ALMOST PERIODIC AND QUASI-PERIODIC
SOLUTIONS OF DIFFERENTIAL EQUATIONS

This thesis is my own work. Results which are not by me are
indicated in the text.

by

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STATEMENT

This thesis is my own work. Results which are not my own are indicated in the text.
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CHAPTER I
INTRODUCTION

1.1.

Recently there has been a renewed interest in almost periodic solutions of ordinary differential equations and in particular in quasi-periodic solutions. The study of such solutions is natural in the theory of non-linear oscillations, yet attention had until the last decade been largely restricted to periodic solutions. These can be considered as quasi-periodic solutions with only one basic frequency. However, the investigation of quasi-periodic solutions with more than one basic frequency is plagued by a subtle problem which has long been famous in celestial mechanics, the so called "small divisor problem".

This problem arises in particular from the fact that integration may lead out of the class of almost periodic functions. If \( f(t) \) is an almost periodic function then the indefinite integral

\[
\int_0^t f(s) \, ds
\]

need not be almost periodic, even if one assumes that \( f(t) \) has mean value zero. It is well known that boundedness is a necessary and sufficient condition for the almost periodicity of this integral but in applications such a condition may be very hard to verify. In fact the early articles on almost periodic solutions of linear differential equations, for example Bohr and Neugebauer [41], contained such an assumption. Even in the restricted case of a quasi-periodic function the same situation holds.

Suppose now \( f(t) \) is quasi-periodic with Fourier series
where \((k, \omega) = k_1 \omega_1 + \ldots + k_m \omega_m\). Then the indefinite integral has the Fourier series
\[
\sum_{k \neq 0} a_k e^{i(k, \omega)t}
\]
and even if \(f(t)\) is holomorphic it is possible to find an \(m\)-vector \(\omega\) whose entries are so close to being rationally dependent that the Fourier series diverges. If, however, we suppose that \(\omega\) satisfies the inequality
\[
| (k, \omega) | \geq \gamma |k|^{-\tau}, \quad \gamma > 0, \quad \tau > 0
\]
for all integral \(m\)-vectors \(k \neq 0\) then the above series converges and we can do away with the boundedness condition on the integral. Moreover such an irrationality condition is meaningful in the sense that provided \(\tau > m - 1\) such a constant \(\gamma\) exists for almost all \(\omega\).

It follows that in certain cases the theory of quasi-periodic solutions is more tractable then the theory of arbitrary almost periodic solutions. Furthermore since quasi-periodic functions describe oscillations with just finitely many basic frequencies, this should be sufficient for most practical purposes, particularly since there is no simple way of characterising those almost periodic functions with just finitely many rationally independent frequencies which are not quasi-periodic.

The now famous work of Siegel, Kolmogoroff, Arnol'd and Moser introduced many new techniques and provided the first methods of overcoming small divisor difficulties after nearly two generations of very little progress. At the same time as this was being done
topological dynamics provided an impetus to the qualitative study of non-linear oscillations. Miller, Sell and others recognised the importance of compactness type arguments and considered along with the given equation all of the other equations in the closed hull. These ideas led to the extension of the concept of a dynamical system to cover almost periodic equations, and many of the ideas involved in studying continuous flows were applied to almost periodic equations and solutions.

The first half of this thesis concerns such qualitative results while the last two chapters examine some small divisor problems.

1.2.

The next chapter is concerned with restrictions which can be placed on an almost periodic solution of the almost periodic differential equation

$$x' = \psi(x, t)$$

where $x \in \mathbb{R}^p$. In particular, how are the frequencies of the almost periodic solution related to the frequencies of the almost periodic function $\psi(x, t)$?

It is very interesting to find that there can only be a finite number ($< p$) of frequencies additional to those of $\psi(x, t)$ in any frequency basis of an almost periodic solution. This was discovered by Cartwright, but our proof is shorter and simpler, and constructs a whole family of almost periodic solutions related to the given solution.

1.3.

Chapter III examines a result suggested by Cherry nearly fifty years ago which asserts that the solutions in a compact $p$-dimensional
invariant set of a holomorphic autonomous differential equation with an integral invariant,

\[ x' = f(x) \]

where \( x \in \mathbb{R}^D \), are necessarily quasi-periodic. Cherry himself expressed doubts about his method and while we have not been able to find a counter-example we present some partial counter-examples which cast considerable doubt on his results. Cherry also stated that in the case of holomorphic Hamiltonian systems, which have the integral invariant \( I \), the quasi-periodic solutions had at most \( p/2 \) basic frequencies. We examine quasi-periodic solutions of Hamiltonian systems and show that results similar to Cherry's are true in this case under much weaker assumptions.

1.4.

We turn to a problem of small divisors in Chapter IV. Here we consider the linear system

\[ x' = Ax + P(\psi)x \]
\[ \psi' = \omega \]

where \( P(\psi) \) is holomorphic, the eigenvalues of \( A \) are real and distinct and

\[ |(k, \omega)| \geq \gamma |k|^{-\tau} \]

for all \( k \neq 0 \) and some \( \gamma > 0, \tau > 0 \). The problem is to reduce this system to a system with constant coefficients for which we can write down the fundamental matrix. It turns out that there is a quasi-periodic transformation which does this and so gives all of the solutions. That is, we have a kind of Floquet theory for such systems.

The reduction is obtained by an iterative procedure and at each
step we are led to consider the partial differential equation

\[ \frac{\partial U}{\partial \phi} + U A - A U = P(\phi). \]

In estimating bounds for solutions \( U \) of this equation we need to evaluate sums of the form

\[ \sum_{k \neq 0} \frac{1}{|\langle k, \omega \rangle|} e^{-\delta |k|}. \]

In order to do this we use the fact first noted by Siegel [366] and subsequently studied by Moser [369] and Rüssmann [379] that only some of the terms in such an expression will have very small denominators. This enables us to obtain quite accurate estimates for such sums. However it is still necessary to employ a method of accelerated convergence to obtain convergence of the iterative procedure.

By accelerated convergence we mean that at the \( r \)-th stage of the iteration we require the error to be less than some power \( x \) of the error at the \((r-1)\)-th stage, where \( x > 1 \). In this Chapter we take \( x = \frac{3}{2} \).

The above problem was first considered by Mitropol'skii and Samoilenko [331], [357]. We obtain better estimates for the bounds of the series with small divisors and hence explicitly computable constants at all stages in our result.

We then extend the above result to systems where the eigenvalues of \( A \) satisfy

\[ |\lambda_\alpha - \lambda_\beta + i \langle k, \omega \rangle| \geq \gamma |k|^{-\tau} \]

for all \( k \neq 0 \) and some \( \gamma > 0, \tau > d \).

1.5.

In Chapter V we formulate and prove theorems analogous to those
in Chapter IV but for the case in which \( P(\phi) \) is differentiable, rather than holomorphic. In applying the method of accelerated convergence to the proof of invariant curves of area preserving annulus mappings Moser originally used a smoothing operator which approximated \( C^2 \) functions by functions in \( C^\infty \) where the error became extremely small if \( \ell \) was large. Later Moser suggested, \([369]\) and \([374]\), that similar approximation techniques could be successfully used to reduce the differentiable case to the holomorphic one. This makes extensive use of the ideas of Bernstein who characterised the differentiable functions by their approximation properties by holomorphic functions. Rüffmann [379] used this idea together with the very precise estimates for sums of series with small divisors mentioned above to considerably improve Moser's result on annulus mappings.

Mitropol'skiy and Samoilenko [358] first treated this problem in the differentiable case. They simply truncated the Fourier series to obtain holomorphic approximations to the differentiable functions. We use the more precise holomorphic approximations of Moser [369] and Rüffmann [379] and by combining these with precise estimates for the sums of series with small divisors obtain a considerable improvement over the results of Mitropol'skiy and Samoilenko. As is always the case in these kinds of problems there is some loss in derivatives. The result of Mitropol'skiy and Samoilenko is considerably wasteful in this respect and they require that \( P(\phi) \) be at least \( \ell \)-times differentiable with

\[
\ell > \frac{x}{(x-1)(2-x)} \left( x(m+1)+2m+2 \right)
\]

where \( 1 < x < 2 \) with \( x \) the exponent used in the method of accelerated convergence. In our result by using the methods mentioned
above we only require \( z > \tau \).

Moreover using the Bernstein type of approximation enables us to formulate and prove the result for non-integral \( z \), and to obtain precise estimates for the constants involved in the reduction to an equation with constant coefficients.

1.6.

The bibliography is divided into three main sections. The first section contains general references which are not concerned with almost periodic or quasi-periodic solutions or with the method of accelerated convergence and its applications.

The second section contains references to almost periodic solutions of ordinary differential equations, although there are some notable omissions. Firstly it was not considered appropriate to include references to differential equations with retarded arguments. Secondly there is no attempt to cover references to integral manifolds as there is a very extensive bibliography of these in Palmer [257]. Finally differential equations in abstract spaces and functional differential equations with almost periodic solutions are not included because there are covered by the long bibliography in Amerio and Prouse [27]. This second section of our bibliography is further subdivided into areas of special interest.

The final section contains references to writings on the method of accelerated convergence and its applications.
CHAPTER II

THE FREQUENCY BASIS OF ALMOST PERIODIC SOLUTIONS

OF ALMOST PERIODIC DIFFERENTIAL EQUATIONS

2.1.1.

This chapter is concerned with the frequency basis of an almost periodic solution of the differential equation

\[(2.1.1.1) \quad x' = \psi(x, t)\]

where \(x' = \frac{dx}{dt}\), \(x \in \mathbb{R}^p\) and \(\psi(x, t)\) is almost periodic in \(t\), uniformly for \(x\) in any bounded subset of \(\mathbb{R}^p\). When \(\psi(x, t)\) satisfies sufficient conditions for the solutions of (2.1.1.1) to be uniquely determined by their initial conditions we show that the co-dimension of the frequency basis of \(\psi(x, t)\) with respect to the frequency basis of any almost periodic solution of (2.1.1.1) is less than \(p\). This includes two special cases of particular interest. When equation (2.1.1.1) is autonomous the frequency basis of an almost periodic solution contains no more than \(p - 1\) elements and when \(\psi(x, t)\) is periodic, and, hence has only one element in its frequency basis, then the frequency basis of any almost periodic solution contains no more than \(p\) elements.

2.1.2.

This problem, with equation (2.1.1.1) Lipschitzian, has been considered recently by Cartwright [120], [121], [122] and [123]. The autonomous and periodic cases are dealt with in [121] while [122] is devoted to the general almost periodic case. Cartwright uses the techniques of topological dynamics (see [8]) and examines the properties of certain almost periodic flows. In [122] the technical
difficulties encountered in using flows for non-autonomous equations as developed by Sell [19] and [20], and others, are overcome by finding a particular flow associated with an almost periodic solution of (2.1.1.1) which allows the general case to be reduced to an application of the results of [121] for the autonomous equation.

Here by extending a technique due to W.A. Coppel, we are able to give a short, direct proof of Cartwright's result. Our method has the additional advantages of not using topological dynamics or the concept of translation numbers of an almost periodic function. Furthermore the means by which we avoid considering all equations in the closed hull of (2.1.1.1) can be applied to other problems associated with almost periodic solutions of equation (2.1.1.1).

The general result is not proved by reducing it to the autonomous case, as in [122]. Nevertheless the simpler autonomous case provides a concise introduction to our method and shortens the proof of the general result. Therefore we will prove the autonomous case first.

2.2.1. Since translation numbers play no part in this chapter we adopt Bochner's criterion as our definition of an almost periodic function. A continuous (vector valued) function $\phi(t)$ is said to be almost periodic if every sequence $\{k_n\}$ of real numbers contains a subsequence $\{k_{n'}\}$ such that $\phi(t+k_{n'})$ converges uniformly on the whole real axis $\mathbb{R}$.

The set of all almost periodic functions is a Banach space with respect to the uniform norm

$$\|\phi\| = \sup_{-\infty < t < \infty} |\phi(t)|.$$
The closed hull of an almost periodic function \( \varphi(t) \) is the set of all almost periodic functions \( \gamma(t) \) such that \( \varphi(t+k_v) \to \gamma(t) \), with respect to the above norm, for some real sequence \( \{k_v\} \), that is, uniformly on \( \mathbb{R} \).

To an almost periodic function \( \varphi(t) \) in \( \mathbb{R}^P \) there corresponds a unique Fourier expansion

\[
\varphi(t) = \sum_{v=1}^{\infty} c_v e^{i\lambda_v t},
\]

where the Fourier coefficients \( c_v \) are non-zero vectors in \( \mathbb{R}^P \), the numbers \( \lambda_v \) are real and distinct, and the number of terms is finite or countably infinite. The numbers \( \lambda_v \) are called the frequencies. A set of real numbers \( \beta_1, \beta_2, \ldots \), is called a basis for the set of frequencies \( \{\lambda_v\} \) or a frequency basis for the almost periodic function \( \varphi(t) \), if they form a basis for the vector space generated by the frequencies \( \lambda_v \) over the field of rational numbers. Thus each \( \lambda_v \) can be uniquely expressed in the form

\[
\lambda_v = \sum_{\mu=1}^{P_v} r_{v\mu} \beta_\mu,
\]

where the \( r_{v\mu} \) are rational and \( r_{vP_v} \neq 0 \). Each member of the closed hull of an almost periodic function \( \varphi(t) \) has the same frequencies as \( \varphi(t) \). An almost periodic function does not have a unique frequency basis, but for definiteness we select a standard basis as follows.

Let \( \psi(t) \) be an almost periodic function with Fourier expansion
(2.2.1.3) \[
\psi(t) = \sum_{\nu=1}^{\infty} c_\nu e^{i\lambda_\nu t},
\]
where the frequencies \( \lambda_\nu \) are real and distinct and the coefficients
\( c_\nu \) are non-zero vectors in \( \mathbb{R}^2 \). Put \( \nu_1 = 1 \) and \( \beta_1^* = \lambda_\nu^* \). Let
\( \nu_2 \) be the least integer \( \nu > 1 \) such that \( \lambda_\nu^* \) is not a rational
multiple of \( \beta_1^* \) and put \( \beta_2^* = \lambda_\nu^* \). In general, having defined
\( \beta_1^*, \ldots, \beta_n^* = \lambda_\nu^n \), let \( \nu_{n+1} \) be the least integer \( \nu > \nu_n \)
such that \( \lambda_\nu^* \) is rationally independent of \( \beta_1^*, \ldots, \beta_n^* \) and put
\( \beta_{n+1}^* = \lambda_\nu^{n+1} \). In this way we define a finite or infinite sequence
\( \{\beta_k^*\} \) of rationally independent frequencies such that every frequency
\( \lambda_\nu^* \) can be uniquely expressed in the form
(2.2.1.4) \[
\lambda_\nu^* = s_{\nu_1}^* \beta_1^* + s_{\nu_2}^* \beta_2^* + \ldots + s_{\nu_{\nu_\nu}}^* \beta_{\nu_\nu}^*
\]
with rational coefficients \( s_{\nu_\nu}^* \) and \( s_{\nu_{\nu}}^* \neq 0 \). That is, we obtain
a basis for the set of frequencies, each member of the basis being
itself a frequency.

We extend the standard frequency basis for \( \psi(t) \) to a frequency
basis for \( \varphi(t) \) and define the standard additional basis of \( \{\lambda_\nu^*\} \)
with respect to \( \{\lambda_\nu^*\} \) as follows. Let \( \nu_1 \) be the least integer \( \nu \)
such that \( \lambda_\nu^* \) is rationally independent of \( \beta_1^*, \beta_2^*, \ldots, \), and put
\( \beta_1^* = \lambda_\nu^* \). Let \( \nu_2 \) be the least integer \( \nu > \nu_1 \) such that \( \lambda_\nu^* \) is
rationally independent of \( \beta_1^*, \beta_1^*, \beta_2^*, \ldots, \), and put \( \beta_2^* = \lambda_\nu^* \). In
general, having defined $\beta_1 = \lambda_{\nu_1}, \ldots, \beta_n = \lambda_{\nu_n}$ let $\nu_{n+1}$ be the least integer $\nu > \nu_n$ such that $\lambda_\nu$ is rationally independent of $\beta_1, \beta_2, \ldots, \beta^*_1, \beta^*_2, \ldots$, and put $\beta_{n+1} = \lambda_{\nu_{n+1}}$. In this way we define a finite or infinite sequence $\{\beta_k\}$ of rationally independent frequencies, which, together with the sequence $\{\beta^*_k\}$, form a basis for $\{\lambda_\nu\}, \{\lambda^*_\nu\}$. Moreover every frequency $\lambda_\nu$ can be uniquely expressed in the form

$$\lambda_\nu = \sum_{\mu=1}^{P_\nu} r_{\nu\mu} \beta_\mu + \sum_{\mu=1}^{P^*_\nu} r^*_{\nu\mu} \beta^*_\mu$$

with rational coefficients $r_{\nu\mu}, r^*_{\nu\mu}$ and $r_{\nu P_\nu} \neq 0$, for each $\lambda_\nu$ which is not rationally dependent on the elements of the set $\{\beta^*_k\}$.

It should be observed that the standard additional basis $\{\beta_k\}$, alone, is not in general a basis for $\{\lambda_\nu\}$ without the elements of the basis $\{\beta_k\}$ for $\{\lambda^*_\nu\}$.

All of the above applies without change to almost periodic functions depending uniformly on parameters (see, for example, [26] Chapter II). We consider the differential equation (2.1.1.1) and suppose that $\psi(x, t)$ has the Fourier series

$$\psi(x, t) \sim \sum_{\nu=1}^{i\lambda^*_\nu t} c_\nu^* (x) e^{i\lambda^*_\nu t}$$

where the coefficients $c_\nu^*$ are not identically zero. We suppose that $\psi(t)$ is an almost periodic solution of (2.1.1.1) and we prove the result by showing that the standard additional basis has less than $p$ elements. Thus, without loss of generality we assume that $r_{\nu P_\nu} \neq 0$.
in (2.2.1.5) for at least one value of $\nu$.

2.2.2.

We will need to use Kronecker's Theorem in this chapter and in the next. For convenience we will simply quote here the version of the theorem which we require.

If $\theta_1, \theta_2, \ldots, \theta_k$ are linearly independent over the integers, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are arbitrary real numbers and $T$ and $\varepsilon$ are positive, then there is a real number $t$, and integers $p_1, p_2, \ldots, p_k$ such that

$$t > T$$

and

$$|t \theta_m - p_m \alpha_m| < \varepsilon \quad (m = 1, 2, \ldots, k).$$

(Hardy and Wright [9], Theorem 444, p. 382.)

2.3.1.

For the autonomous equation

$$x' = \psi(x)$$

where $x' = \frac{dx}{dt}$, $x \in \mathbb{R}^p$, let $\psi(x)$ satisfy sufficient conditions for the solutions of this equation to be uniquely determined by their initial conditions. Let $\varphi(t)$ be an almost periodic solution with Fourier series

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n e^{i \lambda_n t}.$$ 

Then the set $\{\beta_n^\ast\}$ is empty and the standard additional basis is just the standard basis. Thus we need only show that this basis contains less than $p$ elements.
Since the equation is autonomous, translates of \( \varphi(t) \) are also solutions of the equation. We use this property to show that all of the functions \( \varphi(t, \tau_1, \tau_2, \ldots) \) with Fourier series

\[
\varphi(t, \tau_1, \tau_2, \ldots) \sim \sum_{\nu=1}^{\infty} c_{\nu} e^{i\nu \lambda t},
\]

which are in the closed hull of \( \varphi(t) \), are also solutions. In particular, by considering the initial values of these solutions we then show that \( \mathbb{R}^p \) contains a homeomorphic image of the "cube"

\[
C : 0 \leq \tau_k \leq \frac{\pi}{\beta_k}, \quad k = 1, 2, \ldots, m,
\]

for any \( m \leq \sup_{\nu} p_{\nu} \). Hence \( \sup_{\nu} p_{\nu} \leq p \). A further argument shows that equality cannot hold.

2.3.2. THEOREM.

Let \( \varphi(t) \) be an almost periodic solution of the autonomous differential equation

\[
(2.3.2.1)\quad x' = \psi(x)
\]

where \( x' = \frac{dx}{dt}, \ x \in \mathbb{R}^p \), with Fourier expansion

\[
\varphi(t) \sim \sum_{\nu=1}^{\infty} c_{\nu} e^{i\nu \lambda t}.
\]

Let \( \psi(x) \) satisfy sufficient conditions for the solutions of this equation to be uniquely determined by their initial values. Then for any sequence of real numbers \( \{\tau_k\} \) there exists an almost periodic function \( \varphi(t, \tau_1, \tau_2, \ldots) \) in the closed hull of \( \varphi(t) \) with Fourier expansion
Moreover \( \varphi(t, \tau_1, \tau_2, \ldots) \) is also a solution of equation (2.3.2.1).

Proof. For each positive integer \( N \), we write

\[
P_N = \max_{\nu=1,2,\ldots,N} P_{\nu} \quad \text{and} \quad f^j_{\nu} \quad \text{for the lowest common multiple of the}
\]

quotients of the rational numbers \( r_{\nu j} \), \( \nu = 1, 2, \ldots, N \). The numbers \( \left[ \frac{2\pi f^j_{\nu}}{\beta_{\nu j}} \right]^{-1} \), \( j = 1, 2, \ldots, P\), are linearly independent over the integers. Thus according to Kronecker's Theorem, in the form given in §2.2.2, to each integer \( N \) and any \( \delta > 0 \) there corresponds a real number \( t_N \) and a set of integers \( k_j \) such that

\[
|t_N - \tau_{\nu j} - \frac{2\pi}{\beta_{\nu j}} f^j_{\nu} k_j| < \delta, \quad j = 1, 2, \ldots, P_N.
\]

That is

\[
|t_N - \tau_{\nu j}| < \delta \left( \mod \frac{2\pi f^j_{\nu}}{\beta_{\nu j}} \right), \quad j = 1, 2, \ldots, P_N.
\]

By selecting \( \delta \) small enough we obtain

\[
|e^{i(r_{\nu 1} \beta_{\nu 1} t_1 + \ldots + r_{\nu \nu} \beta_{\nu \nu} t_\nu)} - e^{i \lambda_{\nu} t_N}| \leq \frac{1}{N}, \quad \nu = 1, 2, \ldots, N.
\]

Since these inequalities continue to hold when the sequence \( \{t_N\} \)

is replaced by any subsequence we can suppose that \( \varphi(t + t_N) \) converges uniformly for all real \( t \), as \( N \to \infty \). The limit \( \chi(t) \), say, is in the closed hull of \( \varphi(t) \) and has the Fourier series
where
\[ x(t) = \sum_{\nu=1}^{\infty} a_\nu e^{i\lambda_\nu t} \]

It remains now to prove that
\[ x(t) = \varphi(t, \tau_1, \tau_2, \ldots) \]
is also a solution of (2.3.2.1). It is sufficient to show, that if
the real sequence \( \{t_N\} \) is such that
\[ t_N = \varphi(t_N) + \xi \quad \text{as} \quad N \to \infty \]
then \( \varphi(t+t_N) \) converges uniformly to the solution \( \omega(t) \) of (2.3.2.1) with
initial value \( \xi \) at \( t = 0 \).

In fact \( \varphi(t+t_N) \) is the almost periodic solution of (2.3.2.1)
which takes the value \( \xi_N \) for \( t = 0 \). Any subsequence of \( \{t_N\} \)
contains a further subsequence \( \{t'_N\} \) such that \( \varphi(t+t'_N) \) converges
uniformly for all real \( t \). Because of the continuous dependence of
solutions on initial values (Coppel [5], Theorem 3) the limit function
must be \( \omega(t) \). Since the limit is independent of the choice of subsequence
the whole sequence \( \varphi(t+t_N) \) must converge uniformly to \( \omega(t) \).

2.3.3.

Thus if we denote by \( M \) the closure of the range of \( \varphi(t) \) there
is a function in the closed hull of \( \varphi(t) \) which is a solution of
(2.3.2.1) and has as initial value any point of \( M \). The next result
shows that for integers \( m \leq \sup_p \nu \) there is a local homeomorphism
between the initial values of the solutions
\[ \varphi(t, \tau_1, \tau_2, \ldots, \tau_m, 0, 0, \ldots) \] and the points \( (\tau_1, \tau_2, \ldots, \tau_m) \).

### 2.3.4. LEMMA.

For any \( m \leq \sup_{\nu} p_{\nu} \) the set \( M \) contains a homeomorphic image of the cube 

\[ C : 0 \leq \tau_k \leq \frac{\pi}{B_k} \quad (k = 1, 2, \ldots, m). \]

**Proof.** If 

\[ \varphi(0, \tau_1', \tau_2', \ldots) = \varphi(0, \tau_1'', \tau_2'', \ldots) \]

then the solutions \( \varphi(t, \tau_1', \tau_2', \ldots) \) and \( \varphi(t, \tau_1'', \tau_2'', \ldots) \) of (2.3.2.1) coincide because they have the same initial value. Therefore they have the same Fourier coefficients. That is

\[ (2.3.4.1) \quad r_{\nu 1} \beta_1 (\tau_1' - \tau_1'') + \ldots + r_{\nu p_{\nu} \nu} p_{\nu} (\tau_1'' - \tau_1'') \equiv 0 \quad (\text{mod } 2\pi) \]

for all \( \nu \). Since \( \beta_n = \lambda_{\nu n} \) is rationally independent of the preceding \( \beta \)'s we have

\[ r_{\nu n} = 1, \quad r_{\nu k} = 0 \quad \text{for } k \neq n. \]

Therefore taking \( \nu = n \) in the preceding congruence we obtain

\[ \tau_n' - \tau_n'' \equiv 0 \pmod{\frac{2\pi}{B_n}}. \]

Let \( m \) be a fixed positive integer not exceeding the number of elements in the basis \( \beta_n \), that is \( m \leq \sup_{\nu} p_{\nu} \). Suppose \( 0 \leq \tau_k', \tau_k'' \leq \frac{\pi}{B_k} \) for \( k = 1, \ldots, m \) and \( \tau_k' = \tau_k'' = 0 \) for \( k > m \). Then if (2.3.4.1) holds we must have

\[ \tau_k' = \tau_k'', \quad k = 1, \ldots, m. \]

Thus the map...
of the "cube"

\[ C : 0 \leq \tau_k \leq \frac{\pi}{b_k} \quad (k = 1, \ldots, m) \]

into \( \mathbb{R}^p \) is one-to-one. That this map is also continuous may be seen in the following way.

For any \( \varepsilon > 0 \), there exists a trigonometric polynomial (for example the Bochner-Fejér polynomial, see Besicovitch [24], 46-51)

\[ p(t) = \sum_{\nu=1}^{M} \rho_\nu a_\nu e^{i\lambda_\nu t} \]

with rational coefficients \( \rho_\nu \) depending only on \( \varepsilon \) and \( \lambda \)'s and satisfying \( 0 \leq \rho_\nu \leq 1 \), such that

\[ |\varphi(t) - p(t)| \leq \varepsilon \quad \text{for} \quad -\infty < t < \infty . \]

Moreover if we replace \( \varphi(t) \) by any function \( \chi(t) \) in its closed hull and the Fourier coefficients \( a_\nu \) of \( \varphi(t) \) by the corresponding Fourier coefficients of \( \chi(t) \) then this inequality continues to hold.

We can now choose \( \delta = \delta(\varepsilon) > 0 \), so that if

\[ |\tau_k' - \tau_k''| \leq \delta \quad (k = 1, \ldots, m) \]

the first \( M \) Fourier coefficients of \( \varphi(t, \tau_1', \ldots, \tau_m', 0, 0, \ldots) \) and \( \varphi(t, \tau_1'', \ldots, \tau_m'', 0, 0, \ldots) \) satisfy

\[ |a_\nu'^{-1} - a_\nu''^{-1}| \leq \frac{\varepsilon}{M} \quad (\nu = 1, 2, \ldots, M) . \]

Then the corresponding trigonometric polynomials satisfy

\[ |p'(t) - p''(t)| \leq \varepsilon , \quad \text{for} \quad -\infty < t < \infty , \]

and hence

\[ |\varphi(t, \tau_1', \ldots, \tau_m', 0, 0, \ldots) - \varphi(t, \tau_1'', \ldots, \tau_m'', 0, 0, \ldots)| \leq 3\varepsilon \]

for \( -\infty < t < \infty \).
Since the "cube" $C$ is compact, the inverse map $T^{-1}$ is also continuous. Therefore the set $M$ contains a homeomorphic image of $C$.

We can now proceed to the statement and proof of the theorem for the autonomous case.

2.3.5. THEOREM.

Let $\psi(t)$ be an almost periodic solution of the autonomous differential equation

$$x' = \psi(x)$$

where $x' = \frac{dx}{dt}$, $x \in \mathbb{R}^p$. Let $\psi(x)$ satisfy sufficient conditions for all of the solutions of this equation to be uniquely determined by their initial conditions. Then the frequency basis of $\psi(t)$ contains less than $p$ elements.

Proof. It follows from Lemma 2.3.4 that the dimensions of the sets $C$ and $M$ satisfy the inequality

$$m = \dim C \leq \dim M.$$ 

On the other hand $\dim M \leq p$ since $M \subseteq \mathbb{R}^p$. Therefore $m \leq p$. Thus $\{\beta_k\}$, the frequency basis of $\psi(t)$ contains at most $p$ elements.

It remains to show that this basis contains strictly less than $p$ elements. Otherwise, if $\dim C = p$, $C$ contains a non-empty open subset of $\mathbb{R}^p$. Therefore $M$ contains a non-empty open subset $G$ of $\mathbb{R}^p$. Let $\eta$ be a point on the boundary of the compact set $M$ and let $\chi(t)$ be the solution of (2.3.2.1) which takes the value $\eta$ for $t = 0$. Since the range of $\chi(t)$ is dense in $M$ there is an $\xi_0 \in G$

$\dagger$ For the properties of dimension which we require see Hurewicz and Wallman [11], in particular Theorems 3.1, 4.1 and 4.3.
and some \( t_0 \in \mathbb{R} \) such that \( \chi(t_0) = \xi_0 \). For any \( \xi \in \mathbb{G} \) let \( \varphi(t; \xi) \) denote the solution of (2.3.2.1) which takes the value \( \xi \) for \( t = 0 \). In particular \( \varphi(t; \xi_0) = \chi(t+t_0) \). The map
\[
\xi \mapsto \varphi(-t_0; \xi)
\]
is a homeomorphism of a neighbourhood of \( \xi_0 \) onto a neighbourhood of \( \varphi(-t_0, \xi_0) = \eta \). But this contradicts our choice of \( \eta \) as a boundary point of \( M \).

This completes the proof of Theorem 2.3.5.

2.4.1.

We now proceed to the corresponding result for the non-autonomous case
\[
x' = \psi(x, t).
\]
As in §2.3 we consider a whole family of solutions which belong to the closed hull of the given almost periodic solution. However, in this case these solutions are generated from the additional frequency basis of the solution. The following analogue of Theorem 2.3.2, while in principle unchanged, becomes more complicated to prove, since only particular sequences of translates of the given solution converge to solutions of the equation.

2.4.2. THEOREM.

Let \( \varphi(t) \) be an almost periodic solution of the almost periodic differential equation
\[
(2.4.2.1) \quad x' = \psi(x, t)
\]
where \( x' = \frac{dx}{dt} , x \in \mathbb{R}^p \), \( \psi(x, t) \) is almost periodic in \( t \), uniformly with respect to \( x \) for \( x \) in any bounded subset of \( \mathbb{R}^p \) and \( \varphi(t) \) and \( \psi(x, t) \) have Fourier series...
\[ \varphi(t) \sim \sum_{\nu=1}^i c_{\nu} e^{i\lambda_{\nu} t} \]

and

\[ \psi(x, t) \sim \sum_{\nu=1}^i c_{\nu}(x)e^{i\lambda_{\nu} t} \]

where

\[
\begin{align*}
\lambda_{\nu} &= \sum_{\mu=1}^{P_{\nu}} r_{\nu\mu}\beta_{\mu} + \sum_{\mu=1}^{P_{\nu}^*} r_{\nu\mu}^*\beta_{\mu}^*, \\
\lambda_{\nu}^* &= \sum_{\mu=1}^{q_{\nu}} s_{\nu\mu}\beta_{\mu}^*,
\end{align*}
\]

\(\{\beta_k\}^\ast\) is the standard frequency basis of \(\psi(x, t)\), and \(\{\beta_k\}^\ast\) is the standard additional basis of \(\varphi(t)\). Let \(\psi(x, t)\) satisfy sufficient conditions for the solutions of (2.4.2.1) to be uniquely determined by their initial values. Then, for any real numbers \(\{\tau_k\}\) there exists an almost periodic function \(\varphi(t, \tau_1, \tau_2, \ldots)\) in the closed hull of \(\psi(t)\) with Fourier expansion

\[ \varphi(t, \tau_1, \tau_2, \ldots) \sim \sum_{\nu=1}^i c_{\nu} e^{i\left(\sum_{j=1}^{P_{\nu}^*} r_{\nu\mu}^*\beta_{\mu}^* + \sum_{j=1}^{P_{\nu}} r_{\nu\mu}\beta_{\mu}\right) t} \]

Moreover \(\varphi(t, \tau_1, \tau_2, \ldots)\) is also a solution of (2.4.2.1).

Proof. Write \(P_N = \max_{\nu=1, 2, \ldots, N} P_{\nu}\), \(P_N^* = \max_{\nu=1, 2, \ldots, N} \{P_{\nu}^*, q_{\nu}\}\), \(f_j^\nu\) for the lowest common multiple of the quotients of the rational numbers \(r_{\nu j}^\nu\), \(v = 1, 2, \ldots, N\) and \(g_j\) for the lowest common multiple of the quotients of the rational numbers \(r_{\nu j}^\nu, s_{\nu j}^\nu\), \(v = 1, 2, \ldots, N\).

Set
The numbers \( q_1^{-1}, q_2^{-1}, \ldots, q_n^{-1} \) are linearly independent over the integers. From Kronecker's Theorem, in the form given in §2.2.2, corresponding to any \( \delta > 0 \), there exist integers \( k_1, \ldots, k_n \) and a real number \( \xi \) such that

\[
|\xi - \alpha_i - k_i q_i^{-1}| < \delta, \quad i = 1, 2, \ldots, n.
\]

Setting \( \xi = t_N^i \) and \( k_i' = k_i + P_N \) we have

\[
|t_N - \tau_i - \frac{2\pi}{B_i} f_i k_i'| < \delta, \quad \text{for } i = 1, 2, \ldots, P_N,
\]

and

\[
|t_N - \frac{2\pi}{B_i} g_i k_i'| < \delta, \quad \text{for } i = 1, 2, \ldots, P_N^*.
\]

That is,

\[
|t_N - \tau_i| < \delta \left[ \text{mod} \frac{2\pi}{B_i} f_i \right], \quad \text{for } i = 1, 2, \ldots, P_N,
\]

and

\[
|t_N| < \delta \left[ \text{mod} \frac{2\pi}{B_i^*} g_i \right], \quad \text{for } i = 1, 2, \ldots, P_N^*.
\]
By combining these inequalities, and by selecting $\delta$ small enough

$$
(2.4.2.2) \left| \sum_{\nu=1}^{N} \beta_{\nu} \left( e^{-i\lambda_{\nu} t} \right)^{\nu} - 1 \right| < \frac{1}{N}
$$

for $\nu = 1, 2, \ldots, N$

and

$$
(2.4.2.3) \left| \sum_{\nu=1}^{N} \beta_{\nu} \left( e^{-i\lambda_{\nu} t} \right)^{\nu} - 1 \right| < \frac{1}{N}
$$

for $\nu = 1, 2, \ldots, N$.

Since these inequalities continue to hold when the sequence $\{t_{N}\}$ is replaced by any subsequence we can suppose that both $\psi(x, t+t_{N})$ and $\varphi(t+t_{N})$ converge uniformly for all real $t$ as $N \to \infty$. The function $\psi(x, t+t_{N})$ converges to $\psi(x, t)$. For, the Fourier series of $\psi(t, t+t_{N})$ is

$$
\sum_{\nu=1}^{\infty} a_{\nu}(x) e^{i\lambda_{\nu} t}_{N}
$$

and the Fourier series of $\lim_{N \to \infty} \psi(x, t+t_{N})$ is

$$
\sum_{\nu=1}^{\infty} a_{\nu}(x) e^{i\lambda_{\nu} t}_{N}
$$

therefore, since the Fourier series of an almost periodic function uniquely determines the function

$$
\psi(x, t+t_{N}) \to \psi(x, t), \text{ uniformly.}
$$

Similarly

$$
\varphi(t+t_{N}) \to \chi(t), \text{ uniformly,}
$$

where $\chi(t)$ has the Fourier series

$$
\chi(t) \approx \sum_{\nu=1}^{\infty} a_{\nu} e^{i\lambda_{\nu} t}_{N}.
$$
Evidently we have
\[ \varphi(t, 0, 0, \ldots) = \varphi(t). \]

Let \( \{t_N\} \) be any real sequence such that \( \psi(x, t+t_N) + \psi(x, t) \)
uniformly as \( N \to \infty \). We will show that if \( \xi_N = \varphi(t_N) + \xi \) as \( N \to \infty \)
then \( \varphi(t+t_N) \) converges uniformly on \( \mathbb{R} \) to \( \omega(t) \) where \( \omega(t) \) is
the solution of (2.4.2.1) such that \( \omega(0) = \xi \).

In fact \( \varphi(t+t_N) \) is the almost periodic solution of
\[ x' = \psi(x, t+t_N) \]
which takes the value \( \xi_N \) for \( t = 0 \). It follows from a standard
theorem (Coppel [5], Theorem 3) that \( \varphi(t+t_N) \) converges uniformly on
every compact interval of \( \mathbb{R} \). But any subsequence \( \{t'_N\} \) of \( \{t_N\} \)
contains a further subsequence \( \{t''_N\} \) such that \( \varphi(t+t''_N) \) converges
uniformly for all real \( t \) and since the limit \( \omega(t) \) is independent
of the choice of subsequences the whole sequence \( \varphi(t+t_N) \) must converge
to \( \omega(t) \) uniformly on \( \mathbb{R} \).

Hence the functions \( \varphi(t, \tau_1, \tau_2, \ldots) \) are all almost periodic
solutions of (2.4.2.1). Observe that these functions in general form
a smaller set than in the autonomous case, since we only admit
functions obtained from \( \varphi(t) \) by translations by sequences \( \{t_N\} \) for
which \( \psi(x, t+t_N) + \psi(x, t) \) uniformly with respect to \( x \), for \( x \) in
any bounded subset of \( \mathbb{R}^p \).
2.4.3.

As in §2.3.3 we denote by \( \mathcal{M} \) the closure of the range of \( \varphi(t) \).

Then the initial values of the solutions \( \varphi(t, \tau_1, \tau_2, \ldots) \) for all sequences \( \{\tau_1, \tau_2, \ldots\} \) form a subset of \( \mathcal{M} \). The statement and proof of Lemma 2.3.4 now hold for this situation.

We now proceed to the statement and proof of the general theorem.

2.4.4. THEOREM.

Let the differential equation

\[
x' = \psi(x, t)
\]

where \( x' = \frac{dx}{dt} \), \( x \in \mathbb{R}^n \) and \( \psi(x, t) \) is almost periodic in \( t \), uniformly for \( x \) in any bounded subset of \( \mathbb{R}^n \), satisfy conditions sufficient for its solutions to be uniquely determined by their initial values for all \( (x, t) \in \mathbb{R}^n \times \mathbb{R} \). Let \( \varphi(t) \) be an almost periodic solution of this differential equation. Then the codimension of the frequency basis of \( \psi(x, t) \) with respect to the frequency basis of \( \varphi(t) \) is at most \( p - 1 \).

Proof. It follows from Lemma 2.3.4 applied to this situation that

\[
m = \dim \mathcal{C} \leq \dim \mathcal{M}.
\]

On the other hand \( \dim \mathcal{M} \leq p \), since \( \mathcal{M} \subset \mathbb{R}^p \). Therefore

\[
m \leq p.
\]

It remains to show that \( m \neq p \). The argument of Theorem 2.3.5 applies unchanged. This completes the proof of the Theorem.

2.4.5.

If \( \psi(x, t) \) satisfies a Lipschitz condition with Lipschitz
constant independent of $t$ then every $\Psi(x, t)$ in the closed hull of $\psi(x, t)$ satisfies a Lipschitz condition and the solutions of each differential equation

$$(2.4.5.1) \quad x' = \Psi(x, t)$$

are uniquely determined by their initial values. It then follows from the proof of Theorem 2.4.4 that if $\Phi(t)$ is an almost periodic solution of any differential equation (2.4.5.1) then the codimension of the frequency basis of $\psi(x, t)$ with respect to the frequency basis of $\Phi(t)$ is at most $p - 1$.

Cartwright's version of Theorem 2.4.4 assumes that $\psi(x, t)$ is Lipschitzian but not in an essential way.

2.4.6.

In [127] Cartwright shows that in the autonomous case when $k = p - 1$ the frequency basis is an integral basis. Thus

$$\lambda_v = \eta_{v1} \beta_1 + \eta_{v2} \beta_2 + \cdots + \eta_{vp-1} \beta_{p-1},$$

$v = 1, 2, \ldots$, with $\eta_{v\mu}$ integers,

and the almost periodic solution $\Phi(t)$ is quasi-periodic. In other words,

$$\Phi(t) = \tilde{\Phi}(t, t, \ldots, t)$$

where $\tilde{\Phi}(t_1, t_2, \ldots, t_{p-1})$ is periodic in $t_i$ with period $\frac{2\pi}{\beta_i}$,

$i = 1, 2, \ldots, p-1$. Since the proof of the result for the almost periodic equation (2.3.2.1) is reduced to the autonomous case this implies that in [122] when $k = p - 1$ the additional frequency basis is integral, and we have for the frequencies of the almost periodic solution
These results follow from a theorem of Kodaira and Abe [12] which states that the torus group is the only $(p-1)$ dimensional compact connected separable abelian group which can be embedded in $\mathbb{R}^p$. Thus this proof will apply to our theorems if we can show that $M$ can be given the structure of a compact connected separable abelian topological group.

It follows from Theorems 2.4.2 and 2.4.4 that if we restrict our attention to sequences of the kind considered in Theorem 2.4.2 then, if $x_0$ is any fixed point in $M$, $x$ and $y$ any points in $M$ and $\varphi(t)$ the solution of (2.4.2.1) with $\varphi(0) = x_0$ we can suppose that

\[
x = \lim_{n \to \infty} \varphi(s_n) \quad \text{and} \quad y = \lim_{n \to \infty} \varphi(t_n).
\]

We define their product by

\[
xy = \lim_{n \to \infty} \varphi(s_n + t_n).
\]

To justify the definition we must show that this limit exists, depends only on the points $x, y$ and is independent of the particular sequences $\{s_n, t_n\}$.

Let $\varphi_1(t), \varphi_2(t)$ be solutions of (2.4.2.1) from the closed hull of $\varphi(t)$ with $\varphi_1(0) = x, \varphi_2(0) = y$. Then

\[
\varphi(t+s_n) - \varphi_1(t) \to 0, \quad \varphi(t+t_n) - \varphi_2(t) \to 0,
\]

uniformly in $t$ as $n \to \infty$. Hence
\[
\varphi(s_n + t_n) - \varphi_1(t_n) \to 0 \\
\varphi(s_n + t_n) - \varphi_2(s_n) \to 0
\]

and therefore

\[
\varphi_1(t_n) - \varphi_2(s_n) \to 0.
\]

By restricting attention to suitable subsequences we can suppose that \(\varphi_1(t_n)\) and \(\varphi_2(s_n)\) converge. Since their limits must always be the same, this shows that the sequences themselves converge to the same limit, \(z\) say, and this is independent of the choice of sequences \(\{s_n\}, \{t_n\}\). Then

\[
\varphi(s_n + t_n) \to z.
\]

It follows immediately that multiplication is commutative and associative and that

\[
x_0 = \varphi(0)
\]

is an identity element. If \(x = \lim_{n \to \infty} \varphi(s_n)\) then by a similar argument

\[
y = \lim_{n \to \infty} \varphi(-s_n)
\]

also exists and

\[
x y = x_0.
\]

Thus \(y\) is the inverse of \(x\).

For the topology of \(M\) we take the topology induced by \(R^P\). \(M\) is obviously compact, connected and separable. Moreover multiplication and inversion are continuous operations. Finally by selecting a different base point \(x_0\) we obtain another topological group isomorphic to the above.

Now Cartwright's argument applies directly to our case (Theorems 9 and 10 of [121]).
Lerman and Shnirelman [146] have recently announced some results in which they have also discovered the above method of Theorems 2.4.2, 2.4.4. However they restrict themselves to the consideration of quasi-periodic solutions of (2.4.2.1) and consequently they find an \( r \)-parameter family of solutions (our functions \( \Phi(t, r_1, r_2, \ldots, r_m) \)) forming an integral manifold which is homeomorphic to the tube \( T^r \times R \), where \( T^r \) is the \( r \)-dimensional torus. This is precisely the result which occurs in our case when the almost periodic solutions are quasi-periodic, for example, when \( m = p - 1 \).
CHAPTER III
QUASI-PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS
AND CHERRY’S THEOREM

3.1.1.

We consider the conservative (autonomous) Hamiltonian system

\[ J\dot{x}' = \frac{\partial H}{\partial x} \]

where \( x' = \frac{dx}{dt}, \ x \in \mathbb{R}^{2n}, \ H : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) and \( J \) is an invertible real skew-symmetric matrix. Unless stated to the contrary we assume that \( H \in C'(\mathbb{R}^{2n}) \). Without loss of generality \( J \) can be taken to be of the form

\[ J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \]

It follows immediately that

(i) \( J^*J = I_{2n} \),

(ii) \( (J^*)^{-1} = J \),

(iii) \( J^2 = -I_{2n} \),

(iv) \( J^{-1} = -J \).

We denote the inner product \( x^*Jy \) by \( [x, y] \). For all vectors \( x, y \) in \( \mathbb{R}^{2n} \)

\[ [x, y] = [-y, x], \]

\[ [x, x] = 0, \]

and for all real or complex numbers \( \alpha \)

\[ [\alpha x, y] = \bar{\alpha}[x, y] = [x, \bar{\alpha}y]. \]
3.1.2.

Let the vector-valued function \( \Phi(t_1, t_2, \ldots, t_k) \) be continuous and periodic with period \( 2\pi/\beta_i \) in \( t_i \) for \( i = 1, 2, \ldots, k \), where the real numbers \( \beta_1, \beta_2, \ldots, \beta_k \) are linearly independent over the integers. Then

\[
\varphi(t) = \Phi(t, t, \ldots, t)
\]

is said to be a quasi-periodic function.

It follows that \( \varphi(t) \) has the Fourier series

\[
\varphi(t) \sim \sum_{|n|=0}^{\infty} a_n e^{i(n, \beta)t},
\]

where \( n = (n_1, n_2, \ldots, n_k) \) is a vector in \( \mathbb{R}^k \) with integral components, \( |n| = |n_1| + |n_2| + \ldots + |n_k| \), and \( (n, \beta) \) is the scalar product \( n_1 \beta_1 + n_2 \beta_2 + \ldots + n_k \beta_k \). That is \( \varphi(t) \) is an almost periodic function with a finite, integral frequency basis. The function \( \Phi(t_1, t_2, \ldots, t_k) \) is often called the spatial extension of \( \varphi(t) \). See Besicovitch [24], p. 36.

3.1.3.

A measurable function \( M : \mathbb{R}^P \to \mathbb{R} \) is called an integral invariant for the autonomous differential equation

\[
(3.1.3.1) \quad x' = f(x)
\]

where \( x \in \mathbb{R}^P \), if, for any measurable set \( A \subset \mathbb{R}^P \) and for

\[
A_t = \{ \varphi(x, t) : x \in A \},
\]

where \( \varphi(x, t) \) is the solution of (3.1.3.1) satisfying \( \varphi(x, 0) = x \),
\[ \int_A M dx = \int_A M dx \]

for all \( t \in \mathbb{R} \).

Liouville's Theorem states that if \( \text{div} f = 0 \) equation (3.1.3.1) has \( M = 1 \) as an integral invariant, that is the equation preserves Lebesgue measure. In particular, when \( p = 2n \) and equation (3.1.3.1) is Hamiltonian, and so has the form (3.1.1.1), Liouville's Theorem holds.

A set \( \Omega \subset \mathbb{R}^P \) is called invariant for equation (3.1.3.1) whenever
\[ \varphi(x, t) \in \Omega, \]
for all \( x \in \Omega \) and all \( t \in \mathbb{R} \), where \( \varphi(x, t) \) is the solution of (3.1.3.1) satisfying \( \varphi(x, 0) = x \).

3.2.1.

In [4] Cherry set out to prove the following result.

Let the differential equation
\[(3.2.1.1) \quad x' = f(x), \]
where \( x' = \frac{dx}{dt}, x \in \mathbb{R}^P \), satisfy the following conditions:

(i) \( f \) is holomorphic on an open set \( X \subset \mathbb{R}^P \),

(ii) the differential equation has a non-negative integral invariant,

(iii) the differential equation has a \( p \)-dimensional 'cube'
\[ \Omega \subset X, \]
as an invariant set.

Then all of the solutions of (3.2.1.1) with initial values in \( \Omega \) are quasi-periodic with less than \( p \) basic frequencies, and are of the form
\[(3.2.1.2) \quad \varphi(\omega t + z_1, z_2) \]
where \( \omega, z_1 \in \mathbb{R}^k, z_2 \in \mathbb{R}^{p-k}, \varphi(t_1, t_2, \ldots, t_p) \) is holomorphic.
in each of its arguments and periodic with period $2\pi$ in $t_1, t_2, \ldots, t_k$, and the numbers $\frac{1}{\omega_1}, \frac{1}{\omega_2}, \ldots, \frac{1}{\omega_k}$ are integrally independent. Further, writing $z = (z_1, z_2)$, the Jacobian

$$\left| \frac{\partial z}{\partial t} \right|$$

is not identically zero.

Moreover, when equation (3.2.1.1) is Hamiltonian these quasi-periodic solutions have no more than $\frac{L}{2}$ basic frequencies.

Cherry admitted that his methods were open to criticism and he does not seem to have been aware of Bohr's theory of almost periodic functions which had just begun to appear. The following sections contain the best result we have been able to prove along the lines of Cherry's result. These are followed by a detailed discussion and comparison of the two results.

3.3.1.

The following result is similar to the standard theorem on the differentiability of solutions with respect to parameters and initial conditions (see Coppel [5], Theorem 6). However we need a result concerning a one-parameter family of solutions to a single differential equation. Such a result, although easy to prove, and probably well known, does not appear in the standard texts.

The assertions of this lemma are local and hence need only be proved in the interiors of the sets concerned.

3.3.2 LEMMA.

Let $x = \psi(t, \eta)$ where $\psi : U \to \mathbb{R}^n$, $U \subset \mathbb{R} \times \mathbb{R}^m$, be a solution of
(3.3.2.1) \[ x' = f(t, x) \]

for some open \((t, n)\)-set, \(V\) say, with \(x_0 = \psi(t_0, n_0)\) where \((t_0, n_0) \in V\). Let \(f\) be continuously differentiable on some open \((t, x)\)-set containing \((t_0, x_0)\). Let \(\psi_n(t, n)\) exist and be continuous at \(n = n_0\). Then \(\psi_n(t, n_0)\) is a solution of the variational equation

(3.3.2.2) \[ y' = f_x[t, \psi(t, n_0)]y. \]

Proof. Without loss of generality we can take the sets concerned to be convex. Then

\[
\frac{d}{dt}(\psi(t, n) - \psi(t, n_0)) = f[t, \psi(t, n)] - f[t, \psi(t, n_0)]
\]

\[
= \int_0^1 \frac{d}{dt} \left[ f_x[t, \psi(t, n_0)] + \theta(\psi(t, n) - \psi(t, n_0)) \right] d\theta \cdot [\psi(t, n) - \psi(t, n_0)]
\]

\[
= \int_0^1 \left( f_x[t, \psi(t, n_0)] + o(1) \right) \{\psi(t, n) - \psi(t, n_0)\}
\]

\[
= \int_0^1 \left( f_x[t, \psi(t, n_0)] \psi_n(t, n_0) \right) [n_0 - n_0] + o(|n - n_0|).
\]

Furthermore, by the differentiability of \(\psi_x(t, n) = f(t, \psi)\)

\[
\frac{d}{dt}(\psi(t, n) - \psi(t, n_0)) = \psi_x(t, n) - \psi_x(t, n_0)
\]

\[
= \int_0^1 \psi_{xn}(t, n_0 + \varepsilon(n - n_0)) d\varepsilon \cdot (n - n_0)
\]

\[
= \psi_{xn}(t, n_0) \cdot (n - n_0) + o(|n - n_0|)
\]

\[
= \frac{d}{dt}(\psi_n(t, n_0)) \cdot (n - n_0) + o(|n - n_0|).
\]

A comparison of these two expressions gives the required result.
3.4.1 LEMMA.

A finite or infinite set of vectors \( x_1, x_2, \ldots \) is called isotropic if \( [x_i, x_j] = 0 \) for \( i, j = 1, 2, \ldots \). When these vectors are also linearly independent the following result is true.

An isotropic set of linearly independent vectors in \( \mathbb{R}^{2n} \) contains, at most, \( n \) vectors.

Proof. Suppose that \( x_1, x_2, \ldots, x_m \) is the isotropic set of linearly independent vectors. Write \( V = \mathbb{R}^{2n} \).

The set \( S_1 \) of all vectors \( y \) such that \( [x_k, y] = 0 \), \( k = 2, \ldots, m \) has dimension \( 2m - m + 1 \). The set \( S_2 \) of all vectors \( y \) such that \( [x_k, y] = 0 \), \( k = 1, \ldots, m \) has dimension \( 2n - m \).

Therefore \( S_2 \) is a proper subset of \( S_1 \) and there exists a \( y_1 \) such that \( [x_1, y_1] \neq 0 \) and \( [x_k, y_1] = 0 \), \( k = 2, \ldots, m \). By multiplying \( y_1 \) by a scalar we can suppose that \( [x_1, y_1] = 1 \).

Consider the set of all vectors \( x \) such that

\[
[x_1, x] = [y_1, x] = 0.
\]

This set is a subspace \( V_1 \), say, of \( V \) and the intersection of \( V_1 \) with the subspace \( W_1 \) spanned by \( x_1, y_1 \) is \( 0 \).

For any \( v \in V \) write

\[
v_1 = v - [v, y_1]x_1 - [x_1, v]y_1.
\]

Then

\[
[y_1, x_1] = [v_1, y_1] = 0
\]

and \( v_1 \in V_1 \), hence \( V = V_1 \oplus W_1 \). Now \( x_2, \ldots, x_m \in V_1 \) and
dim\( V_1 = 2n - 2 \).

For any \( z_1 \in V_1 \), \( [z_1, z] = 0 \) \( \forall z \in V_1 \Rightarrow z_1 = 0 \), because
\[
[z_1, z] = 0, \ \forall z \in V_1,
\]
and
\[
[z_1, w] = 0, \ \forall w \in W_1,
\]
imply that
\[
[z_1, x] = 0, \ \forall x \in V.
\]
Therefore \( z_1 = 0 \).

Repeating the process we get vectors \( y_1, \ldots, y_m \) such that
\[
[x_i, y_j] = 1,
\]
\[
[x_i, y_j] = 0, \ y \neq i,
\]
and \( x_1, \ldots, x_m, y_1, \ldots, y_m \) are linearly independent.

Hence \( m \leq n \).

3.5.1 LEMMA.

The proof of the following lemma is modelled after a similar result in Federer [7], p. 229.

Let \( \varphi \) be a \( C^1 \) homeomorphism of an open subset \( D \) of \( \mathbb{R}^k \) into \( \mathbb{R}^m \) \((m > k)\), then \( \varphi_x \) has rank \( k \) at some point of \( D \).

Proof. Suppose rank \( \varphi_x < k \) at all points of \( D \). Let \( \nu \) be the maximum rank of \( \varphi_x \) and let \( a \) be a point at which this maximum is attained.

By constant transformations of \( \mathbb{R}^k \) and \( \mathbb{R}^m \) we can assume that,
at $x = a$,

$$
\phi_x = \begin{pmatrix}
I_y & 0 \\
0 & 0
\end{pmatrix}.
$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be the corresponding partitions of points $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$ and similarly write $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$. Consider the map $g : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$
g(x) = \begin{pmatrix} \varphi_1(x) \\ x_2 \end{pmatrix}.
$$

Then

$$
\varphi_x = \begin{pmatrix}
\frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} \\
0 & I_{k-2}
\end{pmatrix}.
$$

In particular,

$$
g_x(a) = I_k.
$$

Therefore by the inverse function theorem $g$ maps a neighbourhood $U$ of $a$ diffeomorphically onto a neighbourhood $V$ of $b = \begin{pmatrix} \varphi_1(a) \\ a_2 \end{pmatrix}$. The composite map

$$
\psi = \varphi \circ g^{-1} : V \to \mathbb{R}^m
$$

has Jacobian...
\[
\begin{pmatrix}
\frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} \\
\frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \psi_1}{\partial x_1}^{-1} & \frac{\partial \psi_1}{\partial x_1}^{-1} \frac{\partial \psi_1}{\partial x_2} \\
0 & I_{k-n}
\end{pmatrix}
= \begin{pmatrix}
I_{\nu} & 0 \\
X & Y
\end{pmatrix}, \text{ say.}
\]

But \(\psi\) has rank \(\leq \nu\) near \(b\), so \(Y = 0\) near \(b\) and the value of \(\psi_2\) depends only on its first \(\nu\) coordinates. Consider now the map
\[
h : \mathbb{R}^m \to \mathbb{R}^m
\]
defined by
\[
h(y) = \begin{pmatrix}
y_1 \\
y_2 - \psi_2(y_1, \alpha_2)
\end{pmatrix}.
\]

Then
\[
h_y = \begin{pmatrix}
I_{\nu} & 0 \\
-\frac{\partial \psi_2}{\partial y_1} I_{m-\nu}
\end{pmatrix}.
\]

In particular, \(h_y(\phi[a])\) is invertible and hence \(h\) maps a neighbourhood of \(\phi(a)\) diffeomorphically onto a neighbourhood of
\[
h(\phi[a]) = \begin{pmatrix}
\phi_1(a) \\
0
\end{pmatrix}.
\]

The composite map
\[
f = h \circ \phi \circ g^{-1}
\]
has Jacobian of the form
\[
\begin{pmatrix}
I_{\nu} & 0 \\
-\frac{\partial \psi_2}{\partial y_1} I_{m-\nu}
\end{pmatrix}
\begin{pmatrix}
I_{\nu} & 0 \\
\frac{\partial \psi_2}{\partial y_1} & 0
\end{pmatrix}
= \begin{pmatrix}
I_{\nu} & 0 \\
0 & 0
\end{pmatrix}
\]

near \(b\).

Thus the partial derivatives of
vanish near \( b \). Therefore this function is a constant. But at \( b \) it value is \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), hence

\[
f(x_1, x_2) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.
\]

Thus \( f \) maps \( V \) onto a set of dimension \( v \). But \( g^{-1} \) maps \( V \) onto \( U \). Since \( \varphi \) is a homeomorphism of \( U \) onto a subset \( W \) of \( R^n \) and \( h \) is a diffeomorphism of \( W \) into \( R^n \)

\[
\dim f(V) = \dim h(W) = \dim \varphi(U) = k.
\]

This contradicts our initial hypotheses that \( v < k \).

### 3.6.1 Theorem

Let \( \varphi(t) \) be a quasi-periodic solution of the conservative Hamiltonian system

\[
(3.6.1.1) \quad Jx' = \frac{\partial H}{\partial x}
\]

where \( x' = \frac{dx}{dt} \), \( J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \), \( x \in R^{2n} \) and \( H : R^{2n} \to R \) has a continuously differentiable derivative \( \frac{\partial H}{\partial x} \). Let \( \Phi(t_1, \ldots, t_k) \), the spatial extension of \( \varphi(t) \), be periodic with period \( 2\pi/\beta_i \), and continuously differentiable in each \( t_i \), \( i = 1, 2, \ldots, k \), where the numbers \( \beta_1, \beta_2, \ldots, \beta_k \) are linearly independent over the integers. Then \( \varphi(t) \) has a frequency basis containing no more than \( n \) elements, that is \( k \leq n \).

**Proof.** For any real vector \( (\tau_1, \tau_2, \ldots, \tau_k) \),

\[
\varphi(t, \tau) = \Phi(t+\tau_1, t+\tau_2, \ldots, t+\tau_k)
\]
is a quasi-periodic function in the closed hull of $\varphi(t)$ and consequently has $\beta_1, \beta_2, \ldots, \beta_k$ as a frequency basis. Since, by Kronecker's Theorem (§2.2.2) corresponding to any integer $v$ there exists a real number $t_v$ such that, putting $\delta_v = \left(\sum_{j=1}^{k} |\beta_j|^{-1}\right)^{-1}$,

$$t_v - \frac{2\pi}{\beta_{i_v}} p_i - \tau_i < \delta_v, \quad i = 1, 2, \ldots, k,$$

where the $p_i$ are integers. Then

$$|e^{i(n_1 \beta_1 \tau_1 + \ldots + n_k \beta_k \tau_k)} - e^{i(n, \beta)t_v} - 1| < \frac{1}{v}$$

for $|n| \leq v$.

Taking $v = 1, 2, \ldots$ we obtain a sequence $\{t_v\}$. Since the inequalities continue to hold when $\{t_v\}$ is replaced by any subsequence we can suppose that $\varphi(t + t_v)$ converges uniformly for all real $t$ as $v \to \infty$. This limit is in the closed hull of $\varphi(t)$ and has the Fourier series

$$\sum_{|n| = 0}^{\infty} a_ne^{i(n, \beta)t} = \sum_{|n| = 0}^{\infty} a_n^* e^{i(n, \beta)t},$$

say. Hence this limit is $\varphi(t, \tau)$.

We now prove that $\varphi(t, \tau)$ also satisfies the differential equation (3.6.1.1). We will show that if the real sequence $\{t_v\}$ is such that $\xi_v = \varphi(t_v) + \xi$ as $v \to \infty$ then $\varphi(t + t_v)$ converges uniformly for all real $t$ to $X(t)$, where $X(t)$ is the solution of (3.6.1.1) which has the initial value $X(0) = \xi$.

Certainly $\varphi(t + t_v)$ is a quasi-periodic solution of equation (3.6.1.1) with initial value $\xi_v$. Any subsequence of $\{t_v\}$ contains
a further subsequence \( \{t''_v\} \) such that \( \varphi(t+t''_v) \) converges uniformly for all \( t \). By the continuous dependence of solutions on initial values this limit must be \( X(t) \) and since this is independent of the choice of subsequence, \( \varphi(t+t'_v) \) must converge uniformly to \( X(t) \).

Hence \( \varphi(t, \tau) \) is a solution of (3.6.1.1) for any real vector \( \tau \). By hypothesis \( \varphi_{\tau}(t, \tau) \) exists and is continuous. Since \( \varphi(t, 0) = \varphi(t) \) it follows immediately from Lemma 3.3.2 that \( \varphi_{\tau}(t, \tau) \) is a solution of the variational equation for (3.6.1.1),

\[
(3.6.1.5) \quad Jy' = H_{xx}(\varphi(t, \tau))y .
\]

If \( y_1, y_2 \) are any two solutions of the variational equation (3.6.1.5) for the conservative Hamiltonian system (3.6.1.1) then

\[
[y_1, y_2] = \text{constant} ,
\]

since

\[
\frac{d}{dt} [y_1, y_2] = \frac{d}{dt} (y_1^*Jy_2) = y_1^*Jy_2 + y_1^*Jy_2' = y_1^*H_{xx}(\varphi^{-1})^{*} Jy_2 + y_1^*H_{xx}y_2 = 0 .
\]

We show that this constant is zero for the solutions

\[
y_1 = \varphi_{\tau}(t, \tau) \quad \quad \quad \quad y_2 = \varphi_{\tau}(t, \tau) .
\]

This is trivial for \( i = j \). For \( i \neq j \) we have
\[ [\dot{y}_1, \dot{y}_2] = [\varphi_{\tau_1}(t, \tau), \varphi_{\tau_2}(t, \tau)] \]

\[ \sum_{|n|=0}^{\infty} a_n \eta_n \beta_i e^{i(n, \beta)t}, \sum_{|m|=0}^{\infty} a_m \eta_m \beta_j e^{i(m, \beta)t} \]

\[ = \sum_{|n|=0}^{\infty} \sum_{|m|=0}^{\infty} [a_n', a_m'] \eta_n \eta_m \beta_i \beta_j e^{i(n-m, \beta)t}. \]

Since \( \beta_1, \ldots, \beta_k \) are linearly independent over the integers, and \([\dot{y}_1, \dot{y}_2]\) is a constant it follows from the uniqueness of the Fourier series that

\[ [\dot{y}_1, \dot{y}_2] = \sum_{|n|=0}^{\infty} [a_n', a_m'] \eta_n \eta_m \beta_i \beta_j = 0. \]

Thus the set of solutions

\[ \varphi_{\tau_1}(t, \tau), \varphi_{\tau_2}(t, \tau), \ldots, \varphi_{\tau_k}(t, \tau) \]

of the variational equation is an isotropic set. We will now show that it is also a linearly independent set.

If

\[ \varphi(0, \tau') = \varphi(0, \tau'') \]

then the solutions \( \varphi(t, \tau'), \varphi(t, \tau'') \) of (3.6.1.1) coincide because they have the same initial values. Thus

\[ \varphi(t+\tau'_1, \ldots, t+\tau'_{k}) = \varphi(t+\tau''_1, \ldots, t+\tau''_{k}). \]

Hence

\[ \tau'_i = \tau''_i \mod \frac{2\pi}{\beta_i}, i = 1, \ldots, k. \]

Suppose \( |\tau'_1|, |\tau''_1| \leq \frac{\pi}{2\beta_i}, i = 1, \ldots, k \), then the map

\[ T: (\tau_1, \tau_2, \ldots, \tau_k) \mapsto \varphi(0, \tau_1, \tau_2, \ldots, \tau_k) \]
of the "cube"

\[ C : |\tau_i| \leq \frac{\pi}{2B_i} \quad (i = 1, \ldots, k) \]

into \( \mathbb{R}^{2n} \) is one-to-one. That this map is continuous may be seen in the following way.

For any \( \varepsilon < 0 \), there exists a trigonometric polynomial (for example, the Bochner-Fejér polynomial, see Besicovitch [24], pp. 46-51)

\[ p(t) = \sum_{|n|=0}^{M} \rho_n e^{i(n,\beta)t} \]

with rational coefficients \( \rho_n \) depending only on \( \varepsilon, n \) and \( \beta_1, \ldots, \beta_k \) and satisfying \( 0 \leq \rho_n \leq 1 \), such that

\[ |\phi(t) - p(t)| \leq \varepsilon \quad \text{for} \quad -\infty < t < \infty. \]

Moreover if we replace \( \phi(t) \) by any function \( X(t) \) in its closed hull and the Fourier coefficients \( \sigma_n \) of \( \phi(t) \) by the corresponding Fourier coefficients of \( X(t) \) then this inequality continues to hold.

We can now choose \( \delta = \delta(\varepsilon) > 0 \), so that if

\[ |\tau'_i - \tau''_i| < \delta \quad (i = 1, \ldots, k) \]

the first \( (2M+1)^k \) Fourier coefficients of \( \phi(t, \tau') \) and \( \phi(t, \tau'') \) satisfy

\[ |\sigma'_n - \sigma''_n| \leq \frac{\varepsilon}{(2M+1)^k} \quad (|n| = 0, 1, \ldots, M). \]

Then the corresponding trigonometric polynomials satisfy

\[ |p'(t) - p''(t)| \leq \varepsilon \quad \text{for} \quad -\infty < t < \infty \]

and hence

\[ |\phi(t, \tau') - \phi(t, \tau'')| \leq 3\varepsilon \quad \text{for} \quad -\infty < t < \infty. \]

Since the cube \( C \) is compact the inverse map \( T^{-1} \) is also continuous.
Therefore the map $\tau \mapsto \varphi(0, \tau)$ maps $C$ homeomorphically onto a subset of $\mathbb{R}^{2n}$. Since this map is $C'$, Lemma 3.5.1 ensures that $\varphi_\tau(0, \tau)$ has rank $k$ for some $\tau = \tau_0$, say. Thus

$$\varphi_{\tau_1}(0, \tau_0), \varphi_{\tau_2}(0, \tau_0), \ldots, \varphi_{\tau_k}(0, \tau_0)$$

satisfy the conditions of Lemma 3.4.1. Hence $k \leq n$.

3.7.1.

In order to compare Theorem 3.6.1 with Cherry's result (§3.2.1) we will consider the latter to be split up into two parts. The first part of the result is that an autonomous holomorphic differential equation in $\mathbb{R}^p$ with a non-negative integral invariant and a $p$-dimensional compact invariant set can have only quasi-periodic solutions within this invariant set. Moreover the spatial extension of these solutions is holomorphic and the Jacobian is non-zero.

The theorems of Chapter II guarantee that any almost periodic solution has at most $p - 1$ basic frequencies, but only in the case of $p - 1$ basic frequencies do they guarantee that the solution must be quasi-periodic. We will in a moment give an example of an autonomous equation in $\mathbb{R}^3$ with a positive integral invariant and a 3-dimensional compact invariant set which contains an almost periodic but non-quasi-periodic solution. This equation is not holomorphic, but its existence certainly throws some doubt on this part of Cherry's result.

The second part of the result indicates that in the case of a Hamiltonian system the number of basic frequencies is no more that $\frac{p}{2}$. Our theorem proves this result under much less stringent
hypotheses on the solutions and on the equation.

3.8.1.

We now discuss some results related to Cherry's Theorem of §3.2.1 and compare them with it. By Poincare's recurrence theorem, it follows from conditions (ii) and (iii) of Theorem 3.2.1 that solutions of (3.2.1.1) with initial values in $\Omega$ return arbitrarily closely to their initial values after a finite time. Cherry calls such solutions "quasi-periodic" in [4], p. 47, in contrast with his description of the solutions of the form (3.2.1.2) as "quasi-periodic".

Earlier in [2], Birkhoff had considered a more restrictive class of functions. He defined $f(t) : \mathbb{R} \to \mathbb{R}^2$ to be recurrent if corresponding to each $\varepsilon > 0$ there is a number $T(\varepsilon)$ such that, for any $t_1, t_2$ there is a $t_3 \in [t_1, t_1 + T(\varepsilon)]$ such that

$$|f(t_2) - f(t_3)| < \varepsilon.$$ 

Although those recurrent motions which he constructed for holomorphic systems were quasi-periodic he did construct a non-holomorphic system with "discontinuous" recurrent motions which were not quasi-periodic and he conjectured that holomorphic systems with such solutions did exist.

Various authors constructed such systems. Littlewood [16] and [17] produced a complicated result which Levinson [14] managed to simplify considerably. Levinson showed that the equation

$$y'' + p(y)y' + y = \cos t,$$

with $p(y)$ some polynomial, had discontinuous recurrent solutions inside its invariant set. Therefore the autonomous system
has a 4 dimensional invariant set, and has recurrent non-quasi-periodic solutions. Furthermore this system is holomorphic. Although this equation does not seem to have an integral invariant this example also casts some doubt on Cherry's result.

In Chapter II we saw that an almost periodic solution of

\[ (3.8.1.1) \]

\[ x' = \psi(x) \]

where \( x \in \mathbb{R}^D \) and \( \psi(x) \) satisfies sufficient conditions for the solutions of (3.8.1.1) to be uniquely determined by their initial conditions has less than \( p \) elements in its frequency basis. However this basis is not necessarily integral, that is the solution is not necessarily quasi-periodic. Only when the frequency basis contains precisely \( (p-1) \) elements is the basis integral and the almost periodic solution quasi-periodic.

For \( p = 2 \) the classical Poincaré-Bendixon theory (see Hartman [10], Chapter 7) asserts that the only recurrent solutions, and hence the only almost periodic solutions of (3.8.1.1) are periodic, but the proof depends on the Jordan curve theorem, a topological peculiarity of the plane. For \( p = 3 \) we have the results of Denjoy [6] who examined the solutions of (3.8.1.1) on a torus (a 2 dimensional manifold without the Jordan curve property). He found that there were three types of recurrent motions:

(i) periodic solutions;
(ii) quasi-periodic solutions with two basic frequencies;
(iii) non-almost-periodic solutions.

Denjoy showed that a sufficiently strong condition of differentiability on the right hand side of equation (3.8.1.1) excluded case (iii). However this does not suffice for $p \geq 4$ and it seems unlikely that any differentiability condition (even holomorphicity) can exclude non-almost periodic solutions in the general case.

In the next section we give an example of an autonomous equation in $R^3$ satisfying conditions (ii) and (iii) of Cherry's Theorem which has periodic solutions with arbitrarily long periods, quasi-periodic solutions with two basis frequencies and an almost periodic solution with a one term non integral frequency basis (that is, an almost periodic solution which is not quasi-periodic). This example is based on a suggestion in Cartwright [121].

3.8.2.

The example mentioned above will be given implicitly by defining a continuous flow on a subset of $R^3$ whose orbits are the above mentioned solutions.

More precisely, let $X \subset R^p$. Suppose that for each $t \in R$ and for each $x \in X$

$$\tilde{x} = \varphi(x, t)$$

is a homeomorphism of $X \to R^p$, such that

(i) $x = \varphi(x, 0), \ x \in R^p,$
(ii) $\varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2)$,
(iii) $\varphi(x, t)$ is continuous in $x$ and $t$. 

Then \( \psi(x, t) \) defines a continuous flow on \( X \). The set

\[ O(x_0) = \{ \psi(x_0, t) : t \in \mathbb{R} \} \]

is called the orbit of \( x_0 \).

3.8.3.

Let \( x_1, x_2 \) be coordinates in \( \mathbb{R}^2 \). In the \( (x_1, x_2) \) plane we consider the sequences of circles

\[ C^0, C^1_2, \ldots, C^n_2, \ldots, C^n_2 \]

and

\[ S^0, S^1_2, \ldots, S^n_2, \ldots, S^n_2 \]

defined inductively by the following formulae:

\[ C^i_2 : \left( x_1 - d^i_2 \right)^2 + x_2^2 = r^2, \quad i = 1, 2, \ldots, 2^n \]

\[ S^i_2 : \left( x_1 - d^i_2 \right)^2 + x_2^2 = \left( \frac{r}{8} \right)^2, \quad i = 1, 2, \ldots, 2^n, \]

where

\[ d^0 = 2 \]

\[ d^1_1 = \frac{3}{2} \]

\[ d^1_2 = \frac{5}{2} \]

and

\[ d^i_2 = \begin{cases} \frac{d^{i-1}_2 - \frac{1}{2^{n-1}}}{2}, & \text{odd}, \\ \frac{d^{i-1}_2 - \frac{1}{2^{n-1}}}{2}, & \text{even}, \end{cases} \]

Clearly, the sequences \( C^i_2 \) are pairwise disjoint. The circles \( C^i_2 \) are illustrated in Figure 3.3.4.

For each \( n \) and a given point \( x \) we define a homeomorphism \( \psi_n(x_1, x_2, t) \) of the \( x_1, x_2 \) plane into itself, as follows.
and where

\[ r_n = \frac{r_{n-1}}{2^2}, \]

\[ r_0 = 1. \]

Then it follows that

\[ r_n = 2^{-2n} \]

and that \( d_i^n, \ i = 1, 2, \ldots, 2^n, \) assumes each of the \( 2^n \) values

\[ 2\left(1 \pm \frac{1}{2^2} \pm \frac{1}{2^4} \pm \cdots \pm \frac{1}{2^{2n}}\right). \]

Clearly

\[ C_i^n \subset C_i^r, \] for all \( n \) and all \( i, \)

\[ C_i^{n+k} \subset C_i^{n}, \] for \( 2^k(i_0 - 1) + 1 \leq i \leq 2^k i_0, \)

and for each \( n \)

\[ C_1^n, C_2^n, \ldots, C_{2^n} \]

are pairwise disjoint. The circles \( C_i^n, C_i^r, C_i^{n+1}, C_i^{n+1}, C_i^{n+1}, C_i^{n+1}, C_i^{n+1}, \) are illustrated in figure 3.8.1.

For each \( n \) and a given value of \( t \) we define a homeomorphism \( \varphi_t^n(x_1, x_2, t) \) of the \( (x_1, x_2) \) plane into itself, as follows

\[
\varphi_t^n(x_1, x_2, t) = \begin{cases} 
\frac{x_1 + \rho \cos \left( \frac{\alpha_n (\rho_n^t) t}{2^{n+1}} + \theta \right)}{2^{n+1}} \\
\frac{x_2 + \rho \sin \left( \frac{\alpha_n (\rho_n^t) t}{2^{n+1}} + \theta \right)}{2^{n+1}}
\end{cases}
\]
where

$$\begin{pmatrix}
    x_1 \\
    x_2
  \end{pmatrix} =
  \begin{pmatrix}
    d_n + \rho_n \cos \theta \\
    \rho_n \sin \theta
  \end{pmatrix}$$

and

$$\rho_n = \sqrt{\left(x_1 - d_n \frac{2}{i}\right)^2 + x_2^2}$$

and we will next consider the action of $E = G \ltimes \mathbb{R}$ on the $(x_1, x_2)$ half-plane.
Since the discs $C_1^n, C_2^n, \ldots, C_{2^n}^n$ are pairwise disjoint then the composition homeomorphism

$$
\phi_n = \varphi_1^n \circ \varphi_2^n \circ \ldots \circ \varphi_{2^n}^n
$$

maps the discs inside the circles $S_i^n$ into themselves with a constant angular rotation $\frac{t}{2}$ which smoothly decreases to 0 over the annulus between $S_i^n$ and $C_i^n$ for $i = 1, 2, \ldots, 2^n$. All points outside

$$
\bigcup_{i=1}^{2^n} C_i^n
$$

remain unaltered by $\phi_n$.

Now consider the composition homeomorphism

$$
\psi_n = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1 \circ \phi_0
$$

and the homeomorphism of $R^3$ onto itself engendered by the rotation $\mathbb{E}$, where

$$
\mathbb{E}(x_1, x_2, 0) = \begin{bmatrix}
  x_1 \cos t \\
  x_2 \\
  x_1 \sin t
\end{bmatrix}.
$$

We will next consider the action of $\mathbb{E} \circ \psi_n$ on the $(x_1, x_2)$ half plane.
plane $x_1 \geq 0$ for $0 \leq t \leq 2\pi$ and see that this implicitly defines a continuous flow.

We observe immediately that points inside the circles $S^2_i$ are rotated through an angle

$$t\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\right) = t(1-2^{-n})$$

and that $E \circ \psi^n$ is $C^\infty$.

This homeomorphism rotates all points outside and on $C^0$ about the $x_2$ axis through an angle $t$. Hence as $t$ varies from 0 to $2\pi$, $C^0$ describes a torus $T^0$ and the points on $C^0$ describe circles about the $x_2$ axis. Thus this torus will be an invariant set for our flow. The points in the annulus between $C^0$ and $S^0$ remain at the same distance from the common centre of these circles and, as $t$ increases from 0 to $2\pi$, describe either a periodic or quasi-periodic orbit according as to whether $a_0(\rho)$ is rational or irrational for that radius $\rho$. All of the points inside and on $S_0$ but not inside $C_1$ or $C_2$, with the exception of $d^0$ describe periodic orbits with period $4\pi$. Furthermore the circle $C_1'$ is mapped onto $C_2'$, for $t = 2\pi$, which in turn is mapped onto $C_1'$, for $t = 2\pi$, in such a way that for $t = 4\pi$ the points of $C_1'$ are mapped back onto themselves.

That is, we have an invariant "torus" which "twists" twice around the $x_2$-axis.

At the $n$-th stage all of the points in the annulus between $C^1_{i-1}$ and $S^1_{i-1}$ remain at the same distance from the common centre of
these circles and define a periodic orbit or a quasi-periodic orbit with two basic frequencies, depending on whether the angular rotation is a rational multiple of \( t \), that is, whether

\[
\frac{a_0(\varphi_0)}{2} + \frac{a_1(\varphi_1)}{2^2} + \ldots + \frac{a_{n-1}(\varphi_{n-1})}{2^n}
\]

is rational. All of the points inside and on \( S_{i}^{n-1} \) but not inside \( C_{2i-1}^{n} \) or \( C_{2i}^{n} \), with the exception of \( d_{i}^{n-1} \), describe periodic orbits with period \( 2^{n+1}\pi \). The orbit through \( d_{i}^{n-1} \) has period \( 2^n\pi \).

\( C_{2i-1}^{n} \) is mapped in succession onto each circle in the set \( \{c_{i}^{n}: i = 1, 2, \ldots, 2^n\} \), although not necessarily in order, and for \( t = 2^{n+1}\pi \) the points on \( C_{2i-1}^{n} \) return to their initial positions. Thus \( C_{2i-1}^{n} \) describes an invariant "torus" \( T_{n} \), say, which "twists" around the \( x_2 \)-axis \( 2^n \) times.

Now every point in \( R^3 \) is the unique consequent of a point in the \( (x_1, x_2) \) half plane \( x_1 \geq 0 \) for some \( t \). The group property of rotations thus yields a homeomorphism of \( R^3 \) into itself (which is too complicated to write down explicitly) and its action on the half plane \( x_3 = 0, x_1 \geq 0 \) is precisely \( \Sigma \circ \Psi_{n} \).

Going over to the limit as \( n \to \infty \) it follows that the limit is a continuous mapping of the torus \( T^0 \) into itself. We will obtain an expression for the orbit of \( \lim_{n \to \infty} d_{i}^{n} \), which is the only orbit which may be neither periodic nor quasi-periodic.
For the orbit of $d^0$, we have

$$O_0 : x_1 = 2\cos t, \ x_2 = 0, \ x_3 = 2\sin t.$$ 

For the orbit of $d_2^1$, we have

$$O_1 : x_1 = 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2}\right)\cos t$$

$$x_2 = \frac{2}{2^2} \sin \frac{t}{2}$$

$$x_3 = 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2}\right)\sin t.$$ 

For the orbit of $d_4^2$, we have

$$O_2 : x_1 = 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2} + \frac{1}{2^4} \cos \frac{t}{2^2}\right)\cos t$$

$$x_2 = 2\left(\frac{1}{2^2} \sin \frac{t}{2} + \frac{1}{2^4} \sin \frac{t}{2^2}\right)$$

$$x_3 = 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2} + \frac{1}{2^4} \cos \frac{t}{2^2}\right)\sin t.$$ 

For the orbit of $d_{2n}^n$, we have

$$O_n : x_1 = 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2} + \ldots + \frac{1}{2^{2n}} \cos \frac{t}{2^n}\right)\cos t$$

(3.8.3.3) $$x_2 = 2\left(\frac{1}{2^2} \sin \frac{t}{2} + \ldots + \frac{1}{2^{2n}} \sin \frac{t}{2^n}\right)$$

$$x_3 + 2\left(1 + \frac{1}{2^2} \cos \frac{t}{2} + \ldots + \frac{1}{2^{2n}} \cos \frac{t}{2^n}\right)\sin t.$$ 

It follows immediately, as noted above, that the points $d_1^n, d_2^n, \ldots, d_{2n}^n$ all lie on the orbit $O_n$, and all of the points on this orbit have period $2^{n+1}\pi$ for $n = 0, 1, 2, \ldots$.

We will now show that the limit of the set of orbits $O_n$, as
is almost periodic. This follows from expression (3.8.3.3). However we will also show it directly. Corresponding to any \( \varepsilon > 0 \) choose \( n \) so that \( 2^{2n-2} < \varepsilon \). If \( x_0 \) is the limit of the set \( d^n_{2n} \) then \( x_0 \in T_n \) and hence the orbit of \( d^n_{2n} \) approximates the orbit of \( x_0 \). When \( t = 0 \), \( x_0 \) and \( d^n_{2n} \) both belong to the disc with boundary \( C^n_{2n} \). Hence denoting the limit homeomorphism defined above by \( \Psi(x, t) \)

\[
\sup_{-\infty < t < \infty} \left| \Psi(x_0, t) - \Psi(x_0, t + q2^n\pi) \right| < \varepsilon
\]

for all integers \( q \). Hence \( \Psi(x_0, t) \) is almost periodic. It follows immediately that this almost periodic motion cannot have an integral frequency basis.

It remains to show that there is a non-negative integral invariant. It follows from (3.8.3.1), (3.8.3.2) and (3.8.3.3) that

\[
M = \left( x_1^2 + x_2^2 \right)^{-\frac{1}{2}}
\]

is such an invariant.

The above example has the three different kinds of almost periodic motions allowed by Theorem 2.3.5 and §2.4.6 of Chapter II, namely quasi-periodic solutions with one basic frequency (periodic solutions), quasi-periodic solutions with two \( (2 = p - 1) \) basic frequencies and an almost periodic solution with one \( (1 < p - 1) \) basic frequency which is not quasi-periodic. In this case there are no other recurrent solutions but such solutions are not necessarily excluded by the compact \( p \)-dimensional invariant set and the integral invariant.
CHAPTER IV

REDUCIBILITY AND QUASI-PERIODIC SOLUTIONS
FOR HOLOMORPHIC QUASI-PERIODIC LINEAR SYSTEMS

4.1.1.

In this chapter we are concerned with the linear differential equations

\[ x' = Ax + P(\varphi)x \]
\[ \varphi' = \omega \]

where \( x \in \mathbb{R}^n \), \( \varphi \in \mathbb{R}^m \), \( A \) is a constant \( n \times n \) matrix, \( \omega \) is a constant vector in \( \mathbb{R}^n \) and the \( n \times n \) matrix \( P(\varphi) \) is holomorphic and periodic in \( \varphi \) with period \( 2\pi \) for \( i = 1, \ldots, m \). Thus we are considering a quasi-periodic system. When \( A, P(\varphi) \) and \( \omega \) satisfy certain conditions we prove the existence of quasi-periodic solutions. Our proof uses the method of accelerated convergence to overcome the so-called "small divisors" difficulty.

4.1.2.

We define the norm of a column vector \( x = (x_1; \ldots; x_m) \) by

\[ |x| = \sum \alpha |x_\alpha| \]

We define the norm of a row vector \( k = (k_1, \ldots, k_m) \) by

\[ |k| = \max \alpha |k_\alpha| \]

Then for the scalar product

\[ (k, x) = k_1x_1 + \ldots + k_mx_m \]

we have the obvious inequality
Finally we define the norm of a matrix

\[A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\]

by

\[|A| = \max_{\beta} \sum_{\alpha} |a_{\alpha\beta}|.\]

4.1.3.

In the course of finding such a reduction of the given system of differential equations we will be led to consider, at each step, matrix equations of the form

\[(4.1.3.1) \left( \frac{\partial U}{\partial \psi}, \omega \right) + UA - AU = C(\psi)\]

where \( \left( \frac{\partial U}{\partial \psi}, \omega \right) \) represents \( \sum_{\alpha=1}^{m} \frac{\partial U}{\partial \psi} \omega_\alpha \). We will do this by trying to find a solution with Fourier series

\[U(\psi) \sim \sum_{|k|=0}^{\infty} U_k^i(k, \psi),\]

and by equating coefficients we obtain equations of the form

\[\{i(k, \omega)I - A\}U_k^i + U_kA = C_k.\]

That is we have the general linear matrix equation

\[A_1X + XB_1 = C_1.\]

This equation has a unique solution if and only if \(A_1\) and \(B_1\) do not have any common eigenvalues.
4.2.1 THEOREM.

Suppose

(i) \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is a real diagonal matrix with distinct eigenvalues;

(ii) \( P(\varphi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \varphi \) which is real for real \( \varphi \), has period \( 2\pi \) in each coordinate \( \varphi_\alpha \) and which is holomorphic and satisfies the inequality

\[ |P(\varphi)| \leq M \]

in the strip \( |\text{Imp}| < \rho \); 

(iii) \( \omega \) is a real \( m \)-vector such that for all integral \( m \)-vectors \( k \neq 0 \)

\[ |(k, \omega)| \geq \gamma |k|^{-\tau} \]

where \( \tau > m \) and \( \gamma > 0 \).

Then the partial differential equation

\[
\left( \frac{\partial U}{\partial \varphi}, \omega \right) + UA - AU = P(\varphi)
\]

has a solution which is real for real \( \varphi \), has period \( 2\pi \) in each coordinate \( \varphi_\alpha \) and which is holomorphic and satisfies the inequality

\[ |U(\varphi) - U_0| \leq c\gamma^{-1}\delta^{-\tau}M, \]

in the strip \( |\text{Imp}| \leq \rho - \delta \) \((\delta < 1)\), where

\[ U_0 = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} U(\varphi) \, d\varphi_1 \cdots d\varphi_m \]

is the mean value of \( U(\varphi) \) and \( c = c(m, \tau) > 0 \).

Proof. Let

\[ P(\varphi) = \sum_k P_k \delta(\varphi, k, \varphi) \]
be the Fourier expansion of \( P(\phi) \), so that

\[
P_k = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P(\phi) e^{-i(k,\phi)} \, d\phi_1 \cdots d\phi_m.
\]

By shifting the lines of integration to \( \text{Im} \alpha = \pm \rho \alpha \), where \( \sum_{\alpha} \rho_\alpha \leq \rho \), we obtain

\[
|P_k| \leq M e^{-\rho |k|}.
\]

We look for a solution of (4.2.1.1) with the Fourier expansion

\[
U(\phi) = \sum_k U_k e^{i(k,\phi)}.
\]

Substituting in (4.2.1.1) and equating coefficients we obtain

\[
U_k \{A + i(k, \omega)\} - AU_k = P_k.
\]

Thus if we write

\[
P_k = \left\{ \begin{array}{c} P_{\alpha \beta} \\ \{U_{\alpha \beta}\} \end{array} \right\}, \quad U_k = \left\{ \begin{array}{c} \{U_{\alpha \beta}\} \\ \{u_{\alpha \beta}\} \end{array} \right\},
\]

then

\[
u_{\alpha \beta} = \frac{P_{\alpha \beta}}{\lambda_{\beta} - \lambda_{\alpha} + i(k, \omega)}, \quad \alpha, \beta = 1, \ldots, n.
\]

Since \( P_{-k} = \overline{P_k} \) it follows that \( U_{-k} = \overline{U_k} \). Also for \( k \neq 0 \)

\[
|u_{\alpha \beta}(k)| \leq \frac{|P_{\alpha \beta}|}{|(k, \omega)|}, \quad \alpha, \beta = 1, \ldots, n,
\]

and hence

\[
|U_k| \leq \frac{|P_k|}{|(k, \omega)|}.
\]

It follows that for \( |\text{Im} \alpha| \leq \rho - \delta \)

\[
|U(\phi) - U_0| \leq M \sum_{k \neq 0} \frac{1}{|(k, \omega)|} e^{-\delta |k|}.
\]

It remains to establish the convergence of the series on the right and obtain a sharp estimate for its sum.
Let \( K_j \) denote the set of all integral vectors \( k \) such that

\[
2^{-j-2} < |(k, \omega)| \leq 2^{-j-1} \quad (j = 0, 1, \ldots)
\]

Every non-zero integral vector belongs to one and only one set \( K_j \).

Hence

\[
S \equiv \sum_{k \neq 0} \frac{1}{|k(\omega)|} e^{-\delta|k|} = \sum_{j=0}^{\infty} \sum_{k \in K_j} \frac{1}{|k(\omega)|} e^{-\delta|k|}
\]

\[
+ \sum_{|k(\omega)| > \frac{1}{2}} \frac{1}{|k(\omega)|} e^{-\delta|k|}
\]

\[
\leq \sum_{j=-1}^{\infty} 2^{j+2} \sum_{k \in K_j} e^{-\delta|k|}.
\]

For \( k \in K_j \) and \( j = 0, 1, \ldots \)

\[
\gamma|k|^{-\tau} \leq |(k, \omega)| \leq 2^{-j-1} < 2^{-j}
\]

and hence

\[
|k| \geq a_j = (2^j \gamma)^{1/\tau}.
\]

For distinct \( k_1, k_2 \in K_j \)

\[
\gamma|k_1 - k_2|^{-\tau} \leq |(k_1 - k_2, \omega)|
\]

\[
\leq |(k_1, \omega)| + |(k_2, \omega)|
\]

\[
\leq 2^{-j}
\]

hence \( |k_1 - k_2| \geq a_j \).

For any \( k \in K_j \) let \( W_k \) denote the open cube with centre \( k \) and sides parallel to the axes of length \( a_j \). These cubes are disjoint since \( y \in W_{k_1} \cap W_{k_2} \) implies

\[
a_j \leq |k_1 - k_2| \leq |y - k_1| + |y - k_2| < \frac{1}{2}a_j + \frac{1}{2}a_j = a_j.
\]
Let \( v_{jL} \) denote the number of \( k \in K_j \) such that
\[
2a_j \leq |k| < (2l+1)a_j \quad (l = 1, 2, \ldots).
\]
The corresponding cubes \( W_k \) lie in the set
\[
(l-1/2)a_j < |y| < (l+1/2)a_j
\]
which has volume
\[
[(2l+3)^m - (2l-1)^m]a_j^m.
\]
Since each cube has volume \( a_j^m \) this gives
\[
v_{jL} \leq (2l+3)^m - (2l-1)^m \leq 4m(2l+3)^{m-1}
\]
by the mean value theorem. Since \( 2 + \frac{3}{l} \leq 5 \) it follows that
\[
v_{jL} \leq 5^m m^{m-1}.
\]
Therefore
\[
\sum_{k \in K_j} e^{-\delta |k|} \leq \sum_{l=1}^{\infty} v_{jL} e^{-\delta a_j} \leq 5^m \sum_{l=1}^{\infty} l^{m-1} e^{-\delta a_j}.
\]
Put \( q = e^{-\delta a_j} \). Since \( t^{m-1} q \) takes its maximum value for
\[
\frac{1}{2} t \log \frac{1}{q} = m - 1 \quad \text{we have}
\]
\[
\sum_{l=1}^{\infty} l^{m-1} q^l = \sum_{l=1}^{\infty} (l^{m-1} q^{l/2}) q^{l/2}
\]
\[
\leq \left( \frac{m}{\log q} \right)^{m-1} \sum_{l=1}^{\infty} q^{l/2}
\]
\[
= \left( \frac{m}{\log q} \right)^{m-1} (q^{-1/2} - 1)^{-1}
\]
and hence
The number of $k$ with $|k| = r$ is

$$n_k \leq (2r+1)^m - (2r-1)^m$$

$$\leq 4m(2r+1)^{m-1}$$

$$\leq 4.3^{m-1}m^{r-1}.$$  

Therefore, when $j = -1$,

$$S_{-1} = 2 \sum_{|(k,\omega)| > 2} e^{-\delta|k|}$$

$$\leq 8m \cdot 3^{m-1} \sum_{r=1}^{\infty} r^{m-1} e^{-\delta r}$$

$$\leq 8m \cdot 3^{m-1} \left( m \frac{1}{\delta} \right)^{m-1} (e^{2\delta} - 1)^{-1}.$$  

It follows from the definition of $a_j$ that

This proves that $U(\omega)$ is holomorphic and estimates the estimate for $U(\omega)$. 

$$S - S_{-1} \leq (5m)^m \sum_{j=0}^{\infty} 2^{j+2} (\delta a_j)^{1-m} \left( e^{\delta / 2} - 1 \right)^{-1}$$

$$= (5m)^m \sum_{j=0}^{\infty} 2^{j+2} (2^{j+1})$$

where

$$g(t) = \left[ 5(\gamma t/2)^{1/\tau} \right]^{1-m} \left[ e^{2\delta (\gamma t/2)^{1/\tau}} - 1 \right]^{-1}.$$  

Since $2^{j+2} = 4(2^{j+1} - 2^j)$ and $g$ is a decreasing function this implies
\[ S - S_{-1} \leq 4(5m)^m \int_{1}^{\infty} g(t)dt \]

\[ \leq 8(5m)^m \tau^{-1} \delta^{-\tau} \int_{0}^{\infty} g \left\{ 2Y^{-1} \left[ \frac{v}{\delta} \right] \right\} v^\tau - 1 dv \]

\[ = 8(5m)^m \tau^{-1} \delta^{-\tau} \int_{0}^{\infty} v^{\tau - m} (v/2 - 1)^{-1} dv \]

\[ = a_1 \gamma^{-1} \delta^{-\tau}, \]

since \( \tau > m \).

Similarly

\[ S_{-1} \leq \frac{16}{3} \left( \frac{3m}{2} \right)^m \left( \frac{\delta}{2} \right)^{1-m} (\delta/2 - 1)^{-1} \]

\[ \leq a_2 \delta^{-m}. \]

Thus, for \( \delta < 1 \) it follows that

\[ S \leq a \gamma^{-1} \delta^{-\tau}. \]

This proves that \( U(\phi) \) is holomorphic and establishes the estimate for \( U(\phi) - U_0 \).

4.2.2.

Consider now the system of ordinary differential equations

\[ x' = Ax + P(\phi)x \]

(4.2.2.1)

\[ \phi' = \omega \]

where \( A, P(\phi) \) and \( \omega \) satisfy the hypothesis of Theorem 4.2.1. Thus

\[ \min_{\alpha \neq \beta} |\lambda_{\alpha} - \lambda_{\beta}| \geq r > 0. \]

Let \( D(\phi) \) be the diagonal matrix with the same diagonal elements as \( P(\phi) \) and let \( D_0 \) be its mean value. As in Theorem 4.2.1 we can find a matrix \( U(\phi) \) such that
\[ \left\{ \frac{\partial U}{\partial \varphi}, \omega \right\} + UA - AU = P(\varphi) - D_0 \]

Its mean value \( U_0 \) must satisfy

\[ U_0A - AU_0 = P_0 - D_0. \]

If \( P_0 = \begin{pmatrix} 0 \\ P_{\alpha\beta} \end{pmatrix}, ~ U_0 = \begin{pmatrix} u^{0} \\ u^{\alpha\beta} \end{pmatrix} \)

this equation has the solution

\[
\begin{cases} 
    P_{\alpha\beta} \\
    \lambda_{\alpha\beta} = \frac{P_{\alpha\beta}}{\lambda_{\alpha\beta}} \\
    u^{0}_{\alpha\beta} = \begin{cases} 
        0 & \text{for } \alpha = \beta \\
        \frac{P_{\alpha\beta}}{\lambda_{\alpha\beta}} & \text{for } \alpha \neq \beta.
    \end{cases}
\end{cases}
\]

Thus

\[ |U_0| \leq \frac{1}{r} |P_0| \leq \frac{M}{r}. \]

The remaining Fourier coefficients of \( U(\varphi) \) are unchanged and thus we obtain

\[ |U(\varphi)| \leq (r^{-1} + \sigma \gamma^{-1} \delta^{-1} \tau^{-1}) M \text{ for } |\Im \varphi| \leq \rho - \delta. \]

The change of variables

\[ x = [I + U(\varphi)] \tilde{x} \]

transforms \( (4.2.2.1) \) into a system

\[ \tilde{x}' = \tilde{A} \tilde{x} + \tilde{P}(\varphi) \tilde{x}, \]

\[ \varphi' = \omega \]

(4.2.2.2)

of the same form, where

\[ \tilde{A} = A + D_0 \]

\[ \tilde{P} = (I + U)^{-1} \left[ PU - UD_0 \right]. \]

Thus if \( |U| \leq \frac{1}{2} \) then
Moreover if \( \alpha \neq \beta \)

\[
|\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta| \geq |\lambda_\alpha - \lambda_\beta| - |P_\alpha P_\beta - P_\beta P_\alpha| \\
\geq r - 2M.
\]

Let \( \varepsilon \) be any number such that

\[
0 < \varepsilon < \min(1, \rho, \frac{1}{2}r).
\]

Choose \( \theta \) \((0 < \theta < 1) \) so small that if \( \theta_j = \theta (\frac{1}{2})^j \) then

\[
\sum_{j=0}^{\infty} \theta_j < \frac{1}{4} \varepsilon
\]

\[
\sum_{j=0}^{\infty} \theta_j^{1/\tau} < \frac{\varepsilon}{\alpha} \quad \text{where} \quad \alpha = (8\sigma/\gamma)^{1/\tau}
\]

\[
\sum_{j=0}^{\infty} \theta_j < \varepsilon/p \quad \text{where} \quad p = \prod_{j=0}^{\infty} (1 + \theta_j).
\]

Put

\[
\rho_j = \rho - \alpha \left( \theta_0^{1/\tau} + \ldots + \theta_{j-1}^{1/\tau} \right) > 0,
\]

\[
r_j = r - 2(\theta_0 + \ldots + \theta_{j-1}) > 0,
\]

and suppose that the system \((4.2.2.1)\) satisfies the above conditions with \( M, \rho, r \) replaced by \( \theta_j^2, \rho_j, r_j \). We will show that the transformed system \((4.2.2.2)\) satisfies the same conditions with \( M, \rho, r \) replaced by \( \theta_{j+1}^2, \rho_{j+1}, r_{j+1} \). Thus we take \( \delta = \alpha \theta_j^{1/\tau} \).

By the definition of \( \alpha \) we have for \( \Im \Phi \leq \rho_{j+1} \),

\[
|U_{j+1}| \leq \left( r_{j+1}^{-1} \theta_j^{-1} \right) \theta_j^2 \leq \frac{1}{2} \theta_j.
\]
since \( r_j \geq \theta_j \). Thus \( |U_{j+1}| \leq \frac{1}{2} \) and hence

\[
|P_{j+1}| \leq 4\theta_j^2 |j_j = \theta_j^2 \leq \theta_{j+1}.
\]

Moreover

\[
\min_{\alpha \neq \beta} |\lambda_\alpha - \lambda_\beta| \geq r_j - 2\theta_j^2 \geq r_j - 2\theta_j = r_{j+1}.
\]

It follows that if in the original system (4.2.2.1), \( M \leq \theta^2 \) then the above transformation can be repeated indefinitely. We will now prove the existence of a limit system. We have

\[
x_j' = A_j x_j + P_j(\varphi)x_j
\]

where

\[
\varphi' = \omega
\]

where

\[
A_j = A + D_0^{(1)} + \ldots + D_0^{(j)}.
\]

Since \( |P_j(\varphi)| \leq \theta_j^2 \) for \( |\text{Im} \varphi| \leq \rho_j \), it is obvious that as \( j \to \infty \),

\[
P_j(\varphi) \to 0 \quad \text{uniformly for} \quad |\text{Im} \varphi| < \rho_\infty = \rho - \alpha \sum_{s=0}^{\infty} \frac{\theta_{j+1}}{s}. \]

Since

\[
|D_0^{(j)}| \leq \theta_j^2 \quad \text{the existence of the limit}
\]

\[
B = \lim_{j \to \infty} A_j
\]

is also assured and \( |B - A| < \varepsilon \). Finally we have \( x = T_j(\varphi)x_j \) where

\[
T_j = (I+U_1) \ldots (I+U_j).
\]

Suppose that for some \( j \)

\[
|T_j| \leq \prod_{s=0}^{j-1} (1+\frac{1}{s} \theta_s).
\]

With the convention \( T_0 = I \) this is certainly true for \( j = 0 \). Since

\[
T_{j+1} = T_j(I+U_{j+1}).
\]
it follows that

\[ |T_{j+1}^j| \leq |T_j^j| |(I+U^j_{j+1})| \leq \frac{j}{s=0} (1+\theta_s^j) . \]

Therefore for \( |\text{Im}\phi| < \rho_\infty \) we have

\[ |T_j^j(\phi)| \leq p \quad \text{for all } j . \]

Hence

\[ |T_{j+1}^j-T_j^j| \leq |T_j^j| |U_{j+1}^j| \leq \frac{1}{4} p \theta_j^j . \]

Thus the series \( \sum (T_{j+1}^j(\phi)-T_j^j(\phi)) \) converges uniformly for

\( |\text{Im}\phi| < \rho_\infty \). The sequence \( \{T_j^j(\phi)\} \) converges uniformly in the same

strip. Its limit \( T(\phi) \) is holomorphic, is real for real \( \phi \), and has

period \( 2\pi \) in each coordinate. Moreover, the partial derivatives

\( \frac{\partial T^j}{\partial \phi_\alpha} \) converge uniformly to \( \frac{\partial T}{\partial \phi_\alpha} \). From

\[ T_{j+1}^j - I = T_j^j - I + T_j^j U_{j+1}^j \]

we obtain

\[ |T-I| \leq \sum_{j=0}^\infty |T_j^j| |U_{j+1}^j| \leq \frac{1}{4} p \sum_{j=0}^\infty \theta_j^j < \epsilon . \]

Hence the matrix \( T(\phi) \) is invertible and

\[ x_j + y = T^{-1}(\phi)x \quad \text{as } j \to \infty . \]

Differentiating the relation \( x = T_j^j x_j \) we obtain

\[ x' = \left( \frac{\partial T^j}{\partial \phi}, \omega \right) x_j + T_j^j x_j' . \]

Hence, letting \( j \to \infty \)

\[ x' = \left( \frac{\partial T}{\partial \phi}, \omega \right) y + T\dot{y} . \]

Differentiating the relation \( x = Ty \) we obtain
Comparison of these two expressions for $x'$ gives $y' = By$. Thus the change of variables $x = T(\phi)y$ transforms the system (4.2.2.1) into the autonomous system

$$y' = By$$

$$\varphi' = \omega.$$

Summarizing the above we have proved

**THEOREM 4.2.2.**

Suppose

(i) $A = [\lambda_1, \ldots, \lambda_n]$ is a real diagonal matrix with

$$\min_{\alpha \neq \beta} |\lambda_\alpha - \lambda_\beta| > r > 0;$$

(ii) $P(\phi)$ is an $n \times n$ matrix function of the $m$-vector $\phi$ which is real for real $\phi$, has period $2\pi$ in each coordinate $\varphi_\alpha$, and which is holomorphic and satisfies the inequality

$$|P(\phi)| \leq M \text{ for } |\text{Im} \phi| \leq \rho \ (\rho > 0);$$

(iii) $\omega$ is a real $m$-vector such that for all integral $m$-vectors $k \neq 0$

$$|k, \omega| \geq \gamma |k|^{-\tau} \ (\tau > m, \gamma > 0).$$

Then for any $\epsilon$ such that $0 < \epsilon < \min(1, \rho, \frac{1}{2}r)$ there exists an (explicitly computable) absolute constant $M_0 = M_0(m, \tau, \gamma, \epsilon)$ with the property that if $M \leq M_0$ the quasi-periodic linear system

$$x' = [A + P(\omega t)]x$$

has a fundamental matrix of the form

$$X(t) = [I + U(\omega t)]e^{tB},$$
where

(i) \( B = [\mu_1, \ldots, \mu_n] \) is a real diagonal matrix with

\[ |B-A| \leq \varepsilon \quad \text{and hence} \quad \min_{\alpha \neq \beta} |\mu_\alpha - \mu_\beta| \leq r - 2\varepsilon > 0 ; \]

(ii) \( U(\phi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \phi \) which is real for real \( \phi \), has period \( 2\pi \) in each coordinate \( \phi_\alpha \), and which is holomorphic and satisfies the inequality

\[ |U(\phi)| \leq \varepsilon \quad \text{for} \quad |\text{Im}\phi| \leq \rho - \varepsilon . \]

4.3.1 THEOREM.

Suppose

(i) \( A \) is a real constant \( n \times n \) matrix with eigenvalues 

\[ \lambda_\alpha, \; \alpha = 1, \ldots, n ; \]

(ii) \( P(\phi, \xi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \phi \) and the \( n \times n \) matrix \( \xi \), which is real for real \( \phi \) and \( \xi \), has period \( 2\pi \) in each coordinate \( \phi_\alpha \) and which is holomorphic in \( \phi \) and \( \xi \) and satisfies the inequality

\[ |P(\phi, \xi)| \leq M \]

in the cylinder

\[ |\text{Im}\phi| \leq \rho, \; (\rho > 0) , \]

\[ |\xi| \leq \sigma ; \]

(iii) \( \omega \) is a real \( m \)-vector such that for all integral \( m \)-vectors \( k \neq 0 \)

\[ |\lambda_\alpha - \lambda_\beta + i(k, \omega)| \geq \gamma |k|^{-\tau } , \; \alpha, \beta = 1, 2, \ldots, n , \]

where \( \tau d > m \) and \( \gamma > 0 \).
Then the partial differential equation

\[(4.3.1.1) \quad \left( \frac{\partial U}{\partial \phi}, w \right) + UA - AU = P(\phi, \xi) \]

has a solution which is real for real \( \phi \) and \( \xi \), has period \( 2\pi \) in each coordinate \( \phi \) and which is holomorphic and satisfies the inequality

\[|U(\phi, \xi) - U_0(\xi)| \leq cY^{-d_0} \cdot 1_d M\]

in the cylinder

\[|\text{Im}\phi| \leq \rho - \delta, \quad |\xi| < \sigma\]

where

\[U_0(\xi) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} U(\phi, \xi) d\phi_1 \ldots d\phi_m\]

is the mean value of \( U(\phi, \xi) \), \( c = c(m, \tau, A) > 0 \) and \( d \) is the square of the order of the largest block in the Jordan decomposition of \( A \).

To avoid interrupting the proof of this theorem we will prove a preliminary lemma.

4.3.2 LEMMA.

Under the hypotheses of Theorem 4.3.1, for \( |\text{Im}\phi| \leq \rho - \delta \),

\[(4.3.2.1) \quad |U(\phi, \xi) - U_0(\xi)| \leq cM \frac{\pi^2}{2} \sum_{j=1}^{n^2} e^{-\delta |k|} |\pi_j+i(k, \omega)|^{-d_0},\]

where

\[d_0 = \begin{cases} d, & |\lambda_{ij} - \lambda_j + i(k, \omega)| < 1 \\ l, & |\lambda_{ij} - \lambda_j + i(k, \omega)| \geq 1 \end{cases}\]

and \( \pi_j, j = 1, 2, \ldots, n^2 \), takes each of the values \( \lambda_{ij} - \lambda_j \) in turn.
Proof. Choose the non-singular \( n \times n \) matrix \( C \) so that
\[
A' = C^{-1}AC
\]
is the Jordan canonical form and write
\[
U' = C^{-1}UC
\]
\[
P' = C^{-1}PC.
\]
Let
\[
P'(\varphi, \xi) \sim \sum_k P'_k(\xi)e^{i(k, \varphi)}
\]
be the Fourier expansion of \( P'(\varphi, \xi) \) so that
\[
P'_k(\xi) = \left(\frac{1}{2\pi}\right)^m \int_0^{2\pi} \cdots \int_0^{2\pi} P'(\varphi, \xi)e^{-i(k, \varphi)}d\varphi_1 \cdots d\varphi_m.
\]
By shifting the lines of integration to \( \text{Im} \varphi_\alpha = \pm \rho_\alpha \) where \( \sum \rho_\alpha = \rho \)
we obtain
\[
|P'_k(\xi)| \leq |C^{-1}| |C| |M_\xi e^{-\rho}| |k| , \quad |\xi| < \sigma.
\]
we look for a solution of (4.3.1.1) with Fourier expansion
\[
U(\varphi, \xi) \sim \sum_k U'_k(\xi)e^{i(k, \varphi)}.
\]
Substituting in (4.3.1.1) and equating coefficients we obtain
\[
(4.3.2.2) \quad U'_k(A' + i(k, \omega)) - A'U'_k = P'_k.
\]
It follows from condition (iii) of Theorem 4.3.1 that the eigenvalues \( [A' + i(k, \omega)] \) and \( A' \) are distinct. Therefore the matrix equation (4.3.2.2) has a unique solution. Thus
\[
A' = \text{diag}\left\{ \begin{pmatrix} \lambda_{\mu} I + r_1 \gamma_{l_1} Z_1 \\
\vdots
\end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{\mu} I + r_\mu \gamma_{l_\mu} Z_\mu \end{pmatrix} \right\}
\]
where \( \lambda_{\mu} \) is a multiple eigenvalue of \( A \) of order \( r_\mu \), \( I_{r_\mu} \) is the
$r_\mu \times r_\mu$ unit matrix and $Z_\mu$ is the $r_\mu \times r_\mu$ nilpotent matrix

$$Z_\mu = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$ 

We obtain, for the corresponding $(\mu, \nu)$-th block of (4.3.2.2),

$$(4.3.2.3) \ U_\mu \nu \left[ \lambda_\mu I_\mu + Z_\mu + i(k, \omega) I_\mu \right] - \left[ \lambda_\nu I_\nu + Z_\nu \right] U_\mu \nu = P_\mu \nu$$

where $U_\mu \nu$ is the $(\mu, \nu)$-th block of the matrix $U_\mu'$ and is of order $r_\mu \times r_\nu$ and $P_\mu \nu$ is the corresponding $(\mu, \nu)$-th block of the matrix $P_\mu'$ and is of order $r_\mu \times r_\nu$. We write $u_\alpha, p_\alpha$ for the $\alpha$-th rows of $U_\mu \nu$ and $P_\mu \nu$ respectively and

$$J_p(\lambda) = \lambda I_p + Z_p.$$ 

Then (4.3.2.3) reduces to

$$(4.3.2.4) \ \begin{cases} u_\alpha J_\mu \left[ \lambda_\mu - \lambda_\nu + i(k, \omega) \right] = p_\alpha + u_{\alpha - 1} , \ \alpha = 1, \ldots, r_\nu \\ u_0 = 0 \end{cases}$$

This system of equations has the solution

$$u_\alpha = \sum_{s=1}^{\alpha} \left[ J_{\mu}^{-1} (\lambda_\mu - \lambda_\nu + i(k, \omega)) \right] p_{\alpha - s + 1}. $$

Therefore taking into account

$$J_p^{-1}(\lambda) = \frac{I}{\lambda} - \frac{Z_p}{\lambda^2} + \cdots + (-1)^{p-1} \frac{Z_{p-1}}{\lambda^p} \frac{Z_p}{\lambda^p}$$

it follows that

$$|u_\mu \nu| \leq \left| (\lambda_\mu - \lambda_\nu + i(k, \omega))^{-1} \right| \cdot |p_\mu \nu|$$

where
\[ d_1 = \begin{cases} 1, & |\lambda - \lambda \nu + i(k, \omega)| \geq 1 \\ n, & |\lambda - \lambda \nu + i(k, \omega)| < 1 . \end{cases} \]

Hence
\[
|u'_k| \leq \max_{\mu, \nu} \left\{ |\lambda - \lambda \nu + i(k, \omega)|^{d_0} \right\}_{F_k'}
\leq \left\{ \min_{\mu, \nu} |\lambda - \lambda \nu + i(k, \omega)| \right\}^{-d_0} |F_k'|
\]

It follows that for \( |\text{Im}\phi| \leq \rho - \delta \)
\[
|U(\phi, \xi) - U_0(\xi)| \leq |C|^2 |c^{-1}|^{2M} \sum_{j \neq 0} \left\{ \min_{\mu, \nu} |\lambda - \lambda \nu + i(k, \omega)| \right\}^{-\delta} |k|^{d_0}
\leq \alpha M \sum_{j=1}^{n^2} \sum_{j \neq 0} e^{\delta |k|} |\pi \nu + i(k, \omega)|^{-d_0}
\]
since \( \lambda - \lambda \nu \) can only assume a finite number of values, \( \pi \nu \), \( j = 1, 2, \ldots, n^2 \), say.

4.3.3.

Proof of Theorem 4.3.1. It remains to establish the convergence of the series on the right hand side of (4.3.2.1) and to obtain a sharp estimate for its sum. It follows from Lemma 4.3.2 that we need only estimate
\[
\sum_{|k| \neq 0} e^{\delta |k|} |\pi \nu + i(k, \omega)|^{-d_0}
\]
Let \( K_j \) \((j = -2, -1, 0, 1, \ldots)\) denote the set of all integral vectors \( k \) such that
\[
2^{-j-2} < |\pi + \nu(k, \omega)| \leq 2^{-j-1}, \ (j = 0, 1, \ldots),
2^{-1} < |\pi + \nu(k, \omega)| < 1, \ (j = -1),
1 < |\pi + \nu(k, \omega)|, \ (j = -2).
\]
Every non-zero integral vector belongs to one and only one set $K_j$. Hence

$$S \equiv \sum_{|k| \neq 0} e^{-\delta |k|} |\pi+i(k, w)|^{-d_0}$$

$$= \sum_{k \in K_{-2}} e^{-\delta |k|} |\pi+i(k, w)|^{-1} + \sum_{j=-1}^{\infty} \sum_{k \in K_{j}} e^{-\delta |k|} |\pi+i(k, w)|^{-d}$$

$$\leq 2 \sum_{k \in K_{-2}} e^{-\delta |k|} + \sum_{j=0}^{\infty} 2^{(j+2)d} \sum_{k \in K_{j}} e^{-\delta |k|}.$$  

It follows from the proof of Theorem 4.2.1 that for the first sum

$$(4.3.3.1) \quad S_{-1} = 2 \sum_{k \in K_{-2} \cup K_{-1}} e^{-\delta |k|} \leq c_2 \delta^{-m}.$$  

We estimate the second sum in the following way.

For $k \in K_j$, $j = 0, 1, \ldots ,$

$$\gamma |k|^{-\tau} \leq |\pi+i(k, w)| \leq 2^{-j-1} < 2^{-j}$$

hence

$$|k| \geq a_j = (2^j \gamma)^{1/\tau}.$$  

For distinct $k_1, k_2 \in K_j$

$$\gamma |k_1-k_2|^{-\tau} \leq |\pi+i(k_1, w) - (\pi+i(k_2, w))|$$

$$\leq |\pi+i(k_1, w)| - |\pi+i(k_2, w)|$$

$$\leq 2^{-j}$$

hence

$$|k_1-k_2| \geq a_j.$$  

For any $k \in K_j$ let $W_k$ denote the open cube with centre $k$ and sides parallel to the axes of length $a_j$. These cubes are disjoint.
since \( y \in \omega_k \cap \omega_{k_2} \) implies
\[
a_j \leq |k_1 - k_2| \leq |y - k_1| + |y - k_2| < \frac{1}{2} a_j + \frac{1}{2} a_j = a_j.
\]

Let \( \nu_j \) denote the number of \( k \in K_j \) such that
\[
2a_j \leq |k| < (l+1)a_j, \quad (l = 1, 2, \ldots).
\]
The corresponding cubes \( \omega_k \) lie in the set
\[
(l - \frac{1}{2})a_j < |y| < (l + \frac{3}{2})a_j
\]
which has volume
\[
[(2l+3)^m - (2l-1)^m] a_j^m.
\]
Since each cube has volume \( a_j^m \) this gives
\[
\nu_j \leq (2l+3)^m - (2l-1)^m \leq 4m(2l+3)^{m-1}
\]
by the mean value theorem. Since \( c + \frac{3}{2} \leq 5 \) it follows that
\[
\nu_j \leq 5ml^{m-1}.
\]
Therefore
\[
\sum_{k \in K_j} e^{-\delta |k|} \leq \sum_{l=1}^{\infty} \nu_j e^{-\delta la_j} \leq 5m \sum_{l=1}^{\infty} l^{m-1} e^{-\delta la_j}.
\]
Put \( q = e^{-\delta a_j} \). Since \( m-1 \log q \) takes the maximum value for
\[
\frac{1}{2} t \log \frac{1}{q} = m - 1 \quad \text{we have}
\]
\[
\sum_{l=1}^{\infty} l^{m-1} q^l = \sum_{l=1}^{\infty} (\frac{l^{m-1} q^{l/2}}{q^{l/2}}) q^{l/2} \leq \left( \frac{m \log \frac{1}{q}}{1 - \log \frac{1}{q}} \right)^{m-1} q^{1/2} = \left( \frac{m \log \frac{1}{q}}{1 - \log \frac{1}{q}} \right)^{m-1} (q^{1/2} - 1)^{-1}
\]
and hence
It follows from the definition of $a_j$ that

$$S - S_{-1} = (5m)^m \sum_{j=0}^{\infty} 2^{(j+2)d} \delta a_j \left( e^{\frac{\delta a_j}{2}} - 1 \right)^{-1},$$

where

$$g(t) = \left[ \delta \left( \frac{t}{2} \right)^{1/\delta} \right]^{1-m} \left[ e^{\frac{1}{2} \delta \left( \frac{t}{2} \right)^{1/\delta}} - 1 \right]^{-1}.$$ 

Since $2^{(j+2)d} = \frac{2^{2d}(2(j+1)d-2j^d)}{2^{2d-1}}$ and $g$ is a decreasing function this implies that

$$S - S_{-1} \leq \frac{(5m)^m 2^{2d}}{2^{2d-1}} \sum_{j=0}^{\infty} \frac{2^{(j+1)d-2j^d} g_j(2(j+1)d)}{2^{2d-1}}$$

$$= \frac{(5m)^m 2^{2d}}{2^{2d-1}} \int_1^{\infty} g(t) dt$$

$$\leq \frac{(5m)^m 2^{2d}}{2^{2d-1}} \int_1^{\infty} \delta \left( \frac{t}{2} \right)^{1/\delta} \left[ e^{\frac{1}{2} \delta \left( \frac{t}{2} \right)^{1/\delta}} - 1 \right]^{-1} s^{d-1} ds$$

$$\leq c_1 \int_0^{\infty} v \tau d^{-m} \left[ e^{\frac{1}{2} \tau v - 1} \right]^{-1} dv, \quad \left( \tau = \left[ \frac{2(v)}{Y(\delta)} \right] \right),$$

$$< \infty, \text{ provided } \tau d > m,$$

where $c_1 = \frac{(5m)^m 2^{3d} d^{\tau}}{(2^{d-1})^{1/\tau} \delta^{1/\delta} v^d}.$

Thus combining this estimate with (4.3.3.1), it follows for $\delta < 1$, that
This proves that $U(\phi)$ is holomorphic and establishes the estimate for $U(\phi) - U_0$. The fact that $U(\phi)$ is real for real $\phi$ and $\xi$ follows from the corresponding property for $P(\phi, \xi)$.

4.3.4.

Consider now the system of equations

$$
\begin{align*}
x' &= Ax + [P(\phi, \xi) + \xi]x \\
\phi' &= \omega
\end{align*}
$$

(4.3.4.1)

where $A$, $P(\phi, \xi)$ and $\omega$ satisfy the hypotheses of Theorem 4.3.1. Let

$$
\begin{align*}
P_0(\xi) &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P(\phi, \xi) \, d\phi_1 \cdots \, d\phi_m
\end{align*}
$$

be the mean value of $P(\phi, \xi)$. As in Theorem 4.3.1 we can find a matrix $U(\phi, \xi)$ such that

$$
\left( \frac{\partial U}{\partial \phi}, \omega \right) + UA - AU = P(\phi, \xi) - P_0(\xi).
$$

Its mean value $U_0(\xi)$ is zero. Thus we obtain

$$
|U(\phi, \xi)| \leq \sigma^{-d_0 - d}M
$$

for $|\text{Im}\phi| \leq \rho - \delta$ and $|\xi| \leq \sigma$.

The change of variables

$$
x = [I + U(\phi, \xi)]\tilde{x}
$$

transforms (4.3.4.1) into a system

$$
\begin{align*}
\tilde{x}' &= A\tilde{x} + [P_1(\phi, \xi) + \xi]\tilde{x} \\
\phi' &= \omega
\end{align*}
$$

(4.3.4.2)

of the same form, where

$$
P_1(\phi, \xi) = (I + U)^{-1}(PU - UP_0 + \xi U - U\xi).
$$

The proof carries over to vector or matrix functions.
We will show that (4.3.4.4) can be inverted to obtain $\xi = \tilde{\xi}(\xi)$ and consequently

$$P(\phi, \tilde{\xi}) = P_1(\phi, \xi(\tilde{\xi}))$$

is holomorphic in $\tilde{\xi}$. Differentiating (4.3.4.4)

$$\frac{d\tilde{\xi}}{d\xi} = 1 + \frac{\partial P_0}{\partial \xi}, \text{ for } |\xi| \leq \sigma$$

and applying Cauchy's estimate

$$\left| \frac{\partial P_0}{\partial \xi} \right| \leq \frac{Mn^2}{\sigma^2} \text{ for } |\xi| \leq \frac{\sigma}{2}.$$ 

Then, provided that

(4.3.4.5) \hspace{1cm} M \leq \frac{\sigma}{4n^2}

we find that

$$\left| \frac{d\tilde{\xi}}{d\xi} - 1 \right| \leq \frac{1}{2}. $$

Therefore the mapping $\xi \rightarrow \tilde{\xi}$ is one-to-one for $|\xi| \leq \frac{\sigma}{2}$. By Theorem 116, Littlewood [15] the values $\tilde{\xi}(\xi)$ for $|\xi| < \frac{\sigma}{2}$ fill a simply connected domain $\Delta$ and there exists an inverse function $\xi(\tilde{\xi})$, which is holomorphic and one-to-one and maps $\Delta$ onto the disc $|\xi| < \frac{\sigma}{2}$. The circle $|\xi| = \frac{\sigma}{2}$ is mapped onto the Jordan curve $\Gamma$, the boundary of $\Delta$. The points $\tilde{\xi}$ of $\Gamma$ satisfy

$$|\tilde{\xi}| \geq \frac{\sigma}{2} - M = R,$$

say.

Also the point $\tilde{\xi}(0)$ lies in the disc $D : |\tilde{\xi}| \leq M$ and also in $\Delta$. If the whole disc $D$ did not lie in $\Delta$ then the boundary $\Gamma$ of $\Delta$ would contain a point of $D$. At this point

$\dagger$ The proof carries over to vector or matrix functions.
\[ \frac{\sigma}{2} - M \leq |\tilde{\xi}| \leq M, \]
which is a contradiction of \( \sigma > 4M \).

Therefore \( \xi(\tilde{\xi}) \) is holomorphic in

\[ |\tilde{\xi}| \leq M, \text{ say,} \]

and in this region

\[ |\xi| \leq 2M \]

and

\[ |\xi - \tilde{\xi}| \leq M. \]

Thus, if \( |U| \leq \frac{1}{2} \) and \( |\tilde{\xi}| \leq M \),

\[ |P| \leq |(I+U)^{-1}| \cdot |U| \cdot (|P| + |\xi| + |\tilde{\xi}|) \leq 8M|U|. \]

Let \( \varepsilon \) be any number such that

\[ 0 < \varepsilon < \min(1, \rho). \]

Suppose \( \sigma < 1 \) and choose \( \theta \) \( (0 < \theta < \sigma(2\pi)^{-4}) \) so small that if

\[ \theta_j = \theta \left( \frac{3}{2} \right)^j \text{ then} \]

\[ \sum_{j=0}^{\infty} \theta_j^p < \frac{8\rho}{p}, \text{ where } p = \sum_{j=0}^{\infty} \left( 1 + \frac{\theta_j}{\theta} \right) \]

\[ \sum_{j=0}^{\infty} \theta_j^{1/\tau d} < \frac{\varepsilon}{\alpha}, \text{ where } \alpha = \left( \frac{8\rho}{\chi^2} \right)^{1/\tau d} \]

\[ \sum_{j=0}^{\infty} \theta_j^2 < \frac{\varepsilon}{2q}, \text{ where } q = \left( \frac{3}{2} \right)^2 \sum_{j=0}^{\infty} \left( 1 + \theta_j \right) \cdot (1 + \theta_j) \cdot \theta_j. \]

Put \( \rho_j = \rho - a \left( \theta_0^{1/\tau d} + \ldots + \theta_{j-1}^{1/\tau d} \right) > 0 \), and suppose that the system (4.3.4.1) satisfies the above conditions with \( M, \rho, \sigma \) replaced by \( \theta_j^2, \rho_j, \theta_j^2 \). We will show that the transformed system satisfies the same conditions with \( M, \rho, \sigma \) replaced by \( \theta_{j+1}^2, \rho_{j+1}, \theta_j^2 \). Thus we
take \( \delta = a \theta_j^{1/\tau_d} \). Put \( \tilde{U}(\varphi, \xi_{j+1}) = U[\varphi, \xi_j(\xi_{j+1})] \). Then, by the definition of \( a \), we have for

\[ |\text{Im} \varphi| \leq \rho_j + 1 \text{, } |\xi_{j+1}| \leq \theta_j^2, \]

\[ |\tilde{U}_{j+1}| \leq \frac{1}{8} \theta_j^{-1} \theta_j^2 = \frac{1}{8} \theta_j. \]

Thus \( |\tilde{U}_{j+1}| \leq \frac{1}{2} \) and hence

\[ |\rho_{j+1}| \leq 8 \theta_j^2 \theta_j = \theta_j^3 = \theta_j^{2+1}. \]

Moreover in this region

\[ |\xi_j| \leq 2 \theta_j^2 \leq \frac{\theta_j^{j-1}}{2} \]

and

\[ |\xi_{j+1} - \xi_j| \leq \theta_j^2. \]

It follows that if in the original system (4.3.4.1) \( M \leq \theta_j^2 \) the above transformation can be repeated indefinitely. We will now prove the existence of a limit system. We have

\[ x_j' = Ax_j + [P_j(\varphi, \xi_j) + \xi_j]x_j \]

\[ \varphi' = \omega. \]

Since \( |P_j(\varphi, \xi_j)| \leq \theta_j^2 \) for \( |\text{Im} \varphi| \leq \rho_j \), \( |\xi_j| \leq \theta_j^{j-1} \) it is obvious that as \( j \to \infty \), \( P_j(\varphi, \xi_j) \to 0 \) uniformly for

\[ |\text{Im} \varphi| < \rho_\infty = \rho - \alpha \sum_{j=0}^{\infty} \theta_j^{1/\tau_d}. \]

It remains to find the value of \( \xi \) which is transformed into the origin and to examine the corresponding transformation.

We have

(4.3.4.6) \( \xi = \xi(\xi_1(\ldots (\xi_j)\ldots)) = f_j(\xi_j) \) , say,
(4.3.4.7) \( x = [I + U_1(\varphi, \xi_1)] \cdots [I + U_j(\varphi, \xi_{j-1}(\xi_j))] x_j \)
\[ = T_j x_j \], say,

and we can rewrite the transformation \( T_j \) as

\[ T_j(\varphi, \xi_j) = \prod_{s=1}^{j-1} [I + U_s(\varphi, \xi_{s-1}(\xi_s(\cdots (\xi_j \cdots )))]) \]

(with the convention \( \xi = \xi_0 \)).

Rewriting equation (4.3.4.4) as

(4.3.4.8) \( \xi_j = \xi_{j-1}(\xi_j) + \overline{F}_{j-1}(\xi_{j-1}(\xi_j)) \)

and differentiating

\[ I = \frac{\partial \xi_{j-1}}{\partial \xi_j} + \frac{\partial \overline{F}_{j-1}}{\partial \xi_{j-1}} \frac{\partial \xi_{j-1}}{\partial \xi_j} . \]

That is

\[ I = \left| \frac{\partial \xi_{j-1}}{\partial \xi_j} + \frac{\partial \overline{F}_{j-1}}{\partial \xi_{j-1}} \frac{\partial \xi_{j-1}}{\partial \xi_j} \right| \geq \left| \frac{\partial \xi_{j-1}}{\partial \xi_j} \right| \left( 1 - \frac{\partial \overline{F}_{j-1}}{\partial \xi_{j-1}} \right) , \]

since, applying Cauchy's estimate

\[ \left| \frac{\partial \overline{F}_{j-1}}{\partial \xi_{j-1}} \right| \leq \frac{2n^2 \theta^2_{j-1}}{\theta^2_{j-2}} = 2n^2 \theta_{j-2} < \frac{1}{2} \theta_{j-3} < \frac{1}{2} \]

for \( |\xi_j| < \theta^2_{j-1} \). Therefore

\[ \left| \frac{\partial \xi_{j-1}}{\partial \xi_j} \right| \leq 1 + \theta_{j-3} \]

and

\[ \frac{\partial f_j}{\partial \xi_j} = \left| \frac{\partial \xi_1}{\partial \xi_1} \right| \cdots \left| \frac{\partial \xi_{j-1}}{\partial \xi_j} \right| \]
\[ \leq \left( \frac{3}{2} \right)^2 \prod_{s=0}^{j-3} (1 + \theta_s) \leq \left( \frac{3}{2} \right)^2 \prod_{s=0}^{\infty} (1 + \theta_s) \]
\[ = q . \]
Therefore by the mean value theorem

\[ |f_{j+1}(0) - f_j(0)| = |f_j(\xi_{j+1}(0)) - f_j(0)| \leq q|\xi_{j+1}(0)| \]

That is

\[ |f_{j+k}(0) - f_j(0)| \leq 2q \sum_{s=0}^{k-1} \theta_j^2 \]

and \( f_j(0) \) converges to \( \xi^0 \), say, and

\[ |f_j(0) - \xi^0| \leq \varepsilon. \]

In order to examine the convergence of \( T_j(\varphi, \xi_j) \) we need to consider the functions

\[ f_{j,k}(\xi_k) = f_{j}(\xi_{j+1}(\ldots (\xi_k) \ldots)) \]

which are holomorphic and satisfy the inequality \( |f_{j,k}(\xi_k)| \leq \theta_j^2 \) for \( |\xi_k| \leq \theta_{j-1}^2 \). Rewriting (4.3.4.8) we obtain in this region

\[ f_{j,k}(\xi_k) = f_{j-1,k}(\xi_k) + \overline{f}_{j-1}(f_{j-1}(\xi_k)) \]

Hence setting \( \xi_k = 0 \) and going over to the limit we obtain

\[ \eta_0 = \lim_{k \to \infty} f_k(0) = \xi^0 + P_0(\xi^0) \]

\[ \eta_1 = \lim_{k \to \infty} f_{2k}(0) = \eta_0 + \overline{P}_1(\eta_0) \]

\[ \vdots \]

\[ \eta_j = \lim_{k \to \infty} f_{j+1,k}(0) = \eta_{j-1} + \overline{P}_j(\eta_{j-1}) \]

Moreover

\[ |\eta_j| \leq 2\theta_j^2 \leq \tfrac{\theta_{j-1}^2}{2}. \]

Put
\[ T_j(\varphi) = \prod_{s=1}^{j} \left[ I + U_s(\varphi, \eta_{j-1}) \right] = T_j(\varphi, \eta_j). \]

Suppose that for some \( j \)
\[
|T_j(\varphi)| \leq \prod_{s=0}^{j-1} \left( 1 + \frac{1}{8} \theta_s \right)
\]
in the strip \(|\text{Im}\varphi| < \rho_s\) with the convention that \( T_0 = I \); this is certainly true for \( j = 0 \). Since
\[
T_{j+1}(\varphi) = T_j(\varphi) \left[ I + U_{j+1}(\varphi, \eta_j) \right]
\]
it follows that
\[
|T_{j+1}(\varphi)| \leq |T_j(\varphi)| \cdot |I + U_{j+1}(\varphi, \eta_j)| \leq \prod_{s=0}^{j} \left( 1 + \frac{1}{8} \theta_s \right).
\]
Therefore for \(|\text{Im}\varphi| < \rho_{\infty}\) we have
\[
|T_j(\varphi)| \leq p \text{ for all } j.
\]
Hence
\[
|T_{j+1}(\varphi) - T_j(\varphi)| \leq |T_j(\varphi)| \cdot |U_{j+1}(\varphi, \eta_j)| \leq \frac{1}{8} p \theta_j.
\]
Thus the series \( \sum \{T_{j+1}(\varphi) - T_j(\varphi)\} \) converges uniformly for
\(|\text{Im}\varphi| < \rho_{\infty}\). The sequence \( \{T_j(\varphi)\} \) converges uniformly in the same strip. Its limit \( T(\varphi) \) is holomorphic, is real for real \( \varphi \) and has period \( 2\pi \) in each coordinate. Moreover the partial derivatives
\[
\frac{\partial T_j}{\partial \varphi_a} \text{ converge uniformly to } \frac{\partial T}{\partial \varphi_a}.
\]
From
\[
T_{j+1} - I = T_j - I + T_j U_{j+1}
\]
we obtain
\[
|T - I| \leq \sum_{j=0}^{\infty} |T_j| |U_{j+1}| \leq \frac{1}{8} p \sum_{j=0}^{\infty} \theta_j < \varepsilon.
\]
Hence the matrix $T(\phi)$ is invertible and

$$x_j \to y = T^{-1}(\phi)x \text{ as } j \to \infty.$$  

Differentiating the relation $x = T_j x_j$ we obtain

$$x' = \left( \frac{\partial T_j}{\partial \phi}, \omega \right) x_j + T_j x'_j.$$  

Hence, letting $j \to \infty$

$$x' = \left( \frac{\partial T_j}{\partial \phi}, \omega \right) y + Ta y.$$  

Differentiating the relation $x = T y$ we obtain

$$x' = \left( \frac{\partial T}{\partial \phi}, \omega \right) y + Ty'.$$  

Comparison of these two expressions for $x'$ gives $y' = Ay$. Thus the change of variables $x = T(\phi)y$ transforms the system (4.3.4.1) into the autonomous system

$$y' = Ay$$

$$\phi' = \omega.$$  

Summarizing the above we have proved:

**THEOREM 4.3.4.**

Suppose

(i) $A$ is a real, constant $n \times n$ matrix with eigenvalues $\lambda_{\alpha}$, $\alpha = 1, \ldots, n$, and $d$ is the square of the order of the largest block in the Jordan decomposition of $A$;  

(ii) $P(\phi, \xi)$ is an $n \times n$ matrix function of the $m$-vector $\phi$ and the $n \times n$ matrix $\xi$, which is real for real $\phi$ and $\xi$, has period $2\pi$ in each coordinate $\phi_{\alpha}$ and which is holomorphic in $\phi$ and $\xi$ and satisfies the inequality

$$|P(\phi, \xi)| \leq M.$$
in the cylinder

\[ |\text{Im} \varphi| \leq \rho \ (\rho > 0) \]
\[ |\xi| \leq \sigma < 1 ; \]

(iii) \( \omega \) is a real \( m \)-vector such that for all integral \( m \)-vectors \( k \neq 0 \)

\[ |\lambda^\alpha - \lambda^\beta + i(k, \omega)| \geq \gamma|k|^{-\tau}, \ \alpha, \beta = 1, 2, \ldots, n, \]

where \( \tau d > m \) and \( \gamma > 0 \).

Then for any \( \epsilon \) such that \( 0 < \epsilon < \min(1, \rho) \) there exists an (explicitly computable) absolute constant \( M_0 = M_0(m, n, \tau, \gamma, \epsilon, \sigma, A) \)

with the property that if \( M \leq M_0 \) there exists a constant \( n \times n \)

matrix \( \xi_0 \) with \( |\xi_0| \leq \sigma \) such that the quasi-periodic linear system

\[ x' = [A + \xi_0 + P(\omega t, \xi_0)]x \]

has a fundamental matrix of the form

\[ X(t) = [I + U(\omega t)]e^{At}, \]

where \( U(\varphi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \varphi \) which is real for real \( \varphi \), has period \( 2\pi \) in each coordinate \( \varphi_\alpha \) and which is holomorphic and satisfies the inequality

\[ |U(\varphi)| \leq \epsilon \]

in the strip

\[ |\text{Im} \varphi| \leq \rho - \epsilon . \]

4.3.5.

The condition

\[ |(k, \omega)| > \gamma|k|^{-\tau} \]

of §4.2.1 is a necessary condition for
\[ |\lambda_a - \lambda_B + i(k, \omega)| > \gamma |k|^{-\tau} \]

and in the particular case when the eigenvalues of \( A \) are real then it is also sufficient. When the eigenvalues of \( A \) are real and distinct then \( A \) may be transformed into diagonal form. However Theorem 4.3.4 applies, with \( d = 1 \), to any diagonal matrix \( A \).
CHAPTER V
REDUCIBILITY AND QUASI-PERIODIC SOLUTIONS FOR
SMOOTH QUASI-PERIODIC LINEAR SYSTEMS

5.1.1.

In this chapter we will again be concerned with reducing the equations

\[ x' = Ax + P(\phi)x \]
\[ \phi' = \omega \]

to a constant system. Now we assume that \( P(\phi) \) possesses only a certain number of derivatives. We will use a technique suggested by Moser [370], [375] where instead of truncating the Fourier series for \( P(\phi) \) we approximate \( P(\phi) \) by a sequence of holomorphic functions. This method has been refined considerably by Rüssmann [380] and we will use some of his techniques.

5.1.2.

A (matrix or vector-valued) function \( P(x) \) is said to be of class \( P^l_{MK} \) if it satisfies the following conditions:

(i) \( P \) is defined for all \( x \in \mathbb{R}^m \), has period \( 2\pi \) in each \( x_i \), \( i = 1, \ldots, m \) and satisfies the inequality

\[ |P(x)| \leq M, \quad x \in \mathbb{R}^m. \]

(ii) \( P \) is \([l]\)-times differentiable with respect to \( x_i \),

\[ i = 1, \ldots, m, \] and satisfies the inequalities
In this case we have the following theorem based on Rüssmann's Theorem 3. The proof will follow Rüssmann directly, since his article is not readily accessible.

5.1.3. THEOREM.

Let \( f \in \mathcal{P}_M^L \) and \( \{ \rho^L \} \) be a monotone sequence of positive real numbers decreasing to zero, with \( \rho^L_0 \leq 1 \). Then there exists a sequence of functions \( f_k \) in the \( m \) complex variables \( x_1, \ldots, x_m \) with the properties

(i) \( f_k \) is holomorphic in the region \( |\text{Im} x| < \rho^L_0 \), is real valued for real \( x \) and has period \( 2\pi \) in \( x_i \),

\( i = 1, \ldots, m \).

(ii) \( f_k \) satisfies the following inequalities

\[
(5.1.3.1) \quad |f_k(x)| \leq 3^m M, \quad |\text{Im} x| < \rho^L_0, \quad k = 0, 1, \ldots
\]

\[
(5.1.3.2) \quad |f_k(x) - f_{k-1}(x)| \leq 3^m M \cdot m \rho^L_{k-1}, \quad |\text{Im} x| < \rho^L_0, \quad k = 1, 2, \ldots
\]

\[
(5.1.3.3) \quad |f(x) - f_k(x)| \leq 3^m + 2 \cdot m \rho^L_k, \quad x \in \mathbb{R}^m, \quad k = 0, 1, \ldots
\]

\[
(5.1.3.4) \quad |f^\theta f_k(x)| \leq \left( 2\rho^L_0 \right)^\theta 3^m M, \quad x \in \mathbb{R}^m, \quad k = 0, 1, \ldots
\]
5.1.4.

If \( f \) is any continuous function with period \( 2\pi \) in each of the \( m \) real variables \( x_1, \ldots, x_m \), corresponding to any integer \( n_j \), we can form the \( n_j \)-th Fejér sum with respect to \( x_j \),

\[
\sigma_{n_j} f(x) = \frac{1}{\pi n_j} \int_{-\pi/2}^{\pi/2} f(x_1, \ldots, x_{j-1}, x_j + 2t, x_{j+1}, \ldots, x_m) \left( \frac{\sin n_j t}{n_j} \right)^2 dt.
\]

Now \( \sigma_{n_j} f(x) \) is a trigonometric polynomial of \( (n_j - 1) \)-th order whose coefficients are continuous functions with period \( 2\pi \) in each of \( x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m \). Since

\[
\frac{1}{\pi n_j} \int_{-\pi/2}^{\pi/2} \left( \frac{\sin n_j t}{n_j} \right)^2 dt = 1
\]

it follows that

\[
(5.1.4.1) \quad |\sigma_{n_j} f(x)| \leq \sup_{\xi \in \mathbb{R}^m} |f(\xi)| = M, \quad (x \in \mathbb{R}^m, j = 1, \ldots, m).
\]

We now form the \( n_j \)-th de la Vallée-Poussin sum with respect to \( x_j \),

\[
\tau_{n_j} f(x) = 2\sigma_{2n_j} f(x) - \sigma_{n_j} f(x).
\]

It is a trigonometric polynomial in \( x_j \) with order no greater than \( (2n_j - 1) \). Thus

\[
(5.1.4.3) \quad |\tau_{n_j} f(x)| \leq 3M, \quad (x \in \mathbb{R}^m, j = 1, \ldots, m).
\]

We define

\[
\tau_{n_j \cdot n_{j+1} \cdot \ldots \cdot n_m} f(x) = \tau_{n_j} \left( \tau_{n_{j+1}} \left( \ldots \left( \tau_{n_m} f(x) \right) \ldots \right) \right), \quad j = 1, 2, \ldots, n-1.
\]

Therefore it follows from (5.1.4.3) that

\[
(5.1.4.4) \quad |\tau_{n_1 \cdot \ldots \cdot n_m} f(x)| \leq 3^m M, \quad (x \in \mathbb{R}^m).
\]
5.1.5. LEMMA.

If \( f \in P_{MK}^l \) then for any positive integers \( n_1, n_2, \ldots, n_m \)

\[
(5.1.5.1) \quad |f(x) - \tau_{n_1} \cdots \tau_{n_m} f(x)| \leq 3^{m+2} K \sum_{j=1}^{m} n_j^{-l}, \quad (x \in \mathbb{R}^m).
\]

Proof. From a theorem of de la Vallée-Poussin we have

\[
(5.1.5.2) \quad |f(x_1) - \tau_{n_1} f(x_1)| \leq 4 E_{n_1}^l (f), \quad (x_1 \in \mathbb{R})
\]

where \( E_{n_1}^l (f) \) is the best approximation of \( f \) by a trigonometric polynomial of order no greater than \( n_1 \). On the other hand it follows from Jackson's Theorem (see Achieser [1]) that

\[
E_{n_1}^l (f) \leq c n_1^{-l}
\]

where \( f \in P_{MK}^l \) is a function of one variable and \( c = 3 \). It follows then that

\[
(5.1.5.3) \quad |f(x_1) - \tau_{n_1} f(x_1)| \leq 3^{m-1} K n_1^{-l};
\]

thus the lemma is true for \( x \in \mathbb{R}^1 \).

We suppose now that the lemma is true for \( x \in \mathbb{R}^{m-1} \) where \( m-1 \geq 1 \). Because \( f \in P_{MK}^l \), a function defined on \( \mathbb{R}^m \), is for any fixed value of \( x_1 \) a function which maps \( (x_2, \ldots, x_m) \mapsto f(x) \) then it follows from the induction hypothesis that

\[
(5.1.5.4) \quad |f(x) - \tau_{n_2} \cdots \tau_{n_m} f(x)| \leq 3^{m+1} K \sum_{j=1}^{m} n_j^{-l}.
\]

On the other hand for fixed \( (x_2, \ldots, x_m) \), \( f(x) \) is a function which maps \( x_1 \mapsto f(x) \) and therefore (5.1.5.3) holds. Since the operator
Combining (5.1.5.5) and (5.1.5.3) we obtain (5.1.5.1) and the induction is proved.

5.1.6.

Proof of Theorem 5.1.3. For the given sequence \( p_0, p_1, \ldots \) we define integers \( m_k \) by the inequalities

\[
(5.1.6.1) \quad m_k - 1 \leq \rho_k^{-1/2} < m_k, \quad (k = 0, 1, \ldots).
\]

Since \( \rho_k \leq \rho_0 \leq 1 \) it follows that

\[
(5.1.6.2) \quad m_k \geq 2.
\]

Define

\[
f_k(x) = \tau_{m_k} \cdots \tau_{m_k} f(x), \quad (k = 0, 1, \ldots).
\]

Then \( f_k(x) \) is a trigonometric polynomial in the \( m \) variables \( x_1, \ldots, x_m \) of order no greater than \( (2m - 1) \) in each variable.

Now it follows from (5.1.4.4) that

\[
(5.1.6.3) \quad |f_k(x)| \leq 3^m M, \quad (x \in R^m, \ k = 0, 1, \ldots)
\]

and from (5.1.5.1) that

\[
(5.1.6.4) \quad |f(x) - f_k(x)| \leq 3^{m+2} m \rho_k, \quad (x \in R^m, \ k = 0, 1, \ldots).
\]

Since \( \rho_{k-1} \leq \rho_k \) it follows that

\[
|f_k(x) - f_{k-1}(x)| \leq 3^{m+3} m \rho_{k-1}, \quad (x \in R^m, \ k = 0, 1, \ldots).
\]
Now we apply Bernstein's inequality to the trigonometric polynomials $f_k$ and $f_k - f_{k-1}$. It follows from (5.1.6.3) that

$$|D^s f_k(x)| \leq 3^m M \prod_{j=1}^{m} (2m_j - 1)^{s_j}$$

where

$$D^s = \left( \frac{\partial}{\partial x_1} \right)^{s_1} \cdots \left( \frac{\partial}{\partial x_m} \right)^{s_m}$$

for any non-negative integers $s_1, \ldots, s_m$ and from (5.1.6.2)

(5.1.6.5) $|D^s f_k(x)| \leq 3^{m+s_1+\ldots+s_m} M \prod_{j=1}^{m} (m_j - 1)^{s_j}$.

Similarly

(5.1.6.6) $|D^s f_k(x) - D^s f_{k-1}(x)| \leq 3^{m+s_1+\ldots+s_m} M \prod_{j=1}^{m} (m_j - 1)^{s_j}$.

We now use Taylor's Theorem to extend these estimates for real $x$ to the strip

$$|\text{Im} x| < (m_k - 1)^{-1},$$

that is, by (5.1.6.1), to the strip

$$|\text{Im} x| < \varphi_k^{1/2}.$$

We obtain

$$|f_k(x)| = |f_k(\text{Re} x + i \text{Im} x)|$$

$$\leq \sum_{|s| \leq 0} |D^s f_k(\text{Re} x)| \prod_{j=1}^{m} (m_j - 1)^{-s_j}.$$

Substituting (5.1.6.5) we obtain

$$|f_k(x)| \leq 3^m . M . e^{3m} \leq 3^{4m} . M.$$

Similarly from the Taylor series for $(f_k - f_{k-1})(\text{Re} x + i \text{Im} x)$ we obtain
Thus Theorem 5.1.3 is proved.

We now prove the converse theorem.

5.1.7. **THEOREM.**

Let \( \{p_k\} \) be a monotone sequence of positive real numbers decreasing towards zero such that the series

\[
\sum_{k=1}^{\infty} p_{k-1}^{-q}
\]

converges for at least one \( q > 0 \). Let \( q_0 \) be the least upper bound of such \( q \). For each \( k = 0, 1, \ldots \), let there be given a holomorphic function, \( f_k(x) \) which is defined for

\[
|\text{Im}x| < \frac{1}{p_k^{\frac{1}{q_0}}},
\]

is real for real \( x \) and has period \( 2\pi \) in \( x_1, \ldots, x_m \) such that

\[
|f_0(x)| \leq L_0, \quad |\text{Im}x| \leq \frac{1}{p_0^{\frac{1}{q_0}}},
\]

(5.1.7.1)

\[
|f_k(x) - f_{k-1}(x)| \leq L_1 p_{k-1}^{\frac{1}{q_0}}, \quad |\text{Im}x| \leq \frac{1}{p_k^{\frac{1}{q_0}}},
\]

for positive constants, \( L_0 \) and \( L_1 \).

Then

\[
f(x) = \lim_{k \to \infty} f_k(x)
\]

is a real function with period \( 2\pi \) in \( x_1, \ldots, x_m \) which possesses continuous partial derivatives \( \partial^s f(x) \) of all orders \( s = (s_1, \ldots, s_m) \) with
where $s! = s_1! \cdots s_m!$ and

$$L(q) = L_0 \rho^{-q} + L_1 \sum_{k=1}^{\infty} \rho_{k-1} \rho^{-q}_k$$

when

$$0 < \frac{\alpha}{h} < q_0 - \frac{|s|}{h} < \frac{1}{h}.$$ 

$
\partial^m f(x)$ is Hölder continuous in each variable $x_j$ with exponent $\alpha$ and Hölder constant

$$s! \left( s_j+2 \right) L \left( \frac{\alpha+|s|}{h} \right).$$

Proof. It follows from (5.1.7.1) and Cauchy's estimates that

$$\left| \partial^0 f_0(x) \right| \leq s! L_0 \rho_0 \frac{|s|}{h}, \quad (x \in R^m)$$

$$\left| \partial^0 f_k(x) - \partial^0 f_{k-1}(x) \right| \leq s! L_1 \rho_{k-1} \rho_k \frac{|s|}{h}, \quad k = 1, 2, \ldots.$$ 

For arbitrary integral $m$-vectors $s$. Then from (5.1.7.2) it follows that

$$\sum_{k=1}^{\infty} \left| \partial^m f_k(x) - \partial^m f_{k-1}(x) \right|$$

converges uniformly for $x \in R^m$. Therefore $\partial^m f(x)$ exists and is continuous for all $s$ satisfying (5.1.7.2) and we have

$$\partial^m f(x) = \lim_{k \to \infty} \partial^m f_k(x).$$
Similarly, the estimate (5.1.7.3) follows from (5.1.7.7) and (5.1.7.4).

For the second half of the theorem put

\[ y = \{x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_m\} \]

Then we apply the mean value theorem to (5.1.7.7) to obtain

\[ |D^\alpha f_0(x) - D^\alpha f_0(y)| \leq s! (s_j + 1) L_0 \rho_0^{\frac{1}{n}} (1 + |s|) |x_j - y_j| \]

and

\[ |D^\alpha (f_{j-1}^j f_{k-1}^j)(x) - D^\alpha (f_{j-1}^j f_{k-1}^j)(y)| \leq s! (s_j + 1) L_1 \rho_{k-1}\rho_k^{\frac{1}{n}} (1 + |s|) |x_j - y_j| \]

and therefore for

\[ |x_j - y_j| \leq \frac{1}{n} \rho_0^\alpha, \quad |x_j - y_j| \leq \frac{1}{n} \rho_k^\alpha \]

respectively, we obtain

\[
\begin{align*}
|D^\alpha f_0(x) - D^\alpha f_0(y)| &
\leq s! (s_j + 2) L_0 \rho_0^{\frac{1}{n}} (\alpha + |s|) |x_j - y_j|^\alpha \\
|D^\alpha (f_{j-1}^j f_{k-1}^j)(x) - D^\alpha (f_{j-1}^j f_{k-1}^j)(y)| &
\leq s! (s_j + 2) L_1 \rho_{k-1}\rho_k^{\frac{1}{n}} (\alpha + |s|) |x_j - y_j|^\alpha, \quad x \in \mathbb{R}^m, \ y_j \in \mathbb{R}
\end{align*}
\]

(5.1.7.9)

When

\[ |x_j - y_j| > \frac{1}{n} \rho_0^\alpha, \quad |x_j - y_j| > \frac{1}{n} \rho_k^\alpha \]

respectively, the estimates (5.1.7.9) follow directly from (5.1.7.7) and the triangle inequality.

Thus (5.1.7.9) is true for \( k = 1, 2, \ldots \) and together with (5.1.7.5) and (5.1.7.7) gives the Hölder constant (5.1.7.6) for \( D^\alpha f(x) \).
Theorem 5.1.3 seems to draw no distinction between functions belonging to $PM_{MK}$ which possess mixed derivatives and those which do not. However, in view of Hartog's Theorem (Bochner and Martin [3]) which guarantees that a function which is holomorphic in each variable $x_1, \ldots, x_m$ separately, is holomorphic, the result is not surprising.

5.2.1.

In Theorem 5.1.3 we set

$$\rho_k = 2^{-k}.$$  

Then the estimates in the statement of the theorem become

$$|f_k'(x)| \leq 3^{\mu_m} M, \quad |\text{Im}x| < \frac{k}{l},$$

$$|f_k(x) - f_{k-1}(x)| \leq 2.3^{\mu_{m+2}m} 2^{-k}, \quad |\text{Im}x| < \frac{k}{l},$$

$$|f(x) - f_k'(x)| \leq 3^{m+2m^2} 2^{-k}, \quad x \in \mathbb{R}^m,$$

$$|D^2f_k(x)| \leq 2^3 3^{\mu_m} 2^{k/2}, \quad x \in \mathbb{R}^m.$$  

For integral vectors $r, l$ we will use the abbreviation

$$\sum_{r=0}^{l} \binom{l}{r} \left( \frac{\ell}{r} \right)$$

for

$$\sum_{r_1=0}^{l_1} \cdots \sum_{r_m=0}^{l_m} \left( \frac{l_1}{r_1} \right) \cdots \left( \frac{l_m}{r_m} \right).$$

For convenience we recall in this section, that if

$$|D^kg| \leq K$$

and
then
\[ |g| \leq M , \]

has the solution
\[ D^\alpha g \leq \frac{\pi^\alpha}{2^\alpha n!} k^k , \quad \alpha = 0, 1, \ldots, k . \]

5.2.2.

Consider now the system of ordinary differential equations

\[ x' = Ax + P(\phi)x \]
\[ \phi' = \omega \]

(5.2.2.1)

where \( A, \omega \) satisfy the hypotheses of Theorem 4.2.1 and \( P(\phi) \in P^2_{MK} \).

From Theorem 5.1.3 there is a holomorphic function \( f_k(\phi) \) satisfying the estimates (5.2.1.1). Therefore Theorem 4.2.1 guarantees that the partial differential equation

\[ \{ \frac{\partial U}{\partial \phi}, \omega \} - AU + UA = f_k - \bar{D} , \]

(5.2.2.2)

where

\[ \bar{D} = \text{diag}(P_{11}, \ldots, P_{nn}) \]

with \( P_{\alpha\beta} \), the \((\alpha, \beta)\)-th entry of the matrix

transforms (5.2.2.1) into a unique solution \( U(\phi) \).

This solution \( |U(\phi)| \) is holomorphic in the strip

\[ |\text{Im} \phi| < \frac{\pi}{2} - \delta , \]

(5.2.2.3)

and satisfies there

\[ |U(\phi) - U_0| \leq e^{\frac{t}{\delta}} |f_k| . \]

From the construction of \( f_k \) in Theorem 5.1.3
then the equation

\[ U_0 \tilde{A} - A U_0 = \overline{\tilde{P}} - \overline{D} \]

has the solution

\[ u_{\alpha \beta}^0 = \begin{cases} \frac{\tilde{P}_{\alpha \beta}}{\lambda^\beta - \lambda^\alpha}, & \text{for } \alpha \neq \beta, \\ 0, & \text{for } \alpha = \beta. \end{cases} \]

Thus

\[ |U_0| \leq \frac{1}{\mu} |\tilde{P}|. \]

Take \( \delta = 2^\frac{k}{1} - 1 \). Then

\[ |U(\varphi)| \leq \left( \mu^{-1} + \alpha y^{-1}, \mu^{-1} 2^{-T} \right) \cdot M, \]

for \( |\text{Im} \varphi| < 2^\frac{k}{1} - 1 \).

The change of variables

\[ x = [I + U(\varphi)] \tilde{x} \]

transforms (5.2.2.1) into a system

\[ \tilde{x}' = \tilde{A} \tilde{x} + \tilde{P}(\varphi) \tilde{x} \]

(5.2.2.4)

\[ \varphi' = \omega \]

of the same form, where

\[ \tilde{A} = A + \bar{D} \]

(5.2.2.5)

\[ \tilde{P} = (I + U)^{-1} (P - U_0 - \bar{U} + \bar{P} f_{k'}) . \]

It also follows from the proof of Theorem 5.1.3 that \( f_{k'}(\varphi) \), and hence \( U(\varphi) \) are trigonometric polynomials of order no greater than
Therefore from Bernstein's inequality

\[ |D^g U| \leq 2^{\frac{a}{\ell}+1} |U| \quad (5.2.2.6) \]

From the second of (5.2.2.5) it follows that \( \tilde{P} \) has the same order of differentiability as \( P \). In particular \( D^l \tilde{P} \) exists and we have

\[
D^l \tilde{P} = \sum_{r=0}^{l} \left( \binom{l}{r} D^r (I+U)^{-1} \right) \left\{ D^{l-r} P \cdot (P - UD - f_\kappa) + \sum_{s=0}^{l-r} \binom{l-r}{s} D^s P \cdot D^{l-r-s} U \right\}.
\]

Therefore

\[
|D^l \tilde{P}| \leq \sum_{r=0}^{l} \left( \binom{l}{r} (I+U)^{r-1} \right) |D^r U| \left\{ |D^{l-r} P| + |D^{l-r} U| \cdot |\overline{D}| \right\}
\]

\[
+ \left| D^{l-r} f_\kappa \right| + \sum_{s=0}^{l-r} \binom{l-r}{s} \left| D^s P \right| \cdot \left| D^{l-r-s} U \right| \quad (5.2.2.7)
\]

that is,

\[
\leq \sum_{r=0}^{l} \left( \binom{l}{r} (I+U)^{r-1} \right) 2^r \cdot 2^{\frac{kr}{\ell}} \left| \frac{\pi}{2} \right| \left| \frac{1}{2} \right| \left| \frac{r}{\ell} \right| |D^l \tilde{P}|.
\]

(5.2.2.7)

\[
+ 2^{l-r} \frac{k}{\ell} 2^{l-r} \left| \frac{l-r}{2} \right| |U| \cdot |P| + 2^{l-r} \cdot 2^{l-r} \frac{k}{\ell} (l-r)^{3m} |P|
\]

\[
+ \sum_{s=0}^{l-r} \left( \binom{l-r}{s} \left| \frac{l-r}{2} \right| \left| \frac{1}{2} \right| \left| \frac{s}{2} \right| \right) \left| \frac{k}{\ell} (l-r-s) \right| |U| \right\},
\]

with the convention that inside the summation \( l = \lfloor l \rfloor \), \( r = \lfloor r \rfloor \) and \( s = \lfloor s \rfloor \).

Now if \( |U| \leq \frac{1}{2} \),

\[
|\tilde{P}| \leq (I+U)^{-1} \left( |U| \left( |P| + |\overline{D}| \right) + |P - f_\kappa| \right)
\]

\[
\leq 2 \left( 2MU + 3^m + 2^m K^2 - k \right) .
\]

Moreover if \( \alpha \neq \beta \)

\[
r = |\tilde{\alpha} - \tilde{\beta}| \geq |\lambda_\alpha - \lambda_\beta| - |P_{\alpha \alpha} - P_{\beta \beta}|
\]

\[
\geq r - 2M .
\]
We put

\[ c_1 = 8\gamma^{-1} \alpha \gamma L \beta \tau \]

\[ c_2 = \alpha \gamma L \beta \tau \]

\[ c_3 = 64 \sum_{r=0}^{L} \left\{ \frac{2r+1}{2^L} + 2 \sum_{r=0}^{L} \left( \frac{r+3 \alpha \gamma}{L} \right) \right\} \]

where

\[ L = \max(K, 1) \]

and write

\[ \frac{k}{L} = \psi, \quad \theta = \psi^{-\beta}. \]

Then provided that \( 0 < \theta < 1, \quad 1 < x < 2, \quad \beta > \frac{1}{\gamma} \quad \text{and} \quad l > x \frac{8}{x-1} \)

we can choose \( \theta \) so small that

\[
\begin{align*}
\theta_{x} - x & \leq \frac{r}{16}, \\
\theta_{x} - \frac{x}{\beta} & \leq \frac{c_1}{l}, \\
\theta_{x} - \frac{x}{\beta} & \leq \left[ \frac{c_2}{l} \right]^{-1}, \\
\left( x-1 \right) \frac{2}{\beta} - x & \leq \frac{c_1}{l}, \\
\theta_{x} - \frac{1}{\beta} & \leq \frac{1}{2}, \quad \text{and} \\
\theta^{x-1} & \leq \frac{c_3}{l}.
\end{align*}
\]

(5.2.2.8)

It follows that for \( M \leq \theta \)
\[ |U| \leq \left( r^{-1} + c_1 \psi \right) \theta \leq \frac{\theta^{x-1}}{8} \text{ for } |\text{Im} \varphi| < \frac{1}{2} \psi^{-1} \]

\[ |P| \leq \frac{\theta^x}{2} + c_2 \psi^{-l} \leq \theta^x \]

and

\[ |D^l P| \leq c_3 \psi^l \theta^{x-1} \leq \psi^l. \]

Let \( \varepsilon \) be any number such that \( 0 < \varepsilon < \min(1, \frac{1}{2\pi}) \). Choose \( \theta \) \((0 < \theta < 1)\) so small that all of (5.2.8.8) are satisfied and that, if \( \theta_j = \theta^{x_j} \),

\[ \sum_{j=0}^{\infty} \theta_j < \frac{\varepsilon}{8} \]

and

\[ \sum_{j=0}^{\infty} \theta_j^{x-1} < \frac{\varepsilon}{p}, \]

where

\[ p = \prod_{j=0}^{\infty} \left( 1 + \frac{1}{8} \theta_j^{x-1} \right). \]

Put \( r_j = r - 2 \left( \theta_{j-1}^{x-1} + \ldots + \theta_0^{2-x} \right) > 0 \) and suppose that the above system satisfies the above conditions with \( M, K, r \) replaced by \( \theta_j, \psi_j, r_j \), then we will show if we set \( \psi_j = 2^{k+l} \) the transformed equation satisfies the same conditions with \( M, K, r \) replaced by \( \theta_{j+1}, \psi_j, r_{j+1} \).

Since \( r_j \geq 16 \theta_j^{2-x} \)

\[ |U_{j+1}| \leq \left( r_j^{-1} + c_1 \psi_j \right) \theta_j \leq \frac{\theta_{j+1}^{x-1}}{8}. \]
Thus $|U_{j+1}| \leq \frac{1}{2}$ and hence

$$|P_{j+1}| \leq \frac{\theta_j}{2} + \sigma_2 \psi_{j-1}^{j-1} \leq \theta_j$$

and

$$|D^j P_{j+1}| \leq \sigma_3 \psi_j^{j-1} \psi_j \leq \psi_j.$$  

Moreover

$$\min_{\alpha \neq \beta} \left| \lambda^\alpha - \lambda^\beta \right| \geq r_j - 2 \theta_j \geq r_j - 2 \theta_j^{j-1} = r_j^{j+1}.$$ 

It follows that if in the original system (5.2.2.1), $M \leq 0$ the above transformation can be repeated indefinitely. We will now prove the existence of a limit system. We have

$$x_j' = A_j x_j + P_j(\varphi)x_j,$$

$$\varphi' = \omega$$

where

$$A_j = A + D_1 + \ldots + D_j.$$ 

Since $|P_j(\varphi)| \leq \theta_j$ for all real $\varphi$ it is obvious that $P_j(\varphi) \to 0$ uniformly for real $\varphi$. Since $D_j \leq \theta_j$ the existence of the limit

$$B = \lim_{j \to \infty} A_j$$

is also assured and $|B-A| < \epsilon$. Finally we have $x = T_j x_j$ where

$$T_j = (I + U_1) \ldots (I + U_j).$$

Suppose that for some $j$

$$|T_j| \leq \frac{j-1}{s=0} \left[ 1 + \frac{\theta_j^{j-1}}{s} \right]$$

with the convention $T_0 = I$ this is certainly true for $j = 0$. Since

$$T_{j+1} = T_j (I + U_{j+1})$$
it follows that
\[ |T_{j+1}^j| \leq |T_j^j||(I+U_{j+1}^j)| \leq \frac{j}{s=0}\left(1+\frac{1}{8}\psi^{-1}\right).\]

Therefore for \(|\Im\varphi| \leq \frac{1}{2}\psi^{-1}\)
\[ |T_{j+1}^j| \leq \varphi.\]

Hence
\[ |T_{j+1}^j-T_j^j| \leq \frac{1}{2}\varphi \psi^{-1}\]
\[ \text{for } |\Im\varphi| < \frac{1}{2}\psi^{-1}, \]
\[ = \frac{1}{2}\varphi \psi^{-1}, \]
that is, for \(|\Im\varphi| \leq \frac{2}{\psi^{-1}}\).

Thus we can apply Theorem 5.1.7 with \(\rho_j = \psi_j^{-1}\) and we obtain \(L_0 = 1, L_1 = \frac{1}{2}\) and \(h = \frac{1}{2}x(\kappa-1)\beta\). It follows from a consideration of the convergence of \(\sum \rho_k^{-1}\) that \(q_0 = \frac{1}{\kappa}\), (see Theorem 5.1.7). Hence
\[ T(\varphi) = \lim_{j \to \infty} T_j^j(\varphi) \]
is a real function with period \(2\pi\) in \(\varphi_1, \ldots, \varphi_m\) which possesses continuous partial derivatives of all orders \(s = (s_1, \ldots, s_m)\) with
\[ |s| = m \sum_{j=1}^m s_j = \frac{1}{2}(\kappa-1)\beta. \]

Moreover the estimates (5.1.7.3), (5.1.7.4), (5.1.7.5) and (5.1.7.6) also hold.

From
\[ T_{j+1}^j - I = T_j^j - I + T_j^j U_{j+1} \]
we obtain...
Hence the matrix $T(\varphi)$ is invertible and
\[ x_j + y = T^{-1}(\varphi)x \quad \text{as} \quad j \to \infty. \]

The partial derivatives $\frac{\partial T}{\partial \varphi}$ converge to $\frac{\partial T}{\partial \varphi}$ uniformly for real $\varphi$. Differentiating the relation $x = T_j x_j$, we obtain
\[
x' = \left( \frac{\partial T}{\partial \varphi}, \omega \right) x_j + T_j x'_j.\]

Hence letting $j \to \infty$
\[
x' = \left( \frac{\partial T}{\partial \varphi}, \omega \right) y + T\dot{y}.\]

Differentiating the relation $x = Ty$ we obtain
\[
x' = \left( \frac{\partial T}{\partial \varphi}, \omega \right) y + Ty'.\]

Comparison of these two expressions for $x'$ gives $y' = By$.

Thus the change of variables $x = T(\varphi)y$ transforms the system (5.2.2.1) into the autonomous system
\[
y' = By, \quad \varphi' = \omega.\]

Summarizing the above we have proved.

**THEOREM.**

Suppose

(i) $A = [\lambda_1, \ldots, \lambda_n]$ is a real diagonal matrix with
\[ \min_{\alpha \neq \beta} |\lambda_\alpha - \lambda_\beta| \geq r > 0; \]

(ii) $P(\varphi)$ is a $n \times n$ real matrix function of the real m-vector $\varphi$ of class $P^l_{MK}$.
(iii) \( \omega \) is a real \( m \)-vector such that for all integral \( m \)-vectors \( k \neq 0 \)

\[ |(k, \omega)| \geq \gamma |k|^{-\tau}, \quad (\tau > m, \gamma > 0). \]

Then provided that \( l > \tau \), for any \( \epsilon \) such that \( 0 < \epsilon < \min(1, \frac{1}{2r}) \)

there exists an absolute constant \( M_0 = M_0(m, \tau, \gamma, \epsilon, K) \) with the

property that if \( M \leq M_0 \) the quasi-periodic linear system

\[ x' = [A + P(\omega t)]x \]

has a fundamental matrix of the form

\[ X(t) = [I + U(\omega t)]e^{tB} \]

where,

(i) \( B = [\mu_1, \ldots, \mu_n] \) is a real diagonal matrix with

\[ |B - A| \leq \epsilon \] and hence

\[ \min_{\alpha \neq \beta} |\mu_\alpha - \mu_\beta| \geq r - 2\epsilon > 0; \]

(ii) \( U(\phi) \) is an \( n \times n \) real matrix function of the \( m \)-vector \( \phi \) which has period \( 2\pi \) in each coordinate \( \varphi_\alpha \), and which satisfies the inequality

\[ |U(\phi)| \leq \epsilon \] for all \( \phi \).

Moreover \( U(\phi) \) possesses continuous partial derivatives of all orders \( s = \{s_1, \ldots, s_m\} \) with

\[ |s| < \frac{1}{2} \left( \frac{x-1}{x} \right)^2 l, \]

where \( x \) is the exponent chosen for the accelerated convergence.

Furthermore, for such \( s \), we note that

\[ |D^s U(\phi)| \leq s! L \left( \frac{2|s|}{x(x-1)^{s}} \right) \]
where \( L(q) \) is defined by (5.1.7.4). When
\[
0 < \frac{2\delta}{x(x-1)\beta} < \frac{1}{x} - \frac{2|s|}{x(x-1)\beta} < \frac{2}{x(x-1)\beta},
\]
\( D^\delta U(\varphi) \) is Hölder continuous in each variable \( \varphi_\alpha \) with exponent \( \delta \)
and Hölder constant
\[
s!\left(\frac{s+2}{s+2}\right)^{\frac{\delta+s}{s}}\left(\frac{s+2}{s+2}\right)^{\frac{\delta+s}{s}}.
\]

5.3.1.

The proof of Theorem 5.2.2 can easily be extended to give the result corresponding to Theorem 4.3.2 for the equation
\[
x' = [A + \xi_0 + P(\omega t, \xi_0)]x.
\]
Although the proof is quite lengthy there are no new complications and the loss of derivatives is precisely that found in Theorem 5.2.2. We obtain the following result.

**THEOREM.**

Suppose

(i) \( A \) is a real, constant, \( n \times n \) matrix with eigenvalues \( \lambda_\alpha, \alpha = 1, \ldots, n \) and \( d \) is the square of the order of the order of the largest block in the Jordan decomposition of \( A \);

(ii) \( P(\varphi, \xi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \varphi \) and the \( n \times n \) matrix \( \xi \), which is real for real \( \varphi \) and \( \xi \), has period \( 2\pi \) in each coordinate \( \varphi_\alpha \) and is of class \( p^J_{MK} \) in the cylinder
\[
|\text{Im} \varphi| \leq \rho, \quad (\rho > 0)
\]
\[
|\xi| \leq \sigma < 1.
\]
(iii) \( \omega \) is a real \( m \)-vector such that for all integral \( m \)-vectors \( k \neq 0 \)
\[
|\lambda_\alpha^\gamma - \lambda_\beta^\gamma + i(k, \omega)| \geq \gamma |k|^{-\tau}, \quad \alpha, \beta = 1, 2, \ldots,
\]
where \( \tau d > m \) and \( \gamma > 0 \).

Then provided that \( l > \tau d \) for any \( \varepsilon \), such that \( 0 < \varepsilon < \min(1, \rho) \)
there exists an absolute constant \( M_0 = M_0(m, n, \tau, \gamma, \varepsilon, \sigma, A, K) \)
with the property that if \( M \leq M_0 \) there exists a constant \( n \times n \)
matrix \( E_0 \) with \( |E_0| \leq \sigma \) such that the quasi-periodic linear system
\[
x' = [A + E_0 + F(\omega t, E_0)]x
\]
has a fundamental matrix of the form
\[
X(t) = [(1 + U(\omega t))e^{At}\]
where \( U(\phi) \) is an \( n \times n \) matrix function of the \( m \)-vector \( \phi \) which
is real for real \( \phi \), has period \( 2\pi \) in each coordinate \( \phi_\alpha \) and
which satisfies the inequality
\[
|U(\phi)| \leq \varepsilon
\]
for all real \( \phi \). Moreover \( U(\phi) \) possesses continuous partial
derivatives of all orders \( s = (s_1, \ldots, s_m) \) with
\[
|s| < \frac{1}{2} \frac{(\kappa - 1)^2}{\kappa} l,
\]
where \( \kappa \) is the exponent chosen for the accelerated convergence.
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