The Free Boundary Equilibrium Problem for Helically Symmetric Plasmas

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The work presented in this thesis is entirely my own except where explicitly stated otherwise.

Henry Gardner

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**I dedicate this thesis to my parents,
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Abstract

An iterative technique is described for solving the helically symmetric plasma equilibrium equations with a free boundary. The method involves an application of Green's theorem, and has been formulated for the geometry of a heliac. Provision is made to incorporate the field of the external solenoid into the model, and the limits of applicability of the conventional Fourier-Bessel series representation of that field are investigated.

The free boundary model is implemented as part of an inverse equilibrium version of the PEST code and is applied to the SHEILA heliac. Mercier and ballooning stability are examined for fixed and free boundary equilibria as a function of pressure and a deformation parameter which is taken to be the current in an extra helical winding near the centre of the configuration. The free boundary stability properties depend almost entirely on the existence of a vacuum magnetic well.

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Chapter 1

Introduction

The dream of a clean and safe form of nuclear power with a virtually limitless source of fuel has inspired one of the greatest scientific projects in history. The goal of electricity generation by the controlled use of thermonuclear fusion reactions has been approached by scientists and governments around the world in an extraordinary spirit of cooperation (Teller, 1981). After thirty years, although we finally seem to be at the point of a demonstration of the scientific feasibility of a fusion reactor, the dream has been qualified.

The high neutron fluxes expected from the first generation of fusion nuclear reactors, which will almost certainly be closed magnetic “bottles” employing the reaction between deuterium and tritium (Sheffield *et al.*, 1986), will make them dangerous to work with (although it should be emphasised that the possibility of a reactor “meltdown” does not exist). Although fusion power technology will not, like fission power, be dogged by the problem of disposing of long lived radioactive waste, the decommissioning of “hot” reactor materials may pose a local environmental hazard (Lidsky, 1983). Fusion reactors (like fission reactors) could be used to manufacture materials necessary for the construction of nuclear weapons, and may encourage their proliferation (Lidsky, 1983). The advantages of having a “limitless” energy source (the limiting material will most probably be lithium) does not, at the moment, appear to favour fusion over fast-breeder fission reactors (Holdren, 1978).

Today’s “qualifications” reflect the comparative maturity of scientific research on the subject. They, combined with the complexity and large capital cost of a fusion reactor, reduce the prospects for stand-alone fusion power in the short term (Holdren, 1978). In fact, some experts feel that the first commercial application of the controlled fusion of deuterium and tritium will be as a fusion-fission hybrid reactor – in which case the pros

and cons of the two technologies will have to be assessed together (Teller, 1981, Moir, 1981). The prospects for fusion power in the medium to long term (i.e. not before well into the next century) are more optimistic (Sheffield *et al.*, 1986). In particular, the possible development of "advanced fuel" reactors would realise virtually neutron-free operation which would solve many of the economic and safety problems outlined above (Lidsky, 1983).

In the magnetic confinement approach to controlled thermonuclear fusion a plasma of electrons and light ions is contained by a magnetic "bottle" while it is heated to temperatures considerably hotter than the centre of the sun (Bateman, 1978). Our understanding of the basic physics of plasmas in magnetic fields has been advanced enormously by comparing experimental measurements with the theoretical predictions of some surprisingly simple models (Grimm, 1984). In this thesis we shall address some of the issues of plasma equilibrium and stability within the framework of ideal magnetohydrodynamics (MHD) which is, in many ways, the simplest of these models. To set the scene for what is to follow, I shall briefly outline the theory of ideal MHD here and discuss some of the history of its application to magnetically confined plasmas. The theoretical development of this subject has been symbiotic with the evolution of the major fusion experiments, and these will also be mentioned briefly.

Ideal MHD treats the plasma as single, inviscid and adiabatic fluid of infinite conductivity. The equations of continuity and motion and the adiabatic equation of state of an ideal fluid of density ρ , velocity $\mathbf{v}$ and pressure p in the presence of a magnetic field $\mathbf{B}$ and current $\mathbf{J}$ are (Freidberg, 1982):

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1.1)$$

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p, \quad (1.2)$$

$$\text{and} \quad \rho^\gamma \frac{d}{dt} (p \rho^{-\gamma}) = 0. \quad (1.3)$$

These fluid equations are combined with Maxwell's equations in the absence of displacement current (Blank *et al.*, 1957):

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (1.4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.5)$$

$$\text{and} \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (1.6)$$

and Ohm's law in the presence of infinite conductivity:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \quad (1.7)$$

To apply Eqs. (1.1) to (1.7) to a magnetic confinement geometry of interest, one usually considers the linearised perturbations about a stationary equilibrium where

$$\mathbf{J} \times \mathbf{B} = \nabla p, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (1.8)$$

implying $\mathbf{B} \cdot \nabla p = \mathbf{J} \cdot \nabla p = 0$ (Grimm, 1984).

The first step in the application of ideal MHD to a physical situation is to find the solution of the equilibrium equations [Eqs. (1.8)] self consistently with the equations for the applied magnetic field. This is a major problem in itself (McNamara, 1976, Johnson *et al.*, 1979). If the plasma geometry of interest can be assumed to possess an “ignorable coordinate” (if it is, for example, cylindrically, axially, or helically symmetric) then the equilibrium equations are essentially two dimensional. Although this results in a considerable simplification, the problem remains highly non-linear and must be solved by iteration (Lackner, 1976). Since the late 1960’s there has been considerable interest in the axisymmetric “tokamak” configuration (Furth, 1981). Accordingly, some elaborate computer codes have been developed to treat axisymmetric equilibria where, when expressed in terms of a cylindrical coordinate system (r, ϕ_{tor}, z) with its z axis along the major axis of the torus, all scalar quantities are independent of the angle ϕ_{tor} . Here the magnetic field can be expressed in terms of a stream function ψ :

$$\mathbf{B} = \nabla \phi_{tor} \times \nabla \psi + g \nabla \phi_{tor}. \quad (1.9)$$

The finite difference form of the equilibrium equation

$$r \partial_r (r^{-1} \partial_r \psi) + \partial_z^2 \psi = -r J_\phi, \quad (1.10)$$

with

$$J_\phi = \mu_0 r p'(\psi) + r^{-1} g g'(\psi), \quad (1.11)$$

is usually solved on a fixed computational domain with Dirichlet boundary conditions (Johnson *et al.*, 1979). It turns out that ψ is proportional to the “poloidal” magnetic flux – or the flux going the short way around the torus. The “toroidal” current density, flowing the long way around the torus, is equal to J_ϕ . The plasma pressure is given by $p(\psi)$ and $g(\psi)$ is proportional to the total poloidal current. A consequence of the existence of an ignorable coordinate is that the current and magnetic field lines will lie on nested surfaces of constant pressure. In this case ψ will also be constant on these surfaces and is commonly used as a surface label. (They are often called “flux surfaces”.)

The solution of Eq. (1.10) necessitates some prescription for finding ψ on the boundary of the computational domain. This can be included in an iterative procedure by integrating the appropriate Green's function over the toroidal currents both within and external to the plasma (Greene and Johnson, 1961). The currents within the plasma must in turn be calculated from Eq. (1.11) using ψ at an earlier iteration level. This Green's function integral over the plasma cross section can, by an application of Green's theorem, be shown to be equivalent to a line integral over the boundary of the computational mesh – once again involving ψ at the earlier iteration level (Johnson *et al.*, 1979). In free boundary models of this nature the surface of the plasma is defined to be a separatrix of the magnetic field or the surface which is tangent to some physical limiter.

Calculations of the properties of equilibria consisting of nested toroidal magnetic surfaces are simplified by the use of a coordinate system (ψ, θ, η) made up of a surface label, ψ , and poloidal and toroidal angles θ and η respectively. The angle coordinates define a grid within each constant- ψ surface. The converged solution to the equilibrium equations is usually mapped to such a flux coordinate system before performing a stability analysis (described shortly), but some codes have been written which solve the equilibrium equations in flux coordinates directly. The so-called “inverse equilibrium” free boundary code of Delucia *et al.* (1980) puts $\eta = \phi_{tor}$ and iterates the mapping from $(\psi, \theta, \phi_{tor})$ to (r, ϕ_{tor}, z) within the plasma region until a converged (axisymmetric) equilibrium is obtained. At each iteration level ψ_{new} is found by solving Eqs. (1.9) and (1.10) on a curvilinear mesh defined by ψ_{old} and θ_{old} at the previous iteration level. The outer flux contour is taken as defining the plasma boundary which is then reestimated (once the “fixed boundary” iterations have converged) by contouring the new values of the “boundary” fluxes given by Green's theorem. The formulation is similar to that described above except that in this case the boundary integral is taken over the plasma itself.

Having solved the equilibrium equations, the stability of the equilibrium can be determined by solving Eqs. (1.1) to (1.7) in terms of the Lagrangian displacement vector $\xi(\mathbf{r}, t) \equiv \xi(\mathbf{r}) \exp(i\omega t)$ (such that $d\xi/dt \equiv \delta \mathbf{v}$ gives the first order perturbed fluid velocity). It turns out that ξ satisfies an eigenvalue equation

$$-\rho\omega^2\xi = \mathbf{F}(\xi) \quad (1.12)$$

where $\mathbf{F}$ is an Hermitian, time independent partial differential operator (Bernstein *et al.*, 1958). The Hermiticity of $\mathbf{F}$ implies that the eigenvalues, ω^2 , are real so that the normal modes are either purely oscillatory or exponentially growing (or decaying). Also, it follows

that the total perturbed energy is constant in time:

$$\partial_t \left\{ \int \left[\frac{1}{2} \rho \dot{\xi}^2 + \frac{1}{2} \xi \cdot \mathbf{F}(\xi) \right] dV \right\} = 0. \quad (1.13)$$

Another consequence of the ideal MHD equations is that of the “freezing” of the magnetic field lines to the fluid, with the result that the topology of the magnetic field is conserved under perturbative departures from an equilibrium (Bateman, 1978).

A number of techniques have been developed to solve the eigenvalue problem of Eq. (1.12). Over the past decade, two large computer packages, the PEST (Grimm *et al.*, 1976) and the ERATO (Appert *et al.*, 1975) codes, which combine equilibrium solvers with techniques for solving Eq. (1.12) in its variational form, have been widely applied to model fusion experiments. They have often been combined with models of particular types of perturbation which would not be well resolved by a full spectral analysis – the localised ideal and resistive interchange modes (Mercier and Luc, 1974, Glasser *et al.*, 1975) and the “high-*n*” (or large toroidal mode number) ballooning modes (Gruber, 1981a, Dewar *et al.*, 1984). Once these or other methods have been used to verify stability on the time-scale of ideal MHD (typically measured in microseconds), a complete analysis of a plasma configuration should include an examination of its stability against resistive modes (typically of a time-scale of milliseconds) and then look at the quasistationary evolution brought about by dissipative processes within the plasma or by changing external conditions (Grimm, 1984).

In a tokamak, the helical twist of the magnetic field lines is essentially due to the large, externally produced, axisymmetric, toroidal magnetic field in combination with the poloidal field due to the toroidal plasma current. An alternative concept is the stellarator, which is a topologically toroidal magnetic bottle in which the magnetic field line twist is essentially due to non-axisymmetric currents external to the plasma (Shafranov, 1983, Miyamoto, 1978). No net toroidal plasma current is required. (Recent stellarator designs have emphasised the advantages of “net-current-free” operation for MHD stability as well as for the possibility of steady state operation – as opposed to the tokamak which is pulsed.) The inherently three-dimensional stellarator geometry makes theoretical MHD equilibrium and stability analysis much more complicated than for axisymmetric systems. The way forward has traditionally been to consider an asymptotic expansion about a two dimensional configuration in powers of the “inverse aspect ratio” (the ratio of the average plasma radius to its distance from the major axis of the torus) which results in a series of equations similar to those solved for axisymmetric systems (Greene and John-

son, 1961). Recently the PEST equilibrium and stability code has been modified to treat stellarator equilibria using this approach (Anania and Johnson, 1983), and some fully three-dimensional computer codes have begun to receive considerable attention (Bauer *et al.*, 1984). These and other theoretical and experimental advances, including the construction of non-ohmic heating schemes and the design of modular coil systems, have improved the outlook for stellarators to the point where they are (arguably) the leading alternative route to fusion power (Johnson, 1983).

Stellarator equilibria may be divided into two classes. Those which rely on "helical", or equivalent, windings to produce the magnetic field line twist have magnetic surfaces which are usually centred on an axis which is (almost) planar. In the second class, the magnetic axis makes large helical excursions from this central ring. The second class of equilibria in particular may be well approximated, in the limit of large aspect ratio or large number of helical periods, as being helically symmetric. This once again reduces the equilibrium problem to two dimensions. The magnetic field can be expressed in terms of a stream function, ψ , resulting in an equilibrium equation similar to Eq. (1.10).

The PEST and ERATO codes have been modified to treat helically symmetric as well as axisymmetric systems. The helically symmetric version of ERATO has been used to model a number of equilibria of the "heliac" type where the strong " $l = 1$ " curvature is combined with bean-shaped magnetic surface cross-sections (Gruber *et al.*, 1981b, Merkel *et al.*, 1983). They examined a number of fixed boundary equilibria for Mercier stability, high- n ballooning modes and low- n internal and external modes. Equilibria were obtained which were completely ideal MHD stable up to a volume average β (the ratio of plasma pressure to "magnetic pressure") of 30%. The average β ($\langle\beta\rangle$) is considered to be a good figure of merit for fusion devices and this value is a dramatic improvement on the presently forecasted ceiling of $\langle\beta\rangle \approx 5\%$ for the JET tokamak (Bickerton and Keen, 1985) which has been designed to achieve reactor conditions. The ERATO results have been supported by Monticello *et al.* (1984) who used a helical version of PEST in combination with a "Mercier-and-ballooning-stability" code (Dewar *et al.*, 1984) to examine a different parameter range of fixed boundary equilibria. Their equilibrium code, which is a helical version of the inverse equilibrium code of Delucia *et al.* (1980), is especially appropriate for the very indented heliac plasma. The indications of three dimensional equilibrium calculations are that the $\langle\beta\rangle$'s obtainable in heliac plasmas may be limited to about 10% for machines currently being planned (Guasp *et al.*, 1985, Strauss *et al.*, 1986). Even

with this caveat, the prognoses for heliacs have been sufficiently good to encourage a deal of experimental interest in this design (HX1 proposal, 1983, Perea *et al.*, 1985, Blackwell *et al.*, 1985, Blackwell *et al.*, 1986). It should be pointed out, as a postscript to the discussion at the beginning of this chapter, that the road to an advanced fuel reactor is likely to require a plasma configuration of high $\langle\beta\rangle$.

The primary goal of this thesis is to extend the helically symmetric version of PEST to solve the free boundary equilibrium problem. This is a logical carry-over of the work on axisymmetric systems, and, in particular, of the inverse equilibrium code of Delucia *et al.* (1980). The present work is motivated by the increasing experimental interest in helical axis stellarators. It is to be expected that future design studies for these devices will become more concerned with the influence of particular coil designs on the shape of the plasma boundary at significant pressure. The formalism presented here, which involves a detailed application of Green's theorem to a helically symmetric geometry, will be applied to model a simple "straight heliac" with a circular cross-section outer solenoid and two central "core" conductors, and with zero net longitudinal current in the plasma. One of the results of the model turns out to be that even the solenoidal field in vacuum is more complicated than seems to have been noted previously.

Chapter 2 of this thesis reviews the basic mathematics and physics of ideal helically symmetric equilibria. The various quantities in the defining equilibrium equation are interpreted physically, and the general solution to the vacuum equation is presented. In a similar pedagogical vein the geometrical consequences of the straight heliac are reviewed in Chapter 3. The aim of this chapter is to introduce some of the terminology in popular use in heliac research and to explain the basic shape of the vacuum configuration. Toroidal corrections to the helically symmetric model are mentioned briefly.

In order to be of value, a free boundary model must include an accurate determination of the external magnetic fields. In Chapter 4, the problem of finding the scalar potential of an infinite solenoid is described formally. A numerical method is then applied to determine the solenoidal field of a straight heliac in terms of an eigenfunction expansion. The limitations of this procedure are highlighted.

The Green's function for the helically symmetric equilibrium equations is presented and proved to be correct in Chapter 5. Its correspondence with the Green's functions for toroidally and cylindrically symmetric equilibria is demonstrated. In Chapter 6, Green's theorem is applied to a heliac-type "model-problem" which incorporates the external coil

system and a possible longitudinal current in the plasma. This formalism is then used to interpret the numerical results of Chapter 4.

The numerical implementation of an algorithm for solving the free boundary problem is described in Chapter 7. The iterative procedure for free boundary heliac equilibria necessitates a number of improvements on the axisymmetric algorithm of Delucia *et al.* (1980) – due in the main to the very indented nature of the flux surfaces. The code is verified by monitoring its long run behaviour and by comparing the results of a fixed and a free boundary equilibrium at zero pressure. In Chapter 8, the free boundary code is used together with a Mercier-and-ballooning-stability code to examine the stability of the first functioning heliac – the SHEILA heliac at the Australian National University. Finally, a brief summary of the work is presented in Chapter 9 and possible extensions are suggested.

Chapter 2

Introduction to helical symmetry

When we refer to plasmas or magnetic fields as having “helical symmetry”, we mean that all scalar physical quantities may be expressed as functions of the two coordinates, ζ and r , where $\zeta = l\phi - hz$ and (r, ϕ, z) are the usual cylindrical coordinates. The constants l and h define the periodicity of the system. Helically symmetric plasmas with a single magnetic axis may be divided into two classes. Those of the first type have straight magnetic axes and integer l values greater than one. The second class comprises $l = 1$ plasmas with spatial magnetic axes such as the flux tube shown in Fig. (2.1). In both cases the periodicity in the z -direction is $\Delta z = 2\pi/h$, and the plasma column will rotate once in l longitudinal periods.

The mathematics of helically symmetric fields is well known and is reviewed in Dewar

Figure 2.1: An $l = 1$ helically symmetric magnetic surface showing the cylindrical coordinate system.

et al. (1984). First note that for any scalar field $F \equiv F(r, \zeta)$:

$$\begin{aligned} \frac{\partial F}{\partial \zeta} &= \frac{1}{l} \frac{\partial F}{\partial \phi} = -\frac{1}{h} \frac{\partial F}{\partial z} \\ \Leftrightarrow \frac{\partial F}{\partial \phi} &= -\frac{l}{h} \frac{\partial F}{\partial z} . \end{aligned} \quad (2.1)$$

From $\mathbf{B} = \nabla \times \mathbf{A}(r, \zeta)$, it is possible to show that any helically symmetric magnetic field can be written in the form

$$\mathbf{B} = lh\mathbf{u} \times \nabla\psi + lhg\mathbf{u} , \quad (2.2)$$

where

$$\psi = -\left(\frac{1}{h}A_z + \frac{r}{l}A_\phi\right) , \quad (2.3)$$

$$\mathbf{u} = \frac{l\mathbf{e}_z + hre_\phi}{l^2 + h^2r^2} , \quad (2.4)$$

and $g(r, \zeta)$ will be evaluated shortly. The vector $\mathbf{u}$ is often referred to as the "symmetry vector". It has the property

$$\mathbf{u} \cdot \nabla F(r, \zeta) = (\mathbf{u} \cdot \nabla r) \partial_r F + (\mathbf{u} \cdot \nabla \zeta) \partial_\zeta F = 0 . \quad (2.5)$$

Also $\nabla \times \mathbf{u} = 2l\kappa h^{-1}\mathbf{u}$ where

$$\kappa = \frac{h^2}{l^2 + h^2r^2}$$

is the sum of the squares of the curvature, $k = h^2r/(l^2 + h^2r^2)$, and torsion, $\tau = lh/(l^2 + h^2r^2)$, of a helical line of pitch h/l .

2(i) Physical interpretation of ψ and g

Consider a helical ribbon defined by $\zeta = \zeta_0$ (a value between 0 and 2π), $0 \leq r \leq r_0$ and $0 \leq z \leq 2\pi lh^{-1}$. The symmetry vector is tangent to the plane of this ribbon. Denote by $\mathbf{e}_n$ the unit normal in the sense of ϕ increasing (as in Fig. (2.2)), then the total magnetic flux through the ribbon is given by

$$\begin{aligned} \int \mathbf{B} \cdot \mathbf{e}_n dA &= lh \int (\mathbf{u} \times \nabla\psi) \cdot \mathbf{e}_n dA \\ &= -lh \int \nabla \times (\psi\mathbf{u}) \cdot \mathbf{e}_n dA \\ &= 2\pi l [\psi(r_0, \zeta_0) - \psi(r_0 = 0)] , \end{aligned} \quad (2.6)$$

where the direction of the line integrals is defined in accordance with Stokes' law, and we note that the differential arc length, dS , along the helical line ($r = r_0, \zeta = \zeta_0$) is defined by

$$\begin{aligned} (ds)^2 &= r_0^2(d\phi)^2 + (dz)^2 = (h^2r_0^2 + l^2)l^{-2}(dz)^2 \\ &= (dz)^2 l^{-2} |\mathbf{u}|^{-2} . \end{aligned} \quad (2.7)$$

Figure 2.2: Path integral to define the helical flux.

The generalisation to configurations with a spatial magnetic axis may be demonstrated with two ribbons and the vanishing divergence of $\mathbf{B}$. The flux through any helical ribbon with one edge on the magnetic axis and the other on a line ($r = r_0, \zeta = \zeta_0$) and of length $2\pi l h^{-1}$ in the z direction (Dewar *et al.*, 1984) is equal to $2\pi l(\psi(r_0, \zeta_0) - \psi(r_{ma}, \zeta_{ma}))$. ψ is known as the “helical flux function”. Only its gradient is directly measurable .

The quantity

$$\begin{aligned} g(r, \zeta) &= \frac{\mathbf{B} \cdot \mathbf{u}}{lh|\mathbf{u}|^2} \\ &= \frac{lB_z + hrB_\phi}{lh} \end{aligned} \quad (2.8)$$

is a measure of the component of $\mathbf{B}$ in the symmetry direction . It is related to the current by Ampère’s law :

$$\begin{aligned} \mu_0 \mathbf{J} = \nabla \times \mathbf{B} &= \nabla \times (hl\mathbf{u} \times \nabla\psi + hl g\mathbf{u}) \\ &= \frac{hl}{\kappa} \mathbf{u} [\nabla \cdot (\kappa \nabla\psi)] + 2l^2 \kappa g \mathbf{u} + hl \nabla g \times \mathbf{u} \end{aligned} \quad (2.9)$$

Consider the total current through a helical ribbon bounded by the magnetic axis and the line ($r = r_0, \zeta = \zeta_0$) as above – measured in the sense of ϕ increasing:

$$\begin{aligned} I &= \int \mathbf{J} \cdot \mathbf{e}_n dA \\ &= \mu_0^{-1} \int (\nabla \times \mathbf{B}) \cdot \mathbf{e}_n dA \\ &= 2\pi l \mu_0^{-1} [g(r_{ma}, \zeta_{ma}) - g(r_0, \zeta_0)] . \end{aligned} \quad (2.10)$$

Because $g(r, \zeta)$ is directly measurable, it is better to define it in terms of the current through a helical ribbon between one line at ($r = r_0, \zeta = \zeta_0$) and another line at infinity:

$$\bar{I} = 2\pi l \mu_0^{-1} [g(r_0, \zeta_0) - g_\infty] . \quad (2.11)$$

Figure 2.3: *Line integral for calculating g_∞ .*

The value of g_∞ is proportional to the total longitudinal (or z -directed) current. This can be shown by integrating $\mathbf{B}$ along a closed path at infinity as shown in Fig. (2.3). One side of the contour is in the symmetry direction and the return path is parallel to the z axis. Because $\mathbf{B} \cdot \mathbf{e}_z = 0$ at infinity for a closed system, we have

$$\begin{aligned} \mu_0 I_z &= \oint_{C_\infty} \mathbf{B} \cdot d\mathbf{s} \\ &= \int \mathbf{B} \cdot \frac{\mathbf{u}}{|\mathbf{u}|} ds \\ &= h l g_\infty \frac{2\pi l}{h} \frac{1}{l}. \end{aligned}$$

Thus

$$2\pi l g(r_0, \zeta_0) = \mu_0 I_z + \mu_0 \bar{I}(r_0, \zeta_0). \quad (2.12)$$

From Eq. (2.9) we can also derive a local expression for g :

$$\begin{aligned} \mu_0 \mathbf{u} \times \mathbf{J} &= h l |\mathbf{u}|^2 \nabla g \\ \Leftrightarrow \nabla g &= \frac{\mu_0 \mathbf{u} \times \mathbf{J}}{h l |\mathbf{u}|^2}. \end{aligned} \quad (2.13)$$

2(ii) ψ and g for some simple systems

Consider a straight wire carrying a current I along the z -axis. Such a wire is of course cylindrically symmetric, however it may be part of any helically symmetric configuration. The periodicities, l and h , are arbitrary. From Ampère's law,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi$$

and

$$g = \frac{\mu_0 I}{2\pi l}.$$

The helical flux function, ψ_1 , must be a function of r only, and

$$\begin{aligned} \mathbf{B} &= h\mathbf{u} \times \mathbf{e}_r \frac{d\psi_1}{dr} + hlg\mathbf{u} \\ &= hl \frac{d\psi_1}{dr} \left(\frac{l\mathbf{e}_\phi - hr\mathbf{e}_z}{l^2 + h^2r^2} \right) + \frac{\mu_0 I h}{2\pi} \left(\frac{l\mathbf{e}_z + hr\mathbf{e}_\phi}{l^2 + h^2r^2} \right). \end{aligned}$$

This implies

$$\psi_1 = \frac{\mu_0 I}{2\pi h} \ln r + C_1, \quad (2.14)$$

where C_1 is a constant of integration.

Our other "fundamental" configuration is a circular-cylindrical solenoid which is centred on the z -axis and carries a surface current i_0 per unit length confined to the $\mathbf{e}_\phi$ direction. This may also be treated as a helically symmetric geometry with arbitrary l and h . Now

$$\mathbf{B} = \mu_0 i_0 \mathbf{e}_z$$

inside the solenoid, and

$$g = \frac{\mu_0 i_0}{h}.$$

Once again, ψ is a function of r only and the equation relating $\mathbf{B}$ and ψ may be integrated to give

$$\psi_2 = -\frac{\mu_0 i_0}{2l} r^2 + C_2. \quad (2.15)$$

Equation (2.13) implies g will be constant throughout any connected vacuum region. Thus, for example, we are able to find g for an ($l = 1$) helical current-carrying wire by considering its asymptotic behaviour as $r \rightarrow \infty$. At distances large compared to the wire radius, the field will approach that of a tightly wound solenoid:

$$\mathbf{B} \longrightarrow \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi \quad \text{as} \quad r \longrightarrow \infty.$$

So $g = \mu_0 I / 2\pi$, where I is the total current in the wire.

2(iii) The helical Grad-Shafranov equation

Taking the $\mathbf{u}$ component of Eq. (2.9) we have

$$\nabla \cdot (\kappa \nabla \psi) = -\frac{2l\kappa^2}{h} g + \frac{\mu_0 \kappa}{hl} (lJ_z + hrJ_\phi). \quad (2.16)$$

Equation (2.16) enables us to express ψ in terms of a given current distribution. Suppose we are dealing with an ideal-MHD equilibrium configuration which comprises closed,

nested magnetic surfaces. A convenient surface label is the helical flux, and the equilibrium condition

$$\nabla p = \mathbf{J} \times \mathbf{B} \quad (2.17)$$

implies that $\mathbf{B} \cdot \nabla p = 0$ and that p is a function of ψ only.

Noting that,

$$\mathbf{u} \times (\mathbf{u} \times \nabla \psi) = -|\mathbf{u}|^2 \nabla \psi, \quad (2.18)$$

we have

$$\mathbf{J} \times \mathbf{B} = \frac{lh}{\mu_0} |\mathbf{u}|^2 [-hl\kappa^{-1} \nabla \cdot (\kappa \nabla \psi) \nabla \psi - 2l^2 \kappa g \nabla \psi - gh l \nabla g]. \quad (2.19)$$

Equations (2.19) and (2.17) imply that $g(\psi)$ is also a surface quantity and that

$$\nabla \cdot (\kappa \nabla \psi) = -\frac{2l\kappa^2}{h} g(\psi) - \kappa g(\psi) g'(\psi) - \frac{\mu_0 p'(\psi)}{l^2}. \quad (2.20)$$

The solution of Eq. (2.20) is the principal topic of this thesis. We shall refer to it as the helical Grad-Shafranov equation (HGSE) in analogy with the equilibrium equation for axisymmetric plasmas.

2(iv) General solution of the HGSE in vacuum.

In the absence of current, g must be constant and $p'(\psi) = 0$. Either of Eqs. (2.16) or (2.20) gives us

$$\mathcal{L}(\psi) \equiv \nabla \cdot (\kappa \nabla \psi) = -2l\kappa^2 h^{-1} g.$$

We first consider the homogenous equation. Using

$$\frac{d\kappa}{dr} = \frac{d}{dr} \left(\frac{h^2}{l^2 + h^2 r^2} \right) = -2\kappa^2 r, \quad (2.21)$$

we have

$$\mathcal{L}(\psi) = -2\kappa^2 r \partial_r \psi + \kappa \left[\frac{1}{r} \partial_r (r \partial_r \psi) + \frac{l^2 + h^2 r^2}{r^2} \partial_\zeta^2 \psi \right]. \quad (2.22)$$

Suppose $\psi = \mathcal{R}(r)Z(\zeta)$ to be a product of functions of r and ζ only, then $\mathcal{L}(\psi) = 0$ is equivalent to

$$-2r\kappa^2 \frac{\mathcal{R}'}{\mathcal{R}} + \frac{\kappa}{r\mathcal{R}} \partial_r (r\mathcal{R}') + \frac{h^2}{r^2} \frac{Z''}{Z} = 0.$$

Choose a separation constant n^2 such that

$$\frac{Z''}{Z} = -\frac{n^2}{l^2}$$

then the general solution for Z is

$$Z = a \cos(n\zeta/l) + b \sin(n\zeta/l) . \quad (2.23)$$

Note that the l -fold periodicity in ϕ implies that n may range over all of the positive integers and zero, so that the complete solution for the ζ equation is

$$Z(\zeta) = \sum_{n=0}^{\infty} a_n \cos(n\zeta/l) + b_n \sin(n\zeta/l) . \quad (2.24)$$

The equation for $\mathcal{R}(r)$ is now

$$-2r\kappa^2 \mathcal{R}' + \kappa r^{-1} \partial_r(r \mathcal{R}') - n^2 h^2 l^{-2} r^{-2} \mathcal{R} = 0 , \quad (2.25)$$

which has two linearly independent solutions

$$\mathcal{R}_n^i = A_n r I_n'(nhr/l)$$

and

$$\mathcal{R}_n^k = B_n r K_n'(nhr/l)$$

where $I_n'(z) = dI_n/dz$, and I_n and K_n are the modified Bessel functions of the first and second kind respectively. The solutions may be verified by using the identities

$$\begin{aligned} (\mathcal{R}_n^i)' &= \frac{d}{dr} \mathcal{R}_n^i = A_n \frac{n(l^2 + h^2 r^2)}{lhr} I_n(nhr/l) , \\ (\mathcal{R}_n^k)' &= B_n \frac{n(l^2 + h^2 r^2)}{lhr} K_n(nhr/l) , \\ \frac{1}{r} \partial_r [r(\mathcal{R}_n^i)'] &= A_n \frac{2nh}{l} I_n + A_n \frac{n^2(l^2 + h^2 r^2)}{l^2 r} I_n' \\ \text{and } \frac{1}{r} \partial_r [r(\mathcal{R}_n^k)'] &= B_n \frac{2nh}{l} K_n + B_n \frac{n^2(l^2 + h^2 r^2)}{l^2 r} K_n' . \end{aligned} \quad (2.26)$$

Leaving for the moment the $n = 0$ term, we have as the general solution to $\mathcal{L}(\psi) = 0$:

$$\bar{\psi} = r \sum_{n=1}^{\infty} [A_n I_n'(nhr/l) + B_n K_n'(nhr/l)] [a_n \cos(n\zeta/l) + b_n \sin(n\zeta/l)] . \quad (2.27)$$

Two particular integrals of the inhomogenous equation are $\psi_1 = \alpha \ln r$ and $\psi_2 = \beta r^2$. For the former

$$\mathcal{L}(\psi_1) = -2\kappa^2 r \frac{\alpha}{r} = -\frac{2l\kappa^2}{h} g$$

giving $\alpha = gl/h$. Similarly

$$\mathcal{L}(\psi_2) = -2\kappa^2 r 2\beta r + \kappa r^{-1} \partial_r(2\beta r^2) = -2\kappa^2 l h^{-1} g$$

gives $\beta = -gh/2l$. Note that $\mathcal{L}(\psi_1 - \psi_2) = 0$ implies that the $n = 0$ term in the general solution may be written $\psi_0 = a_0(\psi_1 - \psi_2)$. The complete expression for ψ now reads

$$\psi = \gamma\psi_1 + (1 - \gamma)\psi_2 + a_0(\psi_1 - \psi_2) + \bar{\psi} \quad (2.28)$$

where $0 \leq \gamma \leq 1$.

Questions relating to the above formalism will occupy Chaps. 4 to 6 of this thesis. In particular, we shall consider the limits of applicability of the expansion for ψ in vacuum of Eq. (2.28). As the principal physical application of interest is to heliac configurations, we shall be assuming $l = 1$ unless stated otherwise. This has a pedagogical advantage, in that it is difficult enough to visualise a spatially twisting magnetic coordinate system – without the confusion of incorporating the $l > 1$ case into the algebra. To transform

$$\psi(r, \zeta = \phi - hz) \longrightarrow \psi(r, \zeta = l\phi - hz) = \psi(r, \zeta' = l(\phi - hz/l))$$

one simply transforms

$$h \longrightarrow h/l$$

and

$$\zeta \longrightarrow \zeta/l$$

where they occur explicitly in the expressions for ψ .

Chapter 3

The geometry of the heliac magnetic field

The basic heliac configuration as proposed by Boozer *et al.* (Boozer *et al.*, 1983) is shown in Fig. 3.1. It consists of a set of toroidal field coils and a central current-carrying

Figure 3.1: *The basic heliac configuration (Boozer et al., 1983).*

Figure 3.2: *Cylindrical coordinate system.*

ring.

If one considers a coordinate system $(\rho, \theta, \phi_{tor})$ defined with respect to the central conductor with θ measuring the poloidal and ϕ_{tor} the toroidal angle, then the centres of the toroidal field coils trace out a helical path which is (essentially) given by $(\rho = \rho_0, \theta = \pm N\phi_{tor})$ where N is the number of periods of the device. The currents in the toroidal field coils describe circles in planes of constant- ϕ_{tor} and by themselves produce the magnetic field of an asperator (Nagao *et al.*, 1980). Under certain circumstances the superposition of this field with that due to a current in the central conductor will produce nested tubes of magnetic flux, the cross-sections of which resemble curved (kidney) beans (see Fig. (3.1)).

3(i) Helically symmetric model

In the helically symmetric limit the central conductor is taken as defining the axis of a cylindrical coordinate system as in Fig. (3.2). The external coils constitute a continuous solenoid of circular cross-section wound around the helix ($r = r_0, \zeta = \phi - hz = \zeta_0$). The parameter h gives the helical pitch of the solenoid.

In order to describe the qualitative features of the straight heliac we shall assume that the helical flux function, ψ^i , in the interior of the solenoid can be written

$$\psi^i = -\frac{b_0 r^2}{2} + \frac{\mu_0 I_c}{2\pi h} \ln r + \sum_{n=1}^{\infty} a_n^i r I_n'(nhr) \cos(n\zeta). \quad (3.1)$$

Here $b_0 = \mu_0 i_0$ where i_0 is the solenoidal surface current per unit increment in z (and is positive if the curl of the current is in the $\mathbf{e}_z$ direction), and I_c is the z -directed current

in the central conductor. These secular terms are needed to satisfy Ampère's law along longitudinal and azimuthal paths respectively.

The secular terms are dominant in the expression for ψ^i . The terms in the power series may be considered as helical perturbations due to the $l = 1$ periodic displacement of the solenoid. Of these it is to be expected that the "fundamental" perturbation will make the greatest contribution and that the effects of the "higher harmonics" $[ra_n^i I_n'(nhr) \cos(n\zeta)]$ for $n \geq 2$ will decay exponentially.

We can show that Eq. (3.1) is consistent with the magnetic field of the straight heliac if the latter may be assumed to be made up of a background field $\mathbf{B}^0$ and a perturbation $\mathbf{B}^1$ where

$$\mathbf{B}^0 = \mu_0 i_0 \mathbf{e}_z + \frac{\mu_0 I_c}{2\pi r} \mathbf{e}_\phi \quad (3.2)$$

and

$$\mathbf{B}^1 = \nabla \left(\sum_{n=1}^{\infty} a_n^i I_n(nhr) \sin(n\zeta) \right). \quad (3.3)$$

The zeroth-order field has nested circular-cylindrical flux surfaces, and a rotational transform per (arbitrary) longitudinal period $2\pi/h$ given by

$$\tau_0 = \frac{\Delta(\text{transverse flux})}{\Delta(\text{longitudinal flux})} = \frac{2\pi h^{-1} B_\phi \Delta r}{2\pi r B_z \Delta r} = \frac{I_c}{2\pi h r^2 i_0}. \quad (3.4)$$

The flux function

$$\psi^0 = -\frac{\mu_0 i_0 r^2}{2} + \frac{\mu_0 I_c}{2\pi h} \ln r \quad (3.5)$$

satisfies $\mathbf{B}^0 \cdot \nabla \psi^0 = 0$. It is a function of r only, and, if i_0 and I_c have the same sign, has one turning point at $r = \sqrt{(I_c/2\pi h i_0)}$ which corresponds to $\tau_0 = 1$. Also, $2\pi[\psi^0(r=r_2) - \psi^0(r=r_1)]$ is the flux through a helical ribbon of radial width $r_2 - r_1$ and length $\Delta z = 2\pi/h$.

The solution of the equation for the first-order perturbations,

$$\mathbf{B}^0 \cdot \nabla \psi^1 = -\mathbf{B}^1 \cdot \nabla \psi^0, \quad (3.6)$$

is, to within a constant,

$$\psi^1 = \sum_{n=1}^{\infty} a_n^i r I_n'(nhr) \cos(n\zeta). \quad (3.7)$$

An interesting derivation of this result is via a Fourier expansion of Eq. (3.6) in the variables ϕ and hz where, following Solov'ev and Shafranov (1970, p. 85), we put

$$\psi^1 = \Re \sum_{m,n>0} \psi_{m,n}(r) \exp[i(m\phi - nhz)] \quad (3.8)$$

and

$$-\mathbf{B}^1 \cdot \nabla \psi^0 = -B_r^1 \partial_r \psi^0 = \Re \sum_{m,n} S_{m,n}(r) \exp[i(m\phi - nhz)] , \quad (3.9)$$

where

$$S_{m,n} = \frac{i}{2} \mu_0 i_0 r (\tau_0 - 1) a_n^i n h I_n'(nhr) \delta_{m,n} .$$

Equation (3.6) gives

$$\psi_{m,n}(r) = -\frac{i S_{m,n}(r)}{\mu_0 i_0 h (m \tau_0 - n)} = \frac{1}{2} \delta_{m,n} r a_n^i I_n'(nhr) . \quad (3.10)$$

The Kronecker delta $\delta_{m,n}$, restricting the summation to harmonics with $m = n$, is necessary to maintain the helical symmetry of the first-order perturbations. The denominator will go to zero at rational values of τ_0 which would lead to unboundedness in ψ_{mn} except that $S_{mn}(r)$ also vanishes. In other words, an infinite resonance of the perturbed flux function when $\tau_0 = 1$ is avoided by $\partial_r \psi^0$ going to zero simultaneously.

From Eqs. (3.3) and (3.7) it is straightforward to show that $\mathbf{B}^1 \cdot \nabla \psi^1 = 0$, implying that there are no higher terms in the expansion for ψ , i.e. if the perturbing magnetic field has the form of Eq. (3.3) then ψ is given exactly by the solution to the first order equation.

3(ii) Rotational transform of the bean

The characteristic "bean" shape of the magnetic surfaces of the straight heliac may be understood by considering the influence of the dominant $l = 1$ harmonic (Boozer *et al.*, 1983, HX1 proposal, 1983). Consider for a start the points of intersection of field lines of the "background" field,

$$\mathbf{B}^0 = \mu_0 i_0 \mathbf{e}_z + \frac{\mu_0 I_c}{2\pi r} \mathbf{e}_\phi ,$$

with constant- z planes of period $\Delta z = 2\pi/h$. At the rational surface $\tau_0 = 1$, all field lines will close on themselves exactly after one period. Field lines on surfaces interior to this, where the rotational transform is greater, will rotate once azimuthally in a shorter longitudinal distance. When these planes of intersection are superimposed (as in Fig. 3.3) the points interior to the rational surface appear to move anti-clockwise as z increases, whereas successive points on exterior surfaces move in a clockwise direction.

Under the influence of a small radial perturbation $B_r^1 = |a_1| h I_1'(hr) \sin \zeta$, field lines in the top half plane will be caused to move slowly outwards. Their points of intersection will move in the direction of increasing r and ϕ until they reach a region where $\tau_0 \approx 1$

Figure 3.3: *Formation of closed magnetic surfaces. The dotted lines contour the vacuum field. The dashed line is the surface $t_0 = 1$. Three field lines placed along the mid-plane will rotate as shown after one helical period. The closed figure results from a perturbation with one helical harmonic: $I_c/i_0 = 0.5890$, $a_1/\mu_0 i_0 = 1.9531 \times 10^{-3}$ and $h = 16$.*

where they will experience the clockwise shift of lines on exterior surfaces. In the lower half-plane the sign of the radial perturbation is reversed, causing the locus to close on itself.

The perturbed flux function

$$\psi^1 = -\frac{\mu_0 i_0 r^2}{2} + \frac{\mu_0 I_c}{2\pi h} \ln r + |a_1| r I_1'(hr) \cos \zeta \quad (3.11)$$

will (for $i_0, I_c > 0$ and $|a_1|$ "small") have three turning points, at which $\partial_r \psi = 0$ and $\partial_\zeta \psi = 0$ simultaneously (McNamara *et al.*, 1966) (see Fig.(3.4)). The first zero of $\partial_r \psi = 0$ for $\zeta = 0$ corresponds to a point at the centre of a set of nested beans. It is an isolated maximum of ψ and is the magnetic axis of the configuration. The other two turning points correspond to saddle points of ψ , one or both of which will lie on the bounding surface or "separatrix".

From the results of Chap. 2, ψ^1 measures the flux through any helical ribbon which has one edge on the magnetic axis and the other on a particular flux surface and which

Figure 3.4: *Contour plot of ψ for a straight heliac with one helical harmonic: $I_c/i_0 = 9.8175 \times 10^{-2}$, $a_1/\mu_0 i_0 = 2.4414 \times 10^{-2}$ and $h = 16$. Dotted lines represent contours outside the separatrix. The axes of the bottom diagram have the same scale. The top diagram is a plot of ψ across the mid-plane.*

rotates about the axis once per period. The “helical transform” is given by $\tau_h = d\psi/d\Phi_B$ (where Φ_B is the longitudinal flux) and measures the average rotation of a field line about the magnetic axis relative to a poloidal coordinate system which twists with the bean. The rotational transform per period relative to a poloidal angle which does not twist with the plasma is given by (Dewar *et al.*, 1984)

$$\tau_p = \frac{d\psi_{pol}}{d\Phi_B} = \frac{d(\Phi_B + \psi)}{d\Phi_B} = 1 + \tau_h, \quad (3.12)$$

where ψ_{pol} is the “poloidal” flux, i.e. the flux through a ribbon which does not encircle the magnetic axis (and which can be considered as the sum of a helical ribbon and a transverse cut). The total transform, τ_p , is the sum of that with respect to the twisting frame and that due to the rotation of the frame itself. It is apparent that the direction of rotation of a field line around the bean is opposite to the direction of rotation of the magnetic axis. The helical transform of a heliac is therefore always negative, a result which also follows from the observation [Fig.(3.4)] that the magnetic axis will correspond to a local maximum of ψ in a configuration with positive net longitudinal flux.

3(iii) Some consequences of the toroidal heliac

The rotational transform of the toroidal heliac is

$$\tau = N(1 + \tau_h) \quad (3.13)$$

where N is the number of periods of the machine. The contribution of the torsion of the magnetic axis makes the heliac a stellarator of comparatively large rotational transform. Because the twisting of the field lines is influenced chiefly by the central conductor, the local rate of change of field line pitch between magnetic surfaces (or “local shear”) will be large, however it will change in opposite senses on the inside and outside of the bean-shaped contour. The averaged or “global” shear is expected to be small, especially near the magnetic axis.

The effects of toroidal curvature may be envisaged as contributing perturbative terms to an expression similar to that of Eq. (3.8). In this case the periodic coordinates would be the poloidal and toroidal angles, and the “unperturbed” rotational transform is the fully toroidal $\tau = N(1 + \tau_h)$. There is no guarantee that the effects of “resonance denominators” will cancel out in the fashion of the helically symmetric treatment, and the presence of low order resonances could be catastrophic for the quality of the magnetic

surfaces. This is an important consideration in heliac design (HX1 study, 1983) and is aided by the smallness of the global shear.

A helical solenoid wound on a central conductor was originally proposed (Furth *et al.*, 1966) as one configuration possessing a "mean magnetic well". If V measures the volume of a magnetic flux tube and Φ_B the total longitudinal flux, then a configuration of nested surfaces is said to possess a mean magnetic well if the absolute specific volume, $dV/d|\Phi_B|$, exhibits a net decrease when moving from the magnetic axis to the bounding surface. In the presence of plasma with pressure $p(\Phi_B)$ the condition that (Furth *et al.*, 1966)

$$\frac{dp}{d\Phi_B} \frac{d^2V}{d\Phi_B^2} < 0,$$

can be shown to be a stability criterion against some interchange instabilities. It is equivalent to stating that the pressure gradient is, on average, in the opposite sense to the field line curvature. This is often referred to as "favourable average curvature". It is possible to design heliac magnetic fields for which the specific volume decreases (McNamara *et al.*, 1966) monotonically (or "uniformly") from the magnetic axis to the separatrix.

The existence of a deep and uniform magnetic well is expected to contribute significantly to the ability of the heliac to confine plasma at large β (see Chap. 1). This ability is further enhanced by the large rotational transform which has the effect of decreasing the distance along a field line (or "connection length") between regions of good and bad (local) curvature. However, since helical curvature is comparable with toroidal curvature, it is desirable to have both τ and τ_h being large which will necessitate compromises.

3(iv) Extensions to the basic design

There are a number of aspects of the design of a toroidal heliac which are inherently three dimensional and can not be treated by the helically symmetric model. Chief amongst these is the addition of a vertical field (HX1 study, 1983) which can affect the position and shape of the flux surfaces and the depth of the magnetic well (and may be considered as a trimming field for the toroidal perturbations) – see Chap. 8.

There have been several proposals for modification of the central coil. Yoshikawa (1985) has suggested that it be split into two helices, whereas Harris *et al.* (1985) have advocated that one helical line current be added to a configuration which retains the axial conductor (see Chap. 8). Recently, Sy has proposed that the central coil be split into four helical currents arranged as a twisting "swiss cross" (W. N-C. Sy, private communication).

The direction of the single axial current may be reversed and result in nested surfaces (McNamara *et al.*, 1966, Furth *et al.*, 1966), however these do not exhibit as favourable magnetic well properties.

By suitably indenting the shape of solenoidal field coils it is possible to dispense with the central conductor (McNamara *et al.*, 1966, Reiman and Boozer, 1983, Bauer *et al.*, 1984). These "modular coils" have also featured in the design of conventional stellarators (Rau *et al.*, 1981, Brossman *et al.*, 1983).

Chapter 4

Computation of the magnetic field of a helical solenoid

A serious design study of the straight heliac must necessarily employ an accurate determination of the magnetic field of the helical solenoid. This may be achieved by direct integration of the Biot-Savart law, or by using an (equivalent) Green's function formulation. Alternatively, it may be desirable to utilise an approximate representation of the field in the form of an easily computed series.

The latter approach is the subject of this chapter. We shall evaluate the coefficients of a series representation of the magnetic scalar potential by boundary collocation on the external coil. The formulation in terms of the scalar potential has been chosen for the sake of variety as much as simplicity. Its correspondence with the helical flux is straightforward, and will be demonstrated here and in subsequent chapters.

4(i) Existence and uniqueness of solutions

Consider an infinitely long solenoid, γ , which divides space into an exterior region, A , and an interior region, B , and which includes the z -axis (Fig. 4.1). Suppose there is a distribution of surface currents on γ having a constant net longitudinal circulation of $\mu_0 i_0$ inside and a constant net azimuthal circulation of $\mu_0 I$ outside. (In most practical calculations this will be the consequence of a combination of longitudinal and azimuthal (i.e. helical) currents flowing on a solenoidal surface which satisfies particular symmetry constraints. We shall consider the cases of particular symmetries in a moment.) From

Figure 4.1: The solenoidal surface γ divides space into an exterior region A and interior region B .

Ampère's law, the scalar magnetic potential, Φ , (satisfying $\mathbf{B} = \nabla\Phi$) may be written as

$$\begin{aligned} \Phi^i &= \mu_0 i_0 z + U & \text{in } A, \\ \text{and } \Phi^e &= \frac{\mu_0 I}{2\pi} \phi + V & \text{in } B, \end{aligned}$$

where U and V are single-valued, harmonic potentials. The magnetic field must satisfy the jump conditions across the surface,

$$[[\mathbf{B}]] \cdot \mathbf{e}_n \equiv (\mathbf{B}^e - \mathbf{B}^i) \cdot \mathbf{e}_n|_{\gamma} = 0, \quad (4.1)$$

and

$$\mathbf{e}_n \times [[\mathbf{B}]] = \mu_0 \mathbf{i}, \quad (4.2)$$

where $\mathbf{e}_n$ is the unit outward normal, and $\mathbf{i}$ is the surface current per unit length.

The boundary conditions may be rephrased in terms of the normal and longitudinal derivatives of U and V to give:

$$\mathbf{e}_n \cdot \nabla(V - U) \equiv \partial_n V - \partial_n U = \partial_n \left(\mu_0 i_0 z - \frac{\mu_0 I}{2\pi} \phi \right), \quad (4.3)$$

Figure 4.2: Cross-section of a cylindrically symmetric solenoid.

and

$$V(\mathbf{p}) - U(\mathbf{p}) = V(\mathbf{p}_0) - U(\mathbf{p}_0) + \int_{\mathbf{p}_0}^{\mathbf{p}} d\mathbf{l} \cdot \nabla(V - U) \equiv f(\mathbf{p}), \quad (4.4)$$

where the line integral in the surface is taken between the points, $\mathbf{p}$ and $\mathbf{p}_0$. If the surface current distribution, $\mathbf{i}$, is known then $f(\mathbf{p})$ may be calculated from Eq. (4.2) to within a constant determined only by the arbitrary choice of $V(\mathbf{p}_0) - U(\mathbf{p}_0)$ (and not the path between $\mathbf{p}_0$, and $\mathbf{p}$).

4(i-a) Cylindrically symmetric solenoid

Suppose that the solenoidal surface, γ , and the current distribution, $\mathbf{i}$, have translational symmetry along the z -axis. The potentials U and V are functions of (r, ϕ) only and satisfy Laplace's equation in the plane. Let C_γ denote the cross-section of γ in a constant- z plane with unit outward normal $\mathbf{e}_{n'}$ as in Fig. 4.2.

Theorem : The two dimensional, boundary-value problem

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{outside } C_\gamma, \\ \nabla^2 U &= 0 \quad \text{inside } C_\gamma, \\ \partial_n(V - U) &= -\partial_n \left(\frac{\mu_0 I}{2\pi} \phi \right) \quad \text{on } C_\gamma, \\ \text{and } V(\mathbf{p}) - U(\mathbf{p}) &= f(\mathbf{p}) \quad \text{on } C_\gamma \end{aligned} \quad (4.5)$$

where U and V are regular within their respective domains, has a unique solution (to within a constant).

[A harmonic function is regular within a bounded domain if it has continuous second derivatives. Regularity of a harmonic function in an unbounded n -dimensional domain carries the additional requirement that it be regular throughout the region obtained by an inversion of the coordinate system with respect to a hypersphere centred at the origin. This implies that regular functions in three dimensions must fall off at least as fast as $1/R$

(where R is the distance from the origin of a spherical coordinate system) as $R \rightarrow \infty$. In two dimensions boundedness at infinity is sufficient to guarantee regularity (Courant, 1962, p.243).]

Proof : A solution may be constructed as the sum of a single layer potential

$$s(r_0, \phi_0) = \oint_{C_\gamma} \rho \ln \left(\frac{1}{D} \right) dl$$

where

$$\rho = -\frac{1}{2\pi} \partial_{n'} (V - U)|_{C_\gamma}$$

and the double layer potential

$$d(r_0, \phi_0) = \oint_{C_\gamma} \sigma \partial_{n'} \ln \left(\frac{1}{D} \right) dl$$

where

$$\sigma = \frac{f(\mathbf{p})}{2\pi}$$

providing C_γ is sufficiently smooth (Courant, 1962, p.257). Here $D = \sqrt{[r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)]}$ is the distance, in polar coordinates, between the observer position (r_0, ϕ_0) and a source point (r, ϕ) on C_γ . To show uniqueness, we note that the difference between two solutions of Eq. (4.5) is harmonic and regular throughout the plane. Because any regular harmonic function in two dimensions can only have a local extremum on the boundary, this difference function must be a constant (Courant, 1962, p.257). $\square$

4(i-b) Spatial Solenoid

Consider once again the spatial solenoid of Fig. 4.1.

Theorem : The three dimensional boundary value problem

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{in } A, \\ \nabla^2 U &= 0 \quad \text{in } B, \\ \partial_n (V - U) &= \partial_n \left(\mu_0 i_0 z - \frac{\mu_0 I}{2\pi} \phi \right) \quad \text{on } \gamma, \\ \text{and } V(\mathbf{p}) - U(\mathbf{p}) &= f(\mathbf{p}) \quad \text{on } \gamma \end{aligned} \tag{4.6}$$

where U and V have continuous second derivatives and V is bounded (i.e. V tends to a constant as $r \rightarrow \infty$) has a unique solution (to within a constant).

This problem could be treated by considering γ to be the limit of a periodic solenoid as the period length is taken to infinity. The following proof, however, uses the properties of Laplace's equation in both two and three dimensions.

Proof : Suppose the entire surface γ to be covered by a charge distribution, ρ , and dipole distribution σ where

$$\rho = -\frac{1}{4\pi} \partial_n (V - U)|_\gamma$$

and

$$\sigma = \frac{f(\mathbf{p})}{4\pi}.$$

The single and double layer potentials due to ρ and σ respectively are

$$s(r_0, \phi_0, z_0) = \int_\gamma \frac{\rho}{D} dS$$

and

$$d(r_0, \phi_0, z_0) = \int_\gamma \sigma \partial_n \left(\frac{1}{D} \right) dS$$

where D is the distance between the observer point (r_0, ϕ_0, z_0) and points on the surface. The sum of s and d gives a harmonic function which is a solution of Eq. (4.6). It has continuous second derivatives throughout the interiors of A and B and satisfies the jump conditions across the solenoid at any point where γ is sufficiently smooth (i.e. the curvature is continuous (Courant, 1962, p.259)). The field due to $s + d$ will fall off as $1/r$ (where r is the cylindrical coordinate) as $r \rightarrow \infty$. The difference between this solution and any other function which satisfies Eq. (4.6) will be harmonic and have no source terms apart from the boundary condition at infinity which is cylindrically symmetric. Hence, from Sec. 4(i-a), this function must be a constant everywhere. $\square$

4(ii) Choice of the expansion set

A Fourier-Bessel expansion for a harmonic potential, U , which is regular at the z -axis but not as $r \rightarrow \infty$ is (Jackson, 1962, p.117):

$$U = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} A_n^i(k) \exp(in\phi) \exp(ikz) I_n(|k|r) dk. \quad (4.7)$$

Here $I_n(|k|r)$ is the modified Bessel function of the first kind of order n and (r, ϕ, z) are the usual cylindrical coordinates. The discrete sum over values of n is required to ensure that U is 2π -periodic in ϕ . Putting $k' = k + nh$ and $\zeta = \phi - hz$ into Eq. (4.7) we have,

$$U = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} A_n^i(k' - nh) \exp(in\zeta) \exp(ik'z) I_n(|k' - nh|r) dk'. \quad (4.8)$$

Specialising now to helically symmetric potentials, the requirement

$$\partial_z U|_{r,\zeta} = 0 \quad (4.9)$$

is satisfied only if

$$A_n^i(k' - nh) = A_n^i\delta(k') , \quad (4.10)$$

yielding

$$U = \sum_{n=-\infty}^{\infty} A_n^i I_n(nhr) \exp(in\zeta) . \quad (4.11)$$

For helically symmetric potentials which are bounded at infinity, but not as $r \rightarrow 0$, the expansion in terms of modified Bessel functions of the second kind gives

$$V = \sum_{n=-\infty}^{\infty} A_n^e K_n(nhr) \exp(in\zeta) . \quad (4.12)$$

Let us postulate that the functions U and V of Eqs. (4.11) and (4.12) may be used to describe the scalar potential of a helically symmetric solenoid which includes the z -axis and which we further restrict to having reflection symmetry about the helical ribbon ($\zeta = 0 \cup \zeta = \pi$). Then

$$\begin{aligned} \Phi^i &= \mu_0 i_0 z + \sum_{n=1}^{\infty} a_n^i I_n(nhr) \sin(n\zeta) , \\ \text{and } \Phi^e &= \frac{\mu_0 I}{2\pi h} \phi + \sum_{n=1}^{\infty} a_n^e K_n(nhr) \sin(n\zeta) \end{aligned} \quad (4.13)$$

where the a_n are real constants.

The restriction to odd functions of ζ is best illustrated by consideration of the corresponding representations of the helical flux:

$$\begin{aligned} \psi^i &= -\frac{\mu_0 i_0 r^2}{2} + r \sum_{n=1}^{\infty} a_n^i I_n'(nhr) \cos(n\zeta) \\ \text{and } \psi^e &= \frac{\mu_0 I}{2\pi h} \ln r + r \sum_{n=1}^{\infty} a_n^e K_n'(nhr) \cos(n\zeta) . \end{aligned} \quad (4.14)$$

[The equivalence of Eqs. (4.13) and (4.14) can be demonstrated from the expression for the magnetic field:

$$\mathbf{B} = hu \times \nabla\psi + hgu = \nabla\Phi$$

and the expressions for g inside and outside the solenoid (see Sec. 2(ii))].

In a geometry where the currents have reflection symmetry about the line ($\zeta = 0 \cup \zeta = \pi$), the magnetic surfaces must also have the same symmetry. Hence, because ψ is constant on a magnetic surface, we have the restriction to the cosine series in Eq. (4.14) and the sine series in Eq. (4.13).

Each field harmonic in the expansions of Eqs. (4.13) and (4.14) is regular throughout either the exterior or the interior region. Providing the series can be shown to converge, they would appear to be a good representation of the magnetic field of a displaced helical solenoid which includes the z axis.

4(iii) Boundary methods

Consider a domain, D , with bounding surface γ . Given the elliptic boundary value problem

$$\mathcal{L}(v) = 0 \quad \text{in } D, \quad (4.15)$$

$$\mathcal{B}(v) = f \quad \text{on } \gamma, \quad (4.16)$$

one may choose a trial solution

$$v^{(N)} = \sum_{i=1}^N a_i v_i \quad (4.17)$$

where each of the v_i satisfy Eq. (4.15). In the Method of Weighted Residuals (MWR), the coefficients a_i are determined by forcing the surface integral of the residual to be zero when weighted by each member of a given set of functions. That is,

$$\int_{\gamma} w_j \mathcal{R} dS = 0 \quad \text{for } \{w_j : j = 1, 2 \dots N\},$$

where $\mathcal{R} = \mathcal{B}(v^{(N)}) - \mathcal{B}(v)$.

The theory of MWR, along with its historical development, is reviewed by Finlayson (Finlayson, 1972). The choice of weighting functions is commonly made in one of four ways:

- (a) In the collocation method, the $\{w_j\}$ are chosen to be displaced delta functions, $w_j = \delta(\mathbf{x} - \mathbf{x}_j)$. In other words, the residual is set to zero at a discrete set of points $\{\mathbf{x}_j\}$ on γ .
- (b) In the least squares - collocation method, the number of points where the residual condition is matched is greater than the number of trial functions. The square of the residual of the resulting overdetermined matrix system is then minimised.
- (c) In the least squares method the weighting functions are chosen to be $w_j = \partial \mathcal{R} / \partial a_j$, so that

$$I = \int_{\gamma} \mathcal{R}^2(a_j) dS$$

is minimised with respect to the constants a_j .

- (d) In the Galerkin method the weighting functions are chosen to be the trial functions themselves, $w_j = v_j$. If the set of trial functions is complete on γ , then in the limit of $N \rightarrow \infty$ the residual will vanish.

The boundary collocation methods, (a) and (b), have been applied to a variety of problems in physics and engineering since the late 1950's (Finlayson, 1972). They are comparatively easy to implement, and have the flexibility of being able to place the collocation points in the region of greatest interest. Thus Ojalvo and Linzer (Ojalvo and Linzer, 1965) were able to use the technique to approximate the solution of a problem with non-smooth boundary conditions by avoiding the jump-points. If used in conjunction with a maximum principle, the collocation points may be positioned to minimise the maximum error on the boundary and hence for the region as a whole. The least squares - collocation method will minimise some average of the error taken over the boundary. As the number of collocation sites is increased it approaches the least squares method.

Although the least squares method leads to more cumbersome equations, it can be thought of as being "optimal" in the sense that the mean square error estimate is optimised. Dommaschk (Dommaschk, 1981) has applied it to the solution of two types of magnetic field problems. In the first, he calculates the field exterior to a topologically toroidal current surface given the field inside. The current is then found from the contours of the differences in the scalar potential across the boundary. In the second problem, he calculates an interior field which is tangential to a given surface. An important feature of his work has been the development of a set of toroidal harmonic functions to model the three-dimensional interior field of a stellarator.

The Galerkin formulation of MWR is closely related to variational principles, and has been widely developed for eigenvalue problems as well as engineering stress calculations.

In the remainder of this chapter, we shall examine the application of boundary collocation methods to the solution of the potentials U and V of Sec. 4(i). The boundary conditions on the magnetic field, across the surface γ will be matched using the expansion sets of Eq. (4.13) as trial functions.

4(iv) Error estimates from Green's theorem

Consider approximate solutions, U^N and V^N , to the equations for a displaced helical solenoid (Eqs. (4.6) with γ helically symmetric) which are themselves helically symmetric harmonic potentials. The error functions $\xi_U = U - U^N$ and $\xi_V = V - V^N$ satisfy the equations:

$$\nabla^2 \xi_V = 0 \quad \text{in } A,$$

Figure 4.3: The closed helical tube and open annular region formed by cutting the helical solenoid, γ , by planes perpendicular to the z -axis.

$$\begin{aligned}\nabla^2 \xi_U &= 0 \quad \text{in } B, \\ \partial_n(\xi_V - \xi_U) &= \xi_1 \quad \text{on } \gamma, \\ \xi_V - \xi_U &= \xi_2 \quad \text{on } \gamma.\end{aligned}\tag{4.18}$$

Theorem : The error functions ξ_U and ξ_V take their extremal values on the surface γ .

The proof of this theorem is complicated by the consideration of the point at infinity. It has as a corollary the result that the extremal values of U , U^N , V and V^N also occur on γ .

Proof : Suppose that we have constructed the unique solution to Eqs. (4.18) by superimposing single and double layer potentials as was done in Sec. 4(i-b), and note that a local extremum in the interior of a helically symmetric system implies that the value of ξ_U on a helical line is greater or less than all of the helical lines in its immediate neighbourhood. Consider the closed tube formed by cutting one helical period of the region B by planes parallel to the z -axis as in Fig. 4.3. The property of a regular three dimensional harmonic potential that its value at any interior point be the arithmetic mean of its values on any sphere which is centred at that point (Courant, 1962, p.254) implies that local extrema cannot exist in the interior or (equivalently) on the ends, S_1 and S_2 .

The result can be shown to hold for ξ_V in an annular volume obtained by cutting region A as above. In order to treat the point at infinity we invert this cut region with respect to the unit cylinder centred on the z -axis (which we assume to lie wholly inside γ). The coordinate transformation establishes a correspondence between ξ_V and $\bar{\xi}_V$ where

$\overline{\xi_V}(r, \phi, z) = \xi_V(1/r, \phi, z)$. One may verify that $\overline{\xi_V}$ satisfies the equation

$$\left[\nabla^2 + \left(\frac{1}{r^2} - 1 \right) \frac{\partial^2}{\partial z^2} \right] \overline{\xi_V} = 0$$

inside the cylinder. A Dirichlet problem may be constructed for $\overline{\xi_V}$ as above. The theorem can then be proved by invoking the condition of helical symmetry in combination with a general "minimax principle" which applies to any second order partial differential operator which is "uniformly elliptic" (Finlayson, 1972, p.383). [An operator

$$\mathcal{L}(u) \equiv \sum_{i,j=1}^3 a_{ij}(\mathbf{r}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^3 b_i(\mathbf{r}) \frac{\partial u}{\partial x_i},$$

defined with respect to Cartesian coordinates $\mathbf{r} = (x_1, x_2, x_3)$, is uniformly elliptic in a volume V if there exists a constant μ_0 such that for any point in V and real numbers t_i

$$\sum_{i,j=1}^3 a_{i,j} t_i t_j \geq \mu_0 \sum_{i=1}^3 t_i^2.$$

In this case we have

$$t_1^2 + t_2^2 + \frac{t_3^2}{r^2} \geq \mu_0(t_1^2 + t_2^2 + t_3^2)$$

which is true for $\mu_0 = 1$ for points inside the cylinder $r \equiv \sqrt{(x_1^2 + x_2^2)} = 1$.] $\square$

The continuous dependence of the error functions on the boundary conditions

$$\begin{aligned} \partial_n(\xi_V - \xi_U) &= \xi_1, \\ \text{and} \quad \xi_V - \xi_U &= \xi_2, \end{aligned}$$

may be demonstrated using Green's theorem. Consider a "helical Green's function" (to be derived in Chap. 5) which satisfies

$$\nabla^2 \mathcal{G}^0(r, \zeta | r_0, \zeta_0) = -\frac{4\pi}{r} \delta(r - r_0) \delta(\zeta - \zeta_0) \quad (4.19)$$

and apply Green's third identity to the truncated helical tube of Fig. 4.3:

$$\int_B (U \nabla^2 \mathcal{G}^0 - \mathcal{G}^0 \nabla^2 U) dV = \int_{\gamma + S_1 + S_2} (U \partial_n \mathcal{G}^0 - \mathcal{G}^0 \partial_n U) dS. \quad (4.20)$$

The surface integrals over S_1 and S_2 cancel because $\mathbf{n}_1$ and $\mathbf{n}_2$ are equal and opposite and $\mathcal{G}^0$ and U are helically symmetric. Equations (4.19) and (4.20) taken together give

$$8\pi^2 h^{-1} U(r_0, \zeta_0) = \int_{\gamma} (\mathcal{G}^0 \partial_n U - U \partial_n \mathcal{G}^0) dS \quad (4.21)$$

for (r_0, ζ_0) in region B and

$$0 = \int_{\gamma} (\mathcal{G}^0 \partial_n U - U \partial_n \mathcal{G}^0) dS \quad (4.22)$$

Figure 4.4: Cross-section of a helical solenoid perpendicular to the z -axis.

for (r_0, ζ_0) in region A. (The surface integrals are understood to be taken over one period of the solenoid.) Similar results can be demonstrated for V, U^N, V^N, ξ_U and ξ_V where the dominant $1/r$ fall-off of the exterior potentials is necessary to eliminate the integral at infinity (see Chaps. 5 and 6). Combining the integral expressions involving ξ_U and ξ_V for points in region B we have:

$$\begin{aligned}
 8\pi^2 h^{-1} \xi_U(r_0, \zeta_0) &= \int_{\gamma} \left[\mathcal{G}^0 \partial_n (\xi_U - \xi_V) - (\xi_U - \xi_V) \partial_n \mathcal{G}^0 \right] dS \\
 \Leftrightarrow 8\pi^2 h^{-1} |\xi_U(r_0, \zeta_0)| &\leq \text{MAX} |\partial_n (\xi_U - \xi_V)| \left| \int_{\gamma} \mathcal{G}^0 dS \right| \\
 &\quad + \text{MAX} |\xi_U - \xi_V| \left| \int_{\gamma} \partial_n \mathcal{G}^0 dS \right|. \quad (4.23)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 8\pi^2 h^{-1} |\xi_V(r_0, \zeta_0)| &\leq \text{MAX} |\partial_n (\xi_U - \xi_V)| \left| \int_{\gamma} \mathcal{G}^0 dS \right| \\
 &\quad + \text{MAX} |\xi_U - \xi_V| \left| \int_{\gamma} \partial_n \mathcal{G}^0 dS \right| \quad (4.24)
 \end{aligned}$$

for (r_0, ζ_0) in region A.

These results show that the maximum values of $|\xi_U - \xi_V|$ and $|\partial_n (\xi_U - \xi_V)|$ on the boundary may be considered as measures of the absolute error. Their relative weighting will depend on the Green's function integrals of Eqs. (4.23) and (4.24).

4(v) Derivation of the boundary conditions

Consider a helical solenoid, γ , which includes the z -axis as in Fig. 4.4. Let γ carry a surface current which is confined to planes of constant z , and is a constant, i_0 , per unit length increment in z . The symmetry vector

Figure 4.5: Area element on the surface of a helical solenoid.

$$\mathbf{u} = \frac{\mathbf{e}_z + h r \mathbf{e}_\phi}{1 + h^2 r^2}, \quad |\mathbf{u}| = \frac{1}{\sqrt{1 + h^2 r^2}} \quad (4.25)$$

is tangential to γ .

The unit tangent vector in a constant- z plane is given by

$$\begin{aligned} (dl)\mathbf{e}_t &= r(d\phi)\mathbf{e}_\phi + (dr)\mathbf{e}_r \\ \Leftrightarrow \mathbf{e}_t &= r \frac{d\phi}{dl} \mathbf{e}_\phi + \frac{dr}{dl} \mathbf{e}_r \end{aligned} \quad (4.26)$$

where $dl = \sqrt{[r^2(d\phi)^2 + (dr)^2]}$ measures the arc length along the contour of Fig. 4.4. Equation (4.26) may be rewritten in terms of any quantity, χ , which parametrises the distance along this contour:

$$\mathbf{e}_t = \frac{r\dot{\phi}\mathbf{e}_\phi + \dot{r}\mathbf{e}_r}{\dot{l}}, \quad (4.27)$$

(with the dot representing differentiation with respect to χ). An outward normal to γ may be written as

$$\begin{aligned} \mathbf{n} &= \mathbf{e}_t \times \mathbf{u} = \frac{r\dot{\phi}\mathbf{e}_r - \dot{r}\mathbf{e}_\phi + h r \dot{r} \mathbf{e}_z}{(1 + h^2 r^2)\dot{l}}, \\ |\mathbf{n}| &= \frac{\sqrt{(r^2\dot{\phi}^2 + \dot{r}^2 + h^2 r^2 \dot{r}^2)}}{(1 + h^2 r^2)\dot{l}}. \end{aligned} \quad (4.28)$$

Let $\mathbf{e}_u$ and $\mathbf{e}_n$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{n}$. The area element on the surface is given by the area of the parallelogram formed by $\mathbf{e}_u$ and $\mathbf{e}_t$. From Fig. 4.5,

$$\begin{aligned} \sin \beta &= |\mathbf{e}_u \times \mathbf{e}_t|, \\ \cos \alpha &= \mathbf{e}_u \cdot \mathbf{e}_z = \frac{1}{\sqrt{1 + h^2 r^2}}, \end{aligned}$$

and

$$\begin{aligned}
 dS &= \frac{(dz)}{\cos \alpha} (dl) |\mathbf{e}_u \times \mathbf{e}_t| \\
 &= (dz)(d\chi) \frac{dl}{d\chi} \sqrt{(1 + h^2 r^2)} |\mathbf{e}_u \times \mathbf{e}_t| \\
 &= (dz)(d\chi) \sqrt{(r^2 \dot{\phi}^2 + \dot{r}^2 + h^2 r^2 \dot{r}^2)} .
 \end{aligned} \tag{4.29}$$

The differential operator $(dS)\mathbf{e}_n \cdot \nabla$ is given by

$$\begin{aligned}
 (dS)\mathbf{e}_n \cdot \nabla &= (dz)(d\chi) \dot{l} \sqrt{(1 + h^2 r^2)} |\mathbf{e}_u \times \mathbf{e}_t| \frac{\mathbf{n} \cdot \nabla}{|\mathbf{u}| |\mathbf{e}_u \times \mathbf{e}_t|} \\
 &= (dz)(d\chi) \left(r \dot{\phi} \partial_r - \frac{\dot{r}}{r} \partial_\phi + h r \dot{r} \partial_z \right) ,
 \end{aligned} \tag{4.30}$$

which reduces to

$$(dS)\mathbf{e}_n \cdot \nabla = (dz)(d\chi) \left(r \dot{\phi} \partial_r - \frac{\dot{r}}{r} (1 + h^2 r^2) \partial_\zeta \right) , \tag{4.31}$$

when operating on helically symmetric functions.

The boundary conditions on the magnetic field across γ are

$$(\mathbf{B}^e - \mathbf{B}^i) \cdot \mathbf{e}_n = 0 , \tag{4.32}$$

$$\text{and } (\mathbf{B}^e - \mathbf{B}^i) \cdot \mathbf{e}_t = 0 , \tag{4.33}$$

where we note that it is sufficient to resolve the tangential condition along one direction in the surface.

Upon substituting the expressions for Φ^i and Φ^e from Eq. (4.13), and noting that the azimuthal circulation is zero, Eq. (4.32) becomes

$$r \dot{\phi} \partial_r (V - U) - \frac{\dot{r}}{r} (1 + h^2 r^2) \partial_\zeta (V - U) = \mu_0 i_0 h r \dot{r} \tag{4.34}$$

and from Eq. (4.33) we obtain

$$\dot{r} \partial_r (V - U) + \dot{\phi} \partial_\zeta (V - U) = 0 . \tag{4.35}$$

Consider now a helical solenoid with a circular cross-section of radius a and displacement b (Fig. 4.6).

The equation of the solenoid is

$$\begin{aligned}
 a^2 &= r^2 + b^2 - 2rb \cos \zeta \\
 \Leftrightarrow r &= b \cos \zeta + \sqrt{(a^2 - b^2 \sin^2 \zeta)} .
 \end{aligned} \tag{4.36}$$

Figure 4.6: *Helical solenoid of circular cross-section, radius a and displacement b .*

If we parametrise a cross-section in the $z = 0$ plane by $r(\zeta)$, and $\phi(\zeta)$ we have

$$\dot{r} = -\frac{rb \sin \zeta}{r - b \cos \zeta}, \quad \dot{\phi} = 1, \quad (4.37)$$

and, from Eqs. (4.34) and (4.35),

$$(r - b \cos \zeta) r \partial_r (V - U) + b \sin \zeta (1 + h^2 r^2) \partial_\zeta (V - U) = -\mu_0 i_0 h r^2 b \sin \zeta \quad (4.38)$$

and

$$-rb \sin \zeta \partial_r (V - U) + (r - b \cos \zeta) \partial_\zeta (V - U) = 0. \quad (4.39)$$

Upon combining Eqs. (4.36), (4.38) and (4.39) we have

$$\frac{(a^2 + h^2 r^2 b^2 \sin^2 \zeta)}{(r - b \cos \zeta)} \partial_r (V - U) = -\mu_0 i_0 h r b \sin \zeta, \quad (4.40)$$

$$\text{and} \quad \frac{(a^2 + h^2 r^2 b^2 \sin^2 \zeta)}{r b \sin \zeta} \partial_\zeta (V - U) = -\mu_0 i_0 h r b \sin \zeta. \quad (4.41)$$

4(vi) Results

The substitution of the trial solutions

$$U^N = \sum_{n=1}^N a_n^i I_n(nhr) \sin(n\zeta), \quad (4.42)$$

$$V^N = \sum_{n=1}^N a_n^e K_n(nhr) \sin(n\zeta), \quad (4.43)$$

into Eqs. (4.40) and (4.41) results in two equations in $2N$ unknowns which must hold at any point on the boundary. A set of N collocation points will give a square matrix which may be inverted to solve for the coefficients $\{a_n^e\}$ and $\{a_n^i\}$.

A program, CALPOT, has been written to achieve these ends. Matrix inversion is carried out using the IMSL routine LLSQF (IMSL, 1984) which solves least squares problems

by QR decomposition. The modified Bessel functions have been calculated using the IMSL routine MMBSIN and the NAG functions S18ACF and S18ADF (NAG, 1984).

The code was run in double precision on a UNIVAC 1100/82 computer (with an approximate floating-point accuracy of 18 decimal places) and a FACOM M360 (16 decimal places).

Dimensionless relative error estimates have been defined in the spirit of Eqs. (4.23) and (4.24). Firstly, we note that the form of the expansion sets of Eqs. (4.42) and (4.43) implies that we are seeking solutions which are such that $U = V = 0$ when $\zeta = 0$ and $\zeta = \pi$. Thus, from helical symmetry and the tangential condition (Eq. (4.33)), we have $U = V$ everywhere on γ , and $|\xi_U - \xi_V| = |U^N - V^N|$ on the coil.

Also

$$\begin{aligned}
 |\partial_n(\xi_U - \xi_V)| &= |\partial_n(U - V) - \partial_n(U^N - V^N)| \\
 &= \left| \frac{-\mu_0 i_0 h r b \sin \zeta}{\sqrt{(a^2 + h^2 r^2 b^2 \sin^2 \zeta)}} - \partial_n(U^N - V^N) \right| \\
 &= \left| (r - b \cos \zeta) \partial_r(U^N - V^N) \right. \\
 &\quad \left. + b r^{-1} \sin \zeta (1 + h^2 r^2) \partial_\zeta(U^N - V^N) + \mu_0 i_0 h r b \sin \zeta \right| \\
 &\quad \div \left| \sqrt{(a^2 + h^2 r^2 b^2 \sin^2 \zeta)} \right|, \tag{4.44}
 \end{aligned}$$

where we have substituted

$$\mathbf{n} = -h r b \sin \zeta \mathbf{e}_z + (r - b \cos \zeta) \mathbf{e}_r + b \sin \zeta \mathbf{e}_\phi \tag{4.45}$$

$$\text{and } |\mathbf{n}| = \sqrt{(a^2 + h^2 r^2 b^2 \sin^2 \zeta)}. \tag{4.46}$$

The relative error estimates ξ^r and ξ_n^r have been defined as

$$\xi^r = \frac{\text{MAX}|U^N - V^N|}{\text{MAX}|V^N|}, \tag{4.47}$$

and

$$h \xi_n^r = \frac{\text{MAX}|\partial_n(\xi_U - \xi_V)|}{\text{MAX}|V^N|} \tag{4.48}$$

where the maxima are chosen from the set $\{\mathbf{p}_i\}$ of points on the boundary mid-way between the collocation points.

Performance of three different collocation methods

The results of trials of three types of collocation technique are shown in Fig. 4.7. The relative error estimates, ξ^r , have been plotted against the dimension of the expansion set,

Figure 4.7: Relative error estimates, ξ^r , against the dimension of the expansion set, N , for three collocation methods as described in the text. The solenoid has radius $a = 1$ m, displacement $b = .5$ m and pitch $h = 1$ m⁻¹.

N , for a circular coil of radius $a = 1$ m, displacement $b = .5$ m, pitch $h = 1$ m⁻¹ and current per unit length $i_0 = 5 \times 10^4$ A.m⁻¹. For two of the methods the number of expansion functions was the same as the number of collocation points – with the latter being distributed in equal arcs (method #1) or in equal increments of ζ (method #2) (see Fig. 4.8). Method #3 used least squares - collocation with 30 expansion functions and 90 collocation points distributed in equal arcs. The first collocation method was inaccurate at all values of N . The other two showed an (approximately) exponential decrease in ξ^r with N up to $N = 30$ which is what one would expect for an expansion in terms of a basis of smoothly varying functions. The decrease in accuracy of method #2 beyond $N = 40$

ξ^r		
Method #1	Method #2	Method #3
5.63×10^{-11}	2.19×10^{-7}	4.24×10^{-12}

Table 4.1: Comparison of the three collocation methods of Fig. 4.7 for a coil displacement of $b = .15$ m.

Figure 4.8: Two methods of placing the collocation points. In the top picture, the arc length between the points is a constant (Δ). The intercepts with the x -axis are automatically included by the form of the expansion sets. The distance to the first collocation point on either side is $\frac{1}{2}\Delta$. In the bottom picture the collocation points are placed in equal increments of the cylindrical angle ϕ .

coincided with the matrix system becoming ill-conditioned. The condition number for a square matrix system may be estimated from its behavior with iterative improvement (Dahlquist *et al.*, 1974) providing the residuals are accumulated in addition precision. Using the quadruple-precision facility of the FACOM-M360, I found that, whereas the $N = 40$ case was well conditioned, the $N = 50$ and $N = 60$ cases were so ill-conditioned that iterative improvement was not possible.

These results should be contrasted with the data of Table 4.1 which compares the three techniques with $N = 20$ at a coil displacement of $b = .15$ m. For these small displacements the first square matrix method actually out-performs the second. It fails

when the displacement is large – presumably because it does not give sufficient weight to points on the interior of the coil (see Fig. 4.8). The least squares - collocation method gives good results over a range of coil displacements, and appears to be better conditioned than the square matrix methods when N is large. Distributing the points of method three in equal intervals of ϕ produced little change in ξ^r .

Accuracy versus coil displacement

Figure 4.9 displays the relative error estimates for a least squares - collocation method with 90 collocation points and 30 expansion functions applied to a solenoid of radius $a = 1$ m, pitch $h = 1$ m⁻¹ and $i_0 = 5 \times 10^4$ A.m⁻¹. The values of ξ^r and ξ_n^r have been plotted against the coil displacement. Their range encompasses some 16 orders of magnitude! The maximum absolute error on the boundary for the exterior field, $|V^N - V|$, has been estimated using a Green's function integral expression which will be derived in Sec. 6(iv). The point with the maximum value of $|U^N - V^N|$ was chosen for the comparison, and a relative error was defined by $\xi_{abs}^r = |V^N - V_{gf}|/|V^N|$. The Green's function integration routine promises a relative error of order 10^{-4} (see Chap. 6) which is probably an over-estimate as the ξ_{abs}^r curve levels out at about 4×10^{-6} . Note that for these values of a and h , ξ_{abs}^r appears to be bracketted by ξ^r and ξ_n^r .

Figure 4.9 also plots ξ^r for a least squares - collocation method with $N = 10$ expansion functions.

4(vii) Discussion

It is possible to approximate the vacuum field of a circular cross-section, helically-symmetric solenoid by a series of products of modified Bessel and trigonometric functions. Boundary collocation will give excellent accuracy up to moderate values of the coil displacement. We have considered a few of the large number of possible collocation techniques. Of these, the least squares - collocation methods gives more consistent results than arbitrarily chosen square matrix methods.

So far we have not considered the convergence of the expansion sets as $N \rightarrow \infty$. This important question will be addressed in Chap. 6 following the development of the Green's function formalism. We preempt this discussion here by remarking that the decreasing accuracy with increasing coil displacement, and the ill-conditioning of the matrix system at large N are consistent with an expansion set which does not converge. This point is

Figure 4.9: Relative error estimates for a helical solenoid with $a = 1. \text{ m}$, $h = 1. \text{ m}^{-1}$, and $i_0 = 5. \times 10^4 \text{ A.m}^{-1}$ against the coil displacement. Least squares - collocation was used with 90 collocation points. The group of three curves on the right hand side of the graph represent ξ^r (dashed line), ξ_n^r (dotted) and a Green's function estimate, ξ_{abs}^r (squares), for a case with 30 expansion functions. The remaining dashed line represents ξ^r for a case with 10 expansion functions.

Figure 4.10: *Ratio of the last two terms in the exterior series for a circular coil as described in the text. Least squares collocation is used with 90 collocation points and 30 expansion functions.*

further illustrated by Fig. 4.10 which plots the ratio

$$\mathcal{R} = \frac{|a_n^e K_n(nhr)|}{|a_{n-1}^e K_{n-1}((n-1)hr)|}$$

against n for $r = 0.5$ m, $a = 1.$ m, $b = .5$ m, $i_0 = 5 \times 10^4$ A.m⁻¹ and $h = 1.$ m⁻¹. If $\mathcal{R}$ is less than unity in the limit of n large, then the series will converge. This is certainly true beyond $n = 22$, however the “tying-down” of the higher order terms of the series which the figure displays is an artifact of the collocation technique. $\mathcal{R}$ shows signs of tending to a limit for $n < 15$, however this limit is greater than one.

Finally, Fig. 4.11 shows the results of an application of a least squares - collocation technique with 90 points and 30 expansion functions to a coil of dimensions $a = .065$ m, $b = .025$ m, $h = 16.$ m⁻¹, $b_0 = \mu_0 i_0 = .2048$ T. This model will be referred to subsequently in this thesis. The figure plots the contours of the scalar potential, Φ , and the helical flux, ψ , in a constant- z plane.

Figure 4.11: *Contours of the scalar potential (upper) and helical flux (lower) for a helical solenoid with $a = .065$ m, $b = .025$ m, $h = 16. \text{ m}^{-1}$, $b_0 = .2048$ T. Ninety collocation points were matched with 30 expansion functions. Axis measurements are in cm.*

Chapter 5

Green's function for the helical Grad-Shafranov equation

5(i) Green's function for the helical Laplace equation

The free-space Green's function for Laplace's equation in three dimensions is

$$G^0(\mathbf{r}|\mathbf{r}_0) \equiv \frac{1}{R} = \frac{1}{|\mathbf{r} - \mathbf{r}_0|}. \quad (5.1)$$

$G^0(\mathbf{r}|\mathbf{r}_0)$ is symmetric in its arguments – the position vectors $\mathbf{r}$ and $\mathbf{r}_0$. It satisfies the equation

$$\nabla^2 G^0(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0), \quad (5.2)$$

where the differentiation is carried out with respect to $\mathbf{r}$, and

$$\begin{aligned} \delta(\mathbf{r} - \mathbf{r}_0) &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \\ &= \frac{1}{r}\delta(r - r_0)\delta(\phi - \phi_0)\delta(z - z_0) \\ &= \frac{1}{r}\delta(r - r_0)\delta(\zeta - \zeta_0)\delta(z - z_0) \\ &= \frac{1}{J}\delta(u - u_0)\delta(v - v_0)\delta(w - w_0). \end{aligned} \quad (5.3)$$

Here $J = [\nabla \mathbf{u} \cdot (\nabla \mathbf{v} \times \nabla \mathbf{w})]^{-1}$ is the Jacobian of the transformation from Cartesian coordinates (x, y, z) to the curvilinear coordinate system (u, v, w) . The helical coordinates $(r, \zeta = \phi - hz, z)$ have been defined elsewhere.

In the fashion of Sec. 4(ii), one may expand G^0 as a Fourier-Bessel series in terms of (r, ζ, z) . The helical "Fourier transform" of G^0 , constructed by integrating over $(z - z_0)$

holding $(\zeta - \zeta_0)$ fixed (Merkel, 1982), is then

$$\begin{aligned} G_k^0(r, \zeta | r_0, \zeta_0) &= \int_{-\infty}^{\infty} G^0(r | r_0) \exp(-ikz) d(z - z_0) \\ &= 2 \sum_{n=-\infty}^{\infty} b_n(r | k - nh, r_0 | k - nh) \exp[in(\zeta - \zeta_0)] , \end{aligned} \quad (5.4)$$

where

$$b_n(r, r_0) = \begin{cases} I_n(r) K_n(r_0) & r \leq r_0 \\ I_n(r_0) K_n(r) & r > r_0 \end{cases} .$$

Because $I_n(r) = I_{-n}(r)$ and $K_n(r) = K_{-n}(r)$ we may write Eq. (5.4) as

$$G_k^0 = 2b_0(r | k, r_0 | k) + 4 \sum_{n=1}^{\infty} b_n(r | k - nh, r_0 | k - nh) \cos[n(\zeta - \zeta_0)] . \quad (5.5)$$

Define

$$\mathcal{G}^0(r, \zeta | r_0, \zeta_0) = \lim_{k \rightarrow 0} G_k^0 ,$$

then, as (Abramowitz and Stegun, 1972, p.375)

$$\begin{aligned} \lim_{z \rightarrow 0} I_0(z) &= 1 \\ \text{and} \quad \lim_{z \rightarrow 0} K_0(z) &= -\ln z , \end{aligned}$$

we have

$$\mathcal{G}^0(r, \zeta | r_0, \zeta_0) = -2 \ln r_{>} + r \sum_{n=1}^{\infty} I_n(nhr_{<}) K_n(nhr_{>}) \cos[n(\zeta - \zeta_0)] , \quad (5.6)$$

where $r_{>} = \max(r, r_0)$. We will now show that $\mathcal{G}^0$ satisfies

$$\nabla^2 \mathcal{G}^0(r, \zeta | r_0, \zeta_0) = -\frac{4\pi}{r} \delta(r - r_0) \delta(\zeta - \zeta_0) \quad (5.7)$$

where ∇^2 is the Laplacian in helical coordinates. The demonstration is accomplished by comparing coefficients of $\cos[n(\zeta - \zeta_0)]$ in the expansion

$$\delta(\zeta - \zeta_0) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos[n(\zeta - \zeta_0)] . \quad (5.8)$$

$n > 0$ terms

The term $t = b_n(nhr, nhr_0) \cos[n(\zeta - \zeta_0)]$ is continuous in r and ζ . There is a discontinuity in the derivative with respect to r at $r = r_0$, given by the Wronskian condition

$$\mathcal{W}(z) \equiv I'_\nu(z) K_\nu(z) - I_\nu(z) K'_\nu(z) = \frac{1}{z} , \quad (5.9)$$

where $I'_\nu(z) = dI_\nu(z)/dz$. Now,

$$\frac{\nabla^2 t}{\cos[n(\zeta - \zeta_0)]} = \frac{1}{r} \partial_r \{ r \partial_r [I_n(nhr_<) K_n(nhr_>)] \} - \frac{n^2(1 + h^2 r^2)}{r^2} I_n(nhr_<) K_n(nhr_>) ,$$

which is equal to zero for $r \neq r_0$, from the identity

$$\frac{d}{dr} [r b'_n(nhr, nhr_0)] = \frac{n(1 + h^2 r^2)}{rh} b_n(nhr, nhr_0) , \quad (5.10)$$

where

$$b'_n(nhr, nhr_0) = \begin{cases} I'_n(nhr) K_n(nhr_0) & r \leq r_0 \\ I_n(nhr_0) K'_n(nhr) & r > r_0 . \end{cases} \quad (5.11)$$

To complete the proof it is necessary to integrate over the discontinuity in $\partial_r t$:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{r_0-\epsilon}^{r_0+\epsilon} r \frac{\nabla^2 t}{\cos[n(\zeta - \zeta_0)]} dr &= \lim_{\epsilon \rightarrow 0} \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{\partial_r(r \partial_r t)}{\cos[n(\zeta - \zeta_0)]} dr \\ &= \lim_{\epsilon \rightarrow 0} [nhr b'_n(nhr, nhr_0)]_{r_0-\epsilon}^{r_0+\epsilon} \\ &= -nhr_0 \mathcal{W}(nhr_0) = -1 \quad \text{as required.} \end{aligned}$$

$n = 0$ term

We need to show

$$\nabla^2(\ln r_>) = \frac{\delta(r - r_0)}{r} .$$

Firstly,

$$\frac{d}{dr} \left(r \frac{d}{dr} \ln r \right) = 0$$

verifies the equality when $r \neq r_0$. The discontinuity in the derivative gives us:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \ln r_> \right) r dr &= \lim_{\epsilon \rightarrow 0} \left[r \frac{d}{dr} \ln r_> \right]_{r_0-\epsilon}^{r_0+\epsilon} \\ &= 1 , \end{aligned}$$

which completes the demonstration.

Notice that in this case we have the same result if we substitute $-\ln r_<$ for $\ln r_>$. We distinguish these two linearly independent solutions by writing

$$\begin{aligned} \mathcal{G}_1^0(r, \zeta | r_0, \zeta_0) &= -2 \ln r_> + 4 \sum_{n=1}^{\infty} I_n(nhr_<) K_n(nhr_>) \cos[n(\zeta - \zeta_0)] \\ \text{and } \mathcal{G}_2^0(r, \zeta | r_0, \zeta_0) &= 2 \ln r_< + 4 \sum_{n=1}^{\infty} I_n(nhr_<) K_n(nhr_>) \cos[n(\zeta - \zeta_0)] . \end{aligned} \quad (5.12)$$

5(ii) Green's function for the helical Grad-Shafranov equation

The Green's functions

$$\begin{aligned} \mathcal{G}_1(r, \zeta | r_0, \zeta_0) &= -\frac{2}{h^2} \left(\ln r_{>} + \frac{1}{2} h^2 r_{>}^2 \right) - 4rr_0 \sum_{n=1}^{\infty} I'_n(nhr_{<}) K'_n(nhr_{>}) \cos[n(\zeta - \zeta_0)] \\ \text{and } \mathcal{G}_2(r, \zeta | r_0, \zeta_0) &= \frac{2}{h^2} \left(\ln r_{<} + \frac{1}{2} h^2 r_{<}^2 \right) \\ &\quad - 4rr_0 \sum_{n=1}^{\infty} I'_n(nhr_{<}) K'_n(nhr_{>}) \cos[n(\zeta - \zeta_0)] \end{aligned} \quad (5.13)$$

are solutions of the "fundamental" helical Grad-Shafranov equation (HGSE)

$$\mathcal{L}(\mathcal{G}) \equiv \nabla \cdot (\kappa \nabla \mathcal{G}) = -\frac{4\pi}{r} \delta(r - r_0) \delta(\zeta - \zeta_0), \quad (5.14)$$

where $\kappa = h^2/(1 + h^2 r^2)$. We shall show this by considering cases as above.

$n > 0$ terms

Expanding the left side of Eq. (5.14) we have

$$\mathcal{L}(\mathcal{G}) = -2r\kappa^2 \partial_r \mathcal{G} + \kappa \left[\frac{1}{r} \partial_r (r \partial_r \mathcal{G}) + \frac{(1 + h^2 r^2)}{r^2} \partial_\zeta^2 \mathcal{G} \right]$$

Define

$$C_n \equiv \begin{cases} I_n(r) K'_n(r_0) & r \leq r_0 \\ I'_n(r_0) K_n(r) & r > r_0 \end{cases} \quad (5.15)$$

Then, using Eq. (5.10) and writing $C'_n(x, x_0) \equiv dC_n(x, x_0)/dx$ we have

$$\begin{aligned} \partial_r [r C'_n(nhr, nhr_0)] &= \frac{n(1 + h^2 r^2)}{hr} C_n(nhr, nhr_0) \\ \text{and } \frac{1}{r} \partial_r [r \partial_r (r C'_n(nhr, nhr_0))] &= 2nh C_n + \frac{n^2(1 + h^2 r^2)}{r} C'_n, \end{aligned}$$

so that $\mathcal{L}(r C'_n \cos[n(\zeta - \zeta_0)]) = 0$ for $r \neq r_0$.

Once again the demonstration is completed by considering the integration over the discontinuity in the derivative:

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{\kappa}{r} \partial_r \{ r \partial_r [r r_0 I'_n(nhr_{<}) K'_n(nhr_{>})] \} r dr \\ &= \lim_{\epsilon \rightarrow 0} \frac{h^2}{1 + h^2 r_0^2} \left[\frac{r r_0 n(1 + h^2 r^2)}{hr} C_n(nhr, nhr_0) \right]_{r_0-\epsilon}^{r_0+\epsilon} \\ &= 1. \end{aligned}$$

$n = 0$ term

Let $u(r)$ and $v(r) \equiv u'(r)$ be functions of r only. Then

$$\mathcal{L}(u) = -2r\kappa^2 v + \frac{\kappa}{r} \frac{d}{dr}(rv),$$

and $\mathcal{L}(u) = 0$ is equivalent to

$$\begin{aligned} \frac{dv}{dr} &= (2r\kappa - r^{-1})v \\ \Leftrightarrow \int \frac{dv}{v} &= \int \frac{h^2 r^2 - 1}{r(1 + h^2 r^2)} dr \end{aligned}$$

This gives

$$\ln v = \ln[(1 + h^2 r^2)/r] + C_0,$$

and

$$u = C_1 \left(\ln r + \frac{1}{2} h^2 r^2 \right) + C_2,$$

where C_0, C_1 and C_2 are constants.

Put $u_1 = -2/h^2(\ln r_> + \frac{1}{2}h^2 r_>^2)$, then u_1 is continuous and $\mathcal{L}(u_1) = 0$ if $r \neq r_0$. The integration over the singularity gives

$$\begin{aligned} \int \mathcal{L}(u_1) r dr &= \lim_{\epsilon \rightarrow 0} \int_{r_0-\epsilon}^{r_0+\epsilon} \kappa \frac{d}{dr} \left(r \frac{du_1}{dr} \right) dr \\ &= \lim_{\epsilon \rightarrow 0} \frac{h^2}{1 + h^2 r_0^2} \left[-\frac{2r}{h^2} \frac{d}{dr} \left(\ln r_> + \frac{1}{2} h^2 r_>^2 \right) \right]_{r_0-\epsilon}^{r_0+\epsilon} \\ &= -2 \end{aligned}$$

which is the required normalisation. The same is true for $u_2 = 2/h^2(\ln r_< + \frac{1}{2}h^2 r_<^2)$.

5(iii) ψ due to a helical current filament

Consider a helical current, I , localised to $r = r_0$ and $\zeta = \zeta_0$. Its current density will have the form

$$\mathbf{j} = \alpha \delta(r - r_0) \delta(\zeta - \zeta_0) (\mathbf{e}_z + h r \mathbf{e}_\phi),$$

where α may be found from

$$I = \int \int \mathbf{j} \cdot d\mathbf{S} = \int \int \mathbf{j} \cdot \mathbf{e}_z r dr d\phi = \alpha r_0.$$

Recall that, from Sec. 2(ii), $g = \mu_0 I / 2\pi$ for a helical current I . The defining equation for ψ , Eq. (2.16), is thus

$$\begin{aligned} \mathcal{L}(\psi) &= -\frac{\kappa^2 \mu_0 I}{\pi h} + \frac{\mu_0 h I}{r} \delta(r - r_0) \delta(\zeta - \zeta_0) \\ &= -\frac{\mu_0 I h}{4\pi} \left(\frac{4\kappa^2}{h^2} + \mathcal{L}(g) \right), \end{aligned} \tag{5.16}$$

where $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G}_2$.

From Sec. 2(iv), there are two linearly independent particular solutions to $\mathcal{L}(\psi) = 4\kappa^2/h^2$ namely $s_1 = r^2$ and $s_2 = -(2/h^2) \ln r$. The appropriate choice of s_1 and s_2 must give the required asymptotic behaviour of ψ . As $r \rightarrow 0$ we require $\mathbf{B} \rightarrow B_0 \mathbf{e}_z$, which will be met by any $\psi : |\nabla\psi| \sim r^n$ with $n \geq 1$ as $r \rightarrow 0$. As $r \rightarrow \infty$ we require that $\mathbf{B} \rightarrow B_\phi \mathbf{e}_\phi$ which is possible for any $\psi : |\nabla\psi| \sim r^n$ with $n \leq 0$ as $r \rightarrow \infty$. If we choose $\mathcal{G} = \mathcal{G}_1$, then the divergence of $\mathcal{G}_1$ as $r \rightarrow \infty$ must be countered by adding s_1 , giving

$$\psi = -\frac{\mu_0 I h}{4\pi} \left(-\frac{2}{h^2} \ln r_> - r_>^2 + r^2 - 4rr_0 \sum_{n=1}^{\infty} I'_n(nhr_<) K'_n(nhr_>) \cos[n(\zeta - \zeta_0)] \right).$$

The solution may be made continuous at $r = r_0$ by adding the constant r_0^2 within the parentheses:

$$\psi = -\frac{\mu_0 I h}{4\pi} \left(-\frac{2}{h^2} \ln r_> + r_<^2 + \dots \right). \quad (5.17)$$

Similarly, if we put $\mathcal{G} = \mathcal{G}_2$, the divergence as $r \rightarrow 0$ is countered by adding s_2 which, when made continuous at $r = r_0$, returns Eq. (5.17).

The equations

$$\begin{aligned} \psi &= -\frac{\mu_0 I h}{4\pi} (\mathcal{G}_1 + s_1) \\ &= -\frac{\mu_0 I h}{4\pi} (\mathcal{G}_2 + s_2) \end{aligned} \quad (5.18)$$

may be inverted to give a physical interpretation of the Green's functions $\mathcal{G}_1$ and $\mathcal{G}_2$. From Eq. (2.15), we see that $\psi_1 = -(\mu_0 I h/4\pi)s_1$ is the flux inside a cylindrical solenoid with current per unit length equal to $i_0 = Ih/2\pi$. This is also the limit of the interior flux of a helical current filament as the radius of the helix is taken to infinity. Also, from Eq. (2.14), we see that $\psi_2 = -(\mu_0 I h/4\pi)s_2$ is the flux due to a current I flowing along the z axis. It follows that $\mathcal{G}_1$, is the flux due to a helical current $I = -(4\pi/\mu_0 h)$ with a return path at infinity and $\mathcal{G}_2$ is the flux due to a helical current $I = -(4\pi/\mu_0 h)$ with a return path along the z axis.

It is possible to derive $\mathcal{G}_1$ and $\mathcal{G}_2$ by integrating the Biot-Savart law along the appropriate current loops. For instance, one may demonstrate the equivalence of the series expansion for $\mathcal{G}_1$ and the integral expression (R.L. Dewar and W.N-Sy, private communication):

$$\mathcal{G}_1(r, \zeta | r_0, \zeta_0) = -(r^2 + r_0^2) + \frac{1}{h^2} \int_{-\infty}^{\infty} \left\{ \frac{[1 + h^2 r r_0 \cos(\phi - \phi_0)]}{|\mathbf{r} - \mathbf{r}_0|} - \frac{1}{\sqrt{(z - z_0)^2 + a^2}} \right\} dz_0 \quad (5.19)$$

where $\phi - \phi_0 = \zeta - \zeta_0 + h(z - z_0)$. The second term under the integral, involving the arbitrary constant a , is added to help it converge.

5(iv) Toroidal and cylindrical limits of the HGSE

The HGSE for $l = 1$ systems remains meaningful in the two limits $h \rightarrow 0$ and $h \rightarrow \infty$. As $h \rightarrow 0$ we have:

$$\begin{aligned}\zeta &\rightarrow \phi, \\ \mathbf{u} &= \frac{\mathbf{e}_z + h r \mathbf{e}_\phi}{1 + h^2 r^2} \rightarrow \mathbf{e}_z, \\ \psi &= -\frac{1}{h}(A_z + h r A_\phi) \rightarrow -\frac{A_z}{h}, \\ \kappa &= \frac{h^2}{1 + h^2 r^2} \rightarrow h^2, \\ \mathbf{B} &= h \mathbf{u} \times \nabla \psi + h g \mathbf{u} \rightarrow -\mathbf{e}_z \times \nabla A_z + B_z \mathbf{e}_z, \\ \text{and } \mathcal{L}(\psi) &\rightarrow h^2 \nabla^2 \psi.\end{aligned}$$

In the last equation ∇^2 is the Laplacian for polar coordinates. ψ is proportional to the azimuthal flux in a cylindrically symmetric system. The limit of Eq. (2.16) is

$$\begin{aligned}h^2 \nabla^2 \left(-\frac{A_z}{h} \right) &= -2h^3 g + \mu_0 h J_z \\ \Leftrightarrow \nabla^2(A_z) &= -\mu_0 J_z,\end{aligned}\tag{5.20}$$

which is the correct equation for $A_z(r, \phi)$.

As $h \rightarrow \infty$:

$$\begin{aligned}\zeta &\rightarrow -hz, \\ \mathbf{u} &\rightarrow \frac{1}{h} \nabla \phi, \\ \psi &\rightarrow -r A_\phi, \\ \kappa &\rightarrow \frac{1}{r^2}, \\ \mathbf{B} &\rightarrow \nabla \phi \times \nabla \psi + B_\phi \nabla \phi, \\ \text{and } \mathcal{L}(\psi) &\rightarrow \frac{1}{r^2} [r \partial_r (r^{-1} \partial_r \psi) + \partial_z^2 \psi] = \frac{1}{r^2} \Delta^* \psi,\end{aligned}$$

Where Δ^* is the Stokes operator (Dewar, 1976). In this case, the limit of ψ is proportional to the poloidal flux, and Eq. (2.16) becomes

$$\Delta^* \psi = \mu_0 r J_\phi,\tag{5.21}$$

which is the defining equation for ψ in toroidal geometry.

5(iv-a) Toroidal and cylindrical limits of the helical Green's function

Many helical systems will not tend to their "toroidal" limit as $h \rightarrow \infty$. In fact, the behaviour in this limit of a helical filament at $(r = r_0, \zeta = \zeta_0)$ is qualitatively different from its cylindrical limit, $h \rightarrow 0$. If we keep one point fixed then as $h \rightarrow 0$ the cylindrical limit (of a straight line parallel to the z -axis) is approached uniformly. However, as $h \rightarrow \infty$ the filament approaches an infinite continuous solenoid, which is topologically distinct from the loop $(r = r_0, z = z_0)$ which might be said to exist at $h = \infty$. On the other hand, it is possible to obtain a toroidal flux tube from a helical one (Dewar *et al.*, 1984) as $h \rightarrow \infty$ providing a cross-section in a constant- ϕ plane is held fixed. It is possible to use both Eqs. (5.20) and (5.21) as tests of the implementation of a helically symmetric equilibrium code.

The difficulty with the toroidal limit is reflected in the fact that there is no simple transformation between the helical and toroidal Green's functions. One may, however, derive both of them from the "generic" Green's function:

$$G(\mathbf{r}|\mathbf{r}_0) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp[ik(z - z_0)] \left\{ \frac{2}{h^2} \ln(r_{<}) - 2r_{<}r_{>} I'_0(r_{<}|k|) K'_0(r_{>}|k|) \right. \\ \left. - 2r_{<}r_{>} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} I'_n(r_{<}|k - nh|) K'_n(r_{>}|k - nh|) \exp[in(\zeta - \zeta_0)] \right\} \quad (5.22)$$

In the same way that $\mathcal{G}_1^0$, was obtained from $1/|\mathbf{r} - \mathbf{r}_0|$ for the helical Laplace equation (Merkel, 1982), one may obtain $\mathcal{G}_2$ from $G(\mathbf{r}|\mathbf{r}_0)$ by integrating over the ignorable direction for helical symmetry:

$$\mathcal{G}_2 = \lim_{k' \rightarrow 0} \int_{-\infty}^{\infty} G(\mathbf{r}|\mathbf{r}_0) \exp[-ik'(z - z_0)] d(z - z_0) . \quad (5.23)$$

One may verify Eq. (5.23) using

$$\delta(k - k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i(k - k')(z - z_0)] d(z - z_0) , \quad (5.24)$$

and noting that, for the $n = 0$ term, (Abramowitz and Stegun, 1972)

$$\lim_{z \rightarrow 0} I_\nu(az) K_\nu(bz) = \frac{1}{2\nu} \left(\frac{a}{b} \right)^\nu , \quad (5.25)$$

gives

$$\lim_{k \rightarrow 0} -2r_{<}r_{>} I'_0(r_{<}|k|) K'_0(r_{>}|k|) = 2r_{<}r_{>} \lim_{k \rightarrow 0} I_1(r_{<}|k|) K_1(r_{>}|k|) \\ = r_{<}^2 . \quad (5.26)$$

This gives Eq. (5.13):

$$\mathcal{G}_2(r, \zeta | r, \zeta_0) = \frac{2}{h^2} \left(\ln r_{<} + \frac{1}{2} h^2 r_{<}^2 \right) - 4r_{<} r_{>} \sum_{n=1}^{\infty} I'_n(nhr_{<}) K'_n(nhr_{>}) \cos[n(\zeta - \zeta_0)] .$$

Now consider the cylindrical limit of $\mathcal{G}_2$. Using

$$\lim_{h \rightarrow 0} -4r_{<} r_{>} I'_n(r_{<} nh) K'_n(r_{>} nh) = \frac{2}{nh^2} \left(\frac{r_{<}}{r_{>}} \right)^n$$

we find

$$\begin{aligned} G_z(r, \phi | r_0, \phi_0) &\equiv \lim_{h \rightarrow 0} (h^2 \mathcal{G}_2) \\ &= 2 \ln r_{<} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r_{<}}{r_{>}} \right)^n \cos[n(\phi - \phi_0)] . \end{aligned} \quad (5.27)$$

G_z is a Green's function for Laplace's equation in polar coordinates:

$$\nabla^2 G_z(r, \phi | r_0, \phi_0) = -\frac{4\pi}{r} \delta(r - r_0) \delta(\phi - \phi_0) . \quad (5.28)$$

To obtain the toroidal Green's function, we must integrate Eq. (5.22) over one period in ζ with z fixed, and then take the limit $h \rightarrow \infty$ to kill the secular term:

$$\begin{aligned} G_\phi(r, z | r_0, z_0) &\equiv \lim_{h \rightarrow \infty} \int_0^{2\pi} G d(\zeta - \zeta_0) \\ &= -2r_{<} r_{>} \int_{-\infty}^{\infty} I'_0(|k|r_{<}) K'_0(|k|r_{>}) \cos[k(z - z_0)] dk . \end{aligned} \quad (5.29)$$

G_ϕ satisfies

$$\frac{1}{r^2} \Delta^* G_\phi = -\frac{4\pi}{r} \delta(r - r_0) \delta(z - z_0) , \quad (5.30)$$

and is the correct Green's function for the field of a current loop (Dewar, 1976). Note that the azimuthally symmetric form of the operator

$$\nabla \cdot (\kappa \nabla \psi) = -2r\kappa^2 \psi + \kappa[r^{-1} \partial_r (r \partial_r \psi) + \partial_z^2 \psi]$$

has $r^{-2} \Delta^* \psi$ as its limit as $h \rightarrow \infty$ which, as we saw in Eq. (5.21), is also the limit of the helically symmetric form.

We observe parenthetically that in the limit of $h \rightarrow \infty$,

$$\begin{aligned} I'_n(nhr_{<}) K'_n(nhr_{>}) &\longrightarrow I'_n(nhr) K'_n(nhr) \\ &\longrightarrow -\frac{1}{2hr} \longrightarrow 0 . \end{aligned}$$

The uniform limit of ψ due to a helical current, Eq. (5.17), is thus

$$\psi \longrightarrow -\frac{\mu_0 I h}{4\pi} r_{<}^2 ,$$

and so $\mathbf{B} \rightarrow \mu_0 I h / (2\pi) \mathbf{e}_z + B_\phi \nabla \phi$. That is, we approach the longitudinal field inside a solenoid of current per unit length equal to $Ih/2\pi$.

Chapter 6

Green's theorem for the straight heliac

6(i) The model problem

In this chapter we shall apply Green's theorem to a "model problem" which includes the essential features of a heliac in the helically symmetric limit. Consider the $l = 1$ helically symmetric configuration shown in Fig. (6.1) in cross-section perpendicular to the z axis. The outer solenoid, γ , has unit outward normal $\mathbf{e}_n$ and carries a surface current $\mathbf{i}$ of magnitude i_0 (per incremental length in the z direction) in the plane of the diagram. There is a straight conductor of current I along the $\mathbf{e}_z$ axis. The plasma surface is denoted by Γ with unit outward normal $\mathbf{e}_N$. The plasma may have a net longitudinal current, I_p ,

Figure 6.1: Configuration for the model problem.

Figure 6.2: *Jump in B across a helically symmetric interface.*

and a surface current i_1 which will not be required to satisfy force balance at this stage. Space is divided into the three regions labelled A, B and C.

From the results of Chap. 2, the equations for ψ are

$$\mathcal{L}(\psi) = -\frac{2\kappa^2}{h}g_A \quad \text{in region A,} \quad (6.1)$$

$$\mathcal{L}(\psi) = -\frac{2\kappa^2}{h}g_B \quad \text{in region B} \quad (6.2)$$

$$\text{and } \mathcal{L}(\psi) = -\frac{2\kappa^2}{h}g_C(\psi) - \kappa g_C g'_C(\psi) - \mu_0 p'(\psi) \quad \text{in region C.} \quad (6.3)$$

The (constant) values of g in the vacuum regions can be calculated from $g = \mathbf{B} \cdot \mathbf{u}/h|\mathbf{u}|^2$, where we note that, because $\mathbf{u} \cdot \nabla f(r, \zeta) = 0$ for any helically symmetric scalar field f , only the secular parts of Φ , needed to satisfy the constraints of Ampère's law, will contribute. Thus

$$g_A = \frac{\mu_0(I + I_p)}{2\pi}, \quad (6.4)$$

and

$$g_B = g_A + \mu_0 i_0 h^{-1}. \quad (6.5)$$

Consider now the boundary conditions on B across a helically symmetric interface. Let $\mathbf{e}_n$ be the unit normal to the interface. Two mutually perpendicular tangent vectors are $\mathbf{u}$ and $\mathbf{u} \times \mathbf{e}_n$, as shown in Fig. (6.2). The jump conditions on B are

$$[[\mathbf{B}]] \cdot \mathbf{e}_n = 0, \quad (6.6)$$

$$[[\mathbf{B}]] \cdot \mathbf{u} = \mu_0 \mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u}), \quad (6.7)$$

$$[[\mathbf{B}]] \cdot (\mathbf{u} \times \mathbf{e}_n) = \mu_0 \mathbf{i} \cdot \mathbf{u}, \quad (6.8)$$

where $[[\mathbf{B}]] \equiv \mathbf{B}_e - \mathbf{B}_i$, and $\mathbf{i}$ is the surface current. From Eq. (6.6) and $\mathbf{u} \cdot \mathbf{e}_n = 0$, we

have

$$\begin{aligned} \mathbf{e}_n \cdot \mathbf{u} \times [[\nabla\psi]] &= 0 \\ \Leftrightarrow \mathbf{u} \cdot \mathbf{e}_n \times [[\nabla\psi]] &= 0. \end{aligned} \quad (6.9)$$

Note that because $\mathbf{e}_n$ and $\nabla\psi$ are both perpendicular to $\mathbf{u}$, $\mathbf{e}_n \times [[\nabla\psi]]$ must be in the direction of $\mathbf{u}$. This is contradicted by Eq. (6.9), implying that the jump in $\nabla\psi$ is always perpendicular to the surface (i.e. the tangential components of $\nabla\psi$ are continuous), and we are free to stipulate

$$\psi_e = \psi_i. \quad (6.10)$$

From Eq. (6.7)

$$\begin{aligned} h[[g]]|\mathbf{u}|^2 &= \mu_0 \mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u}) \\ \Leftrightarrow [[g]] &= \frac{\mu_0 \mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u})}{h|\mathbf{u}|^2}. \end{aligned} \quad (6.11)$$

It is an interesting exercise in pedantry to apply Eq. (6.11) to calculate $g_A - g_B$. This is carried out in Appendix A. In this Appendix, we also derive the following expression for the surface current on the helical solenoid, γ :

$$\mathbf{i} = \frac{i_0 \sqrt{(r^2 \dot{\phi}^2 + \dot{r}^2)}}{\sqrt{(r^2 \dot{\phi}^2 + \dot{r}^2 + h^2 r^2 \dot{\phi}^2)}} \mathbf{e}_t \quad (6.12)$$

where $\mathbf{e}_t(r, \phi)$ is the tangent vector in the z plane which has been calculated in Eq. (4.27).

From Eq. (6.8),

$$\begin{aligned} h[[\mathbf{u} \times \nabla\psi]] \cdot (\mathbf{u} \times \mathbf{e}_n) &= \mu_0 \mathbf{i} \cdot \mathbf{u} \\ \Leftrightarrow h[[\mathbf{u} \cdot \mathbf{u} \nabla\psi \cdot \mathbf{e}_n - \mathbf{u} \cdot \mathbf{e}_n \nabla\psi \cdot \mathbf{u}]] &= \mu_0 \mathbf{i} \cdot \mathbf{u} \\ \Leftrightarrow \mathbf{e}_n \cdot [[\nabla\psi]] &= \mu_0 \mathbf{i} \cdot \mathbf{u} / h|\mathbf{u}|^2. \end{aligned} \quad (6.13)$$

Because the plasma boundary is necessarily a flux surface, an additional boundary condition for $\nabla\psi$ is $\mathbf{e}_T \cdot \nabla\psi|_r = 0$. In summary, the following equations must hold for the flux function ψ for the model problem:

$$\mathcal{L}(\psi) = -\frac{2\kappa^2}{h} \left(\frac{\mu_0(I + I_p)}{2\pi} \right) \quad \text{in } A, \quad (6.14)$$

$$\mathcal{L}(\psi) = -\frac{2\kappa^2}{h} \left(\frac{\mu_0(I + I_p)}{2\pi} + \frac{\mu_0 i_0}{h} \right) \quad \text{in } B, \quad (6.15)$$

$$\mathcal{L}(\psi) = -\frac{2\kappa^2}{h} g_C(\psi) - \kappa g_C g'_C(\psi) - \mu_0 p'(\psi) \quad \text{in } C, \quad (6.16)$$

$$\mathbf{e}_T \cdot \nabla\psi|_r = 0, \quad (6.17)$$

$$[[\psi]]_r = [[\psi]]_\gamma = 0, \quad (6.18)$$

$$\psi \sim \frac{\mu_0 I}{2\pi h} \ln r \quad \text{as } r \rightarrow 0, \quad (6.19)$$

$$\psi \sim \frac{\mu_0(I + I_p)}{2\pi h} \ln r + c_1 \quad \text{as } r \rightarrow \infty, \quad (6.20)$$

$$\mathbf{e}_n \cdot [[\nabla \psi]]_\gamma = \frac{\mu_0 \mathbf{i} \cdot \mathbf{u}}{h|\mathbf{u}|^2}, \quad (6.21)$$

$$\mathbf{e}_N \cdot [[\nabla \psi]]_\Gamma = \frac{\mu_0 \mathbf{i}_1 \cdot \mathbf{u}}{h|\mathbf{u}|^2}, \quad (6.22)$$

$$[[g]]_\Gamma = \frac{\mu_0 \mathbf{i}_1 \cdot (\mathbf{e}_N \times \mathbf{u})}{h|\mathbf{u}|^2}. \quad (6.23)$$

Equation (6.18) implies that there is one arbitrary constant in the equation set (if we assume that the right hand side of Eq. (6.16) remains unchanged when a constant is added to ψ_C) which has been denoted by c_1 in Eq. (6.20). Note that the asymptotic behaviour of ψ as $r \rightarrow 0$ and $r \rightarrow \infty$ depends only on the longitudinal currents (see Sec. 2(ii)). As usual

$$[[f]]_\gamma = (f_A - f_B)|_\gamma$$

and $[[f]]_\Gamma = (f_B - f_C)|_\Gamma.$

6(ii) Green's theorem for the model problem

From

$$\begin{aligned} G\mathcal{L}(\psi) - \psi\mathcal{L}(G) &= G\nabla \cdot (\kappa\nabla\psi) - \psi\nabla \cdot (\kappa\nabla G) \\ &= \nabla \cdot (G\kappa\nabla\psi - \psi\kappa\nabla G), \end{aligned} \quad (6.24)$$

one obtains the identity

$$\int_V [G\mathcal{L}(\psi) - \psi\mathcal{L}(G)] dV = \int_S \kappa(G\partial_n\psi - \psi\partial_n G) dS, \quad (6.25)$$

for two functions, G and ψ , which are suitably defined over the three dimensional volume V and bounding surface S . As usual, ∂_n represents the gradient in the direction of the unit outward normal to S .

We suppose G to be a Green's function satisfying

$$\mathcal{L}[G(\mathbf{r}|\mathbf{r}_0)] = -\frac{4\pi}{r}\delta(\mathbf{r} - \mathbf{r}_0)\delta(\zeta - \zeta_0),$$

and apply Eq. (6.25) to region C of the model problem integrated over one helical period. Then, noting that the integrals over the end surfaces will cancel (as in Sec. 4(iv)), we have

$$\begin{aligned} \psi(r_0, \zeta_0) 4\pi \frac{2\pi}{h} &= \int_\Gamma \kappa(G\partial_N\psi - \psi\partial_N G) dS \\ &\quad - \int_C G(r, \zeta | r_0, \zeta_0) \mathcal{L}(\psi) dV \end{aligned}$$

for (r_0, ζ_0) in C . Taking each of the three regions separately one obtains,

$$\begin{aligned}
I_\Gamma(G, \psi_C) - \int_C G \mathcal{L}(\psi) dV &= \begin{cases} 8\pi^2 h^{-1} \psi(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } C \\ 0 & (r_0, \zeta_0) \text{ outside } C \end{cases} \\
I_\gamma(G, \psi_B) - I_\Gamma(G, \psi_B) + I_0(G, \psi) - \int_B G \mathcal{L}(\psi) dV \\
&= \begin{cases} 8\pi^2 h^{-1} \psi(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } B \\ 0 & (r_0, \zeta_0) \text{ outside } B \end{cases} \\
I_\infty(G, \psi) - I_\gamma(G, \psi_A) - \int_A G \mathcal{L}(\psi) dV \\
&= \begin{cases} 8\pi^2 h^{-1} \psi(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } A \\ 0 & (r_0, \zeta_0) \text{ outside } A \end{cases}
\end{aligned}$$

where we have defined four bilinear, antisymmetric operators:

$$I_\Gamma(G, \psi) \equiv \int_\Gamma \kappa(G \partial_N \psi - \psi \partial_N G) dS, \quad (6.26)$$

$$I_\gamma(G, \psi) \equiv \int_\gamma \kappa(G \partial_n \psi - \psi \partial_n G) dS, \quad (6.27)$$

and I_0 and I_∞ , which are the contributions from the surface integrals around $r = 0$ and $r = \infty$ (whose neighbourhoods are assumed excluded from A and B , so there are no implicit δ function contributions to the volume integrals arising from $\mathcal{L}$ acting on a discontinuous function). Assuming the limits to exist, I_0 and I_∞ take the form:

$$\begin{aligned}
I_0(G, \psi) &\equiv -\lim_{r \rightarrow 0} \frac{2\pi}{h} \int_0^{2\pi} \kappa(G \partial_r \psi - \psi \partial_r G) r d\zeta \\
&= -2\pi h \lim_{r \rightarrow 0} \int_0^{2\pi} r (G \partial_r \psi - \psi \partial_r G) d\zeta, \quad (6.28)
\end{aligned}$$

and

$$\begin{aligned}
I_\infty(G, \psi) &\equiv \lim_{r \rightarrow \infty} \frac{2\pi}{h} \int_0^{2\pi} \kappa(G \partial_r \psi - \psi \partial_r G) r d\zeta \\
&= \frac{2\pi}{h} \lim_{r \rightarrow \infty} \int_0^{2\pi} \frac{1}{r} (G \partial_r \psi - \psi \partial_r G) d\zeta. \quad (6.29)
\end{aligned}$$

Because I_Γ , I_γ , I_0 and I_∞ are linear in ψ one may combine the above expressions to get

$$\begin{aligned}
8\pi^2 h^{-1} \psi(r_0, \zeta_0) &= -\int_{A+B+C} G(r, \zeta | r_0, \zeta_0) \mathcal{L}(\psi) dV - I_\gamma(G, [[\psi]]) \\
&\quad - I_\Gamma(G, [[\psi]]) + I_0(G, \psi) + I_\infty(G, \psi) \quad (6.30)
\end{aligned}$$

for (r_0, ζ_0) throughout $A+B+C$, where $A+B+C$ denotes the union of A , B and C with the neighbourhoods of γ , Γ , $r = 0$ and $r = \infty$ deleted.

6(ii-a) Evaluation of ψ using auxiliary flux functions

The solution to the model problem must obey all of Eqs. (6.14) to (6.23) as well as the integral expression of Eq. (6.30). To avoid having to perform the volume integral in Eq. (6.30) explicitly, we define, as an "auxiliary" flux function in regions A and B , a function ψ_{ext} which obeys the same partial differential equation as ψ in $A + B$ (but not C) and the same boundary conditions on γ (but not Γ). Such a function may be constructed by superimposing known vacuum solutions. Take

$$\psi_{ext} \equiv \psi_{sol} + \psi_{cc} + \hat{\psi},$$

where ψ_{sol} and ψ_{cc} correspond to the vacuum fields of the helical solenoid and the central conductor, and $\hat{\psi}$ is any solution of the vacuum HGSE with $g = \mu_0 I_p / 2\pi$ (see Sec. 6(ii-b)). Then ψ_{ext} satisfies (taking ψ and ψ_{ext} to be continuous across γ)

$$\mathcal{L}(\psi - \psi_{ext}) = 0 \quad (6.31)$$

$$\text{and} \quad I_\gamma(G, [[\psi - \psi_{ext}]]) = 0. \quad (6.32)$$

As the second auxiliary flux function we choose any $\tilde{\psi}$ such that $\mathcal{L}(\psi - \tilde{\psi}) = 0$ inside C . Replacing ψ in Eq. (6.30) with $\psi - \psi_{ref}$, where

$$\psi_{ref}(r, \zeta) = \begin{cases} \psi_{ext}(r, \zeta) & (r, \zeta) \text{ in } A + B \\ \tilde{\psi}(r, \zeta) & (r, \zeta) \text{ in } C \end{cases},$$

and taking ψ to be continuous across the solenoid, we get

$$\begin{aligned} 8\pi^2 h^{-1} \psi(r_0, \zeta_0) = & \begin{cases} 8\pi^2 h^{-1} \psi_{ext}(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } A + B \\ 8\pi^2 h^{-1} \tilde{\psi}(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } C \end{cases} \\ & + I_\Gamma(G, \psi_{ext} - \tilde{\psi}) - \int_\Gamma G \kappa \frac{\mu_0 \mathbf{i}_1 \cdot \mathbf{u}}{h|\mathbf{u}|^2} \\ & + I_0(G, \psi - \psi_{ext}) + I_\infty(G, \psi - \psi_{ext}). \end{aligned} \quad (6.33)$$

Consider the value of $\psi(r_0, \zeta_0)$, evaluated from Eq. (6.33), as the observer point approaches a point (r_s, ζ_s) on Γ . Grouping the surface integrals together as $\alpha(r_0, \zeta_0)$ we have

$$\psi(r_0, \zeta_0) = \alpha^+(r_0, \zeta_0) + \psi_{ext}(r_0, \zeta_0) \text{ for } (r_0, \zeta_0) \text{ outside } \Gamma \quad (6.34)$$

$$\text{and } \psi(r_0, \zeta_0) = \alpha^-(r_0, \zeta_0) + \tilde{\psi}(r_0, \zeta_0) \text{ for } (r_0, \zeta_0) \text{ inside } \Gamma. \quad (6.35)$$

As $(r_0, \zeta_0) \rightarrow (r_s, \zeta_s)$ the left hand sides of Eqs. (6.34) and (6.35) will approach $\psi(r_s, \zeta_s)$ smoothly. To balance the switch from ψ_{ext} to $\tilde{\psi}$ on the right hand sides there must be a

discontinuity in the values of α either side of the boundary and we have

$$\lim_{(r_0, \zeta_0) \rightarrow (r_s, \zeta_s)} (\alpha^+ - \alpha^-) = \tilde{\psi}(r_s, \zeta_s) - \psi_{ext}(r_s, \zeta_s) .$$

[This will presumably be due to the integration involving $(\psi_{ext} - \tilde{\psi})\partial_n G(r, \zeta | r_0, \zeta_0)$. The dominant singularity in G is logarithmic (as is shown in Appendix C). The integral of the normal derivative of G will therefore look something like a two dimensional double potential near the surface. The double potential is, in fact, approached in the cylindrical limit $h \rightarrow 0$.]

In common with standard potential theory, the value of α for (r_0, ζ_0) on the surface is taken to be

$$\begin{aligned} \alpha(r_s, \zeta_s) &= \alpha^- + \frac{1}{2}(\alpha^+ - \alpha^-) \\ &= \alpha^-(r_s, \zeta_s) + \frac{1}{2}(\tilde{\psi}(r_s, \zeta_s) - \psi_{ext}(r_s, \zeta_s)) \end{aligned}$$

so that, from Eq. (6.35),

$$\begin{aligned} \psi(r_s, \zeta_s) &= \alpha(r_s, \zeta_s) - \frac{1}{2}(\tilde{\psi}(r_s, \zeta_s) - \psi_{ext}(r_s, \zeta_s)) + \tilde{\psi}(r_s, \zeta_s) \\ &= \alpha(r_s, \zeta_s) + \frac{1}{2}(\tilde{\psi}(r_s, \zeta_s) + \psi_{ext}(r_s, \zeta_s)) . \end{aligned}$$

This gives the complete expression for ψ :

$$\begin{aligned} 8\pi^2 h^{-1} \psi(r_0, \zeta_0) &= I_\Gamma(G, \psi_{ext} - \tilde{\psi}) \\ &\quad + I_0(G, \psi - \psi_{ext}) + I_\infty(G, \psi - \psi_{ext}) \\ &\quad + \begin{cases} 8\pi^2 h^{-1} \psi_{ext}(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } A+B \\ 8\pi^2 h^{-1} \tilde{\psi}(r_0, \zeta_0) & (r_0, \zeta_0) \text{ in } C \\ 4\pi^2 h^{-1} (\psi_{ext} + \tilde{\psi}) & (r_0, \zeta_0) \text{ on } \Gamma \end{cases} \\ &\quad - \int_\Gamma G \kappa \frac{\mu_0 \mathbf{i}_1 \cdot \mathbf{u}}{h|\mathbf{u}|^2} . \end{aligned} \tag{6.36}$$

Note that it is straightforward to site extra filamentary currents in the vacuum region. In this case g_A and g_B must be adjusted accordingly and extra delta function source terms will appear on the right hand side of the HGSE (see Eq. (5.16)). If the helical flux due to these additional currents in vacuum is added to ψ_{ext} then the rest of the formalism proceeds as above.

6(ii-b) Choice of $\hat{\psi}$

Suppose that $\hat{\psi}_0$ is the helical flux due to a current I_p somewhere in region C [see Eq. (5.17)]. The flux functions $\hat{\psi}_0, \hat{\psi}_1 = (\mu_0 I_p / 2\pi h) \ln r$ and $\hat{\psi}_2 = -(\mu_0 I_p / 2\pi h)(h^2 r^2 / 2)$ all

satisfy

$$\mathcal{L}(\hat{\psi}_0) = \mathcal{L}(\hat{\psi}_1) = \mathcal{L}(\hat{\psi}_2) = -\frac{2\kappa^2}{h} \left(\frac{\mu_0 I_p}{2\pi} \right)$$

in $A+B$, but their asymptotic limits are different.

Two candidate Green's functions are $\mathcal{G}_1$ and $\mathcal{G}_2$ (see Chap. 5), which behave in their asymptotic limits as:

$$\begin{aligned} \mathcal{G}_1(r, \zeta | r_0, \zeta_0) &\sim -\frac{2}{h^2} \left(\ln r + \frac{h^2 r^2}{2} \right) \text{ as } r \rightarrow \infty, \\ \mathcal{G}_1 &\rightarrow -\frac{2}{h^2} \left(\ln r_0 + \frac{h^2 r_0^2}{2} \right) \text{ as } r \rightarrow 0, \\ \mathcal{G}_2 &\rightarrow \frac{2}{h^2} \left(\ln r_0 + \frac{h^2 r_0^2}{2} \right) \text{ as } r \rightarrow \infty, \\ \text{and } \mathcal{G}_2 &\sim \frac{2}{h^2} \ln r \text{ as } r \rightarrow 0. \end{aligned}$$

From Eq. (5.17)

$$\begin{aligned} \hat{\psi}_0 &\sim -\frac{\mu_0 I_p}{2\pi h} \frac{h^2 r^2}{2} = \hat{\psi}_2 \text{ as } r \rightarrow 0 \\ \hat{\psi}_0 &\sim \frac{\mu_0 I_p}{2\pi h} \ln r = \hat{\psi}_1 \text{ as } r \rightarrow \infty. \end{aligned} \quad (6.37)$$

We now evaluate the integrals I_0 and I_∞ in Eq. (6.36). First note that the asymptotic behaviour of $\hat{\psi}_0$ is such that $\psi - \psi_{ext,0} \sim 0$ as $r \rightarrow 0$ or $r \rightarrow \infty$ (where $\psi_{ext,0} = \psi_{sol} + \psi_{cc} + \hat{\psi}_0$). Thus we may replace ψ by $\psi_{ext,0}$ in I_0 and I_∞ giving (upon cancelling $\psi_{sol} + \psi_{cc}$):

$$\begin{aligned} I_0(G, \psi - \psi_{ext}) &= I_0(G, \hat{\psi}_0 - \hat{\psi}) \\ \text{and } I_\infty(G, \psi - \psi_{ext}) &= I_\infty(G, \hat{\psi}_0 - \hat{\psi}). \end{aligned}$$

The values of I_0 and I_∞ in Eq. (6.36) for all possible combinations of $\mathcal{G}_1$, $\mathcal{G}_2$, $\hat{\psi}_1$ and $\hat{\psi}_2$ can now be shown to be:

$$\begin{aligned} I_0(\mathcal{G}_1, \hat{\psi}_0 - \hat{\psi}_1) &= -\frac{4\pi\mu_0 I_p}{h^2} \left(\ln r_0 + \frac{h^2 r_0^2}{2} \right) \\ &= -\frac{8\pi^2}{h} (\hat{\psi}_1 - \hat{\psi}_2), \\ I_\infty(\mathcal{G}_1, \hat{\psi}_0 - \hat{\psi}_1) &= 0, \\ I_0(\mathcal{G}_1, \hat{\psi}_0 - \hat{\psi}_2) &= 0, \\ I_\infty(\mathcal{G}_1, \hat{\psi}_0 - \hat{\psi}_2) &= 0, \\ I_0(\mathcal{G}_2, \hat{\psi}_0 - \hat{\psi}_1) &= 0, \\ I_\infty(\mathcal{G}_2, \hat{\psi}_0 - \hat{\psi}_1) &= 0, \end{aligned}$$

$$\begin{aligned}
I_0(\mathcal{G}_2, \hat{\psi}_0 - \hat{\psi}_2) &= 0, \\
\text{and } I_\infty(\mathcal{G}_2, \hat{\psi}_0 - \hat{\psi}_2) &= \frac{4\pi\mu_0 I_p}{h^2} \left(\ln r_0 + \frac{h^2 r_0^2}{2} \right) \\
&= \frac{8\pi^2}{h} (\hat{\psi}_1 - \hat{\psi}_2). \tag{6.38}
\end{aligned}$$

We are free to substitute any of the expressions for $\hat{\psi}$ into ψ_{ext} in Eq. (6.36). Although there is no real restriction on the choice of Green's function to be used in the remaining integral expressions of Eq. (6.36), it is more "natural" to use $\mathcal{G}_2$ for $\hat{\psi}_1$ and $\mathcal{G}_1$ for $\hat{\psi}_2$ since the contributions of the integrals at $r = 0$ and $r = \infty$ both vanish. The use of either of $\hat{\psi}_1$ or $\hat{\psi}_2$ for $\hat{\psi}$ saves the evaluation of a Bessel function series and is thus more efficient numerically. There is also no need to define the location of region C when siting the current filament, $\hat{\psi}$, which may be an important consideration when solving a free boundary problem by iteration. In the applications of the formalism to date (Chap. 8), I_p has been taken to vanish (so that $\hat{\psi} = 0$).

6(ii-c) The free boundary problem

Even the question of existence and uniqueness of solutions to Eqs. (6.14) through (6.23) with a fixed plasma boundary, Γ , is complicated by the fact that the equation for ψ in region C is generally non-linear. If, however, we suppose that Eq. (6.16) is replaced by

$$\mathcal{L}(\psi) = f = \mathcal{L}(\tilde{\psi}) \quad \text{in } C, \tag{6.39}$$

where f is some known function, then providing $\tilde{\psi}$ exists, Eq. (6.36) proves the existence of ψ . In deriving this integral expression, we have used Eqs. (6.14), (6.15), (6.18) to (6.22) and (6.39). The surfaces γ , Γ and currents $\mathbf{i}$, $\mathbf{i}_1$ are input to the model. Uniqueness can be shown by considering the difference between two solutions, $\Delta\psi = \psi_1 - \psi_2$, which has continuous second derivatives, satisfies $\mathcal{L}(\Delta\psi) = 0$ everywhere and tends to zero as $r \rightarrow 0$ and $r \rightarrow \infty$. Hence, because both $\mathcal{L}(\psi)$ and its inversion with respect to the unit cylinder are uniformly elliptic (see Sec. 4(iv)), $\psi_1 = \psi_2$ everywhere.

The fixed boundary equilibrium problem for $\tilde{\psi}$ may be defined as

$$\begin{aligned}
\mathcal{L}(\tilde{\psi}) &= -\frac{2\kappa^2}{h} g_c(\tilde{\psi}) - \kappa g_c g'_c(\tilde{\psi}) - \mu_0 p'(\tilde{\psi}) \quad \text{in } C \\
\tilde{\psi}|_\Gamma &= 0. \tag{6.40}
\end{aligned}$$

The amount of non-linearity of this equation depends on the form of the g and p profiles. It may be solved by iterative techniques similar to those employed for toroidal equilibria.

It has been shown (Lackner, 1976, Johnson *et al.*, 1979) that one-step iterative schemes, where the right hand side at level n depends on $\tilde{\psi}$ at level $n - 1$, can be designed to converge to physically meaningful equilibria. These algorithms do not specify the right hand side, $f^n(\tilde{\psi}^{n-1}, r, \zeta)$, explicitly, but define it to depend on a number of parameters, $f^n(\tilde{\psi}^{n-1}, r, \zeta, p_1, p_2, \dots, p_k)$, which are calculated to hold an equal number of physical quantities constant. (The g profile, for example, may contain a parameter which is calculated to hold the total toroidal current constant.) Fixed boundary helical equilibrium codes (Gruber *et al.*, 1981a, Dewar *et al.*, 1984) have been written and have shown themselves to give physically reasonable solutions for a variety of heliac equilibria.

Regardless of the linearity or otherwise of the fixed boundary problem, the free boundary problem, where Γ must be determined from compatibility with the external fields, is highly non-linear (Lackner, 1976). It may be solved by iteration as follows. Suppose that I , I_p , i_0 and a prescription for finding the right hand side of Eq. (6.40) are specified, and that the forms of the $g_C(\psi)$, $p_C(\psi)$ profiles are such that the boundary conditions $p_C = 0$ and $g_C = g_B$ both apply at the plasma edge. The pressure must be continuous across the equilibrium plasma boundary (and the pressure in regions $A + B$ equal to zero) if $i_1 = 0$ as well (Bateman, 1978, p. 76). For an initial guess, Γ_1 , we solve (Eq. 6.40) for $\tilde{\psi}$. The Green's function integral expression for ψ , Eq. (6.36) with i_1 set to zero, may now be contoured to find a new approximation for Γ and the procedure repeated. This procedure, which is similar to that of Delucia *et al.* (1980) for the toroidal case, necessitates some prescription for choosing which new ψ value to contour. We may, for instance, dictate that one point of the plasma boundary have contact with a physical limiter, or that it be limited by a separatrix. If the iteration converges, then $\psi = \tilde{\psi}$ (to within a constant) in region C and Γ will be a flux surface. The computational implementation of this algorithm is the subject of the next chapter.

In summary, we cannot show the a priori existence or uniqueness of solutions to the free boundary problem. It is not even guaranteed that the fixed boundary problem will have a unique solution for all choices of the p_C and g_C profiles. We can, however, set up an algorithm which, if it converges, will result in a real free boundary equilibrium. This is so if the contours of ψ coincide with those of $\tilde{\psi}$ within the plasma. The possibility exists that there may be more than one plasma position for which the equilibrium equations are satisfied, but each solution could correspond to a real physical situation, depending on considerations, of the stability of the configuration as well as its accessibility from a

particular discharge sequence, which are outside the scope of this discussion.

6(iii) Green's theorem for the helical solenoid

We return to the problem of finding the scalar magnetic potential exterior and interior to a helical solenoid. From Sec. 4(iv), we have $\Phi^i = \mu_0 i_0 z + U$ inside, and $\Phi^e = V + \mu_0 I \phi / 2\pi$ outside, where

$$\left. \begin{array}{l} U(r_0, \zeta_0) \\ V(r_0, \zeta_0) \end{array} \right\} = \frac{h}{8\pi^2} \int_{\gamma} \left[\mathcal{G}_1^0 \partial_n (U - V) - (U - V) \partial_n \mathcal{G}_1^0 \right] dS \quad (6.41)$$

where $\mathcal{G}_1^0(r, \zeta | r_0, \zeta_0)$ is given by Eq. (5.12),

$$\begin{aligned} \mathcal{G}_1^0 &= \int_{-\infty}^{\infty} \frac{dz}{|\mathbf{r} - \mathbf{r}_0|} \\ &= -2 \ln r_{>} + 4 \sum_{n=1}^{\infty} I_n(nhr_{<}) K_n(nhr_{>}) \cos[n(\zeta - \zeta_0)] . \end{aligned}$$

and the surface integral is taken over one helical period.

Consider a solenoid of circular cross-section having a wholly "azimuthal" surface current i_0 (so that $I = 0$). Suppose the cross-section in the $z = 0$ plane to be parametrised by an angle variable χ , then, from Eqs. (4.31) to (4.34), we can write

$$(dS) \partial_n = (d\chi) [\dot{r} \phi \partial_r - \dot{r} r^{-1} (1 + h^2 r^2) \partial_{\zeta}] , \quad (6.42)$$

and

$$[\dot{r} \phi \partial_r - \dot{r} r^{-1} (1 + h^2 r^2) \partial_{\zeta}] (U - V) = -\mu_0 i_0 h r \dot{r} , \quad (6.43)$$

and

$$\mathbf{e}_t \cdot \nabla (U - V) = 0 \Rightarrow U = V + c \quad \text{on } \Gamma , \quad (6.44)$$

where $\dot{r} = dr/d\chi$ and $\dot{\phi} = d\phi/d\chi$ and c is an arbitrary constant which we set to zero. The substitution of Eqs. (6.42) to (6.44) into Eq. (6.41) gives

$$\left. \begin{array}{l} U(r_0, \zeta_0) \\ V(r_0, \zeta_0) \end{array} \right\} = -\frac{\mu_0 i_0 h}{4\pi} \int_{-\pi}^{\pi} \dot{r} \dot{\phi} \mathcal{G}_1^0(r, \zeta | r_0, \zeta_0) d\chi . \quad (6.45)$$

Suppose that the angle χ is defined in the conventional sense (as in Figs. 6.3 and 6.4 for example), and that the solenoid cross-section has reflection symmetry about $\chi = 0$. Then $\dot{r}$ is an odd function of χ and, as only even terms in the integral will contribute, we may neglect the leading term of $\mathcal{G}_1^0$, as well as all terms involving $\cos(n\zeta) \cos(n\zeta_0)$. It

Figure 6.3: *Cross-section of a helical solenoid which encloses the z axis.*

Figure 6.4: *Cross-section of a helical solonoid which doesn't enclose the z axis.*

is apparent that when $r \geq r_0$ or $r \leq r_0$ over the whole of the contour the integral may be written in the form of the series used as trial functions in Sec. 4(vi). In Fig. 6.3, the displacement, b , is less than the coil radius, a . The solution for U has the form

$$U = \sum_{n=1}^{\infty} a_n^i I_n(nhr_0) \sin(n\xi_0) \quad (6.46)$$

inside the largest interior cylinder $r = a - b$. Outside the smallest exterior cylinder $r = a + b$, V has the form

$$V = \sum_{n=1}^{\infty} a_n^e K_n(nhr_0) \sin(n\xi_0) . \quad (6.47)$$

In the example of Fig. 6.4 with $b > a$, V may be written as a series of I_n Bessel functions inside the tangent cylinder on the left hand side of the solenoid, and as a K_n series outside the tangent cylinder to the right hand side.

With reference to Fig. 6.3 and the discussion of Sec. 4(vii) one might pose the following question: Given that the series of partial sums of Eqs. (6.46) and (6.47) will converge to U and V for $r \leq a - b$ and $r \geq a + b$, under what circumstances will they converge throughout their respective (interior or exterior) regions? Dewar (Dewar and Gardner, 1986) has evaluated the asymptotic forms of the integral expressions for a_n^i and a_n^e obtained

h	a	b	r_c^e	r_c^i
.8	1.0	0.5	1.00	2.08
1.0	1.0	0.25	0.63	2.28
1.0	1.0	0.3	0.73	2.11
1.0	1.0	0.5	1.10	1.62
1.0	1.0	0.7	1.41	1.30
2.0	1.0	0.5	1.36	0.87
∞	a	b	$a + b$	$a - b$

Table 6.1: *Examples of radii of convergence from Eq. (6.48) for the exterior (r_c^e) and interior (r_c^i) series.*

from Eq. (6.45) as $n \rightarrow \infty$. He parametrized the cross-section of the solenoid by $z = r^2$ (rather than χ) and considered the analytic extension of the integral into the complex- z plane. The resulting contour integral was taken over a path which included a saddle point of the argument of an exponential term, which was itself obtained by combining the large- n approximation to the Bessel functions with the cosine term in ζ . The argument of the exponential was then approximated by the expansion to second order about this saddle point. It turns out that the radius of convergence, defined as the limiting values of r_0 for which $a_n^i I_n(nhr_0)$ and $a_n^e K_n(nhr_0)$ tend to zero asymptotically with n , may be found by solving the equations

$$\begin{aligned}
\eta(r_c) &= C_+ \quad \text{for } V, \\
\text{and } \eta(r_c) &= -C_- \quad \text{for } U \\
\text{where } \eta(r) &\equiv \sqrt{1 + h^2 r^2} + \ln \left(\frac{hr}{1 + \sqrt{1 + h^2 r^2}} \right), \\
C_+ &\equiv h(a + \mu b) + \ln(\mu - (hb)^{-1}), \\
C_- &\equiv -h(a - \mu b) + \ln(\mu - (hb)^{-1}), \\
\mu &\equiv \sqrt{1 + (hb)^{-2}}
\end{aligned} \tag{6.48}$$

and a , b and h are respectively the radius, displacement and pitch of the solenoid. The series represent the analytic continuation of U and V from the regions where they are identical with the Green's function expression. Because the magnetic field is analytic away from the coil, the analytic expressions must be correct within their radii of convergence.

Table 6.1 presents r_c for a range of values of a , b and h . The scaling of these results has been verified by evaluating the ratios of integral expressions for some higher order

coefficients for the two series.

From Table 6.1 we see that the exterior series with $h = 1.0$ and $a = 1.0$ fails to converge all the way to the inner edge of the coil ($r = a - b$) once the displacement is larger than some number between 0.25 and 0.3. However, the interior series will converge throughout the inside region (up to $r = a + b$) for displacements not exceeding a value between 0.5 and 0.7. The fact that the exterior series is divergent for $b = 0.5$, $h = 1.0$ and $a = 1.0$ may explain the loss of accuracy and ill-conditioning at large N observed in Fig. 4.7 and the behaviour of the coefficients in Fig. 4.10. The fact that the inner series is convergent here may be part of the explanation for the excellent accuracy of some collocation techniques. That there exist displacements $b < a$ for which either series does not converge is a consequence of the application of a scheme involving cylindrical harmonics to a bounding surface which is not cylindrically symmetric. The phenomenon is the same as that noted by Brazier-Smith (1984). He has considered the application of spherical harmonics to problems with azimuthal, but aspherical, symmetry.

Some "absolute" estimates of the error in the collocation technique were plotted in Fig. 4.9. These were obtained by comparing V^N with V from Eq. (6.45). One must be careful to integrate the singular parts of the Green's function integral correctly when the observer point is on the boundary. I separated $\mathcal{G}_1^0$ into regular and singular parts using the method due to Merkel (Merkel, 1982) (see also Appendix C), and integrated each with the NAG routine DOIAJF (NAG, 1984). As noted in Chap. 4, the integration is achieved with a relative accuracy of better than 10^{-5} .

In conclusion, the Green's function integral expression, Eq. (6.45), may reliably be applied over a range of coil displacements with good accuracy. It may even be used to evaluate the potentials when the coil does not encircle the z -axis. For small to moderate displacements, however, the computer time needed to repeatedly evaluate this integral may be dramatically reduced by employing a Fourier-Bessel series as in Eqs. (6.46) and (6.47). If a series can be shown to converge throughout its region of application, then its coefficients should be evaluated by explicit integration. Otherwise a boundary collocation technique can be used to give an approximate series solution. This should be tested against the Green's function integral to estimate the maximum error on the boundary. Brazier-Smith is unduly pessimistic of the consequences of using series solutions in problems "beyond their limit of applicability".

As a postscript, note that the coefficients of the Fourier-Bessel series may be expanded

in b/a for small displacements. For example,

$$a_1^i = 2\mu_0 i_0 h \pi^{-1} \int_0^\pi f(\zeta) K_1(kr) \sin \zeta \, d\zeta \quad (6.49)$$

where

$$f(\zeta) = -r\dot{r}(\zeta) = r^2 b \sin \zeta (r - b \cos \zeta)^{-1}$$

for a circular coil.

Now

$$\begin{aligned} r &= b \cos \zeta + \sqrt{a^2 - b^2 \sin^2 \zeta} \\ &= a \left(1 + \frac{b}{a} \cos \zeta - \frac{b^2}{2a^2} \sin^2 \zeta + \dots \right) \end{aligned}$$

giving

$$\begin{aligned} f(\zeta) &= ba^2 \sin \zeta \left(1 + \frac{2b}{a} \cos \zeta + \dots \right) a^{-1} \left(1 - \frac{b^2 \sin^2 \zeta}{2a^2} + \dots \right)^{-1} \\ &\approx ba \sin \zeta. \end{aligned} \quad (6.50)$$

If we set $K_1(hr) \approx K_1(ha)$ then

$$\begin{aligned} a_1^i &\approx 2\mu_0 i_0 h \pi^{-1} ba K_1(ha) \int_0^\pi \sin^2 \zeta \, d\zeta \\ &= \mu_0 i_0 hab K_1(ha). \end{aligned} \quad (6.51)$$

This result has been derived by (W.N.C. Sy, private communication) by a different method. He suggests that it might be a convenient rule of thumb for calculating the interior magnetic field when the coil displacement is small. From Eq. (6.51) one may, for example, write

$$B_r^i \approx \alpha B_0 I_1'(hr) \quad (6.52)$$

with $B_0 = \mu_0 i_0$ and $\alpha = h^2 ab K_1(ha)$.

Chapter 7

Code description: FEQ2.5(FR)

The combination of the Green's function integral expression, Eq. (6.36) with $i_1 = 0$, with an algorithm for solving the helically symmetric equilibrium equations with a fixed boundary, Eq. (6.40), will give an iterative technique which should converge to a solution of the free boundary problem (see Sec. 6(ii-c)). The operation of the "inverse-equilibrium" code FEQ2.5 is described in Appendix B. It expresses the converged fixed boundary equilibrium as a mapping $X(\Theta, \Psi)$, $Z(\Theta, \Psi)$ between the coordinates ($X = r \cos(\zeta/l)$, $Z = r \sin(\zeta/l)$, $Y = hz$) and the flux coordinates (Ψ , Θ , Y). It forms an inner loop of the free boundary code FEQ2.5(FR) as follows:

STEP A: Initialise the code, and read in an initial guess for the $X(\Theta, \Psi)$, $Z(\Theta, \Psi)$ grid.

This is usually set up by using the auxiliary code BNDFEQ to contour the vacuum field.

STEP B: Calculate a converged fixed boundary equilibrium using FEQ2.5 (level k).

STEP C: Find the position of the minimum value of ψ^{k+1} on the limiter using the integral expression Eq. (6.36) with $i_1 = 0$. Contour this minimum value of ψ^{k+1} as the new outer surface.

STEP D: If a convergence criterion is satisfied then exit, otherwise set up a new $X(\Theta, \Psi)$, $Z(\Theta, \Psi)$ grid within the plasma and repeat from STEP B.

7(i) Contouring the vacuum field

The initial guess for the plasma boundary must enable the subsequent free boundary "evolution" to be captured without a sudden distortion of the (Θ, Ψ) grid (see Sec. 7(v)). It

Figure 7.1: A typical ψ -contour in the top half-plane.

is often convenient to choose the boundary to be the flux contour of the vacuum magnetic field which is tangent to some limiting structure. The auxiliary code BDNFEQ assumes the limiter to be a helical solenoid of circular cross-section. When modelling toroidal devices without a physical limiter, this might be thought of as defining the region which is free of symmetry breaking perturbations (Monticello *et al.*, 1984).

The input deck to BDNFEQ is the same as for FEQ2.5(FR) and is shown in Table B.1 of Appendix B. The vacuum field, ψ_{vac} , is specified by the series of Eq. (B.4). The surface of section is taken to be the $z = 0$ plane.

The location and magnitude of the minimum value of ψ_{vac} on the limiter (ψ_{edge}) is found in the subroutine FINDLM using a bisection algorithm. The position of the magnetic axis on the mid-plane ($\zeta = 0$), assuming reflection symmetry, is found as a solution of the equation $\partial_r \psi_{vac} = 0$ using the Newton-Raphson technique. The intercepts of the other surfaces ($J=2,3 \dots$ NPSI) along the mid-plane are found by solving $\psi_{vac} - \psi_{edge} = \text{PSIVAL}(J)$, where the array of ψ values is set up in accordance with Eq. (B.3).

In order to fill in the remaining points on the boundary, BDNFEQ uses an efficient contouring algorithm which is based on the observation that for many of the common heliac-type magnetic geometries $\phi(r)$ (such that $\zeta = \phi(r)$ defines contours in the $z = 0$ plane) is single-valued over the cross-section in the top half-plane (see Fig. 7.1). It is apparent that $\partial_\zeta \psi = 0$ at the intercepts, p_A and p_B , on the mid-plane. The single-valuedness of $\phi(r)$ is ensured if $\partial_\zeta \psi \neq 0$ anywhere else, which will be the case, for surfaces inside the separatrix, if the I_1' harmonic is sufficiently dominant. The contouring algorithm uses Newton-Raphson iteration to find values of ϕ satisfying $\psi_{vac}(r, \phi) - \psi_{edge} = 0$ with r stepping from r_{maz} (point p_A of Fig. 7.1) to r_{min} (point p_B) in equal increments. The

contour is then rezoned to equal arcs using cubic interpolation on the X and Z coordinates in subroutine SETBND. For a typical SHEILA contour (see Chap. 8) the application of SETBND five times reduces the average deviation of the segment length from the mean to one part in 10^{-10} .

Setting up a complete initial coordinate grid using this contouring procedure would be very time-consuming. On the other hand, some naive algorithms (such as drawing straight lines from the magnetic axis to the outer contour) can cause the subsequent fixed boundary iterations, Step B above, to become unstable – particularly if the outer flux surface is very indented. I have found that a quick and robust algorithm for specifying the initial grid is to explicitly contour the surfaces 16, 32, 48 and 56 (for $NPSI=56$), rezone each to equal arcs and then interpolate the intermediate surfaces linearly using the x intercepts as the independent variable. Note that the free boundary code does not care whether either of the vacuum field or ψ^k obeys the restriction on the single-valuedness of $\phi(r)$, as long as the algorithm remains stable. However all of the fixed boundary SHEILA runs of Chap. 8 were also initialised as described here. Comparisons with the free boundary results are made in Sec. 7(iv).

7(ii) Calculation of the boundary integral

In Step C of the free boundary algorithm the outer surface for ψ at the next iteration level is found by contouring Eq. (6.36) with $i_1 = 0$:

$$8\pi^2 h^{-1} \psi^{k+1}(r_0, \zeta_0) = \int_{\Gamma} \kappa \left[G \partial_N (\psi_{ext} - \psi^k) - (\psi_{ext} - \psi^k) \partial_N G \right] dS \\ + \begin{cases} 8\pi^2 h^{-1} \psi_{ext} & (r_0, \zeta_0) \text{ in } A + B \\ 8\pi^2 h^{-1} \psi^k & (r_0, \zeta_0) \text{ in } C \\ 4\pi^2 h^{-1} (\psi_{ext} + \psi^k) & (r_0, \zeta_0) \text{ on } \Gamma \end{cases}, \quad (7.1)$$

[assuming $I_p = 0$ or a choice of G and ψ_{ext} such that $I_0 = I_{\infty} = 0$ (see Sec. 6(ii-b))]. The integration, which is taken over the plasma boundary at level k , involves the unsubscripted variables r and ζ . From Eq. (4.29),

$$(dS) = (dz)(d\Theta) \sqrt{[r^2 \dot{\zeta}^2 + \dot{r}^2 (1 + h^2 r^2)]} \quad (7.2)$$

Where $\dot{r} = dr/d\Theta$. The evaluation of this boundary integral has the potential to be very computationally expensive. FEQ2.5(FR) attempts to minimise this by using a trapezoidal rule integrator and

- (a) saving the various quantities $[r, \zeta, \dot{r}, \dot{\zeta}, \psi_{ext}$ and $\partial_N(\psi_{ext} - \psi^k)]$ needed to calculate the boundary integral,
- (b) storing values of the Bessel function series as they are calculated for the observer and the boundary points and
- (c) using a convergence accelerating algorithm in the calculation of the Green's function and its derivatives.

The convergence accelerating algorithm is that used by Merkel (1982) and Barnes and Cary (1984) for the Green's function for the helical Laplace equation. Its generalisation to the HGSE is presented in Appendix C. The use of the trapezoidal rule enables us to restrict the integration net to the set of $\psi^k(\Theta)$ values obtained for the fixed boundary code. Even the interpolated Θ grid of 276 points (discussed shortly) involves many fewer evaluations than some of the fancy integration packages used in Sec. 6(iii). As the observer point approaches a point on the boundary, integrable singularities develop in G and $\partial_n G$. It is (tacitly) assumed by Delucia *et al.* (1980) that the accuracy of the trapezoidal integration is maintained to one radial (ψ) grid step either side of the boundary. This may not be the case for a beany heliac plasma, because of the variation in physical size of the Ψ mesh with Θ , unless the number of radial points is somewhat less than the number of azimuthal points. The algorithm employed by FEQ2.5(FR), which will be described shortly, uses a buffer zone of variable width around the boundary. The value of the integral for observer points on the boundary may be estimated by integrating the singular parts analytically (Delucia *et al.*, 1980, Merkel, 1982, Barnes and Cary, 1984). My experience is that a much more accurate technique involves interpolating these values from grid points either side of the boundary. This will also be described presently.

The various quantities needed to calculate the boundary integral are tabulated in the subroutine TABBND. As the analytic form of ψ_{ext} is known, the calculation of $\partial_N \psi_{ext}$ is possible using Eq. (4.30). On the other hand, $\partial_N \psi^k$ is calculated using the second order finite difference formula (Delucia *et al.*, 1980):

$$(\partial_N \psi^k)_{I,NPSI} = |\nabla \Psi|_{I,NPSI} \frac{-4\psi_{I,NPSI-1}^k + \psi_{I,NPSI-2}^k}{2(\delta \Psi)} \quad (7.3)$$

where the subscripts (I,J) denote the (Θ, Ψ) grid point and $|\nabla \Psi|$ is passed through from the subroutine METRIC of FEQ2.5 (Dewar *et al.*, 1984):

$$|\nabla \Psi|^2 = \frac{1}{h^2 j^2} \left[(1 + h^2 X^2) \partial_\Theta X + (1 + h^2 Z^2) \partial_\Theta Z + 2h^2 XZ \partial_\Theta X \partial_\Theta Z \right], \quad (7.4)$$

(for $l = 1$).

J	$\psi_{\text{FEQ}} \times 10^4$	$(\psi_{\text{FEQ}} - \psi_{gf}) \times 10^4$		
		NB=56	NB=166	NB=276
1	-.512127	.0027	.0031	.0034
34	-.327762	.0030	.0042	.0036
53	-.054345	.45	.0034	.0034
54	-.036568	1.9	.010	.0034
55	-.018454	9.7	.47	.032

Table 7.1: *Comparison of the trapezoidal rule Green's function integration of Eq. (7.5) with a vacuum run of FEQ2.5. NB is the number of Θ points on the boundary, varied by cubic interpolation.*

The buffer-zone method

The application of Green's theorem to the plasma (region C) of the model problem integrated over one helical period (as in Sec. 6(ii)) gives

$$8\pi^2 h^{-1} \psi(r_0, \zeta_0) = \int_{\Gamma} \kappa (G \partial_N \psi - \psi \partial_N G) dS - \int_C G \mathcal{L}(\psi) dV.$$

Choose any auxiliary function $\hat{\psi}$ such that $\mathcal{L}(\psi) = \mathcal{L}(\hat{\psi})$ in region C , then

$$\psi(r_0, \zeta_0) = \hat{\psi}(r_0, \zeta_0) + h(8\pi^2)^{-1} \int_{\Gamma} \kappa [G \partial_N (\psi - \hat{\psi}) - (\psi - \hat{\psi}) \partial_N G] dS \quad (7.5)$$

for points (r_0, ζ_0) in the plasma.

Equation (7.5) may be used to estimate the absolute accuracy of the boundary integration for a given vacuum configuration. Either of the auxiliary functions $\hat{\psi}_1$ or $\hat{\psi}_2$ from Chap. 6 may be normalised to give $\mathcal{L}(\hat{\psi}) = -2\kappa^2 g_{vac}/h$. The right hand side of Eq. (7.5), which we shall denote as ψ_{gf} , is then compared with the known values of $\psi(r_0, \zeta_0)$ inside region C .

Table 7.1 compares three methods of calculating ψ_{gf} with the results of a fixed boundary SHEILA equilibrium, ψ_{FEQ} , with a "near-vacuum" average beta of $\langle \beta \rangle = 2 \times 10^{-6}$ (see Chap. 8). As usual a (Θ, Ψ) grid size of 56×56 points in the top half-plane was used in FEQ2.5, but two of the trapezoidal integrations used an interpolated net of Θ points on the boundary. The subroutine INTTHE used cubic interpolation on each of $X(\Theta, \Psi)$ and $Z(\Theta, \Psi)$ separately to place 2 or 4 new points (giving a total of NB=166 or NB=276 points) between the equilibrium Θ sites. The three methods were contrasted for observer points on the $(\Theta = 0)$ line (along the mid-plane to the right of the magnetic axis as in Fig.

Θ	.628	1.26	1.88	2.68	3.14
$(\psi_{\text{FEQ}} - \psi_{gf}) \times 10^4$	.0030	.0029	.0031	.0023	.028

Table 7.2: Accuracy of the NB=276 integration as a function of Θ at the surface J=54.

8.3) between the magnetic axis (surface J=1) and the edge (J=56). The magnitude of the total variation of ψ across the plasma is given by $\Delta\psi = |\psi_{\text{FEQ}}(J=1)|$. The results of the fixed boundary code by itself differed from the series expansion for the vacuum field, ψ_{vac} , by 0.16% of $\Delta\psi$ at the magnetic axis and by 0.14% of $\Delta\psi$ at surface J=34 (see Sec. 7(iv)). These values did not change when the average beta was decreased to $\langle\beta\rangle = 2 \times 10^{-9}$.

The superiority of the uninterpolated integration (NB=56) at the magnetic axis (see Table 7.1) reflects the interpolation error. However, both the NB=56 and NB=166 methods become inaccurate as the boundary is approached. In contrast the NB=276 method has a similar systematic error over all of the interior surfaces at $\Theta = 0$ apart from the last surface before the boundary. Table 7.2 shows the error in the NB=276 integration as a function of Θ at the surface J=54. This becomes large at the extreme left hand side ($\Theta = \pi$) where the radial grid spacing is the smallest (see Fig. 8.3).

In FEQ2.5(FR) a buffer-zone of variable width (as a function of Θ) is used to exclude observer points from the region of large inaccuracy. For contours of the SHEILA series an *ad hoc* rule is used to relate the half-width of the buffer zone to the average distance between the interpolated Θ points on the boundary (denoted as DARC). At a particular value of Θ the half-width is taken to be NBUFF surfaces where NBUFF is the smallest integer greater than 1 which satisfies

$$\text{NBUFF} * \text{DS}/\text{DARC} > 2.5$$

where DS is the physical distance between radial grid points at this value of Θ . In practice this gives NBUFF=3 at the right hand edge, NBUFF=2 over the region near the tips and NBUFF=4 at the left hand edge. The accuracy of this technique has been verified by cubic interpolation of ψ_{FEQ} and ψ_{vac} (outside region C) to give ψ on the boundary, and by showing that the errors are comparable to the systematic error in ψ_{gf} in the interior.

7(iii) Subroutine NEWBND

The algorithm used to find the new estimate of the plasma boundary is similar to that of Delucia *et al.* (1980). The position of a point on the surface of section is defined

with respect to the old (Θ, Ψ) grid (see Fig. 8.3) and the linear extrapolation of the constant- Θ lines into the vacuum. The subroutine NEWBND is initialised (in PREBND) by interpolating the boundary (in INTTHE) and tabulating (in TABBND) the quantities needed for the Green's function integration. The subroutine INTLIM is then used to find the position and value of the minimum of ψ_{new} (from the Green's function integration) on the circular limiter as follows: A grid of points corresponding to the intercepts of the Θ lines with the limiter is defined in the neighbourhood of the previous minimum. The values of ψ_{new} are calculated on these points – either directly (by the function CHIN which calls the subroutine GREEN) or by cubic interpolation (for points inside the buffer zone). The derivative of ψ_{new} with respect to the arc length along the limiter is then calculated by cubic interpolation and the position of the turning point estimated – also by cubic interpolation. (The use of a bisection algorithm here as in BNDFEQ would entail many evaluations of the boundary integral.) The position of the minimum is found to an accuracy of better than .1% of the distance between the Θ points on the limiter.

For each constant (equilibrium) Θ line, NEWBND now checks to see whether the limiting value of ψ_{new} lies in the buffer zone. If it doesn't, the algorithm steps along the Θ line in units of two surfaces. ψ_{new} is assumed to vary monotonically with distance along this line. The position of the new boundary point is estimated by cubic interpolation of X and Z with respect to ψ_{new} . (Cubic interpolation is used in order to weight points on either side of the buffer zone equally.)

Once a new outer boundary is found, an initial grid for the next fixed boundary iteration must be chosen which will be stable to the rezoning of the (Θ, Ψ) mesh. In practice, this means that the grid must be reasonably "smooth". Sharp angles in either the Θ or the Ψ direction may lead to a crossing of coordinate lines when the quadratic or cubic interpolating routines are used to respace the mesh (see Appendix B). FEQ2.5(FR) distinguishes the early iterations, where a significant boundary shift might be expected, and interpolates linearly between the new boundary (which is rezoned to equal arcs using SETBND) and surfaces 1, 10, 20, 30 and 40 of the old coordinate grid (for NPSI=56). After the third outer loop, it is assumed that the iteration has stabilised sufficiently to use all but the outer surface of the old grid.

7(iv) Code verification

Table 7.3 shows the long-run free-boundary behaviour of an equilibrium with finite

ITOUT	1	2	3	4	5
$x_{ma} \times 10$	.343382	.350932	.351311	.351357	.351361
$\psi_{ma} \times 10^4$	-.474562	-.480616	-.482700	-.482788	-.482827
ITOUT	6	7	8	9	10
$x_{ma} \times 10$	.351362	.351362	.351362	.351362	.351362
$\psi_{ma} \times 10^4$	-.482807	-.482831	-.482810	-.482831	-.482810

Table 7.3: Position of, and value of ψ at, the magnetic axis (in mks units) for a free boundary SHEILA equilibrium (with $I_{zc} = 1.7$ kA; $\langle\beta\rangle = 6.54\%$ and $\alpha = 1.2$ - see Chap. 8) as a function of the outer loop number.

ITOUT	8	9	10	11	12
$\psi_{ma} \times 10^4$	-.482820	-.482826	-.482829	-.482831	-.482833
ITOUT	13	14	15	16	17
$\psi_{ma} \times 10^4$	-.482833	-.482834	-.482834	-.482835	-.482834

Table 7.4: Restart of the free boundary run of Table 7.3, placing the new boundary points half-way between the old boundary position and that returned by the Green's function integration at each step of the subsequent iterations, showing damping of the oscillations in ψ_{ma} .

pressure. The position of the magnetic axis has stabilised to six significant figures after six outer loops. The limiting value of ψ_{ma} oscillates about $(-.48282 \pm .00001) \times 10^{-4}$. This oscillation may be damped by placing the new boundary in between the old surface and that calculated by NEWBND - as is shown in Table 7.4.

The degree of confidence we can expect to have in the code is demonstrated by the data in Tables 7.5 and 7.6 which compare two free boundary "vacuum" runs with the fixed boundary results (which were also initialised using BNDFEQ) and the analytic expression for the external field. The external field is that of the SHEILA heliac with (Table 7.5) and without (Table 7.6) current in the extra coil. The outer boundaries are shown in Fig. 8.5. The "beanier" (more indented and elongated) equilibrium of Table 7.6 has a larger oscillation error after 7 outer loops (shown as the uncertainty in ψ_{ma} and x_{ma}). The relative error $\epsilon(\psi_{ma})$ is equal to the difference between the value of ψ_{vac} at its magnetic axis and ψ_{ma} divided by ψ_{ma} . The maximum error on the boundary $\epsilon(\psi_b)$ is equal to the maximum difference in ψ_{vac} between all of the boundary points divided by ψ_{ma} . The

	ψ_{vac}	Fixed b.	Free b. ITOUT=7
$x_{ma} \times 10$	.330950	.330967	.330889 $\pm$.000001
$\psi_{ma} \times 10^4$	-.505477	-.505671	-.507085 $\pm$.000002
$\epsilon(\psi_{ma})$		.04%	.4%
$\epsilon(\psi_b)$		.3%	.6%

Table 7.5: Comparison of the positions of, and values of ψ at, the magnetic axis as given by a series expansion for the magnetic field, a fixed boundary solution to the equilibrium equations and a free boundary solution for a "vacuum" SHEILA configuration with $I_{zc} = 2$ kA (see Chap. 8).

	Ψ_{vac}	Fixed b.	Free b. ITOUT=7
$x_{ma} \times 10$	.362453	.362395	.36239 $\pm$.00003
$\psi_{ma} \times 10^4$	-.404926	-.405456	-.4066 $\pm$.0001
$\epsilon(\psi_{ma})$		.1%	.4%
$\epsilon(\psi_b)$		.4%	.9%

Table 7.6: Comparison of the accuracy of a fixed and free boundary solutions for a "vacuum" SHEILA configuration with no extra coil. This is a "beanier" contour than the one in Table 7.5 - see Fig. 8.5.

observation that $\epsilon(\psi_b)$ for the fixed boundary runs is comparable for both tables, whereas the corresponding $\epsilon(\psi_{ma})$ is larger for Table 7.6, implies that the accuracy of the contouring algorithm in BNDFEQ is similar for both surface shapes, but that the beanier equilibrium is calculated less accurately by the fixed boundary equilibrium code. The final column of Table 7.6 indicates that we can reasonably expect an accuracy of 1% for the free boundary code with this mesh spacing (of 56 points in Θ and Ψ). This is also about what we get from adding the error of the trapezoidal boundary integration for points in the interior (Table 7.1) to the difference between ψ_{FEQ} and ψ_{vac} at the magnetic axis for that equilibrium (i.e. the procedure of Table 7.1 appears to give a good estimate of the accuracy of the free boundary code).

Unfortunately, it is not possible to provide an absolute test of the accuracy of the free boundary code for an equilibrium with significant pressure. Instead, the validation of a code of this nature is an ongoing process (Johnson *et al.*, 1979) as it is applied to realistic physical situations and compared with other computational, analytic and experimental results. The simplest convergence criterion is to compare x_{ma} and ψ_{ma} between successive outer loops. The implementation of the code on the FACOM-M360R at A.N.U. makes extensive use of checkpoint files. I usually judge when some limiting oscillation has been reached and terminate the program manually, although the code can be set up to terminate when the difference in ψ_{ma} or x_{ma} between outer loops is less than EPSBND (Card 14 of input deck – see Appendix B) or after MAXBND loops. A typical fixed boundary loop with a 56×56 grid and moderate pressure ($\langle \beta \rangle \approx 19\%$) takes about 30 minutes of CPU time on the FACOM-M360R. The evaluation of the new boundary using NEWBND with NB=270 takes 9 minutes. The use of either a CRAY 1 or a CYBER 205 super-computer without vectorisation will reduce CPU time by a factor of 10.

7(v) Conclusion

The greatest problems in the operation of the free boundary code occur when there is a dramatic change in the plasma shape between iterations. It may be necessary to extrapolate the flux coordinate system a large distance into the vacuum region where it could become singular. Even if this were not the case, the initial guess for the coordinate grid at the next iteration level may be sufficiently ill-constructed to become unstable when rezoned by the fixed boundary code. To avoid these difficulties it may be necessary to step up the pressure (and current) gradually from some vacuum configuration. This may be

easily be done by initialising the free boundary code with another converged equilibrium rather than using BNDFEQ.

In the final stages of the free boundary iteration all of the new boundary points will lie in the buffer-zone. The width of the buffer zone can be chosen to make the interpolation error unimportant for the boundary shapes which I have tested. It is possible, nevertheless, to avoid these interpolations across the buffer-zone during the final stages by modifying the fixed boundary code to include a vacuum region about the plasma. To do this one need only to set g' and p' to zero outside the surface which is tangential to the limiter (which must be estimated for each inner loop of the fixed boundary code). The Green's function integration can then be carried out over the interpolated plasma boundary to find a new outer vacuum surface. This method would have less accuracy inside the plasma for a fixed grid size. However, if it was felt that (say) quadratic interpolation was now sufficiently accurate to find the new plasma boundary, it could lead to a reduction in the time needed to calculate the new boundary position.

The convergence of the free boundary code with an increasing number of grid points is guaranteed by the convergence of the fixed boundary code (Delucia *et al.*, 1980) and the convergence of the trapezoidal integration (Table 7.1). In the parameter study of the next chapter, I have chosen to keep the grid size constant as seems to be the fashion for exercises of this nature (Monticello *et al.*, 1984).

Chapter 8

The SHEILA heliac

Recently a small prototype heliac has been built at the Australian National University (Blackwell *et al.*, 1985). It has been shown to confine a low pressure plasma with well-formed bean-shaped magnetic surfaces. Although the question of high beta stability is unlikely to arise for this particular machine, it serves as a starting point to begin to study the relevance of free boundary effects to a real device.

8(i) Specification of the external field

The main toroidal field in SHEILA (Blackwell *et al.*, 1985) is provided by 24 pairs of copper plates, each of which is 15 cm square with an 11 cm diameter circular aperture. The coil-pairs are connected in series and arranged in a 3-period toroidal helix which is (usually) centred on a planar axis of radius $R_0 = 18.75$ cm and displaced from it by a distance of 2.5 cm. The "core-current", which is responsible for the main poloidal field, flows along the central axis and is carried by four (planar) turns of 7 mm-square copper. It is maintained at a constant fraction (4:48 ampere-turns) of the toroidal field current. The vertical field needed to complete the configuration is provided by two single turn coils, each carrying one quarter of the core current, of radius 31.3 cm located 19 cm above and 14.5 cm below the mid-plane. Some typical vacuum magnetic surfaces of the toroidal heliac, obtained by a field-line tracing code, are shown in Blackwell *et al.* (1985) and are reproduced in Fig. 8.1.

In our helically symmetric model we must neglect the vertical field coils and the spatial extent of the core-conductors. For a typical pulse, the peak current in all the toroidal field

Figure 8.1: Flux surfaces for the SHEILA heliac at two toroidal angles (ϕ_{tor}). The flux surface label X_ψ has been defined to be the distance (in cm) from the magnetic axis to the right hand intercept of a surface with the mid-plane at $\phi_{tor} = 0$ (Blackwell et al., 1985).

coils is

$$\begin{aligned} I_T &= 24 \times 2 \times 4 \times 10^3 \text{ A.turns} \\ &= 192 \text{ kA.turns.} \end{aligned}$$

This is modelled as a continuous helical solenoid with average current per unit length of

$$|i_0| = \frac{I_T}{2\pi R_0} = 1.6297 \times 10^5 \text{ A.m}^{-1}$$

which provides a zeroth-order longitudinal field of

$$B_0 = \mu_0 |i_0| = .2048 \text{ T.}$$

The pitch, h , of the solenoid is 16 m^{-1} and the total core current, I_c , is 16 kA.

Our choice of the direction of the z axis must be consistent with the PEST mapping convention (Dewar *et al.*, 1984, Grimm *et al.*, 1976) which has a positive Jacobian. If the "poloidal" and "toroidal" magnetic coordinates (Θ and Y of Chap. 7) are defined to increase in the (cylindrical) ϕ -like and z -like directions respectively, then ψ must increase when moving away from the magnetic axis. Thus i_0 and I_c are both negative (see Fig. 3.4) and the vacuum value of g inside the configuration is

$$g = \frac{\mu_0 i_0}{h} + \frac{\mu_0 I_c}{2\pi} = -.016 \text{ T.m.}$$

The radius of the "circular-limiter" (see Chap. 7) is estimated to be the distance beyond which well-formed, nested, magnetic surfaces will not exist in the toroidal case. I generally take this to be about 4 cm.

The extension of the toroidal field coils in the radial direction could be modelled using a set of nested solenoids. For the purposes of this preliminary study it was felt to be sufficient to use a Fourier-Bessel series for the interior field which had an estimated maximum error of 1% (obtained by comparison on the circular limiter with the Green's function integral expression for a circular coil located in the middle of the distributed solenoid). The sensitivity of the results of the free boundary code to changes in the model for the solenoidal field is discussed in Sec. 8(vi).

8(ii) Low pressure equilibrium

Figures 8.2 and 8.3 show some results from an application of the fixed boundary code FEQ2.5 (see Appendix B) to a low pressure SHEILA equilibrium. The computational procedure will be discussed in detail in Sec. 8(iv). The BNDFEQ code, described in Sec. 7(i),

was used to choose a limiting flux surface of the model for the external field described above. A circular limiter of radius 4.3 cm has a tangent surface whose width along the mid-plane (3.5 cm) is an average of the two outer surfaces shown in Fig. 8.1. The magnetic axis of the helically symmetric plasma is 3.62 cm which is also an average of the distances of the inside and the outside of the toroidal magnetic axis from the central conductor as shown in that diagram.

The poloidal and the toroidal fields of a toroidal heliac are stronger on the inside of the torus. An increase in the magnitude of either of these fields causes the surfaces to move inwards and become more elongated and more indented (a change in shape which we shall refer to as becoming more "beany"). One would thus be tempted to ascribe the asymmetry in the inside and outside surface shape to these effects, but they are overwhelmed by the vertical field which may be adjusted to dramatically increase the beaniness of the inside contour at the expense of the outside one and vice versa. The vertical field value of 18 mT was chosen (Blackwell *et al.*, 1985) to give a reasonable balance between the two contours.

In Fig. 8.3 the pressure profile from the equilibrium code is compared with some of the data of Blackwell *et al.* (1985). It has the analytic form of Eq. (B.4) with $\alpha = 1.2$ and P_0 and $P_{\min}$ being taken from the line of best fit to the experimental data. The profile agrees well with measurements taken at a toroidal angle of $\phi_{\text{tor}} = 0^\circ$ corresponding to the right hand contour Fig. 8.1.

We cannot expect the pressure profile at high β to be identical to that of Fig. 8.3. However, because the measured profile is fairly broad, I have chosen to include some studies with a pressure index of $\alpha = 1.2$ along with those for a peakier $\alpha = 2$. I have also set the pressure at the edge of the plasma to zero to study the free boundary evolution in the absence of surface currents.

Figure 8.3 also plots the rotational transform per period and the specific volume $V'(\Phi_B)$ (where Φ_B is the longitudinal flux), as a function of relative radius (see Sec.8(iv)) across the mid-plane. The rotational transform is related to the helical transform by (see Chap. 3)

$$\tau = 1 - \left| \tau_h \right|. \quad (8.1)$$

Recall that a magnetic well is said to exist across a set of closed magnetic surfaces if the specific volume exhibits a net decrease when moving from the magnetic axis to the edge (see Sec. 3(iii)). Alternatively, a well at the magnetic axis exists if $V''(\Phi_B) < 0$ there. The field-line tracing results of (Blackwell *et al.*, 1985) give τ as being just less than 0.4 on

axis, rising to about 0.45 at the edge, and a net decrease in $V'(\Phi_B)$ of 4%. The helically symmetric model has τ ranging from 0.41 on axis to 0.466 at the edge, and a net drop of 2% in $V'(\Phi_B)$.

The $g(\psi)$ profile for this equilibrium has been calculated to give zero net longitudinal current on each magnetic surface. We can see from Fig. 8.2 that longitudinal currents do flow in the plasma, but they do so in opposite senses on the inside and the outside of the magnetic surfaces. In this case there are two current filaments in the $(-z)$ direction on the inside.

Figure 8.2: *Flux surfaces (top) and longitudinal current (bottom) for a low pressure, $I_{xc} = 0$, $\alpha = 1.2$ equilibrium. Some of the radial (Θ) lines are shown in the top diagram. The solid current contours flow in the $(+z)$ direction. Dotted contours denote $(-z)$ directed current.*

Figure 8.3: Rotational transform, specific volume and pressure for the equilibrium of Fig. 8.2. The points on the pressure profile are measurements taken by Blackwell et al. (1985) along the mid-plane at $\phi_{tor} = 0^\circ$ outside (crosses) and inside (squares) of the magnetic axis.

8(iii) Addition of an extra helical coil

Recently it has been suggested that an extra $l = 1$ helically-symmetric core winding be added to the basic heliac design. Harris *et al.* (1985) have found that such a winding will provide a degree of experimental control over the rotational transform profile. It may also lead to a deepening of the magnetic well.

The incorporation of an extra $l = 1$ core winding into SHEILA is currently in progress. It has been designed to run independently of the current in the central conductor, so that the total core-current will vary as the current in the extra coil varies. We shall include this extra winding in our helically symmetric model as a realistic deformation parameter for the plasma shape.

The flux due to a helical "extra coil" of current I_{zc} , radius a and pitch h may, in the exterior region, be written (see Eq. (5.17))

$$\psi = \frac{\mu_0 I_{zc}}{2\pi h} \ln r + \frac{\mu_0 I_{zc} h r a}{\pi} \sum_{n=1}^{\infty} I'_n(nha) K'_n(nhr) \cos(n\zeta). \quad (8.2)$$

We shall suppose that the extra coil has a radius of 1 cm. Its contribution to the vacuum value of g is $\Delta g = \mu_0 I_{zc}/2\pi$. It is straightforward to incorporate the Fourier-Bessel series of Eq. (8.2) into the expression for the vacuum field. It is important to do this accurately when examining the marginal stability of equilibria whose bounding surface is close to this extra coil. When evaluated at $r = 1.2$, the absolute sum of the first 20 terms of this series was accurate to 0.1% when compared with a 21 term sum.

The outer contours for four values of I_{zc} ranging from 0 to +2 kA (in the opposite direction to the main core current) are shown in Fig. 8.4. A limiter of radius 3.9 cm has been chosen.

8(iv) Computational methodology: fixed boundary

The fixed boundary operation of the code FEQ2.5 is described in Appendix B. The helical Grad Shafranov equation is solved within the plasma on a grid which is adjusted to divide each of the magnetic surface cross-sections into equal arc lengths. The spacing of the ψ surfaces (for $J = 1, 2 \dots \text{NPSI}$) is chosen according to Eq. (B.3):

$$\psi(J) = \text{PSIMIN} \left[1 - \left(\frac{J-1}{\text{NPSI}-1} \right)^{\text{RJAC}+1} \right], \quad (8.3)$$

with $\psi(1) = \text{PSIMIN}$ (less than 0) at the magnetic axis and $\psi(\text{NPSI}) = 0$ at the plasma edge. Because ψ changes slowly with distance near the magnetic axis, RJAC is set equal

to one to give adequate resolution in this region.

In the following studies I have used a (Θ, Ψ) grid size of 56×56 in the top half-plane (see Appendix B). The pressure profile was as has been described in Sec. 8(ii), and the $g(\psi)$ profiles were calculated to set the net longitudinal current to zero surface by surface and to match to the vacuum value of g at the edge. The initial grid was set up using the BNDFEQ code to contour the series for the vacuum field.

In order to test for stability against interchange and ballooning modes, the equilibria were mapped to magnetic coordinates using the code FMAP2.5 (Dewar *et al.*, 1984). I used Hamada coordinates with a 97×128 (radial $\times$ azimuthal) grid. The Mercier criteria against ideal and resistive interchanges were evaluated surface by surface using the code BAL2.5 (Dewar *et al.*, 1984). For surfaces where a discrete ballooning spectrum may exist, BAL2.5 finds the smallest eigenvalue of the Euler equation of the line averaged Lagrangian for (ideal) ballooning modes. I initially tested the innermost 9 surfaces plus every third surface between the 12th and the edge. The graphs of τ , $V'(\Phi_B)$, $p(\psi)$, D_I and D_R have been taken from this code. D_I and D_R are the usual Mercier coefficients (Dewar *et al.*, 1984) and indicate stability against ideal and resistive interchanges when negative. Note that the plots of V' and p are normalised to their values at the magnetic axis. All of the physical quantities are plotted against a flux surface label which is defined as the ratio of the distance, R , from the magnetic axis to the right-hand intercept of the surface to the distance, R_p , to the plasma edge.

Figure 8.4: Surface shapes for several values of the current in the extra coil. The dotted lines represent the circular limiter, and the cross represents the position of the extra coil.

8(v) Stability diagram for a fixed boundary SHEILA

Previous computational studies of the equilibrium and stability of the helically symmetric net-current-free heliac have considered sequences of analytically specified (fixed boundary) equilibria. These studies have used either the quasi-analytical HASE equilibrium code in combination with HERA, the helically symmetric version of ERATO (Gruber *et al.*, 1981b, Merkel *et al.*, 1983), or the FEQ2.5 and BAL2.5 codes described above (Monticello *et al.*, 1984). In the HASE code the equilibrium equation is solved by determining the off-diagonal coefficients in a series expansion for ψ in terms of the polar coordinates $\bar{\rho}$ and $\bar{\phi}$ centered on, and in the plane normal to, the magnetic axis:

$$h\psi = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} S_{mn} \bar{\rho}^m \cos n\bar{\phi}.$$

The coefficients S_{mn} are input to the model, as are the p' and g profiles. The latter have been taken to be linear in ψ ,

$$hg = 1 - h\bar{p}_0\psi$$

and

$$p'(\psi) = h(\bar{p}_0 + h\bar{p}_1\psi),$$

where the constant $\bar{p}_0$ is used to vary β and the constant $\bar{p}_1$ is calculated to approximate the condition of vanishing net longitudinal current. In the study of Gruber *et al.* (1981b) S_{11} and S_{22} were fixed (corresponding, respectively, to a prescription of the curvature of the magnetic axis, k , and the half-axis ratio, $\bar{e}$, of the innermost flux surfaces in a plane normal to the magnetic axis) and S_{33} and $\bar{p}_0$ were adjusted together to vary $\langle\beta\rangle$. Merkel *et al.* (1983) fixed $\langle\beta\rangle$, $\bar{e}$, and S_{nn} for $n \geq 3$ and varied the curvature k .

Monticello *et al.* (1984) prescribed the shape of the outer flux surface in the constant- z plane as input to FEQ2.5 (see Appendix B). We shall henceforth refer to this paper as MDFR. The authors used an analytic expression, described in Sec. 8(vii), which included a parameter, B , which, when increased, would “deform” the cross-section from a roughly circular shape to an indented bean. For each such surface the equilibrium is completely specified by prescribing the form of the pressure profile and the magnitude of the helical pitch and calculating the g profile to set the longitudinal current to zero surface by surface. A set of families of equilibria were investigated, each member of which could be “parametrized” by the values B and $\langle\beta\rangle$ (the latter being varied by adjusting the input parameter P_0). Each family of equilibria was defined by the helical pitch, h , the pressure index, α , and the “aspect ratio”, which was defined as the ratio of the distance from the

Figure 8.5: Critical $\langle \beta \rangle$ for stability against ideal interchange and/or ideal ballooning modes for a fixed boundary SHEILA with the two pressure profiles $\alpha = 1.2$ and $\alpha = 2$. The plasma is stable above the curves and unstable below. The free boundary stability boundary for both pressure profiles appears to be almost horizontal.

z axis to the mid-plane mid-point to the mid-plane half-width of the outer contour. The mid-plane half-width was always held constant.

I have studied the stability properties of two families of fixed boundary SHEILA equilibria, each of which is parametrised by I_{zc} , the current in the extra coil, and $\langle \beta \rangle$. The two families differ in their values of α . Figure 8.5 shows the boundary for stability against ideal interchange and/or ballooning modes for these equilibria. To facilitate comparisons with the results of MDR the vertical axis increases in the direction of increased beaniness of the plasma boundary. Note, however, that the plasma half-width is not held constant as I_{zc} varies.

We shall now examine the qualitative behaviour of the fixed boundary SHEILA equilibria with reference to the "picture show" of Figs. 8.6 to 8.17. Figures 8.6 and 8.7 portray typical moderate β equilibria on the completely stable and completely unstable side of the stability diagram. The D_I and D_R profiles are smooth with a singular point at the axis and limiting values of $-\frac{1}{4}$, for D_I , and 0, for D_R , at the plasma edge. As the pressure increases from zero, the region of large $|D_I|$ and $|D_R|$ becomes broader. For moderate to large β the profiles show more structure and the central region of large D_I tends to move back inwards. However, the configurations remain either Mercier stable ($D_I < 0$) or unstable ($D_I > 0$) as the case may be. The latter case seems to hold over the rest of the stability diagram. I could not find a transition to a Mercier stable regime from an unstable equilibrium, for these fixed boundary cases, except perhaps at $I_{zc} = 1\text{kA}$ as

discussed later. Unstable ballooning modes are observed at moderate to large $\langle\beta\rangle$'s. The $I_{zc} = 2$ kA, $\alpha = 2$ equilibrium becomes ballooning unstable at about 22% $\langle\beta\rangle$.

The rotational transform across the plasma in Fig. 8.6, where $I_{zc} = 0$, is greater than that for the fatter profile in Fig. 8.7. This, from Eq. (8.1), is equivalent to a slower average rate of twisting of the field lines in the helical reference frame. The terms "elongation" (or "half-axis ratio" which is related to the "ellipticity") and "indentation" (which is related to the "triangularity") are often used to describe bean shaped flux surface shapes. I shall define them here with respect to the cross-section in the constant- z plane as:

$$\epsilon_l = \text{elongation} = \frac{(\text{half height})}{(\text{half width}) \text{ across the (mid plane)}}$$

and

$$\text{indentation} = \frac{\text{extension}}{\text{extension} + (\text{full width})}$$

where the "extension" is the maximum horizontal distance from the tip to the closest intercept on the mid-plane. Note that at the magnetic axis the cosine of the angle between $\mathbf{u}$ and $\mathbf{e}_z$ is equal to $1/\sqrt{(1 + h^2 r_{ma}^2)}$, so that for the innermost surfaces ϵ_l is related to $\bar{\epsilon}$ by $\epsilon_l = \sqrt{(1 + h^2 r_{ma}^2)} \bar{\epsilon}$. Increasing both the elongation and the indentation of the flux surfaces increases the total rotational transform. As the $\langle\beta\rangle$ increases, the magnetic axis shifts away from the z axis, the inner surfaces become more elongated and the rotational transform on axis increases. At moderate to high pressures turning points in the τ profile lead to resonances in D_I and D_R as in Fig. 8.10.

The magnitude of the axis shift with pressure for the fixed boundary SHEILA equilibria is much less than that of the analytic profiles studied by MDFR. For example, at $\langle\beta\rangle = 24\%$ the axis of a "fat" unstable $I_{zc} = 2$ kA, $\alpha = 2$ equilibrium had shifted 19% of the distance from the vacuum magnetic axis to the right hand edge of the contour. This equilibrium had an on axis helical transform of $\tau_h = -0.617$ and an aspect ratio (defined above) of $A = 1.93$. The formula for nearly circular flux surfaces (Monticello *et al.*, 1984, Yoshikawa, 1981)

$$\delta \equiv \frac{\Delta R}{r_p} = \frac{1}{2} \langle\beta\rangle A (\tau_h)^2 \quad (8.4)$$

predicts a shift of 60%. The data presented in Fig. 13 of MDFR show that Eq. (8.4) overestimates the shift for beany plasmas. One of their profiles for which the outer surface had an elongation of 2.2 and an indentation of .13 had a 40% shift at these pressures. Our plasma surface has an elongation of 2.1 and an indentation of 0.26. Although, according to Eq. (8.4), rounder plasma profiles have a greater axis shift with pressure at constant A

and τ_h , the weighting of the aspect ratio and the rotational transform causes the axis shift for the beanier (small I_{zc}) SHEILA contours that I have measured to be greater than that of the fatter (large I_{zc}) profiles. An $I_{zc} = 0$, $\alpha = 2$, $\langle\beta\rangle = 24\%$, $A = 2.41$ equilibrium with $\tau_h = -0.520$ on axis and an indentation of 0.33 and elongation of 2.5, had an axis shift of 22%.

The data points on Fig. 8.5 represent the estimated stability boundaries to Mercier and/or ballooning modes for both the $\alpha = 1.2$ and $\alpha = 2$ pressure profiles. A typical transition is that for $I_{zc} = 1.7$ kA, $\alpha = 2$ which is shown in Figs. 8.8 to 8.10. Figure 8.8 shows the flux surfaces and current contours for an equilibrium which is on the stable side of the transition alongside those for one at high $\langle\beta\rangle$. The D_I and D_R profiles for the stable configuration (Fig. 8.9) show the characteristic divergence at the magnetic axis. At the transition point this part of the curve flips over and the region of Mercier instability grows outwards. The depth of the magnetic well increases with pressure. At high $\langle\beta\rangle$ such as in Fig. 8.10 the region of large D_I near the axis stops broadening and turning points in the τ profile lead to resonances in the Mercier coefficients, this equilibrium was ballooning unstable. With one exception, the critical $\langle\beta\rangle$ plotted in Fig. 8.5 was estimated by interpolation as that at which the innermost surface went unstable to ideal interchanges. The exception is the point at $I_{zc} = 1.0$ kA, $\alpha = 2$, where the transition to Mercier instability appears to remain confined to a very narrow region near the magnetic axis (see Fig. 8.11). It does not grow outwards, and disappears at very high $\langle\beta\rangle$ as in Fig. 8.12. I have estimated the critical $\langle\beta\rangle$ here as that for the onset of ballooning modes. It has an accuracy of $\pm 2\%$. An alternative interpretation might be that this sequence represents a transition to a second region of stability to Mercier modes. The transition to Mercier instability at the axis occurs at an $\langle\beta\rangle$ of about 18%.

The rotational transform at the magnetic axis can, for a configuration with zero net longitudinal current, be related to the elongation as (Solov'ev and Shafranov, 1970)

$$\begin{aligned}\tau &= 1 - \frac{2\bar{e}}{\bar{e}^2 + 1} \frac{1}{\sqrt{(1 + h^2 r_{ma}^2)}} \\ &= 1 - \frac{2\epsilon_l}{\epsilon_l^2 + 1 + h^2 r_{ma}^2}.\end{aligned}\tag{8.5}$$

The influences of ellipticity, triangularity and curvature of the magnetic axis on the stability are shown, to some extent, in the limiting form for the Mercier criterion near the magnetic axis for equilibria of the type examined by Gruber *et al.* (1981b) and Merkel *et*

al. (1983):

$$\begin{aligned}
& -(\bar{\epsilon}^2 + 1) + 4(1 - k^2) \frac{\bar{\epsilon}^2}{\bar{\epsilon}^2 + 1} + \frac{3k^2}{4\bar{\epsilon}^2} (\bar{\epsilon}^2 - 1)(\bar{\epsilon}^2 + 2) \\
& - \frac{3k}{\bar{\epsilon}} (\bar{\epsilon}^2 - 1) S_{33} - \frac{\bar{p}_0 k^2}{4(1 - k^2)} \frac{(\bar{\epsilon} - 1)^2 (\bar{\epsilon} + 3)(\bar{\epsilon} + 1)^2}{\bar{\epsilon}^3 (\bar{\epsilon} + 1)} > 0.
\end{aligned} \tag{8.6}$$

For their plasma cross-sections S_{33} was always negative. An increase in the absolute value of S_{33} caused the flux surfaces to become more indented and was stabilising. The first line of Eq. (8.6) was dominated by the first term and was negative definite for the range of elongations and curvatures considered, by Merkel *et al.* (1983). An increase in k by itself also caused the flux surfaces to become increasingly indented and was also stabilising. The last term on the second line was stabilising for their pressure profiles (a behaviour termed “self-stabilisation with pressure”). The units in Eq. (8.6) have been chosen so that the length of one helical period of the magnetic axis is 2π .

It is possible, with caution, to use Eq. (8.6) to illustrate some of the qualitative trends for SHEILA. Because the axis shift was so small for the equilibria examined in Fig. 8.5, the effects of changes in k will be secondary to those of increases in the elongation of the innermost surfaces which will drive the destabilising terms in the first line (at the large elongations considered here – see Fig. 8.16). Increasing indentation (or triangularity) has a stabilising effect. [These comments are somewhat speculative as the heliac equilibria in the HASE studies had different pressure and D_I profiles from those considered here. They had non-zero pressure gradients at the edge of the plasma and Mercier instability first set in at surfaces away from the magnetic axis. They also had the result that the stability boundaries for Mercier and ballooning modes almost coincided, which is not the case for the present study.]

I have included Figs. 8.13 to 8.17, which describe a sequence of unstable equilibria at $I_{zc} = 2$ kA, $\alpha = 2$ for comparison with the free-boundary results. The vacuum configuration (in Fig. 8.13) has $V''(\Phi_B) > 0$ at the magnetic axis although there is a small “net” magnetic well across the plasma.

Figure 8.6: *SHEILA* : $I_{xc} = 0$; $\alpha = 1.2$; $\langle \beta \rangle = 10.4\%$; fixed boundary.

Figure 8.7: *SHEILA* : $I_{xc} = 2$ kA; $\alpha = 1.2$; $\langle \beta \rangle = 10.4\%$; fixed boundary.

Figure 8.8: *SHEILA* : $I_{zc} = 1.7$ kA; $\alpha = 2$; fixed boundary; $\langle \beta \rangle = 2.94\%$ (left) and $\langle \beta \rangle = 26.7\%$ (right).

Figure 8.9: SHEILA : $I_{zc} = 1.7$ kA; $\alpha = 2$; $\langle \beta \rangle = 2.94\%$; fixed boundary.

Figure 8.10: SHEILA : $I_{zc} = 1.7$ kA; $\alpha = 2$; $\langle \beta \rangle = 26.7\%$; fixed boundary.

Figure 8.11: SHEILA : $I_{zc} = 1$ kA; $\alpha = 2$; $\langle \beta \rangle = 21.9\%$; fixed boundary.

Figure 8.12: SHEILA : $I_{zc} = 1$ kA; $\alpha = 2$; $\langle \beta \rangle \approx 40\%$; fixed boundary.

Figure 8.13: *SHEILA* : $I_{xc} = 2$ kA, $\alpha = 2$; nearly vacuum.

Figure 8.14: *SHEILA* : $I_{xc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 17.8\%$; fixed boundary.

Figure 8.15: *SHEILA* : $I_{zc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 24.1\%$; fixed boundary.

Figure 8.16: *Elongations of equilibria with $I_{zc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 0, 17.8\%$ and 24.1% .*

Figure 8.17: *Indentations of equilibria with $I_{xc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 0, 17.8\%$ and 24.1% .*

8(vi) Stability diagram for a free boundary SHEILA

A selection of points on the parameter space of Fig. 8.5, at various values of I_{xc} , has been investigated using the free boundary code FEQ2.5(FR). The fixed boundary loops were set up to run as described in Sec. 8(iv). The boundary grid was interpolated from 56 to 276 points for the Green's function integration and a variable buffer zone of 2-3 surfaces was used. The algorithm was initialised with a vacuum mesh, and run for between 7 and 9 fixed boundary iterations or until the magnetic axis position had stabilised to 5 significant figures. The current in the external coils was held constant.

As the pressure is increased the plasma shifts to the right and broadens. The change in shape of the outer flux surface, shown in Fig. 8.18, is not dramatic. Figure 8.19, however, shows that the half-width and mid-point of the plasma and the magnetic axis position all change smoothly with $\langle\beta\rangle$. This strengthens confidence in the implementation of the free-boundary code. All of the free boundary equilibria which have a vacuum magnetic well on axis appear to remain Mercier stable as the $\langle\beta\rangle$ increases. There is a transition to ballooning instability, but this occurs at higher $\langle\beta\rangle$ than for the fixed boundary case.

The transition line between Mercier stable and Mercier unstable free boundary SHEILA equilibria slopes (shallowly) downwards. Consider Fig. 8.20 which is for an $I_{xc} = 2$ kA, $\alpha = 1.2$, free boundary equilibrium. At low pressures, the D_I and D_R profiles are similar to the typical unstable equilibrium of Fig. 8.7. The $\langle\beta\rangle = 18.8\%$ case, however, is completely Mercier stable. This continues to be the situation as the pressure is increased further. [The bump on the edge of the D_I and D_R profiles is probably spurious, being due to the inability of the equilibrium grid to catch the rapid fall off in $p'(\psi)$. The Mercier criterion includes a difference of terms proportional to p' and $(p')^2$ which become unbalanced in this region. The position of the bump changes if the Jacobian of the equilibrium code is altered.]

Figure 8.21 shows the transition to Mercier stability for the $\alpha = 2$, $I_{xc} = 2$ kA case. Only the region near the magnetic axis is unstable. The transition to stability occurs at a larger pressure than for the $\alpha = 1.2$ series. The equilibrium at $\langle\beta\rangle = 25.66\%$ (Fig. 8.22) is completely Mercier stable but is ballooning unstable.

Comparing Fig. 8.16 with Fig. 8.23, we see that the boundary shift has the effect of decreasing the elongation of the surfaces across the plasma. From Figs. 8.14 and 8.15 compared with Figs. 8.21 and 8.22 we see that the rotational transform decreases in the central regions but not at the edge. The magnetic well across the plasma is greater for the

free boundary equilibria. The indentation profiles (see Figs. 8.17 and 8.24) are remarkably similar.

Monticello *et al.* (1984) have shown that a rigid shift of the plasma boundary away from the z axis should be stabilising. In the present case the stabilisation coincides with a decrease in the elongation of the innermost surfaces, an increase of the curvature of the magnetic axis and a broadening of the flux surfaces. We shall attempt to resolve these effects in the next section.

The qualitative trends shown in the τ , $V'(\Phi_B)$, D_I and D_R profiles appear to hold if the model for the solenoidal field is varied within the 1% error estimate on the circular limiter (see Sec. 8(i)). In particular, the sensitivity of the profiles of Fig. 8.21, a free boundary run at "marginal" Mercier stability, has been examined. A 1% variation in the solenoidal field produces no observable change in any of the profiles (for the scales of Fig. 8.21) save for the first point in the D_R profile which is shifted down by about two point diameters.

Figure 8.18: *SHEILA* : $I_{xc} = 2$ kA; $\alpha = 1.2$; free boundary; $\langle \beta \rangle = 3.2\%$ (left) and $\langle \beta \rangle = 31.4\%$ (right).

Figure 8.19: Midpoint ($\bar{x}$), width (Δx) and magnetic axis position (x_{ma}) as a function of $\langle \beta \rangle$ for $I_{xc} = 2$ kA; $\alpha = 2$; free boundary equilibria.

Figure 8.20: SHEILA : $I_{zc} = 2 \text{ kA}$; $\alpha = 1.2$; $\langle \beta \rangle = 18.8\%$; free boundary.

Figure 8.21: SHEILA : $I_{zc} = 2 \text{ kA}$; $\alpha = 2$; $\langle \beta \rangle = 18.5\%$; free boundary.

Figure 8.22: *SHEILA* : $I_{xc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 25.7\%$; free boundary.

Figure 8.23: Elongations for free boundary equilibria with $I_{xc} = 2$ kA; $\alpha = 2$; $\langle \beta \rangle = 3.3\%$, 7% , 18.8% and 25.6% .

Figure 8.24: Indentations for free boundary equilibria with $I_{xc} = 2 \text{ kA}$; $\alpha = 2$; $\langle \beta \rangle = 3.3\%$, 7% , 18.8% and 25.6% .

Figure 8.25: Analytic profiles (dotted) with $a = 0.03$, $b = 0.5783$, $\bar{x} = 0.48567 \times 10^{-2}$ and $B = 0.6, 1.2$, and 1.55 (the best fit). The solid curve is for SHEILA with $I_{xc} = 1.7$.

8(vii) Parametrizing free boundary equilibria

Monticello *et al.* (MDFR) have used an analytic form to describe the outer flux surface in the $x - y$ plane:

$$\begin{aligned} x &= \rho \cos \gamma + \bar{x}, \\ y &= \rho \sin \gamma, \\ \gamma &= B \sin(\pi - \theta), \\ \rho &= a[1 - b \cos(\pi - \theta)], \end{aligned} \tag{8.7}$$

where θ runs from 0 to 2π . The width across the mid-plane of such a contour is $\Delta x = 2ab$ and the mid-point is $x_{mid} = \bar{x} + a$. One procedure for approximating a given heliac contour by Eq. (8.7) is to select a value of a , solve for b and $\bar{x}$ and then choose the value of B which best models the elongation and indentation of the bean. The “reference heliac” of MDFR had $a = b = 0.5$, $\bar{x} = 0.2$ and $h = 0.8$. SHEILA has a helical pitch of $h = 16$ which gives a convenient length scaling factor of 20. By juggling the values of a and B by eye, I obtained the fit to the $I_{xc} = 1.7$ kA vacuum contour shown in Fig. 8.25. Here $a = 0.03$, $b = 0.5783$, $\bar{x} = 0.48567 \times 10^{-2}$, and $B = 1.55$.

One may very roughly follow the boundary change with pressure by holding a and B constant and fitting b and $\bar{x}$ to the curves shown in Fig. 8.19. The vacuum equilibrium of Fig. 8.25 is Mercier stable and has a magnetic well across the plasma. When the $\langle \beta \rangle$ is 19.9% as in Fig. 8.26 the configuration is Mercier unstable as expected. In Fig. 8.27 the $\langle \beta \rangle$ is 21.1%, but b and $\bar{x}$ have been calculated from a free boundary run at this pressure. The plasma has stabilised, except for the region very near the magnetic axis. In Fig. 8.28

only $\bar{x}$ has been increased, with the plasma width held constant. This "shifted" contour is the most stable of all. The aspect ratios for the "fixed", "free" and shifted equilibria are 2.009, 2.056 and 2.128 respectively.

Comparing Figs. 8.26 through 8.30, we see that the shifted equilibrium has the largest rotational transform. Its elongation near the magnetic axis is slightly greater than that of the fixed boundary equilibrium. This and the increase in r_{ma} associated with the outwards shift is expected to increase τ from Eq. (8.5). The "free boundary" equilibrium has the lowest rotational transform and smallest elongation. It is also significantly less indented than the other two.

For this series of equilibria, the most elongated plasma cross-section is the most stable. This apparent paradox in the light of Eq. (8.6) may be resolved by considering the $V'(\Phi_B)$ profiles, and recalling that the deepening of the vacuum magnetic well with beaniness is associated with increasing elongation. To calculate $V''(\Phi_B)$ on axis it is necessary to estimate $dV'/d\psi$ there and multiply by $\tau = d\psi/d\Phi_B$. I estimate that the magnetic wells on axis for the "free boundary" and the shifted contours are deeper than that of the "fixed boundary" equilibrium at the same $\langle\beta\rangle$ by 13% and 17% respectively. These results show that the most stable equilibrium is that with the deepest magnetic well at $\langle\beta\rangle \approx 20\%$. It also has the largest aspect ratio. The stabilisation with increasing aspect ratio is an effect which has been predicted by MDFR.

8(viii) Discussion

The major result of this first study with the free boundary code is that the finite- β stability of SHEILA heliac equilibria against localised Mercier and ideal ballooning modes is determined almost entirely by the existence of a vacuum magnetic well – there is very little dependence on β . The β dependence is even weaker than that of the fixed boundary equilibria, because the increase in aspect ratio with pressure will stabilise equilibria on the "beany" side of the marginal point. There is a narrow range of values of I_{zc} for which the shift will stabilise a configuration which has $V''(\Phi_B) > 0$ in vacuum. None of these results is at variance with those of the other studies cited above, but further work needs to be done before the mechanisms are fully understood. A deep magnetic well is associated with a large triangularity and ellipticity. However the driving mechanism for Mercier instability with pressure appears to be the increased ellipticity of the surfaces near the magnetic axis. It should be kept in mind that the concept of stabilisation being due to a magnetic

well, curvature or triangularity is, in Bateman's words, merely an interpretation based on certain rearrangements of the mathematical terms in the Mercier criterion (Bateman, 1978). The criterion needs to be presented in a form which highlights the results discussed above.

This study has not included the stability properties of low- n global modes. The results of MDFR and the HASE studies make it appear that the global modes will conform to the stability diagram for Mercier and ballooning modes. This assumption may, however, be specific to the equilibrium sequences considered by them.

Figure 8.26: *Analytic profile; $a = 0.03$; $b = 0.5783$; $\bar{x} = 0.48567 \times 10^{-2}$; $B = 1.55$; $\langle \beta \rangle = 20\%$.*

Figure 8.27: *Analytic profile; $a = 0.03$; $b = 0.5987$; $\bar{x} = 0.69245 \times 10^{-2}$; $B = 1.55$; $\langle \beta \rangle = 21\%$.*

Figure 8.28: Analytic profile; $a = 0.03$; $b = 0.5783$; $\bar{x} = 0.69245 \times 10^{-2}$; $B = 1.55$; $\langle \beta \rangle = 21\%$.

Figure 8.29: Elongations for the analytic contour: "vacuum"; "shifted"; "free boundary"; "fixed boundary".

Figure 8.30: Indentations for the analytic contour: "vacuum"; "shifted"; "free boundary"; "fixed boundary".

Chapter 9

Conclusions

In this chapter, I shall present a summary of the major results of this thesis and suggest some future directions for research.

9(i) Summary

The thesis proper began, in Chapter 2, with a review of the ideal MHD equilibrium equations in their helically symmetric limit. The general solution of the helical Grad Shafranov equation (HGSE) for the helical flux function in the case of pressureless equilibria was presented. The results of this chapter were general to geometries with multiple periodicity in the (cylindrical) ϕ as well as in the z directions. The rest of the thesis was specialised to systems (with spatial magnetic axes) which were 2π -periodic in ϕ .

A model for the straight heliac was presented in Chapter 3. The perturbing effect of the displacement of the helical solenoid on a uniform longitudinal field was represented as a series of products of modified-Bessel and trigonometric functions. This model was used to illustrate the formation of the bean-shaped magnetic surfaces, the significance of the rotational transform and the concept of resonances. Some of the consequences of the fully toroidal heliac were also mentioned.

The problem of calculating the scalar potential of this solenoidal field was formally posed in Chapter 4. A numerical technique was investigated in which the solution was expressed in terms of the Fourier-Bessel series of Chapter 3. It was noted that the accuracy of the technique decreased as the coil displacement increased.

In Chapter 5, it was verified that the, already known, Green's function for the helical Laplace equation does satisfy its defining equation. Two new Green's functions were

introduced and were shown to be "fundamental solutions" for the HGSE. Their physical interpretation was discussed, as was their correspondence with the Green's functions for the toroidal and cylindrical limits of the equilibrium equation.

A "generic", $l = 1$, helical geometry was described formally in Chapter 6. An integral expression for the helical flux function, ψ , was derived by an application of Green's theorem. The derivation was more complicated than the axisymmetric case (Delucia *et al.*, 1980) because of the form of the source terms for the HGSE. The case with net longitudinal current necessitated the additional consideration of the limiting integrals at zero and infinity for some choices of the auxiliary flux function. An iterative technique for finding a free boundary equilibrium was described. It was based on setting the plasma current to zero (or other specified values) in the integral expression for ψ .

The consideration of the asymptotic form of Green's theorem applied to the helical solenoid of Chapter 4 showed that the limit of applicability of the Fourier-Bessel series does, indeed, depend on the magnitude of the coil displacement. This is because the surfaces of the (r, ζ, z) coordinates are cylindrically symmetric whereas the solenoid is a helically symmetric "coordinate surface". The failure of the eigenfunction expansion to converge in this situation is, perhaps, to be expected (Brazier-Smith, 1984). However, these results showed that there are some (moderate) coil displacements for which the series will converge. Also, the earlier results of Chapter 4 demonstrated that approximate numerical methods can be quite accurately be applied to larger coil displacements where it will not.

In Chapter 7, the free boundary code FEQ2.5(FR) was described. As well as incorporating the iterative algorithm suggested in Chapter 6, this code contains several improvements on the axisymmetric code of Delucia *et al.* (1980). The setting up of the initial grid and the evaluation of points near the old plasma boundary are achieved by some new algorithms which have been especially developed for beany heliac plasmas. The code also allows the physical limiter to be circular rather than a discrete set of points.

In Chapter 8, a helically symmetric model for the SHEILA heliac was presented and compared with experimental measurements. A fixed boundary stability diagram was constructed using the FEQ2.5 and BAL2.5 codes. The differences of this diagram with those of previous authors reflect the effects of choosing a different path through the parameter space. When the plasma boundary was allowed to move, the results of the FEQ2.5(FR) code showed that the outwards shift with pressure, though not spectacular to the eye, was

sufficient to stabilise all configurations which had an on-axis magnetic well in vacuum. To the best of my knowledge, these are the first completely free boundary results which have been presented for diffuse-profile helically symmetric equilibria. The stabilising mechanism appears to be related to the effect of the outwards movement on the magnetic well at finite pressure. [This is as opposed to the effect of the reduction in ellipticity of the innermost surfaces which should also be stabilising provided it is not accompanied by a reduction in triangularity.]

9(ii) Future directions

Some immediate extensions to the present work present themselves. With a little effort, the free boundary code FEQ2.5(FR) could be used to model $l > 1$ systems as well as equilibria with net longitudinal current. It could then be used to verify some of the work on low- β straight-axis helical equilibria from the mid 1970's (Barbiero-Corsetti, 1973, Hamada *et al.*, 1976). One would also be able to test the prediction (Barnes and Cary, 1984) that a net longitudinal current should be stabilising for some straight axis stellarator equilibria. To carry out a proper stability analysis, however, the entire helically symmetric PEST package will have to be completed. This will necessitate including the $n = 0$ terms in the global energy functional as well as incorporating some model of the vacuum energy (Dewar *et al.*, 1984). The most expedient way of doing the latter would probably be to use the package developed by Merkel (1982) for ERATO. These projects could probably be completed in eighteen months using the computing facilities at the A.N.U.

The heliac parameter study in Chapter 8 could be extended and applied to the H1 and TJ-II heliacs presently under construction (Blackwell *et al.*, 1986, Guasp *et al.*, 1985). One of the promises of this free boundary model is that the effects of various coil designs should be able to be modelled accurately. An implication of the results of the collocation methods of Chapter 4 is that Fourier-Bessel series should be able to be used to approximate the fields of variously shaped external solenoids. The series could be determined by collocating the values of the Green's function integral over some interior surface (if the collocation of the jump conditions across the coil itself proved to be unstable). It should be pointed out that a free boundary parameter study of any magnitude would best be carried out on a super-computer.

Weimer *et al.* (1975) have demonstrated the equivalence of a form of the Mercier criterion which was obtained by the tokamak expansion for a non-circular plasma cross-

section with an expansion about the magnetic axis. It is possible that such a result may be shown to apply to helically symmetric plasmas (suggested by J.M. Greene, private communication). In any case, it would be instructive to derive a form of the criterion for helical systems which highlights the influence of the magnetic well at finite pressure.

At present, there is much interest internationally in solving the fully three-dimensional stellarator equilibrium problem (Hender *et al.*, 1982, Bauer *et al.*, 1984, Shestakov and Mirin, 1984, Strauss *et al.*, 1986). The most important developments in stellarator theory over the next decade is likely to come from this direction. However, two dimensional studies will continue to be important while available computer hardware restricts the 3D codes to relatively coarse mesh sizes, and they remain limited in their ability to compute free boundary equilibria (Koniges and Johnson, 1985). The availability of sophisticated stability codes to treat two dimensional systems is another point in their favour.

In the future it may be desirable to use the complete free boundary helically symmetric version of PEST to examine aspects of the control theory of stellarators (Dewar, 1978, Jardin, 1978). It may also be possible to use such a code to model Alfvén wave heating (Bernstein, 1983), or aspects of the evolution of kink modes in tokamaks (by looking at the stability of a kinked plasma column).

The free boundary helically symmetric equilibrium model which has been presented in this thesis is likely to find a modest niche in the world fusion effort for some time to come.

Appendix A

The jump in g across a helical solenoid

The geometry we are considering is that of the model problem of Chap. 6. From Eq. (6.11)

$$[[g]] = \frac{\mu_0 \mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u})}{h|\mathbf{u}|^2}$$

where $\mathbf{i}$ is the current on the helically symmetric surface with unit normal $\mathbf{e}_n$. First note that we may also obtain this expression from Eq. (2.13) for ∇g :

$$\begin{aligned} \mathbf{e}_n \cdot \nabla g &= \lim_{\delta n \rightarrow 0} \frac{g_e - g_i}{\delta n} = \frac{\mu_0 \mathbf{j} \cdot (\mathbf{e}_n \times \mathbf{u})}{h|\mathbf{u}|^2} \\ \Leftrightarrow \quad g_e - g_i &= [[g]] = \lim_{\delta n \rightarrow 0} \frac{\mu_0 (\delta n \mathbf{j}) \cdot (\mathbf{e}_n \times \mathbf{u})}{h|\mathbf{u}|^2} \\ &= \frac{\mu_0 \mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u})}{h|\mathbf{u}|^2}. \end{aligned}$$

Now, from Eq. (4.28), the equation of the unit normal to the solenoid γ is

$$\mathbf{e}_n = \frac{r\dot{\phi}\mathbf{e}_r - r\dot{\mathbf{e}}_\phi + hr\dot{\mathbf{r}}\mathbf{e}_z}{\sqrt{(r^2\dot{\phi}^2 + \dot{\mathbf{r}}^2 + h^2r^2\dot{\mathbf{r}}^2)}} = \frac{\mathbf{e}_t \times \mathbf{u}}{|\mathbf{e}_t \times \mathbf{u}|}$$

where

$$\mathbf{e}_t = \frac{r\dot{\phi}\mathbf{e}_\phi + r\dot{\mathbf{e}}_r}{\sqrt{(r^2\dot{\phi}^2 + \dot{\mathbf{r}}^2)}}$$

is the tangent vector in a constant- z plane. The dots represent differentiation with respect to any variable which parametrizes the cross-section. The surface current is assumed to lie in the direction of $\mathbf{e}_t$ such that $\mathbf{i} = |\mathbf{i}|\mathbf{e}_t$, where $|\mathbf{i}|$ is equal to the surface current per length increment perpendicular to $\mathbf{e}_t$. The current which flows through the length ΔL , measured along the solenoid perpendicular to $\mathbf{e}_t$ as shown in Fig. A.1, is that which flows through $\Delta L \cos \theta = \Delta L(\mathbf{e}_n \times \mathbf{e}_t) \cdot \mathbf{e}_z$ measured along the $\mathbf{e}_z$ direction.

Figure A.1: Geometry for calculating $|i|$.

Thus, if the solenoid has a constant current per unit length in the z direction equal to i_0 then

$$|i|\Delta L = i_0\Delta L \cos \theta$$

$$\Leftrightarrow |i| = \frac{i_0\sqrt{(r^2\dot{\phi}^2 + \dot{r}^2)}}{\sqrt{(r^2\dot{\phi}^2 + \dot{r}^2 + h^2r^2\dot{r}^2)}}$$

and

$$\frac{\mathbf{i} \cdot (\mathbf{e}_n \times \mathbf{u})}{|\mathbf{u}|^2} = \frac{|i|\mathbf{e}_n \cdot (\mathbf{u} \times \mathbf{e}_t)}{|\mathbf{u}|^2}$$

$$= -\frac{|i|(1 + h^2r^2)\sqrt{(r^2\dot{\phi}^2 + \dot{r}^2 + h^2r^2\dot{r}^2)}}{(1 + h^2r^2)\sqrt{(r^2\dot{\phi}^2 + \dot{r}^2)}}$$

$$= -i_0,$$

whence $g_e - g_i = -\mu_0 i_0/h$.

Appendix B

Code description: FEQ2.5

FEQ2.5 is a version of the flux coordinate equilibrium code of Delucia *et al.* (1980) which has been modified to handle either toroidal or helical symmetry (Dewar *et al.*, 1984). For helical systems of arbitrary l the equation

$$\nabla \cdot (\kappa \nabla \psi) = -\frac{2l\kappa^2}{h}g(\psi) - \kappa g g'(\psi) - \frac{\mu_0 p'(\psi)}{l^2} \quad (B.1)$$

[where $\kappa = h^2/(l^2 + h^2 r^2)$] is solved on a surface of section of a helically symmetric flux tube. I shall take this section to be in the $z = 0$ plane and assume that the plasma has reflection symmetry about the x axis. The code is also able to treat sections through planes of constant ϕ (Dewar *et al.*, 1984). The code uses the coordinates (X, Z, Y) which are related to the usual helical coordinates by

$$X = r \cos(\zeta/l), \quad Z = r \sin(\zeta/l), \quad Y = hz, \quad (B.2)$$

to specify the equilibrium in the constant- z plane. The converged solution is expressed as $X(\Theta, \Psi)$ and $Z(\Theta, \Psi)$ where (Ψ, Θ, Y) is a right handed set of flux coordinates. The surface label Ψ is related to the helical flux via

$$\Psi = (1 + m\pi)^{\frac{1}{(1+m)}} \left(1 - \frac{\psi}{\psi_{min}}\right)^{\frac{1}{(1+m)}} \quad (B.3)$$

where ψ increases from ψ_{min} (variable PSIMIN) at the magnetic axis to 0 at the edge, and I set m (variable RJAC), which can be used to change the mapping between ψ and Ψ , equal to 1. The Θ coordinate divides each of the flux surfaces into equal arcs, and all scalar quantities are independent of Y .

In terms of (Ψ, Θ, Y) the left hand side of the equilibrium equation becomes (Dewar *et al.*, 1984):

$$l^2 \nabla \cdot (\kappa \nabla \psi) = \frac{1}{J} \left[\partial_\Psi \left(h^{\Psi\Psi} \partial_\Psi \psi + h^{\Theta\Psi} \partial_\Theta \psi \right) + \partial_\Theta \left(h^{\Theta\Psi} \partial_\Psi \psi + h^{\Theta\Theta} \partial_\Theta \psi \right) \right]$$

$$\begin{aligned}
\text{where } h^{\Psi\Psi} &= \frac{l^2 \kappa}{h^2 J} \left[(\partial_\Theta Z)^2 |\nabla X|^2 + (\partial_\Theta X)^2 |\nabla Z|^2 - 2 \partial_\Theta Z \partial_\Theta X \nabla X \cdot \nabla Z \right], \\
h^{\Theta\Psi} &= -\frac{l^2 \kappa}{h^2 J} \left[\partial_\Theta Z \partial_\Psi Z |\nabla X|^2 + \right. \\
&\quad \left. \partial_\Theta X \partial_\Psi X |\nabla Z|^2 - (\partial_\Theta Z \partial_\Psi X + \partial_\Psi Z \partial_\Theta X) \nabla X \cdot \nabla Z \right], \\
h^{\Theta\Theta} &= \frac{l^2 \kappa}{h^2 J} \left[(\partial_\Psi Z)^2 |\nabla X|^2 + (\partial_\Psi X)^2 |\nabla Z|^2 - 2 \partial_\Psi Z \partial_\Psi X \nabla X \cdot \nabla Z \right] \\
\text{and } J^{-1} &= \nabla \Psi \cdot (\nabla \Theta \times \nabla Y).
\end{aligned}$$

The equilibrium equation is solved using the following algorithm:

- STEP A:** Initialise the code and read in an initial guess for the coordinate system within the plasma.
- STEP B:** Set up the various metric quantities, $h^{\Psi\Psi}$ and so on, at the current level of the iteration (level n).
- STEP C:** Calculate the right hand side of Eq. (B.1).
- STEP D:** Solve Eq. (B.1) by successive over-relaxation (SOR) to give Ψ at level $n+1$.
- STEP E:** Contour Ψ and Θ at the new iteration level (level $n+1$).
- STEP F:** Repeat steps B to E either a specified number of times (parameter NUMIT in the input deck) or until the number of SOR iterations is less than a specified amount.

B(i) Initialising the code

The outer plasma boundary and the initial guess for the flux coordinate system are set up by the auxiliary code BNDFEQ as described in Chap. 7. The input deck to the present version of FEQ2.5 (which is the same as that for FEQ2.5(FR)) is shown in Table C.1. Many of the options of the original code have been set as defaults. The numbers in the first column flag the card types. The input format is REAL. The input file must begin with a title card and end with Card 99.

The vacuum field of the external coils is stored as the coefficients of a cylindrical harmonic expansion in the array CHIEXT which must be entered after Card 14. The

Title card							
0	0.	0.	0.	0.	0.	0.	IEQRST
1	0.	0.	NOPCEN	0.	IFUNC	0.	0.
2	NTHE	NPSI	RJAC	0.	0.	0.	OMF
3	OMEGA	FACIMP	NIMAX	NUMIT	0.	0.	TSF
4	ALPHA	0.	P0	PMIN	0.	0.	0.
5	0.	0.	0.	0.	0.	RCOIL	DGUESS
6	0.	0.	0.	0.	0.	AL	H
12	G0	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	ITOOFF	JORGN	0.	0.
14	NEXTI	NEXTK	NBMAX	IGOPT	FFAC	MAXBND	EPSBND
Coefficients of the series of the vacuum field							
99	0.	0.	0.	0.	0.	0.	0.

Table B.1: *Input deck for the code FEQ2.5(FR).*

elements of CHIEXT are related to the field coefficients by

$$\begin{aligned}
\psi_{vac} = & \text{CHIEXT}(1,1) \ln r - \frac{r^2}{2} \text{CHIEXT}(2,1) + \\
& r \sum_{n=1}^{\text{NEXTI}-2} \text{CHIEXT}(n+2,1) * I'_n(nhr) \cos(n\zeta) * \text{FFAC} + \\
& r \sum_{n=1}^{\text{NEXTK}} \text{CHIEXT}(n,2) * K'_n(nhr) \cos(n\zeta) * \text{FFAC} .
\end{aligned} \tag{B.4}$$

The vacuum surface which is tangent to the circular limiter (of radius RCOIL and displacement DGUESS) is taken to define the plasma boundary.

The parameters NTHE and NPSI give the dimension of the (Θ, Ψ) mesh within the plasma. Typically NTHE=NPSI=56. The form of the pressure profile, for the option IFUNC=1, is

$$\frac{p(\psi) - p_{min}}{p_0 - p_{min}} = \left(\frac{\psi_e - \psi}{\psi_e - \psi_{ma}} \right)^\alpha \tag{B.5}$$

where ψ_e and ψ_{ma} are the values of ψ at the edge and the magnetic axis, p_0 (input variable P0) and p_{min} (PMIN) are the peak and minimum values of p , and $\alpha > 0$ (input variable ALPHA) determines the slope of the curve. The peak and minimum pressures are input in mks units multiplied by μ_0 . All other quantities are in mks units. The input parameter G0 is the value of g in the vacuum, the parameters AL and H are equal to l and h (in $\zeta = l\phi - hz$) and IEQRST indexes the checkpointing files. The remaining input parameters

apply to the SOR iterations (explained in Sec. B(iii)) or to the free boundary code. The parameter NBMAX on Card 14 limits the number of terms in the series for the Green's functions $\mathcal{G}_1$ (IGOPT=1) or $\mathcal{G}_2$ (IGOPT=2) of Sec. 5(ii) (see also Appendix C). MAXBND and EPSBND can be used to set termination conditions for the free boundary code (as in Sec. 7(iv)).

B(ii) Calculation of the right hand side

The form of the $g(\psi)$ profile is calculated to set the net longitudinal current to zero as follows: Denote by $\langle \dots \rangle$ an average over the flux coordinate Θ , then q may be expressed in terms of X and Z as (Dewar *et al.*, 1984)

$$q = g \langle l^2 \kappa J \rangle - \langle l^2 \kappa C \rangle \quad (B.6)$$

where

$$lC = X \partial_{\Theta} Z - Z \partial_{\Theta} X. \quad (B.7)$$

If the longitudinal current is set to zero surface by surface then

$$\begin{aligned} \delta J &= -(\langle J \rangle p' + q g') \delta \psi = 0 \\ \Leftrightarrow g' q &= -\langle J \rangle p'. \end{aligned} \quad (B.8)$$

Given the $X(\Theta, \Psi)$, $Z(\Theta, \Psi)$ grid at the n^{th} iteration level, and the form of $p'(\psi)$, the subroutine NEWJ solves Eqs. (B.6) and (B.8) to find $g(\psi)$ and $g'(\psi)$. The value of g at the edge is set equal to g in the vacuum. The value of g at the next surface in is then found by solving the quadratic equation resulting from replacing $g'(\psi)$ by its finite difference form in Eq. (B.8). Thereafter it is straightforward to calculate g' and g for the interior surfaces. The right hand side of the equilibrium equation is accumulated in the array VECR.

B(iii) Successive over-relaxation of the equilibrium equation

The second order accurate finite difference approximation to the term $\partial_{\Psi}(h^{\Psi\Psi} \partial_{\Psi} \psi)$ is given by

$$\partial_{\Psi}(h^{\Psi\Psi} \partial_{\Psi} \psi)_{I,J} = (\delta \Psi)^{-2} \left[h_{I,J+\frac{1}{2}}^{\Psi\Psi} (\psi_{I,J+1} - \psi_{I,J}) - h_{I,J-\frac{1}{2}}^{\Psi\Psi} (\psi_{I,J} - \psi_{I,J-1}) \right]$$

where the subscript (I,J) labels the (Θ, Ψ) grid point. The metric quantities at the intermediate sites are calculated in the subroutine METRIC. The complete finite difference

form of the equilibrium equation expresses $\psi_{I,J}$ in terms of all of the four neighbouring sites on the grid. This is solved by successive over-relaxation in the subroutine ITER.

When the variable OMF is negative a constant relaxation parameter OMEGA, where $0 < \text{OMEGA} < 2$ (Dahlquist *et al.*, 1974), is used. The value of ψ at $(I,J) = (1,1)$, which is a singular point of the flux coordinate system, is estimated by extrapolation from the neighbouring surfaces – from surfaces 2,3 and 4 if the parameter NOPCEN is equal to 1. The iteration terminates after NIMAX loops or when the maximum increment in ψ over the grid is less than FACIMP multiplied by the sum of the absolute values of ψ on the grid. The update to ψ at level $n+1$ is damped by the parameter TSF:

$$\psi_{I,J}^{n+1} = \overline{\psi_{I,J}^{n+1}} * \text{TSF} - (1 - \text{TSF}) * \psi_{I,J}^n, \quad (\text{B.9})$$

where $\overline{\psi^{n+1}}$ denotes the result of the SOR iteration at level $n+1$. A typical input deck has $\text{OMF} < 0$, $\text{OMEGA} = 1.9$, $\text{FACIMP} = 1 \times 10^{-9}$, $\text{NIMAX} = 300$ and $\text{TSF} = 0.4$.

B(iv) Rezoning the coordinate system

The subroutine PSISUR calls the subroutine FINDMN to find the position and value of ψ^{n+1} at the magnetic axis. Quadratic interpolation with respect to the position along the mid-plane is used to estimate the minimum value of ψ^{n+1} . The arc length along each Θ line is then interpolated, using ψ^{n+1} as the independent variable, to estimate the position of the Ψ^{n+1} surfaces according to Eq. (B.3).

For the first ITOOFF iterations, the present version of the code modifies the calculation of the Ψ grid by respacing some of the inner surfaces equidistantly along straight lines drawn from the magnetic axis. This cushions the effects of large movements in the magnetic axis. The number of surfaces treated in this way is:

$$\text{nsurf} = \text{MAX} \left[5, \text{JORG} \left(\frac{\text{ITOOFF} - \text{IT}}{\text{ITOOFF}} \right) \right]$$

where IT is the iteration number n , and, typically, $\text{ITOOFF} = 7$ and $\text{JORG} = 20$.

Cubic interpolation is used in the subroutine MAP3 to find Θ^{n+1} such that it defines equal arcs on all of the surfaces. The X and Z arrays are interpolated separately.

Appendix C

Convergence acceleration of the Green's function series

Consider the series (with standard notation)

$$S(r, \zeta | r_0, \zeta_0) = rr_0 \sum_{n=1}^{\infty} I'_n(nhr_<) K'_n(nhr_>) \cos[n(\zeta - \zeta_0)] . \quad (C.1)$$

As the "observer point" (r_0, ζ_0) approaches the "source point" (r, ζ) , S will diverge logarithmically. The effects of this divergence may be cushioned by separating the logarithmic part using a method due to Merkel (Merkel, 1982) for the helical Laplace equation. The notation here is similar to that used by Barnes and Cary (1984). The asymptotic form of the term $S_n = rr_0 I'_n(nhr_<) K'_n(nhr_>)$ as $n \rightarrow \infty$ is (Abramowitz and Stegun, 1970, p. 378).

$$S_n \sim -\frac{1}{2nh^2} \frac{1}{\sqrt{(T_< T_>)}} u^n \left(1 + \sum_{k=1}^{\infty} \frac{v_k(T_<)}{n^k} \right) \left(1 + \sum_{k=1}^{\infty} \frac{-v_k(T_>)}{n^k} \right) \quad (C.2)$$

where $T_> = 1/\sqrt{(1 + h^2 r_>^2)}$,

$$u = \frac{r_< [1 + \sqrt{(1 + h^2 r_>^2)}]}{r_> [1 + \sqrt{(1 + h^2 r_<^2)}]} \exp[\sqrt{(1 + h^2 r_<^2)} - \sqrt{(1 + h^2 r_>^2)}] ,$$

$$v_0(T) = 1 ,$$

and $v_1(T) = (-9T + 7T^3)/24 .$

The positive quantity u is less than 1 except when $r = r_0$. Noting that

$$\sum_{n=1}^{\infty} \frac{-1}{2h^2 \sqrt{(T_< T_>)}} \frac{u^n}{n} \cos[n(\zeta - \zeta_0)] = \frac{1}{2h^2 \sqrt{(T_< T_>)}} \Re \ln(1 - Z) \quad (C.3)$$

where $Z = u \exp[i(\zeta - \zeta_0)]$, enables us to sum the divergent parts of S_n separately (except when $r = r_0$). In the limit of $h \rightarrow 0$,

$$\Re \ln(1 - z) \rightarrow \ln(r_>^2 + r_<^2 - 2r_>r_< \cos(\phi - \phi_0)) - \ln r_>$$

that is we recover the characteristic logarithmic singularity of the Green's function for Laplace's equation in the plane.

The series for the first derivatives of S with respect to r and ζ will diverge in the first two orders of the series for the asymptotic form:

$$S_n \sim \frac{-1}{2nh^2\sqrt{(T_<T_>)}} \left(u^n + \frac{bu^n}{n} \right) \quad (C.4)$$

where $b = v_1(T_<) - v_1(T_>)$

$$= \frac{-9(T_< - T_>) + 7(T_<^3 + T_>^3)}{24}.$$

We find

$$\frac{dS_n}{dr} \sim -\frac{1}{2h^2\sqrt{(T_<T_>)}} \left[-\frac{D_T}{n} + D_u \left(1 + \frac{b}{n} \right) \right] \quad (C.5)$$

$$\text{where } D_T = \frac{1}{\sqrt{(T_<T_>)}} \frac{d}{dr} \sqrt{(1 + T_<T_>)} \quad (C.6)$$

$$\text{and } D_u = \frac{1}{u} \frac{du}{dr}.$$

So that

$$\sum_{n=1}^{\infty} \frac{dS_n}{dr} \cos[n(\zeta - \zeta_0)] = \frac{-1}{2h^2\sqrt{(T_<T_>)}} \left[D_u \Re \left(\frac{Z}{1-Z} \right) + (D_T - b) \Re \ln(1-Z) \right] \quad (C.7)$$

with Z as above. Also

$$\sum_{n=1}^{\infty} -nS_n \sin[n(\zeta - \zeta_0)] = \frac{1}{2h^2\sqrt{(T_<T_>)}} \left[\Im \left(\frac{Z}{1-Z} \right) - b \Im \ln(1-Z) \right]. \quad (C.8)$$

Subtracting the series of Eqs.(C.3), (C.7) and (C.8) from the summation of the Green's function for the HGSE and its derivatives greatly accelerates the convergence when the observer point is near the source point. For a trial in which $(r_0, \zeta_0) = (1.0, 0.2)$ and $(r, \zeta) = (1.1, 0.25)$ the separated series for $\mathcal{G}_1$ converged to 6 decimal places in 22 iterations compared with about 79 iterations of the unseparated series. The situation for $d\mathcal{G}_1/dr$ and $d\mathcal{G}_1/d\zeta$ was similar.

In FEQ2.5(FR) the Green's function series are evaluated by calling the IMSL routine MMBSIN (IMSL, 1984) and the NAG functions S18ACF and S18ADF (NAG, 1984). The series is truncated either at $n = 50$ or when the incremental difference in $d\mathcal{G}/dr$ is less than $10^{-3}\%$. Generally between 30 and 50 members of the series are needed for source points on the old plasma boundary in the later stages of a free boundary iteration.

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