COVERAGE PROCESSES

BY

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The material contained in this thesis is the product of my own work.

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Introduction and Summary

Suppose that a collection of points is randomly distributed in a subregion A of k-dimensional Euclidean space $\mathbb{R}^k$ according to some random process $\mathcal{P}$. For technical reasons A is assumed to have strictly positive Lebesgue measure. In some systematic fashion, denote the location of these points by the random, rectangular co-ordinate vectors $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$. Independently of $\mathcal{P}$ produce statistically independent copies $S_1, S_2, \ldots$ of the random shape S. As set out in Matheron (1975), a simple axiomatic method of defining random and closed (or open) shapes is adopted in this project. This definition is useful because it does not lead to difficulties of "well definition". The shape S_i centred at $\underset{\sim}{X}_i$ is denoted by $\underset{\sim}{X}_i + S_i$: the set of points in S_i translated through $\underset{\sim}{X}_i$ (some authors write this as $\underset{\sim}{X}_i \oplus S_i$). The collection of random sets $\underset{\sim}{X}_i + S_i$, $i \geq 1$, is denoted by $\mathcal{C}$ and referred to as a stochastic coverage process.

The main aim of this project is to present an historical review of coverage processes and their applications, and to construct tests of the hypothesis of uniformity of the underlying point process controlling $\mathcal{C}$ when the exact locations of the points are unknown.

In many statistical texts a stochastic process $\{X\}$, indexed by a vector $\underset{\sim}{x} \in \mathbb{R}^k$, is called a random field. Suppose for all $\underset{\sim}{x}$, $X(\underset{\sim}{x})$ is a non negative integer-valued random variable. Then $\{X\}$ could be referred to as a coverage process, which is a more general definition than the one adopted here. It is, however, easy to construct random fields which do not fit into our definition of a coverage process. Naturally, difficulties do arise with such a restrictive definition. These shortcomings are considered briefly from a practical point of view later.

A special case of $\mathcal{C}$ occurs when $\mathcal{P}$ is a Poisson point process. Although the definition varies somewhat, in the applied sciences such a model is sometimes known as a mosaic process. The term binary mosaic has been used for the derived stochastic process $\{I\}$ taking the values

$$I(\underset{\sim}{x}) = \begin{cases} 1 & \text{if for all } i \geq 1 \text{ , } \underset{\sim}{x} \notin \underset{\sim}{X}_i + S_i \text{ , and} \\ 0 & \text{otherwise,} \end{cases}$$

for each $\underset{\sim}{x} \in \mathbb{R}^k$. That is, $I(\underset{\sim}{x})$ is the indicator of the event : $\underset{\sim}{x}$ is uncovered. Alternatively, theoreticians commonly use the term Bernoulli Model in place of binary mosaic. Serra (1982) has used this term. Several applications of coverage processes are now considered.

Suppose a virus, approximately circular in shape, enters an organism. To protect itself the organism releases "cigar shaped" antibodies, which attach themselves end-on to the virus. When attached, each antibody physically prevents a circular cap region on the surface of the virus from coming in contact with any of the host's cells. If enough antibodies are spread over the virus's surface, then it becomes impossible for the virus to infect any cell. Moran and Fazekas de St Groth (1962) proposed the following coverage model for the situation described above. On the surface of a sphere, which represents the virus, randomly and uniformly distribute n points. To model the protected areas on the virus's surface, centre circular caps of the same radius as the sphere at the random points. Protection of the organism from the virus corresponds to the complete coverage of the sphere by circular caps. The probability of completely covering a region A by n random shapes is one of the topics of discussion in chapter 1.

Another topic, also discussed in the same chapter, is percolation. In particle physics an interaction can take place between two particles if they are placed in close proximity to each other. Interactions can extend to groups of one or more particles, which we call a cluster. In particular, the event of an infinite number of interactions in at least one cluster is referred to as percolation. Seager and Pike (1974) have used a Bernoulli model to represent impurity conduction in a semiconductor. In the simplest situation the impure particles are represented by fixed radius spheres. An electron can pass from one particle to another only if their associated spheres overlap. Conduction in the semiconductor corresponds to percolation in this model. Even though a more complex model involving dependence between adjacent particle locations would be more realistic in this situation, the Bernoulli model has been used with a fair degree of success in practical situations.

In a slightly different vein, Diggle (1981) used a coverage model, or more specifically a binary mosaic, to analyse the growth pattern of heather in a field. Heather plants grow from seedlings reaching a maximum radius of about 50 cms. The branches of adjacent plants intermingle if the ground they occupy overlaps. Viewed from above, the heather plants may be represented by discs of random radii, the centres of which form a Poisson field. Perhaps this model is far too simplistic, as the analyses of Diggle suggest. In reality, we would expect the sizes of adjacent trees to be dependent on each other. However, the binary mosaic is advantageous in two respects : it is much simpler to define, and theoretically easier to analyse.

Vacancy in the coverage process C is defined to be the Lebesgue measure of the subregion of A not covered by any random set $\underset{\sim}{X}_i + S_i$, $i \geq 1$. In one respect vacancy is a geometric mean, for if A is partitioned into any countable collection of Lebesgue measurable sets, then the vacancy per unit area is just the weighted average of the vacancies per unit area within each set. As discussed previously, antibody protection from a virus may be represented as the complete coverage of a sphere by randomly placed spherical caps. It is clear that if any region A is completely covered by random sets, then vacancy is zero. For all the situations considered in this project the converse is also true, except on a set of probability zero. In Diggle's application of coverage to heather growing in a field, we may be interested in testing the hypothesis that the underlying Poisson process is uniform. If the original seedling locations are known, then a test based on nearest neighbours could be used. However, such information is commonly unavailable. In this case a test based on vacancy can be constructed. Indeed Hall (1984b) has constructed a test by partitioning A into a regular lattice of rectangles, measuring vacancies therein and using a "chi-squared" approach. Unfortunately, vacancy is only a simple summary statistic and can only measure certain aspects of a coverage process. The percolation example best illustrates this, for percolation may occur even when the proportion of covered content is zero (see for example the case of random line segments in the plane).

This project has been divided into two chapters. The first deals with the history and practical applications of stochastic coverage processes. Included in our discussion are sections on probabilities of complete coverage, the distribution of vacancy and a survey of applications. As illustrated in an earlier example, many of the applications assume that the underlying spatial point process P is uniform.

Hall (1984b) has presented a thorough analyses of tests of uniformity based on vacancy, in the case where P is a Poisson process and the shapes random radius spheres. In chapter 2 we present the corresponding theory for the case of a fixed number of bounded random shapes independently distributed in the region A . The power properties of a simple test based on vacancy, against a sequence of alternatives converging to the null hypothesis, are presented. As previously mentioned, vacancy is not the only statistic available. In the one-dimensional situation, where arcs of equal length are randomly distributed on the circumference of a circle, Hüsler (1982) and Hall (1983) have investigated the asymptotic properties of two statistics : the number of uncovered spacings, and the length of the largest spacing. Tests based on these two statistics are constructed and their "local power" is compared with that of the test based on vacancy.

Chapter I Historical Review and Applications

In this chapter we review the history and development of mathematical theory in stochastic coverage processes. Due to the large amount of literature, it would be difficult to present a comprehensive survey here. Rather, a fairly thorough review of a selection of topics is presented. The later part of this chapter is confined to practical applications of coverage processes.

As described in the introduction, a coverage process consists of a collection of independently distributed random shapes located at points, independently and randomly distributed throughout a region A . Thus, work relating to the smallest covering convex hull of the random points is excluded from our discussion. Research work may be divided into several broad topics. The chapter is made up of four sections, each section containing different topics.

In section 1 we review work concerned with finding the probability of completely covering the region A by a collection of random sets. When A is completely covered the content of the vacant, or uncovered, region is zero. The distribution of vacancy is considered in section 2. Continuum percolation occurs when the random sets overlap to form a connected set of "infinite size". We describe the theory of continuum percolation in subsection 3.1. Rather than derive the probability of complete coverage we could attempt to find the distribution of the smallest number of random sets required to completely cover A . This and related problems are referred to as sequential coverage problems in this project. They are the topic of discussion in subsection 3.2. In subsection 3.3 we review the history of particle counting problems. The theory covered has applications in determining the concentration of airborne dust particles from a microscope slide sample. The overlapping particles may clump resulting in a gross underestimate of the concentration. Finally, in section 4 a more thorough review of practical applications is given, including a subsection devoted to the military applications of coverage processes.

§1. Probability of Complete Coverage

The present section consists of a review of the literature on probabilities of complete coverage in a stochastic coverage process. In the first subsection the special case where the coverage region is one dimensional is dealt with. Being distinct from the one dimensional case, at least in the theoretical approach to the subject, discussion of the higher dimensional case is deferred until the next subsection.

§1.1. The One Dimensional Case

In one dimensional problems we shall sometimes take the fixed coverage region A to be the unit interval $[0,1]$. In other cases A is just the perimeter of a circle of circumference one. The two are essentially the same because the unit interval is obtained by unwrapping the perimeter of the circle. Wrapping the interval around the circle removes "edge effect" problems, which shall be discussed later. The covering shapes are intervals (or arcs with the same radius as the circle) each of length a . A collection of shapes is randomly placed on A and the probability of complete coverage is the probability that A is contained in the union of random shapes.

Perhaps one of the first authors who attempted to find the probability of complete coverage was Stevens (1939). Taking as evidence the large number of authors who later cited his work, it may also be considered as one of the most influential in the subject.

The geometrical construction investigated by Stevens is as follows. Fix the origin 0 at some point on the perimeter of the circle. Place $n-1$ points on the circle's perimeter uniformly and at random. Moving in an anticlockwise direction, place the first arc so that it begins at 0 , the second arc so that it begins at the first random point, and so on until n random arcs are placed on the circle.

Let k be the integer part of $1/a$. The number of uncovered gaps can range from zero, in the case of complete coverage, to at most k . If $a \leq n^{-1}$ then it is possible for all pairs of distinct arcs to not overlap. If $a > n^{-1}$ then this is impossible.

Stevens found the probability distribution of the number of gaps. In particular, he showed that the probability of complete coverage is

$$(1.1) \qquad 1 - \binom{n}{1}(1-a)^{n-1} + \ldots + (-1)^k \binom{n}{k} (1-ka)^{n-1}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-1)!} , \qquad \text{if} \quad 0 \leq k \leq n$$

and

$$\binom{n}{k} = 0 \qquad \text{otherwise.}$$

We shall prove Stevens' formula for the probability of complete coverage, using an argument similar to his.

Let $f(1)$ be the probability that there is a gap after the r'th arc, where $1 \leq r \leq n$. If there is a gap after the r'th arc then we can rotate the arcs $r+1, r+2, \ldots, n$ clockwise a distance a resulting in a legitimate configuration. That is, the order of the arcs remains the same. In the new configuration all $n-1$ random points lie in an arc starting at 0 and ending at a point measured anticlockwise a distance $1-a$ from 0. We can transform the new configuration back into the old by rotating the arcs $r+1, r+2, \ldots. n$ anticlockwise a distance a. Furthermore, the probability density function has the same value for the two configurations. Therefore, $f(1)$ equals the probability that $n-1$ random points lie inside a proportion $1-a$ of the perimeter. This probability is :

$$f(1) = (1-a)^{n-1} .$$

Using an analogous argument we can show that the probability that gaps occur after m specified arcs, where $1 \leq m \leq k$, is the same as the probability that all points lie inside an arc beginning at 0 and ending at a point measured a distance $1 - ma$ anticlockwise from 0.

Letting $f(m)$ be this probability we have

$$(1.2) \qquad f(m) = (1-ma)^{n-1}, \qquad 1 \leq m \leq k .$$

Now

$$\begin{aligned} &\Pr(A \text{ is completely covered}) \\ &= 1 - \Pr(A \text{ is not completely covered}) \\ (1.3) \quad &= 1 - \Pr(\cup_{r=1}^{n} \{\text{a gap occurs after the } r\text{'th arc}\}). \end{aligned}$$

We may use the inclusion-exclusion formula to obtain :

$$\begin{aligned} &\Pr(A \text{ is completely covered}) \\ (1.4) \quad &= 1 - \binom{n}{1}f(1) + \binom{n}{2}f(2) \ldots \pm \binom{n}{n}f(n) . \end{aligned}$$

Stevens' result at (1.1) is obtained by substituting (1.2) into (1.4) , and noting that $f(m) = 0$ when $m > k$ and $\binom{n}{m} = 0$ when $m > n$. $\Box$

Stevens used a similar technique to find the probability distribution of the number of gaps. In 1941 Fisher related Stevens' formula to a test of significance in harmonic analysis. We shall discuss this relationship in greater detail in section 4.

As mentioned previously it is possible to unwrap the perimeter of a circle to form the unit interval $[0,1]$. In covering $A = [0,1]$ by random intervals we must consider what to do with those intervals which overlap either 0 or 1 . We shall call this the problem of "edge effects". Stevens overcame this problem by introducing the part of the interval which overlaped an edge at the other end of the unit interval. A similar technique may be employed in higher dimensional space as illustrated by Gilbert (1965).

We shall investigate the technique in greater detail when discussing Gilbert's work in the next subsection. There is, however, another way of overcoming edge effects.

Let P be a Poisson process of uniform intensity λ on the real line $\mathbb{R}$. At each point of P centre an interval of length a . Coverage of A will occur if sufficiently many points are close to each other in a neighbourhood of A . This approach overcomes the problem of edge effects by forcing A to be a subset of a larger space in which the geometrical process is defined.

Domb (1943) obtained probabilities of complete coverage in the set-up described in the previous paragraph. His work also involved finding the distribution of the number of non-overlapping intervals, and the distribution of vacancy which we shall discuss in section 2 .

Domb was interested in the coverage of the interval $[0,y]$ by random intervals of length a . The number of non-overlapping intervals, including no overlap at the endpoints, can range from zero up to k , where k is the integer part of y/a . Each of these events has positive probability. When determining the probability of complete coverage there is no loss of generality in assuming $y = 1$. It is clear that the probability of completely covering $[0,y]$ is the same as the probability of covering $A = [0,1]$ by arcs of length a/y centred according to a Poisson process of intensity λy .

Even though this geometrical problem differs considerably from Stevens', there are many similarities. For example, there is a positive probability that all the intervals in A do not overlap. Furthermore, the number of gaps can range from zero in the case of complete coverage, up to at most $k+1$.

However, the most striking difference between the two problems is that the number of intervals intersecting A is random. To make explicit the difference between a fixed and random number of intervals with centres in A we shall use N , in the latter case, to denote the random number of points in A .

Let $F(x)$ be the probability that the covered portion of $[0,1]$ is less than or equal to x . Now F has discontinuities at $0, a, 2a,\ldots,ka$. This introduces a problem in defining the "density" of F . Domb overcomes this problem by using Dirac's δ-function notation. That is, if F has a discontinuity of size K at $x=k$ then its differential form has a δ-function singularity at $x=k$ and contains the term $K\,\delta(x-k)$. Domb obtained the probability of complete coverage by inverting the Laplace transform of the density of F . Unlike Stevens' proof, the technique involved is analytic rather than geometric. Domb obtained the following expression for the probability of complete coverage :

$$(1.5) \qquad 1 - e^{-\lambda a}(1+\lambda) + e^{-2\lambda a}\{\lambda(y-a) + \lambda^2(y-a)^2/2!\} - \ldots$$

$$+ (-1)^{k+1} e^{-(k+1)\lambda a}\{\lambda^k(y-ka)^k/k! + \lambda^{k+1}(y-ka)^{k+1}/(k+1)!\} .$$

Expression (1.5) does bear some resemblance to Stevens' probability. There is only one extra term in the series and the signs of adjacent terms oscillate.

Consider the situation where $\lambda = EN = n$. Then Stevens' and Domb's results are directly comparable.

Table 1.1 below gives the two probabilities for a large range of n and a .

TABLE 1.1

PS = STEVENS' PROBABILITY OF COMPLETE COVERAGE.
PD = DOMB'S PROBABILITY OF COMPLETELY COVERING THE UNIT INTERVAL
FOR COMPARISON PURPOSES LAMBDA HAS BEEN SET TO n.
NOTICE THAT a HAS BEEN CHOSEN SO THAT PS IS FIXED FOR EACH n.

		n					
		5	10	20	30	50	100
PS							
0.1	PD	0.142	0.135	0.128	0.124	0.120	0.114
	a	0.320	0.205	0.128	0.096	0.066	0.039
0.2	PD	0.214	0.216	0.216	0.214	0.213	0.210
	a	0.353	0.226	0.141	0.105	0.072	0.042
0.3	PD	0.280	0.292	0.298	0.300	0.302	0.302
	a	0.382	0.244	0.151	0.112	0.077	0.045
0.4	PD	0.344	0.366	0.380	0.386	0.391	0.395
	a	0.409	0.261	0.161	0.120	0.081	0.047
0.5	PD	0.410	0.442	0.463	0.472	0.480	0.488
	a	0.437	0.279	0.171	0.127	0.086	0.050
0.6	PD	0.480	0.521	0.549	0.561	0.572	0.582
	a	0.468	0.298	0.183	0.135	0.091	0.053
0.7	PD	0.555	0.606	0.640	0.654	0.667	0.679
	a	0.505	0.322	0.196	0.145	0.097	0.056
0.8	PD	0.641	0.699	0.737	0.752	0.766	0.779
	a	0.553	0.352	0.214	0.158	0.106	0.060
0.9	PD	0.742	0.807	0.845	0.860	0.872	0.883
	a	0.624	0.401	0.243	0.178	0.119	0.067

It is clear from the values shown in Table 1 that Domb's probability is larger than Stevens' when Stevens' probability is large, and vice-versa when Stevens' probability is small. Also when n is large there is not much difference between the two probabilities.

Stevens' and Domb's contributions constitute two important but contrasting approaches to finding probabilities of complete coverage. One contribution may be more suitable than the other in some practical situations. For example, it may be more realistic to have a random number of intervals intersecting A . In this case Domb's work would apply.

In other situations, n (or N) may be very large, and a small. In this case an approximation to Steven's probability of complete coverage can be obtained. The main issue here is how to define a limit theoretic approach because in reality both n and a are fixed. We can, however, view the observed event as a realization from a sequence of coverage processes, indexed by n . That is, the n'th coverage process consists of n arcs of angular radius a_n placed at random on the perimeter of a circle of circumference one. Notice that there is no constraint upon members of the sequence, such as mutual independence.

In the situation described above, let the vacancy, V_n , be the proportion of the perimeter remaining uncovered. Complete coverage of A occurs if and only if $V_n = 0$, while the probability of complete coverage is $P(V_n = 0)$.

Suppose that for each n we fix $P(V_n = 0) = \gamma$, where $0 < \gamma < 1$. Then a_n is determined by γ . Siegel (1979) obtained the following result which allows us to approximate a_n for fixed γ .

Theorem 1.1 (Siegel (1979))

Let $\beta = \log(1/\gamma)$. Then

$$a_n = \frac{1}{n} \log \left(\frac{n}{\beta}\right) + o\left(\frac{1}{n}\right)$$

as $n \to \infty$.

Conversely, for fixed n and a_n we may invert this result to obtain a first order approximation for γ . The approximation is :

(1.6) $$\gamma = P(V=0) = \exp(-n\, e^{-na}) ,$$

where the subscript n has been dropped for convenience.

Table 1.2 below lists the approximation at (1.6) for a range of γ and n . For fixed γ , a was obtained using the Newton-Raphson method on Stevens' formula for the probability of complete coverage.

We can see that for large values of γ the approximation tends to underestimate the true probability of complete coverage. This may be desirable in practical situations. Suppose we wish to find values of n and a that will give us a large probability of complete coverage, γ_o say. Using Siegel's approximation will ensure that $P(V=0) \geq \gamma_o$. In this respect the approximation may be considered as conservative. In other situations we may want to ensure that the probability of complete coverage is smaller than a given small positive constant. Again, Siegel's approximation leads to a conservative result.

It is clear, however, that Siegel's approximation is fairly crude especially when n is small.

TABLE 1.2

PS = STEVENS' PROBABILITY OF COMPLETE COVERAGE
PSL = SIEGEL'S APPROXIMATION

PS		n = 5	10	20	30	50	100
0.1	PSL	0.365	0.277	0.214	0.187	0.162	0.139
	a	0.320	0.205	0.128	0.096	0.066	0.039
0.2	PSL	0.426	0.353	0.300	0.277	0.255	0.234
	a	0.353	0.226	0.141	0.105	0.072	0.042
0.3	PSL	0.476	0.417	0.376	0.358	0.341	0.325
	a	0.382	0.244	0.151	0.112	0.077	0.045
0.4	PSL	0.523	0.479	0.448	0.436	0.424	0.414
	a	0.409	0.261	0.161	0.120	0.081	0.047
0.5	PSL	0.570	0.540	0.521	0.514	0.508	0.503
	a	0.437	0.279	0.171	0.127	0.086	0.050
0.6	PSL	0.618	0.602	0.595	0.594	0.593	0.594
	a	0.468	0.298	0.183	0.135	0.091	0.053
0.7	PSL	0.670	0.670	0.674	0.678	0.682	0.687
	a	0.505	0.322	0.196	0.145	0.097	0.056
0.8	PSL	0.730	0.744	0.759	0.767	0.775	0.784
	a	0.553	0.352	0.214	0.158	0.106	0.060
0.9	PSL	0.802	0.833	0.857	0.867	0.876	0.885
	a	0.624	0.401	0.243	0.178	0.119	0.067

In the following subsection we investigage the work of authors who found probabilities, and approximations to probabilities, of complete coverage in the higher dimensional case.

In recent literature on the subject of probabilities of complete coverage, several authors have attempted to extend the results of Stevens (1939) to the case of random arc lengths.

Siegel and Holst (1982) considered the following model. Suppose n arcs are uniformly and independently distributed on a circle A, of circumference one. The arc lengths are assumed to be independently drawn from a distribution F on $[0,1]$. Furthermore, assume that the arc lengths and the distribution controlling the arcs' locations are independent.

We can see that this model contains Stevens' as a special case. Just allow F to be the step function :

$$(1.7) \qquad F(x) = \begin{cases} 0 & \text{when } x < a \text{ , and} \\ 1 & \text{when } x \geq a \text{ ,} \end{cases}$$

where $a \in [0,1)$. In other words, each random arc is of length a with probability one.

Let $1 \leq k \leq n$, $\underset{\sim}{u}_k' = (u_1,\ldots,u_k)$,

$$\xi_o = 1 \quad , \quad \text{and ,}$$

$$\xi_k = (k-1)! \int_{\{\Sigma_{i=1}^{k} u_i = 1\}} \{\prod_{i=1}^{k} F(u_i)\}\{\sum_{j=1}^{k} \int_0^{u_j} F(\nu)d\nu\}^{n-k} d\underset{\sim}{u} \text{ .}$$

Siegel and Holst have shown that the probability of complete coverage is :

$$\Pr(A \text{ is completely covered})$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \xi_k \text{ .}$$

This simplifies to Stevens' result (1.1) when F is the step function given at (1.7). Following Stevens, Siegel and Holst obtained the probability of exactly m gaps occurring on the circle's perimeter. Indeed, if $0 \le m \le n$,

$$P(\text{exactly } m \text{ gaps}) = \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{k-m} (-1)^{k-m} \xi_k .$$

Of course, $m=0$ if and only if the circle is completely covered. Let us unwrap the circle onto the interval $[0,1)$. Jewell and Romano (1982) have generalized the results of Siegel and Holst, on the probability of complete coverage, to a situation where the midpoint x and length ℓ of each arc follow a bivariate distribution, F, on $[0,1) \times [0,1)$, which is continuous in x.

It is assumed that the n arcs are independently drawn from F. However, we do not require that the length and location of an arc be independent. Jewell and Romano obtained the required probability by showing that the event of complete coverage is equivalent to the event that a random convex hull of n independent points from a bivariate distribution G, determined by F, contains a disc.

The convex hull of a sample of size n is the smallest-area convex set containing all n points. Necessarily the set is the interior of a convex polygon. The vertices of the polygon are a subset of the n points. We shall describe in a simple example how a random convex hull may be used to determine the probability of complete coverage.

Suppose that $\ell = \frac{1}{2}$ with probability one, so that each arc is a semicircle. Then, complete coverage is determined by the location of midpoints around the circle. Let $\underset{\sim}{O}$ be the centre of the circle. If all n midpoints lie on one side of a line passing through $\underset{\sim}{O}$, then the circle is <u>not</u> completely covered. Conversely, if the circle is not completely covered, then there is a line passing through $\underset{\sim}{O}$ with all midpoints on one side of it. Clearly, in this situation, the convex hull of the n midpoints does not contain $\underset{\sim}{O}$. Therefore, the circle is completely covered if and only if the convex hull of the n midpoints contains $\underset{\sim}{O}$. Without too much difficulty, it may be shown that the probability of this event is :

$$P = 1 - n\left[\int_0^{\frac{1}{2}} \{F(u) + 1 - F(u + \tfrac{1}{2})\}^{n-1}\,du \right.$$

$$\left. + \int_{\frac{1}{2}}^{1} \{F(u) - F(u - \tfrac{1}{2})\}^{n-1}\,du \right] ,$$

where $F(u) = F\big|_{x=u,\ \ell=\frac{1}{2}}$.

In general, however, Jewall and Romano have obtained formulae for the probability of complete coverage of the circle. The solution is not given here as its form is very complex.

§1.2 The Higher Dimensional Case

We shall call the coverage set A k-dimensional if there exists a one-to-one topological transformation, T , such that T(A) has non-zero content in k-dimensional space. It is possible for A to be a subset of a higher dimensional space. For example, the perimeter of a circle of circumference one is a subset of $\mathbb{R}^2$. However, by unwrapping the perimeter we may map the circumference into a subset of $\mathbb{R}^1$. Therefore, Stevens' (1939) problem is one-dimensional. In this project we shall never explicitly state the form of T because in most cases the dimensionality will be obvious.

A coverage process consists of randomly placed sets on A . As in the one-dimensional case, A is completely covered if it is contained in the union of the random sets.

To see why probabilities of complete coverage are often more difficult to find in the higher dimensional case, consider the following argument. Take a 1-dimensional section through A . If A is covered then so is any section. However, in order that A is completely covered we require that <u>all</u> sections through A be completely covered. There are uncountably many such sections and even though it may be possible to find the probability that any one is covered, it would be much more difficult to find the probability that all are covered simultaneously. Therefore, it is not at all surprising to find that many of the proofs in the higher dimensional case involve ingeneous geometric and analytic arguments. To illustrate this point we shall shortly present a proof from Gilbert (1965). Gilbert found the exact probability that n randomly placed hemispheres covered a sphere.

Two early authors to obtain significant results in higher dimensional coverage problem were Moran and Fazekas de St Groth (1962). Suppose n points are independently and uniformly distributed on the surface of a sphere. On the surface of the sphere are placed n circular caps, each subtending an angle 2α, their poles coinciding with the random points. Moran and Fazekas de St Groth used this model to solve a problem which arises in virology.

Suppose an approximately spherical virus enters an organism. Cigar shaped antibodies attach themselves end-on to the virus. Each antibody prevents a circular cap area on the surface of the virus from attacking the host's cells. If enough antibodies are spread over the virus then it will be impossible for the virus to infect any cell. This situation corresponds to complete coverage of the sphere by circular caps, each of angular radius $\alpha = 53°$. We shall again refer to this interesting application of coverage in section 4. In the following paragraphs we outline the heuristic method used by Moran and Fazekas de St Groth to find the probability of complete coverage.

Suppose the sphere has radius $(4\pi)^{-1/2}$ and surface area 1. The vacancy V is the surface area of the uncovered portion of the sphere. Let $P = P(V=0)$. When $V = 0$ we say that the sphere is completely covered. The distribution of V, given $V > 0$, is continuous with first and second moments μ_1 and μ_2, say. The probability that the sphere is not completely covered can be shown to be

$$1 - P = \frac{\mu_2}{\mu_1^2} \frac{\{E(V)\}^2}{E(V^2)} .$$

It is possible to derive closed form expressions for the first two moments of vacancy. In section 2 we discuss a general method of finding moments of vacancy developed by Robbins (1945) . Moran and Fazekas de St Groth showed that

$$E(V) = (1 - a)^n$$

and

$$E(V^2) = \tfrac{1}{2} \int_0^{\pi} \{1 - f(\phi)\} \sin(\phi) d\phi \quad ,$$

where a is the area of a circular cap, and $f(\phi)$ is the area of the union of two circular caps separated by angle ϕ . The moments of the continuous part of V are found in the following way.

When a is small n will be large before complete coverage occurs. If the sphere is almost completely covered, then there will be a number of small uncovered regions. The areas of the uncovered regions will be approximately independent random variables. Furthermore, it is not unreasonable to assume that their number has an approximate Poisson distribution.

Since the uncovered regions are small, their perimeters will consist of almost straight lines. The perimeter of a given region can be modelled by a random network of lines in the plane. This random line network is a special case of a Poisson plane process. See, for example, Miles (1970 a,b, 1971 and 1972) . We shall discuss Miles' work in greater detail later on in this subsection. Moran and

Fazekas de St Groth were able to utilize results on the expectation and variance of the area of regions generated by random line networks to obtain approximations for μ_1 and μ_2 , and hence P .

In the virus example discussed above, this approach led to the following approximation to the probability of complete coverage :

$$P \simeq \exp\{-\frac{\pi^2}{2}(2\,EV(1 + 0.025666\,^2))^{-1}\} .$$

The analytic techniques used by Moran and Fazekas de St Groth differ greatly from the methods employed in the one-dimensional case. For example, the possibility of ordering the intervals in the one-dimensional case allowed Stevens to obtain a simple formula for the exact probability of complete coverage. There is no simple way of ordering sets in higher dimensional problems.

Perhaps the major concern with Moran and Fazekas de St Groth's work is the heuristic nature of some of their proofs. To a great extent Gilbert (1965) overcame these problems. He also realized that an exact expression for the probability of completely covering a sphere could be obtained when $\alpha = 90°$.

When $\alpha = 90°$ the spherical caps become hemispheres. Using an argument similar to Gilbert's we shall prove that the probability of complete coverage is

$$(1.8) \qquad 1 - (n^2 - n + 2)2^{-n} .$$

In an earlier work Wendel (1962) showed that if n points are randomly scattered on the surface of a unit sphere in k dimensional space, then the probability that all the points lie in some hemisphere is :

(1.9) $$2^{-n+1} \sum_{j=0}^{k-1} \binom{n-1}{j} .$$

The n hemispheres will be randomly and uniformly located on the surface of the sphere if the poles of the hemisphere are located at independent, uniformly distributed points. We shall show that the events : n points lying inside some hemisphere, and the sphere is not completely covered are the same. It will follow that the probability of complete coverage is given by (1.9) .

Without loss of generality assume that the spheres is centered at the origin $\underset{\sim}{0}$. Suppose the n points lie inside a hemisphere with pole $\underset{\sim}{z}$. Then the point $-\underset{\sim}{z}$, also on the surface of the sphere, is not covered. Conversely, if there exists a point $\underset{\sim}{z}$ which is not covered, then the hemisphere with pole at $-\underset{\sim}{z}$ covers all n points. Therefore, the two events are the same. In three dimensions $k=3$ and (1.9) specializes down to (1.8) . $\Box$

A crossing is defined to be the intersection of the boundaries of two caps. The crossing is said to be covered if any random cap, excluding the two which define it, contain the crossing. Let $G(n)$ be the expected number of uncovered crossings and $U(n)$ the expected number of uncovered crossings when not all crossings are covered.

Gilbert showed that $G(n) = 4n(n-1)a(1-a)^{n-1}$, and that $G(n)$ and $U(n)$ are related by

$$U(n) = (1 - P)\ G(n)\ .$$

Hence

$$(1.10) \qquad P = 1 - \{G(n)/U(n)\}\ .$$

However, the usefulness of (1.10) lies in the fact that $U(n) \to 4$ as $n \to \infty$. Gilbert proved this for $\alpha = 90°$. Miles (1969) later established this for all values of α satisfying $0 < \alpha \leq 90°$. The asymptotic approximation derived from this result and expression (1.10) is

$$P \simeq 1 - n(n-1)a(1-a)^{n-1}\ .$$

So far we have only considered coverage probabilities for circular caps placed at random on the surface of a sphere. Miles obtained generalizations of Gilbert's results for spherical polygons which are not only randomly located on the surface of a sphere, but are also randomly rotated according to a uniform distribution. This may be defined in strict terms as follows.

Let S be an arbitrary set on the surface of a sphere A. Suppose the surface area of A is 1. A random copy of S is made so that

(i) an arbitrary point in S goes to a point which is uniformly distributed on the sphere's surface, and independently of this

(ii) the orientation of S about this point is uniform.

Let $S_1, S_2, \ldots, S_n$ be n independent random copies of S. For an arbitrary point $\underset{\sim}{x}$ on A define $H(\underset{\sim}{x})$ as

the number of S_i , $1 \leq i \leq n$, which cover $\underset{\sim}{x}$,
$\overline{H} = \sup_{\underset{\sim}{x} \in A} H(\underset{\sim}{x})$ and $\underline{H} = \inf_{\underset{\sim}{x} \in A} H(\underset{\sim}{x})$. We may interpret $\underline{H}$ and $\overline{H}$ as the number of coverings on the least and most covered regions of the sphere's surface, respectively. It is clear that the sphere is completely covered if and only if $\underline{H} > 0$.

Miles generalized Gilbert's results in two different ways. Firstly, as previously mentioned, he obtained asymptotic approximations for the probability of complete coverage for spherical polygonal shapes. Secondly, he obtained approximations for $P(\underline{H} = m)$ and $P(\overline{H} = n-m)$ for fixed values of m. Miles' result is set out in Theorem 1.2 below.

<u>Theorem 1.2</u> (Miles (1969))

<u>Let the area of</u> S <u>be</u> a , <u>and its perimeter be</u> b . <u>Then, as</u> $a \to \infty$, <u>both</u>

$$P(\underline{H}=m) \text{ and } P(\underline{H} \leq m) \sim \binom{n}{m+2}(m+1)(m+2)b^2 \frac{a^m(1-a)^{n-m-2}}{4\pi} ,$$

<u>while both</u>

$$P(\overline{H} \geq n-m) \text{ and } P(\overline{H}=n-m) \sim \binom{n}{m+2} (m+1)(m+2)b^2 \frac{a^{n-m-2}(1-a)^m}{4\pi} .$$

<u>where</u> $m \geq 0$.

Theorem 1.2 continues to hold for shapes which can be approximated by spherical polygons. A circular cap is one such shape.

As mentioned in subsection 1.1, it is possible to construct a coverage process on the unit interval rather than the perimeter of a circle. Likewise, in two dimensions, the coverage region could be a rectangle rather than the surface of a sphere. However, when the coverage region is rectangular, edge effect problems occur. Miles used a simple method which overcame edge effects. Essentially it converts the rectangle into a "topological torus". Since the topological torus is of importance in the theoretical section of this project we shall describe it in detail here.

Suppose the rectangle, A, has sides of lengths ℓ_1 and ℓ_2. Orientate A so that its left hand corner is at the origin and its sides are parallel to the co-ordinate axes. For simplicity, assume the basic shape S is bounded by a circle with radius no greater than $\min(\ell_1, \ell_2)$. A random copy of S is defined so that

(i) an arbitrary point of S is at a uniform random point in A, and
(ii) the orientation of S about this point is uniform.

For illustrative purposes we shall only deal with one random copy of S. The generalization to n independent random copies shall be obvious.

Let S_1 be a random copy of S. Diagram 1.1 shows a typical S_1 for a non-spherical basic shape. Now, translate S_1 to eight different positions by moving it the width of A left and right, the height of A up and down and any other combination of these movements. Diagram 1.1 also illustrates these translations. The intersection of each translated shape, and S_1, with A gives us the final mutated image of one random copy of S. Diagram 1.2 illustrates the "topological torus" for a disc in two

Diagram 1.1:

How the "torus topology" is applied to a shape S_1 .

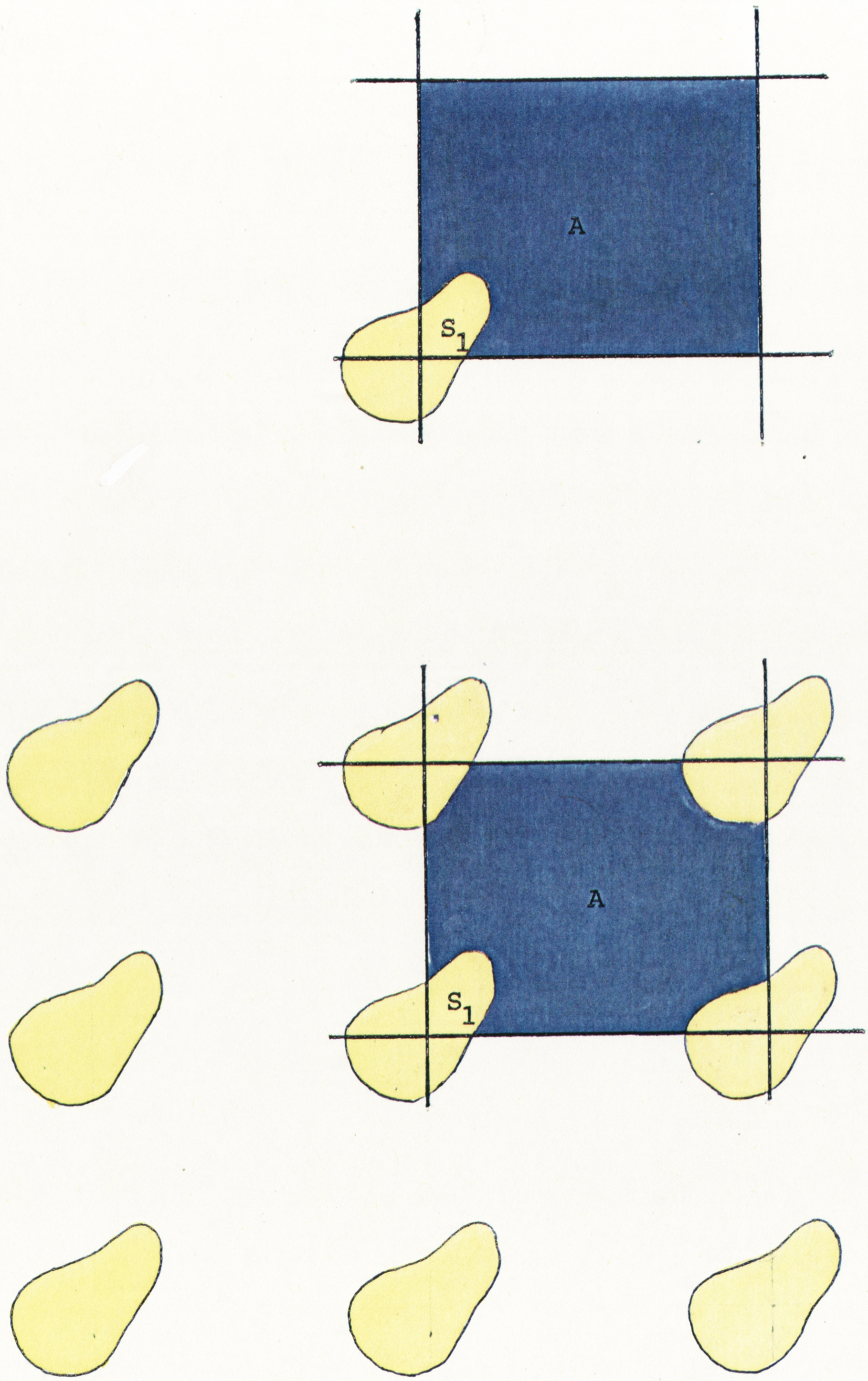


Diagram 1.2:

The Torus topology in 2-dimensions.

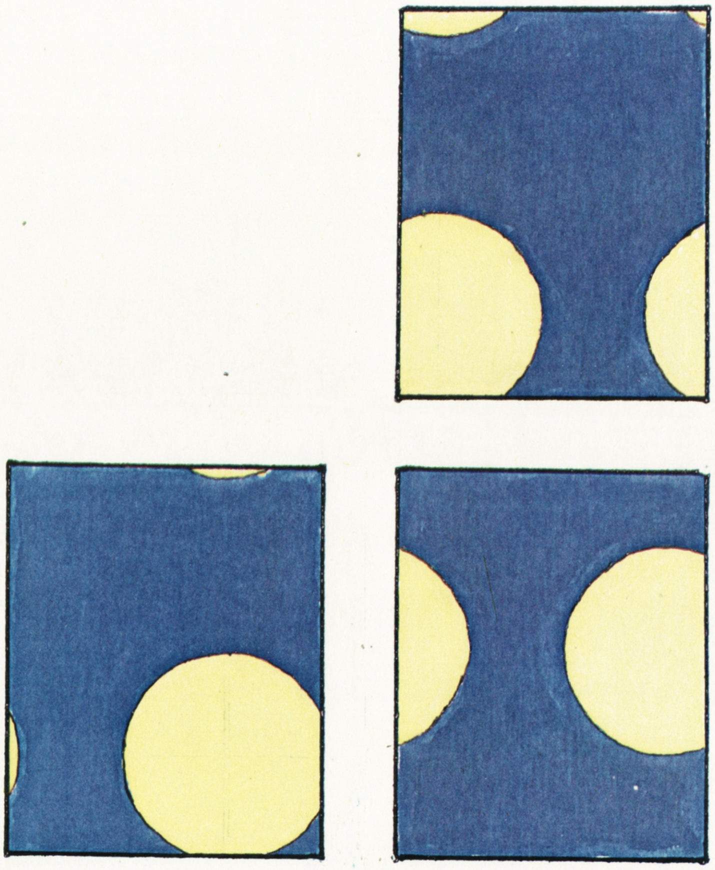
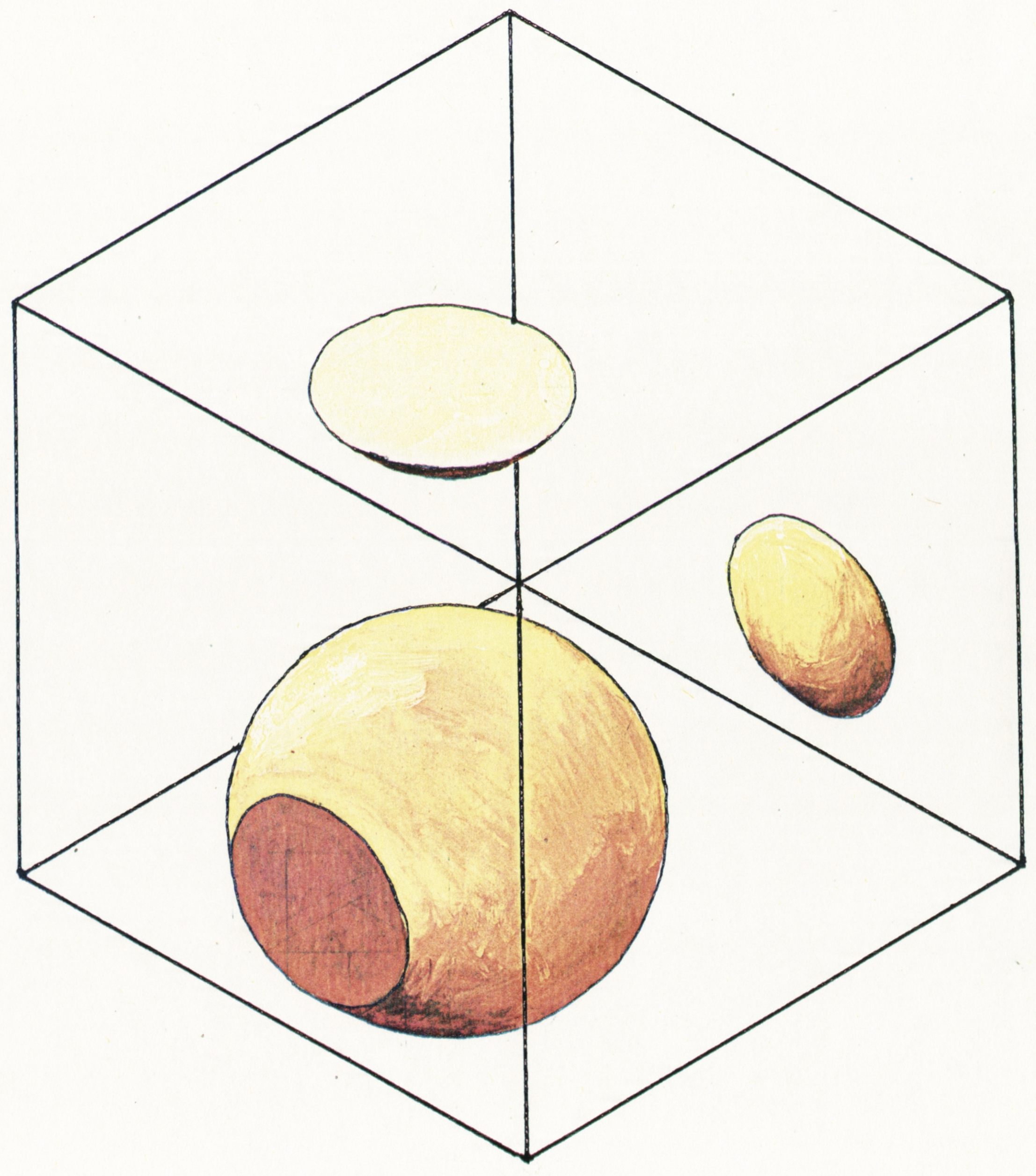


Diagram 1.3

The torus topology in 3-dimensions.



dimensions.

It is possible to extend the topological torus technique to higher dimensional space. The method is very similar to the two dimensional case described above, so we refer the interested reader to Gilbert's (1965) paper. Diagram 1.3 shows the effect of the torus topology on a sphere inside a cube.

In a k-dimensional rectangle, $\underline{H}$ and $\overline{H}$ may be defined in a similar way to before. Imposing a torus topology on the rectangle, Miles obtained results which described the asymptotic behaviour of $P(\underline{H} = m)$ and $P(\overline{H} = n-m)$ as n tends to infinity. These results are in essence generalizations of the theory for the spherical case. (There are some minor complexities introduced in defining a uniform random rotation in greater than two dimensions.) For the types of random covering shapes Miles considered, the following approximation to the probability of completely covering the rectangle was obtained :

$$P \simeq 1 - \frac{\{\Gamma(\frac{1}{2}k)\}^k}{\pi^{\frac{1}{2}} 2^k \{\Gamma(\frac{1}{2}(k+1))\}^{k-1}} \binom{n}{k} \frac{b^k(\|A\| - a)^{n-k}}{\|A\|^{n-1}} ,$$

where a is the content of S, b its surface area, and $\|A\|$ is the content of A.

Let us now explore the ideas discussed by Moran and Fazekas de St Groth (1962) on the size and structure of the small vacant regions obtained when a sphere is almost completely covered by random caps.

As Miles had done in his 1969 paper we can generalize the coverage problem into a k dimensional setting. Suppose, however, the sequence of "random shapes" $\{S_i\}$ are now centred at the points of a Poisson process of uniform intensity, λ, on $\mathbb{R}^k$. (In Chapter 2 of this project we shall describe in a theoretical manner what is meant by the term "random shape".) The structure of the uncovered gaps can be approximated by convex regions formed by a Poisson field of $(k-1)$-dimensional planes in $\mathbb{R}^k$, which we now describe.

Let $\xi_1, \xi_2, \ldots$ be points of a Poisson process on $[0,\infty)$ and $\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \ldots$ be unit vectors uniformly and independently distributed on the surface of the k-dimensional unit sphere centred at $\underset{\sim}{0}$. Let π_i be the k-1 dimensional plane whose normal to the origin has length ξ_i and inclination $\underset{\sim}{\theta}_i$. The Poisson field of k-1 dimensional planes in $\mathbb{R}^k$ is formed by the planes π_i, $i \geq 1$. The planes π_i, $i \geq 1$, partition $\mathbb{R}^k$ into a sequence of convex polygons, each identically distributed.

Miles (1970 a,b, 1971 and 1972) has investigated in detail Poisson plane processes, but in a more general context. The random planes, or flats as described by Miles, are s-dimensional where $1 \leq s \leq k$. A stochastic flat process of intensity $\mu > 0$ in $\mathbb{R}^k$ is defined so that it is stochastically invariant under any translation or rotation. Let X be an arbitrary (k-s)-dimensional subset in $\mathbb{R}^k$ with (k-s)-dimensional content $\|X\|$. The stochastic process is such that the number of s-flats intersecting X has a Poisson distribution with mean $\mu\|X\|$.

When $s = k-1$ the stochastic flat process of Miles reduces to the Poisson plane process used to describe the small vacant regions in a coverage process. When $s = 0$ the stochastic flat process is just a Poisson point process in $\mathbb{R}^k$. We may therefore view the stochastic flat processes as a natural generalization of a Poisson process.

§2 Vacancy

In the previous section we spoke of the coverage region, A , as a fixed subset of k-dimensional space. The coverage process consists of randomly placing shapes upon A . Let us define an indicator function I as follows. For each $\underset{\sim}{x} \in \mathbb{R}^k$, let

$$I(x) = \begin{cases} 1 & \text{if } \underset{\sim}{x} \text{ is not covered by any random shape, and} \\ 0 & \text{otherwise.} \end{cases}$$

The vacancy, or the content of the uncoverd region in A is

$$(2.1) \qquad V \equiv \int_A I(\underset{\sim}{x})\, d\underset{\sim}{x} \, .$$

In all situations investigated in this project, the random shapes and A are always Lebesgue measurable subsets of $\mathbb{R}^k$. Thus, if we interpret the right hand side of (2.1) as a Lebesgue integral and assume that the content of A is finite, then V is well defined as a random variable.

In some situations, it may be more convenient to investigate the properties of the Lebesgue measure of the covered region in A , defined by

$$C \equiv \int_A \{1 - I(\underset{\sim}{x})\}\, d\underset{\sim}{x} \, .$$

Notice that $C + V = \|A\|$, whether the content of A is finite or not.

Suppose that A is completely covered by random shapes. Then $I(\underset{\sim}{x}) = 0$ for all $\underset{\sim}{x}$ in A , and $V = 0$. However, the converse is not true; for it is possible that a set of Lebesgue measure zero could be left uncovered. Such problems do not occur frequently in practice because the probability of a zero measure set, except in artificial examples, is zero.

A large amount of work on vacancy in stochastic coverage processes is concerned with either finding the moments of vacancy, the exact distribution of vacancy or approximations to the distribution of vacancy through limit theoretic methods. Any particular research paper usually deals with more than one of these topics because the results in one area are quite likely to depend on the results in another. Thus, it was decided not to discuss each topic in a different subsection, but rather deal with the work in a chronological fashion. However, this will tend to segregate the topics, as the early work is primarily concerned with moments and the exact distribution of vacancy, while more recent work has concentrated on the limit theoretic approach. As was the case for probabilities of complete coverage, exact results for the distribution of vacancy are usually confined to the simplest geometrical models. The reasons for this are made clear by the following.

If the random process controlling the location of shapes in A is such that no two shapes can overlap, or no shape can intersect the boundary of A, then the vacancy is solely determined by the content of A and of each shape. On the other hand, if the random shapes are allowed to overlap, then vacancy depends on the location of shapes. Thus, in the one dimensional situation, vacancy can be described in terms of the starting position of each interval and their lengths. However, in the higher dimensional situation there is the difficulty of arranging the random shapes in order, and hence of simply describing vacancy. Domb (1943) was one of the first authors to attempt to find the exact distribution of vacancy in a one dimensional situation.

The coverage process he studied was described in subsection 1.2. We repeat the description here for convenience.

Intervals of length a are centred at the points of a Poisson process of uniform intensity λ. Domb was interested in vacancy in the interval $[0,y]$. As previously noted, it is sufficient to consider the coverage of the unit interval $A = [0,1]$.

Let F be the distribution function of the covered portion, C, in A. Let $x > 0$ and $[x]$ be the integer part of x. It is possible for A to contain r non-overlapping intervals none of which overlap the endpoints of A, where $r = 0,1,2,\ldots,[1/a]$. Therefore, F possesses discontinuities at $r = o,a,2a,\ldots,[\frac{1}{a}]a$. Using a Dirac δ-function notation, Domb wrote down the Laplace transform of the "density of F", which may be expanded to find an exact expression for F. Due to the complexity of the final result, he never did this. However, Domb showed that the size of the r'th discontinuity to be :

$$P(C = ra) = 2^r(1-ra)^r \, e^{-\lambda(a+1)}/r!$$

where $r = 0,1,\ldots,[1/a]$.

Even in the one dimensional situation, studied by Domb, exact expressions for the distribution of vacancy are difficult to obtain. However, there is a simple technique available for finding the moments of vacancy, which easily generalizes to the higher dimensional case. Robbins (1944) expressed the problem in terms of the moments of the measure of a random set.

Let Ω be the set of all possible Lebesgue measurable subset of $\mathbb{R}^k$. A probability distribution of random sets, δ , is defined so that the probability that $x \in \Omega$ belongs to a δ-measurable subset, S , of Ω is :

$$Pr(X \in S) = \int_{\Omega} \chi_s(X) \; d\delta(X) \; ,$$

where χ_s is the characteristic function of S .

Let μ be Lebesgue measure in $\mathbb{R}^k$. The measure of X may be re-written as :

$$\mu(X) = \int_{\Omega} I_X(x) \; d\mu(x) \; , \tag{2.2}$$

where

$$I_X(x) = \begin{cases} 1 & \text{if } x \in X \text{, and} \\ 0 & \text{if } x \notin X \text{.} \end{cases}$$

Taking the expectation inside the integral on the right hand side of (2.2) gives

$$E\{\mu(X)\} = \int_{\Omega} E\{I_X(x)\} \; d\mu(x)$$

$$= \int_{\Omega} P(x \in X) \; d\mu(x)$$

as the expected measure of X . Using a similar method it is possible to show that

$$(2.3)\quad E\{\mu^m(X)\}' = \int_\Omega P(x_1 \in X,\ x_2 \in X, \ldots, x_m \in X)\ d\mu(x_1) \ldots d\mu(x_m)$$

where $m \geq 1$. We may easily apply Robbins' results to vacancy; for vacancy is just the measure of the random uncovered region in A . Robbins applied his results to find the expectation and variance of vacancy in an interval $[0,y]$ covered by n randomly and uniformly located intervals of length a . To overcome edge effects he extended the distribution controlling the location of intervals beyond the endpoints of $[0,y]$. Bronowski and Neyman (1945) used a similar method to overcome edge effects in a related two dimensional set-up.

Let N be a non-negative integer-valued random variable with probability generating function ϕ ,defined by :

$$\phi(s) = \Sigma_{n=0}^{\infty} s^n P(N=n) .$$

Suppose that the coverage region, A , is rectangular with side lengths b and c . Now N points are uniformly and independently distributed in a concentric rectangle, A' , of side lengths $b+2\gamma$, and $c+2\gamma$, where $\gamma > 0$. Let rectangles of side lengths α and β , where $\alpha < 2\gamma$ and $\beta < 2\gamma$, be centered at the random points. Each random rectangle is oreinted in the same direction so that their sides are parallel to those of A . Finally, write $a = \alpha\beta$ for the area of each random rectangle and $||A|| = bc$ for the area of A . The conditions $\alpha < 2\gamma$ and $\beta < 2\gamma$ makes it possible for a random rectangle to not intersect A . See diagram 2.1 below. In this sense the edges of A' do not affect the coverage of A . This is sufficient to remove edge effects from the problem.

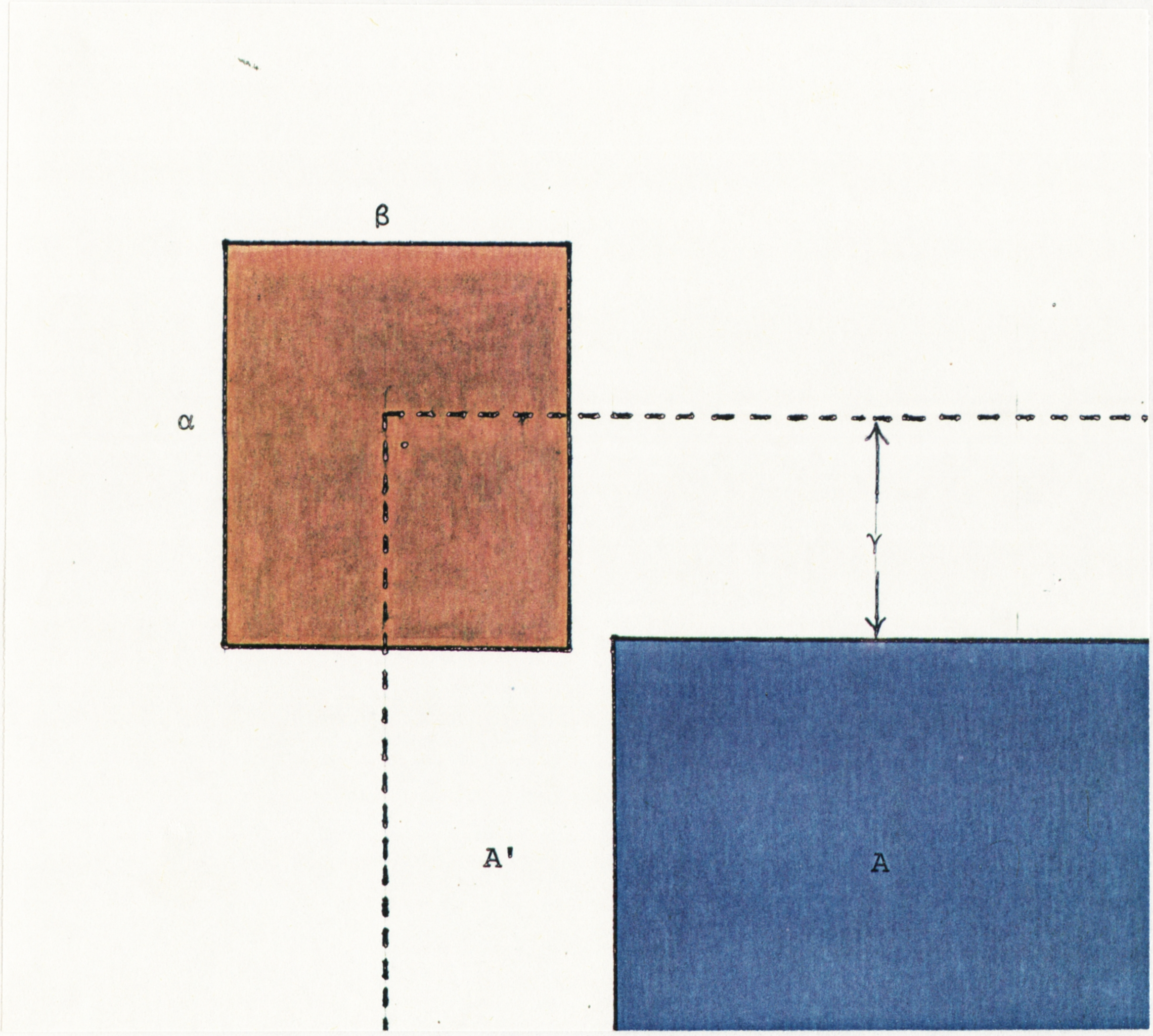
Under the above conditions, Bronowski and Neyman showed that the first moment of vacancy in A is :

$$E(V) = \|A\| \ \phi(1 - a/\|A'\|) \tag{2.4}$$

and a much more complex expression holds for the variance of V. Expression (2.4) includes the cases where the centres are distributed according to a Poisson process in A' , and when there is a fixed number of centres in A' .

Diagram 2.1 :

A rectangle centred close to the boundary of A' does not intersect A.



The method of finding the moments of vacancy used by Bronowski and Neyman differs from the general methods of finding moments of the measure of a random set suggested by Robbins (1944). Bronowski and Neyman considered vacancy on a sequence of lattice points which, if made sufficiently fine, approximated vacancy, and hence expected vacancy, in the region of A. However, as was previously shown by Robbins, the moments of vacancy can be obtained through the integral expression (2.3).

In a subsequent paper, Robbins (1945) used the methods he developed in 1944 to generalize the results of Bronowski and Neyman to a k-dimensional situation. Essentially, N k-dimensional shapes are independently and uniformly distributed over a region, A', which contains the rectangular coverage region, A. Each random rectangle is oriented in the same direction so that their sides are parallel to the sides of A. The region A' is suitably chosen to remove edge effect problems. Again, a probability generating function is introduced for N.

In the abovedescribed situation, Robbins obtained expressions for mean and variance of vacancy, which involved considerably less technical derivations than used by Bronowski and Neyman.

Robbins also found expressions for the mean and variance of vacancy when N circles of area a were independently and uniformly distributed on A.

For rectangular shapes the expression for variance depends on the area of the intersection of two shapes whose centres are separated by a vector $\underset{\sim}{x}$. However, for circular shapes, the area of intersection depends only on the distance that separates the two centres, $r = |\underset{\sim}{x}|$, say. Thus, the variance expression for circular shapes is somewhat simpler than for rectangular shapes, and, indeed, for more general random covering shapes. In the later sections of this project we shall develop formulae for the variance of vacancy in a very general framework.

The work of Bronowski and Neyman (1945) and Robbins (1944, 1945) suggests that formulae for expectation and variance can be obtained for a more general class of shapes, which have been distributed at random throughout a given region, A'. Garwood (1947) proved this to be the case for any shape S bounded by a simple closed curve.

The context in which the theory was developed, was in application to a bombing problem. The area destroyed by a single bomb is represented by a circular region of radius r on the ground. A building is represented by a rectangular region A. The amount of the building destroyed by a cluster of bombs is just the area in A covered by n randomly positioned circles. We shall discuss this application in greater detail in section 4 of the present chapter.

Garwood investigated vacancy for a variety of covering shapes and regions. We shall, however, describe the general coverage process in which each special case can be represented. Let A be the interior of a simple closed curve and S the interior of another closed curve. Fix a point $\underset{\sim}{c}$ in S and call it the centre. Let A' be a region of bounded area and f a density function on $A' \times [0,2\pi]$. Let $(\underset{\sim}{X},\Theta)$ be a random element from a distribution with density f, and N a non-negative integer-valued random variable independent of $(\underset{\sim}{X},\theta)$. A random copy of S is obtained by rotating S through θ about $\underset{\sim}{c}$ and then translating the new set through $\underset{\sim}{X}$. Let $S_1, S_2, \ldots$ be a sequence of independent random copies of S. The coverage process is formed by the random shapes $S_1, \ldots, S_N$, and we are interested in the vacancy V in A.

According to Robbins (1944), to obtain the first moment of vacancy we require the probability that $\underset{\sim}{x} \in A$ is not covered by a single random shape. Let $\overline{S}(\underset{\sim}{y},\theta)$ be the set S centred at $\underset{\sim}{y}$ and rotated through $180° + \theta$. Conditional on it being inclined at θ, S_1 does not cover $\underset{\sim}{x}$ if its centre does not fall in the region $A' \setminus \overline{S}(\underset{\sim}{x},\theta)$. Therefore, the probability that S_1 does not cover $\underset{\sim}{x}$ is :

$$p(\underset{\sim}{x}) = \int_0^{\pi} \int_{A' \setminus \overline{S}(\underset{\sim}{x},\theta)} f(\underset{\sim}{x},\theta) d\underset{\sim}{x}\, d\theta .$$

If N has probability generating function ϕ, the probability that $\underset{\sim}{x}$ belongs to the vacant region in A is $\phi(p(\underset{\sim}{x}))$. Therefore, the expected vacancy is

$$E(V) = \int_A \phi\{p(\underset{\sim}{x})\}\, d\underset{\sim}{x} .$$

Garwood also obtained the following formula for the second moment of vacancy using an argument similar to that above :

$$E(V^2) = \int_{A^2} \phi\{p(\underset{\sim}{x},\underset{\sim}{y})\}\, d\underset{\sim}{x}\, d\underset{\sim}{y}$$

where $p(\underset{\sim}{x},\underset{\sim}{y})$ is the probability that neither $\underset{\sim}{x}$ nor $\underset{\sim}{y}$ is covered by a single random shape.

Garwood applied these results to the following problems :

(i) Uniformly and at random, place N disc of area a on a rectangle. In this case A is a rectangle, A' is a region whose boundary is a distance $\sqrt{a/\pi}$ outside the sides of A and S is a disc of radius $\sqrt{a/2\pi}$.

(ii) Both A and A' are as above, but S is rectangular with sides parallel to the sides of A .

(iii) The set S is as in (i), but both A and A' are circular.

Cases (i) and (ii) are previously investigated by Robbins (1945).

Garwood (1947) had shown that with some effort it is possible to calculate the moments of vacancy for many coverage processes. However, it is of considerable interest, both practically and theoretically, to know the exact form of the distribution of vacancy. In many circumstances this was far too difficult to determine, and so a compromise had to be reached.

This compromise involved approximating the distribution of vacancy using limit theoretic methods. We shall consider one of many practical applications of this theory when introducing the work of Moran (1973 a,b). Let us firstly investigate some of the earlier developments of Ailam (1966, 1968 and 1970).

In the three research papers Ailam defined a general framework for coverage processes. Under several regularity conditions he showed that a standardized version of vacancy converged to a normal distribution as the size of the "viewing" region increased. Unfortunately, his formulation is very difficult to understand and the regularity conditions are virtually impossible to check. This forced subsequent authors to develop their own limit results.

Moran (1973 a,b) obtained results which have application in the theory of vapour-liquid phase transitions. Melnyk and Rowlinson (1971) proposed the following coverage process to model that situation. In a large region $A \subset \mathbb{R}^3$ n spheres are distributed independently and at random. Assume that each sphere has radius r and volume a. The distribution of the coverage C is of considerable interest in the theory of thermodynamics. The density of the gas is defined to be $\rho \equiv na/\|A\|$, which is taken to be fixed. Melnyk and Rowlinson conjectured that as $\|A\| \to \infty$, $(C - EC)/\sqrt{\text{Var } C}$ converged to a standard normal distribution.

Moran proved this to be the case in two different situations : when n is fixed, and when the number of spheres has a Poisson distribution. Let us consider the latter situation first.

Moran assumed that the centres of the spheres formed a Poisson process of intensity λ in $\mathbb{R}^3$. Let A be a cube and N be the number of centres in A. Then $EN = \lambda\|A\|$. So the density may be redefined as $\rho = \lambda a$. We shall prove a central limit result for vacancy, but first we must find the mean and variance of V.

As before define

$$I(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \underset{\sim}{x} \in A \text{ is not covered by a sphere,} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(V) = \int_A E\{I(\underset{\sim}{x})\}d\underset{\sim}{x} = \|A\|\, e^{-a\lambda} = \|A\|\, e^{-\rho} \; .$$

Let $W(t)$ be the volume of the union of two spheres whose centers are separated by a distance t . It is an easy matter to show that

$$W(t) = \begin{cases} 2a & \text{when } t > 2r \text{ and} \\ a + \pi(t\,r^2 - \frac{1}{12}\,r^3) & \text{when } 0 \le t \le 2r \; . \end{cases}$$

Now, if $x,y \in A$, then both $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are left uncovered if no centre occurs in the union of two spheres of radius r centered at $\underset{\sim}{x}$ and $\underset{\sim}{y}$. Therefore,

$$E(I(\underset{\sim}{x})\; I(\underset{\sim}{y})) = e^{-\lambda W(|\underset{\sim}{x}-\underset{\sim}{y}|)} \; .$$

Hence, the variance of vacancy is

$$\text{var}(V) = E\int_{A^2} (I(\underset{\sim}{x})\; I(\underset{\sim}{y}) - [E\{I(\underset{\sim}{x})\}]^2)\; d\underset{\sim}{x}\; d\underset{\sim}{y}$$

$$= \int_{A^2} [E\{I(\underset{\sim}{x})\; I(\underset{\sim}{y})\} - e^{-2\rho}]\; d\underset{\sim}{x}\; d\underset{\sim}{y}$$

$$= \int_{A^2} \{e^{\lambda W(|\underset{\sim}{x}-\underset{\sim}{y}|)} - e^{-2\rho}\} d\underset{\sim}{x}\; d\underset{\sim}{y} \; .$$

We shall now establish the asymptotic normality of $\{V - E(V)\}/\{Var(V)\}^{\frac{1}{2}}$, as $||A|| \to \infty$ and ρ remains fixed.

Let A have sides of length $D = nd + 2(n-1)r$, where d is a constant which remains fixed as $n \to \infty$. Divide A into n^3 smaller cubes, each of volume d^3 and each separated by a distance $2r$. Let A_i be the i'th small cube, where the labelling is done in some systematic fashion. Let V_i be the vacancy in A_i and R the vacancy in $A \setminus \cup_{i=1}^{n^3} A_i$. Then,

$$V = \Sigma_{i=1}^{n^3} V_i + R .$$

Since each A_i is separated by a distance of at least $2r$, and the point process controlling the centers of spheres is Poisson, then $\{V_i\}$ forms a sequence of independent and identically distributed random variables. Therefore,

$$\Sigma_{i=1}^{n^3} \{V_i - E(V_i)\}/\sqrt{n^2 Var V_i} \to \infty , \quad N(0,1)$$

as $n \to \infty$. It is easy to show that $\limsup_{n\to\infty} Var(R)/n^3 \, Var(V_1)$ may be made arbitrarily small by choosing d large.

A lemma of Bernstein's (1926-7) says that if $X_n = Y_n + Z_n$ are random variables with variances V_x^2, V_y^2 and V_z^2, where V_x^2 and V_y^2 tends to infinity, and V_z^2/V_y^2 tends to zero, and if $Y_n V_y^{-1}$ tends to a standard normal distribution, then so does $X_n V_x^{-1}$. Moran's result follows on letting

$$X_n = V - E(V) \quad ,$$

$$Y_n = \Sigma_{i=1}^{n^3} \{V_i - E(V_i)\} \quad , \quad \text{and}$$

$$Z_n = R - E(R) \quad .$$

Melnyk and Rowlinson posed their vapour-liquid phase transition model in terms of a fixed number of centres in A . The problem of finding the asymptotic distribution of vacancy is much more difficult in this case because the V_i's are no longer independent. Thus, the ordinary central limit theorem does not hold for $\Sigma_{i=1}^{n^3} V_i$. The lack of independence was overcome by Moran (1973b) in the following manner.

Suppose $Y_1,\ldots,Y_p$ has a joint multinomial distribution with $\Sigma_i Y_i = Y$, and some marginal distribution of each Y_i is the same. Let $Z_1,\ldots,Z_p$ be random variables such that Z_i depends on Y_i , $i = 1,\ldots,p$. Moran proved a central limit for $\Sigma_{i=1}^{p} Z_i$ under several mild regularity conditions on the moments of the conditional distribution of Z_i given Y_i . We now describe how this result may be applied to the coverage problem described in the previous paragraph.

The centres of n spheres of radius r are independently and uniformly distributed in a large cube A of side length md , where d is a positive constant. Now divide A into m^3 subcubes of side length d . To overcome edge effect problems, concentric subsubcubes of side length d-r are constructed. The vacancy in the i'th subsubcube is denoted by V_i and the number of centres it contains is denoted by N_i . The vacancy in the remaining portion of A , not covered by the subsubcubes, is denoted by R . It follows that

$$V = \Sigma_i V_i + R \ .$$

Now $N_1, \ldots, N_{m^3}$ has a joint multinomial distribution and V_i depends on N_i . Suppose that $||A||$ and n converge to infinity in such a way that $n^{-1} \; ||A||$ remains constant. On letting

$$Y_i = N_i, \; Z_i = V_i \quad \text{and} \quad p = m^3 \;, \qquad i = 1, \ldots, p,$$

Moran showed that these new variables satisfied the regularity conditions of his central limit theorem. As before, the remainder term becomes insignificant as $d \to \infty$, and so the standardized version of V tends to normality.

As can be seen, the methods of obtaining a central limit result for vacancy differ considerably when N is Poisson, and when $N = n$ is fixed. In chapter 2 we present a simpler proof of the second result which relies on the Berry-Esseen theorem. According to Davy (1980), it is also possible to obtain Moran's (1973b) result by applying a theorem obtained by Ailam (1970).

Up until 1974, most research work had concentrated on the cases where the distribution controlling the location of shapes is uniform, or where a Poisson process of uniform intensity determined their location. Practical situations arise where the centres of covering shapes are not described by a uniform distribution. For example, when a salvo of bombs is fired at a target the spread of their points of impact could be approximated by a normal distribution. Moran (1974) has investigated the three dimensional counterpart to this problem, where the centres of a cluster of spheres are independent observations from a circular normal distribution, with zero mean and unit variance.

Suppose that each sphere has radius r and volume a. Moran considered two separate situations : when the number of centres N has a Poisson distribution with mean λ, and when conditioning on $N = n$. In both cases the total volume C covered by all spheres, after standardization, converges in distribution to normality. For the Poisson case asymptotic normality is obtained as $\lambda \to \infty$, and as $n \to \infty$ when the number of centres is fixed. It is interesting to compare means and variances in the two situations.

Let B be the region covered by a sphere of radius r and centre $\underset{\sim}{z}$, and $F(z)$ be the integral of the circular normal density function over B, where $z = |\underset{\sim}{z}|$. When N has a Poisson distribution

$$E(C) = 4\pi \int_0^\infty z^2 \{1 - e^{-F(z)}\} \, dz ,$$

and a more complex expression holds for the variance of C. Now when $N = n$ is fixed,

$$E(C) = 4\pi \int_0^\infty z^2 [1 - \{1 - F(z)\}^n] \, dz . \tag{2.5}$$

Thus, if we replace n by λ in (2.5), the two formulae are asymptotically equivalent as $\lambda \to \infty$. The same can be said of the variances. In fact, if we set $n = \lambda$, either expectations and variance may be used as normalizing constants in the central limit results.

Recall the coverage problem investigated by Stevens (1939) and discussed in section 1. The process consisted of the random and uniform placement of n arcs of length a on the perimeter of a circle of circumference one. Stevens obtained the probability of complete coverage. Siegel (1978a) derived an exact expression for the distribution of vacancy, which includes Stevens' result as a special case. Let V be the vacancy on the circle's perimeter, and F the distribution function for V. Then,

(2.6)

$$F(t) \equiv P(V \leq t) = 1 + \sum_{\ell=1}^{n} \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n}{\ell} \binom{\ell-1}{k} \binom{n-1}{k} t^k (1-\ell a-t)_+^{n-k-1},$$

where $(x)_+ = x$ when $x > 0$ and

$= 0$ otherwise.

Now F has a discontinuity of size p, say, at $t_o = (1-na)_+$. When $t_o = 0$ the size of the discontinuity corresponds to Stevens' probability for complete coverage of the circle. See expression (1.1). When $t_o > 0$, complete coverage of the perimeter of the circle is impossible, but it is possible for all random arcs to be disjoint. In this case $p = (1-na)^{n-1}$.

When $t < t_o$, $F(t) = 0$, and when $t > t_o$, $F(t)$ has a derivative. Thus V is a mixture of an absolutely continuous and degenerate distributions. The probability density function for the continuous part of V is :

(2.7)

$$f(t) = \frac{n}{1-p} \sum_{\ell=1}^{n} \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n-1}{\ell-1} \binom{n-1}{k} \binom{\ell-1}{k} t^{k-1} (1 - \ell a - t)_+^{n-k-1} .$$

Siegel also obtained the following formula for the m'th moment of vacancy :

$$E(V^m) = \binom{m+n-1}{n} \sum_{\ell=1}^{m} \binom{m}{\ell} \binom{n-1}{\ell-1} (1 - \ell a)_+^{m+n-1} \quad ,$$

where $m \leq 1$. Using a simpler argument than Siegel, Holst (1980a) derived the same expressions for the moments and distribution of vacancy.

As can be seen, the exact form of the distribution of vacancy (2.6) is fairly complex. This led Siegel (1979) to search for approximations through limit theoretic methods, which could be useful when the exact distribution is too difficult to evaluate numerically.

Moran (1973 a,b) looked at limit theory for vacancy when the size of the region A increased and everything else was held fixed. However, Siegel examined vacancy in a coverage process where the number of shapes increased out of proportion to the size of the region. Indeed, if n arcs are placed uniformly and at random on the perimeter of a circle it is always possible to choose the common arc length a_n so that the probability of complete coverage remains fixed as $n \to \infty$.

Suppose for some constant $\beta > 0$, the probability of complete coverage is $e^{-\beta}$. Let V_n be the vacancy on the circle's perimeter. Then $P(V_n = 0) = e^{-\beta}$ implies that

(2.8) $$n\,a_n = \log(n/\beta) + o(1) \qquad \text{as} \quad n \to \infty \;.$$

We saw how (2.8) may be used to approximate the probability of complete coverage in section 1 . Siegel obtained an interesting limit result for vacancy, which is summarized in Theorem 2.1 below.

Theorem 2.1 (Siegel (1979)).

The limiting distribution of vacancy, V_n, where the arc lengths a_n are chosen so that the coverage probability remains fixed at $e^{-\beta}$, is given by

$$n\,V_n \to^{\mathcal{D}} \chi_0^2(\beta) \;, \text{ as } \; n \to \infty \;,$$

where

$$\chi_0^2(\beta) = 0 \quad \text{with probability} \quad e^{-\beta} \;, \text{ and}$$
$$= z \quad \text{with probability} \quad 1 - e^{-\beta}$$

and z is absolutely continuous with density

(2.9) $$f_\beta(t) = \frac{1}{e^\beta - 1} \sum_{\ell=1}^{\infty} \frac{\beta^\ell}{\ell!} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t} \;.$$

A non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter β is a Poisson mixture of central chi-squares $\chi^2_{\nu+2k}$, where $P(k=\ell) = e^{-\beta}\,\beta^\ell/\ell_\circ^\nabla$. See chapter 2.4 of Searle (1971) . The limiting distribution in theorem 2.1 is obtained by setting $\nu = 0$. Thus, we may interpret $\chi_0^2(\beta)$ as a non-central chi-squared distribution with zero degrees of freedom, and non-centrality parameter β .

The proof that Siegel used relied on showing that the moments of $n V_n$ converge to the moments of a $\chi_0^2(\beta)$ distribution. Holst (1981), however, proved that the moment generating function converged, which implies Siegel's result as well as convergence of all the moments of $n V_n$. This technique is quite powerful and has since been used by Hüsler (1982), and others, to prove limit results about vacancy. We shall discuss Hüsler's work later on in this section.

Consider, for example, the following application of the moment generating function technique. Suppose that n points are distributed independently and uniformly on the perimeter of a circle of circumference one. Let $S_{(1)} < \ldots < S_{(n)}$ be the ordered spacings, as measured around the perimeter, so that $\Sigma_i S_{(i)} = 1$. Holst proved that

$$E(\exp(t(n S_{(n)} - \ell n\, n))) \to \Gamma(1 - t)$$

as $n \to \infty$. This implies that $n S_{(n)} - \ell n(n)$ converges in distribution to the double exponential distribution, with distribution function F defined by $F(x) = \exp(e^{-x})$, as $n \to \infty$.

In the above described model the vacancy $V_n = \Sigma_{k=1}^n (S_{(k)} - a_n)_+$. Therefore

$$\begin{aligned} P(V_n = 0) &= P\{\Sigma_{k=1}^n (S_{(k)} - a_n)_+ = 0\} \\ &= P(S_{(n)} \le a_n) \\ &= P\{n S_{(n)} - \ell n(n) \le n a_n - \ell n(n)\} . \end{aligned}$$

Thus, if for some constant $\beta > 0$,

(2.10) $$n\, a_n = \log(n/\beta) + o(1)$$

as $n \to \infty$, then

$$P(V_n = 0) = P\{n\, S_{(n)} - \ell n(n) \leq n\, a_n - \ell n(n)\}$$

(2.11) $$= P\{n\, S_{(n)} - \ell n(n) \leq -\log(\beta) + o(1)\}$$

$$= e^{-\beta} + o(1)$$

as $n \to \infty$. By the earlier result of Siegel at (2.8) , we see that (2.10) and (2.11) are equivalent.

Furthermore, $P(V_n = 0) \to 0$ as $n \to \infty$, is equivalent to

(2.12) $$n\, a_n = \ell n\, n - \lambda_n ,$$

where $\lambda_n \to \infty$, as $n \to \infty$.

Holst obtained a limit theorem for vacancy for a special case of (2.12) . We state his results below:

<u>Theorem 2.2</u> (Holst (1981))

<u>Suppose</u> $n \to \infty$ <u>and</u> $a_n \to 0$ <u>in such a way that</u> $n\, a_n - \ell n\, n \to -\infty$, <u>and</u> $\liminf n\, a_n > 0$. <u>Then, when</u> $n \to \infty$,

$$E(\exp(t(n\, V_n - n\, e^{-n\, a_n})/\sigma_n)) \to e^{t^2/2} ,$$

<u>for all sufficiently small</u> $|t|$, <u>where</u>

$$\sigma_n^2 = 2n(e^{-n\, a_n} - e^{-2n\, a_n}(1 + n\, a_n + (n\, a_n)^2/2)) .$$

Thus, $(n V_n - n e^{-n a_n})/\sigma_n$ is asymptotic to a normal $N(0,1)$ distribution.

In general Hüsler (1982) has shown that there are three cases for which different convergence results hold. The first (A) was considered by Siegel (1979); see Theorem 2.1. A special case of the second was obtained by Holst (1981); see Theorem 2.2.

In summary, the cases are :

(A) $n a_n = \log(n/\beta) + o(1)$, where $0 < \beta < \infty$;

(B) $n a_n = \log n - \lambda_n$, where $\lambda_n \to \infty$ and $n^2 a_n \to \infty$, and

(C) $n^2 a_n \to a$, where $0 < a < \infty$,

each as $n \to \infty$.

It follows from Holst's result (2.10), that if a_n converges to zero any slower than in (A) , then the probability of complete coverage tends to one. Furthermore, if $n^2 a_n \to 0$ as $n \to \infty$, then the probability that any two random arcs intersect tends to zero. Thus, (A) - (C) cover the complete spectrum of convergence rates that are of theoretical interest.

Hüsler extended Holst's Theorem 2.2 to include all of case (B) , and also used a moment generating function argument to show that an adjusted version of vacancy tends in distribution to a compound Poisson distribution. For completeness we state Hüsler's result below.

Theorem 2.3 (Hüsler (1982)).

In case (C) above, for $t > 0$

$$E(\exp(t\, n^2(V_n - 1 + n\, a_n)) \to \psi(t)$$

where

$$\psi(t) = \exp(-a + (e^{at} - 1)/t)$$

is the moment generation function for $Y = \Sigma_{k=1}^{K} W_K$, K has a Poisson distribution with mean a, W_K is uniform on $[0,a]$, and $K, W_1, W_2, \ldots$ are independent.

Two other statistics studied by Hüsler (1982) are the number of gaps or uncovered regions on the perimeter, which we shall denote by L_n, and the length, M_n, of the maximum gap. Under suitable scaling and location changes, the distributions of L_n and M_n tend asymptotically to non-degenerate distributions. Table 2.1 below gives these limits along with references where exact formulations can be found.

Table 2.1 Asymptotic Distributions

Case	Statistic L_n	M_n	V_n
A	Poisson. Darling (1953)	Mixture with double exponential component. Lévy (1939)	Non-central chi squared with zero degrees of freedom. Siegel (1979)
B	Normal Hüsler (1982)	Double exponential. Hüsler (1982)	Normal. Siegel (1979), Holst (1981) & Hüsler (1982)
C	Poisson. Darling (1953)	Double Exponential Hüsler (1982)	Compound Poisson Hüsler (1982)

It can be seen that Hüsler has given a fairly complete description of the asymptotic distribution of L_n, M_n and V_n when arcs are randomly and <u>uniformly</u> distributed on the perimeter of a circle. Hall (1983) has extended the scope of these results to non-uniform distributions on a unit circle.

Suppose that the endpoints of arcs are independent observations from a distribution, F . Furthermore, assume that F possesses a unique minimum at m , so that $1 > f(m) > 0$. As n converges to infinity f remains fixed. Clearly, the class of F satisfying $f(m) < 1$ does not include the uniform distribution. It seems necessary to draw this distinction as one would expect the distributions of L_n, M_n and V_n to be very dependent on the behaviour of F near m . Hall has shown this to be the case by obtaining their limiting distributions. As before the limit depends on the rate of convergence of the arc lengths a_n . We replace A, B and C by the following convergence rates :

(A') $\quad n\,a_n\,f(m) = \log(n/\beta) - \tfrac{1}{2}\log\log n + o(1)$,
where $0 < \beta < \infty$;

(B') $\quad n\,a_n\,f(m) = \log(n) - \tfrac{1}{2}\log\log n - \lambda_n$,
where $\lambda_n \to \infty$ and
$n^2 a_n \to \infty$, and

(C') $\quad n^2 a_n \to a$, where $0 < a < \infty$,

each as $n \to \infty$. To give a complete description of the possible limits, (B') must be further subdivided into three cases :

(B') (a) $n a_n = \log n - \frac{1}{2} \log\log n - \lambda_n$,

where $\lambda_n \to \infty$ and $n a_n \to \infty$;

(b) $n a_n \to a$, where $0 < a < \infty$, and

(c) $n a_n \to 0$ and $n^2 a_n \to \infty$.

A scale factor σ_n and location factor μ_n are defined so that for a sequence of random variables $\{X_n\}$, $(X_n - \mu_n)/\sigma_n$ has a non-degenerate distribution. In all the situations we consider μ_n and σ_n are well defined.

A brief description of the form of the limiting distributions of L_n, V_n and M_n is given in Table 2.2 below. For scale and location parameters see Hall (1983). It is interesting to compare Table 2.1 with Table 2.2. We can see that the limiting distributions in Table 2.1 and the corresponding limits in the respective cells of Table 2.2 are always of the same form. However, the scale and location parameters required to obtain these limits differ considerably between the uniform and non-uniform cases. To illustrate this point, location factors have been given in Table 2.3 below for vacancy. Notice that the location factors for the uniform case can be obtained by putting $f \equiv 1$ in the non-uniform factors.

Table 2.2 Limiting distributions in the Non-Uniform Case

Case	L_n	M_n	V_n
A'	Poisson	A mixture including a relocated and truncated double exponential distribution.	A non-central chi-squared distribution with zero degrees of freedom.
B' (a)-(c)	Normal	A relocated double exponential distribution.	Normal
C'	Poisson	"	Poisson.

Table 2.3 Location Factors for Vacancy

Cases	Uniform	Non-uniform
A	0	0
B	$e^{-n a_n}$	$\int_0^1 \exp\{-n a_n f(x)\}dx$
C	$1 - n a_n$	$1 - n a_n$

When the coverage region A is a proper subset of k-dimensional space, and $k > 1$, there is no obvious analogue of L or M. However, it is possible to generalize some of the results for vacancy to higher dimensional space.

Following Moran (1973a), Hall (1984b) considered the properties of vacancy V in the region A, when hyperspheres are centred at the points of a Poisson process in $\mathbb{R}^k$. Unlike Moran though, Hall assumed that the radii of the spheres were independent observations from a predefined distribution, and that the Poisson process was inhomogenous.

Hall shows that the variance of vacancy varies inversely with $E(N)$, the expected number of centres in A, if and only if the content covered by a single random sphere has finite second moment. Furthermore, in the case of finite second moment, a central limit result holds for V as $E(N) \to \infty$ and $a^{-1} EN \to \rho$, where $0 < \rho < \infty$. Expected vacancy reaches a maximum for a homogeneous Poisson process. Therefore, tests based solely on V can be constructed. Hall suggests a sequence of local alternatives to show that the test based on V is not very powerful. Subsequently, the chi-squared argument is used to construct a more powerful vacancy-based test.

It is also possible to obtain limit results for vacancy when conditioning on the number of points in A. Suppose n points are distributed in the k-dimensional cube A according to a distribution with density f. Spheres of fixed content a_n are centred at the points and we assume a torus topology on A . See section 1.2 and Miles (1969) for a description of a topological torus.

Hall (1984a) obtained a central limit theorem for V as $n a_n \to \rho$, where $0 < \rho < \infty$. Also, when $n^2 a_n \to \rho$ a standardized version of vacancy is asymptotic to a compound Poisson distribution. These results represent generalizations of the work of Hüsler (1982) and Moran (1973b). Whereas Moran used a characteristic function argument and results of Renyi (1962) and Holst (1972), Hall applied the Berry-Esseen theorem for sums of independent random variables allowing a simpler moment argument in his proofs.

§3 Continuum Percolation, Sequential Coverage and Counting Problems

§3.1 Continuum Percolation

In particle physics, interaction can take place between two particles if they are located in close proximity to each other. Interactions may extend to one or more particles, which we shall call a cluster. In particular, we are interested in an infinite number of interactions occurring in at least one cluster. This event is referred to as percolation and has important applications in both the physical and chemical sciences. As an example, percolation may be used to describe the conduction of electricity in a crystalline semiconductor. We shall describe this and other examples in more detail later. A more vigorous definition of percolation in a continuum is presented below.

Let P be a Poisson process of intensity λ in $\mathbb{R}^k$. Suppose spheres are centred at the points of P. Assume that the radii of spheres are independent and identically distributed random variables, which are also independent of P.

A cluster is a connected region in $\mathbb{R}^k$ formed by overlapping random spheres. The size of a cluster, which we shall denote by K, is the number of spheres it contains. Notice that K is not observable because a random sphere may be obscured by other random spheres in the same cluster. Percolation occurs when clusters of infinite size are formed. Of much practical importance is the intensity λ_c

at which the probability of percolation becomes positive. Some of the applications of percolation theory are presented below. We have not attempted to be comprehensive in our survey of the literature, as percolation is not the major topic of this project.

Seager and Pike (1974) have computed the electrical conductivity of several model materials, including some which simulate impurity conduction in a semiconductor. Essentially, impurity particles are represented by points of a Poisson process. An electron can pass between two particles if they are sufficiently close; that is, if two spheres of a fixed radius centred at the points overlap. Therefore, conduction in the semiconductor can only occur when the concentration of impurity is sufficient to cause percolation.

Gawlinski and Stanley (1981) mention another application. Percolation theory may be used to model condensation of polymers in a chemical solution. In recent years, lattice rather than continuum percolation has been used to model this type of situation. Since lattice has been used to approximate continuum percolation we describe it in more detail below.

Unlike continuum percolation, lattice percolation models require that the particles be placed at the vertices of a regular grid. In the simplest models the grid is either a square lattice in two dimensions or a rectangular lattice in three dimensional space. Adjacent particles are connected with probability p. It is clear that as p increases the probability of percolation increases.

Also, for $p=1$ every particle is included in an infinite clump with probability one. Therefore, there exists a critical probability p_H beyond which infinite clumps can form, and a probability p_T beyond which clumps of infinite expected size can occur. Now $p_T \leq p_H$. For a square lattice in two dimensional space Seymour and Welsch (1978) have shown that $p_H + p_T = 1$, and finally Kesten (1980) proved that $p_H = p_T = \frac{1}{2}$.

Other types of grids are also of practical interest. For example, there are two ways to optimally pack spheres of common fixed radius in three dimensional space. In crystallography these structures are referred to as hexagonal close packing and body centered cubic packing. Corresponding to each of these packings is a lattice which we illustrate in diagram 3.1 below. Critical probabilities may be computed for these and other crystal structures. See, for example, table I in Pike and Seager (1974). We now briefly describe some of the methods used by authors to estimate critical intensities of continuum percolation in two and three dimensional problems.

Most work has concentrated on the case where spheres of fixed radius are centred at the points of P . Some authors have considered other shapes, including rectangles and squares in two dimensions and cubes in three dimensional space. These departures from the norm shall be mentioned as we progress through the literature. For reasons of consistency, we assume that the random spheres or circles are of radius one.

Gilbert (1961) noted that there were distinct similarities between continuum and lattice percolation. Quite often in lattice percolation the critical probabilities at which clumps of infinite size begin to form and where expected clump size is infinite correspond. Gilbert was able to plot estimated expected clump size against intensity, λ , and thus estimate the point at which the graph diverged. Similar techniques were later employed by Roberts (1967) and Domb (1972).

The second method, used by Pike and Seager (1974), Gawlinski and Stanley (1981) , and others, was to simulate continuum percolation in a very large rectangular region. Of course, clumps of infinite size cannot form in this situation. Therefore Pike and Seager identified percolation as the occurrence of at least one cluster which extended from one side of the rectangle to the other. Pike and Seager estimated λ_c by starting with subcritical values of λ and slowly increasing it until percolation occurred. Other authors have obtained estimates of λ_c by extrapolating functions of the cluster size K . For example, Galwinski and Stanley defined the function $R(\lambda)$ as the probability that a sphere chosen at random belonged to an infinite cluster. They suggested that the behaviour of R in the vicinity of λ_c can be described by

$$R(\lambda) = B(\lambda - \lambda_c)^{\beta}, \quad \lambda > \lambda_c ,$$

where B and β are positive constant. Their method of estimating λ_c was to perform non-linear regression of an estimate of $R(\lambda)$ against λ. Both papers mentioned above consider percolation for non-spherical shapes.

Pike and Seager obtained critical values for squares, which are all oriented in the same direction, and for sticks which are randomly and uniformly oriented about their centres. Galwinski and Stanley have obtained critical values for cubes in the three dimensional problem and squares in the two dimensional problem.

We summarise the estimates of critical values obtained by various authors in table 3.1 below.

We shall now demonstrate that percolation occurs whenever the expected content of each k-dimensional sphere centred at the points of P is infinite.

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ be the points of P numbered in any systematic order and $S_1, S_2, \ldots$ independent random spheres centred at the origin, $\underset{\sim}{0}$. Furthermore, assume that $S_1, S_2, \ldots$ are independent of P. The coverage process consists of the randomly located spheres :

$$\underset{\sim}{X}_1 + S_1 = \{x \in \mathbb{R}^k : \underset{\sim}{x} - \underset{\sim}{X}_i \in S_i\} \quad .$$

Table 3.1

Estimates of critical values for

continuum percolation.

The shapes are chosen to have a volume or area of one.

Authors	Spheres	Cubes	Circles	Squares
Gawlinski and Stanley (1981)			0.36	
Vicsek and Kertész (1981)			0.37	
Haan and Zwanzig (1977)	0.084	0.077	0.37	0.35
Fremlin (1976)	0.081		0.35	
Pike and Seager (1974)	0.079		0.36	
Gayda and Ottavi (1974)	0.079			0.35
Ottavi and Gayda (1974)			0.33	
Kurkijarvi (1974)	0.082			
Holcomb et al (1972)	0.070			
Domb (1972)	0.081		0.36	
Roberts (1967)			0.31	

Now for each $\underset{\sim}{x} \in \mathbb{R}^k$,

$$P(\underset{\sim}{x} \text{ is not covered by any set } \underset{\sim}{X}_i + S_i)$$

$$= P(\underset{\sim}{0} \text{ is not covered by any set } \underset{\sim}{X}_i + S_i)$$

$$= \exp(-\lambda a) \quad ,$$

where a is the expected content of S_1 . Therefore, the expected vacancy in $A \in \mathbb{R}^k$ is :

$$E(V) = \int_A P(\underset{\sim}{x} \text{ is not covered by any set } \underset{\sim}{X}_i + S_i) \, d\underset{\sim}{x}$$

$$= \|A\| \exp(-\lambda a) \quad .$$

Thus, $P(V = 0) = 1$ if and only if $a = \infty$. It follows that percolation occurs for all positive λ whenever $a = E(\|S\|) = \infty$. Hall (1984c) has provided a more complete description of when percolation can occur. We describe his results below.

Let λ_o be the critical value for the formation of infinite clusters with positive probability, and λ_1 the vlaue at which expected cluster size $E(K)$ becomes infinite. Clearly $0 \leq \lambda_1 \leq \lambda_o \leq \infty$. Theorems 3.1 and 3.2 below give conditions under which both λ_o and λ_1 are strictly positive, and finite.

Theorem 3.1 (Hall (1984c))

Assume that $k \geq 2$ and $E(||S||) < \infty$. There exists $\lambda_o > 0$ such that the expected number of spheres in an arbitrary clump is finite whenever $0 < \lambda < \lambda_o$ if and only if $E(||S||^2) < \infty$. Indeed, if $E(||S||^2) = \infty$ then the expected number of spheres which are in the same cluster as a given sphere and distant no more than one sphere away from that sphere is infinite for all values of λ.

Furthermore, if $k \geq 1$ and $E(||S||^{2-(1/k)}) < \infty$ then for all sufficiently small λ, the number of spheres in each cluster is finite with probability one.

Theorem 3.2 (Hall (1984c))

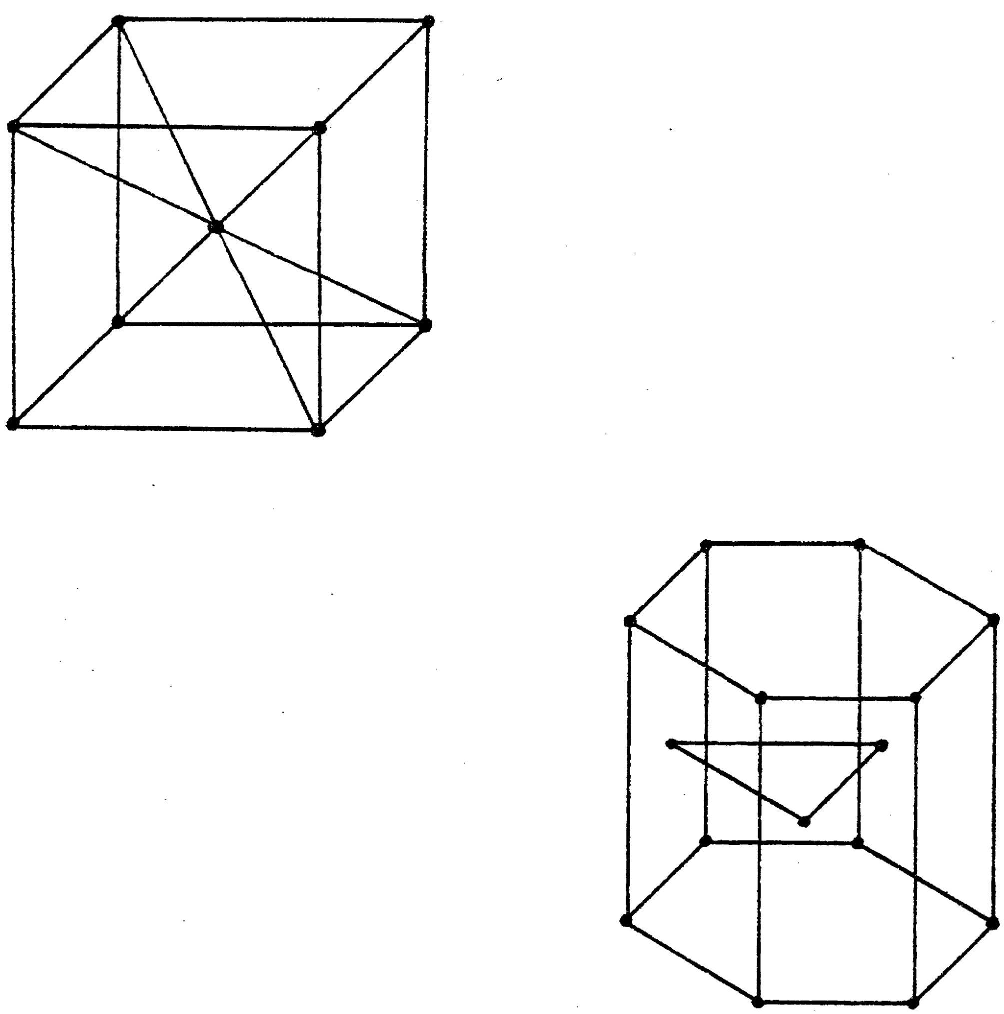
Assume $k \geq 2$. If $E(||S||) > 0$ then for all sufficiently high intensities, the probability that a given random sphere is part of an infinite clump is strictly positive.

The case $k=1$ is dealt with separately by Hall. He shows that percolation is then possible only when $E(||S||) = \infty$. In other words, if $E(||S||) < \infty$ then $\lambda_o = \lambda_1 = \infty$.

In fact, Hall considered random shapes more general than spheres in the formulation of theorem 3.2. Indeed, he assumed that $S_1, S_2, \ldots$ are independent copies of a random and closed set S, as defined by Matheron (1975). In order for theorem 3.2 to follow we replace the condition $E(||S_1||) > 0$ by $E(\underset{\sim}{S}(S)) \geq 0$, where $\underset{\sim}{S}(S)$ is the content of the largest sphere contained in S , and it is assumed that S is connected.

For $k = 2$ Gilbert showed that $0.13 < \lambda_1 \leq \lambda_0 \leq 1.39$. Hall improved these bounds to $0.174 < \lambda_1 \leq \lambda_o < 0.843$ when S is a disc of unit radius

Diagram 3.1 : Two ways of optimally packing spheres of equal radius in 3-dimensional space.



§3.2 Sequential Coverage

Suppose that we sequentially and independently place arcs of length a on the perimeter of a circle of circumference one according to a uniform distribution. In subsection 1.1 we reveiwed literature concerned with the probability of complete coverage when the number of arcs in the perimeter is fixed. In this section we review literature concerned with the "converse" problem : that of finding the minimum number of arcs required to completely cover the circle's perimeter. Since the arcs are placed sequentially, we shall call this a sequential coverage problem.

When exactly n arcs have been placed on the circle's perimeter we know that, except for a set of probability zero, complete coverage occurs if and only if the vacancy, V_n , is zero. One random variable of concern in this section is

$$N_a \equiv \inf\{n : V_n = 0\} .$$

It is clear from this definition that, except on a set of probability zero, the events $\{N_a \leq n\}$ and $\{V_n = 0\}$ are equivalent. When $a > 0$, N_a is well defined as a random variable, for

$$\begin{aligned} P(N_a < \infty) &\geq \liminf_{n\to\infty} P(N_a \leq n) \\ &= \liminf_{n\to\infty} P(V_n = 0) \\ &= 1 , \end{aligned}$$

the final step following, for example, from theorem 2.1 obtained by Siegel in 1979.

As discussed in subsection 1.1, Stevens (1939) obtained an exact expression for the probability of completely covering the circle. Using Stevens' result, Flatto and Konheim (1962) obtained the following expression for the expectation of N_a :

$$(3.1) \qquad E(N_a) = 1 - \sum_{1\leq j\leq a^{-1}} (-1)^j \frac{(1-ja)^{j-1}}{(ja)^{j+1}} .$$

In the same paper they also proved that, as $a \to 0$,

$$(3.2) \qquad E(N_a) \sim \frac{1}{a} \log(\frac{1}{a}) .$$

Subsequently, Steutel (1967) proved that

$$(3.3) \qquad E(N_a) = \frac{1}{a}\{\log(\frac{1}{a}) + \log\log(\frac{1}{a}) + \gamma + o(1)\} ,$$

as $a \to 0$, where γ is Euler's constant.

For purposes of comparison we have listed the two approximations, (3.2) and (3.3) , against the exact value of $E(N_a)$, for a variety of a . Both approximations are poor when a is large. However, when a is small, Steutel's approximation is reasonably good, and certainly much better than Flatto's and Konheim's .

TABLE 3.2

EXPECTED NUMBER OF ARCS REQUIRED TO COMPLETELY COVER THE CIRCLE

a	EXACT SOLUTION	FLATTO'S AND KONHEIM'S APPROXIMATION	STEUTEL'S APPROXIMATION
0.01	713.80	460.52	670.96
0.02	316.42	195.60	292.66
0.025	242.61	147.56	222.86
0.033	171.72	102.04	156.08
0.05	104.80	59.91	93.40
0.1	44.05	23.03	37.14
0.2	17.84	8.05	13.31
0.25	13.20	5.55	9.16
0.333	8.88	3.30	5.31
0.5	5.00	1.39	1.81

Since N_a and V_n are so closely related, and a non-degenerate limiting distribution exists for vacancy, we would expect a limit theorem to hold for N_a. Indeed, allow n_a to be the integer part of

$$\frac{1}{n}(\log(\frac{1}{a}) + \log\log(\frac{1}{a}) + x) ,$$

where x is an arbitrary real number. Then

$$\begin{aligned} n_a &= \frac{1}{a}(\log(\frac{1}{a}) + \log\log(\frac{1}{a}) + x) + o(1) \\ &= \frac{1}{a}\log(\frac{1}{a}) + o(n_a) , \quad \text{as } a \to 0 . \end{aligned} \tag{3.4}$$

Let $\beta = e^{-x}$. From (3.4) it follows that

$$an_a = \log(\frac{1}{a}\log(\frac{1}{a})/\beta) + o(1)$$

$$= \log((n_a + o(n_a))/\beta) + o(1)$$

(3.5) $$= \log(n_a/\beta) + o(1) \quad , \quad \text{as} \quad a \to 0 .$$

Now as $a \to 0$, $n_a \to \infty$. Thus, by the argument following (2.10), (3.5) is equivalent to

(3.6) $$P(V_{n_a} = 0) \to e^{-\beta} \quad \text{as} \quad a \to 0 .$$

However,

$$P(N_a \leq \frac{1}{a}(\log(\frac{1}{a}) + \log\log(\frac{1}{a}) + x))$$

$$= P(N_a \leq n_a)$$

$$= P(V_{n_a} = 0) .$$

Therefore, by (3.6), $aN_a - \log(\frac{1}{a}) - \log\log(\frac{1}{a})$ has a limiting extreme value distribution with distribution function $\exp(-e^{x})$. Flatto (1973) has obtained a slightly more general result.

Allow $N_{a,m}$ to be the minimum number of randomly and uniformly distributed arcs of length a required to cover the circle m times. Then, as $a \to 0$,

$$a\,N_{a,m} - \log(\frac{1}{a}) + m\log\log(\frac{1}{a}) \to^{\mathcal{D}} X ,$$

where X has the relocated extreme value distribution :

$$P(X \leq x) = \exp\{-e^{-x}/(m-1)!\} .$$

To prove this result Flatto expressed $N_{a,m}$ in terms of the spacings between $n-1$ points, independently and uniformly distributed on $[0,1]$. Specifically, let $L_o, L_1, \ldots, L_{n-1}$ be the lengths of the successive spacings, and let $L_{i+n} = L_i$ for $i \geq 0$. If $m < n$ and $S_i = L_i + L_{i+1} + \ldots + L_{i+m-1}$, then

$P(N_{a,m} \leq n) = P(S_i \leq a,\ 0 \leq i \leq n-1) = P(S_{(n)} \leq a)$, where $S_{(n)}$ is the largest S_i, $0 \leq i \leq n-1$. The limiting distribution for $N_{a,m}$ is obtained by finding a limit for the extreme order statistic, $S_{(n)}$. Flatto's result does not allow us to find the behaviour of the k'th moment of $N_{a,m}$ as $a \to 0$. Edens (1975) investigated this problem

Let $X_a = a N_{am} - \log(\frac{1}{a}) - m \log\log(1/a)$. Edens showed that for $0 \leq t < 1$,

$$E(e^{t|X_a|}) \to E(e^{t|X|})$$

as $a \to 0$. One consequence of this result is that the k'th moment of X_a converges to the k'th moment of X. That is,

$$\lim_{a \to 0} E(X_a) = (-1)^k [\frac{d^k}{dx^k} (\exp(\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j} x^j))] , \quad k \geq 1 ,$$

where ζ is Riemann's ζ-function. Thus, as $a \to 0$,

$$E(N_{a,m}) = \frac{1}{a}\{\log(\frac{1}{a}) + m \log\log\frac{1}{a} + \gamma - \log(m-1)! + o(1)\}$$

and

$$\text{(3.7)} \qquad Var(N_{a,m}) = \frac{1}{a^2}(\frac{\pi^2}{6} + o(1)) .$$

It is interesting to note that the right hand side of (3.7) does not depend on m .

In a slightly different vein, Flatto and Newman (1977) generalized to k-dimensions the results of Flatto and Konheim (1962), on the expected number of random arcs required to cover a circle.

Let A be a k-dimensional sphere with surface area one. Suppose circular caps of the same radius as A are sequentially and uniformly distributed on the surface of the sphere. Suppose the surface area of each cap is a . We can see that this geometrical situation is similar to that studied by Moran and Fazekas de St Groth (1962), and Gilbert (1965) . However, in the later cases, the number of random caps is fixed. In the present situation we let $N_{a,m}$ be the minimum number of caps required to cover A m times. Flatto and Newman proved that, as $a \to 0$,

$$E(N_{a,m}) = \frac{1}{a}\{ \log(\tfrac{1}{a}) + (k+m-1) \log\log(\tfrac{1}{a}) + o(1) \} .$$

Kaplan (1978) and Holst (1980b) have obtained limit results for $N_{a,m}(p)$, the minimum number of arcs required to cover a proportion $0 < p < 1$ of the circle's perimeter m times.

Let ρ be a fixed real number, $k \geq 1$ and $p = 1 - ka$. Suppose $\{n_a\}$ is a sequence satisfying , as $a \to 0$,

$$n_a \to \infty$$

and

$$\frac{(n_a e^{-an_a} - n_a a k)}{\sqrt{n_a a k}} \to \rho$$

Kaplan proved that

$$\lim_{a\to 0} P\{N_{a,1}\ (1 - K_a) \le K_a) \le n_a\} = \frac{1}{\sqrt{4\pi}} \int_{\rho}^{\infty} e^{-z^2/4}\, dz\ ,$$

while Holst showed that for fixed p ,

$$a^{-\frac{1}{2}}\{ a\ N_{a,m}(p) - G^{-1}(p)\}$$

converges in distribution to a normal law where

$$G(p) = \sum_{j=0}^{m-1} p^j\, e^{-p}/j!\quad .$$

In other words, if p converges to 1 fairly slowly, or remains fixed at a value less than 1 , then $N_{a,m}(p)$ may be approximated by a normal law.

In subsection 1.1 we discussed the work of Jewell and Romano (1982) and Siegel and Holst (1982), concerned with the complete coverage of a circle by randomly and uniformly distributed arcs of random length. In this case Janson (1983) investigated the asymptotic properties of N_a , the minimum number of arcs required to completely cover the circle.

Let L be a positive random variable and a a positive constant. Assume that the lengths of the random arcs are independent copies of aL , independent of the underlying uniform distribution controlling the location of arcs. Janson established the following limit theorem for N_a :

Theorem 3.1 (Janson (1983))

Suppose that $E(L) = 1$ *and* $E(L-t)_+ = O(1/\log t)$ *as* $t \to \infty$. *Then, as* $a \to 0$,

$$aN_a - \log(1/a) - \log\log(1/a) \xrightarrow{\mathcal{D}} X$$

where X *has the extreme value distribution* $\exp(-e^{-x})$.

Provided the distribution of the tail of L is sufficiently well behaved, the asymptotic distribution of N_a depends only on $E(L)$, and not on the form of the distribution of L . In this sense Janson's results represent a generalization of Flatto's (1973) work.

§3.3 Dust Counting Problems

A coverage process consists of shapes placed at random on a region A . In many practical situations it is useful to determine the concentration of the number of shapes per unit area. For example, a sample of dust particles may have been taken on a microscope slide, and we wish to find the concentration of dust particles per unit area. When examined under a microscope, particles group together forming clumps. Some clumps may overlap the edges of the viewing region. In order to estimate the number, and hence concentration of particles, we must compensate for the effects of clumping. We shall refer to this, and related problems, as dust counting problems. Some other practical applications will be discussed in section 4.

In the present subsection we shall review literature on methods of adjusting for the undercounts.

Irwin, Armitage and Davies (1949) and Armitage (1949) ave used the following model to describe the random overlapping of dust particles on a microscope slide. In a large region A, n discs of area a are randomly and uniformly distributed. A clump is a connected set of overlapping discs, having empty intersection with all other discs. The size of a clump is measured by the number of discs it contains, C, say. Let $m = E(C)$ be the expected clump size, and $\psi = na/\|A\|$.

Irwin, Armitage and Davies derived the following approximate formula for the case where n increases, and $a/\|A\|$ decreases :

(3.8) $$m \simeq 4\psi/(1-e^{-4\psi}) .$$

The argument used in deriving (3.8) involved the assumption that, in every clump, every disc overlaps every other one. This is, in fact, not true and leads to an underestimate of the true value of m. Armitage (1949), later attempted to improve this formula by the following means.

Let n_1, n_2 and n_3 be the expected number of clumps of size one, two and three, and n_3' the expected number of clumps of size at least three. Armitage derived formulae for n_1, n_2 and m_3 and argued that

$$n_3' = n_3 + n\,O(\psi^3)$$

as $n \to \infty$ and $\psi \to 0$. He therefore concluded that the expected number of clumps, ignoring edge effects, is :

$$n_1 + n_2 + n_3'$$

$$= \quad n\{1 - 2\psi + 2\ \frac{(4\pi - 3\sqrt{3})}{3\pi}\psi^2 + O(\psi^3)\}\ .$$

An approximate formula for m is obtained from the relationship $m \simeq \{n_1 + n_2 + n_3'/n\}^{-1}$. Armitage also extended the result at (3.7) obtained by Irwin, Armitage and Davies (1949) to discs of random radii, and rectangular particles (or shapes) of random size and random orientation.

Let μ_1 and μ_2 be the first and second moments of the square root of the area of a particle. Then, Armitage showed that m could be well approximated by :

$$m \simeq 4\psi/(1 - e^{-4\psi})\ ,$$

where

$$\psi = n(\mu_2 + 2.5\ \mu^2)/2||A||\ .$$

This relationship may be used to estimate the mean clump size as follows.

Let $K = (\mu_2 + 2.5\mu^2)$ and N the observed number of clumps. Then $n \simeq mN$, and so

$$m \simeq 4\psi/(1 - e^{-4\psi})$$

$$= \frac{2K\,m\,N/||A||}{1 - \exp(-2Kmn/||A||}\ .$$

Hence, Armitage suggested that

$$\hat{m} = \frac{||A||}{2\,KN}\ \ell n(\frac{1}{1 - 2KN/||A||})$$

be used to estimate the mean clump size.

Various authors have noted that the dust particles can vary greatly in their shape and size. The approximation to their shape by a disc or rectangle, as proposed by Armitage, may indeed be too rough. In light of this argument, Mack (1954) has suggested a more general model, which we now describe.

In a region A' , n convex shapes (or particles) are independently and uniformly distributed with random and uniform orientation. Suppose that n_r of the shapes have area a_r and perimeter s_r , $1 \leq r \leq k$, where $\Sigma_{r=1}^{k} n_r = n$. Mack was interested in finding the expected number of clumps, m , and the expected number of clumps of size one (or expected number of isolated shapes), m_1 , say, in the region $A \subset A'$. A' is chosen sufficiently large to overcome edge effect problems near the boundary of A .

Clearly, Mack's model contains Armitage's as a special case, for both discs and rectangles are convex. As we shall see, Mack chose convex shapes for reasons of mathematical convenience.

The essential feature of Mack's model is that all the shapes have independently undergone a random uniform orientation. In many practical situations this is fairly realistic. Of interest in Mack's formulae for m and m_1 is the fact that they depend on the convex shapes only through their mean perimeter and area.

Mack proved that, for large n_r and $\|A'\|$

$$(3.9) \qquad m_1 \simeq \frac{\|A\|}{\|A'\|} \sum_{r=1}^{k} n_r \exp(-\sum_{u=1}^{k} n_u b_{ur})$$

where

$$b_{ur} = \|A'\|^{-1}(a_u + a_r + s_u s_r/2\pi) .$$

In 1956 he further showed that

$$(3.10) \qquad m \le \frac{\|A\|}{\|A'\|} \sum_{r=1}^{k} n_r \exp(-\tfrac{1}{2} \sum_{u=1}^{k} n_u b_{ur}) .$$

The techniques used by Mack in deriving (3.9) and (3.10) are very similar. We shall present an heuristic proof of (3.9) using a technique similar to his.

Suppose that each shape has a centre. The probability that this centre is in A is $\|A\|/\|A'\|$. Let p_r be the probability that a random convex shape of area a_r is isolated. Clearly,

$$m_1 = \frac{\|A\|}{\|A'\|} \sum_{r=1}^{k} n_r p_r .$$

Now $p_r = \prod_{u=1}^{k} p_{ru}$, where p_{ru} is the probability that a shape of area a_r is not intersected by any shape of area a_u. To complete the proof we need only show that for large n_u,

$$p_{ru} \simeq \exp(-n_u b_{ur}) .$$

Let S_r be the shape of area a_r. For the moment fix its orientation, and its centre at a point $P \in A$. Let $S_{\underset{\sim}{x}}$ be one of the random convex shapes of area a_u, and centre $\underset{\sim}{x}$. Let A be the set of all points $\underset{\sim}{x} \in \mathbb{R}^2$ such that $S_{\underset{\sim}{x}}$ intersects S_r. Now S is a convex set whose boundary is the set of all points $\underset{\sim}{x}$ such that $S_{\underset{\sim}{x}}$ touches S_r . Let Q be the point $\underset{\sim}{x}$ where $S_{\underset{\sim}{x}}$ just touches S_r at T . See diagram 3.2. Move $S_{\underset{\sim}{x}}$ an infinitesimal amount along the boundary of S_r , maintaining its orientation. Now T is translated to T', Q to Q' and the new point of contact is U .

<u>Diagram 3.2</u> Translate of $S_{\underset{\sim}{x}}$ just touching S_r .

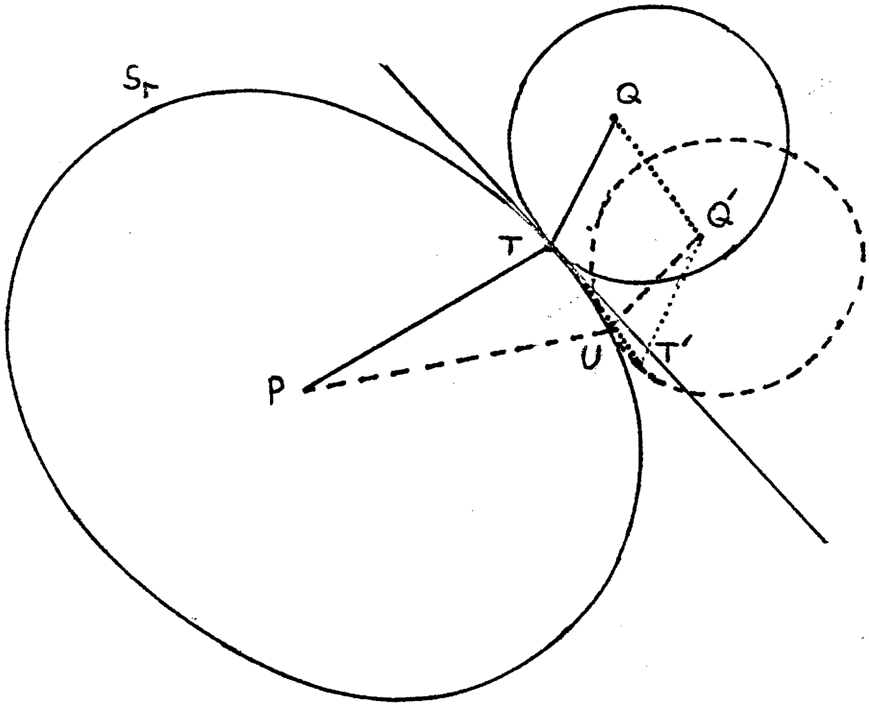
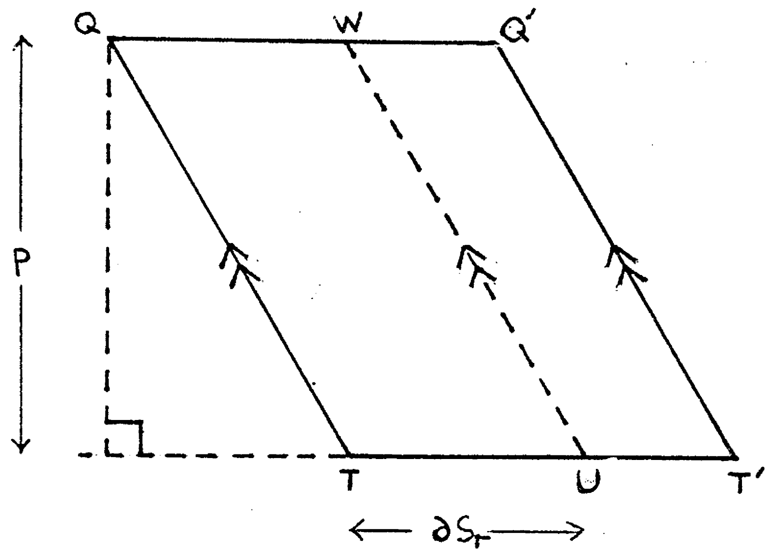


Diagram 3.3 Details of the parallelogram in diagram 3.2



In order to calculate p_{ru} we need to find $\|A\|$. This can be done by integrating around the perimeter of A , the area of the element P T Q Q'U .

It is clear from diagram 3.2 that

(3.11) area(P TQ Q'U) = area (P TU) + area (T Q Q'U) .

Let W divide QQ' in the ratio $|TU| : |TT'|$, where $|XY|$ is the length of the chord XY . Let p be the perpendicular distance from TT' to Q . Refer to diagram 3.3. Since the orientation of $S_{\tilde{x}}$ was maintained in the infinitesimal translation, T Q Q' T' is a parallelogram. Therefore,

(3.12) area(T Q Q' U) = area (Q W T U)

+ area (U Q'T') .

Let ∂S_r be an element of the boundary of S_r .

The infinitesimal area (P T U) is an element of the area of S_r . and the infinitesimal area (Q'T'U) is an element of the area of $S_{\tilde{x}}$. Thus on combining (3.11) with (3.12) and integrating, we obtain

$$\|A\| = \|S_r\| + \|S_{\tilde{x}}\| + \int_{\partial S_r} p\, ds$$

(3.13)

$$= a_u + a_r + \int_{\partial S_r} p ds \; .$$

In the argument just given the orientation of $S_{\tilde{x}}$ remained fixed. In Mack's set-up it is given a uniform orientation on $(0,2\pi)$. On integrating the final term of (3.13) over $(0,2\pi)$ we obtain :

$$\int_{(0,2\pi)} d\theta \int_{\partial S_r} p\, ds = \int_{\partial S_r} ds \int_{(0,2\pi)} p\, d\theta$$

$$= s_r\, s_u \; ,$$

the final step following because $p\, d\theta$ is a small element on the perimeter of $S_{\tilde{x}}$. Hence, the mean area in which shapes of area a_u can intersect S_r is :

$$(2\pi)^{-1} \int_0^{2\pi} \|A(\theta)\|\, d\theta$$

(3.14)
$$= a_v + a_r + s_r\, s_u/2\pi$$

$$= \|A'\|\, b_{ur} \; .$$

The RHS of (3.14) does not depend on the orientation of S_r, so the formula still holds if S_r is given a random uniform rotation. It follows from (3.14), that the probability of a shape of area a_u not intersecting a given shape of area a_r is $1 - b_{ur}$. Therefore,

$$p_{ru} = (1 - b_{ur})^{n_u} \simeq \exp(-n_u b_{ur})$$

for large n_u, as required. $\square$

Mack's formula may also be deduced from the so-called "Fundamental Formula of Integral Geometry", see Blaschke (1949), Santaló(1953,1976) and Miles (1974).

Mack (1956) has generalized the expressions for expected number of clumps, and expected number of clumps of size one, to a three dimensional situation.

Again, convex shapes are randomly and uniformly distributed in a region $A' \subset \mathbb{R}^3$, and given a random and uniform orientation. There are n_r shapes of volume a'_r and surface area s'_r, $1 \leq r \leq k$. The mean perpendicular for a convex shape $S \subset \mathbb{R}^3$ is defined as follows.

Let $r(\theta,\phi)$ be the distance from a pre-defined centre in S, to a point P on the boundary of S, whose tangent plane has a normal with spherical angular coordinates θ,ϕ. The mean radius is

$$R = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r(\theta,\phi)\sin\theta \; d\theta \; d\phi.$$

Formulae (3.9) and (3.10) continue to hold in the three dimensional case if we replace the previous definition of

b_{ur} by

$$b_{ur} = \| A' \|^{-1} (a'_u + s'_u R_r + s'_r R_u + a'_r) ,$$

where R_r is the mean radius for the random convex shapes of area a_r .

There are two minor shortcomings of the formulae obtained by Mack. Firstly, they are only asymptotic approximations. However, they become exact if we assume that the shapes derive from a coverage process in which shapes are centred at points of a Poisson process. In this case, the quantities n_r are means of random variables. Secondly, (3.10) only provides an upper bound for the mean number of clumps. If some prior knowledge of b_{ur} , and the relative proportions of the n_r , $1 \leq r \leq k$, is given, then (3.9) can be used to estimate the average number of shapes per unit area.

By only looking at the number of clumps we are ignoring a large amount of information. Extra information can be obtained by also counting the number of enclosed, or entrapped voids. Kellerer (1983) has obtained an exact formula for the expected number of clumps minus enclosed voids, when the shapes are distributed in a Poisson field.

Let $\mathcal{S}$ be a random shape with mean area a , mean perimeter S and mean Euler characteristic χ . The Euler characteristic of a shape is $\alpha - \beta$, where α is the number separate domains and β the number of enclosed voids. Any convex shape has an Euler characteristic of one. Let S be obtained from $\mathcal{S}$ by giving it a random uniform rotation, about a predefined centre. Let $S_1, S_2, \ldots$ be independent copies of S . Let $S_1, S_2, \ldots$ be centred at the points of a Poisson process of intensity λ .

The mean number of clumps minus voids in a region A of area $||A||$, perimeter $|\partial A|$ and Euler characteristic X is :

$$\exp(-a\lambda)\{\lambda(\chi||A|| + s|\partial A|/2\pi) - \lambda^2||A|| s^2/4\pi - 1\} + 1$$

when $\chi = X = 1$. This result is obtained quite easily from standard results on the mean curvature of random closed circuits.

§4. Applications

Thus far, our review has concentrated on the large body of literature concerned with theoretical analyses of coverage. Research works were divided into various sections, according to their theoretical content. In the present section we attempt to group research works together on the basis of their practical applications.

To review all research works and applications would be a monumental task indeed. Rather, we have chosen a subset of topics, and shall discuss them in enough detail for the reader to appreciate the importance of their application.

§4.1 Military Applications

In the present subsection we shall review some of the military applications of coverage processes to the destruction of point and area targets. For the interested reader, thorough surveys of the literature relating to military applications can be found in three excellent review papers : Guenther and Terrango (1964), and Eckler (1969).

In military applications the following model is commonly used. It is assumed that the bombs' impact points are statistically independent. The random location of an impact point is described by the probability density function $p : \mathbb{R}^k \to \mathbb{R}^+$. Usually we have $k = 2$ or 3 . The damage function d is defined so that the conditional probability that a point target at the origin is destroyed by a bomb impacing at $\underset{\sim}{x}$, is $d(\underset{\sim}{x})$.

One of the simplest situations investigated in the literature is the case where the target, A , is a fixed point. Without any loss in generality assume that $A = \{\underset{\sim}{0}\}$. The probability that a single bomb destroys A is

$$(4.1) \qquad P = \int_{\mathbb{R}^k} d(\underset{\sim}{x})\, p(\underset{\sim}{x})\, dx \quad ,$$

while the probability that at least one of $n \geq 1$ bombs destroys A is $1 - (1 - P)^n$.

In many practical situations the following simplifying assumptions are made. In $k = 2$ dimensions, for some value $R > 0$,

$$(4.2) \qquad d(x,y) = \begin{cases} 1 & \text{when } x^2+y^2 \leq R^2 , \\ 0 & \text{otherwise, and} \end{cases}$$

$$(4.3) \qquad p(x,y) = \frac{1}{2\pi\ \sigma_x \sigma_y} \exp\left\{ - \frac{(x - x_o)^2}{2\sigma_x^2} - \frac{(y - y_o)^2}{2\sigma_y^2} \right\} \quad ,$$

where (x_o, y_o) is the expected point of impact, σ_x^2 is the variance along the x-axis and σ_y^2 is the variance along the y-axis.

The damage function at (4.2) represents the situation where a bomb destroys everything within radius R of its point of impact. A large amount of literature is concerned with tabulating P against the various parameters : x_o, y_o, σ_x, σ_y and R. For many references see Eckler (1969). When $x_o = y_o = 0$ and $\sigma_x = \sigma_y = \sigma$, a closed form solution exists for P. In this case

$$P = \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2<R^2} \exp(-\frac{x^2+y^2}{2\sigma^2})\ dx\ dy$$

$$= \frac{2}{\pi\sigma^2} \iint_{x^2+y^2<R^2, x\geq 0, y\geq 0} \exp(-\frac{x^2+y^2}{2\sigma^2})\ dx\ dy\ ,$$

which on substituting $r = \sqrt{(x^2+y^2)}$ and $s = x/y$ simplifies to

$$\frac{2}{\pi\sigma^2} \int_0^R r \exp(-\frac{r^2}{2\sigma^2})\ dr \int_0^\infty \frac{1}{1+s^2}\ ds$$

$$= \int_0^R \frac{r}{\sigma^2} \exp(-\frac{r^2}{2\sigma^2})\ dr$$

$$= 1 - \exp(-R^2/2\sigma^2)\ .$$

Sometimes it is necessary to represent the target by an area rather than a point. In this case it is reasonable to determine the expected fraction of the target destroyed by a salvo of n bombs.

The expected fraction of a general target, A, destroyed by a single bomb is

$$\iint_{(x_t,y_t)\in A} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d(x-x_t,y-y_t)p(x,y)\,dxdy\; dx_t\; dy_t,$$

where p and d are as previously defined. This may be rewritten as

$$\iint_{(x_t,y_t)\in A} P(x_t,y_t)\; dx_t\; dy_t\;,$$

where

$$P(x_t,y_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d(x-x_t,y-y_t)\; p(x,y)\; dx\,dy$$

is the probability that a single bomb destroys a point target located at (x_t,y_t) .

The probability that at least one of n independent bombs destroys a point target located at (x_t,y_t) is $1 - \{1 - P(x_t,y_t)\}^n$. We must integrate this over the target area to find the expected proportion destroyed. That is,

$$C_n = \frac{1}{\|A\|}\int_A [1 - \{1 - P(\underset{\sim}{x})\}^n]\; d\underset{\sim}{x}\quad,$$

is the expected fraction destroyed.

Most work has concentrated on the case where A is a disc of radius K, and d and p are as defined at (4.2) and (4.3), respectively. Again, a large amount of literature is concerned with tabulating or graphing C_n against the various parameters. As these works are of little interest here, we only mention that many references can be found in the review papers of Eckler (1969), and Guenther and Terrango (1964). There does, however, arise one interesting problem.

Suppose that $x_o = y_o = 0$ and $\sigma_x = \sigma_y = \sigma$. If the precision of the bomb aiming device is perfect then $\sigma = 0$. In this case the fraction of the target destroyed is R^2/K^2, which is less than one when $R < K$. However, as σ increases, C_n initially increases and then approaches zero. Therefore, there must be an optimum value of σ, which maximizes the expected proportion of the target destroyed. Knowing the optimum value would be of practical use if an aiming device could be designed, which allowed us to arbitrarily choose the spread, σ, of impact points. Walsh (1956) has investigated a more general optimization problem, which does not require the assumption of normally distributed impact points.

Walsh defines the function $D : \mathbb{R}^k \to [0,1]$ as follows. The probability that a bomb with impact point $\underset{\sim}{y}$ destroys a given target $A \subseteq \mathbb{R}^k$ is $D(\underset{\sim}{y})$. Let $p(\underset{\sim}{x}-\underset{\sim}{y})$ be the conditional probability density function of the hit location, $\underset{\sim}{y}$, from a bomb with expected hit location, $\underset{\sim}{x}$, and $q(\underset{\sim}{x})$ the density function of $\underset{\sim}{x}$. The conditional probability that a bomb with expected hit location $\underset{\sim}{x}$ destroys A is

$$H(\underset{\sim}{x}) = \int_A D(\underset{\sim}{y})\, p(\underset{\sim}{x} - \underset{\sim}{y})\, d\underset{\sim}{y} ,$$

while

$$H_n = \int_{\mathbb{R}^k} [1 - \{1 - H(\underset{\sim}{x})\}^n] q(\underset{\sim}{x})\, d\underset{\sim}{x}$$

is the probability that at least one bomb in a salvo of n bombs destroys the target.

We see that the notation used by Walsh is not quite the same as that introduced at the beginning of this section. However, it is convenient in practical situations, for the function D may be defined arbitrarily. For example, D could equal one if a predefined proportion of the target was destroyed. Suppose we also fix the function q. Then, H_n only depends on the hit location function p. Walsh described the following procedure for obtaining the optimum, or maximum, value of H_n.

Let $\bar{H}_n$ be the optimum of H_n and $\bar{H}(\underset{\sim}{x})$ the corresponding optimum for $H(\underset{\sim}{x})$. Firstly, determine the value of c satisfying

$$\int_{\{\underset{\sim}{x}:q(\underset{\sim}{x})\geq c\}} \{1-(c/q(\underset{\sim}{x}))^{1/(n-1)}\}\, d\underset{\sim}{x}$$

$$= \frac{1}{||A||}\int_A D(\underset{\sim}{y})\, d\underset{\sim}{y} .$$

Then,

$$\bar{H}_n = 1 - c\{\int_{q(\underset{\sim}{x})\geq d\underset{\sim}{x}} d\underset{\sim}{x} \ , - \frac{1}{||A||}\int_A D(\underset{\sim}{y})\, d\underset{\sim}{y}\}$$

$$- \int_{q(\underset{\sim}{x})<c} q(\underset{\sim}{x})\, d\underset{\sim}{x} ,$$

while

$$\overline{H}(x) = \begin{cases} 1 - \{c/q(\underset{\sim}{x})\}^{1/(n-1)} & \text{for } \underset{\sim}{x} \text{ satisfying } q(\underset{\sim}{x}) \geq c , \\ 0 & \text{otherwise.} \end{cases}$$

Determination of the optimum p is done by an iterative 'cut and try' method.

In more refined models one part of the target may be more important than another. We could measure the importance by a density function A . In this case, the expected total value of the target destroyed by a salvo of n bombs is

$$(4.4) \qquad C_n = \int_{\mathbb{R}^k} A(\underset{\sim}{x})[1 - \{1 - P(\underset{\sim}{x})\}^n] \, d\underset{\sim}{x} \quad ,$$

where $p(\underset{\sim}{x})$ is the probability that a single bomb destroys a point target at $\underset{\sim}{x}$.

Expression (4.4) is also the probability that a point target, placed at random according to the density A , is destroyed by a salvo of n bombs. McNolty (1967) has obtained simple expressons for C_n when the bombs' location density, p , and damage function, d , are as defined at (4.3) and (4.2), respectively, and when the target's random location is distributed as a circular normal distribution. Other related problems, which have been the subject of investigation are :

a) the probability of destroying the target exactly m times, where $m \leq n$;

b) the expected number of bombs required to destroy the point target exactly once;

and

c) the probability that n or fewer bombs are required to destroy the target exactly once.

Many more complex models exist. However, little insight is gained through describing them. In the following subsection we investigate physical and biological applications of coverage processes.

§4.2 Other Applications

In the present subsection we shall describe several applications including : applications in image analysis; fibre counting problems; the modelling of a problem in virology; vapour to liquid phase transitions, and the spatial pattern of heather in a field. For each application we describe the coverage model that has been applied and state results where necessary.

Serra (1982), in his book titled "Image Analysis and Mathematical Morphology", considered the practical applications of what he called a Boolean model. As described in the introduction, a Boolean model is a coverage process consisting of a collection of independently and identically distributed random shapes placed at the points of either an homogeneous or inhomogenous Poisson process. Serra described a method of testing for a Boolean model based on the functional Q defined by :

$$Q(B) \equiv P(B \text{ does not intersect any of the random sets}) ,$$

where B is a Lebesgue measureable subset of $\mathbb{R}^k$. Such work has applications in minerology, for example, the modelling of ferrite crystals in iron sinter, or biological applications such as the random distribution of trees in a forest.

In subsection 3.3 we described the mathematical techniques used for adjusting undercounts of dust particles caused by clumping. We now discuss some important applications of this theory.

In some industries airborne asbestos fibres result as a product of the manufacturing process. It is important to determine whether the concentration of these fibres is sufficient to be a health risk. The concentration can be estimated by examining an asbestos-laden membrane filter under a microscope.

Iles and Johnston (1983) described a procedure which requires that all fibres less than $3\mu m$ in diameter be counted. Larger fibres are not respirable and hence do not represent a health risk. If a fibre is in contact with a particle greater than $3\mu m$ in diameter, then it is ignored. The problem is to determine the respirable fibre concentration from the membrane filter sample. Due to overlap, the count of respirable fibres on the membrane must be adjusted upwards.

Since asbestos fibres are long and narrow, the model which is commonly used, consists of a collection of elongated rectangular shapes, which have been independently and uniformly thrown on a region, A . Independently of this, each rectangle is given an independent and uniform rotation around its centre.

Iles and Johnston have shown how to calculate the concentration of respirable airborne fibres, with reasonable accuracy, from counts of the number of respirable fibre clumps on the membrane. Their work is closely related to the theoretical papers of Irwin, Armitage and Davies (1949) , Armitage (1949) and Mack (1954, 1956), in which approximate formulae are derived for the mean number of clumps. See section 3.3.

Mack's work allows for a more general definition of the shape distribution of fibres. As only small fibres represent a health risk, it is important to be able to estimate the distribution. Schneider, Holst and Skottle (1983) discuss the relationship between the shape distribution and influencing factors such as : the manufacturing process which creates the dust clouds, air supply and ventilation, and the aerodynamic properties of the fibres.

The concentration of airborne fibres is estimated by firstly counting the number of fibre clumps on a filter. In practical situations this is an onerous and time consuming task. Attfield and Beckett (1983) described a simple technique to ease the task of fibre counting.

Briefly, microscope samples are examined for the presence or absence of fibres. If m samples are void of fibres, then the estimated density of fibres is $\ln(n/m)$. The derivation of this formula, which assumes a Poisson distribution of fibres, is described in detail by Attfield and Beckett.

The presence of dust particles, other than asbestos fibres, in the workplace is of major concern. These particles can represent a health hazard to the workers. In some cases the shapes of particles may be better represented by convex regions other than rectangles. Mack (1956) has developed a general theory for adjusting for undercounts in such circumstances.

Particle counting is also of importance in the biological sciences. For example, we may wish to count the number, n , of bacteria colonies on a microscope slide. The colonies can overlap and are approximately circular in shape. Thus, we could use a model consisting of discs spread uniformly and at random throughout the region A . As discussed in subsection 3.3, Armitage (1949) derived a formula for estimating n .

Moran and Fazekas de St Groth (1962) studied the following problem in virology, which can be modelled by a simple coverage process. The problem arises from the way that antibodies prevent a virus from attacking a cell. The thin cigar-shaped antibodies attach themselves end-on to the approximately spherical virus, preventing a circular cap region on the surface of the virus from touching any cell. If n antibodies randomly attach to the virus, then the probability of complete coverage of the virus's surface by the associated circular caps is the probability that the virus will not infect any cell. In subsection 1.2 we discussed in some detail how Moran and Fazekas de St Groth and subsequently Miles (1969) derived approximations to this probability. The interested reader may refer to the discussion in subsection 1.2.

Yet another application of coverage arises in physical chemistry. Widom and Rowlinson (1970) studied the applicability of the following coverage model for liquid to vapour phase transitions. The model consists of n molecules, uniformly and randomly distributed in a large region A. Each molecule is represented by a sphere of volume a. According to Widom and Rawlinson, the potential energy of the system is

$$U \equiv (\|A\| - V - na)\varepsilon/a \quad ,$$

where V is the vacancy and $\varepsilon > 0$ is an energy constant. The quantity U can be used to determine the phase state of the material. Ultimately, this will depend on the distribution of V. However, the exact form of this distribution is unknown, so Melynk and Rowlinson searched for an approximation. Using Monte Carlo methods, they showed that $(\|A\| - V)/\|A\|$ was approximately normal in distribution. As discussed in section 2, this result was later justified in a theoretical manner by Moran (1973b).

An early application of Stevens' (1939) work, on the complete coverage of a circle by n uniformly distributed arcs, arose in harmonic analysis. Suppose that each arc is of length a. We proved in section 1.1 that the probability of complete coverage is

$$(4.5) \qquad 1 - n(1-a)^{n-1} + \ldots + (-1)^k \binom{n}{k} (1-ka)^{n-1} \quad ,$$

where k is the integer part of $1/a$. Fisher (1940) noted a remarkable similarity between Stevens' result at (4.5) , and the probability of Type One error in a test of significance of the largest harmonic component in a series of normally distributed observations. This led Fisher to the discovery of how his problem in harmonic analysis related to Stevens' model in geometrical probability.

In section 3.1 we described in some detail the application of coverage theory to continuum percolation. We shall briefly mention one important application here.

Suppose that impure conducting particles in a semiconductor are represented by spheres. An electron may pass between two impure particles if their associated spheres overlap. Thus, conduction occurs when a clump of spheres of infinite size forms : that is, when percolation of the spheres occurs. Clearly, the concentration of impure particles plays an important role in determining conduction. Using a Poisson point process to represent the spheres' centres, Gilbert (1961) proved that there exists a critical concentration above which percolation, and hence conduction, occurs.

Diggle (1981) employed a coverage model to model the growth pattern of heather in a field. Heather plants grow from seedlings reaching a maximum radius of about 50 cms. The branches of adjacent trees intermingle if the regions they occupy overlap. Under these restrictions, Diggle proposed the following model. Viewed from above, the heather plants may be represented by random radius circles, and the centres of the bushes by a Poisson process with intensity λ . Figure 3 below, which appears in Diggle (1983), shows a 1:100 scale map of heather, coded as a 100×200 binary matrix. The shaded area represents heather, while the light area represents the vacant region.

Diggle's primary concern was to find a model which fitted the data well. This involved estimating the parameter λ , and the density function , f , for the heathers' radii. One simplifying assumption made by Diggle was that f is a relocated Weibull distribution, so that

$$f(r) = k\rho(r-\delta)^{k-1} \exp\{\rho(r-\delta)^{k}\}, \qquad r > \delta ,$$

where k,ρ and δ are positive constants. Let us define $G(u)$ as the distribution functions of the distance of an arbitrary point in the plane to the nearest heather plant , and $\gamma(u)$ as the covariance between points a distance u apart , coding 1 for points occupied by heather and zero otherwise.

Diggle showed that the fit obtained by using γ was much better than that obtained by using G . The reason γ works better than G is that G depends only on the first two moments of disc radius, while γ depends on all properties of disc radius. Hence there is much more information contained in γ than in G . Even the model obtained by using γ is not entirely adequate as illustrated in diagram 4.2. Comparing diagram 4.1 and 4.2 we see that there are distinctly fewer clumps in the raw data than the simulated data.

Diagram 4.1: Map of heather in a 10m x 20m field.
Heather is represented by the shaded region.

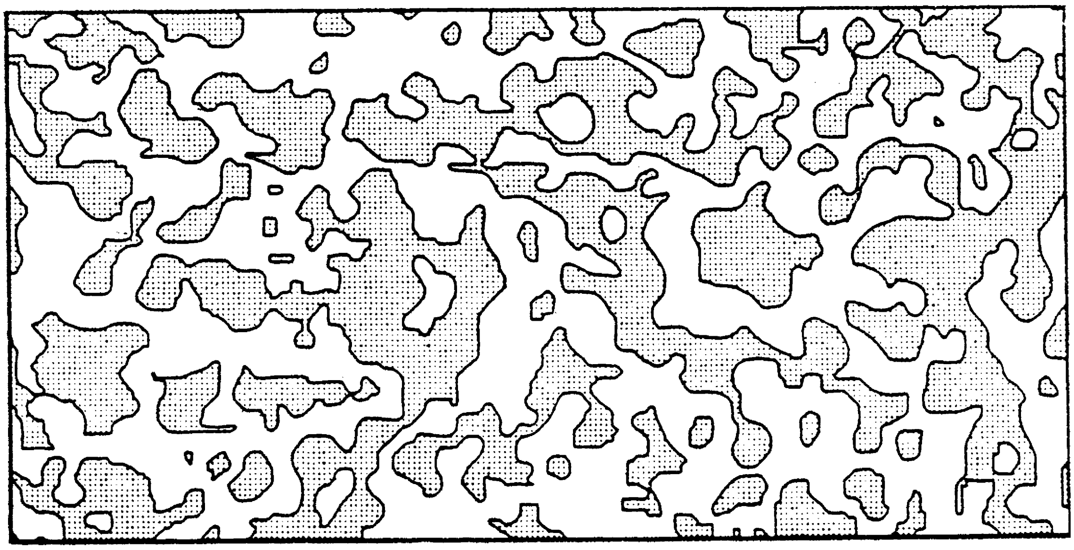
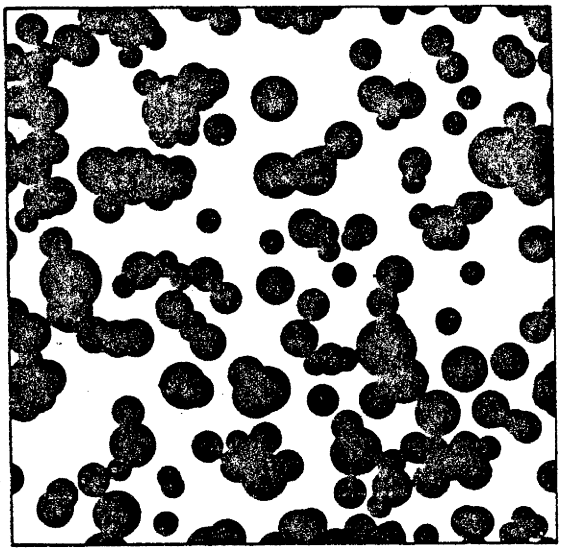


Diagram 4.2: Simulated heather pattern from the model obtained by fitting γ .



Diagrams 4.1 and 4.2 were reproduced with the kind permission of the Biometrics Society.

Chapter 2. Testing the Hypothesis of Uniformity

Recall the definition of a coverage process $\mathcal{C}$ described in the Introduction and Summary. In a region $A \subset \mathbb{R}^k$ points $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$ are distributed according to some random process $\mathcal{P}$. By definition $\mathcal{C}$ consists of independently distributed copies $S_1, S_2, \ldots$ of the random set S relocated at the points of $\mathcal{P}$. It is also assumed that S_i, $i \geq 1$, and $\mathcal{P}$ are statistically independent. Due to overlaping of random sets the location of all the points $\underset{\sim}{X}_i$, $i \geq 1$, is in general non-observable.

In this chapter the aim is to construct tests of the hypothesis that the underlying point process $\mathcal{P}$ is uniform on A, and to analyse the asymptotic power properties of these tests.

In order to construct a theory several simplifying assumptions are made. Assume that $\mathcal{P}$ consists of exactly $n \geq 1$ points, which are independently and identically distributed on the region A. To avoid the difficulties of "edge effects" let A be a k-dimensional unit cube which is also topologically a torus. The properties of a topological torus were discussed in section 2 of chapter 1. Essentially, if a random shape protrudes beyond one side of A it is introduced at the other side. This technique is frequently used in geometrical probability. See for example Miles (1969). To overcome the problems associated with a random set completely wrapping around A and overlapping itself, assume that, with probability one, S is contained inside a sphere of diameter one. The notation $\|.\|$

is often used throughout this chapter to denote the Lebesgue measure of the argument set.

As was seen in the section on applications in the previous chapter, the assumption of uniformity was frequently made. Such an assumption could be well justified in many situations. For example, one would expect gas particles to be uniformly distributed in the vapour-liquid phase transition model used by Melnyk and Rowlinson (1971). However, in the heather plant application considered by Domb (1981) , it would be useful to test the hypothesis of uniformity in the early stages of model fitting. Hall (1984b) constructed a test of uniformity based on vacancy assuming that P was a Poisson point process. Some of Hall's work has direct bearing on the ensuing theory. This is clearly indicated at appropriate stages throughout the chapter. In order to obtain limit results for tests of uniformity Hall used a "change in perspective" concept which is now briefly described.

Initially, let us confine attention to $k=2$ dimensions and uniformly distributed shapes. Suppose a survey camera is mounted in a helicopter, which is flying above a forest. Let A be the square glass plate on the back of the camera. The image of the forest projected on the plate may be represented by a coverage process. The mean size of a tree on the plate is the expected area of a random shape. If the helicopter was to increase its altitude by a factor δ^{-1} , where $\delta < 1$, then the expected area of a random shape would decrease by a factor δ^2 , and the number of shapes per unit area would approximately increase by a factor δ^{-2} . Suppose we were to replace the generating shape S in C

by $\delta S = \{\underset{\sim}{x} : \delta \underset{\sim}{x}^{-1} \in S\}$, then the situation described above corresponds to letting $\delta \to 0$ and $n \to \infty$ in such a manner that $n\delta^2 \to \rho$, where $0 < \rho < \infty$. In the k-dimensional situation allow $\delta \to 0$ and $n \to \infty$ so that $n\delta^k \to \rho$. The associated coverage process for uniformly distributed shapes is denoted by $C(\delta,n)$. When the $\underset{\sim}{X}_i$'s are distributed according to a distribution with density f the coverage process is denoted by $C_f(\delta,n)$. Under the above convergence criteria a central limit theorem for vacancy in a 3-dimensional cube in which spheres of common radius are uniformly distributed was obtained by Moran (1973b).

In section 1 we generalize Moran's result to the case of random and bounded shapes in a k-dimensional set-up. From this theory an asymptotically consistent test of uniformity of the underlying point process based on vacancy is constructed in subsection 1.2.

For increasing n a sequence of density functions f_n , which converge to the uniform density, will be defined. In subsection 2.2 the asymptotic power, or so called local power, of the test based on vacancy is determined for a variety of convergence rates for f_n . It will be shown that the test can only distinguish differences of order $O(n^{-\frac{1}{4}})$ from the null hypothesis of uniformity as $n \to \infty$. In the one dimensional situation Hüsler has considered two other statistics : the total number of uncovered spacings, and the length of the maximum spacing.

Simple tests of uniformity will be constructed from these statistics, and their local power derived. The local power of the test based on the total number of uncovered spacings is comparable to the test based on vacancy.

However, its asymptotic consistency can only be assured when $\rho \leq \{E(\|S\|)\}^{-1}$. As so happens, the test based on the length of the maximum spacing is poor in comparison and can only detect differences of order $O[\{\log(n)\}^{-1}]$ from the null hypothesis as $n \to \infty$.

§1 Uniformly Distributed Random Shapes.

Throughout this section it is assumed that $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ are independently and uniformly distributed on A. Under this hypothesis, in subsection 1 formulae, and asymptotic approximations, are obtained for the expectation and variance of vacancy. Of practical importance in constructing an asymptotically consistent test of the hypothesis of uniformity, is a central limit theorem for vacancy.

Moran (1973b) proved such a result in the special case where n spheres of equal radius are independently and uniformly distributed within a 3-dimensional cube. In subsection 2 we prove a more general limit theorem for random and uniformly bounded shapes in a k-dimensional setting. The proof is considerably different to Moran's and does not use his characteristic function argument, but rather a simple application of the Berry-Esseen theorem.

1.1. Expectation and Variance of Vacancy.

Our first result gives exact formulae for the expectation and variance of vacancy. The Lebesgue measurable set S rescaled by a factor δ and translated through $\underset{\sim}{x}$ is denoted by

$$\underset{\sim}{x} + \delta S \equiv \{\underset{\sim}{x} + \delta \underset{\sim}{y} : \underset{\sim}{y} \in S\} .$$

Theorem 1

For the coverage process C and vacancy as defined in the introduction

(1.1) $$E(V) = \{1 - E(\| S \|)\}^n$$

and

(1.2) $$\mathrm{Var}(V) = \int_{A^2} [\{1 - 2E(\| S \|) + E(\|S \cap (S + \underset{\sim}{x}_1 - \underset{\sim}{x}_2)\|\}^n$$

$$- \{1 - 2E(\| S \|) + (E(\| S \|))^2\}^n] \, d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2 .$$

Proof

The probability that $\underset{\sim}{x} \in A$ is covered by $\underset{\sim}{X}_i + S_i$ is $E(\| S \|)$. Thus $E(\chi(\underset{\sim}{x})) = \{1 - E(\| S \|)\}^n$, where $\chi(\underset{\sim}{x})$ is 1 when $\underset{\sim}{x}$ is uncovered and zero otherwise, and so

$$E(V) = \{1 - E(\| S \|)\}^n ,$$

as required.

Since A is topologically a torus, the stochastic properties of C are invariant under translations. Thus,

$$E\{\chi(\underset{\sim}{x}_1) \; \chi(\underset{\sim}{x}_2)\}$$

$$= P(\text{for } 1 \le i \le n , \; \underset{\sim}{x}_1 \notin \underset{\sim}{X}_i + S_i \text{ and } \underset{\sim}{x}_2 \notin \underset{\sim}{X}_i + S_i)$$

$$= P(\text{for } 1 \le i \le n , \; \underset{\sim}{X}_i \notin (\underset{\sim}{x}_1 - \underset{\sim}{x}_2 + S_i) \cup S_i)$$

$$= \{1 - 2E(\| S \|) + E(\| S \cap (S + \underset{\sim}{x}_1 - \underset{\sim}{x}_2) \|)\}^n .$$

Hence

$$\mathrm{Var}(V) = \int_{A^2} [E\{\chi(\underset{\sim}{x}_1)\chi(\underset{\sim}{x}_2)\} - E\{\chi(\underset{\sim}{x}_1)\}E\{\chi(\underset{\sim}{x}_2)\}]\,d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$= \int_{A^2} [\{1 - 2E(||S||) + E(||S \cap (S + \underset{\sim}{x}_1 - \underset{\sim}{x}_2)||)\}^n$$

$$-\{1 - 2E(||S||) + (E(||S||))^2\}^n]\,d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

which completes the proof of theorem 1. $\square$

In the introduction to this chapter a method of obtaining a limit theorem for vacancy was described. The method was discussed in terms of a change in perspective. Essentially, if n is the number of shapes in A and the expected size of a random shape is proportional to δ, we allow $n \to \infty$ and $\delta \to 0$ in such a manner that $n\delta^k \to \rho$, where $0 < \rho < \infty$. The coverage process corresponding to this situation is as follows.

Let A be a k-dimensional unit cube and assume that A is topologically a torus. As before let $S_1, S_2, \ldots$ be independent and identically distributed copies of the random set S, independent of $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots$, which are independently and uniformly distributed points on A. Furthermore, assume that S is contained inside a sphere of diameter one with probability one. For $\delta < 1$, define $C(\delta, n)$ as the collection of random sets $\underset{\sim}{X}_i + \delta S_i$, $1 \le i \le n$.

The following theorem gives asymptotic approximations for the expectation and variance of vacancy.

Theorem 2.

Let $V \equiv V(\delta, n)$ *denote the vacancy in the unit cube* A *arising from the coverage process* $C(\delta, n)$. *If* $\delta \to 0$ *as* $n \to \infty$, *in such a manner that* $\delta^k n \to \rho$, *where* $0 < \rho < \infty$, *then*

$$E|V - \exp\{-\rho E(||S||)\}| \to 0$$

as $n \to \infty$ *and*

$$n \text{ var}(V) \to \tau^2$$

$$\equiv \rho \, e^{-2\rho E(||S||)} \int_{\mathbb{R}^k} [\exp\{\rho E(||S \cap (S+\underset{\sim}{z})||)\} - 1] d\underset{\sim}{z}$$

$$- \rho^2\{E(||S||)\}^2 \, e^{-2\rho E(||S||)}$$

as $n \to \infty$.

Proof

The proof is trivial when $E(||S||) = 0$. For the rest of the proof assume that $E(||S||) > 0$. By replacing S by δS in (1.1) we obtain

$$E(V) = \{1 - \delta^k E(||S||)\}^n \to \exp\{-\rho E(||S||)\}$$

as $n \to \infty$. Furthermore, $|V| < 1$, so to prove the theorem it suffices to show that

(1.3) $$\text{var}(V) = E\{|V - E(V)|^2\}$$

$$= \tau^2/n + o(1/n)$$

as $n \to \infty$. Since S is bounded by a sphere of diameter one and $\delta < 1$, $E\{\|\delta S \cap (\delta S + \underset{\sim}{x}_1 - \underset{\sim}{x}_2)\|\} = 0$ when $|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| > \delta$. Thus, to find an asymptotic estimate of (1.2) we replace S by δS and the integrate in two parts: $|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| > \delta$ and $|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta$. Writing $a = E(\|S\|)$, the first of these integrals is :

$$\int_A d\underset{\sim}{x}_1 \int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| > \delta,\ \underset{\sim}{x}_2 \in A} \{(1-2\delta^k a)^n - (1-2\delta^k a + \delta^{2k} a^2)^n\}\, d\underset{\sim}{x}_2$$

$$= \|A\|^2 (1+o(1))\ [(1-2\delta^k a)^n - \{(1-2\delta^k a)^n$$

$$+ n\delta^{2k} a^2 (1-2\delta^k a)^{n-1} + O(n^{-2})\}]$$

$$= -n^{-1}\rho^2 a^2 e^{-2a\rho} + o(n^{-1}) \quad \text{as } n \to \infty . \tag{1.4}$$

As for the integral over $|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta$

$$\Bigg| \int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta,\ \underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A} [\{1 - 2\delta^k a + \delta^k E(\| S \cap (S + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta)\|)\}^n$$

$$- \{1 - 2\delta^k a + \delta^{2k} a^2\}^n]\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$- \int_{|x_1 - x_2| < \delta\ \ \underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A} [\exp\{-2n\delta^k a + n\delta^k E(\|S \cap (S + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta)\|)\}$$

$$- \exp\{-2n\delta^k a\}]\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2 \Bigg|$$

$$\leq C\, n^{-1} \int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta\ ,\ \underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A} d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

(1.5) $$= o(n^{-1})$$

as $n \to \infty$, where C is a constant independent of n .

However,

$$\int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta,\ \underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A} [\exp\{n\delta^k E(\| S \cap (S + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta))\} - 1]\ d\underset{\sim}{x}_1 d\underset{\sim}{x}_2$$

$$= (1 + o(1)) \int_{\underset{\sim}{x}_1 \in A} d\underset{\sim}{x}_1 \int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta, \underset{\sim}{x}_2 \in A} [\exp\{\rho\ E(\|S \cap (S + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta)\|)\} - 1]\, d\underset{\sim}{x}_2$$

(1.6)

$$= \delta^k (1 + o(1)) \int_{\underset{\sim}{x}_1 \in A} d\underset{\sim}{x}_1 \int_{\{\underset{\sim}{z}\ :\ \underset{\sim}{x}_1 - \delta \underset{\sim}{z} \in A \text{ and } |\underset{\sim}{z}| < 1\}} [\exp\{\rho E(\|S \cap (S + \underset{\sim}{z})\|)\} - 1]\ d\underset{\sim}{z} \quad .$$

The difference between the multiple integral on the RHS of (1.6) and the same integral over $\{\underset{\sim}{x}_1, \underset{\sim}{z} : \underset{\sim}{x}_1 \in A$ and $|\underset{\sim}{z}| < 1\}$ is bounded by a constant multiple of

$$\int_{\underset{\sim}{x}_1 \in A} d\underset{\sim}{x}_1 \int_{|\underset{\sim}{z}| < 1,\ \underset{\sim}{x}_1 - \delta \underset{\sim}{z} \notin A} d\underset{\sim}{z}$$

$$\leq \int_{A^*} d\underset{\sim}{x} \ ,$$

where $A^* = \{\underset{\sim}{x} : \underset{\sim}{x} \in A$ and $\underset{\sim}{x} - \underset{\sim}{z} \notin A$ for some $|\underset{\sim}{z}|<\delta\}$. The measure of this set is bounded by a constant times δ , which converges to zero as $n \to \infty$. Thus,

$$\int_{|\underset{\sim}{x}_1-\underset{\sim}{x}_2|<\delta\ ,\ x_1,x_2 \in A} [\exp\{n\delta^k E(\|S \cap (S+(\underset{\sim}{x}_1-\underset{\sim}{x}_2)/\delta)\|)\} - 1]\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$= n^{-1} \rho(1+o(1)) \int_{\underset{\sim}{x}_1 \in A} d\underset{\sim}{x}_1 \int_{\{\underset{\sim}{z} : |\underset{\sim}{z}|<1\}} [\exp\{\rho E(\|S \cap (S+\underset{\sim}{z})\|)\} - 1] d\underset{\sim}{z}$$

(1.7)

$$= n^{-1} \rho \int_{\mathbb{R}^k} [\exp\{\rho E(\|S \cap (S+\underset{\sim}{z})\|)\} - 1]\, d\underset{\sim}{z} + o(n^{-1}) \quad .$$

as $n \to \infty$.

Consequently, by (1.2), (1.4), (1.5) and (1.7)

$$\mathrm{var}(V) = n^{-1} \rho\, e^{-2a\rho} \int_{\mathbb{R}^k} [\exp\{\rho E(\|S \cap (S+\underset{\sim}{z})\|)\} - 1]\, d\underset{\sim}{z}$$

$$- n^{-1}\rho a^2 e^{-2a\rho} + o(n^{-1}) ,$$

which proves (1.3) and completes the proof of theorem 2. $\square$

The special case resulting when $E(\|S\|) = 0$ is of little practical importance, and hence is omitted from any future discussion. That is, assume throughout the rest of this chapter that $E(\|S\|) > 0$.

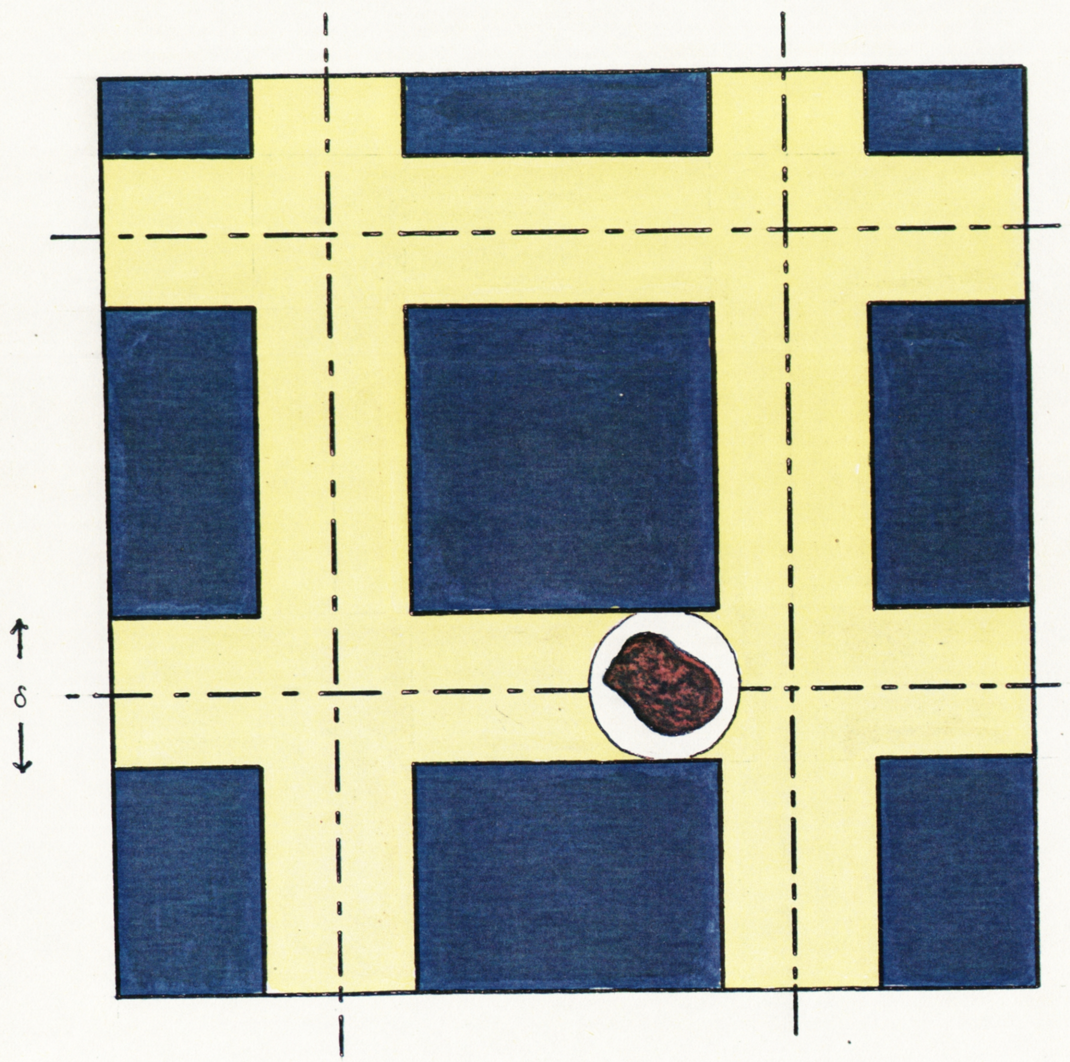
In the following subsection a central limit theorem for vacancy $V(\delta,n)$ will be obtained.

1.2 A Central Limit Result.

The technique used to prove a central limit result is as follows. Divide A up into a regular lattice of subcubes and concentric subsubcubes, so that each subsubcube is separated by a distance of at least δ . See diagram 1.1 below. Since the random sets $\underset{\sim}{X}_i + \delta S_i$, $i \geq 1$, are bounded they can intersect, at most, one subsubcube. However, the vacancies within each subsubcube are not statistically independent because the number of centres occuring in one subcube depends upon the number of centres occuring in all other subcubes. To overcome this problem we condition upon the number of centres occuring within each subcube. Then it is quite easy to apply the Berry-Esseen theorem. The limit result is consequently established by showing that the vacancy in A may be closely approximated by the sum of vacancies within the subsubcubes.

The central limit result is stated next. However, the proof is prefaced with two useful lemmas. The techniques used in the proof are very similar to those used by Hall (1984a) .

Diagram 1.1 : A bounded random shape can intersect at most one subsubcube.



Theorem 3

For $\delta < 1$ let $V = V(\delta,n)$ be the vacancy in the unit cube A resulting from $C(\delta,n)$. Assume that with probability one, S is contained inside a sphere of diameter 1 . If $\delta \to 0$ as $n \to \infty$ in such a manner that $\delta^k n \to \rho$, where $0 < \rho < \infty$, then

$$\sqrt{n}\{V - E(V)\} \xrightarrow{\mathcal{D}} N(0,\tau^2) ,$$

that is convergent in distribution to a normal law with mean zero and variance τ^2 as defined in Theorem 2.

Lemma 1

Let A and B be two Lebesgue measurable subsets of $\mathbb{R}^k$. Then ,

$$\int_{\mathbb{R}^k} \|(x + A) \cap B\| \, d\underset{\sim}{x} = \|A\| \, \|B\|$$

Proof

Since A and B are Lebesgue measureable,

$$\int_{\mathbb{R}^k} \|(\underset{\sim}{x} + A) \cap B\| \, d\underset{\sim}{x}$$

$$= \int_{\mathbb{R}^k} d\underset{\sim}{x} \int_{(x + A) \cap B} d\underset{\sim}{y}$$

$$= \int_B d\underset{\sim}{y} \int_{\underset{\sim}{y} - A} d\underset{\sim}{x}$$

$$= \int_B ||A|| \; d\underset{\sim}{x} = ||A|| \, . \, ||B|| \; ,$$

as required. $\square$

Lemma 2

Let $\{X_n\}$ be a sequence of random variables. Assume that there exists random variables $Y_n(\xi)$ and $Z_n(\xi)$ such that

$$X_n = Y_n(\xi) + Z_n(\xi) \; ,$$

where $\xi > 0$. Suppose that for random variables $Y(\xi)$ and Y ,

(1.8) $\qquad Y_n(\xi) \to^{\mathcal{D}} Y(\xi) \qquad$ as $n \to \infty$,

(1.9) $\qquad Y(\xi) \to^{\mathcal{D}} Y \qquad$ as $\xi \to \infty$,

and for all $\varepsilon > 0$,

(1.10) $\qquad \lim_{\xi\to\infty} \limsup_{n\to\infty} P(|Z_n(\xi)| > \varepsilon) = 0$.

Then,

$$X_n \to^{\mathcal{D}} Y$$

as $n \to \infty$.

Proof

Let $\varepsilon > 0$ be arbitrary. Then,

$$P(X_n \leq x) = P(Y_n(\xi) + Z_n(\xi) \leq x)$$

$$\leq P(Y_n(\xi) - \varepsilon \leq x) + P(Z_n(\xi) < -\varepsilon)$$

$$\leq P(Y_n(\xi) \leq x + \varepsilon) + P(|Z_n(\xi)| > \varepsilon) .$$

Hence, by (1.8)

$$\limsup_{n\to\infty} P(X_n \leq x) \leq \limsup_{n\to\infty} P(Y_n(\xi) \leq x+\varepsilon)$$

$$+ \limsup_{n\to\infty} P(|Z_n(\xi)| > \varepsilon)$$

$$\leq P(Y(\xi) \leq x + 2\varepsilon) + \limsup_{n\to\infty} P(|Z_n(\xi)| > \varepsilon) .$$

Allow $\xi \to \infty$ on the RHS of this expression.
Then by (1.9) and (1.10) ,

$$\limsup_{n\to\infty} P(X_n \leq x) \leq P(Y \leq x + 3\varepsilon) .$$

Suppose that x is a continuity point Y . Since $\varepsilon > 0$ is arbitrary ,

$$\text{(1.11)} \qquad \limsup_{n\to\infty} P(X_n \leq x) \leq P(Y \leq x) .$$

The reverse inequality for $\lim_n \inf$ may be obtained as follows. For $\varepsilon > 0$,

$$P(X_n \le x) = P(Y_n(\xi) + Z_n(\xi) \le x)$$

$$\ge P(Y_n + \varepsilon \le x) - P(Z_n(\xi) \ge \varepsilon)$$

$$\ge P(Y_n(\xi) \le x - \varepsilon) - P(|Z_n(\xi)| \ge \varepsilon)$$

By taking $\liminf_{n\to\infty}$ on both sides of this expression and then letting $\xi \to \infty$ we obtain

(1.12) $$\liminf_{n\to\infty} P(X_n \le x) \ge P(Y \le x)$$

at every continuity point x of Y. Lemma 2 follows from expressions (1.11) and (1.12). |_|

Proof of theorem 3

We partition the unit cube A as follows. Let c be a large positive constant and m the largest integer less than or equal to $[(c+1)\delta]^{-1}$. Define b by $m = [(b+1)\delta]^{-1}$. Divide A into a lattice of m^k subcubes each of side length $(b+1)\delta$. Within each subcube construct a concentric subsubcube of side length $b\delta$. Let N_i be the number of centres that occur in the i'th subcube and V_i the vacancy in the corresponding subsubcube. Let R be the vacancy in the region remaining after removing all subsubcubes from A. Then,

$$V = \Sigma_i V_i + R$$

and so

$$V-E(V) = \Sigma_i\{V_i - E(V_i|N_i)\}$$

$$+ \Sigma_i\{ E(V_i|N_i) - E(V_i)\} + \{R - E(R)\} .$$

The proof is divided into several parts. In step (i) we show, using the Berry-Esseen theorem, that

$$\sqrt{n}\ \Sigma_i\ \{V_i - E(V_i|N_i)\} \to^{\mathcal{D}} N(0,\tau_1^2)$$

as $n \to \infty$, while in step (ii) we prove, using a theorem of Holst (1972), that

$$\sqrt{n}\ \Sigma_i\{E(V_i|N_i) - E(V_i)\} \to^{\mathcal{D}} N(0,\tau_2^2) ,$$

as $n\to\infty$, where τ_1^2 and τ_2^2 are positive constants. In step (iii) of the proof we show, as a consequence of steps (i) and (ii) , that

$$\sqrt{n}\ \Sigma_i\{V_i - E(V_i)\} \to^{\mathcal{D}} N(0,\tau_1^2 + \tau_2^2) .$$

According to lemma 2 it suffices to show

$$\tau_1^2 + \tau_2^2 \to \tau^2$$

as $b \to \infty$, and

$$\lim_{b\to\infty}\ \limsup_{n\to\infty} n\ \mathrm{Var}(R) = 0$$

to complete the proof of theorem 3 . This is done in step (iv) below.

Step (i)

Let us first show that

(1.13) $$n \Sigma_i \operatorname{var}(V_i|N_i) \to \tau_1^2$$

in probability as $n \to \infty$, where

$$\tau_1^2 = (b+1)^{-k} b^{2k} \rho e^{-2\rho E(\|S\|)} \int_{A^2} (\exp[\rho E\{\| S \cap (S + b(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)) \| \}] - \exp[\rho\{E(\|S\|)\}^2 (b+1)^{-k}]) d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2 \, .$$

To do this it suffices to show that the expectation of the left hand side of (1.13) converges to τ_1^2 and its variance converges to zero as $n \to \infty$.

Let A_i be the region covered by the i'th subsubcube. Exactly as in the proof of theorem 1,

(1.14) $$\operatorname{Var}(V_i|N_i) = \int_{A_i^2} ([1 - 2E(\|S\|)(b+1)^{-k} + E\{\| S \cap (S + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta) \| \}(b+1)^{-k}]^{N_i}) - [1 - 2E(\|S\|)(b+1)^{-k} + \{E(\|S\|)\}^2 (b+1)^{-2k}]^{N_i} \, d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2 \, .$$

Now $\underset{\sim}{N} = (N_1, \ldots, N_{m^k})$ has a multinomial distribution.

Thus, writing $p_n = (b+1)^k \delta^k$,

$$P(N_i = r_i, N_j = r_j) = \frac{n!}{r_i!\, r_j!\, (n-r_i-r_j)!}$$

$$p_n^{r_i+r_j} (1 - p_n)^{n-r_i-r_j}$$

when $i \neq j$. Furthermore, for positive constants α_i and α_j ,

(1.15) $$E(\alpha_i^{N_i}) = \{1 + (\alpha_i - 1)p_n\} \quad \text{and}$$

$$E(\alpha_i^{N_i} \alpha_j^{N_j}) = \{1 + (\alpha_i + \alpha_j - 2)p_n\} .$$

Let $a = E(\|S\|)$ and $B(\underset{\sim}{z}) = E\{\|S \cap (S + \underset{\sim}{z})\|\}$. Then by (1.14) and (1.15) ,

$$E\{n\, \Sigma_i \operatorname{var}(V_i | N_i)\}$$

(1.16) $$= n\, \Sigma_i \int_{A_i^2} ([1 - 2\delta^k a + \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]^n$$

$$- \{1 - 2\delta^k a + \delta^k a^2 (b+1)^{-k}\}^n)\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$= (1 + o(1))n(b+1)^{-k}\, \delta^{-k} \int_{A_1^2}$$

$$(\exp[-2n\delta^k a + n\delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]$$

$$- \exp\{-2n\delta^k a + n\delta^k a^2 (b+1)^{-k}\})\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$= (1 + o(1))n(b+1)^{-k}\, b^{2k} \delta^{+k} \int_{A^2}$$

$$(\exp[-2n\delta^k a + n\delta^k B\{b(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\}]$$

$$- \exp\{-2n\delta^k a + n\delta^k a^2(b+1)^{-k}\})\,d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2$$

$$= (1+o(1))\ (b+1)^{-k}b^{2k}\rho\ e^{-2\rho a}\int_{A^2} (\exp[\rho B\{b(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\}]$$

$$- \exp\{\rho a^2(b+1)^{-k}\})\,d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2$$

(1.17) $$= (1 + o(1))\tau_1^2\ .$$

Similarly,

$$E\{\Sigma_i\ \mathrm{var}(V_i|N_i)\}^2$$

$$= \Sigma_i\ \Sigma_j\ E\{\mathrm{var}(V_i|N_i)\ \mathrm{var}\ (V_j|N_j)\}$$

$$= \Sigma_i\ \Sigma_j \int_{A_i^2\ A_j^2} E\{([1 - 2\delta^k a\ p_n^{-1}$$

$$+ \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}p_n^{-1}]^{N_i}$$

$$- (1 - 2\delta^k a\, p_n^{-1} + \delta^{2k} a^2 p_n^{-2})^{N_i})$$

$$([1 - 2\delta^k a\, p_n^{-1} + \delta^k B\{(\underset{\sim}{x}_3 - \underset{\sim}{x}_4)/\delta\}p_n^{-1}]^{N_j}$$

$$- (1 - 2\delta^k\ a\ p_n^{-1} + \delta^{2k} a^2 p_n^{-2})^{N_j})\}d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4$$

(1.18)

$$= \sum_{i\neq j} \int_{A_i^2 A_j^2} ([1 - 4\delta^k a + \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\} + \delta^k B\{(\underset{\sim}{x}_3 - \underset{\sim}{x}_4)/\delta\}]^n$$

$$- \quad 2[1 - 4\delta^k a + \delta^{2k} a^2 p_n^{-1} + \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]^n$$

$$+ \quad (1 - 4\delta^k a + 2\delta^{2k} a^2 p_n^{-1})^n) \, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4$$

$$+ \sum_i \int_{A_i^4} \{1 + ([1 - 2\delta^k a p_n^{-1} + p_n^{-1} \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]$$

$$[1 - 2\delta^k a p_n^{-1} + p_n^{-1} \delta^k B\{(\underset{\sim}{x}_3 - \underset{\sim}{x}_4)/\delta\}] - 1) p_n\}^n$$

$$- 2\{1 + ([1 - 2\delta^k a p_n^{-1} + p_n^{-1} \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]$$

$$(1 - 2\delta^k a p_n^{-1} + \delta^{2k} a^2 p_n^{-1}) - 1) p_n\}^n$$

$$+ \{1 + [(1 - 2\delta^k a p_n^{-1} + \delta^{2k} a^2 p_n^{-2})^2 - 1] p_n\}^n \, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4 \quad .$$

Let

$$u_{jk} = -2 \delta^k a + \delta^k B\{(\underset{\sim}{x}_j - \underset{\sim}{x}_k)/\delta\} \quad \text{and}$$

$$v = -2 \delta^k a + \delta^{2k} a^2 p_n^{-1} \; .$$

Then by (1.16) and (1.18) ,

$$\operatorname{var}\{\Sigma_i \operatorname{var}(V_i|N_i)\}$$

$$= E\{\Sigma_i \operatorname{var}(V_i|N_i)\}^2 - [E\{\Sigma_i \operatorname{var}(V_i|N_i)\}]^2$$

(1.19)

$$= \sum_{i\neq j}\int_{A_i^2A_j^2} \{(1+u_{12}+u_{34})^n - 2(1+u_{12}+v)^n$$

$$+ (1+2v)^n\}\, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4$$

$$+ \sum_i \int_{A_i^4} \{(1+u_{12}+u_{34}+p_n^{-1}u_{12}u_{34})^n$$

$$- 2(1+u_{12}+v+p_n^{-1}u_{12}v)^n$$

$$+ (1+2v+p_n^{-1}v^2)^n\}\, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4$$

$$- \Sigma_i\Sigma_j \int_{A_i^2A_j^2} \{(1+u_{12}+u_{34}+u_{12}u_{34})^n$$

$$- 2(1+u_{12}+v+u_{12}v)^n$$

$$+ (1+2v+v^2)^n\}\, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4 \,.$$

Now each of the terms

$$|(1+u_{12}+v)^n - (1+u_{12}+v+u_{12}v)^n| ,$$

$$|(1+2v)^n - (1+2v+v^2)^n| \quad \text{and}$$

$$|(1+u_{12}+u_{34})^n - (1+u_{12}+u_{34}+u_{12}u_{34})^n|$$

is uniformly bounded on $A_i^2 A_j^2$ by a term of order $O(n^{-1})$ as $n \to \infty$. Furthermore,

$$|(1+u_{12}+u_{34}+u_{12}u_{34})^n - (1+u_{12}+u_{34}+p_n^{-1}u_{12}u_{34})^n| ,$$

$$|(1+u_{12}+v+u_{12}v)^n - (1+u_{12}+v+p_n^{-1}u_{12}v)^n| \text{ and}$$

$$|(1+2v+v^2)^n - (1+2v+p_n^{-1}v^2)^n|$$

are of the order $O(1)$ uniformly in A_i^4 as $n \to \infty$. Also, $\|A_i^4\|$ and $\|A_i^2\| \cdot \|A_j^2\|$ are of order $O(n^{-4})$, the number of terms in Σ_i is of order $O(n)$ and the number of terms in $\Sigma_{i\neq j}$ is of order $O(n^2)$ as $n \to \infty$. Hence by (1.19),

$$\text{Var}\{\Sigma_i \text{ var}(V_i|N_i)\} = O(n^{-3})$$

as $n \to \infty$. This result and (1.17) prove (1.13).

To complete the proof of part (i) we use the Berry-Esseen theorem, which asserts that for any sequence $\{X_i\}$ of independent random variables with finite third moment, and for any $m \geq 1$,

$$\sup_{-\infty<t<\infty} \left| P\left(\sum_{i=1}^{m} (X_i - EX_i) \le t\{ \sum_{i=1}^{m} \mathrm{var}(X_i)\}^{\frac{1}{2}}\right) - \Phi(t) \right|$$

$$\le C\{ \sum_{i=1}^{m} E|X_i - E(X_i)|^3\} \,/\, \{ \sum_{i=1}^{m} \mathrm{var}(X_i)\}^{3/2} ,$$

where C is a positive constant not dependent on m, and Φ denotes the standard normal distribution function (See Petrov (1975), p 111).

Let F_n be the sigma field generated by $\underset{\sim}{N}$. By the Berry-Esseen theorem,

$$\sup_{-\infty<t<\infty} \left| P(\Sigma_i\{V_i - E(V_i|N_i)\} \le t\{\Sigma_i \mathrm{var}(V_i|N_i)\}^{\frac{1}{2}} | F_n) - \Phi(t) \right|$$

(1.20)

$$\le C\ \Sigma_i E\{|V_i - E(V_i|N_i)|^3 |N_i\}/\{\Sigma_i \mathrm{var}(V_i|N_i)\}^{3/2}$$

$$= R_n \text{ , say.}$$

Following from result (1.13) , $\Sigma_i \mathrm{var}(V_i|N_i) = O_p(n^{-1})$ as $n \to \infty$. Also, since $V_i \le ||A_1||$,

$$\Sigma_i E\{|V_i - E(V_i|N_i)|^3 | N_i\}$$

$$\le ||A_1|| \ \Sigma_i E\{|V_i - E(V_i|N_i)|^2 | N_i\}$$

$$= b^k \delta^k \ \Sigma_i \mathrm{var}(V_i|N_i)$$

$$= O_p(n^{-2})$$

as $n \to \infty$. Consequently, R_n is of order $O(n^{-\frac{1}{2}})$ in probability. Clearly, R_n does not exceed one and so by the Dominated Convergence theorem, and (1.20)

$$\sup_{-\infty<t<\infty} \left| P\left(\frac{\Sigma_i\{V_i - E(V_i|N_i)\}}{\{\Sigma_i \text{ var } (V_i|N_i)\}^{\frac{1}{2}}} \leq t\right) - \Phi(t)\right|$$

$$= \left| E\{P(\Sigma_i\{V_i - E(V_i|N_i)\} \leq \{\Sigma_i \text{ var}(V_i|N_i)\}^{\frac{1}{2}}|F_n) - \Phi(t)\}\right| \tag{1.21}$$

$$\leq E\{\min(R_n,1)\} \to 0$$

as $n \to \infty$. Therefore, by (1.13) and (1.21) ,

$$\sqrt{n}\ \Sigma_i\{V_i - E(V_i|N_i\}) \xrightarrow{\mathcal{D}} N(0,\tau_1^2) \tag{1.22}$$

as $n \to \infty$, which completes step (i) of the proof.

Step (ii)

In this part it is our intention to show that

$$\sqrt{n}\ \Sigma_i\ \{E(V_i|N_i) - E(V_i)\} \xrightarrow{\mathcal{D}} N(0,\tau_2^2) \ , \tag{1.23}$$

where

$$\tau_2^2 = \left(\frac{b}{b+1}\right)^{2k} \rho e^{-2\rho E(||S||)}\left(-\{E(||S||)\}^2 + (b+1)^k[e^{\rho\{E(||S||)\}^2(b+1)^{-k}} - 1]\right).$$

As in step (i) of the proof, write $p_n = (b+1)^k\ \delta^k$ for the content of any subcube and $a = E(\| S \|)$. Let

$$f(N_i) = n\, E(V_i | N_i) - E(V_i)$$

$$= n\{(b\delta)^k (1 - \delta^k a p_n^{-1})^{N_i} - (b\delta)^k (1 - \delta^k a)^n\} .$$

Now $\underset{\sim}{N}$ has a multinomial distribution. Furthermore, the regularity conditions of theorem 1 of Holst (1972) are met so if we define

$$(1.24) \qquad \sigma_n^2 \equiv \Sigma_i \operatorname{var}\{f(X_i)\} - n^{-1}[\Sigma_i \operatorname{cov}\{X_i, f(X_i)\}]^2$$

where X_i has a Poisson distribution with mean np_n , then

$$(1.25) \qquad \sigma_n^{-1} \Sigma_i f(N_i) \xrightarrow{\mathcal{D}} N(0,1)$$

as $n \to \infty$.

If X has a Poisson distribution with mean γ then

$$E(a^X) = e^{\gamma(a-1)} ,$$

$$\operatorname{Var}(a^X) = e^{\gamma(a^2-1)} - e^{2\gamma(a-1)} \quad \text{and}$$

$$\operatorname{Cov}(X, a^X) = \gamma(a-1)\, e^{\gamma(a-1)} .$$

Thus,

$$(1.26) \qquad \operatorname{Var}\{f(X_i)\} = n^2 (b\delta)^{2k} \operatorname{Var}\{(1 - \delta^k a\, p_n^{-1})^{X_i}\}$$

$$= n^2 (b\delta)^{2k} (e^{-2n\delta^k a + n\delta^{2k} a^2 p_n^{-1}}$$

$$- e^{-2n\delta^k a})$$

$$= n^2(b\delta)^{2k}\ e^{-2n\delta^k a}(e^{n\delta^{2k}a^2 p_n^{-1}} - 1)$$

and

$$\text{(1.27)} \qquad \mathrm{Cov}(X_i, f(X_i)) = n(b\delta)^k \mathrm{cov}\{X_i, (1 - \delta^k a\, p_n^{-1})^{X_i}\}$$

$$= n(b\delta)^k(-n\delta^k a)\ e^{-n\delta^k a}\ .$$

Hence by (1.24), (1.26) and (1.27) ,

$$\sigma_n^2 = (b+1)^{-k}\ \delta^{-k}\ n^2(b\delta)^{2k}\ e^{-2n\delta^k a}(e^{n\,\delta^{2k}\,a^2\ p_n^{-1}} - 1)$$

$$- n^{-1}\{(b+1)^{-k}\delta^{-k}\ n(b\delta)^k(-n\delta^k a)e^{-n\delta^k a}\}^2$$

$$= n(b+1)^{-k}\ b^{2k}\ \rho\ e^{-2\rho a}\{e^{\rho a^2(b+1)^{-k}} - 1\}$$

$$- n\ \rho^2(b+1)^{-2k}\ b^{2k}\ a^2\ e^{-2\rho a} + o(n)$$

$$= n\left(\frac{b}{b+1}\right)^{2k} \rho\, e^{-2\rho a}[-\ a^2\rho + (b+1)^k\{e^{\rho a^2(b+1)^{-k}} - 1\}]$$

$$+ o(n) \quad \text{as} \quad n \to \infty\ .$$

Result (1.23) follows from this expression and (1.25) .

Step (iii)

The results (1.22) and (1.23) suggest that

$$\text{(1.28)} \qquad \sqrt{n}\ \Sigma_i\{V_i - E(V_i)\} \to^{\mathcal{D}} N(0, \tau_1^2 + \tau_2^2)$$

as $n \to \infty$. To prove this, as before, let F_n be the sigma field generated by $\underset{\sim}{N}$. Then for any $t \in \mathbb{R}$,

$$\left| P(n^{\frac{1}{2}} \Sigma_i \{V_i - E(V_i)\} \leq t) - \Phi\left(\frac{t}{\sqrt{\tau_1^2+\tau_2^2}}\right) \right|$$

$$= \left| E\left[P\left(\frac{\sqrt{n}\,\Sigma_i\{V_i - E(V_i|N_i)\}}{\sqrt{n}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}} \leq \frac{t - n^{\frac{1}{2}}\Sigma_i\{E(V_i|N_i)-E(V_i)\}}{n^{\frac{1}{2}}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}} \middle| F_n\right)\right]\right.$$

$$- \Phi\left(\frac{t - n^{\frac{1}{2}}\Sigma_i\{E(V_i|N_i) - E(V_i)\}}{n^{\frac{1}{2}}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}}\right) + \Phi\left(\frac{t-n^{\frac{1}{2}}\Sigma_i\{E(V_i|N_i) - E(V_i)\}}{n^{\frac{1}{2}}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}}\right)$$

$$\left. - \Phi\left(\frac{t}{\sqrt{\tau_1^2 + \tau_2^2}}\right) \right|$$

(1.29)

$$\leq E\{\min(R_n,1)\} + \left| E\left[\Phi\left(\frac{t - n^{\frac{1}{2}}\Sigma_i\{E(V_i|N_i)-E(V_i)\}}{n^{\frac{1}{2}}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}}\right) - \Phi\left(\frac{t}{\sqrt{\tau_1^2+\tau_2^2}}\right)\right]\right|,$$

where R_n is as defined in step (i) of the proof.

As was shown in step (i), the first term on the RHS of (1.29) converges to zero as $n \to \infty$. According to the results (1.13) and (1.23), and since Φ is bounded,

$$E\left[\Phi\left(\frac{t - n^{\frac{1}{2}}\Sigma_i\{E(V_i|N_i) - E(V_i)\}}{n^{\frac{1}{2}}\{\Sigma_i \operatorname{var}(V_i|N_i)\}^{\frac{1}{2}}}\right)\right]$$

$$\to E\left\{\Phi\left(\frac{t - \tau_2 Z_2}{\tau_1}\right)\right\}, \qquad n \to \infty,$$

(1.30) $$= E\left\{P\left(Z_1 \leq \frac{t - \tau_2 Z_2}{\tau_1} \middle| Z_1\right)\right\},$$

where Z_1 and Z_2 are independent standard normal random variables. The RHS of (1.30) equals

$$E\{P(\tau_1 Z_1 + \tau_2 Z_2 \leq t | Z_1)\}$$

$$= P(\tau_1 Z_1 + \tau_2 Z_2 \leq t)$$

$$= \Phi\left(\frac{t}{\sqrt{\tau_1^2+\tau_2^2}}\right)$$

Consequently, the second term on the RHS of (1.29) converges to zero as $n \to \infty$. Thus, (1.28) follows from (1.29) and the argument given above.

Step (iv)

Our next task is to show that

$$\tau_1^2 + \tau_2^2 \to \tau^2$$

as $b \to \infty$.

Using the notation of step (i) ,

$$\tau_1^2 + \tau_2^2 = (b+1)^{-k} b^{2k} \rho e^{-2\rho a} \int_{A^2} (\exp[\rho B\{b(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\}]$$

$$- \exp\{\rho a^2 (b+1)^{-1}\}) d\underset{\sim}{x}_1 d\underset{\sim}{x}_2$$

$$+ (b+1)^{-2k} b^{2k} \rho e^{-2\rho a}[-a^2\rho + (b+1)^k\{e^{\rho a^2 (b+1)^{-k}} - 1\}]$$

$$= \rho e^{-2\rho a} b^k \int_{A^2} (\exp[\rho B\{b(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\}] - 1) d\underset{\sim}{x}_1 d\underset{\sim}{x}_2$$

$$- \rho^2 a^2 e^{-2\rho a} + o(1)$$

$$= \rho\, e^{-2\rho a} \int_{bA} (\exp\{\rho B(\underset{\sim}{z})\} - 1)\, d\underset{\sim}{z}$$

$$- \rho^2 a^2 e^{-2\rho a} + o(1)$$

$$= \rho\, e^{-2\rho a} \int_{\mathbb{R}^k} (\exp\{\rho B(\underset{\sim}{z})\} - 1)\, d\underset{\sim}{z}$$

$$- \rho^2 a^2 e^{-2\rho a} + o(1)$$

$$= \tau^2 + o(1)$$

as $b \to \infty$, as required.

Finally, we shall show that

$$(1.31) \qquad \lim_{b\to\infty} \limsup_{n\to\infty} n \ \mathrm{var}(R) = 0 \ .$$

Let $\mathcal{A}$ be the subregion of A not covered by any of the subsubcubes. Clearly R is the vacancy within $\mathcal{A}$. The variance of R may be shown to be bounded above as follows :

$$\mathrm{Var}(R) = \int_{\mathcal{A}^2} ([1 - 2\delta^k a + \delta^k B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\}]^n$$

$$- (1 - 2\delta^k a + \delta^{2k} a^2)^n d\underset{\sim}{x}_1 d\underset{\sim}{x}_2$$

$$\leq C_1 \int_{\{\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \mathcal{A} \,:\, |\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta\}} d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$\leq C_1 \int_{\mathcal{A}} d\underset{\sim}{x}_1 \int_{\{\underset{\sim}{x}_2 \,:\, |\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta\}} d\underset{\sim}{x}_2$$

$$\leq C_2 \, \|\mathcal{A}\| \, n^{-1} ,$$

where C_1 and C_2 are positive constants independent of n. Hence,

$$n\ \mathrm{Var}(R) \leq C_2\{1-(b\delta)^k\ (b+1)^{-k}\delta^{-k}\}$$

(1.32)

$$= C_2\{1-b^k(b+1)^{-k}\}\ .$$

The result at (1.31) is an immediate consequence of expression (1.32). This completes the proof of Theorem 3. $\Box$

§2. A test of Uniformity Based on Vacancy

Suppose that $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ are independently distributed on A according to a distribution with density f. For $\delta < 1$, the coverage process $C_f(\delta,n)$ is defined to be the collection of random sets $\underset{\sim}{X}_i + \delta S_i$, $i \geq 1$, where, as in the introduction, $S_1,S_2,\ldots,S_n$ are independent copies of the random shape S. Recall that S is bounded by a sphere of diameter one, and A is topologically a torus.

In this section it is our aim to construct and analyse a test of the null hypothesis of uniformity :

$$H_o : f = f_o \equiv 1 \text{ against } H_A : f \neq f_o ,$$

when the locations of $\underset{\sim}{X}_1,\underset{\sim}{X}_2,\ldots,\underset{\sim}{X}_n$ are unknown. Included in the null hypothesis are density functions which only differ from f_o on a set of Lebesgue measure zero. In general the variable n is not observable, however, we shall assume throughout the rest of this chapter that it is known *a priori* .

The statistic which is considered in this section is the total vacancy V within A . In subsection 1 a simple test based on V is constructed, and in subsection 2 the power of this test is found, against a sequence of alternatives converging to the null hypothesis as $n \to \infty$.

2.1 Constructing a Test.

The following theorem justifies the use of vacancy to test the null hypothesis H_o . The second part of the proof is due to Hall (1984d).

Theorem 1

Let $V = V_f(\delta,n)$ be the vacancy in A resulting from the coverage process $C_f(\delta,n)$ defined just above. Let f' be the vector of first derivatives of f and assume that for some finite constant C , and all $\underset{\sim}{x}$, $f(\underset{\sim}{x}) < C$ and $|f'(\underset{\sim}{x})| < C$. Suppose $\delta \to 0$ and $n \to \infty$ in such a manner that $\delta^k n \to \rho$, where $0 < \rho < \infty$. Then ,

$$(2.1) \qquad E\left|V - \int_A \exp\{-\rho E(\|S\|)\, f(\underset{\sim}{x})\}d\underset{\sim}{x}\right| \to 0$$

as $n \to \infty$. Also, for any density f not in the null hypothesis,

$$\mu(f) \equiv \int_A \exp\{-\rho\, E(\|S\|) f(\underset{\sim}{x})\}\, d\underset{\sim}{x}$$

$$> \exp\{-\rho\, E(\|S\|)\}$$

$$= \mu(f_o) .$$

Proof.

Let the random vector $\underset{\sim}{X}$ be independent of S and have density f. The probability that $\underset{\sim}{x}$ is not covered by the random set $\underset{\sim}{X} + \delta S$, conditional on S, is :

$$P(\underset{\sim}{x} \notin \underset{\sim}{X} + \delta S | S) = P(\underset{\sim}{X} \notin \underset{\sim}{x} - \delta S | S)$$

$$= 1 - \int_{\delta S} f(\underset{\sim}{x} - \underset{\sim}{y})\, d\underset{\sim}{y} \quad .$$

Hence, the expected vacancy in A is :

$$E(V) = \int_A \{P(\underset{\sim}{x} \notin \underset{\sim}{X} + \delta S)\}^n\, d\underset{\sim}{x}$$

$$= \int_A [1 - E\{\int_{\delta S} f(\underset{\sim}{x} - \underset{\sim}{y})\, d\underset{\sim}{y}\}]^n\, d\underset{\sim}{x}$$

Let us expand $f(\underset{\sim}{x} - \underset{\sim}{y})$ in a Taylor series to the second term :

$$f(\underset{\sim}{x} - \underset{\sim}{y}) = f(\underset{\sim}{x}) + \underset{\sim}{y}^T f'(\underset{\sim}{x}_o) \ ,$$

where $|\underset{\sim}{x} - \underset{\sim}{x}_o| \leq |\underset{\sim}{x} - \underset{\sim}{y}|$, and $\underset{\sim}{y}^T$ is the transpose of $\underset{\sim}{y}$. Since S is contained in a sphere of diameter one,

$$| \; E\{ \int_{\delta S} \underset{\sim}{y}^T \; f'(\underset{\sim}{x}_o) \; d\underset{\sim}{y}\} \; |$$

$$\leq E[\int_{\delta S} \{\underset{\sim}{y}^T \; \underset{\sim}{y}|^{\frac{1}{2}} \; \left| \frac{\underset{\sim}{y}^T}{|\underset{\sim}{y} \; \underset{\sim}{y}|^{\frac{1}{2}}} \; f'(\underset{\sim}{x}_o) \right| \} d\underset{\sim}{y} \;]$$

$$\leq \; \delta \, E(\int_{\delta S} c \; d\underset{\sim}{y})$$

$$= c \; \delta^{k+1} \; E(\| S \|) \; = o(n^{-1})$$

as $n \to \infty$, uniformly in $\underset{\sim}{x}$. Thus ,

$$E(V) = \int_A [1 - E\{ \int_{\delta S} f(\underset{\sim}{x}) d\underset{\sim}{y}\} + o(n^{-1})]^n \; d\underset{\sim}{x}$$

$$= \int_A \{1 - \delta^k \; f(\underset{\sim}{x}) \; E(\|S\|) \; + o(n^{-1})\}^n \; d\underset{\sim}{x}$$

$$= \int_A \exp\{- \rho \, f(\underset{\sim}{x}) \; E(\|S\|)\} \; d\underset{\sim}{x} + o(1) \; . \tag{2.2}$$

In view of the fact that $V \leq 1$ and result (2.2) , we need to show that $\text{var}(V) \to 0$ as $n \to \infty$ in order to prove (2.1) .

Let $B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$ be the set $(\underset{\sim}{x}_1 + \delta S) \cap (x_2 + \delta S)$. Then for $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A$,

$$P(\underset{\sim}{x}_1 \text{ and } \underset{\sim}{x}_2 \notin \underset{\sim}{X} + \delta S \mid S)$$

$$= 1 - P(\underset{\sim}{X} \in \underset{\sim}{x}_1 - \delta S \text{ or } \underset{\sim}{X} \in \underset{\sim}{x}_2 - \delta S \mid S)$$

$$= 1 - \sum_{i=1}^{2} \int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y} + \int_{B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f(\underset{\sim}{y})\, d\underset{\sim}{y}$$

Hence,

$$E(V^2) = \int_{A^2} [P(\underset{\sim}{x}_1 \text{ and } \underset{\sim}{x}_2 \notin \underset{\sim}{X} + \delta S)]^n \, d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2$$

$$= \int_{A^2} [E\,\{1 - \sum_{i=1}^{2} \int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y}$$

$$+ \int_{B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f(\underset{\sim}{y})\, d\underset{\sim}{y}\}]^n \, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2.$$

Consequently,

$$\mathrm{Var}(V) = \int_{A^2} [\{1 - \sum_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y}$$

$$+ E(\int_{B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f(\underset{\sim}{y})\, d\underset{\sim}{y})\}^n$$

$$- \{1 - \sum_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y}) + \prod_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})\}^n] d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 .$$

(2.3)

$$= \int_A d\underset{\sim}{x}_1 \int_{\{\underset{\sim}{x}_2 \in A : |\underset{\sim}{x}_1 - \underset{\sim}{x}_2| \geq \delta\}} [\{1 - \Sigma_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})\}^n$$

$$- \{1 - \sum_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})$$

$$+ \prod_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})\}^n] d\underset{\sim}{x}_1 d\underset{\sim}{x}_2$$

$$+ \int_A d\underset{\sim}{x}_1 \int_{\{\underset{\sim}{x}_2 \in A : |\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta\}} [\{1 - \sum_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_1 - \underset{\sim}{y})d\underset{\sim}{y})$$

$$+ E(\int_{B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f(\underset{\sim}{y})d\underset{\sim}{y})\}^n$$

$$- \{1 - \sum_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})$$

$$+ \prod_{i=1}^{2} E(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})d\underset{\sim}{y})\}^n] d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 .$$

Since

$$\prod_{i=1}^{2} E\left(\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y} \right) \le C^2 \delta^{2k} \{E(\|S\|)\}^2$$

$$= o(n^{-1})$$

as $n \to \infty$, uniformly in $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A$, the first term on the RHS of (2.3) converges to zero as $n \to \infty$. Likewise

$$E\left(\int_{B(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f(\underset{\sim}{y})\, d\underset{\sim}{y} \right) \le C\, \delta^k\, E(\|S\|)$$

$$= O(n^{-1}) ,$$

as $n \to \infty$, uniformly in $\underset{\sim}{x}_1, \underset{\sim}{x}_2$. Hence the second term on the RHS of (2.3) is bounded by a constant multiple of

$$\int_A d\underset{\sim}{x}_1 \int_{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| < \delta} d\underset{\sim}{x}_2 = o(1)$$

as $n \to \infty$. We conclude that $\mathrm{var}(V) \to 0$ as $n \to \infty$, which proves (2.1) as required.

Let $a = E(\|S\|)$. The final part of the proof consists of showing that the minimum of

$$\int_A e^{-\rho a f(\underset{\sim}{x})}\, d\underset{\sim}{x} \tag{2.4}$$

subject to the constraint $\int_A f(\underset{\sim}{x})d\underset{\sim}{x} = 1$ is achieved

$f = f_o$ almost everywhere. Using Lagrange multipliers we search for a turning point in f of the functional equation

$$\int_A e^{-\rho a f(\underset{\sim}{x})} d\underset{\sim}{x} + \lambda\left(\int_A f(\underset{\sim}{x})d\underset{\sim}{x} - 1\right) .$$

Using the calculus of variation and differentiating with respect to f we obtain

$$- a\rho e^{-\rho a f} + \lambda = 0 ,$$

which implies f = constant. Thus $f = f_o$ gives a minimum of (2.4). $\square$

Allow $\tau_n^2 = n \operatorname{var}(V_{f_o}(\delta,n))$. According to theorems 2 and 3 of section 1

$$\frac{\sqrt{n}\{V - E(V_{f_o})\}}{\tau_n} \xrightarrow{\mathcal{D}} N(0,1)$$

as $n \to \infty$, under the null hypothesis. It follows from Theorem 1 that for any $f \in H_A$ satisfying the regularity conditions of that theorem, $\{V_f - E(V_{f_o})\}$ converges in probability to $\mu(f) - \mu(fo) > 0$ as $n \to \infty$.
Thus ,

$$\text{(2.5)} \qquad \frac{\sqrt{n}\{V - E(V_{f_o})\}}{\tau_n} \to + \infty$$

in probability as $n \to \infty$, under H_A . Let $z_\alpha = \Phi^{-1}(1-\alpha)$ be the $1-\alpha$ point from the inverse normal cumulative density function. Thus, the test which rejects H_o in favour of H_A when

$$(2.6) \qquad V > E(V_{fo}) + \tau_n z_\alpha/\sqrt{n}$$

is asymptotically of level α .

It follows from (2.5) that the test (2.6) is consistent in the sense that for all $f \in H_A$, satisfying the regularity conditions of theorem 1 , the power is asymptotically equal to one. In the following subsection we shall investigate the power of the test (2.6) against a sequence of "local alternatives" , which converge to the null hypothesis at the rate $O(n^{-\varepsilon})$ as $n \to \infty$, where $\varepsilon > 0$.

2.2 The Power Against Local Alternatives

Pitman efficiency is a measure of the asymptotic power of a test against a sequence of local alternatives from the alternative hypothesis, which converge to the null hypothesis. See for example Bickel and Doksum (1977).

In the present chapter we shall investigate the power of various test statistics against local alternatives. For example, consider a sequence of independent and identically distributed random variables $\{Y_i\}$ with common distribution F_θ , where $\theta \in \mathbb{R}$. Let $\{\delta_n\}$ be a sequence of tests of the null hypothesis $\theta = \theta_o$.

Let ε_o be the supremum over all $\varepsilon > 0$ for which the asymptotic power of δ_n is one against the local alternatives $\theta = \theta_o + n^{-\varepsilon}c$, $c \in \mathbb{R}$. The discriminating power of the test statistic δ_n increases with increasing ε_o . The concept of a local alternative is now extended to testing the hypothesis of uniformity described in subsection 2.1.

Let ℓ be a function defined on A satisfying

$$(2.7) \qquad |\ell(\underset{\sim}{x})| < 1$$

for all $\underset{\sim}{x} \in A$, and

$$(2.8) \qquad \int_A \ell(\underset{\sim}{x}) = 0 \ .$$

For $\varepsilon > 0$ and $n \geq 1$ define the local alternative density function f_n by

$$(2.9) \qquad f_n(\underset{\sim}{x}) = 1 + n^{-\varepsilon} \ell(\underset{\sim}{x}) \ , \quad \underset{\sim}{x} \in A \ .$$

The constraints (2.7) and (2.8) ensure that f_n satisfies the usual conditions of a probability density function.

In the situation of testing for uniformity of the coverage process $C_f(\delta,n)$, the "best" tests available when the central locations of shapes $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ are known give a critical value of $\varepsilon = \varepsilon_o = \frac{1}{2}$. That is, they can detect differences of order $O(n^{-\frac{1}{2}})$ away from the null hypothesis . In the present subsection we shall show that the test described at (2.6) can only distinguish differences of order $O(n^{-\frac{1}{4}})$ away from the null hypothesis. However, in the later situation the location of $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ need not be known.

Hall (1984b) has described how to construct a more intricate test, based on vacancies within a regular grid of squares, which has a critical value $\varepsilon_o = \frac{1}{2}$. His theory, however, is developed for random radii spherical shapes centred at the points of a Poisson process; whereas our theory is for arbitrary, but bounded, random shapes centred at a fixed number of points within A .

In its ability to detect differences of order $O(n^{-\frac{1}{4}})$ the test of (2.6) has power comparable to several popular spacings-based tests for uniformity of n random points on a unit interval. See for example Cressie (1978), Seuthuraman and Rao (1970) and Weiss (1957) . The theory which allows us to say that for the test based on V , $\varepsilon_o = \frac{1}{4}$, is given in theorems 2 and 3 below.

<u>Theorem 2</u>

<u>Let</u> V <u>be the vacancy within the unit cube</u> A <u>resulting from the coverage process</u> $Cf_n\,(\delta,n)$, <u>where</u> f_n <u>is defined at</u> (2.9) . <u>As well as the conditions</u> (2.7) <u>and</u> (2.8) <u>assume that for some constant</u> $C > 0$ <u>and</u> <u>for all</u> $\underset{\sim}{x} \in A$ <u>the vector of first derivatives</u> ℓ' <u>of</u> ℓ <u>exists,</u> $|\ell'(\underset{\sim}{x})| < C$, <u>and</u>

$$\int_A \ell^2(\underset{\sim}{x})\, d\underset{\sim}{x} < \infty \quad .$$

<u>If</u> $\delta \to 0$ <u>and</u> $n \to \infty$ <u>in such a manner that</u> $n\delta^k \to \rho$, $0 < \rho < \infty$, <u>then for</u> $0 < \varepsilon < 1$,

$$E(V) = \{1 - \delta^k E(\|S\|)\}^n [1 + \frac{n}{2}^{-2\varepsilon} \{\rho\ E(\|S\|)\}^2 \int_A \ell^2(\underset{\sim}{x}) d\underset{\sim}{x}$$

$$+ o(n^{-2\varepsilon})]$$

as $n \to \infty$.

Proof.

As in the proof of theorem 1 , and since A is topologically a torus,

$$E(V) = \int_A [1 - E\{\int_{\delta S} f_n(\underset{\sim}{x} - \underset{\sim}{y}) d\underset{\sim}{y}\}]^n d\underset{\sim}{x}$$

(2.10)

$$= \int_A [1 - \delta^k E(\|S\|) - n^{-\varepsilon} E\{\int_{\delta S} \ell(\underset{\sim}{x} - \underset{\sim}{y}) d\underset{\sim}{y}\}]^n dx .$$

The condition that $|\ell(\underset{\sim}{x})| < 1$ for all $\underset{\sim}{x} \in A$ implies

$$[E\{\int_{\delta S} \ell(\underset{\sim}{y})\ d\underset{\sim}{y}\}]^r = O(n^{-r}) \quad \text{for any} \quad r > 0$$

as $n \to \infty$. Therefore, writing $a = E(\|S\|)$, and since $0 < \varepsilon < 1$,

$$[1 - \delta^k a - n^{-\varepsilon} E\{\int_{\delta S} \ell(\underset{\sim}{x} - \underset{\sim}{y})\ d\underset{\sim}{y}\}]^n$$

(2.11)

$$= (1 - \delta^k a)^n (1 - \frac{n^{1-\varepsilon}}{(1-\delta^k a)} E\{\int_{\delta S} \ell(\underset{\sim}{x} - \underset{\sim}{y})\ d\underset{\sim}{y}\}$$

$$+ \frac{n^{2(1-\varepsilon)}}{2} [E\{\int_{\delta S} \ell(\underset{\sim}{x} - \underset{\sim}{y})\ d\underset{\sim}{y}\}]^2 + o(n^{-2\varepsilon}))$$

where the small order terms hold uniformly on $\underset{\sim}{x} \in A$.
On interchanging integrals, and by (2.8)

$$\int_A E\{ \int_{\delta S} \ell(\underset{\sim}{x} - \underset{\sim}{y})\ d\underset{\sim}{y}\}\ d\underset{\sim}{x}$$

$$= E\{ \int_A \ell(\underset{\sim}{y})\ d\underset{\sim}{y} \int_{\underset{\sim}{y}+\delta S} d\underset{\sim}{x}\}$$

(2.12)

$$= \int_A \ell(\underset{\sim}{y})\ d\underset{\sim}{y}\ \delta^k\ a = 0$$

and so by (2.10), (2.11) and (2.12)

$$(2.13) \qquad E(V) = (1 - \delta^k a)^n \ (1 + \frac{n^{2(1-\varepsilon)}}{2} \int_A [E\{ \int_{\delta S} \ell(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}]^2\ d\underset{\sim}{x} + o(n^{-2\varepsilon}))\ .$$

Let $C(\underset{\sim}{z}) = E\{\| S_1 \cap (S_2 + \underset{\sim}{z})\|\}$. The integral within the large curved brackets on the RHS of (2.13) equals

$$\int_A d\underset{\sim}{x}\ .\ E\{\int_{\underset{\sim}{x} - \delta S_1} \ell(\underset{\sim}{y}_1)\,d\underset{\sim}{y}_1\}.\ E\{\int_{\underset{\sim}{x}-\delta S_2} \ell(\underset{\sim}{y}_2)\,d\underset{\sim}{y}_2\}$$

$$= E\{\int_A \ell(\underset{\sim}{y}_1)\, d\underset{\sim}{y}_1 \int \ell(\underset{\sim}{y}_2)\, d\underset{\sim}{y}_2 \cdot \int_{(\underset{\sim}{y}_1+\delta S_1)\,\cap\,(\underset{\sim}{y}_2+\delta S_2)} d\underset{\sim}{x}\}$$

$$= \delta^k \int_{A^2} \ell(\underset{\sim}{y}_1)\, \ell(\underset{\sim}{y}_2)\, C\{(\underset{\sim}{y}_1 - \underset{\sim}{y}_2)/\delta\}\, d\underset{\sim}{y}_1\, d\underset{\sim}{y}_2$$

(2.14)

$$= \delta^k \int_A \ell(\underset{\sim}{y}_1)\, d\underset{\sim}{y}_1 \int_{\{\underset{\sim}{y}_2 \in A\,:\,|\underset{\sim}{y}_1 - \underset{\sim}{y}_2| \le \delta\}} \ell^2(\underset{\sim}{y}_2)\, C\{(\underset{\sim}{y}_1-\underset{\sim}{y}_2)/\delta\}\, d\underset{\sim}{y}_2\,,$$

the final step following because $C(\underset{\sim}{z}) = 0$ when $|\underset{\sim}{z}| > \delta$. Expansion of $\ell(\underset{\sim}{y}_2)$ in a Taylor series around $\underset{\sim}{y}_1$ gives

$$\ell(\underset{\sim}{y}_2) = \ell(\underset{\sim}{y}_1) + (\underset{\sim}{y}_2 - \underset{\sim}{y}_1)^T\, \ell'(\underset{\sim}{y}_o)\,,$$

where $|\underset{\sim}{y}_1 - \underset{\sim}{y}_o| \le |\underset{\sim}{y}_1 - \underset{\sim}{y}_2|$. Under the regularity conditions imposed upon ℓ and ℓ',

$$\left|\int_A \ell(\underset{\sim}{y}_1)\, d\underset{\sim}{y}_1 \int_{\{\underset{\sim}{y}_2 \in A\,:\,|\underset{\sim}{y}_1 - \underset{\sim}{y}_2| \le \delta\}} (\underset{\sim}{y}_2 - \underset{\sim}{y}_1)^T \ell'(\underset{\sim}{y}_o)\, C\{(\underset{\sim}{y}_1 - \underset{\sim}{y}_2)/\delta\}\, d\underset{\sim}{y}_2\right|$$

$$\le C\,\delta\, a \int_{|\underset{\sim}{y}| \le \delta} d\underset{\sim}{y} = o(n^{-1})$$

as $n \to \infty$. Hence by (2.14),

$$\int_A d\underset{\sim}{x}\ [E\{\int_{\delta S} \ell(\underset{\sim}{x}-\underset{\sim}{y})\ d\underset{\sim}{y}\}]^2$$

$$= \delta^k \int_{A^2} \ell^2(\underset{\sim}{y}_1)\ C\{(\underset{\sim}{y}_1-\underset{\sim}{y}_2)/\delta\}\ d\underset{\sim}{y}_1\ d\underset{\sim}{y}_2$$

$$(2.15) \qquad = \delta^{2k} \int_A \ell^2(\underset{\sim}{y}_1)\ d\underset{\sim}{y}_1 \int_{\{\underset{\sim}{z}\,:\,\underset{\sim}{y}_1-\delta\underset{\sim}{z}\in A \text{ and } |\underset{\sim}{z}|<1\}} C(z)\ dz$$

$$(2.16) \qquad = \delta^{2k}\{1+o(1)\} \int_A \ell^2(\underset{\sim}{y}_1)\ d\underset{\sim}{y}_1 \int_{\mathbb{R}^k} C(\underset{\sim}{z})\ d\underset{\sim}{z}\ ,$$

where (2.16) follows from (2.15) in precisely the same way that (1.7) follows from (1.6) . However, by lemma 2 of section 1 ,

$$\int_{\mathbb{R}^k} C(\underset{\sim}{z})\ d\underset{\sim}{z} = E\{\int_{\mathbb{R}^k} \| S_1 \cap (S_2+\underset{\sim}{z}) \|\ dz\}$$

$$(2.17) \qquad = E(\| S_1\|\ \| S_2\|) = a^2\ .$$

Combining the estimate (2.17) with (2.16) we see that

$$(2.18) \qquad \int_A d\underset{\sim}{x}\ [E\{\int_{\delta S} \ell(\underset{\sim}{x}-\underset{\sim}{y})\ d\underset{\sim}{y}\}]^2 = \frac{\rho^2 a^2}{n} \int_A \ell^2(\underset{\sim}{x})\ d\underset{\sim}{x} + o(n^{-2})\ ,$$

as $n \to \infty$. Hence by the results (2.13) and (2.18) , for $0 < \varepsilon < 1$,

$$E(V) = (1 - \delta^k a)^n \{1 + \frac{n^{-2\varepsilon}}{2} \rho^2 a^2 \int_A \ell^2(\underset{\sim}{x}) d\underset{\sim}{x} + o(n^{-2\varepsilon})\}$$

as $n \to \infty$, as required. $\square$

Theorem 3

Let, as in theorem 2, V be the vacancy within the unit cube A resulting from the coverage process $C_{f_n}(\delta, n)$, and assume that $\delta \to 0$ and $n \to \infty$ in such manner that $\delta^k n \to \rho$, where $0 < \rho < \infty$. Then, for all $\varepsilon > 0$,

$$\sqrt{n}\{V - E(V)\} \xrightarrow{\mathcal{D}} N(0, \tau^2)$$

as $n \to \infty$, where τ^2 is as defined in theorem 2 of section 1 .

Proof

In some parts, the proof is identical to the proof of the central limit result in subsection 1.2 and so we make reference to it as necessary. In other parts the present proof is analogous and so we only give an abbreviated argument. The central limit result in subsection 1.2 is referred to as theorem 1.3 .

Define c, b and m as in the proof of theorem 1.3, and partition A in exactly the same way : that is, into a lattice of subcubes of side length $(b+1)\delta$ and concentric subsubcubes of side length $b\delta$. Let N_i

be the number of centres that occur in the i'th subcube and V_{ni} the vacancy within the corresponding subsubcube. Let R be the vacancy in the region remaining after removing all subsubcubes from A. Then

$$V - E(V) = \Sigma_i \{V_{ni} - E(V_{ni}|N_i)\}$$

$$+ \Sigma_i \{E(V_{ni}|N_i) - E(V_{ni})\} + \{R - E(R)\}.$$

The proof is divided into several parts. In step (i) we show that

$$(2.19) \qquad \sqrt{n}\, \Sigma_i \{V_{ni} - E(V_{ni}|N_i)\} \to^{\mathcal{D}} N(0,\tau_1^2)$$

as $n \to \infty$, while in step (ii) we show that

$$(2.20) \qquad \sqrt{n}\, \Sigma_i \{E(V_{ni}|N_i) - E(V_{ni})\} \to^{\mathcal{D}} N(0,\tau_2^2),$$

as $n \to \infty$, where τ_1^2 and τ_2^2 are as defined in the proof of theorem 1.3. The result

$$\sqrt{n}\, \Sigma_i \{V_{ni} - E(V_{ni})\} \to^{\mathcal{D}} N(0,\tau_1^2 + \tau_2^2)$$

as $n \to \infty$ follows in exactly the same manner as (1.28) does in part (iii) of the proof of theorem 1.3. Consequently, its proof is omitted here. Likewise justification of the fact that

$$\tau_1^2 + \tau_2^2 \to \tau^2$$

as $b \to \infty$ is omitted. To complete the proof, we show in step (iii) that

(2.21) $$\lim_{b\to\infty} \limsup_{n\to\infty} n \operatorname{var}(R) = 0 .$$

Step (i)

Firstly, let us show that

(2.22) $$n \Sigma_i \operatorname{var}(V_{ni}|N_i) \to \tau_i^2$$

in probability as $n \to \infty$.

Let C_i be the region covered by the i'th cube and A_i the region covered by the corresponding subsubcube. Let

$$p_i = \int_{C_i} f_n(\underset{\sim}{y})\, d\underset{\sim}{y} \qquad \text{and}$$

$$\beta(\underset{\sim}{x}_j, \underset{\sim}{x}_k) = (\underset{\sim}{x}_j + \delta S) \cap (\underset{\sim}{x}_k + \delta S) .$$

It may be shown, without much trouble, that

(2.23)
$$\operatorname{var}(V_{ni}|N_i) = \int_{A_i^2} \Big([1 - p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})\, d\underset{\sim}{y}\}$$

$$- p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})\, d\underset{\sim}{y}\} + p_i^{-1} E\{\int_{\beta(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f_n(\underset{\sim}{y}) d\underset{\sim}{y}\}]^{N_i}$$

$$- [1 - p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})\, d\underset{\sim}{y}\} - p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})\, d\underset{\sim}{y}\}$$

$$+ p_i^{-2} E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})\, d\underset{\sim}{y}\} E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})\, d\underset{\sim}{y}\}]^{N_i} \Big)\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2 .$$

Now $\underset{\sim}{N} = (N_1, \ldots, N_{m^k})$ has a multinomial distribution with

$$P(N_i = r_i) = \frac{n!}{r_i!(n-r_i)!} p_i^{r_i}(1-p_i)^{n-r_i} .$$

Thus, for positive constants α_i and α_j,

$$E(\alpha_i^{N_i}) = \{1 + (\alpha_i - 1)p_i\}$$

and

$$E(\alpha_i^{N_i}\alpha_j^{N_j}) = \{1 + (\alpha_i + \alpha_j - 2)p_i\} .$$

Hence, by (2.23)

$$\begin{aligned} &E\{n\ \Sigma_i\ \mathrm{var}(V_{ni}|N_i)\} \\ (2.24)\quad &= n\ \Sigma_i \int_{A_i^2} ([1 - E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})d\underset{\sim}{y}\} - E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})d\underset{\sim}{y}\} \\ &+ E\{\int_{\beta(\underset{\sim}{x}_1, \underset{\sim}{x}_2)} f_n(\underset{\sim}{y})d\underset{\sim}{y}\}]^n - [1 - E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})d\underset{\sim}{y}\} \\ &- E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})d\underset{\sim}{y}\} + p_i^{-1}E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y}_i)d\underset{\sim}{y}\} \\ &\qquad E\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})d\underset{\sim}{y}\}]^n)\,d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 . \end{aligned}$$

As before, let $a = E(\|S\|)$ and $B(\underset{\sim}{z}) = E\{\|S \cap (S + \underset{\sim}{z})\|\}$. Also define

$$u_{jk} \equiv - E\{\int_{\delta S} f_n(\underset{\sim}{x}_j - \underset{\sim}{y})\,d\underset{\sim}{y}\} - E\{\int_{\delta S} f_n(\underset{\sim}{x}_k - \underset{\sim}{y})\,d\underset{\sim}{y}\}$$

$$+ E\{\int_{\beta(\underset{\sim}{x}_j,\underset{\sim}{x}_k)} f_n(\underset{\sim}{y})\,d\underset{\sim}{y}\}$$

and

$$v_{jk} \equiv - E\{\int_{\delta S} f_n(\underset{\sim}{x}_j - \underset{\sim}{y})\,d\underset{\sim}{y}\} - E\{\int_{\delta S} f_n(\underset{\sim}{x}_k - \underset{\sim}{y})\,d\underset{\sim}{y}\}$$

$$+ p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x}_j - \underset{\sim}{y})\,d\underset{\sim}{y}\} E\{\int_{\delta S} f_n(\underset{\sim}{x}_k - \underset{\sim}{y})\,d\underset{\sim}{y}\} .$$

Since $f_n(\underset{\sim}{y}) \to 1$ uniformly in $\underset{\sim}{y} \in A$ as $n \to \infty$,

$$n \log(1 - u_{12}) = -n\, E\{\int_{\delta S} f_n(\underset{\sim}{x}_1 - \underset{\sim}{y})\,d\underset{\sim}{y}\} - n\{\int_{\delta S} f_n(\underset{\sim}{x}_2 - \underset{\sim}{y})\,d\underset{\sim}{y}$$

$$+ n\, E\{\int_{\beta(\underset{\sim}{x}_1,\underset{\sim}{x}_2)} f_n(\underset{\sim}{y})\,d\underset{\sim}{y}\}$$

(2.25) $$= -2\rho a + \rho\, B\{(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)/\delta\} + o(1)$$

as $n \to \infty$, where the small order term holds uniformly in $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A$. Likewise,

(2.26) $$n \log(1 - v_{12}) = -2\rho a + (b+1)^{-k} \rho a^2 + o(1)$$

as $n \to \infty$, uniformly in $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in A$. Placing the estimates (2.25) and (2.26) in (2.24) gives

$$E\{n\Sigma_i \operatorname{var}(V_{ni}|N_i)\} = (1+o(1))n(b+1)^{-k}\delta^{-k}$$

$$\int_{A_1} (\exp[-2\rho a + \rho B\{(\underset{\sim}{x}_1 - \underset{\sim}{x})/\delta\}]$$

$$- \exp\{-2\rho a + \rho a^2 (b+1)^{-k}\})d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

(2.27) $$= (1+o(1))\tau_i^2 ,$$

the final step following from (1.16) down to (1.17).

In an analogous fashion to the way (1.19) was derived,

$$\operatorname{var}\{\Sigma_i \operatorname{var}(\ {}_{ni}|N_i)\}$$

$$= \sum_{i \neq j} \int_{A_i^2 A_j^2} \{(1+u_{12}+u_{34})^n - (1+u_{12}+v_{34})^n$$

$$-(1+u_{34}+v_{12})^n + (1+v_{12}+v_{34})^n\}d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_4$$

$$+ \sum_i \int_{A_i^4} \{(1+u_{12}+u_{34}+p_i^{-1} u_{12} u_{34})^n$$

$$- (1+u_{12}+v_{34}+p_i^{-1} u_{12} v_{34})^n$$

$$- (1+u_{34}+v_{12}+p_i^{-1} u_{34} v_{12})^n$$

$$+ (1+v_{12}+v_{34}+p_i^{-1} v_{12} v_{34})^n\}d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2\, d\underset{\sim}{x}_3\, d\underset{\sim}{x}_4$$

$$- \Sigma_i \Sigma_j \int_{A_i^2 A_j^2} \{(1 + u_{12} + u_{34} + u_{12} u_{34})^n$$

$$- (1 + u_{12} + v_{34} + u_{12} v_{34})^n$$

$$- (1 + u_{34} + v_{12} + u_{34} v_{12})^n$$

$$+ (1 + v_{12} + v_{34} + v_{12} v_{34})^n\} \, d\underset{\sim}{x}_1 \, d\underset{\sim}{x}_2 \, d\underset{\sim}{x}_3 \, d\underset{\sim}{x}_4 \, .$$

The term $|(1 + u_{12} + u_{34})^n - (1 + u_{12} + u_{34} + u_{12}u_{34})^n|$ may be bounded by a constant multiple of

$$n[E(\|\delta S\|) + E(\|\delta S\|) + E\{\|\delta S + \underset{\sim}{x}_1) \cap (\delta S + \underset{\sim}{x}_2)\|\}]$$

$$\times [E(\|\delta S\|) + E(\|\delta S\|) + E\{\|\delta S + \underset{\sim}{x}_3) \cap (\delta S + \underset{\sim}{x}_4)\|\}]$$

$$\leq 9 n \delta^{2k} a^2 = O(n^{-1})$$

as $n \to \infty$, uniformly for $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \underset{\sim}{x}_4 \in A$. Likewise, each of the terms

$$|(1+u_{12}+v_{34})^n - (1+u_{12}+v_{34}+u_{12}v_{34})^n| \, ,$$

$$|(1+u_{34}+v_{12})^n - (1+u_{34}+v_{12}+u_{34}v_{12})^n| \quad \text{and}$$

$$|(1+v_{12}+v_{34})^n - (1+v_{12}+v_{34}+v_{12}v_{34})^n|$$

is of order $O(n^{-1})$ as $n \to \infty$ uniformly for $\underset{\sim}{x}_1,..,\underset{\sim}{x}_4 \in A$.

Similarly,

$$|(1+u_{12}+u_{34}+u_{12}\ u_{34})^n - (1+u_{12}+u_{34}+p_i^{-1}\ u_{12}\ u_{34})^n| \ ,$$

$$|(1+u_{12}+v_{34}+u_{12}\ v_{34})^n - (1+u_{12}+v_{34}+p_i^{-1}\ u_{12}\ v_{34})^n| \ ,$$

$$|(1+u_{34}+v_{12}+u_{34}\ v_{12})^n - (1+u_{34}+v_{12}+p_i^{-1}\ u_{34}\ v_{12})^n|$$

and

$$|(1+v_{12}+v_{34}+v_{12}\ v_{34})^n - (1+v_{12}+v_{34}+p_i^{-1}\ v_{12}\ v_{34})^n|$$

are uniformly bounded by a term of order $O(1)$ as $n \to \infty$. Thus (as in the proof of theorem 1.3),

$$\text{(2.28)} \qquad \operatorname{var}\{\Sigma_i \operatorname{var}(V_{ni}|N_i)\} = O(n^{-3})$$

as $n \to \infty$. The results (2.27) and (2.28) imply (2.22).

Let F_n be the sigma field generated by $\underset{\sim}{N}$ and define

$$R_n \equiv \sup_{-\infty<t<\infty} \Big| P(\Sigma_i\{V_{ni} - E(V_{ni}|N_i)\} \le t\{\Sigma_i \operatorname{var}(V_{ni}|N_i)\}^{\frac{1}{2}} | F_n) - \Phi(t) \Big| \ .$$

It may be shown, in percisely the same manner as (1.20) down to (1.21), that

$$\sup_{-\infty<t<\infty} \Big| P(\frac{\Sigma_i\{V_{ni} - E(V_{ni}|N_i)\}}{\{\Sigma_i \operatorname{var}(V_{ni}|N_i)\}^{\frac{1}{2}}} \le t) - \Phi(t) \Big|$$

$$\text{(2.29)} \qquad \le E\{\min(R_n, 1)\}$$

and

(2.30) $$E\{\min(R_n, 1)\} \to 0$$

as $n \to \infty$. The result (2.19) follows from (2.29), (2.30) and the probability estimate (2.22).

<u>Step (ii)</u>

Define

$$f(N_i) \equiv n\{E(V_{ni}|N_i) - E(V_{ni})\}$$

$$= n \int_{A_i} [1 - p_i^{-1} E\{\int_{\delta S} f_n(\underset{\sim}{x} - \underset{\sim}{y})\, d\underset{\sim}{y}\}]^{N_i}\, d\underset{\approx}{x}$$

$$- n \int_{A_i} [1 - E\{\int_{\delta S} f_n(\underset{\sim}{x} - \underset{\sim}{y})\, d\underset{\sim}{y}\}]^{n}\, d\underset{\approx}{x} .$$

Then, according to theorem 1 of Holst (1972),

(2.31) $$\sigma_n^{-1} \Sigma_i f(N_i) \to^{\mathcal{D}} \mathcal{N}(0,1)$$

as $n \to \infty$, where

$$\sigma_n^2 = \Sigma_i \operatorname{var}\{f(X_i)\} - n^{-1}[\Sigma_i \operatorname{cov}\{X_i, f(X_i)\}]^2$$

and X_i has a Poisson distribution with mean

$$np_i = n \int_{C_i} f_n(\underset{\sim}{y})\, d\underset{\sim}{y} .$$

If X has a Poisson distribution with mean γ, then for any $\alpha > 0$,

$$E(\alpha^X) = \exp\{\gamma(\alpha-1)\} \quad \text{and}$$

$$\operatorname{Cov}(X, \alpha^X) = \gamma(\alpha-1)\exp\{\gamma(\alpha-1)\} .$$

Thus,

$$E\{f(X_i) + n\, E(V_{ni})\} = n\int_{A_i} \exp[-nE\{\int_{\delta S} f_n(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}]\,d\underset{\sim}{x}$$

$$= n\int_{A_i} \exp[-n\ \delta^k a - n^{1-\varepsilon} E\{\int_{\delta S} \ell(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}]\,d\underset{\sim}{x}$$

$$(2.32)\qquad = \rho\, b^k\, e^{-\rho a} + o(1)$$

as $n \to \infty$, uniformly in i. Likewise,

$$E[\{f(X_i) + nE(V_{ni})\}^2] = n^2\int_{A_iA_j} \exp[-\ n\ E\{\int_{\delta S} f_n(\underset{\sim}{x}_1-\underset{\sim}{y})\,d\underset{\sim}{y}\}$$

$$-\ n\ E\{\int_{\delta S} f_n(\underset{\sim}{x}_2-\underset{\sim}{y})\,d\underset{\sim}{y}\}$$

$$+\ n\ p_i^{-1}\ E\{\int_{\delta S} f_n(\underset{\sim}{x}_1-\underset{\sim}{y})\,d\underset{\sim}{y}\}\ E\{\int_{\delta S} f_n(\underset{\sim}{x}_2-\underset{\sim}{y})\,d\underset{\sim}{y}\}]\,d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2$$

$$(2.33)\qquad = \rho^2\ b^{2k}\ \exp\{-\ 2\,\rho a\ +\ \rho a^2 (b+1)^{-k}\}\ +\ o(1)$$

as $n \to \infty$, again uniformly in i. Consequently, by (2.32) and (2.33),

$$\Sigma_i\ \mathrm{var}\{f(X_i)\} = \Sigma_i [\rho^2\, b^{2k}\, e^{-2\rho a}\{e^{\rho a^2 (b+1)^{-k}} - 1\} + o(1)]$$

$$= n(b+1)^{-k}\ b^{2k}\ \rho\, e^{-2\rho a}\{e^{\rho a^2 (b+1)^{-k}} - 1\}\ +\ o(n)$$

as $n \to \infty$. Also, using an argument very similar to that above,

$$\text{cov}\{X_i,\ f(X_i)\} = n\ \text{cov}(X_i, \int_{A_i} [1 - p_i^{-1}\ E\{\int_{\delta S} f_n(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}]^{X_i} d\underset{\sim}{x})$$

$$= n\int_{A_i} (-nE\{\int_{\delta S} f_n(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}$$

$$- \exp[-n\,E\{\int_{\delta S} f_n(\underset{\sim}{x}-\underset{\sim}{y})\,d\underset{\sim}{y}\}])\ d\underset{\sim}{x}$$

$$= -\ \rho^2\, a\, b^k\ e^{-\rho a} + o(1)$$

uniformly in i . Consequently,

$$\sigma_n^2 = \Sigma_i\ \text{var}\{f(X_i)\} - n^{-1}[\Sigma_i\ \text{cov}(X_i, f(X_i)]^2$$

$$= n(b+1)^{-k}\ b^{2k}\ \rho\, e^{-2\rho a}\ \{e^{\rho a^2 (b+1)^{-k}} - 1\}$$

$$- n(b+1)^{-2k}\ b^{2k}\ \rho^2\ a^2\ e^{-2\rho a} + o(n)$$

$$= n\{1 + o(1)\}\tau_2^2\ ,$$

as $n \to \infty$. This result and (2.31) imply (2.20) .

Step (iii)

For a general subregion A of $\mathcal{A}$, the variance of vacancy within A is

$$\operatorname{var}\{V_n(A)\} = \int_{A^2} \operatorname{cov}\{I(\underset{\sim}{x}_1),\ I(\underset{\sim}{x}_2)\}\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$(2.34) \qquad = \int_{A^2} \Big([1 - \sum_{i=1}^{2} E\{\int_{\delta S} f_n(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y}\} + E\{\int_{B(\underset{\sim}{x}_1,\underset{\sim}{x}_2)} f_n(\underset{\sim}{y})\, d\underset{\sim}{y}\}]^n - [1 - \sum_{i=1}^{2} E\{\int_{\delta S} f_n(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y}\} + \prod_{i=1}^{2} E\{\int_{\delta S} f(\underset{\sim}{x}_i - \underset{\sim}{y})\, d\underset{\sim}{y}\}]^n \Big)\, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 .$$

When $|\underset{\sim}{x}_1 - \underset{\sim}{x}_2| > \delta$, $B(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) = \emptyset$ with probability one, and so the integrand on the RHS of (2.34) is negative. When $|\underset{\sim}{x}_1 - x_2| \le \delta$ the integrand is bounded by some constant C, independent of n. Therefore,

$$\operatorname{var}\{V_n(A)\} \le C \int_{\{A^2 : |\underset{\sim}{x}_1 - \underset{\sim}{x}_2| \le \delta\}} d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2$$

$$(2.35) \qquad \le C_1\, \|A\|\, n^{-1} ,$$

where C_1 does not depend on n.

Let A be the subregion of $\mathcal{A}$ not covered by $\cup_i A_i$. Then, $R = V_n(A)$ and by (2.35)

$$n \operatorname{var}(R) \leq C_1\{1 + b^k/(b+1)^k\} ,$$

which converges to zero as $b \to \infty$. This proves (2.21) and completes the proof of theorem 3. $\square$

Recall from (2.6) that our test statistic rejects the null hypothesis of uniformity $(H_o : f = f_o)$, when

$$\sqrt{n}\, \frac{\{V - E(V_{f_o})\}}{\tau_n} > z_\alpha , \tag{2.36}$$

where $E(V_{f_o}) = \{1 - \delta^k E(||S||)\}^n$ and $n^{-1}\tau_n^2$ are respectively the expectation and variance of vacancy under H_o . In theorem 2 of section 1 we saw that $\tau_n^2 \to \tau^2$ as $n \to \infty$, and so by theorems 2 and 3 above

$$\frac{\sqrt{n}\{V - E(V_{f_o})\}}{\tau_n}$$

$$= \frac{\sqrt{n}\{V - E(V_{f_n})\}}{\tau_n} + \sqrt{n}\,\tau_n^{-1}\{1 - \delta^k E(||S||)\}^n [\tfrac{n}{2}^{-2\varepsilon}\{\rho E(||S||)\}^2$$

$$\times \int_A \ell^2(\underset{\sim}{x})\, d\underset{\sim}{x} + o(n^{-2\varepsilon})]$$

(2.37)

$$= Z_n + (1 + o(1))\tau^{-1}\, e^{-\rho E(||S||)}\, \tfrac{1}{2} n^{(\frac{1}{2})-2\varepsilon}$$

$$\rho^2\{E(||S||)\}^2 \int_A \ell^2(\underset{\sim}{x})\, d\underset{\sim}{x} ,$$

where Z_n converges in distribution to the standard normal law as $n \to \infty$. From (2.37) we may deduce that :

(a) if $\varepsilon < \frac{1}{4}$, then the power of the test at (2.36) tends to 1 as $n \to \infty$;

(b) if $\varepsilon > \frac{1}{4}$, then the power tends to the significance level, α, and

(c) if $\varepsilon = \frac{1}{4}$ then the power tends to a value lying strictly between α and 1 .

Indeed, in case (c) , the asymptotic power is

$$1 - \Phi(z_\alpha - \tau^{-1} e^{-\rho E(||S||)} \tfrac{1}{2} \rho^2 \{E(||S||)\}^2 \int_A \ell^2(\underset{\sim}{x}) d\underset{\sim}{x}) \quad ,$$

where Φ is the standard normal distribution function.

In the one-dimensional case, where n arcs of equal length are randomly distributed on the perimeter of a circle, several others statistics are available, which do not have obvious counterparts in the higher dimensional situation. These include the total number of spacings and the arc length of the maximum spacing. In the following sections we shall investigate the asymptotic power properties of simple tests based on these two statistics.

§3. Other Tests in the One-Dimensional Case

For the coverage process $C_f(\delta,n)$, defined in the previous section, let the dimension be $k=1$ and assume that the random shape S is an interval of length $1 \geq a > 0$. Then A is the unit interval and also topologically a torus, so if a random interval protrudes beyond one end of A , it is introduced at the other. Thus our model is equivalent to one in which n arcs of radial length δa are randomly placed around the perimeter of a circle of circumference one. When $f \equiv 1$ this model has been the subject of considerable attention in the literature. For example, Stevens (1939) obtained a formula for the probability of complete coverage. The theory in this section will always be presented in terms of A being the unit interval $[0,1]$.

Our aim is to construct tests based on two statistics : the number of uncovered spacings in A , and the length of the largest uncovered spacing. The first is denoted by L and the second by M . The two spacings at the endpoints of A are treated as one, if they appear. Both L and M do not easily generalise to the k-dimensional case.

Hüsler (1982) has summarized the range of asymptotic distributions for L and M when the intervals are uniformly distributed on A , for a wide variety of convergence rates for a . See the discussion in section 2 of chapter 1 . Hall (1983) has extended these limit results to the case where the arcs are independently distributed on A according to a non-uniform distribution. Much of the ensuing theory employs techniques similar to those of Hall. We shall also quote some of Hüsler's results where necessary.

3.1 A Test Based on the Number of Uncovered Spacings

Let $X_1,..,X_n$ be n points independently distributed on $A = [0,1]$ according to a distribution with density f . Let $0 < a < 1$ and $\delta < 1$. For $1 \leq i \leq n$ construct the interval (shape) $[X_i, X_i + \delta a]$. If the i'th interval extends beyond the endpoint 1 introduce it at the other end so that it consists of the union of the two intervals $[X_i, 1]$ and $[0, X_i - 1 + \delta a]$. We shall denote this coverage process by $C_f^{(1)}(\delta,n)$. In the subsequent theory we allow $\delta \to 0$ and $n \to \infty$ in such a manner that $\delta n \to \rho$, where $0 < \rho < \infty$.

Let $X_{n1} \leq \ldots \leq X_{nn}$ be the order statistics corresponding to $X_1,..,X_n$, $D_{ni} = X_{n,i+1} - X_{ni}$ for $1 \leq i \leq n-1$ and $D_{no} = X_{n1} + (1 - X_{nn})$. We define L by

$$(3.1) \qquad L \equiv \sum_{i=0}^{n-1} I(D_{ni} > \delta a) ,$$

where I is the indicator function .

The following adaptation of Rényi's representation is fundamental to the proof of our theorems. Let $H(x) = F^{-1}(e^{-x})$ and $\{Z_j, j \geq 1\}$ be a sequence of independent identically distributed exponential random variables with mean 1 . Then,

$$X_{n,n-i+1} = H\{\sum_{j=1}^{i} Z_j/(n-j+1)\} , \quad 1 \leq i \leq n .$$

If f is continuous and bounded away from zero, and H' is the first derivative of H , then for some $0 < \theta_1 < 1$,

$$D_{n,n-i} = H\{\sum_{j=1}^{i} Z_j/(n-j+1)\} - H\{\sum_{j=1}^{i+1} Z_j/(n-j+1)\}$$

(3.2)

$$= \frac{-Z_{i+1}}{n-i} H'\{\sum_{j=1}^{i} \frac{Z_j}{(n-j+1)} + \theta_1 \frac{Z_{i+1}}{(n-i)}\} ,$$

where $1 \leq i \leq n-1$. Furthermore, if f' is continuous, then for some $0 < \theta_2 < 1$,

$$(3.3) \qquad D_{n,n-i} = \frac{-Z_{i+1}}{(n-i)}(H'\{\sum_{j=1}^{i} (n-j+1)^{-1}\}$$

$$+ \{\sum_{j=1}^{i} \frac{(Z_j-1)}{n-j+1} + \theta_1 \frac{Z_{i+1}}{(n-i)}\}$$

$$\times H''[\sum_{j=1}^{i} (n-j+1)^{-1}$$

$$+ \theta_2\{\sum_{j=1}^{i} \frac{Z_j-1}{(n-j+1)} + \theta_1 \frac{Z_{i+1}}{(n-i)}\}]) ,$$

where H'' is the second derivative of H .

Our next result ensures that it is possible to construct a test based on L of the null hypothesis

$$H_o : f = f_o \equiv 1 ,$$

at least when $\rho a \leq 1$. However, the counterexample following the theorem shows that it is not always possible to construct a test of H_o under all circumstances.

Theorem 1

Let $L = L_f(\delta,n)$ be the number of spacings in $[0,1]$ resulting from the coverage process $C_f^{(1)}(\delta,n)$ defined just above. Assume that f is continuous on $[0,1]$, and bounded away from zero. Suppose that $\delta \to 0$ and $n \to \infty$ in such a manner that $\delta n \to \rho$, where $0 < \rho < \infty$. Then,

$$(3.4) \qquad E \left| n^{-1} L - \int_0^1 f(x)\, e^{-\rho a f(x)}\, dx \right| \to 0$$

as $n \to \infty$. If $a > 0$ and $\rho a \leq 1$, then for any density f not in the null hypothesis H_o ,

$$\mu(f) = \int_0^1 f(x)\, e^{-\rho a f(x)}\, dx$$

$$< e^{-\rho a} = \mu(f_o) .$$

The proof is prefaced with three useful lemmas. It is necessary to describe a notational form used henceforth. If $\{Y_n\}$ is a sequence of random variables and $\{\xi_n\}$ a sequence of non-negative constants such that

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} P(|Y_n| > \lambda\,\xi_n) = 0 ,$$

then we say that Y_n is of order $O_p(\xi_n)$ in probability as $n \to \infty$. If, for all $\lambda > 0$, $P(|Y_n| > \lambda\,\xi_n) \to 0$ as $n \to \infty$, then we say Y_n is of order $o_p(\xi_n)$ in probability.

Lemma 1

Let $\{Z_j, j \geq 1\}$ be a sequence of independent and identically distributed random variables with mean 1. Given $0 < \gamma < 1$,

$$\sup_{1\leq i\leq(1-\gamma)n} \left| \sum_{j=1}^{i} (Z_j-1)/(n-j+1) \right| = O_p(n^{-\frac{1}{2}}) ,$$

as $n \to \infty$.

Proof

The result follows if we note that for $\lambda > 0$,

$$P(\sup_{1\leq i\leq(1-\gamma)n} \left| \sum_{j=1}^{i} (Z_j-1)/(n-j+1) \right| > \lambda\, n^{-\frac{1}{2}})$$

$$\leq \lambda^{-2}\, n \sum_{1\leq j\leq(1-\gamma)n} E\{(Z_j-1)^2\}/(n-j+1)^2$$

$$\leq \lambda^{-2} n \int_{n\gamma}^{n} \frac{1}{x^2} \, dx$$

$$\leq C \lambda^{-2} ,$$

where C does not depend on n . $\Box$

Lemma 2

Let $\{Z_j, j \geq 1\}$ *be a sequence of independent and identically distributed exponential random variables with mean* 1 . *Then, for* $0 < \gamma < 1$,

$$\sup_{1 \leq i \leq (1-\gamma)n} Z_{i+1}/(n-1) = o_p\{n^{-1} \log(n)\} ,$$

as $n \to \infty$.

Proof.

For $\lambda > \gamma^{-1}$,

$$P\left(\sup_{1 \leq i \leq (1-\gamma)n} Z_{i+1}/(n-i) \leq \lambda n^{-1} \log(n) \right)$$

$$\geq P \left(\text{for all } 1 \leq i \leq n , \; Z_i \leq \lambda\gamma \log(n)\right)$$

$$= (1 - n^{-\lambda\gamma})^n$$

$$\to 1$$

as $n \to \infty$. Lemma 2 follows. $\Box$

Lemma 3

Given $0 < \gamma < 1$,

$$\sup_{1 \le i \le (1-\gamma)n} \left| \sum_{j=1}^{i} (n-j+1)^{-1} + \log(1-i/n) \right| = O(n^{-1})$$

as $n \to \infty$.

Proof

The result follows from the inequality ,

$$\left| \sum_{j=1}^{i} (n-j+1)^{-1} + \log(1 - i/n) \right|$$

$$= \left| \sum_{j=1}^{i} (n-j+1)^{-1} - \int_{n-i}^{n} x^{-1}\, dx \right|$$

$$= \sum_{j=n-i+1}^{n} \int_{j-1}^{j} \frac{1}{x}\, dx - \frac{1}{j}$$

$$\le \sum_{j=n-i+1}^{n} \left(\frac{1}{j-1} - \frac{1}{j}\right)$$

$$\le \frac{1}{\gamma n} - \frac{1}{n} = O(n^{-1})$$

as $n \to \infty$.

Proof of theorem 1

Let F be the distribution function corresponding to f and suppose $0 < \gamma < 1$ is given. Now

$H'(x) = e^{-x}/f\{H(x)\}$ and since f is continuous on $[0,1]$ and bounded away from zero, both f and H' are uniformly continuous. Thus, by the representation at (3.2), the fact that f is bounded, and lemmas 1 - 3,

$$D_{n,n-i} = -\frac{Z_{i+1}}{(n-i)} H'\left\{ \sum_{j=1}^{i} \frac{Z_j}{(n-j+1)} + \theta_1 \frac{Z_{i+i}}{(n-i)} \right\}$$

$$= -\frac{Z_{i+1}}{(n-i)} H'\Big[\log(1 - i/n) + \Big\{ \sum_{j=1}^{i} (n-j+1)^{-1} - \log(1-i/n)\Big\}$$

$$+ \sum_{j=1}^{i} \frac{(Z_j-1)}{(n-j+1)} + \theta_1 \frac{Z_{i+1}}{(n-i)}\Big]$$

$$= -\frac{Z_{i+1}}{(n-i)} H'\{\log(1 - i/n)\} + o_p(n^{-1})$$

$$= \frac{Z_{i+1}}{n} [f\{F^{-1}(1 - i/n)\}]^{-1} + o_p(n^{-1}) \tag{3.5}$$

as $n \to \infty$, uniformly in $1 \leq i \leq (1-\gamma)n$. According to (3.5),

$$\sum_{1 \le i \le (1-\gamma)n} P(D_{n,n-i} > \delta a)$$

$$= \sum_{1 \le i \le (1-\gamma)n} P(Z_{i+1} > n\delta a f\{F^{-1}(1 - i/n)\}) + o(n)$$

$$= \sum_{1 \le i \le n(1-\gamma)} \exp[-n\delta a f\{F^{-1}(1 - i/n)\}] + o(n)$$

$$= \int_1^{n(1-\gamma)} \exp[-n\delta a f\{F^{-1}(1 - x/n)\}]dx + o(n)$$

(3.6)
$$= n \int_{F^{-1}(\gamma)}^{1} f(y)\ e^{-\rho a f(y)}\ dy + o(n)$$

as $n \to \infty$, the second last step following by virtue of the fact that f is uniformly continuous.
It follows from (3.6), on letting $\gamma \to 0$, that

$$E(L) = \sum_{i=0}^{n-1} P(D_{n,n-i} > \delta a)$$

$$= n \int_0^1 f(y)\ e^{-\rho a f(y)}\ dy + o(n)$$

as $n \to \infty$. Thus, to prove (3.4) we need to show that $\mathrm{var}(L) = o(n^2)$ as $n \to \infty$.

For $0 < \gamma < 1$,

$$\sum_{1\leq i,j\leq(1-\gamma)n} P(D_{n,n-i} > \delta a \ , \ D_{n,n-j} > \delta a)$$

$$= \sum_{\substack{i\neq j \\ 1\leq i,j\leq(1-\gamma)n}} P(Z_{i+1} > n\delta af\{F^{-1}(1-i/n)\})$$

$$P(Z_{j+1} > n\delta af\{F^{-1}(1-j/n)\})$$

$$+ \sum_{1\leq i\leq(1-\gamma)n} P(Z_{i+1} > n\delta af\{F^{-1}(1-i/n)\}) + o(n^2)$$

(3.7)

$$= \{n \int_{F^{-1}(\gamma)}^{1} f(y)e^{-\rho af(y)} \, dy\}^2 + o(n^2), \quad \text{as} \quad n \to \infty \ .$$

Hence, by (3.7) and (3.6) ,

(3.8) $$\sum_{1\leq i,j\leq(1-\gamma)n} \operatorname{cov}\{I(D_{n,n-i} > \delta a) \ ,$$

$$I(D_{n,n-j} > \delta s)\} = o(n^2)$$

as $n \to \infty$. The left hand side of (3.8) converges to var(L) as $\gamma \to 0$. Consequently, we have proved (3.4).

It remains to show that the unique minimum of $\mu(f)$ is achieved when $f = f_o \equiv 1$ almost everywhere, provided $\rho a \leq 1$. Since $y\,e^{-y}$ is maximised uniquely at $y=1$, the result is clear if $\rho a = 1$.

Now assume $0 < \rho a < 1$. Suppose that there exists a density f, not in the null hypothesis, such that

$$(3.9) \qquad \int_0^1 f(x)\, e^{-\rho a f(x)}\, dx \geq e^{-\rho a} .$$

Let $E = \{x : f(x) > \rho^{-1} a^{-1}\}$ and $E \subset E' \subset (0,1)$ be such that

$$(3.10) \qquad \rho^{-1}\, a^{-1} \int_{E'} dx = \int_{E'} f(x)\, dx .$$

Define

$$g(x) = \begin{cases} (\rho a)^{-1} & \text{on } E' \\ f(x) & \text{on } (0,1) \setminus E' . \end{cases}$$

Then by (3.10), g is also a density. Notice that if E' has measure zero, then $g = f$ everywhere. Hence, g is not essentially equal to 1. Furthermore, $g\, e^{-\rho a g} \geq f\, e^{-\rho a f}$ and so

$$(3.11) \qquad \int_0^1 g(x)\, e^{-\rho a g(x)}\, dx \geq e^{-\rho a} .$$

That is, if g leads to a contradiction in (3.11), then f leads to a contradiction in (3.9). Thus, we assume without any loss in generality that f is bounded above by $(\rho a)^{-1}$.

Let X be a random variable with density f. The function $h(x) = e^{-\rho a/x}$ is convex on $(\rho a, \infty)$. It follows from Jensen's inequality, and the fact that

f is bounded by $(\rho a)^{-1}$, and is essentially non-constant, that

$$\int_0^1 f(x)\, e^{-\rho a f(x)}\, dx = \int_0^1 f(x) h\{1/f(x)\} dx$$

$$= E(h[\{f(X)\}^{-1}\, I\{f(X) > 0\}])$$

$$< h(E[\{f(X)\}^{-1}\, I\{f(X) > 0\}])$$

$$= h[\int_{\{f(x)>0\}} dx]$$

$$\leq h(1) = e^{-\rho a} .$$

But this contradicts (3.9) and consequently proves the result. □

<u>Theorem 2</u>

If $\rho a > 1$ then there exist densities f_1, f_2 and f_3, not essentially constant, such that

$$\mu(f_1) > \mu(f_o) ,$$

$$\mu(f_2) = \mu(f_o) \quad \text{and}$$

$$\mu(f_3) < \mu(f_o) .$$

We may choose f_1, f_2 and f_3 so they are continuous and differentiable on $[0,1]$.

Proof

Let $\beta = \rho a$ be fixed and define the function f by

$$f(x) = \begin{cases} b & \text{on } [0,\gamma_1)\ , \\ \beta^{-1} & \text{on } [\gamma_1,\gamma_2) \quad \text{and} \\ \beta & \text{on } [\gamma_2,\ 1]\ , \end{cases}$$

where $\gamma_1 \leq \gamma_2$ and b are chosen so that f is a density. That is, $b\gamma_1 + \beta^{-1}(\gamma_2-\gamma_1) + \beta(1-\gamma_2) = 1$. Then,

$$(3.12) \qquad \int_0^1 f(x)e^{-\beta f(x)}\, dx = b\gamma_1\, e^{-\beta b} + \beta^{-1}(\gamma_2-\gamma_1)e^{-1} + \beta(1-\gamma_2)e^{-\beta^2}\ .$$

If we allow $\gamma_2 = 1$, and $\gamma_1 \to 0$ the right hand side of (3.12) converges to $\beta^{-1}\, e^{-1} > e^{-\beta}$ for all $\beta > 1$. Thus f_1 may be constructed by choosing $\gamma_2 = 1$ and $\gamma_1 = \gamma_1^*$, where γ_1^* is small and positive.

If, however, we allow $\gamma_2 = \gamma_1 = 1 - \beta^{-1}$, then the right hand side of (3.12) equals $e^{-\beta^2} < e^{-\beta}$ for all $\beta > 1$. Thus f_2 nay be constructed by choosing $\gamma_2 = \gamma_1 = 1 - \beta^{-1}$.

Since the right hand side of (3.12) is continuous in γ_1 and γ_2 , there must exist a $\gamma_1^* \leq \gamma_1 \leq 1 - \beta^{-1}$ and a $1 - \beta^{-1} \leq \gamma_2 \leq 1$ such that the left hand side of

(3.12) equals $e^{-\beta}$. We have therefore constructed a function satisfying the properties of f_2.

Finally, by "rounding the corners" of f_1, f_2 and f_3 we may ensure that the smoothness criteria are met. $\Box$

Allow $\nu_n^2 = n^{-1} \operatorname{var}\{L_{f_o}(\delta,n)\}$. It follows from corollary 2.3 of Hüsler (1982) that

$$\frac{n^{-\frac{1}{2}}\{L - E(L_{f_o})\}}{\nu_n} \to^{\mathcal{D}} N(0,1)$$

as $n \to \infty$, under the null hypothesis. Provided $0 < \rho a \leq 1$, it follows from theorem 1 that for any density f in the alternative hypothesis, satisfying the conditions of that theorem $n^{-1}\{L_f - E(L_{f_o})\}$ converges in probability to $\mu(f) - \mu(f_o) < 0$. Also, ν_n is order $O(1)$ as $n \to \infty$, and so

$$\frac{n^{-\frac{1}{2}}\{L - E(L_{f_o})\}}{\nu_n} \to -\infty \tag{3.13}$$

in probability as $n \to \infty$, under H_A. If Z_α is the $1-\alpha$ point from the standard normal distribution function, the test which rejects H_o when

$$L < E(L_{f_o}) - \nu_n Z_\alpha \sqrt{n} \tag{3.14}$$

is asymptotically level α . As for vacancy (3.13) implies that the test at (3.14) is asymptotically consistent.

When $\rho a > 1$, theorem 2 shows that the test at (3.14) is not asymptotically consistent for all distributions in the alternative hypothesis.

The local alternative density function is defined as the one-dimensional counterpart of (2.9). That is, for $\varepsilon > 0$ and $n \geq 1$

$$f_n(x) = 1 + n^{-\varepsilon} \ell(x) \tag{3.15}$$

where ℓ is a function satisfying

$$|\ell(x)| < 1 \quad \text{and} \quad \int_A \ell(x)\,dx = 0 \tag{3.16}$$

whenever $x \in A = [0,1]$.

Our next two results are the counterparts of theorems 2 and 3 of section 2 . They allow us to precisely quantify the power of the test (3.13), based on L , against local alternatives.

Theorem 3

Let L be the number of uncovered spacings in $A = [0,1]$ resulting from the coverage process $C^{(1)}_{f_n}(\delta,n)$, where f_n is defined at (3.15) . As well as the conditions at (3.16) assume

$$\int_A \ell^2(x)\,dx < \infty .$$

<u>If</u> $\delta \to 0$ <u>and</u> $n \to \infty$ <u>in such a manner that</u> $n\delta \to \rho$, $0 < \rho < \infty$, <u>then</u>

$$n \int_0^1 f_n(x) e^{-n\delta a f_n(x)} \, dx = n \, e^{-n\delta a}\{1 + \frac{\rho a}{2} (\rho a - 2) n^{-2\varepsilon} \int_0^1 \ell^2(x) dx + o(n^{-2\varepsilon})\}$$

<u>as</u> $n \to \infty$.

<u>Proof</u>

By expansion it may be shown that

$$f_n(x) e^{-n\delta a f_n(x)} = e^{-n\delta a}\{1 + n^{-\varepsilon} \ell(x)(1 - n\delta a) + n^{-2\varepsilon} \ell^2(x) \frac{n\delta a}{2}(n\delta a - 2) + o(n^{-2\varepsilon})\} ,$$

uniformly in $0 \leq x \leq 1$, as $n \to \infty$. Theorem 3 easily follows from this expansion. $\Box$

<u>Theorem 4</u>

<u>As in theorem</u> 3 <u>let</u> L <u>be the number of uncovered spacings in</u> A <u>resulting from</u> $C_{f_n}^{(1)}(\delta, n)$. <u>Assume that the first derivative</u> ℓ' <u>of</u> ℓ <u>exists and is continuous on</u> A . <u>If</u> $\delta \to 0$ <u>and</u> $n \to \infty$ <u>in such a manner that</u> $\delta n \to \rho$, <u>where</u> $0 < \rho < \infty$, <u>then for all</u> $\varepsilon > 0$,

$$n^{-\frac{1}{2}}[L - n \int_0^1 f_n(x) \exp\{-n\delta a \, f_n(x)\} dx]$$

<u>converges in distribution to a normal law with mean zero and variance given by</u>

$$\nu^2 = e^{-\rho a}\{1 - (1+\rho^2 a^2)e^{-\rho a}\}$$

<u>Proof</u>

Let $0 < \gamma < 1$ be given. Using the notation introduced earlier in this subsection, and the definition (3.1),

$$L = \sum_{1 \le i \le (1-\gamma)n} I(D_{ni} > \delta a) + I(D_{no} > \delta a)$$

$$+ \sum_{(1-\gamma)n < i \le n-1} I(D_{ni} > \delta a) .$$

To prove the limit theorem we divide the argument up into three steps. Fundamental to the proof is the estimate:

$$(3.17) \qquad D_{n,n-i} = n^{-1} Z_{i+1} [f_n\{F_n^{-1}(1 - {}^{i}/_{n})\}$$

$$\{1 + \sum_{j=1}^{i} \frac{(Z_j - 1)}{(n-j+1)}\} (1 + U_{ni})]^{-1}$$

for $1 \le i \le (1-\gamma)n$, where F_n is the distribution function corresponding to f_n and U_{ni} is a random variable satisfying

$$\sup_{1 \le i \le (1-\gamma)n} |U_{ni}| = o_p(n^{-\frac{1}{2}})$$

as $n \to \infty$. The result (3.17) is established in step (i).

Using (3.17) and the martingale central limit theorem we show in step (ii) for some constants $\nu(\gamma)$ and $c_n(\gamma)$,

$$(3.18) \qquad n^{-\frac{1}{2}}\{\sum_{1\le i\le(1-\gamma)n} I(D_{n,n-1} > \delta a) - c_n(\gamma)\} \to^{\mathcal{D}} N\{0,\nu^2(\gamma)\}$$

as $n \to \infty$.

According to lemma 2 of section 1 we need only show that

$$(3.19) \qquad \lim_{\gamma\to 0} \nu^2(\gamma) = \nu^2 ,$$

$$(3.20) \qquad \limsup_{\gamma\to 0} \left|c_n(\gamma) - n\int_0^1 e^{-n\delta a f_n(y)} f_n(y)\, dy\right| = o(n^{\frac{1}{2}})$$

as $n \to \infty$, and for all $\zeta > 0$,

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} P\{n^{-\frac{1}{2}}\left|\sum_{(1-\gamma)n<i\le n-1} I(D_{n,n-1} > \delta a) + c_n(\gamma) - n\int_0^1 e^{-n\delta a f_n(y)} f_n(y)\,dy\right| > \zeta\}$$

to complete the proof. This is done in step (iii).

Step (i)

Let $H_n(x) = F_n^{-1}(e^{-x})$, and H_n' and H_n'' be the first and second derivatives of H_n . Then

$$H_n'(x) = -e^{-x}/f_n\{H_n(x)\} \qquad \text{and}$$

$$H_n''(x) = [e^{-x} - f_n'\{H_n(x)\}\{H_n'(x)\}^2]/f_n\{H_n(x)\}.$$

Since the first derivative of ℓ is bounded, H_n'' is uniformly bounded on $[0,\infty)$ for all n sufficiently large. Also $H_n'(x)$ converges uniformly to $-e^{-x}$ on $[0,\infty)$ as $n \to \infty$. Thus, by lemma 3

$$H_n'\{\Sigma_{j=1}^{i} (n-j+1)^{-1}\} = H_n'\{-\log(1 - i/n)\}$$

$$+ \{\Sigma_{j=1}^{i}(n-j+1)^{-1} - \log(1 - i/n)\}$$

$$\times H_n''[-\log(1 - i/n)$$

$$+ \theta\{\Sigma_{j=1}^{i}(n-j+1)^{-1} - \log(1 - i/n)\}]$$

(3.21) $$= H_n'\{-\log(1 - i/n)\} + O(n^{-1})$$

(3.22) $$= -(1 - i/n) + o(1)$$

uniformly in $1 \le i \le (1-\gamma)n$, as $n \to \infty$, where $0 < \theta < 1$.

Likewise $H_n''(x)$ converges uniformly to e^{-x} and $H_n''(x)$ is uniformly continuous on $[0,\infty)$. It follows by lemmas 1 - 3 that for arbitrary $0 < \theta_1, \theta_2 < 1$,

$$H_n''[\sum_{j=1}^{i} (n-j+1)^{-1} + \theta_2\{\sum_{j=1}^{i} \frac{(Z_j-1)}{(n-j+1)} + \theta_1 \frac{Z_{i+1}}{(n-i)}\}]$$

$$= H_n''[-\log(1 - {}^i/_n) + O_p(n^{-\frac{1}{2}})]$$

(3.23)

$$= (1 - {}^i/_n) + o_p(1)$$

as $n \to \infty$, uniformly in $1 \leq i \leq (1-\gamma)n$.

On substituting the estimates at (3.21), (3.22) and (3.23) into (3.3), and applying lemmas 2 and 3, we obtain :

$$D_{n,n-i} = \frac{-Z_{i+1}}{n-i}[H_n'\{-\log(1 - {}^i/_n)\} + O(n^{-1})]$$

$$\times [1 + \{\sum_{j=1}^{i} \frac{(Z_j-1)}{(n-j+1)} + \theta_1 \frac{Z_{i+1}}{(n-i)}\} \frac{\{(1-{}^i/_n) + o_p(1)\}}{\{-(1-{}^i/_n) + o(1)\}}]$$

$$= \frac{Z_{i+1}}{n}[f_n\{F_n^{-1}(1 - {}^i/_n)\}\{1 + \sum_{j=1}^{i} \frac{(Z_j-1)}{(n-j+1)}\} (1 + U_{ni})]^{-1}$$

as $n \to \infty$, where $\sup_{1 \le i \le (1-\gamma)n} |U_{ni}| = o_p(n^{-\frac{1}{2}})$, which is

(3.17) .

<u>Step (ii)</u>

If $\{X_n\}$ is a sequence of random variables satisfying $X_n \to 0$ in probability as $n \to \infty$ then there exists constants e_n, $n \ge 1$ such that $P(|X_n| > e_n) \to 0$ and $e_n \to 0$ as $n \to \infty$. Choose the sequence $e_1, e_2, \ldots$ so that

$$P(n^{\frac{1}{2}} \sup_{1 \le i \le (1-\gamma)_n} |U_{ni}| > e_n) \to 0$$

as $n \to \infty$ and set

$$\xi_n = n^{-\frac{1}{2}} e_n \, .$$

For $i \ge 1$, define

$$I_{i+1} = I[Z_{i+1} > n\delta a(1+\xi_n) f_n\{F_n^{-1}(1 - i/n)\} \cdot \{1 + \sum_{k=1}^{i} \frac{(Z_k - 1)}{(n-k+1)}\}] ,$$

and

$$\Delta_n = \sum_{1 \le i \le (1-\gamma)n} \{I(D_{n,n-i} < \delta a) - I_{i+1}\}.$$

Hence,

$$P(\Delta_n = 0) \geq P\{ \sup_{1 \leq i \leq (1-\gamma)n} | I(D_{n,n-1} > \delta a) - I_{i+1} | = 0\}$$

$$\geq 1 - P\{ \text{for some } 1 \leq i \leq (i-\gamma)n ,$$

$$I(D_{n,n-i} > \delta a) - I_{i+1} = 1\}$$

$$- P\{ \text{for some } 1 \leq i \leq (1-\gamma)n ,$$

$$I_{i+1} - I(D_{n,n-i} > \delta a) = 1\}$$

$$\geq 1 - 2P(\bigcup_{1 \leq i \leq (1-\gamma)n} |U_{ni}| > n^{-\frac{1}{2}} e_n)$$

$$\rightarrow 1$$

as $n \rightarrow \infty$. Thus,

(3.24) $$\sum_{1 \leq i \leq (1-\gamma)n} I_{i+1} + \Delta_n = \sum_{1 \leq i \leq (i-\gamma)n} I(D_{n,n-i} > \delta a)$$

and

(3.25) $$P(\Delta_n = 0) \rightarrow 1$$

as $n \rightarrow \infty$.

Notice that if I_{i+1} is conditioned on $Z_1, \ldots, Z_i$, a sequence of independent random variables is obtained for $i \geq 1$. By appropriate choice of constants a martingale can be constructed . We eventually use the martingale central limit theorem to prove (3.18) .

For $0 \le k \le (1-\gamma)n$ let

$$P_{i+1} = E(I_{i+1} \mid Z_1, \ldots, Z_i) ,$$

$$b_{ni} = n\delta a \; f_n\{F_n^{-1}(1 - {}^i/_n)\} ,$$

$$\alpha_{nk} = (n-k)^{-1} \sum_{i=k+1}^{(1-\gamma)n} \rho \; a \; e^{-b_{ni}} \quad \text{and}$$

$$c_n(\gamma) = \sum_{1 \le i \le (1-\gamma)n} e^{-b_{ni}} .$$

Since $\xi_n = o(n^{-\frac{1}{2}})$ and $b_{ni} \to \rho a$ uniformly in $1 \le i \le (1-\gamma)n$ as $n \to \infty$,

$$P_{i+1} = \exp[-n\delta a(1 + \xi_n) f_n\{F_n^{-1}(1 - {}^i/_n)\} \{1 + \sum_{k=1}^{i} \frac{(Z_k - 1)}{(n-k+1)}\}]$$

$$= e^{-b_{ni}} \exp[- \xi_n \; b_{ni}\{1 + \sum_{k=1}^{i} \frac{(Z_k-1)}{(n-k+1)}\} - b_{ni} \sum_{k=1}^{i} \frac{(Z_k-1)}{(n-k+1)}]$$

(3.26)

$$= e^{-b_{ni}} \{1 - \rho a \sum_{k=1}^{i} \frac{(Z_k-1)}{(n-k+1)}\} + o_p(n^{-\frac{1}{2}})$$

uniformly in $1 \leq i \leq (1-\gamma)n$ as $n \to \infty$, where the last step follows from lemma 1 . Thus by (3.26) ,

$$\sum_{1 \leq i \leq (1-\gamma)n} I_{i+1} - c_n(\gamma) = \sum_{1 \leq i \leq (1-\gamma)n} (I_{i+1} - P_{i+1})$$

$$- \sum_{1 \leq i \leq (1-\gamma)n} e^{-b_{ni}} \rho a \sum_{k=1}^{i} \frac{(Z_k - 1)}{(n-k+1)} + o_p(n^{\frac{1}{2}})$$

$$= \sum_{1 \leq i \leq (1-\gamma)n} (I_{i+1} - P_{i+1})$$

$$- \sum_{1 \leq k \leq (1-\gamma)n} \frac{(Z_k - 1)}{(n-k+1)} \rho a \sum_{k \leq i \leq (1-\gamma)n} e^{-b_{ni}} + o_p(n^{\frac{1}{2}})$$

(3.27)

$$= \sum_{1 \leq i \leq (1-\gamma)n} \{(I_{i+1} - P_{i+1}) - (Z_{i+1} - 1) \alpha_{ni}\} + o_p(n^{\frac{1}{2}}) .$$

The series on the right hand side of (3.27) forms a martingale because

(3.28) $$E\{(I_{i+1} - P_{i+1}) + (Z_{i+1} - 1)\alpha_{ni} \mid Z_1, \ldots, Z_i\} = 0$$

almost surely for $1 \leq i \leq (1-\gamma)n$. Next we derive the asymptotic conditional variance of the sequence in (3.28).

According to (3.26) ,

(3.29) $$\sum_{1 \leq i \leq (1-\gamma)n} E\{(I_{i+1} - P_{i+1})^2 \mid Z_1, \ldots, Z_i\} = n(1-\gamma)e^{-\rho a}(1-e^{-\rho a}) + o_p(n) ,$$

as $n \to \infty$. Likewise,

$$\sum_{1\le i\le(1-\gamma)n} E[\{(Z_{i+1}-1)\alpha_{ni}\}^2 | Z_1,\ldots,Z_i]$$

$$= \sum_{1\le i\le(1-\gamma)n} \alpha_{ni}^2$$

$$= \sum_{1\le i\le(1-\gamma)n} [\frac{\{(1-\gamma)n-i\}}{(n-i)} \rho\, ae^{-\rho a}]^2 + o(n)$$

$$= (\rho a)^2\, e^{-2\rho a}\int_0^{(1-\gamma)n} [\frac{\{(1-\gamma)n-x\}}{(n-x)}]^2\, dx + o(n)$$

$$= (\rho a)^2\, e^{-2\rho a}\int_\gamma^1 (\frac{x-\gamma}{x})^2\, dx + o(n)$$

(3.30)

$$= (\rho a)^2\, e^{-2\rho a}\{1-\gamma^2+2\gamma\ell n(\gamma)\} + o(n)$$

as $n \to \infty$. In a fashion very similar to the way (3.30) was derived,

$$\sum_{1\le i\le(1-\gamma)n} E\{(I_{i+1}-P_{i+1})(Z_{i+1}-1)\alpha_{ni} | Z_1,\ldots,Z_i\}$$

$$= \sum_{1\le i\le(1-\gamma)n} \alpha_{ni}\int_0^\infty (y-1)e^{-y}[I\{y>\rho a+o_p(1)\} - e^{-\rho a + o_p(1)}]\, dy$$

$$= \rho\, a\, e^{-\rho a} \sum_{1\le i\le(1-\gamma)n} \alpha_{ni} + o_p(n)$$

(3.31)

$$= (\rho a)^2 e^{-2\rho a}\, n\{1-\gamma+\gamma\, \ell n(\gamma)\} + o_p(n).$$

Combining the estimates (3.29) , (3.30) and (3.31) we have

$$n^{-1} \sum_{1\le i\le (1-\gamma)n} \operatorname{var}[\{(I_{i+1} - P_{i+1}) - (Z_{i+1} - 1)\alpha_{ni}\} | Z_i, \ldots, Z_i]$$

(3.32)

$$\to e^{-\rho a}(1 - e^{-\rho a})(1-\gamma) - (\rho a)^2 e^{-2\rho a}(1-\gamma)^2 = \nu^2(\gamma), \quad \text{say,}$$

in probability as $n \to \infty$. Also the conditional fourth moment of $n^{-\frac{1}{2}}\{(I_{i+1}-P_{i+1})-(Z_{i+1}-1)\alpha_{ni}\}$, given $Z_1,\ldots,Z_i$, is bounded by a constant multiple of

$$\sum_{1\le i\le (1-\gamma)n} n^{-2}(Z_{i+1}+1)^4 \to 0 \text{ in probability as } n \to \infty ,$$

by Markov's inequality. Thus by the results (3.27) and (3.32) , and corollary 3.1 of Hall and Heyde (1980),

$$n^{-\frac{1}{2}} \{ \sum_{1\le i\le (1-\gamma)n} I_{i+1} - c_n(\gamma) \} \to^{\mathcal{D}} N(0,1)$$

as $n \to \infty$. The result (3.18) follows from (3.24) and the above expression.

<u>Step (iii)</u>

The result (3.19) follows simply from expression (3.32).

Now

$$c_n(\gamma) = \sum_{1\le i\le (1-\gamma)n} e^{-b_{ni}}$$

$$= \sum_{1\le i\le (1-\gamma)n} e^{-n\delta a f_n\{F_n^{-1}(1 - {}^i/_n)\}}$$

$$= \int_0^{(1-\gamma)n} e^{-n\delta a f_n F_n^{-1}(1-x/n)} dx + o(n^{\frac{1}{2}})$$

(3.33)
$$= n \int_{F_n^{-1}(\gamma)}^{1} f_n(y) e^{-n\delta a f_n(y)} dy + o(n^{\frac{1}{2}})$$

as $n \to \infty$, where the small order term holds uniformly in γ. Result (3.20) follows.

To complete the proof we need to show that for all $\zeta > 0$,

(3.34)
$$\lim_{\gamma\to 0} \limsup_{n\to\infty} P\Big(n^{-\frac{1}{2}} \Big| \Big\{ \sum_{(1-\gamma)n<i\leq n-1} I(D_{n,n-i} > \delta a) + c_n(\gamma) - n \int_0^1 f_n(y)\, e^{-n\delta a\, f_n(y)} dy \Big\} \Big| > \zeta \Big)$$

Let D^*_{ni} be the distance between the i'th and (i+1)'th order statistics resulting from randomly distributing points on the unit interval according to a distribution with density g_n defined by :

$$g_n(x) = \begin{cases} f_n(1 - x - \delta a) & \text{when } 0\leq x\leq 1-\delta a \\ f_n(2 - x - \delta a) & \text{otherwise.} \end{cases}$$

In view of (3.33), establishing (3.34) is the same as proving for all $\zeta > 0$,

(3.35)
$$\lim_{\gamma\to 0} \limsup_{n\to\infty} P\Big(n^{-\frac{1}{2}} \Big| \Big\{ \sum_{1\leq i<n} I(D^*_{n,n-i} > \delta a) - c_n(1-\gamma) \Big\} \Big| > \zeta + o(1) \Big) = 0 .$$

Now $g_n(x) = f_n(1-x) + o(n^{-\varepsilon})$ as $n \to \infty$, where the small order term holds uniformly in $x \in [0,1]$. One may proceed precisely as in steps (i) and (ii) to prove that

$$n^{-\frac{1}{2}} \{\nu(1-\gamma)\}^{-1}\{ \sum_{1<i<n\gamma} I(D^*_{n,n-i} > \delta a)$$

$$- c_n(1-\gamma)\} \to^{\mathcal{D}} N(0,1) .$$

The result (3.35) follows from this and the fact that

$$\nu(1-\gamma) = e^{-\rho a}(1-e^{-\rho a})\gamma - (\rho a)^2 e^{-2\rho a}\gamma^2 \to 0$$

as $\gamma \to 0$. This completes the proof of theorem 3. $\square$

Recall that our test statistic based on L rejects the null hypothesis H_o when

$$\text{(3.36)} \qquad \frac{n^{-\frac{1}{2}}\{L - ne^{-n\delta a}\}}{\nu_n} < -z_\alpha .$$

It is clear that $\nu_n \to \nu$ as $n \to \infty$, and so by theorems 3 and 4,

$$n^{-\frac{1}{2}}\{L - ne^{-n\delta a}\} / \nu_n$$

$$= n^{-\frac{1}{2}}[L - n\int_0^1 f_n(x)\exp\{-n\delta a\ f_n(x)\}dx]/\nu_n$$

$$+ n^{-\frac{1}{2}}\nu_n^{-1}\ ne^{-n\delta a}\{\frac{\rho a}{2}(\rho a - 2)n^{-2\varepsilon}\int_0^1 \ell^2(x)dx + o(n^{-2\varepsilon})\}$$

(3.37)

....

$$= Z_n + (1+o(1))\nu^{-1}\, e^{-\rho a}\, \frac{\rho a}{2}\, (\rho a-2)\, n^{(\frac{1}{2})-2\varepsilon} \int_0^1 \ell^2(x)\,dx \ ,$$

where Z_n converges to the standard normal law as $n \to \infty$. From (3.37) we may deduce that for $0 < \rho a \leq 1$:

(a) if $\varepsilon < \frac{1}{4}$, then the power of the test at (3.36) tends to 1 as $n \to \infty$;

(b) if $\varepsilon > \frac{1}{4}$, then the power of the test tends to the significance level α , and

(c) if $\varepsilon = \frac{1}{4}$ then the power tends to a value lying strictly between α and 1 .

As was shown in theorem 2 , the test at (2.36) is not always consistent when $\rho a > 1$.

3.2 A Test Based on the Length of the Largest Uncovered Spacing.

As in subsection 3.1, define D_{ni} as the length of the i'th ordered spacing. The length of the largest uncovered spacing M is defined by :

$$M \equiv \sup_{0 \leq i \leq n-1} (D_{ni} - \delta a)_+ \ ,$$

where $(x)_+ = x$ if the argument is positive, and zero otherwise.

The following theorem is used to construct a test of the null hypothesis against the alternative H_A , consisting of all essentially non constant densities on [0,1] possessing a unique, non zero minimum. The result follows via a theorem of Hall (1983) .

Theorem 5

Let M *be the length of the largest uncovered spacing in* $A = [0,1]$ *resulting from* $C_f^{(1)}(\delta,n)$. *Suppose that* f *is essentially non-constant on* A *and possesses a unique non-zero minimum at* $m \in A$. *If* $\delta \to 0$ *and* $n \to \infty$ *in such a manner that* $\delta n \to \rho$, *where* $0 < \rho < \infty$, *then*

$$nM + n\delta a - \log(n) \to +\infty$$

in probability as $n \to \infty$.

Proof

Since f is not constant, $0 < f(m) < 1$. Now

$$\begin{aligned} & nM + n\delta a - \log n \\ (3.38) \quad & = f(m)[nM + \delta na - \{\log(n) \\ & \quad - \tfrac{1}{2}\log\log(n)\}/f(m)] \\ & \quad + [\{f(m)\}^{-1} - 1]\log(n) \\ & \quad + \tfrac{1}{2}\{f(m)\}^{-1}\log\log(n) . \end{aligned}$$

According to theorem M , page 7 of Hall (1983), the first term on the right hand side of (3.38) converges to a proper distribution as $n \to \infty$. The second and third terms converge to $+\infty$ as $n \to \infty$. Theorem 5 follows. $\Box$

A consequence of corollary 3.2 of Hüsler (1982) is that under the null hypothesis

$$n M + n\delta a - \log(n) \xrightarrow{\mathcal{D}} X$$

as $n \to \infty$, where X has the extreme value distribution $P(X \leq x) = \exp(-e^{-x})$. Thus, by theorem 5 , the test which rejects H_o in favour of H_A when

$$(3.39) \qquad M > n^{-1}\log(N) - n^{-1}\log\{-\log(1-\alpha)\} - \delta a$$

is asymptotically level α , where $0 < \alpha < 1$. It also follows from theorem 5 that as $n \to \infty$ the test at (3.39) is asymptotically consistent. We shall now examine the "local power" of this simple test based on M .

In subsection 1 the local alternative density function was defined by $f_n(x) = 1 + n^{-\varepsilon} \ell(x)$, where $\varepsilon > 0$ is fixed and the funtion ℓ satisfies the two conditions set out at (3.16) . The simple tests based on V and L , investigated in section 2 and subsection 3.1 respectively, can distinguish differences of order $O(n^{-\frac{1}{4}})$ from the null hypothesis. It so happens that the test based on M performs poorly in comparison to these tests. Its asymptotic power against the sequence of local alternatives equals the significance level for all $\varepsilon > 0$. To analyse its power properties in more detail we construct a new local alternative density function that converges to $f_o \equiv 1$ much more slowly than f_n .

Let ℓ be a function satisfying the conditions at (3.16) . For each $\varepsilon > 0$ define

(3.40) $$g_n(x) = 1 + \{\log(n)\}^{-\varepsilon} \ell(x) ,$$

where $x \in A$. In order to establish our theory we need to impose several regularity conditions upon the function ℓ. Namely :

(3.41) ℓ is essentially nonconstant on A ;

(3.42) the first derivative, ℓ' of ℓ, exists and is bounded, and

(3.43) there exists an $m \in (0,1)$ such that the second derivative, ℓ'' of ℓ, exists within a neigbbourhood of m, is continuous at m, $\ell''(m) > 0$ and $\ell'(m) = 0$. Furthermore, assume that for each $\eta > 0$,

$$\inf_{|x-m|>\eta} \ell(x) > \ell(m) .$$

The following theorem allows us to describe the asymptotic power properties of the test at (3.39) against the local alternatives at (3.40).

<u>Theorem 6</u>

<u>Let</u> M <u>be the length of the largest uncovered spacing in</u> $A = [0,1]$ <u>resulting from</u> $C^{(1)}_{g_n}(\delta,n)$. <u>As well as the conditions at</u> (3.16), <u>assume that</u> ℓ <u>satisfies</u> (3.41) <u>and</u> (3.42). <u>If</u> $\delta \to 0$ <u>and</u> $n \to \infty$ <u>in such a manner that</u> $\delta n \to \rho$, <u>where</u> $0 < \rho < \infty$ <u>then for</u> $\varepsilon > 1$,

(3.44) $$nM + n\delta a - \log(n) \to^{\mathcal{D}} X ,$$

where X has the extreme value distribution

$$P(X \le x) = \exp(-e^{-x}) ,$$ and when $\varepsilon = 1$,

(3.45) $$nM + n\delta a - \log(n) \to^{\mathcal{D}} X + \log[\int_0^1 \exp\{-\ell(x)\}dx]$$

Furthermore, if we assume the conditions at (3.43) hold, when $0 < \varepsilon < 1$

(3.46) $$nM + n\delta a - \{g_n(m)\}^{-1}\{\log(n) - (1-\varepsilon)\log\log(n)/2\}$$

$$\to^{\mathcal{D}} X + \tfrac{1}{2}\log\{2\pi/\ell''(m)\}$$

as $n \to \infty$.

The proof is prefaced with the following useful lemma.

Lemma 4

Let η be a fixed positive constant and suppose $\xi(y)$ is a positive function on $[-\eta,\eta]$ which converges to one as $y \to 0$. If λ_n is a sequence of constants which converges to infinity as $n \to \infty$, then

$$\int_{-\eta}^{\eta} \exp\{-\tfrac{1}{2} y^2 \lambda_n \xi(y)\}dy \sim \sqrt{\frac{2\pi}{n}}$$

as $n \to \infty$.

Proof

Choose $0 < \eta_1 < \eta$, and η_2 such that for all $-\eta \leq y \leq \eta$, $\xi(y) > \eta_2$. Then ,

$$\left(\int_{-\eta}^{\eta} - \int_{\eta_1}^{\eta_1}\right) \exp\{-\tfrac{1}{2} y^2 \lambda_n \xi(y)\} dy$$

$$\leq 2\int_{\eta_1}^{\eta} \exp(-\tfrac{1}{2} \eta^2 y^2 \lambda_n) dy$$

$$\leq 2 (\eta-\eta_1) \exp\{-\tfrac{1}{2} \eta_2\eta_1^2\lambda_n\} = o(\lambda_n^{-\frac{1}{2}})$$

as $n \to \infty$. Hence, η_1 may be chosen arbitrarily small. Given $\eta_3 > 0$, there exists a $\eta_1 > 0$ such that $\xi(y) \geq 1 - \eta_3$ for $-\eta_1 \leq y \leq \eta_1$. Thus,

$$\int_{-\eta_1}^{\eta_1} \exp\{-\tfrac{1}{2} y^2 \lambda_n \xi(y)\} dy$$

$$\leq \int_{-\eta_1}^{\eta} \exp\{-\tfrac{1}{2} y^2 \lambda_n(1 - \eta_3)\} dy$$

$$\sim \sqrt{\frac{2\pi}{\lambda_n(1-\eta_3)}}$$

as $n \to \infty$. The asymptotic lower bound $\sqrt{2\pi/\{\lambda_n(1+\eta_3)\}}$ can be obtained in a similar way. Since $\eta_3 > 0$ is arbitrary, lemma 4 follows. $\square$

Proof of Theorem 6

Let $0 < \gamma < 1$ be given. Following from (3.17) and lemma 1,

$$(3.47) \qquad D_{n,n-i} = n^{-1} Z_{i+1} [g_n\{G_n^{-1}(1-i/n)\}(1 + U_{ni})]^{-1}$$

where G_n is the distribution function corresponding to g_n, and

$$\sup_{1 \leq i \leq (1-\gamma)n} |U_{ni}| = O_p(n^{-\frac{1}{2}}) .$$

The representation at (3.47) forms the basis of our proof. For clarity the proof is divided into three steps : the limit result at (3.44) is proved in step (i); (3.45) is proved in step (ii); and (3.46) in step (iii).

Step (i)

In this step $\varepsilon > 1$. According to (3.47) we may choose $1 < \eta < \varepsilon$ such that, with probability approaching one as $n \to \infty$,

$$n\, D_{n,n-i} \geq Z_{i+1}[1 + \{\log(n)\}^{-\eta}]^{-1}$$

uniformly in $1 \leq i \leq (1-\gamma)n$. Therefore, for each $\beta > 0$,

$$P\{n\, D_{n,n-i} - \log(n) \leq \log(\beta) \quad \text{for all} \quad 1 \leq i \leq (1-\gamma)n\}$$

$$\leq P(\text{for all} \quad 1 \leq i \leq (1-\gamma)n ,$$

$$Z_{i+1} \leq [1 + \{\log(n)\}^{-\eta}]\log(n/\beta)) + o(1)$$

$$= \{1 - \exp(-[1+\{\log(n)\}^{-\eta}]\log(n/\beta))\}^{n(1-\gamma)} + o(1)$$

$$= \{1 - \beta n^{-1} + o(n^{-1})\}^{n(1-\gamma)} + o(1)$$

$$= \exp\{-(1-\gamma)\beta\} + o(1)$$

as $n \to \infty$. It follows that

$$\limsup_{n\to\infty} P\{ \text{ for all } 1 \le i \le (1-\gamma)n, \ nD_{n,n-i} - \log(n) \le -\log(\beta)\}$$

$$\le \exp\{-(1-\gamma)\beta\}$$

The same lower bound for $\liminf_{n\to\infty}$ may be obtained by using the inequality

$$D_{n,n-i} \le n^{-1} Z_{i+1}[1 - \{\log(n)\}^{-\eta}]^{-1},$$

with probability approaching one as $n \to \infty$, uniformly in $1 \le i \le (1-\gamma)n$.

Thus, for $-\infty < x < \infty$,

(3.48) $$\lim_{n\to\infty} P\{\text{for all } 1 \le i \le (1-\gamma)n , \ n D_{n,n-i} - \log(n) \le x\} = \exp(e^{-x}).$$

Let

$$\Delta = \max_{1\le i\le n} \{n D_{n,n-i} - \log(n)\} - \max_{1\le i\le (1-\gamma)n} \{n D_{n,n-i} - \log(n)\}.$$

Then, for any fixed λ ,

$$P(\Delta\neq 0) \leq P\{ \max_{1\leq i\leq (1-\gamma)n} nD_{n.n-i} - \log(n) \leq \lambda , \quad \text{or}$$

$$\max_{(1-\gamma)n<i\leq n} nD_{n,n-i} - \log(n) > \lambda\}$$

(3.49)

$$\leq P\{ \max_{1\leq i\leq (1-\gamma)n} nD_{n,n-i} - \log(n) \leq \lambda\}$$

$$+ P\{ \max_{(1-\gamma)n<i\leq n} n\, D_{n,n-i} - \log(n) > \lambda\}$$

The first term on the RHS of (3.49) converges to $\exp\{-(1-\gamma)e^{-\gamma}\}$ as $n\to\infty$. Using a reflection argument similar to the one in step (iii) of the proof of theorem 4, the second term on the RHS of (3.49) converges to $1 - \exp\{-\gamma e^{-\lambda}\}$ as $n\to\infty$. Consequently,

$$\lim_{\gamma\to 0} \limsup_{n\to\infty} P(\Delta \neq 0) \leq \exp(-e^{-\lambda}) \to 0$$

as $\lambda \to -\infty$. Thus, by (3.48) and lemma 2 of section 1 , we see that

$$(3.50) \qquad \lim_{n\to\infty} P\{\text{for all } 1\leq i\leq n ,\ nD_{n,n-i} - \log(n) \leq x\}$$

$$= \exp(e^{-x}) .$$

Note, however, that

$$n\, M = \max[\max_{1\leq i\leq n} \{n\, D_{n,n-i} - \log(n)\} + \log(n) - n\delta a,\ 0]$$

and $\log(n) - n\delta a \to \infty$ as $n \to \infty$. Result (3.44) follows from this and (3.50) .

Step (ii)

From the estimate at (3.47) , with probability approaching one as $n \to \infty$,

$$n\, D_{n,n-i} \geq Z_{i+1}[g_n\{G_n^{-1}(1 - i/n)\}\ (1 + n^{-1/4})]^{-1}$$

uniformly in $1 \leq i \leq (1-\gamma)n$. Let
$c = \log[\int_0^1 \exp\{-\ell(x)\}dx]$.

For any $\beta > 0$,

$$P\{\text{for all } 1 \leq i \leq (1-\gamma)n\ ,\ n\, D_{n,n-i} - \log(n) - c \leq -\log(\beta)\}$$

$$\leq P[\text{for all } i \leq i \leq (1-\gamma)n\ , Z_{i+1} \leq (1 + n^{-1/4})\ g_n \{G_n^{-1}(1 - i/n)\} \times \{\log(n/\beta) + c\}] + o(1)$$

$$= \prod_{1\leq i\leq(1-\gamma)n} (1 - \exp[-(1+n^{-1/4})g_n\{G_n^{-1}(1 - i/n)\}\{\log(n/\beta)+c\}]) + o(1)$$

(3.51)

$$= \exp(- \sum_{1\leq i\leq(1-\gamma)n} \exp[-(1+n^{-1/4})g_n\{G_n^{-1}(1-i/n)\}\{\log(n/\beta)+c\}]) + o(1)\ .$$

The summation may be approximated to a term of order $o(1)$ by the integral expression :

$$\int_0^{(1-\gamma)n} \exp[-(1+n^{-\frac{1}{4}})g_n\{G_n^{-1}(1-x/n)\}\{\log(n/\beta)+c\}]dx$$

$$= n\{1+o(1)\}\exp[-(1+n^{-\frac{1}{4}})\{\log(n/\beta)+c\}]$$

$$\int_\gamma^1 \exp[-(1+n^{-\frac{1}{4}})\{\log(n)\}^{-1}$$

$$\times\ \ell(y)\{\log(n/\beta)+c\}]\ g_n(y)\ dy$$

(3.52)

$$= \{1+o(1)\}\ \beta e^{-c}\int_\gamma^1 \exp[-(1+n^{-\frac{1}{4}})\ell(y)$$

$$\{\log(n)\}^{-1}\{\log(n/\beta)+c\}]$$

$$\times\ g_n(y)\ dy, \qquad \text{as} \quad n\to\infty .$$

Let $\eta > 0$ be arbitrary. For n sufficiently large

$$\int_\gamma^1 \exp[-(1+n^{-\frac{1}{4}})\ell(y)\{\log(n)\}^{-1}\{\log(n/\beta)+c\}]g_n(y)dy$$

$$\leq \int_\gamma^1 \exp\{-\ell(y)\}\ e^\eta(1+\eta)dy .$$

A similar lower bound may be found with η replaced by $-\eta$. But $\eta > 0$ is arbitrary and so by (3.52) and the argument just proceeding (3.51) ,

$$\limsup_{n\to\infty} P\{\text{for all } 1 \le i \le (1-\gamma)n ,$$

$$n D_{n,n-i} - \log(n) - c \le -\log(\beta)\}$$

$$\le \exp\left[-\beta \left\{\int_0^1 e^{-\ell(x)} dx\right\}^{-1} . \left\{\int_\gamma^1 e^{-\ell(x)} dx\right\}\right] .$$

Using an analogous argument, the same lower bound can be found for $\liminf_{n\to\infty}$. Consequently,

$$\lim_{n\to\infty} P\{\text{for all } 1 \le i \le (1-\gamma)n ,$$

$$n D_{n,n-i} - \log(n) - c \le x$$

$$= \exp\left[-\beta \left\{\int_0^1 e^{-\ell(x)} dx\right\}^{-1} \left\{\int_\gamma^1 e^{-\ell(x)} dx\right\}\right]$$

Result (2.45) may be derived in a manner similar to the way (2.44) was derived from (2.48) .

Step (iii)

A more intricate bounding argument is required to prove (3.46) . Suppose that for some $\eta > 0$, ℓ'' exists and is bounded away from zero in the interval $[m - 3\eta, m + 3\eta] \subset (0,1)$. Restrict γ to the range $0 < \gamma < m - 3\eta$ and set

$$i_1 = \inf\{i : 1 - i/n \le m - 2\eta\} \qquad \text{and}$$

$$i_2 = \sup\{i : 1 - i/n > m + 2\eta\} .$$

Then $i_1 \ge i_2$ and

$$\inf_{i \le i_2} G_n^{-1}(1 - {}^i/_n) \ge \inf_{i \le i_2} (1 - {}^i/_n) - \{\log(n)\}^{-\varepsilon}$$

$$> m + \eta$$

for some constant $c > 0$ and n sufficiently large.

Likewise,

$$\sup_{i_1 \le i_2} G_n^{-1}(1 - {}^i/_n) \le m - \eta$$

for sufficiently large n. Following from (3.47) there exists a $\xi > 0$ such that with probability approaching one as $n \to \infty$,

$$n\, D_{n,n-i} \le Z_{i+1}[1 + \{\log(n)\}^{-\varepsilon}(1-2\xi)\ell(m)]^{-1}$$

whenever $1 \le i \le i_2$ or $i_1 \le i \le n$. Now $0 < \varepsilon < 1$, and so for each fixed x,

$$P[\text{for all } 1 \le i \le i_2 \text{ and } i_1 \le i \le (1-\gamma)n,$$

$$nD_{n,n-i} - \frac{1}{g_n(m)}\{\log(n) - \frac{(1-\varepsilon)}{2} \log\log(n)\} \le x]$$

$$\ge P(\text{for all } 1 \le i \le n,\ Z_{i+1} \le [1 + \{\log(n)\}^{-\varepsilon}(1-2\xi)\ell(m)]$$

$$\times [x + \frac{1}{g_n(m)} \{\log(n) - \frac{(1-\varepsilon)}{2}\log\log(n)\}])$$

$$+ o(1)$$

$$\ge P(\text{for all } 1 \le i \le n,\ Z_{i+1} \le [1 - \xi\ell(m) \log(n)\}^{-\varepsilon}] \log(n))$$

$$+ o(1)$$

$$= (1 + n^{-[1 - \xi \ell(m) \{\log(n)\}^{-\varepsilon}]})^n + o(1)$$

$$\to 1$$

as $n \to \infty$. Therefore

$$P[\text{for all } 1 \le i \le (1-\gamma)n ,$$

$$n D_{n,n-i} - \frac{1}{g_n(m)}\{\log(n) - \frac{(1-\varepsilon)}{2}\log \log(n)\} \le x]$$

(3.53)

$$= P[\text{for all } i_2 < i < i_1,$$

$$n D_{n,n-i} - \frac{1}{g_n(m)}\{\log(n) - \frac{(1-\varepsilon)}{2} \log \log(n)\} \le x]$$

$$+ o(1)$$

as $n \to \infty$.

Let $b_n = \{g_n(m)\}^{-1}\{\log(n) - 2^{-1}(1-\varepsilon)\log \log(n)\}$.

According to (3.47) , for any $\beta > 0$,

$$P\{\text{for all } i_2 < i < i_1, \ n D_{n,n-i} - b_n \le - \log(\beta)/g_n(m)\}$$

$$\le P[\text{for all } i_2 < i < i_1, \ Z_{i+1} \le (1+n^{-\frac{1}{4}}) g_n\{G_n^{-1}(1-{}^i/_n)\}$$

$$\{b_n - \log(\beta)/g_n(m)\}] + o(1)$$

$$= \exp(- \sum_{i_2<i<i_1} \exp[-(1+n^{-\frac{1}{4}}) g_n\{G_n^{-1}(1-{}^i/_n)\}\{b_n - \log(\beta)$$

$$/g_n(m)\}]) + o(1)$$

(3.54)

$$= \exp(-\int_{i_2}^{i_1} \exp[-(1+n^{-\frac{1}{4}}) g_n\{G_n^{-1}(1 - x/n)\}$$

$$\{b_n - \log(\beta)/g_n(m)\}]dx + o(1) .$$

Also,

$$\int_{i_2}^{i_1} \exp[-(1+n^{-\frac{1}{4}}) g_n\{G_n^{-1}\ (1 - x/n)\}\{b_n - \log(\beta)/g_n(m)\}]dx$$

(3.55)

$$= n\{1+o(1)\}\int_{-2\eta}^{2\eta} \exp[-(1+n^{-\frac{1}{4}})\frac{g_n(y+m)}{g_n(m)}$$

$$\{\log(\tfrac{n}{\beta}) - (\tfrac{1-\varepsilon}{2}) \log\log(n)\}]$$

$$\times\ g_n(y+m)\ dy.$$

Since g_n'' is continuous at m and $\ell'(m) = 0$,

$$g_n(m+y) - g_n(m) = y^2 \int_0^1 (1-t) g_n''(m+ty)\ dt$$

$$= \tfrac{1}{2}\ y^2[g_n''(m) + \{\log(n)\}^{-\varepsilon} h(y)] ,$$

where h is a function satisfying $h(y) \to 0$ as $y \to 0$. Thus, by lemma 4 ,the integral in expression (3.55) equals

$$\exp[-(1 + n^{-\frac{1}{4}})\{\log(\tfrac{n}{\beta}) - (\tfrac{1-\varepsilon}{2})\log\log(n)\}]$$

$$\times \int_{-2\eta}^{2\eta} \exp[-(1+n^{-\frac{1}{4}})\tfrac{1}{2}y^2\frac{\{\log(n)\}^{-\varepsilon}}{g_n(m)}\ \{\ell''(m) + h(y)\}$$

$$\times\ \{\log(\tfrac{n}{\beta}) - (\tfrac{1-\varepsilon}{2})\log\log(n)\}]g_n(y+m)dy$$

$$\sim \frac{\beta}{n}\ \log(n)^{(1-\varepsilon)/2}\sqrt{\frac{2\pi}{\ell''(m)\{\log(n)\}^{1-\varepsilon}}} = \frac{\beta}{n}\sqrt{\frac{2\pi}{\ell''(m)}}\ .$$

By this result, (3.54) and (3.55), for any $\beta > 0$,

$$\limsup_{n\to\infty} P\{ \text{ for all } i_2 < i < i_1,\ n D_{n,n-i} - b_n \le \log(\beta)/g_n(m) \}$$

$$\le \exp\{ - \beta \sqrt{\frac{2\pi}{\ell''(m)}} \} .$$

The same lower bound can be found for $\liminf_{n\to\infty}$ using an analogous argument. Hence by (3.53) ,
for $-\infty < x < \infty$,

$$\lim_{n\to\infty} P\{\text{for all } 1 \le i \le (1-\gamma)n\ ,\ g_n(m)(nD_{n,n-i} - b_n) \le x\}$$

$$= \exp\{-\sqrt{\frac{2}{\ell''(m)}}\, e^{-x}\}.$$

Results (3.46) now follows using a very similar argument to the way result (3.44) was derived from (3.48) . This completes the proof of theorem 6. $\square$

The following conclusions may be drawn from theorem 6 :

(a) if $\varepsilon < 1$ in g_n , then the power of the test at (3.39) converges to 1 as $n \to \infty$;

(b) if $\varepsilon > 1$, then the power of the test tends to the significance level α , and

(c) if $\varepsilon = 1$ then the power tends to a value lying strictly between α and 1 .

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