MULTITYPE GALTON-WATSON PROCESSES

by

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PREFACE

The material in this thesis is the result of my original research, and has not, to the best of my knowledge, been previously published or written by any other person, except where due reference is made in the text. Parts of chapter I, and most of the results of sections II, 1 and V, 3 were given in Quine (1970 a). The rest of chapter V is an extended version of Quine (1970 b). A version of chapters III and IV has been submitted for publication (Quine, 1971).

Parts of this thesis provide generalizations of my earlier work, which as a consequence is not included here. Specifically, theorem IV, 2 generalizes theorem 3 of Fahady, Quine and Vere-Jones (1971), which in turn subsumes theorem A of Quine and Seneta (1969). Theorem B of this latter paper is generalized by the present theorem IV, 2.

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CONTENTS

SUMMARY

This dissertation is concerned with aspects of the behaviour of multitype Galton-Watson processes. In particular, it contains (in chapter II, and section IV, 1) a complete basic asymptotic theory for processes of this kind, with immigration, which are positively regular. "Heavy traffic" theorems (as they are sometimes called) are given for positively regular processes both with (IV, 2) and without (III, 2) immigration. The moment structure is also considered.

Specifically, after an introductory chapter, chapter II analyzes certain positively regular processes with immigration. Section 1 deals with the subcritical case. A necessary and sufficient condition on the immigration component is given for such a process to have a limiting stationary distribution. Section 2 deals with the supercritical case. It is shown that the process behaves largely as it would if there were no immigration effect (this latter behaviour is described in Kesten and Stigum (1966 a)). In both sections, it is shown how earlier single-type results are improved on by the present work. Some pathologies are discussed in the second section.

Chapter III introduces the notion of a class of "paracritical" processes (without immigration), *i.e.* a class of positively regular processes with ρ near unity, which satisfy certain moment conditions. The main results are stated in section 2. Theorem 1 gives results concerning the iterates of the generating functions, showing in particular (III, 2.3) that they can be

uniformly approximated by rational linear functions. Theorem 2 gives expressions for the probability that a process is not extinct by the n-th generation, and its expected value at this time conditional on non-extinction. The principal result (theorem 3) indicates that the limit theorem for a critical process, given by Joffe and Spitzer (1967, theorem 6), continues to hold in an approximate sense for these paracritical processes. The error estimate is shown to be uniform for all processes within the class which have parameter ρ lying within a given distance of unity. The proof of these results is long, and is broken down into a series of lemmas. Of these, the observation (lemma 2) that the class K is compact in a certain metric is crucial to the analysis in this multitype situation. Lemma 7, which concerns the limit of a product of matrices, is of some independent interest.

The first section of chapter IV completes the triad of theorems for positively regular processes with immigration by showing that, if certain natural moment conditions are satisfied, a critical process with immigration, normed by n^{-1}, tends in law to a multivariate gamma distribution which is concentrated on a line, whose direction does not depend on the nature of the immigration component. The second section contains the "paracritical" version of this result.

Chapter V examines a different facet of multitype Galton-Watson processes, namely, their moment structure. After an introductory section, it is shown, in sections 2 and 3, that by using Kronecker (direct) products, linear recurrence relations of a simple form can be derived for the first and second (and more in some cases)

moments of successive generations. With the additional assumption of positive regularity, these are iterated to give limiting expressions, and rates of convergence, for the first and second moments of both sorts of process, in the three cases $\rho \gtreqless 1$.

ACKNOWLEDGEMENTS

My sincere thanks are due to Dr E. Seneta for his careful help and patient guidance during the course of this research. I should also like to thank my other supervisor, Professor P.A.P. Moran, for suggesting branching processes as a research topic and for his continued interest in my work. I am grateful to all those people, too numerous to mention, who, by one means or another, have contributed in some measure to the work in this thesis. Finally, I wish to thank Mrs B. Geary for her excellent work in typing this thesis.

CHAPTER I

INTRODUCTION

1. Historical background

This thesis is concerned with certain Markov processes which generalize a process originally analysed by J. Bienaymé (1845)[†], although commonly referred to as the "Galton-Watson process". The basic results for the original (single-type) process are contained in Harris (1963, ch. 1); however, Kendall (1966) has a fuller account of the history of the early development of the process from 1874 (*i.e.* the time of Watson's analysis). No attempt will be made here to cover the considerable literature which now exists for this single-type process; it is hoped, however, that all relevant works are cited where appropriate.

Less work has been done on the multitype process. According to Harris (1963, ch. 2), the earliest results, concerning the functional iterates of the probability generating function, first and second moments, and extinction probabilities, appeared between 1946 and 1948. Generalizations of Yaglom's important triad of theorems (Yaglom, 1947) (which concern the asymptotic behaviour of the subcritical, critical and supercritical single-type processes) to the multitype case, appeared as follows. In 1957, Jirina gave a partial result for the subcritical multitype Galton-Watson

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[†] A historical study including Bienaymé's knowledge of the fundamental theorem of branching process theory is in preparation by C.C. Heyde, H.O. Lancaster and E. Seneta.

process. The critical theorem was given by Cistyakov (1959) for the continuous time multitype process, by Mullikin (1963) for a discrete time process on a very general state space; and for the multitype Galton-Watson process, under natural conditions, by Joffe and Spitzer (1967). This latter paper, to which we will refer as JS, also generalized Jirina's subcritical result, as well as giving other results for processes with $\rho \leq 1$. At about the same time, Kesten and Stigum (1966a, 1966b, 1967) gave analogues of the supercritical result for the cases when the expectation matrix M is primitive, irreducible, and reducible (we use the term "supercritical" here to indicate that the spectral radius of M is greater than one). The results mentioned earlier generally assumed primitivity of M.

Treatment of processes in near critical condition appears to have been monopolised by soviet mathematicians until fairly recently. The first papers on this subject were published by Sevastyanov (1957a, 1959) and were concerned with single-type branching processes in continuous time. Cistyakov (1961) extended the result to multitype continuous time processes, and then in 1966, Nagaev and Muhammedhanova published a discrete-time version of Sevastyanov's result. The latter paper, and subsequent work, are discussed in Fahady *et al.* (1971); aspects of Cistyakov's paper are discussed in chapter III of this thesis.

Generalization of the single-type Galton-Watson process to allow a stochastic immigration component at each generation was first effected (in special cases) by Smoluchowski (*circa* 1915), and later by Haldane (1949). An account of the former work is

given in Chandrasekhar (1943, ch. III); details of Haldane's model and conclusions, and of subsequent papers on this topic, may be found in Seneta (1969). Further discussion is deferred to chapters II and IV.

2. Notation

Matrices and vectors will be depicted in boldface type; their elements in italics. For example, $\mathsf{M} \equiv \|M_{\alpha\beta}\|$. We list the symbols which occur most frequently:

X : the set of all $k \times 1$ vectors $\mathsf{i} = \{i_1, \ldots, i_k\}'$ whose elements are non-negative integers;

$X^{(r)}$: $\{\mathsf{i} \in X : i_\alpha < r \text{ for } \alpha = 1, \ldots, k\}$;

$CX^{(r)}$: $X \setminus X^{(r)}$;

$\mathsf{0}$: the $k \times 1$ vector $\{0, 0, \ldots, 0\}'$;

$\mathsf{1}$: the $k \times 1$ vector $\{1, 1, \ldots, 1\}'$;

e_α : the $k \times 1$ vector whose α-th element is unity, all others zero;

R^k : k-dimensional Euclidean space;

C : the k-dimensional unit cube, *i.e.*

$$C = \left\{\mathsf{s} \equiv \{s_1, \ldots, s_k\}' \in R^k : 0 \le s_\alpha \le 1, \alpha = 1, \ldots, k\right\} ;$$

X_0 : $X \setminus \{0\}$;

C_0 : $C \setminus \{0\}$;

C_1 : $C \setminus \{1\}$;

s^{i} : $\prod_{\alpha=1}^{k} s_\alpha^{i_\alpha}$, $s \in C$, $\mathrm{i} \in X$;

$|\mathrm{i}|$: $\sum_{\alpha=1}^{k} i_\alpha$, $\mathrm{i} \in X$;

E : the expectation operator.

For two vectors or matrices A, B , we write $A \geq B$ (*resp.* $A > B$) if each element of A is not less than (*resp.* greater than) the corresponding element of B .

3. Definitions and preliminaries

We shall be concerned throughout with two sorts of processes. The first, which we shall refer to as the "ordinary" process, and denote (Y_n) , $n = 0, 1, 2, \ldots$, where $Y_n = \{Y_{n,1}, \ldots, Y_{n,k}\}'$, is a temporally homogeneous Markov process on X . We interpret $Y_{n,\alpha}$ as the number of "particles" of the α-th type at the n-th generation. The transition law of the process is as follows. If $Y_0 = e_\alpha$, for some $\alpha = 1, \ldots, k$, then $Y_1 = \mathrm{i} \in X$ with probability $f_\alpha(\mathrm{i})$. We define probability generating functions (p.g.f.'s):

$$F_{1,\alpha}(s) \equiv E\left[s^{Y_1} \mid Y_0 = e_\alpha\right] = \sum_{i \in X} f_\alpha(i) s^i ,$$

$$s \in C, \ \alpha = 1, \ldots, k ; \quad (3.1)$$

the reason for the first subscript on F will soon become apparent. If $Y_n = i \in X$, then Y_{n+1} is the sum of $|i|$ independent random vectors, of which i_α have p.g.f. $F_{1,\alpha}(s)$, $\alpha = 1, \ldots, k$. If we define the vector

$$F(s) \equiv F_1(s) = \{F_{1,1}(s), \ldots, F_{1,k}(s)\}' , \quad s \in C , \quad (3.2)$$

which we also refer to as a p.g.f, then it is easy to show (Harris, 1963, p. 36) that the p.g.f. of Y_n, conditional on $Y_0 = e_\alpha$, is the n-th functional iterate of $F_1(s)$. Thus, if

$$F_{n,\alpha}(s) \equiv E\left[s^{Y_n} \mid Y_0 = e_\alpha\right] ,$$

then in vector notation, for each $s \in C$,

$$F_n(s) = \{F_{n,1}(s), \ldots, F_{n,k}(s)\}' = \begin{cases} s , & n = 0 \\ F_{n-1}\{F(s)\} , & n = 1, 2, \ldots . \end{cases} \quad (3.3)$$

We also have

$$F_{n+m}(s) = F_n\{F_m(s)\} , \quad s \in C , \quad (3.4)$$

which will be of occasional use.

The $k \times k$ expectation matrix M, whose general element is given by

$$M_{\alpha\beta} \equiv E[Y_{1,\beta} \mid Y_0 = e_\alpha] \equiv \sum_{i \in X} f_\alpha(i) i_\beta = \partial F_{1,\alpha}(1)/\partial s_\beta, \quad (3.5)$$

is a useful means of classifying these processes. In chapters II, III and IV we shall restrict ourselves to consideration of the case where M is primitive (irreducible and aperiodic), *i.e.* where $\infty > (M^N)_{\alpha\beta} > 0$, $1 \le \alpha$, $\beta \le k$, for some integer N . A process with this property is said to be *positively regular*. In this case, the Perron-Frobenius theory (Gantmacher, 1959) assures us of the existence of an eigenvalue, ρ , say, which is positive, and greater than the modulus of any other eigenvalue. We define associated left and right eigenvectors v and u which satisfy

$$v'M = \rho v' \ , \quad Mu = \rho u \ , \quad u'v = u'1 = 1 \ . \quad (3.6)$$

It is known that the value of ρ determines the behaviour of (Y_n) in the same way that the mean, m , determines the behaviour of the single-type process.

The second process we shall consider will be denoted as (Z_n) , $n = 0, 1, 2, \ldots$, where $Z_n \equiv \{Z_{n,1}, \ldots, Z_{n,k}\}'$ is also a temporally homogeneous Markov process on X . The transition law of the process is as follows. If $Z_n = i \in X$, then Z_{n+1} is the sum of $|i| + 1$ independent random vectors, of which i_α have p.g.f. $F_{1,\alpha}(s)$, $\alpha = 1, \ldots, k$, and one has p.g.f. $B(s)$, where

$$B(s) \equiv E\left[s^{Z_1} \mid Z_0 = 0\right] = \sum_{i \in X} b(i) s^i \ , \quad s \in C \ . \quad (3.7)$$

This process, which we shall refer to as the "process with immigration", is usually regarded as consisting of an underlying "offspring process" $\{Y_n\}$, with p.g.f. $F(s)$, augmented at each generation by an independent random immigration component with p.g.f. $B(s)$.

At this point it will be convenient to derive a recurrence relation, analogous to (3.3), for the p.g.f. of Z_n. Given vectors $i, j \in X$, we define transition probabilities

$$p_{ij} \equiv P[Z_n = j \mid Z_{n-1} = i]$$

$$= \sum_{\substack{h \in X \\ h \le j}} f_1^{(i_1)} * f_2^{(i_2)} * \dots * f_k^{(i_k)}(j-h)b(h) ,$$

where $f_\alpha^{(n)}$ is the n-fold convolution of f_α with itself

(*e.g.* $f_\alpha^{(2)}(i) \equiv f_\alpha * f_\alpha(i) \equiv P[Z_1 = i \mid Z_0 = 2e_\alpha]$

$$= \sum_{\substack{h \in X \\ h \le i}} f_\alpha(i-h) f_\alpha(h)) .$$

It follows that the p.g.f. of the n-th generation of Z_n, given that $Z_0 = i \in X$, satisfies for each $s \in C$

$$P_n(\mathrm{i},\ \mathrm{s}) \equiv \sum_{\mathrm{j}\in X} \mathrm{s}^{\mathrm{j}} P[Z_n = \mathrm{j} \mid Z_0 = \mathrm{i}]$$

$$= \sum_{\mathrm{j}\in X} \sum_{\mathrm{h}\in X} \mathrm{s}^{\mathrm{j}} p_{\mathrm{hj}} P[Z_{n-1} = \mathrm{h} \mid Z_0 = \mathrm{i}]$$

$$= \sum_{\mathrm{h}\in X} \{\mathrm{F}(\mathrm{s})\}^{\mathrm{h}} B(\mathrm{s}) P[Z_{n-1} = \mathrm{h} \mid Z_0 = \mathrm{i}]$$

$$= \begin{cases} B(\mathrm{s})\{\mathrm{F}(\mathrm{s})\}^{\mathrm{i}} & n = 1 \\ \\ B(\mathrm{s})P_{n-1}\{\mathrm{i},\ \mathrm{F}(\mathrm{s})\} & n = 2,\ 3,\ \ldots\ . \end{cases} \tag{3.8}$$

This recurrence relation was derived in the single-type case (*i.e.* $k = 1$) , subject to the condition $i = 1$, by Heathcote (1965). Iteration of (3.8) yields

$$P_n(\mathrm{i},\ \mathrm{s}) = \{\mathrm{F}_n(\mathrm{s})\}^{\mathrm{i}} \prod_{r=0}^{n-1} B\{\mathrm{F}_r(\mathrm{s})\}\ . \tag{3.9}$$

At present, it will suffice to define only the mean immigration vector, *viz.*

$$\lambda = E[Z_1 \mid Z_0 = 0]\ , \tag{3.10}$$

where

$$\lambda_\alpha = \frac{\partial B(1)}{\partial s_\alpha}\ .$$

CHAPTER II

THE NON-CRITICAL PROCESS WITH IMMIGRATION

Introduction

This chapter provides generalizations to the multitype case of results given by Heathcote (1966), and Seneta (1970a), concerning respectively the sub- and super-critical single-type Galton-Watson process with immigration. Both of the present theorems, when considered as results about the single-type process, slightly improve the original versions.

Throughout this chapter, we assume

A: $$\sum_{\alpha,\beta} M_{\alpha\beta} < \infty \; ; \quad M \text{ is primitive.}$$

1. The case $\rho < 1$

In this subcritical situation, it is known that the process without immigration, (Y_n), reaches the absorbing state $\{0\}$ almost surely (Harris, 1963, p. 41). However, because of the persistent immigration, the process (Z_n) behaves differently, as the following theorem indicates.

THEOREM 1. *If* M *satisfies conditions* A, *and if in addition* $\rho < 1$ *and* $B(0) < 1$, *then a necessary and sufficient condition for the process* (Z_n) *to satisfy for each* $j \in X$

$$\lim_{n\to\infty} P[Z_n = j \mid Z_0 = i] = \pi(j) ,$$

independently of $i \in X$, *where* $\sum_{j\in X} \pi(j) = 1$, *is*

$$\sum_{j\in X_0} b(j) \log |j| < \infty .$$

The proof of the theorem follows two preliminary lemmas.

LEMMA 1. *For any* $s \in C$,

$$1 - M^n.1 \le F_n(s) \le 1 - H^n(1-s) , \tag{1.1}$$

and the matrix H *is irreducible and aperiodic.*

Proof. We use an expansion of $F(s)$ which occurs in JS (p. 415), *viz.*

$$1 - F(s) = \{M-E(s)\}(1-s) , \quad s \in C . \tag{1.2}$$

This expansion, together with various extensions, will be used again in chapter III. The matrix $E(s)$ has in particular the following properties:

$$0 \le E(s) \le M , \tag{1.3}$$

$$t \le s \Rightarrow E(t) \ge E(s) , \tag{1.4}$$

$$E_{\alpha\beta}(0) = M_{\alpha\beta} - \sum_{i\in X_0} f_\alpha(i) i_\beta . \frac{1}{|i|} . \tag{1.5}$$

Using the functional relation $F_n(s) = F\{F_{n-1}(s)\}$, (see I, 3.4),

we obtain from (1.2) for $n \geq 1$

$$1 - F_n(s) = \prod_{r=1}^{n} \left\{M-E\left(F_{n-r}(s)\right)\right\}(1-s) . \tag{1.6}$$

If we define $H = M - E(0)$ then (1.3), (1.4) and (1.6) combine to give (1.1). Furthermore, we note from (1.5) that

$$H_{\alpha\beta} = \sum_{i \in X_0} f_\alpha(i) i_\beta . \frac{1}{|i|} ,$$

so that for each pair of indices (α, β) , $M_{\alpha\beta} > 0 \Rightarrow H_{\alpha\beta} > 0$. We note further that $H \leq M$ so that by a corollary of the Perron-Frobenius theorem (Debreu & Herstein, 1953, p. 598) the spectral radius θ of H satisfies $0 < \theta \leq \rho < 1$.

LEMMA 2. *For a non-negative primitive* $k \times k$ *matrix* D *with spectral radius* $\delta < 1$, *and for a given* $s \in [0, 1)^k$ *and any* $i \in X_0$,

$$(i) \quad \sum_{n=1}^{\infty} \left\{1-\left[1-D^n(1-s)\right]^i\right\} \geq a_1(s) + (-\log \delta)^{-1}\log|i| .$$

Furthermore, there exists an integer n_0 *such that for any* $i \in X_0$,

$$(ii) \quad \sum_{n=n_0}^{\infty} \left\{1-\left[1-D^n.1\right]^i\right\} \leq a_2 + (-\log \delta)^{-1}\log|i| ,$$

and $a_1(s)$, a_2 *are independent of* i .

Proof. We use the well known result (Gantmacher, 1959) that there is associated with δ a right eigenvector $\mathbf{y} > \mathbf{0}$ which is unique up to a constant multiplier.

(i) For a given $\mathbf{s} \in [0, 1)^k$, set $\max_\nu y_\nu = \min_\nu (1-s_\nu)$.

Then

$$0 < y_\nu \le 1 - s_\nu , \quad \nu = 1, \ldots, k$$
$$\le 1 .$$

Hence $\mathbf{e}_\nu' \mathbf{D}^n(\mathbf{1}-\mathbf{s}) \ge \mathbf{e}_\nu' \mathbf{D}^n \mathbf{y} = y_\nu \delta^n \ge \delta^n \alpha_1$, where $0 < \alpha_1 = \min_\nu y_\nu \le 1$.

It follows that for $\mathbf{i} \in X$,

$$[\mathbf{1}-\mathbf{D}^n(\mathbf{1}-\mathbf{s})]^{\mathbf{i}} \le \left(1-\alpha_1\delta^n\right)^{|\mathbf{i}|} ,$$

so that

$$\sum_{n=1}^{\infty} \left\{1-[\mathbf{1}-\mathbf{D}^n(\mathbf{1}-\mathbf{s})]^{\mathbf{i}}\right\} \ge \sum_{n=1}^{\infty} \left\{1-\left(1-\alpha_1\delta^n\right)^{|\mathbf{i}|}\right\}$$
$$= S_{\mathbf{i}} \text{ , say.}$$

We now use the same integral test that Heathcote (1966, p. 215) used in the single-type case, *viz.*

$$S_{\mathsf{i}} \geq \int_1^\infty \left\{1-\left(1-\alpha_1\delta^x\right)^{|\mathsf{i}|}\right\}dx$$

$$= (-\log\,\delta)^{-1}\int_{1-\alpha_1\delta}^{1} \frac{\left(1-z^{|\mathsf{i}|}\right)}{1-z}\,dz$$

$$= (-\log\,\delta)^{-1}\sum_{r=1}^{|\mathsf{i}|}\left\{r^{-1}-\left(1-\alpha_1\delta\right)^r r^{-1}\right\} .$$

Thus

$$S_{\mathsf{i}} \geq (-\log\,\delta)^{-1}\left\{\sum_{r=1}^{|\mathsf{i}|} r^{-1} - \frac{\left(1-\alpha_1\delta\right)}{\alpha_1\delta}\left(1-\left[1-\alpha_1\delta\right]^{|\mathsf{i}|}\right)\right\}$$

$$\geq (-\log\,\delta)^{-1}\log|\mathsf{i}| + a_1(\mathsf{s}) ,$$

where $a_1(\mathsf{s})$ depends on s (because α_1 does) but does not depend on i . Note that for a given $\mathsf{s} \in [0, 1)^k$, $-\infty < a_1(\mathsf{s}) < 0$.

(ii) Set $\min_\nu y_\nu = 1$. Then

$$D^n.1 \leq D^n.y = \delta^n y .$$

Hence

$$e_\nu' D^n.1 \leq \delta^n\alpha_2 , \quad \nu = 1, \ldots, k ,$$

where $\alpha_2 \geq 1$. Since $\delta < 1$, there exists an integer n_0 such that $\alpha_2\delta^n \leq 1$ for $n \geq n_0$. Put $\alpha_3 = \alpha_2\delta^{n_0-1}$. For any $\mathsf{i} \in X$,

$$(1-D^n.1)^{\mathsf{i}} \geq \left(1-\alpha_2\delta^n\right)^{|\mathsf{i}|} \qquad n \geq n_0 ,$$

and hence, using the same test as in the first part,

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \left\{1-(1-D^n.1)^{\mathsf{i}}\right\} &\leq \sum_{n=n_0}^{\infty} \left\{1-\left(1-\alpha_2\delta^n\right)^{|\mathsf{i}|}\right\} \\
&= \sum_{n=1}^{\infty} \left\{1-\left(1-\alpha_3\delta^n\right)^{|\mathsf{i}|}\right\} \\
&\leq 1 - (1-\alpha_3\delta)^{|\mathsf{i}|} \\
&\qquad + (-\log\delta)^{-1} \sum_{r=1}^{|\mathsf{i}|} \left\{r^{-1}-r^{-1}(1-\alpha_3\delta)^r\right\} \\
&\leq 1 - (1-\alpha_3\delta)^{|\mathsf{i}|} + (-\log\delta)^{-1} \sum_{r=1}^{|\mathsf{i}|} r^{-1} \\
&\leq a_2 + (-\log\delta)^{-1} \log|\mathsf{i}| .
\end{aligned}$$

Proof of the theorem. Since $F_n(s) \to 1$ uniformly on C as $n \to \infty$ (JS, p. 412), we may choose an integer r_0 such that $F_{r_0}(s) > 0$ for every $s \in C$. From (I, 3.8) it follows that for each $s \in C$ and each $n > r_0$,

$$P_n(\mathsf{i}, s) = \left\{F_n(s)\right\}^{\mathsf{i}} \left(\prod_{r=0}^{r_0-1} B\left\{F_r(s)\right\}\right)\left(\prod_{r=r_0}^{n-1} B\left\{F_r(s)\right\}\right)$$

which leads us to examine the asymptotic behaviour of the product $\prod_{r=r_0}^{n-1} B\left\{F_r(s)\right\} (> 0)$ as $n \to \infty$ with a view to determining the

asymptotic properties of $P_n(\mathsf{i}, \mathsf{s})$. This product converges to a positive limit for any $\mathsf{s} \in [0, 1)^k$ if and only if

$$\sum_{r=1}^{\infty} \left\{1-B\left[\mathsf{F}_r(\mathsf{s})\right]\right\} < \infty \tag{1.7}$$

(for notational purposes it is convenient to sum from $r = 1$) . This sum can be expressed as

$$\begin{aligned}\sum_{r=1}^{\infty} \left\{1-B\left[\mathsf{F}_r(\mathsf{s})\right]\right\} &= \sum_{r=1}^{\infty} \sum_{\mathsf{i}\in X} b(\mathsf{i})\left\{1-\left[\mathsf{F}_r(\mathsf{s})\right]^{\mathsf{i}}\right\} \\ &= \sum_{\mathsf{i}\in X} b(\mathsf{i}) \sum_{r=1}^{\infty} \left\{1-\left[\mathsf{F}_r(\mathsf{s})\right]^{\mathsf{i}}\right\} \end{aligned} \tag{1.8}$$

by Fubini's theorem. Identifying H with D and using (1.4), (1.8) and part *(i)* of lemma 2, we obtain for a given $\mathsf{s} \in [0, 1)^k$ the lower bound

$$\begin{aligned}\sum_{r=1}^{\infty} \left\{1-B\left[\mathsf{F}_r(\mathsf{s})\right]\right\} &\geq \sum_{\mathsf{i}\in X} b(\mathsf{i}) \sum_{r=1}^{\infty} \left\{1-\left[1-\mathsf{H}^r(1-\mathsf{s})\right]^{\mathsf{i}}\right\} \\ &\geq \sum_{\mathsf{i}\in X_0} b(\mathsf{i})\left\{a_1(\mathsf{s})+(-\log\,\theta)^{-1}\,\log|\mathsf{i}|\right\} . \end{aligned} \tag{1.9}$$

To obtain the corresponding upper bound we identify M with D , decompose the sum over r in (1.8), and use (1.7) and part *(ii)* of lemma 2 to obtain

$$\sum_{r=1}^{\infty} \left\{1-B\left[F_r(s)\right]\right\} \geq n_0 - 1 + \sum_{i\in X} b(i) \sum_{r=n_0}^{\infty} \left\{1-\left[F_r(s)\right]^i\right\}$$

$$\geq n_0 - 1 + \sum_{i\in X_0} b(i)\left\{a_2+(-\log \rho)^{-1} \log|i|\right\} . \quad (1.10)$$

We can combine (1.7), (1.9) and (1.10) to show that for any

$s \in [0, 1)^k$, $\prod_{r=r_0}^{\infty} B\left[F_r(s)\right] > 0$ if and only if

$$\sum_{i\in X_0} b(i) \log|i| < \infty . \quad (1.11)$$

It follows that if condition (1.11) is not satisfied,

$\lim_{n\to\infty} P_n(i, s) = 0$ for each $s \in [0, 1)^k$, and any $i \in X$, so that

the "limiting distribution" is one degenerate at infinity.

On the other hand, suppose (1.11) holds. Then, defining

$P(s)$ by

$$P(s) = \lim_{n\to\infty} P_n(i, s) , \quad s \in C ,$$

we see from (I, ~~2~~3.8) that $P(s)$ is independent of i , and

certainly positive at least for $s \in C$, $s > 0$, and from

(I, ~~2~~3.7) satisfies

$$P(s) = B(s)P[F(s)] , \quad s \in C . \quad (1.12)$$

In this case it remains only to show that the limiting distribution

is proper, which, by the continuity theorem, will be the case if

$P(s) \to 1$ as $s \to 1$, $s \in [0, 1)^k$. We proceed as follows.

Letting $n \to \infty$ in (I, 3.~~8~~9), we obtain

$$P(s) = \prod_{r=0}^{\infty} B[F_r(s)] \ , \quad s \in C \ . \tag{1.13}$$

Further, (1.12) may be written as

$$P(s) = P[F_n(s)] \prod_{r=0}^{n-1} B[F_r(s)] \ , \quad s \in C \ . \tag{1.14}$$

Now, select a fixed $s_0 > 0$, $s_0 \in C$. Then from (1.13) and (1.14) it follows that

$$0 < P(s_0) = \lim_{n\to\infty} P[F_n(s_0)]P(s_0)$$

so that $\lim_{n\to\infty} P[F_n(s_0)] = 1$. Now, consider a sequence $\{s(n)\}$ such that $s(n) \in [0, 1)^k$ and $s(n) \to 1$ as $n \to \infty$. Then in the manner of JS (p. 422) we may select a subsequence $k(n)$ of the positive integers (not necessarily strictly increasing) such that $F_{k(n)}(s_0) \le s(n)$ and $k(n) \to \infty$. It is obvious from (1.13) that $P(s)$ is monotonic, hence

$$P[F_{k(n)}(s_0)] \le P[s(n)] \le 1$$

so that, as $n \to \infty$,

$$P[s(n)] \to 1 \ .$$

This implies that $P(s) \to 1$ as $s \to 1$ within $[0, 1)^k$, otherwise a contradiction would result. This completes the proof of the theorem.

It should be noted that this theorem does not involve any assumptions about the irreducibility or aperiodicity of the Markov process $\{Z_n\}$ (as opposed to the expectation matrix M). That these may be avoided in the single-type case was pointed out by Seneta (1969, ch. 5). Furthermore, it is evident that in that case, *i.e.* $k = 1$, the assumptions that M is primitive and that $\rho < 1$ are equivalent to $0 < m < 1$, where m is the offspring mean. We permit $B(0) = 0$, in which case some individuals are present at every generation; this case has been excluded in previous single-type discussions with $m < 1$.

2. The case $\rho > 1$

We point out first that, under conditions A on M, it follows from (1.7) and the preceding remarks, and from the fact that, for $s \in C_1$, $F_n(s) \to q < 1$ as $n \to \infty$ (Harris, 1963, p. 41), that in this supercritical case, as $n \to \infty$,

$$P_n(i, s) \to 0 , \quad s \in C_1 ,$$

i.e. the distribution is degenerate at infinity (as in the subcritical case when the logarithmic moment of the immigration component is infinite). However, theorem 2 below gives an idea of the rate of explosion. The proof of the theorem utilises an idea of Sevastyanov (1957b), by considering the k-type process with immigration, $\{Z_n\}$, as a k+1-type process without immigration, $\{Y_n^*\}$, say, with p.g.f.

$$G^*(s, t) \equiv \{G_1^*(s, t), \ldots, G_{k+1}^*(s, t)\}' , \quad s \in C , \quad t \in [0, 1] ,$$

where

$$G_\alpha^*(s, t) = \begin{cases} F_{1,\alpha}(s) , & \alpha = 1, \ldots, k , \\ tB(s) , & \alpha = k + 1 . \end{cases}$$

If $Y_0^{*\prime} = \{e_\alpha, 0\}$ for some $\alpha = 1, \ldots, k$, then it is not difficult to see that $Y_{n,k+1}^* = 0$, $n = 0, 1, 2, \ldots$, while the reduced process $(Y_{n,1}^*, \ldots, Y_{n,k}^*)$, $n = 0, 1, 2, \ldots$, is identical to the ordinary k-type Galton-Watson process (Y_n) with p.g.f. $F(s)$ and $Y_0 = e_\alpha$. However, if $Y_0^{*\prime} = \{0, 1\}$, then $Y_{n,k+1}^* = 1$, $n = 0, 1, 2, \ldots$, while the reduced process $(Y_{n,1}^*, \ldots, Y_{n,k}^*)$, $n = 0, 1, 2, \ldots$, is identical to the k-type Galton-Watson process with immigration, (Z_n) , with $Z_0 = 0$, and with p.g.f.'s $F(s)$ and $B(s)$.

The expectation matrix of (Y_n^*) is given by

$$M^* = \begin{bmatrix} M & 0 \\ \lambda' & 1 \end{bmatrix} ,$$

which is reducible. The eigenvalues of M^* consist of those of M , and an additional value of unity. Let v^* and u^* be $(k+1) \times 1$ vectors satisfying

$$v^{*\prime}M^* = \rho v^* , \quad M^*u^* = \rho u^* , \quad v^{*\prime}u^* = [1', 1]u^* = 1 .$$

Then it is easily seen (*cf*. Kesten and Stigum, 1967, eq. 2.12) that

$$v^{*\prime} = \{v', 0\} , \quad u^{*\prime} = \left\{u', \frac{\lambda' u}{\rho-1}\right\} . \tag{2.1}$$

If M is primitive, and $\rho > 1$, then M^* satisfies the conditions of theorem 2.1 of Kesten and Stigum (1967), which can be stated in our terminology as follows:

Assume that $Y_0^{*\prime} = \{0, 1\}$. If in addition to assumptions A on M we have $0 < |\lambda| < \infty$, then there exists a random variable ϕ such that

$$\lim_{n\to\infty} \frac{Y_n^*}{\rho^n} = \phi v^* \quad \text{almost surely.}$$

In addition, either

$$E(\phi) = \frac{\lambda' u}{\rho-1} \quad (> 0) \tag{2.2}$$

or

$$\phi = 0 \quad \text{almost surely.}$$

Furthermore, (2.2) holds if and only if

$$E\left[Y_{1,\beta}^* \log Y_{1,\beta}^* \mid Y_0^{*\prime} = \{e_\alpha, 0\}\right] < \infty , \quad 1 \le \alpha, \beta \le k . \tag{2.3}$$

Finally, if (2.3) holds, and if there is at least one α , $\alpha = 1, \ldots, k$, such that $\sum_{\beta=1}^{k} Y_{1,\beta}^* u_\beta$ can take at least two

values with positive probability, given $Y_0^{*\prime} = \{e_\alpha, 0\}$, then the distribution of ϕ is continuous, and has a continuous density on $(0, \infty)$. With the aid of this result we can prove the following theorem:

THEOREM 2. *If conditions* A *on* M *are satisfied, if* $0 < |\lambda| < \infty$, *and if* $Z_0 = i \in X$, *then there exists a scalar random variable* ξ *such that*

$$\lim_{n\to\infty} \left(\frac{Z_n}{\rho^n}\right) = \xi v \quad \textit{almost surely.}$$

In addition, either

$$E[\xi] = \frac{\lambda' u}{\rho - 1} + i' u \ , \tag{2.4}$$

or

$$\xi = 0 \quad \textit{almost surely.} \tag{2.5}$$

Furthermore, (2.4) *obtains if and only if*

$$\sum_{i \in X} f_\alpha(i) i_\beta \log i_\beta < \infty \ , \quad 1 \le \alpha, \beta \le k \ . \tag{2.6}$$

Finally, if (2.6) *holds, and if for some* α , $1 \le \alpha \le k$,

$$\sum_{i \in X, i' u = \theta} f_\alpha(i) > 0 \quad \textit{for at least two values of} \ \theta \ , \tag{2.7}$$

then the distribution of ξ *is continuous, and has a continuous density on* $(0, \infty)$.

Proof. We decompose the process (Z_n) with $Z_0 = i \in X$ into the sum of i_1 independent processes $\left\{Y_n^{(1)}\right\}, \ldots, \left\{Y_n^{(i_1)}\right\}$ with $Y_0^{(\alpha)} = e_1$, $\alpha = 1, \ldots, i_1$; i_2 independent processes $\left\{Y_n^{(i_1+1)}\right\}, \ldots, \left\{Y_n^{(i_2)}\right\}$ with $Y_0^{(\alpha)} = e_2$, $\alpha = i_1+1, \ldots, i_2$; and so on up to i_k; and one process $\left\{Z_n^{(1)}\right\}$ with $Z_0^{(1)} = 0$. Each of the first $|i|$ processes satisfies the conditions of the theorem of Kesten and Stigum (1966a), *i.e.* for $\nu = 1, \ldots, |i|$,

$$\frac{Y_n^{(\alpha)}}{\rho^n} \to \eta_\alpha v \quad \text{almost surely}$$

and $E(\eta_\nu) = u_\nu$ or $\eta_\nu = 0$ almost surely, depending on whether or not (2.6) holds. If (2.7) holds for some α, then the distribution of η_ν has a jump at the origin, and a continuous density on $(0, \infty)$. From the comments about the augmented process (Y_n^*), it follows that the process $\left\{Z_n^{(1)}\right\}$ satisfies

$$\frac{Z_n^{(1)}}{\rho^n} \to \phi v \quad \text{almost surely,}$$

and ϕ has the properties previously mentioned. It should be observed that (2.3) is equivalent to (2.6).

The relations between (2.4), (2.5) and (2.6) can now be deduced from the above remarks and the fact that

$$\xi = \sum_{\nu=1}^{|i|} \eta_\nu + \phi \quad \text{almost surely.}$$

To prove the assertions about the distribution function of ξ, we denote the distribution function of η_ν by $G_\nu(x)$, $1 \le \nu \le |i|$; that of ϕ by $H(x)$, and that of ξ by $I(x)$. Then $I(x)$ can be written as the convolution

$$I(x) = H * G_1 * G_2 * \dots * G_{|i|}(x) .$$

We know that $H(x)$ has those properties that we wish to exhibit in $I(x)$. If we can show that $H * G_1(x)$ inherits the properties of $H(x)$, then the result follows by iteration, since $G_2, \dots, G_{|i|}$ have the same properties as G_1. We do this in the following lemma.

LEMMA 3. *Let* $H(x)$, $G(x)$ *be distribution functions such that*

$$H(x) = \begin{cases} \int_0^x h(u)du & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$G(x) = \begin{cases} \int_0^x g(u)du & x > 0 \\ \gamma \quad (> 0) & x = 0 \\ 0 & x < 0 , \end{cases}$$

where $h(u)$ *and* $g(u)$ *are non-negative and continuous on* $(0, \infty)$. *Then*

$$F(x) \equiv H * G(x)$$

$$= \begin{cases} \int_0^x f(u)du & x > 0 \\ 0 & x \leq 0 \end{cases},$$

and $f(u)$ *is non-negative and continuous on* $(0, \infty)$.

Proof. By definition,

$$F(x) = \int_{-\infty}^{\infty} G(x-u)dH(u) = \begin{cases} \int_0^x G(x-u)dH(u) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$= \begin{cases} \int_0^x G(x-u)h(u)du & x > 0 \\ 0 & x \leq 0 \end{cases}$$

by Natanson (1955, theorem 2, p. 231). Now, define

$$F^*(x) = \begin{cases} \int_0^x f(u)du & x > 0 \\ 0 & x \leq 0 \end{cases},$$

where, for $u > 0$, $f(u)$ is the non-negative continuous function defined by

$$f(u) = \int_0^u g(u-y)h(y)dy \ .$$

Then for $x > 0$,

$$F^*(x) = \int_0^x \int_y^x g(u-y)h(u)\,du\,dy$$

$$= \int_0^x h(y) \int_0^{x-y} g(z)\,dz\,dy$$

$$= \int_0^x h(y)G(x-y)\,dy$$

$$= F(x) \ ,$$

and the equality is obvious for $x \leq 0$. This concludes the proof of the lemma.

REMARKS. It should be pointed out that theorem 2 is little more than a specialization of theorem 2.1 of Kesten and Stigum (1967), and is included here mainly for the sake of completeness. An incomplete version of theorem 2 is contained in Mode (1971). Kesten and Stigum's theorem can also be used to obtain results in more general situations - for instance, Seneta (1970d) considers a model where one of a number of immigration distributions may be selected at each generation. In the same vein, Cistyakov (1970) has dealt with a model in continuous time similar to Seneta's, but with only one type of particle.

Let us now consider the implications of theorem 2 in the single-type case. The basic result in that situation is given in Seneta (1970a). The present theorem shows that, under Seneta's conditions, the distribution of the limiting random variable has a continuous density on $(0, \infty)$. The present approach also raises the question: what happens if (2.7) fails to hold? In the single-type case, this involves investigating the process with a

deterministic offspring mechanism $\{i.e.\ F(s) = s^m$, for some $m > 1\}$. Examination of the methods of Seneta (1970a) leads to the following partial conclusions (we assume for simplicity of exposition that $Z_0 = 0$).

(i) If $B(s) = s^\lambda$, then the limiting distribution function of Z_n/m^n is concentrated at $\lambda/(m-1)$.

(ii) If $B(s) \neq s^\lambda$, then the Laplace transform of the limiting distribution is given by

$$\prod_{\alpha=1}^{\infty} B\left[e^{-t/m^\alpha}\right] ,$$

and, from the comments in Seneta (1970a), it follows that the limiting distribution function is continuous.

The analysis of a multitype process with immigration which fails to satisfy (2.7) for $\alpha = 1, \ldots, k$ would appear to be more difficult. We remark that such a process need not have a deterministic offspring mechanism, as the following simple two-type example shows: let

$$F_{1,1}(s) = \frac{1}{4} s^{i^{(1)}} + \frac{3}{4} s^{i^{(2)}} ,$$

$$F_{1,2}(s) = \frac{1}{2} s^{j^{(1)}} + \frac{1}{2} s^{j^{(2)}} ,$$

where

$$i^{(1)} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}, \quad i^{(2)} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}; \quad j^{(1)} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \quad j^{(2)} = \begin{bmatrix} 14 \\ 3 \end{bmatrix}.$$

Then

$$M = \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix}, \quad \rho = 10, \quad u = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix},$$

and

$$i^{(1)'}u = i^{(2)'}u = \rho u_1 = 10/3,$$

$$j^{(1)'}u = j^{(2)'}u = \rho u_2 = 20/3,$$

so that (2.7) fails to hold for both $\alpha = 1$ and $\alpha = 2$. At the same time, however, the following observation applies regardless of the number of types: suppose we have an ordinary multitype process (Y_n), with $Y_0 = i \in X_0$, and with transition probabilities which satisfy (2.6). It is a simple consequence of the theorem of Kesten and Stigum (1966a) (see also the proof of theorem 2) that there exists a random variable η such that

$$\frac{Y_n}{\rho^n} \to \eta v \quad \text{almost surely;}$$

and in addition, that (2.7) *holds for some* α *if and only if the distribution of* η *is not concentrated at one point.* Further deductions concerning the nature of condition (2.7) can be made in the light of comments in Kesten and Stigum (1966a), and it seems likely that results similar to the single-type ones outlined above obtain for the multitype process with immigration in the case of

failure of (2.7) for all α .

Finally, we remark that Seneta (1970b) has shown that if, in the single-type case, (2.6) and (2.7) hold, then a necessary and sufficient condition for Z_n/m^n (where m is the offspring mean) to converge almost surely to a random variable with a continuous distribution function is $\sum_{i=1}^{\infty} b(i) \log i < \infty$; if this sum diverges, then Z_n/m^n diverges almost surely to infinity. We note that, according to theorem 1, this condition is also necessary and sufficient for ergodicity in the subcritical single-type case, as long as (2.7) holds and $0 < \lambda \leq \infty$. No multitype analogue of Seneta's result has so far been given; it would seem that a different approach to the present one would be required.

CHAPTER III

PARACRITICAL PROCESSES WITHOUT IMMIGRATION

Introduction

The principal result of this chapter (theorem 3) generalizes to the multitype process a result of Nagaev and Muhammedhanova (1966). Their result is correct, although the proof contains some errors; this prompted the paper by Fahady *et al.* (1971), where an alternative proof is provided. Although the third theorem of this chapter is proved along the lines of the last named paper, it was found necessary to probe more deeply in certain respects (in particular, into the properties of the class $K(a, b, c, U)$).

The results of this chapter indicate that the limit theorem of Joffe and Spitzer (1967, theorem 6), for a positively regular multitype Galton-Watson process with $\rho = 1$, continues to obtain, in an approximate sense, for a class of such processes with ρ close to one; and that the same is true of other of their results. We remark that, even when considered as a result for single-type processes, theorem 3 generalizes theorem B of Quine and Seneta (1969). The term "paracritical" has been coined here because it is felt that previous descriptions of the phenomenon (*e.g.* "transient behaviour", "heavy traffic") were not entirely satisfactory.

Cistyakov (1961) obtained results analogous to some of those given here, for multitype branching processes in continuous time. There are, however, certain flaws in his article; in particular,

the proof of the boundedness of the elements of the matrix Π^{-1} (pages 30-31) is incorrect, and the appeal to lemmas 1 and 2 of Sevastyanov (1959) does not appear to be justified. It should be pointed out, however, that the author is aware of these discrepancies, and has submitted a paper containing corrections, and results similar to those given in this chapter, to Teoriya Veroiatnostei†.

1. Definitions

Recalling the definition of the offspring p.g.f. $F(s)$ (see I, 3.1 and I, 3.2), we denote by K the class of all such p.g.f.'s which are *proper*, *i.e.* $\sum_{i \in X} f_\alpha(i) = 1$ for $\alpha = 1, \ldots, k$. We shall make use of the following moments of a process $\{Y_n\}$ with p.g.f. $F \in K$:

$$M \equiv M(F) \equiv \|M_{\alpha\beta}\| : M_{\alpha\beta} \equiv M_{\alpha\beta}(F) = \partial F_{1,\alpha}(1)/\partial s_\beta ;$$

$$b^{(\alpha)}_{\beta\gamma} \equiv b^{(\alpha)}_{\beta\gamma}(F) = \partial^2 F_{1,\alpha}(1)/\partial s_\beta \partial s_\gamma ;$$

$$c^{(\alpha)}_{\beta\gamma\delta} \equiv c^{(\alpha)}_{\beta\gamma\delta}(F) = \partial^3 F_{1,\alpha}(1)/\partial s_\beta \partial s_\gamma \partial s_\delta ,$$

for $1 \le \alpha, \beta, \gamma, \delta \le k$.

Given a positive integer U, and constants $a > 0$, $b > 0$, $c < \infty$, let $K \equiv K(a, b, c, U) \subset K$ be the class of all those

† V.P. Cistyakov (personal communication, 1971).

p.g.f.'s $F \in K$ which satisfy

$$\left.\begin{array}{ll} \text{(i)} & \{M^U(F)\}_{\alpha\beta} \geq a \ , \quad 1 \leq \alpha, \beta \leq k \ ; \\ \text{(ii)} & \sum\limits_{\alpha\beta\gamma} b^{(\alpha)}_{\beta\gamma}(F) \geq b \ ; \\ \text{(iii)} & \sum\limits_{\alpha,\beta,\gamma,\delta} c^{(\alpha)}_{\beta\gamma\delta}(F) \leq c \ . \end{array}\right\} \tag{1.1}$$

The first condition implies that $M(F)$ is primitive, in a uniform sense, for all $F \in K$. Condition (ii) implies in particular that any process with p.g.f. $F \in K$ and spectral radius $\rho_F = 1$ (ρ_F will be defined below) will be "non-singular" in the sense of Harris (1963, p. 39).

Condition (1.1, i) on $M(F)$ implies that for each $F \in K$, $M(F)$ has a positive eigenvalue ($\rho \equiv \rho_F$, say) which is greater than the modulus of any other eigenvalue (this has been pointed out already in section I,3). Let $v = v(F)$ and $u = u(F)$ be vectors satisfying

$$\left.\begin{array}{l} v'(F)M(F) = \rho_F v'(F) \ ; \\ M(F)u(F) = \rho_F u(F) \ ; \\ u'(F)v(F) = u'(F)1 = 1 \ . \end{array}\right\} \tag{1.2}$$

For each $F \in K$, $s \in C$ and $\alpha = 1, \ldots, k$, we define the quadratic form

$$q_\alpha[\mathsf{s}] = \frac{1}{2} \sum_{\nu,\mu} b_{\nu\mu}^{(\alpha)}(\mathsf{F}) s_\nu s_\mu \ , \tag{1.3}$$

and the weighted sum

$$\left.\begin{aligned} Q[\mathsf{s}] &= \sum_\alpha v_\alpha(\mathsf{F}) q_\alpha[\mathsf{s}] \ , \\ &\text{and in particular,} \\ Q \equiv Q_{\mathsf{F}} &= Q[\mathsf{u}(\mathsf{F})] \ . \end{aligned}\right\} \tag{1.4}$$

We put

$$\pi_n \equiv \pi_n(\rho) = \left\{\begin{array}{ll} \sum_{r=1}^{n} \rho^{r-2} \ , & n = 1, 2, \ldots \\ 0 \ , & n = 0 \ , \end{array}\right\} \tag{1.5}$$

and

$$h_n(\mathsf{s}) = \frac{\rho^n \mathsf{v}'\mathsf{s}}{1+\pi_n Q \mathsf{v}'\mathsf{s}} \ , \quad \mathsf{s} \in C \ . \tag{1.6}$$

We shall use the symbol $o(n, \rho; \mathsf{F}, \mathsf{s})$ to denote a quantity which behaves in the following way: given $\varepsilon > 0$, there is a $\delta > 0$ and a positive integer N such that $|o(n, \rho; \mathsf{F}, \mathsf{s})| < \varepsilon$ for $n > N$, all those $\mathsf{F} \in K$ which satisfy $|1-\rho_{\mathsf{F}}| < \delta$, and for all s in some specified subset of C . The symbol $\mathsf{o}(n, \rho; \mathsf{F}, \mathsf{s})$ will be used to denote a $k \times 1$ vector of such quantities. The argument ρ is not strictly necessary, since it is a function of F , but is included to emphasize the crucial role it plays. Each of the other arguments will only be included where appropriate.

For $x \in R^k$, and $\xi > 0$, we denote by $T(\xi, x)$ the k-dimensional distribution function

$$T(\xi, x) = \begin{cases} \displaystyle\int_0^{\min_\alpha x_\alpha} \frac{1}{\Gamma(\xi)} \eta^{\xi-1} e^{-\eta} d\eta , & x > 0 \\ 0 , & \text{otherwise.} \end{cases}$$

2. Principal results

THEOREM 1.

$$1 - F_n(s) = o(n, \rho; F, s) \quad , \textit{for } s \in C , \tag{2.1}$$

$$\frac{1-F_n(s)}{v'\{1-F_n(s)\}} = u + o(n, \rho; F, s) \quad , \textit{for } s \in C_1 , \tag{2.2}$$

$$1 - F_n(s) = h_n(1-s)\{u+o(n, \rho; F, s)\} , \textit{for } s \in C . \tag{2.3}$$

THEOREM 2. *For a fixed vector* $i \in X_0$,

$$\pi_n P[Y_n \neq 0 \mid Y_0 = i] = \rho^n [i'uQ^{-1}+o(n, \rho; F)] , \tag{2.4}$$

$$E[Y_n \mid Y_n \neq 0, Y_0 = i] = \pi_n Qv\{1+o(n, \rho; F)\} . \tag{2.5}$$

THEOREM 3. *Let* Y_n^* *denote the vector of normed random variables* $Y_{n,\alpha}^* = Y_{n,\alpha}/\{Q\pi_n v_\alpha\}$, $\alpha = 1, \ldots, k$. *For any fixed vector* $i \in X_0$,

$$\sup_{x \in R^k} |P[Y_n^* \leq x \mid Y_0 = i, Y_n \neq 0] - T(1, x)| = o(n, \rho; F) . \tag{2.6}$$

3. Preliminary lemmas

For $F \in K$, we denote by $\sigma \equiv \sigma_F$ the maximum of the moduli of the $k-1$ eigenvalues of $M(F)$ other than ρ_F. For $F, G \in K$ we define the metric $\Delta(F, G) = \max_{\alpha} \sup_{i \in X} |f_\alpha(i) - g_\alpha(i)|$, where the $g_\alpha(i)$ are the probability elements of G. We can verify that Δ is a proper metric as follows: obviously,

$$\text{(i)} \quad \Delta(F, G) = \Delta(G, F) \geq 0 \quad \text{for all } F, G \in K .$$

Furthermore, $F = G$ if and only if $f_\alpha(i) = g_\alpha(i)$ for all $i \in X$ and for $\alpha = 1, \ldots, k$, so that

$$\text{(ii)} \quad \Delta(F, G) = 0 \quad \text{if and only if } F = G .$$

Finally, if $H \in K$ has probability elements $h_\alpha(i)$,

$$\begin{aligned}
\text{(iii)} \quad \Delta(F, G) &= \max_{\alpha} \sup_{i \in X} |f_\alpha(i) - g_\alpha(i)| \\
&\leq \max_{\alpha} \sup_{i \in X} \{|f_\alpha(i) - h_\alpha(i)| + |h_\alpha(i) - g_\alpha(i)|\} \\
&\leq \max_{\alpha} \left[\sup_{i \in X} |f_\alpha(i) - h_\alpha(i)| + \sup_{i \in X} |h_\alpha(i) - g_\alpha(i)|\right] \\
&\leq \max_{\alpha} \sup_{i \in X} |f_\alpha(i) - h_\alpha(i)| + \max_{\alpha} \sup_{i \in X} |h_\alpha(i) - g_\alpha(i)| \\
&= \Delta(F, H) + \Delta(H, G) .
\end{aligned}$$

It follows from (i), (ii) and (iii) that Δ is indeed a metric.

LEMMA 1. *For* $1 \le \alpha, \beta, \gamma \le k$, *the quantities*

(i) $M_{\alpha\beta}(F)$;

(ii) $b^{(\alpha)}_{\beta\gamma}(F)$;

(iii) ρ_F ;

(iv) σ_F ;

(v) $u_\alpha(F)$;

(vi) $v_\alpha(F)$,

are continuous functions of $F \in K$.

Proof. *(i)*: Condition (1.1, iii) on K implies the existence of a null sequence $S \equiv (S_n)$, $n = 0, 1, 2, \ldots$, which majorizes all of the measures:

$$\left.\begin{array}{ll} \text{(i)} & \sum\limits_{i \in CX^{(n)}} f_\alpha(i) \; ; \\[2ex] \text{(ii)} & \sum\limits_{i \in CX^{(n)}} i_\beta f_\alpha(i) \; ; \\[2ex] \text{(iii)} & \sum\limits_{i \in CX^{(n)}} i_\beta i_\gamma f_\alpha(i) \; , \end{array}\right\} \qquad (3.1)$$

for $1 \le \alpha, \beta, \gamma \le k$. For instance, if (iii) is false, then there is some $\varepsilon_0 > 0$ such that for any integer R , no matter

how large, we can find some p.g.f. $F^{(R)} \in K$ (with probability elements $f_{\alpha}^{(R)}(i)$) so that for some triplet $\{\alpha, \beta, \gamma\}$,

$$\sum_{i \in CX^{(R)}} i_\beta i_\gamma f_\alpha^{(R)}(i) > \varepsilon_0 .$$

Since $CX^{(R)} \subset CX^{(n)}$ for $n = 1, 2, \ldots, R-1$, it follows that

$$\sum_{i \in CX^{(n)}} i_\beta i_\gamma f_\alpha^{(R)}(i) > \varepsilon_0 , \quad n = 1, 2, \ldots, R .$$

This in turn implies that

$$\sum_{\mu=1}^{k} \sum_{i \in Y(n,\mu)} i_\beta i_\gamma f_\alpha^{(R)}(i) > \varepsilon_0 , \quad n = 1, 2, \ldots, R ,$$

where we use $Y(n, \mu)$ to denote the set of vectors $i \in X$ which satisfy $i_\mu \geq n$. On the other hand,

$$\begin{aligned}
\sum_{\mu=1}^{k} \sum_{i \in X} i_\mu i_\beta i_\gamma f_\alpha^{(R)}(i) &= \sum_{\mu=1}^{k} \sum_{r=1}^{\infty} \sum_{i \in Y(r,\mu)} i_\beta i_\gamma f_\alpha(i) \\
&\geq \sum_{\mu=1}^{k} \sum_{r=1}^{R} \sum_{i \in Y(r,\mu)} i_\beta i_\gamma f_\alpha(i) \\
&\geq R\varepsilon_0 .
\end{aligned}$$

Since R is arbitrarily large, this implies that the third moments of K are unbounded, which contradicts condition (1.6, iii).

Consider any sequence $\{F^{(n)}\} \subset K$ converging to $F \in K$,

i.e. $\Delta(F^{(n)}, F) \to 0$. For each $\{\alpha, \beta\}$,

$$|M_{\alpha\beta}(F)-M_{\alpha\beta}(F^{(n)})| \le \sum_{i\in X} i_\beta |f_\alpha(i)-f_\alpha^{(n)}(i)|$$

$$\le \sum_{i\in X^{(r)}} i_\beta |f_\alpha(i)-f_\alpha^{(n)}(i)| +$$

$$\sum_{i\in CX^{(r)}} i_\beta \left\{f_\alpha(i)+f_\alpha^{(n)}(i)\right\} .$$

The second term on the right is majorised by $2S_r$ (by 3.1, ii).

Choose r_0 so large that $S_{r_0} \le \frac{1}{4}\varepsilon$. Then

$$|M_{\alpha\beta}(F)-M_{\alpha\beta}(F^{(n)})| \le \sum_{i\in X^{(r_0)}} i_\beta |f_\alpha(i)-f_\alpha^{(n)}(i)| + \frac{1}{2}\varepsilon$$

$$< \varepsilon$$

if $\Delta(F, F^{(n)}) < \frac{1}{2}\varepsilon/r_0^{k+1}$, which holds for all n sufficiently large. Thus $M_{\alpha\beta}(F^{(n)}) \to M_{\alpha\beta}(F)$, which is equivalent to the assertion of lemma 1 *(i)*, since $(F^{(n)})$ is arbitrary. The proof of lemma 1 *(ii)* is similar, using (3.1, iii) instead of (3.1, ii). We deduce parts *(iii)* and *(iv)* of lemma 1 from part *(i)*, since the eigenvalues of a finite dimensional matrix are continuous functions of its elements. Another consequence of (1.1, i) and the Perron-Frobenius theory is that for each $F \in K$, in view of the norming in (1.2), $v(F)$ and $u(F)$ are unique, and in addition,

$$v(F) > 0 , \quad u(F) > 0 . \tag{3.2}$$

The solution to the equation

$$\begin{bmatrix} 1 \\ \cdots \\ M(F) \end{bmatrix} u(F) = \begin{bmatrix} 1 \\ \cdots \\ \rho_F u(F) \end{bmatrix}$$

will give each $u_\alpha(F)$ as the ratio of two functions of F (via M and ρ). It then follows from lemma 1 parts *(i)* and *(iii)*, and the fact (from (1.2) and (3.2)) that $u_\alpha(F) \le 1$ for each $\alpha = 1, \ldots, k$, that $u_\alpha(F)$ is continuous.

The last part of lemma 1 can be proved similarly, because $v(F)$ is the unique solution of

$$v'(F)[M(F) \vdots u(F)] = [\rho_F v'(F) \vdots 1] .$$

LEMMA 2. *K is compact.*

Proof. For any sequence $(F^{(n)}) \subset K$ we can use Helly's diagonal method to select a subsequence (n_j) for which $f_\alpha^{(n_j)}(i)$ tends to a limit, $f_\alpha^*(i)$, say, as $n_j \to \infty$, for each $i \in X$, and $\alpha = 1, \ldots, k$. It is obvious that

$$f_\alpha^*(i) \ge 0 , \quad \sum_{i \in X} f_\alpha^*(i) \le 1 . \tag{3.3}$$

For each $\{\alpha, \beta\}$, and for $j, r = 1, 2, \ldots$,

$$\sum_{i \in X^{(r)}} f_\alpha^{(n_j)}(i) \ge 1 - S_r ,$$

by (3.1, i), so that, letting $n_j \to \infty$,

$$\sum_{i \in X^{(r)}} f^*_\alpha(i) \geq 1 - S_r \ , \tag{3.4}$$

and, since $X^{(r)} \subset X$,

$$\sum_{i \in X} f^*_\alpha(i) \geq 1 - S_r \ .$$

Letting $r \to \infty$,

$$\sum_{i \in X} f^*(i) \geq 1 \ ,$$

which, with (3.3), gives

$$\sum_{i \in X} f^*_\alpha(i) = 1 \ , \quad 1 \leq \alpha \leq k \ , \tag{3.5}$$

Defining $F^* \equiv F^*(s)$ in terms of the $f^*_\alpha(i)$ in the usual way, we have $F^* \in K$ from (3.3) and (3.5). We use parts *(i)* and *(ii)* of lemma 1, together with (1.1), to show

$$\{M^U(F^*)\}_{\alpha\beta} \geq a \ , \quad 1 \leq \alpha, \beta \leq k \ ,$$

and

$$\sum_{\alpha,\beta,\gamma} b^{(\alpha)}_{\beta\gamma}(F^*) \geq b \ .$$

Since for each $\{\alpha, \beta, \nu, \mu\}$,

$$c^{(\alpha)}_{\beta\nu\mu}\{F^{(n)}\} = \sum_{i\in X} f^{(n)}_{\alpha}(i)i_{\beta}\{i_{\nu}-\delta(\nu,\ \beta)\}\{i_{\mu}-\delta(\nu,\ \mu)-\delta(\nu,\ \beta)\}\ ,$$

where $\delta(\cdot,\ \cdot)$ is the Kronecker delta, we can use Fatou's lemma and (1.1, iii) to show $\sum_{\alpha,\beta,\nu,\mu} c^{(\alpha)}_{\beta\nu\mu}(F^*) \le c$. It follows that $F^* \in K$.

Finally, for any $\alpha = 1, \ldots, k$, and any $\varepsilon > 0$,

$$\sup_{i\in X}\left|f^{(n_j)}_{\alpha}(i)-f^*_{\alpha}(i)\right| \le \sup_{i\in X^{(r)}}\left|f^{(n_j)}_{\alpha}(i)-f^*_{\alpha}(i)\right| + \sup_{i\in CX^{(r)}}\left\{f^{(n_j)}_{\alpha}(i)+f^*_{\alpha}(i)\right\} .$$

The second term on the right is majorized by $2S_r$ (by 3.1, i). Choosing r_0 so large that $S_{r_0} \le \frac{1}{4}\varepsilon$, we have

$$\sup_{i\in X}\left|f^{(n_j)}_{\alpha}(i)-f^*_{\alpha}(i)\right| \le \sup_{i\in X^{(r_0)}}\left|f^{(n_j)}_{\alpha}(i)-f^*_{\alpha}(i)\right| + \frac{1}{2}\varepsilon$$
$$\le \varepsilon\ ,$$

if n_j is sufficiently large, since $f^{(n_j)}_{\alpha}(i) \to f^*_{\alpha}(i)$ for each of the finite number of $i \in X^{(r_0)}$. Thus $\Delta\left(F^{(n_j)},\ F^*\right) \to 0$ as $n_j \to \infty$, which concludes the proof of lemma 2.

LEMMA 3. *There exist constants* $0 < \theta \equiv \theta(a,\ U)$; $d \equiv d(c,\ U) < \infty$ *such that for all* $F \in K$, $1 \le \alpha,\ \beta,\ \gamma,\ \delta \le k$,

and $s \in C_0$ *, the following inequalities hold:*

(i) $\rho_F \geq \theta$;

(ii) $\sigma_F/\rho_F \leq 1 - \theta$;

(iii) $u_\alpha(F) \geq \theta$;

(iv) $v_\alpha(F) \geq \theta$;

(v) $v_\alpha(F) \leq \sum_\alpha v_\alpha(F) \leq \theta^{-1}$;

(vi) $\dfrac{s'1}{s'v(F)} \leq \theta^{-1}$;

(vii) $M_{\alpha\beta}(F) \leq d$;

(viii) $b^{(\alpha)}_{\beta\gamma}(F) \leq d$;

(ix) $c^{(\alpha)}_{\beta\gamma\delta}(F) \leq d$;

(x) $\theta^3 b \leq Q_F \leq d\theta^{-1}$.

Proof. Lemma 1 *(iii)* and lemma 2 imply ρ_F attains its lower bound (θ_1 , say) at some point $F^{(1)} \in K$, so that by (1.1, i) and the Perron-Frobenius theory, $\theta_1 > 0$. It now follows that σ_F/ρ_F is continuous (recalling lemma 1, parts *(iii)* and *(iv)*), so that, by lemma 2, this ratio attains its upper bound

$(1-\theta_2$, say) at some point $F^{(2)} \in K$. The Perron-Frobenius theory assures us that $\theta_2 > 0$. Similarly, we can show $u_\alpha(F) \geq \theta_{2+\alpha}$, $\alpha = 1, \ldots, k$, and $v_\alpha(F) \geq \theta_{2+k+\alpha}$, $\alpha = 1, \ldots, k$, where $\theta_\beta > 0$, $\beta = 3, 4, \ldots, 2k+2$. We set $\theta = \min_\alpha \{\theta_\alpha\}$. To prove *(v)* we use *(iii)* and (1.2): $\sum v_\alpha(F) \leq \sum v_\alpha(F) u_\alpha(F) \theta^{-1} = \theta^{-1}$. *(vi)* follows easily from *(iv)*, and *(vii)*, *(viii)* and *(ix)* are consequences of (1.1, iii). To prove *(x)*, we first note that

$$\begin{aligned} Q_F &= \sum_\alpha v_\alpha(F) \sum_{\beta,\gamma} b^{(\alpha)}_{\beta\gamma}(F) u_\beta u_\gamma \\ &\leq d \sum_\alpha v_\alpha(F) \sum_{\beta,\gamma} u_\beta u_\gamma && \text{by } (viii) \\ &= d \sum_\alpha v_\alpha(F) && \text{by } (1.2) \\ &\leq d\theta^{-1} && \text{by } (v), \end{aligned}$$

and secondly,

$$\begin{aligned} Q_F &\geq \theta^3 \sum_{\alpha,\beta,\gamma} b^{(\alpha)}_{\beta\gamma}(F) && \text{by } (iii) \text{ and } (iv) \\ &\geq \theta^3 b && \text{by } (1.1, \text{ii}) . \end{aligned}$$

LEMMA 4.

(i) $\pi_n^{-1} = o(n, \rho; F)$;

(ii) $\rho^n \pi_n^{-1} = o(n, \rho; F)$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\left(\frac{1}{2}, \frac{1}{2}\varepsilon\right)$. For $F \in K$ with $|1-\rho_F| < \delta$,

$$\pi_n(\rho) \equiv \rho^{-1}(1+\rho + \ldots + \rho^{n-1}) \geq \frac{2}{3}\,\frac{1-(1-\delta)^n}{\delta}$$

$$\geq \frac{4\{1-(1-\delta)^n\}}{3\varepsilon}$$

$$\geq \varepsilon^{-1}$$

for all n sufficiently large. This proves part *(i)*, and in addition,

$$\rho^{-n}\pi_n(\rho) = \rho^{-3}\sum_{r=1}^{n}\rho^{r+1-n} = \rho^{-3}\pi_n\left(\frac{1}{\rho}\right)$$

$$= 1/o(n, \rho; F) \quad \text{by } (i) .$$

LEMMA 5. *There is a positive null sequence* (δ_n) *such that for all* $F \in K$,

$$(1-\delta_n)uv' \leq (M/\rho)^n \leq (1+\delta_n)uv' . \tag{3.6}$$

Proof. It is obvious that the matrix M/ρ has one eigenvalue equal to unity, and no other eigenvalue with modulus greater than $1 - \theta$, for any $F \in K$ (by lemma 3 *(ii)*). Lemma 3 parts *(i)* and *(vii)* imply $\{M/\rho\}_{\alpha\beta} \leq d/\theta$, $1 \leq \alpha, \beta \leq k$, for all $F \in K$. The lemma is now a consequence of the theorem of Buchanan and Parlett (1966)†.

† I am indebted to Dr G.N. de Oliveira (personal communication) for bringing this paper to my attention.

We show now that $F_n(s)$, the n-th functional iterate of $F(s)$, is uniformly continuous in the Δ metric both for $F \in K$ and $s \in C$.

LEMMA 6. *Given* $\varepsilon > 0$, *and a positive integer* n , *there exists a positive constant* $\kappa(n)$ *such that for any* $F, G \in K$ *satisfying* $\Delta(F, G) < \kappa(n)$,

$$\sum_{\alpha} |F_{n,\alpha}(s)-G_{n,\alpha}(s)| < \varepsilon , \quad s \in C .$$

Proof. First, let $n = 1$. For any $F, G \in K$ and $s \in C$,

$$\begin{aligned} |F_{1,\alpha}(s)-G_{1,\alpha}(s)| &= \left| \sum_{i \in X} \{f_\alpha(i)-g_\alpha(i)\} s^i \right| \\ &\le \sum_{i \in X} |f_\alpha(i)-g_\alpha(i)| \\ &< \varepsilon \end{aligned}$$

if $\Delta(F, G)$ is small enough $(< \kappa(1)$, say) by the same sort of argument as that used at the beginning of lemma 1. Thus lemma 6 is true for $n = 1$.

Since for each $\{\alpha, \beta\}$ and each $F \in K$, $\partial F_{1,\alpha}(s)/\partial s_\beta \le M_{\alpha\beta}(F) \le d < \infty$ (by lemma 3 *(vii)*), a straightforward extension of an argument in Courant (1936, p. 54) to k dimensions can be used to show that for given $\varepsilon > 0$, we can find $\delta > 0$ such that for any $s, t \in C$ satisfying $\sum_{\alpha} |t_\alpha - s_\alpha| < \delta$, and any $F \in K$,

$$\sum_{\alpha} |F_{1,\alpha}(s)-F_{1,\alpha}(t)| < \varepsilon . \tag{3.7}$$

Bearing this in mind, we can complete the proof of the lemma by an induction argument. Suppose the lemma is true for some n. Then

$$\sum_{\alpha} |F_{n+1,\alpha}(s)-G_{n+1,\alpha}(s)| \le \sum_{\alpha} |F_{1,\alpha}\{F_n(s)\}-G_{1,\alpha}\{F_n(s)\}| + \\ + \sum_{\alpha} |G_{1,\alpha}\{F_n(s)\}-G_{1,\alpha}\{G_n(s)\}| .$$

The first sum on the right can be made arbitrarily small if $\Delta(F, G)$ is small enough (since the lemma is true for $n = 1$), and the second term can also be made arbitrarily small (by the induction hypothesis, and (3.7)). Thus lemma 6 is proved.

We now prove a lemma concerning the products of certain matrices, which extends lemma 1 of JS. We adopt the convention $\prod_{j=n-m}^{n} A_j \equiv A_n A_{n-1} \ldots A_{n-m}$. Suppose we have a sequence of non-negative, primitive $k \times k$ matrices $\{P(r)\}$ with unit spectral radii and associated left and right eigenvectors $v(r)$, $u(r)$, normed so that $v'(r)u(r) = u'(r)1 = 1$. We set $R(n) = u(n)v'(n)$. We assume there is a positive, null sequence $\{\delta_n\}$ such that

$$\{1-\delta_m\}R(n) \le P^m(n) \le \{1+\delta_m\}R(n) , \tag{3.8}$$

for $n, m = 1, 2, 3, \ldots$. Let $A(n, m)$ be a double sequence of $k \times k$ matrices satisfying

$$0 \le A(n, m) \le P(n) , \quad n, m = 1, 2, \ldots ,$$

and assume there is a positive double sequence $\{p_{n,m}\}$ which for fixed n is a non-increasing sequence in m ; for fixed m , let the sequence $\{p_{n,n-m}\}$, $n = m+1, m+2, \ldots$, be null. We assume that

$$A(n, m) \le p_{n,m}R(n) \ , \quad n,m = 1, 2, \ldots ,$$

and define

$$B(n) = \prod_{j=1}^{n} \{P(n)-A(n, j)\} \ .$$

LEMMA 7. *For any* $\varepsilon > 0$, *and any* $x \ge 0$ *for which* $B(n)x \ne 0$, $n = 1, 2, \ldots$, *there exists* $N_\varepsilon < \infty$ *such that for* $n > N_\varepsilon$,

$$-\varepsilon < \left(\frac{B(n)x}{v(n)B(n)x} - u(n)\right)_\alpha < \varepsilon \ , \quad 1 \le \alpha \le k \ .$$

Proof. First, we use an induction argument to show that for any integer n , and for all $1 \le m \le n$,

$$\prod_{j=n-m+1}^{n} \{P(n)-A(n, j)\} \ge P^m(n) - \sum_{j=n-m+1}^{n} p_{n,j}R(n) \ . \quad (3.9)$$

Suppose this is true for some m , $1 \le m < n$. Then

$$\prod_{j=n-m}^{n} \{P(n)-A(n, j)\} \geq P^{m+1}(n)$$

$$- \sum_{j=n-m+1}^{n} p_{n,j}R(n)P(n) - P^{m}(n)A(n, n-m)$$

$$\geq P^{m+1}(n) - \sum_{j=n-m}^{n} p_{n,j}R(n) .$$

Since (3.9) is obviously true for $m = 1$, it must be true for $m = 1, \ldots, n$. Next, (3.8) and (3.9) give†

$$\left\{1 - \delta_m - \sum_{j=n-m+1}^{n} p_{n,j}\right\}R(n) \leq \prod_{j=n-m+1}^{n} \{P(n)-A(n, j)\}$$

$$\leq (1+\delta_m)R(n) . \qquad (3.10)$$

Given $x \geq 0$, define $w = \prod_{j=1}^{n-m} \{P(n)-A(n, j)\}x$, so that

$B(n)x = \prod_{j=n-m+1}^{n} \{P(n)-A(n, j)\}w$. Then (3.10) gives

$$\frac{\left\{1-\delta_m - \sum_{j=n-m+1}^{n} p_{n,j}\right\}R(n)w}{v'(n)(1+\delta_m)R(n)w} \leq \frac{B(n)x}{v'(n)B(n)x}$$

$$\leq \frac{(1+\delta_{\cancel{n}\,m})R(n)w}{v'(n)\left(1-\delta_m - \sum_{j=n-m+1}^{n} p_{n,j}\right)R(n)w} .$$

For any w , we have $R(n)w/v'(n)R(n)w = u(n)$, so that for $\alpha = 1, \ldots, k$,

† The second inequality in JS (eq. 4.10) is not valid, since $P - \alpha_n R$ may have negative elements. The present approach can be used to prove their principal assertion at that point.

$$\left|\left(\frac{B(n)x}{v'(n)B(n)x} - u(n)\right)_\alpha\right| \le \frac{2\delta_m + \sum_{j=n-m+1}^{n} p_{n,j}}{1-\delta_m - \sum_{j=n-m+1}^{n} p_{n,j}} .$$

Given $\varepsilon > 0$, we choose m so large that $3\delta_m < \min\left(1, \frac{1}{3}\varepsilon\right)$, and then choose $N_\varepsilon > m$ so large that $3p_{n,n-m} < \delta_m/m$ for $n \ge N_\varepsilon$, *i.e.* $3 \sum_{j=n-m+1}^{n} p_{n,j} < \min\left(1, \frac{1}{3}\varepsilon\right)$.

We conclude this section with some Taylor-type expansions for $F(s)$. The standard multivariate analogue of Taylor's theorem (see, *e.g.* Apostol (1957, p. 124)) concerns the expansion of a function around a point in the interior of a region, and is therefore inappropriate here, since we wish to expand around $s = 1$. We utilise and extend results in JS. Assuming only finite first moments, it is shown there that

$$1 - F(s) = M(1-s) - E(s)(1-s) , \quad s \in C \tag{3.11}$$

where

$$E_{\alpha\beta}(1-s) = \sum_{i \in X} f_\alpha(i) i_\beta \left\{1 - \int_0^1 \frac{\Pi(1-\eta s_\beta)^{i_\nu}}{1-\eta s_\beta} d\eta\right\} , \tag{3.12}$$

and

$$0 \le E(s) \le M , \quad t \le s \Rightarrow E(t) \ge E(s) . \tag{3.13}$$

Assuming finite second moments, it is shown in JS that

$$1 - F(s) = M(1-s) - \hat{q}[1-s] \ , \quad s \in C \tag{3.14}$$

where the vector $\hat{q}[\cdot]$ has elements

$$\hat{q}_\alpha[s] = \sum_{i \in X} f_\alpha(i) \sum_{\nu,\mu} s_\nu s_\mu i_\nu \big(i_\mu - \delta(\nu, \mu)\big) I(\nu, \mu, s, i) \ , \tag{3.15}$$

and

$$I(\nu, \mu, s, i) = \int_0^1 \frac{\Pi\big(1-\eta s_\beta\big)^{i_\beta}(1-\eta)d\eta}{\big(1-\eta s_\nu\big)\big(1-\eta s_\mu\big)} \ .$$

It is easy to show $0 \le I(\cdot) \le \frac{1}{2}$, and defining

$$\hat{b}^{(\alpha)}_{\nu\mu}(s) = 2 \sum_{i \in X} f_\alpha(i) i_\nu \big(i_\mu - \delta(\nu, \mu)\big) I(\nu, \mu, s, i) \ ,$$

we have

$$0 \le \hat{b}^{(\alpha)}_{\nu\mu}(s) \le b^{(\alpha)}_{\nu\mu} \equiv \sum_{i \in X} f_\alpha(i) i_\nu \big(i_\mu - \delta(\nu, \mu)\big) \ ,$$

and by Fubini's theorem,

$$\hat{q}_\alpha[s] = \frac{1}{2} \sum_{\nu,\mu} \hat{b}^{(\alpha)}_{\nu\mu}(s) s_\nu s_\mu \ . \tag{3.16}$$

Examination of the integral in (3.12) shows that for any $i \in X$ with $i_\beta \ge 1$,

$$\int_0^1 \frac{\Pi\big(1-\eta s_\nu\big)^{i_\nu}}{1-\eta s_\beta} d\eta = \int_0^1 \left\{ 1 + \int_0^\eta \frac{d}{d\zeta} \frac{\Pi\big(1-\zeta s_\nu\big)^{i_\nu}}{1-\zeta s_\beta} d\zeta \right\} d\eta \ ,$$

i.e.

$$1 - \int_0^1 \frac{\Pi(1-\eta s_\nu)^{i_\nu}}{1-\eta s_\beta}\, d\eta = \sum_\gamma s_\gamma\{i_\gamma - \delta(\gamma,\ \beta)\} \int_0^1 \int_0^\eta \frac{\Pi(1-\zeta s_\nu)^{i_\nu}}{(1-\zeta s_\gamma)(1-\zeta s_\beta)}\, d\zeta d\eta$$

$$= \sum_\gamma s_\gamma\{i_\gamma - \delta(\gamma,\ \beta)\} \int_0^1 \frac{\Pi(1-\zeta s_\nu)^{i_\nu}}{(1-\zeta s_\gamma)(1-\zeta s_\beta)}\, (1-\zeta) d\zeta$$

$$= \sum_\gamma s_\gamma\{i_\gamma - \delta(\gamma,\ \beta)\} I(\gamma,\ \beta,\ \mathsf{s},\ \mathsf{i})\ ,$$

the second line being obtained by changing the order of integration, which is valid since the integrand is continuous for each $\mathsf{s} \in C$ and for any $\mathsf{i} \in X$ with $i_\beta \geq 1$, $i_\gamma \geq 1$. Thus

$$E_{\alpha\beta}(1-\mathsf{s}) = \sum_{\mathsf{i}\in X} f_\alpha(\mathsf{i}) i_\beta \left\{\sum_\gamma s_\gamma [i_\gamma - \delta(\gamma,\ \beta)]\right\} I(\gamma,\ \beta,\ \mathsf{s},\ \mathsf{i})$$

$$= \sum_\gamma s_\gamma \sum_{\mathsf{i}\in X} f_\alpha(\mathsf{i}) i_\beta [i_\gamma - \delta(\alpha,\ \beta)] I(\gamma,\ \beta,\ \mathsf{s},\ \mathsf{i})$$

$$= \frac{1}{2} \sum_\gamma s_\gamma \hat{b}^{(\alpha)}_{\gamma\beta}(\mathsf{s})\ .$$

This alternative expression for $\mathsf{E}(\mathsf{s})$ enables us to deduce the uniform bound (bearing in mind lemma 3 *(viii)*)

$$E_{\alpha\beta}(\mathsf{s}) \leq \frac{1}{2}\, d \sum_\nu (1-s_\nu) \tag{3.17}$$

for $1 \leq \alpha,\ \beta \leq k$, $\mathsf{s} \in C$, and $\mathsf{F} \in K$.

The technique used in JS can be extended to give a third order expansion, which we now do. For $\mathsf{i} \in X$ and $\mathsf{t} \in C$, we define

$$\psi(t) = (1-t)^{i} .$$

The expansion rests on the following identity:

$$\psi(t) - 1 = \psi(t) - \psi(0)$$

$$= \int_0^1 \frac{d}{d\xi}\,\psi(\xi t)d\xi$$

$$= \left[\xi\,\frac{d\psi}{d\xi}\,(\xi t)\right]_0^1 - \int_0^1 \xi\,\frac{d^2\psi}{d\xi^2}\,(\xi t)d\xi \quad \text{on integration by parts}$$

$$= \left[\frac{d\psi}{d\xi}\,(\xi t)\right]_0^1 + \frac{d\psi}{d\xi}\,(\xi t)\Big|_{\xi=0} - \int_0^1 \xi\,\frac{d^2\psi}{d\xi^2}\,(\xi t)d\xi$$

$$= \frac{d\psi}{d\xi}\,(\xi t)\Big|_{\xi=0} + \int_0^1 (1-\xi)\,\frac{d^2\psi}{d\xi^2}\,(\xi t)d\xi$$

$$= \frac{d\psi}{d\xi}\,(\xi t)\Big|_{\xi=0} - \frac{1}{2}\left[(1-\xi)^2\,\frac{d^2\psi}{d\xi^2}\,(\xi t)\right]_0^1 + \frac{1}{2}\int_0^1 (1-\xi)^2\,\frac{d^3\psi}{d\xi^3}\,(\xi t)d\xi ,$$

on integration by parts again. Thus

$$\psi(t) - 1 = \left(\frac{d\psi}{d\xi}\,(\xi t) - \frac{1}{2}\,\frac{d^2\psi}{d\xi^2}\,(\xi t)\right)\Big|_{\xi=0} + \frac{1}{2}\int_0^1 (1-\xi)^2\,\frac{d^3\psi}{d\xi^3}\,(\xi t)d\xi$$

$$= -\sum_{\alpha} t_\alpha i_\alpha + \frac{1}{2}\sum_{\alpha,\beta} t_\alpha t_\beta i_\alpha\big(i_\beta - \delta(\alpha,\beta)\big)$$

$$- \frac{1}{2}\int_0^1 (1-\xi)^2 \sum_{\alpha,\beta,\gamma} \frac{\prod_{\mu}(1-\xi t_\mu)^{i_\mu}}{(1-\xi t_\alpha)(1-\xi t_\beta)(1-\xi t_\gamma)} \times$$

$$\times\, i_\alpha\big(i_\beta - \delta(\alpha,\beta)\big)\{i_\gamma - \delta(\alpha,\beta) - \delta(\beta,\gamma)\}d\xi . \quad (3.18)$$

If we now define $s = 1 - t$, and sum with respect to the measure $f_\mu(i)$, we obtain

$$F_{1,\mu}(s) \equiv \sum_{i\in X} f_\mu(i)s^i = 1 - \sum_\alpha M_{\mu\alpha}(1-s_\alpha) + q_\mu[1-s] - \hat{r}_\mu[1-s] \ , \quad (3.19)$$

say, where $q_\mu[\cdot]$ was defined at (1.3), and

$$\hat{r}_\mu(s) = \frac{1}{6} \sum_{\alpha,\beta,\gamma} s_\alpha s_\beta s_\gamma \hat{c}^{(\mu)}_{\alpha\beta\gamma}(s) \ ,$$

$$\hat{c}^{(\mu)}_{\alpha\beta\gamma}(s) = 3 \sum_{i\in X} f_\mu(i) i_\alpha [i_\beta - \delta(\alpha,\ \beta)][i_\gamma - \delta(\alpha,\ \beta) - \delta(\beta,\ \gamma)] \times$$

$$\times \int_0^1 \frac{(1-\xi)^2 \prod_\nu (1-\xi s_\nu)^{i_\nu}}{(1-\xi s_\alpha)(1-\xi s_\beta)(1-\xi s_\gamma)} d\xi \ .$$

It should be noted that in order to pass from (3.18) to (3.19), we need to use Fubini's theorem; this is valid since the integral in the definition of $\hat{c}^{(\mu)}_{\alpha\beta\gamma}(s)$ lies in the range $[0,\ 1/3]$, so that

$$0 \le \hat{c}^{(\mu)}_{\alpha\beta\gamma}(s) \le c^{(\mu)}_{\alpha\beta\gamma} \le d < \infty$$

by virtue of lemma 3 *(ix)*. We express (3.19) in vector form as

$$1 - F(s) = M(1-s) - q[1-s] + \hat{r}[1-s] \ . \quad (3.20)$$

We define

$$\hat{Q}[s] = \sum_\alpha v_\alpha \hat{q}_\alpha[s] \ , \quad \hat{R}[s] = \sum_\alpha v_\alpha \hat{r}_\alpha[s] \ .$$

Premultiplying (3.20) by v' gives

$$v'\{1-F(1-s)\} = \rho v's - Q[s] + \hat{R}[s] \ ,$$

which we write as

$$\frac{1}{v'\{1-F(1-s)\}} - \frac{1}{\rho v's} = \frac{Q[s]}{(\rho v's)^2} + L[s] \ . \tag{3.21}$$

LEMMA 8. *$L[s]$ is uniformly bounded for $F \in K$ and $s \in C_0$, and $L[s] \to 0$ uniformly for $F \in K$ as $s \to 0$ in C.*

Proof. Premultiplying (3.14) by v',

$$v'\{1-F(1-s)\} = \rho v's\left(1 - \frac{\hat{Q}[s]}{\rho v's}\right) \ . \tag{3.22}$$

By definition and lemma 3 parts *(v)* and *(viii)*,

$$\hat{Q}[s] \le \frac{1}{2}\, d\theta^{-1}\left(\sum s_\nu\right)^2 ,$$

so it follows from lemma 3 parts *(i)* and *(vi)* that for $s \in C_0$,

$$\begin{aligned}\hat{Q}[s]/\rho v's &\le \frac{1}{2}\, d\theta^{-2}\left(\sum s_\nu\right)\left\{\frac{s'1}{s'v}\right\} \\ &\le \frac{1}{2}\, d\theta^{-3}\left(\sum s_\nu\right) .\end{aligned}$$

For any $s \in C_0$ with $\sum s_\nu \le \theta^3/d$, then, $\hat{Q}[s]/\rho v's \le \frac{1}{2}$. It follows from (3.22) that in this case,

$$\frac{1}{v'\{1-F(1-s)\}} = \frac{1}{\rho v's}\left\{1 + \frac{\hat{Q}[s]}{\rho v's} + \frac{\{\hat{Q}[s]/\rho v's\}^2}{1-\hat{Q}[s]/\rho v's}\right\} ,$$

and from (3.21),

$$L[s] = \frac{\hat{Q}[s]-Q[s]}{(\rho v's)^2} + \frac{\{\hat{Q}[s]/\rho v's\}^2}{\rho v's-\hat{Q}[s]} = \frac{\hat{R}[s]}{(\rho v's)^2} + \frac{\{\hat{Q}[s]/\rho v's\}^2}{\rho v's-\hat{Q}[s]} \ .$$

Hence for any $s \in C_0$ with $\sum s_\nu \le \theta^3/d$,

$$|L[s]| \le \frac{1}{6}\frac{d}{(\theta v's)^2}\sum v_\alpha\left(\sum s_\nu\right)^3 + \frac{\left\{\frac{1}{2}d\theta^{-3}\sum s_\nu\right\}^2}{\frac{1}{2}\theta v's}$$

$$\le \text{constant.} \sum s_\nu , \tag{3.23}$$

by lemma 3 parts *(i)*, *(v)*, *(vi)* and *(ix)*.

For any $s \in C$ with $\sum s_\nu > \theta^3/d$ (if any exist), choose $t \in C$ with $t < s$ and $\sum t_\nu = \theta^3/d$. The "monotonicity" of F $\left(i.e.\ t < s \Rightarrow F(t) \le F(s)\right)$ gives

$$\begin{aligned} v'\{1-F(1-s)\} &\ge v'\{1-F(1-t)\} \\ &= \rho v't\{1-\hat{Q}[t]/\rho v't\} \\ &\ge \frac{1}{2}\theta^5/d \end{aligned}$$

by lemma 3 parts *(i)* and *(vi)*. Hence from (3.21), and lemma 3 parts *(i)*, *(v)*, *(vi)* and *(viii)*,

$$\begin{aligned} |L[s]| &\le 2d\theta^{-5} + d\theta^{-5} + \frac{Q[s]}{(\theta v's)^2} \\ &\le 4d\theta^{-5} . \end{aligned}$$

This, together with (3.23), proves the lemma.

4. Proof of theorems and discussion

If theorem 1 (2.1) is false for $s = 0$, then there is some $\varepsilon > 0$ and some sequence $\{F^{(n)}\} \subset K$ for which $\rho_F(n) \to 1$, but $\sum_\alpha \left\{1 - F^{(n)}_{n,\alpha}(0)\right\} > \varepsilon$ for $n = 1, 2, \ldots$. Since $F_{n,\alpha}(0) = F_{m,\alpha}\{F_{n-m}(0)\} \geq F_{m,\alpha}(0)$, $m = 1, \ldots, n$, we have $\sum_\alpha \left\{1-F^{(n)}_{m,\alpha}(0)\right\} > \varepsilon$, $1 \leq m \leq n$, $n = 1, 2, \ldots$. By lemma 2 we can select a subsequence $\{n_j\}$ such that $\Delta\left(F^{(n_j)}, F^*\right) \to 0$ as $n_j \to \infty$ for some $F^* \in K$. Lemma 1 parts *(i)*, *(ii)* and *(iii)* imply that $M(F^*)$ is primitive, $\sum_{\alpha\beta\gamma} b^{(\alpha)}_{\beta\gamma}(F^*) \geq b > 0$, and $\rho_{F^*} = 1$. It follows (Harris (1963, p. 41)) that the vector of extinction probabilities associated with F^*, q^*, say, is equal to 1. The monotone convergence of $F^*_{n,\alpha}(0)$ to q^*_α (Harris (1963, p. 41)) implies the existence of an m_0 such that for $\alpha = 1, \ldots, k$, $1 - F^*_{m_0,\alpha}(0) < \frac{1}{2}\varepsilon/k$. Thus

$$\sum_\alpha \left\{1-F^{(n)}_{m_0,\alpha}(0)\right\} \leq \frac{1}{2}\varepsilon + \sum_\alpha \left|F^*_{m_0,\alpha}(0)-F^{(n)}_{m_0,\alpha}(0)\right|, \quad n = 1, 2, \ldots,$$

and since $\Delta\left(F^{(n_j)}, F^*\right) \to 0$, lemma 6 implies the existence of an integer $m_1 \in \{n_j\}$, $m_1 \geq m_0$, such that

$$\sum_\alpha \left|F^*_{m_0,\alpha}(0)-F^{(m_1)}_{m_0,\alpha}(0)\right| < \frac{1}{2}\varepsilon .$$

Thus

$$\sum_{\alpha} \left\{1-F^{(m_1)}_{m_0,\alpha}(0)\right\} \le \varepsilon ,$$

and the contradiction proves (2.1) for $s = 0$. The full assertion of (2.1) follows since $F_{n,\alpha}(0) \le F_{n,\alpha}(s) \le 1$, $s \in C$.

Because of (2.1), any sequence $\{F^{(n)}\} \subset K$ for which $|1-\rho_F(n)| \to 0$ must satisfy, for arbitrary fixed m ,

$$\sum_{\alpha} \left\{1-F^{(n)}_{n-m,\alpha}(0)\right\} \to 0 \quad \text{as} \quad n \to \infty . \tag{4.1}$$

For such a sequence, define

$$A(n, m, s) = \rho^{-1}_{F^{(n)}} E\left[F^{(n)}_{m-1}(s)\right] , \quad s \in C ,$$

and

$$P(n) = \rho^{-1}_{F^{(n)}} M\left(F^{(n)}\right) ,$$

where $E(\cdot)$ is defined at (3.12). $P(n)$ is non-negative, primitive, and has unit spectral radius. Its principal eigenvectors ($v(n)$ and $u(n)$, say, with $u'(n)v(n) = u'(n)1 = 1$) coincide with those of $M\left(F^{(n)}\right)$. Let $R(n) = u(n)v'(n)$. It is evident from lemma 5 that $P(n)$ satisfies (3.8), and lemma 3 parts *(iii)* and *(iv)* give $\{R(n)\}_{\alpha\beta} \ge \theta^2$, $1 \le \alpha, \beta \le k$, $n = 1, 2, \ldots$. By (3.13), and the monotonicity of $F_{n,\alpha}(0)$, we have for fixed n ,

$$m_1 > m_2 \Rightarrow A(n, m_1, 0) \le A(n, m_2, 0) . \tag{4.2}$$

In the same vein, since $F_n(0) \le F_n(s)$, $s \in C$,

$$A(n, m, s) \le A(n, m, 0) , \quad s \in C . \tag{4.3}$$

For fixed m , $A(n, n-m, 0) \to 0$ as $n \to \infty$ by (3.13) and (4.1). If we put $p_{n,m} = \theta^{-2} \max_{\alpha,\beta} A_{\alpha\beta}(n, m, 0)$, that it is easy to see that $p_{n,m}$ is positive for each $\{n, m\}$, that for fixed m , $p_{n,n-m} \to 0$ as $n \to \infty$ (by (4.1)), and that for fixed n , $\{p_{n,m}\}$ is a non-increasing sequence in m (by (4.2)). From (4.3) ,

$$A(n, m, s) \le p_{n,m} R(n) . \tag{4.4}$$

If for any $s \in C$ we put $A(n, m) = A(n, m, s)$, then all the conditions of lemma 7 are satisfied.

Iteration of the expansion (3.11) gives, for each n ,

$$\begin{aligned} 1 - F_n^{(n)}(s) &= \left\{\prod_{j=1}^{n}\left[M\left(F^{(n)}\right) - E\left(F_{j-1}^{(n)}(s)\right)\right]\right\}(1-s) \\ &= \rho_{F^{(n)}}^{n}\left\{\prod_{j=1}^{n}[P(n) - A(n, j, s)]\right\}(1-s) \\ &= \rho_{F^{(n)}}^{n} B(n)(1-s) , \end{aligned}$$

in the notation of lemma 7. For $s \in C_1$, $1-s \ge 0$, and $B(n)(1-s) \ne 0$, so by lemma 7, given $\varepsilon > 0$, we can find an integer N_ε so large that for all $n > N_\varepsilon$, and $\alpha = 1, \ldots, k$,

$$\left| \frac{1-F^{(n)}_{n,\alpha}(s)}{v'(n)\left\{1-F^{(n)}_{n}(s)\right\}} - u_{\alpha}(n) \right| < \varepsilon . \tag{4.5}$$

Since the bound in (4.4) is independent of $s \in C$, (4.5) is valid uniformly for $s \in C_1$.

If (2.2) is false, we can find some $\varepsilon > 0$, and sequences $\{F^{(n)}\} \subset K$ with $|1-\rho_{F^{(n)}}| \to 0$ as $n \to \infty$, $\{s(n)\} : s(n) \in C_1$, and $\{\alpha(n)\} : 1 \le \alpha(n) \le k$, such that

$$\left| \frac{1-F^{(n)}_{n,\alpha(n)}(s(n))}{v'(n)\left\{1-F^{(n)}_{n}(s(n))\right\}} - u_{\alpha(n)}(n) \right| > \varepsilon , \; n = 1, 2, \ldots .$$

Since the sequence $\{F^{(n)}\}$ and the constant ε in (4.5) are both arbitrary (as long as $|1-\rho_{F^{(n)}}| \to 0$) we must eventually arrive at a contradiction. Thus (2.2) is proved.

It is easy to see that for any scalar a, $Q[as] = a^2 Q[s]$. Using this, the definition of $F_n(s)$ above lemma 6, and (3.21),

$$\begin{aligned}
\frac{\rho^n}{v'\{1-F_n(s)\}} - \frac{1}{v'\{1-s\}} &= \sum_{j=1}^{n} \rho^j \left\{ \frac{1}{v'\{1-F_j(s)\}} - \frac{1}{\rho v'\{1-F_{j-1}(s)\}} \right\} \\
&= \sum_{j=0}^{n-1} \rho^{j+1} \left\{ \rho^{-2} Q\left[\frac{1-F_j(s)}{v'\{1-F_j(s)\}} \right] + L[1-F_j(s)] \right\} \\
&= \sum_{j=0}^{n-1} \rho^{j+1} \left\{ \rho^{-2} Q[u+o(j, \rho; F, s)] + \right. \\
&\qquad \left. + L[1-F_j(s)] \right\} ,
\end{aligned}$$

from (2.2) which, together with lemma 3 *(vi)*, indicates that the elements of the error vector are all bounded in modulus by $1 + \theta^{-1}$, uniformly for all $F \in K$, $s \in C_1$, and $j = 0, 1, 2, \ldots$. Expanding $Q[\cdot]$ and rearranging,

$$\frac{\rho^n}{v'\{1-F_n(s)\}} - \frac{1}{v'\{1-s\}} - \pi_n Q$$

$$= \frac{1}{2}\sum_{j=0}^{n-1} \rho^{j-1}\left\{\sum_{\nu} v_\nu \sum_{\alpha\beta} b_{\alpha\beta}^{(\nu)}\left[u_\alpha+u_\beta+1\right]\right\} o(j, \rho; F, s) +$$

$$+ \sum_{j=0}^{n-1} \rho^{j+1} L\left[1-F_j(s)\right] , \quad (4.6)$$

where $|o(j, \rho; F, s)| \le 1 + \theta^{-1}$. Consider the quantity $\pi_n^{-1} \sum_{j=0}^{n-1} \rho^{j-1} o(j, \rho; F, s)$. Given $\varepsilon > 0$, we can choose N so large and $\delta > 0$ so small that $|o(j, \rho; F, s)| < \frac{1}{2}\varepsilon$ for $j > N$ and $F \in K$ with $|1-\rho_F| < \delta$. Then for any $n > N$,

$$\left|\pi_n^{-1} \sum_{j=0}^{n-1} \rho^{j-1} o(j, \rho; F, s)\right| \le \pi_n^{-1}(1+\theta^{-1}) \sum_{j=0}^{N} \rho^{j-1} + \frac{1}{2}\varepsilon$$

$$\le \varepsilon$$

for all sufficiently large n and all $F \in K$ with $|1-\rho_F|$ sufficiently small, by lemma 4. A similar argument (bearing in mind (2.1) and lemma 8) shows

$$\pi_n^{-1} \sum_{j=0}^{n-1} \rho^{j+1} L\left[1-F_j(s)\right] = o(n, \rho; F, s) .$$

Since the term in curly brackets in (4.6) is bounded uniformly for $F \in K$, we have for $s \in C_1$,

$$\pi_n^{-1}\left[\frac{\rho^n}{v'(1-F_n(s))} - \frac{1}{v'(1-s)}\right] - Q = o(n, \rho; F, s) .$$

Substituting for $h_n(s)$,

$$\frac{h_n(1-s)}{v'\{1-F_n(s)\}} = \rho^n\left\{v'[1-F_n(s)]\left[\frac{1}{v'(1-s)} + \pi_n Q\right]\right\}^{-1}$$

$$= \left\{\frac{1}{v'(1-s)} + \pi_n[Q+o(n, \rho; F, s)]\right\}\left\{\frac{1}{v'(1-s)} + \pi_n Q\right\}^{-1}$$

$$= 1 + \frac{o(n,\rho;F,s)}{\pi_n^{-1}/v'(1-s)+Q}$$

$$= 1 + o(n, \rho; F, s) ,$$

using lemma 3 *(x)*. This result coupled with (2.2) gives, for $s \in C_1$,

$$1 - F_n(s) = \frac{h_n(1-s)}{1+o(n,\rho;F,s)}\{u+o(n, \rho; F, s)\}$$

$$= h_n(1-s)\{u+o(n, \rho; F, s)\} ,$$

which concludes the proof of (2.3), since it is trivial for $s = 1$.

From (2.3),

$$1 - F_n(0) = \frac{\rho^n}{\left(\sum v_\alpha\right)^{-1}+\pi_n Q}\{u+o(n, \rho; F)\} ,$$

i.e.

$$\pi_n\{1-F_{n,\alpha}(0)\} = \frac{\rho^n\{u_\alpha+o(n,\rho;F)\}}{\pi_n^{-1}\left(\sum v_\alpha\right)^{-1}+Q}$$

$$= \frac{\rho^n u_\alpha\{1+o(n,\rho;F)\}}{Q\{1+o(n,\rho;F)\}}$$

$$= \rho^n u_\alpha Q^{-1}\{1+o(n,\ \rho;\ F)\}\ , \qquad (4.7)$$

which is equivalent to the assertion of (2.4) with $i = e_\alpha$. To extend (2.4) to any $i \in X_0$, we use a contradiction argument. If (2.4) is false, then there is an $\varepsilon > 0$ and some $i \in X_0$, a sequence $\{F^{(j)}\} \subset K$ and a sequence $\{n_j\}$, such that as $j \to \infty$, $n_j \to \infty$ and $\rho_{F^{(j)}} \to 1$ (so that if we define

$\varepsilon_j = \pi_{n_j}^{-1}\left(\rho_{F^{(j)}}\right)\rho_{F^{(j)}}^{n_j}$, $\varepsilon_j \to 0$ by lemma 4), and

$$\left|\varepsilon_j^{-1}P\left[Y_{n_j} \neq 0 \mid Y_0 = i\right] - i'u\left(F^{(j)}\right)/Q_{F^{(j)}}\right| > \varepsilon\ ,$$

$$j = 1,\ 2,\ \ldots\ . \qquad (4.8)$$

However, we know from (4.7) that as $j \to \infty$,

$$\varepsilon_j^{-1}\left[1-F_{n_j}^{(j)}(0)\right] - u\left(F^{(j)}\right)/Q_{F^{(j)}} \to 0\ .$$

The bounds $\theta \le u_\alpha(F) \le 1$ and lemma 3 *(x)* imply the existence of a subsequence, (j'), say, for which $u\left(F^{(j')}\right)/Q_{F^{(j')}}$ tends to a limit, x, say, as $j \to \infty$. We now put $x(j) = \varepsilon_j^{-1}\left[1-F_{n_j}^{(j)}(0)\right]$,

so that $x(j') \to x$. Then

$$\begin{aligned}\varepsilon_{j'}^{-1}P\left[Y_{n_{j'}} \neq 0 \mid Y_0 = i\right] &= \varepsilon_{j'}^{-1}\left\{1 - \left[F_{n_{j'}}^{(j')}(0)\right]^i\right\} \\ &= \varepsilon_{j'}^{-1}\left\{1 - \left[1-\varepsilon_{j'}x(j')\right]^i\right\} \\ &= i'x(j') + o(1) \\ &\to i'x \ .\end{aligned}$$

Thus by the triangle inequality,

$$\left|\varepsilon_{j'}^{-1}P\left[Y_{n_{j'}} \neq 0 \mid Y_0 = i\right] - i'u\left(F^{(j')}\right)/Q_{F^{(j')}}\right| \leq \varepsilon$$

for j' sufficiently large, which contradicts (4.8).

We now prove (2.5). For fixed $i \in X_0$,

$$\begin{aligned}&E[Y_n' \mid Y_n \neq 0,\ Y_0 = i] \\ &\quad = E[Y_n' \mid Y_0 = i]/P[Y_n \neq 0 \mid Y_0 = i] \\ &\quad = i'M^n\left[\rho^n\pi_n^{-1}Q^{-1}i'u\{1+o(n,\ \rho;\ F)\}\right]^{-1} \quad \text{by (2.4)}, \\ &\quad = i'\left(\frac{M}{\rho}\right)^n\pi_n Q(i'u)^{-1}\{1+o(n,\ \rho;\ F)\} \\ &\quad = \pi_n i'\{u[v'+o(n;\ F)]\}Q(i'u)^{-1}\{1+o(n,\ \rho;\ F)\} \quad \text{by lemma 5}, \\ &\quad = \pi_n Qv'\{1+o(n,\ \rho;\ F)\} \ ,\end{aligned}$$

which concludes the proof of theorem 2.

The proof of theorem 3 involves the use of k-dimensional Laplace transforms. We will assume that t is a fixed point in

$[0, \infty)^k$. We define ℓ_n as the vector with elements

$$\ell_{n,\alpha} = \exp\left\{-t_\alpha/(v_\alpha \pi_n Q)\right\} , \quad \alpha = 1, \ldots, k .$$

We show that the Laplace transform of the conditional distribution function in (2.6) can be approximated uniformly for large n , and $F \in K$ with small $|1-\rho_F|$, by $(1+t'1)^{-1}$. From (2.3),

$$\frac{i'\{1-F_n(\ell_n)\}}{i'\{1-F_n(0)\}} = \left\{\frac{i'[u+o(n,\rho;F,\ell_n)]}{i'[u+o(n,\rho;F)]}\right\}\left\{\frac{(v'1)^{-1}+\pi_n Q}{[v'(1-\ell_n)]^{-1}+\pi_n Q}\right\} .$$

It is easy to verify that the first ratio in curly brackets differs from unity by an error of $o(n, \rho; F)$. In addition, from the definition of ℓ_n ,

$$1 - \ell_{n,\alpha} \le t_\alpha/Q\pi_n v_\alpha \le t_\alpha/\theta^4 b\pi_n = o(n, \rho; F) , \qquad (4.9)$$

and a three-term expansion of the elements of ℓ_n gives

$$|v'(1-\ell_n)-t'1/Q\pi_n| \le \frac{1}{2} \sum v_\alpha t_\alpha^2/(Q\pi_n v_\alpha)^2 ,$$

so that

$$\begin{aligned} |\pi_n Q v'(1-\ell_n)-t'1| &\le \frac{1}{2} \sum t_\alpha^2/Q\pi_n v_\alpha \\ &\le \frac{1}{2} \sum t_\alpha^2 \pi_n^{-1}\theta^{-4} b^{-1} \\ &= o(n, \rho; F) . \end{aligned} \qquad (4.10)$$

From the above remarks,

$$\frac{i'\{1-F_n(\ell_n)\}}{i'\{1-F_n(0)\}} = \frac{(v'1)^{-1}v'(1-\ell_n)+\pi_n Q v'(1-\ell_n)}{1+\pi_n Q v'(1-\ell_n)}\{1+o(n,\ \rho;\ F)$$

$$= \frac{o(n,\rho;F)+t'1}{1+t'1+o(n,\rho;F)}\{1+o(n,\ \rho;\ F)\}$$

$$= \frac{t'1}{1+t'1}\{1+o(n,\ \rho;\ F)\}\ .$$

If we denote by $\Phi(n,\ F,\ t)$ the Laplace transform of Y_n^*, conditional on $Y_0 = i$ and $Y_n \neq 0$, then

$$\Phi(n,\ F,\ t) = \frac{i'\{F_n(\ell_n)-F_n(0)\}}{i'\{1-F_n(0)\}}$$

$$= 1 - \frac{t'1}{1+t'1}\{1+o(n,\ \rho;\ F)\}$$

$$= (1+t'1)^{-1} + o(n,\ \rho;\ F)\ .$$

The final steps in the proof of theorem 3 are omitted, since they follow the pattern of the more general argument in the proof of theorem IV.2.

Remark. As pointed out in section I.1, results similar to theorem 3, for a fixed critical process, were given by Mullikin (1963) (for a process with a more general state space) and Joffe and Spitzer (1967) (for a multitype Galton-Watson process). Both these results had a factor of $\sqrt{k}$ in the exponent of the distribution function. This is incorrect, as has already been noted by Weiner (1970), who gave the corresponding result for the multitype age dependent branching process.

CHAPTER IV

FURTHER RESULTS FOR PROCESSES WITH IMMIGRATION

Introduction

In this chapter, we give further results for the process $\{Z_n\}$, which was previously examined in chapter II. In the following section, we shall be concerned with the asymptotic behaviour of such a process with an underlying offspring process which is positively regular and critical. Theorem 1 can be regarded as a generalization of the theorem of Seneta (1970 c) to the critical multitype process $\{Z_n\}$. At the same time, it is analogous to theorem 6 of JS, which generalized Yaglom's theorem to the critical process $\{Y_n\}$. It is of interest to note in this latter context that the direction of the line of support of the limiting distribution is the same (*viz.*, through 0 and 1) for either $\{Y_n\}$ or $\{Z_n\}$ when normed by $\{Qnv_\alpha\}^{-1}$, as long as we condition on extinction in the former case (a similar remark is relevant when comparing theorem III.3 with theorem 2 of the present chapter).

The second theorem in this chapter is the multitype version of theorem 3 of Fahady *et al.* (1971) which, as pointed out there, subsumes theorem A of Quine and Seneta (1969).

1. The critical process

In this section we assume that we have a process $\{Z_n\}$ with offspring p.g.f. $F \in K$, and immigration p.g.f. $B(s)$, satisfying the conditions

(i) M is primitive;

(ii) $\rho_F = 1$;

(iii) $0 < Q_F < \infty$;

(iv) $B(1) = 1$;

(v) $0 < |\lambda| < \infty$. (1.1)

THEOREM 1. *Under the conditions* (1.1), *if* Z_n^* *denotes the vector of normed random variables* $Z_{n,\alpha}^* = Z_{n,\alpha}/\{nQv_\alpha\}$, *then for fixed* $i \in X$,

$$\sup_{x \in R^k} \left| P[Z_n^* \le x \mid Z_0 = i] - T(\lambda' u/Q, x) \right| \to 0$$

as $n \to \infty$.

Proof. By taking logarithms of both sides of (I, 3.9) we obtain

$$\log P_n(i, s) = \log\{F_n(s)\}^i + \sum_{r=0}^{n-1} \log B\{F_r(s)\} . \qquad (1.2)$$

As in the last chapter, we use Laplace transforms. To this end

define $\mathbf{p}_n$ as the vector whose α-th element is $p_{n,\alpha} = \exp\{-t_\alpha/(nQv_\alpha)\}$, where, as in the previous chapter, $\mathbf{t}$ is regarded as a fixed point in $[0, \infty)^k$. We note that as $n \to \infty$,

$$\left.\begin{array}{ll} \text{(i)} & \mathbf{p}_n \to \mathbf{1} \ ; \\ \text{(ii)} & nQ\mathbf{v}'(\mathbf{1}-\mathbf{p}_n) \to \mathbf{t}'\mathbf{1} \ . \end{array}\right\} \tag{1.3}$$

We shall be concerned with the asymptotic behaviour of $P_n(\mathbf{i}, \mathbf{p}_n)$. We continue to use the expression $h_r(\mathbf{s})$ which was introduced at (III, 1.6) (p. 32), although of course in this section we have $\rho = 1$ and $\pi_n = n$. The bulk of the proof consists of showing that the difference between each of the quantities

$$-\sum_{r=0}^{n-1} \log B\{\mathbf{F}_r(\mathbf{p}_n)\} \ ,$$

$$\sum_{r=0}^{n-1} [1-B\{\mathbf{F}_r(\mathbf{p}_n)\}] \ ,$$

$$\sum_{r=0}^{n-1} \lambda'\{\mathbf{1}-\mathbf{F}_r(\mathbf{p}_n)\} \ ,$$

$$\sum_{r=0}^{n-1} \lambda'\mathbf{u}h_r(\mathbf{1}-\mathbf{p}_n) \ ,$$

$$\lambda'\mathbf{u}Q^{-1} \log(1+\mathbf{t}'\mathbf{1})$$

is asymptotically negligible. Working in reverse order, we first note that for $r = 1, 2, \ldots$,

$$h_r(\mathsf{s}) - h_{r-1}(\mathsf{s}) = \frac{1}{(\mathsf{v}'\mathsf{s})^{-1}+rQ} - \frac{1}{(\mathsf{v}'\mathsf{s})^{-1}+(r-1)Q}$$

$$= \frac{-Q}{\{(\mathsf{v}'\mathsf{s})^{-1}+rQ\}\{(\mathsf{v}'\mathsf{s})^{-1}+(r-1)Q\}}$$

so that for each $\mathsf{s} \in C$, $h_r(\mathsf{s})$ is monotonic in r . Let us therefore define the integral

$$i(n, \mathsf{s}) \equiv \int_0^n \frac{1}{(\mathsf{v}'\mathsf{s})^{-1}+Qx}\, dx$$

$$= Q^{-1} \log\{1+Qn\mathsf{v}'\mathsf{s}\} . \qquad (1.4)$$

For any $\mathsf{s} \in C$,

$$\left|\sum_{r=0}^{n-1} h_r(\mathsf{s})-i(n-1, \mathsf{s})\right| \le h_0(\mathsf{s}) + h_{n-1}(\mathsf{s}) .$$

Now from (1.3),

$$h_0(1-\mathsf{p}_n) = \mathsf{v}'(1-\mathsf{p}_n) \to 0 ,$$

and

$$h_{n-1}(1-\mathsf{p}_n) = [\mathsf{v}'(1-\mathsf{p}_n)+(n-1)Q]^{-1}$$

$$\le [(n-1)Q]^{-1}$$

$$\to 0$$

as $n \to \infty$, so that

$$\sum_{r=0}^{n-1} h_r(1-\mathsf{p}_n) \to Q^{-1} \log\{1+Q(n-1)\mathsf{v}'(1-\mathsf{p}_n)\}$$

$$\to Q^{-1} \log(1+\mathsf{t}'1) , \qquad (1.5)$$

using both parts of (1.3). This assertion of course still obtains if we multiply both sides by $\lambda'\mathbf{u}$, which gives us the first of our approximations.

Our next task is to show that as $n \to \infty$,

$$\sum_{r=0}^{n-1} \lambda' \mathbf{u} h_r(1-\mathbf{p}_n) \to \sum_{r=0}^{n-1} \lambda'\{1-\mathbf{F}_r(\mathbf{p}_n)\} . \tag{1.6}$$

To do this, we first rearrange and combine equations (3.13) and (3.3) of JS to obtain (in our notation)

$$\begin{aligned} 1 - \mathbf{F}_n(\mathbf{s}) &= \mathbf{v}'\{1-\mathbf{F}_n(\mathbf{s})\}[\mathbf{u}+\mathbf{r}(n)] \\ &= \frac{\mathbf{v}'(1-\mathbf{s})[\mathbf{u}+\mathbf{r}(n)]}{1+nQ\mathbf{v}'(1-\mathbf{s})} \\ &= h_n(1-\mathbf{s})[\mathbf{u}+\mathbf{r}(n)] , \end{aligned} \tag{1.7}$$

where the elements of $\mathbf{r}(n)$ tend to zero as $n \to \infty$. Given $\varepsilon > 0$, let N be so large that $|r_\alpha(n)| < \frac{1}{4}\varepsilon Q/\log(1+\mathbf{t}'\mathbf{1})$, $\alpha = 1, \ldots, k$, and (in view of (1.5))

$$\sum_{r=0}^{n-1} h_r(1-\mathbf{p}_n) \le 2Q^{-1} \log(1+\mathbf{t}'\mathbf{1}) ,$$

for $n > N$. Then, in view of (1.7), for $n-1 > N$,

$$\begin{aligned} \left| \sum_{r=N+1}^{n-1} \{1-F_{r,\alpha}(\mathbf{p}_n)\} - \sum_{r=N+1}^{n-1} u_\alpha h_r(1-\mathbf{p}_n) \right| \\ &\le \frac{\varepsilon Q}{4 \log(1+\mathbf{t}'\mathbf{1})} \sum_{r=N+1}^{n-1} h_r(1-\mathbf{p}_n) \\ &\le \frac{1}{2}\varepsilon . \end{aligned}$$

Furthermore, for $r = 0, 1, \ldots, N$,

$$h_r(1-p_n) = \{v'(1-p_n)\}^{-1+rQ} \le v'(1-p_n) \to 0 ,$$

from (1.3, i), and

$$1 - F_{r,\alpha}(p_n) \le \left\{M^r(1-p_n)\right\}_\alpha \quad \text{by iteration of (III, 3.11),}$$

$$\le k^{N-1} m_0^N \sum_\beta (1-p_{n,\beta}) \to 0 , \tag{1.8}$$

where $m_0 \equiv \max_{\alpha,\beta} (M_{\alpha\beta})$ is finite because of (1.1, iii). The above remarks indicate that

$$\sum_{r=0}^{n-1} u_\alpha h_r(1-p_n) \to \sum_{r=0}^{n-1} \{1-F_{r,\alpha}(p_n)\} , \tag{1.9}$$

which is equivalent to (1.6).

The next step requires the two term expansion

$$\left.\begin{aligned} &1 - B(s) = \lambda'(1-s) - D[s](1-s) , \quad s \in C , \\ &\text{where, for } 1 \le \alpha \le k , \\ &0 \le D_\alpha[s] \le \lambda_\alpha ; \quad D_\alpha[s] \to 0 \text{ as } s \to 1 \text{ in } C . \end{aligned}\right\} \tag{1.10}$$

This expansion can be deduced from the argument in JS at equations (4.2) and (4.3), and summing with respect to the measure $b(i)$. The detailed derivation of a similar expansion is given in the next section of this chapter. Since $F_r(s) \to 1$ uniformly for $s \in C$ as $r \to \infty$ (JS eq. (3.2)), there is some positive integer

N such that for $n > N$,

$$D_\alpha|F_r(p_n)| < \frac{1}{4}\,\varepsilon Q/\{u_\alpha \log(1+t'1)\}\ , \quad 1 \le \alpha \le k\ . \qquad (1.11)$$

In view of (1.9) and (1.5), we may assume without loss of generality that N is so large that we have, for $\alpha = 1, \ldots, k$, and $n > N$,

$$\sum_{r=0}^{n-1} \{1-F_{r,\alpha}(p_n)\} \le 2u_\alpha Q^{-1} \log(1+t'1)\ . \qquad (1.12)$$

Combining (1.10)-(1.12), we see that for $n > N+1$,

$$\begin{aligned}
&\left|\sum_{r=0}^{n-1} |1-B\{F_r(p_n)\}| - \sum_{r=0}^{n-1} \lambda'\{1-F_r(p_n)\}\right| \\
&\quad = \left|\sum_{r=0}^{n-1} D[F_r(p_n)]\{1-F_r(p_n)\}\right| \\
&\quad \le \sum_{r=0}^{N} \lambda'\{1-F_r(p_n)\} + \frac{\varepsilon Q}{4\log(1+t'1)} \sum_{r=N+1}^{n-1} \sum_{\alpha=1}^{k} \{1-F_{r,\alpha}(p_n)\} \\
&\quad \le \sum_{r=0}^{N} \lambda'\{1-F_r(p_n)\} + \frac{1}{2}\,\varepsilon\ , \text{ since } \sum u_\alpha = 1\ , \\
&\quad \le \varepsilon \qquad (1.13)
\end{aligned}$$

for n sufficiently large, by (1.8).

We finish the string of approximations as follows. Given $\varepsilon > 0$, put $\varepsilon^* = \min[1/2,\ Q\varepsilon\ /\ \{2\lambda' u \log(1+t'1)\}]$. It is evident from (1.10) that for $s \in C$,

$$\begin{aligned}
1 - B\{F_r(s)\} &\le \lambda'\{1-F_r(s)\} \\
&\le \varepsilon^*\ , \qquad (1.14)
\end{aligned}$$

if r is greater than some positive integer N, by JS (eq. 3.2). At the same time, we can use (1.8) to choose another integer N^* so that

$$1 - B\{F_r(p_n)\} \le \lambda'\{1-F_r(p_n)\} \le \varepsilon^* \tag{1.15}$$

for $r = 0, 1, \ldots, N$, and $n > N^*$. In view of (1.14), (1.15) in fact holds for all r if $n > N^*$. Furthermore, equations (1.13), (1.6) and (1.5) indicate that as $n \to \infty$,

$$\sum_{r=0}^{n-1} [1-B\{F_r(p_n)\}] \to \lambda' u Q^{-1} \log(1+t'1) ,$$

i.e. for some integer N^+,

$$\sum_{r=0}^{n-1} [1-B\{F_r(p_n)\}] < 2\lambda' u Q^{-1} \log(1+t'1)$$

for $n > N^+$. Since for $x \in [0, 1)$,

$$\begin{aligned} |\log(1-x)+x| &= \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \ldots \\ &\le \tfrac{1}{2}x^2/(1-x) , \end{aligned} \tag{1.16}$$

it follows from (1.15) that if $n > N^*$,

$$\begin{aligned} \left|\sum_{r=0}^{n-1} \log B\{F_r(p_n)\} + \sum_{r=0}^{n-1} [1-B\{F_r(p_n)\}]\right| &\le \tfrac{1}{2}\sum_{r=0}^{n-1} [1-B\{F_r(p_n)\}]^2/\tfrac{1}{2} \\ &\le \varepsilon^* \sum_{r=0}^{n-1} [1-B\{F_r(p_n)\}] \\ &\le \varepsilon^* \, 2\lambda' u Q^{-1} \log(1+t'1) \\ &\le \varepsilon , \end{aligned} \tag{1.17}$$

the third inequality holding if n is also greater than N^+, and the last inequality being a consequence of the definition of ε^*.

The successive approximations at (1.5), (1.6), (1.13) and (1.17), together with (1.2) and the fact (from (1.1, iii)), which implies $F(s) \neq Ms$, and JS (eq. 3.2) that $\{F_n(p_n)\}^i \to 0$ as $n \to \infty$, combine to give

$$\log P_n(i, p_n) \to \lambda' u Q^{-1} \log(1+t'1) ,$$

i.e.

$$P_n(i, p_n) \to (1+t'1)^{-\lambda' u/Q}$$

as $n \to \infty$. The theorem now follows from the continuity theorem. A demonstration of the correspondence between the above Laplace transform and the distribution function $T(\lambda' u/Q, x)$ is deferred until the next section.

Remark. The behaviour of (Z_n), when conditions (1.1, iii) or (1.1, v) are relaxed, is not known in the multitype case. Some results in this direction have been obtained for the single-type situation by Seneta (1970 c). He shows that, if $Q_F = \infty$, a limiting distribution may sometimes exist for the unnormed random variables Z_n, and if $\lambda = \infty$, a norming other than n^{-1} is sometimes appropriate.

2. Paracritical processes with immigration

In this section we give a version of theorem 1 of this chapter which extends it (at the expense of stronger conditions) in the same sort of way that theorem 3 of chapter III extends theorem 6 of JS. Thus, we will be concerned with processes $\{Z_n\}$ which have offspring p.g.f.'s $F \in K$ (K was defined in chapter III). At the same time, we will show that "uniform" convergence for the class of processes to a class of limiting distributions still obtains over a class of immigration p.g.f.'s as well. To this end, we shall exhibit the dependence of the vector mean, λ, on the immigration p.g.f. $B \equiv B(s)$ by writing $\lambda \equiv \lambda(B)$, and $\lambda_\alpha \equiv \lambda_\alpha(B)$.

Given constants $d_1 > 0$, $d_2 < \infty$, we denote by $J \equiv J(d_1, d_2)$ the class of all p.g.f.'s $B(s)$ which satisfy

$$\left.\begin{array}{ll} \text{(i)} & B(1) = 1 \ ; \\ \text{(ii)} & |\lambda(B)| \geq d_1 \ ; \\ \text{(iii)} & \sum\limits_{\alpha,\beta} \partial^2 B(1)/\partial s_\alpha \partial s_\beta \leq d_2 \ . \end{array}\right\} \tag{2.1}$$

It follows from the last of these conditions, and from the facts that $u_\alpha(F) \leq 1$ for $\alpha = 1, \ldots, k$, and $F \in K$, that

$$\lambda'(B)u(F) \leq \lambda'(B)1 \leq d_3 \tag{2.2}$$

for all $F \in K$ and $B \in J$, for some $d_3 < \infty$.

The arguments of the function $o(\cdot)$, which was introduced below (III, 1.6), will be extended in this section to include B. This will indicate that the behaviour of $o(n, \rho; \mathsf{F}, \mathsf{s}, B)$ is the the same as that of $o(n, \rho; \mathsf{F}, \mathsf{s})$, uniformly for $B \in J$. Similar remarks apply to the vector $\mathsf{o}(\cdot)$.

THEOREM 2. *For the class of processes* $\{\mathsf{Z}_n\}$ *with* $\mathsf{F} \in K$ *and* $B \in J$, *let* Z_n^* *denote the vector of normed random variables* $Z_{n,\alpha}^* = Z_{n,\alpha}/\{\pi_n \; Qv_\alpha\}$, $\alpha = 1, \ldots, k$. *Then for any fixed vector* $\mathsf{i} \in X$,

$$\sup_{\mathsf{x} \in R^k} \left| P[\mathsf{Z}_n^* \le \mathsf{x} \mid \mathsf{Z}_0 = \mathsf{i}] - T(\xi, \mathsf{x}) \right| = o(n, \rho; \mathsf{F}, B),$$

where $\xi = \lambda'(B)\mathsf{u}(\mathsf{F})/Q_{\mathsf{F}}$.

Proof. The proof follows the general lines of that of theorem 1 of this chapter. However, the differences in detail are so great that nothing short of a full account is satisfactory. This will now be given. As before, the starting point is (1.2). We recall the definition of ℓ_n (given above (III, 4.9)), and its properties (III, 4.9) and (III, 4.10). We shall show that the difference between each of the quantities $-\sum_{r=0}^{n-1} \log B\{\mathsf{F}_r(\ell_n)\}$,

$$\sum_{r=0}^{n-1} [1-B\{\mathsf{F}_r(\ell_n)\}], \quad \sum_{r=0}^{n-1} \lambda'\{1-\mathsf{F}_r(\ell_n)\}, \quad \sum_{r=0}^{n-1} \lambda'\mathsf{u}h_r(1-\ell_n),$$

$\lambda'\mathsf{u}Q^{-1} \log(1+\mathsf{t}'1)$ is of order $o(n, \rho; \mathsf{F}, B)$. Working in reverse order, first consider the difference

$$h_r(s)-h_{r-1}(s)$$

$$= \frac{\rho^r}{(v's)^{-1}+\pi_r Q} - \frac{\rho^{r-1}}{(v's)^{-1}+\pi_{r-1}Q}$$

$$= \left[\frac{\rho^{r-1}}{\left\{(v's)^{-1}+\pi_r Q\right\}\left\{(v's)^{-1}+\pi_{r-1}Q\right\}}\right]\left\{\rho(v's)^{-1}+\rho\pi_{r-1}Q-(v's)^{-1}-\pi_r Q\right\} .$$

The first factor is always positive; the second is equal to $(v's)^{-1}(\rho-1) - Q\rho^{-1}$, which is non-negative if $v's \le \rho(\rho-1)Q^{-1}$ and negative if $v's > \rho(\rho-1)Q^{-1}$. In either case, then, $h_r(s)$ is monotonic in r . Define

$$i(n, \rho, s) = \begin{cases} \int_0^n \rho^x\left[(v's)^{-1} + \frac{Q(1-\rho^x)}{\rho(1-\rho)}\right]^{-1} dx , & \rho \ne 1 , \\ \int_0^n \left[(v's)^{-1}+Qx\right]^{-1} dx , & \rho = 1 . \end{cases}$$

For any v and s ,

$$\left|\sum_{r=0}^{n-1} h_r(s)-i(n-1, \rho, s)\right| \le h_0(s) + h_{n-1}(s) .$$

The integral can easily be evaluated as

$$i(n, \rho, s) = \begin{cases} \frac{\rho(\rho-1)}{Q \log \rho} \log\{1+Q\pi_n v's\} , & \rho \ne 1 , \\ Q^{-1} \log\{1+Q\pi_n v's\} , & \rho = 1 . \end{cases}$$

From (III, 4.9),

$$h_0(1-\ell_n) = v'(1-\ell_n) = o(n, \rho; F) ,$$

and

$$h_{n-1}(1-\ell_n) = \rho^{n-1}\left[\{v'(1-\ell_n)\}^{-1}+\pi_{n-1}Q\right]^{-1}$$
$$\leq \rho^{n-1}\pi_{n-1}^{-1}/Q$$
$$= o(n, \rho; F) ,$$

from lemmas III, 3 *(x)* and III, 4 *(ii)*. Thus

$$\sum_{r=0}^{n-1} h_r(1-\ell_n) - i(n-1, \rho, 1-\ell_n) = o(n, \rho; F) . \qquad (2.3)$$

For $\rho \neq 1$, the mean value theorem implies

$$\log \rho = \frac{1}{\theta_\rho}(\rho-1)$$

for some θ_ρ with $\rho < \theta_\rho < 1$ or $1 < \theta_\rho < \rho$, depending on ρ , so that

$$\frac{\rho(\rho-1)}{\log \rho} = \rho\theta_\rho = 1 + o(\rho; F) .$$

Inspection of (III, 4.9) shows

$$\pi_{n-1}(1-\ell_{n,\alpha}) \leq \pi_n(1-\ell_{n,\alpha}) \leq t_\alpha/\theta^4 b ,$$

so that $\log\{1+Q\pi_{n-1}v'(1-\ell_n)\}$ is uniformly bounded, and hence

$$|i(n-1, \rho, 1-\ell_n)-\log\{1+Q\pi_{n-1}v'(1-\ell_n)\}| = o(n, \rho; F) . \qquad (2.4)$$

Bearing (III, 4.10) in mind, use of a triangle inequality gives

$$
\begin{aligned}
|\pi_{n-1}Qv'(1-\ell_n)-t'1| &\le (\pi_n-\pi_{n-1})Qv'(1-\ell_n) + o(n, \rho; F) \\
&= \rho^{n-2}Qv'(1-\ell_n) + o(n, \rho; F) \\
&\le t'1\rho^{n-2}\pi_n^{-1} + o(n, \rho; F) \\
&= o(n, \rho; F) , \qquad (2.5)
\end{aligned}
$$

the second inequality being a consequence of (III, 4.9). Combining (2.3) - (2.5), and bearing in mind (2.2),

$$\sum_{r=0}^{n-1} \lambda' u h_r(1-\ell_n) - \lambda' u Q^{-1} \log(1+t'1) = o(n, \rho; F, B) . \qquad (2.6)$$

The next step is to show

$$\sum_{r=0}^{n-1} \lambda' u h_r(1-\ell_n) - \sum_{r=0}^{n-1} \lambda'\{1-F_r(\ell_n)\} = o(n, \rho; F, B) . \qquad (2.7)$$

We can choose $\{N_1, \delta_1\}$ so that the error term in (III, 2.3) is elementwise bounded by $\frac{1}{4}\varepsilon\theta^3 b/\log(1+t'1)$ for $s \in C$, $n > N_1$, $F \in K : |1-\rho_F| < \delta_1$. By (2.1), (2.6) and lemma III, 3 parts *(iii)* and *(x)*, we can find $\{N_2, \delta_2\}$ such that $\sum_{r=0}^{n-1} h_r(1-\ell_n)$ is bounded by $2 \log (1+t'1)/\theta^3 b$ for $n > N_2$, $F \in K : |1-\rho_F| < \delta_2$. Now it follows from (III, 2.3) that for $n : n-1 > N \equiv \max(N_1, N_2)$ and $F \in K$ with $|1-\rho_F| < \delta \equiv \min(\delta_1, \delta_2)$,

$$\left| \sum_{r=N+1}^{n-1} u_\alpha h_r(1-\ell_n) - \sum_{r=N+1}^{n-1} \{1-F_{r,\alpha}(\ell_n)\} \right|$$

$$\leq \frac{1}{4} \sum_{r=N+1}^{n-1} h_r(1-\ell_n)\varepsilon\theta^3 b/\log(1+t'1)$$

$$\leq \frac{1}{2}\varepsilon .$$

Furthermore, for $r = 0, 1, \ldots, N$,

$$h_r(1-\ell_n) = \frac{\rho^r}{\{v'(1-\ell_n)\}^{-1}+\pi_r Q} \leq \rho^r v'(1-\ell_n) = o(n, \rho; F) ,$$

from (III, 4.9), and

$$1 - F_{r,\alpha}(\ell_n) \leq \left\{M^r(1-\ell_n)\right\}_\alpha \leq k^{N-1}d^N \sum_\beta (1-\ell_{n,\beta})$$

$$= o(n, \rho; F) , \quad (2.8)$$

where the first inequality is obtained by iterating (III, 3.11), and the second from lemma III, 3 *(vii)* . Thus

$$\sum_{r=0}^{n-1} u_\alpha h_r(1-\ell_n) - \sum_{r=0}^{n-1} \{1-F_{r,\alpha}(\ell_n)\} = o(n, \rho; F) ,$$

which is equivalent to (2.7) because of (2.2).

At this point, we make use of the assumption (2.1, iii) to obtain a two-term expansion of $B(s)$ in which the remainder term satisfies (2.11) below. The uniformity of its behaviour with respect to $B \in J$ is the only reason for the presence of (2.1, iii). If we were considering the paracritical situation for a single immigration p.g.f. $B(s)$, then the assumption (1.1, v) would

suffice. We recall that during the derivation of (III, 3.18) we obtained the identity (with $\psi(t) \equiv (1-t)^i$)

$$\psi(t) - 1 = \frac{d}{d\xi}\psi(\xi t)\Big|_{\xi=0} + \int_0^1 (1-\xi)\frac{d^2}{d\xi^2}\psi(\xi t)d\xi$$

$$= - t'i + \sum_\alpha \sum_\beta t_\alpha t_\beta i_\alpha [i_\beta - \delta(\alpha, \beta)] I(\alpha, \beta, 1-t, i) \quad (2.9)$$

where $I(\cdot)$ is defined below III, 3.15. If we define $D^*[s]$ as the vector whose α-th element is given by

$$D^*_\alpha[1-s] = \sum_{i \in X} b(i) i_\alpha \left\{ \sum_\beta s_\beta [i_\beta - \delta(\alpha, \beta)] I(\alpha, \beta, s, i) \right\} ,$$

then replacing t by $s \equiv 1 - t$ in (2.9), summing over the measure $b(i)$, and using Fubini's theorem (valid in view of (2.1, iii)), we obtain

$$B(s) - 1 = - \lambda'(1-s) + D^*[s](1-s) . \quad (2.10)$$

Since $0 \le I(\cdot) \le \frac{1}{2}$,

$$0 \le D^*_\alpha[s] \le \frac{1}{2} \sum_\beta (1-s_\beta) \partial^2 B(1)/\partial s_\alpha \partial s_\beta$$

$$\le \frac{1}{2} d_2 \sum_\beta (1-s_\beta) , \quad (2.11)$$

by (2.1, iii). It should be noted that with a little more work (*c.f.* the argument between (III, 3.16) and (III, 3.17)) we could show that $D^*[s]$ is identical to the quantity $D[s]$ in (1.10). The extra work was necessary in chapter III since it was required that the remainder $E(s)$ satisfy both (III, 3.17) and have the

property $t \le s \Rightarrow E(t) \ge E(s)$. The present remainder term does in fact have this "monotonicity" property, but we do not require it.

To show that

$$\sum_{r=0}^{n-1} \lambda'\{1-F_r(\ell_n)\} - \sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}] = o(n, \rho; F, B) , \qquad (2.12)$$

we note that

$$\left|\sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}] - \sum_{r=0}^{n-1} \lambda'\{1-F_r(\ell_n)\}\right|$$

$$\le \frac{1}{2} d_2 \sum_{r=0}^{n-1} \left[\sum_{\alpha} \{1-F_{r,\alpha}(\ell_n)\}\right]^2 ,$$

from (2.11). It follows from (2.6) and (2.7) that

$$\sum_{r=0}^{n-1} \{1-F_{r,\alpha}(\ell_n)\} - u_\alpha Q^{-1} \log(1+t'1) = o(n, \rho; F) ,$$

i.e. we can choose $\{N_1, \delta_1\}$ so that

$$\sum_{r=0}^{n-1} \sum_{\alpha} \{1-F_{r,\alpha}(\ell_n)\} \le 2\theta^{-3}b^{-1} \log(1+t'1)$$

for $n > N_1$, $F \in K : |1-\rho_F| < \delta_1$. At the same time, we can choose $\{N_2, \delta_2\}$ so that, by (III, 2.1),

$$\sum_{\alpha} \{1-F_{r,\alpha}(\ell_n)\} < \varepsilon\left[2\theta^{-3}b^{-1}d_2 \log(1+t'1)\right]^{-1}$$

for $r > N_2$, $F \in K : |1-\rho_F| < \delta_2$, and all n . For $\{N, \delta\}$ defined as above, $|1-\rho_F| < \delta$ implies

$$\sum_{r=N+1}^{n-1} \left[\sum_\alpha \{1-F_{r,\alpha}(\ell_n)\}\right]^2$$

$$\leq \sum_{r=N+1}^{n-1} \left\{\varepsilon\left[2\theta^{-3}b^{-1}d_2 \log(1+t'1)\right]^{-1}\right\} \sum_\alpha \left\{1-F_{r,\alpha}(\ell_n)\right\}$$

$$\leq \varepsilon/d_2 ,$$

so that

$$\left|\sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}] - \sum_{r=0}^{n-1} \lambda'\{1-F_r(\ell_n)\}\right|$$

$$\leq \frac{1}{2} d_2 \sum_{r=0}^{N} \left[\sum_\alpha \{1-F_{r,\alpha}(\ell_n)\}\right]^2 + \frac{1}{2}\varepsilon ,$$

and the first term can be made smaller than $\frac{1}{2}\varepsilon$ for n sufficiently large and $|1-\rho_F|$ sufficiently small by (2.8). This concludes the proof of (2.12).

Given $\varepsilon > 0$, let $\varepsilon^* = \min\left[\frac{1}{2}, \frac{1}{2}\varepsilon\theta^3 b/d \log(1+t'1)\right]$. We can choose $\{N_1, \delta_1\}$ by (III, 2.1) so that for $r > N_1$, $F \in K : |1-\rho_F| < \delta_1$, $1 - F_{r,\alpha}(s) < \varepsilon^*/d$, for $\alpha = 1, \ldots, k$, and $s \in C$, so that by (2.10), (2.11) and (2.2),

$$1 - B\{F_r(s)\} \leq \lambda'(1-F_r(s)) < \varepsilon^* . \qquad (2.13)$$

We can use (2.8), (2.10) and (2.11) to choose $\{N_2, \delta_2\}$ so that

for $n > N_2$, $F \in K$: $|1-\rho_F| < \delta_2$, and $r = 0, 1, \ldots, N_1$,

$$1 - B\{F_r(\ell_n)\} \leq \varepsilon^* . \tag{2.14}$$

It is evident from (2.13) that (2.14) is in fact true for all r if $n > N_2$ and $F \in K$ is such that $|1-\rho_F| < \min(\delta_1, \delta_2)$.

From (2.6), (2.7) and (2.12) we see that the difference between $\sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}]$ and $\lambda' u Q^{-1} \log(1+t'1)$ is of order $o(n, \rho; F, B)$. Hence for n $(> N_2)$ sufficiently large, and $F \in K$ with $|1-\rho_F|$ $(> \min(\delta_1, \delta_2))$ sufficiently small (recalling (2.2) and lemma III, 3 *(x)*),

$$\sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}] \leq 2d\theta^{-3}b^{-1} \log(1+t'1) , \tag{2.15}$$

and, from (1.17),

$$\left|\sum_{r=0}^{n-1} \log B\{F_r(\ell_n)\} + \sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}]\right| \leq \frac{1}{2} \sum_{r=0}^{n-1} [1-B\{F_r(\ell_n)\}]^2/\frac{1}{2}$$

$$\leq \sum_{r=0}^{n-1} \varepsilon^* [1-B\{F_r(\ell_n)\}] ,$$

both inequalities following from (2.14),

$$\leq \varepsilon \tag{2.16}$$

from (2.15) and the definition of ε^* . The successive approximations in (2.6), (2.7), (2.12) and (2.16), combined with

(1.2), and the fact that (from III, 2.1) $\log\{F_n(\ell_n)\}^i = o(n, \rho; F)$, give

$$\log P_n(i, \ell_n) + \lambda' u Q^{-1} \log(1+t'1) = o(n, \rho; F, B) ,$$

or, equivalently,

$$P_n(i, \ell_n) = (1+t'1)^{-\lambda' u/Q} + o(n, \rho; F, B) . \qquad (2.17)$$

To show that theorem 2 is a consequence of (2.17), we first take a closer look at the distribution function $T(\xi, x)$, defined in section 1 of chapter III (p. 33). We consider k random variables $V_1, \ldots, V_k$ which satisfy

(i) $P[V_1 = V_2 = \ldots = V_k] = 1$,

(ii) $$P[V_1 \le x] = \begin{cases} \int_0^x \frac{1}{\Gamma(\xi)} \eta^{\xi-1} e^{-\eta} d\eta & x > 0 \\ 0 & x \le 0 . \end{cases}$$

The joint Laplace transform of $V \equiv \{V_1, \ldots, V_k\}'$ is

$$E[e^{-t'V}] = E\left[e^{-V_1 \sum t_\alpha}\right]$$

$$= \int_0^\infty \frac{1}{\Gamma(\xi)} \eta^{\xi-1} e^{-\eta\left(1+\sum t_\alpha\right)} d\eta$$

$$= \left(1 + \sum t_\alpha\right)^{-\xi} .$$

Furthermore, the joint distribution function is easily seen to be given by

$$P[V \le x] = P\left[V_1 \le \min_{\alpha} x_\alpha\right] = T(x, \xi) \ , \quad x \in R^k \ .$$

If theorem 2 is false, then there exists some ancestral vector $i \in X$, and $\varepsilon > 0$, and sequences $\{F^{(n)}\} \subset K$, $\{B^{(n)}\} \subset J$ such that $\rho_{F^{(n)}} \to 1$ as $n \to \infty$, but

$$\sup_{x \in R^k} |T(x, \xi_n) - F(x, n, F^{(n)}, B^{(n)}, i)| > \varepsilon \ ,$$

$$n = 1, 2, \ldots , \qquad (2.18)$$

where $\xi_n = \lambda'(B^{(n)})u(F^{(n)})/Q_{F^{(n)}}$, and $F(\cdot)$ is the joint distribution function of Z_n^*, conditional on $Z_0 = i$. From (2.1, ii), (2.2) and lemma III, 3 parts *(iii)* and *(x)*,

$$\theta^2 d_1/d \le \xi_n \le d_3/\theta^3 b \ , \quad n = 1, 2, \ldots ,$$

so that there is a subsequence $\{n_j\}$ for which $\xi_{n_j} \to \xi^*$, say, as $n_j \to \infty$. For each $t \in R^k$, $(1+t'1)^{-\xi_{n_j}} \to (1+t'1)^{-\xi^*}$, and, since this latter quantity is the Laplace transform of $T(x, \xi^*)$ (as shown above), we must also have $T\left(x, \xi_{n_j}\right) \to T(x, \xi^*)$ at every point $x \in R^k$, by the continuity theorem. This is well known to imply (see, for example, Rao (1965))

$$\sup_{x \in R^k} \left|T\left(x, \xi_{n_j}\right) - T(x, \xi^*)\right| \to 0 \ . \qquad (2.19)$$

It follows from (2.17) that for the sequences $\{F^{(n)}\}$, $\{B^{(n)}\}$,

$$P_{n_j}\left[i, \ell_{n_j}\right] \to (1+t'l)^{-\xi^*}$$

as $n_j \to \infty$, for each $t \in [0, \infty)^k$. The continuity theorem implies

$$\sup_{x \in R^k} \left| F\left(x, n_j, F^{(n_j)}, B^{(n_j)}, i\right) - T(x, \xi^*) \right| \to 0 ,$$

which, combined with (2.19), gives

$$\sup_{x \in R^k} \left| F\left(x, n_j, F^{(n_j)}, B^{(n_j)}, i\right) - T\left(x, \xi_{n_j}\right) \right| \to 0 ,$$

which contradicts (2.18) and proves theorem 2.

CHAPTER V

THE MOMENT STRUCTURE

1. Introduction

In this chapter we give recursions and asymptotic expressions for some moments of both (Y_n) and (Z_n). We define covariance matrices for the offspring mechanism for $1 \le \alpha, \nu, \mu \le k$:

$$(V_\alpha)_{\nu\mu} = \frac{\partial^2 F_{1,\alpha}(1)}{\partial s_\nu \partial s_\mu} + \delta(\nu, \mu) M_{\alpha\nu} - M_{\alpha\nu} M_{\alpha\mu} ,$$

and for the immigration mechanism the covariance matrix

$$(W)_{\nu\mu} = \frac{\partial^2 B(1)}{\partial s_\nu \partial s_\mu} + \delta(\nu, \mu)\lambda_\nu - \lambda_\nu \lambda_\mu .$$

Except where otherwise indicated, we assume that these matrices are elementwise finite.

For two matrices

$$A = \|A_{\alpha\beta}\| \quad \alpha = 1, \ldots, m ; \quad \beta = 1, \ldots, n$$

$$B = \|B_{\alpha\beta}\| \quad \alpha = 1, \ldots, p ; \quad \beta = 1, \ldots, q ,$$

we define the $mp \times nq$ Kronecker product (Marcus & Minc, 1964, p. 8) as

$$A \times B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix} .$$

For any array $(A)_{i_1 i_2 \cdots i_s}$ which has $s \geq 2$ subscripts which each run from 1 through k, we define $\mathcal{A}$ (denoted by the corresponding script capital) to be the $k^s \times 1$ column vector consisting of the elements of A written in lexicographical order. For example, if $s = k = 2$, then

$$\mathcal{A}' = [A_{11} \quad A_{12} \quad A_{21} \quad A_{22}] .$$

It is easy to verify that when $s = 2$ (*i.e.* we have arrays A, B which are $k \times k$ matrices), if $G = A'BA$, then

$$\mathcal{G} = (A' \times A')\mathcal{B} . \tag{1.1}$$

In dealing with asymptotic means and covariances in the sequel, we will be concerned with

$$H \equiv \begin{bmatrix} M' & 0 \\ \underline{\underline{V}} & M' \times M' \end{bmatrix} , \tag{1.2}$$

where

$$\underline{\underline{V}} = [V_1, \ldots, V_k] , \tag{1.3}$$

and 0 denotes a zero matrix of order $k \times k^2$. H is evidently

reducible. Its spectrum is the union of the spectra of M and $M \times M$, *i.e.* the eigenvalues of M and the products of all pairs. On the assumption that M is primitive, we list some properties of H :

If $\rho < 1$, then the spectral radius of H is also ρ, and if the corresponding eigenvalues are given by

$$a'H = \rho a' \ , \quad Hb = \rho b \ ,$$

then these equations are satisfied by†

$$a = \begin{bmatrix} u \\ \\ 0 \end{bmatrix} , \quad b = \begin{bmatrix} v \\ \\ (\rho I - M' \times M')^{-1} \underline{\underline{V}} v \end{bmatrix} . \tag{1.4}$$

If $\rho > 1$, then the spectral radius of H is ρ^2, and if the corresponding eigenvalues are given by

$$x'H = \rho^2 x' \ , \quad Hy = \rho^2 y \ ,$$

then it is easy to verify that

$$x = \begin{bmatrix} (\rho^2 I - M)^{-1} \underline{\underline{V}}' (u \times u) \\ \\ u \times u \end{bmatrix} , \quad y = \begin{bmatrix} 0 \\ \\ v \times v \end{bmatrix} . \tag{1.5}$$

Finally, if $\rho = 1$, H will have two unit eigenvalues. This means that it is more difficult to study the behaviour of H^n for large n. We shall therefore adopt a different approach to the analysis of asymptotic moments in this critical case.

† Here, and in the rest of this chapter, we use the symbols 0 and 1 to denote vectors of zeros and ones whose orders depend on context.

2. The process without immigration

This section provides alternative, more general proofs of results due to Pollard (1966), who, working with Galton-Watson processes of a very simple form in a demographic context, showed that the mean, and the second and higher non-central moments of the successive generations of the process satisfy a linear recurrence relation. In addition, he observed that the linear coefficients remain unchanged if the second non-central moments are replaced by the corresponding central moments. The present approach, using p.g.f.'s, confirms these results for the most general process, and gives the coefficients of the recurrence relation in an explicit form which, in any practical situation, could be worked out easily by hand from the p.g.f.'s, for at least the first three moments. Furthermore, it is shown that the substitution of variances and covariances for second order non-central moments in the recursion can be accompanied by a similar change from third order non-central to central moments. Finally, asymptotic moments are considered.

If we denote the column vector of first moments of Y_n by d_n, then it follows from the definition of the process that

$$\mathsf{d}_n \equiv \begin{bmatrix} d_{n,1} \\ \cdot \\ \cdot \\ \cdot \\ d_{n,k} \end{bmatrix} \equiv E[\mathsf{Y}_n] = \mathsf{M}'\mathsf{d}_{n-1} \quad n = 1, 2, \ldots . \tag{2.1}$$

Let us now consider the second moments of Y_n. If we denote by

$\mathrm{Var}[\cdot]$ the covariance matrix operator, then by definition,

$$E[Y_{n+1,\alpha}Y_{n+1,\beta} \mid Y_n] = E[Y_{n+1,\alpha} \mid Y_n]E[Y_{n+1,\beta} \mid Y_n] + (\mathrm{Var}[Y_{n+1} \mid Y_n])_{\alpha\beta}$$

$$= \left(M'Y_nY_n'M + \sum_{\nu} V_\nu Y_{n,\nu}\right)_{\alpha\beta} . \tag{2.2}$$

Hence, taking expectations over Y_n ,

$$C^*_{n+1} \equiv \|E[Y_{n+1,\alpha}Y_{n+1,\beta}]\| = M'C^*_nM + \sum_{\nu=1}^{k} V_\nu d_{n,\nu} , \tag{2.3}$$

or, in view of (1.3) and (2.1),

$$C^*_{n+1} = (M' \times M')C^*_n + \underline{\underline{V}}\, d_n . \tag{2.4}$$

The derivation of (2.3) is exactly that of Harris (1963, p. 37), but we reproduce it here since we need to extend the technique later on. Equations (2.1) and (2.4) can be combined to give the linear recurrence relation

$$\begin{bmatrix} d_{n+1} \\ C^*_{n+1} \end{bmatrix} = \begin{bmatrix} M' & 0 \\ \underline{\underline{V}} & M'\times M' \end{bmatrix} \begin{bmatrix} d_n \\ C^*_n \end{bmatrix} . \tag{2.5}$$

Furthermore, if we denote the covariance matrix of Y_n by C_n , then by definition,

$$C_n = C^*_n - d_nd_n' .$$

Substituting in (2.3), it follows from (2.1) that

$$C_{n+1} = M'C_n M + \sum_{\nu=1}^{k} V_\nu d_{n,\nu} \ . \tag{2.6}$$

It is evident from inspection of (2.3) and (2.6) that the relation (2.5) continues to hold if C^*_{n+1} and C^*_n are replaced by C_{n+1} and C_n .

Next, we show that the recurrence relation (2.5) can be extended to include higher non-central moments without losing the linearity property. This phenomenon was originally pointed out by Pollard (1966 , sec.10). His approach assumes that the generating functions of the process are polynomials, although it can be modified to include the case of infinite power series.† The following derivation has the advantage of providing an explicit expression for each element of the multiplier matrix.

Firstly, we note that for r scalar random variables $Z_1, \ldots, Z_r$, with expectations $\mu_1, \ldots, \mu_r$,

$$\begin{aligned} E\left(\prod_{i=1}^{r} Z_i\right) &= E \prod_{i=1}^{r} (Z_i-\mu_i+\mu_i) \\ &= E \prod_{i=1}^{r} (Z_i-\mu_i) + \sum_i \mu_i \, E \prod_{j\neq i} (Z_j-\mu_j) \\ &\quad + \sum_{i<j} \mu_i\mu_j \, E \prod_{\ell\neq i,j} (Z_\ell-\mu_\ell) + \ldots + \prod_{i=1}^{r} \mu_i \ . \end{aligned} \tag{2.7}$$

Thus

† Pollard, J.H. (personal communication, 1969).

$$E\left[\prod_{i=1}^{r} Y_{n+1,\alpha_i} \mid \mathsf{Y}_n\right]$$

$$= E\left[\prod_{i=1}^{r} \left(Y_{n+1,\alpha_i} - d_{n+1,\alpha_i}\right) \mid \mathsf{Y}_n\right] +$$

$$+ \sum_{i=1}^{r} E\left[Y_{n+1,\alpha_i} \mid \mathsf{Y}_n\right] E\left[\prod_{j\neq i} \left(Y_{n+1,\alpha_j} - d_{n+1,\alpha_j}\right) \mid \mathsf{Y}_n\right]$$

$$+ \sum_{i<j} E\left[Y_{n+1,\alpha_i} \mid \mathsf{Y}_n\right] E\left[Y_{n+1,\alpha_j} \mid \mathsf{Y}_n\right] E\left[\prod_{\ell\neq i,j} \left(Y_{n+1,\alpha_\ell} - d_{n+1,\alpha_\ell}\right) \mid \mathsf{Y}_n\right] +$$

$$+ \ldots + \prod_{i=1}^{r} E\left[Y_{n+1,\alpha_i} \mid \mathsf{Y}_n\right]$$

$$= \sum_{\nu=1}^{k} \Phi_\nu^0(r) Y_{n,\nu} + \sum_i \sum_{\nu_1\nu_2} Y_{n,\nu_1} Y_{n,\nu_2} M_{\nu_1\alpha_i} \Phi_{\nu_2}^{i}(r) +$$

$$+ \sum_{i<j} \sum_{\nu_1\nu_2\nu_3} \left(\prod_{\ell=1}^{3} Y_{n,\nu_\ell}\right) M_{\nu_1\alpha_i} M_{\nu_2\alpha_j} \Phi_{\nu_3}^{i,j}(r) +$$

$$+ \ldots + \sum_{\nu_1\ldots\nu_r} \prod_{i=1}^{r} \left\{M_{\nu_i\alpha_i} Y_{n,\nu_i}\right\} ,$$

where

$$\Phi_\nu^{i_1,\ldots,i_s}(r) = E\left[\prod_{\substack{j=1 \\ j\neq i_1,\ldots,i_s}}^{r} \left(Y_{1,\alpha_j} - M_{\nu\alpha_j}\right) \mid \mathsf{Y}_0 = \mathsf{e}_\nu\right] .$$

Taking expectations over Y_n , we obtain

$$E\left[\prod_{i=1}^{r} Y_{n+1,\alpha_i}\right] = \sum_{\nu} \Phi_{\nu}^{0}(r) d_{n,\nu} + \sum_{i} \sum_{\nu_1\nu_2} E\left[Y_{n,\nu_1} Y_{n,\nu_2}\right] M_{\nu_1\alpha_i} \Phi_{\nu_2}^{i}(r) +$$

$$+ \sum_{i<j} \sum_{\nu_1\nu_2\nu_3} E\left[\prod_{\ell=1}^{3} Y_{n,\nu_\ell}\right] M_{\nu_1\alpha_i} M_{\nu_2\alpha_j} \Phi_{\nu_3}^{i,j}(r) +$$

$$+ \ldots + \sum_{\nu_1\ldots\nu_r} \prod_{i=1}^{r} M_{\nu_i\alpha_i} E\left[\prod_{j=1}^{r} Y_{n,\nu_j}\right] . \qquad (2.8)$$

If we denote by $M_r(n)$ the column vector of all non-central moments of Y_n, up to and including the r-th, written in lexicographical order, (*e.g.* $M_2'(n) = \{d_n' \vdots C_n^{*\prime}\}$), then it follows from (1.8) that

$$M_r(n+1) = H_r M_r(n) \qquad (2.9)$$

which is the required extension of the linear recurrence relation (2.4). It is apparent from (2.1) that $H_1 = M'$, and it can be seen from (2.5) that

$$H_2 = \begin{bmatrix} M' & 0 \\ \underline{\underline{V}} & M' \times M' \end{bmatrix} . \qquad (2.10)$$

For larger r, the elements of H_r can be described as follows. The i-th block on the diagonal is the i-th Kronecker power of M'. The top right hand corner consists entirely of zeros. The first k columns of H_2 are extended for $r = 3, 4, \ldots$ by the third, fourth, ... central moments of Y_1 written in lexicographical order. The other components of H_r, $r = 3, 4, \ldots$

are not so easily described, but can be found from (2.8). So, for example, when $r = 3$, (2.8) is

$$E\left[\prod_{i=1}^{3} Y_{n+1,\alpha_i}\right] = \sum_{\nu} (S_\nu)_{\alpha_1\alpha_2\alpha_3} d_{n,\nu} + \sum_{\nu_1\nu_2} (C_n^*)_{\nu_1\nu_2} \sum_{i=1}^{3} M_{\nu_1\alpha_i} \left(V_{\nu_2}\right)_{\alpha_p\alpha_q}$$

$$+ \sum_{\nu_1\nu_2\nu_3} E\left[\prod_{i=1}^{3} Y_{n,\nu_i}\right] \prod_{i=1}^{3} M_{\nu_i\alpha_i} , \qquad (2.11)$$

where the integers p and q are chosen such that $\{i, p, q\} = \{1, 2, 3\}$, and

$$(S_\nu)_{\alpha_1\alpha_2\alpha_3} = \Phi_\nu^0(3) = E\left[\prod_{i=1}^{3} \left(Y_{1,\alpha_i} - M_{\nu\alpha_i}\right) \mid Y_0 = e_\nu\right] .$$

Thus

$$H_3 = \begin{bmatrix} M' & & 0 \\ V_1 \ldots V_k & M' \times M' & \\ S_1 \ldots S_k & B & M' \times M' \times M' \end{bmatrix}$$

where the $k^3 \times k^2$ matrix B can be described as follows: if $\beta = \alpha_1(k^2-1) + \alpha_2(k-1) + \alpha_3$, and $\mu = (k-1)\nu_1 + \nu_2$ $(0 \le \alpha_1, \alpha_2, \alpha_3, \nu_1, \nu_2 \le k)$, then

$$B_{\beta\mu} = \sum_i M_{\nu_1\alpha_i} \left(V_{\nu_2}\right)_{\alpha_p\alpha_q} .$$

Alternatively, we can express B as a sum of matrices, as follows: denote by V_α^β the β-th column of V_α . Then

$$B = M' \times \underline{\underline{V}} + \underline{\underline{V}} \times M' + \begin{bmatrix} M' \times V_1^1 & M' \times V_2^1 & \dots & M' \times V_k^1 \\ \vdots & \vdots & & \vdots \\ M' \times V_1^k & M' \times V_2^k & \dots & M' \times V_k^k \end{bmatrix} .$$

We shall now show that, contrary to the assertion of Pollard (1966), just as (2.5) continues to hold when the non-central second moments are replaced by the elements of the covariance matrices, so, when $r = 3$, (2.9) continues to hold if $M_3'(n)$ is replaced by $N_3'(n) = \{d_n' \vdots C_n' \vdots T_n'\}$, with a parallel change for $M_3(n+1)$, where

$$(T_n)_{\alpha_1\alpha_2\alpha_3} = E\left[\prod_{i=1}^{3} \left\{Y_{n,\alpha_i} - d_{n,\alpha_i}\right\}\right] .$$

To show this, we note from (2.7) and (2.11) that

$$\begin{aligned}
(T_{n+1})_{\alpha_1\alpha_2\alpha_3} &= \sum_{\nu} (S_\nu)_{\alpha_1\alpha_2\alpha_3} d_{n,\nu} \\
&+ \sum_{\nu_1\nu_2} \left[(C_n)_{\nu_1\nu_2} + d_{n,\nu_1} d_{n,\nu_2}\right] \sum_{i=1}^{3} M_{\nu_1\alpha_i} (V_{\nu_2})_{\alpha_j\alpha_\ell} \\
&+ \sum_{\nu_1\nu_2\nu_3} \left[(T_n)_{\nu_1\nu_2\nu_3} + \sum_{i=1}^{3} d_{n,\nu_i} (C_n)_{\nu_j\nu_\ell} + \prod_{i=1}^{3} d_{n,\nu_i}\right] \prod_{i=1}^{3} M_{\nu_i\alpha_i} \\
&- \sum_{i=1}^{3} d_{n+1,\alpha_i} (C_{n+1})_{\alpha_j\alpha_\ell} - \prod_{i=1}^{3} d_{n+1,\alpha_i} .
\end{aligned}$$

Hence, from (2.1) and (2.6),

$$(T_{n+1})_{\alpha_1\alpha_2\alpha_3} = \sum_{\nu} (S_\nu)_{\alpha_1\alpha_2\alpha_3} d_{n,\nu} + \sum_{\nu_1\nu_2} (C_n)_{\nu_1\nu_2} \sum_{i=1}^{3} M_{\nu_1\alpha_i} (V_{\nu_2})_{\alpha_j\alpha_\ell}$$

$$+ \sum_{\nu_1\nu_2\nu_3} (T_n)_{\nu_1\nu_2\nu_3} \prod_{i=1}^{3} M_{\nu_i\alpha_i} \, . \quad (2.12)$$

This equation differs from (2.11) only inasmuch as the non-critical second and third moments of (Y_n) have been replaced by the corresponding central moments. This implies that the change from $M_3(\cdot)$ to $N_3(\cdot)$ in (2.9) is valid, as asserted.

This behaviour does not occur when $r > 3$, however, as can be verified by working along the above lines. We do not perform the calculations here, since they are straightforward and tedious.

We conclude this section by showing that results given by Pollard (1966, eq. 8 and section 6) concerning the asymptotic behaviour of the first and second moments of Y_n are valid, with a sharper order term, in the general case, and give a brief account of some other relevant results. Although the recursion result (2.5) required only the assumption that $\underline{V}$ be elementwise finite, we shall need to make the further assumption, for the rest of this section, that M is primitive. In this way, we can employ the results concerning the behaviour of H^n given in section 1. We denote by θ a number in $(0, 1)$. Its actual value may differ from one equation to the next.

The results we seek hinge on iteration of (2.5), which, replacing C^* by C, $n+1$ by n, and recalling (1.2), gives

$$\begin{bmatrix} d_n \\ \\ c_n \end{bmatrix} = H^n \begin{bmatrix} d_0 \\ \\ c_0 \end{bmatrix} . \tag{2.13}$$

In the light of the remarks in section 1, we consider three cases:

(i) $\rho < 1$: Using (1.4) and a Jordan decomposition of H, we find

$$\rho^{-n} \begin{bmatrix} d_n \\ \\ c_n \end{bmatrix} = \begin{bmatrix} vu' \\ \\ \{\rho I - M' \times M'\}^{-1} \underline{\underline{V}} vu' \end{bmatrix} d_0 + 1.o(\theta^n) .$$

(ii) $\rho > 1$: Iteration of (2.1), together with an appeal to a spectral decomposition, gives, as in the subcritical case,

$$\rho^{-n} d_n = vu' d_0 + 1.o(\theta^n) .$$

Second moments may be found from (2.13); a similar line of reasoning to the one above, using (1.5), yields

$$\rho^{-2n} c_n = (v \times v)\ (u' \times u') \left\{ \underline{\underline{V}} (\rho^2 I - M')^{-1} d_0 + c_0 \right\} + 1.o(\theta^n) . \tag{2.14}$$

It follows from Kesten and Stigum (1966 a) that, under the present conditions (*i.e.* $\underline{\underline{V}}$ elementwise finite and M primitive), if $Y_0 = i \in X$, then $\rho^{-n} Y_n$ tends almost surely to wv, where w has the properties mentioned in section II, 2; in particular

$$E[w] = u' i .$$

From the functional equations at (2.26) and (2.27) in Kesten and

Stigum (1966 a), by differentiation, and by use of Fatou's lemma on (2.14),

$$\mathrm{Var}[\omega] = (u' \times u')\underline{\underline{V}}(\rho^2 I - M')^{-1} i \ . \tag{2.15}$$

(iii) $\rho = 1$: As mentioned in section 1, H will have two unit eigenvalues in this case. We therefore adopt a different approach. Iteration of (2.1) gives

$$\begin{aligned} d_n &= M'^n d_0 \\ &= vu'd_0 + 1.O(\theta^n) \ . \end{aligned} \tag{2.16}$$

From (1.1) and (2.6) we obtain

$$\begin{aligned} C_n &= (M' \times M')\, C_{n-1} + \underline{\underline{V}} d_{n-1} \\ &= (M' \times M')^n C_0 + \sum_{j=1}^{n} (M' \times M')^{j-1} \underline{\underline{V}} d_{n-j} \ , \end{aligned}$$

by iteration. Thus, using (2.16), and the geometric convergence of $(M' \times M')^j$ to its limit,

$$C_n = (M' \times M')^n C_0 + \sum_{j=1}^{n} (M' \times M')^{j-1} \underline{\underline{V}}\, vu'd_0 + 1.O(1) \ .$$

Finally, it follows from (3.19) below that

$$n^{-1} C_n = (v \times v)(u' \times u')\underline{\underline{V}}\, vu'd_0 + 1.O\left(\frac{1}{n}\right) \ . \tag{2.17}$$

3. The process with immigration

To obtain a recurrence relation for the moments of Z_n, we first note that it is evident from the definition of the process that

$$E[Z_{n+1} \mid Z_n] = M'Z_n + \lambda . \tag{3.1}$$

Taking expectations over Z_n,

$$p_{n+1} \equiv E[Z_{n+1}] = M'p_n + \lambda . \tag{3.2}$$

If we denote by $\mathrm{Var}[\cdot]$ the covariance matrix operator, and by A_n^* the matrix $\|E[Z_{n,\alpha}Z_{n,\beta}]\|$, then

$$\begin{aligned} E[Z_{n+1,\alpha}Z_{n+1,\beta} \mid Z_n] \\ &= E[Z_{n+1,\alpha} \mid Z_n]E[Z_{n+1,\beta} \mid Z_n] + (\mathrm{Var}[Z_{n+1} \mid Z_n])_{\alpha\beta} \\ &= (M'Z_nZ_n'M + \lambda Z_n'M + M'Z_n\lambda' + \lambda\lambda' + \mathrm{Var}[Z_{n+1} \mid Z_n])_{\alpha\beta} \end{aligned}$$

from (3.1), so that, taking expectations over Z_n,

$$A_{n+1}^* = M'A_n^*M + \lambda p_n'M + M'p_n\lambda' + \lambda\lambda' + \sum_{\nu=1}^{k} V_\nu p_{n,\nu} + W . \tag{3.3}$$

Writing the covariance matrix of Z_n as A_n, we have by definition $A_n = A_n^* - p_np_n'$, and it follows from (3.2) and (3.3) that

$$A_{n+1} = M'A_nM + \sum_\alpha V_\alpha p_{n,\alpha} + W , \tag{3.4}$$

or, in view of (1.1) and (1.3),

$$A_{n+1} = (M' \times M')A_n + \underline{V}p_n + W \ . \tag{3.5}$$

Combining (3.2) and (3.5),

$$\begin{bmatrix} p_{n+1} \\ \\ A_{n+1} \end{bmatrix} = H \begin{bmatrix} p_n \\ \\ A_n \end{bmatrix} + \begin{bmatrix} \lambda \\ \\ W \end{bmatrix} . \tag{3.6}$$

This result was previously given by Pollard (1966) under certain restrictive conditions on the offspring p.g.f. . Iteration of (3.6) gives (writing n for $n + 1$)

$$\begin{bmatrix} p_n \\ \\ A_n \end{bmatrix} = H^n \begin{bmatrix} p_0 \\ \\ A_0 \end{bmatrix} + \sum_{r=0}^{n-1} H^r \begin{bmatrix} \lambda \\ \\ W \end{bmatrix} . \tag{3.7}$$

Thus as in the last section, the asymptotic behaviour of p_n and A_n depends essentially on that of H^n . We again consider three cases:

(i) *The spectral radius of* M *is* $r < 1$: In this case, the spectral radius of H will also be r , so that $I - H$ is non-singular. It follows from (3.7) that

$$\begin{bmatrix} p_n \\ \\ A_n \end{bmatrix} = H^n \begin{bmatrix} p_0 \\ \\ A_0 \end{bmatrix} + (I-H)^{-1}(I-H^n) \begin{bmatrix} \lambda \\ \\ W \end{bmatrix} \tag{3.8}$$

$$= (I-H)^{-1} \begin{bmatrix} \lambda \\ \\ W \end{bmatrix} + 1.O(r^n) \ , \tag{3.9}$$

as can be verified from a Jordan decomposition of H .

It is a simple matter to show that (3.9) is equivalent to

$$\begin{bmatrix} P_n \\ A_n \end{bmatrix} = \left[\begin{array}{c:c} (I-M')^{-1} & 0 \\ \hdashline (I-M'\times M')^{-1}\underline{\underline{V}}(I-M')^{-1} & (I-M'\times M')^{-1} \end{array}\right] \begin{bmatrix} \lambda \\ \omega \end{bmatrix} + 1\, O(r^n) \ . \qquad (3.10)$$

Note that (3.10) involves only the second moment conditions given in section 1. If we assume further that M is primitive and $B(0) < 1$, then theorem II.1 assures us of the existence of a limiting distribution for (Z_n) . Use of Fatou's lemma on (3.10) indicates that the first and second moments of this distribution are finite, and it can be verified by taking partial derivatives of (II, 1.12) (p. 16) at $s = 1$ that they are given by the limit of (3.10).

(ii) M *is primitive*, $\rho > 1$: Bearing in mind the remarks in section 1, we can use a spectral decomposition argument to show that there is a constant $K < \infty$ such that†

$$\left| \left(\rho^{-r}M^r - uv'\right)_{\alpha\beta} \right| < K\theta^r \qquad (3.11)$$

for $1 \le \alpha, \beta \le k$. It follows that

† θ has the same interpretation in this as in the previous section.

$$\left|\left(\sum_{r=0}^{n-1} \rho^{-n}M^r - \frac{1}{\rho-1}\, uv'\right)_{\alpha\beta}\right| \le K\rho^{-n}\left(1 + \theta\rho + \ldots + (\theta\rho)^{n-1}\right) + \frac{u_\alpha v_\beta \rho^{-n}}{\rho-1}$$

$$= O\left(\eta^n\right) , \tag{3.12}$$

for some $\eta \in (0, 1)$.

Now, iteration of (3.2) gives (with n for $n + 1$)

$$p_n = M'^n p_0 + \sum_{r=0}^{n-1} M'^r \lambda , \tag{3.13}$$

so that, using (3.11) and (3.12),

$$\rho^{-n} p_n = vu'\left[p_0 + \frac{\lambda}{\rho-1}\right] + 1.O\left(\theta^n\right) . \tag{3.14}$$

Application of the above sort of argument to (3.7) gives (bearing in mind (1.5))

$$\rho^{-2n}\begin{bmatrix} p_n \\ \\ A_n \end{bmatrix} = yx'\begin{bmatrix} p_0 \\ \\ A_0 \end{bmatrix} + \frac{1}{\rho^2-1}\, yx'\begin{bmatrix} \lambda \\ \\ w \end{bmatrix} + 1\ O\left(\theta^n\right) ,$$

and then, from (1.5),

$$\rho^{-2n}A_n$$

$$= (v\times v)\left[(u' \times u')\underline{\underline{V}}\left(\rho^2 I - M'\right)^{-1},\ u' \times u'\right]\left[\begin{pmatrix} p_0 \\ \\ A_0 \end{pmatrix} + \frac{1}{\rho^2-1}\begin{pmatrix} \lambda \\ \\ w \end{pmatrix}\right] + 1\ O\left(\theta^n\right) . \tag{3.15}$$

We remark that if $B(0) = 1$, (3.15) is equivalent to (2.14).

If we assume that $Z_0 = i \in X$ is fixed, then with the present moment conditions (M primitive, second moments finite) we know from theorem II.2 that $\rho^{-n}Z_n$ tends almost surely to a limiting random vector, the mean of which is

$$vu'\left[i + \frac{\lambda}{\rho-1}\right] .$$

We remark briefly that we can decompose Z_n as

$$Z_n = Y_n + Z_n^{(0)} ,$$

where (Y_n) is an ordinary process, with $Y_0 = i$, and $\left(Z_n^{(0)}\right)$ is a process with immigration, with $Z_0^{(0)} = 0$. We already have at (2.15) an equation for the variance associated with the limit of $\rho^{-n}Y_n$. In addition, it is possible to imitate the single-type argument in Seneta (1970 a) to derive a functional equation for the Laplace transform of $\lim \rho^{-n}Z_n^{(0)}$, and then by considering its second partial derivatives, and using Fatou's lemma on (3.15), it is possible to verify that the covariance matrix of the almost sure limit of $\rho^{-n}Z_n$ is given by the limit of (3.15) (with $p_0 = i$) .

(iii) M *is primitive and* $\rho = 1$. It follows from (3.11) that

$$\left|\left(n^{-1}\sum_{r=0}^{n-1} M^r - uv'\right)_{\alpha\beta}\right| < \frac{1}{n}\sum_{r=0}^{n-1} K\theta^r = O\left(\frac{1}{n}\right) \tag{3.16}$$

so that, from (3.13)

$$n^{-1}p_n = vu'\lambda + 1\ O\left(\frac{1}{n}\right) . \tag{3.17}$$

This equation holds with only the assumptions of bounded first moments and primitivity of M . If in addition the offspring second moment measure $Q_F < \infty$, then theorem IV.1 indicates that $n^{-1}Z_n$ tends in law to a gamma distribution, and it can readily be checked that the mean vector is given by the limit of (3.17).

If we now reimpose the second-moment assumptions of section 1, and maintain the assumption that M is primitive, then we can obtain the second-moment analogue of (3.17), as follows. Writing N for $M' \times M'$, iteration of (3.5), with n for $n + 1$, gives

$$A_n = N^n A_0 + \sum_{r=0}^{n-1} N^r \underline{\underline{V}} p_{n-r} + \sum_{r=0}^{n-1} N^r W . \tag{3.18}$$

The analogue of (3.16) for N is easily seen to be

$$\left(n^{-1}\sum_{r=0}^{n-1} N^r - (v \times v)(u' \times u')\right)_{\alpha\beta} = O(n^{-1}) . \tag{3.19}$$

Using this and (3.17), it is not difficult to show that

$$n^{-2}\sum_{r=0}^{n-1} N^r \underline{\underline{V}} p_{n-r} = n^{-2}\sum_{r=0}^{n-1} N^r \underline{\underline{V}}(n-r)vu'\lambda + 1\ O(n^{-1}) . \tag{3.20}$$

Furthermore, the analogue of (3.11) for N implies the existence of some $K_2 < \infty$ and some $\theta < 1$ such that for each $\{\alpha, \beta\}$,

$$n^{-2}\left|\sum_{r=0}^{n-1} r\{N^r-(v \times v)(u' \times u')\}_{\alpha\beta}\right| \le n^{-2} \sum_{r=0}^{n-1} rK_2\theta^r$$

$$\le K_2 n^{-2} \frac{\theta}{(1-\theta)^2}$$

$$= O(n^{-2}) ,$$

so that, using (3.19) again,

$$n^{-2} \sum_{r=0}^{n-1} (n-r)N^r = (v \times v)(u' \times u')\left[1 - \frac{n(n-1)}{2n^2} + O(n^{-1})\right]$$

$$= \frac{1}{2}(v \times v)(u' \times u')[1 + O(n^{-1})] . \qquad (3.21)$$

Applying (3.19)-(3.21) to (3.18) gives

$$n^{-2}A_n = \frac{1}{2}(v \times v)(u' \times u')\underline{V}\, vu'\lambda + 1\, O(n^{-1}) . \qquad (3.22)$$

As usual, it is straightforward to check that the limiting (gamma) distribution of $n^{-1}Z_n$ (which assuredly exists under the present conditions) has its covariance matrix given by the limit of (3.22).

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