PARTICLE-IN-CELL SIMULATION
OF A
BEAM-PLASMA DISCHARGE

by

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This thesis is entirely my own work, except where explicitly indicated

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Abstract

This thesis describes the results of a one-dimensional, electrostatic particle-in-cell code with non-periodic boundary conditions, which is used to simulate the beam-plasma discharge. Computer simulation was chosen as a means of examining the discharge because some experimental results indicated that current theory on the threshold conditions for the discharge may not be valid and further detailed experimental investigations appeared to be difficult. Contrary to the assumptions of the theory, these experiments show that near threshold there may be less than one wavelength in the system and that beam to total electron density ratios, n_b/n_e, may be close to unity.

The code has been used to confirm the dispersion theory of one-dimensional streaming instabilities in a bounded system, providing a test of the code's accuracy. Tests have also been made to determine when non-physical heating of the plasma is negligible.

The long term evolution of the beam-plasma interaction has been studied with im­mobile ions. The rate at which the beam-plasma interaction heats the plasma has been examined. The formation of sheaths on the boundaries has been found to modify the interaction, enabling waves and particles to be reflected at the boundaries. When n_b > 0.8n_e, virtual cathodes form, and the interaction can go through cycles. The accuracy with which the code models virtual cathodes has been checked against the theory for Child-Langmuir flow.

Two ionization models have been tried: one which adds pairs of electrons and singly charged ions uniformly across the system at a constant rate, and another which models ionization due to sufficiently energetic electrons. The first model has been used to check whether the code correctly simulates the ambipolar diffusion process. The second model is able to simulate a one-dimensional beam-plasma discharge, indicating that the wave behaviour is determined by n_b/n_e and the number of wavelengths in the system. These simulations have also produced beam-plasma discharges when there is about one wavelength in the system and n_b/n_e is large, in support of the experimental results.
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Commonly used Symbols and Terms

$I_b$ beam current
$I_c$ critical beam current
$\mathcal{E}_b$ beam energy
$r_b$ beam radius
$P$ neutral pressure
$n_n$ neutral density
$n_b$ electron density of the beam
$n_p$ electron density of the background plasma
$n_e = n_b + n_p$ total electron density
$\eta = n_b/n_e$ beam to plasma density ratio
$n_i$ ion density
$\phi$ electric potential
$E$ electric field
$B_0$ static magnetic field
$m$ electron mass
$M$ ion mass

$\omega_e = \left( \frac{n_e e^2}{m \epsilon_0} \right)^{1/2}$ electron plasma frequency

$\omega_i = \left( \frac{n_i e^2}{M \epsilon_0} \right)^{1/2}$ ion plasma frequency

$\omega_b = \left( \frac{n_b e^2}{m \epsilon_0} \right)^{1/2}$ electron plasma frequency of the beam

$\omega_p = \left( \frac{n_p e^2}{m \epsilon_0} \right)^{1/2}$ electron plasma frequency of the background plasma

$\omega_c = eB_0/m$ electron cyclotron frequency

$\omega_{uh} = (\omega_e^2 + \omega_i^2)^{1/2}$ upper hybrid frequency

$T_e$ plasma electron temperature

$v_{eth} = \left( \frac{2kT_e}{m} \right)^{1/2}$ thermal velocity of plasma electrons

$v_{sp}$ velocity spread of a 'top hat' plasma distribution

$v_b$ beam velocity

$v_{bsp}$ velocity spread of a 'top hat' beam distribution

$r_i$ larmor radius of electrons with a perpendicular velocity of $v_b$

$\lambda_d = v_{eth}/\omega_c$ Debye length

$\lambda_0 = v_b/f_e$

$\lambda_b = v_b/f_b$

$k_i$ spatial growth rate
\( Q \) rate at which the beam-plasma interaction heats the plasma
\( \alpha \) conversion efficiency of the beam energy
\( L_m \) maximum length at which the beam is not space charge limited when no ions are present
\( \phi_{sh} \) sheath potential
\( \tau_d \) characteristic diffusion time
\( v_{esc} \) maximum velocity at which ions leave the system
\( v_{in} \) incident velocity of an electron which makes an ionizing collision
\( v_{scat} \) velocity of an electron after making an ionizing collision
\( f_{scat}(v) \) velocity distribution of the scattered electrons
\( \epsilon_{ion} \) ionization energy
\( v_{ion} = (2\epsilon_{ion}/m)^{1/2} \) ionization speed
\( \sigma_i(v) \) ionization collision cross-section
\( \lambda_{en} \) mean free path between ionizing collisions for beam electrons
\( \Delta_x \) width of one cell
\( \Delta_t \) interval between time steps
\( N_0 \) standard particle density, particles per cell
\( \omega_0 \) standard electron plasma frequency
\( \tau_0 \) standard electron plasma period
\( \Theta = \omega_0^2 \Delta_t^2 \) average electron density in a simulation
\( N_e \) average ion density in a simulation
\( N_t \) number of cells in a simulation
\( N_c \) number of ion and electron pairs added per time step in the preliminary ionization model
\( N_\nu \) ionization collision period of beam electrons (\( \Delta_t \))
\( N_{\infty} \) steady state plasma density in a simulation with ionization

cold velocity distribution condition: \( 0 \sim v_{eth}, v_{sp} \ll v_b \)
warm velocity distribution condition: \( 0 \lesssim v_{eth} \lesssim v_b/2 \)
hot velocity distribution condition: \( v_{eth} \gtrsim v_b/2 \)
weak beam density condition: \( n_b \ll n_p, \text{ie. } \eta \ll 1 \)
Introduction

Over the past decade there has been a revived interest in the discharge produced by the interaction of an electron beam with a low density plasma because of its relevance to space plasma research. Electron beams are used in space as diagnostic probes under conditions where the beam electrons follow single particle trajectories, influenced only by the external electric and magnetic fields. However, these beams can ionize the neutral species which are present and the beam can interact with the resulting plasma via collective electrostatic processes. This can cause a rapid increase in the rate at which ionizing collisions occur and leads to the formation of high density plasmas. It is believed that such a discharge can be used as a means of neutralizing the charge acquired by a space craft. The determination of the threshold condition for creating a beam-plasma discharge (BPD) has therefore become a major concern.

Theories (Rowland et al., 1981; Papadopoulos, 1981,1984) have been developed to explain an empirical law (Bernstein et al., 1979; Boswell and Kellogg, 1983) for the critical beam current above which BPD's occur in laboratory experiments. These theories are based mainly upon linear and quasilinear processes; where nonlinear processes are included, they are accounted for only in a phenomenological manner. Near the threshold to BPD these theories may be inappropriate because of the greater importance of nonlinear effects associated with the finite length of the interaction and high beam to plasma density ratios. A more complete nonlinear model needs to be considered, one which will include the effects of both these conditions.

With this objective, a particle-in-cell (PIC) code has been developed which can simulate a BPD in a one-dimensional system with finite length. PIC codes with non-periodic boundary conditions have only recently been used to study beam-plasma interactions (Turikov, 1978; Abe et al., 1979; Burinskaya and Volokitin, 1983,1984; Crystal and Kuhn, 1985), and none of these codes have been used to study sheaths or have included ionization. In this thesis, therefore, besides examining the beam-plasma interaction and discharge in a bounded, one-dimensional system, considerable attention has been given to checking the
conditions under which the code is accurate. Further, because of the complexity of the processes observed in these simulations, a detailed examination of the threshold conditions for BPD has not yet been possible.

Chapter 1 reviews some of the research into beam-plasma interactions which is relevant to the determination of the threshold to BPD, and outlines the theory which tries to explain the empirical law for the critical beam current. The second chapter examines some of the beam-plasma experiments conducted in the large volume plasma apparatus WOMBAT, and discusses the experimental data in relation to the theory.

Following Chapter 3, which contains a brief description of the PIC technique and the algorithms used in the code, the remaining chapters examine the results of simulations of increasing complexity. Firstly, Chapter 4 discusses those electrostatic instabilities in a bounded system which occur on a cold beam when the ions and plasma electrons are also cold. It is found that the growth rate of the beam-plasma instability is reduced when the interaction length is short.

Chapter 5 examines simulations of hot plasmas when the ions are immobile and there is no beam. As it is essential that the code be able to accurately simulate a hot plasma in order to study a BPD, the simulation parameters necessary to achieve this are discussed. A comparison between the type of sheaths obtained in these simulations and a theoretical model is also given in Chapter 5.

In Chapter 6, simulations are examined which have a beam with density below that of the plasma and immobile ions. The long term evolution of the beam-plasma interaction is studied, specifically to measure the rate at which the interaction heats the plasma. Descriptions are given of various phenomena observed in these simulations, including the occurrence of standing waves, waves with negative phase velocity, and low frequency oscillations due to the bouncing of high energy electrons between the sheaths on the boundaries.

Since experiments in WOMBAT show that before the onset of BPD the beam density exceeds the plasma density, Chapter 7 looks at the results of such simulations with immobile ions. It is shown that the one-dimensional beam-plasma interaction changes significantly when the beam density is more than about four times the plasma density, due to the formation of virtual cathodes.

Finally, in Chapter 8, ion motion and ionization are included in the code which enable a BPD itself to be simulated. It is found that the behaviour of the interaction is determined by the beam to plasma density ratio and the length of the system. From this an empirical simulation law for the critical beam current can be derived, which is shown to be similar to the law derived from laboratory experiments.
Chapter 1

Summary of Research on Beam-Plasma Interactions

This chapter reviews some of the research into beam-plasma interactions which is relevant to the determination of the threshold condition for BPD, as well as explaining some of the phenomena which occur during BPD.

Section 1.1 describes the development of beam-plasma research before 1970, which consisted mainly of dispersion theory and its confirmation by experiment and simulation. This research led to the single wave trapping model (Drummond et al., 1970), discussed in Section 1.2, which explains the behaviour of the beam-plasma interaction during its initial growth and the onset of nonlinear processes.

The early experiments also showed that nonlinear phenomena play an important role in BPD; in Section 1.3, some relevant nonlinear processes are described. Finally, Section 1.4 outlines the theory proposed to explain the empirical law for BPD threshold.

1.1 Early Developments

The study of the interactions of electron beams with plasmas has been pursued since the advent of plasma physics research. In 1925 Langmuir proposed that such an interaction was responsible for the high frequency oscillations which he observed in a hot cathode discharge. Research into the behaviour of electron beams was continued to support the development of microwave sources and other electronic devices.

In some travelling wave tubes unwanted instabilities were observed, and many theoretical investigations were made to explain such oscillations. Pierce (1948) showed that electrostatic ion waves could grow through the interaction of the electrons with the neutralizing cold ions. Haeff (1948) showed that a wave could grow on two electron beams
flowing in opposite directions. It was suggested that such an interaction could be used
as an amplifier (Nergaard, 1948; Haeff, 1949; Pierce and Hebenstreit, 1949). Bohm and
Gross (1949,1950) and Akhiezer and Fainberg (1951) showed that the dispersion relation
for Langmuir oscillations is modified when a plasma is not cold, and that the waves can be
amplified if an electron beam passes through the plasma. The dispersion relation for an
unmagnetized one-dimensional beam-plasma system was given by Bohm and Gross (1950)
as
\[1 = \frac{\omega_p^2 + \frac{3}{2} k^2 v_{eth}^2}{\omega^2} + \frac{\omega_b^2}{(\omega - k v_b)^2 + k^2 v_{bep}^2}, \quad (1.1)\]
where \(v_{bep}\) = spread of velocities for a top hat beam velocity distribution.

In the following years several experiments were carried out to study the beam-plasma
instability by injecting an electron beam into a discharge plasma. Although Looney and
Brown (1954) did not succeed in amplifying waves, they did observe standing waves whose
frequency was determined by the axial boundary conditions: an integral number of half-
wavelengths were found to exist between the sheaths created at the anode and cathode.
Boyd et al. (1958) successfully obtained wave amplification and found that the frequency
of the wave with greatest amplification was near the plasma frequency, as predicted by
the theory.

Smullin and Chorney (1958) derived the dispersion relation for a cold electron beam
passing through a stationary, cold ion plasma in a conducting cylinder with a finite,
axial magnetic field. They showed that ion cyclotron oscillations could occur. Trivelpiece
and Gould (1959) examined the same geometry for an electron plasma, and found two
branches of the dispersion relation. The branch with the higher frequency, near the upper
hybrid frequency, had a negative group velocity. Sturrock (1958) showed that there were
two types of instabilities: those which generate waves with positive group velocity and
spatial growth, which became known as convective instabilities, and those instabilities
which resulted when backward waves were unstable and grew in time, known as absolute
instabilities. In the laboratory absolute instabilities can appear to have very high spatial
growth (Manickam et al., 1975).

The dispersion theory was further developed by Briggs (1964), and by Simpson and
Dunn (1966). For a cold beam of radius \(b\), and cold plasma of radius \(p\), within a conducting
cylinder of radius \(p > b\), and an axial magnetic field, Simpson and Dunn solved numerically
the following dispersion relation, which was based on the work by Smullin and Chorney
(1958), assuming uniformly distributed, neutralizing ions with infinite mass.

\[
\left[1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \frac{\omega_b^2}{(\omega - k v_b)^2 - \omega_c^2}\right] T_b \frac{J_1(T_b)}{J_0(T_b)} =
\]
where $T$ is the radial wave number

$$T^2 = -k^2 \frac{1 - \frac{\omega_0^2}{\omega^2 - \frac{\omega_0^2}{(\omega - kv_b)^2}}} {1 - \frac{\omega_0^2}{\omega^2 - \frac{\omega_0^2}{(\omega - kv_b)^2} - \omega_c^2}}.$$  \hspace{1cm} (1.3)

and $J_0$, $J_1$, $H_0^{(1)}$, $H_1^{(1)}$ are Bessel functions, and Hankel functions respectively, of the first kind.

Equation 1.2 shows that many instabilities can exist in a cold beam, cold plasma system with cylindrical geometry and an axial magnetic field. They arise from the coupling of the fast and slow plasma waves on the beam with the plasma modes, or from coupling of the fast and slow cyclotron waves on the beam with the plasma modes, as shown in Figure 1.1. The frequencies of these waves lie in the range $\max(f_e,f_c) < \omega < \omega_{ua}$, or $\omega_{ua} < \omega < \min(f_e,f_c)$, where $\omega_{ua}$ and $\omega_{lb}$ are the upper and lower hybrid frequencies respectively.

 Getty and Smullin (1963), and Kornilov et al. (1966a,b) were the first to discuss the BPD as well as to make a comparison between experiments and the dispersion theory for a radially bounded, beam-plasma interaction. Getty and Smullin defined a BPD as the enhanced discharge produced when the beam-plasma interaction heats the plasma electrons above the ionization energy of the neutral species present, and thereby increases the rate of plasma production.

These experiments on beam-plasma interactions were inspired by the possibility of heating plasmas in fusion devices with electron beams. They were carried out with strong axial magnetic fields (500-5000 gauss) and energetic beams (1-10 keV). In most cases the BPD was used to generate the plasma so that the beam-plasma interaction could be studied. Oscillations were observed near the electron plasma frequency, the electron cyclotron frequency, and their harmonics (Kharchenko et al., 1960,1962; Getty and Smullin, 1963; Kornilov et al., 1966b; Seidl and Sunka, 1967; Hopman, 1969). Oscillations at the ion plasma frequency were also observed by Vermeer et al. (1967).

Realizing that the existence of a BPD could have significant effects upon the basic parameters of the plasma, Malmberg and Wharton (1969) injected a weak electron beam ($\eta \ll 1.0$) into a separately produced quiescent plasma. They were able to show the modification of the dispersion relation caused by the presence of the electron beam, and found that the growth rate agreed well with that proposed by linear dispersion theory.
During the late 1960's, a number of computer simulations were carried out to examine the beam-plasma interaction. Roberts and Berk (1967) used a water bag model to examine the two-stream instability in one dimension, using periodic boundary conditions. Davis and Bers (1968) examined the spatial growth of the fluctuations on a beam injected into a plasma using a particle model, as did Dawson and Shanny (1968), to examine the temporal growth of a warm weak beam, warm plasma interaction in one dimension. Morse and Nielson (1969) also studied the temporal growth in one dimension using a PIC code with periodic boundary conditions.

The papers by Dawson and Shanny (1968), and Morse and Nielson (1969) were concerned with testing the validity of quasilinear theory (Drummond and Pines, 1962; Vedenov et al., 1962), which predicts the diffusion, in phase space, of the electrons in a weak, warm beam when it interacts with a plasma, producing a broad spectrum of unstable waves. The simulations showed that initially a monochromatic wave was dominant. As the wave grew in amplitude, beam electrons became trapped in its potential troughs. The trapped beam electrons produced harmonics of the initial dominant mode, and the spectrum gradually
broadened as the beam electrons diffused in phase space forming a high energy tail on the electron velocity distribution.

1.2 The Single Wave Trapping Model

O'Neil and Malmberg (1968) studied the transition of the dispersion relation for a cold weak beam as the velocity spread increased. From their work developed the single wave trapping model by Drummond et al. (1970), who showed that if the unstable waves grow from a low noise level, they can e-fold many times before nonlinear effects become important. For example, for a one-dimensional cold beam, cold plasma interaction, after N e-foldings the half-width of the unstable energy spectrum is

$$\delta k_{hw} \simeq 3(\ln 2)^{1/2} 2^{-5/8} \eta^{1/3} N^{-1/2} \omega_p/v_b \simeq 1.4\eta^{1/3} N^{-1/2} \omega_p/v_p.$$  (1.4)

Thus, for $\eta^{1/3} \ll 1$ and $N^{1/2} \gg 1$, the oscillation will be a single sinusoidal wave, growing exponentially. If one uses the method outlined by O'Neil and Malmberg (1968), the dispersion relation found by Bohm and Gross, with $v_{bep} = 0$ and $0 \ll v_{th} \ll 0.2v_b$, yields a maximum temporal growth rate (real $k$)

$$\gamma \simeq \frac{\sqrt{3}}{2} \left( \frac{\eta}{2} \right)^{1/3} \left[ 1 + \frac{1}{4} \left( \frac{v_{th}}{v_b} \right)^2 \right]^2 \omega_p$$  (1.5)

for a wave with a frequency

$$\omega \simeq \left\{ 1 + \frac{3}{4} \left( \frac{v_{th}}{v_b} \right)^2 - \frac{1}{2} \left( \frac{\eta}{2} \right)^{1/3} \left[ 1 + \frac{1}{4} \left( \frac{v_{th}}{v_b} \right)^2 \right] \right\} \omega_p,$$  (1.6)

or for maximum spatial growth (real $\omega$)

$$k_l \simeq \frac{\sqrt{3}}{2} \left( \frac{\eta}{3} \right)^{1/3} \left( \frac{v_b}{v_{th}} \right)^{2/3} \left[ 1 + \frac{1}{4} \left( \frac{v_{th}}{v_b} \right)^2 \right] \omega_p/v_b,$$  (1.7)

with a wave number

$$k \simeq \left\{ 1 + \frac{3}{4} \left( \frac{v_{th}}{v_b} \right)^2 + \frac{1}{2} \left( \frac{\eta}{3} \right)^{1/3} \left( \frac{v_b}{v_{th}} \right)^{2/3} \left[ 1 + \frac{1}{4} \left( \frac{v_{th}}{v_b} \right)^2 \right] \right\} \omega_p/v_b.$$  (1.8)

The phase velocity of the waves is given by

$$v_{ph} = \left\{ 1 - \frac{1}{2} \left( \frac{\eta}{2} \right)^{1/3} \left[ 1 - \frac{1}{2} \left( \frac{v_{th}}{v_b} \right)^2 \right] \right\} v_b$$  (1.9)

for temporal growth, and

$$v_{ph} = \left\{ 1 - \frac{1}{2} \left( \frac{\eta}{3} \right)^{1/3} \left( \frac{v_b}{v_{th}} \right)^{2/3} \left[ 1 - \frac{1}{2} \left( \frac{v_{th}}{v_b} \right)^2 \right] \right\} v_b$$  (1.10)

for spatial growth.
Once the beam electrons become trapped in the potential troughs of the wave, the linear theory is no longer valid. The transition from the exponential growth predicted by linear theory, to the production of a plateau in the velocity distribution as predicted by quasilinear theory, was studied in detail with particle simulations by Krainer et al. (1972). They found that the interaction could be classified into two regimes, high and low beam density, depending upon whether or not merging of the space averaged velocity distributions of the beam and the plasma occurred during the initial exponential growth. For \( \eta < 0.038 \), beam electrons were decelerated to the thermal velocity of the plasma electrons only after the period of linear growth had passed, and they found that such processes could be explained by quasilinear theory. For \( \eta > 0.038 \), the interaction rapidly became nonlinear.

1.2.1 Energy Conservation

Before the beam electrons become trapped they give up some of their kinetic energy to the wave. Once trapped and oscillating in the trough of the wave, these electrons travel slower than the wave for part of the time, and for part of the time faster than the wave. Consequently, there is an exchange between the kinetic energy of the beam, and the potential energy of the wave. This exchange was studied using simulations by O'Neil et al. (1971,1972) and Thompson (1971), who showed that the exchange will occur in both space and time. Experimental confirmation was reported by Gentle and Lohr (1973), and Nyack et al. (1974).

Estimates of the maximum potential energy density vary (Drummond et al., 1970; O'Neil et al., 1971; Thompson, 1971; Krainer et al., 1972), but are all close to

\[
\varepsilon_{\text{max}} \approx \eta^{1/2} \frac{1}{2} n_b m v_b^3,
\]  

(1.11)

The initial amplitude of the oscillations of the average potential energy density are about half the kinetic energy density of the beam. However, phase space diffusion and collisional effects reduce the amplitude (Fukumasa, 1978). Eventually, after the beam electrons become part of a hot tail on the velocity distribution and the waves have been Landau damped, the average potential energy density will fall to the new 'thermal' level.

1.2.2 Velocity Distribution

Some of the beam electrons can reach velocities significantly greater than the initial beam velocity before they are trapped in the troughs of the wave. During the process of diffusion in phase space, the beam electrons can be accelerated to even greater velocities, \( \text{ie.} \)
as the waves are Landau damped some electrons continue to gain kinetic energy. Dawson and Shanny (1968), in their simulations of temporal growth, observed beam electrons with velocities four times their initial velocity when the plateau had thermalized. Gentle and Lohr (1973) observed electrons with velocities greater than 1.5$v_b$ even before the beam distribution had formed into a plateau. Krainer et al. (1972) estimated the maximum velocity by assuming that the final distribution of the beam would be a uniform plateau extending from 0 to $v_{max}$. In this case, conservation of beam energy gives

$$v_{max} = \sqrt{3}v_b.$$  \hspace{1cm} (1.12)

They also estimated the maximum trapping range to be

$$\delta v_{trapping} = 2^{11/12} \eta^{1/3} \left[ 1 - \frac{1}{2} \left( \frac{\eta}{2} \right)^{1/3} \right]^{1/4} v_b.$$ \hspace{1cm} (1.13)

1.2.3 Harmonics

Once the beam electrons begin to be trapped by the wave field, the single frequency wave, which exists during the linear stage, is modified so that harmonics become significant and grow in amplitude as the interaction passes further into the nonlinear regime. The growth of the harmonics was examined theoretically by O'Neil et al. (1971) and observed experimentally by Apel (1969a,b), Gentle and Lohr (1973), and Nyack et al. (1974). As the trapped beam electrons diffuse in phase space, the harmonics get broader and eventually become indistinguishable so that the spectra resembles that predicted by quasilinear theory.

1.3 Nonlinear Phenomena

Seidl et al. (1976), and Jones et al. (1976,1977) studied the nonlinear development of the convective and absolute instabilities, and the interaction between wave modes. It was found that large oscillations developed on the electron beam soon after leaving the electron gun, and that the interaction rapidly reached a nonlinear stage even with weak beams. This process was examined by Goldstein et al. (1977) using simulations, which supported some of the experimental results.

Apel (1969a,b) made a detailed examination of the nonlinear as well as linear processes in an interaction of a weak beam with a plasma produced by a Penning discharge. He observed not only harmonics up to the seventh of the dominant mode, but also turbulent spectra, and suggested the existence of parametric processes. Parker and Throop (1973), and Nyack et al. (1974) observed the parametric excitation of an ion-acoustic wave and
a backward-propagating electron wave by the dominant convective beam-plasma wave. Recently, Whalan and Stenzel (1981,1985) examined the parametric process in which electromagnetic waves are produced when electrostatic plasma waves are scattered from self-consistently produced ion acoustic waves.

Many beam plasma experiments were conducted to investigate the spatial collapse of Langmuir waves in plasma density cavities due to their ponderomotive force, as predicted by Zakharov (1972). These experiments used both quiescent plasmas, with and without magnetic fields (Ikezi et al., 1974,1976; Wong and Quon, 1975; Antipov et al., 1978,1979; Cheung et al., 1982; Cheung and Wong, 1985), and BPD plasmas (Christiansen et al., 1981; Bond, 1982).

Langmuir collapse, both with and without magnetic fields, is characterized by large amplitude 'spikes' of plasma waves in density cavities of up to 40%, (Bond, 1982). If pressure balance is assumed ie.

\[
\frac{\varepsilon_0 E^2}{4n\kappa T_e} \sim \frac{\delta n}{n}
\]

(Antipov et al., 1978), this suggests that very large electric fields are present. These bursts were observed to grow and decay within a period similar to naturally occurring ion modes (Christiansen et al., 1981; Cheung and Wong, 1985), and were found to be more repeatable in weak beam, quiescent plasma experiments when the velocity spread of the beam was small (Antipov et al., 1979; Cheung and Wong, 1985).

1.4 The Beam-Plasma Discharge

Electron beams were first used as part of the diagnostic equipment launched by rockets or on satellites, to study the ionosphere and magnetosphere, in the early 1970's. The electron beams were intended to be used primarily as passive probes of geomagnetic and geoelectric fields. It was assumed that the electrons would obey single particle dynamics throughout their passage, through both the magnetosphere and the ionosphere. However, conflicting results were obtained from several experiments (Cartwright et al., 1974,1978; Cambou et al., 1975,1978; Monson and Kellogg, 1978). It was found that an electron beam does not always behave as a passive probe, but, due to collective effects, can cause large plasma disturbances.

This revived interest in the beam-plasma discharge. The study of BPD in space received high priority (Cambou et al., 1980; Maehlum et al., 1980; Duprat et al., 1983; Szuszczewicz, 1985), and to complement this research, laboratory experiments were conducted (Bernstein et al., 1975,1978,1979,1983; Kellogg et al., 1981; Szuszczewicz et al.,
1982; Boswell and Kellogg, 1983; Kellogg and Boswell, 1986). Many similarities were found between phenomena observed in the laboratory and those in space.

Electrostatic waves within the expected frequency bands for a cold beam, cold plasma interaction have been observed in most laboratory BPD experiments (Kharchenko et al., 1960,1962; Getty and Smullin, 1963; Hopman, 1969; Cabral, 1971; Bernstein et al., 1975,1978,1979; Jain and Christiansen, 1981; Kellogg et al., 1981; Boswell and Kellogg, 1983; Kellogg and Boswell, 1986). Bernstein modes have also been observed to couple with the beam modes (Bernstein et al., 1975; Jain and Christiansen, 1981). A characteristic feature of the oscillations observed in these experiments was their 'bursty' nature; large amplitude waves could grow and decay after only a few tens of oscillations.

Laboratory experiments in weak magnetic fields with beam diameters considerably less than the diameter of the plasma and the vacuum vessel should reflect more closely the behaviour of BPD's in space. In such a laboratory experiment it was found that the amplitude of the bursts peaked near the center of the beam and decreased radially extending many beam radii to the edge of the discharge (Bernstein et al., 1978,1979; Kellogg et al., 1981; Szuszczewicz et al., 1982; Boswell and Kellogg, 1983; Kellogg and Boswell, 1986). Jain and Christiansen (1981) found that the amplitude of wave fields due to some instabilities had radial profiles similar to various Bessel functions.

Typically, the plasma temperature was found to be about a few eV near BPD threshold. However, during strong beam-plasma interactions, the plasma temperature may become a significant fraction of the beam energy, and far greater than the ionization energy (Anderson et al., 1981; Kellogg et al., 1981; Boswell et al., 1984).

Laboratory studies found that the BPD occurs when the electron current in the beam is above some critical level. The critical current \( I_c \) above which the BPD occurs in the laboratory was found to be related to the beam energy \( E_b \), the magnetic field \( B_0 \), the length of the interaction \( L \), and the neutral pressure \( P \), in the following manner

\[
I_c \sim \frac{E_b^{3/2}}{B_0^3LP}
\]

(Bernstein et al., 1979; Boswell and Kellogg, 1983).

Rowland et al. (1981) suggested that this scaling law could be explained by the growth rate of a beam-plasma instability. They showed that, based upon the dispersion relation used by Simpson and Dunn (1966), a particular beam-plasma instability was almost absolute under certain conditions. Under these conditions, therefore, the beam-plasma interaction should be able to heat the plasma despite the finite interaction length. When combined with the balance between ionization and diffusion, they showed that these con-
ditions supported the empirical law for the critical current.

Lebedev et al. (1976), and Papadopoulos (1981) considered the interaction, in one dimension, in terms of particle and energy conservation in the plasma. The basic equations which they studied were

\[
\frac{dn_e}{dt} = n_b(n_n - n_e)(\sigma_i(v)v_b) + n_e(n_n - n_e)(\sigma_i(v)v_e) - \beta n_e^2 - \frac{n_e}{\tau_d} \quad (1.16)
\]

\[
\frac{dT_e}{dt} = -E_{ion}n_n(\sigma_i(v)v_e) - H - \frac{T_e}{\tau_d} + \frac{Q}{n_e} S(n_b, n_e, \tau_d, \ldots), \quad (1.17)
\]

where
- \( n_n \) = neutral density
- \( \sigma_i \) = ionization collision cross section
- \( \beta \) = recombination coefficient
- \( \tau_d \) = characteristic diffusion time
- \( E_{ion} \) = ionization energy
- \( H \) = energy lost due to excitation of neutrals and ion heating
- \( Q \) = rate at which the plasma is heated by the beam-plasma interaction
- \( S \) = switching function, equal to 0 or 1

In equation 1.16, the first term on the right hand side represents the ionization due to the beam electrons, and the second term is ionization due to the plasma electrons. The third term describes losses due to recombination, and the fourth term represents the diffusion losses. The characteristic diffusion time, \( \tau_d \), will be determined by the greater of either radial or axial diffusion.

Equation 1.17 describes the electron heating rate. The first term represents the energy lost due to plasma electrons making ionizing collisions, and the second term represents kinetic energy lost due to excitation of the neutral particles and ion heating. The third term corresponds to convective energy loss with the same characteristic diffusion time. The last term represents the heating due to waves excited by the beam-plasma instability, where the switching function \( S \) is on if the threshold for BPD is exceeded.

Below threshold (neglecting recombination) equation 1.16 becomes

\[
\frac{dn_e}{dt} = n_b(n_n - n_e)(\sigma_i(v)v_b) - \frac{n_e}{\tau_d}, \quad (1.18)
\]

and has the solution

\[
n_e(t) = n_b n_n(\sigma_i(v)v_b)\tau_d[1 - \exp(-t/\tau_d)], \quad (1.19)
\]

with an asymptotic value

\[
\lim_{t \to \infty} n_e = n_b n_n(\sigma_i(v)v_b)\tau_d. \quad (1.20)
\]
Therefore, if the threshold condition is not reached, the plasma density will saturate exponentially to a level where the diffusion of plasma from the system is at the same rate as the production of plasma due to ionization by the beam.

Above threshold it is important to know how the energy is transferred from the wave to the plasma. Lebedev et al. (1976) and Papadopoulos (1981) showed that the heating rate is determined by the effective electron collision frequency, \( \nu_{en} \). For \( \nu_{en} \) much less than the dominant frequency of the beam-plasma interaction,

\[
Q \approx \nu_{en} \frac{\varepsilon_0 E^2}{2}.
\]

Then, from the conservation of energy flux in the system, they obtained

\[
\frac{d \varepsilon_0 E^2}{dt} \frac{2}{2} = \frac{v_b}{L} \alpha n_b \frac{mv_b^2}{2} - \nu_{en} \frac{\varepsilon_0 E^2}{2} - \frac{v_g \varepsilon_0 E^2}{L},
\]

where \( v_g \) is the group velocity of the wave, and \( \alpha \) is the conversion efficiency of the kinetic energy in the beam to potential energy of the waves, which has been determined by linear and quasilinear theory. When \( v_g/L < \nu_{en} \) the asymptotic value of the potential energy density can be derived from equation 1.22, and leads to the following estimate of \( Q \)

\[
Q \approx \alpha n_b \left( \frac{mv_b^2}{2} \right) \frac{v_g}{L}.
\]

Combining equations 1.16, 1.17, and 1.23 Papadopoulos was able to derive a further condition which should be met before a BPD can be attained. However, he needed to include the effects of a small proportion of plasma electrons which are heated to a large fraction of the beam energy by nonlinear collective processes (see references in Papadopoulos, 1981), as well as the presence of a sheath of sufficient amplitude to trap these hot plasma electrons within the interaction region.

This theory combines many of the processes which determine the threshold condition for BPD. However, as will be discussed in the following chapter, experiments in WOMBAT have shown that near the threshold of BPD there are at most only a few wavelengths in the interaction length, and, in the region of the beam, \( \eta \) is not necessarily very small and in some cases \( \eta \approx 1.0 \). For these conditions the theory of Rowland et al. (1981) may not be valid since, even though the instability may be almost absolute, the dispersion relation of Simpson and Dunn (1966) is probably not applicable in very short systems. Further, the theory of Lebedev et al. (1976) and Papadopoulos (1981) may not be valid also since it is based on linear and quasilinear theory. The above estimates of the heating rate would be inappropriate under such conditions. Therefore, \( Q \) needs to be determined for interactions in short systems with large \( \eta \). This is done in Chapter 6.
With larger $\eta$ the effect of the boundary conditions become more important. Some researchers believe that Pierce-type instabilities (i.e. instabilities caused by finite axial boundary conditions), can play a role in beam-plasma interactions in a bounded system (Saeki et al., 1977; Mikhailov et al., 1984; Burinskaya and Volokitin, 1984). The relevance of Pierce-type instabilities is discussed in Chapter 4.
Chapter 2

Beam-Plasma Experiments in WOMBAT

This Chapter describes some of the experimental work conducted in WOMBAT, by myself and others, investigating the wave phenomena leading up to and during BPD. The frequencies of the waves increase from below the ion plasma frequency, up to many times the electron plasma frequency. With low gas pressure ($P$) and low beam current ($I_b$), the beam is not neutralized. However, by increasing either $I_b$ or $P$, the beam can be neutralized and oscillations at the ion plasma frequency and subharmonics are observed, as will be discussed in Section 2.2.

Oscillations near the electron plasma frequency and the upper hybrid frequency are observed when $I_b$, or $P$, are increased further. In Section 2.3 it is shown that initially these high frequency oscillations are modulated by the ion waves, and that $\eta \sim 1$ and probably $L \lesssim \lambda_0$. When BPD is reached the waves have larger amplitude, and $\eta$ is small enough for weak beam theory to be valid even though $L < 10\lambda_0$, as shown in Section 2.4. The spatial growth of these waves is not necessarily very large, which does not support the view that BPD is reached only when an almost absolute instability appears, as suggested by Rowland et al. (1981) and Papadopoulos (1981,1984).

2.1 Apparatus and Diagnostics

The laboratory experiments were conducted in the large volume plasma apparatus, WOMBAT, which consists of a cylindrical non-magnetic stainless steel vessel, 2 meters long and 1 meter in diameter. Figure 2.1 is a schematic diagram of WOMBAT. Within the tank is a solenoid producing an axial magnetic field of up to 200 gauss which is uniform to $\pm 0.3\%$ on axis in the central 100 cm and radially out to 35 cm. There are
some perturbations of the magnetic field outside this region where gaps in the solenoid allow diagnostic probes to enter through ports in the tank wall. The probe ports are 100 cm apart and are on the top and both sides of the tank, as shown in Figure 2.1. A base pressure of $1 \times 10^{-6}$ torr is achieved using a 400 litre/sec turbomolecular pump attached at the end nearer the electron gun. However, the effective pumping speed is reduced to 100 litre/sec by the tubing between the pump and the tank. All the experiments described in this thesis have been carried out in dry air.

The electron gun is suspended on axis from a tube entering through one of the probe ports. A retarding field energy analyser is situated, also on axis, approximately 1 meter away from the electron gun within the region of uniform magnetic field. Earthed copper plates of 30 cm diameter are attached to the electron gun and the energy analyser so that the axial boundary conditions are well defined. The radial boundary is the aluminium solenoid former which has an inner diameter of 84 cm; however, the discharge rarely occupies a region of more than 20 cm diameter under the experimental conditions used.

Langmuir probes are used to determine the plasma temperature, density, and potential. One probe is on the end of a 30 cm arm which can rotate and sweep the probe through the beam and associated plasma. Another probe can be moved axially on an externally controlled trolley running along the bottom of the tank. In some cases, however, the motion of the probes affected the plasma phenomena, so to avoid this as well as to study the spatial evolution of waves, a fixed array of 16 probes was used, each equally spaced along the beam.
2.1.1 The Electron Gun

The electron gun has a very simple design, shown schematically in Figure 2.2. A tungsten filament is ohmically heated, and the emitted electrons are accelerated by the electric field between the filament and a mesh at the aperture of the gun. The gun is designed so that the supports to which the filament is attached can be easily removed, allowing quick replacement of a damaged filament.

Under typical operating conditions the resistance of the hot filament is 2 ohms with up to 10 V across it and a perveance of up to $1.5 \times 10^{-6}$ A/V$^{3/2}$, which achieves a maximum current of 10 mA. The beam radius at the gun is 0.5 cm, but because the beam has a divergence angle of about $9^\circ$, as found by Kellogg and Boswell (1986), the half-density radius expands to twice the Larmor radii of electrons with a perpendicular velocity calculated from the gun divergence angle:

$$r_b \approx \left( \frac{2mv_b\sin 9^\circ}{eB_0} \right) = 0.31r_l,$$

(2.1)

where $r_l$ is the Larmor radius of electrons with a perpendicular velocity of $v_b$.

The electron gun is insulated from the tank so that the beam current can be estimated as the current passing through the gun. However, during a BPD the plasma density is high and the number of plasma electrons reaching the copper plate attached to the electron gun increases, therefore the accuracy of this method is reduced.

2.1.2 The Retarding Field Energy Analyser

The electron beam is terminated by a retarding field energy analyser which is used to measure the parallel energy distribution of electrons. The analyser consists of three meshes and a collector all of diameter 25 mm. Each is separated by a 3 mm thick P.T.F.E. insulating ring, and contained in a copper cylinder of diameter 31 mm and length 60 mm.
The collector is a copper plate, and all the meshes are of 0.1 mm copper wire with 40 wires/cm, so each has a transmission factor of about 40%.

To measure the energy distribution of the electrons, the potential of the third mesh \(V_3\), adjacent to the collector, is swept from zero to well below \(-\varepsilon_b\), where \(\varepsilon_b\) is the beam energy, so that only the electrons with energy greater than \(V_3\) reach the collector. The potential of the first mesh \(V_1\) is held at zero so that the beam-plasma system is terminated by a grounded plate and mesh. The potential of the second mesh \(V_2\) is sufficiently positive so that any ions entering the analyser are reflected before reaching the collector. The collector is held at a positive potential \(V_c\) to ensure that all secondary electrons released from the collector are attracted back again.

The secondary electron emission from surfaces within the analyser is comprised of reflected primary electrons as well as true secondary electrons. The distribution function of the reflected primary electrons has a peak at the incident energy with a low energy tail, and the true secondary electron distribution has a peak at a few eV and only a small percentage with energy greater than 20 eV (Brown, 1959; MacDaniel, 1964). Therefore \(V_c\) is kept greater than the beam energy.

With the meshes used as described above, spectrally reflected primary electrons and true secondary electrons should not escape from the collector. However, secondary electrons created at the third mesh may still impair the operation of the analyser if they reach the collector. All other secondary electrons should be reflected by the third mesh, except when \(V_3\) is above about \(-20\) Volts.

If there is a sheath on the axial boundaries of the beam-plasma system, the ions will be accelerated by the sheath potential, and so \(V_2\) should be higher than the sheath potential. The second mesh would therefore collect more of the secondary electrons. When the second grid is used as the discriminator more secondary electrons reach the collector, and this is observed as a large increase in the current from the collector \(I_c\) when \(V_2\) is swept above about \(-40\) Volts.

Hysteresis is observed when \(V_3\) is swept near \(\varepsilon_b\) and at other energies where the distribution is high. This is attributed to the formation of virtual cathodes inside the analyser, and can be alleviated to some extent by increasing \(V_2\).

When the beam distribution is altered by the beam-plasma interaction the radial component of the energy distribution will be greater, and this will reduce the energy resolution of the analyser. The Larmor radius of many electrons can be greater than the aperture radius of the analyser, and the dimensions of the meshes would reduce the transmission of electrons with large pitch angle. Any electrons with a pitch angle greater than 76° will
not reach the collector.

2.1.3 Langmuir Probes

The Langmuir probes used are made of 0.8 mm copper wire of length 20 mm, and attached to coaxial cable inside 5 mm diameter tubes. They can be attached to the axially mobile trolley, the rotating arm, or inserted through the side ports. These probes should allow the measurement of local plasma parameters, such as density, temperature, and potential. In a collisionless thermal plasma, free from a magnetic field, analysis of the current flowing to the probe is easy. Between the electron and ion saturation regimes, as described by Chen (1965), the current collected by the probe varies exponentially with the probe potential. However, with the presence of a magnetic field and a non-thermal electron velocity distribution, analysis of the results from Langmuir probes is complicated.

When the distribution is in the form of a beam the electron current reaching the probe is almost proportional to the beam current, and when a magnetic field is present the electrons will be restricted to axial motion along the magnetic field inside tubes with the Larmor radius. This will also apply to the ions, although, for the experimental conditions considered, the Larmor radius of the ions at room temperature would be greater than 10 cm.

During the regime of electron saturation the electron current reaching the probe is complicated by the collisional processes which knock electrons into the region from which the probe collects electrons. This means that the probe current is not necessarily a measure of the local plasma parameters and that as observed, the probe can influence the conditions of the plasma over great distances.

Because of the effect of mobile probes on the stability of the plasma parameters, an array of 16 probes has been used to make simultaneous measurements. These probes are spaced 6 cm apart along the beam. They are made of 0.8 mm copper wire 30 mm long and attached to 3 mm rigid coaxial cable. The output from each probe passes through a rectifying and smoothing circuit which enables the envelope of large amplitude, high frequency bursts to be examined.

2.2 Ion Oscillations - Preceding BPD

At low pressure and low beam energies only low frequency oscillations are observed with $I_0$ limited by the perveance of the electron gun. Figure 2.3 shows the experimental points which correspond to the appearance of the precursor electron oscillations, when
Figure 2.3: Values of $E_b$ below which no electron oscillations are observed (region B). In region A, both electron and ion oscillations are observed together.

$B_0 = 36$ G. In region B only low frequency oscillations are observed, whereas in region A both high and low frequency oscillations are observed.

Complex structures of harmonics and subharmonics of the ion plasma frequency are observed in region B. An examination of the variation of the spectra with beam current $I_b$ was made by Boswell (1985) with dry air at $P = 6 \times 10^{-6}$ torr, $E_b = 50$ eV, and $B_0 = 72$ G, and the results are shown in Figure 2.4. As the beam current was increased towards 3 $\mu$A the oscillation frequency increased rapidly, and the amplitude of the subharmonic at $f/2$ also increased. At about 3 $\mu$A the subharmonic became dominant, and suppressed the higher frequency oscillation. The lower frequency mode had a frequency close to $f_i$, and gradually increased in frequency with $I_b$ until $I_b \sim 10$ $\mu$A when another period doubling occurred. Above about 20 $\mu$A further period doublings occurred until the wave spectrum appeared relatively featureless, with the fundamental and subharmonic at $f_i/2$ just distinguishable above the noise, as shown in Figure 2.5.

When $I_b$ was increased above 40 $\mu$A a sudden change occurred with a stable three cycle oscillation replacing the apparently chaotic motion. The three cycle oscillation persisted over the range 40-160 $\mu$A with little change in the waveform or frequency. At $I_b \sim 160$ $\mu$A a six cycle appeared followed by a return to a chaotic type of spectrum when $I_b$ was increased further. At $I_b \sim 500$ $\mu$A the two subharmonics merged into one so that the spectra were similar to that of Figure 2.5.

Similar behaviour is observed with different $P$, $E_b$, and $B_0$. Although with higher $P$ and $B_0$ the oscillations increase in frequency more rapidly with increasing $I_b$, and electron
Figure 2.4: Variations of the dominant frequencies ($\omega$) as the beam current was increased. The dashed line represents perturbations of rigid rotator equilibrium; solid line the ion plasma frequency for the neutralized electron beam. The shaded region shows where the oscillations are essentially chaotic.

Figure 2.5: A typical spectrum when in the chaotic region.
oscillations are observed.

Throughout this regime the energy analyser shows that the velocity of the beam is barely affected by the oscillations, indicating that an axial interaction between the beam and plasma electrons is not the cause of these oscillations. A possible source is a strong radial electric field, which would produce an azimuthal $E \times B$ shear flow (Boswell, 1985). This explanation depends upon the extent to which the space charge of the beam is neutralized by ions. The potential across the beam cannot be measured by Langmuir probes because the probe characteristics will be dominated by the large number of beam electrons being collected, so that the current collected by the probe will be almost proportional to the current density of the beam.

If it is assumed that there are no ions and the electron density across the beam is proportional to $J_0(\nu_0 r/r_0)$, where $\nu_0 = 2.405$ ie. the first zero of $J_0$, then the minimum potential in the center of the beam will be

$$\phi_{\text{min}} = \frac{-1}{2\pi \nu_0 J_1(\nu_0)} \frac{I_b}{e_0 v_b} \sim -2.5 \times 10^4 \frac{I_b}{e_b^{1/2}}. \quad (2.2)$$

For the experimental conditions used by Boswell (1985), when $I_b = 3 \mu A$, the potential minimum would be about 10 mV if no ions were trapped. Assuming that the ions were $N^+_2$ at room temperature, ie. $T_i \sim 0.03$ eV, only a small fraction of the ions would be trapped by this potential barrier, and it is probable that the beam was not neutralized when $I_b < 3 \mu A$. The low frequency oscillations detected by the probe under these conditions suggest that this was so, since they can be modelled as perturbations to a rigid rotator equilibrium (Boswell and Giles, 1985).

When $I_b > 3 \mu A$, a greater proportion of the ion distribution could be trapped in the region of the beam, and therefore the beam was more likely to be neutralized. The close agreement between the observed oscillation frequencies and the estimated ion plasma frequency, based upon the assumption of beam neutralization, indicates that the beam probably was neutralized.

Linder and Hernqvist (1950) examined the space charge neutralization of an electron beam and claimed that even though the beam may be neutralized, ie. $n_i \sim n_e$, a small difference in charge distribution can still be sufficient to produce a potential trough with a depth of a few tenths of a volt. This was for beam currents of up to a few milliamps and with gas pressures in the range used in the experiments being described here. However, Linder and Hernqvist did not include the effects of a magnetic field, which would enhance the trapping of ions. They also assumed that the distribution of the ions in the region of the beam would remain thermal, when it is possible that the trough just collects the
low energy ions. Both these factors could reduce the amplitude of the potential trough, and yet Langmuir probe measurements of the saturated ion current showed that the ions are distributed over a larger volume than the beam. Consequently, a strong radial electric field can exist even if the beam were neutralized when averaged radially, and the resulting wave behaviour would be a three-dimensional phenomena.

2.3 First Appearance of Electron Oscillations - Transition Region

Electron oscillations first appear when the beam current is increased so that the boundary between regions $A$ and $B$, as in Figure 2.3, is reached. The position of this boundary is dependent upon the beam energy, the magnetic field, and other parameters which affect the electron density. Although the frequency of these oscillations is also dependent upon $E_b$ and $B_0$, this section describes the variation of the frequency with different neutral pressure when $E_b \sim 100$ eV and $f_s \gtrsim 50$ MHz to illustrate the general behaviour.

For pressures below about $10^{-4}$ torr the frequency of the first electron oscillations is less than about 10 MHz, which have an amplitude much less than that of the ion oscillations. As $I_b$ is increased, the spectra, evolving from behaviour similar to that of the chaotic ion oscillations described in the previous section, grow in amplitude and spread to higher frequencies. The highest frequency sometimes reaches about 10 MHz before BPD occurs, and the frequency of the oscillations with maximum amplitude lie in the range 1-10 MHz.

These oscillations are the precursors described by Boswell and Kellogg (1983), who showed that they are at the electron plasma frequency. It was found that on some occasions the electron oscillations occur in groups with the same period as the low frequency ion waves, as shown in Figure 2.6 taken from (Boswell, 1984). The oscillations of the potential are shown on three time scales, and show that the ratio of the frequencies of the precursor oscillations and the ion waves is approximately $(M/m)^{1/2}$, where $M$ is the mass of $N^+_f$ ions. This supports the view that the high frequency oscillations, at 9 MHz, are at about $f_e$, and the low frequency oscillations, at 55 kHz, are at approximately $f_t$.

The value of $\eta$ can be estimated by comparing the precursor frequency with an estimate of the beam electron plasma frequency. The electron density of the beam can be found using:

$$n_b \sim \frac{I_b}{e \pi r_b^2 v_b}$$

(2.3)

For the experimental conditions used to obtain Figure 2.6, using $r_b = 1$ cm, equation 2.3 shows that $n_b \sim 10^6$ cm$^{-3}$. The value of $r_b = 1$ cm is much greater than that of the half-
Figure 2.6: Wave behaviour in the transition region. $E_b = 90$ eV, $B_0 = 72$ gauss, $P = 1.1 \times 10^{-5}$ torr, $I_b = 3.4$ mA. a) 100 $\mu$s full scale; b) 10 $\mu$s full scale; c) 2 $\mu$s full scale.

density radius and would adequately account for a broad electron density profile. This means that the electron plasma frequency of the beam was about 9 MHz, and so $\eta$ must have been close to unity. This would cause considerable modification to the Trivelpiece-Gould modes.

An estimate of the wavelength of these electron oscillations can be made by assuming that their phase velocity is the same as the beam velocity. This suggests that the wavelength of the electron oscillations in Figure 2.6 were about 60 cm so that $L \sim 1.6\lambda_0$. Similarly, when the electron oscillations first appear with a frequency of only a few MHz, they must have $\lambda_0 \gg L$. If this is so the boundary conditions are probably a significant factor determining the wave behaviour of the first electron oscillations.

In a narrow pressure range near $10^{-4}$ torr two types of precursors are observed, one at $f_c$ and the other with a frequency between $f_c$ and $2f_c$. Both types occur in separate and often consecutive bursts. This region was investigated by Kellogg and Boswell (1986), who showed that the oscillations are probably due to the Cherenkov resonances with the two Trivelpiece-Gould modes, one near the plasma frequency and the other near the
upper hybrid frequency. However, they found that the dispersion relation is sufficiently distorted by the beam that an absolute instability is not possible, and that the convective growth rates are not significant enough to cause the observed oscillations. They suggested that the unstable waves may be trapped within the interaction region, so that convective instabilities may exhibit larger growth rates.

For higher $P$, the Cherenkov resonance at $f_{uh}$ is dominant since the frequency of the precursor oscillations is always greater than $f_c$. Initially it is between one and two times $f_c$, as for the intermediary pressure range, but, as the pressure is increased to $10^{-3}$ torr, the frequency steadily increases to between three and four times $f_c$. Figure 2.7a shows the frequency of the precursors for the pressure range $10^{-3}-10^{-2}$ torr. The estimates of the plasma density in Figure 2.7b are based on the assumption that the precursor frequency is at either $f_c$ or $f_{uh}$. If this assumption is correct then the electron density in the transition region is proportional to the neutral pressure.

A linear relationship between the plasma density and pressure can be expected considering equation 1.18, ie.

$$\frac{d n_e}{d t} = n_b (n_n - n_e)(\sigma_i(v) v_b) - \frac{n_e}{\tau_d},$$

which has an equilibrium solution of $n_e = n_b n_n (\sigma_i(v) v_b) \tau_d$. Given that in the pressure
range $10^{-4}$-$10^{-3}$ torr, $n_b$ and $\langle \sigma_t(v) v_b \rangle$ are fairly constant, only a variation in $\tau_d$ would cause $n_e$ not to be proportional to $P$. The diffusion rate is hard to determine since the beam may be only just neutralized when the precursors first appear, and, as discussed before, the degree of neutralization cannot be determined accurately. However, if it is assumed that the beam is neutralized, and that the plasma is trapped by a sheath determined by the plasma electron temperature, then the magnetic field will restrict the radial diffusion of electrons so that plasma electrons leave the system axially far more rapidly than radially. In this case, the characteristic diffusion time will be determined by axial ambipolar diffusion, so that

$$\tau_d \approx \frac{L}{C_s},$$

where $C_s$ is the ion sound speed.

The straight line through the points in Figure 2.7b corresponds to an ion sound speed appropriate to an electron temperature $T_e \sim 1$ eV and an escape area equal to the beam cross section. Considering the accuracy with which the plasma diameter and potential can be measured, this estimate is not inconsistent with other measurements which suggest that $T_e$ is a few eV (Boswell and Kellogg, 1983). If large amplitude oscillations were present then the diffusion rate may be altered significantly, and $n_e$ would probably not be proportional to $P$.

### 2.4 Large Amplitude Electron Oscillations - Beam-Plasma Discharge

Increasing the beam current above the level required to reach the transition region produces a BPD. Several BPD’s have been observed, which are characterized by their associated wave spectra. The following discussion concerns the wave behaviour observed during the BPD’s when $\varepsilon_b = 300$ eV, i.e. $v_b \simeq 10^9$ cm/s, and $B_0 = 36$ G, i.e. $f_e = 100$ MHz. The spectra of BPD1, as termed by Boswell and Kellogg (1983), consist of harmonics of the electron plasma frequency, as shown in Figure 2.8, whereas the spectra of BPD2 consist of a broad peak near the upper hybrid frequency when it is less than $2f_e$. Other discharges have been observed which have dominant frequencies in the ranges $2f_e$ to $3f_e$, $3f_e$ to $4f_e$ etc., which are formed after sudden transitions and not simply by a gradual increase in the upper hybrid frequency. The width of the peaks of the higher BPD are greater, as is the amplitude of the lower frequency waves in the whistler range. A possible mechanism which could cause these changes is parametric decay.

In general, increasing $P$ has the same effect as increasing $I_b$ because the threshold
current for the different BPD's is inversely proportional to $P$. This can be seen in Figure 2.9, which shows the threshold conditions for the different BPD's and the precursor oscillations. Most research has been conducted on BPD1 partly because it is possible to follow the development of the beam-plasma interaction, with less irregularity, from the initial appearance of ion waves through to BPD1 and beyond, and also because BPD1 bursts have the greatest amplitude.

As $I_b$ is increased to the threshold of BPD1, both the amplitude and the frequency of the precursors increase until, at a critical current, bursts of waves are observed with amplitudes 20-30 dB above the precursors. Measurements of the emitted light, and the number of high energy electrons reaching the energy analyser, show that these bursts coincide with sudden increases in the emitted light and the density of high energy electrons, as shown in Figure 2.10.

Each burst is actually a separate discharge during which the beam-plasma interaction increases the plasma density. Because the interaction can saturate in only a few tens of electron plasma periods, the spatial and velocity distribution of the plasma may be modified significantly within this time. Since the quiet intervals between bursts may be thousands of times the transit period of beam electrons, the beam will certainly relax to a fairly unperturbed state long before a subsequent burst occurs. Therefore it is probable that the threshold condition for a subsequent burst is reached after the plasma has relaxed to a particular state.
Figure 2.9: Threshold conditions for different BPD's, when $\epsilon_b = 300$ eV and $B_0 = 36$ G. The numbers indicate the type of BPD and the p stands for the precursors.

Figure 2.10: Correlation between a) envelope of the RF burst, b) current to a photomultiplier tube, and c) current reaching the collector of the energy analyser with $V_3 = -15$ volts, for BPD1 bursts.
Figure 2.11: a) Frequency of the BPD1 bursts with increasing $I_b$.  

**b) Plasma density estimated by assuming that the oscillations are at $f_e$.**

Figure 2.11 shows the general behaviour of the dominant frequency as $I_b$ is increased when $P = 5 \times 10^{-4}$ torr. If the dominant frequency is $f_e$, this figure shows that the plasma density is proportional to the beam current. This can be expected from equation 1.16. If ionization by energetic plasma electrons is included and assumed to be at a constant rate, and the diffusion rate is also assumed to be constant, then the equilibrium solution is:

$$n_e \approx n_b n_n (\sigma_i(v) v_b) \tau_s, \quad \text{where} \quad \frac{1}{\tau_s} = \frac{1}{\tau_d} - n_n (\sigma_i(v) v).$$

(C2.5)

Closer inspection shows that the frequency of consecutive bursts go through an irregular cycle. At the start of a cycle the dominant frequency is generally low, between 30-40 MHz; however, it can increase in subsequent bursts and eventually reach 60-80 MHz, as shown in Figure 2.12. The length of each cycle is irregular, but in general, the 'duty cycle' increases with increasing beam current, as does the average frequency. When the threshold of BPD2 is approached the cycle can be terminated by a burst at 140-160 MHz.

### 2.4.1 Axial Variation of the BPD1 Bursts

Initially, measurements of the axial distribution of the bursts were made with the mobile axial probe; however, it was found that the movement of the probe would modify
the interaction. Therefore the probe array is now used, and it has been found that the amplitudes of different frequency bursts have significantly different spatial dependences.

The bursts with frequency in the range 30-40 MHz, which appear at the leading edge of most cycles, are shown in Figure 2.13. These bursts have a period of less than 10 µsec which is comparable to the ion plasma period (~ 6 µsec), and are probably influenced by the ion oscillations in a similar manner to the precursors. The envelopes resemble those of standing waves, and measurements of a non-rectified signal show that there is a phase difference of 180° between each minimum. Assuming that the oscillation frequency is slightly below the plasma frequency, since the Cherenkov resonance with the lower Trivelpiece-Gould mode is below the plasma frequency, the wavelength of the oscillations should be 25-35 cm. This is about twice the distance between the minima in Figure 2.13. Therefore, the spatial growth in the figure has \( k_z \sim (1-2)\lambda_0^{-1} \), which shows that BPD can be reached without requiring an absolute instability to occur.

The bursts with frequency of about 50 MHz are shown in Figure 2.14 to be located in the center of the interaction region. This suggests that significant nonlinear processes are involved which cause the amplitude of the oscillations to decay almost to the noise level near the analyser. It is unlikely to be due to some form of Langmuir collapse because there are only a few wavelengths in the system, and, as discussed later, because the beam density is not low enough.

When \( I_b \) is low, these bursts are frequently preceded by the bursts at 30-40 MHz by only a microsecond or so, and modulation at about 2 MHz is frequently observed. The
modulation may be a result of the axial boundary conditions favouring oscillations with an integral number of half-wavelengths in $L$, similar to the findings of Looney and Brown (1954). With $L = 100$ cm and $v_b \approx 10^9$ cm/s this would mean that the resonant frequencies would be multiples of 5 MHz, and the modulation frequency would be 2.5 MHz, which is close to the observed modulation frequency. However, because the plasma is not quiescent, standing waves are not likely to be very stable and this modulation may be due to another process.

The bursts at the trailing edge of a cycle, with frequencies above 60 MHz, are shown in Figure 2.15. These bursts are located at the energy analyser end of the interaction region and vary between being distributed over half the interaction region, or being confined to a small region close to the analyser. In general, these bursts have the greatest amplitude and the highest growth rates. Even though the amplitude of the potential oscillations remains below the noise level for over half the interaction length, i.e. about $4\lambda_0$, it rapidly reaches saturation in about $(2-3)\lambda_0$.

If there is a sheath present between the electron gun and the plasma, one might expect the beam to be perturbed as soon as it enters the system, and, as shown in the simulations of Goldstein et al. (1978), large amplitude waves should appear close to the electron gun. Since large amplitude waves are not observed close to the electron gun it suggests that there is no sheath, and yet with such large amplitude waves a hot plasma should be produced and a sheath might be expected to form.
Figure 2.14: Envelopes of the bursts of about 50 MHz measured with 16 probes spaced 6 cm apart along the axis of the electron beam.

Figure 2.15: Envelopes of the 60-80 MHz bursts measured with 16 probes spaced 6 cm apart along the axis of the electron beam.
2.4.2 Comparison with Linear Weak-Beam Theory

Attempts to measure the phase velocity of the waves during BPD1 bursts have been carried out to compare with linear theory. The measurements are made by examining the phase difference of the waves between the rotating probe and the axially mobile probe. Because movement of the probes perturbs the wave motion, the procedure is very difficult. It has been found that, because of the irregularity in the sequence of the different types of BPD1 bursts, the best procedure is to trigger the measuring system on the largest amplitude bursts at the rotating probe. Consequently, all the results obtained are for the 60-80 MHz bursts, which are shown in Figure 2.16.

The phase velocity of the waves in these bursts when $I_b$ is high, is less than $0.8v_b$ which, for one-dimensional linear theory, equation 1.9, would be for a beam with $\eta > 0.1$. This is well above the limit found by Krainier et al. (1972) for merging of the beam and the plasma during the linear stage. However, the wave amplitude of the bursts with high $I_b$ are below the level required to trap beam electrons, ie.

$$\phi < \frac{m}{2e}(v_b - v_p)^2.$$  \hspace{1cm} (2.6)

This may be a result of radial motion of the electrons near the regions of low potential. Because the Larmor radius of electrons with perpendicular velocity $v_b$ is greater than the beam radius, some electrons will move around potential depressions reducing their amplitude compared with a one-dimensional interaction. It may also be due to the interaction
rapidly passing through the trapping stage and reaching a strongly turbulent stage. This is supported by the energy analyser results which show that the distribution of the electrons reaching the energy analyser is broad and has a high energy tail. It should be noted that the escape velocity distribution is averaged over a period of time which inevitably includes many bursts as well as periods between bursts when the beam is not perturbed.

Any comparison of the spatial growth rate observed in experiments with linear theory is complicated by estimates of $\eta$ and the plasma temperature. Malmberg and Wharton (1969) were successful in estimating the spatial growth in a cylindrical beam-plasma system by assuming that the electrostatic modes of the system can be analysed in terms of the lowest order Bessel-like radial eigenmodes. In this case, the value of $\eta$ can be taken as the average across the plasma so that

$$\eta = \frac{\int_0^r r n_b(r) \, dr}{\int_0^r r n_e(r) \, dr}. \quad (2.7)$$

Between bursts, or in the transition region, the diameter of the plasma is about the same as the beam, and so estimates of $\eta$ are large. However, during the bursts, the diameter of the plasma is about six times that of the beam, and so the effective $\eta$ is reduced by a factor of at least 40. Estimates of $\eta$ during the BPD1 bursts are in the range 0.005-0.03. This means that, when $E_b = 300$ eV and $B_0 = 36$ G, the value of $\eta$ is small enough for linear weak-beam theory to be applicable during BPD1, but probably too large for Langmuir collapse to be important (Papadopoulos, 1975).

The spatial growth of a beam-plasma interaction is not only dependent upon $\eta$, but also the plasma temperature. Before a burst $T_e \sim 4$ eV and the beam energy is 300 eV, so that $v_b/v_{eth} \sim 8$. Using equation 1.7, which gives the spatial growth in a one-dimensional system, and setting $v_b/v_{eth} \sim 8$ yields a spatial growth rate of $3\lambda_0^{-1}$ for $\eta = 0.01$. This is only about twice the growth rates measured for the bursts at 30-40 MHz. Even though the estimates of $\eta$ are poor and the plasma temperature is probably greater than 4 eV during a burst, the rough agreement with the experimental spatial growth rate suggests that the single wave trapping model may be valid for such bursts.

During the higher frequency burst the plasma temperature may increase until $v_b/v_{eth} \sim 2$, and so the spatial growth should decrease (equation 1.7 yields $k_i \sim 1.4\lambda_0^{-1}$ when $\eta = 0.01$ and $v_b/v_{eth} = 2$). However, the growth rates of the larger amplitude bursts are higher. This is not surprising since linear theory is not expected to apply to large amplitude oscillations. Further, when $v_b/v_{eth} \sim 2$ a large amplitude sheath is likely to form, so that the beam would be perturbed immediately upon entry into the system, and the burst would probably saturate at a shorter distance from the injection boundary, as was the
Summary

A wide range of different wave behaviour has been observed in beam-plasma experiments in WOMBAT, all of which are influenced by the balance between the ionization and diffusion rates. Once the beam is almost neutralized, ion oscillations are observed prior to BPD under all conditions.

When electron waves first appear, they are modulated to some extent by the ion oscillations, and it is possible that the ion oscillations play an important role in determining the threshold conditions for BPD, but it is hard to estimate their significance since they can never be isolated from the interaction.

The frequencies of the first electron oscillations correspond roughly to the Cherenkov resonances with the Trivelpiece-Gould modes, as expected by linear dispersion theory. However, the values of $\eta$ and $L/\lambda_0$ when these oscillations first appear are such that we would not expect weak beam theory for an infinitely long cylindrical system to apply.

It has been found that when BPD1 is reached, the plasma density increases sufficiently that $\eta$ falls to about 0.01 with $L \sim (4-8)\lambda_0$, and so linear weak-beam theory may be more relevant. The spatial growth rates of the initial BPD1 bursts at 30-40MHz are certainly close to the values predicted by linear one-dimensional theory, and are not indicative of an instability which is almost absolute, as suggested by Rowland et al. (1981) and Papadopoulos (1981).

During the large amplitude BPD1 bursts many nonlinear characteristics are observed, unfortunately the bursts are generally irregular making detailed examination difficult. These bursts are unlikely to be associated with Langmuir collapse since the values of $\eta$ are too large and there are too few wavelengths in the interaction length.
Chapter 3

The Particle-in-Cell Code

Present theories suggest that the threshold conditions for BPD are determined by either the spatial growth rate of the beam-plasma instability (Rowland et al., 1981), or the rate at which the beam-plasma interaction heats the plasma (Lebedev et al., 1976; Papadopoulos, 1981, 1984). However, the experiments in WOMBAT have shown that these theories may not be valid for the typical plasma conditions observed near threshold.

To examine this problem more closely, the particle-in-cell (PIC) technique has been used to model the interaction numerically, and, unlike previous simulations, non-periodic boundary conditions and ionization processes have been included in the model. Section 3.1 outlines the general PIC technique, and Section 3.2 describes the particular algorithms used in this code. A description of the algorithm used to model ionization is left until Chapter 8.

3.1 The PIC Technique

The PIC technique can be used to study wave-particle interactions by simulating the motion of charged particles under the influence of electric and magnetic fields, both applied and self-generated (Potter, 1973; Hockney and Eastwood, 1981; Birdsall and Langdon, 1985). To accomplish this, all motion must occur in discrete steps, and, for the simulations described in this thesis, they occur at constant intervals in time.

Starting from an initial state, in which the position and velocity of each particle are specified, the electric and magnetic fields are determined using Maxwell's equations. Maxwell's equations must be solved using finite difference methods or fourier transforms, and the field is then evaluated at discrete points throughout the interaction region. The discrete points divide the interaction region into 'cells'.

Once the electric and magnetic fields are calculated, the Newton-Lorentz equation
of motion is used to determine the new velocities and positions of each particle at the following instant in time. This procedure is simply repeated to follow the evolution of an interaction.

To simulate our laboratory beam-plasma interactions, a one-dimensional electrostatic, bounded PIC code is used, in which the sequence of calculations is:

**Step 1:** charge density at the spatial array points,

**Step 2:** electric potential at the spatial array points,

**Step 3:** electric field at the spatial array points,

**Step 4:** electric field at each particle,

**Step 5:** new velocity of each particle,

**Step 6:** new position of each particle.

Although such a model is a considerable simplification of the physical processes involved in a laboratory experiment, it can be used, for example, to test the theories of Lebedev et al. (1976) and Papadopoulos (1981), and so provide insight into the dominant processes in a one-dimensional interaction.

Each simulation particle represents a sheet of either electrons, or singly charged ions, on a plane between the injection boundary and the collection boundary. When simulating the beam-plasma interaction in WOMBAT, the surface number density of each sheet will be between $10^3$ and $10^8$ cm$^{-2}$. The ions can be assumed to be immobile and uniformly distributed in space, or they can be mobile, having a mass which is chosen at the start of the simulation to facilitate study of phenomena occurring on the time scale of the ion motion.

To reduce run time and avoid awkward scaling factors, such as the electron mass $m = 9.11 \times 10^{-31}$ kg, the particle and field variables are made dimensionless. If the tilde "~" signifies a real physical quantity, the dimensionless variables used are:

- **particle position:** $x = \frac{\tilde{x}}{\Delta_x}$, where $\Delta_x$ = cell size
- **time:** $t = \frac{\tilde{t}}{\Delta_t}$, where $\Delta_t$ = length of time step
- **particle velocity:** $v = \frac{\tilde{v}}{\Delta_x}$
- **charge density:** $\rho = \frac{\tilde{\rho}}{\varepsilon n_0}$, where $n_0$ = equilibrium plasma density
potential: \[ \phi = \phi_0 \left( \frac{\Delta t}{\Delta x} \right)^2 \]

electric field: \[ E = \tilde{E} \frac{e}{m} \left( \frac{\Delta t}{\Delta x} \right) \]

ion mass: \[ M = \frac{m}{\tilde{M}} \]

These variables are used throughout the following chapters on simulations; if units are not specified when discussing a physical quantity in the following chapters, they will be the appropriate dimensionless unit given above.

The choice of \( \Delta x, \Delta t \), and the number of particles per cell, \( N_0 \), is important because these parameters determine the accuracy of the simulations. One must ensure that \( \Delta x \) is much less than the shortest scale length over which the potential varies, and, similarly, that \( \Delta t \) is much less than the shortest oscillation period. The choice of \( N_0 \) depends upon whether only a cold beam is to be simulated, in which case \( N_0 \) need be only about 5-20. Although, if a hot plasma is to be simulated as well, many more particles per cell are required. This will be discussed more in the following chapters in relation to specific cases. In general it is obvious that, decreasing \( \Delta x \) and \( \Delta t \) reduces the discrete nature of the simulation, and increasing \( N_0 \) makes the particle density in the simulation more like that in the real plasma being simulated. Consequently, the accuracy is improved when such changes are made.

3.2 The Algorithms used in the Code

3.2.1 Initializing the Simulation

At the start of a simulation, before the interaction commences, the plasma particles must be placed in the system. To ensure that the initial field is small, the particles are placed in phase space in an ordered manner, rather than using a random distribution. Both the beam and the plasma may have a variety of velocity distributions, even though they are distributed uniformly in space between the boundaries. The desired velocity distribution is achieved by choosing particle velocities which are at equal increments of the corresponding cumulative distribution (Birdsall and Langdon (1985) pp 393-402). For example, if the desired velocity distribution is 'top hat', the cumulative velocity distribution will be a line with constant gradient, and so equal increments of the cumulative distribution will occur at equal increments in the velocity. Therefore, the velocities of the particles at the start
of the simulation will be $-v_{sp} + (n + 0.5)\delta v$, where $n$ is an integer less than or equal to $N_0$, and $\delta v = 2v_{sp}/N_0$. As the procedure is repeated in each cell, the distribution is comprised of many weak beams, each beam having one particle in every cell. This is shown in Figure 3.1, which is an example of a plasma with a gaussian velocity distribution ($v_{eth} = 0.5$) and an electron beam passing across the system at time $t = 0$ with a top hat velocity distribution ($v_{bep} = 0.02$).

To ensure that the velocity distribution still approximates the desired type even over a fraction of a cell, as in Figure 3.1, the sequence in which the velocities of particles at regular spatial intervals are chosen must be scrambled. The method used to resequence the allocation of particle velocities is based upon bit-reversed numbers (Hammersley and Handscomb, (1964) p 33).

### 3.2.2 Charge Density

The charge density is calculated by assigning a fraction of the charge carried by each particle to the two nearest points of a spatial array, thereby approximating the charge density at discrete points across the system. For example, if a particle is a distance $\delta$ away from the i-th spatial array point, and a distance of $(\Delta_x - \delta)$ from the (i+1)-th spatial array point, as in Figure 3.2, then the fraction $(\Delta_x - \delta)/\Delta_x$ of the charge is assigned to
To calculate the space charge density at points on a spatial array, the charge carried by each particle is assigned to the two nearest array points.

the i-th spatial array point, and the fraction $\delta/\Delta_x$ is assigned to the (i+1)-th spatial array point. Each particle can be thought of as extending over one cell, thus distributing the charge across a 'cloud', rather than just at a point. Strictly speaking, the code uses the cloud-in-cell technique, described by Birdsall and Fuss (1969).

3.2.3 Potential

The potential is calculated using Poisson's equation, transformed for discrete variables. Using the dimensionless variables defined before, one finds that

$$\phi^t_{i+1} - 2\phi^t_i + \phi^t_{i-1} = \Theta^2 \rho^t_i,$$

where $\phi^t_i = \text{dimensionless potential at the i-th spatial array point at time } t\Delta_t$,

$\rho^t_i = \text{dimensionless density at the i-th spatial array point at time } t\Delta_t$,

$$\Theta = \omega_0 \Delta_t = \left(\frac{m\varepsilon_0^2}{m\varepsilon_0^2}\right)^{1/2} \Delta_t.$$

The potentials at the boundaries are held at zero in all the simulations discussed in this thesis, i.e. $\phi^t_1 = 0$ and $\phi^t_{Nc+1} = 0$, where $N_c$ is the number of cells in the simulation. To solve for $\phi^t_i$ given $\rho^t_i$, the inverse of a tridiagonal matrix must be found, and this is done using the method outlined by Potter (1973) pp 88-90.

3.2.4 Electric Field

The potential gradient is evaluated at points lying midway between the spatial points at which the density and potential are evaluated. Using the dimensionless variables defined previously, the electric field will be the difference between the potentials at the two nearest
array points,

$$E_{i+\frac{1}{2}}^t = \phi_i^t - \phi_{i+1}^t.$$ 

If the electric field had been evaluated at the same points as the potential, then the following formula, suggested by Potter (1973), could have been used:

$$E_i^t = (\phi_{i-1}^t - \phi_{i+1}^t)/2.$$ 

However, this would result in the resolution being halved.

Because the electric field is not calculated explicitly at the boundaries, the electric field is considered to extend over them in order that the same algorithm can be used to calculate the electric field at particles near the boundaries. The electric field at the array point located half a cell outside the injection boundary is an extrapolation from the electric field at the points located one half-cell, and three half-cells inside the injection boundary, i.e.

$$E_{-\frac{1}{2}}^t = 2E_{\frac{1}{2}}^t - E_{\frac{3}{2}}^t,$$

and similarly outside the collection boundary:

$$E_{N_x+\frac{1}{2}}^t = 2E_{N_x+\frac{1}{2}}^t - E_{N_x-\frac{1}{2}}^t.$$ 

If the electric field were evaluated on the same spatial array as the density and potential, then to estimate the electric field on a boundary an extrapolation would have to be made from the array points located one or two cells inside the boundaries. This would reduce the accuracy of the electric fields at the boundaries, and could have a significant effect on a simulation because of the importance of the sheath in both confining the plasma particles and accelerating the electron beam.

### 3.2.5 Injecting and Moving the Particles

When injecting a beam it is assumed that the electric field is constant across the first cell, with a value

$$E_0^t = \frac{3}{2}E_{\frac{1}{2}}^t - \frac{1}{2}E_{\frac{3}{2}}^t.$$ 

This is just the linear extrapolation from the electric field array points at $\Delta_s/2$ and $3\Delta_s/2$. Therefore the phase space locations of the electron beam particles injected in the latest time step will be

$$v_m^t = v_b + v_{bperm} + E_0^t.T_m$$

$$z_m^t = (v_b + v_{bperm}).T_m + E_0^t.T_m^2/2,$$
where \( v_b \) = mean beam velocity,

\( v_{b\text{spm}} \) = velocity difference from the mean of the \( m \)-th beam particle,

\( T_m \) = the fraction of a time step which the \( m \)-th particle should have been in the system if motion did not occur in discrete steps.

The sequence in which \( T_m \) is chosen is based upon the sequence used to load the beam particles initially, in order that the continuity of the beam particles is maintained as they move across the system. We also require the gradient of the electric field at the boundary to be small (i.e. \( \rho \sim 0 \)) to maintain continuity of the beam density.

The algorithm used to move all other particles is simply

\[
v_m^{t+\frac{1}{2}} = v_m^{t-\frac{1}{2}} + m_m E_m^t
\]

\[
x_m^{t+1} = x_m^t + v_m^{t+\frac{1}{2}}
\]

where \( v_m^{t+\frac{1}{2}} \) = velocity of the \( m \)-th particle at time \((t + 1/2)\Delta t\),

\( m_m \) = dimensionless mass of the \( m \)-th particle,

\( x_m^t \) = location of the \( m \)-th particle at time \( t\Delta t \),

\( E_m^t \) = electric field at the location of the \( m \)-th particle at time \( t\Delta t \),

and is calculated via linear interpolation between the two nearest electric field array points.

### 3.2.6 Particle Boundary Conditions

The boundary condition for all particles is complete absorption if they reach either boundary, in which case they are neglected in further calculations. Accumulation of charge at either boundary is not considered as it is assumed that there is a perfectly conducting connection between both ends.

### 3.2.7 Kinetic and Potential Energy

The kinetic energy in the system is calculated by summing the kinetic energy of each particle:

\[
T = \sum_{m=1}^{N_{te}N_{ti}} \frac{1}{2} m_m v_m^2
\]

where \( N_{te} \) = total number of electrons in the simulation

\( N_{ti} \) = total number of ions in the simulation
This will give the kinetic energy at times \((t + \frac{1}{2})\Delta t\).

The potential energy in the system is calculated by evaluating one of the following integrals:

\[
W = \frac{1}{2} \int_0^L \phi \rho dz, \quad \text{or} \quad W = \frac{1}{2} \varepsilon_0 \int_0^L E^2 dz.
\]

To do this, the value of each of the integrated variables is linearly interpolated between each spatial array point. Thus for each cell the first integral becomes

\[
\frac{1}{2} \int_{i}^{i+1} \phi \rho dz = \frac{1}{12} (2\phi_{i+1}\rho_{i+1} + \phi_{i+1}\rho_i + \phi_i\rho_{i+1} + 2\phi_i\rho_i).
\]

Since the potential is zero on the boundaries, the total potential energy in the system is just the sum of the potential energy in each cell, as calculated above. If the potential energy in a small region is required then the second integral is used because the boundary potential does not have to be included.
Chapter 4

Electrostatic Instabilities in a Bounded One-Dimensional Beam-Plasma System

To test the accuracy of the PIC code, comparisons have been made between analytic theory and simulations of electrostatic instabilities in a bounded, one-dimensional beam-plasma system. These simulations have also provided new information which assists in explaining the threshold conditions for BPD. Some of the results are examined in this chapter.

Probably the best known of these instabilities is the one first described by Pierce (1944), which in its original form related to electron beams passing between two electrodes held at the same fixed potential. The ions were assumed to be immobile and distributed uniformly, neutralizing the unperturbed beam, and no plasma electrons were included. Section 4.1 examines the Pierce instability, firstly reviewing the theory, and then comparing it with the simulations.

The Pierce instability in cylindrical geometry with an infinite axial magnetic field is examined in Section 4.2. This instability is more representative of the WOMBAT laboratory conditions when the radius of the beam is much less than the system length. Frey and Birdsall (1966) and Mikhailov et al. (1984) considered the existence of this instability, although, they did not solve the dispersion relation completely. To allow a comparison between simulation and theory, an approximate solution of the dispersion relation is derived here which indicates that an electron beam is stable when its radius is below a certain limit.

Considerable research has been carried out to investigate the effects of ion motion
on this instability (Mikhailovskii, 1966; Frey and Birdsall, 1966; Iizuka et al., 1979,1983; Vladimirov et al., 1983; Kolyshkin et al., 1984), and it was shown that the dispersion relation is modified significantly when the ions have low atomic mass. Simulations of this instability, sometimes termed the Pierce-Buneman instability, are discussed in Section 4.3. They show that the particle boundary conditions in the model can modify the interaction, suggesting that similar effects may occur in laboratory experiments.

Finally, in Section 4.4, the beam plasma instability in a bounded, one-dimensional system is examined. The dispersion relation for this instability has to date been evaluated only for \( \eta > 0.99 \) (Kolyshkin et al., 1984; Iizuka et al., 1985; and Jungwirth, 1985). However, simulations with \( \eta = 0.5, 0.1, \) and 0.01, have been made since \( \eta \) is not likely to be so close to unity near the threshold of BPD. These simulations show that the growth rate of the instability is reduced in systems with length less than only a few \( \lambda_0 \).

### 4.1 The Pierce Instability

Pierce (1944) assumed that, in a moderate vacuum, ions produced in the few ionizing collisions would quickly accumulate in the region of the beam until the negative space charge was neutralized. If the ions had high atomic mass, the energy they received from the ionizing collisions would be negligible, and their inertia would be great enough to ensure that they would not move during the growth of the instability, and remain uniformly distributed in space.

Pierce derived the appropriate dispersion relation (PDR) using linearized electrostatic theory. Given that the density of the ions is the same as the density of the unperturbed electron beam, \( n_b \), and the injection velocity of the beam electrons is \( v_b \), then the three equations which determine the interaction are:

\[
\begin{align*}
\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial z} & = \frac{e}{m} \frac{\partial \phi}{\partial z}, \\
\frac{\partial n}{\partial t} + \frac{\partial n v}{\partial z} & = 0, \\
\frac{\partial^2 \phi}{\partial z^2} & = -\frac{e}{\varepsilon_0} (n_b - n).
\end{align*}
\]  

For a system of length \( L \), the boundary conditions are:

\[\phi(L, t) = \phi(0, t) = 0, \quad n(0, t) = n_b, \quad v(0, t) = v_b.\]

With the usual assumption of linear theory, i.e. that small perturbations can be analysed separately by their Fourier components \( v = v_b + v_1 e^{i(kz-\omega t)}, n = n_b + n_1 e^{i(kz-\omega t)}, \phi = \phi_1 e^{i(kz-\omega t)} \), the dispersion relation for an infinite system can be derived from equation 4.1, as

\[
1 - \frac{\omega^2}{(\omega - kv_b)^2} = 0 \quad \text{i.e.} \quad \kappa^\pm = \frac{(\omega \pm \omega_b)}{v_b}.
\]  

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This dispersion relation does not necessarily indicate instability, since \( \omega \) or \( k \) do not always have imaginary parts.

The general solution for a bounded, one-dimensional system must have the form

\[
\phi = (Ax + B + Ce^{ikz} + De^{-ikz})e^{-iw_0t},
\]

because the one-dimensional capacitor field, \( \phi = Ax + B \), is the complementary solution to Poisson's equation for a bounded system. The boundary conditions \( \text{ie.} \)

\[
\phi |_0 = 0, \quad \frac{\partial^2 \phi}{\partial z^2} |_0 = 0, \quad v_b \frac{\partial n}{\partial z} |_0 = -n_e \frac{\partial v}{\partial z} |_0, \quad v_b \frac{\partial v}{\partial z} |_0 = \frac{e}{m} \frac{\partial v}{\partial z} |_0
\]

then place restrictions upon \( A, B, C, \) and \( D \), leading to the following dispersion relation:

\[
\theta_0^2 + \frac{i\theta_0}{2} \left\{ \frac{\theta_0 - \theta}{\theta_0 + \theta} \left[ \exp(i(\theta + \theta_0) - 1) - \frac{\theta_0 + \theta}{\theta_0 - \theta} \left[ \exp(i(\theta - \theta_0) - 1) \right] \right] \right\} = 0 \tag{4.4}
\]

where \( \theta_0 = \omega_b L/v_b = 2\pi L/\lambda_0 \), and \( \theta = \omega L/v_b \), as used by Pierce.

However, if one makes the following transformations:

\[
\gamma = -i\theta/\theta_0 = -i\omega/\omega_b \quad \text{ie. growth rate in units of } \omega_b
\]

\[
s = \theta_0/2\pi = L/\lambda_0 \quad \text{ie. length of the system in units of } \lambda_0,
\]

then the PDR becomes

\[
2\pi s \gamma^2 (\gamma^2 + 1) - 2\gamma \left[ e^{-2\pi s \gamma} \cos(2\pi s) - 1 \right] - (\gamma^2 - 1) e^{-2\pi s \gamma} \sin(2\pi s) = 0 \tag{4.5}
\]

Figure 4.1 is a plot of \( \gamma \) against \( s \), obtained using numerical methods. The solid lines are the real part of \( \gamma \), \text{ie.} the growth rate, and the broken lines are the imaginary part of \( \gamma \), \text{ie.} the frequency. As can be seen, many modes can exist for a given system length, and that generally, when \( L > \lambda_0/2 \), the beam is unstable because there is at least one mode for which \( \gamma \) has a positive real part. However, when \( L = m\lambda_0/2 \), where \( m = 1, 2, \ldots, \) the largest growth rate is zero, and for \( L < \lambda_0/2 \) and a small range of \( L \) just less than \( (2m + 1)\lambda_0/2 \), the largest growth rate is negative, and therefore the beam is stable. To give an indication of the strength of the Pierce instability, the maximum growth rate, which is about 0.16\( \omega_b \) when \( L \sim 0.77\lambda_0 \), is greater than the growth rate of the unbounded, one-dimensional beam-plasma instability with \( \eta > 0.99 \).

**Simulation Results**

It is found using the PIC code, that the dominant modes have the growth rates and frequencies of the fastest growing mode predicted by the PDR. The accuracy of the growth rate can be improved by using a larger value of \( N_0 \) or a smaller value of \( \Theta \). When using only 10 particles per cell and \( \lambda_0 = 10\pi \Delta_x \), \text{ie.} with \( \Theta = 0.2 \) and \( v_b = 1.0 \), the growth
The system is stable in these regions

Figure 4.1: The dispersion relation for the original Pierce instability. The solid lines are the real part of $\gamma$, i.e. the growth rate, and the broken lines are the imaginary part of $\gamma$, i.e. the frequency.

rate is accurate to within $\pm 2\%$, as shown in Figure 4.1. This means that the interaction can be observed while running the code interactively on a medium speed computer, e.g. VAX11/780.

For system lengths in the range $(m - 1/2)\lambda_0 < L < m\lambda_0$, $m = 1,2,\ldots$, where the frequency is zero, the perturbations on the electron beam in the simulation grow without any oscillation or propagation, i.e. they are purely growing modes. The growth of the potential energy in such a simulation with $L = 2.6\lambda_0$ is shown in Figure 4.2.

For systems with lengths in the range $m\lambda_0 < L < (m + 1/2)\lambda_0$, when the PDR predicts that the growth rate is positive, perturbations on the electron beam in the simulations do oscillate as they grow, as shown in Figure 4.3, and a wave propagates across the system.

The regions of stability are also confirmed by the simulations: the decay rate of an initial perturbation on the beam is the minimum decay rate predicted by the PDR.

Crystal and Kuhn (1985) examined the growth of the Pierce instability for $L < 1.5\lambda_0$ in more detail. They showed that, when the initial perturbation on the beam is not an eigenmode of the system, the initial growth is not exponential. Although, as beam electrons pass across the system to the collection boundary, the potential profile approaches that of an eigenmode, and the eigenmode with the fastest growth predicted by the PDR eventually dominates. This can be seen in Figure 4.2 where the growth rate reaches a constant value after about $300\Delta_t$. 

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Figure 4.2: Growth of the potential energy in a simulation with $L = 2.6\lambda_0$. The imaginary part of $\gamma$ is zero, consequently, there is no wave motion.

Figure 4.3: Growth of the potential energy in a simulation with $L = 1.2\lambda_0$. The imaginary part of $\gamma$ is not zero, consequently, the potential energy oscillates while it grows because waves propagate across the system.
In part of the range \((m - 1/2) < \eta < m\) (the region where purely growing modes are dominant), it is found with the PIC code that the instability will saturate and reach a static steady state. This was predicted by Shapiro and Shevchenko (1966), who found that the growth was determined approximately by a nonlinear first order differential equation for small perturbations, and that under certain conditions the nonlinear term would cause the amplitude to saturate. Figure 4.4 shows the steady state reached in the simulation from which Figure 4.2 was obtained.

According to Burinskaya and Volokitin (1983), if a static steady state is reached, it should have the following form

\[ v = v_b(1 - \delta \sin \xi), \quad n = \frac{n_b}{1 - \delta \sin \xi}, \quad \phi = \frac{mv_b^2}{e} \delta \sin \xi(\frac{\xi}{2} \sin \xi - 1), \]

where

\[ \delta = \text{a constant} < 1, \quad \text{and} \quad \xi(x) = 2\pi \int_0^{x/\lambda_0} n(x')dx'. \]

From the boundary conditions it can be shown that, for \(L > \lambda_0\), \(\delta = (\text{mod}(L/\lambda_0, 1) - 1/2)\pi\), and so \(\delta = 1\) when \(L = (m + 1/2 + 1/\pi)\lambda_0 = (m + 0.8183)\lambda_0\). When \((m + 0.8183)\lambda_0 < L < (m + 1)\lambda_0\) the beam will be reflected back to the injection boundary, and a static steady state will not be possible. For all \(L\) in the range \(\lambda_0/2 < L < \lambda_0\) static steady states can be reached.

The steady states reached by the PIC code agree very well with these theories. It is indeed found that steady states are reached only in the regions defined by the theory of Burinskaya and Volokitin provided, however, that the electric field is positive at the injection boundary (i.e. the velocity of the beam electrons must increase as they leave the injection boundary, as in Figure 4.4a). If the electric field is initially negative at the injection boundary, or if the dominant mode is not a purely growing mode, then a virtual cathode forms, resulting in a dynamic steady state, as shown by Burinskaya and Volokitin (1983) and Crystal and Kuhn (1985).

When \(L \rightarrow (m + 0.8183)\lambda_0\) the potential, density, and electron velocity, change very rapidly near the points of minimum potential, and so more cells are required to resolve the rapid variation to the same degree of accuracy. One run has been made with \(L = 1.808\lambda_0\) in which \(N_c = 568\) and \(N_b = 100\). Both the maximum and minimum potentials, and the beam velocity, are within \(\pm 2\%\) of the values predicted by Burinskaya and Volokitin,

Figure 4.4: (Overleaf) Steady state reached in a simulation with \(L = 2.6\lambda_0\). a) electron phase space, b) density, c) potential. (\(\Theta = 0.2, L = 98\Delta_x, v_b = 1.2\Delta_x/\Delta_t, N_b = 10\))
but the maximum electron density is 16.7\(N_0\), which is only 54% of the value predicted by Burinskaya and Volokitin. Such variations in the degrees of accuracy of the different variables are normal in the simulations (e.g. see Figure 4.4). This is to be expected because the potential is the double integral of the charge density. Irregularities in the electron density do not cause large variations in the potential, and consequently the electrons do not depart greatly from the correct trajectory.

4.2 The Pierce Instability in Cylindrical Geometry

Most laboratory beam-plasma experiments are made with electron beams having a radius \((r_b)\) much less than the length of the system, in which case, the interaction will be two-dimensional, or even three-dimensional if azimuthal symmetry cannot be assumed.

However, if there is an infinite, axial magnetic field, and there is no radial variation of the injected beam velocity, then for any \(r_b\) the interaction will be one-dimensional, at least when the amplitude of the potential perturbations are small. To simulate this situation, the radial profile of the charge density and the potential can be represented by Bessel functions, and each electron in the simulation will represent a disc of radius \(r_b\), with the chosen charge density profile. Unfortunately, when the perturbations grow in amplitude, the radial variation of the potential should cause the beam velocity to vary radially and this cannot be included in a one-dimensional model. Nevertheless, during the initial growth of an instability the radial variation can be minimal, and a one-dimensional code with a Bessel radial field profile will model the stability of a thin electron beam.

In a cylindrical system Poisson’s equation is

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{e}{\varepsilon_0} (n_b - n_0) \tag{4.6}
\]

If the beam is inside and extends to the edge of a conducting cylinder, then the potential will be zero on all boundaries. Further, if the radial profiles of the density and potential are given by the first order Bessel function out to the first zero, \(\phi(r) \propto J_0(2.405r/r_b)\), Poisson’s equation becomes

\[
\frac{\partial^2 \phi}{\partial z^2} - k_{\perp}^2 \phi = -\frac{e}{\varepsilon_0} (n_b - n), \quad \text{where} \quad k_{\perp} = \frac{2.405}{r_b}. \tag{4.7}
\]

A finite difference form of this equation can be used in the PIC code to model the one-dimensional interaction with limited radial extent, as was done by Turikov (1978).

An approximate dispersion relation for a bounded, cylindrical beam is now derived to provide a comparison with the simulations results. The dispersion relation for an electron
The solutions for $k$ can be determined approximately by looking for solutions when $k$ is very large and very small. When $r_b$ is very large, these solutions are

$$k^+ = \frac{\omega \pm \omega_b}{v_b}.$$ 

These four solutions can be used to give an approximate general solution of the form

$$\phi = (A e^{ik_+ x/k_0 v_b} + B e^{-ik_+ x/k_0 v_b} + C e^{ik- x} + D e^{-ik- x}) e^{-i\omega t},$$

where $k_0 = \omega_b/v_b = 2\pi/\lambda_0$.

If the boundary conditions are applied, the resultant dispersion relation becomes

$$2\gamma \cosh \frac{2\pi k_+ s \gamma}{k_0} -$$

$$\left[ \left(1 - 3 \left(\frac{k_+}{k_0}\right)^2\right) \gamma^4 + \left(1 + \left(\frac{k_+}{k_0}\right)^2 + \left(\frac{k_+}{k_0}\right)^4\right) \gamma^2 + \left(\frac{k_+}{k_0}\right)^2 \right] \gamma \cosh \frac{2\pi k_+ s \gamma}{k_0} +$$

$$\left[ \left(\gamma^2 - 2 \left(\frac{k_+}{k_0}\right)^2 - \gamma^2 + 1\right) \sin 2\pi s - 2 \gamma \cos 2\pi s \right] e^{-2\pi s \gamma} = 0, \tag{4.10}$$

using $\gamma$ and $s$ as defined before.

Although this dispersion relation is valid only for large $r_b$, it can be used to estimate how the PDR is modified for a finite diameter beam. In the limit $k_+/k_0 \to 0$, equation 4.10 becomes the original PDR. Figure 4.5 is a plot of equation 4.10 for $k_+/k_0 = 0.1$, i.e. $r_b/\lambda_0 = 3.83$, whose features are very similar to that of the PDR. However, the growth rates are significantly smaller than those derived from the PDR, and they continue to decrease with decreasing $r_b/\lambda_0$. For some value of $r_b/\lambda_0$ the beam eventually becomes stable.

**Simulation Results**

The dispersion relation obtained using the PIC code with the modified Poisson solver for $k_+/k_0 = 0.383$, i.e. $r_b/\lambda_0 = 1.0$, is shown in Figure 4.6, in the range $L = 0$ to $1.7\lambda_0$. As before, the solid lines show the growth rate, and the broken lines show the frequency. Both the growth rates and frequencies have been reduced by about 30% compared with those from the PDR.
Figure 4.5: Approximate dispersion relation for a Pierce instability in cylindrical geometry with an infinite axial magnetic field, and $k_\perp/k_0 = 0.1$, ie. $r_b/\lambda_0 = 3.83$.

Figure 4.6: Dispersion relation for the cylindrical Pierce instability for $k_\perp/k_0 = 0.383$, ie. $r_b/\lambda_0 = 1.0$, obtained using simulations.
It can also be seen that the zero $\omega$ solutions have moved from $L = m\lambda_0/2$, to $m\lambda_c/2$, where

$$\lambda_c = \lambda_0 \left( 1 - \left( \frac{2.405\lambda_0}{2\pi r_b} \right)^2 \right)^{-1/2}$$

and these are the zero $\omega$ solutions for the unbounded system. This indicates that the simulations are probably correct since the zero $\omega$ solutions should be the same whether the system is bounded or not.

Further decreases in $r_b$ lead to an increase in $\lambda_c$, in accordance with equation 4.11, and the growth rates of the instabilities observed in the simulations continue to decrease. Figure 4.7 shows the maximum growth rate obtained by simulation in the range $\lambda_c/2 < L < \lambda_c$, upper curve, and the lower curve is the maximum growth rate based on the approximate dispersion relation. Clearly, the approximate dispersion relation departs from the simulation results even for very small $k_\perp/k_0$. However, when $k_\perp/k_0 \rightarrow 1.0$, the growth rates obtained from the simulations still decrease to zero.

Figure 4.8 shows the growth rates of the dominant modes found by the PIC code when $k_\perp$ is increased above $k_0$. The branch of the dispersion relation originating at $(\gamma, L) = (-\infty, 0)$ and passing through $(\gamma, L) = (0, \lambda_c/2)$ never reaches $\gamma = 0$ when $k_\perp = k_0$, ie. $r_b = 0.383\lambda_0$. When $k_\perp$ is increased above $k_0$ the decay rates continue to increase, and since all other modes of the system must have greater decay rates the beam must be stable for $k_\perp > k_0$. 

Figure 4.7: Maximum growth rate of the cylindrical Pierce instability: Simulation results - O, Approximate dispersion relation - x.
Figure 4.8: Growth rates for the cylindrical Pierce instability with $k_L > k_0$.

It is worth noting that to obtain the decay rates shown in Figure 4.8, a large amplitude perturbation was placed on the beam before commencing the simulation. Because these perturbations were not eigenmodes, the initial decay rates were not that of the eigenmodes with the slowest decay rate. If the decay rate is large, it is possible for the amplitude of the perturbation to be too small to observe pure exponential decay before the limit of the computer accuracy is reached. Consequently, the accuracy of some of the points in Figure 4.8 is poor.

4.3 The Pierce-Buneman Instability

If ion motion is included in the interaction, the dispersion relation is modified, and, as shown by Buneman (1958, 1959) for an infinitely long one-dimensional system with unlimited radial extent, it becomes

\[
1 - \frac{\omega_i^2}{\omega_b^2} - \frac{\omega_b^2}{(\omega - k v_b)^2} = 0 \quad \text{ie.} \quad k^\pm = \frac{\omega}{v_b} \pm \frac{\omega_b/v_b}{\sqrt{1 - (\omega_i/\omega)^2}}^{1/2} \quad \text{(4.12)}
\]

This is the dispersion relation for the Buneman instability, which has a maximum growth rate of

\[
\gamma = \frac{\sqrt{3}}{2} \left( \frac{m}{2M} \right)^{1/3} \omega_b, \quad \text{(4.13)}
\]

and if the ions are protons, it is $0.05612\omega_b$. This is more than one third the maximum growth rate of the Pierce instability, and so the motion of light ions cannot be neglected.

The general solution for the Pierce-Buneman instability is of the same form as that for the original Pierce instability ie.

\[
\phi = (A z + B + Ce^{i(k+\pi)} + De^{i(k-\pi)})e^{-i\omega t},
\]
except with different $k^+$ and $k^-$. Since the boundary conditions are the same as those for the PDR, the dispersion relation will be similar. It becomes

$$2\pi s \gamma^2 (\gamma^2 + \beta^2) - \beta \left[ 2\beta \gamma (e^{-2\pi\gamma\cos(2\pi s)} - 1) + (\gamma^2 - \beta^2) e^{-2\pi\gamma\sin(2\pi s\beta)} \right] = 0, \quad (4.14)$$

where $\gamma$ and $s$ are defined as before, and

$$\beta = \left[ 1 - \left( \frac{\omega_1}{\omega} \right)^2 \right]^{-1} = \gamma \left( \gamma^2 + \frac{m}{M} \right)^{-1/2}. $$

This dispersion relation has been solved for various values of $m/M$ (Saeki et al., 1977; Iizuka et al., 1979,1983; Vladimirov et al., 1983; Kolyshkin et al., 1984). The transformation of the PDR to the above dispersion relation as $m/M$ increases can be seen clearly in the paper by Kolyshkin et al.. Figure 4.9 is taken from this paper and it shows the Pierce-Buneman dispersion relation for $M = 1600m$. When the ion motion is taken into consideration, the system is unstable for all lengths.
Figure 4.10: Dominant modes of the Pierce-Buneman instability, obtained from simulations for $M = 1836m$.

Simulation Results

Some problems have been encountered when trying to simulate the Pierce-Buneman instability. It is found that when only a few ions escape from the system, the growth of the instability can be altered significantly. If ions are lost before the perturbation has grown significantly, the growth rate and frequency cannot be determined accurately.

The ions are lost as a result of the particle boundary conditions which assume total absorption for both ions and electrons. The ions are initially stationary and uniformly distributed across the system. They are then left to move under their self-consistent electric fields. Once an ion reaches a boundary it is deleted from the simulation, and no ions are injected into the system to replace it.

To reduce ion loss, only a few electrons and ions are used, typically $N_0 = 10$. In this case, since the ions and electrons are loaded uniformly in space, the nearest ion to the boundary is initially $0.05 \Delta_x$ away. This ensures that no ions leave the system until the interaction has reached a nonlinear stage.

With $N_0 = 10$ and $\lambda_0 = 20\pi \Delta_x$, i.e. with $\Theta = 0.1$ and $v_b = 1.0$, the PIC code has been used to find the modes of the Pierce-Buneman instability with the fastest growth rates, for $M = 1836m$. Figure 4.10 shows the results of the simulations, together with the growth rates of the purely growing modes which have been derived using numerical methods. The simulation results agree to within $\pm 0.5\%$ for the purely growing modes, and the complex solutions agree with the general features shown in Figure 4.9.
In agreement with theory, simulations of short systems support oscillations at the ion plasma frequency which do not grow. Also, systems with length just less than \((m + 1/2)\lambda_0\) which are stable when the ion motion is neglected, are not stable when ion motion is considered. For example, when \(L \sim 1.43\lambda_0\) the beam would be stable if ion motion were neglected; although, with \(M = 1836m\) two oscillatory modes exist which have similar growth rates. However, because of the boundary conditions, both growth rates are less than that of the Buneman instability. In some simulations of this particular case the initial perturbations have first excited the mode with slower growth, and the eventual quenching by the faster growing mode can be seen.

4.4 The Beam-Plasma Instability in a Bounded System

In his original work, Pierce (1944) assumed that the ions, created by ionizing collisions between the beam electrons and the neutral particles, would accumulate in the region of the beam, and eventually neutralize it. Initially, the electrons released by the ionizing collisions would be repelled by the negative space charge of the beam. However, once the beam becomes neutralized, some of the less energetic plasma electrons will remain in the region of the beam. Therefore, more ions may accumulate in the region of the beam depending upon the balance between creation by ionization and loss by diffusion. Consequently, the equilibrium plasma density may be comparable to, or even exceed the beam density.

The modification of the Pierce instability due to the presence of plasma electrons becomes more important as the density of the plasma electrons increases. This effect is particularly relevant to the BPD in a bounded system, since it is believed that the onset of the discharge is dependent upon the plasma density (Rowland et al., 1981; Papadopoulos et al., 1981, 1984).

In general, for plasma electrons to be present at equilibrium in a bounded system, a sheath will exist on the boundaries. If the sheath potential is large, it is unlikely that a beam-plasma instability would grow from a quiet state since the beam would be perturbed by the sheath (Goldstein et al., 1977). Nevertheless, it is worth examining the growth rates of the cold beam, cold plasma instability in a bounded system to provide a comparison with more realistic simulations in following chapters.

The dispersion relation for a cold beam, cold plasma interaction in an unbounded
one-dimensional system is
\[ 1 - \frac{\omega_i^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} = \frac{\omega_b^2}{(\omega - kv_b)^2} = 0 \]
\[ \text{ie. } k^\pm = \frac{\omega}{v_b} \pm \frac{\omega_b/v_b}{\left[1 - (\omega_i^2 + \omega_p^2)/\omega^2\right]^{1/2}}. \] (4.15)

For a bounded system, the boundary conditions will be the same as for the Pierce or Pierce-Buneman instabilities. Therefore, in a bounded system the dispersion relation will be of the same form as for the Pierce-Buneman instability with \( \omega_i^2 \) replaced by \( \omega_i^2 + \omega_p^2 \).

The parameter \( \beta \) introduced for the Pierce-Buneman instability now becomes
\[ \beta = \left[1 - \left(\frac{\omega_i + \omega_p}{\omega}\right)^2\right]^{-1} = \gamma \left(\gamma + \frac{m}{M} + 1 - \eta\right)^{-1/2}. \] (4.16)

When \( 1 - \eta \gg m/M \), ie. \( n_p/n_e \gg m/M \), \( \beta \approx \gamma(\gamma + 1 - \eta)^{-1/2} \), and so the ion motion can be ignored. The minimum value of \( \eta \) examined previously was \( \eta \approx 0.99 \) by Kolyshkin et al. (1984), and the dispersion relation they evaluated is shown in Figure 4.11.

Comparing Figures 4.9 and 4.11, it can be seen that the general features of the dispersion relation have moved to greater \( L/\lambda_0 \) as a result of plasma electrons being present. This trend continues as \((m/M + 1 - \eta)\) increases, due to either a decrease in \( M \) or \( \eta \). The growth rates of the purely growing modes decrease with decreasing \( \eta \) and eventually such modes are never dominant in a system of any length. It is the dominant oscillatory mode which exists in short systems (when \( \eta = 0.99 \) this mode is dominant when \( L \lesssim 0.9\lambda_0 \), as shown in Figure 4.11), which will be the dominant mode for most system lengths when \( \eta \ll 1 \).

Simulation Results

Figure 4.12 shows the dominant modes found with the PIC code for \( \eta = 0.99 \). The purely growing modes shown in Figure 4.12 are found numerically, and the simulation results obtained with \( \lambda_0 = 20\pi\Delta_x \), \( N_b = 99 \), and \( N_0 = 100 \) agree to within \( \pm 2\% \). The oscillatory modes also compare well with the dispersion relation evaluated by Kolyshkin et al. (1984).

Figure 4.13 shows the frequencies and growth rates obtained from the PIC code for \( \eta = 0.5, 0.1, \) and \( 0.01 \). As can be seen, the growth rates for the bounded system are below those for an unbounded system, although as the system length is increased the growth rates approach those for an unbounded system. The frequencies of the instabilities increase when \( \eta \) is decreased, just as predicted by the theory for an unbounded system, but they are higher. For example, when \( \eta = 0.01 \), the theory of an unbounded system, ie. equation 1.6, gives \( f = 0.915f_e \), while the simulations consistently show \( f = 0.975f_e \).
Figure 4.11: Dispersion relation evaluated by Kolyshkin et al. (1984) for the bounded beam-plasma instability, with $\eta \approx 0.99$.

Figure 4.12: Dominant modes found with simulations of the bounded beam-plasma instability, for $\eta = 0.99$. 
Figure 4.13: Growth rates and frequencies of the bounded beam-plasma instability, for $\eta = 0.5, 0.1,$ and 0.01, found with simulations. The spread in the growth rates for $\eta = 0.1$ and 0.5 was due to the formation of sheaths on the boundaries. The growth rates for an unbounded system are shown on the right hand side.

The broad range of growth rates observed in the simulations with $\eta = 0.1$ and 0.01 is a result of the loss of plasma electrons from the system, similar to the effect of ion loss during the simulations of the Pierce-Buneman instability. Since the electrons have less inertia than the ions, greater care must therefore be taken to ensure that they are not lost from the system. However, in practice this is hard to accomplish, and sheaths begin to form on the boundaries before clear exponential growth is observed. As a result, the growth of the perturbations is seldom purely exponential, and the range of the observed growth rates from different simulations is shown in Figure 4.13.

In addition to this difficulty, significant spatial growth across the interaction region must be accounted for. To distinguish between spatial growth and temporal growth, the increase in the potential energy within a region shorter than $\lambda_0$ can be monitored. The temporal growth rates obtained from these measurements have less spread and are
generally at the low end of the range shown in Figure 4.13 because the potential energy in the sheaths can be avoided.

Summary

The simulations in this chapter have shown that the PIC code can model, very accurately, electrostatic instabilities on an electron beam in a bounded, one-dimensional system. Since the accuracy can be improved arbitrarily by simply using more cells or more particles, a choice must be made between the degree of accuracy required and the speed of the simulation. Many simulations in this chapter can be run interactively, following the interaction through to saturation, in only a few minutes on a medium speed computer (ie. VAX11/780).

The simulations confirm the dispersion theory, which shows that an electron beam in a bounded, one-dimensional system is unstable for all lengths when the ions have finite mass or $\eta < 1$. Even in very short systems, small non-growing oscillations at the ion plasma frequency can exist. However, if a sheath forms, nonlinear effects may also cause any perturbation from the equilibrium of a quiescent plasma in very short systems to be amplified. Also, the growth rate of the beam-plasma instability generally increases with increasing system length for the those values of $\eta$ estimated from WOMBAT laboratory experiments when near the threshold of BPD.

Simulations of a thin beam suggest that the system is more stable. Certainly if there is an infinite axial magnetic field, $\eta = 1$, and immobile ions, the system is stable when $r_b < 0.383\lambda_0$. The laboratory experiments in WOMBAT, however, have not been made with a very strong, axial magnetic field. Consequently, the radial motion of the electrons may allow other instabilities to dominate when $\eta \sim 1$, as observed in the WOMBAT experiments.
Chapter 5

Simulating a Hot Plasma

In order to simulate a BPD, the PIC code must be able to accurately simulate a hot plasma. The problems associated with simulating a hot plasma with immobile, uniformly distributed ions, are discussed in this chapter.

All particle simulations are subject to numerical heating and are not collisionless due to inaccuracies in the computational technique. It was shown that the numerical heating is a stochastic process in particle simulations with periodic boundary conditions by Hockney (1971). This has also been found to be true when the simulation is bounded. However, as discussed in Section 5.1, the effects of numerical heating are influenced by the boundary conditions. A sheath forms on the boundaries to retain the hot electrons, and a balance exists between the kinetic energy of the electrons and the potential energy of the sheath.

The initial conditions used in the simulations examined in Section 5.1 produced large amplitude oscillations of the plasma between the sheaths. Fortunately, as shown in Section 5.2, it is possible to start a simulation with a plasma already in equilibrium. It is shown that, with such initial conditions and negligible numerical heating, a realistic quiescent state can be modelled accurately.

5.1 Numerical Heating

Two important effects which occur in particle models of plasmas are an exaggerated collision rate and numerical heating, and both these nonphysical processes will influence the shape and size of the sheath.

The exaggerated collision frequency is due to the smaller number of particles used in such simulations than would be present in a plasma. In a real plasma the electrostatic forces on a charged particle originate from two general regions: the region outside the Debye sphere which determines collective behaviour, and the region inside the Debye
sphere where only binary encounters are important. The forces on a particle due to other particles within its Debye sphere are not shielded. However, the larger the number of particles in this region the closer the average ensemble force approaches zero, approximating collisionless behaviour.

Since there is no easy means of accurately controlling the collision frequency in a simple PIC model, the collision frequency is made as small as practicable to simulate collisionless phenomena. In the dimensionless units used in the code, the Debye length is given by

$$\lambda_d = \frac{\nu_{\text{th}}}{\Theta},$$

so that low collision rates require a large number of particles within the shielding length $\nu_{\text{th}}/\Theta$. In the cloud-in-cell model used here, the particles have some width which also influences the shielding length. Birdsall et al. (1968) showed that the shielding length is the larger of either the Debye length or the particle width. However, with $\lambda_d$ small, even with a large particle width, it has been found that the numerical heating is still determined by the Debye length.

Numerical heating arises from the discrete nature of the simulation technique. It depends in a complicated manner on the ratios $\Delta_z/\lambda_d$ and $\Delta_t/\tau_e$. The actual details of the relationship depend upon the methods by which the charge density, potential, and the electric field are calculated. Hockney (1971) has made a study of the collision and heating times for a variety of periodic, two dimensional particle simulation techniques. He measured the ratio of heating time, $\tau_h$, to collision time, $\tau_c$, for different values of $\Delta_z/\lambda_d$ and $\Delta_t/\tau_e$. The collision time was taken as

$$\tau_c = \frac{\langle v_\parallel(t) \rangle}{d} \quad \text{at} \quad t = 0,$$

and the heating time was measured as the time taken for the average kinetic energy of an electron to increase by $\kappa T_e/2$. He found that the energy increased linearly with time, which is evidence that the heating was stochastic in origin (see the Appendix), and that for smaller $\Delta_z/\lambda_d$ and $\Delta_t/\tau_e$ the ratio $\tau_h/\tau_c$ was reduced.

In a one dimensional model the collision time cannot be defined in the same way. However, the method by which the heating rate can be reduced still applies: $\Delta_z/\lambda_d$ and $\Delta_t/\tau_e$ must be reduced as much as practicable, while still maintaining a large number of particles per $\lambda_d$. Unfortunately, obtaining better resolution of the field parameters and minimizing the numerical errors generally requires more computer time.
5.1.1 Simulation Results

To examine the numerical heating rates in a bounded PIC code, a series of simulations have been made with very simple initial conditions. The electrons are placed uniformly in phase space in an ordered manner, approximating a top hat velocity distribution with uniform spatial distribution, as discussed in Chapter 3. The ions are assumed to be immobile, and no electrons are injected into the system. The electrons are then left to move under the influence of their self-consistent fields.

Initially there is no electric field, and those electrons which are close to a boundary and moving towards it, leave the system. Electrons also move away from the boundary, initially at the velocity assigned when loaded. Since no electrons are injected into the system in these simulations to replenish the region of phase space which is vacated, and, as the ions cannot move, the net charge can only increase. As the electron density decreases near the boundary a sheath is created.

During the initial stages, the central part of the plasma remains almost undisturbed. The electrons continue to move at their initial velocity, remaining in an ordered array in phase space. Their charge is distributed uniformly between the grid points, and because of the symmetry of the system, a constant potential is maintained in the central region.

However, the distribution becomes disordered at the boundaries, due to the emergence of the sheath, and the disorder propagates into the central region carried by the electrons coming from the boundary. The distribution loses all its original order after approximately $N_c/2v_{sp}$ time steps, where $N_c$ is the number of cells in the simulation and $v_{sp}$ is the initial velocity spread of the electrons.

By this time, the sheath potential has reached an amplitude

$$\phi_{sh} \approx \frac{v_{\text{max}}^2}{2},$$

where $v_{\text{max}}$ is the velocity of the fastest moving electrons. The sheath potential increases until it is just able to trap the fastest electrons, forcing them to rotate in phase space corresponding to their bouncing between the sheaths at either end. This trapping condition is reached in only a few time steps; in fact, for simulations with initial parameters chosen to reduce the numerical errors, it requires much less than $N_c/2v_{sp}$.

An estimate of the time required for the establishment of the sheath can be made if the effects of the developing sheath are neglected until the trapping condition is met, i.e. we assume that all electrons move at their initial velocity until the charge at the boundaries is sufficient to reflect all electrons. In such a situation the region of phase space which is vacated will be triangular, and, with an initial top hat velocity distribution, the electron
density will increase linearly from $N_0/2$ on the boundary to $N_0$ at some point $x_{sh}$ from the boundary.

It can be shown that to obtain $\phi = v_{sp}^2/2$ at $z = x_{sh}$, we must have $x_{sh} = (6/\Theta)^{1/2}v_{sp}$. The time taken to reach this condition will be independent of $v_{sp}$ because the time taken for the fastest moving electrons to reach $x_{sh}$ is $x_{sh}/v_{sp}$. Thus the sheath condition should be met in $(6/\Theta)^{1/2}\Delta_t$, which is typically much less than $N_c/2v_{sp}$.

Of course, the effect of the electric field cannot be neglected even if the evolution of the sheath occurs on a very short time scale. The sheath will retard the escape of electrons from the system once it becomes greater than the potential fluctuations due to numerical errors. The simulations show that for $v_{sp} > 0.01\Delta_x/\Delta_t$, the initial rapid loss of electrons creates a sheath with a potential $\phi_{sh} \approx v_{sp}^2/2$ in approximately $\tau_e/2$, ie. $(\pi/\Theta)\Delta_t$. However, due to the unphysical initial conditions, when the initial velocity spread is greater than $0.1\Delta_x/\Delta_t$, too many electrons are lost, and the plasma oscillates at the electron plasma frequency.

**Effect of varying $v_{sp}$**

The first series of simulations have all been started with $2 \times 10^4$ electrons, 100 cells, and $\Theta = 0.2$, ie. $N_0 = 200$ and $\tau_e = 10\pi\Delta_t$, but the velocity $v_{sp}$ is varied from $10^{-3}$ to $2\Delta_x/\Delta_t$. The general features revealed by these simulations are, an initial rapid loss of electrons followed by an approximately linear growth, or decay, of the kinetic and potential energy in the system, as shown in Figure 5.1. The rate of change of the kinetic and potential energy depend upon $v_{sp}$ since $\lambda_d \approx v_{sp}/\Theta$. Some of these simulations are now discussed in more detail because they illustrate the problems associated with the PIC model.

When $v_{sp} = 10^{-3}$, 5000 time steps, ie. $160\tau_e$, are insufficient to allow the electrons to move far from their original positions. The furthest that an electron could move is approximately $5\Delta_x$, leaving the central 90 cells with a relatively ordered distribution in phase space. However the spatial fluctuations of the potential have approximately the same magnitude across the whole system with the random variation in $(\phi_i - \phi_{i+1})$ of order $10^{-5}$. This is far greater than the sheath potential required to trap electrons with a velocity $10^{-3}$; $\phi_{sh} = 5 \times 10^{-7}$ is all that is required.

When making computations with eight significant figures (ie. single precision numbers on a VAX11/780), the minimum random electric field is about $2 \times 10^{-8}$. However, with a large number of electrons per cell, the systematic numerical errors caused by numbers being truncated can accumulate so that the initial charge density in a quiet start is not exactly zero, even though it should be theoretically.
Figure 5.1: Change in the energy content of the system for different $v_{sp}$ with $N_0 = 200$. When $v_{sp} < 0.2$, i.e. $\lambda_d \lesssim 1.0$, the total energy content increases, but when $v_{sp} > 0.2$ it remains constant.

With 10 electrons per cell the density has a maximum value of $2 \times 10^{-7}$, creating a potential with a maximum value of $2 \times 10^{-7}$. With 500 electrons per cell the maximum density is $5 \times 10^{-8}$ and the maximum potential is $1.6 \times 10^{-4}$. With 200 electrons per cell, as in the series of simulations being described, the charge density has a minimum value of $-3 \times 10^{-9}$ and an initial potential which is roughly parabolic with a central minimum of $-1.2 \times 10^{-4}$. Since the potential required to trap electrons with $v_{sp} = 10^{-3}$ is much less than this value, the initial potential causes the plasma to oscillate at the plasma frequency, and the oscillation gradually destroys the original ordered distribution.

When double precision numbers are used the inaccuracies of calculating the initial field variables are reduced. However, with 200 electrons per cell the initial potential profile has a maximum value of $1 \times 10^{-6}$, a reduction by only a factor of 100. Electrons are still
lost from the system at an unphysically high rate, and eventually the electric field error reaches the same order of magnitude as that observed in the single precision simulations. This is to be expected since it is the number of electrons per Debye length which is the limiting factor in determining the accuracy of the field variables, and when $N_0 = 200$ and $v_{sp} = 10^{-3}$ there is only about one electron per Debye length.

The plasma oscillations force many more electrons from the system than is necessary to achieve a sheath potential of $\phi_{sh} \sim v_e^2/2$. According to the simple model discussed before, which assumed that the motion of electrons is not influenced by the electric field initially, only 0.6 of an electron out of the initial total of $2 \times 10^4$ need leave the system to achieve $\phi_{sh} = 5 \times 10^{-7}$. However, 66 electrons are lost in the first 1000 time steps, which results in a steady reduction of the kinetic energy in the system. After 1000 time steps, the spatial potential fluctuations increase as the initial order of the quiet start is reduced, and the numerical heating rate increases.

When $v_{sp} = 10^{-2}$, the kinetic energy initially decreases, but after approximately 100 time steps it begins to increase, also as a result of the initial perturbation destroying the ordered distribution. The kinetic energy then increases approximately linearly for the next 5000 time steps, but over the following 5000 time steps the numerical heating rate increases almost exponentially. This is indicative of an instability caused by the interaction of electrons of different velocities behaving as separate beams (Gitomer and Adam, 1976). After 10,000 time steps the kinetic energy is increasing at a rate of 150% of the initial value in only 1000 time steps.

With $v_{sp} = 0.1$, the kinetic energy drops by 1.3% in the first 16 time steps (ie. $\approx \tau_e/2$). It then oscillates by ±0.1% about the new level for approximately 500 time steps, ie. $N_e/2v_{sp}$, in which time all order in the distribution is lost because the faster electrons have moved $50\Delta_*$ carrying the disorder to the centre of the system. The kinetic energy then grows at a linear rate of 0.7% of the initial level every 1000 time steps. As the kinetic energy increases, more electrons pass through the sheaths, causing $\phi_{sh}$ to increase to keep $\phi_{sh} \sim v_{max}^2/2$.

As the sheath amplitude increases, so does its width and the potential energy in the system. In this example the potential energy density increases at approximately the same rate as the kinetic energy, causing the total energy in the system to increase at a rate of 1.5% of its initial value every 1000 time steps. After 5000 time steps the sheath extends over $(2-5)\Delta_*$, which is about $(4-10)\lambda_d$. The variation is caused by oscillations of the plasma resulting from the initial conditions.

The initial conditions enable many more electrons to be lost from the system than
Figure 5.2: Potential oscillations at $x = 20$ when $v_{sp} = 1.0$. The initial conditions cause the plasma to oscillate, and, since the behaviour is almost collisionless, there is little attenuation.

would be necessary to establish a sheath with sufficient amplitude to trap the electrons with velocity $v_{sp}$. When $v_{sp} = 1.0$ the potential at $x = 20$, which is near the edge of the sheath, is 25% greater than $v_{sp}^2/2$ in half an electron plasma period. This causes the plasma to oscillate, and, since the behaviour is almost collisionless, the oscillations continue with little attenuation over thousands of time steps, as shown in Figure 5.2. The amplitude of the oscillations increase with greater $v_{sp}$ until the sheaths reach the center of the system. Since the sheath width is generally $(4-10)\lambda_d$, the sheaths meet when $v_{sp} \gtrsim 1.5$ in this series of simulations.

These oscillations are a significant feature of the temporal evolution of the simulations, and would probably influence the development of a beam-plasma interaction. Fortunately they can be reduced significantly by having an initial sheath; this is discussed later in Section 5.2.

Despite the oscillations, these simulations still show that the total energy contained in the system remains relatively constant when $v_{sp} > 0.2$. During each oscillation, when $v_{sp} = 1.0$, the total kinetic energy in the system oscillates by $\pm1.3\%$ and the total potential energy oscillates by $\pm25\%$, yet approximate energy conservation is maintained because the total energy oscillates by only $\pm0.15\%$. After the initial rapid loss of energy, as a result of too many electrons leaving the system in the first $\tau_e/2$, the total energy averaged over one oscillation varies by as little as $0.01\%$ every 1000 time steps. However, the kinetic energy
Figure 5.3: Since $\lambda_d \approx v_{sp}/\Theta$, the number of electrons per Debye length is increased when $v_{sp}$ is increased. Consequently, the numerical heating rate decreases and $v_{\text{max}} \rightarrow v_{sp}$ when $v_{sp}$ is increased.

decreases, reaching a steady rate of 0.05% each 1000 time steps, as shown in Figure 5.1. This is equivalent to 0.16% in 100$\tau_e$, with about 1000 electrons per $\lambda_d$.

With $v_{sp} \gtrsim 0.2$ the velocity of the fastest electrons, $v_{\text{max}}$, is approximately equal to $v_{sp}$, as shown in Figure 5.3. Therefore, there are many electrons with almost sufficient kinetic energy to pass through the sheath. Some of these electrons can leave the system as a result of minimal numerical heating, causing a nett decrease of the kinetic energy contained in the system as observed. However, with $v_{sp} \lesssim 0.2$ the difference between $v_{\text{max}}$ and $v_{sp}$ is greater, and the velocity distribution reached is more gaussian because of the collisional effects. Therefore, the numerical heating can increase the average kinetic energy without causing so many electrons to be lost from the system, hence the observed nett increase in the kinetic energy when $v_{sp} \lesssim 0.2$.

Effect of varying $N_0$

Further runs have been conducted to compare the change of kinetic and total energy in simulations with different $N_0$. Using 100 cells, $\Theta = 0.2$, and no initial sheath as before, simulations have been run with $N_0 = 10$, 40, 200, and 400. The velocity distributions used are top hat with $v_{sp} = 5 \times 10^{-2}$ and $5 \times 10^{-1}$.

For all the values of $N_0$ tried, when $v_{sp} = 0.05$ the total kinetic energy increases with time, and when $v_{sp} = 0.5$ the total kinetic energy decreases. Figure 5.4 shows that for both values of $v_{sp}$ used, the rate of change of kinetic energy decreases when $N_0$ is increased.

However, when $v_{sp} = 0.5$ and $N_0 = 10$, i.e. about 25 electrons per Debye length, the
Figure 5.4: Change in the energy content of the system with different \( N_0 \) with \( v_{ep} = 0.05 \), and 0.5. When \( N_0 \) is increased, the rate of change of the kinetic energy decreases. However, when \( v_{ep} = 0.5 \), the total energy contained in the system barely changes.

The rate of change of the kinetic energy is not constant. The heating and collisional effects change the velocity distribution so that it becomes more gaussian having fewer electrons with velocity \( v \simeq (2\phi_{sh})^{1/2} \), and, as a result, more time is required to heat the electrons so that some escape. Despite this, the changes in the kinetic energy contained in the system after \( 5000\Delta_t \) is still approximately proportional to \( N_0^{-1/2} \) with \( v_{ep} = 0.5 \) (shown in Figure 5.5), as expected for stochastic heating (Hockney, 1971).

The effect on the total energy in the system of varying \( N_0 \) when \( v_{ep} = 0.5 \) is not as simple as when \( v_{ep} = 0.05 \). The total energy remains at about the same level, but the amplitude of the variation, other than that caused by the plasma oscillations, generally increases when \( N_0 \) is decreased. The variation is negligible even with only 25 electrons per Debye length, as shown in Figure 5.4.
Figure 5.5: Despite the effect of the sheath on the partition of energy, the rate of change of kinetic energy in the system is approximately proportional to $N_0^{-1/2}$, as expected for a stochastic process.

5.2 Simulations with Initial Sheaths

To overcome the plasma oscillations caused by the initial rapid loss of electrons in the above simulations, the density of the electrons near the boundaries can be reduced so that there is a sheath already present when the simulation commences. The density can be reduced in such a way that the sheath is of the correct form for a plasma in equilibrium, with a given velocity distribution.

For example, Poisson’s equation for a plasma with immobile ions and electrons with a top hat velocity distribution, in dimensionless variables, is

$$
\frac{\partial^2 \phi}{\partial z^2} = \Theta \left[ \left( \frac{2\phi}{v_{sp}} \right)^{1/2} - 1 \right].
$$

The solution to this equation, $\phi(z)$, can be used as a separatrix passing across the system, discriminating between electrons which will escape, and those which will remain trapped by the sheath. Electrons with kinetic energy greater than the separatrix value for a given position should escape from the system. Therefore, after loading the electrons in phase space using the same algorithm as before, yet prior to moving any electrons, those electrons with too much kinetic energy are discarded. Once this is done, the calculations proceed as before, first calculating the density and then the potential. If the simulation is accurate, the potential calculated in the simulation should be the same as the solution to equation 5.4 with the correct boundary conditions.

Equation 5.4 is easily solved numerically, and the boundary conditions are well defined when the velocity distribution is top hat. Both the charge density and the electric field...
must be zero, while the potential must be $v_{sp}^2/2$ on the plasma side of the sheath. At the boundary, both the potential and the electron density should be zero.

The potential calculated by the code for the top hat distribution is very similar to the numerical solution of equation 5.4, although an underestimate is always made. For example, with $v_{sp} = 1.0$ and $\Theta = 0.2$, the sheath potential should be 0.5, and have a width of about $25\Delta_x$, but, with $N_0 = 200$, the PIC code has an initial sheath potential of only 0.472, though the sheath width is correct. However, neglecting the effect of numerical heating, and averaging over the first one hundred plasma periods, the difference between the numerical solution and the sheath potential found by the code is only 0.1%.

The lower initial sheath potential is a result of too few electrons being discarded at the start, and is caused by the discrete distribution of electrons in phase space and the curvature of the separatrix. The accuracy can be improved by increasing $N_0$, or by using a smaller $\Theta$. A decrease in $\Theta$ increases the width of the sheath ($\lambda_{sh}$), as $\lambda_{sh} \propto \lambda_d \propto v_{sp}/\Theta$ according to equation 5.4. With $\Theta = 0.2$ and $N_0 = 400$ the initial sheath potential is 0.475, and with $\Theta = 0.1$ and $N_0 = 400$ the sheath potential is 0.489.

5.2.1 Comparison with Simulations without an Initial Sheath

With a top hat electron velocity distribution, the amplitude of the oscillations of the sheath potential is approximately the difference between the initial sheath potential, as calculated by the code, and the correct value: $v_{sp}^2/2$. This is true for almost any type of sheath, even one formed by discarding all electrons in a few cells next to the boundaries. However, the heating rate in simulations with initial sheaths is approximately the same as in the corresponding simulation without an initial sheath. This is to be expected since the numerical heating rate is not due to bulk plasma oscillations but random errors in the field variables.

The steady state electron velocity distribution depends upon the initial conditions. When the number of electrons per Debye length is greater than about 200, the numerical heating and collisional effects can be neglected, and the velocity distribution is determined mainly by the initial and boundary conditions. For example, the velocity distribution cannot have electrons with energy greater than the sheath potential, so that a true gaussian distribution is not possible, although, if $v_{sth}^2 < \phi_{sh}$, the difference is insignificant.

The truncation of an initially gaussian distribution is more noticeable in simulations without initial sheaths because a larger proportion of high energy electrons leave the system before a sheath sufficient to trap these electrons has formed. Even after such a sheath has formed, the plasma oscillations favour the escape of high energy electrons,
Figure 5.6: Average electron velocity distributions for a variety of initial distributions when the numerical heating is negligible. a) Gaussian. b) Triangular. c) Top Hat. d) Top Hat when the simulation is started with a sheath of the correct form.

since, according to Liouville's theorem for a collisionless plasma, the low energy electrons should remain in phase space orbits which would not come close to the boundaries.

Similar behaviour has been observed with a triangular distribution, as well as the top hat distribution. Figure 5.6 shows the average electron velocity distributions, in the region with approximately uniform potential between the sheaths, from simulations with $\Theta = 0.2$, $N_0 = 200$, and $L = 100$ cells. The distributions are determined by averaging over $160\tau_e$ not including the start of the simulation. In the three simulations shown in Figure 5.6 without an initial sheath the number of high energy electrons decrease. The few electrons in the simulation with an initial top hat distribution with $v > v_{sp}$ have been accelerated while the sheath is being formed, and are not a result of numerical heating.

When a simulation is started with a sheath, fewer high energy electrons are able to leave the system than when there is no initial sheath. As a result, the velocity distribution remains closer to its initial form, as shown in Figure 5.6d for an initial top hat distribution. Figure 5.6d also shows that, because the plasma oscillations are reduced, there are fewer electrons with $v > v_{sp}$.

The velocity distributions shown in Figure 5.6 can be used to make a comparison between the average sheath potential calculated by the code when there no initial sheath, and an approximate numerical solution. When the distribution is top hat with a small
Figure 5.7: Comparison between the average sheath potential calculated in simulations without an initial sheath, and numerical solutions. Even though the average electron velocity distributions in the simulations are not truly top hat or gaussian, an estimate of the sheath can be made. Since the distributions are not modified significantly, the form of the sheaths are still representative of the initial distribution.

higher energy component, equation 5.4 is no longer strictly valid, although it can be used to estimate the sheath potential. The few electrons with energy greater than \( v_{sp}^2 / 2 \) have phase space orbits which pass closest to the boundary. Therefore, the electron density in some of the boundary cells will be lower than if the distribution had been constant up to \( v_{max} \). Thus, if \( v_{max} \) is used instead of \( v_{sp} \) when solving equation 5.4, an underestimate of \( \phi_{sh} \) is expected. Nevertheless, the solution of equation 5.4 with \( v_{sp} = 1.09 \) (taking \( v_{max} = 1.09 \) from Figure 5.6c) differs from the average potential calculated by the code by only 5%, as shown in Figure 5.7.

A similar comparison can be made for the simulation with a gaussian distribution, using Poisson's equation for a gaussian distribution:

\[
\frac{\partial^2 \phi}{\partial z^2} = \Theta \left[ \exp \left( \frac{2\phi}{v_{eth}^2} \right) - 1 \right].
\]  

(5.5)

Although this equation is not strictly valid when the distribution is truncated, the discrepancy is found to be small if we assume that \( \phi_{sh} = v_{max}^2 / 2 \) and that \( v_{eth} \) is unchanged. Using \( v_{max} = 1.65 \) (from Figure 5.6a) in equation 5.5, with \( v_{th} = 1.0 \), we find that there is only 4% difference between the average potential calculated in the simulation and the appropriate solution of equation 5.5, as shown in Figure 5.7. A plot of the sheath potential for a top hat velocity distribution with \( v_{sp} = 1.65 \) is also shown in Figure 5.7 to show that
the form of the sheath in the simulation corresponds more closely to that for a gaussian
distribution than a top hat distribution.

Summary

To simulate a hot plasma, it has been found that at least 200 electrons per Debye
length are necessary to reduce the numerical heating rate to below 1% per 100τe. This
means that on a medium speed computer, eg. a VAX11/780, many hours of computer
time are required to run some simulations over 100τe. If the numerical heating rate is
high, and if λSH ≫ ΔZ, then the kinetic energy decreases.

If the simulations are not started with a self-consistent sheath, the plasma oscillates
between the boundaries producing large amplitude variations in the sheath potential,
which can modify the beam-plasma interaction. The amplitude of these oscillations can
be reduced by providing some form of initial sheath. With a top hat electron velocity
distribution, it is possible to start the simulation with the electrons spatially distributed
to produce a sheath of the correct form for the plasma to be in a quiescent state. Provided
that such initial conditions are used, the amplitude of the oscillations is the difference
between the initial sheath potential calculated by the code and the analytic value. This
difference can be reduced by either increasing N0 or decreasing Θ.
Chapter 6

The Bounded Beam-Plasma Interaction with $n_b < n_p$

This chapter examines the long term evolution of the beam-plasma interaction in a bounded system, with immobile ions, and no ionization. Emphasis is placed upon the rate at which the beam-plasma interaction heats the plasma because it will be a significant factor which determines the threshold of BPD. The validity of equation 1.23 derived by Lebedev et al. (1976), which shows that the heating rate is dependent upon $\eta$, $L$, and the velocity distribution of the plasma, is investigated.

In previous simulations which studied the spatially growing beam-plasma interaction (Davis and Bers, 1968; Abe et al., 1979; Jones et al., 1983) the heating rate of the beam was either ignored or neglected. Davis and Bers were interested in the velocity distribution of the beam electrons upon reaching the collection boundary. Jones et al. were interested in the linear stage when there is growth both in time and space. Abe et al. were mainly concerned with confirming the theory of O'Neil et al. (1972) which describes the spatial exchange of energy between the beam and the wave, and the spatial variation of the harmonic structure.

Abe et al. used a program written for a bounded system, but were able to simulate a semi-infinite system by having many wavelengths in the interaction length. They were aware of plasma electrons, but tried to avoid it by keeping $\eta \ll 1$. They also used reflecting boundary conditions for the plasma particles so that a sheath did not form on the boundaries and impair the exponential growth near the injection boundary.

The simulations described in this chapter develop sheaths on the boundaries because of the loss of plasma electrons due to the beam-plasma interaction, and, as a result, the beam-plasma interaction is modified. Goldstein et al. (1977) showed that even with $\eta \ll 1$, large
oscillations rapidly develop on the beam when there is a sheath. The simulations described in this chapter had \( \eta \geq 0.01 \) and exhibit similar behaviour. However, when \( \eta \to 0.5 \) (ie. \( n_b \to n_p \)), the interaction is further modified because of the boundary conditions. Section 6.1 examines the modifications of the interaction giving a brief account of the change in the velocity distribution of the plasma and the exchange of energy in the system. The wave behaviour has also been described because it is believed that it can determine the threshold condition for BPD (Rowland et al., 1981).

Section 6.2 examines the heating rate in simulations with length less than \( 5\lambda_0 \) because near the threshold of a BPD in WOMBAT, when the plasma density is low, the number of wavelengths in the system is small. The density ratio, \( \eta \), is varied in order to check the heating rate from the nearly linear regime (\( \eta \sim 0.01 \)), to the highly nonlinear regime (\( \eta \sim 0.5 \)). The importance of the velocity distribution on the heating rate is also examined.

6.1 General Description of the Simulation Results

The beam-plasma interaction has been found to vary considerably for \( \eta = 0.01-0.5 \), and to classify the behaviour of the interaction this range has been divided into three regimes, \( \eta \leq 0.05 \), \( 0.05 \leq \eta \leq 0.25 \), and \( 0.25 \leq \eta \leq 0.5 \), characterized mainly by the wave behaviour. The boundaries between the different regimes are not clearly defined, as the behaviour of the interaction changes smoothly between the regimes. The behaviour can also change during a simulation as the plasma temperature increases.

6.1.1 \( \eta \leq 0.05 \)

For \( \eta < 0.01 \) the growth rate of the beam plasma instability is low, and the length of the system is important in determining the behaviour of the interaction. For long systems when the plasma temperature is low the interaction is similar to that observed by Abe et al. (1979), ie. exponential spatial growth of the wave followed by the trapping of beam electrons in the potential troughs of the wave. However, when the plasma is hot or the system is short, the effects of the boundaries cannot be ignored. The initial conditions are also important in this regime because the beam does not modify the state of the system rapidly. If a simulation is started without a sheath the resulting plasma oscillations persist for many plasma periods. Therefore, the simulations discussed in this section have been started in a state of equilibrium with an initial sheath and a top hat plasma velocity distribution.
Figure 6.1: *Plot of the evolution of the average energy density in a system from a simulation with \( \eta = 0.01, L = 1.59\lambda_0, \) and \( v_{sp} = 0.5v_b, \) showing that the heating rate is irregular.*

\( (\Theta = 0.1, \ L = 100\Delta_x, \ v_b = 1.0\Delta_x/\Delta_t, \) and \( N_0 = 400) \)

**Energy Content**

The rate at which the beam-plasma interaction heats the plasma varies significantly depending upon the oscillations of the plasma. Figure 6.1 shows the variation in time of the average energy density in a system with \( \eta = 0.01, v_{sp} = 0.1v_b, \) and \( L = 1.59\lambda_0. \) The energy in the system increases on average, but it also fluctuates with a period of about \( 12\tau_e. \) It is believed that this is due to variations in the wave behaviour since there is correlation between the heating rate and the wave behaviour; however this point needs to be investigated more thoroughly.

When the initial velocity distribution has great spread, a sheath of large amplitude is generated which increases the potential energy content of the system. The beam-plasma interaction causes the potential energy to increase more rapidly than the kinetic energy, as shown in Figure 6.1. This is because many plasma electrons are lost due to the oscillations and few beam electrons are trapped in the system. With \( \eta \geq 0.05, \) since more beam electrons are trapped, the change in the kinetic energy becomes the greater fraction of the total change in the energy content, even though the change in the potential energy content continues to increase with increasing \( \eta. \) Figure 6.2 compares the change in the energy content of systems with \( L = 3.18\lambda_0 \) for \( \eta = 0.01, \) and \( \eta = 0.05 \) over \( 160\tau_e. \)

The average spatial potential energy density in these simulations reveals that the
growth of a wave from the injection boundary is irregular. The potential energy density does increase moving away from the sheath at the injection boundary, however, the growth saturates close to the injection boundary, and the difference in amplitude between the maxima and minima is negligible. This suggests that the growth of the wave may be nonlinear and that the phase of the wave near the injection boundary varies. Figure 6.3 shows the average spatial variation of the potential energy density in a simulation with \( \eta = 0.01 \), \( v_{sp} = 0.1v_b \), \( L = 3.18\lambda_0 \). This figure shows that, even with a cool plasma, the potential energy density of the sheath is considerably greater than the potential energy density of the plasma.

**Velocity Distribution**

The change in the distribution of the beam in longer systems can be followed spatially from the exponential growth stage, through trapping of the beam electrons in the waves, and into the turbulent stage when there is a high energy tail. In short systems the beam...
Figure 6.3: Average spatial variation of the potential energy density in a simulation with $\eta = 0.01$, $L = 3.18\lambda_0$, and $v_{sp} = 0.1v_b$. Even when the sheath potential is small the potential energy density of the sheath is still much greater than the potential energy density of the waves. ($\Theta = 0.1$, $L = 200\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 400$)

is not always trapped by the wave before reaching the collection boundary. When $L < \lambda_0$ the beam is never reflected at the collection boundary because the spatial growth of the wave is too low.

If a sheath of great amplitude develops on the boundaries, the beam is accelerated to velocities significantly greater than the beam velocity at the injection boundary, and the beam density is reduced in accordance with the continuity equation. According to linear theory (ie. equation 1.7) the spatial growth rate should therefore be reduced, and the probability of beam trapping should decrease. However, because the sheath perturbs the beam as it enters the system, trapping of the beam may occur closer to the injection boundary. Figure 6.4 shows a phase space plot of a beam with $\eta = 0.01$ interacting with a plasma which initially had $v_{sp} = 0.5$. All the beam electrons are plotted, but only about 1/40-th of the plasma electrons are shown. Immediately after injection the beam is accelerated by the sheath, and the perturbation imparted to the beam enables the waves to reach a large amplitude very rapidly. The beam is trapped by the wave before reaching the collection boundary, and occasionally some beam electrons are reflected by the sheath at that boundary.

The velocity distribution of the plasma electrons does not change significantly from its initial form. There are few electrons, either trapped beam electrons or heated plasma
electrons, with velocities greater than the initial maximum velocity. The beam velocity oscillates about a mean which is determined by the sheath amplitude because of energy conservation, i.e. the mean 'local' beam velocity, $v_{bl}$ should be given by:

$$\frac{1}{2} v_{bl}^2 \approx \frac{1}{2} v_{b}^2 + \phi_{sh}.$$  \hspace{1cm} (6.1)

Figure 6.5 shows the average electron velocity distribution for the simulation examined in Figure 6.4, in the region just after the sheath at the injection boundary. The velocity spread of the plasma remains at $0.5\Delta z/\Delta t$, but the mean local beam velocity is 40% greater than $v_b$. Since the sheath potential is 0.5 in dimensionless units, this is in agreement with the above relation.

Wave Behaviour

The interaction of the plasma with the beam and the boundaries is complex. Not only can waves travel with the beam, but they can also be reflected by the sheath at the collection boundary, and waves with negative phase velocity are common. In longer systems, $L \gtrsim 2\lambda_0$, the wave motion becomes more complicated when the beam is trapped by the wave. Figure 6.6 is a colour contour plot of the potential from a simulation with $\eta = 0.01$, $v_{ep} = 0.01v_b$, and $L = 3.98\lambda_0$. The troughs (dark) and peaks (light) of the waves
Figure 6.5: *Average velocity distribution in the region just after the sheath at the injection boundary from a simulation with $\eta = 0.01$, $L = 3.98\lambda_0$, and $v_{sp} = 0.5v_b$ over a period of $80\tau_e$. The velocity of the beam is at $v_b \sim (v_b^2 + 2\phi_{zh})^{1/2}$. ($\Theta = 0.1$, $L = 250\Delta_z$, $v_b = 1.0\Delta_z/\Delta_t$, and $N_0 = 400$)

move away from the injection boundary with a velocity greater than $v_b$ (ie. the slope of the lines is greater than 1). However, within the first 100 cells the velocity decreases and the beam is trapped. The wave structure beyond this point is due to the bouncing of beam electrons in the troughs of the waves. The waves can be seen to travel slower than $v_b$ at one point, and then increase to velocities greater than $v_b$. The sheath is negligible in this simulation and the boundary effects are minimal. It is found that increasing the temperature of the plasma in long systems further complicates the wave behaviour as oscillations of the plasma between the boundaries become more common.

In short systems, $L \lesssim 2\lambda_0$, the beam-plasma interaction can force the plasma to oscillate between the sheaths, and standing waves can result as shown in Figure 6.7a with $\eta = 0.01$, $v_{sp} = 0.5$, and $L = 1.59\lambda_0$. Waves with negative phase velocity are frequently observed and occasionally, when $\eta \sim 0.05$, they have been observed when there are no

Figure 6.6: *(Overleaf) Contour plot of the potential varying in space and time from a simulation with $\eta = 0.01$, $L = 3.98\lambda_0$, and $v_{sp} = 0.01v_b$, showing the trapping of beam electrons in the potential troughs of the waves. ($\Theta = 0.1$, $L = 250\Delta_z$, $v_b = 1.0\Delta_z/\Delta_t$, and $N_0 = 400$)*
CONTOUR PLOT OF THE POTENTIAL

DISTANCE

TIME
Spectra of the Potential Oscillations

With a weak beam the spectra consist of narrow peaks near the harmonics of the fundamental. The amplitude and width of these harmonics increase at greater distances from the injection boundary. This is consistent with the theories of O'Neil et al. (1971) who showed that the amplitude of the harmonics increase as the interaction passes out of the linear stage and the beam starts to become trapped.

An increase in the amplitude of the harmonics and width of the peaks is also observed when the sheath potential is increased. This can be seen in Figure 6.8, which shows the spectra in the center of the system from two simulations with $\eta = 0.01$, and $L = 3.18\lambda_0$; one initially had $v_{sp} = 0.1v_b$ and the other $v_{sp} = 0.5v_b$. This indicates that the interaction passes out of the linear stage more rapidly when the sheath amplitude is increased.

6.1.2 $0.05 \leq \eta \leq 0.25$

Energy Content

The heating rate is irregular and the increase in the energy contained in the system is due more to the increase in the kinetic energy rather than the potential energy. Figure 6.9 shows the growth of energy in a simulation with $\eta = 0.1$, $v_{sp} = 0.5v_b$, and $L = 3.18\lambda_0$. Because the heating rate is faster, the sheath potential is generally higher than when $\eta < 0.05$, and the sheath contributes most of the potential energy.
CONTOUR PLOT OF THE POTENTIAL

DISTANCE

TIME

ABOVE 0.660
0.619 - 0.660
0.578 - 0.619
0.536 - 0.578
0.495 - 0.536
0.454 - 0.495
0.413 - 0.454
0.371 - 0.413
0.330 - 0.371
0.289 - 0.330
0.248 - 0.289
0.206 - 0.248
0.165 - 0.206
0.124 - 0.165
0.083 - 0.124
0.041 - 0.083
BELOW 0.041
CONTOUR PLOT OF THE POTENTIAL
Figure 6.8: Spectra from simulations with $\eta = 0.01$ and $L = 3.18\lambda_0$. a) $v_{sp} = 0.1$, at $z = 1.59\lambda_0$, and b) $v_{sp} = 0.5$, at $z = 1.59\lambda_0$. The peaks are broader when the sheath potential is higher. ($\Theta = 0.1$, $L = 200\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 400$)
Figure 6.9: *Plot of the evolution of the average energy density in a system from a simulation with* $\eta = 0.1, L = 3.18\lambda_0, and v_{ap} = 0.1v_b$, *showing that the heating rate is irregular.*

(\( \Theta = 0.1, L = 200\Delta x, v_b = 1.0\Delta_x/\Delta t, \) and \( N_0 = 200 \))

**Velocity Distribution**

In this regime, when the wave traps some of the beam electrons, a large number of plasma electrons are accelerated by the wave because of the size of $\eta$, and holes in electron phase space can be formed. These electron holes propagate to the collection boundary as a single entity, and the electrons around them are frequently reflected by the sheath at the collection boundary. Consequently, the plasma gains many high energy electrons, and the density of the plasma becomes very inhomogeneous, as shown in Figure 6.10, which is a plot of phase space from a simulation with $\eta = 0.2, L = 3.18\lambda_0$, and with $v_{sp} = 0.2v_b$ initially.

The reduction in the spatial growth rate of the instability is more apparent in this regime because the plasma is heated to higher 'temperatures', leading to higher sheath potentials, and therefore greater local beam velocities. Longer systems are required to ensure that the beam is trapped by the wave before reaching the collection boundary. However, even if the system length is too short for the wave to trap the beam, the beam is still occasionally reflected by the sheath at the collection boundary. This is due to the inhomogeneity of the plasma producing large amplitude fluctuations of the sheath potential. Beam electrons are more likely to be reflected when the sheath potential increases at the collection boundary. The bunches of reflected beam electrons make the plasma
Figure 6.10: Phase space plot of a simulation with $\eta = 0.2$, $L = 3.18\lambda_0$, and $v_{sp} = 0.2v_b$ after a period of $80r_e$, showing an electron hole at $x = 150$. The plasma is inhomogeneous, and many plasma electrons have been accelerated up to $v_b$. ($\Theta = 0.1$, $L = 200\Delta_s$, $v_b = 1.0\Delta_s/\Delta_t$, and $N_0 = 200$)

more inhomogeneous, and bounce between the sheaths on the boundaries. Despite the inhomogeneity, however, the time averaged electron velocity distributions in the center of systems in this regime reveal the high energy plateau predicted by the single wave trapping model. Figure 6.11 is an example taken from a simulation with $\eta = 0.1$, $v_{sp} = 0.2v_b$, and $L = 3.18\lambda_0$.

Wave Behaviour

The wave behaviour in this regime is strongly influenced by the bunches of high energy electrons bouncing between the ends of the system. The potential peaks due to the electron holes are obscured by the chaotic oscillations caused by the inhomogeneity of the plasma. The motion of electron holes can only be followed clearly during the early stages of a simulation, as in Figure 6.12. In this figure the electron holes lie in the regions of high potential at times less than $800\Delta_t$. To achieve this result it is necessary for the beam to be passing through the system initially to avoid the large perturbation caused by injecting a beam of high density into a quiescent plasma. Once a number of electron holes have formed after the initially quiet conditions, the inhomogeneity of the plasma density is sufficient to make the wave behaviour chaotic.

The bouncing of trapped high energy plasma electrons then becomes important, caus-
Velocity

Figure 6.11: Average velocity distribution in the region just after the sheath at the injection boundary from a simulation with $\eta = 0.1$, $L = 3.18\lambda_0$, and $v_{sp} = 0.2v_b$ over a period of $80\tau_e$. The high energy tail extends almost uniformly up to $v_H$. ($\Theta = 0.1$, $L = 200\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$).

Figure 6.12: (Overleaf) Contour plot of the potential varying in space, and time from a simulation with $\eta = 0.2$, $L = 3.18\lambda_0$, and $v_{sp} = 0.1v_b$. Electron holes lie in the regions of high potential at times less than 800 time steps. Once a number of electron holes have formed during the initially quiet conditions, the bouncing of trapped beam electrons between the boundaries can be seen with a period of about 350 time steps. ($\Theta = 0.1$, $L = 200\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)

The spatial variation of the spectra is reduced when compared with the spectra from simulations with $\eta < 0.05$. The fundamental peak at $f_e$ is broader at greater distances from the injection boundary. However, even close to the injection boundary the width of the peaks is sufficient to ensure that the harmonics are barely detectable. This indicates that the interaction is far from linear.

A peak at low frequencies due to the bouncing of the high energy electrons between
Figure 6.13: Spectrum from a simulation with \( \eta = 0.1, L = 3.18\lambda_0, \) and \( \nu_{sp} = 0.1, \) at \( z = 0.795\lambda_0, \) showing the low frequency peak due to the bouncing of high energy electrons between the boundaries. \( (\Theta = 0.1, L = 200\Delta_x, \nu_b = 1.0\Delta_x/\Delta_t, \) and \( N_0 = 200) \)

the boundaries is commonly observed. Close to the boundaries the amplitude of the low frequency oscillations are comparable to the oscillations at the plasma frequency, as shown in Figure 6.13, which is typical for most simulations with \( \eta \sim 0.1. \)

6.1.3 \( 0.25 \leq \eta \leq 0.5 \)

Energy Content

In this regime, \( 160r_e \) is sufficient time for the beam-plasma interaction to heat the plasma so that an upper limit of the energy content of the system is reached. The system is modified very rapidly and reaches a steady state which is independent of the initial conditions (either a hot or cold plasma, with or without an initial sheath, and having the beam injected after \( t = 0, \) or having the beam passing through the system at \( t = 0). \)

In short systems the steady state is very unstable as the energy content fluctuates widely, depending upon the number of plasma electrons in the system. The plasma electrons gradually escape as they are heated by the interaction, resulting in an increase of the sheath potential. This process continues until a point is reached when the beam is reflected at the collection boundary. The energy then oscillates as many high energy electrons bounce between the sheaths. Figure 6.14 shows the average energy density in a simulation with \( \eta = 0.5, L = 1.59\lambda_0, \) and with \( \nu_{sp} = 0.5 \) initially.
Figure 6.14: Plot of the evolution of the average energy density in a simulation with \( \eta = 0.5, L = 1.59\lambda_0, v_{tp} = 0.5v_b \). The beam is reflected at the collection boundary at \( t \sim 130, 230, 330, 380, 590, \) and 920. \( (\Theta = 0.1, L = 100\Delta_x, v_b = 1.0\Delta_x/\Delta_t, \) and \( N_0 = 200 ) \)

The potential energy in such a system is contained almost solely in the sheaths; see Figure 6.15, which shows the average spatial variation of the potential energy density. Because of this, the average energy density of systems at equilibrium have a greater contribution from potential energy in shorter systems, as shown in Figure 6.16. This figure also shows that the average energy density is greater in longer systems. This is because in longer systems the sheath potential is greater and therefore the local beam velocity is greater and the plasma is 'hotter'.

**Velocity Distribution**

The beam is accelerated to very high velocities because of the sheath. The time averaged plasma distribution is relatively uniform up to the velocity of the beam, see Figure 6.17. Although, in systems where the beam is frequently reflected by the sheath at the collection boundary, the bunches of trapped beam electrons make the plasma very inhomogeneous. For example see Figure 6.18, which shows an electron phase space distribution from a simulation with \( \eta = 0.5 \) and \( L = 1.59\lambda_0 \).

In systems with \( L \lesssim \lambda_0 \), beam trapping is not observed because the beam-plasma interaction rapidly heats the plasma electrons, enabling many to escape from the system.

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Figure 6.15: Average spatial variation of the potential energy density from a simulation with $\eta = 0.5$, $L = 3.98\lambda_0$, and $v_{sp} = 0.1v_b$. Almost all of the potential energy is contained in the sheath. ($\Theta = 0.1$, $L = 250\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)

Figure 6.16: Average energy density in systems at equilibrium with different length and $\eta = 0.5$. The average energy density is greater in longer systems because the sheath potential is higher and the plasma is 'hotter'. ($\Theta = 0.1$, $L = 250\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)
Figure 6.17: Average velocity distribution in the central region of a system with $\eta = 0.01$, $L = 1.59\lambda_0$, and $v_{sp} = 0.5v_b$ over a period of $20\tau_e$. The distribution is fairly uniform up to the local beam velocity. ($\Theta = 0.1$, $L = 100\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)

Figure 6.18: Phase space plot from a simulation with $\eta = 0.5$, $L = 1.59\lambda_0$, and $v_{sp} = 0.5v_b$ after a period of $160\tau_e$. Bunches of trapped beam electrons can be seen near the injection boundary. ($\Theta = 0.1$, $L = 100\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)
The few plasma electrons remaining in the system oscillate between the large potential barriers at the boundaries and have little effect on the beam. The beam is accelerated across the system and reaches the collection boundary with a velocity which oscillates about \( v_b \).

**Wave Behaviour**

The potential oscillations are dominated by the bouncing of trapped beam electrons between the sheaths. The bounce period of the system is observed clearly in some simulations just after the beam is reflected at the collection boundary, as in Figure 6.19. The strips of low potential passing between the boundaries corresponds to the position of a bunch of trapped beam electrons. This regular wave behaviour is most common in systems where the bounce period is similar to \( \tau_e \).

Figure 6.19: *(Overleaf)* Contour plot of the potential varying in space and time from a simulation with \( \eta = 0.5, L = 1.59\lambda_0 \), and \( \nu_{sp} = 0.5v_b \). The strips of low potential passing between the boundaries corresponds to the positions of the bunches of trapped beam electrons. *(Q = 0.1, \( L = 100\Delta_x \), \( v_b = 1.0\Delta_x/\Delta_t \), and \( N_0 = 200 \))*

**Spectra of the Potential Oscillations**

For these values of \( \eta \) the spectra lack fine structure, a broad peak near the plasma frequency is the norm. Figure 6.20 shows the spectra from a simulation with \( \eta = 0.5 \), and \( L = 3.18\lambda_0 \), at \( x = 1.59\lambda_0 \). The width of both the low frequency peak and the peak at the plasma frequency are broader than when \( \eta < 0.25 \) due to the large variations in the plasma density and the velocity of the trapped beam electrons. Further, the amplitudes of the low frequency waves due to the bouncing of trapped beam electrons are larger. This is most evident in shorter systems, and it applies to spectra taken at all point across the system.
Figure 6.20: Spectrum from a simulation with $\eta = 0.5$, $L = 3.18\lambda_0$, and $v_{ep} = 0.1$, at $x = 1.59\lambda_0$, showing the lack of fine structure. ($\Theta = 0.1$, $L = 200\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200$)

6.2 The Heating Rate

As discussed in Chapter 1, Lebedev et al. (1976), and Papadopoulos (1981) showed that for a weak beam in a bounded system, when linear theory applies, the heating rate can be expressed as

$$Q \sim \alpha n_b \left( \frac{m v_b^2}{2} \right) \frac{v_b}{L}. \quad (6.2)$$

To test the validity of this relation in a bounded system, with $L \lesssim 5\lambda_0$ and with beam densities too large for linear theory to be applicable, a number of simulations have been made. Firstly the system length is kept constant and $\eta$ is varied from 0.01 to 0.5, and in the second series of simulations the system length is varied from $L = 0.796\lambda_0$ to $4.775\lambda_0$.

6.2.1 The relationship between $\eta$ and $Q$

For the simulations examined in this section the system length is kept at 100 cells with $\Theta = 0.2$ and $v_b = 1.0\Delta_x/\Delta_t$, therefore $L = 3.18\lambda_0$. The initial plasma velocity distribution is top hat with $v_{ep}$ taking the values 0.1, 0.2, 0.5, and $1.0\Delta_x/\Delta_t$. The simulations are initialized with the beam already passing across the system, and there is no initial sheath.

During the first $\tau_e/2$ some of the background plasma electrons escape, and a sheath is formed. The sheath then accelerates the beam electrons to velocities which in some cases

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Figure 6.21: Changes in the average energy density of the system over the initial 160τₑ, from simulations with θ = 0.2, and vₛ = 1.0Δₓ/Δₜ. The heating rate is proportional to η except when η ≈ 0.5

are significantly greater than vₛ, as discussed before. The beam-plasma interaction then proceeds to heat the background plasma.

The changes in the average energy density in the simulations over 160τₑ (i.e. 5000 time steps) for different beam densities and initial plasma velocity distributions, are shown in Figure 6.21. There is an approximately linear relationship between Q and η, although when η ≤ 0.2 there is considerable spread in the results for the different values of vₛ. In general, the heating rate is greater when vₛ is larger. This is because with larger vₛ, the sheath potential is higher and therefore the local beam velocity is greater, and causes the increased heating rate in agreement with equation 6.2.

The points in Figure 6.21 represent the average heating rate over 160τₑ. They do not account for the fact that the heating rate fluctuates considerably throughout most simulations, for example see Figures 6.1, 6.9, and 6.14. The heating rate is generally at its maximum soon after the sheath has formed, and as the plasma is heated the heating rate generally decreases. The first 160τₑ includes the period of most rapid heating, and when η > 0.25 the interaction reaches a point where the average total energy content of the system saturates.

When η = 0.5 the energy density reaches the saturation level in about 40τₑ, and the results in Figure 6.21, therefore, do not indicate the relative heating rates before saturation is reached. However, if the energy change were examined over the first 40τₑ, there would be even greater variation in the energy change found in simulations with weak beams, as in some cases the energy change is negligible during this period.

The initial heating rate in interactions with η ≈ 0.5 is much greater than the value
shown in Figure 6.21, indicating that the linear and quasi-linear theory of Lebedev et al. and Papadopoulos does not apply. It is probable that the increased heating rate is due to the increasing sheath potential providing a feedback mechanism which accelerates the beam electrons to greater velocities and enables higher energy electrons to be trapped.

6.2.2 The relationship between $L$ and $Q$

The series of simulations carried out to examine the relation between the heating rate and the interaction length have $\eta = 0.01$, $0.1$, and $0.5$, with $\Theta = 0.1$, $v_b = 1.0$. The initial plasma velocity distribution is top hat with $v_{sp} = 0.5$. The simulations are initialized without the beam passing across the system, and there is an initial sheath.

The change in the average energy density in simulations with $\eta = 0.01$ and 0.1 over $160\tau_e$ (i.e. $10^4$ time steps), are shown in Figures 6.22 and 6.23, whereas the change in the average energy density in the simulations with $\eta = 0.5$ over $40\tau_{pe}$ are shown in Figure 6.24. As indicated by Figures 6.1, 6.9, and 6.14, the energy content of the system changes irregularly with time, and the corresponding variations in the average heating rate are represented by the error bars in Figures 6.22, 6.23, and 6.24.

There are irregular variations in the average heating rate between simulations with different lengths, possibly because of different wave processes, so that the change in the heating rate with different system lengths is not necessarily smooth. This restricts the accuracy in determining the length dependence of the average heating rate. For $\eta = 0.01$ or 0.1, Figures 6.22 and 6.23 suggest that the $L^{-1}$ dependence may hold for $L > 3\lambda_0$.

![Figure 6.22: Change in the average energy density in systems over a period of 160$\tau_e$ from simulations with $\eta = 0.01$, $v_{sp} = 0.5v_b$, and $L < 4.77\lambda_0$. The maximum heating rate is observed with $L \sim 1.2\lambda_0$, and it decreases in longer systems. ($\Theta = 0.1$, $L < 300\Delta_\pi$, $v_b = 1.0\Delta_\pi/\Delta t$, and $N_0 = 200 - 400$)]
However, for $\eta = 0.5$ the heating rate is almost independent of system length above $3\lambda_0$.

With system lengths below $3\lambda_0$ many plasma electrons can be accelerated out of the system producing a sheath with $\phi_{sh} \gg v_{sp}^2/2$. However, the heating rate decreases in shorter systems, after reaching a maximum between $\lambda_0$ and $2\lambda_0$. This is probably associated with a decrease in the growth rate of the beam-plasma interaction in shorter systems, similar to that discussed in Chapter 4 for the initial exponential growth of a cold beam interacting with a cold plasma.

Figure 6.23: Change in the average energy density in systems over a period of $160\tau_e$ from simulations with $\eta = 0.1$, $v_{sp} = 0.5v_b$, and $L < 4.77\lambda_0$. ($\Theta = 0.1$, $L < 300\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 200 - 400$)

Figure 6.24: Change in the average energy density in systems over a period of $40\tau_e$ from simulations with $\eta = 0.5$, $v_{sp} = 0.5v_b$, and $L < 4.77\lambda_0$. The heating rate varied very little above $L \sim 1-2\lambda_0$. ($\Theta = 0.1$, $L < 300\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_0 = 100 - 200$)
Summary

The simulations have shown that the average $Q$ is approximately proportional to $\eta/L$ for $\eta \lesssim 0.1$ and $L \gtrsim 3\lambda_0$. However, $Q$ can vary significantly because of the reflection of beam electrons at the collection boundary, and possibly because of the wave behaviour. Below $L \sim 2\lambda_0$ the heating rate begins to drop, probably because the growth rate of the beam-plasma instability is reduced in shorter systems.

The boundary conditions are seen to cause some major changes. The development of a sheath on the boundaries means that the beam electrons are accelerated as they enter the system and have an average velocity $v_M \sim (v_b^2 + \phi_{sh})^{1/2}$ as they passed through the plasma. This results in average plasma electron velocity distributions which are either gaussian-like with a high energy tail extending almost uniformly up to $v_M$ when $\eta \lesssim 0.1$, or almost uniform up to $v_M$ when $\eta \sim 0.5$.

It is probable that the feedback of the initial rapid increase of the sheath potential in simulations with $\eta = 0.5$ (causing $v_M$ to increase rapidly during the initial stages of such an interaction) is responsible for the large heating rates observed. It is also probable that the increased energy density in longer systems, at equilibrium, is also due to this feedback because in longer systems the sheath potential is greater.

The other important effect of the sheath is the reflection of beam electrons initially at the collection boundary, followed by the subsequent bouncing of high energy electron bunches between the sheaths at both boundaries. This results in a low frequency peak appearing in the spectra of simulations with $\eta \gtrsim 0.05$.

The spectra also indicate the degree to which the interaction has passed out of the linear regime. For $\eta \lesssim 0.05$ the amplitude of the harmonics are larger at greater distances from the injection boundary as the beam electrons becomes progressively trapped in the potential trough of a wave. For $\eta \gtrsim 0.05$ the plasma is more inhomogeneous and the perturbations imparted to the beam as it enters the system result in the spectra having a broad peak near $f_e$, and no easily distinguishable harmonics.
Chapter 7

The Bounded Beam-Plasma Interaction with \( n_b > n_p \)

Experiments in WOMBAT have shown that before BPD the plasma electron density is low and that \( \eta \), as defined by Malmberg and Wharton for a cylindrical thin beam system, can be greater than 0.5. This is certainly true when the beam behaves as a rigid rotator and probably also when there are no electron oscillations. Therefore, simulations with \( \eta > 0.5 \) have been made with uniformly distributed immobile neutralizing ions.

It is believed that Pierce-type instabilities may play an important part in the beam-plasma interaction when the diameter of the beam is comparable to the distance between the axial boundaries (Saeki et al., 1977; Burinskaya and Volokitin, 1984; Mikhailov et al., 1984). Burinskaya and Volokitin (1983), and Crystal and Kuhn (1985) made some PIC simulations of the Pierce instability, following the interaction beyond the linear stage. Both papers showed that virtual cathodes form under certain conditions, as discussed in Chapter 4, and when they do, dynamic steady states are attained.

In this Chapter the dynamic steady states are examined in more detail because in some cases low frequency oscillations are observed. These low frequency oscillations may play a role in a one dimensional BPD which is similar to the role of \( E \times B \) drift waves in a WOMBAT BPD. A complete analytic explanation of the behaviour is not possible as some interactions are highly nonlinear. The results of the simulations are described so that a comparison can be made later with simulations which have mobile ions and include ionization.

In Section 7.1 simulations which have no positive space charge are examined because they provide a means of testing the model when simulating virtual cathodes. A comparison is made with the theory of Child (1911) and Langmuir (1913) for space charge limited flow.
which shows that that virtual cathodes can be modelled accurately.

In Section 7.2 simulations with uniformly distributed, immobile neutralizing ions are discussed. Firstly, the behaviour of virtual cathodes in simulations with \( \eta = 1 \) and \( L < \lambda_0 \) is examined in Section 7.2.1. It is shown that the oscillations of virtual cathodes are not always near \( f_b \) and that subharmonics may appear. In Section 7.2.2, simulations with \( \eta = 1 \) and \( L > \lambda_0 \) are discussed. It has been found that these virtual cathodes can move across the system, and that the frequent passage of the virtual cathodes across the system produces large amplitude low frequency potential oscillations.

Finally, in Section 7.2.3, the minimum value of \( \eta \) for which virtual cathodes form is found approximately using simulations. Some simulations with \( \eta < 1 \) are described to show how the beam-plasma interaction is modified as \( \eta \) decreases to 0.5.

### 7.1 Child-Langmuir Flow

The flow of charged particles between two electrodes in a vacuum has been studied thoroughly earlier this century. A paper by Child (1911) examined the flow of ions from a thermionic anode to a negatively biased collector. In this paper Child derived the law determining the maximum current density, \( j_m \), which can flow in a one-dimensional system when the current is space charge limited. It related \( j_m \) to \( \phi \), the potential between the anode and the cathode, and the distance between these electrodes, \( d \). This relation has become known as the Child-Langmuir law, and is given by

\[
  j_m = \frac{4}{9} \varepsilon_0 \left( \frac{2e}{m} \right)^{1/2} \frac{\phi^{3/2}}{d^2}.
\]  

(7.1)

Many authors followed in this field of research, examining the flow of electrons between electrodes because of the opportunities offered by such systems for the amplification of electronic signals. Fay et al. (1939) showed that the maximum current which can pass between two electrodes held at the same potential, when the current is not space charge limited, is double that given by the Child-Langmuir law. Therefore, including the factor of two, the maximum density of a beam at the injection boundary which will not be reflected can be expressed more appropriately as

\[
  n_m = \frac{16 \varepsilon_0 m}{9} \frac{\left( v_b \right)^2}{d}. 
\]  

(7.2)

This equation becomes an expression determining the maximum length of such a system given \( \Theta \), \( v_b \), and \( \eta \), where \( \eta \) in these simulations is \( N_b/N_0 \), and \( N_0 \) is the electron density used in the scaling of the code. In the dimensionless units of the code, the above
equation gives $L_m$ in units of $\Delta_x$ as

$$L_m = \frac{4}{3} \frac{v_b}{\eta^{1/2}/\Theta} \quad (= 0.212\lambda_0). \quad (7.3)$$

This relation has been tested and confirmed with simulations; as an example of the code's accuracy, when $L_m = 119.257\Delta_x$ the beam is reflected when $L = 120\Delta_x$ but not when $L = 119\Delta_x$, with $\Theta = 0.05$ and $N_b = 10$.

### 7.1.1 Electron Flow with a Cold Beam

To simulate such electron streams, the system has no electrons between the electrodes initially; they are injected when the simulations start with a fixed density and at a velocity $v_b = 1.0\Delta_x/\Delta_t$. For $j < j_m$ the space charge between the electrodes increases until it reaches a maximum when a steady state is attained. Once the steady state is reached the beam electrons are decelerated, reaching a minimum velocity at the center, and then accelerated out of the diode. They pass the second electrode at the same velocity with which they passed through the first, and the energy of each electron is conserved across the system.

When $j$ is increased to $j_m$, the space charge between the electrodes increases to the point where the minimum value of the potential is just sufficient to reflect some of the beam electrons. Once some of the electrons are reflected, the space charge increases rapidly as more electrons are reflected and more electrons continue to flow into the system. The potential drops and the point of reflection of the beam moves towards the injection boundary.

As this happens, the flow of electrons out of the system, i.e. across the injection and the collection boundaries, becomes greater than the flow of electrons into the system, and the space charge between the boundaries decreases. The minimum potential increases until the beam is no longer reflected and is able to pass across to the collection boundary. The space charge in the system therefore increases again as more electrons approach the collection boundary. Eventually sufficient space charge accumulates between the electrodes for the beam to be reflected, and so the process repeats itself. Figure 7.1 is a contour plot of the potential across such a system showing the motion of the minimum potential.

Figure 7.1: (Overleaf) Contour plot of the potential from a simulation with no ions, $L = 1.006L_m$, and $v_{bep} = 0.001v_b$, showing the oscillation of the virtual cathode as it moves back and forth near the injection boundary. ($\Theta = 0.05$, $N_e = 120\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, $N_b = 10$, and $\eta = 0.05$)
CONTOUR PLOT OF THE POTENTIAL

DISTANCE

TIME

ABOVE  -0.036
-0.071  -0.036
-0.107  -0.071
-0.143  -0.107
-0.178  -0.143
-0.214  -0.178
-0.250  -0.214
-0.285  -0.250
-0.321  -0.285
-0.357  -0.321
-0.392  -0.357
-0.428  -0.392
-0.464  -0.428
-0.499  -0.464
-0.535  -0.499
-0.571  -0.535
BELOW  -0.571
Figure 7.2: Phase space plot from a simulation with no ions, $L = 1.006L_m$, and $v_{bsp} = 0.001v_b$, at a time of maximum electron density. ($\Theta = 0.05$, $N_c = 120\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, $N_b = 25$, and $\eta = 0.05$)

The beam is alternately completely reflected or transmitted past the potential barrier, and the current across the system fluctuates widely, although the motion of the virtual cathode ensures that the current does not drop to zero. Figure 7.2 is a plot of phase space just before the electron density starts to decrease, and Figure 7.3 is a plot of phase space just before the electron density starts to increase. The flux of electrons which reach the collection boundary are spread about $v_b$ due to the oscillations, as shown in Figure 7.4.

When the system is many times longer than the critical length, the oscillations of the virtual cathode have a frequency of $2.6f_b$, where $f_b$ is the plasma frequency of the injected beam density. However, when $L \rightarrow L_m$ from above, the frequency of the oscillations decreases to about $1.6f_b$, and when $L \rightarrow L_m$ from below, the plasma frequency of the beam at the point of maximum density increases from $f_b$ to about $1.6f_b$. This can be seen in Figure 7.5, which shows the oscillation frequency of the virtual cathode as a function of the length of the system.

It is believed that the increase in the frequency for longer systems is partly due to the virtual cathode forming closer to the injection boundary because it would therefore require less time for the space charge of the virtual cathode to be forced out of the system. Although, the maximum electron density certainly increases if the virtual cathode forms closer to the injection boundary, and this results in an increase in the plasma frequency of the region near the injection boundary.
Figure 7.3: Phase space plot from a simulation with no ions, $L = 1.006L_m$, and $v_{bsp} = 0.001v_b$, at a time of minimum electron density. ($\Theta = 0.05$, $N_c = 120\Delta z$, $v_b = 1.0\Delta z/\Delta t$, $N_b = 25$, and $\eta = 0.05$)

Figure 7.4: Flux of electrons which reach the collection boundary from a simulation with no ions, $L = 1.006L_m$, $v_{bsp} = 0.001v_b$, and averaged over 20 oscillations. The distribution has two peaks due to the oscillations of the virtual cathode. ($\Theta = 0.05$, $N_c = 120\Delta z$, $v_b = 1.0\Delta z/\Delta t$, $N_b = 25$, and $\eta = 0.05$)
7.1.2 Electron Flow with a Warm Beam

A warm beam is more stable than a cold one because when the beam is cold a small perturbation in the potential will cause all electrons to be reflected, whereas when the beam has a broad velocity distribution the same perturbation may reflect only a small fraction of the total beam current. If the beam is not cold, oscillations can be damped and a quiet steady state will result. In such a steady state part of the beam has sufficient energy to pass through the potential barrier and the remainder does not, as shown in Figure 7.6.

The transmitted current depends upon the velocity distribution of the beam as well as the injected current, and is not always easy to calculate. Fay et al. (1939) made some calculations to determine the fraction of a cold beam which should be reflected on average. Using the Child-Langmuir relation to determine the potential on both sides of the virtual cathode they showed that

\[
\frac{L}{L_m} = (2 - Z)^{-1/2} + Z^{-1/2},
\]

and

\[
\frac{L_{vc}}{L} = \frac{Z^{-1/2}}{(2 - Z)^{-1/2} + Z^{-1/2}},
\]

where \(Z\) is the fraction of the current that is transmitted and \(L_{vc}\) is the location of the minimum potential. If the beam is cold the minimum potential should be \(-v_0^2/2\), i.e. just sufficient to reflect the beam.
Figure 7.6: Phase space plot from a simulation with no ions, $L = 1.006L_m$, and $v_{bR} = 0.1v_b$, showing a quite steady state where part of the beam is reflected and part is transmitted. However, some composite beams do not reach a quiet steady state, as is the case for the ninth slowest composite beam in this figure, but the steady state of the virtual cathode is barely perturbed. ($\Theta = 0.05$, $N_c = 120\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, $N_b = 25$, and $\eta = 0.05$)

Although the PIC code can be used to examine the steady state currents for different beam velocity distributions, there are some properties of the code which can impair the simulation results. Firstly, in order to ensure that a quiet steady state is attained, more particles must be injected each time step than for simulations of cold beams.

Due to the method in which electrons are injected, the beam consists of a number of cold beams with different velocities distributed about $v_b$, as shown Figure 7.6. The number of separate cold beams is just the number of beam electrons injected per time step, $N_b$, when $v_b = 1.0\Delta_x/\Delta_t$. The density of each composite beam is a fraction $1/N_b$ the total beam density. If only a few electrons are injected per time step it is very unlikely that the correct steady state current across the system could be carried by an integral number of the composite cold beams. A few composite beams would be alternately reflected then transmitted, as shown in Figure 7.6, and a quiet steady state would not be reached.

If a larger number of particles are injected per time step, the fraction of the beam which should be transmitted is more likely be an integral number of composite beams. Although, even if an integral number of composite beams do not carry the correct current to the collection boundary, the density of the few beams which are alternately reflected
and transmitted is reduced. Consequently, the amplitude of the perturbations from a steady state can be made negligible by increasing the number of beam particles.

With a warm beam $L_m$ is not the same as for a cold beam. If there are some electrons with velocities much slower than the mean, the potential barrier created by the accumulation of space charge does not have to be so great to reflect these electrons. Therefore, for a beam with the same initial density but with greater velocity spread, $L_m$ is shorter and a virtual cathode forms more rapidly. This can be seen in Figures 7.7a and 7.7b, which are contour plots of the potential in similar simulations as in Figure 7.1; the only difference being the values of $v_{bap}$, which are 0.01$v_b$ and 0.1$v_b$ respectively, compared with 0.001$v_b$ in Figure 7.1. The warmer the beam, the greater the damping rate of the oscillations, and the sooner a quiet steady state is reached. These figures show that the virtual cathode forms further away from the injection boundary with increased $v_{bap}$, even though $L_m$ decreases.

However, there is a lower limit of the beam velocity spread below which the oscillations are not damped, and the limit is determined by the accuracy of each simulation. There will always be some fluctuation of the minimum potential due to the random field errors, even though it may be less than 1%. If this small oscillation can cause fluctuations in the number of electrons being either transmitted or reflected, the oscillations will not be damped. The velocity spread of the beam in the simulation which produced Figure 7.1 is below the limit.

As an indication of the accuracy of the code, and to illustrate the above phenomena, a special simulation has been made of a beam with a top hat velocity distribution. Initially $v_{bap} = 0.2v_b$, however, for each subsequent time step the beam velocity spread is reduced, so that

$$v_{bap}^{t+1} = v_{bap}^t \times 0.999,$$

which means that in 1000 time steps the velocity spread is reduced by a factor of 0.368. This simulation shows that the system remains stable until $v_{bap} \approx 0.008v_b$ when $L = 1.186L_m$, with $\Theta = 0.05$, $N_c = 100$, and $N_b = 50$. Figure 7.8 is a plot of the potential at

Figure 7.7: (Overleaf) a) Contour plot of the potential from a simulation with no ions, $L = 1.006L_m$, and $v_{bap} = 0.01v_b$. A steady state is reached in 6000 time steps. ($\Theta = 0.05$, $N_c = 120\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, $N_b = 25$, and $\eta = 0.05$)

b) Contour plot of the potential from a simulation with no ions, $L = 1.006L_m$, and $v_{bap} = 0.1v_b$. A steady state is reached in 2000 time steps. ($\Theta = 0.05$, $N_c = 120\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, $N_b = 25$, and $\eta = 0.05$)
CONTOUR PLOT OF THE POTENTIAL

0 - 5000
0 - 120

DISTANCE
TIME

ABOVE -0.035
-0.070 -0.035
-0.105 -0.070
-0.141 -0.105
-0.176 -0.141
-0.211 -0.176
-0.246 -0.211
-0.281 -0.246
-0.316 -0.281
-0.352 -0.316
-0.387 -0.352
-0.422 -0.387
-0.457 -0.422
-0.492 -0.457
-0.527 -0.492
-0.562 -0.527
BELOW -0.562
Figure 7.8: Potential at \( z = 0.167L \) from a simulation with no ions, with \( L = 1.186L_m \) and decreasing \( v_{bep} \). The virtual cathode starts to oscillate when \( v_{bep} \lesssim 0.008v_b \). (\( \Theta = 0.05 \), \( N_c = 100\Delta_x \), \( v_b = 1.0\Delta_x/\Delta_t \), \( N_b = 50 \), and \( \eta = 0.05 \))

\( z = 0.167L \), which is close to the virtual cathode, on the side of the injection boundary. As the velocity spread of the beam decreases, the virtual cathode moves closer to the injection boundary, and the potential at \( z = 0.167L \) decreases.

Just before the virtual cathode starts to oscillate, i.e. when \( v_{bep} \sim 0.008v_b \), the potential minimum is about \(-0.506\) and has moved to \( 0.25L \). The current reaching the collection boundary has also decreased to about \( Z = 0.17 \), as shown in Figure 7.9. This is in agreement with equations 7.4 and 7.5, which show that for a cold beam when \( L = 1.186L_m \), the virtual cathode should form at \( 0.22L \) with \( Z = 0.15 \).

### 7.2 Virtual Cathodes in a Neutralized System

Pierce (1944) claimed that even in a moderate vacuum, ions would accumulate in the region of the beam if the energy of the beam were greater than the ionization energy of the neutrals, and that the beam would eventually be neutralized. This section therefore examines simulations which have high beam density and immobile neutralizing ions which are distributed uniformly across the system. The possibility of ion motion due to the large electric fields associated with virtual cathodes is not considered here.
Figure 7.9: The transmitted current from a simulation with no ions, with \( L = 1.186L_m \) and decreasing \( v_{thp} \). As the velocity spread of the beam decreases, the transmitted current decreases to \( Z \approx 0.17 \), which is comparable to the value \( Z = 0.15 \) calculated by theory for a cold beam. This figure also shows that the beam is comprised of discrete particles. \( (\Theta = 0.05, N_c = 100\Delta_x, v_b = 1.0\Delta_x/\Delta t, N_b = 50, \text{and} \eta = 0.05) \)

7.2.1 \( \eta = 1.0 \) and \( L < \lambda_0 \)

Consider what would happen after an electron beam begins to propagate through a low density gas. Initially there would be a nett negative space charge and the beam would be decelerated. If \( L < L_m \) a virtual cathode would not form, even when there are no ions in the system, and gradually ions, and possibly electrons, would accumulate in the region of the beam. If \( L > L_m \) a virtual cathode is likely to form before many ions are created and before the system is neutralized, unless the ionization rate is very high.

In a neutralized system the theory of Pierce can be used. When \( L < \lambda_0/2 \) the growth rate of the Pierce instability is negative, and therefore any perturbation on the beam would disappear. However, if a virtual cathode forms prior to neutralization, the linear theory of Pierce would not apply.

When \( \lambda_0/2 < L < \lambda_0 \) the growth rate of the Pierce instability is positive, and reaches a maximum at \( L \approx 0.77\lambda_0 \). Therefore, even if a virtual cathode has not formed prior to neutralization but only a minor deceleration of the beam occurs, a virtual cathode would still form. Thus between \( L = 0.212\lambda_0 \) (ie. \( L = L_m \)) and \( L = \lambda_0 \) one would expect virtual cathodes to form.
Simulation Results

The simulations show that virtual cathodes in this regime are stable in the sense that once they have formed there is always a region of high electron density close to the injection boundary. If the beam is cold the virtual cathode oscillates in just the same manner as the virtual cathodes when no ions are present. Similarly, with a warm beam these oscillations are damped.

There are two methods which can be used to create the virtual cathodes, either by injecting the beam into a plasma, or, when $L > \lambda_0/2$, by placing a negative perturbation on the velocity of some beam particles near to the injection boundary. The two methods produce different results because when $L \gtrsim \lambda_0/3$ there is a region between the virtual cathode and the collection boundary where the potential is positive, and some plasma electrons can become trapped and influence the behaviour of the virtual cathode. This is particularly important when $L < \lambda_0$ because the only way that such electrons can escape from the system is as a result of the oscillations of the virtual cathode, and consequently they remain trapped for a considerable time.

When the beam is cold the oscillations of the virtual cathode have spectra with complex structures of harmonics and subharmonics of the dominant frequency. The subharmonics generally appear at either a third or one half of the fundamental, although in all cases the amplitude of the subharmonics are over an order of magnitude less than the amplitude of the fundamental. An example is shown in Figure 7.10 which is the spectrum of oscillation frequencies from a simulation with $L = 0.6366\lambda_0$.

The subharmonics are due to the trapped electrons, and during the evolution of a simulation, as the number of trapped electrons decreases, the amplitude of the subharmonics decrease. The number of trapped electrons also influence the fundamental oscillation frequency because the average electron density of the virtual cathode is higher when more electrons are trapped. Generally, for a system of fixed length, the more trapped electrons the higher the dominant frequency.

Figure 7.11 is a plot of the dominant frequencies from simulations with $L < \lambda_0$, and it shows that the dominant frequency changes considerably with different length systems. When $L \rightarrow \lambda_0$ the frequency of the dominant mode drops well below the beam plasma frequency, and it is possibly due to the decrease in the growth rate of the Pierce instability as $L \rightarrow \lambda_0$. When $L \rightarrow \lambda_0$ the region of positive potential between the virtual cathode and the collection boundary increases in size and the maximum potential increases. The region of high electron density moves further from the injection boundary and the amplitude of the spatial oscillations of the potential become quite small. Figure 7.12 shows the potential
Figure 7.10: Spectrum at $z = 0.5L$ from a simulation with immobile neutralizing ions with $\eta = 1.0$, $L = 0.637\lambda_0$, and $v_{bep} = 0$, showing the relative amplitude of the subharmonics. ($Q = 0.05$, $N_c = 80\Delta_x$, $v_b = 1.0\Delta_x/\Delta_t$, and $N_b = 100$)

Figure 7.11: Dominant peaks in the spectra from simulation with immobile neutralizing ions with $\eta = 1.0$, $L < \lambda_0$, and $v_{bep} = 0$. The dominant frequency decreases well below $\omega_{pe}$ as $L \rightarrow \lambda_0$. 
Figure 7.12: Potential at $z = 0.5L$ from a simulation with immobile neutralizing ions, $\eta = 1.0$, $L = 0.987\lambda_0$ and $v_{up} = 0$. The potential in the center of the system drops to near zero when the virtual cathode moves away from the injection boundary. At this time the spatial variation becomes almost sinusoidal with a node in the center. ($\Theta = 0.05$, $N_c = 124\Delta_z$, $v_b = 1.0\Delta_z/\Delta_t$, and $N_b = 100$)

at the center of a system with $L = 0.987\lambda_0$, and at a time of minimum potential the spatial variation of the potential is almost sinusoidal. As there are no trapped electrons, linear theory should almost be applicable. The theory of Pierce shows that the growth rate of a perturbation on an electron beam in such a system decreases as $L \rightarrow \lambda_0$. Therefore, the time taken for such an instability to saturate should increase as $L \rightarrow \lambda_0$, and this would correlate with the observed decrease in the oscillation frequency of the virtual cathode. It is believed that, in a similar manner, the growth rate produces the bifurcations of the ion oscillations in WOMBAT at low beam current (Boswell, 1985).

The oscillations of the virtual cathode cause the velocity distribution of the beam electrons which reach the collection boundary to have complex structure. The electrons are confined to values around $v_b$ with no slow electrons reaching the collection boundary, as in the case of Child-Langmuir flow. The escape velocity spread is determined by the amplitude of the oscillations.

Finally, for this regime, a test has been made to estimate the minimum beam velocity spread necessary to damp the virtual cathode oscillations. For a simulation with $L = 0.8\lambda_0$, using $\Theta = 0.05$, $N_c = 100\Delta_z$, and $N_b = 100$, the oscillations are damped when $v_{up} \lesssim 0.025v_b$. The minimum beam velocity spread is larger in this example compared with the example for Child-Langmuir flow because the accuracy of the potential near the minimum is reduced since the virtual cathode in this simulation is only 10 cells from the injection boundary compared with 25 as before.
7.2.2 \( \eta = 1.0 \) and \( L > \lambda_0 \)

When \( L > \lambda_0 \) a virtual cathode will probably form before the one-dimensional system is neutralized since \( L \gg L_m \). However, even if the system is neutralized before a virtual cathode forms, as in the case of the ideal situation examined by Pierce (1944), the simulations show that the only time a virtual cathode is not formed is when either a steady state solution to the Pierce instability is reached, or a dynamic steady state is reached in regimes where the Pierce instability has a negative growth rate. The simulations also show that virtual cathodes are not as stable when \( L > \lambda_0 \) in the sense that the regions of high electron density can propagate across the system to the collection boundary.

Simulation Results

A description of a particular simulation will best illustrate the type of processes which have been found to be prevalent in this regime. Figure 7.13 is a contour plot of the potential from a simulation with \( L = 1.11\lambda_0 \) and \( v_{bap} = 0.1v_b \).

When a virtual cathode propagates across the system, upon reaching the collection boundary, the dense group of electrons force the potential to drop locally. In most simulations in this regime the potential drops sufficiently so that a large fraction of the beam current is reflected back to the injection boundary. This results in a greater electron density across the system causing the potential to drop rapidly. The virtual cathode then moves back to the injection boundary (the dark band passing from the collection boundary to the injection boundary at \( t \sim 800 \) and \( 1900 \) in Figure 7.13).

As the virtual cathode moves towards the injection boundary the electron density decreases and the potential rises on the collection boundary side of the virtual cathode. Some of the beam electrons are then reflected at the collection boundary, because of the rise in the potential, and are trapped. The trapped electrons then bounce between the collection boundary and the virtual cathode, causing the potential to oscillate (see Figure 7.13 around \( t \sim 900 \) and \( 2000 \)).

Gradually the trapped electrons escape to the collection boundary, and the potential between the virtual cathode and the collection boundary increases. As the potential in Figure 7.13: \( (\text{Overleaf}) \) Contour plot of the potential from a simulation with immobile neutralizing ions, \( \eta = 1.0, L = 1.11\lambda_0, \) and \( v_{bap} = 0.1v_b \). When the beam has sufficient velocity spread repeatable cycles are observed. \( (\Theta = 0.05, N_c = 140\Delta_x, v_b = 1.0\Delta_x/\Delta_t, \) and \( N_b = 100 \) )
CONTOUR PLOT OF THE POTENTIAL

DISTANCE

TIME

ABOVE

BELOW
Figure 7.14: Potential at $z = 0.5L$ from a simulation with immobile neutralizing ions, $\eta = 1.0$, $L = 1.27\lambda_0$, and $v_{exp} = 0$. When the beam is cold the oscillations are not repeatable, although the basic features are the same. The large negative peak correspond to the passing of the virtual cathode as it moves back from the collection boundary. ($\Theta = 0.05$, $N_c = 160\Delta z$, $v_b = 1.0\Delta z/\Delta t$, and $N_b = 100$)

this region increases, the fraction of the beam which is reflected by the virtual cathode decreases, and so the electron density of the virtual cathode also decreases. Eventually no electrons are reflected by the virtual cathode and the region of high electron density moves across to the collection boundary (this is the band of lower potential moving from the injection boundary to the collection boundary between $t \sim 200$ to $400$, and again between $t \sim 1300$ to $1500$ in Figure 7.13).

The sequence described above does not repeat itself exactly for all interactions with $L > \lambda_0$. When the beam is cold the virtual cathode near the injection boundary oscillates, and this disrupts the motion of the trapped electrons. The regularity of the process is destroyed, although the general features are retained. Figure 7.14 is a plot of the potential midway between the injection boundary and the collection boundary, from a simulation with length $1.27\lambda_0$. The large negative spikes correspond to the passing of the virtual cathode as it moves back from the collection boundary. Note that each cycle has approximately the same period, although there are variations in the details of each cycle.

When the velocity spread of the beam is large, $\lambda_0$, defined as $v_b/\omega_b$, will be significantly different for the fastest and slowest beam electrons. In simulations where
Figure 7.15: Potential at $x = 0.5L$ from a simulation with immobile neutralizing ions, $\eta = 1.0$, $L = 1.11\lambda_0$, and $v_{nsp} = 0.08v_b$. The oscillations at about $t = 950$ and 1700 are of the repeatable type. ($\Theta = 0.05$, $N_c = 140\Delta z$, $v_b = 1.0\Delta z/\Delta t$, and $N_b = 100$).

$\lambda_{\text{min}} = v_{b_{\text{min}}}/\omega_b < L$, and $\lambda_{\text{max}} = v_{b_{\text{max}}}/\omega_b > L$, the above process repeats itself almost exactly, as in the case of the simulation shown in Figure 7.13. The slower electrons modify the interaction so that a combination of irregular high frequency oscillations and more regular sequences occur with a period similar to the oscillations of the virtual cathode when $L \leq \lambda_0$. The greater the proportion of beam electrons with $L \approx \lambda_0$, the more frequently the regular sequences are observed. Figure 7.15 is a plot of the potential from an interaction the same as in Figure 7.13 but with a reduced velocity spread of 0.08$v_b$, so that $L = 1.03\lambda_{\text{max}}$, instead of 1.01$\lambda_{\text{max}}$. The oscillations at about $t = 950$ and 1700 are of the repeatable type.

If there are no particles in the beam with $L \approx \lambda_0$ then the sequence is irregular, with periods varying by as much as 50%. Figure 7.16 is a plot of the mean and the variation of the time period between successive virtual cathodes, $\tau_{vc}$. The dependence of $\tau_{vc}$ on the system length is unclear since some of the results in Figure 7.16 are from simulations which span only a few $\tau_{vc}$ due to impractical run times. Nevertheless, it is clear that $\tau_{vc}$ increases as $L$ decreases towards $\lambda_0$ when $L$ is less than 1.5$\lambda_0$.

The interaction is more complicated when $L \gtrsim 2\lambda_0$ because in such simulations the virtual cathode is not very stable when it forms near the injection boundary. It could form more easily between the collection boundary and the center of the system, and it generally oscillates in that region. In such cases $\tau_{vc}$ is assigned the period between
successive formations of virtual cathodes near the collection boundary.

The spectra of all simulations with $L > \lambda_0$ are broad-band, and are dominated by the low frequency range, $0 - 0.2f_e$, which have an amplitude at least double that of the higher frequency peaks. Fine structure is only found in the spectra of simulations with warm beams when the signal is more repeatable. In some cases harmonics are clearly resolved above the normal low frequency band.

The higher frequency bands are dependent upon the length of the system, as shown in Figure 7.17, and the amplitude of the different bands depended upon where the signal is measured. Figure 7.17 shows all of the bands that have been found, although all the bands are not observed at every point within the system.

In systems with $L < 2\lambda_0$ the highest frequency band generally has greater amplitude than the mid-frequency bands. The highest frequency band is most prominent near the virtual cathode at the injection boundary, as shown in Figure 7.18a, whereas the mid-frequency band is most prominent between the center of the interaction length and the collection boundary, as shown in Figure 7.18b. The higher frequency bands are due to the bouncing of trapped beam electrons between the virtual cathode and the collection boundary, as well as oscillations of the virtual cathode. As the velocity of the higher energy trapped beam electrons is greater than $v_b$, and the distance travelled in one bounce is comparable to $\lambda_0$, the bounce frequency can certainly be greater than $f_e$. 

Figure 7.16: Period between successive virtual cathodes from simulations with immobile neutralizing ions, $\eta = 1.0$, $L > \lambda_0$, and $v_{sep} = 0$. $\tau_{vc}$ is at a maximum when $L \rightarrow \lambda_0$, and the minimum $\tau_{vc}$ is observed in repeatable cycles.
Figure 7.17: Dominant bands in the spectra from simulations with immobile neutralizing ions, $\eta = 1.0$, $L > \lambda_0$, and $v_{\text{bep}} = 0$.

The highest frequency oscillations are strongest just after the beam is trapped, as shown in Figure 7.14. After many of the higher energy trapped electrons have escaped, the frequency decreases and oscillations at, or below, $f_e$ are observed. The variation of the relative amplitude of the various bands, for signals measured at different positions across the system, supports this view.

With a warm beam the high frequency bands are considerably reduced in amplitude. It is believed that this is due to the stabilizing influence of the warm beam on the high frequency oscillations of the virtual cathode. Also because the distribution of trapped electrons is more homogeneous if a warm beam is trapped, thereby reducing the oscillations of the potential due to the motion of trapped beam electrons.

The flux of escaping electrons for systems with $L > \lambda_0$ shows that slow electrons are reaching the collection boundary, although the bulk of the escaping electrons reach the collection boundary with a velocity close to that of the injection velocity. Figure 7.19 is a typical example. As the system length is increased, the width of the distribution increases with more slow electrons reaching the collection boundary. With a warm beam the fine structure disappears.
Figure 7.18: Spectra from a simulation with $\eta = 1.0$ and $L = 1.75\lambda_0$ at a) $x = 0.14L$, and b) $x = 0.86L$, showing that the highest frequency band has greater amplitude near the virtual cathode, whereas the mid-frequency band has greater amplitude away from the virtual cathode.
Figure 7.19: Flux of electrons reaching the collection boundary from a simulation with immobile neutralizing ions, $L = 3.18 \lambda_0$, and $v_{sep} = 0$, averaged over 5 oscillations, showing that slow electrons reach the collection boundary. ($\Theta = 0.1$, $N_c = 200 \Delta z$, $v_b = 1.0 \Delta z / \Delta t$, $N_b = 100$)

7.2.3 $0.5 < \eta < 1.0$

A virtual cathode becomes less stable when $\eta$ is decreased below 1.0 because the electron density of the virtual cathode may not be sufficient to maintain a minimum potential which will continually reflect part of the beam. If there are a large number of trapped plasma electrons, the potential will be reduced and the virtual cathode may remain stable. However, if the motion of the virtual cathode, or the interaction between the beam and the plasma forces plasma electrons out of the interaction region, the system will become ion rich and a virtual cathode is unlikely to form.

A number of simulations have been carried out with beam densities in the range $0.7 < \eta < 1.0$, with system lengths less than $1.6 \lambda_0$. It is not an exhaustive search of the types of processes which occur in the regime, but more a brief examination to see if any new wave phenomena occurs.

Simulation Results

The simulations show that for $\eta \leq 0.8$ and $L < 1.5 \lambda_0$ a virtual cathode is not stable, i.e. if a virtual cathode is formed initially by injecting a beam into a dense plasma, it passes across to the collection boundary and another does not form. This is true even
when $L < \lambda_0$ unlike simulations with $\eta = 1.0$. It has been found that this limiting value of $\eta$ is weakly dependent upon the length of the system, the limit increasing as the system length decreases. However, a sufficient number of simulations have not been carried out to determine the limiting value of $\eta$ to better than about $\pm 10\%$. It should be possible to determine the limiting value of $\eta$ more accurately using analytic methods.

With $\eta$ above the limit the behaviour of the virtual cathode is very similar to when $\eta = 1.0$, although for $L < \lambda_0$ the oscillations of the virtual cathode are at a lower frequency, and for $L > \lambda_0$ the period between passes of successive virtual cathodes, $\tau_{vc}$, is reduced. For $L \simeq \lambda_0$ repeatable cycles are observed with a cold beam similar to the cycles observed when $\eta = 1.0$, $L \sim \lambda_0$, and when the beam is warm.

When $L \sim 1.5\lambda_0$ a variety of wave processes are observed. Steady state solutions have been found with $L = 1.59\lambda_0$, which are similar to the steady state solutions of the Pierce instability, but with some trapped plasma electrons producing small amplitude oscillations. With $L = 1.43\lambda_0$, $\eta > 0.95$, and with the electron beam passing across the system initially, the beam oscillates significantly with a negative phase velocity without producing a virtual cathode. If $\eta$ is reduced below 0.95 the amplitude of the oscillations grows sufficiently to form a virtual cathode in the center of the system, which propagates back to the injection boundary and returns the system to a quieter state. The oscillations then grow again, eventually forming another virtual cathode only to repeat the process.

Figure 7.20 is a plot of the potential at the center of a system which sustained this 'blocked oscillator' cycle, and Figure 7.21 is a contour plot of the potential over the period $t = 3000$-$7000$. The 'blocked oscillator' process has also been observed in simulations with $\eta \lesssim 0.8$ and lengths in the range $1.3\lambda_0 \lesssim L \lesssim 1.6\lambda_0$. However, the oscillations of the beam are perturbed by the plasma, as shown in Figure 7.22 where $\eta = 0.8$, and the potential is generally higher.

When the beam density is not sufficient to form a virtual cathode, the wave process can still be described as a 'blocked oscillator', although the blocking occurs when the beam is reflected by a sheath at the collection boundary. These potential oscillations are similar to the oscillations observed in simulations of short systems with $\eta = 0.5$, as can be seen in Figure 7.23, which shows the potential at $x = 0.5L$ from a simulation with $\eta = 0.7$ and $L = 1.43\lambda_0$. The sheath potential gradually increases as plasma electrons are forced from the system, but the amplitude of the potential oscillations remains fairly constant until the beam is reflected at the collection boundary. The spread in the escape velocity of the beam electrons due to these constant amplitude potential oscillations increases with increasing sheath potential until eventually some beam electrons are reflected.
Figure 7.20: Potential at \( z = 0.5L \) from a simulation exhibiting a 'blocked oscillator' cycle with immobile neutralizing ions, \( \eta = 0.9 \), \( L = 1.43\lambda_0 \), and \( v_{b+} = 0 \). When the amplitude of the oscillations reached a certain level a virtual cathode forms and returns the system to a quieter state. The greater the amplitude of the oscillation before saturation, the smaller the amplitude of the oscillations in the following quieter state. (\( \Theta = 0.05 \), \( N_c = 180\Delta_z \), \( v_b = 1.0\Delta_z/\Delta_t \), and \( N_b = 90 \))

Figure 7.21: (Overleaf) Contour plot of the potential from a simulation exhibiting a 'blocked oscillator' cycle with immobile neutralizing ions, \( \eta = 0.9 \), \( L = 1.43\lambda_0 \), and \( v_{b+} = 0 \). The oscillations have a negative phase velocity and must be a result of the boundary conditions. (\( \Theta = 0.05 \), \( N_c = 180\Delta_z \), \( v_b = 1.0\Delta_z/\Delta_t \), and \( N_b = 90 \))
CONTOUR PLOT OF THE POTENTIAL
Figure 7.22: Potential at $x = 0.167L$ from a simulation exhibiting a 'blocked oscillator' cycle with immobile neutralizing ions, $\eta = 0.8$, $L = 1.43\lambda_0$, and $v_{bep} = 0$. The plasma electrons complicate the wave behaviour, but a virtual cathode can still form and propagate back to the injection boundary, shown by the negative potential spikes. ($Q = 0.05$, $N_c = 180\Delta z$, $v_b = 1.0\Delta z/\Delta t$, and $N_b = 80$)

Figure 7.23: Potential at $x = 0.5L$ from a simulation with immobile neutralizing ions, $\eta = 0.7$, $L = 1.43\lambda_0$, and $v_{bep} = 0$. Virtual cathodes are not formed, but a 'blocked oscillator' is possible due to the trapping of beam electrons at the collection boundary. ($Q = 0.05$, $N_c = 180\Delta z$, $v_b = 1.0\Delta z/\Delta t$, and $N_b = 70$)
Summary

The simulations of virtual cathodes created when no ions are present agree with theory on the maximum current before space charge limiting occurs to better than 1% when $\lambda_0 \sim 500\Delta_z$. When the beam is cold the virtual cathode oscillates with a frequency between $1.6f_b$, when the beam is only just space charge limited, up to $2.6f_b$ when the beam is severely space charge limited. When the beam is warm a steady state can be reached which agrees closely with the theoretical steady state.

When uniformly distributed, immobile neutralizing ions are used, the behaviour of the virtual cathode depends on the length of the system and the density of the beam. For $\eta = 1.0$ and $L < \lambda_0$ the behaviour of the virtual cathode is similar to when there are no ions. When the beam is cold a dynamic steady state is reached, and when the beam is warm a non-oscillating steady state is reached. The frequency of the oscillations with a cold beam varied from 0 to $1.4f_b$, and it is possible that when $L \to \lambda_0$ the frequency is influenced by the growth rate predicted by the linear theory of Pierce.

When $\eta = 1.0$ and $L > \lambda_0$, the virtual cathodes can move across the system forming near the collection boundary as well as the injection boundary. The motion of the virtual cathode produces oscillations of the potential with frequencies in the range 0.02 to $1.6f_b$. The oscillations of the virtual cathode follow cycles which can be repeated when $L \sim \lambda_0$.

The behaviour of the beam-plasma interaction is very similar when $0.08 \leq \eta \leq 1.0$ and $L < 1.5\lambda_0$. However, when $\eta \lesssim 0.8$ it has been found that virtual cathodes rarely form, and the wave behaviour is similar to that observed in simulations with $\eta \sim 0.5$. 
Chapter 8

Simulations with Ion Motion and Ionization

In this chapter a more complete model of a beam-plasma interaction is described which has been developed in order to include ionization by energetic electrons as well as ion motion. It incorporates most of the phenomena which will determine the threshold conditions for BPD in a one-dimensional system, and therefore provides a good comparison with laboratory experiments.

In Chapter 4 it was shown that ion motion can modify instabilities on an electron beam in a bounded system. In the following chapters ion motion was neglected, although in many cases large amplitude, steady state electric fields form in these simulations, and so ion motion should occur. In Section 8.1, the effects of ion motion in simulations with virtual cathodes and large amplitude sheaths are examined. These simulations indicate that once ion motion is included an ion source must also be included, otherwise the system will eventually reach a state similar to that of Child-Langmuir flow.

Ionization has previously been modelled in a PIC simulation (Machida et al., 1984), but that model had periodic boundary conditions and did not consider diffusion. Therefore, a preliminary model of ionization is discussed in Section 8.2, in which the electron beam is not included so that the balance between ionization and diffusion can be examined in a less dynamic system. This model is then used later to compare with simulations which have ionization due to energetic electrons.

A model of ionization by energetic electrons is described in Section 8.3 which considers energy conservation, the ionization collision cross-section, and the velocity distribution of the scattered electrons. The results of simulations using this model are examined in Section 8.4. A BPD can be simulated by following the interaction initially from the formation of
virtual cathodes during Child-Langmuir flow, to the saturation of the interaction when the diffusion rate equals the ionization rate. It is found that the behaviour of the interaction is influenced strongly by the linear growth rate of a beam-plasma instability in a bounded system. A scaling law for the critical beam current is determined which is similar to that found in laboratory experiments, and many of the wave characteristics are similar to those observed in laboratory experiments.

8.1 Simulations with Mobile Ions

8.1.1 Stability of Virtual Cathodes

The simulations discussed in Chapter 7 show that when a virtual cathode forms in neutralized systems with $\eta = 1.0$ and $L < \lambda_0$, it will remain in a region close to the injection boundary. Therefore, it is inevitable that the spatial distribution of the ions will be modified. In systems with $\eta = 1.0$ and $L > \lambda_0$ the virtual cathode is more mobile, moving between the boundaries in less than $10\tau_e$ in some cases. However, the virtual cathode spends the majority of each cycle near the injection boundary, and so even heavy ions will eventually accumulate in the region of the virtual cathode.

To check this, simulations have been made with mobile ions and run for many ion plasma periods. The simulations are started with cold ions at rest, and with a small perturbation on a cold electron beam. They show that once the virtual cathode forms, ions are drawn towards it, partially neutralizing the negative space charge. Nevertheless, with the continual influx of beam electrons, the system remains electron rich and the ions rotate in phase space about the virtual cathode. The oscillations of the virtual cathode enable many ions to reach the boundaries, and eventually only the low energy ions remain. As the ion density decreases, the ions play less of a role in the interaction, and the electrons eventually follow trajectories similar to that observed in Child-Langmuir flow.

8.1.2 Modification of the Sheath

In one simulation, discussed in chapter 5, a plasma with a truncated gaussian velocity distribution with $v_{eth} = 1.0 A_e/\Delta_t$ is confined between sheaths with a potential of $1.36(m \Delta_x^2/e\Delta_t^2)$ volts. The electric field on the boundaries is so great that if there were protons near the boundaries they would be accelerated from rest and moved approximately $0.1\lambda_d$ in only $10\tau_e$. This suggests that the ion distribution can be altered significantly during the period over which many simulations are run.

To examine this more closely, a series of simulations have been made of hot plasmas
similar to those discussed in Chapter 5, but with mobile ions initially at rest. The electron beam is not included and top hat electron distributions are used with \( v_{ep} = 0.5 \) or 1.0. The simulations are started with sheaths appropriate for top hat plasma electron distributions with \( v_{ep} = 0.5 \) or 1.0 in order to reduce plasma oscillations. The ions are assumed to be singly charged and their mass has been varied over the range \( M/m = 10^{-10^4} \).

During the initial stage of these simulations the ions nearest the boundaries are accelerated from rest to an escape velocity:

\[
v_{esc} \approx \left( \frac{2\phi_{sh} m}{M} \right)^{1/2} \approx \left( \frac{m}{M} \right)^{1/2} v_{ep},
\]

This says that the ions leave the system with kinetic energy gained after being accelerated by the full sheath potential, and it occurs in a period:

\[
\tau_{ini} \approx \left( \frac{M}{m} \right)^{1/2} \left( v_{ep} \Theta \right)^{-1} \approx \frac{\tau_i}{2\pi v_{ep}}.
\]

Few ions and electrons escape during this stage, and the potential across the system changes little. However, once the ions nearest the boundaries have been accelerated to \( v_{esc} \) and begin to leave the system, electrons also begin to leave at the same rate.

When the ions in the boundary cells are accelerated, the charge density begins to decrease in these regions, marking the start of the second stage. The charge density and potential initially decrease nearest the boundaries before the deformation propagates into the center of the system from both boundaries at a constant rate. This can be seen in Figure 8.1, which is a contour plot of the potential from a simulation with \( v_{sp} = 1.0, M = 1836m, \) and \( L \approx 20\lambda_d \).

The potential in the center of the system remains at \( \phi_{sh} \) until the sheaths have expanded and met in the center. The velocity at which an equipotential point moves towards the center of the system during the second stage is proportional to \( v_{esc} \), or the ion sound speed. Figure 8.2 shows the velocity at which the points with a potential of 0.8\( \phi_{sh} \) move during the second stage.

It is found that the escape velocity of the ions remains the same until the potential in the center of the system has decreased by as much as 20%, and that both ions and electrons are lost from the system at the same constant rate during this period. It is also found that the energy of each ion is conserved, and that the total energy in the system decreases at a constant rate during this period.

Figure 8.1: (Overleaf) Contour plot of the potential from a simulation of a hot plasma, with \( v_{sp} = 1.0, \Theta = 0.2, \) and mobile ions with \( M = 1836m \). In this example the first stage ends after only 200 time steps, and the second stage ends after about 3000 time steps.
CONTOUR PLOT OF THE POTENTIAL

DISTANCE

TIME

ABOVE  0.481
0.451 - 0.481
0.421 - 0.451
0.391 - 0.421
0.361 - 0.391
0.331 - 0.361
0.301 - 0.331
0.271 - 0.301
0.241 - 0.271
0.211 - 0.241
0.180 - 0.211
0.150 - 0.180
0.120 - 0.150
0.090 - 0.120
0.060 - 0.090
0.030 - 0.060
BELOW  0.030
Figure 8.2: Velocity of the 0.8\(\phi_{\text{ph}}\) equipotential in simulations of hot plasmas with \(v_{\text{sp}} = 0.5\) and 1.0, with mobile ions of different mass.

The final stage of these simulations extends almost to infinity, the time when all ions and electrons have escaped from the system. Once the potential in the center of the system has decreased by about 20\%, the rate at which the charged particles escape from the system decreases, and asymptotes to zero.

If a beam of electrons were included in these simulations, a virtual cathode would form after the plasma density has decreased to the point where \(\eta \gtrsim 0.8\), and the simulation would eventually reach a state in which the electrons follow trajectories similar to Child-Langmuir flow. Therefore, if ion motion is included in simulations of a beam-plasma interaction, there must be a source of ions which either injects them from the boundaries or creates them within the interaction region.

8.2 Preliminary Model with an Ion Source

In laboratory BPD experiments ions are continually created by the high energy beam electrons. These electrons create ions reasonably uniformly across the system since the number of ionizing collisions is small enough not to significantly deplete the beam current. Therefore, a study has been made of the evolution of a system with a simple uniform plasma source to examine how the PIC code simulates the balance between ionization and diffusion, and to see whether a more complex ionization model can be developed which will have practical run times. As this model provides a useful insight into the diffusion rates observed in the one-dimensional simulations of a BPD, some of the results are discussed here.
It has been found that the characteristic diffusion time is generally very large compared with $\tau_e$, and therefore considerable computer time is required to run these simulations, even with an unrealistically small ion mass. To overcome this problem the ionization rate is kept small so that only a few charged particles are present per cell even near equilibrium. This means a reduction in the accuracy of the simulations, and possibly numerical heating rates which may modify the process significantly. Nevertheless, useful results have still been obtained.

The ionization is modelled by adding a constant number of pairs of electrons and singly charged positive ions to the system each time step in a random manner. It is assumed that the density of the neutral particles does not change, and that they remain uniformly distributed in space. The ions and electrons are also uniformly distributed in space when first added to the system, and the charged particles of each pair are loaded at the same point. The average velocity distributions of the ions and electrons are gaussian when loaded, with $v_{\text{eth}} = 0.001$ and various values of $v_{\text{eth}}$.

If the diffusion rate were constant then the average electron and ion densities should increase with the form given by equation 1.19, ie.

$$N_{e,i} = \frac{N_s \tau_d}{N_c} (1 - \exp(-t/\tau_d)),$$

where $N_s$ = number of charged particle pairs added per time step, $\tau_d$ = characteristic diffusion time.

However, because the simulations are started without charged particles in the system, the characteristic diffusion time is not constant. During the first few time steps the average ion density increases linearly, at a rate $N_i = N_s \Delta t/N_c$ because there is no sheath. The average electron density initially increases at a much lower rate, until the difference in the densities of the electrons and ions creates a sheath which traps many of the electrons. Once a sheath has formed, the number of electrons and ions increase at almost the same rate. Figure 8.3 shows the change in the average electron and ion densities in a simulation with $N_s = 2$, $M = 100m$, $\Theta = 0.2$, $v_{\text{eth}} = 1.0 \Delta u/\Delta t$, $v_{\text{eth}} = 0.001 \Delta u/\Delta t$, and $N_c = 100$. The simulation has not been run until equilibrium is reached, yet an estimate of the characteristic diffusion time can be made by plotting $\ln(1 - N_{e,i}(t)/N_\infty)$, where $N_\infty$ is the estimate of the equilibrium density which produces the straightest line. Figure 8.4 is a plot of $\ln(1 - N_{e,i}/4N_0)$ for the results shown in Figure 8.3. It shows that initially $\tau_d$ increases considerably, but then reaches an approximately constant level.
Figure 8.3: The average ion and electron densities in a simulation with $\Theta = 0.2$, $M = 100m$, $N_e = 2$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_c = 100$.

Most simulations have not been run as close to equilibrium as the one examined in the previous figures, and so the equilibrium density cannot be determined as accurately, leading to a poor estimate of the characteristic diffusion time. However, running a simulation very close to equilibrium will not greatly improve the accuracy of the estimate of $\tau_d$, unless many more charged particles are used, because the random fluctuations of the plasma density cause large fluctuations in $\ln(1 - N_e/N_\infty)$ when $N_{e,i} \sim N_\infty$.

The change in $\tau_d$ can be explained by the change in the sheath potential during these simulations. Figure 8.5 shows the potential at the center of the simulation referred to above. The high frequency oscillations are at $f_e$, and are the field fluctuations at thermal equilibrium. This figure shows that the initial rapid loss of electrons is too great because the sheath potential increases too far, only to decrease back to the equilibrium level. With an excessively large sheath potential, the ions will be accelerated from the system at a greater velocity than at equilibrium, and the diffusion rate will be greater than at equilibrium. Hence the increase in $\tau_d$ with time.

A series of simulations have been made with ions of different mass in the range $M/m = 2-1836$. They show that the diffusion rate increases when the ion mass is decreased. Figure 8.6 shows how the characteristic diffusion time changes with different ion mass for simulations with $\Theta = 0.2$, $N_c = 100$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_c = 2$. It shows that for $M \lesssim 100m$, $\tau_d \propto (M/m)^{1/2}$, consistent with ambipolar diffusion. However, with $M > 100m$, the diffusion rate remains constant within the accuracy of the estimates of $\tau_d$, at about $1000\tau_0$. This can be understood by the increase in the sheath potential with increasing ion mass, as shown in Figure 8.7. For the characteristic diffusion
Figure 8.4: Plot of $\ln(1 - N_{e,i}/4N_0)$ for simulation in Figure 8.3. $\Theta = 0.2$, $M = 100m$, $N_s = 2$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_c = 100$.

Figure 8.5: The potential at the center of a system with $\Theta = 0.2$, $M = 100m$, $N_s = 2$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_c = 100$. 
Figure 8.6: The characteristic diffusion time from simulations with different ion mass in the range $M/m = 2-1836$, and with $\Theta = 0.2$, $N_s = 2$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_e = 100$.

Figure 8.7: The sheath potential in simulations with different ion mass in the range $M/m = 2-1836$, and with $\Theta = 0.2$, $N_s = 2$, $N_0 = 50$, $v_{eth} = 1.0$, $v_{ith} = 0.001$, and $N_e = 100$.

time to remain fairly constant, the ions with greater mass require a greater amplitude sheath in order to accelerate them from the system at a comparable rate to lighter ions.

The velocity distribution of the trapped electrons changes throughout each simulation. As a simulation evolves, the density of the slower electrons increases, while the density of the electrons with kinetic energy greater than $\phi_{sh}$ remains the same. As the sheath potential decreases, the velocity distribution becomes truncated at lower velocities, and eventually the density of the low energy electrons becomes considerably greater than the higher energy tail. In many simulations, the velocity at which the distribution is truncated ($v_{tr}$) is much less than $v_{eth}$, so that the distribution of the trapped electrons is approximately top hat. The distribution of the hot electrons remains almost gaussian because
they can escape. At equilibrium, those electrons which are added to the system with kinetic energy less than $\phi_{eh}$ should remain trapped. However, this would mean that the plasma density should continue to increase since, at equilibrium, $\phi_{eh}$ is constant by definition. The reason that the density does not continue to increase is because the plasma oscillates. Figure 8.5 shows that the potential can oscillate considerably in the center of a system, and such oscillations enable some of the slower electrons to escape. Figure 8.8 shows the flux of electrons which reach one boundary in a simulation with $v_{eth} = 0.5$. It corresponds to an approximately gaussian velocity distribution for $v \gtrsim 0.3$, yet below this there is a significant increase in the number of escaping electrons. Most of these would have been low energy electrons with $v^2/2 < \phi_{eh}$, and were only able to pass through the sheath because of the plasma oscillations.

It has been found that the sheath potential is also dependent upon the temperature of the electrons created by the ionization process. The sheath potentials from a series of simulations with different $v_{eth}$ are shown in Figure 8.9, which indicates that $\phi_{eh}$ increases with increasing $v_{eth}$. The values of the other parameters were: $\Theta = 0.2$, $N_c = 100$, $N_0 = 50$, $M = 100m$, $v_{ith} = 0.001$, and $N_s = 2$. This is an important fact because the amplitude of the sheath influences the extent to which the beam is perturbed upon entering the system. It has also been found that the characteristic diffusion times for this series of simulations do not vary significantly - all are about $1000\tau_0$. 

---

Figure 8.8: Flux of electrons reaching a boundary from a simulation with $\Theta = 0.2$, $M = 100m$, $N_s = 2$, $N_0 = 50$, $v_{eth} = 0.5$, $v_{ith} = 0.001$, and $N_c = 100$. 

---
In all the simulations described in this section the equilibrium velocity distribution is truncated with \( v_{tr} \sim 0.2-0.5 \). Consequently, with \( \Theta = 0.2 \) and \( N_c = 100 \), \( L \sim (40-100)\lambda_d \) and the sheath occupies a large fraction of the simulation length. As a result, the velocity distribution of the escaping ions is broad, although the majority of the ions still reach the boundaries with a velocity \( v_{esc} = (2\phi_{sh} m/M)^{1/2} \), in support of simple models of ambipolar diffusion which have \( L \gg \lambda_{sh} \).

As the plasmas in these simulations behave in a realistic manner, it would appear that the process is being modelled correctly, even though there may be as few as 50 ions and electrons per Debye length when near equilibrium. It should be possible to give a more complete explanation of the dependence of \( \tau_d \), \( \phi_{sh} \), and \( v_{tr} \), on \( M/m \), \( v_{eth} \), and other variables, if more simulations are carried out. The major concern is the need for many hours of computer time on a medium speed computer in order to run these simulations, and this also applies to the following model of ionization by energetic electrons.

### 8.3 Ionization by Energetic Electrons

A more complete model of the beam-plasma interaction has been made by including ionization due to sufficiently energetic electrons. If electrons have an energy greater than the ionization energy of the neutral species, it is possible for ions to be created and other electrons to be released. However, with a one-dimensional model, the three-dimensional nature of an ionizing encounter cannot be included. Only if an almost infinite axial magnetic field were present would transverse motion be negligible, but even then energy conservation should consider transverse velocities. Further, non-ionizing inelastic encounters have not been modelled explicitly because they are believed to play a minor
role in the interaction. Despite these deficiencies, a one-dimensional model can still be made reasonably consistent, as will be discussed in the following sections.

8.3.1 Modelling the Ionization and the Collision Cross-Section

To reduce the number of calculations and account for the finite probability of a particular electron making an ionizing collision, only a randomly selected sample of all the electrons in the system are examined each time step. The number of electrons which are selected each time step determines the ionization rate, and each electron selected may make an ionizing collision with a probability which is determined by its velocity. The \( m \)-th electrons in a simulation are chosen by the following formula:

\[
m = (\text{random integer } < N_v) + nN_v,
\]

where \( N_v \) and \( n \) are integers.

The integer \( N_v \) is effectively the period between ionizing collisions for a beam electron in units of \( \Delta t \), and is chosen so that the mean free path of beam electrons (\( \lambda_{en} \)) in the simulations is comparable to the mean free path of beam electrons in the laboratory experiments. For the experiments conducted in WOMBAT, the mean free path of beam electrons was in the range \( L-100L \). Therefore, if a beam velocity of \( 1.0\Delta \xi/\Delta t \) is used in a simulation with \( L = 100 \) cells, \( N_v \) should be in the range \( 10^2-10^4 \) since \( \lambda_{en} = N_v\nu_b \). For example, if \( N_v = 100 \), on average each beam electron should be chosen once every 100 time steps. This is the time a beam electron would take to pass across the system, and so the mean free path is equal to \( L \).

The probability of a selected electron making an ionizing collision can be determined by comparing its velocity with a random number. The simplest condition used in these simulations is:

\[
v_m^2/2 > \varepsilon_{ion} = v_{ion}^2/2,
\]

where \( \varepsilon_{ion} \) is the ionization energy of the neutral species, and \( v_{ion} \) has been introduced for convenience when examining phase space diagrams. With this condition, any electron in the randomly selected sample group with kinetic energy above the ionization energy will make an ionizing collision. Therefore the collision cross-section is:

\[
(\sigma_i(v)) = \frac{1}{n_n\nu_b} = \frac{1}{n_nN_v\nu_b}, \quad \text{for} \quad v > v_{ion}.
\]

However, if for \( v_{ion} < v < 2v_{ion} \) a further test is applied, such as requiring

\[
\frac{v}{2v_{ion}} > R,
\]
where $R$ is a random number between zero and one, the cross-section will be constant between $v_{\text{ion}}$ and $2v_{\text{ion}}$. Other conditions can be used so that the ionizing collision cross-section can have any velocity dependence. Nevertheless, the simplest condition has been used in most of the simulations discussed in this chapter in order to reduce the run time.

8.3.2 Distribution of the Particles after an Ionizing Collision

When the conditions required for an ionizing collision to occur are met by one of the selected electrons, an electron and an ion are added to the simulation, and these particles must be assigned realistic velocities and positions. It is assumed that neither the electron or the ion created in a collision move far within one time step, and so they are placed in the simulation at the point at which the collision occurs. For similar reasons the incident electron does not change position as a result of the collision. The velocity distributions of the new pairs of ions and electrons are assumed to be gaussian. The ions are given a temperature similar to that of 'room temperature' because little kinetic energy is imparted to the ions from the incident energetic electrons, since the ions are far more massive than electrons. The electrons released by the collisions are assigned a temperature corresponding to about 4 eV because the electron temperature of laboratory plasmas before BPD were found to be a few eV at most (Kellogg et al., 1981; Boswell and Kellogg, 1983; Boswell et al., 1984). The velocity distribution of the scattered energetic electrons, $f_{\text{scat}}(v)$, requires more consideration.

In most laboratory experiments the kinetic energy of the beam electrons is considerably greater than the ionization energy. Therefore, according to the conservation of energy and momentum, the kinetic energy of the scattered beam electrons should not be significantly different from the kinetic energy they had before the collision. If excitation of the neutral species is neglected, the conservation of energy requires that

$$
\frac{1}{2}mv_{\text{b}}^2 + \frac{1}{2}Mv_{n}^2 \simeq \frac{1}{2}mv_{\text{scat}}^2 + \frac{1}{2}mv_{i}^2 + \frac{1}{2}Mv_{i}^2 + \epsilon_{\text{ion}},
$$

where $v_n$ is the velocity of the neutral species and $v_{\text{scat}}$ is the velocity of the scattered beam electron. Assuming that $v_i \simeq v_n$ and $v_e \ll v_i$, then in dimensionless units $v_{b}^2 \simeq v_{\text{scat}}^2 + v_{i}^2$.

However, because of the beam-plasma interaction, some plasma electrons will have kinetic energy just greater than $\epsilon_{\text{ion}}$. When these electrons make ionizing collisions, many of them will be scattered at large angles, and will have a small component of velocity in the direction of the beam. To model such collisions, a range of different velocity distributions has been tried for the scattered electrons. One extreme is to treat the plasma electrons in
the same way that most beam electrons behave, i.e.

\[ v_{\text{scat}} = \frac{v_{\text{in}}}{|v_{\text{in}}|}(v_{\text{in}}^2 - v_{\text{ion}}^2)^{1/2}, \quad (8.9) \]

where \( v_{\text{in}} \) is the incident velocity of a scattered electron. The other extreme considered is to assign low velocities to most of the scattered electrons. This can be achieved by using a gaussian distribution centered about zero with a thermal velocity:

\[ v_{\text{eth}} = (v_{\text{in}}^2 - v_{\text{ion}}^2)^{1/2}/3. \quad (8.10) \]

This choice of \( v_{\text{eth}} \) ensures that almost no scattered electrons have kinetic energy greater than \((v_{\text{in}}^2 - v_{\text{ion}}^2)^{1/2}\). An appropriate \( f_{\text{scat}}(v) \) can be chosen by considering the conservation of energy during an ionizing collision, allowing for the effects of inelastic and elastic collisions.

The energy exchanged during each simulation collision is not necessarily conserved because the velocities of the electrons and ions created in each collision are chosen by different random distributions. However, the average energy exchange can be made to account for inelastic collisions and radiative loss. For example, when the gaussian distribution is used, the average energy exchanged during a collision can be expressed as

\[ \frac{1}{2}v_b^2 \rightarrow \frac{3}{4}v_{\text{eth}}^2 + \frac{5}{8}v_{\text{eth}}^2 + \epsilon_{\text{ion}}. \quad (8.11) \]

Here it has been assumed that the kinetic energy of the ions is equal to that of the neutral particles, and that the velocity distribution of the plasma particles is three-dimensional. Substituting the value of \( v_{\text{eth}} \) gives

\[ \frac{1}{2}v_b^2 \rightarrow \frac{1}{12}v_b^2 + \frac{3}{4}v_{\text{eth}}^2 + \frac{5}{8}v_{\text{eth}}^2 + \epsilon_{\text{ion}}, \quad (8.12) \]

and so if \( v_b \gg \epsilon_{\text{ion}} > v_{\text{eth}} \), a large fraction of the kinetic energy of a beam electron which makes a collision is lost. This energy loss is analogous to the energy lost in laboratory experiments due to inelastic collisions and radiative loss. If the velocity of the scattered electron is given by equation 8.9, the average energy exchange is

\[ \frac{1}{2}v_b^2 \rightarrow \frac{1}{2}v_b^2 + \frac{3}{4}v_{\text{eth}}^2, \quad (8.13) \]

so that energy is added to the system during a collision.

Despite the difference in the energy exchange during ionizing collisions, the results of simulations with the two extreme scattered electron distributions are similar, as will be shown in a later section. This is because the energy exchanged during a collision is, in most cases, only a small fraction of the beam energy injected into the system between
The scattered electron velocity distribution used in most of the simulations discussed in Section 8.4 has a maximum near \((v_{in}/2 - \varepsilon_{ion})^{1/2}\), but still includes electrons which are scattered with some velocity component perpendicular to the direction of the beam.

Each collision. Using the dimensionless units of the simulation, the kinetic energy injected per time step \((\varepsilon_{inj})\) will be \(N_b v_b^2/2\), and the energy lost or gained per time step due to ionization \((\Delta \varepsilon)\) will be approximately \(N_{ion} \delta v_{in}^2/2\), where \(N_{ion}\) is the number of ionizing collisions which occur per time step, and \(\delta\) is the average fraction of the incident kinetic energy which is lost or gained. Thus

\[
\frac{\Delta \varepsilon}{\varepsilon_{inj}} \sim \frac{N_{ion} \delta}{N_b} \left(\frac{v_{in}}{v_b}\right)^2,
\]

which in the worst case is

\[
\frac{\Delta \varepsilon}{\varepsilon_{inj}} \sim \frac{N_c}{N_v}.
\]

In most simulations this ratio is small, and so the errors due to the approximate model of an ionizing collision are small. Fortunately, the errors increase with decreasing \(N_v\) which corresponds to the increase in radiative and excitation losses with increasing neutral density in a laboratory experiment.

The velocity of the scattered electrons used in most of the simulations described in this chapter is

\[
v_{scat} = (v_{in}/|v_{in}|)(v_{in}^2 - v_{ion}^2)^{1/2}(4R - 2R^2 - 1),
\]

where \(R\) is a random number between zero and one. The distribution produced by this formula is shown in Figure 8.10, and is a compromise between the two extremes described above.
8.3.3 Scaling in Simulations with Ionization by Energetic Electrons

The simulations described in the following sections have \( \Delta_x = 1 \text{ cm} \), and \( \Delta_x/\Delta_t = 5.9 \times 10^8 \text{ cm/sec} \). Consequently, the interaction length in WOMBAT is represented by about 100 cells, and a beam energy of 100 eV is equivalent to \( v_b = 1.0\Delta_x/\Delta_t \). With \( \varepsilon_{\text{ion}} \) chosen to be 16eV, \( v_{\text{ion}} = 0.4\Delta_x/\Delta_t \), and, with \( T_e = 4\text{eV}, v_{\text{eth}} = 0.2\Delta_x/\Delta_t \). The thermal velocity of the ions is \( v_{\text{th}} = 0.001\Delta_x/\Delta_t \), which is a little below room temperature since the ion mass is \( M = 1836m \). The velocity spread of the beam is set at \( 0.02\Delta_x/\Delta_t \). Also, since \( \Theta = 0.05 \), it means that \( n_0 = 2.8 \times 10^5 \text{ cm}^{-3} \) and \( f_0 = 4.8 \text{ MHz} \).

To simulate the equivalent ionization rates which occur in a typical experiment, the creation rate of pairs of ions and electrons in a simulation can be as low as one pair every few time steps; it depends on the choice of \( N_0 \). In the simulations to be discussed, \( N_0 = 50 \), and so for example, if \( L = 100\Delta_x \), \( N_\nu = 10^4 \), and \( \eta = 1 \), then \( N_{\text{ion}} = N_\varepsilon N_0/N_\nu = 0.5 \), i.e. one ionizing collision takes place every two time steps.

8.4 Simulation of the Beam-Plasma Discharge

Initially, the beam is injected into a system without any charged particles, and so for the first few time steps these simulations resemble the simulations of Child-Langmuir flow. As more ions are created, the space charge of the beam electrons may be neutralized and a plasma will form.

8.4.1 Neutralization

It is the ionization rate which determines whether the space charge of the beam is neutralized, and the rate at which ionizing collisions occur is determined by \( N_\nu, \sigma_i(v) \), and the beam density and velocity. The time taken to neutralize the beam \( (\tau_n) \) for a given beam current, system length, and \( \sigma_i(v) \), has been found to be proportional to \( N_\nu \) in the range of interest, as shown in Figure 8.11.

This figure also shows that the rate at which ionizing collisions occur is not proportional to the beam density, since beams with different density are not neutralized in the same time. This is because the number of electrons with \( |v| > v_{\text{ion}} \) is not proportional to \( N_\varepsilon \) when a virtual cathode forms. There is only a limited region of the system where \( |v| > v_{\text{ion}} \), as shown in Figure 8.12, and increasing \( N_\varepsilon \) moves the virtual cathode closer to the injection boundary and causes a greater fraction of the beam to be reflected. This reduces the fraction of electrons in the system which can make ionizing collisions.

It has also been found that the ionization rate varies with time because the potential
Figure 8.11: Time taken to neutralize the initial virtual cathode in simulations with, a) $L = 2.37L_m$ ($N_c = 100$, $v_b = 1.0$, and $N_b = 20$), b) $L = 1.68L_m$ ($N_c = 100$, $v_b = 1.0$, and $N_b = 10$), and c) $L = 1.19L_m$ ($N_c = 100$, $v_b = 1.0$, and $N_b = 5$).

Figure 8.12: Phase space distribution of a virtual cathode showing the region where ionization is not possible.
across the system increases as more ions are created, and so the average speed of electrons increases. Consequently, a greater proportion of the electrons in the system retain sufficient kinetic energy to make an ionizing collision when the beam is almost neutralized.

The simulations show that just before \( N_i \sim N_e \) the virtual cathode becomes unstable and propagates across to the collection boundary and leaves the system. After a virtual cathode leaves the system the average ion density can far exceed the average electron density because the electron density of a virtual cathode is greater than \( N_b \). Therefore, if a virtual cathode forms initially, a large positive potential spike will follow immediately after its loss. Fortunately, the ions generally diffuse from the system in less than an ion plasma period, and the potential decreases rapidly back towards zero so that the system evolves from a fairly quiet and almost neutral state. This can be seen in some of the following figures.

8.4.2 Evolution of the Simulations after Neutralization

When the ions are immobile the wave behaviour is determined by \( \eta \) and \( L/\lambda_0 \), as discussed in Chapters 6 and 7, and so the evolution of the one-dimensional discharge created by an energetic electron beam is discussed mainly in relation to these quantities. Throughout each simulation the density of the plasma varies, and, as a result, \( \eta \) and \( \lambda_0 \) will also vary. As the plasma density increases, \( \eta \) will decrease because \( N_b \) is constant, and \( \lambda_0 \) will decrease because \( v_b \) is constant. The state of a simulation can be represented as a point in \((L/\lambda_0, \eta)\) space, and the evolution of a simulation can be represented as a point moving along a line of the form:

\[
\eta = \frac{\omega_b^2}{\omega_e^2} = \left( \frac{L}{\lambda_b} \right)^2 \left( \frac{\lambda_0}{L} \right)^2 ,
\]

where \( \lambda_b = v_b/f_b \). Some of these paths are shown in Figure 8.13, with \( L/\lambda_b = 0.25, 0.5, \) and \( 1.0 \).

Each simulation starts at a point with \( \lambda_0 = \lambda_b \) on the line \( \eta = 1.0 \). This includes the period during which the beam is being neutralized, and any subsequent time when there are no plasma electrons in the system. As more plasma electrons are created and trapped, the state of a simulation moves along a line until the plasma density stops increasing when the loss rate due to diffusion is equal to the creation rate due to ionization.

The behaviour of the interactions in these simulations reflects the behaviour observed in the simulations with immobile ions, in that different processes have been found to be dominant in particular regions of \((L/\lambda_0, \eta)\) space. Four regimes have been determined so far, associated with: quiet beam discharges, beam-plasma oscillations, beam-plasma
Figure 8.13: As the plasma density in a simulation increases, $\eta$ and $\lambda_0$ decrease. Since the beam density and the system length were kept constant, the evolution of a simulation will lie on a line similar to those shown in this diagram.

Figure 8.14: The different types of wave behaviour are found to depend upon $\eta$, and $L/\lambda_0$. The wave behaviour can be divided into four types associated with, quiet beam discharge - QBD, beam-plasma oscillations - BPO, beam-plasma discharge - BPD, and recurring virtual cathodes - RVC discharges, and recurring virtual cathodes. The boundaries between the different regions have not been determined precisely, and are sketched roughly in Figure 8.14.

It is probable that the boundaries cannot be determined more precisely because the distribution of particles after an ionizing collision are chosen randomly. This allows for some variation in the results of different runs with the same initial parameters. The steady states reached in different runs with the same initial parameters appear to be within the same regime, but, because the wave behaviour can vary significantly during their evolution, the boundaries between the different regimes are not clear. This reflects the variable nature of the laboratory experiments.

Given that the growth rate of the beam-plasma instability reaches a maximum when
\( \eta = 0.5 \) and \( L \gg \lambda_0 \), and decreases as \( L \to 0, \eta \to 0, \) or \( \eta \to 1 \), the boundaries between the QBD, BPO, and BPD regions appear to lie close to lines of equal growth rate, or possibly equal heating rate. The boundary between RVC and BPD has been the least investigated; however, it appears to lie near the limit of \( \eta \sim 0.8 \), which has been found to be the minimum \( \eta \) at which a stable virtual cathode can form, as discussed in Chapter 7.

Brief outlines of the characteristic processes observed in the four regimes are given in the following sections.

**Regime 1: Quiet Beam Discharge**

In this regime, the growth rate of the beam plasma instability is not great enough to produce large amplitude oscillations. The sheath potential is also low, and the beam electrons are barely perturbed when they enter the system. The oscillations are at the plasma frequency, which increases as the plasma density increases. This can be seen in Figure 8.15, which shows the potential at the center of a system with \( L = 0.126 \lambda_b \). If \( L < 0.212 \lambda_b \), i.e. \( L < L_m \), as is the case for the simulation shown in Figure 8.15, a virtual cathode will not form, and there is no large excess of ions after neutralization to produce a large positive potential peak.

The simulations in this regime are similar to those discussed in Section 8.2 because

![Figure 8.15: The potential at the center of a system with \( L/\lambda_b = 0.126 \). The simulation remains in the QBD regime and the amplitude of the oscillations remains small. (\( \nu_b = 2.0, N_b = 5, N_e = 100, \) and \( N_\nu = 1000 \))](image-url)
the interaction between the beam and the plasma is weak, and the ionization occurs reasonably uniformly across the system. The flux of electrons which reaches the collection boundary is comprised of the beam electrons at $v_b$, plus some low energy plasma electrons, as shown in Figure 8.16, which indicates that the beam is not interacting with the plasma. This is supported by Figure 8.17, which shows that the beam maintains an almost constant velocity across the system. The distribution of the plasma electrons is almost top hat with $v_r^2/2 \sim \phi_{sh}$, and resembles the distributions in simulations with the preliminary ionization model. Furthermore, Figure 8.18 shows how the plasma density starts to saturate exponentially to an equilibrium level just like the simulations with the preliminary ionization model. The initial $1000 \Delta t$ in Figure 8.18, when $N_e > N_i$, corresponds to the period in which the beam is neutralized.

Regime 2: Beam-Plasma Oscillations

In this regime, the system length is long enough for the beam-plasma interaction to produce large amplitude oscillations, as shown in Figure 8.19. This figure also shows the large potential peak appearing after the virtual cathode leaves the system. The large amplitude oscillations accelerate a significant fraction of the plasma electrons so that many escape, and the sheath potential is increased. The larger sheath increases the diffusion rate so that a steady state is reached with a low density plasma, as indicated by Figure
Figure 8.17: Plot of electron phase space from the same simulation with $L/\lambda_b = 0.126$. The beam is barely perturbed because $\phi_{sh}$ is small and the amplitude of the oscillations remains small. ($v_b = 2.0$, $N_b = 5$, $N_c = 100$, and $N_{\nu} = 1000$)

Figure 8.18: The densities of the electrons and ions in the same simulation with $L/\lambda_b = 0.126$. The start of the exponential saturation of the plasma density can be seen. The initial $1000\Delta_t$ when $N_e > N_i$ corresponds to the period in which the beam is neutralized. ($v_b = 2.0$, $N_b = 5$, $N_c = 100$, and $N_{\nu} = 1000$)
Figure 8.19: The potential at the center of a system with $L = 0.356 \lambda_b$. Immediately after the virtual cathode leaves the system at $t \sim 5000 \Delta t$ there is an excess of ions and a large positive potential peak appears, as discussed in Section 8.4.1. The simulation then reaches the BPO regime and the amplitude of the oscillations remain large. ($v_b = 1.0$, $N_b = 10$, $N_e = 100$, and $N_\nu = 2000$)

Consequently, $\eta$ remains high; for example $\eta \sim 0.25$ for the steady state reached in Figure 8.20, and the amplitude of the oscillations remains fairly constant, as can be seen in Figure 8.19.

These oscillations are standing waves with $L = \lambda/2$, which cause the beam to oscillate in phase space although rarely causing the beam to be reflected at the collection boundary. An example of the electron distribution in phase space during BPO is shown in Figure 8.21. It shows that most of the plasma electrons are trapped near the center of the system, and that many of them have sufficient energy to make ionizing collisions. However, because the ionization rate is not high in simulations which reach a steady state in the BPO regime, the number of hot plasma electrons is not sufficient to overcome the high diffusion rate and increase the plasma density. Figure 8.22 shows the ion distribution in phase space, which indicates that the sheaths extend virtually to the center of the system so that most ions experience a significant acceleration towards the boundaries. This figure also shows that despite the sheath width, the majority of the ions leave the system with the maximum escape velocity $v_{esc} \sim (2 \phi_{th} m/M)^{1/2}$.

The flux of electrons reaching the collection boundary have two peaks due to the beam electrons, and a third and smaller peak due to the plasma electrons, as shown in Figure
Figure 8.20: The densities of the electrons and ions in the same simulation as in Figure 8.19 with $L = 0.356 \lambda_b$. The plasma density saturates at less than four times the beam density, and so $\eta$ is always greater than 0.25. The rapid decrease in the electron density when the virtual cathode leaves the system can be seen at $t \sim 5000 \Delta t$. ($v_b = 1.0$, $N_b = 10$, $N_c = 100$, and $N_u = 2000$)

Figure 8.21: Plot of the electron phase space from the same simulation with $L/\lambda_b = 0.356$. Many plasma electrons have sufficient energy to make ionizing collisions, yet the high diffusion rate stops the plasma density from increasing. ($v_b = 1.0$, $N_b = 10$, $N_c = 100$, and $N_u = 2000$)
8.23. The large amplitude oscillations force the beam to move about in phase space so that more beam electrons reach the collection boundary with velocities given by

\[ v_\pm \simeq \left( v_b^2 \pm \phi_{pp} \right)^{1/2}, \]

where \( \phi_{pp} \) is the potential of the oscillations from peak to peak in the center of the system. For example, Figure 8.23 shows the escaping electron flux from the simulation which produced the oscillations shown in Figure 8.19; the beam velocity was 1.0 and \( \phi_{pp} \simeq 0.5 \), therefore equation 8.18 states that \( v_- \simeq 0.7 \) and \( v_+ \simeq 1.2 \), in agreement with Figure 8.23.

Regime 3: Beam-Plasma Discharge

The boundary of the BPD regime is determined by the first indication that a large fraction of the plasma electrons are making ionizing collisions and cause the plasma density to increase. Only two examples are given in this section to indicate the range of interactions in this regime. A closer examination of the processes observed in simulations which reach BPD is given in Section 8.4.4.

A wide variation in the rate of change of the plasma density is observed in the simulations as they pass into the BPD regime. Simulations with an ionization rate just sufficient to pass through BPO and reach BPD can experience a sudden increase in the plasma
Figure 8.23: The flux of the electrons reaching the collection boundary in a simulation with $L/\lambda_b = 0.356$. ($v_b = 1.0$, $N_b = 10$, $N_c = 100$, and $N_v = 2000$)

density once BPD is reached, whereas simulations with greater beam density, which pass directly from the RVC regime, do not. This is illustrated in Figure 8.24, which shows the change in the plasma density during two simulations in which BPD is reached. In Figure 8.24a the virtual cathode leaves the system at $t \sim 300 \Delta t$, and BPD is reached at $t \sim 17000 \Delta t$ after passing slowly through the BPO regime. The rate at which the density increases when the BPD regime is reached is more than double the rate during BPO. In Figure 8.24b the simulation reaches BPD as soon as the virtual cathode leaves the system. The variation in the rate at which the plasma density increases is due to the different wave behaviour within the BPD regime and the sheath potential.

The wave behaviour in the BPD regime exhibits similar features observed in all the simulations with immobile ions discussed in Chapter 6. The wave behaviour depends upon the beam density because it determines which regimes a simulation will pass through before reaching BPD. Furthermore, it will depend upon the ionization rate since that determines the steady state of a simulation as well as the rate at which a discharge is created. Figure 8.25 shows the central potential from the two simulations shown in Figure 8.24. When a simulation which passes through BPO reaches BPD the potential oscillations remain fairly regular, as shown in Figure 8.25a for $t \gtrsim 15000 \Delta t$. This is because $\eta$ is small and the beam is rarely reflected at the collection boundary. Those simulations with greater beam density reach BPD when $\eta$ is larger. This enables electron holes to form and the
Figure 8.24: The changes in the plasma density due to the onset of BPD vary considerably depending upon the ionization rate and $L/\lambda_0$.

a) $L = 0.356\lambda_0$ and $\lambda_{en} = 10L$. This simulation passes through BPO before reaching BPD at about $17000\Delta_t$, at which time the plasma density increases rapidly. $(v_b = 1.0, N_b = 10, N_e = 100, \text{ and } N_\nu = 1000)$

b) $L = 1.0\lambda_0$ and $\lambda_{en} = 12.5L$. This simulation reaches the BPD regime after requiring about $10^4\Delta_t$ to neutralize the virtual cathode and passing rapidly through the RVC regime. The variation in $dN_{ei}/dt$ is due to variations in the wave behaviour. $(v_b = 1.0, N_b = 20, N_e = 200, \text{ and } N_\nu = 2500)$
plasma becomes inhomogeneous. Combined with the bouncing of trapped beam electrons, this causes the wave behaviour to be chaotic, as shown in Figure 8.25b. As the plasma density increases, the wave behaviour can become more regular, resembling simulations with immobile ions, weak beams, and $L \gg \lambda_0$.

Regime 4: Recurring Virtual Cathodes

A steady state in this regime is reached in simulations with very low ionization rates and with length greater than about $L_m$. The combined effects of the low ionization rate and greater system length means that the plasma density is always low. This is because large amplitude oscillations are produced, which maintain a high diffusion rate, even before the first virtual cathode is neutralized. These potential oscillations are dependent upon the ionization rate, and exhibit similar behaviour to those observed in the simulations discussed in Chapter 7 with immobile ions and $\eta \sim 1$.

The virtual cathode is most stable in simulations with $v_b$ just greater than $v_{ion}$ and $L \gg L_m$. The ions are created in the regions close to the boundaries, as shown in Figure 8.12, and large oscillations near the collection boundary ensure that the diffusion rate of the ions is sufficient to stop neutralization. A second virtual cathode can form in these simulations, but they are less stable than the virtual cathode at the injection boundary, and generally leave the system after moving to the collection boundary.

Simulations with $v_b \gg v_{ion}$ and $L \lesssim \lambda_0/2$ go through a regular cycle in which virtual cathodes are neutralized and pass across the system to the collection boundary only to have another form at the injection boundary. The charge imbalance is so high just after a virtual cathode leaves the system that the ions rapidly diffuse from the system, returning it to an electron rich state in which another virtual cathode forms. The cycle consists of a long period in which the virtual cathode is neutralized followed by a short period of less than $\tau_i$ when the ions are rapidly lost from the system, as shown in Figure 8.26. The period of the cycle is therefore determined mainly by the ionization rate. As observed in the simulations of Child-Langmuir flow, the virtual cathode 'rings' just after it forms, but with $v_{sep} = 0.02$, the oscillations are damped after only a few periods.

In longer systems with $v_b \gg v_{ion}$ the recurrence of virtual cathodes is irregular and not as frequent. The potential fluctuates greatly, not just during a single plasma period, but over longer cycles with a period varying between several $\tau_c$ and a few $\tau_i$, as shown in Figure 8.27. The change in the wave behaviour with increased system length can be attributed to an increase in the characteristic diffusion time. This enables plasma electrons to remain trapped in the system, bouncing between temporary sheaths, and causing the irregularity
Figure 8.25: The central potential from the two simulations shown in Figure 8.24.

a) $L = 0.356 \lambda_b$. The simulation reaches the BPD regime at $t \sim 17000 \Delta t$ after passing through the BPO regime. The sheath potential increases just before the BPD regime is reached, and waves remain comparatively regular within the BPD regime. ($v_b = 1.0$, $N_b = 10$, $N_c = 100$, and $N_u = 1000$).

b) $L = 1.0 \lambda_b$. The simulation passes rapidly through the RVC regime before reaching BPD. The beam is trapped at the collection boundary and the plasma is inhomogeneous causing chaotic wave behaviour. ($v_b = 1.0$, $N_b = 20$, $N_c = 200$, and $N_u = 2500$).
1.5

Figure 8.26: The potential at the center of a system with \( L = 0.5\lambda_b \). A virtual cathode is neutralized at \( t \sim 5000\Delta_t \), but another reforms at \( t \sim 8000\Delta_t \) because the excess ions diffuse rapidly from the system. The period between virtual cathodes is only a fraction of the ion plasma period because the amplitude of the positive potential peak is so large. \((v_b = 1.0, N_b = 20, N_c = 100, \text{ and } N_v = 10^4)\)

Figure 8.27: (Overleaf) Contour plot of the potential in a simulation with \( L = \lambda_b \) in the RVC regime. The diffusion rate is not sufficient to allow a virtual cathode to reform rapidly after the previous virtual cathode has been neutralized. However, the large amplitude low frequency oscillations eventually allow another virtual cathode to form. During the period shown in this figure, \( \tau_e \sim 100\Delta_t \) and \( \tau_l \sim 4000\Delta_t \). \((v_b = 1.0, N_b = 20, N_c = 200, \text{ and } N_v = 5000)\)

in the wave behaviour.

The velocity distribution of the electrons reaching the collection boundary for simulations in this regime is broad, and there is generally a peak near \( v_b \), similar to the escape velocity distributions of the simulations discussed in Chapter 7.

8.4.3 The Effects of Varying \( M, \sigma_i(v), \) and \( f_{\text{scat}}(v) \)

This section briefly discusses how the balance between ionization and diffusion is modified by varying the ion mass, \( M \), the ionizing collision cross-section, \( \sigma_i(v) \), and the velocity distribution of the scattered electrons, \( f_{\text{scat}}(v) \).
CONTOUR PLOT OF THE POTENTIAL

TIME

DISTANCE

ABOVE 1.65
1.51  1.65
1.37  1.51
1.23  1.37
1.10  1.23
0.96  1.10
0.82  0.96
0.68  0.82
0.54  0.68
0.41  0.54
0.27  0.41
0.13  0.27
-0.01  0.13
-0.14  -0.01
-0.28  -0.14
-0.42  -0.28
BELOW -0.42
The dominant ions in the WOMBAT laboratory experiments are \( \text{N}_7^+ \), which have a mass \( M \simeq 28 \times 1836m \). It was therefore thought appropriate to examine the effects of greater ion mass. However, only a few simulations with \( M > 1836m \) have been made because large amounts of computer time are required since some of the interactions have a period proportional to the ion plasma period.

These simulations indicate that the boundaries between the different regimes do not change if the ion mass is varied, although the steady state of a simulation with a given beam current and \( N_v \) will be different. They also show that the steady states may be kept within the same regime by changing the ionization rate by the factor \( (M_{old}/M_{new})^{1/2} \). This suggests that the ions influence the interaction only by regulating the plasma density via the diffusion rate, and that it is mainly the interaction between the beam and the plasma electrons which determines the wave behaviour.

Since the linear growth rate of the beam-plasma instability is modified only marginally by different but realistic ion mass, the shift of the boundaries might be expected to be small and unlikely to be detected with only a few simulations. However, the sheath potential increases with increasing ion mass, in a similar manner to the simulations described in Section 8.2, and therefore linear growth may become less important if the perturbation of the beam upon entry into the system is increased. This needs to be studied further, especially since it appears as though the sheath potential greatly influences the threshold of BPD.

The effect of varying \( \sigma_i(v) \) or \( f_{\text{scat}}(v) \) has been found to be similar to varying the ion mass. The steady state of a particular simulation can change significantly, but the boundaries between the different regimes do not. The few simulations which have been made indicate that the boundaries remain within the range of uncertainty found in the simulations with the initial choices of \( \sigma_i(v) \) and \( f_{\text{scat}} \). It is the balance between ionization and diffusion which is altered by any change in \( \sigma_i(v) \) or \( f_{\text{scat}}(v) \), whereas the boundaries between the different regimes appear to be determined more by the growth rate of the beam-plasma instability.

8.4.4 Examination of the Interaction Near the Threshold to BPD

As stated before, the wave behaviour in these simulations is very similar to that in simulations with immobile ions and no ionization. However, with the inclusion of ionization and ion motion, the plasma density can vary greatly, and so the wave behaviour can vary considerably in a single simulation. In this section the wave behaviour in short systems (ie. \( L \lesssim 1.5\lambda_0 \)) when the plasma density is low, is examined because it is more relevant.
Figure 8.28: The wave amplitude in the center of simulations with $L = 0.356\lambda_0$ for $N_\nu$ in the range $10^2$ to $10^4$. When $N_\nu \geq 2000$ a steady state is reached, and for $N_\nu \leq 1000$ the results are for when $N_e = 2N_0$, i.e. $L = 1.12\lambda_0$. The standing waves with $L \sim \lambda/2$ have a maximum $\phi_{pp} \approx v_b^2/2$ just before BPD is reached, and they are replaced by standing waves with $L \sim \lambda$ when BPD is reached. ($v_b = 1.0$, $N_b = 10$, and $N_e = 100$)

to the determination of the threshold of BPD.

The boundary between QBD and BPO has been defined by the first appearance of standing waves with $L \sim \lambda/2$, which have an amplitude above the noise level. It has been found that for $v_b = 0.5$ to 2.0 the amplitude of $\phi_{pp}$ increases up to about $v_b^2/2$ when the steady state of a simulation lies closer to the boundary between BPO and BPD. Very rarely are oscillations in the BPO regime observed with $\phi_{pp} > v_b^2/2$.

An example of the change in $\phi_{pp}$ can be seen in Figure 8.28, which shows the range of the potential at the center of simulations with $L = 0.356\lambda_0$ ($v_b = 1.0$, $N_b = 10$, and $N_e = 100$) for different $N_\nu$ between $10^2$ and $10^4$. When $N_\nu \geq 2000$ a steady state is reached, and for $N_\nu \leq 1000$ the results are for the time when $N_e$ has increased to $2N_0$, i.e. $L = 1.12\lambda_0$. These simulations lie on the line $\eta = 0.356(L/\lambda_0)^2$, and the when $N_\nu \approx 2000$ the steady state of the simulation lies at the point $L \sim 0.8\lambda_0$, $\eta \sim 0.2$. This is close to the boundary between BPO and BPD, and $\phi_{pp} \approx v_b^2/2$.

If the ionization rate is increased so that BPD is reached, the standing waves with $L \sim \lambda/2$ cannot be sustained, and a transition is observed as oscillations with $L \sim \lambda$ replace them. These oscillations can have either negative or positive phase velocity, and may be standing waves. Generally, the oscillations are smallest near the center of the system, and so $\phi_{pp}$ decreases when the BPD regime is reached, as shown in Figure 8.28
for $N_v \leq 1000$. The oscillations are larger at two points midway between the central node and the boundaries, and have been observed with an amplitude up to $v_b^2/2$.

During this transition the sheath potential increases and is accompanied by a sudden increase in the plasma density. This can be seen when comparing Figures 8.25a and 8.24a, which show the potential at the center, and the plasma density respectively, in a simulation with $N_v = 1000$ ($v_b = 1.0$, $N_b = 10$, and $N_c = 100$). The sheath is formed when low energy plasma electrons are accelerated out of the system by the beam-plasma interaction, and the increased sheath potential enables more hot plasma electrons to become trapped within the system. The velocity distribution of the plasma changes so that a greater proportion have sufficient energy to make ionizing collisions, and causes the sudden increase in the plasma density.

After making only a few simulations with immobile ions and ionization it was thought that the reflection of the beam at the collection boundary caused the sudden increase in the plasma density. However, in the simulation referred to in Figures 8.25a and 8.24a, the beam is not reflected until $L \sim 1.5\lambda_0$ when $\eta \sim 0.06$ which is well into the BPD regime, and no sudden increase in the plasma density is observed at that time.

When the ionization rate is increased, i.e. $N_v < 1000$ for those simulations referred to in Figure 8.28, the sheath potential at the time when $N_e = 2N_0$ is reduced, as shown in Figure 8.28. This is because the plasma density reaches $2N_0$ sooner, and so the beam-plasma interaction has less time in which to heat the plasma. Therefore, as discussed in Chapter 6, even with the same $\eta$ and $L/\lambda_0$ the wave behaviour can be significantly different depending upon the velocity distribution of the plasma, which in these simulations will occur with different ionization rates.

Those simulations with an ionization rate more than sufficient to reach the BPD regime have also been observed to exhibit a sudden increase in plasma density when far within the regime. The sudden increase in the rate at which ionizing collisions are made is again found to occur at the same time as when the sheath potential increases. However, these increases in $dN_e/dt$ are generally not as large as when they occur near the boundary between BPO and BPD. This is to be expected because the heating rate will not be as high since $\eta$ is smaller.

When the beam density is great enough so that $L/\lambda_b \gtrsim 0.7$, the interaction will not pass through the BPO regime, but go directly from the RVC regime to the BPD regime if the ionization rate is high enough. In these simulations, $\eta \gtrsim 0.5$ when the boundary of the BPD regime is reached. Because of the high growth rate of the beam-plasma interaction near the threshold of BPD, a large amplitude sheath is formed rapidly, but this does not
ensure that BPD will be reached. Even in the RVC regime large amplitude sheaths can form, but they are only temporary because the loss of ions due to diffusion is greater than the rate at which ions are formed. If the ionization rate is just great enough for BPD to be reached and $\eta$ stays in the range 0.4-0.7, the sheath potential can fluctuate widely due to the number of trapped beam electrons. With a greater ionization rate the sheath potential reaches an approximately steady level soon after the BPD regime is reached, and $dN_i/dt$ does vary rapidly. However, minor fluctuations in $dN_i/dt$ do correlate with minor variations in $\phi_{sh}$ and the wave amplitude, as can be seen by comparing Figures 8.24b and 8.25b.

The simulations carried out so far suggest that the amplitude of the waves in a particular regime are proportional to the beam energy. It is certainly the case in the BPO regime, and it can be expected considering the scaling in the simulations. For example, to simulate the same interaction with the value of $\Delta_\varphi$ halved, the beam velocity must be doubled and the potential will be increased by a factor of four in dimensionless units. Conversely, if $\Delta_\varphi$ and $L$ are kept constant but $v_b$ is still doubled, the beam density must be increased by a factor of four to maintain the same ratio of $L/\lambda_0$ and model the same interaction. Therefore, the potential should also be increased by a factor of four at the same point in $(L/\lambda_0, \eta)$ space.

However, simply increasing the beam energy will not necessarily result in a proportional increase in the wave amplitude because the steady state may have different values of $\eta$ and $L/\lambda_0$. Consequently, the wave behaviour will probably be different and the potential oscillations at a fixed point in the system can even decrease when the beam energy is increased. To illustrate this, Figure 8.29 shows the range of the potential in the center of simulations with different beam energy. $N_\nu$ and $L$ are kept constant at 2000 and 100$\Delta_\varphi$ respectively, and the beam density is also kept constant, and therefore the simulations with higher beam velocity start with smaller $L/\lambda_0$. The simulation with $v_b = 2.0$ remains in the QBD regime, and the wave amplitude is small, whereas the simulations with $v_b = 1.0$, and 1.5 reach the BPO regime and the wave amplitude is large. The simulation with $v_b = 1.5$ is further from the boundary of the BPD regime, and so the amplitude of the waves are not significantly greater than those in the simulations with $v_b = 1.0$. The simulation with $v_b = 0.5$ reaches the BPD regime, and the combined effects of small $v_b$ and inhomogeneity of the plasma decreases the wave amplitude.

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Figure 8.29: The wave amplitude of the final states of simulations with $N_c = 100$ $N_b = 10$ and different $v_b$ shows that increasing the beam velocity will not necessarily increase the amplitude of the waves because the equilibrium state can change regimes as $L/\lambda_b$ changes. The simulation with $v_b = 2.0$ remained in QBD, whereas the simulations with $v_b = 1.0$ and 1.5 reached the BPO regime. Only the simulation with $v_b = 0.5$ reached BPD.

8.4.5 Scaling Law for the Threshold in the Simulations

Simulations with ionization by energetic electrons have not yet been made specifically to determine a scaling law for the threshold beam current. The first objective was just to simulate BPD, and then possibly examine the wave behaviour near threshold. However, as a number of simulations have been made with different beam currents and ionization rates with $v_b = 1.0$ ($\xi_b = 100$ eV) and $N_c = 100$ ($L = 100$ cm), it is possible to make an estimate of the pressure dependence of the critical current $j_c$.

The final states of these simulations are displayed in Figure 8.30 for different beam current and $N_{c}^{-1}$, and, since $N_{c}^{-1} \propto P$ this diagram is analogous to Figure 2.9 and can be used to determine a threshold condition for BPD. The letters Q, R, O, and D, represent QBD, RVC, BPO, and BPD respectively. If the final state lies on a boundary between different regimes it is marked appropriately. The line with gradient $-1$ roughly separates those simulations which reach a high plasma density from those which do not. This indicates that

$$j_c \propto P^{-a},$$

where $a = 1 \pm 0.2$.

Estimates of the plasma density of the final states reached by the simulations with $N_b v_b = 5$ in Figure 8.30 are shown in Figure 8.31. This figure shows that the plasma density of the steady states increases rapidly as the steady states move through the BPO
Figure 8.30: The final states of simulations with \( v_b = 1.0 \) and \( N_e = 100 \) are shown against beam current \( N_b v_b \) and \( N^{-1}_\nu \). This is equivalent to the diagram in Chapter 2 which shows the threshold condition for the precursors and the different BPD. When \( N_\nu = 1000 \) the mean free path of the beam electrons is \( 10L \).

regime, and the threshold boundary marked in Figure 8.30 occurs while the simulations with \( N_b v_b = 5 \) are still in the BPO regime, as shown in Figure 8.31. Within the BPD regime the plasma density appears to increase proportionally to \( N^{-1}_\nu \), similar to the laboratory experiments.

The estimates of the density for low \( N_\nu \) are based upon the assumption that there will be an exponential saturation with a constant ionization rate. They are poor because these simulations have not been run close to equilibrium since an impractical amount of computer time would be required. Estimates of the plasma density in the steady states of simulations with greater \( N_b v_b \) would be worse because these simulations reach even higher plasma densities and would therefore require even more computer time.

A series of simulations have not yet been made to examine the dependence of the critical current on \( \epsilon_b \) and \( L \); however, an estimate can be made by assuming that BPD follows the first appearance of large amplitude oscillations which would cause some of the plasma electrons to be accelerated above \( v_{ion} \). Large amplitude oscillations first appear in the BPO regime, and so one could take the boundary of QBD and BPO as the threshold for the enhancement of the ionization due to the plasma electrons. Neglecting the region where the boundaries of QBD, BPO and BPD are uncertain (when \( \eta \leq 0.1 \) and \( L \geq 0.8\lambda_0 \)), the boundary between QBD and BPO lies approximately on the line \( \eta = 0.25(\lambda_0/L) \). This says that BPO can be reached only if

\[
\frac{L}{\lambda_b} \geq 0.25.
\] (8.20)
Threshold as in Figure 8.30

Figure 8.31: Plot showing the estimated plasma density of the equilibrium states in simulations with $v_b = 1.0$, $N_c = 100$, and $N_b = 5$

If different $M$, $\sigma_i(v)$, or $f_{\text{scat}}(v)$ are used, this limit might change by $\pm 20\%$. This condition can therefore be used to determine an expression defining the critical current above which BPO can be reached, with

$$j_c \alpha \frac{E_b^{3/2}}{L^2}. \quad (8.21)$$

The beam energy dependence here is the same as in the scaling law determined by laboratory experiments. However, the laboratory experiments suggest that the length dependence is $L^{-1}$.

The above analysis is an over simplification because it firstly neglects the finding that $\phi_{pp} \alpha v_b^2$ in the BPO regime, and secondly, it neglects the uncertainty of the boundary region between QBD, BPO, and BPD. To determine the significance of these two points there must be some clarification of whether BPD is reached - when large amplitude oscillations are observed, when a significant fraction of the plasma electrons can make ionizing collisions, or simply when the steady state plasma density is greater than if there were no beam-plasma interaction.

Large amplitude oscillations certainly increase the number of plasma electrons which can make ionizing collisions when in the BPO regime and $\eta \gtrsim 0.1$. However, the increased amplitude of the sheath raises the ambipolar diffusion rate and the steady state plasma density can therefore be reduced.

Further, if the ionization rate is high, simulations with $L/\lambda_b \lesssim 0.25$ reach a steady state which has a high plasma density, and large amplitude oscillations can still appear
in bursts. This is not surprising since linear theory states that \( k_i \propto \eta^{1/3}/\lambda_0 \) when \( \eta \ll 1 \), so that \( k_i L \propto (L/\lambda_0)^{1/3} \) since \( \eta \propto (\lambda_0/L)^2 \). Therefore the growth of the waves across the system should increase as the plasma density increases.

However, the heating rate derived by Lebedev et al. (1976) and Papadopoulos (1981), \( Q \propto \alpha b \eta \xi v_b / L \), should remain constant since \( N_b, v_b, \) and \( L \) are fixed. When \( \eta \) is small the beam-plasma interaction can produce large amplitude oscillations without causing large perturbations of the plasma distribution. A sheath can be formed which increases the diffusion rate but without accelerating many plasma electrons above \( v_{ion} \). Hence the steady state plasma density can be reduced by the beam-plasma interaction. A few simulations with the preliminary ionization model, with the same ionization rate, \( v_{eth} \), and \( L \), have confirmed this. In fact they indicate that the steady state plasma density is even reduced by the beam-plasma interaction within the BPD regime near the boundary. This will be investigated more closely in the future.

Because of the balance between ionization and diffusion, the threshold of BPD may not lie on a line of the form \( \eta \propto (\lambda_0/L)^2 \), and so the scaling law may be the same as for the laboratory experiments. An examination of the length and beam energy dependence should be made separately, but this will require many more simulations to make an accurate estimate of the threshold condition and it also is left as a future area of research. Likewise, since it has been found that \( \phi_{eh} \) is influenced by \( v_{eth} \) and \( M \), these variables should also be investigated more thoroughly.

**Summary**

Simulations with mobile ions, but without an ion source, show that the ions would eventually leave the system because of ambipolar diffusion. If a beam of electrons is injected into the system, a virtual cathode would eventually form and the trajectories of the electrons would be similar to that of Child-Langmuir flow.

The simulations with the preliminary model of ionization show that the balance between ionization and diffusion is complex and is determined by many variables. Ambipolar diffusion governs the process, determining the sheath potential and the characteristic diffusion time.

The simulations with a model of ionization due to energetic electrons show that the wave behaviour is determined by \( \eta \) and \( L/\lambda_0 \) in the range examined. The wave behaviour is changed by variations in \( v_b, N_b, \) and \( L \), depending upon how \( \eta \) and \( L/\lambda_0 \) are altered. The boundaries between the regions in which the wave behaviour is different appear to
be almost independent of variables such as $M$, $\sigma_e(v)$, and $f_{scat}(v)$, in the few simulations made.

The axial boundary conditions are instrumental in determining the standing wave structure in simulations with $L = \lambda/2$, which appear to have an amplitude proportional to the beam energy for the same values of $L/\lambda_0$ and $\eta$. This relationship between wave amplitude and beam energy is believed to extend to all regions, although insufficient simulations have been made to check this.

The scaling law determined from these simulations is similar to the law determined from experimental results, with the exception of the length dependence. The difference may arise because the law derived from the simulations is based upon the boundaries between regimes with different wave behaviour, and not at an observed discharge threshold. More simulations are required to check this threshold condition.
Conclusion

Experiments in WOMBAT have shown that the wave behaviour in the BPD is complex with nonlinear growth of waves in bursts which are rapidly damped both in space and time. The axial dependence of the wave amplitude of the BPD1 bursts suggests that an absolute instability is not necessary to create a BPD, contrary to the theory of Rowland et al. (1981). Measurement of the phase velocity of the waves was found to be difficult because movement of the probes modified the wave behaviour, and probably because the beam was trapped by the wave causing the wave behaviour to be complex.

The simulations with immobile ions have shown likewise that the wave behaviour can be complex, even when the beam is weak. This is because in most simulations sheaths form on the boundaries which perturb the beam upon entry, and the beam can be trapped by the wave before reaching the collection boundary. Even when \( \eta \lesssim 0.01 \), which is the approximate value of \( \eta \) during BPD1 in WOMBAT, the beam can be trapped by the wave in only a few wavelengths. In the simulations the wave behaviour is further complicated by the reflection of beam electrons at the collection boundary, and, for larger \( \eta \), low frequency oscillations can appear due to the bouncing of trapped beam electrons between the sheaths. Furthermore, in simulations with larger \( \eta \), the plasma is more inhomogeneous and the spectra of the potential oscillations were found to have broader peaks, in contrast to experimental spectra.

The analytic heating rate derived by Lebedev et al. (1976) and Papadopoulos (1981) has been confirmed by the simulations with immobile ions, and found to be valid when \( \eta \lesssim 0.1 \) and \( L \gtrsim 3\lambda_0 \). However, because of the complex wave behaviour and the trapping of beam electrons, the heating rate is not constant and can fluctuate considerably. When \( L \lesssim \lambda_0 \) the average heating rate decreases as \( L \to 0 \), which correlates with the decreasing growth rate in short systems, as shown in Chapter 4.

The PIC technique has also been shown to model virtual cathodes associated with space charge limited flow in one dimension, indicating that virtual cathodes are unstable when \( \eta \lesssim 0.8 \) and the ions have infinite mass. Unfortunately, since a region of high electron
density exists at a virtual cathode, the potential changes with a very short scale length, especially when the beam is cold. Consequently, many cells and particles are required to model the virtual cathodes accurately.

The beam-plasma interaction is further complicated, and more computer time is required, when ionization and ion motion are included in the model. When ion motion is included a source of ions must exist, otherwise the ions will diffuse from the system and eventually the electron beam will be space charge limited and follow Child-Langmuir trajectories. The preliminary ionization model, which simply adds pairs of ions and electrons uniformly across the system at a constant rate, shows that the diffusion process is ambipolar. As expected, many variables such as the ionization rate, the velocity distribution of the plasma source, and the ion mass, influence the diffusion rate. Further study is required to establish the relationship of each variable.

The model of ionization by energetic electrons is more representative of the ionization process during the beam-plasma interaction. It uses the conservation of energy to determine the velocity distribution of the scattered electron, and the ion and electron pair created after an ionizing collision. It also considers the velocity dependence of the ionization collision cross-section. This model shows that the balance between ionization and diffusion is influenced strongly by the beam-plasma interaction because it determines the sheath potential. When a large amplitude sheath is created, the density of hot electrons can increase and cause a sudden increase in the frequency at which ionizing collisions occur. However, if the ionization and heating rates are low, the sheath can cause a reduction in the plasma density. As suggested by Papadopoulos (1981), it is believed that the sheath plays an important role in the one-dimensional BPD, although further investigations are required to assess its full importance.

The laboratory interaction, however, is not one-dimensional; this modifies the wave behaviour and so the sheath and the diffusion process will probably be affected. For example, the ion waves observed before the appearance of electron oscillations in the WOMBAT experiments are probably not longitudinal waves along the beam. They occur when $\eta \sim 1$ and, depending upon the ionization rate, $L$ can be much greater than $\lambda_0$, and should therefore lie in the RVC regime (see Chapter 8). However, in thin beam experiments, virtual cathodes are unlikely to form since, as discussed in Chapter 4, Pierce-type instabilities probably cannot exist on a beam with a radius $r_b \lesssim 0.4\lambda_0$, even with an infinite axial magnetic field. Also, if $r_l$ (the larmor radius of electrons with perpendicular velocity $v_b$) is not much smaller than $r_b$ at the gun aperture, most beam electrons will be deflected perpendicular to the magnetic field to avoid regions of high electron density,
rather than be reflected back to the gun.

In the WOMBAT laboratory experiments the beam is not pulsed, in contrast to the simulations. The beam current is increased at a rate much slower than that of the diffusion process, and so the interaction effectively remains in a steady state. When the beam is first switched on, the interaction has $\eta \approx 1$ and $L/\lambda_0 \sim 0$. As the beam current is increased $L/\lambda_0$ increases, and, depending upon the ionization and diffusion rates, $\eta$ may decrease. Simulations as well as laboratory experiments have shown that near the threshold of BPD $\eta$ will not be very small, and that $L$ is not necessarily much greater than $\lambda_0$. Therefore, dispersion theories of weak beams in infinitely long systems are not likely to be accurate near threshold.

During the transition region, when electron oscillations first begin to play a role in the laboratory beam-plasma interaction, estimates of the diffusion rate indicate that axial diffusion is dominant and occurs within the cross-section of the beam. Since the electron oscillations are low amplitude, it is possible that the the magnetic field is sufficient to ensure that the waves are also axial. Therefore, the one-dimensional simulation may most accurately model the laboratory interaction when it is in the transition region, except that azimuthal ion oscillations cannot be included.

When BPD is reached, either by increasing the neutral pressure, or by increasing $I_b$, the wave amplitude is large, and so the velocity of the beam electrons is likely to vary radially. This may mean that the beam is not trapped to the same extent by the potential troughs of the waves because electrons move radially. The velocity distribution of the beam electrons on a plane perpendicular to the magnetic field may be broad and not remain as narrow as indicated by the simulations. This will cause a reduction in the wave amplitude in the laboratory experiments, and probably mean that the beam is never momentarily reflected at the collection boundary.

To simulate a thin beam-plasma discharge, a two or three-dimensional model is required, although a one-dimensional model will certainly improve the understanding of the BPD. Simulation results from the already enabled the development of ionization models, and provided a means of studying the balance between ionization and diffusion. The ability of the PIC technique to simulate a sheath will be of great use in the study of many other collective plasma processes in bounded systems.
Appendix

It can be shown that the heating rate of a stochastic process increases linearly with time and be proportional to the square magnitude of the random field fluctuations, \( E_r^2 \), by examining the change in the average kinetic energy of an electron.

\[
\Delta K.E. = \frac{1}{2}m(v + \Delta v)^2 - \frac{1}{2}mv^2,
= \frac{1}{2}m(\Delta v)^2 + mv\Delta v.
\]

Thus

\[
\langle \Delta K.E. \rangle = \frac{1}{2}m\langle (\Delta v)^2 \rangle.
\]

Now

\[
\Delta v = v(t) - v(0) = \int_0^t \frac{eE(t_1)}{m} dt_1,
\]

and so

\[
\langle \Delta v \rangle^2 = \int_0^t \frac{eE(t_1)}{m} dt_1 \int_0^t \frac{eE(t_2)}{m} dt_2,
= \frac{e^2}{m^2} \int_0^t dt_1 \int_0^t E(t_2 - \tau)E(t_2)dt_2, \quad \text{where } \tau = t_2 - t_1,
= \frac{e^2}{m^2} \int_0^t dt_1 \langle E_r^2 \rangle \tau_{corr}, \quad \text{where } \tau_{corr} = \text{correlation time},
= \frac{e^2}{m^2} \langle E_r^2 \rangle \tau_{corr}.
\]

Therefore

\[
\langle \Delta K.E. \rangle = \frac{1}{2} \frac{e^2}{m} \langle E_r^2 \rangle \tau_{corr}.
\]
Bibliography


