THE BAROCLINIC INSTABILITY
of the
ZONAL WINDS to SHORT WAVES

by
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A thesis submitted to the A.N.U. for the degree of
Master of Science
in the Department of Applied Mathematics.

Canberra
1964
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Dear Sir,

I have found a few slight typist errors in my M.Sc. thesis, "The Baroclinic Instability of the Zonal Winds to Short Waves." I wonder if you could correct the copy you carry?

Equation (2.7), page 15, should read

\[ \ldots \quad \frac{1}{4P} \left( p^{\frac{4}{3}} - p' \right) \]

Equation (2.22), page 20,

\[ \ldots \quad w_0 \sqrt{P_1} \frac{F'(w_0)}{F(w_0)} - w_0 \sqrt{P_1} \frac{G'(w_0)}{G(w_0)} . \]

Equation (2.24), page 20, should be followed by a comma, viz:

\[ + o \left( \frac{1}{k} \right) , \]

Equation (2.29), page 21, should end with a full stop, viz:

\[ + o \left( \frac{1}{k^3} \right) . \]

and in the pentultimate line of page 18, it should state

\[ \left| \frac{q(w) - q(1)}{w - 1} \right| \]

Thanking you,

Yours sincerely,

Herbert E. Huppert

HEH:ke
PREFACE

The work for this thesis was undertaken between the months of January and December, 1964, at the Australian National University, during which time I held a Research Scholarship.

The thesis deals with a problem suggested to me by my supervisor, Professor J.W. Miles. To him I wish to express my deepest thanks. I would also like to thank Mrs. P. Munns who typed this work, and all other members of the Mathematics Department, A.N.U., each one of whom helped me in some way.

The body of this thesis is my own work except where special reference has been made.
SUMMARY

THE BAROCLINIC INSTABILITY
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This thesis deals with the asymptotic solution of the equations describing a simple harmonic, travelling-wave disturbance of short wavelength in a baroclinic zonal wind.

Chapter 1 describes some of the existing work in this field. The assumptions and approximations pertaining to Charney's model [2] are listed. A qualitative review is given of this paper, culminating in the statement of the existence of a stability boundary. The works of Eady [3] and Green [5] are considered, together with the latter's conjecture of total instability. The work of Miles [7], which leads to the equations on which this thesis is founded, is then described.

Chapter 2 discusses the restriction imposed on the basic distributions of temperature and wind speed. By suitable transformations, the differential equation describing the disturbance
is brought to the form approximating the hypergeometric differential equation. Applying the boundary conditions, the wavespeed of an unstable mode is obtained. A rigorous proof to determine whether or not this is the only unstable mode is not known to the author, although a conjecture that it is the only mode is made. The eigenvalue equations of Charney [2] and Miles [10] are solved for large $\alpha$. 

CHAPTER II

THE FORM OF THE EIGENVALUES

1. The Functions $F(x)$ and $U(x)$.
2. The Eigenfunctions.
3. The Eigenvalue Equation.
4. The Wavespeed Approaching $U(x)$.
5. Comparison with Previous Results.
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Chapter I

INTRODUCTION

1. The Zonal Wind

It is observed that the Earth is enveloped by a westerly wind, known as the zonal wind. It is found in the mid-latitudes, with a speed which increases with height up to the tropopause (about ten kilometres from the earth's surface). This wind has been the subject of investigation of theoretical meteorologists over a considerable period of time.

This thesis is concerned primarily with a mathematical investigation of its stability when subject to a short wavelength disturbance.
2. Charney's Model

The first thorough investigation of the instability of the zonal wind was carried out by Charney [2]. Charney's model was one of the first to be fully baroclinic, and has remained the basis of most subsequent investigations. The model represents the motion for waves whose wavelengths are of the order of between one and ten thousand kilometres in the aforementioned region of the zonal wind.

As well as the supposition that the atmosphere is in static equilibrium, the model is based on the following approximations or assumptions:

(1) The atmosphere is a perfect gas.

(2) The fluid motion is adiabatic.

(3) The gravitational potential is proportional to, and dependent only upon, the true altitude.

(4) The conventional Rossby number is small. This assumption implies that the flow is quasi-geostrophic.

(5) The value of the Richardson number is large. This means that the energy of the disturbance is derived from the potential, rather than the kinetic, energy of the mean flow. By this assumption, the short period gravity waves are eliminated.

(6) The hydrostatic approximation is applicable.
(7) The motion can be considered to take place in the Rossby \( \beta \)-plane, the plane tangent to the Earth at the latitude in question.

(8) The vertical component of the Earth's angular velocity, and its first derivative with respect to latitude, can be represented by their mean values.

(9) The travelling-wave disturbances are directed along parallels.

In order to express the solution of the problem in terms of standard functions, Charney added:

(10) In the troposphere, the wind shear, the temperature and its vertical derivative are replaced by their mean values.

(11) In the stratosphere the temperature and wind speed are independent of altitude.

Obtaining a steady state solution from the equations of geostrophic balance, Charney superimposed a simple harmonic travelling-wave disturbance. After some mathematical manipulation this disturbance was described by a differential equation of the confluent hypergeometric type, to which Charney applied the following boundary conditions:

(1) The normal component of velocity vanishes at the surface of the earth.
(2) The energy density is finite at the limit of the atmosphere.

(3) The velocity components, together with the pressure, density and temperature, are continuous across the tropopause.

These boundary conditions led to the desired eigenvalue equation for the wave speed. Solving this equation by graphical methods, Charney deduced that a stability boundary exists in the wind-shear, wavelength plane, such that stability increases with wind-shear and diminishes with wavelength. Charney concluded this paper with some discussion on the structure of the two possible types of waves: neutral and unstable.
3. Further Models

Subsequently, the main stream of research in the problem of instability of zonal winds was directed towards the elucidation of Charney's model. The two assumptions most queried were those pertaining to the Coriolis parameter, and those defining the wind speed and temperature in the upper limit of the atmosphere. In the model proposed by Charney, the wind speed approached infinity as the altitude became infinite. However he did not claim that the model was representative of very high altitude motion, though, due to the evanescence of the density there, the discrepancy involved in the calculation of the total kinetic energy is no larger than those errors already incorporated in the model. The question of the boundary condition at infinity has been the matter of considerable discussion.

Eady [3] attacked the problem by supposing the motion to take place between two horizontal planes separated by a finite distance \( d \), say, which is small compared with the wavelength of the investigated waves. In order to obtain standard transcendental function solutions, the assumptions of a constant Coriolis parameter and constant wind shear were made. An eigenvalue equation was obtained which yielded unstable disturbances for all waves with greater than a certain wavelength.

Thus by addition of a solid boundary, and the neglect of the
variation of the Coriolis parameter, a "short-wave cut-off" was produced. This can be compared with Howard's result [6]: for a large class of parallel shear flows, the addition of boundaries tends to stabilize the flow. However, no such cut-off was found in Charney's model. Eady mentioned that if \( d \) is taken as the altitude of the tropopause, i.e. if we consider a motion in the troposphere bounded above by a solid tropopause and below by a solid earth, the wavelength of the most unstable waves (predicted by this model) compares favourably with that actually observed.

Following Green's conclusion that there is instability for almost all wavelengths, Berger [1] investigated a semi-infinite model. (11)-(8) of Charney's basic principles were applied, (10) and (11) being replaced by the assumption of a constant positive wind shear and constant temperature over the physical range. The boundary condition at infinity was taken to be the necessity of finiteness of the three kinetic energy components per unit volume. From the solution of transformed form a stronger condition was satisfied.
4. The Spectrum of Unstable Modes

The importance of the boundary conditions is decisively shown in the work of Green [5]. He states that the removal of the upper boundary in Eady's model, or the neglect of the Coriolis parameter variation in Charney's, renders them equivalent; only neutral solutions being possible. He tries to incorporate both of the preceding ideas by considering a model bounded above and below by horizontal planes and allowing a variation in the Coriolis parameter. He states, without derivation, that instability exists for almost all wavelengths; a result contradicting both Eady and Charney. He admits the existence of a neutral curve of a special type, asserting that there is instability, but of a different nature, on both sides of this neutral curve.

Following Green's conclusion that there is instability for almost all wavelengths, Burger [1] investigated a semi-infinite model. (1)-(9) of Charney's basic principles were applied, (10) and (11) being replaced by the assumption of a constant positive wind shear and constant temperature over the physical range. The boundary condition at infinity was taken to be the necessity of finiteness of the three kinetic energy components per unit volume. From the solution it transpired that a stronger condition was satisfied.
Burger proceeded to obtain a confluent hypergeometric equation with two boundary conditions. One reduced the solution to the general confluent hypergeometric function, while the other furnished the relevant eigenvalue equation. Solving this equation, he arrived at the conclusion that instability exists for almost all wavelengths. Burger remarked that the exceptions, for which he has found neutral solutions, were obtained by the consideration of only a discrete spectrum, and anticipates a further paper considering a continuous spectrum, which will prove that these disturbances are algebraically unstable with time.
5. The Equations Governing the Disturbance

Miles [7], in the first of a series of papers proved that only Charney’s assumptions (1)-(5) are really basic; (6)-(8) follow directly from these five, while the restriction (9) can be discarded, since the eigenvalue equation obtained was independent of the direction of propagation of the disturbances. In a later paper [8] he showed that a model incorporating viscosity and heat conduction leads to essentially similar results.

The following variables or parameters were introduced:

- \( c = c_r + i c_i \) dimensionless wavespeed of the disturbance;
- \( p \) pressure, with ground value \( p_0 \);
- \( z \) pressure altitude, \( z = 1 - \frac{p}{p_0} \);
- \( P(z) \) a function defined in the complex \( z \) plane, obtained by analytic continuation of a function dependent on the temperature distribution in the physical range, \( z \in [0,1] \);
- \( U(z) \) basic velocity of the zonal wind, analytically continued from the physical domain into the complex \( z \) plane;
- \( \alpha \) dimensionless wave number;
- \( \beta \) the constant, dimensionless derivative of the Coriolis
parameter with respect to latitude; 
\[ \kappa \]

the dimensionless value of the static stability, to be considered small and positive;
\[ \lambda = \beta - \left( \frac{P_U}{z} \right) \]

\[ \psi \]
dimensionless perturbation potential.

In order to investigate the temporal stability of the dimensionless potential, Miles perturbed the steady state solution by a simple harmonic travelling-wave to obtain the following equations describing the disturbance:

\[ (P'\psi')' + \frac{\lambda \psi}{U-c} - \alpha^2 \psi = 0 \quad (1.1) \]

\[ (U-c)\psi' - (U'-\kappa c)\psi = 0 \quad \text{at} \quad z = 0, \quad (1.2) \]

\[ P\psi' = 0 \quad \text{at} \quad z = 1. \quad (1.3) \]

(Primes denote differentiation with respect to \( z \)).

The general results of [7], [8] and [9] suggest the existence of instability for almost all wavelengths and wavespeeds. However, there may be a transition from parametrically strong to parametrically weak instability across a neutral curve similar to Charney's stability boundary. Also there may exist discrete
curves along which \( c_1 = 0 \). Thus although Charney's curve cannot
be a stability boundary, his region of instability is the dominant
one, containing the unstable waves of largest growth rate.

In a section of [7], Miles examined the solution for large
\( \alpha \), after having neglected \( \kappa U_0 \) with respect to \( U' \), (the zero
subscript implies evaluation at ground level). By a rather
heuristic analysis, in which he applied a boundary condition to
a Green-Liouville expansion in the neighbourhood of one of its
singularities, Miles concluded that, as the wavelength approaches
zero, there exists an unstable mode, for which the wavespeed
approaches the ground speed of the basic flow. A similar result
was given by Green [5] without derivation.

This thesis presents the rigorous analysis of the
Sturm-Liouville system

\[
(P\psi')' + \frac{\lambda \psi}{U-c} - \alpha^2 \psi = 0 ,
\]

\[ (U-c)\psi' - (U'-\kappa c)\psi = 0 \quad \text{at} \quad z = 0 , \quad \text{(1.5a)} \]

\[ P\psi\psi' = 0 \quad \text{at} \quad z = 1 , \quad \text{(1.5b)} \]

as \( \alpha \rightarrow \infty \).
The results obtained in one section verify those previously derived heuristically by Miles, while the discussion in another elaborates on the difficulties to a question posed by him [7].
Chapter II  THE FORM OF THE EIGENVALUES

1. The Functions $P(z)$ and $U(z)$

The solution depends very much on the form of $P(z)$, the weight function in the Sturm-Liouville system. Denoting the temperature distribution by $T(z)$, and the ratio of the specific heats at constant pressure and constant volume by $\gamma$, $P(z)$ is given by

$$P(z) = \frac{K T_0 (1-z)^2}{(1-z)T_z + \frac{1}{\gamma} T}.$$  \hspace{1cm} (2.1)

The denominator in this expression, which is proportional to the static stability of the atmosphere, is assumed to be positive definite for $z \in [0,1]$. The hypothesis of a bounded temperature in the upper atmosphere leads to

$$P(z) \sim P_1 (1-z)^2 \quad \text{as} \quad z \to 1,$$  \hspace{1cm} (2.2)

where $P_1$ is a constant. We further normalize $P(z)$ such that $P(0) = 1$. For the subsequent development we shall require that $P(z)$ be analytic in a domain $D$, which is to be defined below. $D$ includes at least the segment $z \in [0,1]$ as well as the point
\[ z_c \quad [U(z_c) = c] \, . \]

We assume the basic wind velocity, \( U(z) \), to be monotonic increasing over \([0,1]\), and \((U-c)\) to be analytic in \( D \), with a simple zero at \( z = z_c \).

We thus have a singular, self-adjoint, Sturm-Liouville equation, with two singularities, \( z_c \) and 1, in \( D \). There are no further singularities inside \( D \), but there exists at least one singularity, that at infinity, outside \( D \). Invoking the two boundary conditions \((1.5a,b)\), we can obtain the wavespeed \( c = c(\alpha,\beta,\kappa) \). The purpose of this thesis is to answer the question: what is the form of \( c \) as \( \alpha \) tends to infinity?
2. The Eigensolution

Motivated by the exponents of (1.4) at the singularity $z = 1$, we define new variables $\varphi$ and $w$ by

$$\psi = w^{k} P^{-\frac{1}{4}} \varphi(w),$$  \hspace{1cm} (2.3)

$$w = \exp\left[-\frac{1}{2} \int_{z}^{1} P^{-\frac{1}{2}} dz\right],$$  \hspace{1cm} (2.4)

where

$$k = \left\{ \left( \alpha^2 - \frac{\beta}{U_1 - c} \right) \frac{1}{P_1} + \frac{1}{4} \right\}^{\frac{1}{2}}, \quad \Re(k) > 0. \hspace{1cm} (2.5a,b)$$

Substituting (2.3) and (2.4) into (1.4), we find that $\varphi(w)$ satisfies

$$w \varphi''(w) + (2k+1) \varphi'(w) + \frac{q(w)}{1-w} \varphi(w) = 0,$$  \hspace{1cm} (2.6)

where

$$q(w) = \frac{1-w}{w} \left\{ \frac{1}{4} + \frac{1}{P_1} \left( \frac{\lambda}{U-c} - \frac{\beta}{U_1-c} \right) - \frac{1}{4P_1} (P^{-\frac{1}{4}} P')' \right\}. \hspace{1cm} (2.7)$$

We note that $q(0)$ is finite, and denote $q(1)$, equal to

$$P^{-\frac{k}{2}} \frac{\lambda}{U'} \frac{c}{c},$$

by $v$.

For use in the boundary condition (1.5a) it would be sufficient to convert (2.6) into the integral equation
\[ \varphi(w) = 1 + \frac{1}{2k} \int_0^w \left[ 1 - \left( \frac{v}{w} \right)^{2k} \right] \frac{q(v)}{1-v} \varphi(v) \, dv . \] (2.8)

The solution to (2.8), a Volterra integral equation of the second kind, may be expressed as the Neumann series

\[ \varphi = 1 + \sum_{n=0}^{\infty} \frac{\phi_n}{(2k)^n} , \] (2.9)

\[ \phi_{n+1} = \int_0^w \left[ 1 - \left( \frac{v}{w} \right)^{2k} \right] \frac{q(v)}{1-v} \phi_n \, dv \quad n \geq 0 \] (2.10a)

\[ \phi_0 = 1 . \] (2.10b)

In the domain \( \Delta \), for which \( q(w) \) is uniformly modulo-bounded, and \( w \) approaches infinity at most algebraically with \( k \), it can be shown, by induction, that

\[ O\left( \frac{|\phi_{n+1}|}{|\phi_n|} \right) \leq \min \left( \log(1-w) , \log k \right) . \] (2.11)

Thus (2.9) represents an asymptotic expansion in \( \Delta \).

Defining

\[ w_o = \exp \left[ \frac{1}{2} \int_0^{z_c} \left( 1 - \frac{1}{2} \right) dz \right] , \] (2.12)
and substituting (2.3) into the boundary condition (1.5a), the resulting equation can be satisfied for a finite value of \( w_0 \) only if

\[ |w_0 - 1| = O\left(\frac{1}{k}\right). \tag{2.13} \]

However, since

\[ \varphi = 1 + O\left(\frac{\log k}{k}\right) \quad (w \to 1), \tag{2.14} \]

the error in such a solution would not be uniformly bounded with respect to \( k \). In order to obtain a uniform bound on the error we proceed by defining the function \( G(w) \) by

\[ \varphi = \sum_{1}^{\infty} F(a, b; a+b+1; w)G(w) = F(w)G(w), \tag{2.15a,b} \]

where

\[ a = k + \sqrt{(k^2 + \nu)} \tag{2.16a} \]

\[ b = k - \sqrt{(k^2 + \nu)} \tag{2.16b} \]

\[ a+b+1 = 2k+1. \tag{2.16c} \]
We render the determination of \( F(w) \) unique by introducing a cut in the \( w \)-plane, along the real axis, from 1 to \( \infty \).

The transformation (2.15a) is equivalent to selecting the hypergeometric differential equation as a comparison equation for the determination of \( \varphi \). The author, using different transformations to (2.3) and (2.4), has considered a confluent hypergeometric comparison equation about the critical point (where wavespeed equals wind speed) and matched the solution so obtained to a Green-Liouville solution, commenced at \( z = 1 \). Although the results obtained are equivalent to those given below, the argument employed was slightly heuristic.

Substituting (2.15) into (2.3) and (2.6), we obtain

\[
\psi = w^k P_1 F(w) G(w),
\]

(2.17)

where \( G(w) \) satisfies

\[
G(w) = 1 + \int_0^w \frac{\varphi(u)-\varphi(1)}{\varphi(u)} u^{2k-2}(u)G(u)du.
\]

(2.18)

We now define \( D \) as the smallest domain in the \( z \)-plane such that \( |\varphi(w)-\varphi(1)| \) is uniformly bounded in a region \( R \) of the \( w \)-plane, which includes at least the segment \([0, w_0]\).
and the point \( w = 1 \).

From the integral representations of the hypergeometric function (see e.g. Erdélyi et al., [4]), it can be shown that

\[
F(w) = 1 + O\left(\frac{\log k}{k}\right). \tag{2.20}
\]

Using (2.20) in (2.18), we can prove that

\[
G(w) = 1 + O\left(\frac{1}{k}\right) \tag{2.21}
\]

uniformly in \( R \), if \( w_0 \) is bounded uniformly with respect to \( k \). If \( w_0 \) approaches infinity, \( z_c \) approaches 1, a case which will be examined separately below.
3. The Eigenvalue Equation

It remains now only to substitute the eigensolution (2.17) into the boundary condition (1.5a), to obtain

\[ \frac{U'_0 - \kappa c}{c - U_0} = k \sqrt{P_1} + \frac{1}{4} p'(0) + w_o \sqrt{P_1} \frac{F'(w_o)}{F(w_o)} - w_o \sqrt{P_1} \frac{G'(w_o)}{G(w_o)}. \]  

(2.22)

If \( w_o \) is bounded uniformly with respect to \( k \), (2.22) can be satisfied only if

\[ |U_0 - c| = O\left(\frac{1}{k}\right). \]  

(2.23)

This can also be proved by supposing that there exists a critical point uniformly bounded away from both \( z = 0 \) and \( z = 1 \), and applying the boundary conditions (1.5a,b) to a Green-Liouville solution valid away from the critical point. A direct contradiction is obtained, proving that the only permissible wavespeeds must approach one of the wind speed extrema. Assuming that \( c \) may be expressed as a power series in inverse powers of \( k \), and equating like powers of \( k \) in (2.22), we see that

\[ c = U_0 + \frac{U'_0 - \kappa U_0}{k \sqrt{P_1}} + O\left(\frac{1}{k^2}\right), \]  

(2.24)
Expanding $U(z)$ in a Taylor series about $z = 0$, and evaluating this series at $z = z_c$, we see, on comparison with (2.24), that

$$z_c = (1 - \frac{kU^0}{U'^0}) \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

(2.26)

Utilizing this in (2.12),

$$w_0 = 1 + (1 - \frac{kU^0}{U'^0}) \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

(2.27)

Employing standard integral representations [4] for the hypergeometric function, one can show that, in a domain in which $w$ is bounded with respect to $k^*$,

$$\frac{F'(w)}{F(w)} = -\sqrt{\frac{1}{1-tw}}^{2k} \int_0^1 \frac{t^{2k}}{1-tw} dt \left[1 + O\left(\frac{1}{k}\right)\right].$$

(2.28)

Placing this into (2.25) and executing the limit operation, we have

$$c_i \equiv \frac{\lambda \omega \pi}{|k|^2 \sqrt{P_1}} \left(1 - \frac{kU^0}{U'^0}\right) \exp - 2(1 - \frac{kU^0}{U'^0}) + O\left(\frac{1}{|k|^3}\right).$$

(2.29)

* Actually it is sufficient that there exist an $N$, independent of $k$, such that $\lim_{k \to \infty} k^{-N} w = 0$. 
From this we may infer that, subject to the foregoing restrictions on $P(z)$ and $U(z)$, small disturbances of sufficiently small wavelength are unstable. The wave speed of this unstable mode is given by

$$c = U_o + \frac{(U'_o - \kappa U'_o)}{\alpha} + O\left(\frac{1}{\alpha^2}\right)$$  \hspace{1cm} (2.30)

where

$$c_i = \frac{\lambda_o \pi}{\alpha^2} \left(1 - \frac{\kappa U'_o}{U'_o}\right) \exp\left[-2\left(1 - \frac{\kappa U'_o}{U'_o}\right)\right] + O\left(\frac{1}{\alpha^2}\right).$$  \hspace{1cm} (2.31)

We notice that these results, after neglecting $\kappa U'_o$ with respect to $U'_o$, are exactly those obtained heuristically by Miles [7], by continuing a Green-Liouville solution through a singularity of exponents zero and one. We are thus led to conjecture that such a procedure may be generally valid.
4. The Wave Speed Approaching \( U(1) \)

Although the existence of a mode \( * \) in this case seems very unlikely, the author is not able to give a rigorous proof to this conjecture.

The difficulty in solving (1.4) subject to the boundary conditions (1.5) lies mainly in the fact that (1.4) describes a differential equation, all of whose singularities in \( D \) are regular. However in the limit \( z_c \to 1 \), (1.4) has an irregular singularity at \( z = 1 \). We also remark that \( k \), as defined in (2.5a,b) is not necessarily of order \( \alpha \). All that can be said \( \text{a priori} \) is that

\[
\frac{3\pi}{2} < \arg k < 2\pi . \tag{2.32}
\]

The transformation (2.4) is not applicable since, in view of (2.2), \( z_c \to 1 \) implies that \( w(z) \to \infty \) for all \( z \) away from \( z_c \).

With \( P(z) = (1-z)^2 \), we consider approximating (1.4) by

\[
((1-z)^2 \psi')' + \left( \frac{\beta}{U'(z-z_c)} - \alpha^2 \right) \psi = 0 , \tag{2.33}
\]

since away from \( z = z_c \), the coefficient of \( \psi \) is dominantly \( \alpha^2 \), while near \( z = z_c \), we use a linear approximation to \( U - c \). The solution to (2.33) is in the form of a hypergeometric function. However, to rigorously demonstrate that the error in \( \psi \), obtained

\* From Miles [7], theorem 5, we know that if such a mode did exist, it would necessarily have to be unstable.
by approximating (1.4) by (2.33), is uniformly small, is not possible by the usual methods of evaluating error bounds.

We do remark however, that (2.33) is equivalent to the differential equation obtained by considering the baroclinic stability problem with \( U = U'z \) and \( P(z) = (1-z)^2 \). For this problem Miles [10] has shown that there is one, and only one, unstable mode. We conjecture that, since we have already found one unstable mode for the more general problem considered here, this is the only one possible, and that no further mode exists in the region \( z \to 1 \).
5. **Comparison with Previous Results.**

The system (1.4) and (1.5) has been investigated fully for only two particular cases. Charney [2] considered a basic wind profile given by

\[ U = U_0 \log(1-z) , \quad (2.34) \]

while Miles [10] studies the profile

\[ U = z . \quad (2.35) \]

In both these cases, the temperature distribution was such that

\[ T(z) = (1-z)^2 . \quad (2.36) \]

By considering the relevant eigenvalue equations as \( \alpha \) tends to infinity, one can obtain asymptotic representations for the wave speed in agreement with (2.30) and (2.31), viz.
\begin{align}
c &= U_0 + (1 - \kappa U_0)\alpha^{-1} + O(\alpha^{-2}) \\
c_i &= \frac{\pi (\beta+1)}{\alpha^2} (1-\kappa U_0) \exp \left[ -2(1-\kappa U_0) + O\left(\frac{1}{\alpha^3}\right) \right]
\end{align}

\text{for } U = U_0 \log(1-z), \quad (2.38)

and
\begin{align}
c &= \frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right) \\
c_i &= \frac{\pi (\beta+2)}{\alpha^2} e^{-2} + O\left(\frac{1}{\alpha^3}\right)
\end{align}

\text{for } U = z. \quad (2.40)

We remark that there definitely exists only one unstable mode for each of these two cases.
REFERENCES


