection is \(\rho(v)\).

where

\[
\Delta \xi = A\varepsilon_o^2 \cos v \left( \varepsilon_o^{-2} - \cos^2 v \right) \left[ \frac{\sin \Gamma}{C} - 1 \right] \\
\Delta \eta = A\varepsilon_o^2 \sin v \left( 1 - \varepsilon_o^2 \right)^{-\frac{1}{2}} \left( \varepsilon_o^{-2} - \cos^2 v \right) \left[ \frac{\sin \Gamma}{C} - 1 \right]
\]

(8.12)

and \(x(v), y(v)\) are points on the fibre boundary. \(\Delta \xi, \Delta \eta\) change sign where \(C = \sin \Gamma\).
Substituting (8.8) in (8.10) we see that, for tunnelling rays, \( C < \sin \Gamma \) and hence (8.11) and (8.12) predicts that the curve \( \{ \xi(v), \eta(v) \} \) lies outside the elliptical core everywhere. For fixed \( \theta_2 \), \( \{ \xi(v), \eta(v) \} \) intersects the boundary of the core at the point where the tunnelling/refracting ray changes its reflection type. Two examples illustrating the variation of the position of the radiation caustic \( \{ \xi(v), \eta(v) \} \) for a pure tunnelling ray and for a mixed refracting tunnelling ray are shown on Fig. 8.5.

8.2.2 Ray attenuation

Having classified the ray families, we now construct a simple ray optics solution for ray attenuation on an elliptical cylinder. In order to do this it becomes convenient to use elliptical cylindrical...
coordinates \((u,v,z)\) with the \(z\) axis aligned along the longitudinal axis of the guide. They are related to cartesian coordinates by the relations

\[
\begin{align*}
    x &= \sigma \cosh u \cos v \\
    y &= \sigma \sinh u \sin v \\
    z &= z
\end{align*}
\] (8.13)

and illustrated on Fig. 8.6. The surfaces \(u=\) const are confocal ellipses

![Fig. 8.6 Contours defined by the elliptical coordinate system (8.13) are confocal ellipses and hyperbolae. \(OF=OF'=\sigma\).](image)

with foci at \(F,F'\) where \(OF=OF'=\sigma\). The surface \(u=u_o\) defines the surface of the ellipse. The surfaces \(v=\) constant are hyperbolae confocal with \(F,F'\).

If the core cladding boundary is defined by \(u=u_o\) then from Eqn (8.1) we find
As $\sigma \to 0$, $\cosh u_o \to \sinh u_o$ and the ellipse degenerates to the particular case of a circle. The eccentricity $\varepsilon_0$ of the elliptical boundary from 8.2 is

$$\varepsilon_0 = \frac{1}{\cosh u_o} \quad (8.15)$$

Our attention is devoted to a purely tunnelling family of rays having an elliptical inner caustic defined by $u = u_1$, since the rays within the refracting or mixed tunnelling refracting classes have very high attenuation rate and are rapidly lost from the core. This family of rays makes angle $\theta_\omega > \theta_\chi$ with the longitudinal axis of the cylinder. A typical member of this family is represented by the ray PQ reflected at $Q = (u_o, v_o)$ on Fig. 8.7 which also shows the projection $P'^Q$ of PQ onto the cross-sectional plane of the fibre so that $P'^Q = PQ \sin \theta_\omega$. QR is tangential to the boundary at the point of reflection Q. If $P'^Q$ touches the inner caustic at $S = (u_1, v_1)$ in the cross section then $P'^Q$ has the equation

$$y = \frac{\sinh u_1}{\sin v_1} \left[ 1 - \frac{\cos v_1}{\cosh u_1} x \right] \quad (8.16)$$

If $\alpha_1$ and $\alpha_2$ are the angles between $P'^Q$ and the x direction, and between QR and the x direction then

$$\tan \alpha_1 = -\tanh u_o \cot v_o \quad (8.17)$$

$$\tan \alpha_2 = \tanh u_1 \cot v_1 \quad (8.18)$$

Also

$$\tan(\alpha_1 + \alpha_2) = \frac{-\tanh u_o \cot v_o + \tanh u_1 \cot v_1}{1 + \tanh u_o \cot v_o \tanh u_1 \cot v_1} \quad (8.19)$$

Denoting $\theta_\phi = (\alpha_1 + \alpha_2)$ then a little simple algebra gives
Fig. 8.7 Sketch showing ray PQ propagating within an elliptical optical fibre between reflections at both P and Q. The point P' is the projection of P onto the cross section through Q.

\[
\cos \theta_\phi = \frac{\cosh u_0 \sin v_0 \cosh u_1 \sin v_1 + \sinh u_0 \cos v_0 \sinh u_1 \cos v_1}{\sqrt{\sinh^2 u_1 + \sin^2 v_1} \sqrt{\sinh^2 u_0 + \sin^2 v_0}} \quad (8.20)
\]

Also the angle \( \theta_x \) defined in Fig. 8.6 is given by

\[
\cos \theta_x = \frac{QR}{PQ} \frac{PQ'}{PQ} \cos \theta_\phi = \cos \theta_\phi \sin \theta_z \quad (8.21)
\]

The angle of incidence \( \alpha_1 \) in the plane of incidence is therefore

\[
\cos \alpha_1 = \sin \theta_\phi \sin \theta_z \quad (8.22)
\]

The difference between the abscissae of P' and Q we denote \( \Delta x \), and this is readily calculated by solving the quadratic for the intersection of P'Q and the ellipse. Thus, putting \( b = \frac{\text{sinh } u_1}{\text{sin } v_1}, \mu = \tan \alpha_2 \).
\[ \Delta x = 2 \sqrt{\frac{b^2 u^2 - y^2 \tanh^2 y \sinh^2 y}{b^2 - \sinh^2 y}} \frac{\tanh^2 y}{\tanh^2 y + \sinh^2 y} \]  \hspace{1cm} (8.23)

and \( P'Q = \Delta x \sec \alpha_z \). The radius of curvature \( \rho_K \) at \( Q \) is calculated using Eqn (3.12) and, substituting \( \rho_Z = \infty \), we get

\[ \rho_K = \frac{\rho_x \sin^2 \alpha_1}{\cos^2 \theta} \]  \hspace{1cm} (8.24)

where

\[ \rho_x = \frac{\sigma [\sinh^2 u + \sin^2 v_o]^{3/2}}{\sinh u_o \cosh u_o} \]  \hspace{1cm} (8.25)

Only one further relation is necessary. \( v_o \) and \( v_1 \) are related since the point \( Q \) must satisfy (8.16). This leads to a quadratic in \( \cos v_1 \) which we solve to give

\[ \cos v_1 = \cosh u_1 \left( \cosh u_o \cos v_o \sinh^2 u_1 \right) \pm \cosh u_1 \sinh u_o \]

\[ \frac{\sqrt{\cosh^2 u_o \sinh^2 u_1 \cos^2 v_o \sin^2 v_o + \cosh^2 u_1 \sinh^2 u_o \sin^4 v_o - \cosh^2 u_1 \sinh^2 u_1 \sin^2 v_o}}{\cosh^2 u_1 \sinh^2 u_o \sin^2 v_o + \cosh^2 u_o \cos^2 v_o \sinh^2 u_1} \]  \hspace{1cm} (8.26)

Using all these quantities listed above, it is possible to calculate the ray power transmission coefficient \( T \) which varies with position \( v_o \) and gives the fraction of power lost at any point on the circumference of an ellipse where from Eqn (3.31)

\[ T(v_o) = 4 \left( \frac{\cos^2 \alpha_c}{\cos^2 \alpha_1} - 1 \right) \frac{1}{2} \exp \left\{ - \frac{2}{3} \frac{k_1 \rho_K}{\sin^2 \alpha_1} \left( \cos^2 \alpha_c - \cos^2 \alpha_1 \right)^{3/2} \right\} \]  \hspace{1cm} (8.27)

To calculate the overall attenuation of the ray family, it is necessary to weight the ray losses according to the density of rays around the circumference of the fibre cross section. This situation is discussed in Appendix 8A where a ray density distribution \( W(v_o) \) is derived so that the overall attenuation coefficient \( \gamma_e \) is
8.2.3 Ray attenuation in guides with small eccentricity

Considerable simplification occurs in the formulae of the previous section for guides which have only slight ellipticity, i.e. $\varepsilon_0 << 1$. We consider the situation of weak tunnelling rays, where the inner elliptical caustic approaches the fibre boundary, i.e. $u \sim u_0$. Also we consider those rays which have their radiation caustic in the cladding but close to the core cladding interface, as in Fig. 8.5(a), so that the Generalised Fresnel's Law of Chapter 3 is valid. These represent purely tunnelling rays on the elliptical fibre which are weakly leaky and therefore contribute significantly to the overall attenuation rate of power within the guide.

In what follows, the situation can be regarded as a perturbation of the circularly symmetric guide enclosed by the semi-minor axis as shown in Fig. 8.8. The important feature of the perturbation approach is that the angle $\theta_z$ the ray makes with the $z$ direction remains constant. This is the analogous situation to the modal analysis performed in the following section.

For a guide with $\varepsilon_1 \approx \varepsilon_0 << 1$, the length of the semi-minor axes of the core and inner core caustic are $s$ and $r$ respectively, where

\[
\begin{align*}
\sigma \sinh u_1 &= r \\
\sigma \cosh u_1 &= r \left(1 + \frac{\varepsilon_1^2}{2}\right) \\
\sigma \sinh u_0 &= s \\
\sigma \cosh u_0 &= s \left(1 + \frac{\varepsilon_0^2}{2}\right)
\end{align*}
\]
Fig. 8.8 Sketch illustrating the elliptically deformed fibre considered in the text showing the perturbation (shaded) of the circularly symmetric fibre. The radius of the fibre is the semi-minor axis of the ellipse.

from which it follows that

\[ \sigma^2 = r^2 \epsilon_1^2 \approx s^2 \epsilon_o^2 \]

Expanding all the expressions to second order in eccentricity and to second order in \((s - r)\) we get

\[ \phi_0 = s \left[ 1 + \frac{3}{2} \epsilon_o^2 \sin^2 v_o - \frac{\epsilon_o^2}{2} \right] \]

\[ \cos \theta_\phi = \frac{\epsilon_o^2 + \epsilon_1^2}{(1 + \frac{\epsilon_1^2}{2} \sin^2 v_1)(1 + \frac{\epsilon_o^2}{2} \sin^2 v_o)} \sin v_1 \sin v_o + \cos v_1 \cos v_o \]

\[ \tan \alpha_2 = -(1 - \frac{\epsilon_o^2}{2}) \cot v_i \]

\[ \Delta x = 2 \sin v_1 \sqrt{s^2 - r^2} + \frac{\epsilon_o^2 \sin v_1 \sqrt{s^2 - r^2}}{s \cdot r} \left( s^2 - r^2 \right) \cos^2 v_1 + r^2 \sin^2 v_o \]
PQ = \frac{2\sqrt{r^2 - s^2}}{\sin \theta_z} \left[ 1 + \frac{\epsilon_0^2}{2} \left( (s^2 - r^2) \cos^2 v_1 + r^2 \sin^2 v_1 \right) \right] \left[ 1 - \frac{\epsilon_1^2}{2} \cos^2 v_1 \right]

\sin \theta_\phi = \sin(v_0 - v_1) \left[ 1 + \frac{\epsilon_0^2}{2} - \frac{\epsilon_0^2}{2} \sin^2 v_1 - \frac{\epsilon_0^2}{2} \sin^2 v_0 \right]

(8.30)

The attenuation coefficient \( \gamma_e \) is then calculated from (8.28) using the expansions (8.30) when \( r \to s, v_1 \to v_0 \) to give

\[
\gamma_e = \frac{T_{\text{circle}}}{z_{\text{pcircle}}} \int_0^{2\pi} \left[ 1 + \frac{\epsilon_0^2}{2} \left( 1 - 2 \sin^2 v_0 - K_1 + 2K_1 \sin^2 v_0 \right) \right] \exp \left\{ -C \frac{\epsilon_0^2}{2} \left( 3 \sin^2 v_0 - 1 - 3K_1 + 6K_1 \sin^2 v_0 \right) \right\} \, dv_0
\]

(8.31)

where

\[
C = \frac{2}{3} \frac{k_1 \rho}{\cos^2 \theta_x} \left( \cos^2 \alpha_i - \cos^2 \alpha_c \right)^{3/2}
\]

\[
K_1 = \left( \frac{\cos^2 \alpha_i}{[\cos^2 \alpha_c - \cos^2 \alpha_i]} \right)^{1/2}
\]

and \( z_{\text{pcircle}} \) represents the distance between successive reflections on the unperturbed circular guide.

The subscript 'circle' refers to the corresponding values on the inscribed circular fibre indicated on Fig. 8.8. Then using \( \exp(-C\delta) = 1 - C\delta \) we have the overall attenuation for the ray family of

\[
\gamma_e = \gamma_{\text{circle}} \left( 1 - \frac{\epsilon_0^2}{4} C \right)
\]

(8.32)

We now seek to derive an analogous situation for the modal attenuation, but first it is necessary to show how to classify the modes using electromagnetic theory.
8.3 MODAL ANALYSIS

8.3.1 Mode classification

The fields on the guide are deduced from the transverse scalar wave equation, assuming \( \exp(\imath \beta z) \) longitudinal dependence.

\[
(\nabla_t^2 + (n^2 k^2 - \beta^2)) \psi = 0 ,
\]

where \( \beta \) is the propagation constant and \( k = 2\pi/\lambda \), with \( \lambda \) the vacuum wavelength. In elliptic coordinates, \( \nabla_t^2 \) becomes

\[
\nabla_t^2 = \frac{1}{\sigma^2 (\cosh 2u - \cos 2v)} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)
\]

(8.34)

The method of separation of variables\(^5\) can be used to obtain pairs of equations for the solutions in the core and cladding. Putting \( \psi = f(u)g(v) \) in (8.33) gives in the core

\[
\frac{d^2 f}{du^2} - (a - \tilde{q}_1 \cosh 2u)f = 0 \quad u < u_o
\]

(8.35)

\[
\frac{d^2 g}{dv^2} + (a - \tilde{q}_1 \cos 2v)g = 0 \quad u < u_o
\]

(8.36)

[Note that these two equations are transforms of each other. \((v = iu \Rightarrow g = f)\). They represent the Mathieu and modified Mathieu equations respectively\(^4\)].

Outside in the cladding we have

\[
\frac{d^2 f}{du^2} - (b - \tilde{q}_2 \cosh 2u)f = 0 \quad u > u_o
\]

(8.37)

\[
\frac{d^2 g}{dv^2} + (b - \tilde{q}_2 \cos 2v)g = 0 \quad u > u_o
\]

(8.38)

where \( \tilde{q}_1 = \frac{\sigma^2}{2}(n_1^2 k^2 - \beta^2) \), \( \tilde{q}_2 = \frac{\sigma^2}{2}(n_2^2 k^2 - \beta^2) \) and \( a, b \) are separation constants.

We can rewrite these last four equations in a more easily interpretable form
\[ \frac{d^2f}{du^2} + \tilde{q}_1 (\cosh 2u - K) f = 0 \quad u < u_o \]  \hspace{2cm} (8.39)

\[ \frac{d^2g}{dv^2} + \tilde{q}_1 (K - \cos 2v) g = 0 \quad u < u_o \]  \hspace{2cm} (8.40)

\[ \frac{d^2f}{du^2} + \tilde{q}_2 (\cosh 2u - K_2) f = 0 \quad u > u_o \]  \hspace{2cm} (8.41)

\[ \frac{d^2g}{dv^2} + \tilde{q}_2 (K_2 - \cos 2v) g = 0 \quad u > u_o \]  \hspace{2cm} (8.42)

Eqn (8.39) above determines the behaviour of \( f \) in the core which is in turn determined by the coefficient of \( f \).

(i) **Elliptical inner caustic**

If \( K_1 > 1 \) then we put \( K_1 = \cosh 2u_1 \) and an inner field caustic must exist at \( u = u_1 \). The factor multiplying \( g \) in Eqn (8.40) is then always positive definite. This mode therefore corresponds to the previously discussed family of rays within the core with confocal elliptical inner caustic as shown in Fig. 8.2(b).

(ii) **Hyperbolic inner caustic**

If \( K_1 < 1 \) then we put \( K_1 = \cos 2v_1 \), and the factor multiplying \( g \) in Eqn (8.40) is +ve definite only for \( v_1 \leq |v| \leq \frac{\pi}{2} \). The factor multiplying \( f \) in Eqn (8.39) is then always +ve definite. This mode therefore corresponds to the previously derived family of rays with confocal hyperbolic caustics in the core as shown in Fig. 8.2(c).

(iii) **Cladding fields**

In Appendix 8B we show from the continuity of the fields at the core cladding boundary that, for \( u > u_o \) and \( u \approx u_o \), the field variation within the cladding is determined by (B.11) as

\[ \frac{d^2\psi}{du^2} - p^2 \psi = 0 \quad u > u_o \]  \hspace{2cm} (8.43)
where

\[ p = \tilde{q}_2 \cosh 2u + (\tilde{q}_1 - \tilde{q}_2) \cos 2v - \tilde{q}_1 K_1 \]  \hspace{1cm} (8.44)

or

\[ p = 2\tilde{q}_2 (\sinh^2 u + \sin^2 v) - \tilde{q}_1 (K_1 - \cos 2v) \]  \hspace{1cm} (8.45)

Now we consider the following cases

(a) \( n_2 k < \beta < n_1 k \)

In the case of elliptical inner caustics \( K_1 > 1, q_1 > 0, q_2 < 0 \)

and from (8.45), \( p < 0 \) for all \( u \) and the mode is trapped.

In the case of hyperbolic inner caustics \( K_1 < 1 \) but (8.45) again gives \( p < 0 \) for \( |v| \leq \frac{\pi}{2} \) and the mode is trapped.

(b) \( \beta < n_2 k \)

For an elliptical inner caustic \( K_1 > 1, q_1 > q_2 > 0 \), then for \( u_1 \to 0 \)

\[ p \approx -(q_1 - q_2)(\cosh 2u_0 - \cos 2v) \]  \hspace{1cm} (8.46)

which yields evanescent solutions for the cladding fields near the core.

As \( u_1 \) decreases away from the surface \( u_0 \) then \( p \) changes sign. From (8.44) it can be seen that \( p \) is a maximum at \( v = 0, \pi \) and a minimum at \( v = \pm \frac{\pi}{2} \). Therefore \( v \) first changes sign at \( v = 0, \pi \) and a refracting region develops near the ends of the guide. The point where \( v \) changes sign is \( v = v_R \) where

\[ \cos 2v_R = \frac{(\tilde{q}_1 \cosh 2u_1 - \tilde{q}_2 \cosh 2u_0)}{(\tilde{q}_1 - \tilde{q}_2)} \]  \hspace{1cm} (8.47)

We then have a mixed tunnelling and refracting mode.

The position of the radiation caustic occurs at \( u = u_2 > u_0 \)

where
A purely tunnelling mode occurs when the tunnelling region extends completely around the perimeter of the ellipse, i.e. \( p > 0 \) at \( v = 0 \).
This is equivalent to the ray situation shown sketched in Fig. 8.5 which shows the oblong shape of the outer caustic for both purely tunnelling and mixed tunnelling and refracting modes.

When \( p > 0 \) at \( v = \pm \frac{\pi}{2} \) then a purely refracting mode occurs.

For hyperbolic caustics \( K < 1 \), \( q_1 > q_2 > 0 \) and (8.45) shows \( p \) is always +ve at \( v = v_1 \) for any \( u \). No purely tunnelling modes can exist.
As \( v \) increases from \( v_1 \) then \( p \) can change sign. If \( p > 0 \) at \( v = \frac{\pi}{2} \) then the mode is purely refracting. If \( p < 0 \) at \( v = \frac{\pi}{2} \) then tunnelling occurs with the division at \( v = v_R \) where

\[
\cos 2 v_R = \frac{\tilde{q}_1 \cos 2 v_1 - \tilde{q}_2 \cosh 2 u_0}{(\tilde{q}_1 - \tilde{q}_2)}
\]  

and the caustic is at \( u = u_z \) where

\[
cosh 2 u_z = \frac{\tilde{q}_1 \cos 2 v_1 - (\tilde{q}_1 - \tilde{q}_2) \cos 2 v}{\tilde{q}_2}
\]  

for \( v_1 < v < \frac{\pi}{2} \)

This corresponds to a mixed tunnelling and refracting mode equivalently to the ray analysis.

8.3.2 Derivation of the eigenvalue equation in elliptical fibres

In this section an approximate form for the eigenvalue equation is derived in the weakly guiding approximation \( (n_1 \approx n_2) \). This simplified eigenvalue equation is solved for fibres with small ellipticity in the next section.
Following Yeh, we consider an elliptical dielectric fibre with uniform core refractive index \( n_1 \) and cladding index \( n_2 < n_1 \).

Because of the asymmetry, two types of modes may exist. They are the even modes (HE\(_{\text{even}}\)), with longitudinal magnetic and electric fields given by

\[
\begin{align*}
\mathbf{H}_z &= \sum_{m=0}^{\infty} A_m \mathcal{C}_m(u, \gamma_2^2) \mathcal{C}_m(v, \gamma_1^2) \quad u \leq u_0 \\
&= \sum_{m=0}^{\infty} L_m \mathcal{F}_m(u, |\gamma_2|^2) \mathcal{C}_m(v, |\gamma_2|^2) \quad u \geq u_0 \\
\mathbf{E}_z &= \sum_{m=1}^{\infty} B_m \mathcal{S}_m(u, \gamma_2^2) \mathcal{S}_m(v, \gamma_1^2) \quad u \leq u_0 \\
&= \sum_{m=1}^{\infty} P_m \mathcal{G}_m(u, |\gamma_2|^2) \mathcal{S}_m(v, |\gamma_2|^2) \quad u \geq u_0
\end{align*}
\]

and the odd modes (HE\(_{\text{odd}}\)) with longitudinal fields

\[
\begin{align*}
\mathbf{H}_z &= \sum_{m=1}^{\infty} A_m \mathcal{S}_m(u, \gamma_2^2) \mathcal{S}_m(v, \gamma_1^2) \quad u \leq u_0 \\
&= \sum_{m=1}^{\infty} L_m \mathcal{G}_m(u, |\gamma_2|^2) \mathcal{S}_m(v, |\gamma_2|^2) \quad u \geq u_0 \\
\mathbf{E}_z &= \sum_{m=0}^{\infty} B_m \mathcal{C}_m(u, \gamma_2^2) \mathcal{C}_m(v, \gamma_1^2) \quad u \leq u_0 \\
&= \sum_{m=0}^{\infty} P_m \mathcal{F}_m(u, |\gamma_2|^2) \mathcal{C}_m(v, |\gamma_2|^2) \quad u \geq u_0
\end{align*}
\]

where \( e, A_m, e, B_m, e, L \) and \( e, P_m \) are arbitrary constants and \( \mathcal{C}, \mathcal{S}, \mathcal{F}, \mathcal{G} \) are the appropriate Mathieu functions. Also

\[
\begin{align*}
\gamma_1^2 &= (n_1^2k^2 - \beta^2)\sigma^2/4 \\
\gamma_2^2 &= (n_2^2k^2 - \beta^2)\sigma^2/4
\end{align*}
\]

\( \beta \) is the axial propagation constant, \( \sigma \) is the semi-focal distance defined in Fig. 8.6. The position \( u = u_0 \) defines the boundary of the fibre in elliptical coordinates \( (u,v,z) \) as used in the ray analysis. Matching the tangential electric and magnetic fields at the boundary \( u = u_0 \), the exact
eigenvalue equations can be derived as previously\(^2\) as

\[
\begin{pmatrix}
e_{0,1,1}^g & e_{0,1,1}^h & e_{0,3,1}^g & e_{0,3,1}^h \\
e_{0,1,1}^t & e_{0,1,1}^s & e_{0,3,1}^t & e_{0,3,1}^s \\
e_{0,1,3}^g & e_{0,1,3}^h & e_{0,3,3}^g & e_{0,3,3}^h \\
e_{0,1,3}^t & e_{0,1,3}^s & e_{0,3,3}^t & e_{0,3,3}^s \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\(= 0\) \hspace{1cm} (8.61)

and

\[
\begin{pmatrix}
e_{0,0,0}^g & e_{0,0,0}^h & e_{0,2,0}^g & e_{0,2,0}^h \\
e_{0,0,0}^t & e_{0,0,0}^s & e_{0,2,0}^t & e_{0,2,0}^s \\
e_{0,0,2}^g & e_{0,0,2}^h & e_{0,2,2}^g & e_{0,2,2}^h \\
e_{0,0,2}^t & e_{0,0,2}^s & e_{0,2,2}^t & e_{0,2,2}^s \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\(= 0\) \hspace{1cm} (8.62)

where

\[
e_{m,n}^g = \left[ 1 + \frac{\gamma_1^2}{|\gamma_2|^2} \right] F_{\text{ek}} m (u_0, |\gamma_2|^2) \sum_{r=1}^{\infty} x_{r,n} \chi_{r,n} \alpha_{m,r}
\]

\(\text{8.63}\)

\[
e_{m,n}^s = \left[ 1 + \frac{\gamma_1^2}{|\gamma_2|^2} \right] G_{\text{ek}} m (u_0, |\gamma_2|^2) \sum_{r=1}^{\infty} v_{r,n} \beta_{m,r}
\]

\(\text{8.64}\)

\[
e_{m,n}^h = \left( \frac{\omega n \in \Sigma e (u_0, \gamma_2^2)}{\beta} \right) G_{\text{ek}} (u_0, |\gamma_2|^2) + \frac{\gamma_1^2}{|\gamma_2|^2} \frac{\omega n \in \Sigma e (u_0, \gamma_2^2)}{\beta} G_{\text{ek}} (u_0, |\gamma_2|^2)
\]

\(\text{8.65}\)
\[
e_{m,n} = \left( \frac{\omega \mu}{\beta} \frac{C_n'(u_o, \gamma_1^2)}{C_n(u_o, \gamma_1^2)} \text{Fek}_m(u_o, |\gamma_2|^2) + \frac{\gamma_1^2}{|\gamma_2|^2} \frac{\omega \mu}{\beta} \text{Fek}_m'(u_o, |\gamma_2|^2) \right) \alpha_{m,n}
\]

(8.66)

with

\[
\alpha_{r,n} = \frac{2\pi}{\omega \mu} \int_0^{2\pi} c_e(v, -\gamma_2^2) c_e(v, \gamma_1^2) dv \int_0^{2\pi} c_n(v, \gamma_1^2) dv
\]

(8.67)

\[
\beta_{r,n} = \frac{2\pi}{\omega \mu} \int_0^{2\pi} s_e(v, -\gamma_2^2) s_e(v, \gamma_1^2) dv \int_0^{2\pi} s_n(v, \gamma_1^2) dv
\]

(8.68)

\[
\chi_{r,n} = \frac{2\pi}{\omega \mu} \int_0^{2\pi} c_e'(v, \gamma_1^2) s_e(v, \gamma_2^2) dv \int_0^{2\pi} s_n(v, \gamma_1^2) dv
\]

(8.69)

\[
\nu_{r,n} = \frac{2\pi}{\omega \mu} \int_0^{2\pi} s_e'(v, \gamma_1^2) c_e(v, \gamma_2^2) dv \int_0^{2\pi} c_n(v, \gamma_1^2) dv
\]

(8.70)

where the ' denotes differentiation with respect to \( u \) or \( v \) as appropriate.

\( \epsilon_o \) is the permittivity of free space. Expressions for the odd coefficients

\( g_{m,n}, s_{m,n}, h_{m,n} \) and \( t_{m,n} \) are obtained similarly or by replacing \( \text{Fek}_m \) by \( \text{Gek}_m \), \( \text{Gek}_m \) by \( \text{Fek}_m \), \( \text{Se}_m \) by \( \text{Ce}_m \), \( \text{Ce}_m \) by \( \text{Se}_m \), \( \chi \) by \( \nu \), \( \nu \) by \( \chi \), \( \alpha \) by \( \beta \) and \( \beta \) by \( \alpha \) in the above expressions for \( e_{m,n}, e_{m,n}, e_{m,n}, e_{m,n} \). The computation of the propagation constants for modes in elliptical fibres from these exact equations is long and tedious and only practical for the lowest order modes. However, considerable simplification can be achieved if we realise that for an elliptical waveguide with \( n_1 \equiv n_2 \), \( \beta \equiv n_1 k \). To facilitate this, we introduce the following definitions

\[
U_e^2 = \sigma^2 e^{\omega (n_1^2 k^2 - \beta^2)} = 4 e^{\omega \gamma_1^2}
\]

(8.71)

\[
Q_e^2 = \sigma^2 e^{\omega (n_2^2 k^2 - \beta^2)} = 4 e^{\omega \gamma_2^2}
\]

(8.72)

\[
Y_e^2 = U_e^2 - Q_e^2 = \sigma^2 e^{\omega k^2 \left( \frac{n_2^2 - n_1^2}{n_2^2} \right)} = \sigma^2 e^{\omega k^2 \delta}
\]

(8.73)
\[
\theta_p = \sqrt{\delta \frac{V_e}{V_e}} = \left[1 - \frac{\beta^2}{n^2 k^2}\right]^{1/2} \quad (8.74)
\]

which are similar to the parameters defined by Snyder\textsuperscript{3} and those of Yeh.\textsuperscript{2} They reduce to the circular form in the degenerate circle case. The subscript \(e\) is used to denote that these quantities are those associated with the ellipse. We now expand all functions in powers of \(\theta_p^2\)

\[
\gamma_2 = \gamma_1^2 - \frac{1}{4} k_1^2 \delta = \gamma_1^2 + 0(\theta_p^2) \quad (8.75)
\]

\[
\text{ce}_r(v, -\gamma_2) = \text{ce}_r(v, \gamma_1^2) + 0(\theta_p^2) \quad (8.76)
\]

\[
\text{se}_r(v, -\gamma_2) = \text{se}_r(v, \gamma_1^2) + 0(\theta_p^2) \quad (8.77)
\]

\[
\alpha_{r,m} = \int_0^{2\pi} \text{ce}_r(v, \gamma_1^2) \text{ce}_m(v, \gamma_1^2) dv = \int_0^{2\pi} \text{ce}_n(v, \gamma_1^2) dv + 0(\theta_p^2) \quad (8.78)
\]

\[
\beta_{r,m} = \int_0^{2\pi} \text{se}_r(v, \gamma_1^2) \text{se}_m(v, \gamma_1^2) dv = \int_0^{2\pi} \text{ce}_n(v, \gamma_1^2) dv + 0(\theta_p^2) \quad (8.79)
\]

but we must be careful to retain terms in \(V_e^2\) which cannot vanish even though \(\theta_p \to 0\). We note

\[
V_e = 4e \, 2u^o (\gamma_1^2 - \gamma_2^2) \quad (8.80)
\]

Substituting the above into the Eqns (8.63 - 8.66)

\[
e, o_{r,m,n} = C^{(1)} r, o_{m,n} \Delta_{m,n} + 0(\theta_p^2) \quad (8.81)
\]

\[
e, o_{r,m,n} = C^{(2)} r, o_{m,n} \Delta_{m,n} + 0(\theta_p^2)
\]

\[
e, h_{r,m,n} = D^{(1)} r, h_{m,n} \Delta_{m,n} + 0(\theta_p^2)
\]

\[
e, o_{r,m,n} = D^{(2)} r, o_{m,n} \Delta_{m,n} + 0(\theta_p^2)
\]

where \(C^{(1)}, (2)\) and \(D^{(1)}, (2)\) can be easily obtained from Eqns (8.63 - 8.66).
Inserting these into the determinant representing the exact eigenvalue equation we can show that all off diagonal terms are small and of order $\theta^2_p$ and we get for Eqn (8.61)

\[
\begin{vmatrix}
    e, o^g_{1,1} & e, o^h_{1,1} & 0(\theta^2_p) & 0(\theta^2_p) & \cdots & \cdots \\
    e, o^t_{1,1} & e, o^s_{1,1} & 0(\theta^2_p) & 0(\theta^2_p) & \cdots & \cdots \\
    0(\theta^2_p) & 0(\theta^2_p) & e, o^g_{3,3} & e, o^h_{3,3} & \cdots & \cdots \\
    0(\theta^2_p) & 0(\theta^2_p) & e, o^t_{3,3} & e, o^s_{3,3} & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{vmatrix}
\]

\[= 0\]

(8.82)

A similar simplification results for Eqn (8.62). The eigenvalue equation therefore reduces to, for small $\theta_p$,

\[
\left(1 - \frac{1}{\gamma^2_2}\right) F_{ek}(u_o, \gamma^2_2) G_{ek}(u_o, \gamma^2_2) \sum_{r,m} x_{r,m} \alpha \gamma^2 \sum_{r,m} \gamma_{r,m} \beta
\]

\[
+ \left[ n^2 u \frac{\psi}{\sum_{r,m}} \left( F_{ek}(u_o, \gamma^2_2) G_{ek}(u_o, \gamma^2_2) \right) - \frac{\gamma^2_1}{\gamma^2_2} \frac{n^2 u \alpha}{\beta} \right]
\]

\[
\left[ \frac{\omega m}{\beta} \frac{F_{ek}(u_o, \gamma^2_2)}{G_{ek}(u_o, \gamma^2_2)} \right]
\]

\[
= 0
\]

(8.83)

This leads to

\[
\left[ \frac{n^2_1 \psi}{\sum_{r,m}} \right] \left( \frac{F_{ek}(u_o, \gamma^2_2)}{G_{ek}(u_o, \gamma^2_2)} - \frac{\gamma^2_1}{\gamma^2_2} \frac{n^2 u \alpha}{\beta} \right)
\]

\[
\left[ \frac{C_{ek}(u_o, \gamma^2_2)}{G_{ek}(u_o, \gamma^2_2)} - \frac{\gamma^2_1}{\gamma^2_2} \frac{n^2 u \alpha}{\beta} \right]
\]

\[
= m^2 \frac{V^4_e}{u^4_e Q^4_e} \cdot \frac{\gamma^2_1}{\gamma^2_2} \frac{n^2 k^2}{\gamma^2_1}
\]

(8.84)

where we have used the properties of the series $\sum_{r=1}^{\infty} \chi_{r,n} \alpha_{m,r}$ and $\sum_{r=1}^{\infty} \gamma_{r,n} \beta_{m,r}$ derived in Appendix 8C. Note that terms in $V_e$ must be retained on the right hand side of (8.84) while they have been omitted in ref. 2. It now remains to solve (8.84).
8.3.3 Solution of the simplified eigenvalue equation

It is important to note at this stage that the equation (8.84) is of the same form as the eigenvalue equation for a circle,\(^3,10\) except that the Bessel functions used in the circular case are replaced by Mathieu functions for an elliptical guide. This similarity in the form can be exploited to simplify the analysis enormously if the definitions (8.71–8.74) are used.

The expansions of Appendix 8D for the solutions of the modified Mathieu equation relate the \(C_e, S_e, F_e, G_e\) to the corresponding Bessel functions to fourth order in eccentricity \(\varepsilon_o\), when \(\sigma \rightarrow 0\) as

\[
C_e(\mu, \gamma_1^2) \sim p'_m J_m(\sigma e) + O(\varepsilon_o^4) \quad (8.85)
\]

\[
S_e(\mu, \gamma_1^2) \sim s'_m J_m(\sigma e) + O(\varepsilon_o^4) \quad (8.86)
\]

\[
F_e(\mu, \gamma_2^2) \sim q'_m K(i\sigma e) + O(\varepsilon_o^4) \quad (8.87)
\]

\[
G_e(\mu, \gamma_2^2) \sim r'_m K(i\sigma e) + O(\varepsilon_o^4) \quad (8.88)
\]

where \(p'_m, q'_m, s'_m, r'_m\) are known series. If we substitute these functions in the simplified eigenvalue equation, (8.84), then a simpler form results

\[
\left( \frac{n_1}{n_2} \right)^2 \cdot \frac{J'_m(\sigma e)}{J_m(\sigma e)} - \frac{H'_m(\sigma e)}{H_m(\sigma e)} = 0(\varepsilon_o^4)
\]

\[
\frac{J'_m(\sigma e)}{J_m(\sigma e)} - \frac{H'_m(\sigma e)}{H_m(\sigma e)} = 0(\varepsilon_o^4)
\]

\[
\frac{\beta^2 m^2}{n_1^2 k^2} \frac{V^4}{U^4} = 0(\varepsilon_o^4) \quad (8.89)
\]

Then to order \(\sim \varepsilon_o^4\), (8.89) takes the form of the eigenvalue equation for a circle,\(^10\) but with perturbed values of the arguments.
If we define $U, Q, V$ as being the values relating to the inscribed circular guide of Fig. 8.8, then

\[
U_e = U \left( 1 + \frac{\varepsilon_0^2}{4} \right)
\]

\[
Q_e = Q \left( 1 + \frac{\varepsilon_0^2}{4} \right)
\]

(8.90)

Also from a simple WKB argument described in Appendix 8E a relation between $m$ and $\ell$, where $\ell$ is the equivalent value for the inscribed circular guide is

\[
m = \ell \left( 1 + \frac{\varepsilon_0^2}{4} \right)
\]

(8.91)

Therefore the solution follows exactly the solution for the circular case, \(^{10}\) but with $U_e$, $Q_e$, $m$ in place of $U, Q, \ell$. In the tunnelling ray region close to the critical angle ($Q_e \approx m$), the appropriate form for the dimensionless attenuation coefficient $\alpha_e$ is

\[
\alpha_e = \frac{4}{\pi} \frac{1}{V_e} \frac{\sin^2 \theta}{\cos \theta} \frac{1}{|H_m(Q_e)|^2} \exp \left\{ -\frac{2}{3} \frac{(\ell^2 - Q_e^2)^{3/2}}{Q_e} \left( 1 + \frac{\varepsilon_0^2}{4} \right) \right\}
\]

(8.92)

Using $\exp \left( -\frac{C \varepsilon_0^2}{4} \right) = \left( 1 - \frac{C \varepsilon_0^2}{4} \right)$ where $C$ is equivalent to that in the ray analysis (8.31) then

\[
\alpha_e = \alpha \left( 1 - \frac{C \varepsilon_0^2}{4} \right)
\]

(8.94)

and the unnormalised attenuation coefficient on the ellipse $\gamma_e$ is

\[
\gamma_e = \gamma \left( 1 - \frac{C \varepsilon_0^2}{4} \right)
\]

(8.95)
where $\alpha, \gamma$ are the corresponding attenuations when $\varepsilon_0 = 0$. This equation (8.95) is exactly the previous result for the attenuation coefficient derived from a ray analysis (8.32). We have therefore established equivalence of the modal analysis which gives the attenuation of (8.95) and a ray analysis based on the use of ray power transmission coefficients (8.32) for skew tunnelling rays on multimode fibres of slight ellipticity.
APPENDIX 8A: DENSITY OF RAYS AT A POINT ON THE ELLIPTIC BOUNDARY

Consider the projection of the narrow flux tube containing the ray PQ onto the cross section as shown in Fig. 8.9. The arc of surface intersected by the flux tube is $ds$ and the point at which the flux tube touches the inner caustic, $u = u_1$, is $S(u_1, v_1)$. If the angle subtended by $ds$ at the centre of the ellipse is $dv$, then

$$\rho \frac{dv}{ds} = \cos \phi$$  \hspace{1cm} (A.1)

where

$$\cos \phi = \frac{\cosh u_0 \sin^2 v_0 + \cos^2 v_0 \sinh u_0}{(\cosh^2 u_0 - \cos^2 v_0)^{1/2}}$$  \hspace{1cm} (A.2)

If we define the power density at a point on the core boundary as $D(v)$ then
Then the total ray power within the core is

\[ \text{Tot} = \int_0^{2\pi} D(v) \, dv \]  

(A.4)

We define a weighting function \( W(v) \), proportional to the power density, which weights the losses at a point with respect to the total attenuation

\[ W(v) = \frac{2\pi D(v)}{\int_0^{2\pi} D(v) \, dv} \]  

(A.5)

Within the flux tube shown on Fig. 8.9

\[ \text{ds} \cdot g(v) = \text{constant} \]  

(A.6)

where

\[ g(v) = \frac{\sin \theta}{\text{SQ}} \]  

(A.7)

\[ \text{SQ} = \sqrt{\left( \cosh u_1 \cos v_1 - \cosh u_o \cos v_o \right)^2} \]

\[ + \left( \sinh u_1 \sin v_1 - \sinh u_o \sin v_o \right)^2 \]  

(A.8)

and \( \sin \theta \) is given by Eqn (8.20). Since the power within a given flux tube remains constant

\[ D(v) = K \cdot \frac{\rho}{\cos \phi} \cdot g(v) \]  

(A.9)

where \( K \) is constant. Expanding each of the terms of (A.8) for small eccentricity \( \varepsilon_o \)

\[ g(v) = \frac{1}{s} \left[ 1 + \frac{\varepsilon_o^2}{2} - \frac{\varepsilon_o^2}{2} \sin^2 v_o - \frac{\varepsilon_o^2}{2} \sin^2 v_1 \right] \]  

(A.10)

\[ \rho = s \left[ 1 + \frac{\varepsilon_o^2}{2} \left( 1 - \sin^2 v_o \right) \right] \]  

(A.11)

\[ \cos \phi = 1 + \mathcal{O}(\varepsilon_o^4) \]  

(A.12)
then

\[ \int_0^{2\pi} D(v_o) dv_o = 2\pi \left[ 1 + \frac{\varepsilon_o^2}{4} \right] \]  \hspace{1cm} (A.13)

and

\[ W = \left[ 1 + \frac{3\varepsilon_o^2}{4} - \frac{3\varepsilon_o^2}{2} \sin^2 v_o \right] \]  \hspace{1cm} (A.14)
APPENDIX 8B: COEFFICIENTS FOR THE CLADDING FIELDS

Consider the potential function for the fields within the core of an elliptical fibre from Eqn (8.33) to assume the form

\[ \psi^1 = f(u) g(v) \quad (B.1) \]

where \( f(u) \), \( g(v) \) are the appropriate Mathieu functions satisfying (8.35) and (8.36) respectively. The position of the inner caustic \( u = u_1 \) corresponds to the turning point behaviour of (8.35) where

\[ \cosh 2u_1 = \frac{a}{q_1} \quad (B.2) \]

The matching potential function in the cladding is

\[ \psi^0 = \sum_{m=0}^{\infty} \alpha_m f_m^*(u) g_m^*(v) \quad (B.3) \]

where

\[ \frac{d^2f_m^*}{du^2} + (\tilde{q}_2 \cosh 2u - b_m)f_m^* = 0 \quad (B.4) \]

and

\[ \frac{d^2g_m^*}{dv^2} + (b_m - \tilde{q}_2 \cos 2v)g_m^* = 0 \quad (B.5) \]

On \( u = u_0 \) we shall consider continuity of \( \psi \)

\[ \psi^1 = \psi^0 \text{ on } u = u_0 \text{ for all } v \quad (B.6) \]

We need to expand \( g(v) \) in terms of the complete set of \( g_m^*(v) \)'s to satisfy the continuity condition on \( v \)

\[ g(v) = \sum_{m=0}^{\infty} \beta_m g_m^*(v) \quad (B.7) \]

Substituting (B.7) into (B.1) and satisfying (B.6)

\[ f(u_o)\beta_m = \alpha_m f_m^*(u_o) \quad (B.8) \]
Substituting into (B.3)

\[ \psi^0 = f(u_o) \sum_{m=0}^{\infty} \beta_m \frac{f^*(u)}{f^*(u_o)} g^*_m(v) \]  

(B.9)

We examine

\[ \frac{\partial^2 \psi^0}{\partial u^2} = f(u_o) \sum_{m=0}^{\infty} \beta_m \frac{f^*(u)}{f^*(u_o)} \left\{ b_m - \tilde{q}_2 \cosh 2u \right\} g^*_m(v) \]  

(B.10)

From (B.4) and (B.5)

\[ \frac{\partial^2 \psi^0}{\partial u^2} = f(u_o) \sum_{m=0}^{\infty} \beta_m \frac{f^*(u)}{f^*(u_o)} \left\{ - \frac{d^2 g^*_m(v)}{dv^2} \right\} + \tilde{q}_2 \frac{(\cos 2v - \cosh 2u)}{2} \psi^0 \]

If \( u \approx u_o \) so that \( f^*_m(u) \approx f^*_m(u_o) \) then

\[ \frac{\partial^2 \psi^0}{\partial u^2} \approx f(u_o) \sum_{m=0}^{\infty} \frac{\partial^2 (\beta_m g^*_m(v))}{dv^2} + \tilde{q}_2 (\cos 2v - \cosh 2u) \psi^0 \]

\[ = - \frac{d^2}{dv^2} g(v) f(u_o) + \tilde{q}_2 (\cos 2v - \cosh 2u) \psi^0 \]

\[ = \psi^0 \left\{ a - (\tilde{q}_1 - \tilde{q}_2) \cos 2v - \tilde{q}_2 \cosh 2u \right\} \]  

(B.11)
APPENDIX 8C: COEFFICIENTS IN THE SIMPLIFIED EIGENVALUE EQUATION

The coefficients in the simplified eigenvalue equation which we wish to consider are

\[ a) \sum_{r=1}^{\infty} \chi_{r,n} \alpha_{m,r} \]
\[ b) \sum_{r=1}^{\infty} \nu_{r,n} \beta_{m,r} \]

where the terms are defined in terms of angular Mathieu functions as in Eqns (8.67 - 8.70). Some important results for the angular Mathieu functions are

\[
\int_{0}^{2\pi} c_{e m}^{2}(v, \zeta) dv = \int_{0}^{2\pi} s_{e m}^{2}(v, \zeta) dv = \pi \tag{C.1}
\]

\[
\int_{0}^{2\pi} c_{e m}(v, \zeta)c_{e p}(v, \zeta) dv = 0 \quad m \neq p \tag{C.2}
\]

\[
\int_{0}^{2\pi} s_{e m}(v, \zeta)s_{e p}(v, \zeta) dv = 0 \quad m \neq p \tag{C.3}
\]

\[
\int_{0}^{2\pi} c_{e m}(v, \zeta)s_{e m}'(v, \zeta) dv = -\int_{0}^{2\pi} s_{e m}(v, \zeta)c_{e m}'(v, \zeta) dv = \pi \sum_{r=1}^{\infty} \alpha_{r} B_{r}^{(m)} \tag{C.4}
\]

Also we have

\[
\int_{0}^{2\pi} c_{e 2n}^{2}(v, \zeta) dv = 2\pi [A_{2n}^{(0)}]^{2} + \pi \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^{2} \tag{C.5}
\]

\[
\int_{0}^{2\pi} c_{e 2n+1}^{2}(v, \zeta) dv = \pi \sum_{r=0}^{\infty} [A_{2r+1}^{(2n+1)}]^{2} \tag{C.6}
\]

\[
\int_{0}^{2\pi} s_{e 2n+1}^{2}(v, \zeta) dv = \pi \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^{2} \tag{C.7}
\]

\[
\int_{0}^{2\pi} s_{e 2n+2}^{2}(v, \zeta) dv = \pi \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+2)}]^{2} \tag{C.8}
\]

where \( A_{r}^{(m)} \) and \( B_{r}^{(m)} \) are the coefficients of the trigonometric series expansions for the Mathieu functions.\(^4\)\(^5\) It can also be readily shown
that as $\sigma \to 0$ (eccentricity $\epsilon \to 0$) then $A_m^{(m)}$, $B_m^{(m)} \to 1$ while all other $A, B$ tend to zero.

Also using the results for $m >> 0$, $\sigma$ small

$$\sum_{r=0}^{\infty} [A_{2r+1}^{(m)}]^2 = 1$$  \hspace{1cm} (C.9)

$$A_{m+2r}^{(m)} \approx (-1)^r \frac{m!}{r!(m+r)!} t^r$$  \hspace{1cm} (C.10)

where $t = \frac{\epsilon}{4} = O(\epsilon^2)$.

Since we know the principal contributions are from terms either side of the $m$-th term for $\sigma \to 0$ then $A_m^{(m)}$ can be readily estimated as, for large $m$,

$$A_m^{(m)} \approx 1 - \frac{t^2}{m^2} = 1 - O(\frac{\epsilon^2}{m^2})$$  \hspace{1cm} (C.12)

An exactly analogous argument follows for the coefficients $B_m^{(m)}$.

Using these relationships we consider the series a) and b) above in the simplified eigenvalue equation where $m = n$. Substituting we get for the near diagonal terms which are the only ones of interest

$$\sum_{r=1}^{\infty} \chi_{r,m} \alpha_{m,r} = -m$$  \hspace{1cm} (C.13)

$$\sum_{r=1}^{\infty} \nu_{r,m} \beta_{m,r} = m$$  \hspace{1cm} (C.14)
APPENDIX 8D: ASYMPTOTIC SOLUTION OF THE MODIFIED MATHIEU EQUATION

Some simple substitutions in the modified Mathieu equation result in a particularly convenient asymptotic solution for small ellipticity. These substitutions were discovered previously and we present the essential results of the analysis in this appendix.

We begin by seeking a solution of the equation

\[ y'' - (a - q \cosh 2u)y = 0 \]  \hspace{1cm} (D.1)

Putting \( x = -ike^u = -it \) where \( k = \sqrt{\frac{a}{2}} \) it becomes

\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \left( 1 + \frac{a}{2x} + \frac{k^4}{x} \right) y = 0 \]  \hspace{1cm} (D.2)

The second substitution is to put \( y = \omega e^{-x} \) resulting in

\[ \frac{d^2\omega}{dx^2} - \left( 2 - \frac{1}{x} \right) \frac{d\omega}{dx} - \left( \frac{1}{x} + \frac{a}{2x} + \frac{k^4}{x} \right) \omega = 0 \]  \hspace{1cm} (D.3)

A solution to (D.3) is

\[ \omega = \sum_{r=0}^{\infty} (-1)^r c_r e^{-r\frac{x}{2}} \]  \hspace{1cm} (D.4)

with coefficients

\[ c_0 = 1 \]
\[ c_1 = -\left( \frac{4a - 1^2}{8} \right) \]
\[ c_2 = \frac{(4a - 1^2)(4a - 3^2)}{2! 8^2} \]
\[ c_3 = -\frac{(4a - 1^2)(4a - 3^2)(4a - 5^2)}{3! 8^3} - \frac{1}{3!} \frac{k^4}{4!} \]
\[ c_4 = \frac{(4a - 1^2)(4a - 3^2)(4a - 5^2)(4a - 7^2)}{4! 8^4} + \frac{k^4}{2.4!} (4a - 13) \]

...
\[
y = \omega e^{-x} = \frac{e^{i(v + \frac{n}{4})}}{\sqrt{v}} \sum_{r=0}^{\infty} c_r (it)^{-r}
\]
where
\[
X = 1 - c_2 t^{-2} + c_4 t^{-4} - \ldots
\]
\[
Y = c_1 t^{-1} - c_3 t^{-3} + c_5 t^{-5} - \ldots
\]

Now it should be noted that the first terms in the coefficients \(c\) are exactly the terms in Hankel's asymptotic expansion for \(J_m(t)\) where \(m = \sqrt{a}\). Subsequent terms in the series are of order \(k^4\) or higher which results in terms of eccentricity \(\varepsilon^4\) or higher. This gives an even radial Mathieu function

\[
Ce_m(u, \zeta) = p_m J_m(\zeta) + O(\varepsilon^4), \quad \zeta = \sqrt{\frac{e^u}{2}}
\]

Similar results apply to the odd radial solution

\[
Se_m(u, \zeta) = s_m J_m(\zeta) + O(\varepsilon^4)
\]

where \(p_m\), \(s_m\) are coefficients defined in ref. 4 and discussed below.

Differentiation of the functions with respect to \(u\) give the expected results

\[
\frac{d}{du} Ce_m(u, \zeta) = p_m \zeta J_m'(\zeta) + O(\varepsilon^4)
\]

and

\[
\frac{d}{du} Se_m(u, \zeta) = s_m \zeta J_m'(\zeta) + O(\varepsilon^4)
\]

Similar results apply for \(Fey_m(u, \zeta)\) and \(Gey(u, \zeta)\) in terms of the \(Y_m(\zeta)\) Bessel Functions. A pair of functions corresponding to the Hankel
function solutions also exist. They are

\[ \text{Me}_m^{(1),(2)}(u, \zeta) = C_{m}(u, \zeta) \pm i F_{\gamma m}(u, \zeta) \]

\[ \text{Ne}_m^{(1),(2)}(u, \zeta) = S_{m}(u, \zeta) \pm i G_{\gamma m}(u, \zeta) \]  

(D.9)

For the situation \( \zeta < 0 \) we have forms corresponding to the modified Bessel Functions \( I_m(\zeta), K_m(\zeta) \) but the coefficients become more complicated. We only need the results

\[ \pi F_{ek2n}(u, -\zeta) = p_{2n} K_{2n}(\zeta) + O(\epsilon^4) \]

\[ \pi F_{ek2n+1}(u, -\zeta) = s_{2n+1} K_{2n+1}(\zeta) + O(\epsilon^4) \]

\[ \pi G_{ek2n+1}(u, -\zeta) = p_{2n+1} K_{2n+1}(\zeta) + O(\epsilon^4) \]

\[ \pi G_{ek2n+2}(u, -\zeta) = s_{2n+2} K_{2n+2}(\zeta) + O(\epsilon^4) \]

The derivatives follow as before. \( p'_m, s'_m \) are defined in ref. 4 but are of no real consequence here since upon substitution for the Mathieu functions they cancel immediately.
In this appendix an angular quantisation condition derived from Eqn (8.40) is considered for the case of small ellipticity. The phase variation around the inner caustic must be an even number of $2\pi$ radians\(^7\),\(^{11}\) which leads to the quantisation condition\(^7\)

\[
\sqrt{q_1} \int_0^{2\pi} \sqrt{\cosh 2u - \cos 2v} \, dv = 2\pi m
\]  \hspace{1cm} (E.1)

\[
\sqrt{2} \sqrt{q_1} \int_0^{2\pi} \sqrt{\sinh^2 u_1 + \sin^2 v} \, dv = 2\pi m
\]  \hspace{1cm} (E.2)

\[
4 \sqrt{2q_1} \sinh u_1 \, E(ip) = 2\pi m
\]

where $E$ is the complete elliptic integral of the second kind,\(^{12}\)

\[
E(ip) = \frac{\pi}{2} \left[ 1 - \frac{1}{2^2}(ip)^2 - \frac{1^2}{2^2} \cdot 3 \cdot 4 \cdot 5(ip)^6 - \ldots \right]
\]  \hspace{1cm} (E.3)

and

\[
p = \frac{1}{\sinh u_1}
\]

For small ellipticity $\varepsilon$

\[
E(ip) = \frac{\pi}{2} \left( 1 + \frac{\varepsilon^2}{4} \right)
\]  \hspace{1cm} (E.4)

and

\[
\sqrt{2q_1} \sinh u_1 = m \left( 1 - \frac{\varepsilon^2}{4} \right)
\]  \hspace{1cm} (E.5)

This is the modal result for a slightly elliptic guide.

From the ray analysis for a circular\(^{13}\) fibre we have the radius of the inner caustic given by

\[
r_{ic} = \frac{\ell p}{\ell}
\]  \hspace{1cm} (E.6)

\[
\ell = \sqrt{2q_1} \sinh u_1
\]
For the elliptically perturbed guide we can express \( m \) as a perturbation of \( l \) for the circular case

\[
m = l \left( 1 + \frac{e^2}{4} \right)
\]  

(E.7)
REFERENCES


