APPLICATION OF THE GALERKIN METHOD TO
PROBLEMS IN TWO DIMENSIONAL COMPRESSIBLE FLUID FLOW

A thesis submitted for the degree of Master of
Science at the Australian National University

by

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Statement

I hereby declare that to the best of my knowledge, no work previously written or published has been used in this thesis except where referenced in the text.

Y. Wiriyawit
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I wish to express my grateful thanks to my supervisor, Dr. B. Davies for his help, encouragement and patience in the course of this work. Also my thanks to members and friends of the Department of Applied Mathematics in the SGS for their help and assistance. To my typist Anna Zalucki, my thanks for a job well done. Finally, I wish to thank the head of the Department, Professor A. Brown for allowing me to undertake this work.
Abstract

This thesis is devoted to the indirect or design problem for the steady, irrotational, isentropic, two-dimensional motion of an inviscid compressible fluid. The shape of the boundary is determined as part of the complete solution to the problem, assuming that the velocity on the boundary surface is prescribed. To obtain the solution numerically, we make use of the Galerkin Method together with suitable conformal transformations which effect some important simplifications in constructing a trial function.

In the first chapter we give a brief survey of previous work done on cascades using either the direct method, the indirect method or the hodograph method. In chapter II, variational and related methods are discussed in detail. We propose the use of a Galerkin Method using a trial function which does not satisfy the boundary condition. Then we briefly present numerical solutions obtained through the application of the Galerkin Method to a channel flow. Chapter III sets out the conformal transformation used to map the rather complicated potential plane for cascades to a square in which the solution to the problem can be written down in a comparatively simple form. The Galerkin Method explained in chapter II is then applied to cascades in general and some mathematical details are given. Chapter IV gives numerical results obtained from the application of the technique so far developed to the case of a cascade with no circulation.

We then consider the case of cascade flow with circulation, i.e. a cascade which turns the flow. This introduces extra complications in the analytical process, as will be discussed in chapter V. Finally, general observations concerning our technique and the various results, together with suggestions of some possible further studies are given in the concluding chapter.
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subscript B signifies values on the boundary
subscript u and d signify values upstream and downstream respectively
subscript S signifies values at the stagnation point
\( \phi \) velocity potential function
\( \psi \) stream function
\( \phi_S \) value of \( \phi \) at stagnation point
\( \phi_L \) a parameter in \( q_B \) effecting blade shape
\( \Delta \phi \) difference in \( \phi \) between two corresponding points of fluid channel
\( \Delta \psi \) difference in \( \psi \) between opposite walls of channel
\( \Delta \phi_C \) change in \( \phi \) around the closed curve \( C \)
\( \gamma \) local velocity, velocity field
\( u,v \) velocity components
\( c \) local speed of sound
\( a \) stagnation speed of sound
\( q \) fluid speed
\( q(n) \) \( n^{th} \) approximation of \( q \), \( n = 0,1,2,\ldots \)
\( q_B \) fluid speed on boundary
\( q_u, q_d \) upstream and downstream speeds
\( \theta \) fluid direction in the physical plane
\( \Delta \theta \) turning angle
\( \theta_u, \theta_d \) upstream and downstream angles respectively
\( \alpha \) angle at stagnation point
\( x,y \) physical coordinates
\( dn \) differential distance normal to the streamline
\( ds \) differential distance along the streamline

**NOTATION**
\( r_i \quad \) the residuals, \( i = 1,2,3,... \)

\( a_i, a_{ij} \quad \) coefficients in expression for \( q \), \( i = 1,2,... \)

\( a_N^{(n)} \quad \) coefficient vector from \( n \)th iteration with \( N \) trial functions

\( h_i \quad \) trial functions, \( i = 1,2,... \)

\( h \quad \) distance in potential plane between corresponding points on adjacent airfoils

\( h_S \quad \) a function giving required behaviour of \( q \) near stagnation point

\( g_i(\xi), g_j(\eta) \quad \) orthogonal functions in \( \xi \) and \( \eta \) respectively

\( A \quad \) upstream value of \( q \) as determined

\( A_{ij} \quad \) matrix elements when Galerkin Method is used

\( M_{ij} \quad \) matrix elements in Hendry's approach

\( L, M \quad \) differential operators

\( L^+, M^+ \quad \) adjoint operators for \( L \) and \( M \) respectively

\( F_c \quad \) core function in the expression for \( q \)

\( F, G \quad \) variables introduced by Sylvester and Fitch, both are functions of \( q \)

\( N \quad \) number of trial functions used in expression for \( q \)

\( I \quad \) number of \( \zeta \)-functions used in expression for \( q \)

\( J \quad \) number of \( \eta \)-functions used in expression for \( q \)

\( \bar{\gamma} \quad \) ratio of specific heats of the fluid

\( \zeta \quad \) complex variable = \( \gamma + i\mu \)

\( \omega \quad \) complex variable = \( \phi + i\psi \)

\( \omega_0 \quad \phi_0 + i\psi_0 \quad \text{ (incompressible flow)} \)

\( R \quad \) region of interest

\( \partial R \quad \) boundary of \( R \)

\( H \quad \) perpendicular distance between two blades in physical
plane

\( H_1, H_2, R_1, R_2 \) parameters in \( q_B \)

\( T_1, T_2, T_\phi \) factors in \( q_B \) on two blade surfaces

\( T \) temperature

\( p \) pressure

\( \rho \) density

\( \rho_s \) density at stagnation point

\( \Delta, \delta \) Jacobians of transformations

\( \Gamma \) circulation

\( b \) blade thickness at downstream infinity

Thus, there are six basic variables, the physical coordinates \( x, y \), the velocity components \( u, v \), and the potential functions \( \phi, \psi \). This division has led to three main approaches to solving the problem, which are classified as follows.

1) The direct method. Here \( \phi \) and \( \psi \) are treated as the dependent variables which are determined as functions of \( x \) and \( y \), the physical coordinates. Since \( x, y \) are independent variables, it is natural to prescribe the boundary values of \( (x, y) \) as functions of \( (x, y) \) on some boundary in the physical plane. Thus, the physical shape of the boundary is given, and the flow must be computed.

2) The indirect method (or potential plane approach). The essential character of this method is that the local velocity components \( (u, v) \) are treated as the dependent variables, with the potential functions...
CHAPTER 1

INTRODUCTION

The theoretical analysis of two-dimensional compressible potential flow is complicated by the intrinsic non-linearity of the equations, and many different techniques have been developed for obtaining numerical solutions. Potential flow is described by introducing a potential function $\phi$ and a stream function $\psi$, in terms of which the local velocity $\mathbf{v} = (u,v)$ and the density $\rho$ are given by

$$ \mathbf{v} = \begin{pmatrix} \frac{\partial \phi}{\partial x}, & \frac{\partial \phi}{\partial y} \end{pmatrix} $$

$$ \rho \mathbf{v} = \begin{pmatrix} \frac{\partial \psi}{\partial y}, & -\frac{\partial \psi}{\partial x} \end{pmatrix} $$

Thus, there are six basic variables, the physical coordinates $x,y$, the velocity components $u,v$, and the potential functions $\phi,\psi$. This division has led to three main approaches to solving the problem which are classified as follows.

1) The direct method. Here $\phi$ and $\psi$ are treated as the dependent variables which are determined as functions of $x$ and $y$, the physical coordinates. Since $x,y$ are independent variables, it is natural to prescribe the boundary values of $(\phi,\psi)$ as functions of $(x,y)$ on some boundary in the physical plane. Thus, the physical shape of the boundary is given, and the flow must be computed.

2) The indirect method (or potential plane approach). The essential character of this method is that the local velocity components $(u,v)$ are treated as the dependent variables, with the potential functions
as the independent ones. The velocity distribution on the
boundary is prescribed as a function of \( \phi \), but the physical shape
of the boundary must be computed along with the details of the flow.
For this reason, the method is useful as a design tool. However, the
potential plane approach can also be used to solve some direct problems,
by prescribing the flow direction as a function of \( \phi \), but this is
less common.

3) The hodograph method. This entails transformation to the
hodograph plane, where either \((x,y)\) or \((\phi,\psi)\) are the dependent
variables, and the velocity components \((u,v)\) are the independent ones.
Use of the hodograph plane has many advantages for compressible flow.
In particular, the equations which determine the stream and potential
functions are linear in this representation and therefore easy to solve.
However, it is extremely difficult to prescribe the boundary conditions,
either in the form of the shape of the boundary or the boundary velocity
distribution; in addition, complications arise when it becomes necessary
to transform the solution back to the physical plane.

Many papers have been written concerning all three of the above
approaches. Most techniques seem to be successful when the peak velocity
remains subsonic, although the treatment of stagnation points is a
difficulty, particularly with the indirect approach. However, greater
care always seems to be needed when the flow or part of it becomes
supersonic. The area of current activity involves transonic or mixed
flows where both subsonic and supersonic areas coexist. We shall discuss
each of the above main headings in more details below, particularly as it
applies to the "cascade" problem.
1.1 DIRECT METHOD

Various techniques have been proposed to solve the equation for $\phi$ for two-dimensional compressible flow, which has the form

$$\phi_{xx} + \phi_{yy} = \frac{1}{c^2} \left( \phi_x^2 \phi_{xx} + \phi_y^2 \phi_{yy} + 2\phi_x \phi_y \phi_{xy} \right)$$ (1.2)

In addition to the differential equation, the physical shape of the boundary is prescribed on which some boundary condition is specified. The equation is non-linear and hence its solution for the flow around an airfoil must involve numerical methods. We are particularly interested, in this thesis, in the problem of flow through a cascade of aerofoils, by which we mean an infinite set of similar aerofoils at the same incidence, and spaced at equal distance from each other. Then we have, in addition to the prescribed boundary condition, the property that the flow repeats itself periodically because the cascade is periodic. (For more detail, see section 3.2). For the purpose of setting up various approximate methods for the cascade problem, it is convenient to introduce the concept of solidity, which is the ratio of chord to the blade spacing in the cascade. When this ratio is comparable to unity, the cascade is of high solidity; conversely low solidity corresponds to widely spaced blades.

For the flow around a cascade of high solidity, for example a turbine hub, and some guide vane cascades, Stanitz and Prian (Gostelow 1973) gave solutions based on the channel flow approach, i.e. the flow is analysed approximately by the use of the techniques for channel flows. A rounding off process was needed to close the blades at the leading and trailing edges. Therefore, accurate results are not possible in the vicinity of stagnation points.
D.A. Frith (Frith 1973 a,b) suggested mapping the region external to the airfoils to a regular region, the interior of a unit circle. This simplifies the applications of the boundary conditions as well as the establishment of uniform inlet and outlet conditions. The periodic boundary conditions are automatically satisfied. He then used the technique of finite differences (Isaacson and Keller 1966), choosing a suitably spaced grid within the circle, obtaining a solution as a perturbation stream function which, added to the stream function for incompressible flow, gives the solution to the compressible case. This method can be used for high subsonic inlet Mach number which gives rise to supersonic patches on the airfoils; numerical accuracy is good because of the comparatively small magnitude of the perturbation stream function.

For cascades of low solidity, for example some fan blades, the solution can be approached by linearizing the above potential equation either in the physical (x,y) plane or in the hodograph plane. Evidence suggests that for a thin profile this method provides dependable results. Solutions have been given by Stanitz and Prian (Stanitz and Prian 1951) following the assumption that all boundary gradients and perturbations on the inlet velocity vector are small.

For general flow around cascades intermediate between the above two extremes, for example most compressor fan and turbine cascades, Poggi (Poggi 1932) obtained a series solution for the velocity potential function \( \phi \) consisting of the leading term being the known incompressible flow together with a series of terms in powers of the free stream inlet Mach number, the ratio of the local velocity to the speed of sound. Price (Price 1966), on the other hand, obtained the solution in terms of the series for the stream function \( \psi \), the leading term being the
incompressible solution $\psi_0$. His approach appears to give reasonable approximation for local Mach numbers as high as 1.1 provided that the prominent feature controlling the pressure distribution is the transverse pressure gradient due to streamline curvature rather than one dimensional area variations.

Imbach (Imbach 1964) used an iterative method. The source distribution corresponding to the right-hand side of equation (1.2) is firstly obtained from the incompressible flow solution. Then, the velocity components $(u', v')$ in x and y directions due to this source distribution are computed. A line source must be placed around the boundaries such that it exactly cancels all normal velocities. The velocity components $(u', v')$ are then treated as perturbations on the original velocities and hence a new flow is established. This new flow is then treated on the same basis and the procedure is repeated until convergence is obtained. Unfortunately the iterative methods converge slowly as the sonic condition is approached.

Smith and Frost (1969-1970) developed a Matrix method for the general case where a Poisson-type differential equation was solved using finite difference techniques with a ten-point star (Isaacson and Keller 1966). A band matrix solution was chosen so as to ensure efficiency and stability, and the inlet Mach number was increased in gradual steps to the desired value.

The Streamline Curvature technique was proposed by Katsanis (Katsanis 1969). It is based on finite differences, using as a grid the intersections between the streamlines and the "quasi-orthogonals", which are simply lines passing from one channel wall to the other in an arbitrary (usually pitchwise) direction. The equations of motion are used in a form which relates the streamline
velocity gradient directly to the radius of curvature of the streamline, hence the name of the technique. The least satisfactory feature of this method is its inability to give accurate potential flow solution in the vicinity of the blade edges, although it appears to be an efficient operation otherwise and is capable of dealing with velocities above the sonic condition.

1.2 INDIRECT METHOD

In the potential plane the equations of motion take the form

\[
\frac{1}{2} \frac{\partial \rho q}{\partial \phi} + \frac{\partial \theta}{\partial \psi} = 0
\]

(1.3)

\[
\rho \frac{\partial q}{\partial \psi} - \frac{\partial \theta}{\partial \phi} = 0
\]

the derivation of which will be discussed in more detail in section 2.2.

Stanitz (Stanitz 1951) combined the equations to yield

\[
\frac{\partial^2 \log \rho}{\partial \phi^2} + \frac{\partial^2 \log q}{\partial \phi^2} + \frac{\partial \log \rho}{\partial \phi} \left( \frac{\partial \log \rho}{\partial \phi} + \frac{\partial \log q}{\partial \phi} \right) + \rho \frac{\partial \log q}{\partial \psi} \cdot \frac{\partial \log \rho}{\partial \psi} + \rho^2 \frac{\partial^2 \log q}{\partial \psi^2} = 0
\]

(1.4)

He then solved it by either relaxation, matrix or Green's function techniques. The Green's function solution of the Stanitz method has been further developed by Payne (Payne 1964) and is extensively used.

Sylvester and Fitch (Sylvester and Fitch 1974) attempted to calculate the flow on a blade to blade surface of revolution by introducing new dependent variables \( F \) and \( G \), in terms of which the equations (1.3) become
They then used finite differences on a rectangular grid in a new plane mapped from the potential plane. They had difficulties with the leading and trailing edges of the airfoils. Nevertheless, we have made use of their functions $F$ and $G$ in this thesis.

In recent years, variational methods have become popular in solving differential equations numerically. These methods have been applied in Strang and Fix (Strang and Fix 1973) and Whiteman (Whiteman 1973) where the trial functions are piecewise continuous and non-zero over only a small part of the region of interest (that is the Finite Element Method).

On the other hand, the trial functions can be chosen to be non-zero over the whole region of interest, in which case it is termed a Global Variational Method (GVM). The conventional method is to use the trial functions which explicitly satisfy the boundary conditions. However, work has been done to relax such constraints on the trial functions. Then, the variational method reproduces the conditions on the boundary as well as obtaining the solution to the equation in the region of interest.

The attempt along this line can be found in Arthurs' work (Arthurs 1970) on Complimentary Principles and in Hendry and Hennell (Hendry and Hennell 1975). In Davies and Hendry (Davies and Hendry 1976), the method is applied to the problem of indirect design of a compressible fluid channel. In Hendry's recent work, the method is presented in a more general form, so that any differential operator and any boundary conditions can be handled, at least in principle. Attempts to obtain a solution in terms of a trial function which is defined directly in the physical plane were also made by Davies and Hendry, but without such success.

\[
\frac{\partial F}{\partial \Psi} - \frac{\partial \Phi}{\partial \Psi} = 0
\]

\[
\frac{\partial G}{\partial \Phi} + \frac{\partial \Phi}{\partial \Psi} = 0
\]
can be handled, at least in principle. Attempts to obtain a solution in terms of a trial function which is defined directly in the $(\phi, \psi)$-plane were also made by Davies and Hendry, but without much success.

In this thesis, the Ritz-Galerkin Method is applied to the non-linear cascade problem using trial functions which do not explicitly satisfy the boundary conditions. In principle, we ought to consider two stagnation points, one associated with the leading, and the other with the trailing edges of the airfoils. However, we maintain, on physical grounds, that the behaviour at the leading edge is of much greater importance than at the trailing edge, and we therefore consider it a safe enough approximation to ignore the trailing edge stagnation point. Consequently, some rounding or truncating process will have to be used at the trailing edge.

1.3 THE HODOGRAPH METHOD

The use of the hodograph transformation in the theory of plane compressible fluid flow was initiated by Chaplygin in 1904. In this plane the equations of motion are transformed to the linear hodograph equations (Yoshihara 1972), given by

\[(a^2 - u^2) \frac{\partial v}{\partial v} + 2uv \frac{\partial x}{\partial v} + (a^2 - v^2) \frac{\partial x}{\partial u} = 0 \tag{1.6}\]

\[\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u} = 0\]

where $a$ is the stagnation speed of sound.

Because of the linearity, general solutions of (1.6) can be obtained by linear superposition. Such a solution will represent a
meaningful solution only if it is at the same time the solution of
the equations in the physical plane. This is assured by inverting the
transformation, a procedure which turns out to be so complicated that it
offsets the advantages to be obtained from linearity.

The earliest application, to our knowledge, of the hodograph
method to the problem of flow around a body was given by Von Kármán and
Tsien (Von Kármán 1941, Tsien 1939). However, for flow around a body,
the hodograph plane is a multiply sheeted Riemann surface, and this
introduces the problem of how to analytically continue the compressible
flow solution around the various singularities. The exact solution of
transonic flow related to the non-circulatory incompressible flow around
a circular cylinder was given by Goldstein, Lighthill and Craggs (Goldstein,
Lighthill and Craggs 1948), and by Cherry (Cherry 1947). The work was
later extended to the circulatory flow case by Lighthill (Lighthill 1948)
and by Cherry (Cherry 1949a).

A different, more general form of the theory was given by
Lighthill (Lighthill 1947b), in which the transformation from an incompressible
to a compressible flow solution is defined as an operator, for which any
incompressible flow given in the physical plane is admissible as an origin.
However this approach is restricted from the physical point of view because
it is valid only for strictly subsonic flow. Lighthill then proposes
that the solutions can be extended to the transonic flow if the series
expansion of a certain type (a generalized Laurent expansion) for the
analytic potential function in the hodograph of the original flow is
available. Since only a limited number of incompressible flows are known
as yet from which one might be able to obtain such information, this places
an essential restriction on the class of admissible problems that can make
use of this method.
More recently, significant progress has been made on super critical flow over airfoils by Nieuwland and Boerstoel and by Garabedian and Korn (Yoshihara 1972). The solutions as obtained by them are analytic and general flow feature can be obtained in a precise fashion. However, the complexity of these solutions caused in particular by the highly-oscillatory hypergeometric functions used, and the slowness of convergence of the series, makes the task of a detailed numerical evaluation of the solution an extremely difficult one.

The principles of the Global Variational Method are elaborated in section 2.3 in a form which is applicable to a linear operator. However, the equations of section 2.2 are non-linear, and to apply a linear method to this system Hendry made use of linear (or Picard) iteration, and set up a scheme termed the Iterative-Variational Method (Hendry 1976a), which is also discussed in section 2.3.

In section 2.4 we show that the Iterative-Variational Method is closely related to the Dynamic Method (Morison 1975) and propose a new procedure which takes advantage of this fact. Finally we discuss briefly the application of the method to the flow through a channel in section 2.5.

2.2 THE FUNDAMENTAL EQUATIONS

We are continuing our discussion, in case that is, to the study of the steady motion of an incompressible fluid, since in the steady in which the transformation of work done and by viscosity are to be distinguished. The flow will be fully entered when the velocity field $\mathbf{u}$, pressure $p$, and...
2.1 Introduction

In this chapter we reproduce the equations of motion which determine the fluid speed \( q \) and the flow direction \( \theta \) in the potential plane, assuming that the steady compressible flow motion is irrotational, inviscid and isentropic. The problem is stated in two dimensions as a first approximation to the three dimensional case, as found in turbomachinery, for example.

The principles of the Global Variational Method are elaborated in section 2.3 in a form which is applicable to a linear operator. However, the equations of section 2.2 are non-linear, and to apply a linear method to this system Hendry made use of linear (or Picard) iteration, and set up a scheme termed the Iterative-Variational Method (Hendry 1976a), which is also discussed in section 2.3.

In section 2.4 we show that the Iterative-Variational Method is closely related to the Galerkin Method (Marchuk 1975) and propose a new procedure which takes advantage of this fact. Finally we discuss briefly the application of the technique to the flow through a channel in section 2.5.

2.2 The Fundamental Equations

We are confining our attention, in this thesis, to the study of the steady motion of an inviscid fluid, that is the motion in which the transformation of work into heat by viscosity can be disregarded. The flow will be fully defined once the velocity field \( \mathbf{v} \), pressure \( p \), density \( \rho \), and temperature \( T \) are prescribed as functions of the space coordinates in order to satisfy a sufficient set of boundary and initial conditions. The problem is formulated as a statement about the flow of a continuum which is described in the form of the equation of continuity;
density $\rho$, and temperature $T$ are known as functions of the space coordinates so as to satisfy a sufficient set of boundary and initial conditions. In a macroscopic sense the fluid may be regarded as a continuum which is described in the form of the equation of continuity:

$$\text{div}(\rho \mathbf{v}) = 0 \quad (2.1)$$

Now, if we assume that the motion is such that there is no heat conduction from or to the outside and in the fluid itself, we have an isentropic flow, and the entropy is constant. Furthermore, if the total energy of the fluid is constant the motion is irrotational, that is

$$\nabla \times \mathbf{v} = 0 \quad (2.2)$$

Introducing the fluid speed $q$ (in units of the stagnation speed of sound) and the direction $\theta$ in the physical plane, by

$$u = q \cos \theta$$

$$v = q \sin \theta$$

we can define the stream function and the potential function, $\psi$ and $\phi$, as

$$d\psi = \rho q dn \quad (2.3)$$

$$d\phi = q ds \quad (2.4)$$

where $dn$ and $ds$ are the differential distances normal to and along the streamline, respectively, related to the physical coordinates by
density $\rho$, and temperature $T$ are known as functions of the space coordinates so as to satisfy a sufficient set of boundary and initial conditions. In a macroscopic sense the fluid may be regarded as a continuum which is described in the form of the equation of continuity;

$$\text{div}(\rho v) = 0 \quad (2.1)$$

Now, if we assume that the motion is such that there is no heat conduction from or to the outside and in the fluid itself, we have an isentropic flow, and the entropy is constant. Furthermore, if the total energy of the fluid is constant the motion is irrotational, that is

$$\nabla \times v = 0 \quad (2.2)$$

Introducing the fluid speed $q$ and the direction in the physical $\theta$, such that

$$u = q \cos \theta$$

$$v = q \sin \theta$$

we can define the stream function and the potential function, $\psi$ and $\phi$, as

$$d\psi = \rho qdn \quad (2.3)$$

$$d\phi = qds \quad (2.4)$$

where $dn$ and $ds$ are the differential distances normal to and along the streamline, respectively, related to the physical coordinates by
\[ dx = ds \cos \theta \]
\[ dy = ds \sin \theta \]

Under these definitions, therefore, the continuity condition (2.1) for the steady flow and the irrotational condition become, respectively

\[ \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho q) + \frac{\partial \theta}{\partial \psi} = 0 \quad (2.5) \]

\[ \frac{\rho}{q} \frac{\partial q}{\partial \psi} - \frac{\partial \theta}{\partial \phi} = 0 \quad (2.6) \]

Again, assuming the fluid motion is such that viscosity and heat conduction can be neglected, which means that changes of state at a fluid particle are adiabatic, then the pressure is a function of the fluid density only. For the cases we will be dealing with, the pressure-density relationship can be written in the form

\[ p = k\rho \gamma \]

where \( \gamma \) is the ratio of the specific heats of the fluid \( (\gamma \approx 1.4 \text{ for air}) \).

Hence

\[ dp = \gamma k\rho \gamma^{-1} \, d\rho \quad (2.7) \]

and the square of the local velocity of sound is

\[ c^2 = \frac{dp}{d\rho} = \gamma k\rho \gamma^{-1} \]

(2.10)
Making use of Bernoulli's equation,

\[ \int \frac{dp}{\rho} + \frac{1}{2}q^2 = \text{constant} \]  

(2.8)

and inserting equation (2.7), we have

\[ \gamma k \rho \int_{\rho_s}^{\rho} \rho^{\gamma-2}dp + \frac{1}{2}q^2 = 0 \]

(2.9)

with \( \rho_s \) being the density at the stagnation point.

Integrating the above equation gives

\[ \frac{\gamma k}{\gamma-1} (p^{\gamma-1}-\rho_s^{\gamma-1}) + \frac{1}{2}q^2 = 0 \]

which, after rearranging, gives

\[ \rho = \left( 1 - \frac{1}{2}(\gamma-1)q^2 \right)^{\frac{1}{\gamma-1}} \]  

(2.10)

where \( q \) is expressed in units of the stagnation speed of sound and \( \rho \) in the units of stagnation density.

Eliminating \( \theta \) from the equations (2.5) and (2.6), we obtain a non-linear second order partial differential equation for \( q \):

\[ \frac{\partial}{\partial \phi} \left\{ f_1(q) \frac{\partial q}{\partial \phi} \right\} + \frac{\partial}{\partial \psi} \left\{ f_2(q) \frac{\partial q}{\partial \psi} \right\} = 0 \]

(2.11)
where

\[ f_1(q) = \frac{1}{\rho q} \left\{ \frac{1 - \frac{1}{2}(γ+1)q^2}{1 - \frac{1}{2}(γ-1)q^2} \right\} \tag{2.11} \]

and

\[ f_2(q) = \frac{ρ}{q} \tag{2.12} \]

with \( q \) prescribed on the boundary \( \partial R \) of the region of interest \( R \).

### 2.3 Iterative Variational Method

We present here Hendry's approach (Hendry 1976a) in which a Global Variational Method, not requiring the boundary conditions to be met by the trial functions, is used together with Linear (or Picard) Iteration. The variational principles involved can be found in Arthurs' work (Arthurs 1970); they are also discussed and application given in Hildebrand (Hildebrand 1962). Hendry extends them to the present non-linear problem (equation (2.10)), by the use of linear iteration.

Consider, first, the equation

\[ Lq = f \tag{2.13} \]

over some region \( R \) where \( L \) is a linear differential operator acting on \( q \). It has been shown that if

\[ L^*p = f \]

is the adjoint equation and a suitable inner product \( \langle \cdot, \cdot \rangle \) is defined over the region \( R \), by which the Hermitian conjugate operator \( L^* \) is defined in terms of the inner product as...
\[ \langle p, Lq \rangle = \langle L^*_p, q \rangle \]  
(2.14)

then the functional

\[ F(p, q) = \langle p, Lq \rangle - \langle p, f \rangle - \langle f, q \rangle \]  
(2.15)

is stationary about the solution of the original equation (2.13).

This GVM or Rayleigh-Ritz method is a general procedure for obtaining approximate solutions of the problems expressed in variational form. The procedure consists essentially of assuming that the desired stationary function for a given problem can be approximated by a linear combination of a suitably chosen set of functions, as

\[ q(x) = \sum_{i=1}^{N} a_i h_i(x) \]  
(2.16)

where \( N \) and \( a_i \) are constants to be determined. Usually the trial functions \( h_i \) are to be chosen so that the above expression satisfies the specified boundary conditions. However, we will relax this constraint and make use of trial functions which do not explicitly satisfy the boundary conditions.

Suppose therefore that the flow is subject to the boundary condition of the form

\[ Mq = g \]  
(2.17)

on the boundary \( \partial R \) of \( R \), \( M \) being a linear operator. It has been shown that the functional
\[ F(p,q) = \langle p, Lq \rangle - \langle p, f \rangle - \langle f, q \rangle + \beta\{\langle p, Mq \rangle_B - \langle p, g \rangle_B - \langle g, q \rangle_B\} \]

(2.18)

is stationary about the solution of the equation (2.13) subject to (2.17) and the adjoint equations

\[
L^*p = f \quad \text{in } R
\]

(2.19)

\[
M^*p = g \quad \text{on } \partial R
\]

where \(\langle \, , \, \rangle_B\) signifies an inner product defined on \(\partial R\) and the operator \(M^*\) is defined as

\[
\langle M^*p, q \rangle_B = \langle p, Mq \rangle_B
\]

\(\beta\) is an arbitrary parameter which is to be suitably chosen for the problem. Intuitively, though, it is a measure of how much the variational method should feel the effect of the boundary condition. Hendry's results obtained from the channel flow indicated that the value of \(\beta\) is not of critical importance, however the results obtained for cascades (which will be presented in chapters 4 and 5) indicate that it does play an important role in the convergence to a reasonable solution of the problem.

Introducing a complete set \(\{h_i\}\) which do not satisfy the boundary condition, we write

\[ q_t = \sum_{i=1}^{N} a_i h_i \]

(2.20)

\[ p_t = \sum_{i=1}^{N} b_i h_i \]

(2.21)
On substituting these expressions into the functional (2.18) and finding the stationary value of $F(p_t, q_t)$ with respect to $b$, we obtain a set of linear equations for the coefficients $a_i$ which can be written in a matrix notation as

$$Ma = c$$

where $M$ is a matrix with elements

$$M_{ij} = \langle h_i, Lh_j \rangle + \beta \langle h_i, Mh_j \rangle_B$$

and $c$ is a vector with

$$c_i = \langle h_i, f \rangle + \beta \langle h_i, g \rangle_B$$

whilst

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

Now, the idea indicated above is extended to the non-linear problem, i.e. when $L$ is a non-linear operator, as follows. For the flow through a channel, the equation is

$$\frac{\partial}{\partial \phi} \left( f_1(q) \frac{\partial q}{\partial \phi} \right) + \frac{\partial}{\partial \psi} \left( f_2(q) \frac{\partial q}{\partial \psi} \right) = 0$$

(2.23a)

over the region $R$, which is the area internal to the channel. $q$ is prescribed on the boundary $\partial R$ consisting of the two walls of the channel,
together with the values at upstream and downstream infinity. We write this as

\[ q = q_B \quad \text{on } \partial \mathcal{R} \quad \text{(2.23b)} \]

So that we can apply the variational principle to the equation (2.23a), we need the differential operator \( L \) to be linear. This is achieved by assuming, at each stage of the calculation, that \( f_1(q) \) and \( f_2(q) \) are known functions of \( \phi \) and \( \psi \). The method chosen by Hendry is standard linear iteration, in which the equations (2.23) are written as

\[
\begin{align*}
L(n)q(n+1) &= 0 \quad \text{in } \mathcal{R} \\
M(n)q(n+1) &= 0 \quad \text{on } \partial \mathcal{R} \\
\end{align*}
\quad \text{(2.24)}
\]

where \( L(n), M(n) \) are the operators with \( q \) set equal to \( q(n) \), the (approximate) value of \( q \) determined upon the previous iteration. Thus

\[ L(n) = \frac{\partial}{\partial \phi} \left[ f_1(q(n)) \frac{\partial}{\partial \phi} \right] + \frac{\partial}{\partial \psi} \left[ f_2(q(n)) \frac{\partial}{\partial \psi} \right] \quad \text{(2.25)} \]

in the equation (2.24). The variational method can then be applied to the linear system (2.24). A starting value \( q(0) \) must be suitably chosen to commence this scheme.

The function being evaluated, \( q^{(n+1)} \), consists of two parts. So that \( q^{(n+1)} \) will satisfy the condition prescribed upstream and downstream, the first part of \( q^{(n+1)} \), which we call the core function \( F_c \), is some simple function which serves this purpose. The second part is then a suitable linear combination of a complete set \( \{h_i(\phi, \psi)\} \) of functions
which are zero upstream and downstream. Thus, the function takes the form
\[ q(n+1) = F_c + \sum_{i=1}^{N} a_i (n+1) h_i (\phi, \psi) \] (2.26)

Inserting the above expression of \( q(n+1) \) into the functional (2.18) as before, we obtain a set of linear equations for \( a_i (n+1) \) as follows

\[ M_N (n) a_N (n+1) = c_N (n) \quad n = 0, 1, 2, \ldots \] (2.27)

where

\[ M_N (n) (i,j) = \langle h_i, L(n) h_j \rangle + \beta \langle h_i, M(n) h_j \rangle_B \quad i = 1, 2, \ldots, N \] (2.28)

\[ c_N (n) (i) = \beta \langle h_i, q_B - M(n) F_c \rangle_B - \langle h_i, L(n) F_c \rangle \quad i = 1, 2, \ldots, N \] (2.29)

\[ a_N (n+1) = \left\{ a_1 (n+1), a_2 (n+1), \ldots, a_N (n+1) \right\} \]

\[ = \left[ M_N (n) \right]^{-1} c_N (n) \quad n = 0, 1, 2, \ldots \] (2.30)

\( N \) being the number of trial functions used which, once the convergence in \( n \) is reached, we keep increasing until the convergence in \( N \) is reached in turn.

Thus, Hendry's approach is to solve the variational problem corresponding to the system (2.24), with the linear operator given by (2.25), and to iterate (2.30) simultaneously, obtaining \( a_N (n+1) \), and hence \( q_N (n+1) \), from the initial value \( a_N (0) \), until convergence in both \( n \) and \( N \) is achieved.
2.4 THE GALERKIN METHOD

We now propose to replace Hendry's rather indirect approach by the Galerkin Method (Marchuk 1975), which will give a set of linear equations for the coefficients of the expansion (2.16) directly. We want to solve the equation

\[ Lq = f \quad \text{in } R \quad (2.31) \]

subject to

\[ Mq = q_B \quad \text{on } \partial R \quad (2.32) \]

where \( L \) and \( M \) are not necessarily linear operators, and \( \partial R \) is, as before, the boundary of the region \( R \). The method consists of requiring the expressions in (2.31) and (2.32) to be orthogonal to \( N \) linearly independent functions \( k_j \), \( (j = 1,2,\ldots,N) \), over the region \( R \). Explicitly, we require

\[ \langle k_j, Lq \rangle = \langle k_j, f \rangle \quad \text{for } j = 1,2,\ldots,N \]

and

\[ \langle k_j, Mq \rangle_B = \langle k_j, q_B \rangle_B \]

where \( \langle , \rangle \) is some inner product defined on the region \( R \) and \( \langle , \rangle_B \) is that defined on the boundary \( \partial R \). We shall be using the same \( k_j \) as \( h_i \) \( (i,j = 1,2,\ldots,N) \). Rather than solving equations (2.31) and (2.32) simultaneously, we add them with a parameter \( \beta \) and then solve a weaker set of equations.
\[ \langle h_j, L q \rangle + g(\langle h_j, M q-B \rangle_B) = 0 \quad j = 1, 2, \ldots, N \]  

(2.33a)

the function \( f \) being, in fact, zero.

At this stage, it is of certain interest to note the connection between our approach and Hendry's (section 2.3). If we write in (2.24)

\[ q^{(n+1)} = q^{(n)} + \sum_{i=1}^{N} d_i^{(n+1)} h_i \]

(2.24)

where \( d_i^{(n+1)} \) are the coefficients to be determined, then using (2.26), we have

\[ q^{(n+1)} = F_c + \sum_{i=1}^{N} \left[ a_i^{(n)} + d_i^{(n+1)} \right] h_i. \]

By substituting this into (2.27) we find that we have to solve for \( d_i^{(n+1)} \) from a set of linear equations expressed in matrix form as

\[ M_N^{(n)} \left[ a_i^{(n)} + d_i^{(n+1)} \right] = c_N^{(n)} \quad n = 0, 1, 2, \ldots, N \]

with \( M_N^{(n)} \) and \( c_N^{(n)} \) exactly the same as given before in (2.28) and (2.29). Rearranging a little, the above equations become

\[ M_N^{(n)} d^{(n+1)} = - M_N^{(n)} a_N^{(n)} + c_N^{(n)} \]

(2.33b)

where

\[ r_N^{(n)} = L^{(n)}(q^{(n)}) \]
Obviously, Hendry's method converges as $r^{(n)}_\infty$ tends to 0, and equation (2.33b) is an algorithm for finding a sequence of $q^{(n)}$'s which give this convergence.

Now, our approach to solving the equation

$$ r_j(a) = \langle h_j, Lq \rangle + \beta \langle h_j, Mq_B \rangle_B = 0 \quad j = 1, 2, \ldots, N \quad (2.34) $$

is to use Newton's Method (Ortega and Rheinbolt 1970 chapter 7 p.183ff). That is, we express $r(a)$ in a Taylor expansion in powers of the differentials $da$ and then retain only the first term in the expansion. This gives

$$ r(a) + \sum_{i=1}^{N} \frac{\partial r}{\partial a_i} \cdot da_i = r(a') \quad (2.35) $$

where

$$ da = a' - a $$

If we write

$$ q^{(n+1)} = \sum_{i=1}^{N} a_i^{(n+1)} h_i(\phi, \psi) $$

we can then set up an iteration on $n$, starting from some initial values for $a_i^{(0)}$ ($i = 1, 2, \ldots, N$), using the algorithm

$$ r(a^{(n+1)}) = r(a^{(n)}) + \sum_{i=1}^{N} \frac{\partial r}{\partial a_i} \cdot da_i^{(n)} = 0 \quad (2.40) $$

which, in matrix notation, gives
\[
A^{(n)}\delta a^{(n)} = - r^{(n)}
\]

(2.36)

where

\[
A_{ij}^{(n)}(a^{(n)}) = \frac{\partial r_i}{\partial a_j}(a^{(n)})
\]

and

\[
r_i(a^{(n)}) = r_i^{(n)}
\]

\[
= \langle h_i, L^{(n)}q^{(n)} \rangle + \beta \langle h_i, M^{(n)}q^{(n)} - q_B \rangle_B
\]

(2.37)

with \(L^{(n)}\) and \(M^{(n)}\) as defined in (2.24) and (2.25). Note that the matrix \(A^{(n)}\) in our scheme is different from \(M^{(n)}\) used in Hendry's approach. In fact, our scheme gives much more rapid convergence.

The inner products are given by

\[
\langle h_i, h_j \rangle = \iint_R h_i h_j d\phi d\psi
\]

(2.38)

and

\[
\langle h_i, h_j \rangle_B = \int_{\partial R} h_i h_j ds
\]

(2.39)

The boundary inner product is a line integral on the boundary \(\partial R\) of \(R\) which is, in our case, the two walls of the channel. Hence we have

\[
r_i = \int_{\phi=-\infty}^{\phi=\infty} \int_{\psi=0}^{\psi=\Delta \psi} h_i L q d\phi d\psi + \beta \int_{\phi=-\infty}^{\phi=\infty} \{ [h_i (M q - q_B)]_{\psi=0}^{\psi=\Delta \psi} \} d\psi
\]

(2.40)

\[
A_{ij} = \frac{\partial r_i}{\partial a_j}
\]
where $\Delta \psi$ is the difference in the stream function between the opposite walls of the channel. The region $R$ is the semi-infinite strip extending to infinity along the $\phi$ axis in both directions, with the streamlines $\psi = 0$ and $\psi = \Delta \psi$ as the two walls. An iterative procedure has thus been set up consisting of iterating equation (2.24) and solving equation (2.36) to find $a^{(n)}_z$, and consequently $a^{(n+1)}_z$ (and therefore $q^{(n+1)}$), from an initial starting value of $a^{(0)}_z$. The converged solution (in $n$) is also the variational solution to the Linear problem obtained by setting $q$ to the converged solution in $L$. After convergence in $n$ is achieved, the number $N$ of functions used in the expansion for $q_N^{(n)}$ is increased, and the whole iteration process repeated until the convergence with respect to the non-linear iteration and the number of functions in the trial function is reached simultaneously. The algorithm for the overall procedure explained above is given in Table 1.

2.5 A CHANNEL FLOW

For the case of fluid motion through a channel, the problem is to solve the non-linear equation (2.23) with the velocity $q$ given on the channel boundaries $\partial R$. Taking the streamline where the stream function takes the value of zero at the lower wall, and $\Delta \psi$ to be the difference in the stream function between the two walls, we have

$$ r_1 = \int_{-\infty}^{\infty} \int_0^{\Delta \psi} h_1 Lq \psi d\phi - \int_{-\infty}^{\infty} \left[ (q - q_B) f_2 \frac{\partial h_1}{\partial \psi} \right]_B d\phi $$

(2.41)
TABLE I

(i) N starting value
\[ n = 0 \quad a_N^{(n)} = 0 \]

(ii) Calculate \( A_N^{(n)} \) and \( r_N^{(n)} \)

(iii) Solve for \( a_N^{(n)} \)
\[ a_N^{(n+1)} = a_N^{(n)} + da_N^{(n)} \]

(iv) Has \( a_N^{(n+1)} \) converged?

(v) Calculate \( \theta, x, y \)

(vi) Has the solution converged w.r.t. \( N \)?

\[ \text{NO} \]

\[ a_N^{(n)} = a_N^{(n+1)} \]

\[ J = J + 1 \]

\[ n = 0 \quad \text{Set} \quad a_N^{(0)} \]

\[ \text{YES} \]

\[ \text{FINISH} \]
\[ A_{ij} = \frac{\partial r_i}{\partial a_j} \]

\[ = \int_{-\infty}^{\infty} \int_{0}^{\Delta \psi} \left\{ h_{i j} \frac{\partial}{\partial \phi} \left( f_{3} h_{j} \frac{\partial q}{\partial \phi} + f_{1} \frac{\partial h_{i}}{\partial \phi} \right) + \frac{\partial}{\partial \psi} \left( f_{4} h_{j} \frac{\partial q}{\partial \psi} + f_{2} \frac{\partial h_{i}}{\partial \psi} \right) \right\} d\phi d\psi \]

\[ - \int_{-\infty}^{\infty} \left[ h_{i j} \frac{\partial h_{i}}{\partial \psi} \left( f_{2} + (q-q_B) f_{4} \right) \right] d\phi \]

where

\[ f_3 = \frac{\partial f_1}{\partial q} \]

\[ f_4 = \frac{\partial f_2}{\partial q} \]

and

\[ q_N^{(n)} = F_c + \sum_{i=1}^{N} a_i^{(n)} h_i(\phi, \psi), \text{ as in (2.26)} \]

Here, we have chosen functions \( k_i = f_{2} \frac{\partial h_{i}}{\partial \psi} \) (i = 1, 2, ..., N) for the boundary integrals, and set \( \beta = -1 \). This simple choice is possible for channel flow, because of the insensitivity to \( \beta \) already noticed by Hendry. The actual value \( \beta = -1 \) was used by Davies and Hendry in unpublished work involving their "variational iteration" scheme, based on the theory of self-adjoint linear operators.

We simplify the procedure by introducing a new set of coordinates \((\xi, \eta)\) by

\[ \xi = \tanh \lambda \phi \]

\[ \eta = \frac{2}{\Delta \psi} \psi - 1 \quad (2.44) \]

which takes the potential plane into a square plane

\[ -1 \leq \xi, \eta \leq 1 \]
where $\lambda$ is a scaling parameter. Accordingly, the residual $r_i$ becomes

$$
r_i = \frac{\Delta \psi}{2\lambda} \int_{-1}^{1} \int_{-1}^{1} \frac{h_i L q}{(1-\xi^2)} \, d\eta d\xi - \frac{1}{\lambda} \int_{-1}^{1} \frac{h_i}{(1-\xi^2)} \, d\xi \frac{\partial h_i}{\partial \psi} \int_{n=-1}^{1} (q-q_B) f_2 \, d\xi (2.45)
$$

In this new plane, we take the trial function to be of the form

$$
q_N(n) = F_c(\xi) + (1-\xi^2) \sum_{i=1}^{J} \sum_{j=1}^{J} P_i^{(1,1)} P_j(\eta) a_{ij} (2.46)
$$

$F_c(\xi)$ is given the simple form:

$$
F_c = (a + b\xi) q_d
$$

where $a$ and $b$ are constants, and $a + b = 1$. Thus

$$
q_u = \frac{(a-b)}{(a+b)} q_d
$$

In our particular channel problem, the results of which are presented below, the downstream velocity $q_d$ is 0.8; the upstream velocity $q_u = 0.4$, in units of the stagnation velocity of sound. Hence $a = 0.75$ and $b = 0.25$.

The basis set of orthogonal polynomials $P_i^{(1,1)}(\xi)$ and $P_j(\eta)$ are the Jacobi polynomial and the Legendre polynomial of degree $j$, respectively. They do not satisfy the boundary conditions on $\eta = \pm 1$ and the orthogonal functions are used here in order to ensure the numerical stability (Anderssen 1969). It also has been confirmed in practice that such trial functions give better conditioned matrices than
trial functions using monomials. Furthermore, in practice, it has also been found best to have twice as many functions along the channel as across the channel and so, in (2.46),

$$I = 2J$$

Once convergence is reached, information about the flow in the physical plane can be found. In particular, the angle $\theta$ and the $x$ and $y$ coordinates, are given by

$$\begin{align*}
\theta &= \int \frac{\rho}{q} \frac{\partial q}{\partial \psi} \, d\phi = \int f(\xi) \, d\xi \\
\psi &= \text{constant} \quad \eta = \text{constant}
\end{align*}$$

$$\begin{align*}
x &= \int \frac{\cos \theta}{q} \, d\phi = \int \frac{\cos \theta}{q} \frac{d\xi}{\lambda(1-\xi^2)} \\
\psi &= \text{constant} \quad \eta = \text{constant}
\end{align*}$$

$$\begin{align*}
y &= \int \frac{\sin \theta}{q} \, d\phi = \int \frac{\sin \theta}{q} \frac{d\xi}{\lambda(1-\xi^2)} \\
\psi &= \text{constant} \quad \eta = \text{constant}
\end{align*}$$

and the turning angle of the channel is

$$\Delta \theta = \text{downstream angle} - \text{upstream angle}$$

$$\quad = \theta_d - \theta_u$$

Moreover, since we know that

$$d\psi = \rho q d\eta$$
from (2.3), then

\[ \Delta \Psi = \int \rho q \mathrm{d}n = \rho_d q_d \Delta n_d = \rho_d \Delta n_d \]

downstream

\[ \Delta \Psi = \int \rho q \mathrm{d}n = \rho_u q_u \Delta n_u \]

upstream

and hence

\[ \Delta n_u = \frac{\rho_d}{\rho_u q_u} \Delta n_d \]

where \( \Delta n_u \) and \( \Delta n_d \) are the perpendicular distances across the channel upstream and downstream respectively.

The results presented below were tabulated for the case where \( \gamma = 1.4 \), and the boundary conditions are

\[ q_B = \begin{cases} a + b \tanh(\psi - \Delta \Psi) & \text{on } \psi = 0 \\ a + b \tanh(\psi + \Delta \Psi) & \text{on } \psi = \Delta \Psi \end{cases} \]

Computation of the coefficients continued until they converged to an accuracy of \( 10^{-4} \times \) largest coefficient. Figure 2.1 shows the results obtained for the turning angle along various streamlines. For each \( J \), the converged value of \( q \) with respect to the non-linear iteration was used to calculate the turning angles. It is seen that as \( J \) increases, all the estimates for the angles converge to a common value consistent with
the exact value of the turning angle. (This exact value can be calculated in a way which will be described in chapter 5.) From the graphs, the error in the calculation of the turning angles, along the various streamlines, was estimated to be less than .02%. Figure 2.2 shows the local Mach number $M$, at some point in the channel, versus $J$ along the streamlines $\psi = \frac{1}{4} \lambda \psi$ and $\psi = \frac{3}{4} \lambda \psi$. Also shown in the same figure are the corresponding local angle $\theta$. Both $M$ and $\theta$ are converging with the increase in $J$. It can be noted that $\theta$ takes longer to converge than $M$ and this is to be expected since calculations of $\theta$ involve a first derivative of the velocity distribution, while $M$ is tabulated from the local velocity directly.

The shape of the channel is shown in figure 2.3 for the case where $\lambda = .6$ was used. The design of the channel was plotted from the solution obtained for $J = 7$. The channel is seen to narrow at the downstream end, which is in accordance with the compressibility property of the flow.

Figure 2.4 shows the effects different values of the scaling factor $\lambda$ have on the efficiency of the technique. In this figure the maximum errors in the turning angle (from the exact value) across the channel are plotted against $\lambda$, and it is obvious that there is a particular value of $\lambda$ where the best efficiency will be obtained from the technique. Also, figure 2.5 compares the rate of convergence for different values of $\lambda$ by plotting the errors in the turning angle against $J$. From these two figures we can predict that the best value of the scaling parameter $\lambda$ lies somewhere between .5 and .6. We chose $\lambda = .6$ to obtain the results presented in the first 3 figures.

We also include Table II below, which gives values of the leading coefficients $a_1$ and $a_2$, against the number of trial functions $J$. 
for various values of \( \lambda \). It is apparent that we have very fast convergence for all 3 values of the scaling parameter \( \lambda \).

**TABLE II**

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( J )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
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<td>( .4 )</td>
<td>( a_1 )</td>
<td>-.0148297</td>
<td>-.0148467</td>
<td>-.0148470</td>
<td>-.0148470</td>
</tr>
<tr>
<td></td>
<td>( a_2 )</td>
<td>.0482859</td>
<td>.0482922</td>
<td>.0482922</td>
<td>.0482922</td>
</tr>
<tr>
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<td>( a_1 )</td>
<td>-.0201786</td>
<td>-.0202044</td>
<td>-.0202048</td>
<td>-.0202048</td>
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<tr>
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<td>-.0216651</td>
<td>-.0216649</td>
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</tr>
<tr>
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<td>( a_1 )</td>
<td>-.0223063</td>
<td>-.0223382</td>
<td>-.0223388</td>
<td>-.0223389</td>
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<tr>
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### 2.6 CONCLUSION

We have derived the non-linear partial differential equation of motion in the potential plane as

\[ \frac{\partial}{\partial \psi} \left[ f_1(q) \frac{\partial q}{\partial \phi} \right] + \frac{\partial}{\partial \psi} \left[ f_2(q) \frac{\partial q}{\partial \psi} \right] = 0 \]

Then, following Hendry's work, we developed a functional from some suitably defined inner product so that the problem of solving a linear differential equation, subject to boundary conditions, is equivalent to determining the stationary value of that functional. The Iterative Variational Method, using the trial functions not satisfying the boundary conditions, was extended to the non-linear system above with the introduction of a suitable iterative scheme.
We then introduced the Galerkin Method, and applied the technique to the channel flow. The matrices involved are now different from those used by Hendry. The results showed a fast convergence as might be expected. They converge to the same values obtained by Davies and Hendry in their recent paper (Davies and Hendry 1975). Fig. 2.6 gives some comparison between the results obtained by Davies and Hendry and those obtained from the technique proposed by us in section 2.4. The same values for all relevant parameters were used. The turning angles on the streamline $\psi = 0$ and $\psi = \frac{\Delta \psi}{2}$ are plotted against the number of trial functions $J$, for the two methods. The graphs show that we obtain a much better rate of convergence than Davies and Hendry's method.
Figure 2.1: Turning angle $\Delta \theta$ against number of trial functions $J$ for various streamlines.
Figure 2.2: Local Mach number $M$ and angle $\theta$ against number of trial functions $J$ for various streamlines.
\[ \psi = \Delta \psi \]

\[ \psi = 0 \quad \phi = -4 \quad \phi = -3 \quad \phi = -2 \quad \phi = -1 \quad \phi = 0 \quad \phi = 1 \quad \phi = 2 \quad \phi = 3 \quad \phi = 4 \]

- \( a = 0.75 \)
- \( b = 0.25 \)
- \( \lambda = 0.6 \)
- \( J = 7 \)

---

\( \psi \text{ constant} \)
Figure 2.3: Shape of the channel in the (x,y)-plane

--- φ constant

--- ψ constant

a = 0.75, b = 0.25

λ = 0.6

J = 7
Figure 2.4: Maximum error in turning angle across the channel against the scaling parameter $\lambda$. 
Figure 2.5: Maximum error in turning angle against number of trial functions $J$ for various values of $\lambda$. 
Figure 2.6: Turning angle $\Delta \theta$ against number of trial functions $J$

- $\psi = 0$
- $\psi = \frac{1}{2} \Delta \psi$

---

Our result

Hendry's result
CHAPTER III

CASCADES OF AEROFOILS

3.1 INTRODUCTION

In this chapter we proceed to the problem of flow through cascades of aerofoils. The description of the problem, together with the mathematical formulation, is given in section 3.2. Then in section 3.3, we deduce the various conformal transformations to map the complicated potential plane onto a relatively simple region. This effects a substantial reduction in the analytic difficulties attached to the selection of a suitable form for the trial function for the velocity distribution. In this, we make use of the approach adopted by Woods (Woods 1961), who gave a conformal transformation mapping the potential plane of incompressible cascade flow onto a semi-infinite, rectangular strip (figure 3.6).

We then extend this work to the case of compressible cascade flow in section 3.4. The final modified region turns out to be a square, over which the trial function is defined. In section 3.5, the principles of the Galerkin Method, described in the previous chapter, are applied to cascade flow in general. In section 3.6, some relationships amongst the mapping planes are investigated and some of the limiting values of terms which are of relevance are worked out. This information is of considerable use to us in our later work. Lastly, in section 3.7, the general forms of the residuals \( r_i \), and the matrix elements, relevant in the proposed non-linear iterations, are given in detail for the trial function \( q \) of section 3.5.
3.2 DESCRIPTION OF THE PROBLEM

By a cascade of aerofoils, we mean an infinite set of similar aerofoils at the same incidence, spaced at equal distance from each other along the $y$-axis as shown in figure 3.1.

![Diagram of a cascade of aerofoils with labels for $A'$, $B'$, $q_u$, $q_d$, $s$, $t$, and $x$-plane.

$z = (x+iy)$ - plane

Figure 3.1

The solution to this problem is valuable because it is a first approximation to the flow through the blading of an axial compressor, especially at the high pressure end, where the blade height is small compared to its radius. Also, if we let the ratio of the
space between two blades to the length of the blade tend to infinity, the limiting case becomes the problem of a single aerofoil.

As shown in figure 3.1, the aerofoils are equally spaced along the y-axis by a distance $H$. The flow is shown to have the upstream or inlet speed of $u$, with the inlet angle of $\theta_u$. The downstream or outlet speed at infinity is $d$, with the outlet angle of $\theta_d$. The repeat condition is the requirement that the flow at points $(x, y+nH)$ where $n = 0, \pm 1, \pm 2, \ldots$ are identical. The flow is shown to separate from each aerofoil at some point on its upper surface, resulting in a wake of slowly moving turbulent fluid extending to infinity. Different ways of treating this phenomena are discussed briefly in Woods (Woods 1961 p. 19 ff). Our model, however, ignores the trailing edge (as has been mentioned), and the aerofoils, therefore, only close at infinity, conveniently bypassing the displacement property created by the wakes.

Now consider the representation of the flow in the potential plane as shown in figure 3.2. Note that, since the surfaces of the aerofoils are streamlines, for which $\psi$ is constant, they become straight lines in this plane. Let $h$ be the distance in this potential plane between corresponding points on adjacent aerofoils.

Since the incident angle or inlet angle is non-zero, the lines of constant $\psi$ do not coincide with the lines of constant $x$. Consequently, the line connecting all the stagnation points, is shown in figure 3.1 to make an angle $\alpha$ to the $x$ axis. In incompressible flow $\psi = 0$. However, for the compressible case, it is shown in 4.1 that $\tan \alpha = \frac{\sin \theta_u}{\cos \theta_d}$ also, the two points at the trailing edge are shown in figure 3.1 to be not apart. Although, in the case that there is no circulation, the slits representing the aerofoils are closed at the trailing edges.
The following assumptions have been made about the flow:

1) The flow is steady, inviscid and irrotational.
2) The flow is plane two-dimensional and the normal component of velocity is zero at the blade surface.
3) The fluid is a perfect gas and it is assumed that the total temperature remains the entry to the cascade.
4) The cascade consists of an infinite number of equally spaced blades.

To determine the flow corresponding to the prescribed velocity distribution on the blade surface and the downstream condition, we have to solve equation (2.30) subject to the following boundary conditions.

1) $q$ is given on the top and bottom surfaces of the aerofoils.
2) $q$ and $\phi$ are such that the repeat conditions hold elsewhere, that is

$$\omega = (\phi + i\psi) - \text{plane}$$

Since the incident angle or inlet angle is non-zero, the lines of constant $\phi$ do not coincide with the lines of constant $x$. Consequently, the line connecting all the stagnation points, is shown in figure 3.2 to make an angle $\alpha$ to the $\psi$ axis. In incompressible flow $\alpha = \theta_u$. However, for the compressible case, it is shown in 4.1 that $\tan \alpha = \frac{\tan \theta_u}{\rho_u}$. Also, the two points at the trailing edge are shown in figure 3.2 to be set apart. Although, in the case that there is no circulation, the slits representing the aerofoils are closed at the trailing edges.
The following assumptions have been made about the flow.

1) The flow is steady, inviscid and irrotational.
2) The flow is plane two-dimensional and the normal component of velocity is zero on the blade surface.
3) The fluid is a perfect gas, and it is also assumed that the total temperature is uniform across the entry to the cascade.
4) The cascade consists of an infinite number of equally spaced blades.

To determine the blade shape corresponding to the prescribed velocity distribution on the blade surface and the downstream condition, we have to solve equation (2.10), subject to the following boundary conditions.

1) \( q \) is given on top and bottom surfaces of the aerofoils
2) \( q \) and \( \theta \) are such that the repeat conditions hold elsewhere, that is

\[ q(\phi, \psi) = q(\phi + \Delta\phi, \psi + \Delta\psi) \]

where

\[ \Delta\phi = h \sin \alpha \]
\[ \Delta\psi = h \cos \alpha \]

3) \( q \) is given at downstream infinity to be \( q_d \).

3.3 THE CONFORMAL TRANSFORMATIONS

We now introduce a suitable transformation, as developed by Woods (Woods 1961 p.487), for an incompressible flow, to map the rather complicated \( \omega \)-plane shown in figure 3.2 onto a simple \( \zeta \)-plane, in which the solution of the boundary value problem can be written down.
First, we take the particular case of the incompressible flow through the cascade for which both the downstream and upstream angles are zero, as is shown in figure 3.3. The flow in the potential plane is shown in figure 3.4. We note that there is no circulation since the inlet and outlet angles are zero, and therefore the slits representing the aerofoils are closed (see section 4.1 for more detail). $h$ is now the perpendicular distance from one aerofoil to another, and hence equal to $\Delta \psi$ (the jump in $\psi$ from one blade to the next).

\[
\theta_u = 0 \quad q_u \quad S_0 \quad \theta_d = 0 \quad q_d
\]

$z$ - plane

Figure 3.3

We will distinguish the potential plane of this flow from the one with circulation by the subscript $0$, so it is the $\omega_0$-plane which is
shown in figure 3.4. If the length of the slits is \( k \), it can be verified that the transformation

\[
t = 1 - \exp(-2\pi \omega_0/h)
\]  

(3.1)

maps the slits into a single slit of length

\[
t^*_k = 1 - \exp(-2\pi k/h)
\]

in the \( t \)-plane shown in figure 3.5, taking the origin of the \( \omega_0 \)-plane to be at the leading edge \( S \) of one of these slits.

\[\begin{align*}
\text{\( \omega_0 \)-plane} \\
\text{Figure 3.4}
\end{align*}\]

\[\begin{align*}
\text{\( t \)-plane} \\
\text{Figure 3.5}
\end{align*}\]
The \( t \)-plane is in turn mapped into the \( \zeta \)-plane by the transformation

\[
t = \frac{1}{2} \{1-\exp(-2\pi k/h)\}(1-\cos \zeta)
\]

which, in using (3.1) gives

\[
1 - \exp(-2\pi \omega_0/h) = \frac{1}{2} \{1-\exp(-2\pi k/h)\}(1-\cos \zeta)
\]

which may be written as:

\[
\omega_0 = \frac{1}{2} \frac{\hbar}{2\pi} \ln \{\cosh r + \sinh r \cos \zeta\}
\]

where

\[
r = \frac{k\pi}{\hbar}
\]

If we define \( \eta^* \) by,

\[
\cosh \eta^* = \coth r
\]

then the transformation (3.2) takes the point \( \phi_0 = -\infty \) in the \( \omega_0 \)-plane, onto the point \( \mu = \infty \) in the \( \zeta \) plane, while the points \( \phi_0 = \infty \), \( \psi_0 > 0 \) and \( \phi_0 = \infty \), \( \psi_0 < 0 \) are mapped onto the points \( (\gamma, \mu) = (\pi, \eta^*) \) and \( (-\pi, \eta^*) \) respectively. Furthermore, the opposite sides of the trailing edge streamline in the \( t \)-plane, \( t_1 = 0, \ t_1 < t_2 < \infty \),
where \( t = t_1 + it_2 \), are mapped to two separate lines, \( \gamma = +\pi \) and \( \gamma = -\pi \) with \( 0 < \mu < \eta^* \) in the new plane, shown in figure 3.6.
The stagnation points $S_0$ will coincide, in this case, with the origin of the $\zeta$-plane. The streamlines $t = 1 - \exp(-2\pi\omega_0/h)$ are mapped to the lines $\gamma = \pm \pi$ with $\eta < \mu < \infty$. The lines $\mu = 0$, $-\pi < \gamma < 0$ and $0 < \gamma < \pi$ are the opposite sides of the blades.

Now, if we consider the flow in figure 3.2 to be an incompressible flow, then both $\omega$ and $\omega_0$ will be analytic functions of $z$. In the sense used in Woods, $\omega$ will consequently be an analytic function of $\omega_0$, and therefore of $\zeta$.

Defining $\tau$ and $\tau_0$ as
\[ \tau = \ln \left( \frac{q}{u_{d\omega}} \right) = \ln \frac{q}{u} + i\theta \quad (3.3) \]

and
\[ \tau_0 = \ln \left( \frac{q}{u_{d\omega_0}} \right) = \ln \frac{q}{u} + i\theta_0 \quad (3.4) \]

then, again in the sense used in Woods, both \( \tau \) and \( \tau_0 \) are analytic functions of \( \zeta \).

By the repeat condition of the flow, we deduce that \( \tau \) and \( \tau_0 \) are periodic, that is,

\[ \tau(\pi, \mu) = \tau(-\pi, \mu) \]

and
\[ \tau_0(\pi, \mu) = \tau_0(-\pi, \mu) \]

since the points \((\pi, \mu)\) and \((-\pi, \mu)\) would either be the same point in the \( z \)-plane, or the corresponding points separated by the distance \( nH \), where \( n = 0, \pm 1, \pm 2 \), in the \( 0y \)-direction.

The above information enables us to make use of the theory developed in Woods (Woods 1961 §4.3), where the calculus of residues is employed to write

\[ \tau(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_S(\gamma) \cot \frac{1}{2}(\gamma - \zeta) d\gamma \quad (3.5) \]

and
\[ \tau_0(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_0S(\gamma) \cot \frac{1}{2}(\gamma - \zeta) d\gamma \quad (3.6) \]

Here \( \theta_S \) and \( \theta_0S \) are the directions on the surface of the boundaries. It is easy to see that \( \theta_S \) and \( \theta_0S \) will differ only because of the different positions of the stagnation points in the two flows. Supposing,
then, that the two stagnation points \( S \) and \( T \) of the circulating flow map on to the points \( \gamma = -\zeta_1 \) and \( -\pi - \zeta_2 \) respectively, then in terms of \( \theta_{OS} \), \( \theta_S \) will have the form

\[
\theta_S(\gamma) = \theta_{OS}(\gamma) + \pi[H(\gamma - \zeta_1) - H(\gamma)] - \pi[H(\gamma + \pi) - H(\gamma + \pi - \zeta_2)] \tag{3.7}
\]

where \( H(\gamma) \) is the usual unit step function.

Substituting (3.7) into (3.5) and using (3.6) we obtain

\[
\tau(\zeta) = \tau_0(\zeta) + \ln \left\{ \frac{\sin \frac{1}{2}\zeta \cos \frac{1}{2}\zeta}{\sin \frac{1}{2}(\zeta_1 + \zeta) \cos \frac{1}{2}(\zeta_2 - \zeta)} \right\} \tag{3.8}
\]

Now, we know that

\[
\lim_{\mu \to \infty} \tau(\gamma, \mu) = i\alpha, \quad \lim_{\mu \to \infty} \tau_0(\gamma, \mu) = 0
\]

and so, on taking the limit as \( \mu \) tends to \( \infty \) in (3.8), we find

\[
\alpha = \frac{1}{2}(\zeta_1 - \zeta_2) \tag{3.9}
\]

Using (3.3), (3.4), and (3.8) we find that

\[
\frac{d\omega}{d\omega_0} = \frac{\sin \lambda + \cos \alpha \sin \zeta + \sin \alpha \cos \zeta}{\sin \zeta} \tag{3.10}
\]

where \( \lambda = \frac{1}{2}(\zeta_1 + \zeta_2) \) and \( \alpha = \frac{1}{2}(\zeta_1 - \zeta_2) \).

Differentiating \( \omega_0 \) with respect to \( \zeta \), and using (3.10), we obtain,

\[
\omega = \frac{1}{2\pi} \left[ \zeta \sin \alpha + 2 \cos \alpha \tan \frac{\zeta_1 + \zeta_2}{2} \right] \tag{3.14}
\]
\[
\frac{d\omega}{d\zeta} = \frac{h}{2\pi} \left\{ \sin \lambda + \cos \alpha \sin \zeta + \sin \alpha \cos \zeta \right\} \left( \coth r + \cos \zeta \right) \tag{3.12}
\]

which, on integrating, gives

\[
\omega = \frac{h}{2\pi} \left\{ \zeta \sin \alpha + 2(\sin \lambda \sinh r - \sin \alpha \cosh r) \tan^{-1}(e^{-r}\tan \frac{1}{2}\zeta) \right. \\
\left. - \cos \alpha \ln \left( \frac{1 + \tanh \lambda \cos \zeta}{1 + \tanh \lambda \tanh r} \right) \right\} \tag{3.13}
\]

as the final transformation from the flow in figure 3.1 to the \( \zeta \)-plane. The solution to the incompressible cascade flow problem, using this transformation, is also given in Woods (Woods 1961 §12.3).

### 3.4 THE TRANSFORMATION IN THE COMPRESSIBLE CASE

The work done by Woods, as we presented in the last section, is now extended to the compressible case, by saying that this transformation (3.13) is applicable to both compressible and incompressible flow. The proof of this statement is simply to verify in a straightforward manner that the transformation does indeed map the flow in the \( \omega \)-plane in figure 3.2 onto the \( \zeta \)-plane in figure 3.6, and hence it is, in fact, a mapping which has no connection with the compressibility of the flow.

We note further that there are two stagnation points \( S \) and \( T \) involved, as shown in figure 3.6. However, we wish to ignore, at least in this thesis, the trailing edge which is relatively much less important than the leading edge. Taking the limit that \( k \), and consequently \( r \), tends to \( \infty \), we arrive at our model. From (3.13), the limiting transformation as \( r \to \infty \) becomes

\[
\omega = \frac{h}{2\pi} [\zeta \sin \alpha - 2 \cos \alpha \ln \cos \zeta/2] \tag{3.14}
\]
where
\[ \omega = \phi + i\psi, \]
\[ \zeta = \gamma + i\mu. \]

The \(\zeta\)-plane of the present case is shown in figure 3.7, as mapped from the flow shown in figure 3.2 for the potential plane, but with the trailing edge extending to infinity, so that \(T\) and \(T'\) are the points at downstream infinity.
Every streamline with $\psi < 0$ ends at the point $(-\pi, 0)$, and those with $\psi > 0$ end at the point $(\pi, 0)$. The stagnation streamline of course ends at the stagnation point $S$, the position of which is governed by the fact that $\frac{\partial \phi}{\partial \gamma} = 0$ and $\mu = 0$ there.

From (3.14) we know that,

$$\phi = \frac{h}{2\pi} \left[ \gamma \sin \alpha - \cos \alpha \ln \left( \cos^2 \frac{\gamma}{2} + \sinh^2 \frac{\mu}{2} \right) \right] \quad (3.15)$$

Moreover,

$$\psi = \frac{h}{2\pi} \left[ \mu \sin \alpha + 2 \cos \alpha \arctan \left( \tan \frac{\gamma}{2} \tanh \frac{\mu}{2} \right) \right] \quad (3.16)$$

Hence

$$\frac{\partial \phi}{\partial \gamma} = \frac{h}{2\pi} \left[ \sin \alpha + \frac{\cos \alpha \sin \gamma}{\cos \gamma + \cosh \mu} \right] \quad (3.17)$$

and

$$\frac{\partial \phi}{\partial \mu} = -\frac{h}{2\pi} \left[ \frac{\cos \alpha \sinh \mu}{\cos \gamma + \cosh \mu} \right] \quad (3.18)$$

It can be verified also that,

$$\frac{\partial \psi}{\partial \gamma} = -\frac{\partial \phi}{\partial \mu} \quad \text{and} \quad \frac{\partial \psi}{\partial \mu} = \frac{\partial \phi}{\partial \gamma} \quad (3.19)$$

Equating (3.17) to zero, and putting $\mu = 0$, we find that

$$\tan \frac{\gamma}{2} = -\tan \alpha$$

and hence the stagnation point is at

$$y_s = -2\alpha$$
Furthermore, the stagnation streamline is given by $\psi = 0$ and equation (3.16) gives

$$\tan(\gamma_0/2) = -\tan\left(\frac{1}{2} \mu \tan \alpha \right)/\tanh(\mu/2)$$  \hspace{1cm} (3.20)

where $\gamma = \gamma_0(\mu)$ is the function of the stagnation streamline as a function of $\mu$.

Moreover,

$$\gamma_0'\left[\tanh^2\left(\frac{1}{2} \mu \right) + \tan^2\left(\frac{1}{2} \mu \tan \alpha \right)\right]$$

$$= \tan\left(\frac{1}{2} \mu \tan \alpha \right) \text{sech}^2\left(\frac{1}{2} \mu \right) - \tan \alpha \tanh \frac{1}{2} \mu^2 \sec^2\left(\frac{1}{2} \mu \tan \alpha \right)$$  \hspace{1cm} (3.21)

In the limit as $\mu$ tends to infinity, we find that

$$\gamma_0' \sim -\tan \alpha$$  \hspace{1cm} (3.22)

In fact, any streamline, for which $\psi$ is constant, can be shown to possess the same gradient as the stagnation streamline, at a limit as $\mu$ tends to $\infty$. This is seen by putting $\psi$ equal a constant in (3.16), and on differentiating with respect to $\mu$, we obtain exactly the same expression as (3.21) of the stagnation streamline. Hence, we show in figure 3.7 the two streamlines, with $\psi < 0$ and $\psi > 0$, as tending to become parallel to the middle streamline as $\mu$ approaches $\infty$.

In the case where $\alpha$ is non zero, a complication can arise in the evaluation of a line integral along a streamline. For example, in calculating the angle $\theta$ (see chapter 4 for details), we need to integrate
along the stagnation streamline $\psi = 0$. To do this, a change of variable is made to map the semi infinite plane into a rectangular plane, shown in figure 3.8, by defining

$$
\eta = 1 - 2e^{-\mu}
$$

\[ (3.23) \]

which, intuitively, tilts the $\xi$-plane in figure 3.7 so that we have the stagnation streamline $y = \eta$, perpendicular, coinciding with the $\xi$-direction and the stagnation point at the point $(\xi, \eta) = (L, -1)$. Also, the semi infinite region $\mathbb{R}$, now where $\Phi > 0$, so that to the right is where $\Phi < 0$, with the blade boundary extending from $\eta = 1$ to $\eta = 1$ on the line $\eta = -1$.

Figure 3.8

Using the information in (3.22), and remembering the periodic property of the solution, we expect the stagnation streamline to behave as shown in figure 3.8, which gives rise to a singularity at the points close to $\eta = 1$, where the behaviour of the integrands becomes complicated, and the integration rules cannot cope. We remove this complication by introducing a further change of variable as,
\[ n = 1 - 2e^{-\mu} \quad (3.24) \]

which, intuitively, tilts the \( \xi \)-plane in figure 3.7 so that we have the stagnation streamline exactly perpendicular, coinciding with the \( 0\eta \) direction and the stagnation point at the point \( (\xi, \eta) = (\xi_S, -1) \).

Also, the semi-infinite plane is now mapped into a square region \( R \), with the following characteristics:

1) The function \( \psi \) is required to take into account the behaviour of \( n \) near the transition point.

\[-1 \leq \xi \leq 1, \quad -1 \leq \eta \leq 1 \]

The region to the left of the vertical line \( \xi = \xi_S = \gamma_S/\pi \) is now where \( \psi < 0 \), and that to the right is where \( \psi > 0 \), with the blade boundary extending from \( \xi = -1 \) to \( \xi = 1 \) on the line \( \eta = -1 \) for large \( \mu \), giving a constant value for the approach velocity.

We take \( h_0(\xi, \eta) \) to be of the form

\[ h_0(\xi, \eta) = \sin^2 \frac{\theta}{2} \quad (3.25) \]

which gives

\[ \xi + i\eta \quad \text{plane} \]

Figure 3.9
3.5 APPLICATION OF THE GALERKIN METHOD

We now apply the Ritz-Galerkin Method to the general cascade flow problem by writing our trial function for the velocity distribution \( q \) in the form

\[
q = h_S[A + \Sigma a_{ij} g_i(\xi)g_j(\eta)]
\]  

(3.25)

with the following characteristics.

1) The function \( h_S(\xi, \eta) \) is required to take into account the behaviour of \( q \) near the stagnation point,

\[
(\xi, \eta) = (\xi_S, -1)
\]  

(3.26)

where the flow is approximately compressible, and tends to a constant for large \( \mu \), giving a constant value for the upstream velocity.

We take \( h_S(\xi, \eta) \) to be of the form

\[
h_S = \frac{h_0(\xi, \eta)}{k + \xi h_0(\xi, \eta)}
\]  

(3.27)

where \( k + \xi = 1 \) and

\[
h_0^2(\xi, \eta) = \sin^2 \frac{1}{2}(\gamma - \gamma_0) + \cos^2 \frac{1}{2}(\gamma - \gamma_0) \tanh^2 \frac{1}{2} \mu
\]  

(3.28)

which gives
Hence

\[ 2h_0 \frac{\partial h_0}{\partial \gamma} = \frac{1}{2} \sin(\gamma - \gamma_0) \sech^2 \left( \frac{1}{2} \mu \right) \] (3.29)

\[ 2h_0 \frac{\partial h_0}{\partial \mu} = -\frac{\gamma'}{2} \sin(\gamma - \gamma_0) \sech^2 \left( \frac{1}{2} \mu \right) \]

\[ + \cos^2 \frac{1}{2}(\gamma - \gamma_0) \tanh \left( \frac{1}{2} \mu \right) \sech^2 \left( \frac{1}{2} \mu \right) \] (3.30)

Hence

\[ \frac{\partial h_S}{\partial \gamma} = \left( 1 - \frac{\partial h_0}{k + \frac{\partial h_0}{\partial \gamma}} \right) \frac{\partial h_0}{\partial \gamma} \cdot \frac{1}{k + \frac{\partial h_0}{\partial \gamma}} \] (3.31)

and

\[ \frac{\partial h_S}{\partial \mu} = \left( 1 - \frac{\partial h_0}{k + \frac{\partial h_0}{\partial \mu}} \right) \frac{\partial h_0}{\partial \mu} \cdot \frac{1}{k + \frac{\partial h_0}{\partial \mu}} \] (3.32)

We see that \( h_0 \) behaves like \( \frac{1}{2} \sqrt{(\gamma - \gamma_S)^2 + \mu^2} \), and hence \( h_S \) like \( \frac{1}{2k} \sqrt{(\gamma - \gamma_S)^2 + \mu^2} \), near the stagnation point, and also tends to a constant for large \( \mu \) as required (see section 3.6).

ii) The summation sign is of the indices \( i \) and \( j \), ranging from 1 to \( N_1 \) and \( N_2 \) respectively, being integers to be determined from the convergence of our method.

iii) \( g_i(\xi) \) and \( g_j(\eta) \) are orthogonal functions in \((\xi, \eta)\), not satisfying all the boundary conditions. It has been theoretically shown that the use of orthogonal functions gives desirable numerical stability properties. It was also confirmed in practice that trial functions using orthogonal functions gave better conditioned matrices than trial functions using simple monomials.
\( g_1(\xi) \) is chosen to take into account the repeat conditions along the line of constant \( \eta \), which is

\[
g_1(\xi) = g_1(\xi + 2n) \quad n = 0, \pm 1, \pm 2, \ldots
\]

hence we use the orthogonal Trigonometric functions;

\[
g_1(\xi) = 1
\]

\[
g_2(\xi) = \sin \{\pi(\xi - \xi_S)\} \quad (3.33)
\]

\[
g_3(\xi) = \cos \{\pi(\xi - \xi_S)\} \quad (3.34)
\]

\[
g_i(\xi) = \begin{cases} 
\sin \left\{\frac{i\pi}{2}(\xi - \xi_S)\right\} & \text{if } i \text{ even} \\
\cos \left\{\left(\frac{i-1}{2}\right)\pi(\xi - \xi_S)\right\} & \text{if } i \text{ odd}
\end{cases} \quad (3.35)
\]

\( g_j(\eta) \), \( j = 1, 2, \ldots, N_2 \), are chosen to be the Legendre Polynomials of degree \( j \), multiplied by a simple factor \( (1-\eta) \) for the function with \( j = 1 \), and by the factor \( (1-\eta^2) \) for those with \( j \) greater than 1, that is

\[
g_1(\eta) = (1-\eta) \quad (3.37)
\]

\[
g_j(\eta) = (1-\eta^2)P_{j-1}(\eta) \quad j > 1 \quad (3.38)
\]

where \( P_j(\eta) \) is the Legendre Polynomial of degree \( j \).

iv) The coefficients \( A \) and \( a_{ij} \), \( i = 1, 2, \ldots, N_1; \)
\( j = 1, 2, \ldots, N_2 \), are determined by direct computation, but the factors \( (1-\eta) \) and \( (1-\eta^2) \), introduced in iii), ensure that when \( \eta = 1 \) the trial function satisfies
\[ q = A \quad (3.39) \]

since \( h_S \) tends to 1 as \( \eta \) tends to 1 (see section 3.6 for further details). This means that \( q \) has the value \( A \) upstream, and hence, \( A \) is the upstream velocity which is determined during the iteration. Furthermore, if we put \( \eta = -1 \), which would give the value of the trial function on the surface of the blade, i.e. \( -1 \leq \xi \leq 1 \), \( \eta = -1 \), the only coefficients which affect \( q \) are \( A \) and \( a_{11} \), \( i = 1, 2, \ldots, N_1 \).

### 3.6 LIMITING VALUES

In order to understand the behaviour of the various functions involved in later computations, we look at their limiting values at various points of interest. In particular we investigate their asymptotic behaviour for larger \( \mu \).

#### 3.6.1 Relations Amongst the Mapping Planes

The various transformations introduced in sections 3.3 and 3.4 give rise to some relationships of some importance. Denoting \( \Delta \) and \( \delta \) as

\[
\Delta = \left( \frac{\partial \phi}{\partial \gamma} \right)^2 + \left( \frac{\partial \phi}{\partial \mu} \right)^2 \quad (3.40)
\]

\[
\delta = \frac{\partial \xi}{\partial \phi} \cdot \frac{\partial \eta}{\partial \psi} - \frac{\partial \xi}{\partial \psi} \cdot \frac{\partial \eta}{\partial \phi} \quad (3.41)
\]

the Jacobians of two of the transformations, it can be shown that
Furthermore, for the total transformation in (3.24), we have

\[ \frac{\partial \xi}{\partial \eta} = \frac{1}{\pi}, \quad \frac{\partial \eta}{\partial \xi} = \pi \]  

(3.46)

\[ \frac{\partial \xi}{\partial \mu} = -\frac{\gamma_0'}{\pi}, \quad \frac{\partial \mu}{\partial \xi} = 0 \]  

(3.47)

\[ \frac{\partial \eta}{\partial \gamma} = 0, \quad \frac{\partial \gamma}{\partial \eta} = \frac{\gamma_0'}{1-\eta} \]  

(3.48)

\[ \frac{\partial \eta}{\partial \mu} = 1 - \eta, \quad \frac{\partial \mu}{\partial \eta} = \frac{1}{1-\eta} \]  

(3.49)

and therefore

\[ \frac{\partial \xi}{\partial \psi} = \frac{\partial \xi}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial \psi} + \frac{\partial \xi}{\partial \mu} \cdot \frac{\partial \mu}{\partial \psi} \]  

(3.50)

\[ = \frac{1}{\Delta} \left( \frac{1}{\pi} \frac{\partial \xi}{\partial \gamma} - \frac{\gamma_0'}{\pi} \frac{\partial \xi}{\partial \mu} \right) \]  

(3.51)

\[ \frac{\partial \xi}{\partial \psi} = \frac{\partial \xi}{\partial \gamma} \frac{\partial \gamma}{\partial \psi} + \frac{\partial \xi}{\partial \mu} \frac{\partial \mu}{\partial \psi} \]  

(3.52)

\[ = \frac{1}{\Delta} \left( -\frac{1}{\pi} \frac{\partial \xi}{\partial \mu} - \frac{\gamma_0'}{\pi} \frac{\partial \xi}{\partial \gamma} \right) \]  

(3.57)
\[ \frac{\partial \eta}{\partial \phi} = \frac{\partial \eta}{\partial \gamma} \frac{\partial \gamma}{\partial \phi} + \frac{\partial \eta}{\partial \mu} \frac{\partial \mu}{\partial \phi} \]

\[ = \left( \frac{1 - \eta}{\Lambda} \right) \frac{\partial \phi}{\partial \mu} \]  

(3.53)

and

\[ \frac{\partial \eta}{\partial \psi} = \frac{\partial \eta}{\partial \gamma} \frac{\partial \gamma}{\partial \psi} + \frac{\partial \eta}{\partial \mu} \frac{\partial \mu}{\partial \psi} \]

\[ = \left( \frac{1 - \eta}{\Lambda} \right) \frac{\partial \phi}{\partial \gamma} \]  

(3.54)

with the inverse derivatives,

\[ \frac{\partial \psi}{\partial \eta} = \frac{1}{\delta} \frac{\partial \xi}{\partial \phi} \]

(3.60)

\[ \frac{\partial \phi}{\partial \eta} = - \frac{1}{\delta} \frac{\partial \xi}{\partial \psi} \]

(3.61)

\[ \frac{\partial \phi}{\partial \xi} = \frac{1}{\delta} \frac{\partial \eta}{\partial \psi} \]

(3.62)

where \( C_1 \) is a constant, the form of which is rather complicated.

The Jacobian of the total transformation defined in (3.41) becomes

3.6.2 Asymptotic forms for \( \mu \) tending to infinity

As \( \mu \) tends to infinity, \( 1 - \eta \) becomes zero. Also by looking

at (3.16) and (3.17), we see that

\[ \frac{\partial \phi}{\partial \gamma} = \frac{\partial \psi}{\partial \mu} \frac{h \sin \alpha}{2\pi} \]  

(3.56)

\[ \frac{\partial \phi}{\partial \mu} = - \frac{\partial \psi}{\partial \gamma} \frac{h \cos \alpha}{2\pi} \]  

(3.57)
From (3.42) through to (3.45) we see that as $\mu \to \infty$

$$\frac{\partial \gamma}{\partial \phi} = \frac{\partial \mu}{\partial \psi} + \frac{2\pi \sin \alpha}{h}$$  \hspace{1cm} (3.58)$$

$$\frac{\partial \gamma}{\partial \psi} = -\frac{\partial \mu}{\partial \phi} + \frac{2\pi \cos \alpha}{h}$$  \hspace{1cm} (3.59)$$

Consequently, for the total mapping in (3.51) through to (3.54),
the asymptotic forms for large $\mu$ are

$$\frac{\partial \xi}{\partial \phi} \sim C_1 (1-\eta)$$  \hspace{1cm} (3.60)$$

$$\frac{\partial \xi}{\partial \psi} \sim \frac{2}{h \cos \alpha}$$  \hspace{1cm} (3.61)$$

$$\frac{\partial \eta}{\partial \phi} \sim -\frac{2\pi \cos \alpha}{h} (1-\eta)$$  \hspace{1cm} (3.62)$$

$$\frac{\partial \eta}{\partial \psi} \sim \frac{2\pi \sin \alpha}{h} (1-\eta)$$  \hspace{1cm} (3.63)$$

where $C_1$ is a constant, the actual form of which is rather complicated.

The Jacobian of the total transformation defined in (3.41) becomes

$$\delta \sim \frac{4\pi}{h^2} (1-\eta) \quad \text{as} \quad \mu \to \infty$$  \hspace{1cm} (3.64)$$

and so, from (3.55) and (3.60) through to (3.63), we see that

$$\frac{\partial \psi}{\partial \eta} \sim C_2$$  \hspace{1cm} (3.65)$$

$$\frac{\partial \psi}{\partial \xi} \sim \frac{h \cos \alpha}{2}$$  \hspace{1cm} (3.66)$$
\[
\frac{\partial \phi}{\partial \eta} \sim - \frac{h}{2\pi \cos \alpha (1-\eta)}
\] (3.67)

\[
\frac{\partial \phi}{\partial \xi} \sim \frac{h \sin \alpha}{2}
\] (3.68)

### 3.6.3 Asymptotic forms of the trial function for large \( \mu \)

From (3.25) we obtain

\[
\frac{\partial q}{\partial \phi} = \frac{\partial h}{\partial \phi} [A + \Sigma a_{ij} g_i g_j] + h_S \Sigma a_{ij} \left\{ \frac{\partial \xi}{\partial \phi} g'_i g_j + \frac{\partial \eta}{\partial \phi} g_i g'_j \right\}
\] (3.69)

\[
\frac{\partial q}{\partial \psi} = \frac{\partial h}{\partial \psi} [A + \Sigma a_{ij} g_i g_j] + h_S \Sigma a_{ij} \left\{ \frac{\partial \xi}{\partial \psi} g'_i g_j + \frac{\partial \eta}{\partial \psi} g_i g'_j \right\}
\] (3.70)

However, from (3.28),

\[
h_0 = \sqrt{\sin^2 \frac{1}{2} (\gamma-\gamma_0) + \cos^2 \frac{1}{2} (\gamma-\gamma_0) \tanh^2 \frac{1}{2 \mu}}
\] (3.71)

as \( \mu \to \infty \)

\[
\tanh^2 \left( \frac{1}{2 \mu} \right) \to (1 - 2 e^{-\mu})^2
\]

\[
\sim 1 - 4 e^{-\mu}
\]

\[
= 1 - 2(1-\eta)
\]

hence

\[
h_0 \sim \sqrt{\sin^2 \frac{1}{2} (\gamma-\gamma_0) + \cos^2 \frac{1}{2} (\gamma-\gamma_0) [1 - 2(1-\eta)]}
\] (3.72)
that is

$$h_0 \sim 1 - (1-\eta)\cos^2 \frac{1}{2}(\gamma-\gamma_0)$$

Similarly

$$\frac{\partial h_0}{\partial \gamma} \sim \frac{1}{2}(1-\eta)\sin(\gamma-\gamma_0)$$

$$\frac{\partial h_0}{\partial \mu} \sim \frac{1}{2}(1-\eta)\tan \alpha \sin(\gamma-\gamma_0) + (1-\eta)\cos^2 \frac{1}{2}(\gamma-\gamma_0)$$

Hence

$$h_S \sim 1 - (1-\eta)\cos^2 \frac{1}{2}(\gamma-\gamma_0)$$

$$\frac{\partial h_S}{\partial \gamma} \sim \frac{k}{2}(1-\eta)\sin(\gamma-\gamma_0)$$

$$\frac{\partial h_S}{\partial \mu} \sim \frac{k}{2}[(1-\eta)\tan \alpha \sin(\gamma-\gamma_0) + 2(1-\eta)\cos^2 \frac{1}{2}(\gamma-\gamma_0)]$$

Using (3.58) and (3.59) we get

$$\frac{\partial h_S}{\partial \phi} \sim -\frac{2\pi k}{h}(1-\eta)\cos \alpha \cos^2 \frac{1}{2}(\gamma-\gamma_0)$$

and

$$\frac{\partial h_S}{\partial \psi} \sim \frac{\pi k}{h}(1-\eta)\left[\cos \alpha + \sin \alpha \tan \alpha \sin(\gamma-\gamma_0)\right]$$

+ 2 \sin \alpha \cos^2 \frac{1}{2}(\gamma-\gamma_0)$$
in which case, as \( \mu \to \infty \),

\[
q \sim A
\] (3.77)

\[
\frac{\partial q}{\partial \phi} \sim A \frac{\partial S}{\partial \phi} - h_S \left[ \frac{2\pi \cos \alpha}{h} (1-\eta) \sum a_{ij} g_i(\xi) g_j^1(1) \right]
\] (3.78)

\[
\frac{\partial q}{\partial \psi} \sim A \frac{\partial S}{\partial \psi} + h_S \sum a_{ij} \left[ \frac{2}{h \cos \alpha} b_i'(\xi) g_j(1) + \frac{2\pi \sin \alpha}{h} (1-\eta) g_i(\xi) g_j^1(1) \right]
\] (3.79)

Both (3.78) and (3.79) are of the order \((1-\eta)\), since \(g_i(\xi)\) are periodic functions and

\[
g_j(\eta) \sim (1-\eta) \quad \text{as} \quad \eta \to 1
\]

3.6.4 Asymptotic forms near the stagnation point

As we approach the stagnation point, i.e. as \( \mu \to 0 \) and \( \gamma \to \gamma_0 \), \( \gamma_0 = \gamma_S \) we know that

\[
\sin \frac{1}{2} (\gamma - \gamma_0) \to \frac{1}{2} (\gamma - \gamma_S)
\]

\[
\cos \frac{1}{2} (\gamma - \gamma_0) \to 1
\]

\[
tanh \left( \frac{1}{2} \mu \right) \to \frac{1}{2} \mu
\]

and hence

From the relations (3.48) and (3.49), using (3.81), (3.84) and (3.85) we obtain
\[ h_S \sim \frac{1}{2k} \sqrt{(\gamma - \gamma_S)^2 + \mu^2} \quad (3.80) \]

\[ \frac{\partial h_S}{\partial \gamma} \sim \frac{1}{2k} \cdot \frac{\gamma - \gamma_S}{\sqrt{(\gamma - \gamma_S)^2 + \mu^2}} \quad (3.81) \]

\[ \frac{\partial h}{\partial \mu} \sim \frac{1}{2k} \cdot \frac{(\gamma - \gamma_S) \tan \alpha + \mu}{\sqrt{(\gamma - \gamma_S)^2 + \mu^2}} \quad (3.82) \]

Also

\[ \frac{\partial \phi}{\partial \gamma} = \frac{\partial \psi}{\partial \mu} \sim \frac{h}{2\pi} \left[ \sin \alpha + \frac{\cos \alpha \sin(\gamma_S)}{\cos(\gamma_S) + 1} \right] \quad (3.83) \]

\[ h(\gamma - \gamma_S) \sim \frac{4\pi \cos \alpha}{4\pi \cos \alpha} \quad (3.83) \]

\[ \frac{\partial \phi}{\partial \mu} = - \frac{\partial \psi}{\partial \gamma} \sim - \frac{h}{2\pi} \left[ \frac{\cos \alpha \cdot \mu}{\cos(\gamma_S) + 1} \right] \]

\[ = - \frac{h\mu}{4\pi \cos \alpha} \quad (3.84) \]

and hence

\[ \Delta = \left( \frac{\partial \phi}{\partial \gamma} \right)^2 + \left( \frac{\partial \phi}{\partial \mu} \right)^2 \]

\[ \sim \left( \frac{h}{4\pi \cos \alpha} \right)^2 (\mu^2 + (\gamma - \gamma_S)^2) \quad (3.85) \]

From the relations (3.48) and (3.49), using (3.83), (3.84) and (3.85) we obtain
Furthermore, the relations (3.51) through to (3.54), give

\[ \frac{\partial \xi}{\partial \phi} \sim \frac{4 \cos \alpha (\gamma - \gamma_S)}{h(\mu^2 + (\gamma - \gamma_S)^2)} \quad (3.88) \]

\[ \frac{\partial \xi}{\partial \psi} \sim \frac{-4 \cos \alpha \mu}{h(\mu^2 + (\gamma - \gamma_S)^2)} \quad (3.89) \]

\[ \frac{\partial \eta}{\partial \phi} \sim \frac{8 \pi \cos \alpha \mu}{h(\mu^2 + (\gamma - \gamma_S)^2)} \quad (3.90) \]

\[ \frac{\partial \eta}{\partial \psi} \sim \frac{8 \pi \cos \alpha (\gamma - \gamma_S)}{h(\mu^2 + (\gamma - \gamma_S)^2)} \quad (3.91) \]

3.6.5 **Asymptotic forms downstream**

Downstream, as \( \gamma \to \pm \pi \) or \( \xi \to \pm 1 \), with \( \mu = 0 \), we notice that

\[ \sin \frac{1}{2}(\gamma - \gamma_0) \to \sin(\pm \frac{\pi}{2} + \alpha) = \pm \cos \alpha \quad (3.92) \]

and

\[ \cos \frac{1}{2}(\gamma - \gamma_0) \to \mp \sin \alpha \quad \text{as} \ \gamma \to \pm \pi \]
in which case

\[ h_0 \sim \cos \alpha \]

and

\[ \frac{\partial h_0}{\partial \gamma} \sim -\frac{1}{2} \sin \alpha \quad \frac{\partial h_0}{\partial \mu} = 0 \]

Consequently,

\[ h_S \sim \frac{\cos \alpha}{k+\ell \cos \alpha} \tag{3.92} \]

\[ \frac{\partial h_S}{\partial \gamma} \sim -\frac{1}{2} \cdot \frac{k \sin \alpha}{(k+\ell \cos \alpha)^2} \tag{3.93} \]

\[ \frac{\partial h_S}{\partial \mu} = 0 \tag{3.94} \]

From (3.17), (3.18) and (3.19) again we see that as \( \gamma \to \pm \pi \)

\[ \frac{\partial \psi}{\partial \gamma} = \frac{\partial \psi}{\partial \mu} \sim \frac{h \cos \alpha}{\pi (\gamma \mp \pi)} \tag{3.95} \]

\[ \frac{\partial \psi}{\partial \mu} = -\frac{\partial \psi}{\partial \gamma} = 0 \tag{3.96} \]

Thus, it follows that

\[ \Delta \sim \left( \frac{h \cos \alpha}{\pi (\gamma \mp \pi)} \right)^2 \tag{3.97} \]

in which case
and define the
\[
\frac{\partial \gamma}{\partial \phi} = \frac{\partial \mu}{\partial \psi} \frac{\pi (\gamma \pm \pi)}{h \cos \alpha}
\]  
(3.98)

and
\[
\frac{\partial \gamma}{\partial \psi} = -\frac{\partial \mu}{\partial \phi} = 0
\]  
(3.99)

Finally we obtain, for the total transformation, that
\[
\frac{\partial \xi}{\partial \phi} = \frac{(\gamma \pm \pi)}{h \cos \alpha}
\]  
(3.100)

\[
\frac{\partial \xi}{\partial \psi} = 0
\]  
(3.101)

\[
\frac{\partial \eta}{\partial \phi} = 0
\]  
(3.102)

and
\[
\frac{\partial \eta}{\partial \psi} = \frac{2(\gamma \pm \pi)}{h \cos \alpha}
\]  
(3.103)

from the relations (3.51) through to (3.54).

3.7 THE RESIDUALS AND THE MATRIX ELEMENTS

Consider now the partial differential equation with the boundary conditions and the repeat condition, as stated in section 2.2. We write the problem as
\[
L[q] = \frac{\partial}{\partial \phi} (f_1 \frac{\partial q}{\partial \phi}) + \frac{\partial}{\partial \psi} (f_2 \frac{\partial q}{\partial \psi}) = 0
\]

with
\[
\frac{q_{-}}{q_{B}} = 1 \quad \text{on} \quad \eta = -1
\]  
(3.104)
and define the residual by

$$r_i = - \int_{-1}^{1} \int_{-1}^{1} \frac{d\xi d\eta}{\delta} h_i L[q] + \beta \int_{-1}^{1} h_i (1 - \frac{a}{q_B}) d\xi$$  (3.105)

Here $q = \sum_{i=1}^{M} a_i h_i$; the second integral is on the boundary surface of the blade, and $\delta$ is the Jacobian as defined in (3.41).

We see that the equation (3.104) is satisfied if

$$r_i = 0 \quad i = 1, 2, ...$$

and the essence of the Galerkin Method is to require

$$r_i = 0 \quad i = 1, 2, ..., M$$

Integrating by parts,

$$r_i = \int_{-1}^{1} \int_{-1}^{1} d\xi d\eta \left\{ f_1 \frac{\partial q}{\partial \phi} \left[ \frac{\partial \xi}{\partial \phi} \frac{h_i}{\delta} + \frac{\partial \eta}{\partial \phi} \frac{h_i}{\delta} \right] + f_2 \frac{\partial q}{\partial \psi} \left[ \frac{\partial \xi}{\partial \psi} h_i + \frac{\partial \eta}{\partial \psi} h_i \right] \right\}$$

$$+ \int_{-1}^{1} d\eta \left[ \int_{-1}^{1} \frac{\partial \xi}{\delta} h_i \left( \frac{\partial \xi}{\partial \phi} f_1 \frac{\partial q}{\partial \phi} + \frac{\partial \xi}{\partial \psi} f_2 \frac{\partial q}{\partial \psi} \right) \right]_{\xi=+1}^{\xi=-1}$$

$$- \int_{-1}^{1} d\xi \left[ \int_{-1}^{1} \frac{\partial \eta}{\delta} h_i \left( \frac{\partial \eta}{\partial \phi} f_1 \frac{\partial q}{\partial \phi} + \frac{\partial \eta}{\partial \psi} f_2 \frac{\partial q}{\partial \psi} \right) \right]_{\eta=+1}^{\eta=-1}$$

$$+ \beta \int_{-1}^{1} h_i (1 - \frac{a}{q_B}) d\xi$$

Using (3.55) for the double integral, we find that it is equal to
The first boundary integral in (3.106) becomes zero because $q$ and $h_i$ are periodic with $\xi$. For the second boundary integral, on the other hand, the following conditions hold:

a) as $n \to 1$

\[
\left. \frac{\partial}{\partial \xi} \int_{-1}^{1} \frac{d \xi}{\delta} \left\{ f_1 \frac{\partial q}{\partial \phi} \frac{\partial h_i}{\partial \phi} + f_2 \frac{\partial q}{\partial \psi} \frac{\partial h_i}{\partial \psi} \right\} \right|_{\xi=0} = 0
\]

and using the limiting values obtained in (3.66) and (3.78), we find that

\[
\frac{\partial \psi}{\partial \xi} f_1 \frac{\partial q}{\partial \phi} \sim (1-n) \quad \text{as} \quad n \to 1.
\]

Hence, there is no contribution at the point $n = +1$. Also,

\[
\frac{h_i}{\delta} \frac{\partial n}{\partial \psi} f_2 \frac{\partial q}{\partial \psi} = - \frac{\partial \phi}{\partial \xi} f_2 h_i \frac{\partial q}{\partial \psi} \sim (1-n) \quad \text{using (3.68) and (3.79)}
\]

and hence, there is no contribution either.

b) as $n \to -1$

\[
\frac{\partial n}{\partial \phi} = \frac{(1-n)}{\Delta} \frac{\partial \phi}{\partial \mu}
\]
but \[ \frac{\partial \phi}{\partial \mu} \to 0 \] and hence

\[ \frac{\partial \eta}{\partial \phi} \to 0 \quad \text{as } \eta \to -1 \]

Therefore the integral is also zero at \( \eta = -1 \).

The only remaining line integral in (3.106) is, therefore, the integral on the surface of the blade, i.e.

\[
\int_{-1}^{1} d\xi \left[ \frac{h_i}{\partial \psi} \frac{\partial \eta}{\partial \phi} f_2 \frac{\partial q}{\partial \psi} \right]_{\eta=-1} = \int_{-1}^{1} d\xi \left[ \frac{\partial \rho}{\partial \xi} h_i f_2 \frac{\partial q}{\partial \psi} \right]_{\eta=-1}
\]

Consequently, the total residual is

\[
r_i = \int_{-1}^{1} \int_{-1}^{1} \frac{d\xi d\eta}{\partial} \left\{ f_1 \frac{\partial q}{\partial \phi} \frac{\partial h_i}{\partial \phi} + f_2 \frac{\partial q}{\partial \psi} \frac{\partial h_i}{\partial \psi} \right\} + \int_{-1}^{1} \frac{d\xi}{\partial} h_i \left\{ \frac{\partial \rho}{\partial \xi} f_2 \frac{\partial q}{\partial \psi} + \beta \left( 1 - \frac{q}{q_B} \right) \right\}_{\eta=-1}
\]

(3.107)

From the above residual we can obtain the matrix elements \( A_{ij} \) defined by

\[
A_{ij} = \frac{\partial r_i}{\partial a_j}
\]

Differentiating \( r_i \), we obtain
\[
A_{ij} = \int_{-\infty}^{1} \int_{-\infty}^{1} \frac{d\xi d\eta}{\delta} \left\{ f_1 \frac{\partial h_j}{\partial \phi} \frac{\partial h_i}{\partial \phi} + f_2 \frac{\partial h_j}{\partial \psi} \frac{\partial h_i}{\partial \psi} \right\} \\
+ f_3 \frac{\partial q}{\partial h_j} h_j \frac{\partial h_i}{\partial \phi} + f_4 \frac{\partial q}{\partial h_j} h_j \frac{\partial h_i}{\partial \psi}\right\} \\
+ \int_{-\infty}^{1} d\xi \int \frac{\partial \phi}{\partial \xi} \left\{ h_i \left[ f_2 \frac{\partial h_j}{\partial \psi} - \frac{\partial h_j}{\partial q_j} \right] \\
+ f_4 h_j h_i \frac{\partial q_j}{\partial \psi} \right\} \right\}_{\eta=1}
\]

where

\[
h_i = h_s g_{i1}(\xi) g_{j1}(n)
\]

\[
f_3 = \frac{\partial f_1}{\partial q} = -\frac{[1-(\tilde{\gamma}-1)q^2 + \frac{1}{2}(\tilde{\gamma}+1)q^2]^2}{\rho q^2 (1 - \frac{1}{2}(\tilde{\gamma}-1)q^2)^2}
\]

\[
f_4 = \frac{\partial f_2}{\partial q} = -\frac{\rho}{q^2} \cdot \frac{(1 + \frac{1}{2}(3-\tilde{\gamma})q^2)}{(1 - \frac{1}{2}(\tilde{\gamma}-1)q^2)}
\]

In detail, if we write

\[
h_i = h_s g_{i1}(\xi) g_{j1}(n)
\]

\[
h_j = h_s g_{i2}(\xi) g_{j2}(n)
\]
then the residual becomes

\[ r_{i1j1} = \int_{-1}^{1} \int_{-1}^{1} \frac{d\xi d\eta}{\delta} \left\{ f_{1} \frac{\partial q}{\partial \phi} \left[ \frac{\partial (h_s g_{i1})}{\partial \phi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \phi} g_{j1}' \right] + f_{2} \frac{\partial q}{\partial \psi} \left[ \frac{\partial (h_s g_{i1})}{\partial \psi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \psi} g_{j1}' \right] \right\} \\
+ \int_{-1}^{1} d\xi \left[ \frac{d\phi}{d\xi} h_s g_{i1} g_{j1} \left\{ f_{2} \frac{\partial q}{\partial \psi} + \beta (1 - q_{B}) \right\} \right] \right\} \]

and the matrix elements become:

\[ A_{i1j1, i2j2} = \int_{-1}^{1} \int_{-1}^{1} \frac{d\xi d\eta}{\delta} \left\{ f_{1} \frac{\partial (h_s g_{i2})}{\partial \phi} g_{j2} + h_s g_{i2} \frac{\partial n}{\partial \phi} g_{j2}' \right\} \times \left[ \frac{\partial (h_s g_{i1})}{\partial \phi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \phi} g_{j1}' \right] \\
+ f_{2} \left[ \frac{\partial (h_s g_{i2})}{\partial \psi} g_{i2} + h_s g_{i2} \frac{\partial n}{\partial \psi} g_{j2}' \right] \times \left[ \frac{\partial (h_s g_{i1})}{\partial \psi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \psi} g_{j1}' \right] \\
+ h_s g_{i1} \frac{\partial n}{\partial \psi} g_{j1}' \right\} + f_{3} \frac{\partial q}{\partial \phi} h_s g_{i2} g_{j2} \left[ \frac{\partial (h_s g_{i1})}{\partial \phi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \phi} g_{j1}' \right] \\
+ f_{4} \frac{\partial q}{\partial \psi} h_s g_{i2} g_{j2} \left[ \frac{\partial (h_s g_{i1})}{\partial \psi} g_{j1} + h_s g_{i1} \frac{\partial n}{\partial \psi} g_{j1}' \right] \right\} \\
+ \int_{-1}^{1} d\xi \left\{ \frac{d\phi}{d\xi} h_s g_{i1} g_{j1} \left[ f_{2} \left[ \frac{\partial (h_s g_{i2})}{\partial \psi} g_{j2} + h_s g_{i2} \frac{\partial n}{\partial \psi} g_{j2}' \right] \\
- \beta h_s g_{i2} g_{j2}' \right\} + f_{4} h_s g_{i2} g_{j2} + h_s g_{i1} g_{j1} \frac{\partial q}{\partial \psi} \right\} \right\} \]
3.8 CONCLUSION

Woods' conformal transformation, developed for the mapping of the incompressible flow fields in cascades, has been given as

\[ \omega = \frac{h}{2\pi} \left\{ \tan \alpha + 2(\sin \lambda \sinh r - \sin \alpha \cosh r) \tan^{-1}(e^{-r \tan \frac{1}{2} \xi}) \right\} \]

We have extended it and alleged that the transformation is, in fact, useful for both the compressible and incompressible cases, although the functions \( q \) and \( \theta \) are not harmonic for compressible flow. We then proposed to use this transformation, in conjunction with the Galerkin Method, to construct an algorithm for the numerical solution of compressible flow through a cascade of aerofoils. Mathematical details were subsequently discussed in section 3.6 and 3.7 for a trial function of the form

\[ q = h_s[A + \sum a_{ij} g_i(\xi)g_j(\eta)] \]

where \( A \) is the upstream velocity, and \( h_s \) is defined as

\[ h_s = \frac{h_0(\xi, \eta)}{k+zh_0(\xi, \eta)} \]

The circulation around a cascade can be calculated from (3.14) by keeping \( \gamma \) constant. The circulation varies from \( -\infty \) to \( +\infty \) as \( \gamma \) varies from \( -\infty \) to \( +\infty \). The result is

\[ k + z = 1 \]

It is generally a function of two variables. However, we take the limit

\[ k = \nu \text{ in equation (3.15)} \]

to reduce the trailing edge, one of the variables
CHAPTER IV

NON-TURNING CASCADES

4.1 INTRODUCTION

The circulation \( \Gamma \) about any closed curve \( C \) is defined by

\[
\Gamma = \oint_C \vec{v} \cdot d\vec{r}
\]  

(4.1)

where \( d\vec{r} \) is a vector element tangential to the curve \( C \) and \( \vec{v} \) is the velocity vector. It has been proved that when the external forces on the flow are conservative, the circulation about any closed circuit which moves with the fluid is constant (Woods 1961). Further, note that by writing \( \vec{v} \) as

\[
\vec{v} = \vec{v} \sim
\]

(4.2)

for irrotational flow, then the circulation becomes

\[
\Gamma = \oint_C \vec{v} \sim \cdot d\vec{r} = \oint_C d\phi = \Delta \phi_C
\]

(4.3)

Thus, the circulation around a closed curve \( C \) equals the change in the potential function \( \phi \) around \( C \).

The circulation about an aerofoil in a cascade can be calculated from (3.14) by keeping \( \mu = 0 \) and determining the change in \( \phi \) as \( \gamma \) varies from \(-\pi\) to \(\pi\). The result is

\[
\Gamma = \sin \alpha
\]

(4.4)

We observe that the circulation depends only on one variable here. It is generally a function of two variables. However, on taking the limit \( k \to \infty \) in equation (3.13) to remove the trailing edge, one of the variables
is eliminated.

Now, consider the channel formed by two adjacent stagnation streamlines. By (2.3) and (2.4), we have at upstream infinity the conditions

\[ \Delta \phi = \rho u q u \text{d}n \]

and

\[ \Delta \psi = q u \text{d}s \]

However

\[ \tan \theta_u = \frac{\text{d}s}{\text{d}n} \]

Hence

\[ \tan \theta_u = \rho \frac{\Delta \phi}{\rho u \Delta \psi} = \rho u \tan \alpha \quad (4.5) \]

Figure 4.1 demonstrates the geometrical meaning of this relation.

In the case where there is no circulation about the aerofoil in a cascade, by (4.4) the angle \( \alpha \) is equal to zero, and by (4.5) the inlet angle \( \theta_u \) is zero as well. Also, because there is no circulation, \( \Delta \phi = 0 \); the flow does not turn in leaving the aerofoils but continues downstream with
\[ \theta_d = 0 \]

and

\[ \Delta \theta = \theta_d - \theta_u = 0 \]

In the next two sections we show how the flow through such a non turning cascade may be computed using the techniques of the previous chapter (conformal transformation followed by a Galerkin Method). In section 4.2 we develop the necessary formulae; numerical results for a model computation are presented in section 4.3.

4.2 COMPUTATIONAL ASPECTS

With a non turning cascade, \( \alpha = 0 \) and \( \xi_s = 0 \), i.e. stagnation point coincides with the point \((0,-1)\) in the \((\xi,\eta)\) plane, as shown in figure 4.2.

Figure 4.3 shows how the function \( \psi \) appears as a function of \( \theta \)

\[ \psi < 0 \]

\[ \psi = 0 \]

\[ \psi > 0 \]
The region on the left of \( \eta \)-axis is the region where \( \psi < 0 \); that on the right is the region where \( \psi > 0 \). The stagnation streamline coincides with \( \eta \)-axis. The blade boundary extends from \( \xi = -1 \) to \( \xi = 1 \) on the line \( \eta = -1 \). The boundary conditions for our model calculation were taken to be

\[
q_B = \frac{(1-T_1)}{1-H_1} \left\{ 1 - H_1 \tanh \left[ \frac{2\pi(\phi - \phi_\infty)}{h \cos \alpha} \right] \right\}
\]

\[-1 \leq \xi \leq 1, \quad \eta = -1\]  

where

\[
T_1 = \exp \left( \frac{2\pi R_1 \phi}{h \cos \alpha} \right)
\]

\[
R_1 = 2, \quad H_1 = 0.36, \quad \phi_\infty = 0.45
\]

Figure 4.3 shows how the boundary function \( q_B \) appears as a function of \( \phi \).
Since \( q_B \) is symmetrical about \( \xi = 0 \), we see by (4.1) that there is no circulation in this case.

In applying the technique developed in chapter 3, we take the distance \( h \) between the two blades in the potential plane (figure 3.2) to be

\[ h = 1 \] (4.8)

The trial function is as given in section 3.5 with \( \xi_s = 0 \)

\[ q = h_s [A + \sum_{i,j} g_i(\xi)g_j(n)] \] (4.9)

with the downstream velocity given as

\[ q_d = .6 \] (4.10)

This value of \( q_d \), together with the form of \( q_B \), produce supersonic patches on the blade surface where the Mach Number is as high as 1.2.

The value of \( A \) depends on the width of the channel and the closure condition at the trailing edge. It is determined by the potential equation and hence its value depends on the form of the boundary condition. The constants in (4.6) and (4.7) have been chosen to obtain a closure, in which case the upstream velocity should be equal to the downstream velocity and \( A \) should equal 1.

Following the flow chart in table I, we now discuss in more detail the essential features involved in each step. In the starting phase (i), the number of trial functions used initially is decided on. In our case
we chose the number of \( n \)-functions to be 2 at first. \( q_N^{(0)} \) is initially set to the zero vector. This corresponds to the initial choice of trial function

\[
q_N^{(0)} = A h_s, \quad A = 0.4 \tag{4.11}
\]

The matrix elements required in step (ii), as given in section 3.7, are calculated by using a product Gauss-Legendre integration rule (Carnahan, Luther and Wilkes 1969) in the \( \xi \) and \( \eta \) variables, which was chosen to be a 24 by 10 point rule. However, a complication arises from the existence of the stagnation point, in the vicinity of which various factors in the integrands behave rather violently, for example, by (3.81) and (3.82),

\[
\left. \frac{\partial h_s}{\partial \gamma} \right|_\gamma = \frac{\gamma}{\sqrt{\gamma^2 + \mu^2}}
\]

\[
\left. \frac{\partial h_s}{\partial \mu} \right|_\mu = \frac{\mu}{\sqrt{\gamma^2 + \mu^2}}
\]

We subdivide the lines of constant \( \mu = \mu_i \) into 2 or 3 intervals depending on how close we are to the stagnation point \((\xi_i, 0)\). On the blade \((\eta = 1)\) we integrate from \( \xi = \xi_1 \) to \( \xi = \xi_2 \), then from \( \xi = \xi_2 \) to \( \xi = \xi_3 \). On any line \( \eta = \eta_i \) we use a 24-point integration rule in each of the three intervals \((\xi = \xi_1, \xi = \xi_2, \xi = \xi_3)\) more about this situation by putting more points in the integration rule around the stagnation point than in the area further away from it. This scheme is partially explained in Fig. 4.4.
We subdivide the lines of constant \( n \) into 2 or 3 intervals depending on how close we are to the stagnation point \((\xi_s, -1)\). On the blade \((\eta = -1)\) we integrate from \( \xi = -1 \) to \( \xi = \xi_s \) then from \( \xi = \xi_s \) to \( \xi = 1 \). On any lines \( \eta = \eta_i \), we use a 24-point integration rule in each of the three intervals; \((-1, \xi_i^1)\), \((\xi_i^1, \xi_i^2)\) and \((\xi_i^2, 1)\) where

\[
\xi_i^1 = \xi_s - \mu_i \\
\xi_i^2 = \xi_s + \mu_i
\]

(4.12)

with

\[
\mu_i = - \ln \frac{1}{2}(1 - \eta_i)
\]
totalling 72 points on the line $\eta = \eta_1$. However, if

$$\mu_i \geq \min \{1 + \xi_S, 1 - \xi_S\}$$

(4.13)

that is, if we get far enough from the stagnation point, then we only use the 24-point rule in 2 intervals, i.e. $(-1, \xi_S)$ and $(\xi_S, 1)$, totalling 48 points on these lines of constant $\eta$.

We go on to step (iii) in Table I. Here we solve the linear equation (2.36) for the vector $\mathbf{a}_N^{(n)}$ by using a direct method (Gauss elimination).

The convergence test for the non-linear iterations in step (iv) is

$$\|\mathbf{a}_N^{(n)}\| \leq \varepsilon\|\mathbf{a}_N^{(n+1)}\|$$

(4.14)

with $\varepsilon$ a prescribed accuracy. Once this accuracy is achieved after a few iterations we go on to calculate the angle $\theta$ and the shape of the blade, i.e. the $(x,y)$-coordinates to check the convergence of the scheme with respect to the number of trial functions. The angle $\theta$ along the stagnation streamline is expected to take a step of $\pi/2$ at the stagnation point. The number of $\eta$-functions (and hence the number of trial functions used) is increased until a reasonable shape of blades is obtained especially around the stagnation point.

The physical quantities ($\theta, x, y$) can be found by integration (step (vi)). Any numerical integration scheme could be used, but for convenience a rule based on equally spaced finite difference was used. The angle $\theta$ is given as
\[ \theta = \int \frac{\rho}{q} \frac{\partial q}{\partial \psi} \, d\psi \]

\[ \psi = \text{constant} \]

therefore,

\[
\theta = \begin{cases} 
\theta_d + \int_{-1}^{\xi} \frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{d\psi}{d\xi} \, d\xi & \text{on the lower blade-surface} \\
\theta_d + \int_{-1}^{\xi} \frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{d\psi}{d\xi} \, d\xi & \text{on the upper blade-surface} \\
\theta_u + \int_{1}^{\eta} \frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{d\psi}{d\eta} \, d\eta & \text{along the stagnation streamline} 
\end{cases} \tag{4.15}
\]

\[
\xi = \xi_s \]

A modified trapezium rule with equal intervals \((K_h=2)\) was chosen to integrate these integrals along the 3 intervals for

\[
\theta = \begin{cases} 
\theta(kh-1,-1) & \text{on the blade} \\
\theta(\xi_s, kh-1) & \text{on the stagnation streamline.} 
\end{cases} \tag{4.16}
\]

\[ K - 1 > k > 1 \]

Calling

\[
\phi = \begin{cases} 
\frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{d\psi}{d\xi} & \text{on the blade} \\
\frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{d\psi}{d\eta} & \text{on the stagnation streamline.} 
\end{cases} \tag{4.17}
\]
then we have

\[ \theta_{k+1} = \theta_k + \frac{h}{24} \left[ -x_{k-1} + 13x_k + 13x_{k+1} - x_{k+2} \right] + O(h^5) \]

modified at the end points by

\[ \theta_1 = \theta_0 + \frac{h}{12} \left[ 5x_0 + 8x_1 - x_2 \right] + O(h^4) \]

with similar rule at the other end point. The results presented below in section 4.3 are for 50 steps.

Similarly, the \( x, y \) coordinates are given by

\[ x = \int \frac{\cos \theta}{q} \, d\phi \]

\[ \psi = \text{constant} \]

\[ y = \int \frac{\sin \theta}{q} \, d\phi \]

\[ \psi = \text{constant} \]

which can also be separated into 3 line integrals as in the case of the angle \( \theta \).

4.2.1 Asymptotic Analysis for \( \theta \)

Some complication may arise from the behaviour of the integrands in the integrals for \( \theta \) near the end points. It is necessary to look more closely at their limiting values both far upstream and downstream, as well as at the stagnation point.
1) At the stagnation point, for the case where \( \alpha = 0 \), we have from (3.80) and (3.88) that

\[
q = \frac{\sqrt{2 + \mu^2}}{2k} \{ A + \sum a_{ij} g_i g_j \}
\]

On the other hand, at the stagnation point

\[
\frac{\partial \phi}{\partial \xi} \sim \frac{h \mu}{8\pi}
\] (4.18)

and

\[
\frac{\partial \phi}{\partial \eta} \sim \frac{h y}{4}
\]

along the stagnation streamline.

Hence

\[
\chi = \frac{\rho}{q} \frac{\partial q}{\partial \psi} \frac{\partial \phi}{\partial \xi} \quad \text{or} \quad \frac{\rho}{q} \frac{\partial q}{\partial \psi} \cdot \frac{\partial \phi}{\partial \eta} \sim 0
\]

near the stagnation point for all line integrals (4.15), (4.16) and (4.17).

2) Far upstream, \( \mu \to \infty \). We note that for the particular case when \( \alpha = 0 \),

\[
\frac{\partial q}{\partial \psi} = 0
\] (4.20)
on the stagnation streamline $\gamma = \gamma_0$, $\mu > 0$. Therefore, at upstream infinity

$$\chi = 0$$

3) Far downstream, $\gamma \to \pm \pi$ with $\mu = 0$. By (3.92), (3.93), (3.94), (3.98) and (3.95) we have

$$\frac{\partial q}{\partial \psi} \sim \frac{2\pi(\gamma \mp \pi)}{h} \sum a_{ij} g_i g_j'$$

together with (3.100), it becomes clear that

$$x \sim \frac{2\pi \rho_d \sum a_{ij} g_i g_j'}{[A + \sum a_{ij} g_i g_j]}$$

(4.21)

4.2.2 Asymptotic Analysis for $x$ and $y$

Let

$$f_x = \frac{\cos \theta}{q}$$

$$f_y = \frac{\sin \theta}{q}$$

so that we can write

$$x = \int f_x d\phi$$

$$y = \int f_y d\phi$$

$$\psi = \text{constant}$$

(4.22)

At the stagnation point;
\[ \lim_{x \to 0} f_x = \lim_{\phi \to 0} \left\{ -\sin \theta \frac{\partial \theta}{\partial \phi} \right\} \]

using L'Hôpital's theorem. Similarly for \( f_y \). However

\[ \frac{q}{\rho} \frac{\partial \theta}{\partial \phi} = \frac{\partial q}{\partial \psi} \quad \text{by (2.6)} \]

and

\[ \frac{\partial q}{\partial \phi} \sim \frac{2\pi \left( A + \sum_{i=1}^{k} S_i \right)}{h \kappa} \quad \text{by (3.81), (3.86)} \]

(3.82) and (3.87). Using (4.18) and (4.19) we conclude that

\[ f_x \frac{d\phi}{d\xi} \sim 0 \]

and

\[ f_y \frac{d\phi}{d\xi} \sim 0 \]

near the stagnation point.

### 4.3 RESULTS

We choose the case where \( \gamma = 1.4 \). The computation started with the number of \( n \)-functions \( J = 2 \), and continued until the coefficients converged to an accuracy of \( 10^{-4} \times \) largest coefficient.

A few results were obtained for different values of the parameter \( \beta \) which, as has been mentioned, measures the extent at which the problem feels the effects of the boundary conditions. Figure 4.5 shows a graph of the numbers of iterations needed for convergence, and the values of \( A \), when \( J = 2 \) against the parameter \( \beta \). The value of \( k \) was taken to be 2.1. The curves indicate that even though the increase of \( \beta \) improves the
rate of convergence, it at the same time reduces the accuracy of the solution. The value of $A$ diverges from 1 if $\beta$ gets too big or too small. Hence we chose the value $\beta = 2000$ for the rest of the computation which is presented below.

Figure 4.6 shows the variations of $A$, the angle at the stagnation point $\theta_0$ (depicted in figure 4.11), and the maximum error in surface velocities, as a function of $k$ (see section 3.3). We see that the error is minimum (when $J = 2$) if $k$ takes a value close to 2.5. Therefore, for the rest of the computation, $k$ was put equal to 2.55.

In figure 4.7, we compare the convergence of the solution for the upstream velocity with the convergence for the stagnation angle $\theta_0$. The quantities are plotted against $J$, and it is clear that we have a much faster convergence for the velocity than for the stagnation angle. This is to be expected since the calculation of angles involves a first derivative of the velocity distribution, and this behaviour has already been noticed in the case of channel flow.

Figure 4.8 shows two leading coefficients from the vector $a_{(n)}^{n}$; both are seen to be converging with respect to $J$. Figure 4.9 compares the convergence in the upstream and downstream velocities for the case $\beta = 2000$ and $k = 2.55$. Both are converging as $J$ increases but it is seen that the convergence in $q_u$ is better, and this is in agreement with the results found for channels, where the accuracy of the solution nearest to the boundary was found to be the poorest.

In figure 4.10 we show again how it takes longer for solution to converge to a local angle $\theta$ than to the corresponding local velocity $q$. Finally, the shape of the aerofoils in the flow with the boundary conditions
given in (4.6) and (4.10) was tabulated and graphed in figure (4.11). The total number of trial functions used was 73. The step in the angle along the stagnation streamline at the stagnation point was found to be $\theta_0 = 1.56389$, having an error of 0.7%. The upstream velocity was found to be $1.0030 qd$.

4.4 CONCLUSION

The solution to the problem of a cascade of aerofoils with no turning, i.e.

$$\Delta \theta = 0$$
$$\alpha = 0$$

has been obtained. The angle at the stagnation point was found to be

$$\theta_0 = 1.56389$$

which is quite close to the exact value of $\frac{1}{2} \pi = 1.57079$. The error in the surface velocity got down to $0.07 qd = 0.049$. This error is at far downstream where $qd = 0.7$ in units of the stagnation speed of sound. Therefore, the error is only 7% of the exact value. These results are for $J = 6$ which makes the total functions used $N = 73$. Considering that we were dealing with a transonic case, the results are reasonably good, and we did not pursue the calculation to larger values of $J$.
Figure 4.5: Number of iterations \( IT \), and \( A = \frac{q_u}{q_d} \) against parameter \( \beta \).
Figure 4.6: Upstream Mach number $\Lambda$, angle at stagnation point $\theta_0$, and maximum error in surface velocity ERR, against $k$;

$\Lambda = x + \Delta$, $x = 96$

$\theta_0 = x + 2\Delta$, $x = 1.48$

$ERR = x + 10\Delta$, $x = .3$
Figure 4.7: Upstream velocity $A$, and stagnation angle $\theta_0$ against number of trial functions $J$. 

$\beta=2000$

$k=2.55$
Figure 4.8: Two leading coefficients $a_0$ and $a_1$ against $J$. 

$\beta = 2000$

$k = 2.55$
Figure 4.9: Upstream and downstream velocities against $J$.
Figure 4.10: Local velocity and corresponding angle against \( J \).

\[
\begin{align*}
\beta &= 2000 \\
k &= 2.55
\end{align*}
\]
Figure 4.11: Shape of the aerofoils and the stagnation streamline.

\[ \beta = 2000 \quad , \quad k = 2.55 \quad , \quad J = 6 \]

\[ R = 2 \quad , \quad H = .36 \quad , \quad \phi_k = .45 \]

\[ q_d = .6 \quad , \quad \theta_0 = 1.56389 \]
CHAPTER V

CASCADES WITH CIRCULATION

5.1 INTRODUCTION

In the previous chapter we dealt with the case where the circulation about each aerofoil was zero. In this chapter the scheme is extended to the case where the circulation is non-zero, and hence the cascade turns the flow. We make use of work done by Silvester and Fitch in a paper on the design of cascades, giving a relationship from which, once $\alpha$ is fixed, the turning angle $\Delta\theta$ of the compressible flow can be calculated from the upstream and downstream velocities and the knowledge of the velocity distribution on the two surfaces of the blade (Silvester and Fitch 1970).

We are going to derive another relationship, in addition to the one mentioned above, which in turn determines $\alpha$ once $\Delta\theta$ is fixed. These two relationships thus comprise an iterative scheme from which the angle $\alpha$ and the turning angle $\Delta\theta$ may be obtained. Since the value of the upstream velocity must be calculated along with the solution vector $a_{\alpha}^{(n)}_{\alpha}$, a new value of $\alpha$ must be calculated from the scheme for every new value of $A$. Starting off with some arbitrary value for $\alpha$, all the relevant transformations are made and the solution to the flow is obtained through the technique set out in chapter III. This gives us the value of $A$, from which a new value of $\alpha$ can be calculated. This new $\alpha$ is then used to obtain a new solution to the flow. In this way, the values of $\alpha$, $A$ and the solution vector $a_{\alpha}^{(n)}_{\alpha}$ are expected to converge simultaneously with respect to $N$.

The mathematical formulation of the problem is given in sections 5.2, 5.3 and 5.4. The flow chart of the whole scheme is in table III.
The solution for a flow subject to a chosen form of boundary distribution is then presented in section 5.5 in order to illustrate the performance of our proposed method.

5.2 THE TURNING ANGLE

Consider the part of the flow between a pair of blades as shown in figure 5.1, with the inlet angle $\theta_u$ and the outlet angle $\theta_d$. In the potential plane the equations governing the flow have the form

$$\frac{\rho}{q} \frac{\partial q}{\partial \psi} - \frac{\partial \theta}{\partial \phi} = 0 \quad (5.1)$$

$$\frac{1}{\rho q} \left( 1 - \frac{1}{2} (\gamma+1) q^2 \right) \frac{\partial q}{\partial \phi} + \frac{\partial \theta}{\partial \psi} = 0 \quad (5.2)$$

In this plane the part of the flow we are interested in is now shown in Figure 5.2 with the lines BC and HG representing the upper and lower surfaces of the two adjacent blades respectively.

![Figure 5.1](image-url)
The blades extend to infinity downstream. The points G and C are therefore shown in figure 5.2 to be on the same line of constant \( \phi \) downstream. The stagger \( \Delta \psi \) gives rise to the distance KJ and \( \Delta \psi \) is the difference in \( \psi \) between K and A. GE is a line drawn parallel to HB and JA and therefore CE has a length \( \Delta \phi \).

Figure 5.2

Following Silvester and Fitch (Silvester and Fitch 1930), we introduce new variables \( F \) and \( G \) such that

\[
\frac{dF}{dq} = \frac{\rho}{q} \quad (5.3)
\]

\[
\frac{dG}{dq} = \frac{1}{\rho q} \left( \frac{1}{2(\gamma+1)q^2} \right) \quad (5.4)
\]
so that the equations (5.1) and (5.2) become

\[ \frac{\partial F}{\partial \psi} - \frac{\partial \psi}{\partial \phi} = 0 \]  
(5.5)

\[ \frac{\partial G}{\partial \phi} + \frac{\partial \theta}{\partial \psi} = 0 \]  
(5.6)

Using (5.3) and (5.4) and knowing \( \rho \) as a function of \( q \) (see section 2.2) we can integrate directly for \( F \) and \( G \) for some value of \( \bar{y} \). Taking \( \bar{y} = \frac{7}{5} \), we find that

\[ F = \log q + \frac{1}{5} z^5 + \frac{1}{3} z^3 + z - \log \frac{1}{2}(1+z) + \frac{23}{15} \]  
(5.7)

and

\[ G = \log q - z^5 + \frac{1}{3} z^3 - z^{-1} - \log \frac{1}{2}(1+z) - \frac{1}{3} \]  
(5.8)

where

\[ z = (1 - \frac{\rho q}{5}) \]

In each case the constant of integration has been chosen so that \( F \) and \( G \to \log q \) as \( q \to 0 \) (so that the equations (5.5) and (5.6) go back to incompressible case.)

Now integrate equation (5.5) over rectangle \( AKGC \) to get

\[ \int_A^C d\phi \int_A^K d\psi \left( \frac{\partial F}{\partial \psi} - \frac{\partial \psi}{\partial \phi} \right) = 0 \]
(5.12)

that is

\[ \int_K^G F d\phi - \int_A^C F d\phi + \int_A^K \theta d\psi - \int_C^G \theta d\psi = 0 \]
(5.9)
Because of the repeat condition, the flow is the same at any corresponding points \((\phi, \psi)\) and \((\phi+\Delta \phi, \psi+\Delta \psi)\). Therefore, the integrals on AB and JH cancel and we obtain

\[
\int_B^C [F]_{\psi=0+} \, d\phi - \int_B^C [F]_{\psi=0-} \, d\phi = \Delta \psi (\theta_u - \theta_d) + \Delta \phi (F_u - F_d)
\]

or

\[
\theta_d - \theta_u = (F_u - F_d) \tan \alpha - \frac{1}{\Delta \psi} \int_B^C [F(0+) - F(0-)] \, d\phi
\]

\[
\tan \alpha = \frac{\Delta \phi}{\Delta \psi}
\]

Since \(F(0+)\) and \(F(0-)\) both tend to \(F_d = F(q=q_d)\) as \(\phi \to \infty\) we can let \(C\) tend to infinity and the integral remains finite. Thus, we obtain the deflection condition,

\[
\Delta \theta = \theta_d - \theta_u = (F_u - F_d) \tan \alpha - \frac{1}{\Delta \psi} \int_S^{\infty} \int_{\phi}^0 [F]_{\psi=0-} \, d\phi
\]

from which, given \(\alpha\), the turning angle \(\Delta \theta\) can be calculated.

5.3 THE ANGLE \(\alpha\)

We now derive the second relationship which determines \(\alpha\) once \(\Delta \theta\) is known. At upstream infinity we have from (2.3) and (2.4) that

\[
\Delta \psi = \rho u q u \, d\eta
\]

(see figure 5.3)
By integrating (5.12) from $\psi = 0$ to $\psi = \Delta \psi$ we obtain

$$\Delta \psi = \rho_u q_u H \cos \theta_u$$  \hspace{1cm} (5.13)

where $H$ is the distance along the $y$ axis in the physical plane between the streamlines $\psi = 0$ and $\psi = \Delta \psi$. Similarly downstream we have

$$\Delta \psi = \rho_d q_d (H \cos \theta_d - b)$$  \hspace{1cm} (5.14)

where $b$ is the blade thickness at downstream infinity.
On equating (5.13) and (5.14) we obtain

\[ \frac{\Delta \psi}{\rho U q_u \cos \theta_u} = \left( \frac{\Delta \psi}{\rho_d q_d} + b \right) \times \frac{1}{\cos \theta_d} \]

i.e.

\[ K_d \cos \theta_d = K_u \cos \theta_u \]

Putting

\[ \theta_d = \theta_u + \Delta \theta \]

we have

\[ K_d \cos \theta_u = K_d \cos(\theta_u + \Delta \theta) \]

\[ = K_d \left( \cos \theta_u \cos \Delta \theta - \sin \theta_u \sin \Delta \theta \right) \]

Rearranging, we find the following relationship

\[ \tan \theta_u = \frac{K_d \cos \Delta \theta - K_u}{K_d \sin \Delta \theta} \] (5.15)

From (4.5) we have

\[ \tan \alpha = \frac{1}{\rho_u} \tan \theta_u \]

hence

\[ \tan \alpha = \frac{K_d \cos \Delta \theta - K_u}{\rho_u K_d \sin \Delta \theta} \] (5.16)

This relationship (5.16) together with (5.11) gives us the condition which the angle \( \alpha \) must satisfy. The thickness of the blade far downstream, \( b \), must be known before (5.16) can be used. However, we cannot know \( b \) until...
after the solution to the equations of motion has been obtained. On the other hand, the solution to the equations cannot be found unless we know the angle \( \alpha \) beforehand. We therefore propose an iterative scheme in which \( \alpha \) is iterated along with the solution to the complete problem. That is, a starting value is given to \( \alpha \) quite arbitrarily, then \( \alpha \) is used to obtain a solution to the flow subjected to the boundary condition which is dependent on this initial value of \( \alpha \). The solution to the problem, which has the form

\[
q = h_s[A + z \alpha I g_i(5)g_j(n)] ,
\]

is obtained through the technique already discussed in chapters 2 and 3. This solution is then used to compute the shape of the blade using the equations (4.15) to (4.17). \( \beta \) is thus determined. We consequently iterate for \( \alpha \) from (5.11) and (5.16). Thus we keep iterating until both the solution vector \( a_{\alpha}^{(n)} \) and \( \alpha \) converge. Then we can increase the number of trial functions and go through the iterations again. Observe also that every time a new value of \( \alpha \) is obtained, a new value of \( \theta_u \) can be found by using (4.5)

\[
\theta_u = \arctan [\rho_u \tan \alpha]
\]

5.4 COMPUTATIONAL ASPECTS

The flow chart of the whole process is in table III. The solution vector \( a_{\alpha}^{(0)} = \{a_1, a_2, \ldots, a_N\} \), where \( N \) = number of trial functions, are initially taken to be the zero vector. \( A \) is also given some initial value \( A^{(0)} \) as well, and it is to be iterated along with the other coefficients \( a_{ij}^{(n)} \), \( i = 1, 2, \ldots, I \), \( j = 1, 2, \ldots, J \).
From the table we see that the scheme is not much different from the one for the problem with no turning in table II, only here \( \Delta \theta \) and \( \alpha \) have to be re-evaluated every time we have convergence in the solution vector \( a^n \). Once \( \alpha \) converges we then increase the number of the trial functions used in the expression for \( q \).

To calculate \( \Delta \theta \) from the relation (5.11) we are given the boundary condition. In our particular application of the method, the velocity distribution on the blade surface has the form

\[
q_B = \begin{cases} 
\frac{(1-T_1)^b_2}{(1-H_1)} \left\{ 1 - H_1 \tanh \left[ \frac{2\pi}{h \cos \alpha} (\phi - \phi_s) \right] \right\} 
\text{along } \psi = 0^+ \\
\frac{(1-T_2)^b_2}{(1-H_2)} \left\{ 1 - H_2 \tanh \left[ \frac{2\pi}{h \cos \alpha} (\phi - \phi_s) \right] \right\} 
\text{along } \psi = 0^-
\end{cases} 
\]

where

\[
T_{1,2} = \exp \left[ \frac{-2\pi R_{1,2}}{h \cos \alpha} (\phi - \phi_s) \right] 
\]

(5.17)

\( \phi \) is given in (3.15) and \( \phi_s \) is the value of \( \phi \) at the stagnation point given by

\[
\phi_s = \frac{h}{2\pi} [-2\alpha \sin \alpha - 2 \cos \alpha \ln(\cos \alpha)] 
\]

(5.18a)

since \( \gamma_s = -2\alpha \) in (3.21) at the stagnation point. \( h \) is given by

\[
h = \frac{\Delta \psi}{\cos \alpha}
\]
for every new value of \( \alpha \).

The two values \( H_1 \) and \( H_2 \) are related by the fact that we are using the same transformation for the upper and the lower blade surface. We must have that, at the stagnation point,

\[
q_B \sim \left\{ \frac{2\pi R_1}{h \cos \alpha} (\phi - \phi_s) \right\}^{\frac{1}{2}} \left\{ 1 - H_1 \tanh \left( \frac{2\pi \rho}{h \cos \alpha} \right) \right\}
\]

\[
= \left\{ \frac{2\pi R_2}{h \cos \alpha} (\phi - \phi_s) \right\}^{\frac{1}{2}} \left\{ 1 - H_2 \tanh \left( \frac{2\pi \rho}{h \cos \alpha} \right) \right\}
\]

where \( \phi \sim \phi_s \). Hence

\[
\left( \frac{R_1}{R_2} \right)^{\frac{1}{2}} \left( \frac{1 + H_1 T_\phi}{1 - H_2} \right) = 1
\]

(5.19b)

where

\[
T_\phi = -\tanh \left( \frac{2\pi \rho}{h \cos \alpha} \right)
\]

In other words, if

\[
K(H_2) = \left( \frac{R_2}{R_1} \right)^{\frac{1}{2}} \left( \frac{1 + H_2 T_\phi}{1 - H_2} \right)
\]

(5.20)

then

\[
H_1 = \frac{K(H_2) - 1}{K(H_2) + T_\phi}
\]

(5.21)

which determines \( H_1 \) once \( H_2 \) is fixed.
The integrals in (5.11) were evaluated in step (v) of table III using the same quadrature rules as those which have already been described in section 4.2. In step (v) also, we used equations (5.11) and (5.16) to iterate for \( \alpha \) from a given initial value, and then applied Aitken's \( \Delta^2 \)-method for accelerating convergence. That is if \( \alpha_n, \alpha_{n+1}, \alpha_{n+2} \) are any three consecutive linear iterates, then

\[
\alpha' = \alpha_n - \frac{(\alpha_{n+1} - \alpha_n)^2}{\alpha_{n+2} - 2\alpha_{n+1} + \alpha_n}
\]

(5.22)

is taken to be a better approximation to \( \alpha \) than \( \alpha_{n+2} \) (Henrici 1964).

Otherwise, the techniques used in the other steps in table III are all the same as those already described in detail in the case of a non-turning cascade in chapter 4.

However, since \( \alpha \) is non-zero for this case, we expect some finite contribution to the integrals for \( \theta \), \( x \) and \( y \) from the end points. Therefore, we must look more closely into those points of concern.

5.4.1 Asymptotic Analysis for \( \theta \)

As in (4.15) through to (4.17) we need to integrate

\[
\theta = \int \rho \frac{\partial q}{\partial \psi} d\phi
\]

with the end points being either at upstream or downstream infinity or at the stagnation points where we might expect some complication from the fact that the denominator in the integrand is zero.

1) At the stagnation point, \( \gamma = \gamma_S \) then \( \mu \to 0 \), we have
\[ \frac{\partial h_s}{\partial \psi} = \frac{\partial h_s}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial \psi} + \frac{\partial h_s}{\partial \mu} \cdot \frac{\partial \mu}{\partial \psi} \]

\[ \sim 0 \]

From (3.81), (3.82), (3.86) and (3.87). Also

\[ q \sim \frac{\sqrt{\gamma - \gamma_S}^2 + \mu^2}{2k} \{ A + \Sigma a_{i j} g_{i j} g_{i j} \} \text{ from (3.80)} \]

From (3.88) and (3.90) we know that

\[ \frac{\partial \phi}{\partial \xi} \sim \frac{h}{4 \cos \alpha} (\dot{\gamma} - \gamma_S) \]

and

\[ \frac{\partial \phi}{\partial \eta} \sim -\frac{h \mu}{8 \pi \cos \alpha} \]

Hence

\[ \frac{\partial q}{\partial \psi} = \frac{\partial h_s}{\partial \psi} [A + \Sigma a_{i j} g_{i j} g_{i j}] + h_s \left\{ \Sigma a_{i j} \left[ g_{i j} g_{i j} \frac{\partial \xi}{\partial \psi} + g_{i j} g_{i j} \frac{\partial \eta}{\partial \psi} \right] \right\} \]

\[ \sim 0 + \frac{\sqrt{(\gamma - \gamma_S)^2 + \mu^2}}{2k} \left\{ \Sigma a_{i j} \left[ g_{i j} g_{i j} \frac{-4 \cos \alpha \mu}{h \mu^2 + (\gamma - \gamma_S)^2} \right] + g_{i j} g_{i j} \frac{8 \pi \cos \alpha (\gamma - \gamma_S)}{h \mu^2 + (\gamma - \gamma_S)^2} \right\} \]

Therefore
\[ \frac{\rho}{q} \frac{\partial q}{\partial \psi} \cdot \frac{\partial \phi}{\partial \zeta} = \frac{2\pi E_{ij} g_i g_j}{[A^2 + E_{ij} g_i g_j]} \]

\[ \frac{\rho}{q} \frac{\partial q}{\partial \psi} \cdot \frac{\partial \phi}{\partial \eta} \approx \frac{E_{ij} g_i g_j}{2\pi [A^2 + E_{ij} g_i g_j]} \]

\( \rho \) being \( 1 \) at the stagnation point. Therefore, even though \( q \to 0 \) at this point there is a finite contribution to the integral here.

2) Far upstream; \( \mu \to \infty \) while \( \gamma \to \gamma_0 \). From (3.76) we have

\[ \frac{\partial h_s}{\partial \psi} \cdot \frac{2\pi k}{h} \frac{(1-\eta) \sin \alpha}{(1-\eta) \sin \alpha} \]

and

\[ h_s \sim 1 \quad \text{from (3.72)} \]

Moreover

\[ \frac{\partial \phi}{\partial \eta} \sim - \frac{h}{2\pi \cos \alpha (1-\eta)} \]

Hence

\[ \frac{\rho}{q} \frac{\partial q}{\partial \psi} \cdot \frac{\partial \phi}{\partial \zeta} \sim - \frac{\rho u}{A} \left\{ \frac{2\pi k}{h} A (1-\eta) \sin \alpha + E_{ij} \left\{ \frac{2}{h \cos \alpha} g_i (\xi) g_j (\eta) \right. \right. \]

\[ \left. + \frac{2\pi \sin \alpha}{h} (1-\eta) g_i (\xi) g_j (1) \right\} \times \left. \frac{h}{2\pi \cos \alpha (1-\eta)} \right. \]

\[ \nu \frac{\rho u}{\pi A \cos \alpha} \left\{ - \pi k A \sin \alpha + \frac{1}{\cos \alpha} E_{ij} g_i (\xi) g_j (1) - \pi \sin \alpha E_{ij} g_i (\xi) g_j (1) \right\} \]

\[ (5.23) \]

3) Far downstream; \( \gamma \to \pm \pi \), \( \mu = 0 \). From (3.92) to (3.103) we have
and
\[ \frac{\partial q}{\partial \psi} \sim \frac{2\pi}{\hbar \cos \alpha} \sum_{ij} g_i g_j h_S \]

With (3.100), we have
\[ \left( \frac{\partial q}{\partial \psi}, \frac{\partial \phi}{\partial \xi} \right) \sim \frac{2\pi \rho}{\rho q} \sum_{ij} g_i g_j h_S \left[ A^2 + \sum_{ij} g_i g_j \right] \]

5.4.2 Asymptotic Analysis for \( x \) or \( y \)

As in (4.22)
\[ (x, y) = \int \left( \frac{\cos \theta}{q}, \frac{\sin \theta}{q} \right) d\phi \quad \psi = \text{constant} \]

and we also have to take care at the stagnation point where \( q \to 0 \).

From (3.80) and (3.90) we have
\[ \lim_{\mu \to 0, \gamma = \gamma_0} \frac{\cos \theta}{q} \frac{\partial \phi}{\partial \eta} = \frac{\cos \theta S}{(\gamma - \gamma_S)^2 + \mu^2} \frac{2k h \mu}{8\pi \cos \alpha} \frac{1}{[A^2 + \sum_{ij} g_i g_j]} \]

\[ = \frac{hk \cos \theta S}{4\pi \cos \alpha [A^2 + \sum_{ij} g_i g_j]} \]

where \( \theta_S \) is the angle at the stagnation point (which depends on which branch of the streamline is being considered). From (5.46) we find that
\[ \lim_{\gamma + \gamma_0, \mu \to 0} \frac{\cos \theta}{q} \frac{\partial \phi}{\partial \xi} = 2\pi \lim_{\mu \to 0} \frac{\cos \theta}{q} \frac{\partial \phi}{\partial \eta} \]

\[ \frac{\partial h_S}{\partial \psi} \sim 0 \]
Similarly we obtain

\[
\lim_{\gamma \to \gamma_s} \frac{\sin \theta \ \partial \phi}{\partial \zeta} = 2\pi \lim_{\mu \to 0} \frac{\sin \theta \ \partial \phi}{\partial \eta}
\]

\[
= \frac{h \ k \ \sin \ \theta_s}{2 \cos \alpha \ [A + \Sigma \ \lambda_{ij} \ g_i g_j]}
\]

(5.27)

5.5 RESULTS

Here again \( \gamma = 1.4 \) and the computation started with the number

of \( \eta \)-function \( J = 2 \); it continued until the coefficients converged
to an accuracy of \( 10^{-4} \times \) largest coefficient. The starting value of

\( A \) was \( A(0) = 0.6 \). We also arbitrarily chose \( a(0) = 0.2 \). The flow

is subjected to the boundary condition as given in (5.17) with

\[
R_1 = 1 \ , \ R_2 = 13
\]

\[
\phi_k = 1.4 \ , \ H_2 = -0.4
\]

(5.28)

and \( \Delta \psi = 0.5 \)

By (5.21)

\[
H_1 = \frac{K(H_2)^{-1}}{K(H_2)^{+1} + T_{\phi}}
\]

(5.29)

We put \( q_d = 0.7 \) in this particular example. The parameter \( \beta \) remains
equal to 2000 as was found best for the non turning cascade. \( k \) is

1.45 in all the results presented below.
TABLE III

START

i) N starting value
\[ n = 0, a_{\infty}^{(0)} = 0, A_{N}^{(0)} = A^{0}, \alpha_{N}^{(0)} = 0.2, \]

ii) Calculate \( A_{N}^{(n)}(n) \) and \( \alpha_{N}^{(n)}(n) \) and \( r_{N}^{(n)}(\alpha_{N}^{(n)}) \)

iii) Solve for \( da_{N}^{(n)}(n) \)
\[ a_{\infty}^{(n+1)} = a_{\infty}^{(n)} + da_{N}^{(n)} \]
\[ A_{N}^{(n+1)} = A_{N}^{(n)} + 0.1 \tanh(10 da_{N}^{(n)}) \]

iv) Has \( a_{\infty}^{(n+1)}, A_{N}^{(n+1)} \) converged?

v) Calculate \( \theta, x, y, b, a_{N}^{(n+1)} \)

vi) Has \( a_{N}^{(n+1)} \) converged?

vii) Has the solution converged w.r.t. \( N \)?

\[ J = J + 1 \]
\[ n = 0 \text{ Set } a_{\infty}^{(n)} \text{ and } A_{N}^{(n)} \]

NO

YES

NO

YES

FINISH
As before, we used twice the number of \( \eta \)-functions \( g_j(\eta) \), 
\( j = 1,2,\ldots,J \), for the number of \( \xi \)-functions \( g_i(\xi) \) that is

\[
N = 2J^2 + 1
\]

Figure 5.4 shows the boundary condition (5.17) as a function of \( \phi \). The peak of the upper curve is at \( q_B \approx 1.54 \). This means that the velocity becomes as high as

\[
q = 1.54q_d \approx 1.08
\]

Therefore, we have a supersonic patch on the upper blade surface. The form of the boundary condition determines the value of \( A \) which is plotted against the number of trial functions \( J \) in figure 5.5. It is seen that \( A \) is converging as \( J \) increases. The next diagram shows the two leading coefficients \( a_0 \) and \( a_1 \) plotted against \( J \). They are also converging very well with respect to \( J \).

The values of \( \theta_u \) and \( \alpha \) are plotted against \( J \) in figure 5.7. They are converging as \( J \) increases but not as well as \( A \). This is not surprising since their values cannot be calculated until \( A \) is known. Also, a knowledge of the shape of the cascade is needed for their calculation. The turning angle \( \Delta \theta \) is iterated along with the calculation for \( \theta_u \) and \( \alpha \); its value is plotted against \( J \) in figure 5.8.

Figure 5.9 shows the angles \( \theta_0 \) and \( \theta_1 \), which measure a jump in \( \theta \) of \( \frac{1}{2} \pi \) at the stagnation point, against \( N_0 = \) the number of times we go through step (v) in table III. They are seen to be both converging to \( \frac{1}{2} \pi = 1.5708 \) as \( N_0 \) increases. When \( N_0 = 8 \), \( \theta_0 \) and \( \theta_1 \) got down to 1.5628 and 1.5823 respectively.
The last figure is the shape of the cascade, subject to the boundary conditions (5.17), (5.28) and (5.29). It was plotted from the solution obtained when \( J = 5 \). The upstream angle was found to be \( \theta_u = -0.1756 \) whilst \( \alpha = -0.1856 \). The turning angle is

\[
\Delta \Theta = -0.6443 \quad \text{for } J = 5
\]

Hence

\[
\theta_d = \theta_u + \Delta \Theta
\]

\[
= -0.8199
\]

Also \( \theta_1 = 1.5628 \) and \( \theta_2 = 1.5823 \) for \( J = 5 \), with errors within .01 of the expected value, \( \theta_1 + \theta_2 \) which measures the angle of \( \pi \) at the stagnation point (depicted in figure 5.10) was found to be 3.1451 for \( J = 5 \) with an error of .4%. Lastly, \( A = .4856 \) and hence \( q_u = .3399 \) in unit of the stagnation speed of sound.

### 5.6 Conclusion

We have derived the equation

\[
\tan \alpha = \frac{K_d \cos \Delta \Theta - K_u}{\rho \ u^2 K_d \sin \Delta \Theta}
\]

which we can use in conjunction with the deflection condition

\[
\Delta \Theta = (F_u - F_d) \tan \alpha - \frac{1}{\Delta \Psi} \int_{\phi_s}^{\infty} F \left|_{\psi=0^+}^{\psi=0^-} \right. d\phi
\]

to iterate for \( \alpha \), once the upstream velocity \( q_u \) and the blade thickness far downstream are known. The solution for a cascade with
circulation was obtained and the results were presented in section 5.5. The error in the velocity on the blade-surfaces was higher than the case with no circulation, and also the calculation is more time consuming. Nevertheless, we have demonstrated that the method is capable of obtaining solutions to the design problem.
Figure 5.4: Boundary condition on blade surfaces as a function of $\phi$.

$J = 6$

$H_2 = -0.4 \quad \phi_2 = 1.4$

$R_1 = 1 \quad R_2 = 13$
Figure 5.5: \( A \) against number of trial functions \( J \).

\[ \beta = 2000 \]
\[ k = 1.45 \]
Figure 5.6: Two leading coefficients $a_0$ and $a_1$ against number of trial functions $J$. 

$\beta = 2000$

$k = 1.45$
Figure 5.7: Upstream angle $\theta_u$ and $\alpha$ against number of trial functions $J$. 

\[ \beta = 2000 \]
\[ k = 1.45 \]
Figure 5.8: Turning angle $\Delta \theta$ against number of trial functions $J$.

$\beta = 2000$

$k = 1.45$
$\beta = 2000$

$k = 1.45$

Figure 5.9: Angles at stagnation point $\theta_0, \theta_1$ against $N_0$.
Figure 5.10: Design of the cascade

\[ q_u = 0.4856q_d \quad , \quad q_d = 0.7 \]

\[ \theta_0 = 1.5628 \quad , \quad \theta_1 = 1.5823 \]

\[ \theta_u = -0.1756 \quad , \quad \theta_d = -0.8199 \]
CHAPTER VI

CONCLUSION

We have developed a technique with which the design problem for two-dimensional irrotational inviscid flow can be solved numerically. This involves the Galerkin Method which requires the equations governing the flow to be orthogonal to a set of independent functions \( k_j \), \( j = 1,2, \ldots, N \), over the region of interest. The velocity function \( q \) is written down as a linear combination of another set of functions \( h_i \), \( i = 1,2, \ldots, N \), which do not satisfy all the boundary conditions. The coefficients \( a_i \), \( i = 1,2, \ldots, N \), of this linear combination are determined by requiring that \( N \) residuals, \( r_i \), \( i = 1,2, \ldots, N \), should be zero. This is achieved by using Newton's Method, which turns the non-linear problem into the iterative solution of a linear problem. This method gives a much faster convergence to the solution than the Global Variational Method used by Davies and Hendry, as was seen in chapter II. In the case of channel flow we obtain very accurate results.

For cascade flow, the main complication is associated with the existence of stagnation points. To our knowledge, all the techniques which have been developed previously need some rounding off process in the vicinity of stagnation points, where the shape of the blade is guessed, using a combination of intuition and practical experience. With our technique, however, we obtain solutions with good accuracy at the leading edge stagnation point, so that the design of a cascade can be calculated directly. We have developed transformations which map the potential plane, in which the velocity is multiply valued for cascades, into a square with the boundary condition that the flow is periodic across the square. The velocity \( q \) takes a comparatively simple form in this plane,
with a factor $h_s$ which takes care of the behaviour of the flow in the region near the stagnation points. The repeat condition is taken care of by the trial functions which are chosen to be the trigonometric functions in the appropriate direction. This gives a very accurate solution for cascades with no turning.

For cascades with circulation, however, one of the parameter needed to make the transformations can be obtained only after the solution has been found. Hence, this parameter must be iterated along with the solution to the flow. Thus, more computer time is necessary since all the transformations must be re-evaluated every time the parameter takes a different value. In spite of this, a reasonably accurate solution is possible, even near the stagnation points, as can be seen in chapter V. In addition, the technique is capable of dealing with transonic problem, as long as the Mach number does not get too high (it was found to cope as high as 1.16).

We conclude by giving some suggestions for improvements which can be made to the technique.

1) It is seen in the channel flow that more accurate results are possible when more trial functions are used, in which case a more accurate integration rule is needed. Hence we feel that with better choice of integration rules, more accurate results can be obtained by using more functions in the expansion set.

2) The function $h_s$ which ensures that $q$ approaches zero at the stagnation points was chosen to take a very simple form, with a parameter $k$ which must be suitably estimated for the problem. The importance of this parameter is unquestionable as can be seen in chapter IV. This leads us to conclude that if a more complicated function is
used for $h_s$, a better solution can be obtained. The use of the incompressible flow solution instead of $h_s$, for example, might give much faster convergence and better accuracy. In view of the fact that a lot of studies have been done on the incompressible problem; all the information needed can be obtained without any difficulty, at least in principle.

3) Newton's Method was used to solve for the coefficients $a_i$, $i = 1,2,\ldots,N$. There are several other techniques, however, for the solution of simultaneous non-linear equations. With Newton's Method, a matrix equation must be solved in every iteration. This results in $\sim N^3$ steps in the computation, where $N$ is the number of trial functions used. We suggest that with another choice of technique, it is possible that the solution could be obtained more economically.

Natural extensions to this work might be to include the trailing edge stagnation point, or to consider a more complex boundary value problem.

In conclusion, it is hoped that this work might serve as a stimulus in an endeavour to develop a better technique in a field which has currently become of great interest in the real world.
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