"MULTIPLIERS AND DIFFERENTIAL OPERATORS".

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Thesis submitted to the School of General Studies, Australian National University for the Degree of Master of Science, February 1968.
STATEMENT

Except where otherwise indicated this thesis is all my own work.

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During the time the work for this thesis was done I was supported by an award under the Commonwealth Scholarship and Fellowship Plan which was generously supplemented by the Australian National University.

My greatest debt is to Dr. R.E. Edwards. His patience and helpfulness in all things have known no bounds.

My grateful thanks are due to Dr. Anne C. Baker, my supervisor during these last few months when Dr. Edwards was on study leave.

I am grateful to John Price for many useful discussions and to Professors Bernhard and Hanna Neumann for their constant encouragement and consideration.

Special thanks go to Mrs. Felicity Wickland who typed the stencils for this thesis with such artistry.
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**References.**
0.1. This thesis contains a study of a few multiplier problems in harmonic analysis. The following description of a multiplier problem will be adequate for our purposes. Suppose that $Y$ and $Z$ are topological vector spaces of extended real-valued functions over a Hausdorff, locally compact, abelian group $G$. We shall assume that $Y$ and $Z$ are invariant under the translation operators $\tau_x$, $x$ in $G$. The multiplier problem is to characterise those continuous linear maps of $Y$ into $Z$ which commute with all the translation operators. Such maps are called multipliers $Y \rightarrow Z$.

We try to express such a multiplier as a convolution with a fixed generalised function belonging to a known class. Our main concern throughout will be with multipliers of Sobolev spaces into other Sobolev spaces.

Chapter 0 consists of definitions and preliminaries. In chapter 1 we give characterisations of multipliers

(a) $C(G) \cap L^p(G) \rightarrow C(G) \cap L^q(G)$, $1 \leq p, q < \infty$;

(b) between certain classes of weighted Lebesgue spaces;

(c) between certain special Sobolev spaces.

Chapter 2 is devoted entirely to studying multiplier problems in Sobolev spaces. Apart from section 2.1 the Sobolev spaces used in chapter 2 are all on the additive group of finite (but otherwise arbitrary) dimensional Euclidean space. Although some multipliers are characterised therein, sections 2.2 and 2.3 are devoted to laying the
ground for the main result which appears in section 2.4 and which completely characterises the multipliers of the Sobolev spaces

\[ W^{(m)}_p(\mathbb{R}^k) \to W^{(n)}_q(\mathbb{R}^k) \]

for \( 1 < p, q < \infty \).

A simple but incomplete characterisation of similar multipliers is given in section 2.3.
0.2
We now define the concepts used in this thesis and establish our notation.

0.2 DEFINITIONS AND NOTATION

We will denote the set-theoretic difference of two sets A and B by

\[ A \setminus B \quad \text{or} \quad A \sim B, \quad A \cap B' \]

The letter \( \mathbb{R} \) will always denote the additive group of all real numbers equipped with the usual norm topology. By \( \mathbb{R}^k \) we shall mean the product of \( k \) copies of \( \mathbb{R} \) endowed with the topology defined by the norm

\[ ||x|| = \sqrt{\sum_{i=1}^{k} x_i^2} \]

where \( x = (x_1, x_2, \ldots, x_k) \). A complete list of the notation we will adopt in connection with \( \mathbb{R}^k \) appears in subsection 2.2.1.

The usual pointwise derivative of a real-valued function \( f \) defined on \( \mathbb{R} \) will be denoted by \( \partial f \):

i.e. \( \partial f(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \)

The distributional derivative of a distribution \( T \), defined as in Edwards [1], p.303, will be denoted by \( DT \).

We note that for absolutely continuous functions the distributional and pointwise derivatives are identical.
Let $X$ be a locally compact abelian group.

$L^p = L^p(X)$ denotes the usual Lebesgue space of functions whose $p$-th power is integrable with respect to Haar measure on $X$. We will not distinguish notationally between a function and the class of functions which are equal to it except on a set of measure zero. The usual norms will be employed:

$$||f||_{L^p} = \frac{1}{[\int |f(x)|^p dx]^p}, \text{ if } 1 \leq p < \infty;$$

$$||f||_{L^\infty} = \text{ess.sup}|f(x)|.$$

We shall abbreviate $||f||_{L^p}$ to $||f||_p$.

We now define Sobolev spaces. Let $m$ be a positive integer.

$$W^{(m)}_p = W^{(m)}_p(\mathbb{R}^k)$$

is the subspace of $L^p(\mathbb{R}^k)$ of functions whose partial derivatives of order up to and including $m$ (taken in the distributional sense) are also in $L^p$. As before we shall not distinguish notationally between a function and its equivalence class modulo a negligible set. The following equivalent norms will be used:

$$||f||_{W^{(m)}_p} = \max \left\{ ||D^\alpha f||_p \right\}, \ 1 \leq p \leq \infty;$$

$$||f||_{\overline{W}^{(m)}_p} = \sum_{|\alpha| \leq m} ||D^\alpha f||_p, \ 1 \leq p \leq \infty.$$
With these norms $W^p_m$ is a Banach space.

In section 2.1 we discuss $\mathcal{M}_{\text{bd}}^{(m)}$ which consists of those bounded measures whose derivatives of order up to and including $m$ (taken in the distributional sense) are also bounded measures. $\mathcal{M}_{\text{bd}}^{(m)}$ is also a Banach space when equipped with the norm

$$||\mu||_{\mathcal{M}_{\text{bd}}^{(m)}} = \max \{|D_\alpha \mu|_p\}.$$

Let $X$ be a locally compact abelian group.

We will let $C = C(X)$ denote the space of all continuous complex-valued functions on $X$. The natural topology on $C$ is the topology of locally uniform convergence. Unless otherwise stated $C$ will always be taken to have this topology.

$$C_0 = C_0(X)$$ denotes the subspace of $C$ formed of those functions which tend to zero at infinity. The natural topology on $C_0$ is the topology of uniform convergence, defined by restricting to $C_0$ the norm $||\cdot||_\infty$.

$$C_c = C_c(X)$$ denotes the subspace of $C_0$ formed of those continuous functions which have compact supports. The natural topology on $C_c$ is obtained by regarding $C_c$ as the internal inductive limit of its subspaces

$$C_{c,K} = \{f \in C_c : \text{Supp } f \subset K\},$$
K ranging over a base for the compact subsets of X and each \( C_{c,K} \)
being regarded as a Banach space with the supremum norm.

We now turn to functions on \( R^k \). \( \mathcal{E} = \mathcal{E}(R^k) \) will denote
the space of all real-valued, infinitely differentiable functions in
\( R^k \). The natural topology is that of uniform convergence of every
derivative on every compact subset of \( R^k \).

\( \mathcal{D} = \mathcal{D}(R^k) \) will denote the space of all functions in \( \mathcal{E} \)
which have compact supports. The natural topology on \( \mathcal{D} \) is
obtained by regarding \( \mathcal{D} \) as the internal inductive limit of its
subspaces,

\[
\mathcal{D}_K = \{ f \in \mathcal{E} : \text{Supp } f \subset K \}
\]

obtained as K ranges over a base for the compact subsets of \( R^k \) and
each \( \mathcal{D}_K \) being regarded as a Fréchet space with the topology
induced by \( \mathcal{E} \).

We now look at the duals of these spaces.

The dual of the space \( C_c(X) \) is the space \( M \) of all Radon measures
on X. For details see Edwards [1] Chapter 4. The associated weak
topology \( \sigma(M,C_c) \) is the "vague topology of measures."

If \( \mu \) is a Radon measure, the support of \( \mu \), denoted by \text{supp } \mu,
is the complement in X of the largest open subset \( U \) of X satisfying
\( |\mu|(U) = 0 \).
The dual of $C$ is $M_c$, the subspace of $M$ formed of those measures which have compact supports.

The dual of $C_0$ is the space $M_{bd}$ formed of those measures $\mu$ for which

$$||\mu|| = |\mu|(X) < \infty. $$

We will denote the dual of $\mathcal{D}'(R^k)$ by $\mathcal{D}'(R^k)$ and we will call this the space of distributions on $R^k$. The dual of $\mathcal{E}$, denoted by $\mathcal{E}'$ may be identified with the space of distributions with compact supports.

Throughout this thesis $\delta_0$ will denote the Dirac measure at the origin.

The left (and right in our case, since the groups we deal with are abelian) translation operator $\tau_a$ is defined for functions $f$ on $G$ by $\tau_a f(x) = f(a^{-1}x)$ for all $x,a$ in $G$. This definition is extended consistently to measures (respectively distributions) in the following way:

$$\int \phi d(\tau_a \mu) = \int (\tau_{a^{-1}} \phi) d\mu = \int \phi(ax) d\mu(x).$$

[respectively $\tau_a T(\phi) = T_x(\phi(a+x))$] for $\phi$ in $C_c$ [respectively $\mathcal{D}$].

It is immediately clear that

$$||\tau_a f||_p = ||f||_p, \quad 1 \leq p \leq \infty, \quad a \in G.$$
Convolutions are essential in our treatment of the subject.

If \( \lambda \) and \( \mu \) are positive measures on some locally compact space \( X \), the convolution \( \lambda \ast \mu \) is said to exist if and only if the following integrals, (known to have a common value), are finite for each non-negative function \( k \) in \( C_c(X) \):

\[
\int d\lambda(x)\int k(xy)d\mu(y);
\int d\mu(y)\int k(xy)d\lambda(x);
\int \int k(xy)d\lambda(x)d\mu(y).
\]

Then \( \lambda \ast \mu \) is the positive measure defined by setting \( \int kd(\lambda \ast \mu) \) equal to this common value.

If \( \lambda \) and \( \mu \) are complex measures, \( \lambda \ast \mu \) is said to exist if and only if \( |\lambda| \ast |\mu| \) exists in the preceding sense. For more details, see Edwards [1], Section 4.19.

We know that \( \lambda \ast \mu \) exists if either of \( \lambda, \mu \) has a compact support or if both are bounded measures. In the latter case

\[
||\lambda \ast \mu|| \leq ||\lambda|| \cdot ||\mu||.
\]

Convolution defined thus is associative provided that all but at most one of the factors has a compact support or that all are bounded measures.

Let \( S \) and \( T \) be distributions on \( \mathbb{R}^k \). We can define the convolution of \( S \) and \( T \) in a similar way. For any \( \phi \) in \( \mathcal{D}(\mathbb{R}^k) \),

\[
\langle \phi, S \ast T \rangle = \langle \phi(\xi + \eta), S_{\xi}T_{\eta} \rangle
\]

provided that

\[\dagger\] We sometimes use this notation (Edwards [1] p.300) for distributions.
the expression on the right hand side always makes sense (see Schwartz [2] Chapitre VI). This is the case if one of S, T has compact support. When all but at most one of the factors have compact supports this convolution is associative.

We also note that the Chevalley convolution (which has been shown by Shiraishi [1] to be equivalent to the above definition) exists if and only if \((S*\phi).\langle T*\psi \rangle\) is in \(L^1(\mathbb{R}^k)\) for every \(\phi, \psi\) in \(\mathcal{D}(\mathbb{R}^k)\).

With this definition convolution is associative provided that the simultaneous convolution exists. See Shiraishi [1] for details.

We shall make frequent use of a result by Schwartz [2], pp.53-54, which states that if T is a continuous map of \(\mathcal{S}\) into \(\mathcal{S}'\), which commutes with translations then there exists a distribution U such that \(Tf = U*f\) for every \(f\) in \(\mathcal{S}\). For details of the proof see Edwards [1], p.332.

We also note that if we define a map \(S : C_c \rightarrow C\) by \(Sf = V*f\) for \(f\) in \(C_c\), where V is a measure or distribution, then \(S\) certainly commutes with translations. See Edwards [1], p.262.

Suppose that \(X\) is a locally compact abelian group. If \(1 \leq p, q < \infty\) and \(L^p(X), L^q(X)\) are the usual Lebesgue spaces of index \(p\) and \(q\) respectively, with respect to Haar measure, the set of all multipliers of \(L^p(X)\) into \(L^q(X)\) will be denoted by \(L^q_p\).
It is known that if $X$ contains an infinite discrete group and $p > q$ then $L^q_p = \{0\}$. See Gaudry [1] for details.

We note that if $T$ is a multiplier in $L^q_p$ then the adjoint of $T$ is a multiplier in $L^p_{q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and that the two maps coincide on $L^p \cap L^{q'}$. For details see Brainerd and Edwards [1] p.317. We also note that for $1 < p < \infty$

$$L^p_p \supseteq M_{bd}.'$$

This result is proved in Brainerd and Edwards [1] p.316. Substantial results and references for $L^q_p$ appear in Gaudry [1] and Figà-Talamanca and Gaudry [1].
CHAPTER 1.

This chapter is devoted to the solution of four multiplier problems. These problems are not intimately related and we need to devise special tools for the solution of each one. The early part of each section is devoted to developing one of these tools enough to solve the problem that comes immediately afterwards.

§1.1 A characterisation of the multiplier $L^p(G) \rightarrow L^q(G)$ has been given by Figa-Talamanca and Gaudry [1]. We shall make use of their results in determining the multipliers of a subspace of $L^p(G)$.

In what follows, let $G$ be a locally compact, Hausdorff, abelian, topological group. Let us denote by $L^p \cap C$ the space of continuous real-valued functions on $G$ which are also in $L^p(G)$. Further, let this space have the upper bound of the $L^p$-norm topology and the topology of locally uniform convergence. A base at 0 for this topology consists of the sets $\{\{f:|f|_p < \varepsilon\} \cap \{f: \text{Sup}_K |f| < \varepsilon\}\}$, obtained when $\varepsilon$ ranges over the positive real numbers and $K$ ranges over the compact subsets of $G$.

§1.1.1 Lemma.

When $p$ lies in the range $1 \leq p < \infty$, the dual of $L^p \cap C$ consists of all measures of the form $\mu + \phi$ where $\mu \in \mathcal{M}_C$ and $\phi \in L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. 
Proof. Denote by \( J \) the topology on \( L^p \cap C \). We note that \( C_c \) is dense in \( L^p \cap C \). The map \( J \):

\[
[C_c, J] \rightarrow [C_c, \| \cdot \|_p] \times [C_c, \text{loc.unif.}]
\]

defined by \( f \rightarrow (f, f) \) is continuous and 1-1. If \( W = J(C_c), J^{-1} \)
defined on \( W \) is a continuous function. Let \( h \) be any continuous linear functional on \( [C_c, J] \). Then we can define \( h' : W \rightarrow \mathbb{R} \) by

\[
h' = h \circ J^{-1}.
\]

By the Hahn-Banach Theorem for locally convex topological vector spaces, there exists a continuous extension \( h'' \) of \( h' \) defined on the whole of \( [C_c, \| \cdot \|_p] \times [C_c, \text{loc.unif.}] \). Then \( h''(f_1, f_2) = \mu(f_2) + \phi(f_1) \), where \( \mu \in M \) and \( \phi \in L^p \). Now

\[
h = h'' \circ J
\]

and so

\[
h(f) = \mu(f) + \phi(f)
\]

for every \( f \) in \( C_c \).

\section{Theorem.}

Suppose \( 1 \leq p < \infty \) and \( 1 \leq q < \infty \). The multipliers \( T \) of

\[
L^p \cap C \rightarrow L^q \cap C
\]

are of the form
Tf = (µ*f) + (ϕ*f)

for all \( f \) in \( L^p \cap C \), where
\[
\mu \in M \cap L^p \quad \text{and} \quad \phi \in L^{p'} \cap L^p \quad \text{when} \ q = p,
\]
and where \( \mu \in M \) and \( \phi \in L^{p'} \) are such
that \( (\mu+\phi) \in L^q \) when \( q \neq p \).

Conversely every map of type (A) is a multiplier of \( L^p \cap C \) into \( L^q \cap C \).

Proof. Let \( T \) be a multiplier

\[ L^p \cap C \rightarrow L^q \cap C. \]

Since \( C \) is dense in \( L^p \cap C \) we shall confine our attention to the restriction of \( T \) to \( C \).

The map \( \alpha : L^p \cap C \rightarrow \mathbb{R} \), defined by

\[ f \rightarrow Tf(0), \]

is a continuous linear functional on \( L^p \cap C \). [It is obvious that \( \alpha \) is linear. As for the continuity, let \( K \) be any compact set containing 0; the set

\[ N = \{ g \in L^q \cap C : \sup_{K} |g| < 1 \} \]

is open in \( L^q \cap C \). The continuity of \( T \) implies that \( T^{-1}(N) \) is open in \( L^p \cap C \) and clearly \( \alpha \) is bounded on this neighbourhood.]
From the previous lemma

\[ T\phi(0) = \mu \ast \phi(0) + \phi \ast \phi(0) \]

where \( \mu \in \mathcal{M}_c \) and \( \phi \in L^p' \). Applying this result to \( f' = \tau_a f \)
we immediately obtain

\[ T\phi = \mu \ast \phi + \phi \ast \phi. \]

We now make use of the continuity of \( T \) to further restrict \( \mu \) and \( \phi \).

The continuity of \( T \) implies that, for some compact subset \( K \) of \( G \),

\[ \| (\mu + \phi) \ast \phi \|_{q} \leq \text{Const} (\| \phi \|_p + \sup_{K} |\phi|) \]

for each \( \phi \) in \( C_c \). But for each \( \phi \) in \( C_c \) there exists \( a \) in \( G \)
such that \( K \) does not interest the support of \( \tau_a \phi \). This fact, together
with the translation invariance of the \( L^p \) and \( L^q \) norms shows that,

for every \( \phi \) in \( C_c \),

\[ \| (\mu + \phi) \ast \phi \|_{q} \leq \text{Const} \| \phi \|_p. \]

This inequality tells us that \( (\mu + \phi) \) belongs to \( L^q_p \). In the case
\( q = p \) we can further simplify this result. It is known that, for \( 1 \leq p < \infty \),

\[ L^p_p \supset M_{bd} \supset M_c. \]

(See Brainerd and Edwards [1].)
This fact, taken in conjunction with inequality (B), enables us to infer that both \( \mu \) and \( \phi \) belong to \( L^p \).

We now have to show that the continuous extension of \( T/C \) to the whole of \( L^p \cap C \) is of the form

\[
Tf = (\mu*f) + (\phi*f)
\]  

..... (C)

for every \( f \) in \( L^p \cap C \). Let \( f \) be any function in \( L^p \cap C \).

There exists a sequence \( (f_n) \) of continuous functions with compact support, which converges to \( f \) in the \( L^p \cap C \) topology. The continuity of \( T \) means that

\[ (\mu+\phi)*f_n \to Tf \quad \text{in} \quad L^q \cap C \]  

topology,

which implies that

\[ (\mu+\phi)*f_n \to Tf \quad \text{in} \quad \sigma(M,C_c) \]  

Since \( \sigma(M,C_c) \) is a Hausdorff topology, we will obtain (C) if we can show that the convolution \( (\mu+\phi)*f \) exists and that

\[ (\mu+\phi)*f_n \to (\mu+\phi)*f \quad \text{in} \quad \sigma(M,C_c) \]  

The first part is easily settled; \( \mu*f \) exists because \( \mu \) has compact support; \( \phi*f \) exists because \( f \) is a function in \( L^p \) and \( \phi \) is a function in \( L^{p'} \).
For any \( \psi \in C_c \),

\[
| \int (\mu f_n - \mu f) \psi |
\]

\[
\leq \int |\mu \psi| |f - f_n|
\]

\[
\leq \|\mu \psi\|_p \|f - f_n\|_p \quad \ldots \quad (D)
\]

Also

\[
| \int (\phi f_n - \phi f) \psi |
\]

\[
\leq \int |\phi \psi| |f - f_n|
\]

\[
\leq \|\phi \psi\|_p \|f - f_n\|_p \quad \ldots \quad (E)
\]

Expressions (D) and (E) converge to zero as \( n \) tends to infinity and thus we obtain (C). We now show that conversely every map of type (A) is a multiplier

\[
L^p \cap C \rightarrow L^q \cap C.
\]

The following inequalities are valid for every \( f \in C_c \) and any compact subset \( K \) of \( G \):

\[
| |(u+\phi)f| |_q \leq \text{Const} |f|_p \leq \text{Const} \{ |f|_p + \sup_K |f| \};
\]

\[
\sup_K |(u+\phi)f| \leq ||(u+\phi)f| |_{\infty}
\]

\[
\leq |\phi|_p |f|_p + \text{Const} \sup_{K+ \text{Supp}\mu} |f|
\]

\[
\leq \max(|\phi|_p, \text{Const})(|f|_p + \sup_{K+ \text{Supp}\mu} |f|).
\]
Note that the first line follows from the membership of \((\mu + \phi)\) to \(L^q_p\).
Thus the map \(T\) defined by \((\mu + \phi)\) is continuous from \(L^p \cap C\) into \(L^q \cap C\). The properties of convolution ensure also that \(T\) commutes with translations.

§1.2 We now turn to an examination of the multipliers of a class of weighted Lebesgue spaces. In order to obtain the strong conclusion of Theorem 1.2.1, which we use repeatedly, it is necessary to restrict ourselves throughout this section to \(\sigma\)-compact, first countable groups.

§1.2.1 Theorem.

Let \(G\) be a locally compact, separated, first countable, \(\sigma\)-compact, abelian, topological group. Denote the Haar measure on \(G\) by \(\mu\).

Let \(\omega\) be a locally integrable, extended real-valued function on \(G\).

Then there exists a sequence \((\phi_n)\) of continuous, non-negative functions on \(G\), each with compact support, such that:

1. there is a fixed compact set containing the support of every \(\phi_n\);
2. \(\int_{G^n} \phi \, d\mu = 1\) for each \(n\);
3. if \(N\) is any neighbourhood of \(0\) (the identity of \(G\)), there exists an integer \(M\) such that \(n > M\) implies that \(\text{Supp} \phi_n \subset N\);
4. there exists a set \(S\) with \(\mu(S) = 0\) and, for every \(a \in G \cap S'\),

\[ \lim_{n \to \infty} \int_{G^n} \phi (x) \omega(x+a) \, d\mu(x) = \omega(a). \]
Proof. Since $G$ is locally compact and $\sigma$-compact, we can see that there exists a sequence $(\Omega_n)$ of open sets such that:

- $\overline{\Omega_n}$ is compact;
- $\Omega_n \subset \Omega_{n+1} \subset G$;
- $G = \bigcup_{i=1}^{\infty} \Omega_i$.

For each positive integer $k$ we define a function $\omega_k$ on $G$ by

$$\omega_k = \omega \text{ on } \Omega_k,$$

$$\omega_k = 0 \text{ on } G \cap \Omega_k'.$$

Then $\omega_k \in L^1(\mu)$. Let $(\psi_n)$ be any sequence of non-negative functions in $C_c(G)$ satisfying 1, 2, 3. We show how to select a subsequence which satisfies 4. Now

$$\omega_1 \ast \psi_n \rightharpoonup \omega_1 \text{ in the } L^1 \text{-norm topology and hence}$$

$$\omega_1 \ast \psi_n \rightharpoonup \omega_1 \text{ in measure.}$$

Hence, by a theorem of F. Riesz, there exists a subsequence labelled $(\psi_{1,r})$ such that $\omega_1 \ast \psi_{1,r} \rightharpoonup \omega_1$ almost everywhere; say on $G \cap S_1'$, where $\mu(S_1) = 0$. Now we find that

$$\omega_2 \ast \psi_{1,r} \rightharpoonup \omega_2 \text{ in the } L^1 \text{-norm topology}$$

and so we can find a subsequence $(\psi_{2,r})$ of $(\psi_{1,r})$ such that

$$\omega_2 \ast \psi_{2,r} \rightharpoonup \omega_2 \text{ almost everywhere,}$$
say on $G \cap S_2'$, where $\mu(S_2) = 0$. Proceeding in this way, we see that we can find a sequence, the diagonal sequence $(\psi_n, n)$, such that for any positive integer $k$,

$$\lim_{n \to \infty} w_k \ast \psi_{n, n} = \omega_k \text{ almost everywhere;}$$

say on $G \cap S'$. [$S \subseteq \cup S_k$, which implies that $\mu(S) = 0$].

For any $a \in G \cap S'$ there exists $k$ such that $a \in \Omega_k$. If $n$ is large enough we have

$$a + \text{Supp } \psi_{n, n} \subseteq \Omega_k.$$ 

For this and larger $n$,

$$w_k \ast \psi_{n, n}(a) = \omega_k \ast \psi_{n, n}(a).$$

Also

$$\omega_k(a) = \omega(a).$$

Thus

$$\lim_{n \to \infty} w_k \ast \psi_{n, n}(a) = \lim_{n \to \infty} \omega_k \ast \psi_{n, n}(a)$$

$$= \omega_k(a)$$

$$= \omega(a).$$

If we write $\phi_n = \psi_{n, n}$ we obtain a sequence $(\phi_n)$ satisfying 1, 2, 3, 4.
§1.2.2 Definition.

A positive locally integrable function \( \omega \) on \( G \) is said to be moderate (with respect to \( \mu \) and \( G \)) if to each point \( s \in G \) there corresponds a number \( M(s) > 0 \) such that

\[
\int_{sE} \omega \, d\mu \leq M(s) \int_E \omega \, d\mu,
\]

for every bounded integrable set \( E \).

Let \( \omega_1, \omega_2 \) be moderate, positive, lower semicontinuous, locally integrable, extended real-valued functions on \( G \). Denote by \( W_1, W_2 \) the spaces of Haar-measurable functions \( f \) such that \( \omega_1 f \) and \( \omega_2 f \) respectively are integrable. The fact that \( \omega_1 \) and \( \omega_2 \) are locally integrable and moderate implies that the weighted Lebesgue spaces \( W_1 \) and \( W_2 \) are invariant under translations. See Edwards [2] for a proof of this. We endow \( W_1 \) and \( W_2 \) with the norm topologies defined by

\[
||f||_{W_i} = \int_G \omega_i f \, d\mu, \quad i = 1,2.
\]

A little later we will find it convenient to denote \( W_1 \) by \( L^1(\omega_1 \mu) \).

We note that the product \( \omega_1 \mu \) is a positive Radon measure. We now prove some useful lemmas.

Lemma.

The space \( C_c(G) \) is contained in \( W_1 \).
Proof. If $f$ is in $\mathcal{C}_c(G)$ then

$$\int_{\text{supp} f} |f| \, d\mu = \int_{\text{supp} f} \omega_i f \, d\mu \leq ||f||_{\infty} \int_{\text{supp} f} \omega_i \, d\mu < \infty.$$ 

§1.2.3 Lemma.

Every function in $W_1$ is locally integrable.

Proof. Let $f \in W_1$. Let $K$ be any compact set. Firstly we show that there is a positive number $\omega^i_1$, such that

$$\omega_i(x) > \omega^i_1 \text{ for each } x \in K.$$ 

The fact that $\omega_i$ is positive and lower semicontinuous implies that, for every $x$ in $K$, we can find an open set $N_x$ containing $x$ and having the property that

$$\omega_i(y) > \frac{1}{2} \omega_i(x) > 0 \text{ for } y \in N_x.$$ 

Now the family consisting of all such $N_x$ is an open cover of $K$ and, since $K$ is compact, we can select a finite subcover, which we shall call $\{N^i_x\}$. We now select $x_j$ such that

$$\min_{n} \omega_i(x_n) = \omega_i(x_j) > 0.$$
Hence

\[ \omega_1(x) > \frac{1}{2} \omega_1(x_j) = \omega'_1 > 0 \quad \text{for all } x \in \mathbb{K}. \]

Armed with this fact, we obtain the stated result by noting

\[ \infty > \int_{\mathbb{K}} |f| \, du > \omega'_1 \int_{\mathbb{K}} |f| \, du. \]

§1.2.4 Lemma.

The space \( C_c \) is dense in \( W_1 \).

Proof. Identifying \( W_1 \) with \( L^\#(\omega_1 \mu) \), we see that \( C_c \) is dense in \( W_1 \) by construction.

§1.2.5 Lemma.

The topology \( \sigma(W_2, C_c) \) is weaker than the norm topology on \( W_2 \).

Proof. The result follows at once when we note that if \( g \in C_c \) then the map

\[ f \mapsto \int g f \, du \]

is continuous on \( W_2 \) (norm topology).
We take note of the near trivial fact that \( T \) is a multiplier

\[
L^1(\omega_1 \mu) \rightarrow L^1(\omega_2 \mu)
\]

if and only if \( \tau_a T \) is a multiplier

\[
L^1(\tau_a\omega_1 \mu) \rightarrow L^1(\omega_2 \mu).
\]

§1.2.6 Theorem.

If \( T \) is a multiplier

\[
W_1 \rightarrow W_2,
\]

there exists a Radon measure \( \nu \) such that

\[
Tf = \nu* f \quad \text{for every } f \text{ in } C_c.
\]

Proof. Let a be a point in \( G \cap S' \) (where \( S \) is determined for \( \omega_1 \) as in Theorem 1.2.1) such that \( \omega_1(a) < \infty \). We now proceed to characterize the multipliers

\[
L^1(\tau_a \omega_1 \mu) \rightarrow L^1(\omega_2 \mu).
\]

By Lemma 1.2.4, we need only consider the restriction of \( T \) to \( C_c(G) \).

Let \( \phi_n \) be the sequence of functions constructed for \( \omega_1 \) as in Theorem 1.2.1.
If \( f \) is any function in \( C_c \), then

\[(\phi_n^* f) \to f \text{ uniformly.}\]

Also, all \( \phi_n^* f \) have supports contained in a fixed compact set.

This implies that \( (\phi_n^* f) \to f \) in \( W_1 \)-norm because \( \omega_1 \) is locally integrable.

The continuity of \( T \) implies that

\[T(\phi_n^* f) \to Tf \text{ in } W_2 \text{-norm.}\]

But \( T(\phi_n^* f) = (T\phi_n)^* f = \phi_n^* Tf \) and each exists as a Radon measure.

(See Edwards [1], p. 577.) Let \( \mu_n = T\phi_n \). Since \( (\phi_n^* f) \) is bounded in the \( W_1 \)-norm topology (because of the choice of \( a \)), we have that \( (\mu_n) \) is bounded in the \( W_2 \)-norm topology. Let \( K \) be any compact subset of \( G \).

From the proof of 1.2.3 we know that \( \omega_2 \) is bounded below on \( K \).

Thus

\[
\sup_n \|\mu_n\|_{W_2} < \infty
\]

implies that

\[
\sup_n \|\mu_n\|(K) < \infty.
\]

Hence \( \{\mu_n\} \) is equicontinuous as a subset of \( M(G) \), the dual of \( C_c(G) \).

We also note that the space \( C_c(G) \) is separable. Thus the relative topology

\[
\sigma(M(G), C_c(G))|\{\mu_n\}
\]

is metrizable.
Hence we can select a subsequence \((\mu_{n_i})\) from the equicontinuous sequence \((\mu_n)\) which converges to a measure \(\nu\) in the topology \(\sigma(M, C_c)\).

This follows from Theorem 1.11.4(2) of Edwards, [1].

Consequently,

\[\mu_{n_i} * f \rightarrow \nu * f \text{ pointwise,} \quad \cdots \quad (A).\]

But

\[\mu_{n_i} * f \rightarrow T f \text{ in } W_2 \quad \cdots \quad (B).\]

From (B) and the theorem of convergence in measure there exists a subsequence \((\mu_{n_{r_i}})\) of \((\mu_{n_i})\) such that

\[\mu_{n_{r_i}} * f \rightarrow T f \quad \omega_2 \mu - \text{a.e.} \quad \cdots \quad (B)\]

But \((\mu_{n_{r_i}})\) is a subsequence of \((\mu_{n_i})\) and so equation (A) gives

\[\mu_{n_{r_i}} * f \rightarrow \nu * f \text{ pointwise}. \quad \cdots \quad (A).\]

Therefore, since we have equality \(\omega_2 \mu - \text{almost everywhere},\)

\[T f = \nu * f \text{ as an element of } W_2.\]
The next lemma plays an important part in finding a necessary and sufficient condition that a Radon measure $\nu$ determines a multiplier $W_1 \to W_2$.

§1.2.7 Lemma.

For any positive, lower semicontinuous functional $F$ defined on $G$, the function

$$\lambda \mapsto |\lambda|(F)$$

is vaguely lower semicontinuous as a map $M \to \mathbb{R}$.

Proof. By definition,

$$|\lambda|(F) = \text{Sup}\{|\lambda|(g) : g \in C_c, g \preceq F\}$$

$$= \text{Sup}\{|\lambda(f)| : f \in C_c, |f| \leq F\}.$$ 

Take any $\varepsilon > 0$ and let $\lambda$ be any measure. There exists $f \in C_c$ such that

$$|\lambda(f)| > |\lambda|(F) - \frac{\varepsilon}{2}$$

But the map $\lambda \mapsto \lambda(f)$ is continuous for the vague topology on $M$ and hence is lower semicontinuous.

This means that there exists a vague neighbourhood $N$ of $\lambda$ such that
for each $\rho \in \mathbb{N}$.

Hence, for every $\rho$ contained in $\mathbb{N}$,

$$|\rho(f)| > |\lambda(f)| - \frac{\varepsilon}{2}$$

Thus the map is vaguely lower semicontinuous, as asserted.

§1.2.8 Theorem.

Let $W_1, W_2$ be the weighted Lebesgue spaces as defined in section 1.2.2.

The map

$$f \mapsto v * f, \quad v \in M,$$

with domain $C_c$ can be continuously extended into a map $W_1 \rightarrow W_2$ (which map is then a multiplier) if and only if

$$\int_{G} \omega_2(x+a) |v|(x) \leq \text{constant } \omega_1(a)$$

for almost all $a$.

Proof. Suppose (P) holds. Take any $f \in C_c$. 

| (P) |
Then

\[ ||v*f||_{W_2} \leq ||v*|f| ||_{W_2} \]

\[ = \int \omega_2(x) \{ \int |f|(x-t) \, d|v|(t) \} \, dx \]

\[ = \int \omega_2(x) \{ \int |f|(x-t) \, d|v|(t) \} \, dx \]

\[ = \int \omega_2(x) \{ \int |f|(x+t) \, d|v|(t) \} \, dx \]

\[ \leq \int |f|(x) \cdot \text{Const.} \omega_1(x) \, dx \]

\[ = \text{const} \ ||f||_{W_1} . \]

\[ \therefore \ |v*f| \leq \text{Const}||f||_{W_1} \text{ for every } f \text{ in } C_c. \]

Since \( C_c \) is dense in \( W_1 \) and \( W_2 \) is complete, the map \( f \rightarrow v*f \) may be continuously extended into a multiplier \( W_1 + W_2 \). Conversely, since the map \( f \rightarrow v*f \) is continuous \([C_c,W_1\text{top}] \rightarrow W_2\), for every \( \phi_n \) in \( C_c(G) \) we have

\[ \int \omega_2(x) |v*_{a_n} \phi_n(x)| \, dx \leq \text{Const} \int \omega_1(x) |r_{a_n} \phi_n(x)| \, dx. \]

Call this inequality (A). Now let \( (\phi_n) \) be any sequence of non-negative \( C_c \) functions satisfying (for \( \omega_1 \)) conditions 1, 2, 3, 4 of Theorem 1.2.1. We begin by showing that

\[ v*_{r_{a_n}} \phi_n \rightarrow r_{r} v \text{ vaguely.} \]
For any \( g \) in \( C_c \),

\[
| (\tau_a \nu^\phi_n)(g) - \tau_a \nu(g) | \\
= | \int \int g(x) \phi_n(t-x) d(\tau_a \nu)(t) dx - \int g(x) d(\tau_a \nu)(x) | \\
= | \int [g(x) - \int g(x-t) \phi_n(t) dt] d(\tau_a \nu)(x) | \\
\leq \int |g(x) - \int g(x-t) \phi_n(t) dt| d(\tau_a \nu)(x).
\]

The barred term is one of a sequence of \( C_c \) functions, all of which have support contained in a fixed compact set. Also this sequence converges uniformly to zero.

Hence the last integral converges to zero and so

\[
\nu^\phi_n \rightarrow \tau_a \nu \text{ vaguely.}
\]

From Lemma 1.2.7 the map

\[
\lambda \rightarrow |\lambda|(\omega_2)
\]

is vaguely lower semicontinuous. Hence, for every \( \varepsilon > 0 \), there is a vague neighbourhood \( U \) of \( \tau_a \nu \) such that \( \rho \in U \) implies

\[
|\rho|(\omega_2) > |\tau_a \nu|(\omega_2) - \varepsilon.
\]

But we know that \( \nu^\phi_n \) is eventually in this neighbourhood. This means that, for every \( \varepsilon > 0 \),

\[
\lim |\nu^\phi_n|(\omega_2) > |\tau_a \nu|(\omega_2) - \varepsilon;
\]
i.e. \[ \lim \|
u^* \tau a^n \phi_n\| (\omega_2) \geq \|\tau a^n\| (\omega_2). \]

Now \[ \|\tau a^n\| (\omega_2) \leq \lim \|
u^* \tau a^n \phi_n\| (\omega_2) \]

\[ \leq \text{Const} \lim \int \omega_1(x) |\tau a^n \phi_n(x)| dx, \]

from inequality (A),

\[ = \text{Const.} \lim \int \omega_1(x) \phi_n(x-a) dx. \]

But \((\phi_n)\) satisfies condition 4 of Theorem 1.2.1 which enables us to write this last inequality as

\[ \int \omega_2(x+a) d |\nu|(x) \leq \text{Const} \omega_1(a) \]

for almost all \(a\) in \(G\).

§1.3 In the next few pages we find an interesting feature of the space \(\mathcal{D}(\mathbb{R})\); namely the subspace \(D^k \mathcal{L}(\mathbb{R})\) has co-dimension \(k\) in \(\mathcal{L}(\mathbb{R})\) for \(k\) any positive integer. We exploit this knowledge to characterise the multipliers of \(W_1^{(m)}(\mathbb{R})\) into \(W_q^{(n)}(\mathbb{R})\) when \(m \leq n; 1 \leq q \leq \infty\).

§1.3.1 Definition.

For every positive integer \(r\), let us define a map \(J^r : \mathcal{D}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})\) by
\[ (J^{1}f)(x) = \int_{-\infty}^{x} f(x_1)dx_1 \]

\[ (J^{r}f)(x) = \int_{-\infty}^{x} \int_{-\infty}^{x} \ldots \int_{-\infty}^{x} f(x_1)dx_1 \ldots dx_r. \]

We define \( Jf(\infty) \) by

\[ Jf(\infty) = \lim_{x \to \infty} Jf(x). \]

For any point \( y \) which is less than the left extremity of the support of \( f \), \( Jf(y) = 0 \).

\( \S 1.3.2 \) Lemma.

For each pair \( s, t \) of positive integers and for every \( f \) in \( \mathcal{C}^\infty(\mathbb{R}) \) we have

\[ J^s(J^t f) = J^{s+t} f. \]

\( \S 1.3.3 \) Lemma.

The \( r \)-th derivative of the function \( J^r f \) is the function \( f \).

Proof. Apply the fundamental theorem of the integral calculus \( r \) times.

\( \S 1.3.4 \) Lemma.

For every \( f \) in \( \mathcal{C}^\infty(\mathbb{R}) \) we have

\[ J^r(D^r f) = f. \]
Proof. We first note that $J(Df) = f$. This result follows from the mean value theorem when we note that

$$D[J(Df)] = Df$$

and

$$JDF(y) = 0$$

for any number $y$ less than the left extremity of the support of $f$. Now $J^r(D^r_f) = J^{r-1}[JD(D^{r-1}_f)]$

$$= J^{r-1}D^{r-1}_f$$

$$= f$$ by induction.

With these preliminaries behind us we are now ready to prove an important result.

§1.3.5 Theorem.

A function $f$ belonging to $\mathcal{D}(\mathbb{R})$ is an element of $D^n \mathcal{D}(\mathbb{R})$ if and only if

$$0 = Jf(\infty) = J^2f(\infty) = \ldots = J^n f(\infty).$$

Proof. We tackle the only if part first. Suppose $f = D^n \phi$

where $\phi \in \mathcal{D}$. Then

$$J^i f(\infty) = J^i D^n \phi(\infty)$$

$$= J^i D^i (D^{n-i} \phi)(\infty)$$

$$= D^{n-i} \phi(\infty) \text{ for } i = 1, \ldots, n$$

$$= 0 \text{ for } i = 1, \ldots, n.$$
In order to prove the converse, we have only to show that, when \( k \geq 0 \), the two statements, \( f \) is in \( D^k \mathcal{L} \) and \( J^{k+1} f(\infty) = 0 \), together imply that \( f \) is in \( D^{k+1} \mathcal{L} \). Let us take any \( f \) in \( D^k \mathcal{L} \) satisfying \( J^{k+1} f(\infty) = 0 \). From Lemma 1.3.4, \( J^k f \in \mathcal{L} \) and so \( J^{k+1} f \) has the value \( \int_{-\infty}^{\infty} J^k f \) for all \( x \) greater than the right extremity of the support of \( J^k f \). Now \( J^{k+1} f \) is zero for all \( x \) less than the left extremity of the compact support of \( J^k f \).

Hence \( J^{k+1} f \in \mathcal{L} \) and, by Lemma 1.3.3, \( f \) is in \( D^{k+1} \mathcal{L} \).

§1.3.6 Proposition.

When \( 1 \leq p < \infty \) and \( m \geq 0 \), the space \( W^{(m)}_p(\mathbb{R}) \) contains \( \mathcal{L}(\mathbb{R}) \) as a dense subspace.

Proof. Take any \( f \in W^{(m)}_p(\mathbb{R}) \). We will first show that, for every \( \varepsilon \) greater than zero, there exists \( f \in W^{(m)}_p(\mathbb{R}) \) with compact support such that

\[ ||f - f_\varepsilon||_W < \varepsilon. \]

We know, from the definition of the integral, that there exists a positive integer \( N \) such that

\[ ||X_{[-N,N]}^i f||_p > ||D^i f||_p - \frac{\varepsilon}{2} \]

for

\[ i = 0,1, \ldots, m. \]
From the existence of partitions of unity, we know that there exists a sequence \((\psi_k)\) of functions in \(\mathcal{D}(\mathbb{R})\) and a positive number \(K\) such that

\[
\| D^j \psi_k \|_\infty \leq K \quad \text{for} \quad j = 0, 1, \ldots, m
\]

and

\[
\psi_k = 1 \quad \text{on} \quad [-k, k], \quad 0 \quad \text{on} \quad \mathring{\mathbb{C}} [-k-1, k+1].
\]

Since both

\[
\int_{n}^{n+1} |D^j f|^p \, d\chi \quad \text{and} \quad \int_{-n-1}^{-n} |D^j f|^p \, d\chi
\]

tend to zero as \(n\) tends to infinity, there exists an integer \(M\) such that \(n > M\) implies that both integrals are less than

\[
\frac{\epsilon^p}{2^{(m+2)p,K}}
\]

for all non-negative integers \(i\) not exceeding \(m\).

If we take \(k \geq \max(M,N)\) the function \(f_\epsilon = f \psi_k\) has compact support and satisfies

\[
\| f - f_\epsilon \|_W < \epsilon.
\]

This is clear when we note that, for any \(j\) such that \(0 \leq j \leq m\),
\[ ||D^j f - D^j f_\varepsilon||_p < \frac{\varepsilon}{2} + ||\chi_{[-k,k]}D^j f - D^j f_\varepsilon||_p \]
\[ < \frac{\varepsilon}{2} + 2 \cdot \frac{2^j K \varepsilon}{2^{m+2} K} \]
\[ < \varepsilon. \]

We can now obtain the stated result. For \( 0 \leq j \leq m \), \( D^j f_\varepsilon \) is a function in \( L^p \) and has its support contained in \([-k-1, k+1]\).

Let \( (\phi_n) \) be any sequence of non-negative continuous functions satisfying

\[ \text{Support } \phi_n \subset \left[ \frac{1}{n}, \frac{1}{n} \right] \text{ for each } n, \]

\[ \int \phi_n = 1 \text{ for each } n. \]

Then the function \( f_\varepsilon * \phi_n \) has support contained in the fixed compact set \([-k-3, k+3]\) for all \( n \), and

\[ ||D^j f_\varepsilon - D^j (f_\varepsilon * \phi_n)||_p \to 0 \]

as \( n \to \infty \), for all \( j \) between \( 0 \) and \( m \). Thus there exists a positive integer \( n_0 \) such that

\[ ||D^j f_\varepsilon - D^j f_\varepsilon * \phi_n||_p < \varepsilon \]

for all \( n \geq n_0 \) and all \( j \) between \( 0 \) and \( m \). Consequently

\[ ||f - f_\varepsilon * \phi_{n_0}||_W < 2\varepsilon. \]
§1.3.7 Lemma.

The $W^{(m)}_p$-norm $(1 \leq p < \infty)$ induces a weaker topology on $\mathcal{D}(\mathbb{R})$ than the inductive limit topology.

Proof. We establish the result by showing that the identity map $I$ is continuous $\mathcal{D}$(Inductive limit top.) $\rightarrow W^{(m)}_p$. This is equivalent to showing that the restriction of $I$ to $\mathcal{D}_K$ is continuous for every compact set $K$.

However, $(f_n) \rightarrow 0$ in $\mathcal{D}_K$ implies that $(f_n)$ and each of its derivatives converge uniformly to zero. Since $\|D^nf_n\|_p \leq C(K)\|D^nf_n\|_\infty$ for all $i \geq 0$, where $C(K)$ denotes the Lebesgue measure of $K$, we find that $(f_n) \rightarrow 0$ in the $W^{(m)}_p$ topology. Thus it follows that the said restriction is continuous.

§1.3.8 Lemma.

When $g$ is any function in $\mathcal{D}(\mathbb{R})$ and $1 \leq p < \infty$, the map

$$f \mapsto \int f g \, dx$$

is a continuous linear functional on $W^{(m)}_p(\mathbb{R})$. This implies that the norm topology on $W^{(m)}_p(\mathbb{R})$ is stronger than $\sigma(W^{(m)}_p, \mathcal{D})$.

Proof. Using Hölder's inequality, we can see that,

$$|\int f g \, dx| \leq \|f\|_p \|g\|_p' \leq \|g\|_p' \|f\|_{W^{(m)}_p}.$$
§1.3.9 Lemma.

Let $u \in D^{1,\mathcal{C}}(\mathbb{R})$ be such that the support of $u$ is contained in an interval of unit length. For $0 < k \leq i+1$,

$$||J^k u||_\infty \leq ||u||_1.$$  

Proof. We establish this result by induction. First we note that

$$\int_{-\infty}^{\infty} |\int_{-\infty}^{\infty} u(t)dt| dt \leq \int_{-\infty}^{\infty} |u(t)| dt = ||u||_1$$

and so the result is true for $k = 1$. Suppose now that the result is true for $k (< i+1)$. Then

$$|J^{k+1}u(x)| \leq \int_{-\infty}^{\infty} |J^k u(t)| dt \leq \int_{-\infty}^{\infty} |J^k u(t)| dt \leq (\text{length of support } J^k u) ||J^k u||_\infty.$$ 

The last step is obtained by noting that when $k \leq i$ the function $J^k u$ has compact support. The inequalities above combine with the induction hypothesis to tell us that

$$|J^{k+1}u(x)| \leq ||J^k u||_\infty \leq ||u||_1.$$ 

This completes the proof.
§1.3.10

Let \((h_r)\) be the sequence of functions in \(L^2(\mathbb{R})\) defined by

\[ h_r(x) = 0 \quad \text{for} \quad |x| > \frac{1}{2r}, \]

\[ h_r(x) = k_r \exp\left(-\frac{1}{1 - 4r^2x^2}\right) \quad \text{for} \quad |x| < \frac{1}{2r} \]

where

\[ \frac{1}{k_r} = 2 \int_0^{\frac{1}{2r}} \exp\left(-\frac{1}{1 - 4r^2x^2}\right) dx. \]

For any choice whatsoever of scalars \(a_{r,s}\), and any positive integer \(l\), the function

\[ h_r = \sum_{s=0}^{l} a_{r,s} D^s h_1 \]

has support contained in an interval of unit length.

For each positive integer \(r\), we define a sequence of functions \((h_{r,k})\) as follows:

\[ h_{r,0} = h_r - h_1; \]

\[ h_{r,k} = h_{r,k-1} - [J^k h_{r,k-1(\infty)}] D^k h_1 \]

for \(k \geq 1\). We shall call this equation A.

In the light of Theorem 1.3.5, this construction ensures that each \(h_{r,k}\) is a function in \(D^{k+1}(\mathbb{R})\).
§1.3.11 Lemma.

With the notation of the last section,

\[ \| h_{r,s} \|_1 \leq \frac{s}{\prod_{i=0}^{s}} \left( 1 + \| D^i h_1 \|_1 \right) \]

for any non-negative integer \( s \).

Proof. This result will be obtained by induction. Now \( h_{r,0} = h_r - h_1 \),
and so \( \| h_{r,0} \|_1 \leq 1 + \| h_1 \|_1 \). Assume now that the result is true
for \( s = k \geq 0 \); we show this implies the truth of the statement for \( k + 1 \).

\[ h_{r,k+1} = h_{r,k} - [J_{k+2} h_{r,k}^{(\infty)}] D^{k+1} h_1 \]

and therefore

\[ \| h_{r,k+1} \|_1 \leq \| h_{r,k} \|_1 + \| J_{k+2} h_{r,k}^{(\infty)} \| \| D^{k+1} h_1 \|_1 \]

and so, by Lemma 1.3.9,

\[ \| h_{r,k+1} \|_1 \leq \| h_{r,k} \|_1 + \| h_{r,k} \|_1 \| D^{k+1} h_1 \|_1 \]

\[ \leq \frac{\prod_{i=0}^{k+1}}{\prod_{i=0}^{s}} \left( 1 + \| D^i h_1 \|_1 \right). \]

§1.3.12 Lemma.

If we define the scalars \( a_{rs} \) by writing equation A of §1.3.10 as

\[ h_{r,k} = h_r + \sum_{s=0}^{k-1} a_{r,s} D^s h_1, \]

then

\[ |a_{r,s}| \leq \frac{s}{\prod_{i=0}^{s}} \left( 1 + \| D^i h_1 \|_1 \right). \]
Thus for any fixed $s$, $|a_{r,s}|$ is bounded.

**Proof.** The result is immediately obvious when we note that

$$-a_{r,s} = j^{s+1}h_{r,s-1}(\infty)$$

and apply Lemma 1.3.9 and Lemma 1.3.11.

§1.3.13 **Theorem.**

If $m \leq n$, $1 \leq q < \infty$ the multipliers $T$

$$W_1^{(m)}(\mathbb{R}) \to W_q^{(n)}(\mathbb{R})$$

are of the form

$$Tf = S^jf$$

where

$$D^iS \in M_{bd} \text{ for } 0 \leq i \leq n-m \text{ when } q = 1$$

and

$$D^iS \in L^q \text{ for } 0 \leq i \leq n-m \text{ when } q > 1.$$  \hspace{1cm} (A)

Conversely, every such map is a multiplier.

**Proof.** The converse follows easily from the inequality

$$\sum_{j=0}^{n} ||u \ast D^jf||_q \leq ||u||_1 \sum_{j=0}^{m} ||D^jf||_1$$

$$+ ||D^mf||_1 \sum_{j=1}^{n-m} ||D^ju||_q$$

$$\leq (\sum_{j=0}^{n-m} ||D^ju||_q) (\sum_{i=0}^{m} ||D^i f||_1)$$
which is valid for all \( f \) in \( W_1^{(m)} \).

For the main result, we can confine our attention temporarily to the restriction of \( T \) to \( \mathcal{D}(\mathbb{R}) \).

From Lemma 1.3.7 and Lemma 1.3.8 we can see that \( T \) is continuous as a map \( \mathcal{D} \rightarrow \mathcal{D}' \) when both of these spaces have their natural topologies. Schwartz ([2] pp. 53-54) has shown that such a map, commuting with translations has the form

\[
T f = S* f
\]

for a suitably chosen \( S \in \mathcal{L}' \). We now show that if \( T \) is continuous as a map

\[
W_1^{(m)} \rightarrow W_q^{(n)}
\]

then \( S \) satisfies condition (A).

The continuity of \( T \) is equivalent to

\[
\text{Max} \left\{ ||S* D^i f||_q : 0 \leq i \leq n \right\} \leq \text{Const} \text{Max} \left\{ ||D^i f||_1 : 0 \leq i \leq m \right\}
\]

for every \( f \) in \( \mathcal{D} \). This condition implies that

\[
||S* D^j f||_q \leq \text{Const} \sum_{i=0}^{m} ||D^i f||_1, \quad 0 \leq j \leq n.
\]

Again we may rewrite this condition as

\[
||S* D^i f||_q \leq \text{Const} \sum_{j=0}^{m} ||D^j f||_1, \quad 0 \leq i \leq m.
\] (B)

and

\[
||D^j S* D^m f||_q \leq \text{Const} \sum_{k=0}^{m} ||D^k f||_1, \quad 0 \leq j \leq n-m.
\] (C)

If \( f \) has support contained in an interval of unit length we have,
from Lemma 1.3.9, \[ \sum_{i=0}^{m} ||D^i f||_1 = \sum_{i=0}^{m} ||J^i(D^m f)||_1 \leq (m+1)||D^m f||_1. \]

Applying this result to \( h_{r,m-1} \) as defined in §1.3.10, equations (C) become

\[ ||D^{i}S^{h_{r,m-1}}||_q \leq \text{Const}(i) \]

for \( 0 \leq i \leq n-m \).

Equations (B) and (C) also give us, for \( i \geq 0, j \geq 0 \) and \( i + j \leq n \),

\[ ||D^{i}S \ast D^{j}h_{1}||_q \leq \text{Const}(i + j). \]

Lemma 1.3.12 tells us that in

\[ h_{r,m-1} = h_{r} - \sum_{s=0}^{m-1} a_{r,s} D^{s}h_{1}, \]

\[ \sup_{0 \leq s \leq m-1} |a_{r,s}| < \infty. \]

Hence \( ||D^{i}S \ast h_{r}||_q \leq \text{Const}(i) \)

for \( 0 \leq i \leq n-m \).

Thus \( \{D^{i}S^{h_{r}} \} \) is equicontinuous as a subset of \( L^{q} \) [respectively \( M_{bd} \)], the dual of \( L^{q'} \) [respectively \( C_{0} \)]. But \( L^{q'}(\mathbb{R}) \), \( (q > 1) \) and \( C_{0}(\mathbb{R}) \) are both separable. Hence

\[ \sigma(L^{q}(\mathbb{R}), L^{q'}(\mathbb{R})) \{D^{i}S^{h_{r}} \} \]

and

\[ \sigma(M_{bd}(\mathbb{R}), C_{0}(\mathbb{R})) \{D^{i}S^{h_{r}} \} \]

are metrizable topologies.
From Edwards [1], 1.11.4(2) there exists a subsequence \( (n_j) \) of \( (n) \) and \( g \in L^q \) [respectively \( M_{bd} \)] such that

\[
\lim_{n_j} D^i S^* h_{n_j} \to g \quad \text{in} \quad \sigma(L^q, L^{q'}) \quad \text{[resp.} \quad \sigma(M_{bd}, C_0)\text{].}
\]

Therefore

\[
\lim_{n_j} D^i S^* h_{n_j} \to g \quad \text{in} \quad \sigma(L^q, \tilde{L}^{q'}) \quad \text{[resp.} \quad \sigma(M_{bd}, \tilde{C})\text{].}
\]
But \((D^i S^* h) \to D^i S\) in \(\sigma(L^q, L^r)\) and in \(\sigma(M_{bd}, L^r)\). Since these topologies are Hausdorff, \(g = D^i S \in L^q\) [resp. \(M_{bd}\)]. Thus the restriction of \(T\) to \(L^r\) can be represented in the form (A). It only remains to show that the continuous extension of this map to \(W^{(m)}_1\) can be represented in the form (A).

Let \(f\) be any function in \(W^{(m)}_1\). We know there exists a sequence \((f_n)\) of functions in \(L^r\) such that

\[
(f_n) \to f \text{ in the } W^{(m)}_1\text{-norm.}
\]

Since the convolution \(S^* f\) exists as a function in \(W^{(n)}_q\) we only need to show that

\[
(S^* f_n) \to S^* f \text{ in the } W^{(n)}_q\text{-norm.}
\]

For \(0 \leq j \leq m\),

\[
\|D^j (S^* f_n - S^* f)\|_q \leq \|S\|_q \|D^j (f - f_n)\|_1.
\]

For \(m \leq j \leq n\),

\[
\|D^j (S^* f_n - S^* f)\|_q \leq \|D^{j-m} S\|_q \|D^m (f - f_n)\|_1.
\]

Since the right hand sides of both inequalities tend to zero as \(n\) tends to infinity, the proof is now complete.
§1.4. In this section we use some special properties of Hilbert spaces in order to characterise completely the multipliers of the Sobolev spaces associated with $L^2(\mathbb{R})$.

§1.4.1 Theorem.

Let $X$ be a Hilbert space; $A$ a vector subspace of $X$; $Y$ a Banach space and $f$ a continuous linear operator, $f : A \to Y$.

Then there exists an extension $F$ of $f$, $F : X \to Y$, such that

$$||F||_X = ||f||_A.$$ 

Proof. Let $f'$ be the natural extension of $f$ to the closure of $A$ defined by

$$f'(x) = \lim f(x_n) \quad \text{when} \quad (x_n) \to x \quad \text{in} \quad X.$$ 

Then

$$||f'||_A = ||f||_A.$$ 

Since $X$ is a Hilbert space $\overline{A}$ is a direct summand. Let $P$ be the orthogonal projection associated with $\overline{A}$, $P : X \to \overline{A}$. Then $||P||_X = 1$.

Define a map $F : X \to Y$ by

$$F(x) = f'P(x).$$
Corollary. Since we may complete any separated inner-product space to form a Hilbert space, the above result is true for any separated inner-product space.

§1.4.2 Proposition.

Let \( i \) be any positive integer. The space \( D^i \) is dense in \( L^p(\mathbb{R}) \) for any \( p \) satisfying \( 1 < p < \infty \).

Proof. In view of the fact that \( D \) is dense in \( L^p(\mathbb{R}) \), it will be sufficient to prove that, for any positive integer \( i \), \( D^i \) is dense in \( D^{i-1} \) when the latter space is endowed with the \( L^p \)-norm topology.

For each \( n \), let \( \rho_n \) be the function in \( D \) defined by

\[
\rho_n(x) = \begin{cases} 
0, & x \geq n, \\
n \exp\left(-\frac{n^2}{n^2-x^2}\right) & \text{when } |x| < n,
\end{cases}
\]

and where

\[
\frac{1}{k_n} = 2n \int_0^1 \exp\left(-\frac{1}{1-t^2}\right)dt.
\]
By construction \( \int_{-\infty}^{\infty} \rho_n(x) \, dx = 1 \).

Let \( f \) be any function in \( D^{i-1} \). Then, if \( a = J^i f(\infty) \), the function \( f' = f - a D^{i-1} \rho_n \) belongs to \( D^i \). Now
\[
\| D^{i-1} \rho_n \|_\infty \leq \frac{1}{n} \| D^{i-1} \rho_1 \|_\infty \quad \text{and so}
\]
\[
\| D^{i-1} \rho_n \|_p \leq \frac{1}{n^i} \| D^{i-1} \rho_1 \|_\infty (2n)^p \quad \left( \frac{1}{p} - 1 \right) \leq \text{Const.} \quad n \to \infty \text{ since } p > 1, \ i > 1.
\]

Hence \( \| f - f' \|_p \to 0 \) as \( n \to \infty \). i.e. \( D^i \) is dense in \( D^{i-1} \) when the latter has the \( L^p \)-norm topology.

Footnote.

This result can also be obtained using the Hahn-Banach theorem.

§1.4.3.

Let \( S \) be an arbitrary distribution on \( \mathbb{R} \) and let \( n \) be any positive integer.

Define a linear functional \( U \) on \( D^n \) by \( U(D^n f) = (-1)^n S(f) \).

If there exist real numbers \( v > 1 \) and \( w > 1 \) such that
\[
\| U^* D^n f \|_W \leq \text{Const} \| D^n f \|_v
\]
for every \( f \) belonging to \( \mathcal{D} \); we may now define an extension of \( U \) to \( \mathcal{D} \) in such a way that this continuity condition is preserved. Since \( D^n \) is dense in \( \mathcal{D} \) when the latter has the topology induced by the
$L^V$-norm, we define this extension to be the usual extension of $U$ to the closure of $D^\infty$.

This extension $\bar{U}$ also satisfies

$$||\bar{U}^*f||_W \leq \text{Const} ||f||_V$$

for every $f$ in $\mathcal{D}$.

§1.4.4 Definition.

The distribution $\bar{U}$ just defined will be called the $n$-th anti-derivative of $S$ and will be denoted by $(D^{-n}S)_0$. Also, we will denote the restriction of $(D^{-n}S)_0$ to $D^{\infty}_{\mathcal{D}}$ by $D^{-n}S$. We know that, for every $f \in \mathcal{D}$,

$$(-1)^n D^{-n}S(D^n f) = S(f).$$

§1.4.5.

Denote the space $\mathcal{D}((\mathbb{R})$ with topology defined by

$$N^1(f) = ||D^1 f||_2$$ by $S_1$.

Proposition. The continuous linear operators $T$ commuting with translations

$$S_1 \rightarrow W^{(n)}_q$$

are precisely the maps $Tf = U^*f$ where

$$(D^{j-i}U)_0 \in L^q_2$$ for $j = 0, \ldots, n$. 
Proof. We know that $T$ is continuous as a map $\mathcal{D} \to \mathcal{D}'$ and so, from Schwartz's Theorem [2] pp. 53-54, there exists $U \in \mathcal{D}'$ such that

$$Tf = U\ast f \text{ for every } f \text{ in } \mathcal{D}. $$

The continuity of $T$ implies that for every $f \in \mathcal{D}$ we have

$$||U\ast D^j f||_q \leq \text{Const} \cdot ||D^j f||_2 \quad 0 \leq j \leq n;$$

this is equivalent to

$$||D^{j-1} U \ast D^j f||_q \leq \text{Const} \cdot ||D^j f||_2$$

for $0 \leq j \leq n$. This condition can be rewritten as

$$||D^{j-1} U \ast f||_q \leq \text{Const} \cdot ||f||_2, \quad 0 \leq j \leq n,$$

for every $f$ in $\mathcal{D}'(R)$. But, by Proposition 1.4.2, $D^j \mathcal{D}(R)$ is dense in $\mathcal{D}(R)$ when the latter has the $L^2$-norm. Following the construction in §1.4.3 and §1.4.4 we find that

$$||D^{j-1} U \ast f||_q \leq \text{Const} \cdot ||f||_2$$

for every $f$ in $\mathcal{D}(R)$.

Hence $(D^{j-1} U)_0 \in L^q$ for $0 \leq j \leq n$.

Almost trivially, every $U$ satisfying the above condition defines a map which is continuous and commutes with translations.
§1.4.6 Theorem.

If $T$ is a multiplier

$$W^m_2(R) \to W^n_q(R), \quad 1 \leq q < \infty$$

then

$$Tf = \left( \sum_{0 \leq i \leq m} U_i \right) \hat{f} \text{ for every } f \text{ in } \mathcal{D}(R) \text{ where}$$

$$(D^{j-i}U_i)_0 \in L^q_2 \text{ for } 0 \leq i \leq m, \quad 0 \leq j \leq n$$

Conversely, every map satisfying this condition can be extended to a multiplier of $W^m_2(R)$ into $W^n_q(R)$.

Proof. Endow $\prod_{0 \leq i \leq m} S_i$ with the product topology. The map

$$J : [\mathcal{D}, W^m_2 \text{ topology}] \to \prod_{0 \leq i \leq m} S_i, \text{ defined by } f \to (f, f, \ldots f)$$

is continuous. Let $W = J(\mathcal{D})$. Since $J$ is 1-1, $J^{-1}$ with domain $W$ is a map and this map is clearly continuous. Let $T$ be any continuous linear function, commuting with translations, mapping

$$[\mathcal{D}, W^m_2 \text{ topology}] \to W^n_q.$$ 

Now $T' = T \circ J^{-1}$ is also continuous as a map from $W$ into $W^n_q$. By Theorem 1.4.1, there exists an extension $T''$ of $T'$ to the whole of $\prod_{0 \leq i \leq m} S_i$. Now $T''(f_0, f_1, \ldots f_m) = \sum_{0 \leq i \leq m} U_i \hat{f_i}$, from Proposition 1.4.5, where the $U_i$ satisfy condition A.
But $T = T' \circ J$ and so $Tf = \sum_{0 \leq i < m} U_i f$ for every $f$ in $\mathcal{D}$.

The converse of this result is obvious.
In this chapter we shall be concerned mainly with Sobolev spaces.

Section 2.1 is devoted to finding results about the Sobolev spaces over \( \mathbb{R} \); the rest of the chapter is concerned with Sobolev spaces over \( \mathbb{R}^k \).

In sections 2.2 and 2.3 we use results of Lizorkin and Calderón to gain valuable insight into the nature of the Sobolev spaces \( W^{(m)}_p(\mathbb{R}^k) \), when \( 1 < p < \infty \).

The main result of this chapter, Theorem 2.4.3, depends crucially on this insight.

§2.1. In this section we show, when \( m = 1 \) or \( 2 \), how to resolve any bounded measure \( \mu \) on \( \mathbb{R} \) in the form

\[
\mu = \sum_{i=0}^{m} D^i \mu_i
\]

where each \( \mu_i \) is in \( W^{(m)}_{bd}(\mathbb{R}) \).

We then use this decomposition to characterise the multipliers

\[
W^{(m)}_1(\mathbb{R}) \rightarrow W^{(m)}_q(\mathbb{R})
\]

obtaining the same results as in Section 1.3.

We first prove an easy extension of the Hahn-Banach theorem.

2.1.1 Theorem.

Let \( p_1, p_2, \ldots, p_n \) be semi-norms on a linear space \( X \). If \( Y \) is a
subspace of $X$ and $f$ is a linear functional on $Y$ such that there is a constant $K$ so that

$$|f(y)| \leq Kp_i(y)$$

for $i = 1, 2, \ldots, n$ and for every $y$ in $Y$, then there exists an extension $F$ (of $f$) to the whole of $X$ such that

$$|F(x)| \leq Kp_i(x)$$

for $i = 1, 2, \ldots, n$ and for every $x$ in $X$.

Proof. Define $N(x) = \min\{p_1(x), \ldots, p_n(x)\}$ and $\overline{N}(x) = \inf\{a \geq 0 : x \in aV_1\}$ where $V_1 = \text{convex envelope of } \{y \in X : N(y) \leq 1\}$. This seminorm $\overline{N}$ is the largest seminorm majorised by $N$. We can see that this is so when we note that the set $\{x : \overline{N}(x) \leq 1\}$ is the smallest linearly closed convex set containing $\{x : N(x) \leq 1\}$.

With this notation, we can rewrite equation A as

$$|f(y)| \leq KN(y).$$

The map $y \mapsto |f(y)|$ is a seminorm on $Y$ which is majorised by $KN$ and hence by $\overline{KN}$.

Hence, by the Hahn-Banach theorem, there exists an extension $F$ of $f$ such that

$$|F(x)| \leq \overline{KN}(x)$$
for every $x$ in $X$.

Since $\overline{N}(x) \leq \underline{N}(x)$, we have $|F(x)| \leq K\underline{N}(x)$ which implies that

$$|F(x)| \leq Kp_1(x)$$

for every $x$ in $X$ and for $i = 1, 2, ..., n$.

§2.1.2 Lemma.

Let $\mu$ be a bounded measure on $\mathbb{R}^k$ such that the distributional derivative $D_j\mu = \nu_j$ is also a bounded measure. Suppose that $f$ is an indefinitely differentiable function in $L^1(\mathbb{R}^k)$ such that the pointwise derivative $\partial_j f$ belongs to $L^1(|\mu|)$. Then

$$\int_{\mathbb{R}^k} D_j f \, d\mu = - \int_{\mathbb{R}^k} f \, d\nu_j.$$

Proof. Let $(\phi_m)$ be a sequence of functions in $C_c(\mathbb{R}^k)$ such that

$\phi_m = 1$ on the cube $K_m = \{x \in \mathbb{R}^k : |x_i| \leq m, i = 1, ..., n\}$

$\phi_m = 0$ on $\mathbb{R}^n \setminus K_m$ and

$$\|D_j \phi_m\|_\infty \leq P < \infty.$$

Now we know from the definition of the distributional derivative that

$$\int D_j (\phi_m f) \, d\mu = - \int \phi_m f \, d\nu_j.$$
But

$$\int D_j (f \phi_m) d\mu = \int \phi_m D_j f d\mu + \int f D_j \phi_m d\mu$$

and

$$|\int f D_j \phi_m d\mu| \leq |f| |D_j \phi_m| d\mu| \leq |f|_\infty |\int D_j \phi_m| d\mu| \leq |f|_\infty P \int_{B(m)} d|\mu|$$

where $B(m)$ is the set $K_{m+1} \sim K_m$.

But $\int_{B(m)} d|\mu| \to 0$ as $m \to \infty$ because $|\mu| = \sum_{m=0}^\infty \int_{B(m)} d|\mu|$.

Also

$$\int \phi_m D_j f d\mu + \int D_j f d\mu$$

because

$$D_j f \in L^1(|\mu|).$$

Hence

$$\int D_j f d\mu = -\int f d\nu_j.$$

We now introduce some notation which will be used in the next few subsections.

The map $J_1 : \mathcal{C}(R^k) \to C^\infty(R^k) \cap L^\infty(R^k)$ is defined by

$$J_1 f(x_1, x_2, \ldots, x_k) = \int_{-\infty}^{x_1} f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_k) dt.$$

The map $N_0 : \mathcal{C}(R^k) \to R$ is defined by

$$N_0(f) = \text{Min}\{|f|_\infty, |J_1 f|_\infty, \ldots, |J_k f|_\infty\}.$$
We define \( N_1 : \mathcal{D}(\mathbb{R}^k) \to \mathbb{R} \) by

\[
N_1(f) = N_0 \left( \frac{\partial f}{\partial x_1} \right).
\]

Also \( N : \mathcal{D}(\mathbb{R}^k) \to \mathbb{R} \) is given by

\[
N(f) = \max\{N_0(f), N_1(f), \ldots, N_k(f)\}.
\]

The seminorm \( \overline{N}_1 : \mathcal{D}(\mathbb{R}^k) \to \mathbb{R} \) is defined by

\[
\overline{N}_1(f) = \inf\{a > 0 : f \in aC_1\}
\]

where \( C_1 \) is the convex envelope of

\[
\{f : N_1(f) \leq 1\}.
\]

Finally the seminorm \( \overline{N} : \mathcal{D}(\mathbb{R}^k) \to \mathbb{R} \) is defined by

\[
\overline{N}(f) = \max\{\overline{N}_0(f), \overline{N}_1(f), \ldots, \overline{N}_k(f)\}.
\]

Despite the notation, we are not asserting that \( \overline{N} \) is obtained from \( N \) in the same way that \( \overline{N}_1 \) is obtained from \( N_1 \).

§2.1.3 Lemma.

The dual of \( \mathcal{D}(\mathbb{R}^k) \) with the topology defined by the seminorm \( \overline{N}_1 \) is identical to the set of linear functionals on \( \mathcal{D}(\mathbb{R}^k) \) which are majorised by some multiple of \( \overline{N}_1 \).
Proof. Suppose $f$ is in the dual of $[\mathcal{L}, \bar{N}_1$ topology] and $L$ is (a constant) such that $|f(x)| \leq LN(x)$. Since $\bar{N}_1 \subset \bar{N}_1$, we then have $|f(x)| \leq LN(x) \leq LN(x)$.

Suppose $g$ is a linear functional on $\mathcal{L}$ such that $|g(x)| \leq KN(x)$. Now $g^{-1}([-1,1])$ is a convex set in $\mathcal{L}$ which contains the set

$$\{x \in : N_1(x) < \frac{1}{K}\}.$$ 

Hence $g^{-1}([-1,1])$ contains an $\bar{N}_1$ neighbourhood of $0$ and thus $g$ is continuous on $\mathcal{L}$ when this space has the $\bar{N}_1$ topology and

$$|g(x)| \leq KN_1(x).$$

§2.1.4 Proposition.

The dual of $[\mathcal{L}, \bar{N}_0$ topology] is $M_{bd}^{(1)}$ and for $i = 1, \ldots, k$, the dual of $[\mathcal{L}, \bar{N}_i$ topology] is $D_{i, bd} M_{bd}^{(1)}$.

Proof. We consider first the case $i = 0$. We show that $[\mathcal{L}, \bar{N}_0$ topology] $\subseteq M_{bd}^{(1)}$. Take $\mu \in M_{bd}^{(1)}$, $\nu_j = D_j \mu$, $g \in \mathcal{L}$; then

$$|\int g d\mu| = \int |J_j g d\nu_j|$$

$$\leq \frac{\max_{i}(|\mu|_1, \nu_i)_1}{\min_{i}(|g|_\infty, |J_i g|_\infty)}$$

$$\leq \text{Const.} N_0(g),$$
the first step by means of Lemma 2.1.2.

Now we show \( (\mathcal{L}, \mathcal{N}_0 \text{ topology})' \subseteq \mu(1). \) Take \( \ell \in (\mathcal{L}, \mathcal{N}_0 \text{ topology})' \).

Then

\[
|\ell(g)| \leq K \min_{i} \{ |g_i|_\infty, \ldots, |J g|_\infty, \ldots \}
\]

\[
\leq K |g|_\infty.
\]

Hence \( \ell \) is generated by a bounded measure \( \mu \).

The continuity of \( \ell \) entails that

\[
|\int g d\mu| \leq K |J g|_\infty
\]

and so

\[
|\int \frac{\partial f}{\partial x_i} \, d\mu| \leq K |f|_\infty
\]

which means that

\[
|D_i \mu(f)| \leq K |f|_\infty
\]

for every \( f \) in \( \mathcal{L} \).

Hence \( D_i \mu \) is a bounded measure for \( i = 1, \ldots, k \) which means that

\[
\mu \in \mathcal{M}_{\text{bd}}^{(1)}.
\]

We now turn to the case \( i \neq 0 \). First we show that \( (\mathcal{L}, \mathcal{N}_1 \text{ top.})' \supseteq D_1 \mathcal{M}_{\text{bd}}^{(1)} \).

Suppose \( \mu \in \mathcal{M}_\text{bd}^{(1)}, \nu_j = D_j \mu \) etc., and \( g \in \mathcal{L} \). Then

\[
|\int g d\nu_j| = |\int \frac{\partial g}{\partial x_1} \, d\mu|
\]

\[
= |\int (J \frac{\partial g}{\partial x_1}) d\nu_j|
\]

\[
\leq \max\{||\mu||_1, ||D_j \mu||_1\} \mathcal{N}_1(g);
\]
the last equality comes again from Lemma 2.1.2.

Finally we show \([\mathcal{D}, N_1 \text{ top.}]' \subseteq D_1^M(1)\). Take \(\lambda \in [\mathcal{D}, N_1 \text{ top.}]'\).

Now

\[ |\lambda(f)| \leq KN_1(f) \leq K|f|_\infty. \]

Hence \(\lambda\) is a bounded measure. We must now show there exists \(L \in M_{bd}^{(1)}\) such that \(D_1L = \lambda\). Define \(L^0\) on \(D_1\mathcal{D}\) by

\[ L^0(f) = -\lambda(f) \quad \text{for } f \in D_1\mathcal{D}. \]

Hence \(|L^0(f)| \leq K|f|_\infty\) and \(|L^0(f)| \leq K|J_j f|_\infty, \quad j = 1, \ldots, k,\)

for every \(f\) in \(D_1\mathcal{D}\). Now, by Lemma 2.1.1 (Modified Hahn-Banach Theorem), there exists an extension \(L\) of \(L^0\) defined on \(\mathcal{D}\) such that

\[ |L(f)| \leq K|f|_\infty \]

and

\[ |L(f)| \leq K|J_j f|_\infty, \quad j = 1, \ldots, k,\]

for every \(f\) in \(\mathcal{D}\). This means that we can write

\[ |D_j L(f)| = |L \left[ \frac{\partial f}{\partial x_j} \right]| \leq K|f|_\infty \]

for every \(f\) in \(\mathcal{D}\), and so \(L \in M_{bd}^{(1)}\) and the definition of \(L^0\) shows that \(D_1L = \lambda\).
§2.1.5 Theorem.

The dual of \([\mathcal{D}(R^k), \mathcal{N} \text{ topology}]\) is

\[
\{\mu_0 + \sum_{i=1}^{k} D_i \mu_i : \mu_i \in \mathcal{M}(1) \text{ for } i = 0,1,\ldots, k\}.
\]

Proof. Consider the map \(J\)

\[
J : [\mathcal{D}, \mathcal{N} \text{ topology}] \to \bigoplus_{i=0}^{k} [\mathcal{D}, \mathcal{N}_i \text{ topology}]
\]

defined by \(f \to (f,f,\ldots,f)\). This map is 1-1 and continuous for the product topology. Let \(J[\mathcal{D}] = V\). Then \(J^{-1}|V\) is a continuous map. Let \(\ell\) be any continuous linear functional on \([\mathcal{D}, \mathcal{N} \text{ topology}]\) and define \(\ell' : V \to R\) by \(\ell' = \ell \circ J^{-1}\). Define a seminorm \(N'\) on \(\bigoplus_{i=0}^{k} \mathcal{D}\) by

\[
N'(f_0,f_1,f_2,\ldots,f_k) = \sum_{i=0}^{k} \mathcal{N}(f_i).
\]

Then \(|\ell'(f)| \leq MN'(f)|\) on \(V\). By the Hahn-Banach theorem, there exists a continuous extension \(\ell''\) (of \(\ell'\)) to the whole of \(\mathcal{D}\).

We know that for \(i = 0,1,\ldots,k\) there exist \(\phi_i\) in the dual of \([\mathcal{D}, \mathcal{N}_i \text{ topology}]\) such that

\[
\ell''(f_0,f_1,\ldots,f_k) = \sum_{i=0}^{k} \phi_i(f_i)
\]

for

\[
(f_0,f_1,\ldots,f_k) \in \bigoplus_{i=0}^{k} \mathcal{D}.
\]

Now \(\ell = \ell' \circ J = \ell'' \circ J\).
Therefore \( l(f) = \sum_{i=0}^{k} \phi_i(f) \)

\[ = \mu_0(f) + \sum_{i=1}^{k} (\tilde{D}_i \mu_i)(f) \]

where \( \mu_1 \in M_{bd}(1) \).

Our next task is to show that \( \bar{N} \) is equivalent to some otherwise known norm so that we may identify their dual spaces. In the case \( k = 1 \) the equivalent norm is easily found. For the rest of this section we shall restrict ourselves to \( R^1 \).

§2.1.6 Theorem.

The topology defined by \( \bar{N} \) on \( \mathcal{D}(R) \) is equivalent to that defined by the \( L^\infty \)-norm.

Proof. It is obvious that \( ||f||_\infty \geq \bar{N}(f) \) for every \( f \) in \( \mathcal{D} \).

We now show that there can never exist a sequence of functions \( (\phi_n) \) such that \( \phi_n \in \mathcal{D} \) and \( ||\phi_n||_\infty = 1 \), which tends to zero in the \( \bar{N} \) topology.

For \( N(\phi_n) \) to tend to zero we must have

\[ ||J\phi_n||_\infty \to 0 \quad \text{and} \quad \left|\left| \frac{d\phi_n}{dx} \right|\right|_\infty \to 0. \]
Let $x^n_0$ be a point such that $|\phi_n(x^n_0)| = 1$, and let $x^n_1$ and $x^n_2$ be the closest points to $x^n_0$ such that $x^n_1 < x^n_0 < x^n_2$ and $|\phi_n(x^n_i)| = \frac{1}{2}$, $i = 1, 2$. Now, since $||\phi_n||_\infty = 1$, $||J\phi_n||_\infty \to 0$ implies that $|x^n_2 - x^n_1| \to 0$. But

$$\left|\frac{d\phi_n}{dx}\right|_\infty \geq \frac{1}{4|x^n_2 - x^n_1|},$$

and so $\left|\frac{d\phi_n}{dx}\right|_\infty \to \infty$ as $|x^n_2 - x^n_1| \to 0$. Thus there can never exist a sequence $(\phi_n)$ of functions in $\mathcal{D}'(\mathbb{R})$ such that $||\phi_n||_\infty = 1$ for each $n$ and for which $N(\phi_n) \to 0$.

Thus the seminorm $N$ is equivalent to the $L^\infty$-norm on $\mathcal{D}'(\mathbb{R})$.

**Corollary.**

The above result combined with Theorem 2.1.5 tells us that for every $\mu$ in $\mathcal{M}_{bd}(\mathbb{R})$, there exist bounded measures $U, V$ such that

$$\mu = U + DV$$

where

$$U, DU, V, DV$$

are all bounded measures.

§2.1.7 **Theorem.**

Let $M$ be the map $\mathcal{D}'(\mathbb{R}) \to \mathbb{R}$ defined by

$$M(f) = \text{Max} \left\{ \begin{array}{c}
\text{Min}\{||f||_\infty, ||Jf||_\infty, ||J^2f||_\infty\} \\
\text{Min}\{||Df||_\infty, ||f||_\infty, ||Jf||_\infty\} \\
\text{Min}\{||D^2f||_\infty, ||Df||_\infty, ||f||_\infty\} 
\end{array} \right\}$$
Then there exists \( k > 0 \) such that

\[
||f||_\infty \geq M(f) \geq k||f||_\infty
\]

for all \( f \) in \( \mathcal{D}(\mathbb{R}) \).

**Proof.** It is obvious that \( ||f||_\infty \geq M(f) \). The result will be established if we show that it is impossible to construct a sequence \((\phi_n)\) such that \( \phi_n \in \mathcal{L} \), and \( ||\phi_n||_\infty = 1 \) for each \( n \), such that even one of the following happen:

(i) \( ||J\phi_n||_\infty \to 0 \) and \( ||D\phi_n||_\infty \to 0 \);

(ii) \( ||J\phi_n||_\infty \to 0 \) and \( ||D^2\phi_n||_\infty \to 0 \);

(iii) \( ||J^2\phi_n||_\infty \to 0 \) and \( ||D\phi_n||_\infty \to 0 \);

(iv) \( ||J^2\phi_n||_\infty \to 0 \) and \( ||D^2\phi_n||_\infty \to 0 \).

Let \( x^n_0 \) be any point at which \( |\phi_n(x^n_0)| = 1 \) and let \( x^n_2 \) be the least value of \( x \) which is greater than \( x^n_0 \) and which satisfies \( |\phi_n(x^n_2)| = \frac{1}{2} \). Let \( a_n = \frac{1}{3}(x^n_2 - x^n_0) \). Now

\[
2||J^2\phi_n||_\infty \geq \sup_{x \in \mathbb{R}} |J^2\phi_n(x + a_n) - J^2\phi_n(x)|.
\]

Let \( \theta_n \) be the smallest function such that

\[
J^2\phi_n(x + a_n) - J^2\phi_n(x) = a_n \int_{-\infty}^{x + \theta_n(x)a_n} \phi_n(t)dt.
\]

We also know that \( 0 < \theta_n(x) < 1 \). For any \( y_1 > y_2 \) we have

\[
\sup_x \left| a_n \int_{-\infty}^{x + \theta_n(x)a_n} \phi_n \right| \geq \frac{a_n}{2} \left| \int_{-\infty}^{y_1 + \theta_n(y_1)a_n} \phi_n + \int_{-\infty}^{y_2 + \theta_n(y_2)a_n} \phi_n \right|.
\]
Let \( y_1 = x_0^n + 2a_n \), \( y_2 = x_0^n \). But

\[
\begin{aligned}
\left| \int y_1^{\theta_n} (y_1)^{a_n} \right| & \geq \left| \int y_2^{\theta_n} (y_2)^{a_n} \right| \geq \frac{x_0^n + 2a_n}{x_0^n + a_n} \geq \frac{1}{2a_n}.
\end{aligned}
\]

Hence

\[
||J^2 \phi_n||_\infty \geq \frac{1}{3} (a_n)^2.
\]

Thus

\[
||J^2 \phi_n||_\infty \to 0
\]

implies \( a_n \to 0 \). We already know from theorem 2.1.6 that

\[
||J \phi_n||_\infty \to 0 \quad \text{implies} \quad a_n \to 0.
\]

Now

\[
||D \phi_n||_\infty \geq \sup_{x_0^n, x_2^n} |D \phi_n(x)| \geq \frac{1}{6a_n} \to \infty.
\]

Hence (i) and (iii) are impossible. Now \( D \phi_n(x_0^n) = 0 \) for every \( n \) and so for every \( x > x_0 \) we have

\[
D \phi_n(x) = \int_{x_0^n}^x D^2 \phi_n(t) dt.
\]

Then

\[
\sup_{x \in [x_0^n, x_2^n]} |D \phi_n(x)| \leq 3a_n \sup_{x \in [x_0^n, x_2^n]} |D^2 \phi_n(x)|
\]

which means that

\[
||D^2 \phi_n||_\infty \geq \sup_{x \in [x_0^n, x_2^n]} |D^2 \phi_n(x)| \geq \frac{1}{18 a_n} \to \infty.
\]
Hence (ii) and (v) are impossible.

Corollary.

By using a result similar to theorem 2.1.5 we can prove that for every bounded measure \( \mu \) there exist measures \( U, V, W \) in \( M_{bd}^{(2)} \) such that

\[
\mu = U + DU + D^2 V.
\]

§2.1.8.

We can now use the decompositions for \( \epsilon_0 \) (otherwise called the Dirac delta function) given in §2.1.6 and §2.1.7 in order to find the multipliers of \( W_1^{(1)}(R) \) and \( W_1^{(2)}(R) \) into \( W_q^{(n)}(R) \). We obtain the same results as in Theorem 1.3.13.

Theorem.

The multipliers

\[
W_1^{(m)}(R) \rightarrow W_q^{(n)}(R) \quad m = 1, 2; \quad n \geq m
\]

are of the form

\[
Tf = S^i f \quad \text{for } f \text{ in } W_1^{(m)}(R),
\]

where \( D^i S \) is in \( L^q \) for \( 0 \leq i \leq n-m \)

for \( q > 1 \), and \( D^i S \) is in \( M_{bd} \) for \( 0 \leq i \leq n-m \)

for \( q = 1 \).
Proof. Suppose $T$ is a multiplier

$$W^1_1(R) \to W^q_q(R).$$

Now as $T$ is continuous as a map $\mathcal{D} \to \mathcal{D}'$ and commutes with translation, there exists $S$ in $\mathcal{D}'$ such that

$$Tf = S*f$$

for each $f$ in $\mathcal{D}$. The fact that $T$ is continuous $W^1_1 \to W^q_q$ implies that

$$\left| \left| D^j(S*f) \right| \right|_q \leq \text{Const} \sum_{0 < i < m} \left| \left| D^i f \right| \right|_1$$

for every $f$ in $\mathcal{D}$ and $0 \leq j \leq n$.

Now we know that there exist $f_i$ in $\mathcal{M}^m_{\text{bd}}(R)$ such that the measure

$$\epsilon_0 = \sum_{0 < i < m} D^i f_i$$

Since $\mathcal{D}$ is dense in $\mathcal{M}^m_{\text{bd}}(R)$ we can find a sequence $(f_{i,v})$ such that

$$(f_{i,v}) \to f_i$$

in the $\mathcal{M}^m_{\text{bd}}$-norm and each $f_{i,v}$ is in $\mathcal{D}$. Take any $\phi$ in $\mathcal{D}$. Then

$$\left| \left| (f_i - f_{i,v})(\phi) \right| \right|_\infty \leq \left| \left| f_{i,v} - f_i \right| \right|_1 \left| \left| \phi \right| \right|_\infty$$

and so $f_{i,v} \to f_i$ in the sense of $\mathcal{D}'$. Thus $D^i f_{i,v} + D^i f_i$ in the sense of $\mathcal{D}'$. Hence $h_v = \sum_{0 < i < m} D^i f_{i,v} + \epsilon_0$ in sense of $\mathcal{D}'$.  

Those inequalities of (B) for which \( j \leq m \) tell us that

\[
| | S^{D^i f_{1, \nu}} | |_q \leq \text{Const} \sum_{j=0}^{m} | | D^j f_{1, \nu} | |_1
\]

where \( i = 0, 1, \ldots, m \). Adding together all \( m+1 \) of these we get

\[
| | S^h \nu | |_q \leq \text{Const} \sum_{i=0}^{m} \left( \sum_{j=0}^{m} | | D^j f_{1, \nu} | |_1 \right).
\]

But each bracketed term on the right hand side is bounded and so

\[
| | S^h \nu | |_q
\]

is bounded with respect to \( \nu \). We now use the rest of §1.3.13 after the appearance of a similar inequality to conclude that \( T \) is of the form given in (A).

§2.2.

In the present section we shall use a result of P.I. Lizorkin [1] to show that every multiplier of \( W_p^{(m)}(\mathbb{R}^k) \) into \( W_q^{(n)}(\mathbb{R}^k) \),

\( 1 < p, q < \infty \), \( m \leq n \), may be extended to a multiplier of

\[
L^p(\mathbb{R}^k) \text{ into } L^q(\mathbb{R}^k).
\]

§2.2.1.

The following notation will be used for the remainder of the chapter:

\[
x = (x_1, x_2, \ldots, x_k)
\]

will denote a point in \( \mathbb{R}^k \);

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)
\]

will be a \( k \)-tuple of non-negative integers.
Also \( |x| = \left( \sum_{i=1}^{k} x_i^2 \right)^{1/2} \);

\[ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} ; \]

\[ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n ; \]

\( \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k) \).

In the case of derivatives we shall write

\[ D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_k^{\alpha_k} \]

for the DISTRIBUTIONAL derivative and

\[ \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_k^{\alpha_k} \]

for the POINTWISE derivative.

Following Edwards [1] p.375, we define \( \mathcal{F} = \mathcal{F}(R^k) \) to be the subspace of \( C^\infty(R^k) \) formed of those functions \( u \) for which

\[ p_m(u) = \sup \{ (1 + |x|)^m \partial^\alpha u : |\alpha| \leq m, \ x \in \mathbb{R}^k \} \]

is finite for \( m = 1, 2, \ldots \). The natural topology on \( \mathcal{F} \) makes it a Fréchet space, and \( \mathcal{F}' \), the dual of \( \mathcal{F} \) will be called the space of tempered distributions. We may extend the Fourier transform from \( \mathcal{F} \) to \( \mathcal{F}' \) by means of the adjoint map. We will denote the Fourier transform of a tempered distribution \( f \) by \( \mathcal{F} f \) or \( \hat{f} \). We note that \( \mathcal{F} \) is a topological automorphism of \( \mathcal{F}' \).
If $z$ is any real number we define a map $J^z : \mathcal{F}' \to \mathcal{F}'$ as follows:

$$(J^z f)^\wedge = (1 + 4\pi^2 |x|^2)^{-\frac{z}{2}} f.$$ 

Clearly the operations $J^z$ form an additive group. The map $J^z$ in the Bessel potential of order $z$ in the special case when $z$ is real. See Calderón [1].

§2.2.2.

Firstly we state the theorem of P.I. Lizorkin [1].

Let the functional $\phi$ on $\mathbb{R}^k$ be continuous together with the derivatives $D^\beta \phi$ (where $\beta_i = 0$ or 1 for $i = 1, 2, \ldots, k$) in the region $R_0 = \{ x : |x_i| > 0 \text{ for } i = 1, 2, \ldots, k \}$. Then $\phi$ is the transform of a distribution in $L^p$ if there is an absolute constant $M$ such that

$$|x^\beta D^\beta \phi| \leq M$$

for the derivatives, where $\beta_i = 0$ or 1 for $i = 1, 2, \ldots, k$. We note that there are $2^k$ inequalities to be satisfied in (A).

Theorem.

Let $m$ be any positive integer and let $\alpha$ be a $k$-tuple of non-negative integers such that $|\alpha| \leq m$. Then, for $1 < p < \infty$, $D^\alpha J^m$ is a multiplier of $L^p(\mathbb{R}^k)$ into itself.
Proof. In order to establish our theorem we have to show that for every k-tuple of non-negative integers \( \alpha : |\alpha| \leq m \).

\[
[x^\alpha (1 + 4\pi^2|x|^2)^{-\frac{m}{2}}]
\]

satisfies the \( 2^k \) inequalities given in (A).

We note that

\[
\partial^\beta [x^\alpha (1 + 4\pi^2|x|^2)^{-\frac{m}{2}}]
\]

is of the form

\[
\sum_{\gamma \leq \beta} \binom{\beta}{\gamma} C_{\gamma} x^{\alpha - \beta + \gamma} \partial^\gamma [(1 + 4\pi^2|x|^2)^{-\frac{m}{2}}].
\]

But

\[
\partial^\gamma [(1 + 4\pi^2|x|^2)^{-\frac{m}{2}}] = \frac{p_\gamma}{(1 + 4\pi^2|x|^2)^{\frac{m}{2} + |\gamma|}}
\]

where \( p_\gamma \) is a polynomial of degree \( \leq |\gamma| \).

For \(|x| \) large,

\[
x^\beta \partial^\beta [x^\alpha (1 + 4\pi^2|x|^2)^{-\frac{m}{2}}]
\]

\[
= 0 \quad (|x| |\alpha|^{-m})
\]

\[
= 0 \quad (1) \quad {\text{provided}} \quad |\alpha| \leq m.
\]

For \(|x| \) small

\[
x^\beta \partial^\beta [x^\alpha (1 + 4\pi^2|x|^2)^{-\frac{m}{2}}]
\]

\[
= 0 \quad (1).
\]
Hence the result follows.

§2.2.3.

It is an obvious consequence of the definition of $J^u$ that if

$$\Delta = \sum_{i=1}^{k} (D_i)^2$$

then

$$(1 - \Delta)J^u = J^{u-2}.$$
Definition.

Let $u$ be a real number and $1 \leq p < \infty$. We define $L^P_u(R^k)$ to be the image of $L^P(R^k)$ under the map $J^u$.

If $f$ is a function in $L^P_u(R^k)$ then $f = J^u g$ for some $g$ in $L^P(R^k)$. Since the Fourier-Schwartz transform is a bijective map of $\mathscr{S}'$ onto itself we can see that $g$ is unique and is given by

$$g = \mathcal{F}^{-1}((1 + 4\pi^2|x|^2)^{\frac{u}{2}}f).$$

We now define the norm

$$\|f\|_{p,u} \text{ of } f \in L^P_u(R^k)$$

by

$$\|f\|_{p,u} = \|g\|_p.$$

Thus $J^u$ is an isometry of $L^P(R^k)$ onto $L^P_u(R^k)$ which means that $J^u$ is a topological isomorphism $L^P(R^k) \to L^P_u(R^k)$. As a consequence of this homeomorphism $L^P_u(R^k)$ is complete.

The next theorem shows that, when $1 < p < \infty$, $L^P_m(R^k)$ is identical to $W^{(m)}_p(R^k)$.

§2.2.4 Theorem. (Calderón [1]).

Let $m$ be a positive integer and $1 < p < \infty$. Then $f$ is in $L^P_m(R^k)$ if and only if $f$ has distributional derivatives of all orders up to and including $m$ each in $L^P$. The norms $\|\cdot\|_{p,m}$ and

$$\sum_{|\alpha| \leq m} \|D^\alpha\cdot\|_p$$

are equivalent.
Proof. Suppose \( m \) is a positive integer and \( f \in L^p_m(\mathbb{R}^k) \). Then there exists \( g \in L^p(\mathbb{R}^k) \) such that \( f = J^m g \). Also \( D^\alpha f = D^\alpha J^m g \) which, by Theorem 2.2.2, is also in \( L^p \) if \( |\alpha| \leq m \).

Further, noting that \( ||g||_p = ||f||_{p,m} \), we find
\[
||D^\alpha f||_p = ||D^\alpha J^m g||_p \leq \text{Const} ||g||_p \leq \text{Const} ||f||_{p,m}.
\]
Conversely, suppose that \( D^\alpha f \in L^p \) for \( 0 \leq |\alpha| \leq m \). Let \( r \geq 0 \) be an integer such that \( m + r = 2s \).

Then, according to Theorem 2.2.2, \( D^\alpha J^r f \in L^p \) for \( |\alpha| \leq m + r = 2s \). Hence \( J^{-m} f = J^{-2s} J^r f = (1 - \Delta)^s J^r f \in L^p \) and
\[
||J^{-m} f||_p \leq C \sum_{|\alpha| \leq 2s} ||D^\alpha J^r f||_p \leq C' \sum_{|\alpha| \leq m} ||D^\alpha f||_p
\]
by Theorem 2.2.2 again. But
\[
||J^{-m} f||_p = ||f||_{p,m}.
\]
Hence \( f \in L^p_m(\mathbb{R}^k) \). We thus find that the norms \( ||f||_{p,m} \) and \( \sum_{|\alpha| \leq m} ||D^\alpha f||_p \) are equivalent.

The following chain of topological isomorphisms summarises this section so far:
\[
L^p(\mathbb{R}^k) \xrightarrow{J^m} L^p_m(\mathbb{R}^k) \xrightarrow{W^p(m)} (\mathbb{R}^k)
\]
where \( 1 < p < \infty \).
§2.2.5 Theorem.

Let $m$ be any positive integer. For $1 < p < \infty$ each $f$ in $L^p(R^k)$ can be written in the form

$$f = \sum_{0 \leq |\alpha| \leq m} D^\alpha f_\alpha$$

where

$$f_\alpha \in W^{(m)}_p(R^k)$$

for each $\alpha$.

Proof. From the results just obtained we know that $W^{(m)}_p(R^k) = J^m[L^p(R^k)]$.

Suppose now that $m$ is even. The multinomial theorem tells us that

$$(B) \ldots (1 + 4\pi^2|x|^2)^{\frac{m}{2}} = \sum_{|\alpha| \leq m} a_\alpha (2\pi i)^{|\alpha|} x^\alpha$$

where $\{a_\alpha\}$ is a family of constants. For each index $\alpha$ in the above sum we define a map $\phi_\alpha : L^p(R^k) \to W^{(m)}_p(R^k)$ by

$$f \mapsto a_\alpha J^m f.$$

We can now see that

$$\sum_{|\alpha| \leq m} D^\alpha (\phi_\alpha f) = \left[ \sum_{|\alpha| \leq m} D^\alpha a_\alpha J^m \right](f)$$

$$= \int_1^{-1} \left[ \sum_{|\alpha| \leq m} a_\alpha (2\pi i)^{|\alpha|} x^\alpha \right]$$

$$= \frac{1}{(1 + 4\pi^2|x|^2)^{m/2}} f$$

$$= f \text{ from (B)}.$$
Thus when we write \( f = \phi f \) we obtain the stated result.

Suppose now that \( m \) is odd. From the previous result any \( f \) in \( L^p \) can be written

\[
f = \sum_{|\alpha| \leq m+1} D^\alpha (\psi f)
\]

where \( \psi f \in W^{(m+1)}_p (\mathbb{R}^k) \). This sum can also be written in the form

\[
f = \sum_{|\alpha| = m+1} D^\alpha (\psi f) + \sum_{|\alpha| \leq m} D^\alpha (\psi f).
\]

For any \( k \)-tuple \( \alpha \) of non-negative integers, let \( \pi(\alpha) \) be the \( k \)-tuple with 1 in the position of the first non-zero \( \alpha_i \) and 0's elsewhere. We can now write the last equation as

\[
f = \sum_{|\alpha| = m+1} D^{\alpha - \pi(\alpha)} (D^{\pi(\alpha)} \psi f) + \sum_{|\alpha| \leq m} D^\alpha (\psi f).
\]

Define the family \( \{ \phi_\alpha \} \) of maps \( \phi_\alpha : L^p(\mathbb{R}^k) \to W^{(m)}_p (\mathbb{R}^k) \) by

\[
\phi_\alpha = \psi_\alpha \text{ if } |\alpha| < m \text{ and }
\]

\[
\phi_\alpha = \psi_\alpha + \sum D^{\pi(\alpha')} \psi_\alpha, \text{ otherwise},
\]

where the summation is taken over all \( \alpha' \) such that \( \alpha' - \pi(\alpha') = \alpha \). Since \( W^{(m)}_p \) contains \( W^{(m+1)}_p \) (and the injection is continuous) we can write the resolution as

\[
f = \sum_{|\alpha| \leq m} D^\alpha (\phi f) \text{ where } \phi f \in W^{(m)}_p (\mathbb{R}^k).
\]
Corollary.

The $\phi_\alpha$ defined in this section for both $m$ and even and $m$ odd are multipliers of $L^p(R^k)$ into $W_p^{(m)}(R^k)$, $1 < p < \infty$.

This is a consequence of Theorem 2.2.2.

We can now prove the main result of section 2.2.

Theorem 2.2.6.

Let $m, n$ be positive integers such that $m \leq n$, and $1 < p, q < \infty$. Every multiplier of $W_p^{(m)}(R^k)$ into $W_q^{(n)}(R^k)$ may be continuously extended into a multiplier of $L^p(R^k)$ into $L^q(R^k)$.

Proof. Suppose $T$ is a multiplier

$$W_p^{(m)}(R^k) \rightarrow W_q^{(n)}(R^k).$$

The last theorem enables us to write any $f \in L^p(R^k)$ in the form

$$f = \sum_{|\alpha| \leq m} D^\alpha(\phi_\alpha f).$$

Define a map $T' : L^p \rightarrow L^q$ by

$$T'f = \sum_{|\alpha| \leq m} D^\alpha(T\phi_\alpha f).$$

Then $T'$ is a multiplier $L^p \rightarrow L^q$ because $T\phi_\alpha$ is a multiplier $L^p \rightarrow W_q^{(n)}$ and $m \leq n$. The stated result now follows because

$$Tf = T'f \quad \text{whenever } f \in W_p^{(m)}.$$
§2.3

In this section we find how to resolve any bounded measure $\mu$ on $\mathbb{R}^k$ in the form

$$\mu = \sum_{|\alpha| \leq m+r} D^\alpha f_\alpha$$

where $r$ is any positive integer, $f_\alpha$ is in $W^m_p(\mathbb{R}^k)$ and $p$ depends on $r$. We use this result to determine further properties of the multipliers of $W^m_p(\mathbb{R}^k)$ into $W^m_q(\mathbb{R}^k)$.

§2.3.1.

Let us define the following function on $\mathbb{R}^k$ for $0 < \Re z < k+1$

$$G_z(x) = \gamma(z) e^{-|x|^2} \int_0^\infty e^{-|x|^2} t^{(t^2/2)} dt,$$

where the constant $\gamma(z)$ is given by

$$\frac{1}{\gamma(z)} = \frac{k-1}{(2\pi)^2 \Gamma(z/2) \Gamma(k-z+1/2)}.$$

The Fourier transform of $G_z$ is

$$G_z(\xi) = \frac{1}{(1+4\pi^2 |\xi|^2)^{k/2}}.$$

By using the semigroup properties of $G_z$, namely $G_{z+z'} = G_z * G_{z'}$, we may extend our definition to $0 < \Re(z) < 2k+2$ by defining

$$G_{z+z'} = G_z * G_{z'},$$

where $k+1 > \Re(z), z' > 0$, and so on for any finite $z$. Both $G_z$ and $G_{z'}$ are functions in $L^1(\mathbb{R}^k)$ so the convolution always makes sense. We also note that if $\omega = z_1' + z_1 = z_2' + z_2$ where $\Re \omega, z_1, z_2, z_1', z_2' > 0$, then on taking the Fourier transform.
\[ \mathcal{F} (G_z, z G_z) = \frac{1}{(1+4\pi^2 |x|^2)^{\omega/2}} \]
\[ \mathcal{F} (G_z, z G_z) = \frac{1}{(1+4\pi^2 |x|^2)^{\omega/2}} \]

But \( \mathcal{F} \) is 1-1 as a map \( L^1 \rightarrow C_0 \) and so our "definition" uniquely defines \( G_{z+z'} \).

It is proved elsewhere that \( G_z \) is indefinitely differentiable on \( \mathbb{R}^k \setminus \{0\} \). For example see Aronszajn and Smith, "Theory of Bessel Potentials", Annales de l'Institut Fourier, Tome X1, 1961.

§2.3.2.

In this subsection we find some further useful features of the function \( G_z \) defined in the previous subsection.

Calderón ([1], Theorem 4) proves the following inequality for \( x \neq 0, \quad 0 < u < k \):

\[ |\delta^\alpha G_u (x)| \leq C_{u, \alpha} \frac{|x|}{2} |x|^{-k-u-|\alpha|} \]

... (A).

From this we immediately obtain

\[ \delta^\alpha G_u \in L^1 (\mathbb{R}^k) \text{ if } |\alpha| < u < k \]

... (B).

We ask the reader to note carefully that in (A) and (B) the derivatives are taken in a pointwise sense on \( \mathbb{R}^k \setminus \{0\} \).
We also note from the inequalities given on page 416 of Aronszajn and Smith [1] that, for any \( v > 0 \), we have

\[ G_v \in L^p \text{ whenever } \]

\[ 1 < p < \infty \text{ and } v - k + \frac{k}{p} > 0. \] ... (C).

Let \( u \) be a real number between 1 and 2 and suppose \( k > 1 \).

From (B) we find that \( G_u \) and \( \mathfrak{g}_u \) are both in \( L^1(R^k) \).

We also find that there exists a number \( p > 1 \) such that \( G_u \) is in \( L^p(R^k) \).

Applying these results to Appendix A we find that

\[ \mathcal{D}_j G = \mathfrak{g}_j G \in L^1(R^k) \]

for \( j = 1, 2, \ldots, k \).

§2.3.3 Theorem.

Suppose \( k > 1 \). For each positive integer \( m \) and each positive integer \( r \) which does not exceed \( m \) we have

\[ G_m \in W^{(m-r)}_p(R^k) \text{ where } 1 \leq p < \frac{k}{k-r} \text{ when } r < k; \]

\[ G_m \in W^{(m-r)}_p(R^k) \text{ for } 1 \leq p < \infty \text{ when } r \geq k. \]

Proof. Let \( \alpha \) be any \( k \)-tuple of non-negative integers such that \( |\alpha| \leq m-r \). We let \( t = |\alpha| + 1 \) and \( \epsilon \) be any positive number less than \( \frac{1}{2} \). We now write
where
\[ m = u_1 + u_2 + \ldots + u_t \]
and
\[ u_i = 1 + \frac{\varepsilon}{|\alpha|} \quad \text{for} \quad i = 1, 2, \ldots, (t-1). \]
We can also write
\[ \alpha = \alpha^1 + \alpha^2 + \ldots + \alpha^t \]
where \( \alpha^i \) is also a \( k \)-tuple of non-negative integers such that
\[ |\alpha^i| = 1 \quad \text{for} \quad i = 1, 2, \ldots, (t-1) \]
and \( \alpha^t = 0. \)

By the remark immediately preceding the statement of this theorem,
\[ D_{\alpha}^i G_{u_i} = \partial_{\alpha}^i G_{u_i} \in L^1 \]
for \( i = 1, \ldots, (t-1). \) At the same time, equation (C) shows that
\[ G_{u_t} \in L^p \quad \text{provided} \quad 1 \leq p < \infty \quad \text{and} \quad k > p(k-r+\varepsilon). \]
Now
\[ D_{\alpha}^m G_{u_t} = D_{\alpha}^1 G_{u_1} \ast \ldots \ast D_{\alpha}^{t-1} G_{u_{t-1}} \ast G_{u_t}. \]
The convolution makes sense and equals the left-hand side, because all the factors except the last are functions in \( L^1 \) and \( G_{u_t} \) is a function in \( L^p. \)

Hence \( D_{\alpha}^m G_{u_t} \in L^p \quad \text{provided} \quad 1 \leq p < \infty \quad \text{and} \quad k > p(k-r+\varepsilon). \) We remind the reader that \( \varepsilon \) is arbitrarily small.
In the case \( r < k \) we thus find that the relation \( D^\alpha G_m \in L^p \) holds for all \( \alpha : |\alpha| \leq m-r \) when

\[
1 < p < \frac{k}{k-r}.
\]

In the case \( r \geq k \), we see that \( D^\alpha G_m \in L^p \), for all \( \alpha : |\alpha| \leq m-r \), whenever \( 1 \leq p < \infty \), \( k > p(k-r+\varepsilon) \). But in this case \( (k-r+\varepsilon) \) is either negative or tends to zero as \( \varepsilon \to 0 \). Thus \( G_m \in W^{(m-r)}(\mathbb{R}^k) \) for \( 1 \leq p < \infty \).

\section{2.3.4.}

Suppose \( k > 1 \). Let \( m, r \) be any positive integers such that \( r \) does not exceed \( m \). Then we may resolve the measure \( \varepsilon_0 \) in the form

\[
\varepsilon_0 = \sum_{|\alpha| \leq m} D^\alpha f_{\alpha} \quad \text{where} \quad f_{\alpha} \in W^{(m-r)}(\mathbb{R}^k)
\]

and \( p \) satisfies \( \ldots \) \( (A) \)

\[
\begin{cases}
1 \leq p < \frac{k}{k-r} & \text{if } r < k, \\
1 \leq p < \infty & \text{when } r \geq k.
\end{cases}
\]

Proof. We first consider the case \( m \) even. Using the multinomial theorem we find that

\[
(1+4\pi^2|x|^2)^\frac{m}{2} = \sum_{|\alpha| \leq m} a_{\alpha} (2\pi i)^{|\alpha|} x^\alpha.
\]

Now consider \( h = \sum_{|\alpha| \leq m} a_D G_m \). Each term is a tempered distribution and the Fourier-Schwartz transform of \( h \) is \( 1 \), the Fourier-Schwartz transform of \( \varepsilon_0 \). We can now use the inversion theorem to obtain
But, according to Theorem 2.3.3, \( a \in W^{(m-r)}_p \) for \( p \) as in (A), and so we have the stated result. Now consider the case \( m \) odd.

Using the previous result,

\[
\varepsilon_0 = \sum_{|\alpha| \leq m} D^\alpha(a \in C^m_0).
\]

\[
= \sum_{|\alpha| \leq m-1} D^\alpha(a \in C^m_0) + \sum_{|\alpha| = m+1} D^\alpha(a \in C^m_0). \quad \ldots \quad (B)
\]

For any partial differentiation operator \( D^\alpha \) such that \( |\alpha| = m+1 \), there exists a multi-index \( \beta \) such that \( |\beta| = 1 \) and \( D^{\alpha-\beta} \) is still a partial differentiation operator. The second sum of equation (B) equals

\[
\sum_{|\alpha| = m+1} D^{\alpha-\beta}(D^{\beta}a \in C^m_0).
\]

We now note that \( f \in W^{(m)}_p \) implies that \( D^j f \in W^{(m-1)}_p \) when \( |j| = 1 \), and that \( W^{(m)}_p \subseteq W^{(m-1)}_p \). Thus equation (B) can be written in the form asserted in the theorem. When \( k=1 \) the powerful result at the end of subsection 2.3.2 is no longer available. However special properties of \( R \) allow us to resolve \( \varepsilon_0 \) in a similar form to 2.3.4.

§2.3.5 Theorem.

Let \( m \) be any positive integer. Then \( \varepsilon_0 = \sum_{0 \leq i \leq m+1} D^i f \), where \( f \in W^{(m)}_p(R) \), for all \( p \) satisfying \( 1 \leq p < \infty \).
Proof. Let us define a real-valued function $\psi$ on $\mathbb{R}$ by

$$
\psi(x) = 0, \quad x \leq 0;
$$

$$
\psi(x) = \frac{1}{m!}x^m, \quad x > 0.
$$

Note that all pointwise derivatives of $\psi$ of order $\leq m-1$ exist everywhere and are continuous. Also $\psi^m$ exists everywhere except at 0. It is also clear that for $i \leq m$, $\psi^i$ is locally integrable so, from Appendix A, we know that $D^i\psi = \psi^i$ for $i = 1, \ldots, m$.

Let $\phi$ be a function in $C_c^\infty(\mathbb{R})$ which is 1 on a neighbourhood of the origin.

Now since $D^m\psi = H$, the Heaviside function,

$$
D^{m+1}(\phi\psi) = \varepsilon_0 + \sum_{r=1}^{m+1} (m+1-r)D^r\phi D^{m+1-r}\psi
$$

... (A)

If $\theta$ is any function in $C_c^\infty$ and if $n \leq m$, we obtain from Leibniz's theorem

$$
\theta.D^n\psi = D^n(\theta\psi) + \sum_{i=0}^{n-1} \theta_i D^i\psi
$$

where $\theta_i \in C_c^\infty$.

It is also obvious that $\phi\psi \in W_p^{(m)}(\mathbb{R})$ for all $p$ satisfying $1 \leq p < \infty$. These facts, taken with equation (A), show that, for every $p : 1 \leq p < \infty$,

$$
\varepsilon_0 = \sum_{0 \leq i \leq m+1} D^i f_i \quad \text{where} \quad f_i \in W_p^{(m)}(\mathbb{R}).
$$
We are now ready to find some further properties of the multipliers of $W_p^m(R^k)$ into $W_q^n(R^k)$.

§2.3.6 Theorem.

Suppose $1 \leq p < \infty$, $1 \leq q < \infty$, $r = I.P.\left(\frac{k}{p}\right) + 1.$ Then if $n > m+r$ the multipliers of

$$
W_p^m(R^k) \rightarrow W_q^n(R^k)
$$

are of the form

$$
Tf = S*tf \text{ for every } f \text{ in } L^1
$$

where

$$
D^\alpha S \in L^1 \text{ for } 0 \leq |\alpha| \leq n-m-r.
$$

Proof. We know from Theorem 2.3.4 and Theorem 2.3.5 that we may decompose $\varepsilon_0$ in the form

$$
\varepsilon_0 = \sum_{|\alpha| \leq m+r} D^\alpha f_\alpha
$$

where $f_\alpha \in W_p^m(R^k)$ and $1 \leq p < \frac{k}{k-r}$. If $\theta \in C^\infty(R^k)$ and $T \in L'(R^k)$ we have the following analogue of the Leibniz rule for any multi-index $\gamma$

$$
D^\gamma (\theta T) = \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} D^\delta T, \quad D^\gamma_\delta \theta,
$$

whence

$$
\theta D^\gamma T = D^\gamma (\theta T) - \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} D^\delta T D^\gamma_\delta \theta.
$$

If $\phi$ is a function in $L(R^k)$ which is 1 on a neighbourhood of the origin, we may use the last two facts in conjunction with
\( \psi \in \mathcal{D}(\mathbb{R}^k), \ f \in \dot{W}^m_p(\mathbb{R}^k) \) implies that \( \psi f \in \dot{W}^m_p(\mathbb{R}^k) \), to conclude that

\[
\varepsilon_0 = \phi \varepsilon_0 = \sum |\alpha| \leq m+r D^\alpha g_\alpha \quad \text{where} \quad g_\alpha
\]

has a compact support and is contained in \( \dot{W}^m_p(\mathbb{R}^k) \) for each \( p \) satisfying

\[
l \leq p < \frac{k}{k-r}.
\]

We also note that if \( T \) is a multiplier of

\[
\dot{W}^m_p(\mathbb{R}^k) \quad \text{into} \quad \dot{W}^n_q(\mathbb{R}^k)
\]

there exists a distribution \( S \) such that

\[
Tf = S*f \quad \text{for every} \quad f \in \mathcal{D}(\mathbb{R}^k).
\]

We now show that

\[
Tf = S*f \quad \text{for every} \quad f \in \dot{W}^m_p(\mathbb{R}^k)
\]

which has compact support. If \( f \) is any function in \( \dot{W}^m_p(\mathbb{R}^k) \) with compact support then there exists a compact set \( K \) and a sequence of functions \( (f_n) \) in \( \mathcal{D}^t_K \) such that

\[
f_n \to f \quad \text{in the} \quad \dot{W}^m_p-\text{norm}.
\]

Since \( T \) is continuous,

\[
\lim f_n = f \quad \text{in the} \quad \dot{W}^m_q-\text{topology}.
\]

This certainly means that

\[
\lim f_n = f \quad \text{in} \quad \sigma(\mathcal{D}^t, \mathcal{L}).
\]
Now for any $\psi \in \mathcal{D}(\mathbb{R}^k)$

$$(S^*f_n)\ast \psi = (S^*\psi) \ast f_n$$

and

$$(S^*f) \ast \psi = (S^*\psi) \ast f.$$ 

Hence we find

$$\left| \int_{\mathbb{R}^k} \left[ (S^*f_n) \psi - (S^*f) \psi \right] \right|$$

$$= \left| \int_{\mathbb{R}^k} (S^*\psi) (f-f_n) \right|$$

$$\leq ||\chi_K(S^*\psi)||_p \cdot ||f-f_n||_p$$

by Hölder's inequality. The expression in the last line $\to 0$ as $n \to \infty$ and we thus obtain $Tf_n \to S^*f$ in $\sigma(\mathcal{D}', \mathcal{D})$, which implies that $Tf = S^*f$ for each $f$ in $W^{(m)}_p$ with compact support.

Finally

$$S = S^*\varepsilon_0 = \sum_{|\alpha| \leq m+r} D^\alpha(S^*g_\alpha)$$

$$\subset \sum_{|\alpha| \leq m+r} D^\alpha W(n)$$

$$\subset W^{n-m-r}$$

and so $D^\alpha S \in L^q$ for $0 \leq |\alpha| \leq n-m-r$. 
§2.4.

We regard this as the most important section of the thesis. A complete and convenient characterisation of the multipliers

$$
\mu_p^{(m)}(\mathbb{R}^k) \to \mu_q^{(n)}(\mathbb{R}^k)
$$

is found for the case $m > 0$, $n > 0$, $1 < p, q < \infty$.

A solution of this multiplier problem has been found for the case $1 < p, q \leq 2$ by Merlo [1].

Results similar to ours have been obtained by a different method by J.F. Price [1].

We first prove two preliminary lemmas.

§2.4.1. Lemma.

Let $K$ be a compact subset of $\mathbb{R}^k$. If $f$ is in $L^1(\mathbb{R}^k)$ and if

$$
\phi(y) = \int_K f(x+y) \, dx
$$

then $\phi$ is also in $L^1(\mathbb{R}^k)$.

Proof.

We may write $\phi$ as $\chi_K * f$ and since $\chi_K$ and $f$ are $L^1$ functions, so is $\phi$.

For any integer $u$ let us denote $\mathcal{F}^{-1}[\frac{1}{(1+4\pi^2|x|^2)^\frac{u}{2}}]$ by $G_u$. 

Let $A$ be any distribution in $L^q_p(R^k)$, $B$ be any bounded measure on $R^k$, and $C$ be any function in $\mathcal{D}'(R^k)$.

**Lemma 2.4.2.**

The simultaneous convolution $G_n * A * B * G_{-2t} * C$ always exists when $n$ and $t$ are non-negative integers.

**Proof.** Let $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ be arbitrary functions in $\mathcal{D}'(R^k)$. From Shirasiti [1] we know that the simultaneous convolution exists if

$$(G_n * \phi_1(x)).(A * \phi_2(y)).(G_{-2t} * \phi_4(z))$$

$$(C * \phi_5(\omega)).(B * \phi_3(x+y+z+\omega))$$

is always a function in $L^1(R^{4k})$. We let

$$G_n * \phi_1 = e \ (\in L^{p_r} \quad 1 \leq r \leq \infty),$$

$$A * \phi_2 = f \ (\in L^q \cap L^{p'},)$$

$$G_{-2t} * \phi_4 = g \ (\in L^p),$$

$$C * \phi_5 = h \ (\in L^q)$$

and

$$B * \phi_3 = k \ (\in L^1).$$
\[ \int_{-\infty}^{\infty} e(x)f(y)g(z)h(\omega)k(x+y+z+\omega) \, dx \, dy \, dz \, d\omega \]
\[ \leq \text{Const} \int_{-\infty}^{\infty} e(x)f(y)g(z)k'(x+y+z) \, dx \, dy \, dz \]

(where \( k'(u) = \int \text{Supp } h \, k(u+x) \, dx \), and by Lemma 2.4.1)

\[ \text{Supp } h < \text{Const} \int_{-\infty}^{\infty} e(x)f(y)g(z)k''(x+y) \, dx \, dy \]

(here \( k''(u) = \int \text{Supp } g \, k'(u+x) \, dx \))

\[ \leq \text{const} \int |(f \ast k')(x)e(x)| \, dx \]

\[ < \infty \text{ since } f \ast k' \in L^q \]

and \( e \) is certainly in \( L^q \).

Thus the simultaneous convolution exists.

\[ \section{2.4.3 Theorem.} \]

Let \( 1 < p, q < \infty \). Every multiplier

\[ T : W_p^m(\mathbb{R}^k) \rightarrow W_q^n(\mathbb{R}^k) \]

is of the form

\[ Tf = U \ast f, \quad f \in W_p^m \]

where

\[ U = X \ast \frac{1}{(1 + 4\pi^2|x|^2)^{\frac{m-n}{2}}} \]

and

\[ X \in L_p^q. \]

Conversely, every such map is a multiplier, \( W_p^m(\mathbb{R}^k) \rightarrow W_q^n(\mathbb{R}^k) \).
Proof. Let $T$ be a multiplier of $W_p^{(m)}(\mathbb{R}^k)$ into $W_q^{(n)}(\mathbb{R}^k)$.

Consider the following commuting diagram. (Which defines $T'$).

\[
\begin{array}{ccc}
W_p^{(m)} & \xrightarrow{T} & W_q^{(n)} \\
J^m \uparrow & & \uparrow J^n \\
L^p & \xrightarrow{T'} & L^q
\end{array}
\]

Since $J^m$, $J^n$ are topological isomorphisms which commute with translations we find that $T$ is a multiplier $W_p^{(m)} \to W_q^{(n)}$ if and only if $T'$ is a multiplier $L^p \to L^q$.

Now $Tf = J^n \circ T' \circ (J^m)^{-1}f$
\[= J^n \circ T' \circ J^{-m}f.\]

Let $r$ be either 0 or 1 so as to make $-m-r(-2t)$ an even integer.

Then $Tf = J^n \circ T' \circ J^r \circ J^{-2t}f$.

The results from sections 2.2 and 2.3 now show that $Tf$ is equal to the simultaneous convolution
\[G_n \ast T' \ast G_r \ast G_{-2t} \ast f\]
whenever $f$ is such that the simultaneous convolution is defined.

Lemma 2.4.2 shows that when $f$ is in $\mathcal{D}(\mathbb{R}^k)$, the simultaneous convolution exists and
\[Tf = G_n \ast T' \ast G_r \ast G_{-2t} \ast f.\]

The simultaneous convolution enables us to re-arrange and re-bracket this expression as we please.
Hence \( T_f = (G_n * G_r * G_{-2t}) * T' * f \) for every \( f \) in \( \mathcal{D}(R^k) \).

Thus \( T_f = G_{n-m} * T' * f \) for every \( f \) in \( \mathcal{D}(R^k) \).

We now have to show that

\[
T_f = T' * G_{n-m} * f
\]

for every \( f \) in \( W^{(m)}_p(R^k) \). Let \( f \) be any function in

\( W^{(m)}_p(R^k) \) and let \( (f_n) \) be a sequence in \( \mathcal{D}(R^k) \) such that

\( (f_n) \to f \) in \( W^{(m)}_p(R^k) \)-norm. It is easy to see that the simultaneous convolutions exist.

Now \( G_{n-m} * f = J^{n-m} f \). From section 2.2.2 we have that

\( J^{n-m} \) is a multiplier of \( W^{(m)}_p(R^k) \) into \( L^p(R^k) \). Hence \( G_{n-m} * f \to G_{n-m} * f \)

in the \( L^p(R^k) \)-norm. But \( T' \) is a multiplier of \( L^p(R^k) \) into \( L^q(R^k) \)

and so

\[
T' * G_{n-m} * f \to T' * G_{n-m} * f
\]

in the \( L^q \)-norm. But the \( L^q \)-norm topology is Hausdorff and so we find

\[
T_f = T' * G_{n-m} * f
\]

for every \( f \) in \( W^{(m)}_p(R^k) \).

The proof of the converse follows the same lines and is omitted.
Appendix A.

The results in this appendix are due to R.E. Edwards whose assistance is again gratefully acknowledged.
Appendix A

I Suppose \( f \) is a real-valued function on \( \mathbb{R}^k \) with the following properties:

(i) \( f \in L^1_{\text{loc}}(\mathbb{R}^k) \);

(ii) \( \partial_j f \) exists on \( \mathbb{R}^k - \{0\} \) and is continuous on \( \mathbb{R}^k - \{0\} \)

for some \( j, 1 \leq j \leq k \). Then \( \partial_j f = c\delta_0 + \partial_j f \) for some constant \( c \).

Proof. If a function \( \phi \) in \( \mathcal{S}(\mathbb{R}^k) \) vanishes on a neighbourhood of zero, elementary partial integration gives

\[
-\int_{\mathbb{R}^k} f \partial_j \phi \, dx_j = \int_{\mathbb{R}^k} (\partial_j f) \phi \, dx_j.
\]

which gives us

\[
-\int_{\mathbb{R}^k} f (\partial_j \phi) \, dx = \int_{\mathbb{R}^k} (\partial_j f) \phi \, dx.
\]

We will write this as

\[
<\phi, \partial_j f> = <\phi, \partial_j f> \quad \ldots \quad (A)
\]

We now show that if \( \psi \) is a function in \( \mathcal{S}(\mathbb{R}^k) \) which vanishes at \( 0 \) then

\[
<\psi, \partial_j f> = <\psi, \partial_j f>.
\]

Distribution theory then shows that

\[
\partial_j f = c\delta_0 + \partial_j f,
\]
where \( c \) is a constant.

Now let \( \psi \) be any function in \( \mathcal{E}(\mathbb{R}^k) \) such that \( \psi(0) = 0 \) and let \( u \) be a function in \( C^\infty(\mathbb{R}^k) \) such that

\[
0 \leq u(x) \leq 1 \text{ for every } x \text{ in } \mathbb{R}^k;
\]

\[
u(x) = 1 \text{ when } |x| \geq 1;
\]

\[
u(x) = 0 \text{ when } |x| \leq \frac{1}{2};
\]

we note that these conditions imply that \( ||\partial_j u||_\infty < \infty \). We now define \( u_v : \mathbb{R}^k \rightarrow \mathbb{R} \) by \( u_v(x) = u(vx) \) for \( v = 1, 2, \ldots \).

We note that \( 0 \leq u_v \leq 1 \) and

\[
\partial_j u_v(x) = v \partial_j u(vx).
\]

From equation (A) we find

\[
-\int_{\mathbb{R}^k} f \partial_j (u \psi) dx = \int_{\mathbb{R}^k} (\partial_j f) u \psi dx \quad \ldots \quad (B)
\]

for \( v = 1, 2, \ldots \).

The Lebesgue dominated convergence theorem shows that as \( v \rightarrow \infty \), the right-hand side of (B) converges to \( \int_{\mathbb{R}^k} (\partial_j f) \psi dx \). The left-hand side of (B) equals

\[
-\int_{\mathbb{R}^k} f u \psi (\partial_j \psi) dx - \int_{\mathbb{R}^k} f \psi (\partial_j u_v) dx.
\]

The dominated convergence theorem shows that the first term converges to \( -\int_{\mathbb{R}^k} f (\partial_j \psi) dx \) as \( v \rightarrow \infty \).
The second term is such that

\[ |\int f(\theta_j u_j) \psi dx| = |\int f(x) \theta_j u(x) \psi(x) dx| \]

\[ \leq v |\theta_j u_j|_{\infty} \int |f\psi| dx. \]

\[ \frac{1}{2v} \leq |x| \leq \frac{1}{v} \]

Since \( \psi(0) = 0 \), \( \sup_{|x| \leq \frac{1}{v}} |\psi(x)| = 0 \left( \frac{1}{v} \right) \), and so the last expression is

\[ \leq v |\theta_j u_j|_{\infty} 0 \left( \frac{1}{v} \right) \int |f| dx \]

\[ \frac{1}{2v} \leq |x| \leq \frac{1}{v} \]

\[ = o(1) \text{ as } v \to \infty. \]

\[ -\int R k^f(\theta_j \psi) dx = \int R k(\theta_j f) \psi dx. \]

i.e.

\[ \langle \psi, D_j f \rangle = \langle \psi, \theta_j f \rangle. \]

Hence \( D_j f = c \epsilon_0 + \theta_j f \), where \( c \) is some constant.

We now look for ways of ensuring that \( c = 0 \). The following restrictions on \( f \) have proved to be useful. The next case is peculiar to \( k = 1 \).
II Suppose \( f \) is a function on \( \mathbb{R}^1 \) which satisfies the conditions of I and is continuous at the point \( x = 0 \).

Then \( \partial f = Df \).

Proof. We may assume that \( f(x) \to 0 \) as \( x \to 0 \). From I we know that

\[
Df = c \varepsilon_0 + \partial f,
\]

so that we now need to show \( c = 0 \). For any \( \psi \) in \( \mathcal{D}(\mathbb{R}) \) we have

\[
<\partial \psi, f> = -<\psi, Df> = -c\psi(0) - \int \partial f \psi \, dx \quad \ldots \quad (C).
\]

Now let us take a sequence \( (\phi_n) \) of functions in \( \mathcal{D}(\mathbb{R}) \) such that \( \phi_n(0) = 1 \),

\[
\text{Supp } \phi_n \subset \left[ -\frac{1}{n}, \frac{1}{n} \right],
\]

and

\[
\partial \phi_n \leq 0 \text{ on } \left[ 0, \frac{1}{n} \right]
\]

and

\[
\partial \phi_n \geq 0 \text{ on } \left[ -\frac{1}{n}, 0 \right].
\]

Then

\[
\left| <\partial \phi_n, f> \right| = \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} \partial \phi_n \, f \, dx \right|
\]

\[
= -\int_{0}^{\frac{1}{n}} \partial \phi_n \, f \, dx + \int_{-\frac{1}{n}}^{0} \partial \phi_n \, f \, dx.
\]

By the continuity of \( f \),
\[
\frac{1}{n} \int_0^n |f| \, dx = o(1) \int_0^n |\phi| \, dx = o(1) \text{ as } n \to \infty.
\]

Similarly, \[
\frac{1}{n} \int_{-1}^0 |f| \, dx = o(1) \text{ as } n \to \infty.
\]

At the same time,
\[
\frac{1}{n} \left| \int \phi \cdot f \, dx \right| \leq \int \frac{1}{n} \left| \phi \right| \, dx = o(1) \text{ as } n \to \infty.
\]

Thus (C) gives us that
\[
c = o(1) \text{ and therefore } c = 0.
\]

### III

When \( k > 1 \), a very useful criterion is available.

Suppose, in addition to the conditions at the start of I, that \( k > 1 \) and \( f \) is in \( L^p(U) \), where \( U \) is a neighbourhood of \( 0 \) in \( \mathbb{R}^k \) and \( p \) is any number strictly greater than unity. Then \( c = 0 \) and
\[
D_j f = \phi_j f, \quad j = 1, \ldots, k.
\]

Proof. We assume that \( j = 1 \) and we will write \( (x_2, \ldots, x_k) \) as \( y \). Thus \( (x_1, x_2, \ldots, x_k) \) will be \( (x_1, y) \).
We want to find a sequence \((\phi_n)\) of functions in \(\mathcal{D}(\mathbb{R}^k)\) such that \(\phi_n(0) = 1\) and
\[
\langle \partial_1 \phi_n, f \rangle = o(1) \text{ as } n \to \infty.
\]
We take \(\phi_n(x, y) = u_n(x) v_n(y)\) where \(u_n\) is in \(\mathcal{D}(\mathbb{R})\);
\[
u_n(0) = 1;
\]
\(\text{Supp } u_n \subset \left[-\frac{1}{n}, \frac{1}{n}\right];\)
\(\exists u_n < 0 \text{ on } [0, \frac{1}{n}];\)
\(\exists u_n > 0 \text{ on } [-\frac{1}{n}, 0].\)
Also
\[
u_n \text{ is in } \mathcal{D}(\mathbb{R}^{k-1});
\]
\(\nu_n(0) = 1;\)
\(0 \leq \nu_n \leq 1;\)

\(\text{Supp } \nu_n \subset (\text{the } (k-1) \text{ dimensional cube}) [-\delta, \delta]^{k-1}\)

where \(\delta\) is a function of \(n\) to be determined, but which will certainly be less than \(\frac{1}{n}\).

We will denote \([-\delta, \delta]^{k-1}\) by \(Q'_n\), and \(\left[-\frac{1}{n}, \frac{1}{n}\right] \times Q'_n\) by \(Q_n\).

It is clear that eventually \(Q_n \subset U\). Then
\[ |\langle 1 \phi_n, f \rangle| = \left| \int_{\mathbb{R}^n} \int_{Q_n} f(x) \partial u_n(x) v(y) dy dx \right| \]

\[ \leq \left( \int_{Q_n} |f|^{\frac{p}{p'}} \right)^{\frac{1}{p'}} \left( \int_{Q_n} |\partial u_n(x)|^{p'} dx \right)^{\frac{1}{p}} \left( \int_{Q_n} |v(y)|^{p'} dy \right)^{\frac{1}{p}} \]

by Hölder's inequality where \( \frac{1}{p} + \frac{1}{p'} = 1. \)

A routine check (performed at the end of this section) shows that \( u_n \) may be chosen so that

\[ \left| \int_{Q_n} |\partial u_n(x)|^{p'} dx \right|^{\frac{1}{p}} \leq \Lambda \left( \frac{1}{n} \right)^{\frac{1}{p}}. \]

So, since

\[ \left( \int_{Q_n} |v_n|^{p'} \right)^{\frac{1}{p'}} = O(\delta)^{\frac{1}{p'}} \]

as \( n \to \infty \), we find that

\[ |\langle 1 \phi_n, f \rangle| = O(\delta^{\frac{1}{p'}}) \left( \int_{Q_n} |f|^{\frac{p}{p'}} \right)^{\frac{1}{p'}} \]

Since \( f \) is in \( L^p(U) \) the right-hand side is \( o(1) \) as \( n \to \infty \) provided we take \( \delta = \left( \frac{1}{n} \right)^{\lambda} \) where \( \lambda \) is so large that \( \frac{\lambda(k-1)}{p'} > \frac{1}{p} \).

But, since \( \partial_1 f \) is locally integrable,

\[ \langle \phi_n, \partial_1 f \rangle = o(1) \text{ as } n \to \infty, \]

Hence \( c = o(1) \text{ as } n \to \infty \), and so \( c = 0. \)
We now show, as promised, that we can find a sequence \((u_n)\) of functions in \(\mathcal{D}(\mathbb{R})\) such that

\[
\frac{1}{n} \int_{-1}^{1} |\partial u_n(x)|^{p'} \, dx \leq A \left(\frac{1}{n}\right)^p
\]

and which satisfies the other properties demanded of \((u_n)\).

We define \(u_1(x) = \exp(-\frac{1}{2x^2}), |x| < 1,\)

\[
u_1 = 0, \quad |x| \geq 1.
\]

\(u_1\) is in \(\mathcal{D}\); \(0 \leq u_1 \leq 1\); \(\text{Supp } u_1 \subset [-1,1]\); \(u_1\) is even and decreasing on \([0,1]\). Define \(u_n(x_1) = u_1(nx_1)\); Then \(u_n\) is in \(\mathcal{D}(\mathbb{R}), 0 \leq u_n \leq 1, \text{ Supp } u_n \subset [-\frac{1}{n}, \frac{1}{n}]\).

\[
\partial u_n(x) = n \partial u_1(nx).
\]

\[
||u_n'||_{p'} = \left(\int \partial u_n(x)|^{p'} \, dx\right)^{\frac{1}{p'}} = \left(n^{p'} \int |\partial u_1(nx)|^{p'} \, dx\right)^{\frac{1}{p'}} = n \left(\int |\partial u_1(y)|^{p'} \, dy\right)^{\frac{1}{p'}} = (n)^{\frac{1}{p'}} ||\partial u_1||_{p'} = \left(\frac{1}{n}\right)^{-\frac{1}{p'}} ||\partial u_1||_{p'}.
\]
REFERENCES


Shirashi [1], "On the definition of convolution for distributions". Journal of Science of Hiroshima University, 23(1) 1959, 19-32.