FULLY NON-LINEAR PARABOLIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

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The work described in this thesis is my own work except where otherwise indicated.
THESIS ABSTRACT

I study the classical solvability of the first boundary value problem for fully nonlinear parabolic equations of second order in general for the nondegenerate case, and for a particular class of degenerate problems. As tools for this study, I also prove results for nondegenerate linear and quasilinear equations.

The central result required for the development of the necessary estimates is a weak Harnack inequality, which I prove in Chapter 1. This weak Harnack inequality is also of independent interest, since it is applicable to linear and quasilinear equations in general form. In the process of proving this estimate, I also establish a parabolic analogue of the Aleksandrov-Bakelman maximum principle.

Classical solvability of uniformly parabolic equations under "natural" structure conditions is proved in the second chapter. I allow the problems to be posed on non-cylindrical as well as cylindrical domains. This allows more general statements than usual, though the cases of most interest are probably those on cylindrical domains. The non-cylindrical domain problems are perhaps better thought of as degenerate elliptic problems. To this end, I have included in Chapter 2 a proof of the boundary regularity theorem of Krylov adapted to these domains.

The final two chapters are devoted to some model equations. The third chapter deals with a natural extension of a problem considered by Caffarelli, Nirenberg and Spruck, the solvability of equations involving symmetric functions of the eigenvalues of the Hessian matrix. The equations I consider are evolution equations which are related in an obvious way to these problems. I prove that the parabolic problems are solvable under essentially the same conditions as the elliptic ones.
The final chapter considers the parabolic Monge–Ampere type equation,

\[-\frac{\partial u}{\partial t} \det[\partial^2 u/\partial x_i \partial x_j] = f(x, t, u, \nabla u)\]

I show that the methods of the elliptic theory are applicable to the parabolic problem, and conclude the solvability of a wide range of such equations.
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INTRODUCTION

In recent years, the theory of fully non-linear elliptic equations has been extensively studied by many authors, with results on classical solvability established for a large class of problems. Notable works in the general theory include Evans [E 1], Krylov [Kr 2], [Kr 3], and Trudinger [Tr 2]. The theory of the corresponding parabolic case has not been quite so thoroughly examined, though results have been proved by Krylov [Kr 2]. The aim of this thesis is to extend the results known for the elliptic theory to the parabolic setting.

The central result required for the development of the necessary Hölder estimates is a weak Harnack inequality, which I prove in Chapter 1. The results of Krylov for fully non-linear equations do not directly use such an inequality, and consequently his proof appears cumbersome. This weak Harnack inequality is also of independent interest, since it is applicable to linear and quasilinear equations in general form. In the process of proving this estimate, I also establish a parabolic analogue of the Aleksandrov-Bakelman maximum principle. This theorem has been independently proved by Tso [Ts 1] by a different method. My proof is in some sense more elementary, though perhaps also less elegant.

Classical solvability of uniformly parabolic equations under "natural" structure conditions is proved in the second chapter. Here I follow methods used by Trudinger in [Tr 2] and also to be found in [G.T. 1]. These results extend those of Krylov [Kr 2] with regard to the growths allowed, though Krylov has announced further results in this context [Kr 4]. Also I allow the problems to be posed on non-cylindrical domains. This allows more general statements than usual, though the cases of most interest are probably those on cylindrical domains. The non-cylindrical domain problems are perhaps better thought of as degenerate elliptic
problems. To this end, I have included in Chapter 2 a proof of the boundary regularity theorem of Krylov [Kr 3] adapted to these domains.

The final two chapters are devoted to some model equations. The third chapter deals with a natural parabolic extension of a problem considered by Caffarelli, Nirenberg and Spruck [C.N.S. 2], the solvability of non-uniformly elliptic equations involving the eigenvalues of the Hessian matrix. These equations approximate the equations of prescribed curvature of various types in the same sense that Poisson's equation approximates the equation of prescribed mean curvature. The equations I consider are evolution equations which are related in an obvious way to these problems. I prove that the parabolic problems are solvable under essentially the same conditions as the elliptic ones.

The final chapter briefly considers the parabolic Monge-Ampere type equation. Such equations were first considered by Krylov in [Kr 1], and a geometric problem closely related to such equations was considered by Tso in [Ts 2]. I show that the methods of the elliptic theory are applicable to the parabolic problem, and conclude the solvability of a wide range of such equations.

0.1 NOTATION AND DEFINITIONS

Throughout this thesis, I shall use the conventions described in this section. I denote

\[ \Omega \quad \text{a domain in } \mathbb{R}^{n+1} \]

\[ \Omega_t = \{ x \mid (x, t) \in \Omega \} \]
where $\psi$ is a representation of the boundary at $(x, t)$; i.e. locally $\partial \Omega$ is given by

$$\partial \Omega = \{ (y', \psi(y', s), s) | (y', s) \in \tilde{N}(x', t) \}$$

$$D_i u = \frac{\partial u}{\partial x_i} ; D_t u = \frac{\partial u}{\partial t} ; D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

$$Du = (D_1 u, D_2 u, \ldots, D_n u) = \text{grad } u$$

Operators will be printed in boldface;

$$Lu = a^i j D_{ij} u + b^i D_i u + cu - D_t u$$

$$L_0 u = a^i j D_{ij} u + b^i D_i u - D_t u$$

$$Q[u] = A^i j (x, t, u, Du) D_{ij} u + B(x, t, u, Du) - D_t u$$

$$D[u] = \det A$$

$\lambda$ - the minimum eigenvalue of $A$ (or $a$)

$\Lambda$ - the maximum eigenvalue of $A$ (or $a$)
The function spaces used in this work shall be denoted as follows:

- $C^{k,m}$ functions with $k$ continuous derivatives in the space co-ordinates and $m$ continuous time derivatives

- $C^{k,\alpha;m,\beta}$ those functions in $C^{k,m}$ with the relevant derivatives in the Hölder spaces $C^{\alpha}$ and $C^{\beta}$ respectively

- $W^{k,m}_r$ the Sobolev space of functions with $k$ space derivatives and $m$ time derivatives in $L^r$
I. LOCAL ESTIMATES FOR LINEAR AND QUASILINEAR EQUATIONS

In order to prove estimates for nonlinear equations, it is necessary to have estimates for solutions of linear and quasilinear equations in general form, so that these may be applied to the differentiated equations. Such estimates are the purpose of this chapter. Also, the theorems of this chapter are of some independent interest. The estimates to be proved are of three types; a maximum principle similar to the Aleksandrov-Bakelman Maximum Principle for subsolutions of linear equations, a local maximum principle similar to those of Trudinger [Tr 1] (in the elliptic quasilinear case) and Gruber [Gr 1] (in the linear parabolic case) for subsolutions of quasilinear equations, and a weak Harnack inequality similar again to those of Trudinger [Tr 1] and Gruber [Gr 1] for supersolutions of quasilinear equations.

The central ideas in the proofs of these theorems come from papers of Krylov [Kr 1], Krylov and Safonov [K.S. 1], and Trudinger [Tr 1]. The first theorem, the maximum principle, has also been proved independently, by another method, by Tso [Ts 1]. The local maximum principle and weak Harnack inequality were both proved for the linear case by Gruber using stochastic methods, and for the elliptic case by Trudinger. I do not use any probability theory in this thesis; my methods follow closely those of Trudinger, and those of the important paper of Krylov and Safonov.

1.1 A MAXIMUM PRINCIPLE

The maximum principle to be proved here is a slight extension of that proved by Krylov in [Kr 1], and the proof uses a useful calculation from that work. The improvement lies in the estimate depending only on
the restriction of the inhomogeneous term to the upper contact set (in the case of smooth subsolutions). This refinement is necessary for the proof of the local maximum principle of the next section. It also allows one to easily conclude estimates for operators with lower order terms in $L^{n+1}$.

Let $u \in C^0(\Omega_t)$. Then one may define the upper contact set of $u$ by

$$ \Gamma^+ = \Gamma = \{ y \in \Omega_t \mid u(x) \leq u(y) + p.(x-y) \ \forall x \in \Omega_t, \text{ some } \ p = p(y) \in \mathbb{R}^n \} $$

Further, one may define the concave hull of $u$ to be the smallest concave function on $\Omega_t$ lying above $u$, $\psi_u$. Then it is easy to see that

$$ \Gamma^+ = \{ x \in \Omega_t \mid u(x) = \psi_u(x) \}. $$

It is also clear that if $u \in C^2(\Omega_t)$, then $p(y)$ in the definition of $\Gamma^+$ must be taken to be $Du(y)$, while $D^2u(y)$ is negative on $\Gamma^+$. For $u \in C^2.1(\Omega)$, $\Gamma^+$ will denote the upper contact set with respect to $x$; i.e.

$$ \Gamma^+ = \{(x,t) \in \Omega \mid Du(x,t) \in \Gamma^+ u(.,t)\} $$

and $I$ will denote the increasing set of $u$,

$$ I = \{(x,t) \in \Omega \mid Dt u(x,t) > 0\} $$

and we will denote by $E$ the set
The main theorem of this section is

**Theorem 1.1.** Suppose that $b_i = c = 0$, $u \in C^2(\bar{\Omega})$, $\Omega = B_R(0)$, $u|_{\partial \Omega} = 0$. Then

$$\sup_{\Omega} u \leq C(n) R^{n/2} u_{x_1}^{1/2} (L u)^{-1/2} + \| L^{n+1} (E_u)\|_{L^{n+1}}$$

**Proof.** The above estimate is proved by considering the increasing concave hull of $u$, and applying a simple but crucial calculation of Krylov for convex decreasing smooth functions. The increasing concave hull of $u$, $\zeta_u$, is defined to be the smallest function on $\Omega$ which is concave with respect to $x$, nondecreasing with respect to $t$, and which lies above $u$. $\zeta_u$ has the following important properties:

1. $\zeta_u \in W^{2,1}(\Omega)$
2. $D_t \zeta_u \cdot \det D^2 \zeta_u = \begin{cases} D_t u \det D^2 u & \text{a.e. on } E \\ 0 & \text{elsewhere} \end{cases}$

To establish the first of these properties, first observe that $\zeta_u(x, t)$ is given by

$$\zeta_u(x, t) = \psi(x, t)(x) ; u(x, t) = \sup_{s \leq t} u(x, s)$$
It is clear that \( v \), and hence \( t_u \), is Lipschitz with respect to \( t \), with Lipschitz constant bounded by \( \| D_t u \|_\infty \). To obtain the second derivative bound, note that for each \((x_0, t_0)\), there exists a parabola

\[
\theta(x_0, t_0, x) = u(x_0, t_0) - c(x_0, t_0)\|x - x_0\|^2 + Du(x_0, t_0) \cdot (x - x_0)
\]

with \( c \|D^2 u\|_\infty \), such that \( u(x, t_0) \geq \theta(x_0, t_0, x) \) for all \( x \in \Omega_{t_0} \). Then since the sup in the definition of \( v \) must be attained, it follows that \( v \) has the same property, with \( Du(x_0, t_0) \) replaced by some vector bounded by \( \|Du\|_\infty \). Taking the concave hull in \( x \), for fixed \( t_0 \), since the convex hull of a set in \( \mathbb{R}^n \) may be represented as the collection of convex combinations of at most \( n+1 \) points of the original set, it follows that for any \( y \in \Omega_t \) there exist \( y_1, \ldots, y_{n+1} \in \Omega_{t_0} \) and \( \alpha_1, \ldots, \alpha_{n+1} \in (0,1) \) with \( \sum \alpha_i = 1 \), such that \( y = \sum \alpha_i y_i \), and

\[
\Psi_u(. , t)(y, t_0) = \sum \alpha_i \cdot u(y_i, t_0)
\]

Then if \( \theta_i \) is the parabola lying under \( u(y_i, t_0) \), with

\[
\theta_i(y) = u(y_i, t_0)
\]

it follows that \( \sum \alpha_i \theta_i(x) \) is a parabola with the same bound on its second derivatives, lying under \( \Psi_u(. , t_0) \) and equal to it at \( y \). This establishes the second derivative bound from below, while the second derivative bound from above follows immediately from concavity.
To prove the second property, consider for a given function $v$ in $x$, the function $\psi = \psi_v$. If for some point $x_0$, $v(x_0) \neq \psi(x_0)$, then it follows that there exist $\alpha_1, ..., \alpha_{n+1} \in (0,1)$, and $x_1, ..., x_{n+1} \in \mathbb{R}^n$, such that $\sum \alpha_i = 1$, $x_0 = \sum \alpha_i x_i$, and $\psi(x_i) = v(x_i)$, $\psi(x_0) = \sum \alpha_i v(x_i)$. Now set $x = (1-\alpha_1)^{-1} \sum_{i=2}^{n+1} \alpha_i x_i$. It is clear that $\psi(x) = (1-\alpha_1)^{-1} \sum_{i=2}^{n+1} \alpha_i v(x_i)$, and so $\psi(x_0) = \alpha_1 \psi(x_1) + (1-\alpha_1)\psi(x)$.

Then by concavity, $\psi$ is an affine function on the line segment joining $x_1$ and $x$, and so in particular the second derivative at $x_0$ in the direction of this line is zero; i.e. there exists $\xi \in \mathbb{R}^n$ with $D\xi \psi_v(x) = 0$. Since $\psi_v$ is concave, this implies that the maximum eigenvalue of $D^2\psi_v(x)$ is zero, and so $\det D^2\psi(x) = 0$.

The foregoing shows that if $\xi_u(x, t) = v(x, t)$, then det $D^2\xi_u(x, t) = 0$. Now suppose $\xi_u(x, t) = v(x, t) \neq u(x, t)$. Then for some $s < t$, $u(x, s) = v(x, s) = v(x, t)$, and so since $\xi_u$ is increasing, $\xi_u(x, s) = \xi_u(x, t)$, and hence $D_t \xi_u(x, t) = 0$. It remains to check 1.2) at those points where $\xi_u(x, t) = u(x, t)$, but at almost all such points $D_t \xi_u = D_t u$ and $D^2 \xi_u = D^2 u$, by virtue of the following fact:

**Proposition** Suppose $f$ is a Lipschitz function. Then $|Df| = 0$ almost everywhere on the set $\{ f = 0 \}$.
This proposition is easily proved by considering the positive and negative parts of \( f \). To apply it to the problem in hand, \( D_t \zeta_u = D_t u \) is obtained directly by applying the lemma to \( u - \zeta_u \), while the other equation is obtained by first differentiating \( u - \zeta_u \) in any given direction, observing that the resulting function vanishes on \( E \), and then applying the lemma. Thus 1.2) is proved.

In order to use the function \( \zeta_u \) to conclude the statement of the theorem, the following two lemmas are required. Both were proved by Krylov in [Kr 1], but for completeness the proofs are repeated here.

**Lemma 1.1** Suppose \( w \) is a smooth function on \( \Omega = B_R(0) \times (0, T) \) with \( w \mid_{\partial \Omega} = 0 \). Then

\[
\int_{\Omega} D_t w \det D^2 w = \frac{1}{n+1} \int_{B_R(0)} w \det D^2 w \bigg|_{t=T}
\]

**Lemma 1.2** Suppose \( w_1 \) and \( w_2 \) are Lipschitz convex functions on the \( n \)-dimensional ball \( B = B_R(0) \), \( w_1 \mid_{\partial B} = w_2 \mid_{\partial B} = 0 \), \( w_1 < w_2 \) on \( B \). Then

\[
\int_{B_R(0)} w_1 \det D^2 w_1 > \int_{B_R(0)} w_2 \det D^2 w_2
\]

**Proof of lemmas 1.1 and 1.2.** Integrating the left hand side of 1.3) with respect to \( t \) yields
\[ \int_{\Omega} D_t w \det D^2 w = \int_{B(0)} w \det D^2 w \bigg|_{t=T} - \int_{\Omega} w \, D_t \det D^2 w \]

\[ = \int_{B(0)} w \det D^2 w \bigg|_{t=T} - \sum_{i,j=1}^{n} \int_{\Omega} w \, D^3_{t x_i x_j} a/aw_{x_i x_j} \det D^2 w \]

Now for all \( i \),

\[ \sum_{j=1}^{n} D_{x_j} (a(ax_{x_i} x_j \det D^2 w) = 0 \]

and so the second term on the right may be integrated twice by parts with respect to \( x_i \) and \( x_j \) to give

\[ \int_{\Omega} D_t w \det D^2 w = \int_{B} w \det D^2 w \bigg|_{t=T} - \sum_{i,j=1}^{n} \int_{\Omega} D_t w \, D_{ij} w \, a/aw_{x_i x_j} \det D^2 w \]

By Euler's theorem on homogeneous functions, the second term on the right is equal to \( n \) times the term on the left, and 1.3) follows.

Turning to 1.4), it can be seen that 1.3) may also be written as

\[ 1.5) \hspace{1cm} D_t \int_{B} w \det D^2 w = (n+1) \int_{B} D_t w \det D^2 w \]

Now if \( w_1 \) and \( w_2 \) are smooth, one may form

\[ w(x,t) = tw_1(x) + (1-t)w_2(x) \]
and put this in 1.5) to conclude 1.4). To extend 1.4) to convex Lipschitz functions, it suffices to observe that a convex function \( w \) may be uniformly approximated by smooth convex functions \( w_c \) (see for example [A]). Interpreting \( \det D^2w \) as the Jacobian of the normal mapping of the graph of \( w \), one may conclude that the corresponding measures \( \det D^2w_c \) converge weakly to \( \det D^2w \), and so Lemma 1.2 is proved.

Note that Lemma 1.2 yields an estimate for convex functions which vanish on the boundary of \( B_R(0) \) of the following form

1.6) \((-w(x_0))^{n+1} + nR(R^2-\|x_0\|^2)-\delta(n+1) \leq \int_{B_R} w \det D^2w \)

To see this, simply take \( w = w_1 \), and \( w_2 \) the function whose graph is the cone with apex \((x_0, w(x_0))\) and base \( B_R(0) \) in Lemma 1.2. Returning then to \( \xi_u \) from 1.1) it is clear that \( \xi_u \) may be approximated by smooth functions in \( W^{n+1}_p(\Omega) \) which vanish on the parabolic boundary of \( \Omega \), and so Lemma 1.1 yields

\[
\int_{\Omega} D_t \xi_u \det D^2\xi_u = (n+1)^{-1} \int_{B_R(0)} \xi_u \det D^2\xi_u \bigg|_{t=T}.
\]

Then the left hand side may be replaced using 1.2), while the right hand side may be estimated by 1.6), to obtain

1.7) \( u(x_0, T)^{n+1} + nR(R^2-\|x_0\|^2)-\delta(n+1) \leq (n+1) \int_{E_u} D_t u \cdot \det(-D^2u) \)

To obtain the statement of the theorem, apply the matrix inequality
\[ \det A \cdot \det B \leq (n^{-1}\text{trace } AB)^n, \quad A, B \geq 0, \ A, B \in \mathbb{R}^{n \times n} \]

to the matrices
\[ A = \begin{bmatrix} 0 & u \\ 0 & -2u \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & a_{ij} \end{bmatrix} \]
on \( E \) to give
\[ D_t u \det(-D^2 u) \leq D^{-1}(n+1)-1(n+1)(Lu)^{n+1}. \]

Using this in 1.7) concludes the proof of the theorem. \( \square \)

The following are now easy corollaries of Theorem 1.1:

**COROLLARY 1.1** Suppose \( |b| < N_1; \ -c < N_2; \ d(\Omega) \leq R; \)
\( u \in \mathcal{C}^{2,1}(\bar{\Omega}), \) with \( u|_{\partial \Omega} \leq 0. \) Then
\[ \sup u \leq C(n)\exp[N_1T/(R+N_2)] R^n/n! D^{-1/n+1}(Lu)^{-1} \| L^{n+1}(E_u) \]

**COROLLARY 1.2** Suppose \( \| D^{-1/n+1}b \|_{L^{n+1}} \leq N_2, \ c^- < N_1, \ d(\Omega) \leq R \)
\( u \in \mathcal{C}^{2,1}(\bar{\Omega}) \) with \( u|_{\partial \Omega} \leq 0. \) Then
\[ \sup u \leq C(n, N_2, R) \cdot \exp[N_1T] \cdot \| D^{-1/n+1}(Lu)^{-1} \|_{L^{n+1}(E_u)} \]
and so the first corollary is proved.

**COROLLARY 1.3** The conclusion of the above corollaries hold for
\( u \in W^{2+1}_n(\Omega) \) if \( E_u \) is replaced by \( \Omega. \)
Proof: First note that in the proof of Theorem 1.1, if \( u \in C^{2,1}(\bar{\Omega}) \) for some \( \Omega \subset B_R(0) \) with \( u < 0 \) on \( \partial \Omega \), then \( u \) may be extended to \( \tilde{u} \in C^{2,1}(B_R) \) so that \( u < 0 \) off \( \Omega \), \( \tilde{u}|_{\partial \Omega} = 0 \), and then \( E_u = E_\tilde{u} \).

Thus \( \Omega \) may be replaced by any cylinder containing \( \Omega \).

Set \( v = u \cdot \exp \left[-(R^{-1}N_1 + N_2)t\right] \). Applying Theorem 1.1 to \( v \) on \( B_{2R}(0) \):

\[
\sup_{B_{2R}} v \leq C(n)R^{-n/n+1}\|D^{-1/n+1}(L_0v)\|_{L^{n+1}}(E_v)
\]

\[
\leq C(n)R^{-n/n+1}\|D^{-1/n+1}u^{-1}v(L_0v)\|_{L^{n+1}}(E_v) + \|D^{-1/n+1}[b_1D_1v + c^{-1}u - (R^{-1}N_1+N_2)v]\|_{L^{n+1}}(E_v)
\]

Now on \( \Gamma^+ \) (observing that \( E_v \subset E_u \)) by concavity,

\[
\|Du\| \leq d(x,\partial B_{2R})^{-1}u \leq R^{-1}u.
\]

Thus

\[
\sup_{B_{2R}} v \leq C(n)R^{-n/n+1}\|D^{-1/n+1}(L^+v)\|_{L^{n+1}}(E_v)
\]

and so the first corollary is proved.

For the second corollary, set \( v = e^{-N}t_0u \), and apply theorem 1.1 on \( B_{kR}(0) \) to get
Choosing $K$ sufficiently large, depending on $R$, $n$, and $B$, so that the second term is bounded by $\frac{1}{2}\sup u$, the second corollary is proved.

The third corollary follows by standard approximation methods. I should point out that Corollary 1.1, with $C^{2,1}$ replaced by $W^{2,1}_{n+1}$ and $E_u$ replaced by $\Omega$, was proved by Krylov [Kr 1].

1.2 A LOCAL MAXIMUM PRINCIPLE

Here I prove a parabolic version of theorem 6 of [Tr 1]. For the linear case, this is theorem 3.1 of [Gr 1]. The method is entirely analogous to that employed in [Tr 1]. Suppose $Q$ satisfies the following structure conditions:

\begin{align}
\Lambda &\lesssim (a_0\|p\|^N + a)D^{1/n+1} \\
B &\lesssim (b_0\|p\|^{N+1} + b\|p\| + cz + g)D^{1/n+1}
\end{align}

for $(x,t), z, p \in \Omega \times (0, M) \times \mathbb{R}^n$, where $M$ and $N$ are non-negative.
constants, $a_0$, $a$, $b_0$, $b$, $c$, and $g$ are non-negative functions on $\Omega$ with $a_0$, $a \in L^r(\Omega)$, $b_0$, $b$, $c \in L^q(\Omega)$, and $g \in L^{n+1}(\Omega)$ for some $r \geq q > n+1$. Define $r^*$ and $q^*$ by

$$\frac{1}{r} + \frac{1}{r^*} = \frac{1}{n+1}; \quad \frac{1}{q} + \frac{1}{q^*} = \frac{1}{n+1}$$

Finally, suppose

$$D^{-1/n+1} \in L^q(\Omega)$$

This last condition is not required in the elliptic case, but it is easily seen that some such condition is necessary in the parabolic case. The local maximum principle is the following:

**THEOREM 1.2** Let $u \in W^{1,2}_r(\Omega)$ satisfy $Qu \geq 0$, $u < M$ in $\Omega$, and let $B = B_R$. Then for any $p > 0$, $0 < \delta < 1$,

$$\sup_{B_{\delta R}} u \leq C ([|B|^{-1} \int_B (u^+)^p]^{1/p} + R^{n/(n+1)} \|g\|_{L^{n+1}(\Omega)})$$

where $C$ depends on $n$, $p$, $q$, $N$, $\delta$, $R^{-(n+1)/q}$, and $\|h_R\|_{L^q(\Omega)}$, and $h_R$ is given by

$$h_R = (M/R)^N(a_0 + Rb_0) + a + D^{-1/n+1} + R^2c$$

**Proof:** Suppose $R = 1$; the full result will follow by a routine scaling argument. Define
where $\beta$ is to be chosen later, and for $u$ smooth, set $v = \eta u$ on $B$. On the upper contact set $\Gamma_v$ of $v$ in $B$, $u > 0$ and

$$\|Du\| = \eta^{-1}\|Du - uD\eta\| \leq 2(1+\beta)t^{\frac{\beta}{2}\eta^{-1/\beta}}u.$$ 

Thus on $\Gamma_v$:

\[
Lu = A^i_j(x,t,u, Du)D_i j u - D_t u + \eta L u + 2 A^i_j \eta D_i \eta D_j u + uL\eta
\]

\[
\geq -\eta B - 2\Lambda |D\eta| |Du| - \beta u (2t^{\frac{\beta}{2}\eta^{-1/\beta}} + \frac{\eta^{-1/2}}{\beta})
\]

\[
\geq -\eta B - C(1+\Lambda)\eta^{1-2/\beta}u
\]

where $C$ depends on $\beta$. Applying the structure conditions 1.8) gives on $\Gamma_v$:

\[
D^{-1/n+1} \eta B \leq b_0 \eta |Du|^{N+1} + b_\eta |Du| + c\eta u + \eta g
\]

\[
\leq 2(1+\beta)[b_0(2(1+\beta)u)^{N_0} - N/\beta t N/2 + b].t^{\frac{\beta}{2}\eta^{-1/\beta}}u +
\]

\[
+ cu + \eta g
\]

and

\[
(1+\Lambda)D^{-1/n+1} \leq a_0 |Du|^{N} + a + D^{-1/n+1}
\]

\[
\leq a_0(2(1+\beta)u)^N \eta - N/\beta t N/2 + a + D^{-1/n+1}
\]

so
where \( \alpha = (N+2)/\beta \), \( C = C(n, \beta, N) \) and

\[
h^* = (a_0 + b_0)(u^+)^N + a + \frac{1}{n+1} + b + c \leq h_1
\]

Applying the maximum principle Theorem 1.1, this shows that

\[
\sup u \leq C h_1 \eta^{-\alpha} u^+ + g\|L^{n+1}(B)
\]

Choosing \( \beta = (N+2)q^*/p \) and \( \alpha = p/q^* < 1 \), and applying Young’s inequality to \( \sup (u^+) \leq \alpha \), the result follows for \( R = 1 \) and \( u \) smooth. The full result now is obtained by scaling and approximation. The approximation argument is identical to that in [Tr 1], and so is omitted here.

1.3. A WEAK HARNACK INEQUALITY FOR LINEAR OPERATORS

In this section, I shall work with cubes rather than balls. The only
reason for this is that the "crawling inkspots" lemma of Krylov and
Safonov, Lemma 3.1 of [K.S. 1] is stated in such terms. Define

\[ Q(\theta, R) = \{ |x| < R \} \times (-\theta R^2, 0) \]

\[ Q^+ = Q^+(\theta_1, \varepsilon_1, R) = \{ |x| < \varepsilon_1 R \} \times (-\theta_1 R^2, 0) \]

\[ Q^- = Q^-(\theta_2, \varepsilon_2, R) = \{ |x| < \varepsilon_2 R \} \times (-\theta R^2, -\theta_2 R^2) \]

where \( 0 < \varepsilon_1, \varepsilon_2 < 1 \), \( 0 < \theta_1 < \theta_2 < \theta \). The weak Harnack
inequality states that some \( L^p \) "norm" (\( p \) is less than one) on \( Q^- \) of a
positive supersolution \( u \) may be estimated in terms of the infimum of \( u \)
on \( Q^+ \). As a corollary, one gets the usual Harnack inequality for solutions
of parabolic equations. Krylov and Safonov proved the Harnack inequality
for solutions of homogeneous equations without directly obtaining the
weak Harnack. Gruber improved their result to include the theorem to be
proved in this section. I use primarily the method of [K.S. 1], but am able to
avoid the function \( \gamma(\beta) \) considered there by using Theorem J.2.

**THEOREM J.3** Suppose \( L \) is uniformly parabolic; \( |b|, c < N \);
\( u \in W^{2,1}_{n+1}(QR) \); \( u \geq 0 \); \( Lu \geq f \); \( f \in L^{n+1}(QR) \), \( 0 < R < R_0 \).

Then there exist positive constants \( C \) and \( p \) depending only on \( n, \lambda, \Lambda, N, \theta_1, \theta_2, \varepsilon_1, \varepsilon_2 \), and \( R_0 \) such that

\[ \left( |Q^-|^{-1} \int_{Q^-} u^p \right)^{1/p} \leq C \left( \inf_{Q^+} u + R^{n/n+1} \| f \|_{L^{n+1}(QR)} \right) \]

**Proof:** Without loss of generality, assume \( c = 0 \) (this is possible
since one may consider in place of the function \( u \) the function \( e^{-Nt} u \)).
Also suppose \( R = 2 \), \( \varepsilon_2 = 1/2 \), \( \theta_1 = 1 \), \( \theta_2 = 3/4 \) - the general case
will follow by scaling and covering.
The idea of the proof is to estimate the measure of the level sets of $u$ on $Q^-$ in terms of its infimum on $Q^+$. The full estimate then follows by standard theorems. The proof proceeds via three key lemmas. The first, due to Krylov and Safonov, is a comparison type lemma, which says that if $u$ is uniformly large on a cube of radius $\varepsilon$ at some $t$, then it is greater than a power of $\varepsilon$ on a cube of radius one at some larger value of $t$. This allows one to compare the infimum of $u$ on $Q^+$ and the size of cubes in $Q^-$ on which $u$ is uniformly large. The next lemma, proved using the local maximum principle of the last section, states that if $u$ is large on "most" of a cube, then it must be uniformly large on some smaller cube. Finally, the third lemma, again due to Krylov and Safonov, allows one to combine the above two results to conclude an estimate on the size of level sets in terms of the infimum of $u$ on $Q^+$, by covering such a level set with cubes with large intersection with the level set, and building a larger set on which $u$ must be uniformly large.

The first lemma presented here is in fact a slightly simpler version of the comparison lemma of [K.S. 1].

**Lemma 1.3** Suppose $u > 1$ on $\{\|x\| < \varepsilon R, \ t = 0\}$, $R < R_0$, $\varepsilon < \varepsilon_0$. Then there exists a constant $p$, depending only on $R_0$, $\varepsilon_0$, $n$, $\lambda$, $\Lambda$, and $\alpha$, such that for $\|x\| < 2^{-\frac{1}{2}}R$,

$$u(x, \alpha R^2) > \varepsilon^{p/4} - C\|f\|_{L^n+1}$$

where $C$ is the constant of Corollary 1.2.
Proof: Define a subsolution for $L$ as follows:

$$
\psi(x,t) = (\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2)^2 \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q}
$$

for $\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2 \geq 0$. It is clear that $\psi$ may be extended to be zero for larger $\|x\|$, and that then $\psi \in C^2,1(B_R(0,\alpha))$.

Calculating derivatives,

$$
D_i \psi = -4(\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} x_i
$$

$$
D_{ij} \psi = -4(\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} \delta_{ij} + 8 \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} x_i x_j
$$

$$
D_t \psi = (\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} \alpha^{-1}(1-\varepsilon^2).
$$

$$
\{ 2 - q(\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-1} \}
$$

Hence on $((\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) \geq 0)$,

$$
L \psi = \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} \{ 8 a_i j x_i x_j -

- (\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2)(2T + 2b_i x_i + \alpha^{-1}(1-\varepsilon^2)) +

+ \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-1} \alpha^{-1}(1-\varepsilon^2) \}

\geq \left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-q} \{ 8 \lambda \|x\|^2 - 2(2n+2NR_0) +

+ \alpha^{-1}(1-\varepsilon^2)(\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2) +

+ q \alpha^{-1}(1-\varepsilon^2)(\varepsilon^2 R^2 + \alpha^{-1}(1-\varepsilon^2)t - \|x\|^2)^2 \}

\left[ \alpha^{-1}(1-\varepsilon^2)t + \varepsilon^2 R^2 \right]^{-1} \}
Set $C_1 = 2(2n\Lambda + 2NR_0 + \alpha^{-1})$. Then for

$$\|x\|^2 > C_1(\theta \lambda + C_1)^{-1}(\epsilon^2 R^2 + \alpha^{-1}(1-\epsilon^2))$$

the above is positive. Now for

$$\|x\|^2 < C_1(\theta \lambda + C_1)^{-1}(\epsilon^2 R^2 + \alpha^{-1}(1-\epsilon^2))$$

one has

$$L \psi \geq \left[ \alpha^{-1}(1-\epsilon^2)t + \epsilon^2 R^2 \right] \psi \left( \frac{\epsilon^2 R^2 + \alpha^{-1}(1-\epsilon^2)t - \|x\|^2}{\alpha} \right) + \left[ \alpha^{-1}(1-\epsilon^2)t + \epsilon^2 R^2 \right] \psi \left( \frac{\epsilon^2 R^2 + \alpha^{-1}(1-\epsilon^2)t - \|x\|^2}{\alpha} \right)$$

and so choosing $q$ sufficiently large, depending on $C_1$, $\epsilon_0$, and $\alpha$, it follows that $\psi$ is a subsolution for $L$. Furthermore, at $t = 0$,

$$\epsilon^2(2-q)R^2(2-q)\psi(x,0) = \epsilon^2(2-q)R^2(2-q)(\epsilon^2 R^2 - \|x\|^2)^2 \epsilon^{-2q \alpha R^2}$$

while at $t = \alpha R^2$,

$$\epsilon^2(2-q)R^2(2-q)\psi(x,\alpha R^2) = \epsilon^2(2-q)R^2(2-q)(R^2 - \|x\|^2)^2 R^{-2q}$$

Then for $\|x\|^2 < \frac{1}{2} R^2$, 

and so choosing $q$ sufficiently large, depending on $C_1$, $\epsilon_0$, and $\alpha$, it follows that $\psi$ is a subsolution for $L$. Furthermore, at $t = 0$,
Now apply the maximum principle, Corollary 1.2, to the function
\[ u - \epsilon^{2(q-2)} R^{2(q-2)} \psi(x, \alpha R^2) \] to conclude the claim of the lemma with
\[ p = 2(q-2). \]

**Lemma 1.4** There exists a positive constant \( \xi < 1 \) such that if
\[ | \{ (x, t) \in Q(\theta, R) = Q \mid \bar{u} \equiv u + \| f \|_{L^{n+1}}(Q) > k \} | > \xi | Q | \]
then
\[ \inf_{Q_6} u \geq \frac{1}{2} k - C \| f \|_{L^{n+1}}(Q) \]

*Proof of Lemma 1.4:* Applying Theorem 1.2 to \( k - \bar{u} \) with \( p = 1 \) gives
\[ \sup_{Q_6} (k - \bar{u}) \leq C_1 \left[ \frac{1}{|Q|} \int_Q (k - \bar{u})^+ + Rn/n + 1 \| f \|_{L^{n+1}}(Q) \right] \]
which is equivalent to
\[ \inf_{Q_6} u \geq k - C_1 \left[ \frac{1}{|Q|} \int_{\bar{u} < k} k - \bar{u} \right] - \| f \|_{L^{n+1}}(Q) (C_1 Rn/n + 1 - 1) \]
and so
\[ \inf_{Q_6} u \geq k - C_1 k | (\bar{u} < k) | | Q |^{-1} - \| f \|_{L^{n+1}}(Q) C_1 Rn/n + 1 \]

Now choose \( \xi = 1 - (2C_1)^{-1} \) to prove the lemma.

To state and use the central lemma of Krylov and Safonov, the
following construction is required. For any cube $Q$ of the form

$$Q = (x_0, t_0) + Q(1, R) \subset Q(1, 1) = Q_1$$

define the sets

$$Q^1 = \{(t_0-3R^2, x_0) + Q(7/9, 3R)\} \cap Q_1$$
$$Q^2 = \{(t, x) | t_0+R^2 < t < t_0+R^2+4\eta^{-1}R^2, |x_0-x|<3R, |x|<1\}$$

where $\eta$ is some fixed constant. Note that $Q^1 \subset Q_1$, while $Q^2$ may extend beyond $\{t = 0\}$. Now for any set $K \subset Q_1$ and constant $0 < \xi < 1$, define

$$\tilde{G} = \tilde{G}(K, \xi) = \{Q = (t_0, x_0) + Q(1, R) \mid Q \subset Q_1; |QnK| \geq \xi |Q|\}$$

and set

$$\tilde{y}^i = \bigcup_{Q \in \tilde{G}} Q^i$$

for $i = 1, 2$. The next lemma states that the $\tilde{y}^i$ expand $K$ by at least some known factor.

**Lemma 1.5**

1) $|K\setminus \tilde{y}^1| = 0$ and if $|K| \leq \xi |Q_1|$ then

$$|K| \leq \xi |\tilde{y}^1|$$

2) $|\tilde{y}^1| \leq (1 + \eta)|\tilde{y}^2|$

These two statements are lemmas 2.1 and 2.3 in [K.S. 1], where a proof may be found. Notice in particular that if $\eta$ and $\xi$ are such that
Now choose $\xi \gg \xi$, where $\xi$ is the constant of Lemma 1.4 with $R = \frac{1}{2}$, and take

$$K = K_k = \{(x,t) \in Q \mid \bar{u} = u + \|f\|_{L^{n+1}(Q_1)} \geq k\}$$

Then from Lemma 1.4, for any $Q \in \mathcal{G}_k = \mathcal{G}(K_k, \xi)$

$$\inf_{Q_0} u \geq \frac{1}{2}k - \|f\|_{L^{n+1}(Q)}$$

Thus the above inequality holds for $u$ on

$$\|x-x_0\| < \frac{1}{2}, \quad t_0 + R^2 \geq t \geq t_0 + 3R^2/4$$

Applying Lemma 1.3, with $\alpha$ between $1/4$ and $1 + 4/\eta$, one obtains

$$u \geq (\frac{1}{2}k - \|f\|_{L^{n+1}(Q)}).C_n^p/4 - \|f\|_{L^{n+1}(Q_1)}$$

on $Q^1 \cap Q_1$. Here, $p$ depends on $n, \lambda, \Lambda, \eta$, while $C_n = n^{-\frac{1}{2}}/6$ is a constant which arises from changing from balls to cubes. Setting $C_1 = (C_n^p/4 + 1)C$, this can be rewritten as

$$u \geq C_n^p.\frac{k}{\beta} - C_1\|f\|_{L^{n+1}(Q^-)} \quad \text{on} \quad Q^2 \cap Q^-$$

Now suppose $\xi |Q_1| \gg |K_k| \gg \beta, \eta < \xi^{-\frac{1}{2}} - 1$, and set $k^1 = k.(C_1)^{-1} C_n^p \beta$, and let $0 < \theta_0 < 1$ be some fixed constant. Then
applying Lemma 1.5 shows that either

\[ |K_k| > \theta_0 \xi^{-k/\beta} \]

or

\[ (1 - \theta_0) \xi^{-k/\beta} \geq |\tilde{\nu}^2 \setminus Q^2| \]

Now by construction, \( Q^2 \subset \{ |x| < 1 \} \), and so b) can only hold if there exists some \( Q \in \tilde{G}_1 \) such that \( Q^2 \cap \{ t > (1 - \theta_0) \xi^{-k/\beta} \} \) is non-empty.

This is equivalent to the existence of \( Q \in \tilde{G}_1 \) such that

\[ Q = (x_0, t_0) + Q(1, R) \]

\[ t_0 + R^2 + 4R^2/\eta > (1 - \theta_0) \xi^{-k/\beta} ; t_0 < -R^2 \]

from which one immediately sees that also

\[ R^2 > (1 - \theta_0) \xi^{-k/\beta} \]

Now since \( \tilde{u} \geq k^1 \) on \( Q^1 \), one may conclude that at \( t = 0 \) there exists a cube of side length \( (1 - \theta_0) \xi^{-1/4} R^{k/2} \), and hence a ball, \( B^1 \) say, of the same radius, with \( \tilde{u} \geq k^1 \) on \( B^1 \). Now Lemma 1.3 a) may be applied again to conclude that on \( Q^+ \),

\[ \tilde{u} > C \left( 1/4 \cdot k^1 \rho \left( 1 - \theta_0 \right) q/2 \right) q/2 \xi^{-q/4} - C \| f \|_{L^{n+1}(Q(1, 2))} \]

where now \( \rho \) and \( q \) depend on \( \varepsilon_1 \) and \( (\theta_1 - \theta_2) \) from the definition of \( Q^+ \) and \( Q^- \). Consequently, in the case that b) holds, it has been shown that
\[ pq/2 < \bar{c}/C^1 \inf \bar{u} \]

where

\[ \bar{c} = (1+\epsilon)q(1-\theta_0)^{-q/2}q/2. \sqrt{q}.q/4 \]

The foregoing is summarized in the following lemma.

**Lemma 1.6** For any \( k \), setting \( k^1 = \gamma k \), where \( \gamma < 1 \) is determined by given data, one of the following is true:

1. \( |K_k| > \theta_0 \varepsilon^{-\frac{1}{2}}|K_k| \)
2. \( |K_k| < C^1(\inf \bar{u}/k)^q = C^\infty(\inf \bar{u}/k)^q \)
3. \( |K_k| > \varepsilon \)

where \( C^1 \) and \( q \) are determined by given data and \( \theta_0, \varepsilon > 0 \) from Lemma 1.4.

Now in the event that (iii) holds, one may immediately invoke Lemmas 1.3 and 1.4 to conclude that

\[ k < C_0 \inf \bar{u} \]

So choosing \( \theta_0 \) sufficiently close to 1, one obtains the desired bound through iteration in Lemma 1.6; namely that there exist constants \( C_0, C, \) and \( q \) depending only on \( n, \lambda, \Lambda, N \) and the parameters of the statement of Theorem 1.3, such that for \( k > C_0 \inf \bar{u} \)

\[ |K_k| < C(\inf \bar{u}/k)^q \]

It now remains simply to apply a lemma on the distribution function.
of a given function (see, for example, [G.T. 1], lemma 9.7) to conclude the desired estimate. As mentioned earlier, the full statement of the theorem now follows through a routine scaling and covering argument.

Combining Theorems 1.2 and 1.3 yields in the usual way the full Harnack inequality.

**COROLLARY 1.4** Let $Q^+$ and $Q^-$ be as in Theorem 1.3, and suppose $Lu = f \in L^{n+1}$. Then there exists a constant $C$ depending only on $n$, $\lambda$, $\Lambda$, and the parameters in that theorem such that if $u \in W^{2,1}_n(Q(\theta, R))$ then

$$
\sup_{Q^+} u \leq C \left( \inf_{Q^-} u + R^{n+1} \| f \|_{L^{n+1}}(Q(\theta, R)) \right)
$$

I would also like to observe that Hölder estimates may be obtained as in the divergence structure case in elliptic theory in [G.T. 1] chapter 8 with only routine changes.

## 1.4 A WEAK HARNACK INEQUALITY FOR QUASILINEAR OPERATORS

The theorem to be proved here is the natural analogue of Theorem 9 in [Tr 1]. As in that case, the following structure is assumed:

1.10) $B \geq -(b_0 \| p \|^2 + b \| p \| + cz + g)$

for all $(x, t, z, p) \in \Omega \times (0, M) \times \mathbb{R}^n$ where $M$ and $b_0$ are non-negative constants and $b \in L^2(n+1)$; $c, g \in L^{n+1}(\Omega)$. Also $Q$ is assumed to be uniformly parabolic.
For \( B_R \subset \Omega \) as in section 1, denote for \( \varepsilon < 1 \)

\[
B_{6R}^+ = \{ \| x \| < \varepsilon R, \varepsilon R^2 < t < 0 \}
\]

\[
B_{6R}^- = \{ \| x \| < \varepsilon R, -R^2 < t < (1-\varepsilon)R^2 \}
\]

For \( \varepsilon < \frac{1}{2} \), the weak Harnack inequality takes the form:

**Theorem 1.4** Let \( u \in W^{2,1}_0(\Omega) \) satisfy \( 0 < u < M, \mathcal{Q}[u] < 0 \) in \( \Omega \), and suppose \( B_R \subset \Omega \). Then for some \( p > 0 \) and \( C > 0 \),

\[
\left( \frac{1}{B_{6R}^-} \int_{B_{6R}^-} u^p \right)^{1/p} \leq C \left( \inf_{B_{6R}^+} u + R^{n+1} \| g \|_{L^{n+1}_0(\Omega)} \right)
\]

where \( p \) and \( C \) depend only on \( n, \lambda_0, \Lambda_0, \varepsilon, b_0, M \), and \( R, \| b \|^2 + \| c \|_{L^{n+1}_0(\Omega)} \).

This theorem is not stated in its most general form with respect to the two sets \( B_{6R}^+ \) and \( B_{6R}^- \), but the reader can easily see that they may be replaced by any cylinders \( B^+ \) and \( B^- \) with

\[
\sup \{ t \mid (x, t) \in B^- \} < \inf \{ t \mid (x, t) \in B^+ \}
\]

and such that \( B^+ \) and \( B^- \) are relatively compact in \( B_R \).

**Proof of Theorem 1.4:** The proof follows the lines of the elliptic case as in [Tr 1] and the linear parabolic case. In fact, a simple argument may be used to deduce this theorem directly from the linear case. However, the proof presented here differs in some interesting respects from the linear proof.
The non-linearity and the \( b \) term are handled by introducing the auxiliary function

\[
\omega = \int_u^1 (s+g_0)^{-1} \exp[-b_0 s] \, ds
\]

where \( g_0 = \|g\|_{L^{n+1}(\Omega)} \). Clearly \( \{w < 0\} = \{u > 1\} \). The idea of the proof is to obtain a bound on the measure in \( B_{\tilde{c}} \) of \( \{u > 1\} \) in terms of \( \inf u \). As in the linear case, the first step is to show that if \( u \) is large on most of a cylinder, then it is uniformly small (or rather, \( w \) is uniformly large) on some smaller cylinder.

**Lemma 1.7** There exists a constant \( \zeta < 1 \) such that if

\[
|\{x,t\} \in B_R \mid \bar{u} = u + g_0 \mid k \}| > \zeta |BR|
\]

then

\[
\inf_{B_{\bar{c}R}} u \geq \frac{1}{2}k - C g_0
\]

**Proof of Lemma 1.7**: Note that \( u = k - \bar{u} \) satisfies

\[
\begin{align*}
\partial_t \bar{u} + D_t u &= -A^{i,j}D_{ij}u + D_t u \\
A^{i,j}D_{ij}u - D_t u &= 0
\end{align*}
\]

Now follow the proof for \( \bar{u} \) on \( (x,t) \in B_{\bar{c}R} \), then \( \bar{u} \) satisfies...
where

\begin{align*}
A^i_j[v] &= A^i_j(x_1, t, k + g_0 - v, -Dv) \\
B[v] &= B(x, t, k + g_0 - v, -Dv)
\end{align*}

Thus Theorem 1.2 may be applied to \( v \) with \( N = 1, \ a_0 = 0, q = 2(n+1), \ r = \infty, \ p = 1, \) to obtain

\begin{align*}
\sup_{B_R} (k-\bar{u}) &\leq C_1 \left\{ |B_R|^{-1} \int_{\{k-\bar{u} \}} (k-\bar{u}) + R/n+1 g_0 \right\} \\
\iff \inf_{B_R} u &\geq k - C_1 \left\{ |B_R|^{-1} \int_{\{k-\bar{u} \}} (k-\bar{u}) \right\} - C_1 R/n+1 g_0 \\
\Rightarrow \inf_{B_R} u &\geq k - C_1 k \{ k < g_0 \} |B_R|^{-1} - C_1 R/n+1 g_0
\end{align*}

from which the result follows for \( \zeta = 1 - (2C_1)^{-1} \). Note that \( \zeta \) depends only on those quantities determining \( p \) and \( C \) in the statement of Theorem 1.4. Note also that

\begin{align*}
\inf_{B_{6R}} u &\geq \frac{1}{2} k - C g_0 \Rightarrow \inf_{B_{6R}} \bar{u} > (2C)^{-1} (1+g_0) \\
\Rightarrow \sup_{B_{6R}} u &< C
\end{align*}

Thus it has been shown that for some fixed \( \zeta \), and any \( B_R \)

\[ |\{ w < 0 \} \cap B_R | > \zeta |B_R| \Rightarrow w < C \quad \text{on} \ B_{pR} \]

Now I show that if \( w < C \) on \( \{ \|x\| < R, \ t = 0 \} \), then \( w < C' \) on \( \{ \|x\| < 6 R, \ t = 0 \} \), with \( C' \) depending on \( \theta \) and \( \alpha \) as well as \( C \).
Set
\[ \psi(x,t) = (1 + \alpha^{-1}(2\delta^2 - 1)t - \|x\|^2)^2[\alpha^{-1}(2\delta^2 - 1)t + 1]^{-q} \]
for \( 1 + \alpha^{-1}(2\delta^2 - 1)t - \|x\|^2 \geq 0 \), and extend \( \psi \) to be zero elsewhere.

This is the function used as a comparison function in section 3. Here it is used as a cut-off function. As shown in the last section, for \( q \) sufficiently large, depending on \( \lambda_0 \), \( \Lambda_0 \), \( \theta \), and \( \alpha \),

\[ L\psi = \sum_{i,j} a_{ij}(x,t,u,Du)D_{ij}\psi - D_t\psi \geq 0 \]

Thus with \( v = w\psi \),

\[ L\psi = \psi Lw + wL\psi + 2\sum_{i,j} a_{ij}D_i\psi D_j\psi \]

\[ \geq \psi(u+g_0)^{-1}\exp[-b_0u]\{B + (b_0 + (u+g_0)^{-1})a_{ij}D_iuD_ju\} + \]
\[ + 2\sum_{i,j} a_{ij}D_i\psi D_jw + wL\psi \]

\[ \geq -\lambda_0\psi\exp[-b_0u]\{(u+g_0)^{-1}b|Du| + c + \tilde{g}\} + wL\psi + \]
\[ + (u+g_0)^{-1}\exp[-b_0u]\{(u+g_0)^{-1}\psi a_{ij}D_iuD_ju + 2\sum_{i,j} a_{ij}D_i\psi D_ju\} \]

\[ \geq -\lambda_0\psi(b^2 + c + \tilde{g}) - 2\psi^{-1}\sum_{i,j} a_{ij}D_iD_j\psi + wL\psi \]

where \( \tilde{g} = g/g_0 \). Thus on \( \{ v > 0 \} \),
Applying the maximum principle, one obtains

$$\sup \{u(x, \alpha) \mid \|x\| < 2\theta\} \leq \sup u^+(x, 0) + C'$$

from which

$$\sup \{w(x, \alpha) \mid \|x\| < \theta\} \leq 2^{q\theta^{-4}} \left(\sup w(x, 0) + C'\right)$$

$$\leq C(\sup w(x, 0) + 1)$$

A simple scaling argument then gives the result. At this point it is convenient to switch from balls to cubes again. It is clear that by choosing \(\rho\) in Lemma 1.7 appropriately, one may conclude that for fixed \(\alpha > 0\), if

$$|Q_R \cap \{w < 0\}| > \zeta |Q_R|$$

then \(w < C'\) on

$$Q_R' = \{(x, t) \mid 0 < t < 4\alpha^{-1}R^2, \|x\| < 3R\}$$

Here \(\zeta\) and \(C'\) depend on the quantities in the statement of the theorem, and \(C'\) depends also on \(\alpha\). Now the measure theoretic lemma of Krylov and Safonov may be applied. Let

$$\tilde{\mathcal{G}} = \{(x_0, t_0) + Q_R \mid \mathcal{L}(x_0, t_0) + Q_R \cap \{w < 0\} \supset \xi |Q_R|, \ (x_0, t_0) + Q_R \subset Q_1\}$$
and for \( K = (x_0, t_0) + Q_R \in \tilde{G}, \) set

\[
K'' = (x_0, t_0) + Q'_R \cap \{ |x| < 1 \}
\]

Then by Lemma 1.6, if

\[
\bar{y} = U \{ K'' | K \in \tilde{G} \}
\]

then either \( \{ w < 0 \} \setminus \zeta, \) or \( \bar{y} \gg \zeta^{-1}(1+\alpha)^{-1}\{ w < 0 \}. \) Arguing as in the linear case, one may choose \( \alpha \) sufficiently small to conclude that one of the following holds:

i) \( \{ w < C' \} \cap Q_1 \gg \theta_0 \{ w < 0 \} \cap Q_1 \)

or

ii) There exists a ball \( B = x_0 + \{ \|x\| < R \} \subset \{ |x| < 1 \} \) such that \( R \gg \gamma \{ w > 0 \} \cap K_1 \setminus \frac{1}{2} \) for some fixed \( \gamma > 0, \) and \( w \ll C' \) on \( B \setminus \{ 0 \}. \)

In the latter case, the estimate 1.11 may be iterated to see that after \( s \) iterations:

\[
\sup \{ w(x, t) | t = \sum_{j=1}^{S} 2JR^2, \|x-x_0\| < 2^S R \} \ll sC'
\]

from which one may deduce

\[
\sup \{ w(x, 1) | |x| < 1 \} \ll -C' \log R
\]

Now using the fact that \( w \gg -e^{-b_0 M \log (u+g_0)}, \) this may be
exponentiated to give the estimate

\[ ii)' \quad \inf u(x,1) + g_0 \geq \inf_{\{u \geq 1\}} \|u\|^\varphi \cap Q_1 \]

where \( \varphi \) depends only on the data of the theorem. Thus either i) or ii)' holds. Now one may iterate this, replacing \( u \) by \( e^{-C'}u \), to finally obtain for any \( k > 0 \),

\[ \kappa^k \inf_{\{u > k\}} \|u\|^\varphi \leq \inf u(x,1) + g_0 \]

where \( \kappa, \varphi \) depend on the data of the theorem. From this one may conclude the statement of the theorem with \( B_{6R}^+ \) and \( B_{6R}^- \) replaced by \( (0,1) + Q_2 \) and \( Q_1 \). The full statement then follows by standard scaling and covering arguments.

By the usual arguments, Theorems 1.2 and 1.4 may be combined to yield the following Hölder estimate:

**Corollary 1.5** Suppose \( u \in W^{2,1}_n(\Omega) \) satisfies \( Q[u] = 0 \) in \( \Omega \), where \( Q \) is uniformly parabolic, with \( \|B\| \leq (b_0 \|p\|^2 + b_1 p + c |z| + g) \), and that \( u \leq M \) in \( \Omega \). Then if \( B_{6R} \subset \Omega \), for any \( \delta < 1 \),

\[ \text{osc}_{B_{6R}^+} u \leq C_6 \delta^\alpha (M + \|g\|_{L^{n+1}}(\Omega)) \]

where \( C \) and \( \alpha \) are positive constants depending only on \( \lambda_0, \Lambda_0, n, b_0, \text{diam } \Omega, \|b\|_{L^2(n+1)}(\Omega) \) and \( \|c\|_{L^{n+1}}(\Omega) \).
In this chapter, I use the results of the previous chapter to prove the classical solvability of fully nonlinear parabolic equations under the so-called "natural" structure conditions. These conditions extend those of the previous chapter to fully non-linear equations, and are the obvious analogue of those considered in the elliptic case by Trudinger in [Tr 2].

The equations to be considered are of the form:

\[ F[u] - D_t u = F(x, t, u, Du, D^2 u) - D_t u = 0 \]

where \( F \) is a real valued function on \( \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \) with \( S^n \) the \( n(n+1)/2 \) dimensional space of symmetric \( n \times n \) real matrices. The operator \( F - D_t \) will be said to be uniformly parabolic if there exist constants \( 0 < \lambda_0 < \Lambda_0 \) such that

\[ \lambda_0 \text{tr} \eta \leq F(x, t, z, p, r+\eta) - F(x, t, z, p, r) \leq \Lambda_0 \text{tr} \eta \]

for all positive matrices \( \eta \).

The principle results of the chapter are interior estimates for smooth solutions of 2.1) where \( F \) satisfies the following structure conditions:

2.3) \( F \) is a concave function of the \( r \)-variables

2.4) \( |F(x, t, z, p, 0)| \leq \mu_0 (1 + |p|^2) \) for all \( (x, t, z, p) \in U_\kappa \)
2.5) \( |p| \cdot |F_p|, |F_z|, |F_x| \leq \mu_1(1+|p|^2+|r|) \) for all 
\((x,t,z,p,r) \in U_K, r \in S^n, K \in \mathbb{R} \), where \( \mu_1 = \mu_1(K) \in \mathbb{R} \)

2.6) \( |F| \leq \mu_2(1+|r|) \) for all \((x,t,z,p) \in \bar{U}_K, r \in S^n, K \in \mathbb{R} \),
where \( \mu_2 = \mu_2(K) \)

2.7) \( |F_{px}|, |F_{rp}|, |F_{rz}| \leq \mu_3 ; \)
\( |F_{pp}|, |F_{px}|, |F_{pz}|, |F_{zz}|, |F_{zx}|, |F_{xx}| \leq \mu_3(1+|r|) \)
for all \((x,t,z,p) \in \bar{U}_K, r \in S^n, K \in \mathbb{R} \), where \( \mu_3 = \mu_3(K) \).

In the above conditions, \( U_K = \Omega \times (-K,K) \times \mathbb{R}^n \) and \( \bar{U}_K = \Omega \times (-K,K) \times \{ |p| < K \} \). These conditions extend those assumed by Krylov in [Kr 2]. Once the local estimates are proved, one may conclude the classical solvability of the first initial boundary condition problem for these equations by invoking the method of continuity in the manner of Trudinger in [Tr 2].

The conditions above are not all required for each estimate. Accordingly, each condition shall be invoked as it is required. The methods employed are generally the same as those of [Tr 2], with the weak Harnack inequality of the last chapter filling the role of the corresponding result of Trudinger [Tr 1].

2.1 GRADIENT ESTIMATES

In order to prove the gradient estimate, it is necessary to first prove the following Hölder estimate:
THEOREM 2.1 Let \( u \in C^3, 2(\Omega) \) satisfy \( F[u] - D_t u = 0 \) in \( \Omega \) with 2.2) and 2.4) holding. Then

2.8) \( \| u \|_{0, \alpha; \Omega} < C \)

where \( \alpha \) depends on \( n, \lambda_0(M_0), \Lambda_0(M_0), \) and \( \mu_0(M_0), \) and \( C \) depends in addition on \( \text{diam}(\Omega), \) with \( M_0 = |u|_{0, \Omega}. \)

Proof: First assume that \( F \) is differentiable with respect to \( r. \)

Then with \( F_{ij} = F_{r_{ij}} \) the mean value theorem implies

2.8) \( F_{ij}(x, t, u, Du, s).D_{ij} u - F(x, t, u, Du, 0) = 0 \)

where \( s = s(x, t) \in S^n. \) By the H"older estimate, Corollary 1.5, for any cylinder \( B_{\epsilon R} \subset \Omega \) and \( \epsilon \in (0, 1), \)

\[
\text{osc}_{B_{\epsilon R}} u \leq C\epsilon^\alpha
\]

with \( C \) and \( \alpha \) as specified in the theorem. For \( F \) not differentiable with respect to \( r, \) a simple approximation argument gives the result. \( \blacksquare \)

For the interior gradient estimates, \( F \) is assumed to be once differentiable in \( \Gamma, \) and to satisfy in addition 2.5).

THEOREM 2.2 Let \( u \in C^3, 2(\Omega) \) satisfy \( F[u] - D_t u = 0 \) in \( \Omega \) with 2.2), 2.4) and 2.5) holding. Then

2.9) \( \| u \|_{1; \Omega} < C \)
where $C$ depends on $n, \lambda_0, \Lambda_0, \mu_0, \mu_1, M_0$ and $\text{diam}(\Omega)$.

Proof: Set

$$\eta = (R^2 + t)R^{-2}(1 - \|x\|^2).R^{-2}$$
$$v = |Du|^2 = \varphi(u)\bar{v}$$

where $\varphi$ satisfies

$$\varphi \in C^2(m, M); \varphi > 0; \varphi' > 0; \varphi^* < 0$$

and set

$$\Omega' = \Omega \cap B_R(0, 0); m = \inf u; M = \sup u$$
$$w = \eta v, \bar{w} = \eta \bar{v}; \chi = \varphi'.\varphi^{-1}$$
$$B_i = -2\eta^{-1}F_{i\bar{j}}D_i \bar{j} \eta + 2x F_{i\bar{j}}D_i u + F_p$$

Repeating almost word for word the calculations of [Tr 2], one obtains on the set $\Omega_R = \{ (x, t) \in \Omega' \mid w \geq R^{-2} \}$ the differential inequality

$$-\varphi(F_i j D_i j \bar{w} - D_t \bar{w} + B_i D_i \bar{w}) \leq \chi'.F_{i\bar{j}}D_i uD_j u.w + R u w(x^2 + 1)$$

where $A$ depends only on $n, \lambda_0, \Lambda_0, \mu_1$. Consequently, if

$$\text{osc } u = M - m < \pi/2A$$
$$B_{6R}$$

then one may choose
\( x = \tan A(z-m) \)

to obtain

\[
F; \partial_i j \partial \bar{w} - \partial_t \bar{w} + B_i \partial_i \bar{w} \geq 0
\]

in \( \Omega_R \). Applying the classical maximum principle thus yields

\[
|Du(0)| \leq R^{-1}
\]

provided \( R < d(0, a\Omega) \) and \( 2.10 \) holds. Combining this with the Hölder estimate Theorem 2.1 thus yields the desired result. 

The Hölder estimate for the gradient takes the following form:

**THEOREM 2.3** Let \( u \in C^{3,2}(\Omega) \) satisfy \( F[u] - \partial_t u = 0 \) in \( \Omega \) with

2.2), 2.4), and 2.5) holding. Then

2.11) \( \|Du\|_{\alpha, \Omega} \leq C \)

where \( \alpha > 0 \) depends on \( n, \lambda_0(M_1), \Lambda_0(M_1), \mu_1(M_1); M_1 = \|u\|_{1; \Omega}, \) and \( C \) depends also on \( \text{diam}(\Omega) \).

**Proof:** Simply mimicking the elliptic case easily yields, for

\[
\omega^+ = \omega_l^{+} = \pm \partial_t u + cu
\]

the differential inequality
\[ F_{ij} D_{ij} w^\pm - D_t w^\pm \leq C(10w^\pm)^2 + 1 \]

where \( C = C(\mu_1, M_1, \varepsilon) \lambda_0 \). Now suppose

\[ B_{2R} = \{ \|x\| < 2R \} \times (-4R^2, 0) \]

and set

\[ \Omega^\pm = \sup_{B_{2R}} w^\pm \]

Applying the weak Harnack inequality Theorem 1.1 to \( w^\pm \), one obtains

\[ \varphi_{\kappa} (\Omega^\pm - w^\pm) \equiv \sup_{B_R} (\Omega^\pm - w^\pm)^\kappa \]

where \( \kappa > 0 \), and \( C > 0 \) depend on \( n, \lambda_0, \Lambda_0, \mu_1, M_1, \) and \( \varepsilon \). Choosing \( \varepsilon \) sufficiently small, the result follows by standard methods.

\[ 2.2 \text{ TIME DERIVATIVE ESTIMATES} \]

The estimation of \( |D_t u|_{0; \Omega^\tau} \) is achieved in a manner similar to that for the second derivatives in \([Tr 2]\); i.e., \( [D_t u]_{0, \alpha; \Omega^\tau} \) is estimated in terms of \( M_t = \sup_{\Omega^\tau} |D_t u|_0 \), and then an interpolation argument is used. Since the original equation must be differentiated with respect to the time variable, structure condition 2.6) is required in addition to those used in the preceding section. The estimate to be proved in this section is:
THEOREM 2.4  Suppose \( u \in C^{3,2}(\Omega) \) satisfies \( F[u] - D_t u = 0 \) in \( \Omega \) with 2.2), 2.4), 2.5), and 2.6) holding. Then

\[
\begin{align*}
2.12) \quad [D_t u]^{\alpha, \gamma}_{\Omega, \alpha; \Omega} \leq C
\end{align*}
\]

where \( \alpha > 0 \) depends on \( n, \lambda_0, \) and \( \Lambda_0, \) and \( C \) depends in addition on \( \mu_1, \mu_2, M_1, \) and \( \text{diam}(\Omega). \)

**Proof:** Let

\[
\begin{align*}
w^\pm = \theta v + \frac{\kappa}{2} (1 + D_t u(1 + M_t)^{-1})
\end{align*}
\]

where \( v = |Du|^2 \) and \( \theta > 0 \) is a constant to be chosen. Adding the equation differentiated with respect to \( t \) to the equation for \( v \) yields

\[
\begin{align*}
F_{ij}D_{ij} w^\pm - D_t w^\pm + 2\theta F_{ij} D_{ij} u D_\gamma u + F_{\gamma} D_{\gamma} w^\pm + F_z w^\pm &+ \theta F_z u - \frac{\kappa}{2} F_z + \theta F_x D_\kappa u \pm \frac{\kappa}{2}(1+M_t)^{-1} F_t = 0
\end{align*}
\]

Using the structure conditions and Cauchy's inequality thus yields

\[
\begin{align*}
F_{ij}D_{ij} w^\pm - D_t w^\pm \leq -2\theta \lambda_0 |D^2 u|^2 + |Dw^\pm|^2 + \frac{\kappa}{2} \mu_1^2 (1+|D^2 u|^2)
+ 2(1+\theta M_1^2) \mu_2 (1+|D^2 u|^2) + \theta M_1 \mu_1 (1+|D^2 u|^2) + \frac{\kappa}{2} \mu_2 (1+|D^2 u|^2)
\end{align*}
\]

where \( \lambda_0, \mu_1, \mu_2, M_1, \) and \( \text{diam}(\Omega) \) depend on \( n \) and \( u \), and \( C \) depends only on known quantities.
where $C_1$ and $C_2$ depend on $\mu_1, \mu_2,$ and $M_1$. Thus choosing $\theta > \frac{\mu_1^2}{2\lambda_0}$, it follows that

$$F_{ij} D_{ij} w^+ - D_t w^+ \leq \frac{1}{2} |Dw^+|^2 + C_3$$

where $C_3$ depends only on known quantities (and in particular not on $M_t$).

Applying the weak Harnack inequality, there exist $\kappa$ and $C$ positive depending on $\lambda_0, \Lambda_0$, and $n$ such that

$$\varphi^{-1, R}(W^+ - w^+) \leq C(W^+ - \sup_{B_{R}} w^+ + C_3 R^2)$$

where $\varphi$ and $W^+$ are defined as in the previous section. Summing these two inequalities yields

$$2^{1-1/k} \varphi^{-1, R}(W^+ + W^- - w^+ - w^-) \leq C(W^+ + W^- - \sup_{B_{R}} w^+ - \sup_{B_{R}} w^- + C_3 R^2)$$

Now the Hölder estimate for $u$ can be used to see that

$$2^{1-1/k} \omega(2R) \leq C(\omega(2R) - \omega(R) + CR^2) + C'R^B$$

where

$$\omega(2R) = \text{osc } D_t u(1+M_t)^{-1} \quad \omega(R) = \text{osc } D_t u(1+M_t)^{-1}$$

and where $\beta$ is the estimated Hölder exponent for $u = |Du|^2$ and $C'$ depends on $\theta$ and the $C^0$ norm for $u$. Thus

$$\omega(R) \leq 2\omega(2R) + C_3 R^2 + C'R^B$$

where $s < 1$, $\beta$, $C_3$ and $C'$ depend only on known quantities.
Therefore the Hölder estimate is proved; for $\varepsilon < 1$

\[
\text{osc } D_t u (1+M_t)^{-1} \leq C(1+M_t) \varepsilon \alpha (1+C_3 R^2 + C^\alpha R^\beta)
\]

where $\alpha$ depends on $n, \lambda_0, \Lambda_0$, and $p$, and so

\[2.13 \quad \text{osc } D_t u \leq C(1+M_t) \varepsilon \alpha (1+C_3 R^3 + C^\alpha R^\beta)
\]

To establish an estimate for $|D_t u|$ via interpolation, the Hölder estimate for $u$ is used. For any function $v$, define a seminorm by

\[
\|v\|_{D_t u, p; \alpha} = \sup \left\{ d(x,t;y,s) \rho^\alpha [v(x,t) - v(y,s)] |l(x,t) - (y,s)|^{\alpha} \right\}
\]

where the function $d$ is defined by

\[
d(x,t;y,s) = \min \{d((x,t),a\Omega);d((y,s),a\Omega)\}
\]

and the $\sup$ is taken over all $(x,t)$ and $(y,s)$ in $\Omega$. The estimate 2.13 then yields the further estimate

\[2.14 \quad \|D_t u\|_{D_t u; p; \alpha} \leq C(\rho^p + \sup d(x,t) |D_t u(x,t)|)
\]

for any $p > 0$. Further, a standard interpolation argument gives the inequality

\[
\sup_{B_\delta} d(x,t)2^{-\alpha} |D_t u(x,t)| \leq C(\overline{\alpha}, \epsilon) [u]^{\overline{\alpha}; B} + \epsilon \|D_t u\|^{2-\alpha}_{\overline{\alpha}; B}
\]

where

\[\overline{\alpha} = \sup \{[u(x,t) - u(y,s)] |l(x,t) - (y,s)|^{-\alpha} |l(x,t), (y,s) \in B_\delta\}
\]
Thus choosing $p = 2 - \alpha$ and taking $\epsilon$ small, one obtains

$$2.14) \quad \sup_{B_\delta} d(x,t)^{2-\alpha} |D_t u(x,t)| \lesssim C([u]_{\alpha; B} + \delta^{2-\alpha})$$

and hence the local estimate for $D_t u$ is completed. \[ \square \]

### 2.3 SECOND DERIVATIVE ESTIMATES

The estimates of this section are proved as in the elliptic case ([Tr 2]), with the estimates of the last section being needed as well. As in the last section, a Hölder estimate is proved in terms of the second derivative bound, and then the second derivative bound itself is obtained by interpolation. Here the concavity condition is required, as well as growth bounds on the various derivatives of the operator. The estimate to be proved is:

**Theorem 2.5** Let $u \in C^{1,2}(\Omega)$ satisfy $F[u] = D_t u = 0$ in $\Omega$ with conditions 2.2), 2.3), 2.4), 2.5), and 2.6) holding. Then

$$2.15) \quad \langle D u \rangle_\alpha \lesssim C$$

where $\alpha > 0$ depends on $n, \lambda_0, \Lambda_0$, and the Hölder exponent for $D_t u$, while $C$ depends in addition on $\mu_0, \mu_1, \mu_2, \mu_3, M_1, M_2,$ and $\text{diam}(\Omega)$.

**Proof:** Let $\gamma$ be a fixed unit vector in $\mathbb{R}^n$, and for $\Omega' \subset \Omega$, set

$$M_2 = \sup_{\Omega} |D^2 u|$$

$$h_\gamma = \frac{1}{2}(1 + D_\gamma \gamma u(1 + M_2)^{-1})$$

and combine the last two inequalities to get...
Then by differentiating the original equation twice in the direction \( y \), and repeating the calculations of [Tr 2], one obtains the differential inequality

\[
D_t h_Y = F_{ij} D_i j h_Y \leq C \lambda_0 [1D^3 u| + (1 + M_2)^2]
\]

where \( C \) depends on \( n, \mu_1, \mu_2, \) and \( M_1 \). By the lemma of Motzkin and Wasov [M.W. 1] (see [G.T. 1] for a simple proof), one may choose directions \( \gamma_1, \gamma_2, \ldots, \gamma_N \) depending only on the ratio \( \Lambda_0/\lambda_0 \) such that

\[
F_{ij} = \sum_{k=1}^{N} \beta_k \gamma_i \gamma_k
\]

with \( 0 \leq \lambda^* \leq \beta_k \leq \Lambda^* \), where \( \lambda^* \) and \( \Lambda^* \) depend only on \( \lambda_0, \Lambda_0, \) and \( n \). Further one may assume that the co-ordinate directions \( e_1, \ldots, e_n \) and the vectors \( \frac{1}{2}(e_i \pm e_j) \) are included. Thus one has

\[
\sum_k F_{ij} D_i h_k D_j h_k - \frac{1}{2} F_{ij} D_i j u + D_t u \leq C \lambda_0 [1D^3 u| + (1 + M_2)^2]
\]

where \( u = \sum h_k^2 \)

Set

\[
w = w_k = h_k + \epsilon u , \quad k = 1, \ldots, N
\]

and combine the last two inequalities to get

\[
\epsilon \sum F_{ij} D_i h_k D_j h_k - \frac{1}{2} F_{ij} D_i j w + D_t w \leq C \lambda_0 [1D^3 u| + (1 + M_2)^2]
\]

Then by 2.2) and the choice of the \( \gamma_k \),
where

\[ \bar{\mu} = C \varepsilon^{-2} (1 + M^2) \]

and

\[ C = C(n, \lambda_0, \Lambda_0, \mu_1, \mu_2, M_1) \]

Fix a cylinder \( B = B_{2R} \) and define \( B^+ \) and \( B^- \) as earlier. Then set

\[ W_k(2) = \sup_{B_{2R}} w_k, \quad W_k(1) = \sup_{B^+} w_k \]

\[ M_k(2) = \sup_{B_{2R}} h_k, \quad M_k(1) = \sup_{B^+} h_k \]

\[ m_k(2) = \inf_{B_{2R}} h_k, \quad m_k(1) = \inf_{B^+} h_k \]

\[ \omega(2R) = \sum_{B_{2R}} \text{osc } h_k = \sum (M_k(2) - m_k(2)) \]

\[ \omega(R) = \sum_{B_R} \text{osc } h_k = \sum (M_k(1) - m_k(1)) \]

Applying the weak Harnack inequality to \( W_k(2) - w_k \) gives

\[ \varphi_{\kappa, R}(W_k(2) - w_k) \leq C (W_k(2) - W_k(1) + \bar{\mu} R^2) \]

where \( \kappa, C \) are positive constants depending only on \( n, \lambda_0, \Lambda_0 \). Also the following inequalities are clear

\[ W_k(s) - w_k \geq M_k(s) - h_k - 2\varepsilon \omega(s); \quad s = 1, 2 \]

\[ W_k(2) - W_k(1) \leq M_k(2) - m_k(1) + 2\varepsilon \omega(2R) \]
so that one also obtains a corresponding inequality for $h_k$:

$$\phi_{p,R}(M_k^2-h_k) \leq (M_k^2-M_k^1+c\omega(2R)+\mu R)$$

Summing this over $k=1$ yields

$$\phi_{p,R}(\sum_{k=1} M_k^2-h_k) \leq N^1/p\sum_{k=1} \phi_{p,R}(M_k^2-h_k)$$

$$\leq C((1+c)\omega(2R)-\omega(R)+\mu R^2)$$

To obtain a reverse inequality, the concavity of $F$ implies that

$$F_{ij}(y,s,u(y,s),Du(y,s),D^2u(y,s))(D_{ij}u(y,s)-D_{ij}u(x,t))$$

$$\leq F(y,s,u(y,s),Du(y,s),D^2u(x,t)) -$$

$$- F(x,t,u(x,t),Du(x,t),D^2u(x,t)) + D_t u(x,t) -$$

$$- D_t u(y,s)$$

$$\leq \mu_1(1+M_2^1)|x-y|+|u(x,t)-u(y,s)|+|Du(x,t)-Du(y,s)| +$$

$$+ |D_t u(x,t)-D_t y(s)| + \mu_2(1+M_2)|s-t|$$

$$\leq (1+M_2^1)R(1+M_4+M_tR+M_t^2M_2^{1/2}+M_2) + C(1+M_t)R^B +$$

$$+ \mu_2(1+M_2^1)R^2$$

Here the term $M_t^{1/2}M_2^{1/2}$ arises from Lemma 11.3.1 of [L.U.S. 1], where a Hölder estimate for $Du$ is derived in terms of Hölder estimates for $Du$ with respect to $x$ and $u$ with respect to $t$, and the term $C_0(1+M_t)R^B$
comes from the Hölder estimate for $D_t u$ from the previous section. This expression may be simplified to give

$$F_{ij}(y,s,u(y,s),D_1 u(y,s),D^2 u(y,s))(D_{ij} u(y,s)-D_{ij} u(x,t)) \lesssim \bar{C}(1+M_2)^2 R + (1+M_t)C_0 R^\beta$$

where $\bar{C}$ depends on $\mu_1$, $M_1$, $M_t$, and $R$ is supposed to be less than some fixed $R_0$.

Now by the choice of the $\gamma_k$ and $\beta_k$,

$$F_{ij}(y,s,u(y,s),D_1 u(y,s),D^2 u(y,s))(D_{ij} u(y,s)-D_{ij} u(x,t)) = 2(1+M_2)^2 \sum \beta_k (h_k(y,s)-h_k(x,t))$$

and so

$$\sum \beta_k (h_k(y,s)-h_k(x,t)) \lesssim \bar{C}(1+M_2)R + (1+M_t)(1+M_2)^{-1}C_0 R^\beta$$

$$\lesssim \bar{C}(1+M_2)R + C_0 R^\beta$$

since from the equation, $M_t \lesssim \mu(1+M_2)$ for some $\mu$. Hence for fixed $l$,

$$h_l(y,s) - m_l(2) \lesssim 1/\Lambda^x \{C_0 R^\beta + \Lambda^x \sum_{k \neq l} (M_k(2)-h_k(y,s)) + \bar{C}(1+M_2)R\}$$

$$\lesssim \bar{C}((1+M_2)R + R^\beta + \sum_{k \neq l} (M_k(2)-h_k(y,s)))$$

where $\bar{C}$ depends on the same quantities as $\bar{C}$ as well as $\lambda_0$, $\Lambda_0$ and the $C^\beta$-Hölder constant for $D_t u$. Consequently, by 2.14),
Adding this to 2.14) and summing over \( l \) thus yields

\[
\omega(2R) \leq C((1+\varepsilon)\omega(2R) - \omega(R) + \mu R^2 + \tilde{C}(1+M_2)R + \tilde{C}R^\beta)
\]

whence

\[
\omega(R) \leq \tilde{s} \omega(2R) + (\mu R^2 + \tilde{C}(1+M_2)R + \tilde{C}R^\beta + C\varepsilon \omega(2R))
\]

for \( s = 1 - \varepsilon^{-1} \). Finally, choosing \( \varepsilon \) sufficiently small, one gets

\[
\omega(R) \leq \tilde{s} \omega(2R) + (\mu R^2 + \tilde{C}(1+M_2)R + \tilde{C}R^\beta)
\]

where \( 0 < \tilde{s} < 1 \) depends only on known quantities, and not on \( M_2 \).

Applying Lemma 8.23 of [G.T. 1], this yields the desired result; for \( B_R \subset \Omega \),

\[ \sum_k \text{osc } h_k \leq C_6 \varepsilon \alpha(1+\mu R^2 + \tilde{C}(1+M_2)R + \tilde{C}R^\beta) \]

and so

\[ \text{osc } D^2 u \leq C_6 \varepsilon \alpha(1+M_2)(1+\mu R^2 + \tilde{C}(1+M_2)R + \tilde{C}R^\beta) \]

where \( \alpha \) and \( \tilde{C} \) depend only on \( n, \lambda_0, \Lambda_0 \) and \( \beta \). Thus the following estimate is proved:

\[ <D^2 u>_{1, \alpha'; \Omega} \leq C \]
where $\alpha = \alpha(n, \lambda_0, \Lambda_0, \beta)$, $C = C(n, \lambda_0, \Lambda_0, \mu_1, \mu_3, |u_t|_{0, \beta; \Omega}, |u|_{2, \Omega})$

To establish the second derivative bound, the nature of the $M_2$ dependence is crucial. This bound can now be proved in identical manner to that used by Trudinger in [Tr 2], and so is omitted here.

### 2.4 Solvability of the Classical First Boundary Value Problem

The estimates of the preceding three sections allow one to conclude the existence of classical solutions of the first boundary value problem for continuous boundary data on any domain for which the heat equation may be smoothly solved for operators satisfying the conditions (2.1)–(2.6). The argument is virtually identical to that used by Trudinger in [Tr 2] for the elliptic case. First the domain and the prescribed boundary data are smoothed so that boundary estimates exist to the desired order for solutions near the boundary of the heat equation. Then the operator $F - D_t$ is smoothly modified near the boundary to be equal to the heat operator there. Once these approximations are made, the interior estimates of this chapter may be combined with the boundary estimates for the heat operator solutions to provide global estimates in $C^2, \alpha; 1, \alpha^2/2(\Omega)$ for solutions of the approximating equation. Now the method of continuity is invoked to establish the solvability of these equations. However, the interior estimates used are independent of the order of approximation, and so by passing to the limit, one may conclude the existence of a function which solves the equation on the interior of the domain. It then remains to observe that a global modulus of continuity estimate holds by elementary barrier arguments, and so the function created does indeed satisfy the prescribed boundary conditions. Also, higher regularity estimates follow by the usual bootstrap arguments.
All the theorems in the last section of [Tr 2] have natural parabolic analogues. In particular, one may conclude the classical solvability of the parabolic Bellman equation from stochastic control theory for nondegenerate diffusions with stopping at the boundary, and the generalization of this equation to families of quasilinear operators under natural structure conditions. Thus one has the following theorems.

**THEOREM 2.6** Suppose \( \Omega \) satisfies a uniform exterior cylinder condition, \( \varphi \in C^{2,1}(\overline{\Omega}) \), \( F \in C^{2,\alpha}(\Gamma) \) for some \( \alpha > 0 \), and suppose \( F \) satisfies 2.2), 2.3), 2.4), 2.5), 2.6), and 2.7). Then there exists a unique solution \( u \in C^{0,1;0,1/2}(\overline{\Omega}) \cap C^{1,\alpha;2,\alpha/2}(\Omega) \) of the first boundary value problem.

**THEOREM 2.7** Suppose \( \Omega \) satisfies a uniform exterior cylinder condition, \( \varphi \in C^{2,1}(\overline{\Omega}) \), and that \( Q_k \) are quasilinear parabolic operators satisfying conditions 2.2), 2.4), and 2.5) uniformly in \( k \), and with the further conditions that \( A_k^{ij} \) are independent of \( z \), and either \( B_k z \) are uniformly bounded above away from zero, or \( B_k z \not\equiv 0 \) and

\[
|B_k| \leq C(1 + |p|).
\]

Then there exists a unique solution \( u \in C^{0,1;0,1/2}(\overline{\Omega}) \cap C^{2,\beta;1,\beta/2}(\Omega) \), for some \( \beta > 0 \), of the first boundary value problem

\[
F[u] - D_t u = \inf Q_k[u] - D_t u = 0, \quad u = \varphi \text{ on } \partial\Omega.
\]

2.5 Hölder Estimates at the Boundary for Higher Derivatives

Although they are not necessary for the equations of this chapter, it
is useful for the applications of the next two chapters to have estimates for the Hölder norms near the boundary of $D_t u$ and $D^2 u$. The domains to be considered in this chapter are smooth; i.e. I do not consider cylindrical domains. However, the estimates for cylindrical domains are easily proved (provided the usual compatibility conditions hold) by essentially the same methods. The cylindrical case is probably of more general interest, but I specifically require the smooth case in the next chapter. The central element in these estimates is a version of the boundary regularity lemma proved by Krylov in [Kr 3]. By slightly modifying the proof of Caffarelli, I establish this lemma for the domains under consideration.

Suppose $\Omega$ is a $C^\infty$ domain in $\mathbb{R}^n \times \mathbb{R}^+$ such that for each $T > 0$, $\Omega_T$ is simply connected, and such that for $(x, t) \in \partial \Omega$ and $t > 0$, $\tau(x, t)$ is finite. Then the boundary regularity lemma takes the following form:

**Lemma 2.6** Let $u \in W^{2,1}_{n+1, \text{loc}}(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$Lu = a^{ij} D_{ij} u - D_t u = f \in L^\infty(\Omega) \text{ in } \Omega$$

$$u = 0 \text{ on } \partial \Omega$$

where $L$ is uniformly parabolic. Then for $(0, t_0) \in \partial \Omega$, and any $R \leq R_0/8$, and $\varepsilon \in (0, 1]$

$$2.18) \quad \text{osc} \sup_{B_{6R}} D_n u(x, t + t_0) \leq C \varepsilon^\alpha [\sup_{\Omega} |u| + R^{n+1} \sup_{\Omega} |f|]$$

where $B_{6R} = \{(x, t + t_0) \in \partial \Omega \mid |(x, t)| < 6R \}$, $\xi = \sup_{\partial \Omega} \tau^B$, and $C$ and $\alpha$ depend only on $n$, $\lambda$, and $\Lambda$ and $R_0$.

**Proof:** First I construct a supersolution for $L$ on a neighbourhood of the
boundary. For \((x,t) \in \Omega\), set
\[
\bar{d} = \bar{d}(x,t) = \inf_{(y,t) \in \partial \Omega} \|x - y\|
\]

Consider the function
\[
u(x,t) = P^{-1}(e^{-P\bar{d}} - 1)
\]

By uniform parabolicity, for any \(t > 0\), \(P\) may be chosen sufficiently large so that near \((x,t) \in \partial \Omega\), \(Lu > 0\). Now near \(t = 0\), one may suppose that \(\partial \Omega\) is given by
\[
\partial \Omega = \{t = \theta(x)\}
\]

for some \(C^4\) function \(\theta > 0\). Since \(\theta\) achieves its minimum at \(t = 0\), it follows that for \(t < \delta\), \(\delta\) sufficiently small,
\[
L(t - \theta) \leq 0
\]

Now choosing two functions \(\zeta_1(t)\) and \(\zeta_2(t)\) such that
\[
\begin{align*}
\zeta_1' &> 0 \text{ and } \zeta_2' < 0, \\
\zeta_1(t) &\equiv 1 \text{ on } t > \delta; \quad \zeta_1(t) \equiv 0 \text{ on } t < \delta/2 \\
\zeta_2(t) &\equiv 0 \text{ on } t > \delta; \quad \zeta_2(t) \equiv 1 \text{ on } t < \delta/2
\end{align*}
\]
set \(\bar{u} = \zeta_1(t)u + \zeta_2(t)(\theta(x) - t)\), and observe that
\[
L\bar{u} < 0
\]

To extend \(\bar{u}\) to the interior of \(\Omega\), I imitate [C.N.S. 2]. Take \(g\) to be a convex
$C^\infty$ function defined on $s \leq 0$, satisfying:

$$g \equiv -1 \text{ for } s \leq -\epsilon; \quad g(0) = 0; \quad g'(s) > 0 \text{ for } -\epsilon < s < 0$$

and set $w = g(\bar{u})$. Then $Lw < 0$ on $\{-\epsilon/2 \leq \bar{u} \leq 0\}$. Let $\zeta$ have compact support (in the parabolic sense), with $\zeta \equiv 1$ off $\{-\epsilon/2 \leq \bar{u} \leq 0\}$. Define for some $C < 0$

$$\rho = C\zeta(\frac{1}{2}\|ar{u}\|^2 - t) + w$$

Then

$$\rho_{ij} = C\zeta \delta_{ij} + w_{ij} + C(\zeta_i x_j + \zeta_j x_i + (\frac{1}{2}\|ar{u}\|^2 - t)\zeta_{ij})$$

and

$$\rho_t = -C\zeta + w_t + C(\frac{1}{2}\|ar{u}\|^2 - t)\zeta_t$$

Off $\{-\epsilon \leq \bar{u} \leq 0\}$ one has

$$\rho_{ij} = C\delta_{ij} + w_{ij}, \quad \rho_t = -C + w_t$$

so $L\rho < 0$ there. On $\{-\epsilon \leq \bar{u} \leq 0\}$, $-C$ may be chosen sufficiently small so that again $L\rho < 0$. Thus I have constructed $\rho$ such that

$$\Omega = \{0 < \rho\}; \quad L\rho < 0$$

Now by this construction, for large $t$, $D_n\rho(x, t) = 1$, while for small
\( t, \, D_n \rho(x, t) = \tau^{-1}(x, t) \). Then for small \( t \) I may expand by a factor \( \tau \) as follows; set

\[
\tau = \tau(0, 0), \text{ where } (0, 0) \in \partial \Omega,
\]
\( \tilde{\Omega} = \{ 0 < \tilde{\rho}(x, t) \} \)
\( \tilde{g} = \tau^2 g(x/\tau, t/\tau^2) \quad g = \rho, \, u \)

Then

\[
\tilde{\tilde{u}} = \sigma \int(x/\tau, t/\tau^2) \tilde{D}_i j \tilde{u} - \tilde{D}_t \tilde{u} = f(x/\tau, t/\tau^2) \quad \text{on} \, \tilde{\Omega}
\]
\( \tilde{\tilde{\rho}} < -1 \)
\( D_n \tilde{\rho}(0) = 1 = D_t \tilde{\rho}(0) = \tilde{\tau}(0) \)

Thus it may be assumed that on a neighborhood of fixed diameter, \( |\tau| < C \), and \( |D\rho| \in (1/2, 3/2) \). For if the estimate is proved for \( \tilde{u} \), then the estimate for \( u \) follows with an additional factor of \( \tau^{d-1} \).

Now suppose that \( u \gg 0 \) on

\( B_R = \{ (x, t) \in \Omega \mid l(x, t) < R \} \)

and set

\( v = u/\rho ; \, B_{R,S} = \{ (x, t) \in \Omega \mid l(x', t) < R, \, \rho < SR \} \)

where \( R < R_0/8 \), and \( R_0 \) is chosen so that the conditions of the last paragraph are satisfied. Observe that
\[ D_\Omega u(x, t) = \lim_{(y, s) \to (x, t)} u(y, s) \cdot D_\Omega p(y, s) \quad (x, t) \in \partial \Omega \]

I now prove the following inequality:

\[ \inf \{ u(x, t) \mid \| (x', t) \| < R, \ p = 8R \} \leq 2 \left[ \inf_{B_R} u + \lambda^{-1} R \sup_{B_R} |f| \right] \]

for some \( \delta > 0 \). Normalize so that

\[ R = 1; \ \inf \{ u(x, t) \mid \| (x', t) \| < 1, \ p = \delta \} = 1 \]

and define the comparison function \( h \) by

\[ h = \rho [1 - \| (x', t) \|^2 + (1 + \lambda - 1 \delta^{-\frac{1}{2}} \sup |f|)(\rho - \delta) \delta^{-\frac{1}{2}}] \]

Direct computation shows that

\[ Lh \geq -C(1 + \sup |f|) - \rho |C + 2\delta^{-\frac{1}{2}} Lp + 2\lambda \delta^{-\frac{1}{2}} \sup |f| \cdot Lp| + 2\lambda \delta^{-\frac{1}{2}} |Dp|^2 + 2\delta^{-\frac{1}{2}} \sup |f| \cdot |Dp|^2 \]

Thus for \( \delta \) sufficiently small, depending on \( n, \lambda, \) and \( \Lambda \), \( Lh \geq f \) on \( B_R \).

Furthermore on \( \partial \Omega \), \( h \equiv 0 \); on \( \{ p = \delta \} \), \( h = \delta(1 - \| (x', t) \|^2) < u \);

and on \( \{ \| (x', t) \| = 1 \} \), \( h < 0 < u \). Hence \( h < u \) on \( B_1, \delta \), and so

\[ u > [1 - \| (x', t) \|^2 + (1 + \lambda - 1 \delta^{-\frac{1}{2}} \sup |f|) \delta^{-\frac{1}{2}} (\rho - \delta)] \]

on \( B_1, \delta \). Therefore on \( B_{\frac{\delta}{2}}, \delta \), \( u \geq \frac{1}{2} - \lambda^{-1} \sup |f| \); i.e.

\[ 1 \leq 2 \left[ \inf_{B_{\frac{\delta}{2}}} u + \lambda^{-1} \sup |f| \right] \]
Removing the normalization yields 2.19) as required.

It is easily seen that \( v \) satisfies the parabolic equation

\[
L_v = a^{ij} D_{ij} v + b^i D_i v + c v - D_t v = f \cdot p^{-1}
\]

where \( b^i = 2p^{-1}a^{ij} D_j p \) ; \( c = p^{-1}L p \). If I now define

\[
B^+ = \{(x,t) | l(x',t) < R; 8R/2 < \rho < 36R/2; 0 > t > -R^2/4\}
\]
\[
B^- = \{(x,t) | l(x',t) < R; 8R/2 < \rho < 36R/2; -3R^2/2 > t > -R^2\}
\]

I may apply the weak Harnack inequality of Chapter 1 to obtain

\[
sup_{B^-} v \leq C(\inf_{B^+} v + R^{n+1} + \sup_{B^+} |f|)
\]

where \( C \) and \( C' \) depend only on \( n, \lambda, \Lambda \), and the second derivative bounds for \( p \). I now remove the restriction \( u > 0 \), and set

\[
M = \sup_{B_{2R},8} v \quad ; \quad m = \inf_{B_{2R},8} v
\]

and apply the above inequality to \( M - v \) and \( v - m \) to get

\[
sup_{B^-} (v-m) \leq C(\inf_{B^-} (v-m) + R^{n+1} + \sup_{B^-} |f|)
\]
\[
M - \inf_{B^-} v \leq C(\inf_{B^-} (M-v) + R^{n+1} + \sup_{B^-} |f|)
\]
and summing these yields

\[ \text{osc } u \leq \gamma (\text{osc } u + C \frac{n}{n+1} \sup_{\overbar{B}_R/2, R} |f|) \]

where \( 0 < \gamma < 1 \) and \( C > 0 \) depend on \( n, \Lambda, \lambda, \) and \( |\rho| \leq B_R \).

With the aid of the foregoing lemma, I will now establish Hölder estimates for the traces of the first time- and the second space-derivatives on \( \partial \Omega \) for solutions of nonlinear uniformly parabolic equations. Note that in the light of the estimates already proved, it is sufficient to consider operators \( F \) depending only on \( D^2 u \). To this end, I require a further restriction on the domains under consideration. Henceforth I will suppose that the function \( \theta \) satisfies the following condition:

\[ |\partial \Theta| \geq \theta^p \quad \text{for some } p > 0 \]

Without this condition, a modulus of continuity estimate is possible; the power \( p \) ensures that this estimate is in fact a Hölder estimate.

First I estimate a modulus of continuity for \( D_t u \) at \( t=0 \). Note that since \( \theta \) takes its minimum at \( t = 0 \), \( D^2 \theta \), and hence \( D^2 \rho \), is non-negative definite there. Hence sufficiently close to \( t = 0 \),

\[ F(D^2 u) > -\varepsilon \]

for any fixed \( \varepsilon > 0 \). Define \( u(x) \) to be the solution of

\[ F(u(x) D^2 \rho(x)) - u(x) = \psi(x, t) \]

Clearly \( u \in C^2 \), with derivative bounds depending on those for \( \psi, \rho \),
and \( F \), while on \( \{ t = \rho(x) = 0 \} \), \( u = D_t u \). Calculating,

\[
F[\nu p] - u = F[\nu \partial^2 \rho + \rho \partial^2 u + \partial u \otimes \partial \rho + \partial \rho \otimes \partial u] - u
= F[\nu \partial^2 \rho] - u + \theta(\rho + |\partial \rho|)
= \Psi(x, \rho(x)) + \theta(\rho + |\partial \rho|)
= \Psi(x, t) + \theta(\rho-t) + \theta(|\partial \rho|)
\]

Setting \( w^\pm = u \pm C(t_0 + \sup |\partial \rho|). (\rho-t) \), it follows that \( \rho<t_0 \)

\[
G[w^+] \gtrless G[u] + C'(t_0 + \sup |\partial \rho|) \gtrless G[\nu(\rho-t)]
\]

\[
G[w^-] \lesssim G[u] - C'(t_0 + \sup |\partial \rho|) \lesssim G[\nu(\rho-t)]
\]

for \( C' \) chosen appropriately, independent of \( t_0 \), where \( G = F - D_t \).

Then since

\[
u|_{\partial \Omega} = 0 = w^\pm|_{\partial \Omega} = u(\rho-t)|_{\partial \Omega}
\]

it follows that on \( \Omega \)

\[
w^+ \lesssim u(\rho-t) \lesssim w^-
\]

and so

\[
u - C(t_0 + \sup |\partial \rho|) \lesssim (\rho-t)^{-1} u \lesssim v - C(t_0 + \sup |\partial \rho|)
\]

Hence

\[
2.22) \quad |D_t u - v| \lesssim C(t_0 + \sup |\partial \rho|) \lesssim C t_0
\]
Now for \( t < 0 \),
\[
D_t u = D_n u \cdot (D_n \rho)^{-1}. \quad D_t \rho = D_n u \cdot \tau(x, t). D_t \rho
\]
and since I already have an estimate for the \( C^{1,1/2} \) norm of \( D u \), it follows that for \( (x, t), (y, s) \in \partial \Omega \),
\[
|D_t u(x, t) - D_t u(y, s)| \leq C(\|x - y\| + |s - t|^\frac{1}{2}). \max\{ |\tau(x, t)|, |\tau(y, s)| \}
\]
Then from the condition 2.20, one may conclude
\[
|D_t u(x, t) - D_t u(y, s)| \leq C(\|x - y\| + |s - t|^\frac{1}{2}). [1 + (\min (s, t))^{-P}]
\]
Combining this with the estimate above in the usual manner yields the desired Hölder estimate for the trace of \( D_t u \) on \( \partial \Omega \).

Turning to the second derivatives in \( x \), I again begin with a continuity estimate at \( \{ t = 0 \} \). With \( u \) as in 2.21, set
\[
w = u + \psi(\rho - t)
\]
and observe that \( \bar{H}[w](x, 0) = 0 \), while
\[
LD_k w = \psi_k + LD_k (\psi(\rho - t)) \in L^\infty
\]
where \( L \) is the linearization of \( F - D_ho \). For \( (x_0, t_0) \in \partial \Omega \) fixed, suppose that \( e_k \) is tangential to \( \{ \rho(x) = t \} \) at \( (x_0, t_0) \). Then
\[ D_{kk}w = D_{kk}u + D_{kk}(u\rho) \]
\[ = -D_t u \cdot D_{kk}\rho + \nu D_{kk}\rho + 2D_k u \cdot D_k\rho + \rho D_{kk}u \]

so that at \((x_0, t_0)\)

\[ |D_{kk}w(x_0, t_0)| = |D_{kk}\rho(x_0)(u - D_t u)| \lesssim C t_0 \]

for \(t_0\) sufficiently small. Now set

\[ g = D_{kk}w = D_{kk}u + u D_{kk}\rho + \rho D_{kk}u \]

On \(\partial\Omega\), \(g = (D_t u - \nu)D_{kk}\rho\), so if \(l\) set

\[ \tilde{g} = g + [u(x_0) - D_t u(x_0, t_0)]D_{kk}\rho \]

It follows that on \(\partial\Omega\)

\[ \tilde{g}(x, t) \lesssim C l(x, t) - (x_0, t_0)|^{1+\alpha} \]

while

\[ D_n\tilde{g} = D_n g + (u(x_0) - D_t u(x_0, t_0))D_{nk}\rho \]

Let \(d\) be the distance function with respect to an exterior cylinder at \((x_0, t_0)\), and define the barrier

\[ \xi(x, t) = C l(x, t) - (x_0, t_0)|^{1+\alpha} - R d^{1+\alpha} + B d \]
For a sufficiently large, depending on \( \alpha \) and the maximum and minimum eigenvalues of the leading coefficient matrix of \( L \),

\[
L \xi < 0
\]

Now set \( \bar{B} = \{ d < t_0 \} \cap \Omega \), so that

\[
\xi \mid_{d=t_0} = C_l(x, t) - (x_0, t_0) |^{1+\alpha} + t_0 \tilde{B}(B-A \tilde{t}_0 \alpha \beta)
\]

\[
\xi \mid_{\partial \Omega} = C_l(x, t) - (x_0, t_0) |^{1+\alpha} + d(B-A \tilde{d} \alpha)
\]

Since \( \bar{g} \in C_t \alpha \), I have that \( \xi \mid_{\partial \bar{B}} \geq \bar{g} \) if \( B > A \tilde{t}_0 \tilde{\varphi} \) for some \( \tilde{\varphi} > 0 \).

Thus

\[
|D_n \bar{g}(x_0, t_0)| \leq B \leq C_t \varphi
\]

and so

\[
|D_k n \bar{g}(x_0, t_0)| \leq C_t \alpha \nu
\]

This completes the estimate at \( t=0 \). It remains to estimate the Hölder modulus for positive \( t \). Here it is sufficient to imitate the argument used for \( D_t u \), again using the function \( p \) to avoid direct boundary flattening arguments. Thus one obtains the estimate for the trace of the mixed normal-tangential derivatives, and so the equation may be solved to obtain the estimate for the trace of the pure normal second derivatives. The extension of these estimates into the interior is done by the usual barrier arguments.
3. EQUATIONS ARISING FROM SYMMETRIC FUNCTIONS OF THE EIGENVALUES OF THE HESSIAN MATRIX

In [C.N.S. 2], Caffarelli, Nirenberg and Spruck studied the classical solvability of a class of nonlinear partial differential equations obtained by considering symmetric concave functions of the eigenvalues of the Hessian matrix $D^2u$. Amongst other results, they proved that for such a function $F$, with some natural conditions pertaining to ellipticity, the classical Dirichlet problem

$$F(D^2u) = \psi(x) \quad \text{on } \Omega$$

$$u = \psi(x) \quad \text{on } \partial \Omega$$

has a unique elliptic solution for smooth functions $\psi > 0$ and $\varphi$, and for $\Omega$ belonging to a class of domains naturally defined in terms of $F$.

The aim of this chapter is to define and prove the solvability of a corresponding problem in the area of parabolic equations for the same class of functions $F$. For a function $F$ defined on the space of $(n+1)\times(n+1)$ real matrices, I consider equations of the form

$$F(\overline{H}[u]) = \psi(x,t)$$

where for convenience of notation, I shall denote by $\overline{H}[u]$ the matrix

$$\overline{H}[u] = \begin{pmatrix} -D_tu & 0 \\ 0 & D^2u \end{pmatrix}$$

The inspiration for studying this class of equations comes largely from a work of Krylov [Kr 1], in which he proposes as a natural analogue of the
Monge-Ampere equation the equation

\[ \partial_t u \cdot \text{det}(\partial^2 u) = -f(x, t) \]

In that paper, Krylov used this operator to prove the maximum principle referred to in the first chapter. This particular equation will be studied in more detail in the next chapter, where the right hand side will be allowed to depend also on \( u \) and \( Du \).

Most of the arguments required for the results of this chapter follow the lines of [C.N.S. 2] very closely, and so for brevity, the reader will be referred to that work for the most routine extensions.

### 3.1 Definitions and Preliminary Results

First the class of functions to be considered must be defined. Let \( f \) be a real valued function on \( \mathbb{R}^{n+1} \), \( f = f(\lambda_1, \ldots, \lambda_n, \lambda_t) \) satisfying

1. \( f \) is symmetric and smooth in all its arguments
2. \( \frac{\partial f}{\partial \lambda_i} > 0 \) on \( \mathcal{Y} \)
3. \( f \) is concave on \( \mathcal{Y} \)

Also suppose that a positive cone \( \mathcal{Y} \subset \mathbb{R}^{n+1} \) is given with vertex at the origin and containing the cone \( \{ \lambda_i > 0 ; i = 1, \ldots, n+1 \} \) and symmetric in the \( \lambda_i \). Then as extra conditions on \( f \) I require

1. \( \frac{\partial f}{\partial \lambda_i} > 0 \) on \( \mathcal{Y} \)
2. \( f \) is concave on \( \mathcal{Y} \)

Note that 3.2) is simply the requirement that
is parabolic on those \( u \) such that \( \bar{A}[u] \in Y \), where \( \lambda_i[u] \) is defined to be the \( i \)-th eigenvalue of \( D^2u \), and \( \bar{A}[u] \) is defined to be the \( n+1 \)-vector

\[
\bar{A}[u] = (\lambda_1[u], \ldots, \lambda_n[u], -D_t u)
\]

It is also required that \( F \) be locally uniformly parabolic; i.e. I shall require for some of the estimates that

\[
\begin{align*}
3.4) \quad & \text{For every } C > 0 \text{ and compact } K \subset Y \text{ there exists } R = R(C, K) \\
& \text{such that } f(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, A) \leq C \quad \forall \lambda \in K.
\end{align*}
\]

and

\[
3.5) \quad f(R\lambda) \geq C \quad \forall \lambda \in K
\]

The next condition controls the behaviour of \( f \) near the extremal points of the cone \( Y \).

\[
3.6) \quad \exists \overline{\psi}_0 \text{ such that } \lim_{\lambda \to \lambda_0} f(\lambda) \leq \overline{\psi}_0 \quad \forall \lambda_0 \in \partial Y
\]

For the Inhomogeneous term \( \psi \) I require that

\[
3.7) \quad \psi \in C^\infty(\Omega); \quad 0 < \psi_0 = \min_{\overline{\Omega}} \psi \leq \max_{\overline{\Omega}} \psi = \psi_1; \quad \psi_0 > \overline{\psi}_0.
\]

Now I turn to the conditions to be satisfied by the domain \( \Omega \). To obtain the most general solvability theorem, it is necessary to consider
domains which are variable in $t$. To see this, consider the parabolic Monge-Ampere equation of Krylov. It is clear that the class of functions on which this operator is parabolic is the set of functions which are convex in $x$ and decreasing in $t$. Thus on time-independent domains one can only hope to solve such an equation for boundary values which are convex on the initial boundary and which decrease with increasing $t$. The theorems to be formulated in this chapter will establish the solvability for arbitrary smooth boundary values on a somewhat different class of domains; the question for time-invariant, or cylindrical, domains shall be addressed at the end of the chapter.

The domains to be treated will be assumed to be reasonably smooth; i.e. I do not initially consider domains with corners, and so cylindrical domains are not included. Note that this means that the usual compatibility conditions at the corners do not arise. My first assumption is that sufficiently close to $(0,0) \in \partial \Omega$, the surface $\partial \Omega$ may be represented as

$$x_n = p(x',t) = \frac{1}{2} \sum_{\alpha<n} \kappa_{\alpha} x_\alpha^2 - \tau \cdot t + o(|x'|^3 + t^2) \quad \ldots \quad t \ll 0$$

where the positive $x_n$-axis is the interior normal to $\partial \Omega$ with respect to $x$, and $\kappa_{\alpha}$ are the principle curvatures of $\partial \Omega$ at $(0,0)$ with respect to $x$, and $\tau$ is the "time derivative" of $\partial \Omega$ defined in the introduction. The condition tying in the nature of $F$ now becomes

$$3.9) \ V(x,t) \in \partial \Omega, \ \exists R \text{ such that } (\kappa_1, \ldots, \kappa_{n-1}, \tau, R) \in \mathcal{Y}$$

Note that this condition does not make sense where $\kappa_i$ or $\tau$ are not well defined. I therefore assume that for $t$ sufficiently small, $\partial \Omega$ may be represented as
3.8)'  \[ t = \theta(x) \]

and insist that

3.9)'  \[ \theta - t \text{ is an admissible function} \]

where the term "admissible" is defined by the following.

**DEFINITION 3.1**  
A function \( u \in C^{2+1}(\overline{\Omega}) \) is called admissible if at every \((x,t) \in \overline{\Omega}, \ \overline{\lambda}[u](x,t) \in Y \)

Note that with this definition, condition 3.9)' is equivalent to the requirement that

\[ \overline{\lambda}[\theta-t](0,x) = (\theta_1(x), \ldots, \theta_n(x), 1) \in Y \]

Thus the restrictions on \( \partial \Omega \) take on a purely pointwise form. The following lemma may be proved in identical manner to Lemma A of [C.N.S. 2].

**LEMMA 3.2**  
Suppose the positive \( \lambda \) axes are contained in \( Y \) and conditions 3.8), 3.8)', 3.9) and 3.9)' hold. Then \( \partial \Omega \) is connected.

The main result of this chapter is the obvious extension of Theorem 2 of [C.N.S. 2].

**THEOREM 3.3**  
Suppose conditions 3.1) - 3.9)' all hold, and that \( \varphi \in C^\infty(\partial \Omega) \). Then there exists a unique admissible solution \( u \in C^\infty(\Omega) \).
to the boundary value problem

3.10) \( \mathcal{F}[u] = \psi(x, t) \) in \( \Omega \), \( u = \phi \) on \( \partial \Omega \)

The method of proof is the same as that of Chapter 2; apriori estimates are derived for solutions, and then the method of continuity is invoked. In this case, estimates in \( C^{2,1}(\bar{\Omega}) \) suffice, since such estimates imply the uniform parabolicity of the operator, and so the Hölder estimates of the previous chapter may be applied. The usual bootstrapping methods then give the higher regularity in the interior.

As is usual for elliptic and parabolic equations, the first step in proving estimates is a maximum principle. The lemma below is proved in the same manner as in [C.N.S. 2].

**Lemma 3.4 (Maximum Principle)** Suppose \( u \) is an admissible function and \( v \in C^{2,1}(\Omega) \cap C(\bar{\Omega}) \). Suppose also that for every \( (x, t) \in \Omega \),

\[
\{ \lambda \in \mathbb{R}^+ \mid f(\lambda) \geq \psi(x, t) \}
\]

If \( u \leq v \) on \( \partial \Omega \), then \( u \leq v \) on \( \Omega \).

**3.2 Existence of Admissible Functions**

The aim of this section is to prove the existence of admissible subsolutions of problem 3.10). The method is the same as that used in § 2.5. Throughout this section, I assume that the boundary of the domain satisfies conditions 3.8), 3.8'), 3.9), and 3.9'). Assuming \( (0, t_0) \in \partial \Omega \),
and using the representation 3.8), one has at \((0, \tau_0)\)

\[
\overline{\mathcal{A}}(\tilde{\alpha}) = \begin{bmatrix}
-\tau & 0 & \cdots & 0 \\
0 & \kappa_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \kappa_{n-1}
\end{bmatrix}
\]

Set \(v = P^{-1}(e^{-P\tilde{\alpha}} - 1)\), where \(P\) is to be chosen sufficiently large. At \((0, \tau_0)\):

\[
\overline{\mathcal{A}}(v) = -\overline{\mathcal{A}}(\tilde{\alpha}) + PD_1\tilde{d}_0\tilde{d}_0 = \begin{bmatrix}
-\tau & 0 & \cdots & 0 \\
0 & \kappa_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \kappa_{n-1}
\end{bmatrix}
\]

and so by 3.9), for \(P\) sufficiently large, \(\overline{\lambda}(v)(0, \tau_0) \in \mathcal{Y}\). Now \(\tilde{\alpha}\) is \(C^2, 1\) on a neighbourhood of \(\partial \Omega \setminus \{t > \varepsilon\}\) for any \(\varepsilon > 0\), while for \(\varepsilon\) sufficiently small, and \(t < \varepsilon\), one has from 3.9') that \(\theta(x) - t\) is admissible. Now choosing two functions \(\xi_1(t)\) and \(\xi_2(t)\) such that

\[
\begin{align*}
\xi_1' &> 0 \text{ and } \xi_2' < 0, \text{ with} \\
\xi_1(t) &\equiv 1 \text{ on } t > \delta; \quad \xi_1(t) \equiv 0 \text{ on } t < \delta/2 \\
\xi_2(t) &\equiv 0 \text{ on } t > \delta; \quad \xi_2(t) \equiv 1 \text{ on } t < \delta/2
\end{align*}
\]

set \(\overline{u} = \xi_1(t)u + \xi_2(t)(\theta(x) - t)\), and observe that on \(\partial \Omega\), \(\overline{\lambda}([\overline{u}]\) is contained in a compact subset of \(\mathcal{Y}\). Thus on some neighbourhood \(\tilde{N}\) of \(\partial \Omega\), \(\overline{\lambda}([\overline{u}]\) remains in a compact subset of \(\mathcal{Y}\). To extend \(\overline{u}\) to the interior of \(\Omega\), I again imitate [C.N.S. 2]. Take \(g\) to be a convex \(C^\infty\) function defined on \(s < 0\), satisfying:
g \equiv -1 \text{ for } s \leq -\varepsilon; \quad g(0) = 0; \quad g'(s) > 0 \text{ for } -\varepsilon < s < 0

and set \( w = g(u) \). Then \( \overline{X}(w) \in \mathcal{Y} \) on \((-\varepsilon/2, u < 0)\), and \( \overline{X}(w) \in \mathcal{Y} \) on \( \mathbb{N} \). Let \( \zeta \) have compact support (in the parabolic sense), with \( \zeta = 1 \) off \((-\varepsilon/2, u < 0)\). Define

\[
\rho = C\zeta(\frac{1}{2}\|x\|^2 - t) + w
\]

Then

\[
\rho_{ij} = C\zeta \xi_{ij} + w_{ij} + C(\xi_i x_j + \xi_j x_i + (\frac{1}{2}\|x\|^2 - t)\xi_{ij})
\]

and

\[
\rho_t = -C\zeta + \omega_t + C(\frac{1}{2}\|x\|^2 - t)\zeta_t
\]

Off \((-\varepsilon < u < 0)\) one has

\[
\rho_{ij} = C\zeta \xi_{ij} + w_{ij}, \quad \rho_t = -C + \omega_t
\]

so \( \overline{X}(u) \in \mathcal{Y} \) there. On \((-\varepsilon < u < 0)\), \( C \) may be chosen sufficiently small so that again \( \overline{X}(u) \in \mathcal{Y} \). Thus for \( \varphi \equiv 0 \), there exists an admissible function. To obtain an admissible function with general boundary values \( \varphi \), set \( u = A\rho + \varphi \), and choose \( A \) sufficiently large (as in the argument in [C.N.S. 2]). Since \( F[\rho] \) is bounded away from zero, choosing \( A \) large yields the existence of admissible subsolutions for general boundary values.
In this section, I deal with global $C^2$ estimates in terms of the boundary estimates of the next section. The arguments are standard applications of the maximum principle, and follow the lines of [C.N.S. 2].

First note that any solution of the equations under consideration is a subsolution of the corresponding heat equation with the same boundary data; i.e.

$$3.11) \quad \Delta u - D_t u \geq \delta > 0 \quad \text{on } \Omega$$

This follows from 3.1)–3.7) in exactly the same manner as in the elliptic case. Thus setting $v$ to be the solution of

$$3.12) \quad \begin{cases} \Delta v - D_t v = \delta \quad \text{on } \Omega \\ v = \varphi \quad \text{on } \partial \Omega \end{cases}$$

and taking $v$ to be the subsolution constructed in the last section, it follows that

$$v \geq u \geq \varphi \quad \text{on } \Omega$$

$$v = u = \varphi \quad \text{on } \partial \Omega.$$ 

Therefore the following estimates are immediate:

$$3.13) \quad \sup_{\Omega} |u| \leq C$$

$$3.14) \quad \sup_{\partial \Omega} |D u| \leq C$$
3.15) \( \sup_{\partial \Omega} |D_t u| \leq C \)

3.14) and 3.15) are now used to provide global estimates. Using the once differentiated equation and the concavity of \( f \), one gets the inequalities

\[
L(C(u-u) \pm u_t) \geq 0 ; \quad L(C(u-u) \pm u_t) \geq 0
\]

where \( C \) depends on the first derivatives in \( x \) and \( t \) of \( \psi \), and \( L \) is the linearized operator

\[
Lu = F_{ui,j} D_{ij} u - F_{-u_t} D_t u
\]

Thus the functions \( C(u-u) \pm u_t \) take their maxima on \( \partial \Omega \).

To handle the second derivatives in \( x \), suppose \( \gamma \) is a unit vector in \( \mathbb{R}^n \), and differentiate the equation twice in the direction \( \gamma \):

\[
Lu_\gamma = \psi_\gamma
\]

\[
Lu_{\gamma \gamma} \geq \psi_{\gamma \gamma} \geq -C
\]

where the inequality comes from the concavity assumption. Then as above, one obtains that \( C(u-u) + u_{\gamma \gamma} \) takes its maximum on \( \partial \Omega \); i.e.

\[
u_{\gamma \gamma} \leq C
\]
where C now depends on $\sup_{\partial\Omega} |D^2 u|$. Combining this with the original equation and the estimate for $|D_t u|$, I conclude that

$$-u_{yy} \leq (n-1)C$$

and hence

$$|D^2 u| \leq C$$

### 3.4 BOUNDARY ESTIMATES FOR SECOND DERIVATIVES

I present here the estimate for the mixed pure-tangential second derivatives on the boundary, using a slightly simplified version of the proof given in the elliptic case by Caffarelli, Nirenberg and Spruck. Following that, I estimate the pure normal second derivative, again using a simplified version of the proof of [C.N.S. 2]. The main element in the simplification of the proof lies in a more explicit use of the subsolution constructed in section 3.2. Thus from section 3.2 one may assume as given a function $\rho = \rho(x,t)$ such that $\rho$ is admissible, with apriori bounds on its derivatives up to third order in $x$ and second order in $t$, and such that

$$F[\rho] \geq 1; \quad \rho|_{\partial\Omega} = 0$$

Note that the purely tangential second derivatives may be estimated via the formula
3.17) \( D_{\alpha \beta} (u-v) = -D_n (u-v) D_{\alpha \beta} \rho (D_n \rho)^{-1} \)

since from the previous sections, there exists a constant \( C \) such that

3.18) \( |D_n (u-v)| \leq C |D_n \rho| \)

In fact it is this last property which allows the simplification in the proof of the mixed derivative estimate.

By the rotation invariance with respect to \( x \) of the operator \( F \),

\[
L(x_i D_j - x_j D_i) u = (x_i D_j - x_j D_i) \psi
\]

Fix a point \( (x_0, t_0) \in \partial \Omega \); by rotation and translation, one may suppose that \( (x_0, t_0) = (0,0) \), and that \( D_\alpha \psi(0,0) = \psi(0,0) = 0 \). Define an operator \( T \) by

\[
Tw = \rho_n (0) D_\alpha w - \rho_{\alpha \alpha} (0) (x_\alpha D_n - x_n D_\alpha) w
\]

where \( e_n \) is the inner unit normal to \( \partial \Omega \) at \( (0,0) \) with respect to \( x \), and \( e_\alpha \) is some fixed tangential direction. Now

\[
L(T(u-v)) = \rho_n (0) \psi_\alpha - \rho_{\alpha \alpha} (0) (x_\alpha D_n - x_n D_\alpha) \psi - \rho_n (0) L u + \rho_{\alpha \alpha} (0) L(x_\alpha D_n - x_n D_\alpha) u
\]

and so
\[ |L(T(u-v))| \leq C_0(1 + \sum_{i=1}^{n} F_{u_{ii}} - F_u) \]

where here \( C_0 \) depends on third derivative bounds for \( u \), second derivative bounds for \( p \), and first derivative bounds for \( \psi \).

Since \( p \) is admissible, there exists \( \delta > 0 \) sufficiently small so that

\[ w(x,t) = p(x,t) - \delta (\|x\|^2 - 2t) \]

is also admissible, and such that \( F[w] > \frac{1}{2} \). Choosing \( A \) sufficiently large, \( F[Aw] > C \). Therefore by the concavity of \( F \)

\[
C \leq F[Aw] \leq F[u] + L(Aw-u) \]

\[ L(Aw-u) > C - \psi > C' \]

and so

\[ L(Aw - u + \frac{1}{2} A_0 (\|x\|^2 - 2t)) > C' + A_0 (\sum F_{u_{ii}} - F_u) \]

Hence if \( A \) is chosen sufficiently large,

\[ L[Aw + \frac{1}{2} A_0 (\|x\|^2 - 2t) - u \pm T(u-v)] > 0 \]

and so \( Aw + \frac{1}{2} A_0 (\|x\|^2 - 2t) - u \pm T(u-v) \) takes its maximum on \( \partial \Omega \).

I now show that for \( A \) sufficiently large, this maximum is zero. Firstly
\[ T(u-v) = P_n(0)D_\alpha(u-v) - \rho_{\alpha\alpha}(0)(x\alpha D_n-x_n D_\alpha)(u-v) \]

\[ = \rho_n(x,t)D_\alpha(u-v) + (\rho_n(0)-\rho_n(x,t))D_\alpha(u-v) - \]

\[ - \rho_\alpha(x,t)D_n(u-v) + (\rho_\alpha(x,t)-\rho_{\alpha\alpha}(0)x_\alpha)D_n(u-v) + \]

\[ + \rho_{\alpha\alpha}(0)x_n D_\alpha(u-v) \]

Now since \( D_\alpha \) is proportional to \( D(u-v) \) on \( \partial\Omega \), the term

\[ \rho_n(x,t)D_\alpha(u-v) - \rho_\alpha(x,t)D_n(u-v) \]

vanishes there, and \( D_\alpha(u-v) \) may be estimated by \( C|D_\alpha\rho| \). Thus on \( \partial\Omega \),

\[ |T(u-v)| \leq C|\rho_n(0)-\rho_n(x,t)||D_\alpha\rho(x,t)| + \rho_{\alpha\alpha}(0)x_n|D_\alpha\rho(x,t)| + \]

\[ + |\rho_\alpha(x,t)-\rho_{\alpha\alpha}(0)x_\alpha||D_n(u-v)| \]

\[ \leq C(\|x\|^2-t) + C|x_n\rho_n(0)-\rho_n(x,t)| \]

\[ \leq C(\|x\|^2-t) + C\|\rho_\alpha(0)x_n-\rho_n(x,t)\| \]

\[ \leq C(\|x\|^2-t) \]

Since \( \rho \equiv 0 \) on \( \partial\Omega \), \( A \) may therefore be chosen sufficiently large, depending only on derivative bounds on \( \rho \) up to third order in \( x \) and second order in \( t \) so that \( A(w + \frac{1}{2}d(\|x\|^2-2t)) \leq T(u-v) \) on \( \partial\Omega \). Finally, since \( u = \phi \) on \( \partial\Omega \), and \( \partial_\beta \psi(0) = \phi(0) = 0 \) for all \( \beta < n \), it follows that \( \sup_{\partial\Omega} u \leq C(\|x\|^2-2t) \), and so the estimate of the mixed
tangential-normal second derivatives is complete; i.e. I have shown that

\[ |D_n T(u-v)(0)| = |D_n (u-v)_\alpha| = \tilde{A} D_n \rho(0) \]

\[ \Rightarrow D_{n\alpha} u(0) \leq A + |D_{n\alpha} v(0)| \]

To estimate the pure normal second derivative, I prove that the equation is in fact uniformly parabolic, by establishing bounds from below on the tangential second derivatives. By subtracting a linear function from \( \varphi \), one may assume that \( D\varphi(0) = 0 \). Observe that then

\[ D_{\alpha\beta}(u-\varphi)(0) = D_n u(0) \cdot (D_n \rho(0))^{-1} D_{\alpha\beta} \rho(0) \]

\[ D_t (u-\varphi)(0) = D_n u(0) \cdot (D_n \rho(0))^{-1} D_t \rho(0) \]

Let \( Y' \) be the projection of \( Y \) onto the first \( n \) coordinates. If \( Y' \) is all of \( \mathbb{R}^n \), then the bounds already established allow \( D_{nn} u(0) \) to be solved for. If not, then set

\[ s_0 = \sup \{ s \in \mathbb{R} \mid \tilde{H}[\varphi] + s \tilde{H}[\rho] \in \partial Y' \} \]

where \( \tilde{H} \) is the "reduced" Hessian:

\[ \tilde{H}[g] = \begin{bmatrix} -D_t g & 0 \\ 0 & D_{\alpha\beta} g \end{bmatrix} \quad ; \quad \alpha, \beta = 1, 2, \ldots, n-1 \]

My aim is to show that \( D_n u(0) \cdot (D_n \rho(0))^{-1} \geq s_0 + \varepsilon \) for some fixed \( \varepsilon > 0 \). For if this is so, then the projection of \( \bar{X}'[u] \) of \( \bar{X}[u] \) onto \( Y' \) lies in a compact subset of \( Y' \), from which the desired bound may be
deduced. To simplify calculations, set

\[ \tilde{g} = \tau^2 g(x/\tau, t/\tau^2) \quad g = \rho, u, v \]

where \( \tau = \tau(0) \). Clearly \( \tilde{g} \) satisfies the same bounds on its first \( t \)-derivative and second \( x \)-derivatives as \( g \) itself does, while \( \rho_n(0) = -1 \).

Now there exists a \( \delta > 0 \) such that \( \tilde{\rho} - \delta(\frac{1}{2}\|x\|^2 - t) \) is admissible. Define a set of orthonormal vector fields \( b_\alpha(x, t) \) such that \( b_\alpha \) is orthogonal to \( D(\tilde{\rho} - \delta(\frac{1}{2}\|x\|^2 - t)) \) and \( b_\alpha(0) = e_\alpha \). These may be supposed to be \( C^1,2(\tilde{N}) \) in some neighbourhood \( \tilde{N} \) of \((0,0)\), with derivative bounds depending only on the second and higher derivatives in \( x \) and first and higher derivatives in \( t \) of \( \tilde{\rho} \), while \( \tilde{N} \) may be assumed to contain a neighbourhood of the form:

\[ \tilde{N}_h = \{(x,t) \mid \tilde{\rho}(x,t) - \delta(\frac{1}{2}\|x\|^2 - t) > -h\} \]

for some fixed \( h \). Since \( \tilde{\lambda}'[\phi](0) + s_0\lambda'[\rho](0) \in \partial \lambda' \), there exists a supporting hyperplane for \( \lambda' \) at \( \tilde{\lambda}'[\phi](0) + s_0\lambda'[\rho](0) \); i.e. there exist positive numbers \( \mu_1, \ldots, \mu_{n-1}, \mu_t \) such that

\[ \lambda' \subset \{ \lambda \mid \sum_\alpha \mu_\alpha \lambda_\alpha + \mu_t \lambda_t > 0 \} \]

and such that

\[ \mu_t(\lambda_t[\phi] + s_0\lambda_t[\rho]) + \sum_\alpha \mu_\alpha(\lambda_\alpha[\phi] + s_0\lambda_\alpha[\rho]) = 0 \]

I define a degenerate parabolic operator \( \tilde{L} \) on \( \tilde{N} \) by
\( \mathcal{L} g = \sum \mu_{\alpha} b_{\alpha}^i(x,t) b_{\alpha}^j(x,t) D_{ij} g - \mu_t D_t g \)

As in the elliptic case, if \( \mathcal{L} g < 0 \) then \( g \) is not admissible. Now

\( \mathcal{L}(\tilde{\varphi} + s_0 \tilde{\rho}) = m(x) + o(\|x\|^2 - t) \)

where \( m \) is a linear function in \( x \), with bounds on its coefficients depending on the third \( x \)-derivatives of \( \tilde{\varphi} \) and \( \tilde{\rho} \) and on the derivative bounds for \( b_\alpha \).

Defining

\[ w = \tilde{\varphi} + s_0 \tilde{\rho} - (\mathcal{L}\tilde{\rho}(0) + s)^{-1}m(x)\tilde{\rho} \]

it is clear that

\( \mathcal{L}w = o(\|x\|^2 - t) \)

Also,

\( \mathcal{L}[\tilde{\rho} - s(\|x\|^2 - t)]^2 = [\tilde{\rho} - s(\|x\|^2 - t)].\mathcal{L}[\tilde{\rho} - s(\|x\|^2 - t)] \)

\[ \leq C[\tilde{\rho} - s(\|x\|^2 - t)] \]

where \( C \) derives from the admissibility of \( \tilde{\rho} - s(\|x\|^2 - t) \). Thus for \( M \) sufficiently large,

\( \mathcal{L}(w + M[\tilde{\rho} - s(\|x\|^2 - t)])^2 \leq 0 \)

on \( \tilde{N} \). Further, on \( \partial \Omega \cap \tilde{N}_h \), \( \tilde{\rho} = 0 \) and so
\[ w + M[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)]^2 \geq \varphi \]

while on \{[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)] = -h\}, one has

\[ w + M[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)]^2 \geq \tilde{\varphi} + [s_0 - (\tilde{\varphi}(0) + s)^{-1}m(x)]\tilde{\rho} - Mh[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)] \]

and so choosing \( M \) sufficiently large, depending on \( h, s_0, s, \) and the constant from \( o(||x||^2-t) \), it follows that

\[ w + M[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)]^2 \geq \tilde{\varphi} - C\tilde{\rho} \]

and so for \( C \) large as in Section 3.1,

\[ w + M[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)]^2 \geq \tilde{u} - \varepsilon\tilde{\rho} \]

on \( \partial(\tilde{\Omega}_h \cap \Omega) \). Consequently, for \( \varepsilon \) sufficiently small,

\[ F(w + M[\tilde{\rho} - s(\frac{3}{2}||x||^2-t)]^2 + \varepsilon\tilde{\rho}) \leq \psi \]

and so \( -D_n\tilde{u} > -(s_o + \varepsilon)D_n\tilde{\rho} \) as required.

Applying the estimates from the previous chapter now yields all the remaining estimates required.
3.5 INITIAL BOUNDARY VALUE PROBLEMS

To complete this chapter, I wish to make some simple observations regarding time invariant domains. Firstly, as noted in the last chapter, given the natural compatibility conditions, estimates at the boundary follow the same lines as used here. Thus, if the cone $Y$ includes the projection of the usual strictly positive cone onto $\mathbb{R}^n$, then there will exist solutions to the given problem on some time invariant domains. The main exception to this is of course the parabolic Monge-Ampere equation of Krylov. To formulate this more precisely;

**THEOREM 3.2** Suppose $\Omega$ is of the form $\Omega_0 \times (0, T)$ where $\partial \Omega_0$ satisfies

$$(k_1, \ldots, k_{n-1}, 0) \in Y', \quad \text{and suppose that } \varphi \text{ satisfies}$$

$$D^2 \varphi(., 0) \in Y'$$

$$(k_1, \ldots, k_{n-1}, -\mu \frac{\partial}{\partial t} \varphi) \in Y' \quad \text{on } \partial \Omega_0 \quad \text{for some positive } \mu$$

$$F(\varphi) = \psi \quad \text{on } \partial \Omega_0 \times \{t = 0\}$$

Then the first boundary value problem is solvable in $C^{2, \alpha; 1, \alpha/2}(\overline{\Omega})$
4. THE PARABOLIC MONGE-AMPERE EQUATION

This chapter is devoted to the study of the parabolic "Monge-Ampere" equation first considered by Krylov in [Kr 1]. This equation takes the form:

4.1) \(-D_t u \cdot \det D^2 u = f(x,t,u,Du)\)

In [Kr 1], Krylov studies this equation on cylindrical domains with zero boundary values and with the right hand side depending only on \(x\) and \(t\). There he proved results relating to the existence of "generalized solutions", while in subsequent work ([Kr 2], [Kr 3]) he proved further results for classical solvability. A related problem, that of the evolution of convex surfaces with velocity proportional to Gauss curvature, has been studied by Tso [Ts 2]. The problem I deal with here is to extend the main results known for the classical solvability of the Dirchlet problem for Monge-Ampere type equations in the elliptic case to equation 4.1). In particular, I extend the results on classical solvability to be found in [G.T.1], the local second derivative estimate of Trudinger and Urbas [Tr.U.1], and some of the results to be found in the Ph.D. thesis of John Urbas [Ur 1].

The domains to be considered again include both cylindrical domains and smooth domains. The latter case will again be dealt with first. The central reason that this equation is approachable is that local estimates are possible. This is in contrast to the case with the other equations covered in the last chapter, where to my knowledge no method of proving local estimates for second derivatives is known. These local estimates are proved by a direct adaptation of the argument of Pogorelov [P.I] for zero boundary values. Thus for theorems on the existence of solutions, the
operator may be modified near the boundary to an operator for which the boundary estimates of the last chapter hold, and then an approximation argument gives the result.

4.1 SECOND DERIVATIVE ESTIMATES

This section contains the central second derivative estimates analogous to those presented in [G.T. 1] and [T.U. 1] for the elliptic case. First notice the following simple facts about the determinate function:

\[ D_{uij} \log \det D^2 u = u_{ij} = (D^2 u)^{-1} \]

\[ D_{u_{ij}u_{kl}} \log \det D^2 u = -u_{ik}u_{jl} \]

Set

\[ w = \eta(x,t) \exp[\kappa \rho |Du|^2] D_{yy}u \]

where \( \gamma \) is some unit vector in \( \mathbb{R}^n \). Clearly

\[ w^{-1}D_i w = \eta^{-1}D_i \eta + \rho D_{ik}uD_{ik}u + (D_{yy}u)^{-1}D_{iy}u \]

\[ w^{-1}D_{ij}w = w^{-2}D_{ij}wD_{ji}w + \eta^{-1}D_{ij} \eta - \eta^{-2}D_{ij} \eta D_{ji} \eta + \rho D_{ik}uD_{jk}u + \]

\[ + \rho D_{ik}uD_{ijk}u + (D_{yy}u)^{-1}D_{iy}D_{jy}u - (D_{yy}u)^{-2}D_{ij}uD_{jy}u \]

\[ w^{-1}D_t w = \eta^{-1}D_t \eta + \rho D_{kt}uD_{kt}u + (D_{yy}u)^{-1}D_{ty}u \]

Using the twice differentiated equation,
4.2) \((\eta \exp [\|p^1 Du \|^2])^{-1}[D_{t}w + D_{t}uU^{i}jD_{i}jw]\)

\[= D_{yy}u(\eta^{-1}D_{t}\eta + D_{k}uD_{tk}u) + D_{yy}uD_{t}uU^{i}j[w^{-2}D_{i}wD_{j}w + \eta^{-1}D_{i}j\eta + \eta^{-2}D_{i}\eta^{-1}D_{i}j\eta + \rho D_{i}kD_{j}kD_{i}j] + D_{tt}uD_{yy}[\log f(x,t,u,Du)] + (D_{t}u)^{-1}(D_{ty}u)^{2} - D_{t}uF_{i,j,k}D_{i}jyD_{k}lyu - \]

At a maximum for \(w\), \((y_{0},s_{0})\) say, the gradient of \(w\) is zero and hence

\[D_{i}yyu = -D_{yy}u(\eta^{-1}D_{i}\eta + \rho D_{k}uD_{ik}u)\]

and so at \((y_{0},s_{0})\)

4.3) \(0 \leq D_{yy}u(\eta^{-1}(D_{t}\eta + D_{t}uU^{i}jD_{i}j\eta) - \eta^{-2}D_{t}uU^{i}jD_{i}j\eta + \rho D_{t}u.Du) + D_{tt}uD_{yy}uD_{k}uD_{k}[\log f(x,t,u,Du)] + D_{tt}uD_{yy}[\log f(x,t,u,Du)] + D_{tt}u^{k}u^{j}D_{i}jyD_{k}lyu\)

\(\leq \eta^{-1}D_{yy}u(D_{t}\eta + D_{t}uU^{i}jD_{i}j\eta) - D_{tt}u\eta^{-1}U^{i}jD_{i}j\etaD_{yy}u + \rho D_{t}u.Du.D_{yy}u + \rho D_{t}uD_{yy}uD_{k}u[(\log f)_{p_{i}}D_{1}ku + (\log f)_{x}D_{k}u + (\log f)_{y}D_{y}u + D_{tt}u[(\log f)_{p_{i}}D_{1}iyu + (\log f)_{p_{i}}D_{i}yD_{y}u + 2(\log f)_{p_{i}}D_{i}yD_{y}u + (\log f)_{x}D_{y}u + (\log f)_{zz}(D_{y}u)^{2} + 2(\log f)_{z}D_{y}u + (\log f)_{tt}]\)
\[ \begin{align*}
\xi & \eta^{-1} D_{yy} u(D_{tt} \eta + D_{tu} \eta u \eta D_{ij} j^n) - D_{tt} u \eta^{-1} u \eta D_{ij} j^n D_{yy} u + \\
& + \rho D_{tu} \Delta u D_{yy} u - D_{tu} \nu \| \log f \|_{pp} \| D^2 u \|_2 - \\
& - D_{tu} \nu \| D^2 u \| \left( \rho \| D u \|_2 \| \log f \|_z \| + \rho \| D u \| \| \log f \|_t \right) + \\
& + 2 \| D u \| \| \log f \|_{pp} \| D_{zz} \| D_{tt} + \\
& - D_{tu} \nu \| (\log f) \|_{pp} \| \eta^{-1} D_{ij} \eta - D_{tu} \nu \| (\log f) \|_{zz} \| D u \|_2 + \\
& + 2 \| (\log f) \|_{zz} \| D u \|_2 + (\log f)_{tt} \]
\end{align*} \]

For the global estimate, take \( \eta = 1 \) in the above, and divide through by \( D_{tu} \), to obtain at \( (y_0, s_0) \):

\[ 0 > \rho (D_{yy} u)^2 - \| (\log f) \|_{pp} \| D^2 u \|_2 - C(\rho \| D^2 u \| + 1 + \rho) \]

where \( C \) depends on \( D^2(\log f) \) and \( \| u \|_{1,0} \). Thus choosing \( \rho > \| (\log f) \|_{pp} \), an estimate for \( D_{yy} u \) follows:

**THEOREM 4.3** Suppose \( \| \log f \|, \| D(\log f) \|, \) and \( \| D^2(\log f) \| \) are bounded by a constant \( C_0 \) at \( u \). Then

\[ \sup_{\Omega} \| D^2 u \| \leq C + \sup_{\partial \Omega} \| D^2 u \| \]

where \( C \) depends on \( C_0 \) and \( \sup_{\partial \Omega} \| D u \| \).

To obtain the interior estimate, the cut-off function \( \eta \) must be constructed so that the term \( L_\eta \) does not cause a problem. In the case of zero boundary values, one may take \( \eta = \mu \), as is seen in [G.T. 1]. For
inhomogeneous boundary values, a related function is used in [T.U. 1]. It is this latter construction that I now extend to the parabolic case.

Firstly, let \( u(x, t) \) be the convex decreasing hull of the boundary values \( \varphi(x, t)|_{\partial\Omega} \), which is defined in the obvious way analogous to the construction in Chapter 1. It is easily seen, by similar arguments to [Tr.U. 1], that \( u \in C(1, 1; 0, 1)(\Omega) \cap C(0, 1; 0, \frac{1}{2})(\bar{\Omega}) \). Now fix

\[
\Omega' \subset \Omega ; \quad \varepsilon = d(\Omega', \partial\Omega); \quad \Omega'' = \{(x, t) \in \Omega \mid d(x, t) > \frac{\varepsilon}{2}\}; \quad \Omega''' = \{(x, t) \in \Omega \mid d(x, t) > \varepsilon/4\}.
\]

To estimate \( \inf (u - u) \), let \((x_0, t_0) \in \Omega'''\), \( \varepsilon > 0 \), and set

\[
\psi(x, t) = \psi_\varepsilon(x, t) = -\varepsilon((\varepsilon/4)^2 - \|x - x_0\|^2 + t - t_0)
\]

Then

\[
-D_t(u + \psi) \cdot \det D^2(u + \psi) = (-D_t u + \varepsilon) \cdot \det D^2(u + \psi) \leq (-D_t u + \varepsilon) \cdot C(n) \sum_{k=1}^{n} \varepsilon^k M^n - k
\]

where \( M = \sup \|D^2 u\| \). From convexity, \( \sup_{\Omega'''} |Du| \leq 8\varepsilon^{-1} |u|_{0; \Omega} \), and so

\[
\inf_{\Omega'''} f(x, t, u, Du) \geq \varepsilon > 0
\]

Thus I may choose \( \varepsilon \) sufficiently small to conclude that

\[
\inf_{\Omega''} (u - u) \geq (\varepsilon/4)^2
\]
Setting
\[ \eta = \eta_c = u - u - \varepsilon; \ \Omega_c = \{(x,t) \in \Omega \mid \eta(x,t) > 0\} \]

and taking \( \varepsilon = \varepsilon(8/4)^2 \), it follows from 4.3) that \( \Omega' \subset \subset \Omega_c \), while
\[ d(\Omega_c, \partial \Omega) \]
may be estimated from below in terms of \( \varepsilon, \|D\eta\|_{0;\Omega}, \) and a modulus of continuity of \( u \) at \( \partial \Omega \). Consequently one may estimate

\[ \sup_{\Omega_c} (|D(\log f)|, |D^2(\log f)|) \leq C; \inf_{\Omega_c} f \geq C' > 0 \]

where \( C, C' \) depend on \( n, \|u\|_{0;\Omega}, \|\psi\|_{2,1;\Omega}, \Omega, d(\Omega', \partial \Omega), f \), and the modulus of continuity of \( u \).

In order to use this estimate, I return to 4.3), and suppose that \( \gamma \) is a coordinate direction with \( w \) at a maximum with respect to \( \gamma \) as well as \( (x,t) \) to get at \((y_0,s_0)\)

\[
U^i j D_i j \eta = U^i j D_i j u - U^i j D_i j \eta \geq n
\]

\[
\eta^{-1} (D_t u)^{-1} D_t \eta = -\eta^{-1} + (\eta D_t u)^{-1} D_t u \geq -\eta^{-1}
\]

\[
\eta^{-1} D_k u D_k \eta = \eta^{-1} D_k u D_k u - \eta^{-1} \|D\eta\|^2 \geq C\eta^{-1}
\]

Replacing these inequalities in 4.3) and dividing through by \( D_t u \),

\[
D \geq \beta(D_{yy} u)^2 - C(D_{yy} u)^2 - C'(1+\beta)(1+\eta^{-1})D_{yy} u - C''
\]

so choosing \( \beta > \frac{1}{2} C \), it follows that
Thus I have established the following local estimate:

**Theorem 4.4** Suppose the quantities in the assumptions of the last theorem are locally bounded. Then

\[ \sup_{\Omega} |\partial^2 u| \leq C \]

where \( C \) depends on \( d(\Omega', \partial \Omega) \), \( f \), \( \Omega \), \( |\varphi|_{2,1; \Omega} \) and a modulus of continuity for \( u \) on \( \bar{\Omega} \).

### 4.2 Time Derivative Estimates

This short section is devoted to time derivative estimates. There are three estimates to be proved. Firstly, a global bound on the time derivative in terms of a boundary estimate is shown, then a bound away from zero, and finally an estimate local with respect to \( t \). The first of these is necessary for the direct application of the method of continuity, while the second is necessary for the global second derivative estimate. The third estimate allows one to use approximation arguments to prove the solvability of problems not immediately amenable to the method of continuity. The estimate in terms of the boundary estimate is a simple application of the classical maximum principle.

**Theorem 4.5** Suppose \( f_z \geq 0 \) and \( (\log f)_t \leq C \). Then

\[ D_t u(x, t) \geq \inf_{\partial \Omega} e^{C(t-s)} D_t u(y, s) \]
Proof: Differentiating the equation with respect to $t$ gives

$$4.4) \quad D_{tt}u + D_tu D_i J D_{ij} u = [(\log f)_i D_i t u + (\log f)_z D_t u + + (\log f)_t] D_t u$$

Simply using the classical maximum principle (see, for example [L.U.S. 1], Theorem 1.2.1 for the cylindrical case) yields the result. □

Multiplying 4.4) through by $e^{kt}$ and setting $v = e^{kt} D_t u$ one has

$$4.5) \quad D_t v + D_t u D_i J D_{ij} v = D_t u (\log f)_i D_i v + [(\log f)_z D_t u + + (\log f)_t + k] v$$

Choosing $k$ sufficiently large that $(\log f)_z D_t u + (\log f)_t + k > 0$ one sees that at a maximum for $v$ one must have $v > 0$, which is impossible. Thus $v$ does not attain an interior maximum, and so the following estimate is established.

**Theorem 4.6** Suppose $|(\log f)_z|$ and $|(\log f)_t|$ are bounded by some constant $C_0$. Then

$$D_t u(x, t) \geq \sup_{\partial \Omega} e^{k(s-t)} D_t u(y, s)$$

where $k$ depends on $|D_t u|_{0; \Omega}$ and $C_0$. 

The estimates on the boundary for $D_t u$ depend on the barrier constructions for the gradient estimates in the case of the non-cylindrical domains, and are obtained directly from the boundary data in the case of cylindrical domains.

The local estimate may be proved under various structure conditions. Let $\theta(t)$ be some given negative decreasing function, and suppose that $\psi(x)$ is a uniformly concave function. Then with $g = \log f$

\[
L \theta u_t = \theta L u_t + \theta' u_t = \theta u_t (g_t + g_z u_t + g_{p,i} D_{i,t} u) + \theta' u_t
\]

\[
= u_t (\theta' + \theta (g_t + g_{p,i} D_{i,t} u) + \theta g_z u_t)
\]

and

\[
L \psi = u_t U^i j D_{i,j} \psi \leq C u_t \text{tr}[U]
\]

\[
\leq C u_t (\text{det } U)^{1/n} = C (-u_t)^{(n+1)/n} \cdot r^{-1}
\]

Considering the first of these equations, if one has the condition

\[
g_z \geq \mu > 0
\]

then at a maximum for $\theta u_t$ one could conclude that

\[
u_t(x,t) \geq \mu^{-1}(-\theta^{-1} \theta' - g_t)(x,t)
\]

from which one has the estimate
\[ \theta u_t \geq \min \left\{ \inf_{\partial \Omega} \theta u_t; \mu^{-1} \sup_{\Omega} (g_t \theta + \theta') \right\} \]

In the event that (4.6) doesn't hold, one may add \( k \) times the inequality for \( \psi \) to the equation for \( \theta u_t \) to obtain at a maximum for

\[ \theta u_t + k \psi \]

\[ 0 \geq C k (-u_t)^{1/n} + \theta f u_t + \theta f_t + k \phi_i D_i \psi + \theta' f \]

Thus if \( |Df| \leq C \) at \( u \), then one has the estimate \( \theta u_t \leq C \).

4.3 LOWER ORDER ESTIMATES

Estimates for \( |u|_0 \) and \( |D u|_0 \) are proved here by simple comparison arguments. In the case of the gradient bound, for the cylindrical case the argument is taken directly from the barrier argument used in the quasilinear elliptic case in [G.T. 1]. For admissible domains, the gradient bound is proved by a simple modification of that argument. The latter case allows for a power of the gradient one higher than the cylindrical case, which is important, since in this situation the time derivative estimate depends on the gradient bound, while in the cylindrical case it is immediate from the given boundary data.

**Theorem 4.7** Suppose \( f_z \geq 0 \). Then there exists a constants \( k \) and \( k' \) depending only on \( |f|_0 \), \( D_t \varphi \), and \( D^2 \psi \) such that

\[ |u| \leq |\varphi + k \psi - k't| \]
Proof: Set \( w = \psi + k\psi - k't \), where \( k \) is so large that this is admissible. Clearly

\[
F[w] = -D_t w \det D^2 w - f[w]
\]

\[
= k' \cdot \det D^2(\psi+k\psi) - D_t(\psi+k\psi) \cdot \det D^2(\psi+k\psi) - f[w]
\]

\[
\geq k' k^n C - f(x,t,w,D\psi + kD\psi)
\]

and so choosing \( k' \) sufficiently large, one has

\[
F[w] \geq 0
\]

and the maximum principle gives the result. \( \blacksquare \)

**Theorem 4.8** Suppose \( \Omega \) is a cylindrical domain, and \( f \) satisfies the structure condition

\[
0 \leq f(x,t,z,p) \leq \mu(|z|)d^\beta|p|^\alpha
\]

for all \((x,t) \in \bar{N} \) a neighbourhood of \( \partial \Omega \), \( z \in \mathbb{R} \), \( |p| \geq \mu(|z|) \), \( \mu \) a nondecreasing function, \( \beta = \alpha - n - 1 \geq 0 \). Suppose also that \( \psi \) satisfies

\[
\psi_t(x,t) < \mu' < 0
\]

Then

\[
\sup |Du| \leq C
\]
where $C$ depends on $n$, $\mu$, $\mu'$, $\beta$, $\tilde{\Omega}$, $\Omega$, $|u|_{0;\Omega}$, and $|\varphi|_{1;\Omega}$.

Proof: Simply apply an identical argument to that of Chapter 17 of [G.T. 1].

**Theorem 4.9** Suppose $\Omega$ is an admissible domain, and $f$ satisfies the structure condition

$$0 \leq f(x,t,z,p) \leq \mu(|z|)d\beta|p|^\alpha$$

for all $(x,t) \in \bar{\Omega}$ a neighbourhood of $\partial\Omega$, $z \in \mathbb{R}$, $|p| \geq \mu(|z|)$, $\mu$ a nondecreasing function, $\beta = \alpha - n - 2 > 0$. Then

$$\sup |Du| \leq C$$

where $C$ depends on $n$, $\mu$, $\beta$, $\tilde{\Omega}$, $\Omega$, $|u|_{0;\Omega}$, and $|\varphi|_{1;\Omega}$.

Proof: Consider the function

$$\psi = C \log(1 + k(Rt - \|x\|^2))$$

Now one may choose $R$ large, and translate the domain $\Omega$, so that $\psi$ is the defining function of an exterior parabola at the given boundary point $(x_0, t_0)$. I define a barrier $w$ by

$$w = \varphi - \psi$$
Calculating easily yields near \((x_0, t_0)\),

\[
F[w] \geq C_0 |Dw|^\alpha \cdot k^{n+2-\alpha} \cdot c^{n+1-\alpha} \cdot [1 + k(\sqrt{R^2 - \|x\|^2})]^{\alpha-n-2}
\]

\[
\geq C_0 |Dw|^\alpha \cdot d^\beta \cdot c^{n+1-\alpha} \cdot (Rt - \|x\|^2)^{\alpha-\beta-n-2}
\]

Thus choosing \(C\) sufficiently small, for \(R\) fixed, it follows that \(w\) is a subsolution of the original equation. Then choose \(k\) large so that \(w\) lies below \(u\) on the boundary of some neighbourhood of \((x_0, t_0)\). The comparison principle then gives the desired result. 

To obtain a more general existence result in the cylindrical case, since the time derivative on the lateral boundary is given directly by the boundary data, the following Hölder estimate is useful.

**THEOREM 4.10** Suppose \(\Omega\) is a cylindrical domain, and \(f\) satisfies the structure condition

\[
0 \leq f(x, t, z, p) \leq \mu(|z|) d^\beta |p|^\alpha
\]

for all \((x, t) \in \tilde{N}\) a neighbourhood of \(\partial \Omega\), \(z \in \mathbb{R}\), \(|p| \geq \mu(|z|)\), \(\mu\) a nondecreasing function, \(\beta = \alpha - n - 1 \geq 0\). Then

\[
|u(., t)|_{\lambda; \Omega_t} \leq C
\]

where \(C\) and \(\lambda\) depend on \(n, \mu, \mu', \beta, \tilde{N}, \Omega, |u|_{0; \Omega}\), and \(|\phi|_{1; \Omega}\).
Proof: Suppose $\psi$ is the defining function for $\Omega_\alpha$. Then I define a barrier by

$$w(x,t) = \psi - C(1+t)\psi(x)^\lambda$$

Then since $\tilde{d}$ is comparable to $\psi$, on a neighbourhood of the boundary one obtains

$$F[w] > C^{n+1}\lambda \psi^{n+1} - n\text{det}(-D^2\psi + (1-\alpha)\psi^{-1}D_i\psi D_j\psi)$$

Now if $\lambda$ is fixed, then $C$ may be chosen sufficiently large so that in $\tilde{N}$,

$$|Dw| < 2\lambda C\psi^{\lambda-1}$$

Then one obtains

$$F[w] > C_0|Dw|^{\alpha} C^{n+1-\alpha} \lambda^{n-\alpha} (1-\lambda)^\alpha (n+1-\alpha) \psi^{n+1-\alpha} \lambda + \alpha - n + \rho \psi^{\rho - 1}$$

Thus as long as $\gamma > (n+1-\alpha)\lambda + \alpha - n + \rho$, it will follow that $w$ is a barrier for the equation on $\tilde{N}$ if $\psi < \varepsilon$ if $C$ and $\varepsilon$ are chosen so that $\varepsilon^{\rho - 1}C^{n+1-\alpha} > C_1$, where $C_1$ depends on $\lambda$ and known data. To ensure that $w$ lies below $u$, it is also necessary that $Cs^{\lambda} > |u|_0$. These last two inequalities are satisfied for $C$ sufficiently large and

$$\delta = (|u|_0/C)^{1/\lambda} \text{ if } (\delta > n+1)$$

$$\lambda < (\rho - 1)/(n + 1 - \alpha)$$
A Hölder estimate for $u$ with respect to $t$ is also easily established once a gradient bound is known. To see this, consider the function

$$
\psi(x, t) = t^{1/(n+1)}(R-\|x\|^2)
$$

Clearly

$$
F[\psi] = (R-\|x\|^2).1/(n+1).t^{-n/(n+1)}.2^n t^{n/(n+1)}
= 2^n(n+1)^{-1}.(R-\|x\|^2)
$$

Thus by choosing $R$ large, and multiplying $\psi$ by a sufficiently large constant, it follows that $\psi$ is a subsolution of the equation. Moreover, by subtracting $u(0,0) + Du(0,0)$ from $u$, one may assume that $u(0) = \psi(0)$ and $u|_{\partial \Omega} \leq \psi|_{\partial \Omega}$. Hence I have established an estimate for the $C^{1/n+1}$-norm of $u$ with respect to $t$.

### 4.4 Classical Solvability

Combining all the foregoing estimates allows me to formulate some theorems asserting the classical solvability of the first boundary value problem for the parabolic Monge-Ampere type equation under reasonable structure conditions. The following theorems may be proved by a combination of the method of continuity and approximation arguments in exactly the same way as the elliptic case presented in [G.T. 1] and [U 1].

Note that the boundary estimates for second derivatives are exactly as in the last chapter in the case of admissible domains, while the proof of the elliptic case carries over immediately for cylindrical domains when one
has a uniform bound away from zero for the time derivative. In the case of cylindrical domains without a bound away from zero on the time derivative of the boundary data, one may use the estimate of Theorem 4.10 to conclude that classical solutions exist with a uniform Hölder estimate.

**THEOREM 4.10** Suppose $\Omega$ is a $C^2$ admissible domain, $\varphi \in C^{2,1}(\overline{\Omega})$, and $f$ is a positive function in $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfying $f_z \not< 0$ and the structure condition of Theorem 4.9. Then the classical first boundary value problem has a unique convex decreasing solution in $C^{2,1}(\Omega) \cap C^{0,1;0,1/n+1}(\overline{\Omega})$.

**THEOREM 4.11** Suppose $\Omega$ is a $C^2$ cylindrical domain, $\varphi \in C^{2,1}(\overline{\Omega})$ a uniformly convex function with $0 < -\epsilon < 0$, and $f$ is a positive function in $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfying $f_z \not< 0$ and the structure condition of Theorem 4.8. Then the classical first boundary value problem has a unique convex decreasing solution in $C^{2,1}(\Omega) \cap C^{0,1;0,1/n+1}(\overline{\Omega})$. If $\varphi$ is convex and decreasing (not necessarily uniformly), then the classical first boundary problem has a solution in $C^{2,1}(\Omega) \cap C^{0,1/2;0,1/n+1}(\overline{\Omega})$. 
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