GEOMETRIC OPTICS THEORY OF GRADED
INDEX OPTICAL FIBRES

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ERRATA for 'Geometric optics theory of graded index optical fibres'
(A Ankiewicz, ANU, Canberra, 1978)

The following are corrections of typing errors in the text:

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<td>After line 4 add: 'results are compared with wave theory'</td>
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<td>2nd last line: insert 'small' before 'V'.</td>
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ERRATA (continued...)

142 eq (7.16): h(u) should be $\varepsilon h(u)$, as in eq (7.13).

143 eq (7.19): u should be $u_i$ (twice).

162 eq (8.5): final bracket ) should be $]$.  

164 middle of page: $(\bar{\beta}, z)$ should be $F(\bar{\beta} z)$.

164 eq (8.9): $F(\bar{\beta}', \bar{\beta})$ should be $F(\bar{\beta}', z)$.

164 4th line should be $\alpha = \alpha_a + N \sigma_l$.

167 1st eq: (m) should be $K(m)$.

168 on this page $F(\bar{\beta})$ should be $f(\bar{\beta})$ (3 times).

Please note: All corrections are to be included in your own copy of the book.
This dissertation is an account of work carried out in the Department of Applied Mathematics of the Institute of Advanced Studies in the Australian National University between March 1975 and February 1978, under the supervision of Dr. Colin Pask and Dr. Allan Snyder. While I have benefited greatly from discussions with other members of the department, especially my supervisors, the material presented in this thesis is my own, unless specifically stated.

None of the work appearing here has been submitted to any other institution of learning for any degree.

A. Ankiewicz
PUBLICATIONS


5. A. Ankiewicz, "Comparison of wave and ray techniques for solution of graded index optical waveguide problems", *Optica Acta*, in press.
ACKNOWLEDGEMENTS

Our head, Barry Ninham, deserves credit for making our department, in the space of a few years, a centre of excellence in research on fundamental problems of a scientific and technological nature. The fructuous endeavours of members from various disciplines have created an environment most conducive to inquiry.

Scientific research here, ranging from reflections on my own to animated group arguments, has been fascinating. I take this opportunity to say how much I appreciate Colin Pask, who has been cheerful, perspicacious, and always helpful.

Allan Snyder's zest, and stress on concepts has had quite an effect on me. He has encouraged me a lot, and I have been fortunate to have as a supervisor one of the founders and main exponents of the field.

Discussions I have had with John Love have served to explicate and clarify many concepts. I have been assisted and enriched by talks with Jos Beunen, Jacek Duniec and Clive Winkler on topics scientific and otherwise.

I thank Diane Shepherd for her fast and accurate typing of this rather technical material.

I am grateful to the Australian Government and the Australian National University for their financial support.
Our life is frittered away by detail.

..... Simplify, simplify.

- H.D. Thoreau (1817-62)

ό βίος βραχὺς, ἡ τεχνὴ μακρὰ

- Hippocrates
ABSTRACT

Optical fibres have come to the forefront of research on transmission media for communication systems because of their high information carrying capacity, low weight, small size, cheapness of material and ability to guide light around bends. Pulse spreading (dispersion), which limits the achievable bandwidth, is quite large in the uniform core, step index fibres; it can, however, be greatly reduced by tailoring the refractive index profile to partially equalize the velocities of the rays (or modes) within the fibre.

In this thesis it is shown that problems involving excitation, propagation and losses of light in these graded index optical fibres can be solved using suitable geometric optics methods. Wave aspects are incorporated into geometric optics analysis by means of building blocks. At each stage, the analysis can be clearly related to the physics of the problem. The approach is generally simpler and more widely applicable than that of modal approximations.

The ray equation, derived from the eikonal equation, is used in Chapter 2 to find ray paths and transit times in graded fibres. Ray properties reflect the underlying symmetries of the physical system - thus in fibres where the refractive index depends only on the distance from the axis, a ray is specified by two invariants, one axial ($\beta$), and one azimuthal ($\lambda$). Rays are classed as bound, tunnelling or refracting. In a fibre with no material absorption, bound rays propagate unattenuated, whereas the tunnelling rays have an intrinsic loss mechanism, and so
attenuate at a rate depending on $\tilde{\beta}$ and $\tilde{\lambda}$. These two classes are conveniently depicted as regions on a graph with $\tilde{\beta}^2$ and $\tilde{\lambda}^2$ as axes. Refracting rays are lost very close to the source.

The dimensionless parameter $V$ relates the linear dimension of a waveguide to the wavelength of light being used. In Chapter 3, exact solutions are obtained for some structures, mainly in two dimensions. By making comparisons with these results, it is then shown that accurate transit times and eigenvalues can be found using geometric optics, down to $V$ values much lower than those used in practise in multimode guides. Transit times derived using the WKB method are identical with those obtained using the conceptually simpler geometric optics procedure.

In Chapter 4, it is shown that the excitation of rays can be dealt with by determining the functional dependence of the source element effectiveness in launching bound and tunnelling rays. Total bound and tunnelling powers can then be found by integrating these efficiencies over the area of the source being considered. Lateral misalignment of source and fibre and the effects of a non-uniform source are also handled in this manner. For a small, centrally located source almost all the guided energy is in bound rays. Tunnelling rays are mainly excited by source elements in the outer part of the core. The distributions of rays with respect to their invariant parameters are then found, as these are needed in the analysis of pulse propagation.

Chapter 5 commences with a derivation of the transit times of rays in 'power law' profiles, proving that these transit times are independent of $\tilde{\lambda}$. It is found that the optimum profile in this class differs from a parabolic profile by an amount which depends on fibre parameters and material dispersion. RMS and absolute pulse widths give
virtually the same optimum, which is rather sharp and thus possibly
difficult to achieve in practice. Actual pulse shapes are found using
the distribution of rays in $\tilde{\beta}$. Core material absorption is considered.

A technique for including tunnelling rays in the analysis of
propagation is presented in Chapter 6.

Initially, graded fibres have relatively less energy in
tunnelling rays than do step fibres. The tunnelling attenuation coeffi-
cient varies greatly, so that a demarcation line, depending on distance
from the source and fibre parameters, can be drawn to dichotomize the
tunnelling ray region. Tunnelling rays contribute a long tail to the
parabolic fibre impulse response. However, the tunnelling ray power is
quite low for moderate lengths of fibre, and the effective pulse width
is not greatly increased by them, except in short lengths of fibre.
The tunnelling ray influence is less than in step fibres.

Because of manufacturing processes, fibres can have imper-
fections like a central dip, or sinusoidal oscillations superimposed
on their intended refractive index distribution. With such deviations,
the transit times depend on both invariants, thus complicating the
analysis of pulse propagation. In Chapter 7, a convenient and accurate
general technique for handling such cases numerically is presented.
Relatively small deviations cause considerable pulse broadening. As
slight deviations are difficult to eliminate entirely, it may not be
possible to come very close to predicted minimum widths.

Following previous work on the applications of geometric optics
to scattering in step guides, in Chapter 8 an outline is given of
geometric optics methods for solving scattering problems in graded
structures.
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# CHAPTER 8: LIGHT SCATTERING IN WAVEGUIDES

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1.1 Optical communications

As the information carrying capacity of a communications system increases with increasing frequency, engineers have turned to progressively higher frequencies as communications needs have expanded. Thus medium wave radio, VHF, TV and microwave frequencies have been employed, and finally optical systems are now being considered.

The wide availability of new technological devices, such as picturephone, cable TV and home computer terminals, will be contingent upon the existence of transmission links of very high bandwidth. Great changes in society are possible with new communication systems. A picturephone will require a bandwidth of perhaps 1 MHz, as compared with 4KHz for a telephone channel, but innovations such as this and video conferencing can greatly reduce the need to commute, thus saving energy at a time when this is of critical importance.

The amount of information at the finger-tips of the individual can be vastly increased. Data banks can save paper by obviating the need for such things as telephone books and newspapers. High fidelity, wide-screen TV could be introduced by increasing the present TV channel bandwidth. Problems of interference and the crowding of the electromagnetic spectrum can be overcome by the use
of guided media. Indeed coaxial cables carrying TV signals are already being widely used to improve TV reception.

Interest in optical communications was stimulated by the coming of the laser, which produces coherent monochromatic light. In the 1960's it became clear that effects of weather and random disturbances in the atmosphere would prevent use of unshielded laser beams for long distance communications.

Among early proposals for guided transmission was the lens waveguide\(^1\),\(^2\) consisting of a series of lenses in an airtight pipe. However, considerations of installation mean that the pipe cannot be laid perfectly straight, and a laser beam will not follow the required curved path when the lens spacing is large. Decreasing the spacing increases losses substantially. In the gas lens waveguide,\(^2\) carbon dioxide is slowly passed through a heated tube; the gas near the axis is cooler and denser, thus forming a converging lens. The only losses are due to slight scattering by gas molecules. Nonetheless, the very tight tolerances on lens waveguides make them somewhat impractical.

1.2 Research on fibres

As electromagnetic waves can propagate in a dielectric medium it became apparent that a communication system could be made exploiting this phenomenon.\(^3\),\(^4\) Thus, a few years ago, analysis and experimentation on step-index dielectric optical fibres was carried out. These fibres consist of a uniform inner glass cylinder with refractive index greater than that of the surrounding cladding. It is clear that this system will guide light along it by the mechanism of total internal reflection. Work on bending losses\(^5\)–\(^8\) and the generalisation of Fresnel's laws to curved surfaces\(^9\) showed that
losses due to curvature are very slight unless the radius of curvature is of the order of a few centimetres — thus the main difficulty of the lens systems is circumvented.

Apart from the fact that they can be bent around corners, optical fibres have the advantages of being light weight, of small size and potential low cost compared with copper cables. There are several review articles dealing with theories and experiments on step fibres, as well as system and economic considerations. The idea of using a high refractive index liquid as the core material was also investigated.

As the time taken for a zig-zag ray to travel along the fiber (the 'transit' time) is longer than that for a ray travelling along the axis, an impulse of energy spreads out or disperses as it moves along the fibre. This dispersion limits the possible bandwidth. A fibre which is graded, i.e. having a refractive index (optical density) higher on the axis and gradually decreasing away from the axis, can be used to partially equalize the transit times of rays and thus significantly reduce dispersion. This can be achieved because rays are speeded up when they traverse low refractive index regions far from the axis. The refractive index distribution can be effected by varying the composition of the glass with distance from the axis.

Neodymium YAG solid state lasers or gallium arsenide semiconductor lasers can be used to couple light into a fibre and can be modulated at very high bit rates. Laser light, however, is not essential, and the cheaper Ga-Al-As light emitting diodes, which radiate over a wide angle and with a range of frequencies, are useful, especially for lower capacity systems. Silicon avalanche photodiodes
or p-i-n diodes can be used as photodetectors; the former achieve a high multiplication factor by the avalanche process. 18

It is desired to minimise signal degradation due to noise, despite the comparatively poor amplitude linearity of optical repeaters. Digital systems in general require a bandwidth higher than analogue ones, but they give a superior performance, in that the signal can be reconstructed more or less exactly.

In view of the high bandwidth possible with optical fibres, use of pulse code modulations in conjunction with time division multiplexing should be quite a viable proposition.

Various loss mechanisms affect light propagation in optical fibres. Electronic transitions of metal ions and the OH− ion cause absorption. Material absorption is the main reason why ordinary glass has a loss of around 10³ dB/km. High purity glasses made in recent years have made optical fibre systems practical; losses of around 1 dB/km have been obtained at wavelengths near 1 micron. In fibres, both cladding and core absorption occur because part of the guided energy travels outside the core. Rayleigh scattering occurs because of the molecular nature of glass, and thus cannot be entirely eliminated. This loss decreases with increasing wavelength, so sources operating in the infrared end of the spectrum tend to be chosen. Surface roughness and variations in diameter can lead to a coupling of power among the guided modes. When a waveguide is bent, a radiation loss occurs because the energy in the evanescent field on the far side from the centre of curvature is forced to move faster to keep in step with the guided wave; at some distance out it reaches the speed of light in the medium, and any energy crossing this surface radiates out.
Obviously, when the total loss is decreased, the repeaters can be spaced further apart. Fibres will probably be installed in cables in the ground. If losses are low enough, repeaters may be placed in telephone exchanges only, thus eliminating the need for underground repeaters.

The small diameter, step or graded index, monomode fibres give low dispersion but introduce difficulties with regard to coupling light in, and also with joining.

In conclusion it appears that the amelioration of communications which can be brought about by the widespread use of optical fibres will allow many of the advances mentioned earlier to become practicable. Interest in the field is clearly shown by the number of recent international conferences, and planning for field demonstrations.

1.3 Philosophy of this thesis

To use optical fibres effectively, we need to understand light propagation in them in a quantitative manner. Several authors use a mode theory to solve one or more problems relating to fibres. These theories tend to involve many approximations and much algebra. This thesis is concerned mainly with graded multimode fibres, although comparisons are made with step fibres where appropriate. The theme is that geometric optics methods, with wave theory corrections built in when necessary, can be used to provide accurate results to graded fibre problems. Furthermore, from technological and didactic points of view, there is the advantage that the physics of the situation is evident at each stage.

In Chapter 2 we shall derive the basic equations, and discuss the ray paths, transit times, invariants and the division of rays into
bound, tunnelling and refracting categories. With enough basic geometric optics results in hand, we shall embark, in Chapter 3, on an outline and comparison of the main methods for solving optical waveguide problems, with specific waveguides as examples. Geometric optics results in cases where both are known. This Chapter will show that the geometric optics method gives accurate results in multimode guides of practical interest.

Turning to light sources, in Chapter 4 we shall consider the relative effectiveness of sources of several shapes and sizes in launching bound and tunnelling rays. The distribution of energy with respect to the axial invariant will then be analysed. Knowledge of this distribution is most important in determining the shape of a pulse at the end of a fibre, as will be seen in Chapter 5. The distributions of energy with respect to both invariants simultaneously will also be found — this will be made use of in Chapter 7 for profiles with more complicated transit time functions.

In Chapter 5, pulse shapes and widths will be calculated. The optimum profile will be found, and core material absorption will be investigated. Even in a fibre with no absorption, rays classified as "tunnelling" attenuate — some quickly, some slowly. Their effect on pulse propagation will be assessed in Chapter 6, by means of the tunnelling transmission coefficient and a generalised parameter.

Because of the way they are manufactured, fibres may have imperfections such as a dip around the axis or a layered structure. In Chapter 7 we shall present a general numerical technique for efficiently handling such profiles. Using this, the transit time for any ray can be found, as can the impulse response. A number of examples will be given.
In the final chapter a formalism for describing scattering in optical waveguides will be given. This will use concepts developed in the preceding chapters, especially the distribution of rays in terms of an invariant, described in Chapter 4. It is an illustration of the wide applicability of geometric optics techniques.
References


CHAPTER 2

GEOMETRIC OPTICS ANALYSIS OF GRADED INDEX WAVEGUIDES

2.1 Introduction

In this chapter the basic methods for the application of geometric optics to light propagation along graded index fibres are set out. The validity of this approach will be considered in detail in Chapter 3. Our aim in this chapter is to describe and classify the rays in optical fibres and to provide formulae needed in later chapters. In section 2.2 we start from the ray equation and deduce properties of ray paths, which follow from the symmetries of the physical system being studied. The formal solution to the equation is obtained and general forms for the ray path described. As examples and for later use some sample problems are solved in detail in sections 2.3 and 2.4. On the basis of the general results it is convenient to divide rays into three classes: bound rays, tunnelling rays and refracting rays. This classification forms section 2.5. In section 2.6 the formulation is extended to cover energy transit times, with detailed examples for two dimensional structures being presented in 2.7. Finally, in section 2.8, we make some comments concerning notation and the use of formalism and symmetry in the description of physical systems.
2.2 Formalism for ray paths in graded fibres

The ray equation, derived from the eikonal equation, describes the path of a ray in a medium of refractive index $n$:

$$\frac{d}{ds} \left( n \frac{dR}{ds} \right) = V_n \tag{2.1}$$

As shown in fig. 2.1, $s$ is the distance measured along the path, and $R$ is the position vector for a point on the ray path; hence $dR/ds$ is a unit vector tangent to the ray path. The $z$ axis is taken as the fibre axis, and we use cylindrical co-ordinates $(r, \psi, z)$.

Fig. 2.1 Ray trajectory. $s$ is the distance measured along the ray path. $r, \psi, z$ are cylindrical coordinates. $P(R) = P(x, y, z) = P(r \cos \psi, r \sin \psi, z)$. 
Initially, consider

\[ n(R) = n(r, \psi) \]  \hfill (2.2)

This gives the ray path equation

\[ \frac{d}{ds} \left[ n \left( \frac{dr}{ds} e_r + r \frac{d\psi}{ds} e_\psi + \frac{dz}{ds} e_z \right) \right] = \frac{3n}{dr} e_r + \frac{1}{r} \frac{\partial n}{\partial \psi} e_\psi \]  \hfill (2.3)

Now \( n \left( \frac{dr}{ds} e_r + r \frac{d\psi}{ds} e_\psi \right) \) is a vector in the transverse plane; hence its derivative with respect to \( s \) is also a vector in the transverse plane, i.e. it has no component in the \( z \) direction.

Thus,

\[ \frac{d}{ds} \left( \frac{dz}{ds} \right) = 0 \]

i.e. \( \frac{dz}{ds} = \) constant along path.

If the angle between the ray path and the \( z \) axis is \( \theta \), then

\[ \frac{dz}{ds} = \cos \theta \]  \hfill (2.4)

We call the axial invariant \( \tilde{\beta} \)

Thus

\[ \tilde{\beta} = n(r) \cos \theta(r). \]  \hfill (2.5)

In general, there will be one invariant for each direction in which \( \text{grad} n \) is zero, in the co-ordinate system being used.

Since

\[ \frac{dz}{ds} = \frac{\tilde{\beta}}{n} \]  \hfill (2.6)

the operator \( \frac{d}{ds} \) is equivalent to \( \frac{\tilde{\beta}}{n} \frac{d}{dz} \), and (2.3) may be rewritten as

\[ \frac{d}{dz} \left( \frac{dr}{dz} e_r \right) + \frac{d}{dz} \left( r \frac{d\psi}{dz} e_\psi \right) = \frac{n}{\tilde{\beta}^2} \left( \frac{\partial n}{\partial r} e_r + \frac{1}{r} \frac{\partial n}{\partial \psi} e_\psi \right) \]

We need to differentiate the unit vectors \( e_r \) and \( e_\psi \):

\[ e_r = \cos \psi \hat{i} + \sin \psi \hat{j} \]
Denoting differentiation with respect to \( z \) by a dot

\[
\frac{d}{dz} e_r = \dot{e}_r = -\dot{\psi} \sin \psi \hat{i} + \dot{\psi} \cos \psi \hat{j} = \dot{\psi} e_\psi
\]

and similarly

\[
\dot{e}_\psi = -\dot{\psi} e_r
\]

Thus

\[
\ddot{r}\dot{e}_\psi + \ddot{r} e_r - r\dot{\psi}^2 e_r + \frac{d}{dz} (r\dot{\psi}) e_\psi = \frac{n}{\beta^2} \left( \frac{\partial n}{\partial r} e_r + \frac{1}{r} \frac{\partial n}{\partial \psi} e_\psi \right)
\]

The radial components give the equation

\[
\ddot{r} - r\dot{\psi}^2 = \frac{n}{\beta^2} \frac{\partial n}{\partial r} = \frac{1}{2\beta^2} \frac{\partial}{\partial r} (n^2) \tag{2.7}
\]

whereas the azimuthal equation is

\[
2\ddot{\psi} + r\ddot{\psi} = \frac{n}{r\beta^2} \frac{\partial n}{\partial \psi}
\]

i.e.

\[
\frac{d}{dz} (r^2 \dot{\psi}) = \frac{1}{2\beta^2} \frac{\partial}{\partial \psi} (n^2) \tag{2.8}
\]

Where \( n \) varies with \( \psi \), we can either proceed from (2.7) and (2.8), or define a new co-ordinate system in which grad \( n \) varies in only one direction, thus revealing the two invariants; the latter, however, complicates the left side of (2.1).

We now take \( n = n(r) \), as this is the main case of interest.

From (2.8)

\[
r^2 \ddot{\psi} = \text{constant} \tag{2.8a}
\]

The local azimuthal angle \( \phi \) is the angle between the ray projection on the cross section through a point \( P \) and the line in that cross section
perpendicular to OP, as shown in fig. 2.2.

To find the second invariant in terms of $r$, $\theta$ and $\phi$, we note that the length of the projection of a segment $ds$ of path on $z$ axis is $dz = ds \cos \theta$, and on the cross section $(P_{1\ 2})$ is $ds \sin \theta$. 

Fig. 2.2. Geometry of a ray at point $P$ in a graded index fibre aligned with the $z$-axis. $\theta$ is the angle between the ray direction at $P$ and $z$ direction. $\phi$ is the angle between the ray projected onto the cross-section through $P$ and the line in that cross-section which is perpendicular to OP. The fibre has radius $\rho$ and $n_{co}$ and $n_{cl}$ are the core and cladding refractive indices.
Fig. 2.3 Projection of ray path on cross section.

From fig 2.3

\[ rd\psi = ds \sin \theta \cos \phi \]

so

\[ r^2 \frac{d\psi}{dz} = r \tan \theta \cos \phi = \frac{r}{\beta} n(r) \sin \theta \cos \phi \]

Thus the dimensionless quantity \( \tilde{\lambda} \) is an invariant where

\[ \tilde{\lambda} = \frac{r}{\beta} n(r) \sin \theta(r) \cos \phi(r) \quad \text{(2.9)} \]
We can substitute \( \psi = \frac{\bar{\rho}^2}{\beta r^2} \) into (2.7) and obtain the governing differential equation:

\[
\frac{d^2 r}{dz^2} - \frac{\bar{\rho}^2}{\beta^2 r^3} = \frac{1}{2\beta^2} \frac{\partial}{\partial r} (n^2). \tag{2.10}
\]

Integrating both sides with respect to \( r \) gives

\[
\frac{dr}{dz} = \frac{1}{\beta^2} \left[ n^2(r) - \left( \frac{\bar{\rho}^2}{r} \right)^2 \right] + C_1 \tag{2.11}
\]

To find \( C_1 \), we note that from fig 2.3

\[
dr = ds \sin \theta \sin \phi
\]

so

\[
\frac{dr}{dz} = \tan \theta \sin \phi
\]

Substituting this and (2.9) into (2.11) shows that

\[
C_1 = -1
\]

so

\[
\left( \frac{dr}{dz} \right)^2 = \frac{f(r)}{\beta^2} \tag{2.12}
\]

where

\[
f(r) = n^2(r) - \frac{\bar{\rho}^2}{r^2} - \beta^2 \tag{2.13}
\]

Hence

\[
z = \frac{\bar{\beta}}{\beta} \int \frac{dr}{\left[ f(r) \right]^{1/2}} \tag{2.14}
\]

A ray is thus specified by its two invariants \( \bar{\beta} \) and \( \bar{\lambda} \) — the ray path is defined by these alone. For rays of interest in propagation, \( f(r) \) generally has two roots, \( r_{\text{min}} \) and \( r_{\text{tp}} \), in the range \((0, \rho)\), indicating that the ray is confined to the region \( r_{\text{min}} \leq r \leq r_{\text{tp}} \), where \( f(r) \) is positive. If \( \bar{\lambda} = 0 \), then \( r_{\text{min}} = 0 \) and the ray is confined to move in a
plane containing the axis. Such rays are called "meridional", and behave like rays in a two dimensional, or "slab" structure. Hence at times we shall refer to a "slab/meridional" case. We mainly consider cases where $n(r)$ decreases monotonically from $n_0 = n(0)$ to $n_{cl}$ at $r = p$, and then remains constant. A commonly occurring quantity is

$$\gamma^2 = n_0^2 - n_{cl}^2$$  \hspace{1cm} (2.15)$$

Fig. 2.4 Typical profile of a graded index fibre.

$\min$ is the distance of closest approach to the axis, while $r_{tp}$ is the distance of furthest excursion from the axis; the inner and outer turning points, or "caustics" are the surfaces $r = r_{min}$ and $r = r_{tp}$ respectively.

2.3 Slab path equations and lengths

The analysis of ray propagation in slab waveguides, formally equivalent to meridional rays in cylindrical structures, is simplified by the fact that $\tilde{k}$ is equal to zero. Slab/meridional results are
obtained by taking \( r = x \), \( r_{tp} = x_{tp} \), \( r_{min} = 0 \) and \( \tilde{k} = 0 \) in cylindrical results. Thus

\[
f(x) = n(x) - \tilde{\beta}^2 \tag{2.16}
\]

and so

\[
n(x_{tp}) = \tilde{\beta}
\]

(i) Path equations

For many slab profiles it is possible to find the equation of the ray path \( x(z) \) explicitly. In general

\[
z = \tilde{\beta} \int \left[ n^2(x) - \tilde{\beta}^2 \right]^{-\frac{1}{2}} dx
\]

The "power law" profiles

\[
n^2(x) = n_0^2 - \gamma^2 \left( \frac{x}{\rho} \right)^q, \quad |x| \leq \rho
\]

\[
n^2(x) = n_{c1}^2, \quad |x| \geq \rho
\]

are interesting because a wide variety of profiles can be obtained by changing \( q \). For these profiles

\[
z = \frac{\tilde{\beta}}{\gamma} \frac{\rho^{q/2}}{q} \int \left[ a^2 \rho^q - x^q \right]^{-\frac{1}{2}} dx
\]

where

\[
a^2 = \frac{(n_0^2 - \tilde{\beta}^2)}{\gamma^2}
\]

For example, the ray paths in a parabolic \( (q=2) \) slab are sinusoidal:

\[
x = \frac{\rho}{\gamma} \left( n_0^2 - \tilde{\beta}^2 \right)^{\frac{3}{2}} \sin \left( \frac{\gamma z}{\rho \tilde{\beta}} \right)
\]

The ray path in a linear \( (q=1) \) slab consists of parabolic segments:

\[
x = \frac{\gamma z}{\tilde{\beta}} \left[ \frac{(n_0^2 - \tilde{\beta}^2)^{\frac{1}{2}}}{\gamma} - \frac{\gamma z}{4 \tilde{\beta} \rho} \right]
\]
(ii) Path lengths

It is useful to know the geometric path length for calculations involving absorption.

If A is a point at which \( r = r_{\text{min}} \) and B is the following maximum \( (r = r_{tp}) \), then \( z_a \), the axial distance from A to B is

\[
z_a = \beta \int_{r_{\text{min}}}^{r_{tp}} \frac{dr}{[f(r)]^{\frac{1}{s}}} \tag{2.20}
\]

Fig. 2.5 Distance \( r \) from fibre axis to point on ray path. The inner and outer caustics are distance \( r_{\text{min}} \), and \( r_{tp} \) respectively, from the axis. \( z_p \) is the distance between successive outer caustics.
We define the geometric path length, $L_p$, as the ratio of the length of path to the axial length.

Hence,

$$L_p = \int_A^B ds = \int sec \theta \, dz$$

$$= \frac{1}{\beta} \int n \, dz = \int_0^{x_p} \frac{n(x)}{(n^2 - \tilde{\beta}^2)\frac{1}{2}} = P(q, \tilde{\beta}) \text{ say}$$

For the slab power law profile

$$z_a = \frac{\tilde{\beta}}{\gamma} \int [a^2 - \left( \frac{x}{\rho} \right)^q]^{-\frac{1}{2}} dx$$

$$= \frac{\rho \beta \pi^{\frac{1}{2}}}{\gamma^{\frac{2q}{q}}} \frac{1}{(n_0^2 - \tilde{\beta}^2)^{(1/q) - 1/2}} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)} \left(\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)} \right)$$

(2.21)

where $\Gamma$ is the gamma function.

Letting $d^2 = n_0^2 - \tilde{\beta}^2$ for convenience, and taking

$$\left[\frac{y}{d}\right]^q \left[\frac{x}{\rho}\right]^q = \sin^2 \zeta$$

we find

$$P(q, \tilde{\beta}) = \frac{2 \rho d}{q \gamma} \frac{2}{q} \int_0^{\pi/2} \left[\frac{n_0^2}{\sin^2 \zeta} - \frac{1}{2} \right] \sin^2 \zeta \, d\zeta$$

$$= \frac{n_0 \rho}{qd \gamma} \frac{2}{q} \frac{\Gamma(1) \Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)} \left(\frac{\Gamma\left(\frac{1}{q}\right) - \frac{1}{q}}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)} \right) F\left(\frac{1}{q}, -\frac{1}{2}; \frac{1}{q} + \frac{1}{2}; 1 - \frac{\tilde{\beta}^2}{n_0^2}\right)$$

where $F$ is the Gauss hypergeometric function. Using (2.21), the path length is

$$L_p(q, \tilde{\beta}) = \frac{P(q, \tilde{\beta})}{z_a} = \frac{n_0 \rho}{\beta} \left(\frac{1}{q}, -\frac{1}{2}; \frac{1}{q} + \frac{1}{2}; 1 - \frac{\tilde{\beta}^2}{n_0^2}\right)$$

(2.22)

$$n_{cl} \leq \tilde{\beta} \leq n_0$$

Path lengths for various values of $q$ can be obtained as special cases of (2.21), or by direct integration.
\[
L_p(1, \tilde{\beta}) = \frac{1}{2} \left\{ \frac{n_0}{\beta} + \left[ \frac{n_0^2}{\beta^2} - 1 \right]^{-\frac{1}{2}} \operatorname{arcosh} \frac{n_0}{\beta} \right\} 
\]
(2.23)

\[
L_p(2, \tilde{\beta}) = \frac{2n_0}{\pi \beta} \operatorname{E} \left[ k = \left[ 1 - \left( \frac{\beta}{n_0} \right)^2 \right]^{\frac{1}{2}} \right] 
\]
(2.24)

where \( \operatorname{E}(k) \) is the complete elliptic function of the second kind

\[
L_p(\infty, \tilde{\beta}) = \frac{n_0}{\beta} = \sec \theta \quad \text{(step case)} 
\]
(2.25)

Of course, for a straight ray along the axis we find \( L_p(q, n_0) = 1 \).

### 2.4 Parabolic and other power law profiles

We now consider fibre power law profiles (\( q \) is a positive constant)

\[
n^2(r) = n_0^2 - \gamma^2 \left( \frac{r}{\rho} \right)^q 
\]

\[
= n_0^2 \left[ 1 - 2\Delta \left( \frac{r}{\rho} \right)^q \right], \quad 0 \leq r \leq \rho 
\]
\[
= n_0^2 \Delta, \quad r \geq \rho 
\]
(2.26)

Hence

\[
\gamma^2 = 2\Delta n_0^2 
\]

The parabolic (\( q=2 \)) fibre is of special interest because in practice profiles will often be approximately parabolic, and also because mathematical analysis can proceed further with the parabolic fibre than with profiles having \( q \neq 2 \). The explicit path equations, \( r = r(z) \) and \( \psi = \psi(r) \) for a ray \( (\tilde{\beta}, \tilde{z}) \) in a fibre with index \( n(r) \) are, from results in section 2.2.

\[
z - z_0 = \tilde{\beta} \int \left[ n^2(r) - \frac{\rho^2 \gamma^2}{r^2} - \tilde{\beta}^2 \right]^{-\frac{1}{2}} \, dr 
\]
(2.27)

\[
\psi - \psi_0 = \rho \tilde{\kappa} \int r^{-2} \left[ n^2(r) - \frac{\rho^2 \gamma^2}{r^2} - \tilde{\beta}^2 \right]^{-\frac{1}{2}} \, dr 
\]

(2.27) gives \( z \) in terms of sine functions for \( q = 2 \), and elliptic functions for \( q = 2/3, 1, 4 \) and 6. The periodic distance \( z_p = 2z_a \) is the distance between two outer caustics. For the parabolic fibre, \( z_p \) can be found exactly.
For meridional rays (\(a = 0\)), the problem is tractable for any \(q\), the result being given by (2.21). The study below shows that \(z_p\) is not very dependent on \(\tilde{\chi}\) in many cases. For example, when \(1 \leq q \leq 3\), for fixed \(\tilde{\beta}\), \(z_p\) varies by less than 5% as \(\tilde{\chi}\) varies through all possible values.

\(z_p\) is important for the following reason. When the ray is at the radial distance \(r_{tp}\), it is at its turning point and it is associated with an evanescent field which extends beyond the fibre radius \(\rho\) to some extent. Power loss occurs if the surrounding medium is absorbing, or if the ray is classed as a tunnelling ray (see section 2.5), in which case we have optical tunnelling with its concomitant radiation loss.

Thus the total power attenuation suffered by a ray in travelling an axial distance \(z\) depends on the number of times it passes through the outer caustic, and this is given by \(z/z_p\). This will be made use of in Chapter 6.

Using (2.20) with (2.13) we can rewrite \(z_p\) in the form

\[
z_p = \rho \tilde{\beta} \left\{ \frac{n_0^2 - \tilde{\beta}^2}{\gamma^2} \right\}^{1/q} \frac{\sqrt{n_0^2 - \tilde{\beta}^2}}{(n_0^2 - \tilde{\beta}^2)^{1+2/q}} I(q,L)
\]

\[
I(q,L) = \int_{u_1}^{u_2} \left[ (1-u^{q/2})u-L \right]^{-\tilde{\chi}/2} du
\]

\[
L = \frac{\tilde{\chi}^2\gamma^{2/q}}{(n_0^2 - \tilde{\beta}^2)^{1+2/q}}
\]

\(u_1\) and \(u_2\) are now the appropriate zeros of \((1-u^{q/2})u-L\). Note that \(z_p\) is dependent on \(\tilde{\chi}\) only through \(I(q,L)\).

Now \(0 \leq u_1 \leq u_2 \leq 1\) and

\[
0 \leq L \leq L_{\text{max}} = \frac{q}{2(1+q/2)^{1+q/2}}
\]
First fix \( L \) and let \( q \) vary.

\[
I(2,L) = \pi
\]

For large \( q \), since \( u \leq 1 \), \( u^{q/2} \to 0 \) in (2.30) except for \( u = 1 \). Thus \( u \to L, u \to 1 \) and

\[
I(q,L) \sim 2(1-L)^{1/2}, \quad q \text{ large}
\]

Now we fix \( q \) and vary \( L \). For the extremes in \( L \) we find

\[
I(q,0) = \frac{2^{1/2}}{q} \frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)}
\]

\[
I(q,L_{\text{max}}) = \frac{2\pi}{\sqrt{q(1+q/2)^{1/q}}}
\]

Fig. 2.6 The integral \( I(q,L) \), (2.25b), plotted vs \( q \) for the extreme values of \( L \). The approximation \( I_{\text{ap}} \) (2.31) is also shown.
We can now present some numerical results. Fig. 2.6 displays I vs q for the extreme values of L. The behaviour of I(q,L)/I(q,0) as a function of L/L_{\text{max}} is shown in fig. 2.7 for a range of q's useful in fibre optics.

The results in fig 2.7 indicate that for many values of q the integral I(q,L) is not very dependent on L and thus z_p varies little with \ell or ray skewness. As a first approximation, we can ignore the L dependence and take I(q,L) \approx I(q,0) or I(q,L_{\text{max}}) with an error of <5% for 1 \leq q \leq 3.

Fig. 2.7 Behaviour of the integral I(q,L) for a range of q's as L varies over all allowed values. The normalization by I(q,0) means we are comparing L \neq 0, i.e. \ell \neq 0, results with L = \ell = 0 or meridional ray results. Since z_p depends on \ell only through I (see Eq.(2.29), this graph can also be used to determine z_p(\beta,\ell)/z_p(\beta,0).
Approximating the L dependence of $I(q,L)$ by a linear function of $L$ linking $I(q,0)$ and $I(q,L_{\text{max}})$ involves an error of <1% for $1 < q < 10$. The approximation

$$I(q,L) \approx I_{ap} = \frac{\pi}{2} \left( \frac{q+2}{2} \right)^{1/4} \left( \frac{q+2}{q} \right)^{1/2}$$

(2.31)

is based on Gloge's work and fig. 2.6 shows that it provides a useful $L$ independent result roughly midway between the $L = 0$ and $L = L_{\text{max}}$ values.

For the parabolic fibre, the actual path is given by

$$r^2 = \frac{\rho^2}{2y^2} \left[ n_0^2 - \beta^2 + \Delta \sin \frac{2\gamma}{\rho_0} (z - z_1) \right]$$

(2.32)

and

$$\frac{1}{r^2} = \frac{1}{2\rho_0^2 \gamma^2} \left\{ n_0^2 - \beta^2 - \Delta \sin 2(\psi - \psi_1) \right\}$$

(2.33)

where

$$\Delta = \left[ (n_0^2 - \beta^2)^2 - (2\gamma^2) \right]^{1/2}$$

(2.34)

and $z_1, \psi_1$ are determined from initial conditions.

For any refractive index, the turning points are the zeros of $f(r)$. For the parabolic ($q=2$) fibre, they are

$$r_{tp} = \frac{\rho}{\sqrt{2} \gamma} \left( n_0^2 - \beta^2 \pm \Delta \right)^{1/2}$$

(2.35)

$$r_{\text{min}}$$

with $\Delta$ given by (2.34). Eq.(2.33) is the polar form of an ellipse with semiminor and semimajor axes $r_{\text{min}}$ and $r_{tp}$, showing that the path is an elliptical helix with non-rotating axes, i.e. the projection of the path on the cross-section is a closed ellipse. In general the projection of the path is not a closed curve for $q \neq 2$, as is illustrated in fig. 2.8.
Fig. 2.8 Projection of ray path onto fibre cross-section for \( q \neq 2, \ \xi \neq 0 \). For \( q = 2 \), parabolic index fibres, the projection is an ellipse. When \( \xi = 0 \) the ray is meridional, \( r_{\text{min}} = 0 \) and the projection is a straight line.

The path length is

\[
L_p = \frac{P(2, \tilde{\beta}, \tilde{\xi})}{z_a(2, \tilde{\beta}, \tilde{\xi})}
\]

with

\[
P(2, \tilde{\beta}, \tilde{\xi}) = \int f^{-\frac{1}{2}}(r) \ n(r) \ dr
\]

and

\[
z_a(2, \tilde{\beta}, \tilde{\xi}) = \tilde{\beta} \int f^{-\frac{1}{2}}(r) \ dr
\]

\[
= \frac{\tilde{\beta} \rho}{2\gamma} \pi
\]
Thus
\[ L_p = \frac{\sqrt{2}}{\pi \beta} \left( n_0^2 + \beta^2 + \Delta \right)^{\frac{3}{2}} E \left[ k = \left( \frac{2\Delta}{n_0^2 + \beta^2 + \Delta} \right)^{\frac{1}{2}} \right] \] (2.36)

For \( \tilde{\beta} = 0 \) this reduces to the slab/meridional result previously derived (2.24).

2.5 Ray classification in fibres

In this section we investigate the behaviour of ray paths in graded index fibres and show that three classes of rays may be identified. These classes have distinct physical properties.

From (2.12) we have
\[ \frac{dr}{dz} = \frac{1}{\beta} [f(r)]^{\frac{1}{2}} \] (2.37)
indicating that a ray in a graded fibre has a turning point when
\[ f(r) = 0 \] (2.38)

We can distinguish classes of rays on the basis of the behaviour of \( f(r) \). Rays exist when \( f(r) > 0 \), corresponding to an oscillating field, whereas we have an evanescent field when \( f(r) < 0 \). To be trapped, within a fibre of radius \( \rho \), a ray must have \( f(\rho) < 0 \), i.e.
\[ n_2^2 - \tilde{\beta}^2 - \beta^2 \leq 0 \] (2.39)

The trapped rays can be conveniently divided into two groups by observing that rays with \( \tilde{\beta} > n_{c1} \) always satisfy (2.39).

The remaining rays are
Rays with $f(\rho) > 0$ are refracting. They are lost immediately and so play no part in propagation. The ray types are shown in schematic plots of $f(r)$ in fig. 2.9, and of distance from the fibre axis in fig. 2.10.

![Schematic plots of the function $f(r)$](image)

Fig. 2.9 Schematic plots of the function $f(r)$. The values of $\sqrt{f(r)}$ determine $dr/dz$ on a ray path according to Equation (2.37), and so physical domains along the $r$-axis (indicated by heavy lines) occur when $f(r) > 0$. For tunnelling rays this gives a disconnected domain, the significance of which is discussed in the text. The refracting rays have an unbounded domain extending beyond the fibre radius $\rho$. 

Tunnelling rays

\[
\begin{align*}
0 & \leq \beta \leq n_{c1} \\
\lambda^2 & > n_{c1}^2 - \beta^2 \text{ and} \\
\text{consistent with (2.6)}
\end{align*}
\]
Fig. 2.10 Distance $r$ from fibre axis (z-axis) to a ray path in a graded index fibre. The ray path is a helix with axis coinciding with the z-axis. Refracting rays correspond to $f(p) > 0$, while bound and tunnelling rays have $f(p) < 0$. $z_p$ is the periodic distance.
Tunnelling and non-meridional bound rays have two solutions of (2.38) in the range $0 \leq r \leq \rho$; these are the inner and outer caustics. Meridional bound rays have only one solution, $r_{tp}$, and hence no inner caustic. The tunnelling rays have an additional solution, $r_{rad} > \rho$, as well. This is an example of optical tunnelling,\textsuperscript{4,5} where some energy leaks out beyond $r = r_{rad}$. Electromagnetic theory shows that a tunnelling ray is only partially reflected at $r_{tp}$ and that it causes energy to leave the fibre and propagate through space for $r > r_{rad}$.\textsuperscript{6,7,8} Thus an attenuation coefficient must be included when tunnelling rays are being traced along a graded fibre as has been done for step fibres.\textsuperscript{9} This will be dealt with in Chapter 6.

Since they cover such a wide range of shapes, we shall be very interested in the power law fibre profiles, as given by (2.26). Hence

$$\gamma^2 = 2 \Delta n^2_0 = n_0^2 \sin^2 \theta_c (0)$$

$$= n_0^2 - n_{cl}^2$$ \hspace{1cm} (2.42a)

where $\theta_c (r)$ is the local critical angle

$$\cos \theta_c (r) = \frac{n_{cl}}{n(r)}$$ \hspace{1cm} (2.42b)

For power law profiles, the ray regions are:

**bound rays**

$$\begin{cases} n_{cl} \leq \tilde{\beta} \leq n_0 \\ 0 \leq \tilde{\chi} \leq \tilde{\chi}_{\text{max}}(\tilde{\beta}) \end{cases}$$ \hspace{1cm} (2.43)

**tunnelling rays**

$$\begin{cases} \left[ \max \{0, n_{cl}^2 - q\gamma^2 / 2 \} \right]^{\frac{1}{2}} \leq \tilde{\beta} \leq n_{cl} \\ (n_{cl}^2 - \tilde{\beta}^2)^{\frac{1}{2}} \leq \tilde{\chi} \leq \tilde{\chi}_{\text{max}}(\tilde{\beta}) \end{cases}$$ \hspace{1cm} (2.44)

Where
Clearly it is convenient to plot the regions on a plane with $\tilde{\beta}^2$ and $\tilde{\lambda}^2$ as axes, and this is done in fig. 2.9.

\[
\tilde{\lambda}_{\text{max}}^2(\tilde{\beta}) = q \left( \frac{2}{\gamma^2} \right)^{2/q} \left( \frac{n_0^2 - \tilde{\beta}^2}{q + 2} \right)^{1+2/q}
\]  

Fig. 2.11 Bound and tunnelling ray domains in terms of the ray path constants $\tilde{\beta}$ and $\tilde{\lambda}$. It is assured that $n_0^2 > q\gamma^2/2$, as is always the case in practice.

The sets of rays can also be defined in terms of the ray angles $\theta = \theta(r)$, $\phi = \phi(r)$ and radial distance $r$ (see fig. 2.2).
bound rays \[ \begin{cases} 0 \leq \theta \leq \theta_c(r) \\ \text{any } \phi \end{cases} \] \hspace{1cm} (2.46)

tunnelling rays \[ \begin{cases} \theta \text{ and } \phi \text{ satisfying} \\ \theta_c(r) \leq \theta \leq \arcsin \left[ \frac{\sin \theta_c(r)}{(1 - \frac{r^2}{\rho^2} \cos^2 \phi)^{\frac{1}{2}}} \right] \end{cases} \] \hspace{1cm} (2.47)

The tunnelling ray radiation distance \( r_{\text{rad}} \) is

\[ r_{\text{rad}} = \rho \tilde{\lambda} \left( n_c^2 - \tilde{\beta}^2 \right)^{\frac{1}{2}} \] \hspace{1cm} (2.48)

For the meridional rays and any \( q \), we have \( r_{\text{min}} = 0 \) and

\[ r_{\text{tp}} = \rho \left( \frac{n^2 - \tilde{\beta}^2}{\gamma^2} \right)^{1/q} \]

2.6 Transit times in graded fibres

The transit time is the time taken for the energy associated with a ray to traverse a fibre, from source to detector. It can be found by noting the periodicity of the ray path. If \( A \) is a point where \( r = r_{\text{min}} \) and \( B \) is the next outer turning point (as shown in fig. 2.10), then the reciprocal of the average \( z \)-directed speed is

\[ \frac{T}{z} = \frac{T_{AB}}{z_a} \] \hspace{1cm} (2.49)

where axial distance \( z_a \) is given by (2.20) and \( T_{AB} \) is the time of flight from \( A \) to \( B \). The speed in a medium with refractive index \( n(r) \) is \( c/n(r) \), where \( c \) is the speed of light in vacuum, so

\[ T_{AB} = \int \frac{ds}{c/n(r)} = \frac{1}{c} \int n^2 dz \]

\[ = \frac{1}{c} \int_{r_{\text{min}}}^{r_{\text{tp}}} \frac{n^2(r)}{[f(r)]^{\frac{1}{2}}} dr \] \hspace{1cm} (2.50)
In general any population of rays will have a variety of transit times — thus even a delta function pulse will spread out as it travels along the fibre; this major topic will be dealt with in Chapter 5.

2.7 Transit times in two-dimensional structures

(i) Power law profile

The transit time of a ray is found from (2.49), with \( z_a \) given by (2.21), and \( T_{AB} \) from (2.50):

\[
cT_{AB} = \frac{n_0^2 z_a}{\tilde{\beta}} - \gamma^2 \frac{I_2}{2}
\]

with

\[
I_2 = \frac{1}{\gamma q} \int_0^{\infty} \frac{x^p}{\rho} \left[ a^2 - \left( \frac{x}{\rho} \right)^q \right]^{-\frac{1}{2}} dx
\]

\[
= \frac{1}{\gamma q} \pi^2 a^{1+2/q} \frac{\Gamma \left( \frac{1}{q} + 1 \right)}{\Gamma \left( \frac{1}{q} + \frac{3}{2} \right)}
\]

where \( a^2 = \frac{(n_0^2 - \tilde{\beta}^2)}{\gamma^2} \)

Hence

\[
\frac{cT}{z} = \frac{q n_0^2 + 2 \tilde{\beta}^2}{(q+2)\tilde{\beta}}
\]

(2.51)

Taking the limit \( q \to \infty \) corresponds to the step case; this gives

\[
\frac{cT}{z} (\text{step}) = \frac{n_0^2}{\tilde{\beta}} = n_0 \sec \theta
\]

which is clearly the step index result.

(ii) We now show that for the profile

\[
n(x) = n \sech bx
\]

(2.52)

all rays have the same transit time

\[
z = \tilde{\beta} \int \left[ n_0^2 \sech^2 bx - \tilde{\beta}^2 \right]^{-\frac{1}{2}} dx
\]

\[
= b^{-1} \arcsin[\tilde{\beta}(n_0^2 - \tilde{\beta}^2)^{-\frac{1}{2}} \sinh bx]
\]
Hence the equation of the path is

\[ x(z) = b^{-1} \arcsinh \left( \frac{n_0^2}{\beta^2} - 1 \right)^{\frac{1}{2}} \sin b \]

and \( z_a = \frac{\pi}{2b} \)

\[ c_{AB} = \frac{n_0^2}{\beta} \int \text{sech}^2 bx \, dz \]

\[ = 2n_0^2 \beta \int_{0}^{z_a} \frac{dz}{\sqrt{n_0^2 + \beta^2 + (n_0^2 - \beta^2) \cos 2bz}} \]

\[ = \frac{n_0}{b} \arctan \left[ \frac{\beta}{n_0} \tan b \right]_{0}^{\pi/2b} = \frac{n_0 \pi}{2b} \]

Thus

\[ \frac{c_{AB}}{z} = n_0 \]  \hspace{1cm} (2.53)

indicating that there is no dispersion in this case.

2.8 Discussion of basic formalism

In this chapter we have found two invariants relating to light propagation in cylindrical fibres. These have consciously been labelled \( \beta \) and \( \xi \) in order to parallel the electromagnetic theory parameters

\[ \beta = (2\pi/\lambda)n(r) \cos \theta = 2\pi\tilde{\beta}/\lambda \]

\[ \xi = (2\pi/\lambda)n(r) \sin \theta \cos \phi = 2\pi\tilde{\xi}/\lambda \]

There is no \( \lambda \) in the geometric optics parameters since the local plane-wave approximation applies and the form of the fields and their behaviour (e.g. Snell's and Fresnel's laws) is independent of \( \lambda \).

Here, as in mechanics, \( \lambda \) conserved quantities, or invariants, exist because of the symmetry properties of the system. As an alternative to the procedure based on the eikonal equation used here, the
Hamiltonian-Lagrangian formalism can be used to obtain (2.5) and the differential equations of motion (2.8a) and (2.10).\textsuperscript{11}

The formal connection between optics and mechanics has long been known—here \( \tilde{\beta} \) is analogous to an energy.\textsuperscript{12} (2.8a) is clearly of the same form as the conservation of angular momentum, \( mr^2 d\psi/dt \), of an object of mass \( m \) moving under the influence of a central force, e.g. a planet orbiting around the sun. In such cases the Hamiltonian is independent of \( \psi \), and indeed when the Hamiltonian is independent of a given co-ordinate, then the corresponding generalized momentum is conserved.\textsuperscript{10} For a parabolic fibre, as \( z \) is increased, the projection of the path traces out an ellipse, as does a planet moving in the inverse square law gravitational field, with time \( t \) replacing \( z \).

Symmetry principles and their associated group theoretical methods are of great use in physics and chemistry, especially in cases where quantum mechanics is involved. Problems can be simplified very much by exploiting symmetries, as is evident in the present case.
References


CHAPTER 3

COMPARISON OF METHODS FOR SOLVING OPTICAL WAVEGUIDE PROBLEMS

3.1 Principles

We have already discussed the need to have a quantitative understanding of light propagation in optical waveguides, so that they can be used in communication systems. Following analysis of step index fibres using mode or ray methods, it is now known that excellent accuracy can be obtained with geometric optics, and that wave corrections can be added when needed.

For a given slab or cylindrical waveguide, we can attempt to obtain the required quantities by solving Maxwell's equations. However, in general this is quite involved and solutions can be obtained only for a small number of structures. To investigate pulse propagation we are fundamentally interested in transit times, and these can often be found using geometric optics. This has the advantage of simplicity in concept and in calculation, and, as we shall demonstrate, gives excellent agreement with exact results, where both can be found. Eigenvalues can also be calculated accurately.

For weakly guiding media, the scalar wave equation for modes is the same as Schrodinger's wave equation - hence approximate methods from quantum mechanics, such as the WKB approximation can be applied.
to waveguide problems. WKB estimates of the fields can thus be obtained; the accuracy obtained has recently been investigated. The field changes from oscillatory to evanescent at the 'caustic' of geometric optics.

In this chapter we consider slab waveguides with co-ordinates \((x,z)\), index of refraction \(n(x)\), and width \(2\rho\). For fibers, \(\rho\) is the radius and we use cylindrical co-ordinates \((r,\psi,z)\), with the \(z\)-axis corresponding to the fibre axis.

3.2 Solution of wave equations

Slab waveguides support TE and TM modes. The wave equations can be obtained directly from Maxwell's equations. Thus for TE modes \(H_y = 0\) and

\[
\frac{d^2 E_y}{dx^2} + [k^2 n^2(x) - \beta^2] E_y = 0 \tag{3.1}
\]

where \(k = \frac{2\pi}{\lambda}\) and \(\beta\) is the modal propagation constant (the mode field depends on \(z\) as \(\exp(i\beta z)\)). In waveguides of interest, the range in refractive index is quite small, so the TM modes wave equation \((\text{in } H_y)\) is similar.\(^9\)

Given \(n(x)\), we try to solve (3.1) for \(E_y\). From boundary conditions, we see that only particular values (eigenvalues) of \(\beta\) are possible; the corresponding field configurations, or eigenfunctions, are called modes. From wave principles, the inverse of the group velocity is \(\frac{\partial \beta}{\partial \omega}\), where \(\omega\) is the angular frequency — thus, once we know \(\beta\) as a function of \(k\), we can find

\[
\tau = z \nu^{-1} = z \frac{\partial \beta}{\partial \omega} = z \frac{\partial \beta}{c \partial k} \tag{3.2}
\]
If \( n_0 \) is the maximum refractive index (usually at \( r = 0 \), \( x = 0 \)), and \( n_s \) is the minimum, then

\[
\gamma^2 = n_0^2 - n_s^2
\]  

(3.3)

gives a measure of the index difference. When the minimum is the cladding index, we have \( n_s = n_{cl} \) and (3.3) agrees with the definition in Chapter 2. The important quantity is the dimension of the structure relative to the wavelength being used. This dimensionless parameter is called \( V \):

\[
V = k\rho\gamma = 2\pi\rho(n_0^2 - n_s^2)^{\frac{3}{2}}/\lambda
\]  

(3.4)

We now consider some profiles \( n(x) \) for which (3.1) can be solved.

(a) Parabolic profile

For the 'infinite' parabolic profile

\[
n_0^2(x) = n_0^2 - \gamma^2\left(\frac{x}{\rho}\right)^2
\]  

(3.5)

the solutions are of the form [eq. 22.6.20 of ref. 11]

\[
E_y \sim \exp\left[-\frac{k\gamma x^2}{2\rho}\right]H_m\left\{\frac{k\gamma}{\rho}x\right\}
\]  

(3.6)

with modal propagation constants

\[
\beta^2 = k^2n_0^2 - V(2m+1)/\rho^2, \quad m = 0,1,2,...
\]  

(3.7)

Here \( V = k\rho\gamma \) and \( H_m \) is the Hermite polynomial of order \( m \).

From (3.2)

\[
\frac{cT}{z} = \frac{\partial \beta}{\partial k} = \frac{n_0^2 + \tilde{\beta}^2}{2\tilde{\beta}}
\]  

(3.8)

where \( \tilde{\beta} = \beta/k \).
For the 'infinite' radial parabolic profile, if the refractive index changes only slightly over a wavelength, then the solution of the scalar wave equation is a good approximation to the exact solution. The modes can be written as a product of two Hermite-Gaussian functions, each having form (3.6), with propagation constants analogous to (3.7) [page 270 of ref.9]. Hence the transit times so obtained will be given by (3.8).

(b) Pöschl-Teller profile

This structure, with profile

\[ n^2(x) = n_0^2 - \gamma^2 \tan^2(x/h), \quad |x| < \pi h/2 \]  

(3.9)

has eigenvalues \( \beta^2 = k^2 n_0^2 - (m^2 + 2ma + a)/h^2 \), \( m = 0,1,\ldots \)

where \( a \) is the positive root of

\[ a(a - 1) = (kh\gamma)^2 = \gamma^2 \]

Hence

\[
\frac{\partial \beta}{\partial k} = \frac{1}{\beta} \left[ n_0^2 - \gamma^2 \left\{ \frac{1 + (n_0^2 - \beta^2)/\gamma^2}{1 + kV^2} \right\}^{1/2} - 1 \right]
\]

(3.10)

\[ \approx \frac{1}{\beta} \left[ n_0^2 - \gamma^2 \left\{ 1 + \frac{n_0^2 - \beta^2}{\gamma^2} \right\}^{1/2} - 1 \right] + \frac{\gamma^2}{8V^2} \left( 1 + \frac{n_0^2 - \beta^2}{\gamma^2} \right) \]

for large \( V \).

(c) Linear slab

To solve for the linear profile

\[ n^2(x) = n_0^2 - \gamma^2 \frac{|x|}{\rho} \]

(3.11)

we let \( d = k^2 n_0^2 - \beta^2 \) and \( b = k^2 \gamma^2 / \rho \)
For $x \geq 0$ the substitution

$$d - bx = -b^{2/3}t$$

gives

$$E''_y - tE'_y = 0$$

Hence

$$E_y = c_1 \text{Ai}\left(-\frac{d+bx}{b^{2/3}}\right), \quad x \geq 0$$

where $\text{Ai}$ is the Airy function (section 10.4 of ref. 11).

Similarly

$$E_y = c_2 \text{Ai}\left(-\frac{d-bx}{b^{2/3}}\right), \quad x \leq 0$$

A mode must have $E_y$ and $H_z (\sim 3E_y/\partial x)$ continuous at $x=0$, and this means that either

$$\text{Ai}\left(-\frac{d}{b^{2/3}}\right) = 0 \quad \text{(odd modes, } c_1 = -c_2)$$

or

$$\text{Ai}'\left(-\frac{d}{b^{2/3}}\right) = 0 \quad \text{(even modes, } c_1 = c_2)$$

Let the zeros of $\text{Ai}(x)$ be $a_s$ (all are negative).

Then for large $s$

$$\frac{d}{b^{2/3}} = -a_s \approx \left[\frac{3\pi}{8} (4s - 1)\right]^{2/3}$$

and thus the asymptotic odd modes eigenvalue equation is

$$k^2n_0^2 - \beta^2 = \left[\frac{3\pi}{8\rho} k^2\gamma^2 (4s - 1)\right]^{2/3}$$

For even modes, the same result is obtained, with $4s - 1$ replaced by $4s - 3$. Thus the eigenvalue equation including all modes can be written

$$k^2n_0^2 - \beta^2 = \left[\frac{3\pi}{8\rho} k^2\gamma^2 (2m + 1)\right]^{2/3} \quad (3.12)$$
We obtain $\tau$ by differentiating

$$2k_n^2 - 2\beta \frac{\partial \beta}{\partial k} = \frac{4}{3k} (k_n^2 - \beta^2)$$

or

$$\frac{cT}{z} = \frac{\partial \beta}{\partial k} = \frac{n^2 + 2\beta^2}{3\beta} \quad (3.13)$$

The above results are valid only when $m$ is such that the mode field is mainly confined to the region near $x=0$; then the cladding, which exists in real waveguides, has little effect. This limitation is removed when we consider guides with $n^2(x) > 1$ everywhere - a physically meaningful profile of this type which can be analysed completely is given in (d) below.

(d) Sech-squared profile

The 'sech-squared' profile is

$$n^2(x) = n_s^2 + \gamma^2 \text{sech}^2 \left( \frac{2x}{h} \right), \quad n_s^2 > 1 \quad (3.14)$$

Here $n^2(0) = n_s^2 = n_0^2 + \gamma^2$ and $V = kh\gamma$.

The wave equation (3.1) takes the form

$$\frac{d^2E}{dx^2} + \left[ k_n^2 - \beta^2 - k^2 \gamma^2 \text{sech}^2 \left( \frac{2x}{h} \right) \right] E_y = 0 \quad (3.15)$$

and solving it is the problem corresponding to the solution of Schrodinger's equation with a sech squared potential. In the appendix to this chapter, it is shown that the mode fields can be written in terms of the associated Legendre functions, and that

$$\beta^2 = k_n^2 + \frac{1}{h^2} \left[ (V^2 + 1)\frac{1}{2} - (2m + 1) \right]^2 \quad (3.16)$$

$$m = 0, 1, \ldots, m_t$$
where $m_t$ is the largest integer less than or equal to $\frac{1}{2}[(V^2+1)^{\frac{1}{2}} - 1]$.

By differentiating (3.16) we obtain

$$\frac{cT}{z} = \frac{\partial \beta}{\partial k} = \frac{1}{\beta} \left[ n_s + \gamma \left( \frac{\tilde{\beta}^2 - n_s^2}{1+V^{-2}} \right) \right]$$

(3.17a)

$$\approx \frac{1}{\beta} \left[ n_s^2 + \gamma (\tilde{\beta}^2 - n_s^2)^{\frac{1}{2}} - \gamma \frac{\tilde{\beta}^2 - n_s^2}{2V^2} \right]$$

(3.17b)

for large $V$.

(e) Slab perfect focussing profile

We can find the transit time for the profile

$$n(x) = n_0 \text{sech}(2x/h)$$

(3.18)

by taking $\gamma = n_0$, $n_s = 0$ in (d) above

Thus

$$\frac{cT}{z} = n_0 (1+V^{-2})^{-\frac{1}{2}}$$

(3.19)

This is again independent of $\tilde{\beta}$, and approaches the geometric optics value, (2.53), asymptotically for large $V$.

3.3 Accuracy of the WKB approximation for transit times

Eigenvalues obtained from the WKB approximation can be used to find transit times, by making use of (3.2). In this section we investigate the accuracy of these transit times.

If $\nu$ is the mode azimuthal invariant, then for the mode $(\beta, \nu)$ we have the WKB condition

$$\int_{r_{min}}^{r_{tp}} \left[ k^2 n^2(r) - \beta^2 - \frac{\nu^2}{r^2} \right]^{\frac{1}{2}} \, dr = (m+\frac{1}{2})\pi$$

(3.20)
where \( m \) is an integer. This is derived by matching the oscillating and evanescent fields on either side of a caustic.\(^7,13\) Many authors\(^14-16\) use this condition to find transit time. We now prove that the transit time so obtained is identical with the geometric optics value, even when dispersion is included.

Special cases include

(i) dispersionless guide, for which \( \partial n / \partial k = 0 \)

(ii) slab waveguide: \( \nu = 0 \) and \( r_{\text{min}} = 0 \)

First consider the geometric optics transit time. The group velocity is \(^17\)

\[
\frac{v_g}{c} = \frac{c}{n + k \frac{\partial n}{\partial k}}
\]

where \( c \) is the speed of light in vacuo.

By definition

\[
\tau = \int \frac{ds}{v_g}
\]

where \( ds \) is an increment of length along the ray path, and, applying similar reasoning to that used in obtaining (2.50), we find

\[
c_{\text{r}} A B = \int_{r_{\text{min}}}^{r_{\text{tp}}} \frac{n(n + k \frac{\partial n}{\partial k})}{[f(r)]^{1/2}} \, dr
\]

where \( f = f(r) \) is defined by (2.13).

\( z_a \) is given by (2.20) so that

\[
\frac{c_{\text{r}}}{z} = \frac{c_{\text{r}} A B}{z_a} = \frac{\int_{r_{\text{min}}}^{r_{\text{tp}}} f^{-1/2} (n + k \frac{\partial n}{\partial k}) \, dr}{\int_{r_{\text{min}}}^{r_{\text{tp}}} f^{-1/2} \, dr}
\]

(3.22)
To find the WKB value, we differentiate (3.20) with respect to k, and reverse the order of differentiation and integration, noting that the integrand is zero at both limits, and that \( \nu = k \rho \lambda \):

\[
\int_{r_{\text{min}}}^{r_{\text{tp}}} \left( \frac{\partial}{\partial k} \left( k^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right) \right) \, dr = 0
\]

i.e.

\[
\int \left( \left( k^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right)^{-1/2} \left( k^2 n + k^2 n \frac{\partial n}{\partial k} - \beta \frac{\partial \beta}{\partial k} \right) \right) \, dr = 0
\]

so

\[
\beta \frac{\partial \beta}{\partial k} \int f^{-1/2} \, dr = \int f^{-1/2} \left( n + k \frac{\partial n}{\partial k} \right) \, dr \quad (3.23)
\]

Thus, using (3.2) \( \partial \beta / \partial k = \alpha/2 \) from (3.23 gives the same result as the geometric optics value (3.22). We conclude that the WKB approximation gives exactly the same transit times as geometric optics, and therefore it would seem better to use the latter, conceptually simpler approach when finding transit times.

3.4 Accuracy of geometric optics

In this section transit times derived from geometric optics are compared with wave theory results found in section 3.2.

(i) For the power law profiles, we see from (2.51) that

\[
\frac{cT}{z} \quad (q = 1) = \frac{n^2 + 2\tilde{\beta}^2}{3\tilde{\beta}}
\]

\[
\frac{cT}{z} \quad (q = 2) = \frac{n^2 + \tilde{\beta}^2}{2\tilde{\beta}} \quad (3.24)
\]

The second is in agreement with exact results (3.8), while the first is in agreement with the linear profile asymptotic result (3.13).
(ii) Sech-squared profile

The geometric optics transit time for this profile (3.14) can be calculated from (2.20) and (2.50) with

\[ f(x) = n_s^2 - \beta^2 - \gamma^2 \text{sech}^2(2x/h) \]

We find

\[ z_a = \frac{h n_s^2}{4(\beta^2 - n_s^2)^{1/2}} \]  

(3.25)

and

\[ c_T_{AB} = \beta z_a + \frac{h n}{4} [\gamma - (\beta^2 - n_s^2)^{1/2}] \]

Hence

\[ \frac{c_T}{z} = \frac{1}{\beta} [n_s^2 + \gamma (\beta^2 - n_s^2)^{1/2}] \]  

(3.26)

As \( \gamma \) is small, this is quite close to the exact result (3.17a), even for relatively low values of \( V \). Fig. 3.1 shows the relative error in the geometric optics value, for \( n_s \leq \beta \leq n_0 \).

Consider the impulse response for this case; \( \tau \) is a monotonically increasing function of \( \tilde{\beta} \), so the pulse width from geometric optics is

\[ T_g = \tau(n_0) - \tau(n_s) \]

\[ = \frac{z}{c} (n_0 - n_s) \]  

(3.27)

Once \( V \) is given, we can find \( m_t \) and hence the exact result,

\[ T_e = \frac{z}{c} \left\{ \frac{\partial \beta}{\partial k} \bigg|_{\beta(m=0)} - \frac{\partial \beta}{\partial k} \bigg|_{\beta(m=m_t)} \right\} \]  

(3.28)

As long as \( V \) is not too small, \( \beta(m=0) \) is close to \( n_0 \) and \( \beta(m=m_t) \) is close to \( n_s \), so the geometric optics pulse width is very accurate. For example, the low \( V \) value of 7, with \( n_s = 1.5 \) and \( \gamma = 0.15 \) gives \( m_t = 3 \),
Fig. 3.1 Per cent error in transit time calculated by geometric optics versus axial invariant $\tilde{\beta}$ for profile $n^2 = n_S^2 + \gamma^2 \text{sech}^2(2x/h)$. $n_S = 1.5$, $\gamma = 0.15$ (hence $n_0 \approx 1.50748$). Geometric optics is exact for $\tilde{\beta} = n_S$. The curves are labelled by the waveguide parameter $V = k h \gamma$.

and

$$\frac{c}{z} T_e = 1.50722 - 1.50015 = .00707$$

whereas

$$\frac{c}{z} T_g = .00748$$

an error of less than 6%. 
3.5 Waveguide eigenvalues

(i) General formulation

Approximate eigenvalue equations can be obtained using purely plane wave principles and including the phase change at caustics. Love and Snyder do this for step and parabolic \((q = 2)\) index fibres, and it will now be shown that in fact eigenvalue equations can be found for two-dimensional structures whenever the transit time and spatial period are known. The explicit path function is \textit{not} required.

Let
\[
\frac{ct}{z} = g(\beta)
\]  
(3.29)

and consider two consecutive maxima, A and C, both on the same side of the axis, and let B be the axis crossing point following A; thus distance \(z_{AC} = 2z = 4z\_a\). The total phase change can be calculated in two ways. First consider the curved path — the total phase change is the sum of the plane change along the path, \(\Delta \phi_p\), and the phase change due to the caustics, \(\Delta \phi_c\).

By noting that the optical path length is comprised of four equivalent units and that it is equal to \(c\) times the transit time, we find
\[
\Delta \phi_p = k \int_A^C nds = 4k \int_A^B nds
\]
\[
= 4k \left( \frac{ct}{z} \right) z_a = 2kz_p g(\beta)
\]

The phase change at each of the two caustics is \(-\pi/2\), so
\[
\Delta \phi_c = -\pi
\]

The second method involves finding the geometrical transformation that moves A to C in the waveguide and maintains the path invariant.
This is a straight line translation of length $2z_p$. The associated phase change is the product of this length and $\beta$, the axial component of the wave vector or modal propagation constant. Thus

$$\Delta \phi_g = 2\beta z_p$$

The eigenvalue equation is determined by specifying that the total phase changes calculated by two methods differ by an integral multiple of $2\pi$, i.e.

$$\Delta \phi_p + \Delta \phi_c = \Delta \phi_g + 2m\pi$$

Thus

$$2z_p [kg(\tilde{\beta}) - \beta] = (2m+1)\pi \quad (3.30)$$

(ii) Examples

(a) sech-squared profile

For the profile of (3.14), $g(\beta)$ is given by (3.26), and, from (3.25) we have

$$2z_p = \hbar \pi \tilde{\beta} (\tilde{\beta}^2 - n_s^2)^{-\frac{1}{2}}$$

Thus

$$\frac{\hbar \pi \tilde{\beta}}{(\tilde{\beta}^2 - n_s^2)^{\frac{1}{2}}} \left[ \frac{k}{\beta} \left( n_s^2 + \gamma (\tilde{\beta}^2 - n_s^2)^{\frac{1}{2}} \right) - \beta \right] = (2m+1)\pi$$

Noting $\beta = k\tilde{\beta}$ this simplifies to

$$\beta^2 = k^2 n_s^2 + \frac{1}{h^2} \{V - (2m+1)^2\}, \quad m = 0, 1, 2 \ldots \quad (3.31)$$

Fig. 3.2 gives a comparison of the exact eigenvalue equation (3.16) and the approximate one (3.31) for a few low order modes.

Clearly the approximate curves give very accurate results, even for $V$, thus increasing our confidence in this method. The structure is monomode for $V < 2.828$. (Eq (3.31) predicts $V < 3$).
Fig. 3.2 Eigenvalues for the first few modes of a waveguide with a sech-squared refractive index profile versus waveguide parameter $V$. The exact curves (full lines) are from (3.16) and the approximate ones (broken) are from (3.31).
(b) Slab power law profiles

Profiles of the form

\[ n^2(x) = n_0^2 - \gamma^2 (x/\rho)^q, \quad 0 \leq x \leq \rho \]

\[ = n_{c1}^2, \quad x \geq \rho \]

are of considerable interest in studies on optical waveguides.

From (2.21),

\[ 2z_p = \frac{\tilde{\beta} \rho \pi^{1/2}}{\eta_0^{2/q} \Gamma \left( \frac{1}{q} + \frac{1}{2} \right)} \left( n_0^2 - \tilde{\beta}^2 \right) \left( \frac{2-q}{2q} \right) \]

g(\tilde{\beta}) is given by (2.51). Using (3.30) and simplifying, we get

\[ (k^2 n_0^2 - \tilde{\beta}^2)^{2+q/2q} = (2m+1)(q+2)(k\eta_0^2)^{2+q/2q} \eta_0^{1/2} \frac{\Gamma \left( \frac{1}{q} + \frac{1}{2} \right)}{4 \rho \Gamma \left( \frac{1}{q} \right)} \]

\[ m = 0, 1, \ldots \]  \hspace{1cm} (3.32)

Thus for \( q = 1 \) (linear)

\[ k^2 n_0^2 - \beta^2 = \left[ \frac{3\pi}{8\rho} k^2 \gamma^2 (2m+1) \right]^{2/3} \]  \hspace{1cm} (3.33)

This agrees with the asymptotic form derived from wave theory (3.12).

For \( q = 2 \) (parabolic), we obtain

\[ \beta^2 = k^2 n_0^2 - V(2m+1)/\rho^2 \]  \hspace{1cm} (3.34)

which is the same as the exact wave theory result (3.7).

The condition for the slab waveguide to support only a single mode is

\[ V = k\rho \gamma < V_c \]

here \( V_c \) can be obtained from (3.32) with \( \beta = kn_s \) and \( m = 1 \). Thus
\[ V_c \approx \frac{3\pi^{1/2}}{4} (q+2) \frac{\Gamma(\frac{1}{q} + \frac{1}{2})}{\Gamma(\frac{1}{q})} \]  

(3.35)

\( V_c \) decreases from 3.53 when \( q = 1 \) to 3.0 when \( q = 2 \) and 1.84 when \( q = 5 \). Thus single-modedness can be preserved with larger dimensions when lower values of \( q \) are used.

As shown in section 3.2(c), the wave theory linear profile eigenvalue equation for odd modes is

\[ k^2 \beta^2 = -a_\gamma^2 \left( \frac{k^2 \gamma^2}{\rho} \right)^{2/3}. \]

The first zero of \( \text{Ai}(x) \) is \( a_1 = -2.338 \). Thus \( V_c (m=1) = (-a_1)^{3/2} = 3.575 \). When \( V \) is less than this, only the lowest order even mode (\( m=0 \)) can exist. The geometric optics value of 3.53 is in very good agreement with this. For \( q = 2 \), (3.7) gives \( V_c = 3.0 \) of course.

This method can be applied to other commonly occurring profiles.

3.6 Conclusion

The geometric optics techniques explained in this chapter give excellent accuracy in calculations of transit times and eigenvalues in optical waveguides, down to quite low \( V \) values. It seems better to use these methods rather than the commonly used and more cumbersome WKB approach. A wide range of problems involving graded index waveguides which cannot be solved exactly can be solved with good accuracy using the techniques delineated in this chapter. In the remaining chapters we shall use the geometric optics approach, with wave corrections build in where necessary, to investigate light acceptance, propagation and scattering in optical fibres of various refractive indices.
APPENDIX

Modes of Sech-squared Profile Waveguide

The mode fields for waveguide (3.14) satisfy

\[ \frac{d^2 E}{dx^2} + [B - k^2 \gamma^2 \text{sech}^2(2x/h)] E_y = 0 \]

where \( B = k^2 n_s^2 - \beta^2 \).

By taking \( E_y = \tanh(2x/h) \) this is transformed to

\[ (1 - \xi^2) \frac{d^2 E_y}{d\xi^2} + 2\xi(1 - \xi^2) \frac{dE_y}{d\xi} - \frac{1}{4} [B h^2 - V^2(1 - \xi^2)] E_y = 0 \]

i.e.

\[ \frac{d}{d\xi} (1 - \xi^2) \frac{dE_y}{d\xi} - \frac{1}{4} \left[ \frac{B h^2}{1 - \xi^2} - V^2 \right] E_y = 0 \]  \hspace{1cm} (A.1)

After taking \( E_y = (1 - \xi^2)^\eta w \), we can make the coefficient of \( w \) constant in the resulting equation by choosing \( \eta = h B^2 / 4 = \frac{1}{4} h (\beta^2 - k^2 n_s^2)^{1/2} \).

This gives

\[ (1 - \xi^2) \frac{d^2 w}{d\xi^2} + 2\xi \frac{dw}{d\xi} (2\eta + 1) + \left[ \frac{V^2}{4} - 2\eta - 4\eta^2 \right] w = 0 \]

Now the roots of \( \frac{V^2}{4} - 2\eta - 4\eta^2 \) are \( s/2 \) and \( -(s+1)/2 \) where

\[ s = \frac{1}{2} [(V^2 + 1)^{1/2} - 1] \]

Letting \( z = (1 - \xi)/2 \), we obtain the hypergeometric equation (eq. 15.5.1 of ref. 11):

\[ z(1 - z) \frac{d^2 w}{dz^2} + (1 - 2z)(2\eta + 1) \frac{dw}{dz} - 4(\eta - \frac{s}{2})(\eta + \frac{s}{2} + \frac{1}{2}) w = 0 \]

The solution finite for \( x = \infty \) (i.e. \( \xi = 1 \) or \( z = 0 \)) is

\[ w = F(2\eta - s, 2\eta + s + 1; 2\eta + 1; z) \]  \hspace{1cm} (A.2)
This solution must be finite for \( x = -\infty \) (i.e. \( z = 1 \)); (A.2) diverges for \( z = 1 \), unless either \( 2\eta - s \) or \( 2\eta + s + 1 \) is equal to \(-m\), \( m = 0,1,2, \ldots \). The latter is always positive, so we must have

\[
2\eta - s = -m
\]
i.e.

\[
\beta^2 = k^2 \eta^2 + \frac{1}{h^2} \left[ (\nu^2 + 1)^{1/2} - (2m + 1) \right]^2
\]
(A.3)

where \( m \) is the largest integer \( \leq \eta \).

As \( Bh^2 = 4(s-m)^2 \) and \( \nu^2/4 = s(s+1) \), (A.1) becomes

\[
(1-\xi^2) \frac{d^2E}{d\xi^2} - 2\xi \frac{dE}{d\xi} + \left[ s(s+1) - \frac{(s-m)^2}{1-\xi^2} \right] E = 0
\]

showing that the mode fields can be written in terms of associated Legendre functions (cf. 8.1.1 of ref. 11). In fact, using 15.3.3 and 8.1.2 of ref. 11, we find

\[
E_y = 2^{2\eta} c_1 \frac{(1-\xi)}{1+\xi}^\eta F(-\eta, s+1; 2\eta; \frac{1-\xi}{2})
\]

\[
= c_1 2^{-\mu} \Gamma(1-\mu) P_s^{m-s}(\xi)
\]

\[
= c_2 P_s^{m-s} \left( \tanh \frac{2\xi}{h} \right)
\]

(A.4)

where \( c_1 \) and \( c_2 \) are constants.
References


CHAPTER 4

THE EXCITATION OF RAYS IN
OPTICAL FIBRES

This chapter contains an investigation of the power launched into a graded index fibre by sources with various shapes, directional properties and intensity distributions. The relative energies in bound and tunnelling rays are calculated and plotted, and comparisons are made with step fibres. As discussed earlier, rays are described by the invariants $\tilde{\Phi}$ and $\tilde{\lambda}$. Distributions of energy in terms of these invariants and the outer caustics are important in studies of the propagation of pulses, and will be determined in the later sections of this chapter.

4.1 Basic concepts

The efficiency of an element of source $dA$ in exciting bound or tunnelling rays depends on its position relative to the fibre axis. Consider a small element of area $dA$ at position $(r, \Psi)$ on the fibre entrance face, as in fig. 4.1. The power launched into the rays with angles $\theta$ to $\theta + d\theta$, $\phi$ to $\phi + d\phi$ is

$$dP = I dA d\Omega$$  \hspace{1cm} (4.1)

where $d\Omega$ is the solid angle $\sin \theta d\theta d\phi$ and $I$ is the source intensity function. $I$ is taken to depend on $\theta$ but not $\phi$. The angle $\theta$ and the
incidence angle $\theta_1$ are related by Snell's law:

$$n_1 \sin \theta_1 = n(r) \sin \theta$$  \hspace{1cm} (4.2)

where $n_1$ is the refractive index of the incidence medium.

The source directionality is modelled by a parameter $s$:

$$I_s = I_0 \cos^s \theta_1$$  \hspace{1cm} (4.3)

with $I_0$ being a strength constant

$$I_0 = I(r,\psi)$$  \hspace{1cm} (4.4)

$I_s$ gives the power $dP_s$ into $d\Omega_s = \sin \theta_1 d\theta_1 d\phi$.

In wave theory the degree of coherence is the quantity corresponding to the directionality of the source in geometric optics - the higher the value of $s$, the more highly directed the source. As we shall see, the directionality controls the relative bound and tunnelling ray powers; on the other hand the coherence decides the relative powers (see refs. 2-4).

The total power is obtained by integrating (4.1). Using the above source and changing the variable from $\theta_1$ to $\theta$, we get

$$P = \int \int \int I(r,\psi) \frac{n^2(r)}{n_1^2} \left[ 1 - \frac{n^2(r)}{n_1^2} \sin^2 \theta \right]^{s-1} \cos \theta \sin \theta d\theta d\phi dr d\psi$$  \hspace{1cm} (4.5)

The power in a particular class of rays is found by specifying the appropriate limits of integration. Bound ray power is obtained when $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \theta_c(r)$, with $\theta_c(r)$ being the local critical angle (2.42b). In a step index fibre $\theta_c$ is independent of $r$ and so the efficiency of a source element in exciting bound rays does not vary with position.
We wish to know the bound and tunnelling powers, \( P_{br} \) and \( P_{tn} \) respectively, for sources of various shapes. We take \( I_0 \) independent of \( r \) and \( \psi \) and normalize the total energy emitted to unity by taking

\[
I_0 = \frac{s+1}{2\pi A_s}
\]

where \( A_s \) is the area of the source.

The varying effectiveness of a source element for exciting bound rays can be represented by the function

\[
b(r) = \frac{dP_{br}}{dA}
\]

Then

\[
P_{br} = \int b(r)dA \quad (4.6)
\]

where the integral is over the whole source area. For the class of sources given by (4.3)

\[
b(r) = 2\pi I_0 \int_0^\theta_1 (r) \cos \theta_1 \sin \theta_1 d\theta_1
\]

with

\[
\theta_1(r) = \text{maximum } \theta_1 \text{ possible for bound rays at } r
\]

\[
= \text{minimum } \theta_1 \text{ possible for tunnelling rays at } r
\]

Now

\[
\cos \theta_{\text{max}} = \cos \theta_c(r) = n_{ci}/n(r)
\]

so

\[
\sin^2 \theta_1 = \frac{n_1^2(r)}{n_1^2} \quad \sin^2 \theta_{\text{max}} = \frac{[n_1^2(r) - n_{ci}^2]/n_1^2}{n_1^2}
\]

(4.7)

and

\[
b(r) = \frac{2\pi I_0}{s+1} (1 - \cos^{s+1} \theta_1)
\]
\[
\frac{1}{A_s} \left[ 1 - \left( 1 - \frac{\gamma^2}{n_1^2} \left[ 1 - \left( \frac{r}{\rho_c} \right)^q \right] \right) \right]^{s+1 \over 2}
\]

where \( \rho_c \) is the radius of the fibre core, and \( \gamma \) is given by (2.42a).

For \( s \) not too large, say \( s < 10 \), we can expand the term in brackets, obtaining

\[
b(r) \approx \frac{s+1}{2A_s} \frac{\gamma^2}{n_1^2} \left[ 1 - \left( \frac{r}{\rho_c} \right)^q \right]
\]

For a step fibre, the corresponding quantity does not depend on \( r \):

\[
b_{st} = \frac{s+1}{2A_s} \frac{\gamma^2}{n_1^2}
\]

In a similar manner, the relative efficiency of a source element in launching tunnelling rays can be represented by

\[g(r) = \frac{dP_{tn}}{dA}\]

so that

\[P_{tn} = \int g(r) dA\]

where again the integral is over the source area. The upper and lower
limits on \( \theta_1 \) are found from the angular limits for tunnelling rays.
The lower limit, \( \theta_1 (r) \), is the same as the upper limit for bound rays
(4.7), whereas the maximum \( \theta_1 \) for a tunnelling ray at \( r \) with angle \( \phi \) is
\( \theta_2 (r,\phi) \) where

\[\sin^2 \theta_2 = \frac{n_1^2(r) - n_{cl}^2}{n_1^2\left[ 1 - \frac{r^2}{\rho_c^2} \cos^2 \phi \right]}\]

Thus the solid angle of acceptance for bound rays is a circular cone, with angle monotonically decreasing to 0 as \( r \) increases from 0 to \( \rho \). Tunnelling rays are accepted between this circular cone and a larger
elliptical one. The cones have two lines in common \((\phi = \pi/2, 3\pi/2)\), corresponding to meridional rays, which cannot be tunnelling. This variable acceptance is shown in ref. 5.

Hence

\[
g(r) = I_0 \int_0^{2\pi} d\phi \int_0^{\theta_1(r)} \cos^2 \theta_1 \sin \theta_1 d\theta_1
\]

\[
= \frac{I_0}{s+1} \int_0^{2\pi} d\phi (\cos^{s+1} \theta_1 - \cos^{s+1} \theta_2)
\]

\[
\approx \frac{I_0}{2} \frac{\gamma^2}{n_1^2} \left[ 1 - \left( \frac{r}{\rho_c} \right)^q \right] \int_0^{2\pi} \left\{ \left( 1 + \frac{r^2}{\rho_c^2} \cos^2 \phi \right)^{-1} - 1 \right\} d\phi
\]

\[
= \frac{(s+1)\gamma^2}{2A_s n_1^2} \left[ 1 - \left( \frac{r}{\rho_c} \right)^q \right] \left[ 1 - \frac{r^2}{\rho_c^2} \right]^{-1} - 1
\]

(4.11)

For a step fibre the corresponding efficiency is

\[
g_{st}(r) \approx \frac{(s+1)\gamma^2}{2A_s n_1^2} \left[ 1 - \frac{r^2}{\rho_c^2} \right]^{-1} - 1
\]

(4.12)

valid unless \(r\) is very close to \(\rho_c\). \(b(r), b_{st}, g(r)\) and \(g_{st}\) are plotted in fig. 4.1.

In the following sections some applications of the above are presented.

For bound rays in a step fibre we obviously have

\[
\frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2
\]

independent of the shape of the source.
Fig. 4.1 A small Lambertian source element distance r from the fibre centre has total power $\Delta P_S$. The power $\Delta P$ launched into bound (solid curves) or tunnelling rays (broken curves) in step and graded ($q=1,2,4$) index fibres is plotted as a function of $r/\rho$. $\theta_C = 0.1$ has been used in the step index tunnelling ray case. If we assume $n_i = n_0$ and $\theta_C(0) = 0.1$ the scale may be read as percent of source power going into the various rays.
4.2 Source covering the fibre core

To find the ratio of bound to total power in a fibre with a circular source of radius \( \rho \) equal to that of the core, we take \( dA = 2\pi \rho dr \) and integrate (4.6) over the area, using (4.8). This gives

\[
\frac{P_{br}}{P_{tot}} \approx \frac{s+1}{2} \frac{q}{q+2} \left( \frac{\gamma}{n_i} \right)^2
\]

(4.13)

A similar procedure for the tunnelling rays gives

\[
\frac{P_{tn}}{P_{tot}} \approx \frac{s+1}{2} \left( \frac{\gamma}{n_i} \right)^2 \left\{ \frac{q+4}{q+2} - \frac{\Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{q+3}{3}\right)} \right\}
\]

(4.14)

where \( \Gamma \) is the gamma function. (4.13) and (4.14) are exact for a Lambertian source, \( s = 1 \).

For the parabolic index

\[
\frac{P_{tn}}{P_{tot}} \approx \frac{s+1}{12} \left( \frac{\gamma}{n_i} \right)^2
\]

(4.15)

These results show that a more highly directed source, i.e. one with larger \( s \), is more efficient for power launching; the variation with \( s \) is the same for bound and tunnelling rays. Increasing \( q \) increases the power accepted by the bound rays. The ratio of tunnelling to bound ray power is

\[
\frac{P_{tn}}{P_{br}} = \frac{q+2}{q} \frac{\left( \frac{q+4}{q+2} - \frac{\Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{q+3}{3}\right)} \right)}{\frac{\Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{q+3}{2}\right)}}
\]

(4.16)

\( P_{tn}/P_{br} \) increases monotonically from 0.288 at \( q = 1 \), to 1/3 at \( q = 2 \), to 0.370 at \( q = 3 \).
For the step fibre

\[ \frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_i} \right)^2 \]  
(4.17)

and

\[ \frac{P_{tn}}{P_{br}} \approx 2 - \left( \frac{n_i}{n_0} \right)^2 \]

\[ = 1, \quad (n_i = n_0) \]  
(4.18)

(4.18) is the small \( \theta_c \) approximation and in that case is exact for Lambertian sources. Thus step fibres are more efficient power collectors than graded index fibres; \( P_{br}/P_{tot} \) is \((q+2)/q\) times larger for the step, a factor of 2 for the parabolic index case \((q=2)\). Step index fibres also accept more power into the tunnelling rays; \( P_{tn}/P_{br} \) is 1 for the step index, but 1/3 for the parabolic index fibre.

4.3 General circular source

To find the bound ray power in a fibre with a circular source of radius \( \rho_s < \rho_c \) we take \( dA = 2\pi r dr \), and integrate (4.6) from 0 to \( \rho_s \), using (4.8). This gives

\[ \frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_i} \right)^2 \left[ 1 - \frac{2}{q+2} \left( \frac{\rho_s}{\rho_c} \right)^q \right] \]  
(4.19)

The ratio is plotted in fig. 4.2 for \( q = 1, 2 \) and 4.

When the source and core have the same radius, (4.19) reduces to (4.13).

The tunnelling ray power is found by evaluation (4.10) using (4.11). For \( q = 2 \) this ratio is
\[
\frac{P_{\text{tn}}}{P_{\text{tot}}} = (s+1) \left( \frac{\gamma \rho_c}{n_1 \rho_S} \right)^2 \left[ \frac{1}{12} - \frac{1}{3} \left( 1 - \frac{\rho_S^2}{\rho_c^2} \right)^{3/2} + \frac{1}{4} \left( 1 - \frac{\rho_S^2}{\rho_c^2} \right)^2 \right].
\] (4.20)

Fig. 4.2 Power P launched into various ray types by a circular source, \( s = 1 \), radius \( \rho_S \) and total power \( P_S \). Solid curves are for bound rays, and broken curves are for tunnelling rays.
When \( \rho_s = \rho_c \), this gives (4.15). For the purpose of comparison we calculate the step fibre result, using (4.10) and (4.12)

\[
\frac{P_{\text{tn}}}{P_{\text{tot}}} \text{ (step)} = (s+1) \left( \frac{\rho_c}{n_1} \right)^2 \left[ 1 - \frac{1}{2} \left( \frac{\rho_s}{\rho_c} \right)^2 \right] - \left( 1 - \frac{\rho_s}{\rho_c} \right)^2 \quad \text{(4.21)}
\]

Both of the tunnelling results are also plotted in fig. 4.2.

As the source becomes larger it can be seen that the fraction of power going into bound rays decreases, while the fraction going into tunnelling rays increases. This indicates that small circular sources will produce a high ratio of bound to tunnelling power.

### 4.4 Thin strip passing through centre

Consider a thin strip source of length \( 2L \) passing through the centre of the core, as shown in Fig. 4.3. The width is \( a \).

\[
\frac{P_{\text{br}}}{P_{\text{tot}}} = \frac{s+1}{4L \pi L} \left( 1 - \frac{r}{\rho} \right) \left( 1 - R^q \right) \left( 1 - R^2 \right)^{-\frac{1}{2}} \text{ dA}
\]

\[
= \frac{s+1}{2 \pi} \left( 1 - \frac{1}{q+1} \left( \frac{r}{\rho} \right)^q \right) \quad \text{(4.22)}
\]

To find the tunnelling ray power, we evaluate

\[
\frac{P_{\text{tn}}}{P_{\text{tot}}} = \frac{(s+1)}{2L} \left( \frac{\gamma}{n_1} \right)^2 \left[ 1 - \frac{L}{\rho} \right] \left( 1 - R^q \right) \left( 1 - R^2 \right)^{-\frac{1}{2}} \text{ dR}
\]

with \( R = r/\rho \). We find

\[
\frac{P_{\text{tn}}}{P_{\text{tot}}} (L = \rho) = \frac{s+1}{4} \left( \frac{\gamma}{n_1} \right)^2 \left[ 1 - \frac{L^2}{\rho^2} \right] - \frac{2q}{q+1} \quad \text{(4.23)}
\]

and

\[
\frac{P_{\text{tn}}}{P_{\text{tot}}} (q = 2) = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left[ 1 - \frac{L^2}{\rho^2} \right] + \frac{\rho}{2L} \arcsin \frac{L}{\rho} - 1 + \frac{L^2}{3\rho^2} \quad \text{(4.24)}
\]
The corresponding result for the step is

$$\frac{P_{\text{tn}} \text{ (step)}}{P_{\text{tot}}} = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left( \frac{\rho}{L} \arcsin \frac{L}{\rho} - 1 \right)$$  \hspace{1cm} (4.25)

These results are shown graphically in fig. 4.3.

Fig. 4.3 Power P launched into bound rays (solid curves) and tunnelling rays (broken curves) by a thin strip source (s = 1) of variable length 2L.
As the source becomes longer, relatively less power goes into bound rays, with the effect being more marked for lower values of $q$. The fraction of power going into tunnelling rays increases when the source becomes longer, especially for the step fibre.

4.5 Misaligned strip

For a thin strip source of length $2L$, with closest distance of approach to the origin being $d$, we have

$$\frac{P_{br}}{P_{tot}} = \frac{s+1}{2L} \left( \frac{\gamma}{n_1} \right)^2 \int_0^{(d^2+L^2)^{1/2}} \left[ 1 - \left( \frac{r}{\rho} \right)^q \right] \frac{r dr}{(r^2 - d^2)^{1/2}}$$

Thus

$$\frac{P_{br}}{P_{tot}} (q=2) = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left[ 1 - \left( \frac{L^2}{3} + d^2 \right)/\rho^2 \right]$$

(4.26)

The $d=0$ case agrees with the $q=2$ case of (4.22). For tunnelling rays

$$\frac{P_{tn}}{P_{tot}} (q=2) = \frac{s+1}{2L} \left( \frac{\gamma}{n_1} \right)^2 \int_0^{(d^2+L^2)^{1/2}} \left( 1 - \frac{r^2}{\rho^2} \right) \left[ 1 - \left( \frac{r^2}{\rho^2} \right)^{1/2} \right] \frac{r dr}{(r^2 - d^2)^{1/2}}$$

This simplifies to

$$\frac{P_{tn}}{P_{tot}} (q=2) = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left[ \frac{L^2}{3} - X^2 \right] + \frac{X^2}{2L\rho} \left( \arcsin \frac{L}{X} + \frac{L}{X} \left( 1 - \frac{L^2}{X^2} \right)^{1/2} \right)$$

(4.27)

where $X^2 = \rho^2 - d^2$. (4.24) is seen to be the $d=0$ case of this.

In the step fibre, so long as $d$ is not extremely close to the core radius $\rho$, then

$$\frac{P_{tn}}{P_{tot}} \text{ (step) } = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \frac{\rho}{L} \arcsin \frac{L}{\rho^2 - d^2} - 1$$

(4.28)
Alternately, we may wish to consider the displacement of a strip of length $2\rho$. This corresponds to taking $L^2 = \rho^2 - d^2$ and multiplying by the relevant fraction of source energy, viz. $L/\rho$. Thus for the parabolic graded fibre

$$\frac{P_{br}}{P_{tot}} = \frac{s+1}{3} \left( \frac{\gamma}{n_1} \right)^2 \left( 1 - \frac{d^2}{\rho^2} \right)^{3/2}$$

(4.29)

and

$$\frac{P_{tn}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left( 1 - \frac{d^2}{\rho^2} \right) \left\{ \frac{\pi}{3} - 2 \left( 1 - \frac{d^2}{\rho^2} \right)^{3/2} \right\}$$

(4.30)

Here the $d=0$ cases of (4.29) and (4.30) are the same as the $L=\rho$ cases of (4.22) and (4.24) respectively.

For the step fibre

$$\frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left( 1 - \frac{d^2}{\rho^2} \right)^{1/2}$$

(4.31)

and

$$\frac{P_{tn}}{P_{tot}} = \frac{s+1}{2} \left( \frac{\gamma}{n_1} \right)^2 \left( \frac{\pi}{2} - \left( 1 - \frac{d^2}{\rho^2} \right)^{3/2} \right)$$

(4.32)

Again the $d=0$ case of (4.32) is the $L=\rho$ case of (4.25). (4.29) to (4.32) are plotted in fig. 4.4.

The energy in bound rays decreases substantially as the displacement $d$ increases. This reflects the smaller acceptance angle for bound rays far from the axis. The tunnelling ray energy for a misaligned strip is in general greater than the $d=0$ value, a consequence of the fact that very little tunnelling ray energy is accepted near the axis.
Fig. 4.4 Change in ray power $P$ when a thin strip source of length $\approx 2\rho$ is displaced from perfect alignment by a distance $d$. Solid curves are for bound rays, broken for tunnelling.
4.6 Lateral misalignment

The problem of having a source and fibre core of equal area, but with centres displaced \(d_1\), is quite a practical one, as dimensions are small, and exact alignment may be difficult to achieve. The normalized displacement is \(d = d_1 / \rho\).

(i) Step fibre

\[ P_{br} \] depends only on the ratio of the areas, and so is simply

\[
\frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{y}{n_1} \right)^2 \left[ 1 - \frac{1}{\pi} \left( d \left( 1 - \frac{d^2}{4} \right)^{1/2} \right) + 2 \arccos \left( 1 - \frac{d^2}{4} \right)^{1/2} \right]
\] (4.33)

\(P_{tn}\) is found using \(g_{st}(r)\), given by (4.12)

\[
\frac{P_{tn}}{P_{tot}} = \int_0^\rho g_{st}(r)h(r)dA
\]

with

\[
h(r) = 1, \quad 0 \leq r \leq \rho - d_1
\]

\[
h(r) = \frac{\alpha(r)}{\pi}, \quad \rho - d_1 < r < \rho
\] (4.34)

\[
\alpha(r) = \arccos \left( \frac{d^2 + r^2 - \rho^2}{2d_1 r} \right)
\]

(ii) Parabolic fibre

Here

\[
\frac{P_{br}}{P_{tot}} = \int_0^\rho b(r)h(r)dA
\]

with \(b(r)\) given by (4.8) and \(h(r)\) by (4.34). Thus taking \(x = 1 - r^2 / \rho^2\)

we get

\[
\frac{P_{br}}{P_{tot}} = \frac{s+1}{2} \left( \frac{y}{n_1} \right)^2 \left\{ \int_0^1 x \, dx + \frac{1}{\pi} \int_0^{d(2-d)} x \alpha(x) \, dx \right\}
\] (4.35)
with
\[ \alpha(x) = \arccos \left( \frac{\sqrt{d^2 - x}}{2d(1 - x)^{\frac{1}{2}}} \right) \] (4.35)

Similarly,
\[ \frac{P_{\text{tn}}}{P_{\text{tot}}} = \int_0^\rho g(r)h(r)dA \] (4.36)

with \( g(r) \) from (4.11) and \( h(r) \) from (4.34). The step and parabolic fibre results are plotted in fig. 4.5.

Fig. 4.5 Change in ray power \( P \) when a circular source of radius \( \rho \) is displaced from perfect alignment by a distance \( d_1 \). Solid curves - bound rays; broken curves - tunnelling rays.
As tunnelling rays are mainly excited in the outer part of the core, they are more affected by the lateral misalignment than the bound rays — this is particularly significant for the step fibre. Bound rays in a parabolic fibre are mainly excited near the core, and so are influenced only slightly. (see fig. 4.1).

Misalignment can also occur when fibres are joined. For fibre jointing calculations we need the near field, i.e. the light intensity in the fibre cross-section, given as a function of distance from the axis. (The far field distribution is the angular spread of rays at a given cross section.) A Lambertian source has uniform intensity, and directs rays equally in momentum space \((p_x, p_y)\). Thus the density of rays in phase space, \(\rho(x, y, p_x, p_y)\) is initially constant within a certain volume in 4-space. From Liouville's theorem\(^6\), the density of points in this phase space is independent of \(z\), as the volume in 4-space cannot change, the near field and far field distributions retain their form along the fibre. This is the basis of the near field method of profile determination\(^7\), which makes use of the simple relation between intensity distribution and the refractive index function:

\[
p(r) = \frac{2\pi I}{n_i^2} n^2(r) \int_0^\theta c(r) \sin \theta \cos \theta \, d\theta
\]

so

\[
\frac{p(r)}{p(0)} = \frac{n^2(r) - n_{cl}^2}{\gamma^2}
\]  

(4.37)

where \(p(r)\) is the power per unit area at distance \(r\) from the axis. We see from ref. 8 that (4.37) is very accurate for parabolic fibres. It assumes equal attenuation of rays — corrections must be incorporated if tunnelling ray losses are to be accounted for.
In fact a steady state exists in a general graded fibre for the near and far field distributions for almost any input distribution. This is as expected intuitively, because rays have different periods.

4.7 Effects of a non-uniform source

Many sources, such as light-emitting diodes, have non-uniform intensities. In this section the experimental source intensity function is approximated by

\[ I_s = e^{-\alpha r^2} \]  

(4.38)

A Lambertian source is considered, with a parabolic profile fibre. Then applying the methods set out in section 4.1,

\[ \frac{P_{br}}{P_{tot}} = \left(\frac{\gamma}{n_1}\right)^2 \frac{(e^{-\alpha \rho^2} + \alpha \rho^2 - 1)}{\alpha \rho^2 (1 - e^{-\alpha \rho^2})} \]  

(4.39)

\[ P_{br} \] increases as \( \alpha \) increases. The limiting expressions are

\[ \frac{P_{br}}{P_{tot}} \approx \left(\frac{\gamma}{n_1}\right)^2 \left[ 1 - \frac{1}{\alpha \rho^2} \right], \quad \alpha \rho^2 \to \infty \]

\[ \approx \left(\frac{\gamma}{n_1}\right)^2 \frac{1 - \alpha \rho^2/3}{2 - \alpha \rho^2}, \quad \alpha \rho^2 \approx 0 \]

\[ \approx \frac{1}{2} \left(\frac{\gamma}{n_1}\right)^2, \quad \alpha = 0 \]  

(4.40)

Concentrating a source's power at the centre is seen to increase its efficiency for exciting bound rays in a graded fibre. This effect does not occur in the step index fibre. For the tunnelling rays

\[ \frac{P_{tr}}{P_{tot}} = \left(\frac{\gamma}{n_1}\right)^2 \left[ \frac{1}{\alpha \rho^2} - \frac{D(\rho \sqrt{\alpha})}{(1 - e^{-\alpha \rho^2}) \rho \sqrt{\alpha}} \right] \]  

(4.41)
where $D$ is Dawson's integral $^{10}$

$$D(x) = e^{-x^2} \int_0^x e^{-t^2} dt$$

Using the appropriate expansions we then obtain

$$\frac{P_{tr}}{P_{tot}} \approx \frac{1}{2\alpha \rho^2} \left( \frac{\gamma}{n_i} \right)^2, \quad \alpha \rho^2 \to \infty$$

$$\approx \frac{5 - 3\alpha \rho^2}{15(2 - \alpha \rho^2)} \left( \frac{\gamma}{n_i} \right)^2, \quad \alpha \rho^2 \to 0$$

$$\approx \frac{1}{6} \left( \frac{\gamma}{n_i} \right)^2, \quad \alpha = 0 \quad (4.42)$$

From sections 4.3, 4.4 and 4.7 it will be noted that as the (rms) width of the source approaches zero, almost all the power goes into the bound rays, and the power in the tunnelling rays approaches zero.

$$\frac{P_{br}}{P_{tot}} \to \frac{s+1}{2} \left( \frac{\gamma}{n_i} \right)^2, \text{ i.e. the step result.}$$

This is as expected from the considerations of section 4.1. Clearly there will be substantial power in the tunnelling rays only when a significant amount of power is incident on the outer part of the core.

4.8 Power distribution in $\tilde{\beta}$

The distribution of bound rays with respect to invariant parameters is now considered. These distributions are important when considering pulse propagation and absorption, and will be used in the next chapter.

We normalize the total bound power to unity by taking

$$I_0 = \frac{q+2}{q} \left( \frac{n_i}{\gamma \pi \rho} \right)^2$$
for a source of form (4.3). Then

\[ P_{br} = 2\pi \int_0^\rho rdr \int_0^{2\pi} d\phi \int_0^1 I_s(\theta_1) \sin \theta_1 d\theta_1 \]

\[ = \frac{8}{\gamma^2 \pi \rho^2} \frac{q+2}{q} \int_0^\rho r n^2(r) dr \int_0^{\pi/2} d\phi \int_0^\theta(r) \sin \theta \cos \theta \]

\[ \times \left[ 1 - \frac{n^2(r)}{n_1^2} \sin^2 \theta \right]^{s-1/2} d\theta. \] (4.43)

where \( \theta_1(r) \) is given by (4.7) and \( \theta_c(r) \) by

\[ \cos \theta_c(r) = \frac{n}{n_1} \frac{cl/\rho(r)}{n} \]

First, we analyze the distribution of energy with respect to \( \tilde{\beta} \) by defining a density function \( P_{br} \) as \( F_{br}(\tilde{\beta}) d\tilde{\beta} = \) power carried by bound rays with axial invariants \( \tilde{\beta} \) to \( \tilde{\beta} + d\tilde{\beta} \).

(i) For a Lambertian source and general \( q \) we find

\[ P_{br} = \frac{4}{\gamma^2 \rho^2} \frac{q+2}{q} \int_0^\rho rdr \int_0^{n(r)} \tilde{\beta} d\tilde{\beta}. \] (4.44)

By reversing the order of integration, and carrying out the last integration we see that

\[ F_{br}(\tilde{\beta}) = \frac{2\tilde{\beta}}{\gamma^2 + 4/q} \frac{q+2}{q} (n_0^2 - \tilde{\beta}^2)^2/q, \ s = 1 \] (4.45)

(ii) Parabolic profile with general \( s \)

\[ P_{br} = \frac{8}{\gamma^2 \rho^2} \int_0^\rho rdr \int_0^{n(r)} \tilde{\beta} d\tilde{\beta} \left[ 1 - \frac{n^2(r) - \tilde{\beta}^2}{n_1^2} \right]^{s-1/2} \]

\[ = \frac{4n_1}{\gamma^4} \int_0^{n_1} \tilde{\beta} d\tilde{\beta} \int_0^1 du u^{(s-1)/2} \]

\[ \left( \int_0^{n_1} \tilde{\beta} d\tilde{\beta} \right)^{s-1/2} \]
where

\[ u_L = 1 - \frac{n^2 - \tilde{\beta}^2}{n_1^2} \]

Hence

\[ F_{br}(\tilde{\beta}) = \frac{2}{s+1} \left( \frac{2n_1}{\gamma^2} \right)^2 \tilde{\beta} \left[ 1 - \left( 1 - \frac{n^2 - \tilde{\beta}^2}{n_1^2} \right) \right]^{(s+1)/2} \]  \hspace{1cm} (4.46)

For a uniform on-axis beam

\[ F_{br} = \frac{4\tilde{\beta}}{q^2} \frac{(n^2 - \tilde{\beta}^2)(2-q)}{q} \]  \hspace{1cm} (4.47)

The variation of \( F_{br}(\tilde{\beta}) \) with source directionality is shown in fig. 4.6 for \( q = 2 \).

**Fig. 4.6** Distribution of bound rays in a parabolic index \((q = 2)\) fibre with respect to path constant \( \tilde{\beta} \) for various source directionality parameters \( s \). The incident medium refractive index \( n_1 \) has been taken equal to \( n_0 \). The fibre has \( n_{cl}/n_0 \approx 1 - \Delta = 0.99 \). Small changes around \( s = 1 \), i.e. small deviations from a Lambertian source, make negligible changes in the distribution shown.
No matter how directional the source is, rays with all possible values of $\tilde{\beta}$ are still excited. This conclusion also holds for an on-axis beam, in a fibre with any $q$.

For a non-uniform source with $I_0$ from (4.38), the power is more concentrated in the rays with $\tilde{\beta}$ near $n_0$. $F_{br}$ is complicated but $q = 2$, $s = 1$ and $q \rho^2$ small is an illustrative case, for which

$$F_{br}(\tilde{\beta}) = \text{const} \times \tilde{\beta}(n_0^2 - \tilde{\beta}^2) \left[ 1 - \frac{q \rho^2}{2 \gamma^2} (n_0^2 - \tilde{\beta}^2) \right]$$

(4.48)

This has the form (4.46) when $\alpha = 0$; the correction is zero for $\tilde{\beta} = n_0$ and causes a maximum reduction in density at $\tilde{\beta} = n_{cl}$.

### 4.9 Distribution $G(\tilde{\beta}, \tilde{\lambda})$

The distribution of rays with respect to both $\tilde{\beta}$ and $\tilde{\lambda}$ is now found. This is interesting from a fundamental point of view, and also useful in practice, as we shall see in chapter 7. We define $G(\tilde{\beta}, \tilde{\lambda})d\tilde{\beta}d\tilde{\lambda}$ as the energy carried in bound rays in the range $\tilde{\beta}$ to $\tilde{\beta} + d\tilde{\beta}$ and $\tilde{\lambda} + d\tilde{\lambda}$. We consider a Lambertian source, and so take $s = 1$ in (4.43)

$$F_{br} = \frac{8}{\gamma^2 \pi \rho^2} q^{q+2} \int_0^n r n^2(r) dr \int_0^{\pi/2} d\phi \int_0^\rho (r) \sin \theta \cos \theta d\theta$$

We now change the variables from $\theta$ and $\phi$ to $\tilde{\beta}$ and $\tilde{\lambda}$ where

$$\tilde{\beta} = n(r) \cos \theta(r)$$

and

$$\tilde{\lambda} = \frac{\lambda}{\rho} n(r) \sin \theta(r) \cos \phi(r)$$

Then

$$n^2(r) \sin \theta \cos \theta = \tilde{\beta} [n^2(r) - \tilde{\beta}^2]^{1/2}$$

and

$$d\theta d\phi = J(\tilde{\beta}, \tilde{\lambda}) d\tilde{\beta} d\tilde{\lambda}$$
where

\[ J(\tilde{\phi}, \tilde{\lambda}) = \begin{vmatrix} \frac{\partial \theta}{\partial \tilde{\phi}} & \frac{\partial \theta}{\partial \tilde{\lambda}} \\ \frac{\partial \phi}{\partial \tilde{\phi}} & \frac{\partial \phi}{\partial \tilde{\lambda}} \end{vmatrix} \]

is the Jacobian of the transformation.

Now

\[ \frac{\partial \theta}{\partial \tilde{\lambda}} = 0 \]

so

\[ J(\tilde{\phi}, \tilde{\lambda}) = \frac{\partial \phi}{\partial \tilde{\phi}} \frac{\partial \phi}{\partial \tilde{\lambda}} \]

\[ = \frac{\rho}{[n^2(r) - \tilde{\beta}^2]^{1/2}} \{r^2[n^2(r) - \tilde{\beta}^2] - \tilde{\lambda}^2 \rho^2\}^{-1/2} \quad (4.49) \]

Thus

\[ p_{br} = \frac{8}{\gamma^2 \pi \rho} \frac{q+2}{q} \int_0^\rho rdr \int_{n_{cl}}^n \tilde{\lambda}(r) \tilde{\phi} d\tilde{\phi} \int_0^{n(r)} \tilde{\phi} d\tilde{\phi} \int_0^{\tilde{\lambda}(r, \tilde{\phi})} d\tilde{\lambda} [r^2(n^2 - \tilde{\beta}^2) - \tilde{\lambda}^2 \rho^2]^{-1/2} \]

where

\[ \tilde{\lambda}(r, \tilde{\phi}) = \frac{r}{\rho} [n^2(r) - \tilde{\beta}^2]^{1/2} \]

We note, as an aside, that the last integral (over \( \tilde{\lambda} \)) is equal to \( \pi/2\rho \), thus recovering (4.44).

To proceed, we reverse the order of the \( r \) and \( \tilde{\beta} \) integrals, giving

\[ p_{br} = \frac{8}{\gamma^2 \pi \rho} \frac{q+2}{q} \int_{n_{cl}}^{n} \tilde{\phi} d\tilde{\phi} \int_0^{r(\tilde{\beta})} rdr \int_0^{\tilde{\lambda}(r, \tilde{\phi})} d\tilde{\lambda} [r^2(n^2 - \tilde{\beta}^2) - \tilde{\lambda}^2 \rho^2]^{-1/2} \]

where

\[ r(\tilde{\beta}) = \frac{\rho}{\gamma} (n^2 - \tilde{\beta}^2)^{1/2} \]
We now wish to put the $r$ integration on the right by reversing the $r$ and $\tilde{\kappa}$ integrals. We take the parabolic fibre, $q = 2$; the remaining results in this section, therefore, are strictly valid only for this case, although of course they will still be quite accurate for near parabolic profiles. Now

$$\tilde{\xi}(r,\tilde{\beta}) = \frac{r}{\rho} \left[ n_0^2 - \tilde{\beta}^2 - \left( \frac{\gamma r}{\rho} \right)^2 \right]$$

which is zero when $r = 0$ and $r = r(\tilde{\beta})$, and reaches its maximum value of $\frac{n_0^2 - \tilde{\beta}^2}{2\gamma}$ when $r = r(\tilde{\beta})/\sqrt{2}$.

Hence we find

$$P_{br} = \frac{16}{\gamma^2 \pi p} \int_{n_{cl}}^{n_0} \int_{0}^{\tilde{\beta} d\tilde{\beta}} \frac{n_0^2 - \tilde{\beta}^2}{2\gamma} d\tilde{\xi} \frac{r_{tp}(\tilde{\kappa},\tilde{\beta})}{r_{min}(\tilde{\kappa},\tilde{\beta})} \frac{rdr}{[r^2(n_0^2 - \tilde{\beta}^2) - \tilde{\kappa}^2 \rho^2]^{1/2}}$$

(4.50)

with

$$r_{tp}^2(\tilde{\kappa},\tilde{\beta}) = \frac{\rho^2}{2\gamma^2} \left[ n_0^2 - \tilde{\beta}^2 \pm \Delta_1 \right]$$

$$r_{min}^2(\tilde{\kappa},\tilde{\beta})$$

where

$$\Delta_1^2 = (n_0^2 - \tilde{\beta}^2)^2 - 4\tilde{\kappa}^2 \gamma^2$$

$r_{tp}$ and $r_{min}$ are in fact the outer and inner turning points for a ray with invariants $\tilde{\beta}$ and $\tilde{\kappa}$ (cf. 2.35). To find $G(\tilde{\beta},\tilde{\kappa})$ we need to evaluate the final integral, which turns out to be equal to $\pi p/2\gamma$. Thus

$$P_{br} = \frac{8}{\gamma^3} \int_{n_{cl}}^{n_0} \frac{n_0^2 - \tilde{\beta}^2}{2\gamma} \tilde{\beta} d\tilde{\beta}$$
and
\[ G(\tilde{\beta}, \tilde{\lambda}) \, d\tilde{\beta} \, d\tilde{\lambda} = \frac{8}{\gamma^3} \tilde{\beta} \, d\tilde{\beta} \, d\tilde{\lambda}, \quad n_{cl} \leq \tilde{\beta} \leq n_0 \]
\[ 0 \leq \tilde{\lambda} \leq \frac{n_0^2 - \tilde{\beta}^2}{2\gamma} \tag{4.51} \]

As \( \tilde{\beta} \) varies only slightly, the density is almost constant over the whole bound ray area; if the variables taken are \( \tilde{\beta}^2 \) and \( \tilde{\lambda} \), then the density is exactly constant, a fact which will be of use in Chapter 7.

We now attempt to find the distribution for the non-uniform source of section 4.7, viz.
\[ I_s = 2 \left( \frac{n_1}{\gamma \pi \rho} \right)^2 e^{-\alpha r^2} \cos \theta_1 \]
(bound ray power equals unity when \( \alpha = 0 \)). We have
\[ P_{br} = \frac{16}{\gamma^2 \pi \rho} \int_{n_{cl}}^{n_0} \tilde{\beta} d\tilde{\beta} \int_{n_{cl}}^{\frac{n_0^2 - \tilde{\beta}^2}{2\gamma}} d\tilde{\lambda} K(\tilde{\beta}, \tilde{\lambda}) \]
where
\[ K(\tilde{\beta}, \tilde{\lambda}) = \int_{r_{min}}^{r_{tp}} \frac{re^{-\alpha r^2} dr}{r_{min} [r^2(n_0^2 - \tilde{\beta}^2) - \tilde{\lambda}^2 \rho^2]^{1/2}} \]

After some algebra we find the required distributions:
\[ G(\tilde{\beta}, \tilde{\lambda}, \alpha) \approx \frac{8 \tilde{\beta}}{\gamma^3} \left[ 1 - \frac{\alpha \rho^2}{2\gamma^2} (n_0^2 - \tilde{\beta}^2) + \frac{(\alpha \rho^2)}{4\gamma^2} \left\{ 3(n_0^2 - \tilde{\beta}^2) - 4\gamma^2 \tilde{\lambda}^2 \right\} \right] \tag{4.52} \]

This reduces to the uniform source result when \( \alpha = 0 \).

4.10 Turning point contours and ray distribution in terms of outer caustics

If we let \( R_c \) be the normalized value of a caustic, i.e. \( r_{min}/\rho \) or \( r_{tp}/\rho \), then manipulation of the expression giving the caustics in a
parabolic fibre (2.35) shows
\[ \tilde{\xi}^2 = R^2_c(n^2 - \tilde{\beta}^2 - \gamma^2 R^2_c) \] (4.53)

Thus, when plotted on a graph with axes $\tilde{\beta}^2$ and $\tilde{\xi}^2/\gamma^2$, the contours of constant turning point ($R_c = \text{constant}$) are straight lines. For convenience, we define a normalized $\tilde{\beta}$:
\[ B = \frac{n^2 - \tilde{\beta}^2}{\gamma^2} \] (4.54)

and a normalized $\tilde{\xi}$
\[ \Lambda = \left( \frac{\tilde{\xi}}{\gamma} \right)^2 \] (4.55)

Bound rays have
\[ 0 \leq B \leq 1 \] (4.56)
\[ 0 \leq \Lambda \leq \frac{B^2}{4} \]

Then the contour lines are
\[ \Lambda = R^2_c(B - R^2_c) \] (4.57)

so the $R_c$ contour meets the vertical axis, $\tilde{\beta} = n_{cl}$ (i.e. $B=1$), at $\Lambda = R^2_c(1 - R^2_c)$, and the horizontal axis (meridional rays, $\tilde{\xi} = 0$) at $B = R^2_c$. This is shown in fig. 4.7. Contours with $R_c < 1/\sqrt{2}$ are tangent to the top curve ($q=2$ in eq. 2.45)
\[ \Lambda = \frac{B^2}{4} \] (4.58)

at $B = 2R^2_c$, $\Lambda = R^4_c$. Then part of the line to the left gives the position of the inner caustic, $r_{\min}/\rho = R_c$, whereas the part to the right gives the position of the outer caustic, $r_{tp}/\rho = R_c$. 
Fig. 4.7 Contours of constant turning point for bound rays in a parabolic fibre. Full line segments indicate outer caustics, broken lines inner caustics. Each point in the region has two contours through it — the full line gives the position of the outer caustic, the broken one gives the position of the inner caustic. Note that $0 \leq r_{\min} / \rho \leq 1/\sqrt{2}$, and $0 \leq r_{tp} / \rho \leq 1$.

To get an idea of the influence of the cladding, *inter alia*, we need to determine the distribution of rays in terms of the outer caustics. The density $H_{br}$ is defined by

$$H_{br}(r_{tp})dr_{tp} = \text{power carried by bound rays with paths having turning point radii between } r_{tp} \text{ and } r_{tp} + dr_{tp}.$$
We consider the parabolic fibre excited by a Lambertian source, as this reveals the physics. Using the above results and the density function $G$ from section 4.9, we see that

$$H_{br}(r_{tp})dr_{tp} = \int_{\tilde{\beta}_L(r_{tp})}^{\tilde{\beta}_u(r_{tp})} d\tilde{\beta} \left\{ \left| \frac{\partial \tilde{\beta}}{\partial r_{tp}} \right| dr_{tp} G(\tilde{\beta}, \tilde{\lambda}) \right\}$$

where

$$\tilde{\beta}_u^2(r_{tp}) = n^2 - \gamma^2 \left( \frac{r_{tp}}{\rho} \right)^2$$

$$\tilde{\beta}_L^2(r_{tp}) = n_{cl}^2, \quad \frac{r_{tp}}{\rho} \geq \frac{1}{\sqrt{2}}$$

$$= n^2 - 2\gamma^2 \left( \frac{r_{tp}}{\rho} \right)^2, \quad \frac{r_{tp}}{\rho} < \frac{1}{\sqrt{2}}$$

and, from (4.53)

$$\tilde{\lambda}^2 = \frac{r_{tp}^2}{\rho^2} \left[ n_0^2 - \tilde{\beta}^2 - \left( \frac{r_{tp}}{\rho} \right)^2 \right]$$

Letting

$$w = n_0^2 - \tilde{\beta}^2 - \gamma^2 \frac{r_{tp}^2}{\rho^2}$$

we find

$$H(r_{tp}) = \frac{8}{\rho \gamma^2} \left[ \gamma^2 \frac{r_{tp}^2}{\rho^2} - w \right]^{1/2}$$

where

$$w_1 = \gamma^2 \cdot \min \left\{ \frac{r_{tp}^2}{\rho^2}, 1 - \frac{r_{tp}^2}{\rho^2} \right\}$$

Thus

$$H_{br}(r_{tp}) = \frac{16}{3\rho} \left( \frac{r_{tp}}{\rho} \right)^3, \quad 0 \leq \frac{r_{tp}}{\rho} \leq \frac{1}{\sqrt{2}}$$

$$= \frac{8}{3\rho} \left( 4 \frac{r_{tp}^2}{\rho^2} - 1 \right) \left[ 1 - \frac{r_{tp}^2}{\rho^2} \right]^{1/2}, \quad \frac{1}{\sqrt{2}} \leq \frac{r_{tp}}{\rho} \leq 1$$

(4.60)
Once again the bound power is normalized to unity, so

\[ \int_0^\rho H_{br}(r_{tp}) \, dr_{tp} = 1 \]

This function is plotted in fig. 4.8.

---

Fig. 4.8 Distribution of bound rays with respect to their turning point \( r_{tp} \) for \( q = 2 \) fibre excited by a Lambertian source. The units of H are chosen to give unit area under the curve.
$H_{br}$ reaches its maximum value when $r_{tp}/\rho = \sqrt{3}/2 \approx 0.866$. Thus many of the rays never get close to the boundary; in a step fibre on the other hand, all trapped rays are reflected from the core-cladding interface. Integrating the above density function shows that 80.1\% of the rays have $r_{tp}/\rho \leq 0.9$. Thus in graded fibres the evanescent field extending beyond the caustic into the cladding would generally be less than in the step, meaning that the cladding loss would tend to have less effect.\textsuperscript{11,12} Only rays with $r_{tp}$ close to $\rho$ (see fig. 4.7) have a high loss due to the cladding.
References


Having considered the influence of the source type on the distribution of rays, we now turn to the subject of power propagation away from the source, and investigate the time distribution of a pulse at a distance $z$ from the source. The impulse response is the pulse shape derived when the energy is initially peaked as a delta function in time at $z = 0$. We begin by obtaining ray transit times and then consider pulses in terms of absolute widths. We then make use of our knowledge of the ray distribution in $\beta$ (section 4.8) to find rms widths and finally actual pulse shapes. Power law profiles are used. Absorption will be considered at the end of the chapter.

5.1 Derivation of transit time

As explained earlier a ray moves in a curved path from a distance of closest approach to the axis, $r_{\text{min}}$, to a point of furthest departure, $r_{tp}$, in an axial distance $z_a$, taking time $\tau_{AB}$ to do so. It then moves to the next minimum ($r_{\text{min}}$ from the axis), a distance $z_a$ further along the fibre, again taking time $\tau_{AB}$. This pattern repeats, showing that, even for small lengths of fibre, the average velocity overall is the same as the average velocity over an axial length $z_a$, i.e.

$$\frac{\tau}{z} = \frac{\tau_{AB}}{z_a}$$
From (2.20) and (2.50)

\[ cT_{AB} = \begin{cases} \int_{r_{\min}}^{r_p} \frac{n^2(r)}{[f(r)]^{1/2}} dr, \\ r_p \geq r_{\min} \end{cases} \]

and

\[ z_a = \frac{1}{\beta} \int_{r_{\min}}^{r_p} [f(r)]^{-1/2} dr, \]

We consider power law profiles

\[ n^2(r) = n_0^2 - \gamma^2 \left( \frac{r}{\rho} \right)^q, \quad 0 \leq r \leq \rho \]

\[ = n_0^2 \left[ 1 - 2\Delta \left( \frac{r}{\rho} \right)^q \right] \]

\[ = n_0^2 c_1, \quad r \geq \rho \]

so

\[ f(r) = n_0^2 - \tilde{\beta}^2 - \gamma^2 \left( \frac{r}{\rho} \right)^q - \frac{\tilde{\gamma}^2 \rho^2}{r^2} \]

Hence

\[ \frac{cT}{z} = \frac{n_0^2}{\beta} - \frac{\gamma^2}{z_a} I_2 \]

(5.1)

with

\[ I_2 = \frac{1}{\beta} \int_{r_{\min}}^{r_p} \left[ \frac{r}{\rho} \right]^q [f(r)]^{-1/2} dr \]

However, a difficulty now arises in that neither \( I_2 \) nor \( z_a \) can be found explicitly, except in special cases. Nonetheless, \( I_2 \) can be written in terms of \( z_a \) using the substitution

\[ w = r^2 f(r) \]

(5.2)

Then

\[ \gamma^2 (q+2) I_2 = 2(n_0^2 - \tilde{\beta}^2) \left( \frac{r dr}{[w(r)]^{1/2}} \right) - \left( \frac{dw}{w^{3/2}} \right) \]
In general we expect that the transit time of a ray $\tau(\tilde{\beta}, \tilde{\xi})$ will depend on both invariants. (5.3a) is an important result – the surprising fact that the transit time is independent of the second invariant $\tilde{\xi}$ means that the analysis of pulse propagation in power law fibres is simplified. Considering this, the above result should be the same as the $\tilde{\xi} = 0$ (slab/meridional) result previously found (2.51), and this is indeed the case. When the profile deviates from a power law one in some way, the transit time will depend on $\tilde{\xi}$ as well as $\tilde{\beta}$, thus adding an extra dimension to pulse spreading. We shall see this when considering imperfections in fibres in Chapter 7. (5.3a) can be written

$$\frac{ct}{z} = A \frac{\tilde{\beta}}{\tilde{\beta}} + \frac{1}{\tilde{\beta}}$$

(5.3b)

where $A = \frac{2}{q+2}$, $B = \frac{q\eta_0}{q+2}$. 

Transit times are found using the group velocity of a plane wave along a ray – this depends on the dispersion property of the glass.
The above uses the phase velocity \( c/n \) as an approximation to the group velocity.\(^1\)

\[
V_g = \frac{c}{n + \omega \frac{\partial n}{\partial \omega}}
\]

Note that

\[
k \frac{\partial n}{\partial k} = \omega \frac{\partial n}{\partial \omega} = -\lambda \frac{\partial n}{\partial \lambda}
\]

where \( \omega \) is the angular frequency and \( \lambda \) is the wavelength. We now find the transit time using the group velocity.

If we define the on-axis group refractive index \( N_0 \) as

\[
N_0 = n_0 + \omega \frac{dn}{d\omega}
\]

and the dispersion parameter \( y \) as

\[
y = \frac{2n_0 \omega d\Delta}{N_0 d\omega}
\]

where

\[
\Delta = \frac{y^2}{2n_0} = \frac{1}{2} \left( 1 - \frac{n_0^2}{n_0^2} \right)
\]

from (2.42a),

then we can generalize (5.3) by a simple modification to the derivation presented above.

We find

\[
\frac{ct}{z} = A \tilde{\beta} + B \tilde{\beta}^2
\]

where

\[
A = \frac{2N_0}{n_0} \frac{(1 + \frac{y}{4})}{q+2}
\]

\[
B = qr^2 \frac{A}{2} - \frac{N_0 n y}{4}
\]

Where there is no dispersion \((y = 0)\) then (5.7) reduces to (5.3b).
These results are valid for bound and tunnelling rays, but we shall now consider pulse propagation in terms of bound rays only, leaving consideration of the tunnelling ray contribution until chapter 6. When we are a long distance along the fibre the energy remaining in tunnelling rays is small, so bound rays alone provide a good approximation to actuality.

5.2 Pulse widths

We wish to know and minimize the impulse response width \( \tau \). This width gives a good indication of how much the pulse has spread because \( \tau \) depends on \( \tilde{\beta} \) only, and most sources give a distribution of power over all possible values of \( \tilde{\beta} \) for bound rays, i.e. \( n_{cl} \leq \tilde{\beta} \leq n_0 \). Thus

\[
\tau = \tau_{\text{max}} - \tau_{\text{min}} \tag{5.8}
\]

with \( \tau = \tau(\tilde{\beta}) \) given by (5.3) or (5.7).

For the sech profile in a slab (section 2.7), all rays have the same transit time, and hence \( \tau = 0 \); however this result does not hold when \( \tilde{x} \neq 0 \).

Clearly \( \tau \) has a minimum at

\[
\tilde{\beta} = \left( \frac{B}{A} \right)^{\frac{1}{2}} = n_0 \left( \frac{2q-y}{4+y} \right)^{\frac{1}{2}} = \tilde{\beta}_m \text{ say} \tag{5.9}
\]

We wish to obtain the optimum \( q \). From the form of (5.6), this involves having

\[
n_{cl} < \tilde{\beta}_m < n_0
\]

so that the minimum in \( \tau \) occurs within the bound ray region. Now
\[ T = \max \{ \tau(n_0), \tau(n_{cl}, q) \} - \tau(\tilde{\beta}_m, q) \]

and

\[ \frac{c}{z} \tau(\tilde{\beta}_m, q) = 2(AB)^{\frac{1}{2}} = \frac{N_0}{q + 2} \left[ \frac{(4+y)(2q-y)}{4} \right]^{\frac{1}{2}} \]  

(5.10)

In fact \( T \) is minimized by taking \( \tau(n_0) = \tau(n_{cl}, q) \)

i.e.

\[ q_0 = 2 - \frac{n_{cl}}{n_0} + \frac{y}{2} \left( 1 + \frac{n_{cl}}{n_0} \right) \]  

(5.11)

Hence

\[ \frac{c}{z} T(q_0) = N_0 \left\{ 1 - 2 \left[ \frac{n_{cl}}{n_0} \right]^{\frac{1}{2}} \left( 1 + \frac{n_{cl}}{n_0} \right)^{-1} \right\} \]  

(5.12)

For example if \( N_0 = n_0 = 1.5 \) and \( n_{cl}/n_0 = 0.995 \), the dimensionless pulse width \( (c/z)T(q_0) \) is \( 4.71 \times 10^{-6} \). To write \( T \) in terms of \( \Delta \), note

\[ \frac{n_{cl}}{n_0} = (1 - 2\Delta)^{\frac{1}{2}} \approx 1 - \Delta \]

as \( \Delta \) is small.

Thus

\[ q_0 = 2 + y - \Delta \left( 2 + \frac{y}{2} \right) \]  

(5.13)

\[ q_0(y = 0) = 2 - 2\Delta \]

\( T \) is plotted in fig. 5.1 for a fibre with \( \Delta = .01 \) and \( y = 0 \).

From an engineering point of view it is important to note the sharpness of the width minima shown in fig. 5.1. Even a slight deviation from the optimum causes a large increase in the pulse width, and so a significant reduction in bandwidth. This must be remembered when setting profile tolerances and estimating the realizable maximum bandwidth. This problem is taken up again in Chapter 7.
Fig. 5.1 Pulse width divided by $t_0$ vs profile parameter $q$ for fibres with $\Delta = 0.01$. $t_0 = z N_0 / c$ is the transit time of an on-axis ray. The width is measured by the absolute width $T$ or the rms width $\sigma$. The dispersion parameter $y$ (eq. 5.5) is used to label the curves.

The form of the curves in fig. 5.1 can be understood physically by observing the $\tau$ vs $\tilde{\beta}$ curves in fig. 5.2. As stated above, these curves have a minimum at $\tilde{\beta} = \tilde{\beta}_m$. The bound rays cover $n_{cl} \leq \tilde{\beta} \leq n_0$ and so may be all above $\tilde{\beta}_m$ ($\tilde{\beta}_m < n_{cl}$), all below $\tilde{\beta}_m$ ($\tilde{\beta}_m > n_0$), or partly above
and partly below \( n_{cl} \leq \tilde{\beta}_m \leq n_0 \); which condition applies depends on \( q \).

Clearly \( \tilde{\beta}_m \) must be between \( n_{cl} \) and \( n_0 \) to minimize \( T \). Substituting (5.11) into (5.9) shows that

\[
\tilde{\beta}_m(q = q_0) = (n_0 n_{cl})^{\frac{1}{2}}
\]

(5.14)

i.e. that \( \tilde{\beta}_m \) is the geometric mean of \( n_0 \) and \( n_{cl} \) in the optimum case.

---

Fig. 5.2 The lower curves show transit time \( T \) vs path constant \( \tilde{\beta} \) for rays in a power law profile graded index fibre with \( n_0 \) and \( n_{cl} \) fixed. The bound ray \( \beta \) domain is indicated by a heavy line.

As \( q \) varies, the section of \( T \) vs \( \beta \) curve relevant to bound rays alters with respect to the minimum in the curves, and thus variations in the pulse width \( T \), measured by the difference between longest and shortest transit times, can be traced out. When \( q = q_0 \), the optimum value, \( T \) is minimized as shown in the upper curve.
It is worthwhile to compare this value with the optimum value \( q_{\text{ok}} \) obtained by Olshansky and Keck\(^2,^3\) using a modal method and rms width

\[
q_{\text{ok}} = 2 + y - \Delta \frac{(4+y)}{(5+2y)} (3+y)
\]

\[
\approx 2 + y - \Delta \left( \frac{12}{5} + \frac{11}{25} y \right)
\]

(5.15)

where \( y \) is assumed small in the last line. This result is quite close to (5.13).

The roles of source and absorption are not shown by these width results, and so we now take some of the pulse structure into account.

5.3 Pulse rms widths

The root mean square width \( \sigma \) is now considered. If we take \( \langle \rangle \) to indicate average over the pulse, and define

\[
I_m = \langle \tau^m \rangle
\]

(5.16)

then

\[
\sigma^2 = \frac{I_2}{I_0} - \left( \frac{I_1}{I_0} \right)^2
\]

(5.17)

For the power law profiles, \( \tau \) depends on \( \beta \), so

\[
I_m = \int \tau^m(\tilde{\beta})F(\tilde{\beta})d\tilde{\beta}
\]

(5.18)

The limits of integration depend on which rays are considered. For bound rays \( n_{\text{cl}} \leq \tilde{\beta} \leq n_{\text{o}} \) and \( F = F_{\text{br}} \) as given in section 4.8.

Consider a Lambertian source \( (s = 1 \text{ in } 4.3) \) and take \( q \) close to 2. Then

\[
F_{\text{br}}(\tilde{\beta}) \approx \text{const. } \tilde{\beta}(n_{\text{o}}^2 - \tilde{\beta}^2)
\]

(5.19)
After some algebra, we find the minimum $\sigma$ as $q$ varies, to order $A$ at least:

$$q_{\text{opt}} = 2 + y - \Delta \left( \frac{12}{5} + \frac{3}{5} y \right)$$  \hspace{1cm} (5.20)$$

$$\frac{c}{z} \sigma_{\text{opt}} = \frac{\sqrt{3}}{15} N_0 \left( 1 + \frac{1}{4} \right) \frac{\Delta^2}{q_{\text{opt}} + 2}$$

Plots of $\sigma$ versus $q$ are shown in fig. 5.1. $\sigma$ is proportional to $\Delta^2$ around the optimum $q$, but is proportional to $\Delta$ for $q$ well away from the optimum. $\Delta$ is of order $10^{-2}$, so this represents a variation in rms pulse width of 2 orders of magnitude.

Since the energy is spread over the $\tilde{\beta}$ values, the source has little effect on the value of $q$ which should be chosen to minimize the pulse width. For example, when the source directionality parameter $s$ is increased from 1 to 10, the change in $\sigma$ for a $q = 2$ fibre is of order 0.1%. Rays with angles less than $\arccos(n_{cl}/n_0)$ are accepted as bound rays; thus the angles are quite small, and $\cos^8 \theta$ does not vary much when $\theta$ is small. Hence for most bound rays the angular region involved means that the source function varies only slightly as the directionality parameter is changed.

Even in the extreme case of a parallel on-axis beam for the source, the power is spread over the whole range of $\tilde{\beta}$ values. We find

$$\left( \frac{c}{z} \sigma \right)^2 = n_0^2 \frac{\Delta^4}{45}, \hspace{1cm} q = 2 \hspace{1cm} (5.21)$$

The optimum $q_{\text{opt}}$ in (5.20) is almost identical with $q_{\text{ok}}(5.15)$. Olshansky and Keck\textsuperscript{2,3} calculated $q_{\text{ok}}$ using a WKB mode theory and assuming that all modes are equally excited. This assumption is only completely true for an incoherent source of infinite extent, or asymptotically
large waveguide parameter \( V \) (eq. 3.4). A source which is more highly
directed (higher \( s \)) in geometric optics terms, has a higher degree of
coherence in terms of electromagnetic theory.

Note that \( T \) and \( \sigma \) vary in a similar manner with \( q \). The relation
between them for \( q = 2 \) is

\[
\begin{align*}
T &= 3.462 \sigma \quad (y = 0) \\
T &= 4.33 \sigma \quad (y = 0.28)
\end{align*}
\]

where \( y \) is the dispersion parameter. For a rectangular shaped pulse

\[
T = 2\sqrt{3} \sigma = 3.464 \sigma
\]  

(5.22)

As we shall see in the next section, the pulse shape is indeed almost
rectangular for a parabolic fibre.

5.4 Pulse shapes

The complete description of a pulse is given by the pulse shape
function \( Q \) defined by

\[
dP = Q(\tau) d\tau
\]  

(5.23)

where \( dP \) is the power within the time interval \( \tau \) to \( \tau + d\tau \). We have the
density function \( F(\tilde{\beta}) \), so that when there is a one-to-one relationship
between \( \tilde{\beta} \) and \( \tau \),

\[
dP = F(\tilde{\beta}) d\tilde{\beta} = F(\tilde{\beta}) \left| \frac{d\tilde{\beta}}{d\tau} \right| d\tau
\]

and

\[
Q(\tau) = F(\tilde{\beta}) \left| \frac{d\tilde{\beta}}{d\tau} \right|
\]  

(5.24)
The path constant \( \tilde{\beta} \) and time \( \tau \) are related by (5.3) and (5.7) for the profiles considered here. When two values of \( \tilde{\beta} \) correspond to one value of \( \tau \) for part of the range of \( \tilde{\beta} \), \( Q(\tau) \) is the sum of two terms, each having the form of (5.24), but having different time intervals. The impulse responses are given explicitly in the appendix to this chapter.

For a Lambertian source, \( Q \) as given in (5.24), with \( F = F_{br} \) given by (4.45), is plotted in fig. 5.3 for fibres with \( q \) near the optimum. The normalized time variable

\[
\bar{\tau} = \frac{\tau - \tau_{\text{min}}}{\tau_{\text{max}} - \tau_{\text{min}}} \tag{5.25}
\]

has been used, as shape is the main consideration here.

Note that when \( q \) is such that the minimum in the \( \tau \) vs \( \tilde{\beta} \) curve is in the region of interest, \( n_{cl} < \tilde{\beta} < n_0 \) (see fig. 5.2), \( Q(\bar{\tau}) \) becomes infinite at \( \bar{\tau} = 0 \). Mathematically, in terms of \( A \) and \( B \) used in defining \( \tau(\tilde{\beta}) \), we find

\[
Q(\bar{\tau}) \to \infty \text{ as } \bar{\tau} \to 0
\]

when

\[
n_{cl}^2 < B(q)/A(q) < n_0^2
\]

i.e. when

\[
\frac{1}{2} \left\{ y + (4+y) \frac{n_{cl}^2}{n_0^2} \right\} \leq q < 2+y
\]

The pulse shape for \( q = 2 \) is almost rectangular, as anticipated above. The effect of source directionality may be illustrated for the parabolic index (\( q = 2 \)) fibre. In terms of the directionality parameter \( s \), we find

\[
Q(\tau) = \text{const.} \frac{\tilde{\beta}^3}{z(n_0^2 - \tilde{\beta}^2)} \left\{ 1 - \left[ 1 - \left( \frac{n_0^2 - \tilde{\beta}^2}{n_1^2} \right) \right]^{\frac{s+1}{2}} \right\} \tag{5.26}
\]
Fig. 5.3 Impulse response pulse shapes \( Q \) vs normalized transit time \( \bar{\tau} \) (eq. 5.25). Fibre excitation is by a Lambertian source. \( q_0 \) is the optimum \( q \) value (eq. 5.11) and \( y \) is the refractive index dispersion parameter (eq. 5.5).

\[ q_1 = \frac{1}{2}(y + (4+y)n_c/n_0). \]

As \( q \) increases from \( q_1 \) to \( q_0 \) the position of the jump in the curve moves progressively from \( \bar{\tau} = 0 \) to \( \bar{\tau} = 1 \).

where

\[ \tilde{\beta} = \frac{c}{z} \left[ \bar{\tau} - \left\{ \bar{\tau}^2 - \left( \frac{zn_0}{c} \right)^2 \right\}^{1/2} \right] \]

and

\[ \frac{zn_0}{c} \leq \bar{\tau} \leq \frac{zn_0}{2c} \left( \frac{n_{c1}}{n_0} + \frac{n_0}{n_{c1}} \right) \]
For other values of $\tau$, $Q(\tau)$ is zero. The shape for $s=1$ is shown in fig. 5.3 and its form is changed little as $s$ varies, as may be seen from the following numerical results for $n_i = n_0$, $\theta_c(0) = 0.1$:

$$\frac{Q(\tau_{\min})}{Q(\tau_{\max})} = \frac{Q(\tau=0)}{Q(\tau=1)} = 1.015 \text{ for } s = 1$$

$$= 1.020 \text{ for } s = 3$$

$$= 1.038 \text{ for } s = 10$$

For these $s$ values, eq. (5.26) may be replaced by

$$Q(\tau) = \text{const.} \frac{(s+1)^3}{z} \left[ 1 - \frac{s-1}{4n_i^2} (n_0^2 - \beta^2) \right]$$

(5.27)

with an error of less than 0.05%.

5.5 Power attenuation

Power flow in optical waveguides can be attenuated in four main ways: core absorption (discussed below), cladding absorption, optical tunnelling (Chapter 6), and radiation and coupling due to scatterers and imperfections (Chapter 8).

We have expressions for power of the form

$$P = \int f(\xi) d\xi$$

(5.28)

where $\xi$ is one of the variables labelling rays. When no attenuation mechanisms are present, (5.28) gives the power at distance $z$ from the source. When power loss occurs we introduce an attenuation coefficient $\alpha$ such that

$$P(z) = \int f(\xi) \exp[-\alpha(\xi,z)z] d\xi$$

(5.29)

If the loss mechanism is uniform along the fibre, then $\alpha$ is independent of $z$. 
Effect of core material absorption

Assume an absorption coefficient $\alpha_{co}$ such that a ray $i$ suffers a power loss along its path

$$dP_i = -P_i \alpha_{co}(s) ds$$

where $s$ measures the distance along the ray path.

Hence

$$dP_i = -P_i \alpha_{co}(s) \sec \theta \, dz$$

$$= -P_i \alpha_{co}(s) \, ndz/\tilde{\beta}$$

Integrating this, and noting that the path has the path period $z_p$

gives

$$P_i(z) = P_i(0) \cdot \exp\left(-\frac{z}{z_p \tilde{\beta}} \int_0^{z_p} \alpha_{co} \, ndz\right)$$

(5.30)

There are reasons⁵ for expecting $\alpha$ to vary in the same way as $n$, so we set

$$\alpha_{co} = \tilde{\alpha}_{co} n(r)/n_0$$

Thus $\tilde{\alpha}_{co}$ is the absorption coefficient for the on-axis material. The integral (5.30) now involves $\int n^2 dz$, and this is just the integral required to find $\tau(\tilde{\beta})$. This was done in section 5.1, so that for attenuation due to an absorbing core, the coefficient depends only on $\tilde{\beta}$:

$$P(\tilde{\beta},z) = P(\tilde{\beta},0) \exp[-\alpha(\tilde{\beta})z]$$

(5.31)

$$\alpha(\tilde{\beta}) = \frac{2\tilde{\alpha}_{co}}{q+2} \left(\frac{\tilde{\beta}}{n_0} + \frac{qn_0}{2\tilde{\beta}}\right)$$

(5.32)
A second case is $\alpha_{co}$ constant; this requires $\int ndz$.

Note

$$n(r) = n_0 \left[ 1 - 2\Delta(r/p)^q \right]^{1/2} = \frac{n_0}{2} + \frac{n^2(r)}{2n_0}$$

Hence using the above we obtain

$$\alpha(\tilde{\beta}) \approx \frac{\alpha_{co}}{q+2} \left( \frac{\tilde{\beta}}{n_0} + (q+1) \frac{n_0}{\tilde{\beta}} \right) \quad (5.33)$$

The total bound power at distance $z$ from the source is then

$$P_{br}(z) = \int_{n_{cl}}^{n_0} F_{br}(\tilde{\beta}) \exp[-\alpha(\tilde{\beta})z]d\tilde{\beta} \quad (5.34)$$

Since $n_{cl} \approx n_0$, $\alpha(\tilde{\beta})$ is almost constant for bound rays, so to a good approximation

$$P_{br}(z) \approx \exp[-\alpha(\tilde{\beta} = n_0)z] \int_{n_{cl}}^{n_0} F_{br}(\tilde{\beta}) d\tilde{\beta}$$

$$= P_{br}(0) \exp[-\alpha_{co}z] \quad (5.35)$$

While the absolute pulse width $T$ is independent of attenuation mechanisms, the rms pulse width $\sigma$ is not, since it depends on time averages weighted by ray powers. When core absorption is present, the moments giving $\sigma$ must be modified by generalizing (5.18) to

$$I_m(z) = \int \tau^m(\tilde{\beta}) F(\tilde{\beta}) \exp[-\alpha(\tilde{\beta})z]d\tilde{\beta} \quad (5.36)$$

Generally, this leads to very small changes in $\sigma(z)$. For example, when the on-axis ray ($\tilde{\beta} = n_0$) has been attenuated 20dB in a $q = 2$ fibre, $\sigma$ varies from the no loss case by less than 0.05% for $\alpha_{co}$ constant. The effect is even smaller for $\alpha_{co}$ proportional to $n$. 
5.6 Discussion

The formalism presented above allows pulse propagation to be understood in terms of the $\tau$ vs $\tilde{\beta}$ curve (fig. 5.2), and the manner in which the initial power is distributed with respect to $\tilde{\beta}$. The fact that this curve has a minimum is central to the understanding of pulse width reduction in graded index fibres. Analyzing $T$ vs. $q$ or $\sigma$ vs. $q$ gives very similar results, i.e. $T$ and $\sigma$ are minimized simultaneously by $q_{opt}$. This is important, as imperfections in the fibre may cause pulse distortion and a change in $\sigma$, but leave $T$ unchanged.

The pulse widths are very sensitive to $q$ near the optimum value − this means that small imperfections make the achievement of complete optimization difficult − see Chapter 7. Other factors may influence the choice of $q$. Thus the fraction of source power going into the bound rays in proportional to $q/(q+2)$ − see eq. (4.13). However, this is a slowly varying function, changing from 0.33 to 0.6 as $q$ increases from 1 to 3, whereas $T$ varies by two orders of magnitude over this range.

Changing from near parabolic profile to a step index doubles the bound ray power accepted, but the rms pulse width is then given by

$$\sigma_{\text{step}} = \frac{zn}{c} \frac{\Lambda}{2\sqrt{3}} \left( 1 + \frac{5}{6} \Delta \right) + O(\Delta^3)$$

for $s = 1$. This is a larger than the optimum graded index case by a factor of order $10^2$. A highly collimated source must be used to reduce $\sigma_{\text{step}}$, but then coupling and the resultant pulse spreading must be considered. For the step fibre, the absolute (bound) pulse width $T$ is

$$T_{\text{step}} = \frac{zn}{c} \Delta$$
APPENDIX

General forms for the impulse response

To find the impulse response we consider a source producing an impulse of energy at \( \tau = 0 \) with an angular distribution given by (4.3) with \( I_0 = \frac{s+1}{2(\pi s)^2} \) (this normalizes the total energy emitted to unity). Equation (A1) is an excellent approximation when \( s \neq 1 \), and is exact when \( s = 1 \). \( A \) and \( B \) are given by (5.7) when dispersion is included, and by (5.3) when it is not.

Let

\[
Q(\tau) = \frac{(s+1)\tilde{\theta}_3^3}{zn_1^2|\tilde{\theta}^2 - B|} \left( \frac{n_0^2 - \tilde{\theta}^2}{\gamma^2} \right)^{2/q} \tag{A1}
\]

\[
\tau(n_0) = \frac{z}{c} N_0 \tag{A2}
\]

\[
\tau(n_{c1}) = \left( \frac{An_{c1} + \frac{B}{n_{c1}}} \right) z \tag{A3}
\]

\[
\tau(\tilde{\theta}_m) = 2z(AB)^{1/2}
\]

\[
= \frac{N_0 z}{q+2} (4+y)^{1/2}(2q-y)^{1/2} \tag{A4}
\]

\[
\tilde{\theta}_L = \frac{1}{2Az} \left[ \tau - (\tau^2 - 4Abz^2)^{1/2} \right] \tag{A5}
\]

\[
\tilde{\theta}_u = \frac{1}{2Az} \left[ \tau + (\tau^2 - 4Abz^2)^{1/2} \right] \tag{A6}
\]

Then for

\[
q \leq \frac{1}{2} \left[ y + (4+y) \frac{n_{c1}^2}{n_0^2} \right]
\]
the impulse response is given by (Al) with $\tilde{B} = \tilde{B}_u$ given by (A6) and

$$\tau(n_{c1}) \leq \tau \leq \tau(n_0)$$

For other values of $\tau$, $Q(\tau) = 0$.

For

$$\frac{1}{2} \left[ y + (4+y) \frac{n_{c1}^2}{n_0^2} \right] < q < 2 + y$$

the impulse response is

$$Q(\tau) = Q_1(\tau) + Q_2(\tau)$$

$Q_1(\tau)$ is given by (Al) with $\tilde{B} = \tilde{B}_L$ given by (A5), for $\tau(n_m) \leq \tau \leq \tau(n_{c1})$ and $Q_2(\tau) = 0$ for other values of $\tau$.

$Q_2(\tau)$ is given by (Al) with $\tilde{B} = \tilde{B}_u$ given by (A6), for $\tau(\tilde{B}_m) \leq \tau \leq \tau(n_0)$, and $Q_2(\tau) = 0$ for other values of $\tau$.

In this region $\tau(\tilde{B})$ reaches its minimum value when

$$\tilde{B} = \tilde{B}_m = \left[ \frac{B}{A} \right]^{\frac{1}{2}} = n_0 \left[ \frac{2q-y}{4+y} \right]^{\frac{1}{2}}$$

When $q \geq 2+y$, $Q(\tau)$ is given by (Al) with $\tilde{B} = \tilde{B}_L$ given by (A5), for $\tau(n_0) \leq \tau \leq \tau(n_{c1})$, and $Q(\tau) = 0$ for other values of $\tau$. 
References


CHAPTER 6
TUNNELLING RAYS

6.1 Introduction

In the classification of rays in section 2.5, it was pointed out that in general tunnelling rays, as well as bound rays, exist in graded fibres. In chapter 5 pulse propagation was analyzed in terms of bound rays. In this chapter the nature and effect of tunnelling rays are investigated. First the distributions of tunnelling rays with respect to the invariants and outer caustics are found. Tunnelling ray attenuation is handled by means of a generalized parameter which depends on fibre parameters, as well as the distance from the source. This treatment is then used to determine the tunnelling ray influence on pulse widths, optimum profile and finally pulse shapes. The pulse shapes in a step fibre, with the tunnelling ray contribution included, is found and plotted, so that a comparison can be made.

For the power law or q profile with $(n_0/n_{c1})^2 \leq 1 + 2/q$, a condition valid in practical cases, the tunnelling rays occupy the region bounded by

$$\left(1 + \frac{q}{2}\right)n_{c1}^2 - \frac{q}{2}n_0^2 \leq \beta^2 \leq n_{c1}^2$$

$$n_{c1}^2 - \tilde{\beta}^2 \leq \bar{\chi}^2 \leq \bar{\chi}_{\text{max}}^2(\tilde{\beta})$$  \hspace{1cm} (6.1)
where $\tilde{\beta}_{\text{max}}$ is given by (2.45). When the parameter $\Delta$ is small, we find

\[
\text{tunnelling rays: } 1 - \left( 1 + \frac{q}{2} \right) \Delta \lesssim \frac{\tilde{\beta}}{n_0} \lesssim 1 - \Delta \\
\text{bound rays: } 1 - \Delta \lesssim \frac{\tilde{\beta}}{n_0} \lesssim 1 
\]

(6.2)

For a step index profile

\[
\text{tunnelling rays: } 0 \lesssim \frac{\tilde{\beta}}{n_0} \lesssim 1 - \Delta \\
\text{bound rays: } 1 - \Delta \lesssim \frac{\tilde{\beta}}{n_0} \lesssim 1 
\]

(6.3)

Thus for example in the parabolic index profile, the bound and tunnelling rays have a spread in $\tilde{\beta}$ values of width $n_0 \Delta$, whereas in the step fibre the $\tilde{\beta}$ value range is of width $n_0 \Delta$, but for tunnelling rays the spread is $n_0 (1 - \Delta)$. These intervals will be important in considering pulse widths.

As noted in Chapter 4, there are initially relatively fewer tunnelling rays than in the step index fibre — for example, the ratio of tunnelling to bound power at $z = 0$ is $1/3$ for $q = 2$, and 1 for a step profile.

6.2 Distribution of tunnelling rays

We can obtain the density distribution of tunnelling rays in $\tilde{\beta}$ and $\tilde{\lambda}$, or in $\tau_{tp}$, by a process similar to that used for bound rays in Chapter 4, but with integral limits appropriate to tunnelling rays.

Thus for a Lambertian source and a parabolic fibre, if we normalize the bound ray power to unity for convenience, then

\[
P_{tn} = \frac{8}{\gamma^3} \int_{n_0^2 - 2\gamma^2}^{n_{cl}^2} \tilde{\beta} \tilde{\lambda} \sqrt{\frac{n_{cl}^2 - \tilde{\beta}^2}{(n_0^2 - 2\gamma^2)^{1/2}}} \frac{n_0^2 - \tilde{\beta}^2}{(n_{cl}^2 - \tilde{\beta}^2)^{1/2}} d\tilde{\beta} d\tilde{\lambda} 
\]

(6.4)
Hence
\[ G(\tilde{\beta}, \tilde{\xi}) d\tilde{\beta} d\tilde{\xi} = \frac{8}{\gamma^3} \tilde{\beta} d\tilde{\beta} d\tilde{\xi} \] (6.5)

Also
\[ P_{tn} = \frac{8}{\gamma^3} \int_{(n_0^2 - 2\gamma^2)^{1/2}}^{n_{c1}} d\tilde{\beta} \tilde{\beta} [n_0^2 - \tilde{\beta}^2 - (n_{c1}^2 - \tilde{\beta}^2)^{1/2}] \]

Hence
\[ F_{tn}(\tilde{\beta}) = \frac{8}{\gamma^3} \tilde{\beta} [n_0^2 - \tilde{\beta}^2 / 2\gamma - (n_{c1}^2 - \tilde{\beta}^2)^{1/2}] \] (6.6)

\[ P_{tn} = \int_{(n_0^2 - 2\gamma^2)^{1/2}}^{n_{c1}} \frac{F_{tn}(\tilde{\beta}) d\tilde{\beta}}{\gamma^2} = \frac{1}{3} \]

as calculated by a different method in Chapter 4.

The distribution of rays in terms of their outer turning points, \( H_{tr}(r_{tp}) \), is now considered for the parabolic fibre. By taking \( R = r_{tp} / \rho \) and \( R_c = r_{tp} / \rho \) in (4.53) we see that, when plotted on a graph with axes \( \tilde{\beta}^2 \) and \( \Lambda = \tilde{\beta}^2 / \gamma^2 \), the turning point contours in the tunnelling region are straight lines continuous with those in the bound region. With the normalized \( \tilde{\beta} \) and \( \tilde{\xi} \) of eqns 4.54 and 4.55, tunnelling rays have

\[ 1 \leq \beta \leq 2 \]

and

\[ B - 1 \leq \Lambda \leq B^2 / 4 \] (6.7)

This time, contours with \( R_c > \frac{1}{\sqrt{2}} \) are tangent to the top curve at \( B = 2R_c^2 \), with the line segment to the left representing the inner caustic, and that to the right giving the outer caustic. Clearly, for tunnelling rays:

\[ \frac{1}{\sqrt{2}} \leq \frac{r_{tp}}{\rho} \leq 1 \]
so that lines with \( R_c < 1/\sqrt{2} \) represent inner caustics only. Thus a ray with outer caustic \( r_{tp} \) has

\[
1 \leq B < 2R_c^2
\]

i.e.

\[
\tilde{\beta}_t(r_{tp}) \leq \tilde{\beta} < n_{cl}
\]

where

\[
\tilde{\beta}_t^2(r_{tp}) = n^2 - 2\gamma^2 \left( \frac{R_{tp}}{\rho} \right)^2 \tag{6.8}
\]

The contours are shown in fig. 6.1. The \( r_{tp} = \rho \) contour coincides with the line \( \tilde{\kappa}^2 = n_{cl}^2 - \tilde{\beta}^2 \), i.e. \( \Lambda = B - 1 \), whereas the \( r_{tp} = \rho/\sqrt{2} \) "contour" is the point \( \tilde{\beta} = n_{cl}, \tilde{\kappa} = \gamma/2 \), i.e. \( B = 1, \Lambda = 1/4 \).

Thus

\[
H_{tn}(r_{tp})dr_{tp} = \int_{\tilde{\beta}_t(r_{tp})}^{n_{cl}} \frac{d\tilde{\beta}}{\tilde{\beta}_t^2(r_{tp})} \left\{ \frac{\partial \tilde{\kappa}}{\partial r_{tp}} dr_{tp} G(\tilde{\beta},\tilde{\kappa}) \right\}
\]

where \( \tilde{\beta}_t(r_{tp}) \) is given by (6.8), and \( G(\tilde{\beta},\tilde{\kappa}) \) by (4.51).

i.e.

\[
H_{tn}(r_{tp}) = \frac{8}{\gamma^3 \rho} \int_{\tilde{\beta}_t(r_{tp})}^{n_{cl}} \frac{d\tilde{\beta}}{\tilde{\beta}_t^2(r_{tp})} \left\{ \frac{\tilde{\beta}^2 + 2\gamma^2 \frac{r_{tp}}{\rho^2} - n_0^2}{n_0^2 - \tilde{\beta}^2 - \gamma^2 \frac{r_{tp}^2}{\rho^2}} \right\}
\]

So

\[
H_{tn}(r_{tp}) = \frac{8}{3\rho} \left\{ 2 \frac{r_{tp}^3}{\rho^3} - \left( 1 - \frac{r_{tp}^2}{\rho^2} \right)^{1/2} \left( 4 \frac{r_{tp}^2}{\rho^2} - 1 \right) \right\}, \quad \frac{1}{\sqrt{2}} \leq \frac{r_{tp}}{\rho} < 1
\]

\[
= 0, \quad 0 \leq \frac{r_{tp}}{\rho} \leq \frac{1}{\sqrt{2}} \tag{6.9}
\]

This function is plotted in fig. 6.2.
Fig. 6.1 Contours of constant turning point for tunnelling rays in a parabolic fibre. Full line segments represent outer caustics, broken lines inner caustics. Here $0 \leq r_{\text{min}}/\rho \leq 1$ and $1/\sqrt{2} \leq r_{\text{tp}}/\rho \leq 1$. 
Fig. 6.2 Distribution of tunnelling rays with respect to their outer caustics, $r_{tp}$.

Consistency is seen by integrating (6.9):

$$
\int_0^\rho H_{tn}(r_{tp}) dr_{tp} = \frac{1}{3}
$$

We can find the fraction of tunnelling rays with outer turning points in any given range. For example, 88% have their turning points in the range

$$
\frac{\sqrt{3}}{2} \rho \leq r_{tp} \leq \rho
$$
From fig. 6.2 it is evident that most of the tunnelling rays have \( r_{tp} \) close to the core-cladding interface. This means that the associated evanescent fields will be quite significant in the region \( r > \rho \), implying a relatively high cladding absorption (see ref. 1).

6.3 Tunnelling ray attenuation

The power carried by tunnelling rays attenuates as it propagates, even in an ideal, non-absorbing fibre. A tunnelling ray is only partially reflected by a caustic, so at each turning point, \( r_{tp} \), there is a loss of energy due to radiation, and hence energy propagates from \( r = r_{rad} > \rho \). This energy carried away leads to a transmission coefficient \( T_{tr} \).

The transmission coefficient of a tunnelling ray is found by integrating the radial component of the wavector from the outer caustic to the radiation point.\(^2\) Thus

\[
T_{tr} = \exp \left\{ -2 \int_{r_{tp}}^{r_{rad}} |k_y(\xi)| \, d\xi \right\} \tag{6.10}
\]

with

\[
k_y^2(r) = k_0^2 f(r) = k_0^2 \left[ n^2(r) - \beta^2 - \frac{\rho^2 \Gamma^2}{r^2} \right]
\]

\( f(r) \) has been obtained from (2.13).

Hence

\[
T_{tr} = \exp\left\{ -2k_0 \left( I_1 + I_2 \right) \right\}
\]

where

\[
I_1 = \int_{r_{tp}}^{\rho} \left| f(r) \right|^2 dr
\]
and

\[ I_2 = \int_0^{\rho} f(r) \frac{dr}{r} \]

\( I \) is ignored in [3], but by linearizing \( n(r) \) near \( r = \rho \), and using the approximation for \( r_t \) when it is close to \( \rho \), \( I_1 \) can be found for general \( q \):

If

\[ n^2(r) = n_0^2 - \gamma^2 \left( \frac{r}{\rho} \right)^q, \quad r \ll \rho \]

then for \( r \approx \rho \),

\[ n^2(r) \approx n_{cl}^2 + \gamma^2 q \left( 1 - \frac{r}{\rho} \right) \]

and hence

\[ I_1(q) \approx \frac{2\rho}{3(q\gamma^2 - 2\tilde{\beta}^2)} \left( \tilde{\beta}^2 + \tilde{\beta}^2 - n_{cl}^2 \right)^{3/2} \quad (6.11) \]

Now

\[ I_2 = \int_0^{\rho} \frac{dr}{r} \left[ \rho^2 \tilde{\beta}^2 - r^2 (n_{cl}^2 - \tilde{\beta}^2) \right]^{1/2} \]

\[ = (n_{cl}^2 - \tilde{\beta}^2)^{1/2} \int_0^{\rho} \frac{dr}{r} \left( r^2 \frac{\tilde{\beta}^2}{r^2} - 1 \right)^{1/2} \]

\[ = \rho \tilde{\beta} \log \left[ \frac{\rho^2 \tilde{\beta}^2}{r^2} + \left( \frac{\tilde{\beta}^2}{r^2} - 1 \right) \right] - (n_{cl}^2 - \tilde{\beta}^2)^{1/2} (r^2 \frac{\tilde{\beta}^2}{r^2} - \tilde{\beta}^2)^{1/2} \]

\[ = \rho \tilde{\beta} \arccosh \left( \frac{r^2 \frac{\tilde{\beta}^2}{r^2}}{\rho^2} \right) - (n_{cl}^2 - \tilde{\beta}^2)^{1/2} (r^2 \frac{\tilde{\beta}^2}{r^2} - \tilde{\beta}^2)^{1/2} \quad (6.12) \]

where (from eq. 2.48)

\[ r^2 \frac{\tilde{\beta}^2}{r^2} = \frac{\rho^2 \tilde{\beta}^2}{n_{cl}^2 - \tilde{\beta}^2} \]

For convenience, the following variables are introduced:

\[ j = \frac{2(n_{cl}^2 - \tilde{\beta}^2)}{2n_0^2 \theta^2 \left( \frac{v}{c} \right)} \quad (6.13) \]
\[
p = \left[ \frac{2}{\rho} \right]^{1/2} \frac{1}{n_0} \frac{\gamma}{\theta_c}
\]  \hspace{1cm} (6.14)

Then
\[
\frac{r^2 \text{ rad}}{\rho^2} = \frac{p^2}{j}
\]

and
\[
\frac{P}{\rho^{1/2}} = p \arccosh \left( \frac{P}{\sqrt{j}} \right) - (p^2 - j)^{1/2}
\]

Thus
\[
T_{tr} \approx \exp(-2k I_2)
\]

\[
\approx \exp\left\{ - (2q)^{1/2} \left[ p \arccosh \frac{P}{\sqrt{j}} - (p^2 - j)^{1/2} \right] \right\}
\]

i.e.
\[
T_{tr} = g^V
\]

where
\[
g = \exp\left\{ - (2q)^{1/2} \left[ p \arccosh \frac{P}{\sqrt{j}} - (p^2 - j)^{1/2} \right] \right\}
\]  \hspace{1cm} (6.15)

The periodic distance, \( z_p \), as discussed in Chapter 2, gives the axial distance between two outer caustics. The fractional power loss at a reflection is \( T_{tr} \), and the number of outer caustics in distance \( z \) is \( z/z_p \), so
\[
P(z) = P(0) \exp\left\{ - \frac{az}{\rho} \right\} = P(0) \exp\left\{ - T \frac{z}{z_p} \right\}
\]

Now from Chapter 2
\[
z_p \approx \frac{\rho}{\theta_c} \pi N(q)
\]

where
\[
N(q) = 2q^{1/2} \left( 1 + \frac{a}{2} \right)^{-1/q}
\]
so
\[ \frac{z}{\rho p} = \frac{\theta c z}{\rho \pi N(q)} L^V \] say

Thus
\[ L_{gr}(q) = \left( \frac{\theta c z}{(\pi \rho N(q))} \right)^{1/V} \] (6.16)

and
\[ \frac{\alpha z}{\rho} = T \frac{z}{\rho p} = (gL)^V \]

As \( V \) is large, we can draw a line of demarcation, \( gV = 1 \), in the
tunnelling ray area, and take \( \alpha = \infty \) when \( gL > 1 \), and \( \alpha = 0 \) when \( gL < 1 \).

By defining a generalized parameter \( D_{gr} \) for graded fibres
\[ D_{gr}(q) = \ln L_{gr}(q) = \frac{1}{V} \ln \left( \frac{\theta c z}{\pi \rho N(q)} \right) \] (6.17)

we see that the dividing line \( p = p_b(j) \) satisfies
\[ D_{gr}(q) + \ln g = 0 \]
i.e.
\[ D_{gr}(q) = (2q)^{1/2} \left[ \arccosh \left( \frac{p_b(j)}{j^{1/2}} \right) - (p_b - j)^{1/2} \right] \] (6.18)

The line \( p = p_b(j) \), i.e. \( \tilde{t} = \tilde{t}_L(j) \), is shown in the tunnelling region
in fig. 6.3.

The upper boundary on tunnelling rays, in terms of \( p \), is found from (2.45) giving \( \tilde{t}_{\text{max}} \), and (6.14):
\[ p_{\text{max}}(j) = \left( \frac{2+q_1}{2+q} \right) \frac{1}{q} + \frac{1}{2} \] (6.19)

Thus the effective tunnelling ray domain is
\[ 0 \leq j \leq j_b(D) \]
\[ p_b(D,j) \leq p \leq p_{\text{max}}(j) \] (6.20)
Fig. 6.3 Demarcation line $\tilde{\lambda} = \tilde{\lambda}_L(\tilde{\beta})$ for graded index fibre. The curves are correct for $q = 2$, but the results are similar and the diagram is schematically correct for other values of $q$. As the generalized parameter $D$ (eq. 6.17) increases, the $\tilde{\lambda} = \tilde{\lambda}_L(\tilde{\beta})$ curve moves progressively from coincidence with $\tilde{\lambda} = (n_{cl}^2 - n_c^2)\tilde{\beta}^{1/2}$ for $D = -\infty$ (all tunnelling rays present) to finally coincide with the line $\tilde{\beta} = n_{cl}$ when $D = 0$ (only bound rays present). The $\tilde{\lambda} = \tilde{\lambda}_L$ curve shown is an intermediate case corresponding to $D = 0.1$ for $q = 2$.

$p = p_b(j)$ is a monotonically increasing function; it intersects $p = p_{\text{max}}(j)$ at some point $j = j_b(D)$, which can be found by substituting $p_b = p_{\text{max}}(j)$ in (6.18) and solving for $j$. 
Once we know the fibre parameters and distance from source, we find \( D(q) \) using (6.17). Now \( 0 \leq z \leq \infty \) maps to \(-\infty \leq D \leq \infty\), but the physically important parameters are confined to roughly \( 0.2 \leq D \leq 0.5\).

For example, in a parabolic index fibre with \( V = 50, \theta_c = 0.1 \) and \( \rho = 80 \mu m \), we find \( 100m \leq z \leq 10 km \) corresponds to \( 0.211 \leq D \leq 0.304\).

Results for a range of \( D \) values can be interpreted in various ways — e.g. the distance \( z \) can be fixed and we can consider the parameter \( V \) to be changing; or the fibre parameters can be fixed and we can vary the distance from the source.

For \( 0.2 \leq D \leq 0.5 \) and \( 1 \leq q \leq 6 \), an excellent approximation to \( j_b \) is given by

\[
j_b(D) = \exp -[1.07 + 0.035 + 2.84(1 + e^{-1.1q})D] \tag{6.21}\]

Using (6.13) we see that the lowest possible value of \( \tilde{\beta} \) is

\[
\tilde{\beta}_L^2(D) = n_{cl}^2 - \frac{q}{2} n_0^2 \theta_c j_b(D) \tag{6.22}\]

Thus once the generalized parameter is known, the lower limit on \( \tilde{\beta} \) can be found from (6.21) and (6.22). Once \( j \) is known, we find \( p_b(j) \), the lower limit on the tunnelling region and use (6.14) to find \( \tilde{\xi}_L(D, \tilde{\beta}) \).

The rays of importance for a particular \( D \) have now been identified with

\[
\tilde{\beta}_L(D) \leq \tilde{\beta} \leq n_{cl}
\]

\[
\tilde{\xi}_L(D, \tilde{\beta}) \leq \tilde{\xi} \leq \tilde{\xi}_{max}(\tilde{\beta}) \tag{6.23}\]

If we assume that there is no material absorption, then

\[
P(z) = P_{br}(z) + P_{tn}(z) = P_{br}(0) + P_{tn}(z)\]
with

$$P_{tn}(z) = \int_{\tilde{\beta}_{\min}}^{n_{cl} \tilde{\lambda}_{max}} d\tilde{\beta} \int_{\tilde{\lambda}_{\min}}^{\tilde{\lambda}_{max}} d\tilde{\lambda} \ G(\tilde{\beta},\tilde{\lambda}) \ \text{exp}[-\alpha(\tilde{\beta},\tilde{\lambda})z/\rho] \quad (6.24)$$

The lower limits on the integrals are given by (6.1) and $G(\tilde{\beta},\tilde{\lambda})$ by (6.5).

The analysis of this section allows us to replace (6.24) by

$$P_{tn}(z) = P_{tn}(D) = \int_{\tilde{\beta}_{L}(D)}^{n_{cl} \tilde{\lambda}_{max}} d\tilde{\beta} \int_{\tilde{\lambda}_{L}(D)}^{\tilde{\lambda}_{max}} d\tilde{\lambda} \ G(\tilde{\beta},\tilde{\lambda}) \quad (6.25)$$

Clearly the power in a fibre is a function of the generalized parameter $D$ only (see refs. 4,5).

For the purpose of comparison, the effect of tunnelling rays on propagation in step fibres is also considered. The generalized parameter concept also holds for step fibres, where

$$D(\text{step}) = \frac{1}{V} \ln \left( \frac{2\theta}{c \rho} \right) \quad (6.26)$$

and

$$\tilde{\beta}_{\text{LOW}} = n \ (1 - R_b^2 \theta_c^2)^{1/2}$$

with

$$R_b(D(\text{step})) = \left[ 1 + \frac{2}{3D(\text{step})} \right]^{1/2} \quad (6.27)$$

6.4 Pulse widths

The absolute pulse width $T$ for the impulse response is given by

$$T = \tau_{\text{max}} - \tau_{\text{min}}$$

where $\tau = \tau(\tilde{\beta})$ is given by (5.6) and $\tilde{\beta}$ ranges over all values for bound and effective tunnelling rays, i.e.
As $D$ increases, $\tilde{\beta}_L$ increases and reaches $n_{cl}$ for $D = \infty$. Since $\tau$ is proportional to $z$, we can write

$$T = zW(D)$$

(6.29)

Where $W$ is a width function. When $D = \infty$ no tunnelling rays are present and the pulse is the bound ray pulse described Chapter 5. The behaviour of $W(D)/W(\infty)$ is displayed in fig. 6.4 for parabolic and optimum index fibres, assuming no material dispersion.

Note that tunnelling rays are responsible for a pulse broadening of order 20% in practice. For $q = q_0$, the optimum value for which $T$ is minimized when only bound rays are considered, fig. 6.4 shows that the tunnelling ray contribution gives a pulse broadening of order 40-50%. This holds for any value of the dispersion parameter, $\gamma$. In terms of the effective domain parameter $j_b(D)$ (eq. 6.21) we find, for small $\theta_c$

$$\frac{W(D)}{W(\infty)} = \left[1 + 2j_b(D)\right]^2, \quad q = q_0$$

(6.30)

$$\frac{W(D)}{W(\infty)} = \left[1 + j_b(D)\right]^2, \quad q = 2, \gamma = 0$$

(6.31)

Pulse broadening due to tunnelling rays as a function of $q$ for fixed $D$ is now examined. ($D$ itself depends on $q$, but this dependence is very weak for the small variations in $q$ around $q = 2$ which we consider here.) To obtain qualitative results, we observe the part of the $T$ vs $\tilde{\beta}$ curve relevant to the rays with $\tilde{\beta}$ as in (6.28). The curve has a minimum at $\tilde{\beta}_m = (B/A)^{1/2}$, where $A$ and $B$ are defined in (5.7). The pulse width then depends on the position of $\tilde{\beta}_m$ relative to the physical interval (6.28); since $A$ and $B$ depend on $q$, the width also depends on $q$. Some examples are shown in fig. 6.5.
Fig. 6.4 Impulse response width $W$ (eq. 6.29) versus generalized parameter $D$ (eq. 6.17) for fibres with parabolic index, $q = 2$, and optimum profile, $q = q_0$. (Here $q_0$ is based on bound rays alone.) Also shown is the total power $P$ in the $q = 2$ fibre for Lambertian source excitation. $D = \infty$ corresponds to bound rays only. The upper scales show two possible interpretations of $D$ obtained by fixing observation or fibre parameters.
Fig. 6.5 Transit time $\tau$ versus $\tilde{\beta}$ for various values of $q$. The bound rays and the effective tunnelling rays for a given $D$ are in the ranges $n_{cl} \leq \tilde{\beta} \leq n_0$ and $\tilde{n}_L(D) \leq \tilde{\beta} \leq n_{cl}$, respectively. The $\tau$ vs $\tilde{\beta}$ curve has a minimum at $\tilde{\beta} = \tilde{\beta}_m$ which is a function of $q$. $\tau$ indicates the impulse response width and is a minimum when $q = q_o(D)$. For bound rays alone, $q_o(\infty)$ gives the minimum pulse width.
The optimum $q$ value, $q_0(D)$ is the one for which $\tau(\beta_L(D)) = \tau(n_0)$. In fact $q_0$ satisfies

$$(4+y)\beta_L^2(q,D) + n_0^2(2q-y) = 2n_0(q+2)\beta_L(q,D)$$

(6.32)

where

$$\beta_L^2(q,D) = \frac{n_0^2}{c^2} - \frac{q}{2} n_0^2 \sigma_L^2(q,D)$$

Fig. 6.6 Pulse width $W$ (eq. 6.29) versus $q$ for various values of the generalized parameter $D$. For each value of $D$ the rays forming the pulse are given by (6.28). $N$ is the on-axis group index, $c$ is the speed of light, $\Delta = 0.01$ and the dispersion parameter $y = 0.3$. 

Since $\tilde{\beta}_L$ depends very weakly on $q$, we find

$$q_0(D) \approx \frac{2}{n_0} \tilde{\beta}_L(2,D) + \frac{\gamma}{2} \left(1 + \frac{\tilde{\beta}_L(2,D)}{n_0}\right)$$

(6.33)

If necessary, more accurate values can easily be obtained by iteration.

Taking bound rays only ($z \to \infty$ or $D \to \infty$) corresponds to $j_b = 0$ and $\tilde{\beta}_L = n_{c1}$. Then, from (6.33)

$$q_0 = q_0(\infty) = 2 \frac{n_{c1}}{n_0} + \frac{\gamma}{2} \left(1 + \frac{n_{c1}}{n_0}\right)$$

i.e. (5.11), the optimum when only bound rays are considered.

Quantitative results are shown in fig. 6.6 for fibres with dispersion parameter $\gamma = 0.3$ and index difference $\Delta = 0.01$.

For fixed $D$, as $q$ increases $W(D,q)$ decreases to a minimum at $q = q_0(D)$ and then increases again, as expected by consideration of sequence (b) to (e) in fig. 6.5. For $q < q_0(D)$, fig 6.6 shows that there is a common curve for $W(D,q)$, although fig. 6.5(a) indicates that for even smaller $q$ such that $3T > 3\gamma$, the curve will again split up into $D$-dependent branches.

In figs 6.7 and 6.8, $q_0(D)$ and the optimum width $W$

$$W_0 = W(D,q_0(D))$$

are plotted as functions of $D$.

For operating at a given $D$, we see the optimum profile parameter $q$ to be used, and the expected pulse width. Suppose we choose $D = D_1$, and calculate $q_0(D)$. Then from fig. 6.5 we can deduce that, in this fibre, for $D \geq D_1$ we shall have width $W_0(D_1,q_0(D_1))$, while for $D < D_1$ we expect a width $>W_0(D_1,q_0(D_1))$. 
In most cases the changes brought about by including tunnelling rays are small, although of the same order of magnitude as Δ, which is the deviation of $q_0(\infty)$ from 2 in the dispersionless case. In chapter 5 it was shown that consideration of the pulse width is a useful approach for choosing optimal profile parameters when only bound rays are involved;
we now consider the validity of applying this concept when tunnelling rays are also included.

Fig. 6.8 Optimum width $W_0$ given by a fibre with optimum profile parameter $q_0(D)$ vs generalized parameter $D$. $N_0$ is the on-axis group index, $c$ is the speed of light, and the curves are labelled by the index difference $\Delta$, and dispersion parameter $y$. 
6.5 Shape details

For bound rays only, the impulse response near \( q = 2 \) is roughly rectangular, so the optimum profile chosen by minimizing the absolute width is almost the same as that found by procedures taking shape details into account. Fig. 6.4 however, shows that additional tunnelling ray power of 10\% of the bound power causes pulse spreading of 30-60\%. Thus minimization of absolute width, which gives the ultimate limit on energy spread, may not be appropriate - this depends on detection criteria.

As an example fig. 6.9 shows the computed impulse response \( Q \) in a parabolic index fibre with no material dispersion. Clearly the tail due to tunnelling rays is not rectangular in shape.

6.6 Comparison with step index fibres

As pointed out earlier, there are relatively more tunneling rays in step index fibres than in graded fibres with \( q \) around 2 or \( q_o \); also they are spread over a wide range of \( \bar{\beta} \) (and hence \( \tau \)) values. In fig. 6.10 these facts are reflected in tunnelling ray power contributions of order 20\% and pulse spreading by factors of 2 or 3. We have calculated impulse response shapes, and these are shown in fig. 6.11. The long tail due to tunnelling rays is evident.

6.7 Conclusion

In this chapter it has been shown how tunnelling rays can be incorporated into the general geometric optics model for light propagation in graded fibres. Describing attenuation by the generalized parameter formalism allows these rays to be treated like bound rays. The approach is not exact, but it has allowed the behaviour to be
Fig. 6.9 Impulse response $Q$ for a parabolic index, $q = 2$ fibre with zero material dispersion and excited by a Lambertian source. $\tau_b$ is the width of the bound ray contribution to the pulse and $\tau_0 = \frac{z n_0}{c} =$ on-axis transit time. The curves are labelled by the generalized parameter $D$ (6.17) which will be in the range $0.2 < D < 0.5$ in many practical situations. Ignoring tunnelling rays gives the $D = \infty$ result.
Fig. 6.10 Total power $P$ and impulse response pulse width $W$ (eq. 6.29) versus the generalized parameter $D$ (eq. 6.26) for a step index fibre excited by a Lambertian source. The upper scales show two possible interpretations of $D$ obtained by fixing fibre or observation parameters.
The curves are labelled by the generalized parameter \( D \) (eq. 6.26), \( T_0 = \frac{z_{n0}}{c} \) (sec \( \theta_c \) = 1)/c is the width of the bound ray contribution to the pulse.

\[ T_0 = \frac{z_{n0}}{c} \]

\[ \tau = \frac{z_{n0}/c}{c} \]

\[ \tau = \frac{z_{n0}/c}{c} \]

\[ \frac{Q(t)}{Q(\tau_0)} \]
explained, and general conclusions to be drawn. The necessary modifications to bound ray results are small in most cases, especially for long lengths of fibre. However, the presence of tunnelling rays should be considered in the analysis of pulse spreading in short lengths of fibre, and of course corrections are needed in near field methods of profile determination. 3
References


CHAPTER 7

PROFILES WITH MANUFACTURING IMPERFECTIONS

7.1 Motivation

Fibres are generally made by a drawing process from a graded preform. Because of the manufacturing method, the actual refractive index profile obtained is usually not exactly the profile desired. It is commonly observed, for example, that a "dip" often appears near the axis. In the chemical vapour deposition (CVD) process the layering inherent in the method makes itself evident as a small amplitude oscillation superimposed on the intended refractive index profile. It is our purpose in this chapter to investigate the effect of such imperfections on pulse propagation in graded fibres.

7.2 Dip near axis

We consider a profile with an on-axis dip extending out to radius $r_d$, shown in fig. 7.1. Thus in the region $0 \leq r \leq r_d$ the profile deviates from the required form. We begin by calculating how much power is influenced by a deviation in this region.

A ray with invariants $\tilde{\beta}$ and $\tilde{\lambda}$ exists in the region

$$ r_{\text{min}} (\tilde{\beta}, \tilde{\lambda}) < r < r_{\text{tp}} (\tilde{\beta}, \tilde{\lambda}) $$
Fig. 7.1 Profile for fibre having on-axis dip in refractive index. \( r_d \) is the radius of the dip, and \( n_d \) is the on-axis refractive index.

and so to be affected by a dip of radius \( r_d \), a ray will need to have

\[
r_{\min}(\tilde{\beta}, \tilde{\lambda}) < r_d
\]

Hence for a narrow dip, only a small fraction of the energy is affected in any way by the dip. The rays affected will be those which are almost meridional (small \( \tilde{\lambda} \)). Consider a parabolic fibre with the bound ray region plotted on a graph with axes \( \tilde{\beta}^2 \) and \( \tilde{\lambda}^2 \). The constant inner turning point contours are given by the straight lines

\[
\tilde{\lambda}^2 = \frac{r_{\min}^2}{\rho^2} \left( n_0^2 - \tilde{\beta}^2 - \gamma^2 \frac{r_{\min}^2}{\rho^2} \right)
\]

Thus the line \( r_{\min} = r_d \) touches the \( \tilde{\lambda}_{\max} \) curve tangentially when
\[
\left( \frac{n_0^2 - \tilde{\beta}^2}{2\gamma} \right)^2 = \frac{r_d^2}{\rho^2} \left( n_0^2 - \tilde{\beta}^2 - \gamma^2 \frac{r_d^2}{\rho^2} \right) = \tilde{\kappa}^2(r_d) \quad (7.2)
\]

i.e. when
\[
\tilde{\beta}^2 = n_0^2 - 2\gamma^2 \frac{r_d^2}{\rho^2} = \tilde{\beta}_1^2(r_d) \text{ say} \quad (7.3)
\]

All rays with \( \tilde{\beta} > \tilde{\beta}_1(r_d) \) are influenced by the dip, as are those having \( \tilde{\beta} < \tilde{\beta}_1(r_d) \) which are below the line \( r_{\min} = r_d \). These regions are shown in fig. 7.2. The areas are somewhat distorted because the vertical axis is \( \tilde{\kappa}^2 \), not \( \tilde{\kappa} \) — this understates the fraction of rays affected.

Fig. 7.2 Bound ray region in fibre with dip of radius \( r_d \). \( B \) is the normalized axial invariant (4.54).
From Chapter 4, we know that the energy density (with the total bound power normalized to unity) is

\[ G(\bar{\beta}, \tilde{\kappa}) = \frac{8}{\gamma^3} \tilde{\beta} \]

We define \( P(r_d) \) as the power carried by bound rays having \( r_{\min} < r_d \).

This can be calculated using the density result.

\[
P(r_d) = \frac{8}{\gamma^3} \left\{ \int_{n_{cl}}^{\tilde{\beta}_1(r_d)} \tilde{\kappa}(r_d) d\tilde{\beta} + \int_{0}^{n_0} \tilde{\beta} d\tilde{\beta} \int_{0}^{\tilde{\beta}_1(r_d)} \tilde{\kappa}(r_d) d\tilde{\kappa} \right\} \quad (7.4)
\]

where \( \tilde{\kappa}(r_d) \) and \( \tilde{\beta}_1(r_d) \) are given by (7.2) and (7.3).

---

Fig. 7.3 Fraction of power affected by profile dip as function of dip radius \( r_d \), for a parabolic index fibre excited by a Lambertian source.
Evaluating (7.4) shows that

\[ P(r_d) = \frac{4}{3} \frac{r_d}{\rho} \left\{ 2 \left( 1 - \frac{r_d^2}{\rho^2} \right)^{3/2} + \left( \frac{r_d}{\rho} \right)^3 \right\}, \quad 0 \leq \frac{r_d}{\rho} \leq \frac{1}{\sqrt{2}} \]  

(7.5)

The fraction of energy affected is shown in fig. 7.3.

Thus if \( r_d = 0.05\rho \), only 13% of the energy in bound rays is affected. These have \( \tilde{\ell} \) close to zero, but cover the whole \( \tilde{\beta} \) range, \( n_{c1} \leq \tilde{\beta} \leq n_0 \).

Before giving an example, we can make some general observations. For rays having \( r_{\min} \geq r_d \), the transit time is the same as in the parabolic fibre with no dip. A ray entering the lower refractive index dip is speeded up; it also travels at a shallower angle to the fibre axis. Thus the transit times of these rays are reduced, and we expect them to contribute a low amplitude leading edge to the pulse. The extent of the edge will depend on the depth of the dip, while its amplitude will depend on \( P(r_d) \).

A realistic model for a dip is

\[ n^2(r) = n^2_d + \delta^2 \left( \frac{r}{\rho} \right)^2, \quad 0 \leq r \leq r_d \]

\[ = n^2_0 - \gamma^2 \left( \frac{r}{\rho} \right)^2, \quad r_d \leq r \leq \rho \]  

(7.6)

The transit times of rays can be found analytically in this case (see Chapter 2, sect. 6), although the expressions are complicated. As was pointed out in Chapter 4, the energy density in the bound ray region, with axes \( \tilde{\beta}^2 \) and \( \tilde{\ell} \), is constant. This means that we can construct an output pulse histogram by sampling points from this region, in such a way that a given area anywhere in the region contains the same number of
points. The number of points in each time interval can then be totalled and the pulse shape accurately found, if there are sufficiently many points.

Fig. 7.4 Impulse response for parabolic index fibre with central dip in refractive index (eq. 7.6). Here $n_A^2 - n_0^2 = 0.3\gamma^2$ and $r_d = 0.2\rho$. This wide and deep dip has been chosen to emphasize the effect. The low amplitude, long duration pulse before the main pulse exists because of rays entering the dip. For other dip parameters similar results are obtained. $N = 7124$. Note the change of scale on the time axis.
This has been done for the profile (7.6), and a typical example is shown in fig. 7.4. In this chapter, \( N \) is the total number of points taken, and we use the (dimensionless) transit time relative to \( n_0 \) (i.e., the on-axis transit time), and multiply by \( 10^5 \) to obtain convenient numbers. Thus

\[
\frac{c}{z} \tau_n = 10^5 \left( \frac{c}{z} \tau - n_0 \right)
\]  

(7.7)

If there was no dip, then all the energy would be between 0 and 2 in fig. 7.4. From eq. (7.5) and fig. 7.3, it is seen that for the dip in this example, .504 of the energy in rays should be influenced; these rays are mostly speeded up considerably, but slightly more than .496 of the energy should be unaffected, and thus still be in the main pulse. The actual value (i.e., the area under the curve in fig. 7.4 from 0 to 2) is .501, in good agreement.

The pulse shapes predicted in this section are confirmed experimentally by the work of Behm.\(^4\)

Generally, a digital system receiver would include a level detector arranged to respond only to signals above a certain threshold level — this would often be above the low amplitude of the leading edge of the pulse and hence would have very little effect on bandwidth.

7.3 Numerical techniques for profile deviations.

It is clear that many profiles cannot be handled exactly, and so will require numerical evaluation of integrals. We now outline some numerical techniques which will be applied to various profile deviations in later sections.
Consider a power law profile with a deviation described by the function $h\left(\frac{r}{\rho}\right)$:

$$n^2(r) = n_0^2 - \gamma^2 \left[\frac{r}{\rho}\right]^q - \epsilon h\left(\frac{r}{\rho}\right), \quad 0 \leq r \leq \rho$$

(7.8)

where $\epsilon$ is the amplitude constant.

As explained in Chapter 5, the transit time $T$ for a length $z$ of fibre is

$$\frac{cT}{z} = \frac{cT_{AB}}{z_a}$$

where

$$cT_{AB} = \int_{r_{min}}^{r_{tp}} f\left(\frac{r}{\rho}\right) \frac{n^2(r)}{r^{k_2}} \, dr,$$

$$z_a = \tilde{\beta} \int_{r_{min}}^{r_{tp}} f^{-k_2} \, dr$$

and

$$f(r) = n_0^2 - \tilde{\beta}^2 - \gamma^2 \left[\frac{r}{\rho}\right]^q + \gamma^2 \epsilon h\left(\frac{r}{\rho}\right) - \frac{\tilde{\gamma}^2 \rho^2}{r^2}$$

(7.9)

for profiles of the form (7.7). $r_{min}$ and $r_{tp}$ are the zeros of $f(r)$. Some study shows that this problem is analytically intractable for many forms of $h\left(\frac{r}{\rho}\right)$ of interest, hence we look for a convenient numerical method. The following method turns out to be accurate, widely applicable and efficient.

First we change the variable to $u = \frac{r^2}{\rho^2}$ and re-introduce normalized $\tilde{\beta}$ and $\tilde{\gamma}$ (eqns 4.54 and 4.55) for convenience:

$$B = \frac{n_0^2 - \tilde{\beta}^2}{\gamma^2} = \frac{n_0^2 - \tilde{\beta}^2}{n_0^2 - \frac{n^2}{\epsilon}}$$
\[ \Lambda = \frac{\tilde{x}^2}{\gamma^2} \]

Hence, for bound rays, \( 0 \leq B \leq 1 \) and \( 0 \leq \Lambda \leq \frac{B^2}{4} \). The maximum value of \( \Lambda \) is 0.25, and this occurs when \( B = 1 \).

We now define \( c(u) \)

\[ c(u) = \frac{f(u)}{\gamma^2} = \frac{n^2}{\gamma^2} - \frac{\tilde{x}^2}{\gamma^2 u} - u \frac{q/2}{2} + \epsilon h(u) \]

\[ = B - \frac{A}{u} - u \frac{q/2}{2} + \epsilon h(u) \tag{7.10} \]

Then

\[ c'(u) = \frac{\lambda}{u^2} - \frac{q}{2} u \left( \frac{q}{2} - 1 \right) - \epsilon h'(u) \]

and

\[ z_a = \frac{\rho}{\beta} \int_{u_{\min}}^{u_{tp}} \frac{du}{u c\left( u \right)} \]

where \( f(u_{\min}) = f(u_{tp}) = 0 \), i.e. \( u_{\min} \) and \( u_{tp} \) are the roots of (7.10).

Also

\[ c_{AB} = \frac{\rho}{2\gamma} \int_{u_{\min}}^{u_{tp}} \frac{n^2(u) du}{u \frac{q/2}{2} c\left( u \right)} \]

\[ = z a \frac{n^2}{\frac{q}{2}} - \frac{\rho \gamma}{2} \int_{u_{\min}}^{u_{tp}} \frac{u \frac{q/2}{2} - \epsilon h(u)}{u c\left( u \right)} \]

Hence

\[ \frac{c}{z} \tau(B, \Lambda) = \frac{1}{\beta} \left\{ n^2 - \gamma^2 \text{S}_2(B, \Lambda) \right\} \frac{S_2(B, \Lambda)}{S_1(B, \Lambda)} \] \tag{7.11} \]

Hence

\[ S_1(B, \Lambda) = \int_{u_{\min}}^{u_{tp}} \frac{du}{u c\left( u \right)} \] \tag{7.12} \]
The caustics are the roots of (7.10) and they can be found numerically, using Newton's method for example, to great accuracy using only a few iterations on a computer. The unperturbed profile (i.e. \( h(u) = 0 \) in eq. 7.8) gives suitable starting values. Thus for \( q \) close to 2, the parabolic fibre values can be used as initial points (eq. 2.35):

\[
\begin{aligned}
\frac{t_p}{u_{\text{min}}} &= \frac{B}{2} \pm \left( \frac{B}{2}^2 - \Lambda \right)^{1/2} \\
\end{aligned}
\]  

(7.14)

Equipped with accurate values of the roots, we now look for a suitable numerical integration procedure. Since the integrand has a singularity at each endpoint, we re-write (7.12) and (7.13) as follows:

\[
\begin{aligned}
S_{1}(B, \Lambda) &= \int_{u_{\text{min}}}^{t_p} \frac{du}{(u_{\text{tp}} - u)^{1/2}(u - u_{\text{min}})^{1/2}} g(u) \\
S_{2}(B, \Lambda) &= \int_{u_{\text{min}}}^{t_p} \frac{du}{(u_{\text{tp}} - u)^{1/2}(u - u_{\text{min}})^{1/2}} g(u)\left\{ u^{q/2} - h(u) \right\} \\
\end{aligned}
\]  

(7.15) (7.16)

where

\[
\begin{aligned}
g(u) &= \left[ \frac{(u_{\text{tp}} - u)(u - u_{\text{min}})}{u \ c(u)} \right]^{1/2} \\
\end{aligned}
\]  

(7.17)

A Taylor expansion of \( g(u) \) around \( u_{\text{min}} \) and \( u_{\text{tp}} \) shows that both \( g(u_{\text{min}}) \) and \( g(u_{\text{tp}}) \) are finite.

The most appropriate form for evaluating integrals of the form (7.15) and (7.16) is one based on Chebyshev polynomials, as this
has the behaviour of the denominator built in. If we take \( k \) intervals, then the integrals are simply given by

\[
S_1(B, \lambda) = \frac{\pi}{k} \sum_{i=1}^{k} g(u_i)
\]

(7.18)

\[
S_2(B, \lambda) = \frac{\pi}{k} \sum_{i=1}^{k} g(u_i)\left\{u_i^{q/2} - \varepsilon h(u)\right\}
\]

(7.19)

where

\[
u_i = \frac{1}{2}(u_{tp} + u_{min}) + \frac{1}{2}(u_{tp} - u_{min})x_i
\]

with

\[
x_i = \cos \left[ \frac{(2i-1)\pi}{2k} \right]
\]

Great accuracy can be obtained, even for relatively small values of \( k \).

Thus, once the form of the deviation \( h(u) \) is specified, the transit time for any ray can be found by computing \( S_1 \) and \( S_2 \) using (7.18) and (7.19), with \( g(u) \) given by (7.17), and then substituting the ratio of \( S_2 \) to \( S_1 \) into (7.11). This method will be applied in the following sections to several profiles which are likely to occur.

7.4 Parabolic profile with sinusoidal oscillation

The core of a fibre manufactured by the CVD method has a layered structure. If there was no diffusion, the profile would consist of many small steps corresponding to the times when the dopant concentration was altered. Because of the high temperatures, diffusion will smooth out the profile to some extent, so that the fibre is likely to have a sinusoidal oscillation superimposed on the power law profile.

As the number of layers is increased, the amplitude of this oscillation would naturally tend to decrease. The profile can be modelled by
\[ n^2(r) = n_0^2 - \gamma^2 \left[ \frac{r^2}{\rho^2} - \frac{\varepsilon}{m} \sin \left( \frac{m\pi r^2}{\rho^2} \right) \right] \]  \hspace{1cm} (7.20)

where \( \varepsilon \) depends on the diffusion occurring during the fibre drawing process. Note that \( m \) is the number of layers; the index jump per layer is proportional to \( 1/m \), and so the amplitude of the remaining oscillations in \( n(r) \) also varies like \( 1/m \). The method of section 7.3 can be applied, with \( q = 2 \) and \( h(u) = \frac{1}{m} \sin(m\pi u) \). In this and subsequent sections, we use typical values \( n_0 = 1.5, \gamma = 0.15 \).

We can then delineate the bound ray region on a graph of \( B \) versus \( \Lambda \), and find the transit time for a ray corresponding to any point in the region. With the help of an interactive computer terminal, isochronous contours can quickly be drawn. An example of contour plot is shown in fig. 7.5.

It is important to note that departure from a power law profile introduces a dependence on \( \varepsilon \), which is usually quite strong — thus an extra factor is involved in the determination of pulse spreading. As a comparison, in a parabolic fibre with no oscillations, the \( cT/\ell_B \) contours are vertical lines, going from 0 at \( B = 0 \) to 1.8939 along the \( B = 1 \) line. Hence the transit time \( \frac{c}{z} T \) would be \( 1.89 \times 10^{-5} \), whereas in the example of fig. 7.5 it would be about \( 6.6 \times 10^{-5} \).

The program can be extended to produce numerical values for the impulse response by finding the transit time at a large number of points in the bound ray region, and then sorting them into the appropriate time slots. In figures 7.6 and 7.7 some of these pulse shapes are presented in the form of histograms. It will be noticed that as \( m \) becomes large, the pulse shape approaches that of the unperturbed parabolic fibre (i.e. a rectangular pulse, width approximately \( 1.9 \times 10^{-5} \)).
Fig. 7.5 Contours of constant transit time for profile (7.20), \( m = 4, \ v = .004 \). The numbers on contours are values of \( cT_n/z \) (eq. 7.7), i.e. the transit time relative to the on-axis ray (\( B = 0, \Lambda = 0 \)) transit time.
Fig. 7.6 Pulse shape for sine deviation from parabolic profile. 
$\varepsilon = .002$, $m = 2$, $N = 3025$. 
Fig. 7.7 Impulse response for sine deviation from parabolic profile, \( \varepsilon = .01 \). (a) \( m = 4 \) (\( N = 1892 \)); (b) \( m = 10 \) (\( N = 1156 \)).
The transit times obtained using this method are exact within the context of geometric optics, as no further physical approximations have been made. The validity of geometric optics depends on the slope...
refractive index variations in $r$ — see section 7.7.

The problem considered in this section is approached in a different manner by Behm — he applies a perturbation method to the approximate eigenvalues of the parabolic guide, and then finds the group delay by differentiating $\tilde{\beta}$, obtaining a very complicated expression for $\tau$. It would seem that the method we have presented here is simpler.

7.5 Optimum profile with sinusoidal oscillation

The 'optimum' profile discussed in chapter 5 is the power law profile giving the minimum impulse response width; the optimum $q$ is called $q^*_0$ (see section 5.2). For optimum profile, the minimum transit time occurs at $B = 0.5$, whereas the transit time for points on the line $B = 1$ is the same as the on-axis value. For $n_0 = 1.5$, $\gamma = 0.15$, the $q = q^*_0$ pulse width is $0.4735 \times 10^{-5}$, i.e. one-quarter that of the corresponding parabolic profile. Since the refractive index values of the two profiles are very close, it would appear that a relatively small oscillation in a near-optimum profile fibre could influence the pulse propagation considerably, and indeed this is the case.

Thus for the profile

$$n^2(r) = n_0^2 - \gamma^2 \left[ \left( \frac{r}{\rho} \right)^q - \frac{\varepsilon}{m} \sin \left( \frac{m\pi r^2}{\rho^2} \right) \right]$$  \hspace{1cm} (7.21)

we use the method of section 7.3 with $q = q^*_0$ ($\varepsilon = 0$) and

$$h(u) = \frac{1}{m} \sin (m\pi u)$$

Fig. 7.8 shows the optimum profile pulse shape, found by taking $\varepsilon = 0$, and plotted in the form of a histogram.
Fig. 7.8 Impulse response for optimum profile (N = 3645). The previously calculated theoretical width is \(0.4735 \times 10^{-5}\). \(c, q = q_0\) in fig. 5.3.

Fig. 7.9 shows some pulse shapes for \(\varepsilon = .005\). Once again as \(m\) becomes large the effect of the oscillation diminishes and the \(q = q_0\) pulse width is approached.
Fig. 7.9 Impulse response for sinusoidal deviation from optimum profile \( \varepsilon = 0.005 \). (a) \( m = 4 \) (\( N = 1023 \)); (b) \( m = 10 \) (\( N = 1156 \)).
Fig. 7.9 continued. Impulse response for sinusoidal deviation from optimum profile, \( \varepsilon = 0.005 \). (c) \( m = 40 \) \( (N = 1520) \); (d) \( m = 80 \) \( (N = 1803) \).
As the above figures reveal, there is clearly an effective width; we define it as the pulse width at half the maximum height. The absolute widths are obtained from the maximum and minimum transit times, which can be found quickly by studying contour plots. For example, when $\varepsilon = .002$, $m = 2$, the extreme values are $2.37 \times 10^{-5}$ and $-0.73 \times 10^{-5}$, giving a width of $3.1 \times 10^{-5}$. The effective pulse width relative to the optimum profile pulse width is shown in fig. 7.10.

Fig. 7.10 Ratio of effective pulse width to optimum profile pulse width for sine deviation from optimum profile, $\varepsilon = .005$ and .01.
7.6 Fourth order profile

Profiles with a fourth order term may be written

\[
\frac{n^2 (r)}{n_0^2} = \gamma^2 \left\{ \left( \frac{r}{\rho} \right)^2 - \varepsilon \left( \frac{r}{\rho} \right)^4 \right\}
\]  

(7.22)

which is of the general form considered (7.8), with \( q = 2 \) and \( h(u) = u^2 \).

Thus \( g(u) \) is given by (7.17) with

\[
c(u) = B - \frac{\Lambda}{u} - u + \varepsilon u^2
\]

Again transit times and contour plots can be found for various values of \( \varepsilon \). The contours are approximately straight lines. The slopes have large
negative values when \( \varepsilon < 0 \). \( \varepsilon = 0 \) is the parabolic case, where the contours are vertical lines (\( \tau \) is independent of \( \tilde{x} \)). As \( \varepsilon \) increases, the slopes decrease from large positive values for small \( \varepsilon \), to zero when \( \varepsilon = .0066 \). (Typical values \( n = 1.5, \gamma = 0.15 \) are used). We note in passing that this value of \( \varepsilon \) gives the minimum pulse width for a profile of the form (7.22) — it is about \( 6.33 \times 10^{-6} \), i.e. about one third the parabolic fibre width. This is shown in fig. 7.11. As \( \varepsilon \) increases further the contour slopes become increasingly negative, and the pulse width increases substantially. The impulse width as a function of \( \varepsilon \) is shown in fig. 7.12.

For fourth order departures from other power law profiles, \( \tau \) again becomes a function of \( \tilde{x} \) as \( \varepsilon \) changes from zero.

![Graph](image)

**Fig. 7.12** Impulse response width for 4th order profile (eq. 7.22) relative to that of parabolic fibre.
7.7 Discussion

It seems unlikely that small oscillations in refractive index will be removed completely. We have shown in this chapter that even small deviations introduce $\mu$ dependence into the transit times and make pulse widths considerably greater than the optimum power law profile. This means that in practice it may be difficult to obtain pulse widths close to the theoretical minimum. Recent work shows that the central dip in refractive index may well be eliminated.

The calculations in this chapter are for multimode fibres and they assume that the dimensionless parameter $V$ (eq. 3.4) is not too small. For geometric optics to be accurate, the refractive index should not vary greatly over distances of a few wavelengths. For profiles of the form (7.20) and (7.21) the maximum rate of change of $n^2$ with $r$ is independent of $m$, so accuracy is not lost when high values of $m$ are taken.

The relatively simple, conceptual approach presented is quite accurate and is easy to apply to a wide variety of problems for which the full electromagnetic theory is not available; it contrasts with complicated modal approximation methods. The method of section 7.3 can be applied to a very large class of profiles — in (7.8) we simply choose $q$, and the function giving the deviation from a power law profile, $h(r/p)$. Then $c(u)$ is specified by (7.10). Thus transit times and pulse shapes can be found for a wide range of profiles.
References


CHAPTER 3

LIGHT SCATTERING IN WAVEGUIDES

8.1 Introduction

In this final chapter, we show the versatility of geometric optics theory, by demonstrating how scattering phenomena can be handled with it. We derive the scattering formalism and suggest methods of solution; a complete study would yield another thesis! As with other complex problems, the geometric optics method here is conceptually simple, with the physical mechanism being evident at every stage. As Rayleigh scattering\textsuperscript{1,2} presents the ultimate limit to the performance of optical fibres because of the granularity of materials,\textsuperscript{3} it will be the main theme. Geometric optics methods have been successfully applied to scattering in step index guides by Snyder \textit{et al.}\textsuperscript{3,4} hence it is appropriate to investigate their application to graded index structures, making use of the formalism developed earlier in this thesis.

Marcuse's coupled power equation for bound modes\textsuperscript{5} reduces to the ray optical equation for propagation in a fibre with scatterers, provided many modes exist.\textsuperscript{6}

8.2 Scattering concept

Because of scatterers, a ray may change its \( \hat{\beta} \) value many times while travelling along the fibre. This tendency towards the average
transit time means that the pulse width as a function of $z$ will increase at a rate slower than linear.

The differential cross section $\sigma$ is defined as the energy $E$ scattered by a particle per unit solid angle per unit of incident flux impinging on the particle (see ref. 7). Thus

$$\frac{dE}{d\Omega} = \sigma P \quad (8.1)$$

where $d\Omega$ is the solid angle and $P$ is the incident flux, i.e. energy per unit area.

For Rayleigh scattering, the cross section is

$$\sigma = C[1 + \cos^2 \gamma] \quad (8.2)$$

where $\gamma$ is the angle between the scattered direction and the incident beam, and the constant $C$ includes the well known inverse fourth power dependence on the wavelength.

The general formulation for a slab waveguide with graded index $n(x)$ is presented. For cylindrical geometry the physical description is similar but the analysis is more involved. Accordingly, to highlight the basic processes we restrict this chapter to the slab guide.

8.3 Formalism for slab scattering

The distribution of rays, $F$, with respect to the axial invariant $\vec{z}$ depends on the source, as discussed in section 4.8. If scatterers are present, this will depend on $z$, and so will be denoted $F(\vec{z}, z)$. The axial distance from an axis crossing point to the following outer caustic is $z_a$ (see fig 2.5 and eq. 2.20). It is assumed that $F$ varies only slightly over a distance $z_a$. 
Let the density of scatterers be \( N(x,z) \) per unit area. To use (8.1) we need to know the flux \( P(\tilde{\beta},x,z) \) incident on the area \( x \) to \( x + dx \), and \(-x\) to \(-x - dx\) at position \( z \). Even for a source of irregular cross-section, the rays in range \( \tilde{\beta} \) to \( \tilde{\beta} + d\tilde{\beta} \) spread out to cover the area \( |x| \leq x_{tp}(\tilde{\beta}) \) after an initial spatial transient.

Let us assume that the rays with invariant \( \tilde{\beta} \) are contained within \( 2m \) flux tubes (\( m \) large), as shown in fig. 8.1(a), and \( 2m \) mirror-image flux tubes (which are not shown).

![Fig. 8.1](image)

Fig. 8.1 (a) Flux tubes in slab waveguide. Each tube carries energy \( F/2m \). Note \( dz = z/m \).

(b) Parallelogram element, distance \( x \) from axis.
For a fixed value of $x$, the horizontal distance, $dz$, between two neighbouring flux tubes is $z_p/2m$, i.e. $z_a/m$. The flux through element $dx$ at $x$ is the same as that through the mirror-image flux tube through $-x$, hence the flux per unit width at $x$ is $\frac{1}{2} P(\tilde{\beta}, x, z)$. Thus, equating the total flux through $(x, dx)$ to the energy flowing in the tube, we find

$$\frac{1}{2} P(\tilde{\beta}, x, z) dx = \frac{F(\tilde{\beta}, z)}{2m}$$

Now, from fig. 8.1(a)

$$dx = dz \tan \theta$$

$$= \frac{z_a}{m} \frac{dx}{dz}$$

The slope of the ray at $x$ is

$$\frac{dx}{dz} = \tan \theta = \left\{ \frac{n^2(x)}{\tilde{\beta}^2} - 1 \right\}^{1/2}$$

where we have used $\tilde{\beta} = n(x) \cos \theta(x)$.

Thus

$$P(\tilde{\beta}, x, z) = \frac{F(\tilde{\beta}, z)}{z_a(\tilde{\beta})} \frac{dz}{dx}$$

$$= \frac{F(\tilde{\beta}, z)}{z_a(\tilde{\beta})} \left\{ \frac{n^2(x)}{\tilde{\beta}^2} - 1 \right\}^{-1/2}$$

(8.3)

Clearly,

$$\int_0^{x_{tp}} P(\tilde{\beta}, x, z) dx = \frac{F(\tilde{\beta}, z)}{z_a} \int_0^{x_{tp}} \frac{dz}{dx} dx$$

$$= \frac{F(\tilde{\beta}, z)}{z_a} \int_0^{z_a} dz$$

$$= F(\tilde{\beta}, z)$$

(8.4)
(1) Loss term

Consider scattering in the parallelogram of width $dx$ and axial length $dz$ marked out by rays with invariant $\tilde{\theta}$ passing through $x$ and $x + dx$, as shown in fig. 8.1(b). The area encloses $Ndx\,dz$ scatterers, and since the incident flux per unit area perpendicular to the scatterers is $P(\tilde{\theta}, x, z) = \frac{P(\tilde{\theta}, x, z)}{\cos \theta} \, d\tilde{\theta}$, we find that the energy scattered to angles $\theta'$ to $\theta' + d\theta'$ is

$$Ndx\,dz \, P \sec \theta \, d\tilde{\theta} \, \sigma(\theta', \theta) \, d\theta'$$

where

$$\theta = \arccos[\tilde{\theta}/n(x)]$$

Hence the energy scattered to all angles from $(x, x+dx)$ is

$$Ndx\,dz \, \frac{n(x)}{\tilde{\theta}} \, P(\tilde{\theta}, x, z) \, d\tilde{\theta} \, \sigma_t$$

where the total scattering cross section is

$$\sigma_t = \int_0^{2\pi} \sigma(\theta', \theta) \, d\theta'$$

To find the decrease in energy per unit $\tilde{\theta}$, i.e. $dF$, we divide by $d\tilde{\theta}$, and then to obtain the total energy scattered we integrate from the axis to the outer caustic $x_{tp}(\tilde{\theta})$. Thus the loss term is

$$\frac{dF}{dz}_{(\text{loss})} = -\frac{\sigma_t}{\beta} \int_0^{x_{tp}} N(x,z) \, n(x) \, P(\tilde{\theta}, x, z) \, dx$$

$$= -\sigma_t \frac{P(\tilde{\theta}, z)}{z_{a(\tilde{\theta})}} \int_0^{x_{tp}} dx \, \frac{N(x,z) \, n(x)}{[n^2(x) - \tilde{\theta}^2]^{\frac{1}{2}}}$$

(8.5)

(ii) Gain term

The energy scattered from range $(\tilde{\theta}', \tilde{\theta}' + d\tilde{\theta}')$ to $(\theta, \theta + d\theta)$ in region $(x, x + dx)$ is
\[ N \, dx \, dz \, p(\beta', x, z) \sec \theta' \, d\beta' \, \sigma(\theta, \theta') \, d\theta \]

The energy scattered from all incident angles to \((\theta, \theta + d\theta)\) is found by integrating over \(d\beta\); we then divide by \(d\beta\) to find the energy scattered per unit \(\tilde{\beta}\):

\[
\frac{dF}{dz} \left( \text{gain} \right) = - \frac{d\theta}{d\beta} \, N(x, z) \, n(x) \, dx \int_{\text{ncl}}^{\text{cl}} \frac{d\tilde{\beta}'}{d\beta'} \, p(\beta', x, z) \, \sigma(\theta, \theta')
\]

(The negative sign arises because integral limits have been reversed).

Now

\[
\frac{d\theta}{d\beta} = - \left[ n^2(x) - \beta^2 \right]^{-\frac{1}{2}}
\]

so the energy scattered from all values of \(\beta'\) to \((\theta, d\theta)\) from all values of \(x\), per unit \(\tilde{\beta}\) is

\[
\frac{dI}{dz} \left( \text{gain} \right) = \int_{0}^{x} N(x, z) \, n(x) \, dx \int_{\text{ncl}}^{\text{cl}} \frac{d\tilde{\beta}'}{d\beta'} \, p(\beta', x, z) \, \sigma(\tilde{\beta}, \tilde{\beta}', x)
\]

\[
= \int_{0}^{x} N(x, z) \, n(x) \, dx \int_{\text{ncl}}^{\text{cl}} \frac{d\tilde{\beta}'}{d\beta'} \, p(\tilde{\beta}', x, z) \, \sigma(\tilde{\beta}, \tilde{\beta}', x)
\]

\[
= \int_{0}^{n} \int_{\text{ncl}}^{\text{cl}} d\tilde{\beta}' \, p(\tilde{\beta}', z) \frac{x}{z(\tilde{\beta}')} \left[ \frac{\sigma(\tilde{\beta})}{\left[ n^2(x) - \beta^2 \right]^{\frac{1}{2}}} \right] \left[ \frac{\sigma(\tilde{\beta}')}{\left[ n^2(x) - \beta'^2 \right]^{\frac{1}{2}}} \right]
\]

(8.6)

where \(\tilde{\beta} > = \max(\tilde{\beta}, \tilde{\beta}')\).

For Rayleigh scattering

\[
\sigma = c[1 + \cos^2(\theta' - \theta)] \quad (8.6a)
\]

with

\[
\tilde{\beta} = n(x) \cos \theta
\]

\[
\tilde{\beta}' = n(x) \cos \theta'
\]
Defining a coupling coefficient \( S \) allows the gain term to be written in a succinct form:

\[
S(\tilde{\beta}, \tilde{\beta}') = S(\tilde{\beta}', \tilde{\beta}) = \int_0^{x_{tp}(\tilde{\beta})} \frac{dx \, N(x, z) \, n(x) \, \sigma(\tilde{\beta}, x, \tilde{\beta}')}{{[n^2(x) - \tilde{\beta}^2]^2 [n^2(x) - \tilde{\beta}'^2]^2}} \quad (8.7)
\]

We now include a material absorption term and arrive at the final equation for general \( n(x) \):

\[
\frac{dT}{dz} (\tilde{\beta}, z) + \alpha_a F(\tilde{\beta}, z) = \int_{n_{cl}}^{n} \frac{d\tilde{\beta}'}{z_a(\tilde{\beta})} S(\tilde{\beta}, \tilde{\beta}') - \frac{\alpha}{z_a(\tilde{\beta})} F(\tilde{\beta}, z) \int_0^{x_{tp}(\tilde{\beta})} \frac{dxN(x, z)n(x)}{[n^2(x) - \tilde{\beta}^2]^2} \quad (8.8)
\]

Let \( N(x, z) \) be a constant, as would often be the case. Using results from section 2.3(ii), the final term can now be written in terms of the geometric path length \( L_p \), as:

\[
\int_{n_{cl}}^{n} \frac{d\tilde{\beta}'}{z_a(\tilde{\beta})} S(\tilde{\beta}, \tilde{\beta}')
\]

Hence

\[
\frac{dT}{dz} + [\alpha + \sigma \frac{L_p(\tilde{\beta})}{L_p(\tilde{\beta})}] F = \int_{n_{cl}}^{n} \frac{d\tilde{\beta}'}{z_a(\tilde{\beta})} S(\tilde{\beta}, \tilde{\beta}') \quad (8.9)
\]

For a power law profile

\[
z_a(\tilde{\beta}) = \frac{\tilde{\beta} \, \gamma^{2/3}}{2 \, \gamma^{2/3} \Gamma \left[ \frac{1}{q} + \frac{1}{2} \right]} \left( n_0^2 - \tilde{\beta}^2 \right)^{-\frac{1}{2}}
\]

and

\[
x_{tp}(\tilde{\beta}) = \frac{\rho}{\gamma^{2/3}} \left( n_0^2 - \tilde{\beta}^2 \right)^{1/q}
\]

As the angles involved are very small, \( \sigma \) in (8.6a) can be taken as a constant for Rayleigh scattering.
8.4 Solution of scattering equation for step index profile

For a step profile

\[ z_a(\beta) = \beta \rho (n_0^2 - \beta^2)^{-\frac{1}{2}} = \frac{\beta \rho}{n_0 \sin \theta} \]

\[ L_p = \sec \theta = n_0 / \beta \]

and

\[ S(\tilde{\beta},\tilde{\beta}') = N_n \sigma \rho (n_0^2 - \tilde{\beta}^2)^{-\frac{1}{2}}(n_0^2 - \tilde{\beta}'^2)^{-\frac{1}{2}} \]

Now \( \tilde{\beta} \approx n_0 \) so \( \alpha_a + \sigma n \tilde{\beta} / n \tilde{\beta} = \alpha_a + N \tilde{\beta} = \alpha \) say

so

\[ \frac{dF}{dz}(\tilde{\beta},z) + \alpha F(\tilde{\beta},z) = \frac{N_n \sigma}{(n_0^2 - \tilde{\beta}^2)^{1/2}} \int_{n_{cl}}^{n_0} \frac{d\tilde{\beta}'}{\tilde{\beta}'} F(\tilde{\beta}',z) \quad (8.10) \]

This can be solved by assuming a solution

\[ F(\tilde{\beta},z) = F_0(\tilde{\beta}) e^{-\alpha z} + F_1(\tilde{\beta},z) + F_2(\tilde{\beta},z) + \ldots \]

where, for the Lambertian source considered here, the initial distribution is

\[ F_0(\tilde{\beta}) = F(\tilde{\beta},0) = \rho \tilde{\beta}(n_0^2 - \tilde{\beta}^2)^{-1/2} / \gamma \]

\[ P_0 = \int_{n_{cl}}^{n_0} F_0(\tilde{\beta}) d\tilde{\beta} \]

Then

\[ \frac{d}{dz} (F e^{\alpha z}) = \frac{N_n \sigma P_0}{\gamma (n_0^2 - \tilde{\beta}^2)^{1/2}} \int_{n_{cl}}^{n_0} \frac{d\tilde{\beta}'}{\tilde{\beta}'} F(\tilde{\beta}',z) \]

so

\[ F_1 = N_n \sigma \frac{\theta}{c} \gamma^{-1} (n_0^2 - \tilde{\beta}^2)^{-1/2} e^{-\alpha z} \]

Furthermore

\[ \left( \frac{d}{dz} + \alpha \right) F_m = \frac{N_n \sigma}{(n_0^2 - \tilde{\beta}^2)^{1/2}} \int_{n_{cl}}^{n_0} \frac{d\tilde{\beta}'}{\tilde{\beta}'} F_{m-1}, \quad m \geq 2 \]
so

\[ F_m = e^{-\alpha z} (N\sigma z)^m b^m \frac{1}{m!} e^\left( n_0^2 - \beta^2 \right)^{-\frac{1}{2}} \left( \begin{array}{c} \theta \\ n_0 \end{array} \right) \left( \begin{array}{c} n_0 \\ n \end{array} \right) \]

where

\[ b = \log \left( \frac{n + \gamma}{n_{c1}} \right) \]

Hence

\[ F(\tilde{\beta}, z) = \frac{P_0}{\gamma} e^{-\alpha z} (n_0^2 - \tilde{\beta}^2)^{-\frac{1}{2}} \left[ \tilde{\beta} + \frac{\theta}{c \sigma_0} \sum_{m=1}^{\infty} \left( \frac{N\sigma_0 z}{m!} \right)^m \right] \]

\[ = \frac{P_0}{\gamma} e^{-\alpha z} (n_0^2 - \tilde{\beta}^2)^{-\frac{1}{2}} \left[ \tilde{\beta} + \frac{\theta}{c \sigma_0} \left( \exp \left( N\sigma_0 z \right) - 1 \right) \right] \quad (8.11) \]

Direct substitution verifies that (8.11) is indeed a solution of integral equation (8.10)

\[ P(z) = \int_{n_{c1}}^{n_0} F(\tilde{\beta}, z) d\tilde{\beta} \]

\[ = \frac{P_0}{\gamma} e^{-\alpha z} \left[ 1 + \frac{\theta^2 n_0}{c \sigma_0} \left( \exp \left( N\sigma_0 z \right) - 1 \right) \right] \quad (8.12) \]

\[ P(z) \to 0 \text{ as } z \to \infty \text{ as } \alpha > N\sigma_0. \] The physical consequences are discussed by Snyder.\(^3\)

8.5 Analysis of parabolic slab scattering equation

For the parabolic case

\[ n^2(x) = n_0^2 - \gamma^2 \left( \frac{x}{\rho} \right)^2 \]

we have

\[ z_a(\tilde{\beta}) = \frac{\rho \tilde{\beta} \pi}{2\gamma} \]

and

\[ L_p = \frac{2n_0}{n_{\sigma0}} E \left( \frac{1 - \tilde{\beta}^2}{n_0^2} \right) \]

\[ \approx \frac{n_0}{\tilde{\beta}} \left( \frac{3}{4} + \frac{1}{4n_0^2} \right) \approx 1 \]
Hence,

\[
\frac{dF}{dz} + \frac{\alpha}{\rho \pi} F(\tilde{\beta}, z) = \frac{2\gamma}{\rho \pi} \int_{n_{cl}}^{n} \frac{d\tilde{\phi}'}{\tilde{\beta}'} F(\tilde{\beta}', z) S(\tilde{\beta}, \tilde{\beta}')
\]

(8.13)

where

\[
\alpha_p = \alpha_a + \frac{2\gamma N}{\rho \pi n_0}
\]

and

\[
S(\tilde{\beta}, \tilde{\beta}') = \int_0^{x_{tp}(\tilde{\beta}')} dx \frac{n(x)}{[n^2(x) - \tilde{\beta}^2][n^2(x) - \tilde{\beta}'^2]^{1/2}}
\]

\approx \frac{N n_0 \sigma^2}{x_{tp}(\tilde{\beta})} F\left(\frac{\pi}{2}, k = \frac{x_{tp}(\tilde{\beta})}{x_{tp}(\tilde{\beta})}\right)

(8.14)

where \(\tilde{\beta}_\gamma = \min\{\tilde{\beta}, \tilde{\beta}'\}\)

Here \(F\) is the elliptic function of the first kind, and \(K\) the complete function (of the first kind). In accord with standard notation \(^8\) for the argument, \(m = k^2\). One approach to solving this is to attempt a series solution as before. The first approximation to the solution is then

\[
F(\tilde{\beta}, z) = F_0(\tilde{\beta}) e^{-\alpha z}
\]

where

\[
F_0(\tilde{\beta}) = F(\tilde{\beta}, 0) = \frac{2}{\gamma} \tilde{\beta} P_0
\]

for the Lambertian source and parabolic slab being considered.

With

\[
F(\tilde{\beta}, z) = F_0(\tilde{\beta}) e^{-\alpha z} + F_1(\tilde{\beta}, z)
\]
we find
\[ F e^{\alpha z} = \frac{4Nn \sigma P}{\pi Y^2} \int_{n_{cl}}^{n_0} d\beta' \left( \frac{(m)}{(n_0^2 - \beta'^2)^{3/2}} \right) \]

Thus
\[ F(\tilde{\beta}, z) = \frac{2}{\gamma^2} P_0 e^{-\alpha z} \left[ \tilde{\beta} + \frac{2Nn \sigma}{\pi} zF(\tilde{\beta}) + \ldots \right] \]

where
\[ F(\tilde{\beta}) = \int_{n_{cl}}^{\tilde{\beta}} \frac{d\tilde{\beta}'}{(n_0^2 - \tilde{\beta}'^2)^{3/2}} K \left[ m = \frac{n_0^2 - \tilde{\beta}'^2}{n_0^2 - \tilde{\beta}^2} \right] \]
\[ + (n_0^2 - \tilde{\beta}^2)^{-1/2} \int_{\tilde{\beta}}^{n_0} d\tilde{\beta}' K \left[ m = \frac{n_0^2 - \tilde{\beta}'^2}{n_0^2 - \tilde{\beta}^2} \right] \]

To find \( F(\tilde{\beta}) \), an approximation to \( K(m) \) is needed, e.g. \( K(m) = \frac{1}{2\pi} (1 - m)^{-1/2} \) or
\[ K(m) = \frac{1}{2\pi} (1 + 0.3m + 0.3m^2) \]

The latter does not approach infinity as \( m \to 1 \) but this is an artifice anyway, being a consequence of the infinite intensity at a caustic in geometric optics. It is then straightforward to evaluate \( f(\tilde{\beta}) \) using it.

8.6 Conclusion

The formalism for scattering presented here further illustrates the wide applicability of geometric optics in waveguide problems. Alternate methods for scattering are less physically intuitive and more complicated.
References


