Solitons, Modes of Nonlinear Waveguides, and Saturable Nonlinear Couplers

by

JAVID ATAI

A thesis submitted for the degree of Doctor of Philosophy of the Australian National University.

Canberra
Declaration

This thesis is an account of research undertaken in the Optical Sciences Centre within the Research School of Physical Sciences and Engineering at the Australian National University between May, 1991 and May 1994, while I was enrolled for the degree of Doctor of Philosophy.

The research has been conducted under the supervision of Professor A. W. Snyder and Dr. Yijiang Chen. However, unless specifically stated otherwise, the material presented within is my own.

None of the work presented here has ever been submitted for any degree at this or any other institution of learning.

Javid Atai

November 1994
To My Parents
Acknowledgments

I wish to express my sincere thanks and gratitude to Professor Allan W. Snyder and Dr. Yijiang Chen for introducing me to the field and all the guidance and support they have provided throughout the course of my PhD. I would also like to thank Professor Barry Luther-Davies from Laser Physics Centre, for reading portions of this manuscript and his very helpful comments. I am also indebted to Dr. Billy Todd from Research School of Chemistry, for reading the manuscript and providing me with many useful suggestions.

I am grateful to other members of staff in the Optical Sciences Centre for fruitful discussions: Professor D. John Mitchell, Dr. Nail Akhmediev, Dr. Yuri Kivshar, Dr. John D. Love, and Dr. Adrian Ankiewicz, and Dr. Simon Hewlett.

I gratefully appreciate the administrative support provided by Mrs. Andrea Robins and Ms. Rebecca Pallavicini.

I wish to express my thanks to the Australian Government and the Australian Telecommunications and Electronics Research Board for the provision of scholarships without which the completion of this thesis would not have been possible.

Javid Atai
Publications


Conference Papers:

1) Yijiang Chen and Javid Atai, Nonlinear Couplers Composed of Different Nonlinear Cores, *Proceedings of 16th Australian Conference on Optical*


5) Yijiang Chen and Javid Atai, STABILITY OF 3D SELF-TRAPPED BEAMS WITH A DARK SPOT SURROUNDED BY BRIGHT RINGS OF VARYING INTENSITY, *Proceedings of 18th Australian Conference on Optical Fibre Technology*, Hobart, 28 November-1 December 1993, pp. 376-379
Abstract

This thesis is concerned with solitons, modes of nonlinear waveguides and nonlinear dynamics in saturable nonlinear couplers.

Firstly, dark solitons in two and three dimensions are considered. In the case of two dimensional dark solitons it is found that two-photon absorption (intensity dependent loss) leads to broadening and attenuation of dark solitons, whereas gain is shown to lead to their narrowing and amplification. The relations describing the adiabatic evolution of dark solitons in weakly perturbed media are presented and shown to be in excellent agreement with numerical results. In comparison with fundamental bright solitons, dark solitons are shown to be less sensitive to the perturbations. An exact analytical stationary solution representing a dark soliton in the presence of both two-photon absorption and Raman gain is also obtained. In the case of three-dimensional dark solitons (vortex solitons) a Hamiltonian approach is used to find an approximate stationary solution. It is found that in the presence of loss, gain or two-photon absorption the soliton evolves adiabatically. Also, stationary solutions (vortex solitons) in the presence of gain and two-photon absorption are found numerically.

Secondly, the stability of vortex-like stationary solutions of (2+1)-dimensional nonlinear Shrödinger equation in a self-focusing saturating nonlinearity is investigated analytically and numerically. These solutions represent three dimensional self-trapped beams with a dark spot surrounded by bright rings of varying intensity. Using an operator-theoretic approach, It is shown that the fundamental bound state of the family is stable to a symmetric perturbation but unstable to an asymmetric perturbation (that breaks the azimuthal symmetry of the beam, i.e. transverse instabilities). The higher-order states are also found to display transverse (modulation) instabilities. The development of the instabilities is shown to lead to emission of filaments which spiral away as the beam propagates. The number of these filaments are predicted using numerical stability analysis the results of which are in complete agreement with the numerical beam propagation.

Thirdly, Modal characteristics of light waves trapped in a thin self-defocusing
film bounded by an infinite self-defocusing cladding, whose nonlinear coefficient is different from that of the film, are investigated. It is shown that both gray and dark solitary waves can be trapped in the film when the linear refractive index of the film, $n_f$, is smaller than that of the cladding, $n_0$, and the effective index, $n_e$, is $n_f < n_e < n_0$. On the other hand, a series of trapped dark oscillatory solitary waves results when $n_e < \min\{n_f, n_0\}$. Also, stability of the fundamental asymmetric mode trapped in a thin linear film bounded by self-focusing media is investigated numerically. The discrepancy between numerical and analytical methods, which has been reported in the literature, is resolved. The existence of a class of quasi-periodic solutions resulting from perturbation of the asymmetric modes is demonstrated. In addition, the effect of loss on the propagation characteristics of the asymmetric modes is investigated.

Finally, nonlinear couplers composed of two different saturable nonlinear cores are examined. It is found that the detrimental effect of nonlinear saturation in switching can be greatly reduced when the cores of the coupler are nonlinearly mismatched. As a result of the presence of both linear and nonlinear mismatch novel bifurcation diagrams emerge which are utilized to design new devices. In addition, the switching behavior of a saturable nonlinear coupler with gain is investigated. It is found that nonlinear mismatch can improve the switching characteristics of active saturable nonlinear couplers.
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Chapter 1

Introduction

Since its inception with the pioneering works of Bloembergen and his co-workers [1–3], nonlinear optics has evolved to one of the most exciting and rapidly growing sub-fields of physics. Today, the scope of nonlinear optics has extended and its connection with such fields as materials science has greatly strengthened. The evidence of such a tremendous evolution is seen in the development of trans-oceanic telecommunications system, nonlinear waveguides, and all-optical switching devices in recent times.

Much of the progress and development in this field may be attributed to the advent of lasers. Indeed, for the most part of the history of optics it was thought that all optical materials behave linearly in the presence of light. Lasers played a pivotal role in demonstrating that optical media do behave in a nonlinear manner, as embodied in the following observations:

- The principle of superposition does not apply.

- The frequency of light may change as it propagates through the medium.

- When two or more light beams are present they interact.

- The refractive index of the medium, and consequently the speed of light, depends upon the intensity of light.
In this chapter, a brief review of some of the ideas and concepts in nonlinear optics will be presented. Firstly, the origin of nonlinear behavior of optical materials is discussed. This leads to a discussion of the self-focusing and self-defocusing character of nonlinear media. Next, a discussion of optical solitons and their origins is given, followed by a description of the confinement of light in nonlinear waveguides. Finally, the organization of the thesis is presented.

1.1 Nonlinear Optical Media

In a general nonlinear medium each component of the polarization density vector \( \mathbf{P} = \{P_1, P_2, P_3\} \) is a function of the components of the electric field vector \( \mathbf{E} = \{E_1, E_2, E_3\} \). The relationship between \( \mathbf{E} \) and \( \mathbf{P} \) can be written as:

\[
P_i = \varepsilon_0 (\chi_{ij}E_j + \chi^{(2)}_{ijk}E_jE_k + \chi^{(3)}_{ijkl}E_jE_kE_l), \quad i,j,k,l = 1,2,3.
\]  

(1.1)

where \( \varepsilon_0 \) is the permittivity of free space. The coefficients \( \chi_{ij} \), \( \chi^{(2)}_{ijk} \) and \( \chi^{(3)}_{ijkl} \) are tensors which are closely related to the Pockels and Kerr tensors. The first tensor describes the usual linear susceptibility, and the second and third ones define quadratic and cubic nonlinear susceptibilities, respectively.

There are a number of nonlinear processes that are associated with media characterized by the second order (quadratic) nonlinearity. These processes include second-harmonic generation, frequency conversion, parametric amplification and parametric oscillation. Media characterized by third-order nonlinearity exhibit third-harmonic generation, self-phase modulation, self-focusing, self-defocusing, four-wave mixing, optical amplification, and optical phase conjugation. The detailed discussion of these effects is beyond the scope of the present chapter and may be found elsewhere (see for example [4] for a discussion of these effects). Nevertheless, a brief discussion of self-(de)focusing will be given since it is pertinent to the rest of the thesis.
In the case of centrosymmetric materials\(^1\), the \(P - E\) function possesses odd symmetry. The result is that the second order nonlinearity is identically zero and hence the third-order nonlinearity dominates. Unless specified otherwise, in this thesis all the materials are assumed to be centrosymmetric.

### 1.1.1 Physical Origins of Nonlinear Susceptibility

The nonlinear response of optical materials may originate from microscopic or macroscopic causes. Recall that third order nonlinearity manifests itself as a change in the refractive index of the medium. In addition, the total refractive index \(n\) is related to other parameters of the system by the *Clausius-Mossotti relation*\(^2\), i.e.

\[
\frac{n^2 - 1}{n^2 + 2} = \frac{4}{3} \pi \rho \mu. \tag{1.2}
\]

where \(\mu\) is the average polarizability of a molecule and \(\rho\) is the average molecular number density. It follows from (1.2) that nonlinear behavior can be due to (a) changes in \(\rho\) or (b) changes in \(\mu\). The mechanisms associated with (a) are *molecular orientation*, *molecular redistribution*, and *nonlinear electronic polarizability*. On the other hand, (b) is associated with *electrostriction* and *thermal effects* \[^5\].

*Molecular orientation* arises in liquids where a strong applied optical field leads to an alignment of anisotropic molecules. As a result of such molecular orientation the medium becomes anisotropic and the average refractive index increases. *Molecular redistribution* arises due to the fact that the applied field induces a dipole moment in each molecule, which in turn causes extra dipole-dipole forces to occur between molecules in addition to the intra-molecular forces already existing. Consequently, the molecules are "redistributed" in space and this redistribution causes a change in refractive index \[^6\]. *Nonlinear electronic polarizability* applies to any molecule.

\(^1\)Centrosymmetric media are materials with inversion symmetry. The properties of such media are not affected by the transformation \(r \rightarrow -r\).

and originates from a nonlinear distortion of the electronic clouds in molecules. The *electrostrictive* and *thermal effects (heating)* are mechanisms which change the number density of molecules. Thermal effects often produce self-defocusing behavior and are more significant when CW beams are used, whereas electrostriction often leads to self-focusing.

It should also be borne in mind that the nonlinear behavior of matter in the presence of an electromagnetic field is not an instantaneous effect. Rather, there is a time delay associated with the response of the medium. Of the aforementioned causes for the nonlinear response the fastest is that of electronic polarizability (of the order of \(10^{-15}\) seconds) and the slowest is associated with thermal nonlinearity (of the order of \(10^{-1}\) seconds). For most theoretical investigations, however, the assumption of instantaneous response is a good approximation which allows us to concentrate on the basic nonlinear phenomena without the added algebraic complexity.

### 1.1.2 Self-Focusing and Self-Defocusing

**Kerr Nonlinearity**

Consider the propagation of a monochromatic beam of light of frequency \(\omega\) in a centrosymmetric and isotropic third order nonlinear medium. The nonlinear part of the polarization is written as \([7,8]\)

\[
P^{NL}(\omega) = \varepsilon_0 \left( A + \frac{1}{2} B \right) E^2(\omega) E^*(\omega) \exp(ikr) \tag{1.3}
\]

where asterisk represents complex conjugation and

\[
A = 3 \left( \chi^{(3)}_{1122}(\omega, \omega, \omega, -\omega) + \chi^{(3)}_{1212}(\omega, \omega, \omega, -\omega) \right), \tag{1.4a}
\]

\[
B = 6 \chi^{(3)}_{1221}(\omega, \omega, \omega, -\omega). \tag{1.4b}
\]
Far from any resonant behavior, Kleinman symmetry applies, i.e.

$$\chi^{(3)}_{1122} = \chi^{(3)}_{1212} = \chi^{(3)}_{1221} = \frac{1}{3} \chi^{(3)}_{1111}. \quad (1.5)$$

Using (1.5), (1.3) reduces to

$$P_{AL}(\omega) = 3 \varepsilon_0 \chi^{(3)}_{1111}(\omega, \omega, -\omega) |E(\omega)|^2 E(\omega) \exp(ik\cdot r). \quad (1.6)$$

Equation (1.6) represents an incremental change in the susceptibility at frequency \(\omega\), namely

$$\Delta \chi = \frac{P_{NL}}{E(\omega) \exp(ik\cdot r)} = 3 \varepsilon_0 \chi^{(3)}_{eff} |E(\omega)|^2 = \frac{6 \chi^{(3)}_{eff} I}{c \varepsilon_0 n_0}, \quad (1.7)$$

where \(I = \frac{1}{2} c \varepsilon_0 n_0 |E(\omega)|^2\), \(n_0\) is the linear part of the refractive index and \(\chi^{(3)}_{eff} = \chi^{(3)}_{1111}(\omega, \omega, -\omega)\). Using the relationship \(n_0^2 = 1 + \chi\), the change in refractive index is found to be \(\Delta n_0 = \frac{\partial n_0}{\partial \chi} \Delta \chi = \frac{\Delta \chi}{2n_0}\). It readily follows that

$$n = n_0 + n_2 I \quad (1.8)$$

where \(n_2 = \frac{3 \chi^{(3)}_{eff}}{c \varepsilon_0 n_0^2}\). Equation (1.8) shows that the overall refractive index is a linear function of intensity. This relationship, which resembles the electro-optic Kerr effect, is known as the optical Kerr effect.

An important consequence of the intensity-dependent refractive index is that the optical properties of the medium can be altered locally by intense laser beams. Consider for example a laser beam with a Gaussian transverse distribution. Since the intensity of the beam varies across the transverse dimension, the refractive index change induced in the medium will be different in different parts of the beam. This results in a nonuniform phase-shift across the wavefront of the beam thereby causing a wavefront curvature. The sign of this curvature indicates focusing or defocusing of the beam. In the media where the change in refractive index is positive, focusing occurs and when the refractive index change is negative the beam defocuses. Since focusing or defocusing of the beam are induced by the beam itself, these phenomena
are termed as self-focusing and self-defocusing.

Consider a Kerr nonlinear medium. When \( n_2 > 0 \) (i.e. self-focusing) the wave-front distortion in the high-intensity part of the beam forces the beam to contract. Consequently the power increases, which strengthens the nonlinear effects. This in turn causes further focusing to occur. In principle, this process can continue until the beam collapses catastrophically to a singular point. However, in practice, the diffraction of the beam sets a lower limit for the beam diameter. As a result, the beam diameter either remains constant or oscillates as the beam propagates. Also, as the power of the beam increases, the higher order terms in the \( P - E \) function become more significant. This leads to the saturation of nonlinearity which causes self-filamentation of the beam.

A totally different scenario occurs in the case of \( n_2 < 0 \) (i.e. self-defocusing). In this case the deformation of the wavefront causes the beam to diverge, which results in the expansion of the beam. Owing to this expansion, the intensity of the beam decreases and nonlinear effects are lessened.

Before closing this section, it should be mentioned that the effects described thus far are based on the assumption that the susceptibility tensors are real-valued. In general, however, these tensors are complex-valued. As a result, various other effects such as loss or gain, two-photon absorption, Raman gain, etc., do exist in practical situations.

### 1.2 Self-Trapping of Optical beams

#### 1.2.1 Solitons and Solitary Waves

The earliest report of the observation of a self-guided wave goes back to 1844 [9]. It was reported that a wave of water can propagate over long distances and keep its original shape. An interesting characteristic of these waves was their robustness in how they survived collisions, i.e. after collisions they emerged unchanged. In 1965 Zabusky and Kruskal [10] solved the Korteweg-deVries (KdV) equation numerically
and showed that its solution exhibited this character. Since this behavior (surviving the collisions) is very much like two particles undergoing elastic collision, they used the name "soliton" for these waves. Later, Gardner et al. [11], successfully solved the KdV equation exactly for a localized initial condition using the inverse scattering transform. They found that soliton solutions correspond to the bound states of a Schrödinger operator, which mathematically demonstrated the particle-like behavior of solitons.

1.2.2 Solitons in Nonlinear Optics

Spatial Solitons

In 1962 Askaryan predicted that laser beams that propagate in a plasma can be trapped as a result of the balance between the diffraction of the beam and the self-focusing behavior of the plasma [12]. Later, Chiao, Garmire and Townes [13], considered the propagation of a laser beam in an isotropic Kerr medium in both one and two transverse dimensions. They found that in one transverse dimension solitons existed. In this case, as a result of the alteration of the refractive index of the medium, the beam induces a graded-index waveguide in the medium. Under certain conditions, the intensity of the beam has the same spatial distribution as one of the modes of the induced waveguide, which will result in self-guidance of the beam. These beams are often termed as spatial solitons. However, in two transverse dimensions, the beam would undergo catastrophic collapse above the critical self-trapping power.

Mathematically, the self-guiding behavior can be described by the Helmholtz equation:

\[ \nabla^2 E + k^2 n^2 E = 0, \]

(1.9)

where \( k \) is the wavevector and \( n = n_0 + n_2 I \) and \( I = \frac{|E|^2}{2\eta} \). Assuming that \( E = \)

\[ \eta \]

\( \eta \) is known as the impedance of the medium and is defined as \( \eta = \frac{1}{n_0} \sqrt{\frac{\mu_0}{\epsilon_0}} \). The nonlinear coefficient \( n_2 \) is expressed in the units of \( m^2/W \) (MKS units).
\( \psi(x, z) \exp(i\beta z) \), where \( \beta = n_0k \) is the propagation constant, and assuming that the envelope of the wave \( \psi(x, z) \) varies slowly in the \( z \) direction compared with the wavelength, the second derivative with respect to \( z \) can be approximated as

\[
\frac{\partial^2 E}{\partial z^2} \approx \left( 2i\beta \frac{\partial \psi}{\partial z} - \beta^2 \psi \right) \exp(i\beta z)
\] (1.10)

Using this approximation (1.9) becomes

\[
2i\beta \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + k^2 \left[ n^2 - n_0^2 \right] \psi = 0
\] (1.11)

Owing to the fact that the nonlinear effect is small, we have

\[
n^2 - n_0^2 = (n + n_0)(n - n_0) \approx \frac{n_0n_2}{\eta} |\psi|^2
\] (1.12)

Incorporating (1.12) into (1.11), we obtain

\[
i\beta \frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + k^2 \frac{n_0n_2}{2\eta} |\psi|^2 \psi = 0
\] (1.13)

Equation (1.13) is known as nonlinear Schrödinger equation (NLS).

If the medium under consideration is self-focusing \((n_2 > 0)\), NLS admits a hyperbolic secant solution (bright soliton). On the other hand, if the medium is self-defocusing \((n_2 < 0)\) NLS admits a hyperbolic tangent solution which is known as a dark soliton\(^1\).

Recently, it has been shown theoretically and experimentally that spatial solitons can exist in photorefractive materials [14–17]. In the case of photorefractive materials, self-trapping of light occurs as a result of a balance between diffraction and self-scattering of the plane-wave components of the soliton beam. Experimental observation of these solitons has revealed that they may evolve from an arbitrary input, and self-trapping may occur in two transverse dimensions [15,18].

\(^1\)For a review of literature on dark solitons see the article Yu. Kivshar, IEEE J. Quantum Electron., QE-29, 250 (1993).
**Temporal Solitons**

Propagation of a pulse of light in a linear dispersive medium results in shape alteration due to the fact that the constituent frequency components travel at different group-velocities. This effect is known as group-velocity dispersion. In nonlinear media, there is an additional effect of self-phase modulation which changes the phase of the different parts of the beam by unequal amounts. The signs and magnitudes of these two effects govern the pulse compression or expansion. When self-phase modulation is fully counteracted by group-velocity dispersion the pulse propagates without any change in its profile. These solitons are called *temporal solitons*. Hasegawa and Tappert were the first to predict the existence of temporal solitons (both bright and dark) in optical fibers [19,20]. The first experimental demonstration of temporal solitons in optical fibers was reported in Ref. [21].

Although the physical origins of temporal and spatial solitons are different, NLS describes the propagation characteristics of both types. In fact by simply replacing $x$ in (1.13) with $t$ and reinterpreting the coefficients, one obtains the equation for the propagation of a temporal soliton in an optical fiber. As a result the soliton solution for both time and space domains are identical in form (i.e., hyperbolic secant or hyperbolic tangent). However, it should be noted that the derivation of the NLS in time domain is totally different from that in space (for a detailed derivation of NLS in time domain see [22,23]).

A point also should be made with regard to the definition of solitons and solitary waves. Solitary wave refers to a wave which is resistant to perturbations which often disturb other waves. On the other hand, soliton emphasizes the particle-like characteristics of solitary waves, i.e. solitary waves that maintain their original shape after collision. This definition of solitons has arisen from the early work on the KdV equation. In the case of the NLS, it has been shown that it is integrable [24] and therefore solitary wave solutions of the NLS exhibit particle-like behavior. However, in recent years the definition of soliton has been expanded so that it refers to any nonlinear solitary wave which is relatively stable, even if the equation that
is describing the wave is not integrable\textsuperscript{1}.

\subsection{Nonlinear Guided Waves}

The past several years have witnessed an explosion of interest in the subject of nonlinear guided waves owing to the potential of nonlinear waveguides in optical devices. Due to their inherent confinement of light in one dimension, slab waveguides provide the best means to perform efficient nonlinear interactions, particularly nonlinear optical signal processing. All nonlinear guided wave devices function on the basis of the fact that the wavevector depends on the local intensity of the guided wave; namely, when the refractive indices of one or more layered media are intensity-dependent the field and propagation constant can depend on the power.

There are two general categories of nonlinear optical devices that can be identified, based on the aforementioned phenomena. The first category comprises of those devices where nonlinear change in the refractive index is small compared with the difference between the refractive indices of the guiding media. In this case the coupled-mode theory can be used to evaluate the dependence of the wavevector on the power flow; the field profile can be approximated using the linear modes of the waveguide. The nonlinear distributed coupler [25,26] and the nonlinear directional coupler [27] are examples of devices that operate in this regime. The second category consists of devices where the induced refractive index change is comparable with or larger than the difference of refractive indices of the guiding media. This results in the dependence of the field distribution and wavevector on power. In order to investigate this dependence the nonlinear wave equation must be solved subject to the continuity of the tangential electric and magnetic fields across the interfaces. Seaton \textit{et al.} [28] have suggested optical switching devices which operate in this regime.

\footnote{See for example A. Hasegawa, \textit{Optical Solitons in Fibers}, Springer-Verlag, Berlin, 2nd ed. 1989, page 1.}
1.3 Thesis Organization

In this thesis a variety of nonlinear optical phenomena are investigated. The investigations encompass phenomena that occur in bulk media, waveguides and couplers.

1 In Chapter 2 the stability of dark solitons in two and three dimensions under the influence of such effects as two-photon absorption and gain is investigated.

2 Chapter 3 addresses the stability of vortex-type solitary waves in self-focusing saturable media.

3 In Chapter 4 we consider the problem of finding and characterizing the stationary solutions (modes) of waveguides with a self-defocusing core and self-defocusing cladding. These are the modes that belong to the second category of nonlinear guided waves described in 1.2.3.

4 Chapter 5 considers the stability of nonlinear guided-waves in waveguides composed of a linear core and self-focusing cladding. The reported discrepancy in the literature between numerical and analytical stability of the asymmetric modes of this structure is addressed and resolved. Also, the existence of quasi-periodic modes arising from the perturbation of asymmetric modes are demonstrated.

5 Chapter 6 is concerned with the first category of nonlinear guided waves described in 1.2.3, i.e. the modes of the nonlinear directional coupler. In this chapter, nonlinear couplers with saturable nonlinearity are examined. Using the mode diagrams and phase portraits the modes of these couplers are characterized and classified. In addition, the switching properties of these devices in both passive, lossy and active media are investigated.

6 Chapter 7 provides the summary and conclusions for the thesis.
Chapter 2

Black Solitons in Two and Three Dimensions

2.1 Preamble

In the previous chapter, the main characteristics of solitons in passive, lossless uniform media were described. In practice, however, material loss may be unavoidable. This is particularly the case when the nonlinearity is associated with two-photon absorption (TPA). This occurs when one attempts to utilize the enhancement of the nonlinear Kerr coefficient near two-photon resonances in nonresonant nonlinearities [29,30]. Indeed, fundamental black spatial solitons have been observed in ZnSe, which has a defocusing nonlinear component of the nonlinear refractive index at wavelength $\lambda = 532$ nm resulting from the dispersive change associated with two-photon absorption [31,32]. On the other hand, self-trapping may occur at Raman Stokes frequencies [33,34]. Assuming that pump depletion is negligible, the self-guided black beams (or black pulses) will experience a constant gain.

Accordingly, it is necessary to examine the effect of TPA and Raman amplification on the propagation of black solitons in both (1+1) and (2+1) dimensions. Section 2.2 is devoted to the study of these effects on the (1+1)-dimensional black solitons. Firstly, an approximation for the width and amplitude of the (1+1)-dimensional black soliton, based on the adiabatic evolution, i.e. assuming that
TPA and gain are small, is obtained and compared with the numerical calculations of these quantities using the Beam Propagation Method\(^1\). Also, an exact expression for the stationary propagation of black solitons in the presence of both Raman gain and TPA is derived. In section 2.3 a Hamiltonian approach is employed to investigate the adiabatic evolution of the (2+1)-dimensional black solitons in the presence of TPA and gain. In addition, stationary solutions in the presence of gain and TPA are explored.

### 2.2 Black Solitons in (1+1) Dimensions

#### 2.2.1 Adiabatic Evolution

The starting point in the analysis is a modified version of the nonlinear Schrödinger equation [see Equation (1.13)] in dimensionless form which includes the effects of Raman gain and TPA:

\[
\iota \frac{\partial e}{\partial Z} + \frac{\partial^2 e}{\partial X^2} + (-1 + i\alpha_2)|e|^2 e - i\alpha e = 0, \tag{2.1}
\]

where the normalized field amplitude \(e\) is related to the actual field amplitude \(E\) by \(e = \sqrt{n_2}E\exp(-ikn_0z)\), \(\alpha_2 = \frac{\kappa_2}{2kn_2}\) represents the normalized TPA coefficient, and \(\alpha = \frac{g}{2kn_2}\) denotes the normalized Raman gain contribution with no depletion. [See Appendix A for a detailed description of TPA coefficient \((\kappa_2)\) and Raman gain \((g)\)]. \(X = kx\) and \(Z = \frac{kz}{n_0}\) represent the normalized transverse direction and propagation direction, respectively. The refractive index profile for the medium under consideration is given by

\[
n^2 = n_0^2 - n_0n_2|E|^2 + i\alpha_2n_0n_2|E|^2 - i\alpha n_0 \tag{2.2}
\]

---

\(^1\)This numerical scheme is based on the split-step Fourier method which has been described by in the article M. D. Feit and J. A. Fleck, *Appl. Opt.* 7, 3990 (1978). For both (1+1)-dimensional and (2+1)-dimensional cases a Gaussian background, whose FWHM was 100 times that of the black soliton itself, was added. The step size was taken to be \(\Delta z = 0.01\) where \(z\) is dimensionless. The (1+1)-dimensional simulations were done with 1024 sampling points. In the case of (2+1)-dimensional simulations a Cartesian mesh of 1024 x 1024 was used.
where $n_0$ is the linear refractive index of the medium and $k$ is the wave number in free space. In the ideal case of $\alpha = \alpha_2 = 0$, as is well known, (2.1) admits stationary solutions - black soliton solutions [36]

$$e = A \tanh(AX) \exp(-iA^2Z)$$

(2.3)

where $A$ is an arbitrary constant and the power of the beam $\int |e|^2 dX$ is one of the invariants of the system.

The existence of absorption and/or gain in the medium inhibits the stationary propagation of black soliton. That is, the width and the amplitude of the soliton will change during propagation. To address the question of the evolution of black solitons in a medium with gain and/or TPA we solve (2.1) using the Beam Propagation Method. It is found that a black soliton indeed evolves adiabatically in the perturbed media when the absorption and/or gain are weak. This means that a black soliton is stable enough to withstand a change of the induced guiding structure due to the presence of small gain and/or loss. However, a deformation results when perturbations are large. Therefore, for a small perturbation, we assume the field profile of an adiabatically evolving black soliton to be of the form

$$e(X, Z) = A(Z) \tanh \left( \frac{X}{W(Z)} \right) \exp \left[ -i \int_0^Z A^2(Z) dZ \right].$$

(2.4)

Substituting (2.4) into (2.1) and collecting real and imaginary terms, the following equations are obtained:

$$A'(Z) + \alpha_2 A^3(Z) - \alpha A(Z) = 0$$

(2.5a)

$$\frac{A(Z)}{W^2(Z)} = A^3(Z).$$

(2.5b)

1The adiabatic approximation assumes that all changes in profile occur over such large distances that there is negligible change in the power of the local mode. See page 409 of Ref. [141] for more details on adiabatic approximation.
Solving (2.5a) the amplitude $A(Z)$ is found to be given by

$$A(Z) = \frac{A(0) \exp(\alpha Z)}{\sqrt{1 + \frac{A^2(0)\alpha_2}{\alpha}[\exp(2\alpha Z) - 1]}}$$

and it readily follows from (2.5b) that the width is inversely proportional to the amplitude, namely:

$$W(Z) = \frac{1}{A(Z)}. \quad (2.7)$$

We note that in the absence of gain, (2.6) reduces to

$$A(Z) = \frac{A(0)}{\sqrt{1 + 2A^2(0)\alpha_2 Z}}, \quad (2.8)$$

but when TPA is nonexistent, it simplifies to

$$A(Z) = A(0) \exp(\alpha Z). \quad (2.9)$$

Physically, Equations (2.8) and (2.9) show how the amplitude varies as the beam propagates in the presence of TPA and Raman gain respectively. It has been shown that a bright soliton evolves in an active medium with its amplitude increasing (and beam width narrowing) at the rate of $\exp(\pm 2\alpha Z)$ [33-35]. However, here for a black soliton, we find that the amplitude increases and the width $W$ shrinks at the rate of $\exp(\pm \alpha Z)$, much slower than that for a bright soliton. Also, in a medium with only TPA present, (2.8) shows that an adiabatically evolving black soliton expands and its amplitude decreases at the rate of $[1 + 2A^2(0)\alpha_2 Z]^{\pm \frac{1}{2}}$, slower than that of its bright counterpart, the width and amplitude of which vary at the rate of $[1 + 2.7A^2(0)\alpha_2 Z]^{\pm \frac{1}{2}}$ [37]. This indicates that a black soliton is less sensitive to small perturbations than a bright soliton, i.e., the former is more stable than the latter. Physically, this difference originates from the fact that bright solitons are rather sensitive to perturbations since a small change in soliton power leads to a large change in the soliton intensity due to the feedback caused by the inverse relationship between soliton width and soliton intensity. This feedback does not
occur in the case of black (or dark) solitons [38].

To address the question of the accuracy of the approximation, we compare the analytical relations [Equations (2.6) and (2.7)] with numerical solutions of (2.1). The results of these comparisons are shown in Figure 2-1 where in (a) only Raman gain is present, in (b) only TPA is present and in (c) both Raman gain and TPA are present. The values used in Figure 2-1 for $\alpha$ and $\alpha_2$ are chosen to investigate the behavior of black solitons when the perturbation caused by gain and/or TPA is small. To this end, these values are more of theoretical interest than practical. However, it should be mentioned that these small values are realistic for some glasses with enhanced nonlinearities (see Ref. [37]). In all three cases, the amplitude of adiabatically evolving solitons obtained from the analytical estimation (represented by the dotted curves in the figure) agrees very well with that obtained from numerical solutions (denoted by the solid curves in the figure). It should be noted, however, the approximation for the width [see Equation (2.7)] slightly overestimates the broadening of black solitons traveling in the medium associated with TPA [Figure 2-1(b)] or the narrowing of the black solitons propagating in the active medium [Figure 2-1(a)]. Figure 2-1(c) shows an interesting situation, where according to (2.6) and (2.7) the beam width is predicted to remain unchanged [as indicated by the dotted straight line at $W(Z) = 1$ in the figure] while the numerical solution shows a slight variation of the width near $W(Z) = W(0) = 1$ during the evolution, i.e., intensity dependent two-photon absorption is almost balanced by constant Raman gain in this case.
Figure 2-1: Evolution of black soliton intensity $A^2(Z)$ and width $W(Z)$ along the normalized distance $Z$. (a) $\alpha = 0.01$ and $\alpha_2 = 0$ (b) $\alpha = 0$ and $\alpha_2 = 0.01$ and (c) $\alpha = 0.01$ and $\alpha_2 = 0.01$. The solid curves denote numerical results and the dotted curves represent analytical results obtained from (2.6).
2.2.2 Stationary Solutions in the Presence of Absorption and Gain

In the previous section we saw how a black soliton narrows (broadens) in the presence of Raman gain (TPA). We also observed that the simultaneous existence of Raman gain and TPA (equal coefficients) can partially counter-balance each other. Naturally, a question arises as to whether two-photon absorption can be completely counteracted by gain to lead to stationary propagation of black solitary waves in the perturbed media.

To investigate the existence of such a stationary solution we assume that it can take the following form:

\[ e(A', Z) = f(X)\exp(i\beta Z + i\phi(X)) \]  

(2.10)

where \( f(X) \) and \( \phi(x) \) are real functions representing the amplitude and phase of the solution and \( \beta \) denotes the propagation constant. Substituting (2.10) into (2.1) and grouping the real and imaginary parts of the resulting equation leads to:

\[ -\beta f(X) - f(X)[\phi'(X)]^2 + \frac{f''(X)}{2} - f^3(X) = 0 \]  

(2.11a)

\[ f'(X)\phi'(X) + \frac{f(X)\phi''(X)}{2} + \alpha f^3(X) - \alpha f(X) = 0 \]  

(2.11b)

Since we are interested in black solitary solutions of (2.1), \( e(X, Z) \) must behave like a black soliton, namely the intensity must asymptotically approach a constant value at \( \pm\infty \), and \( e(0, 0) = 0 \). Hence, the amplitude function \( f(X) \) can be written as \( f(X) = A\tanh(\nu X) \), where \( A \) and \( \nu \) are constants to be determined. Substituting this expression for \( f(X) \) into (2.11) and solving the resulting differential equations we find that \( \phi(X) = \gamma \ln(\text{sech}(\nu X)) \) and hence the general solution is:

\[ e(X, Z) = A\tanh(\nu X)\exp(i\beta Z + i\gamma \ln(\text{sech}(\nu X))) \]  

(2.12)
where \( A = \sqrt{\frac{\alpha}{\alpha_2}}, \nu = \sqrt{-\beta}, \beta = \frac{2\alpha}{3\gamma}, \) and \( \gamma = \frac{3}{2\alpha_2} - \sqrt{\frac{9}{4\alpha_2^2} + 2}. \) Figure 2-2 illustrates the variation of the width \( W = \frac{1}{\nu} \) and \( \beta \) as the TPA coefficient \( \alpha_2 \) is varied when \( A = 0.5, 1, \) and \( 1.5. \) As is shown, for a given \( A, \) the absolute value of the propagation constant \( |\beta| \) increases with increasing \( \alpha_2 \) while the width \( W \) decreases with increasing \( \alpha_2. \) On the other hand, \( |\beta| \) increases but \( W \) diminishes with increasing \( A \) for a given TPA coefficient \( \alpha_2. \) Therefore, we conclude that gain can completely counter-balance two-photon absorption which leads to stationary propagation of black solitary waves in the perturbed medium.

\[ Figure 2-2: \text{Variation of (a) the propagation constant } \beta \text{ and (b) the width } W \text{ of the black solitary waves in perturbed media versus TPA coefficient } \alpha_2 \text{ for the values } A = 0.5, 1 \text{ and } 1.5. \]

### 2.3 Black Solitons in (2+1)-Dimensions

In 1961 Pitaevskii, while studying superfluidity and superconductivity, discovered the stationary vortex phenomena [39]. The equation describing this phenomena is mappable to a (2+1)-dimensional nonlinear Schrödinger equation with Kerr self-defocusing nonlinearity. An important characteristic of the solutions of this equation is that their phase profile is that of a hydrodynamical vortex, i.e. a \( 2\pi \) helical phase.
ramp [see Equation (2.16)]. This is why the term optical vortex solitons [40–42] is used to describe them. In the following, we will adopt this terminology.

In recent years seminal theoretical and experimental advances have been made in the study of optical vortex solitons. On the theoretical side, various approaches have been used to conclude the existence of optical vortex solitons. Chiao et al. [40] based their analysis on the superfluid analogies of photonics. Snyder et al. [43] independently utilized the linear waveguide theory to argue for the existence of vortex solitons. McDonald et al. [44] used a superfluid analogy to describe the equations of motion of vortex solitons. Experimental observation of vortex solitons have been reported by Swartzlender and Law [41,42].

Like their (1+1)-dimensional counterparts, vortex solitons are also susceptible to the effects of TPA, material loss, and gain. Therefore, it is necessary to investigate how these effects influence the propagation characteristics of vortex solitons. Since there is no exact explicit analytical solution for vortex solitons, first we utilize a Hamiltonian approach to find an approximate analytical solution for these solitons.

### 2.3.1 Approximate Analytical Solution for Vortex Solitons

**A. The Wave Equation**

Assuming that the refractive index of the perturbed medium obeys (2.2) and that the electric field is of the form $E = \Psi(X, Y, Z) \exp(i \beta Z)$, the nonlinear Schrödinger equation in Cartesian coordinates can be written as follows:

$$
2i\beta \frac{\partial \Psi}{\partial Z} + \frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} + \left(k^2 n_0^2 - \beta^2\right) \Psi + (-1 + i\alpha_2)k^2 n_0 n_2 |\Psi|^2 \Psi - ik^2 \alpha n_0 \Psi = 0.
$$

(2.13)

where $\beta(< kn_0)$ is the propagation constant, $n_0$ is the linear part of the refractive index, $n_2$ is the nonlinear coefficient and $k$ is the wavenumber. In dimensionless form (2.13) reads

$$
\frac{i}{\partial z} \frac{\partial \psi}{\partial z} + \nabla_T^2 \psi + \gamma^2 \psi + (-1 + i\alpha_2)|\psi|^2 \psi - i\alpha \psi = 0
$$

(2.14)
where $\psi = \sqrt{n_2} \psi$, $z = \frac{Z k^2 n_0}{2\beta}$, $x = kX \sqrt{n_0}$ and $y = kY \sqrt{n_0}$, and $\gamma^2 = \frac{k^2 n_0^2 - \beta^2}{k^2 n_0}$.

In cylindrical geometry the Laplacian is written as

$$\nabla_T^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}. \quad (2.15)$$

It is well-known from hydrodynamics that a single vortex solution of (2.14) is written

$$\psi(r, \phi, z) = e(r) \exp(\pm i\phi). \quad (2.16)$$

Substituting (2.16) into (2.14) yields

$$i \frac{\partial e}{\partial z} + \frac{\partial^2 e}{\partial r^2} + \frac{1}{r} \frac{\partial e}{\partial r} - \frac{e}{r^2} + \gamma^2 e + (-1 + i\alpha_2)|e|^2e - i\alpha e = 0 \quad (2.17)$$

**B. The Hamiltonian and Approximate Solution**

In the absence of TPA ($\alpha_2 = 0$) and gain or loss ($\alpha = 0$), (2.17) reduces to

$$i \frac{\partial e}{\partial z} + \frac{\partial^2 e}{\partial r^2} + \frac{1}{r} \frac{\partial e}{\partial r} - \frac{e}{r^2} + \gamma^2 e - |e|^2e = 0. \quad (2.18)$$

This equation has been shown numerically to admit optical vortex soliton solutions [47], as is illustrated by the solid curve in Figure 2-3.

Since the system under consideration is a conservative one, it will have invariants, i.e. quantities that are constant during propagation. One such invariant is the Hamiltonian of the system\(^1\)

$$H = \int_0^\infty \left[ \frac{|de|^2}{dr} + \frac{|e|^2}{r^2} + \frac{1}{2}(|e|^2 - \gamma^2)^2 \right] r dr. \quad (2.19)$$

In addition to being an invariant, the Hamiltonian $H$ is stationary with respect to small perturbations about the soliton solution. This is equivalent to the statement

\(^1\)The Hamiltonian can be derived by multiplying (2.18) by $\frac{de^*}{dz}$ and its complex conjugate by $\frac{de}{dz}$ and adding the resulting equations. Integrating this equation with respect to $r$ yields (2.19).
of the variational principle namely, $\delta H = 0$. This property of the Hamiltonian provides a way to obtain an approximate analytical expression for the optical vortex soliton.

Since the one-dimensional analog of (2.18) admits a hyperbolic-tangent solution, we use

$$e = A \tanh \left( \frac{r}{r_0} \right)$$  \hspace{1cm} (2.20)

as a trial function. The boundary condition at infinity requires $A = \gamma$ (which can be varied by changing the propagation constant $\beta^2$ in $\gamma^2 = \frac{k^2 n_0^2 - \beta^2}{k^2 n_0}$). Substitution of (2.20) into (2.19) yields

$$H = \gamma^2 \left[ \left( 1 + \frac{\gamma^2 r_0^2}{2} \right) \frac{2 \ln 2 - 0.5}{3} - \ln \frac{\gamma r_0}{\sqrt{2}} + H_0 \right].$$  \hspace{1cm} (2.21)

where $H_0$ is the constant of integration independent of $r_0$. The variation of $H$ with
respect to \( r_0 \) gives \( r_0 = \frac{1.301\sqrt{2}}{\gamma} \). Thus the hyperbolic-tangent approximation of the vortex soliton is 
\[ e = \gamma \tanh \left( \frac{\gamma r}{1.301\sqrt{2}} \right) \]. As is shown by the dashed curve in Figure 2-3, this solution is in excellent agreement with the numerical solution (the solid curve). It should be mentioned here that Swartzlander and Law [41], using numerical methods, produced an expression of the hyperbolic-tangent function to approximate the vortex soliton with the width \( r_0 = \frac{1.270\sqrt{2}}{\gamma} \). This value of \( r_0 \) is 2.4\% smaller than that obtained by the variational approach.

Another possible candidate to approximate vortex soliton solution is a Gaussian trial function [45], i.e.
\[ e^2 = \gamma^2 \left[ 1 - \exp \left( -\frac{r^2}{r^2_0} \right) \right]. \tag{2.22} \]

The Hamiltonian for this ansatz is found to be
\[ H = \gamma^2 \left( I - \ln \gamma r_0 + \frac{\gamma^2 r^2_0}{8} + H_0 \right) \tag{2.23} \]
where \( I = \frac{1}{2} \int_0^\infty \frac{x \exp(-x)}{\exp(x) - 1} \, dx \) with \( x = \frac{r^2}{r^2_0} \). Variation of (2.23) gives \( r_0 = \frac{2}{\gamma} \). This approximation, as is illustrated in Figure 2-3 by the dotted curve, is in good agreement with the numerical solution, but it is not as good as the hyperbolic-tangent case.

The accuracy of the above approximate solutions is reflected in the value of the Hamiltonian. It is found that the value of \( H \) calculated for the hyperbolic-tangent (Gaussian) approximation differs from that of the numerical solution only by 0.4\% (1.2\%) where the integration in (2.19) has been performed from \( r = 0 \) to \( r = r_{max} = 11 \). This error decreases as \( r_{max} \) increases.

### 2.3.2 Adiabatic Evolution of Vortex Solitons in Perturbed Media

In the unperturbed system discussed above the beam width \( r_0 \) and the background amplitude \( A \) of the vortex soliton are constants. However, the presence of pertur-
bations such as TPA, gain or loss would cause these quantities to vary with the propagation distance. It was shown in 2.3.1 that a hyperbolic-tangent function best describes a vortex soliton in the unperturbed media. Accordingly, we assume that in the presence of TPA and gain the vortex soliton can be expressed as:

\[ e = A(z) \tanh \left( \frac{r}{r_p(z)} \right) \]  

(2.24)

Substituting (2.24) into (2.17) and following the procedure that was outlined for the case of the (1+1)-dimensional case we obtain

\[ A(z) = \frac{A(0) \exp(\alpha z)}{\sqrt{1 + \frac{A^2(0) \alpha_2}{\alpha} [\exp(2\alpha z) - 1]}} \]  

(2.25a)

\[ r_p(z) = \frac{\sqrt{1 + [\exp(2\alpha z) - 1] 3.385\alpha_2}}{r_p(0) \exp(\alpha z)} \]  

(2.25b)

The corresponding \( H \) varies with \( z \) according to \( H(z) = \frac{H(0) A^2(z)}{A^2(0)} \). In Figure 2-4 we compare (2.25) with the numerical solutions of (2.17) for four distinct cases where (a) only constant loss is present, (b) only TPA is present, (c) only constant gain is present and (d) both TPA and gain (\( \alpha = \alpha_2 = 0.01 \)) are present. In all these cases the analytical results (solid curves) are in good agreement with those from numerical simulations (dotted curves). As a further illustration of the accuracy of the analytical approximation, in Figure 2-5 the field profiles obtained from (2.25) at \( z = 10 \) are compared with those from numerical simulations. The agreement between the field profiles are very good. Physically, this means that vortex solitons are stable enough against small perturbations imposed by TPA, loss or gain and they can resist a change in the induced guiding structure due to the presence of such perturbations. Therefore, We conclude that vortex solitons, like their (1+1)-dimensional counterparts, evolve adiabatically in the perturbed media.
Figure 2-f: Evolution of the background intensity and width of the optical vortex soliton in a perturbed self-defocusing Kerr medium; (a) constant loss only where $\alpha = -0.01$ and $\alpha_2 = 0$, (b) two-photon absorption only where $\alpha = 0$ and $\alpha_2 = 0.01$, (c) constant gain only where $\alpha = 0.01$ and $\alpha_2 = 0$ and (d) two-photon absorption and gain where $\alpha = 0.01$ and $\alpha_2 = 0.01$. The solid curves represent analytical evaluation of (2.25) and the dotted curves denote the numerical simulation results.
2.3.3 Stationary Optical Vortex Solitons in Perturbed Media

In 2.3.2 it was shown that the presence of perturbations such as gain, loss, TPA or combinations thereof leads to nonstationary propagation of optical vortex solitons. In particular, Figure 2-4(d) shows that simultaneous existence of TPA and Raman gain leads to diminished effect of attenuation or amplification. In this section we address the question of the existence of stationary vortex solitons when both TPA and Raman gain are present.

In order to investigate the existence of stationary vortex solitons in a medium with TPA and Raman gain we assume that the solution can be written as

$$e(r) = \psi_1(r) + i\psi_2(r).$$  \hspace{1cm} (2.26)

where $\psi_1$ and $\psi_2$ are real functions. Substituting (2.26) into (2.17) and grouping the
real and imaginary parts leads to

\[
\frac{d^2 \psi_1}{dr^2} + \frac{1}{r} \frac{d \psi_1}{dr} - \frac{\psi_1}{r^2} + \gamma^2 \psi_1 - \left( \psi_1^2 + \psi_2^2 \right) \psi_1 - \alpha_2 \left( \psi_1^2 + \psi_2^2 \right) \psi_2 + \alpha \psi_2 = 0 \tag{2.27a}
\]

\[
\frac{d^2 \psi_2}{dr^2} + \frac{1}{r} \frac{d \psi_2}{dr} - \frac{\psi_2}{r^2} + \gamma^2 \psi_2 - \left( \psi_1^2 + \psi_2^2 \right) \psi_2 + \alpha_2 \left( \psi_1^2 + \psi_2^2 \right) \psi_1 - \alpha \psi_1 = 0 \tag{2.27b}
\]

Solving Equations (2.27) using a shooting method we find that indeed stationary solutions in such perturbed media do exist. Figure 2-6 displays some examples of such stationary solutions where \( \alpha_2 = 0.02 \) and \( \alpha = 0.02, 0.016, 0.012, 0.008, 0.004 \) and \( \gamma^2 \) is varied from 0.2 to 1. On the basis of the numerical solution we find that the background intensity \( A_p^2 \) depends on the values of \( \alpha, \alpha_2 \) and \( \gamma^2 \). In general, we find that for self-trapping \( A_p^2 \neq \gamma^2 \). This is in contrast to the unperturbed system where \( A_p^2 = \gamma^2 \). The difference between \( A_p^2 \) and \( \gamma^2 \) increases with increasing \( \alpha \) and \( \alpha_2 \) and \( \gamma^2 \rightarrow \gamma^2 \) when \( \alpha \rightarrow 0 \) and \( \alpha_2 \rightarrow 0 \).

![Figure 2-6: Intensity profiles of the optical vortex soliton in the perturbed medium with \( \alpha_2 = 0.02 \) and \( \alpha = 0.02, 0.016, 0.012, 0.008, 0.004 \) associated with the values of \( \gamma^2 = 1, 0.8, 0.6, 0.4, 0.2 \).](image-url)
Chapter 3

Stability of Three-Dimensional Bright Ring Solitary Waves

3.1 Introduction

Self-trapping of optical beams in a nonlinear medium has been a subject of interest and investigated extensively over the last three decades both experimentally and theoretically [13,48–61]. In their pioneering works, Chiao et al. [13] and Haus [48] demonstrated that in a Kerr-law nonlinear medium, a cylindrically-symmetric beam with the maximum intensity at the center can be a three dimensional self-trapped beam pattern which in theory remains unchanged with the propagation distance. However, in practice it may not be the case as this family of the bound state solutions is not stable to a symmetric perturbation [49]. Above the critical trapping power the beam focuses, and below the critical trapping power the beam diffracts. In a saturable nonlinear medium (as is often the case in practice), the trapped beams demonstrate quite different stability characteristics. The fundamental state of the family becomes stable (against both symmetric and asymmetric perturbations) [55], whereas the higher-order states are unstable against the perturbation that breaks azimuthal symmetry of the beams (transverse instabilities) [56]. These stability results predicted from linear stability analysis have been confirmed by direct numerical simulations of the nonlinear wave equation [61].
The self-trapped beam patterns discovered in Refs. [13, 48] are not the only family of trapped light patterns (with a linear polarization in the x or y direction which are approximations to the possible trapped $HE_{1m}$ light patterns resulting from Maxwell's equations). There actually exists another cylindrically-symmetric family of self-trapped light pattern characterized by a dark spot surrounded by bright rings of varying intensity (TE-type light patterns) which was first reported in Ref. [62] for the Kerr law nonlinearity, and later in Ref. [63] for a saturable nonlinearity. Similar to the trapped light patterns with maximum intensity at the center, in the Kerr law medium this family of TE self-trapped light patterns is unstable against a symmetric perturbation [64]; the beam focuses or diffracts when the power is greater or smaller than the trapping power. In a saturable nonlinear medium, the fundamental state of this TE family was shown to be stable to symmetric perturbation, based on the optical force argument [63]. However, the question of the stability of this fundamental state to the perturbation that breaks the symmetry of the beam (transverse stability) and the stability of higher-order states has been left unaddressed.

In this chapter we examine the stability (including the transverse or modulation stability) of the TE family of the self-trapped beams both by linear stability analysis and by direct numerical simulations on the wave equation. In section 3.3.1 an operator-theoretic analytical stability analysis method is used to investigate the stability of the fundamental modes of this family. In section 3.3.2 a numerical linear stability analysis is used to calculate the growth rates and perturbation eigenmodes. Finally, in section 3.3.3 the results of numerical simulations of the nonlinear wave equation are presented and discussed.

### 3.2 Stationary Solutions

By and large, the propagation of a light beam in a uniform nonlinear medium follows the vector wave equation [64] which normally involves three field components. Here we are considering a case where there is only one field component present, i.e. linearly polarized monochromatic light. In addition, we assume that the medium is
a saturable self-focusing one which is characterized by

\[ \varepsilon (|E|^2) = n_0^2 + n_0 n_2 \frac{|E|^2}{1 + \frac{n_0 n_2 |E|^2}{n_{sat} - n_0^2}} \] (3.1)

where \( n_0 \) denotes the linear relative permittivity, and \( n_2 > 0 \) signifies the nonlinear coefficient, and \( n_{sat} \) is the maximum value of \( \varepsilon \) in presence of nonlinearity. By writing the electric field in the form

\[ E = \Psi(X,Y,Z) \exp(i\beta Z), \] (3.2)

and invoking the slowly varying approximation, the following equation is obtained for the envelope \( \Psi \):

\[ 2i\beta \frac{\partial \Psi}{\partial Z} + \frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} + (k^2 n_0^2 - \beta^2) \Psi + k^2 n_0 n_2 \frac{|\Psi|^2}{1 + \frac{n_0 n_2 |\Psi|^2}{n_{sat} - n_0^2}} = 0. \] (3.3)

In dimensionless form (3.3) can be written as follows:

\[ i \frac{\partial e}{\partial z} + \nabla^2 e - \gamma^2 e + \frac{e^2}{1 + e^2} e = 0. \] (3.4)

where \( \gamma^2 = \frac{k^2}{n_{sat}^2 - n_0^2} \) (\( \gamma \) plays the role of a scaled effective index for the nonlinear wave), \( x = X k \sqrt{n_{sat}^2 - n_0^2}, y = Y k \sqrt{n_{sat}^2 - n_0^2}, z = Z k \frac{(n_{sat}^2 - n_0^2)}{2\beta} \), and \( e = \Psi \sqrt{\frac{n_0 n_2}{n_{sat}^2 - n_0^2}} \). In cylindrical geometry the Laplacian is given by:

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}. \] (3.5)

We are interested in stationary solutions whose envelope depends on both \( r \) and \( \phi \), namely:

\[ e(r, \phi) = \psi(r) \exp(\pm i\phi). \] (3.6)

As was described in Chapter 2, (3.6) represents vortex-type solutions. Therefore,
one might characterize these stationary solutions as “bright” vortex solitary waves. Substituting (3.6) into (3.4) we obtain

\[ i \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} - \gamma^2 \psi + \frac{\psi^2}{1 + \psi^2 } \psi = 0. \]  

The stationary solutions of (3.7) are obtained by setting \( \frac{\partial \psi}{\partial z} = 0 \) and solving the remaining equation numerically\(^1\). In general, there exists an infinite number of solutions (bound states) identified by \( n = 1, 2, 3, \ldots \). As is shown in Figure 3-1, the \( n^{th} \) bound state is characterized by a dark spot [i.e. \( \psi(0, 0) = 0 \)] surrounded by \( n \) bright rings of varying intensity [63,64].

\[ \text{Figure 3-1: Field profiles corresponding to } \gamma^2 = 0.5. \]

The normalized self-trapped power of the \( n^{th} \) bound state is given by

\[ P = \int |e|^2 dS = 2\pi \int_0^\infty \psi_n^2 r dr. \]  

The power carried by each bound state increases with increasing \( n \). Also, for a fixed

\(^1\)In general this equation cannot be solved analytically. Here, a shooting method has been used to obtain the solutions.
$n$, $P$ increases with increasing $\gamma^2$ as is shown in Figure 3-2 for the fundamental bound state, where $P_c = 15.38\pi$ is the critical trapping power for the Kerr law nonlinearity. Note that the dependence of Power on $\gamma^2$ shown in Figure 3-2 arises from the nonlinear saturation. In other words, saturation of nonlinear susceptibility governs the size of the mode. As $\gamma$ approaches unity the effective index $\frac{\beta}{k}$ approaches $n_{sat}$. Since $n_{sat}$ is the maximum refractive index, it is attained at very high power limit, i.e., as $\gamma \to 1$, $P \to \infty$.

![Figure 3-2: Dependence of the trapping power $P$, of the $n = 1$ bound state of the TE family of self-trapped beams, on $\gamma^2$.](image)

3.3 Stability of the Bound States

3.3.1 Analytical Stability Theory

In order to investigate the stability of a bound state $\psi_n$, we seek a perturbed solution of the form

$$\psi = \psi_n + (u + iv) \exp(\delta z). \quad (3.9)$$
Substituting (3.9) into (3.7) and linearizing leads to

\[
\delta u = -L_{0n}v \quad \text{(3.10a)}
\]

\[
\delta v = L_{1n}u, \quad \text{(3.10b)}
\]

where \( L_{0n} = \nabla^2_r - \frac{1}{r^2} - \gamma^2 + \frac{\psi_n^2}{1 + \psi_n^2} \) and \( L_{1n} = L_{0n} + 2\frac{\psi_n^2}{(1 + \psi_n^2)^2} \). It should be noted that if \((\delta, u, v)\) is a solution to (3.10), \((-\delta, u, v)\) is a solution as well. The linearized equations (3.10) can be decoupled and written in terms of \(u\):

\[
L_{0n}L_{1n}u = -\delta^2 u. \quad \text{(3.11)}
\]

When \(\delta \neq 0\), it can be shown that all the solutions of (3.10) or (3.11) are orthogonal to \(\psi_n\). This result can be derived as follows:

\[
\delta \langle u|\psi_n \rangle = -\langle L_{0n}v|\psi_n \rangle
\]

\[
= -\langle v|L_{0n}\psi_n \rangle
\]

\[
= 0
\]

where \(\langle f_1|f_2 \rangle = \int f_1 f_2^* dS\). Insofar as the solutions with \(\delta \neq 0\) are concerned, (3.11) needs to be solved only in the subspace orthogonal to \(\psi_n\). In this subspace, the inverse operator \(L_{0n}^{-1}\) exists, and for the fundamental bound state the operator \(L_{0n}(L_{0n}^{-1})\) is negative-definite [because all the eigenvalues of \(L_{01}\) or \(L_{01}^{-1}\) are negative, whereas \(L_{0n}^{-1} (n > 1)\) admit positive eigenvalues and are not negative-definite]. Therefore, for the fundamental bound state the eigenvalues \(\delta\) are either real or imaginary and the variational principle can be applied for obtaining the largest value \(\delta^2\) of (3.11),

\[
\delta^2 = \max \frac{\langle u|L_{11}u \rangle}{-\langle u|L_{01}^{-1}u \rangle}. \quad \text{(3.12)}
\]

Owing to the negative-definiteness of \(L_{01}^{-1}\), the denominator of (3.12) is a positive quantity. Hence, the sign of the numerator of (3.12), i.e. \(G = \langle u|L_{11}u \rangle\), determines
whether or not there exists an exponential growth resulting from a perturbation of the fundamental bound state. If \( G > 0 \), real \( \delta > 0 \) exists, implying that the fundamental bound state is unstable. On the other hand, \( G < 0 \) means that \( \delta \) is imaginary and no exponential growth results from a perturbation.

From the method of indeterminate Lagrange multipliers [55] it can be shown that maximization of the quantity \( \langle u \mid L_{11}u \rangle \) in (3.12) is equivalent to solving the equation

\[
L_{11}h = \lambda h + q\psi_1
\]

for the largest eigenvalue \( \lambda \). This eigenvalue and the constant \( q \) are determined by the conditions of orthogonality \( \langle h\mid\psi_1 \rangle = 0 \) and normalization \( \langle h\mid h \rangle = 1 \).

We first consider the case where \( q \neq 0 \) which corresponds to symmetric eigenfunctions \( h \). Expanding \( h = \sum_{m = 1}^{\infty} a_m h_m \) and \( \psi_1 = \sum_{m = 1}^{\infty} c_m h_m \) of (3.13) in the complete orthonormalized set of eigenfunctions \( h_m \) of the operator \( L_{11} (L_{11}h_m = \lambda_m h_m) \) gives rise to

\[
h = q \sum_{m = 1}^{\infty} \frac{c_m h_m}{\lambda_m - \lambda}.
\]

where \( c_m = \langle \psi_1\mid h_m \rangle \). Substituting this expansion for \( h \) into the orthogonality condition \( \langle h\mid\psi_1 \rangle = 0 \), leads to an equation in terms of \( \lambda \)

\[
qq(\lambda) = q \sum_{m = 1}^{\infty} \frac{c_m^2}{\lambda_m - \lambda} = 0.
\] (3.14)

As is shown in Figure 3-3, for symmetric eigenfunctions, \( L_{11} \) admits only one positive eigenvalue \( \lambda_1 \) within \( 0 \leq \gamma < 1 \). Since \( \lambda_m, m = 1, 2, 3, \ldots \) are the poles of (3.14), the largest \( \lambda = \lambda_s \) for the symmetric \( h \) (or \( u \)) must lie between \( \lambda_1 \) and the largest negative eigenvalue of the symmetric eigenfunction. In addition, (3.14) indicates that \( \lambda_s > 0 \) when \( g(0) < 0 \), and \( \lambda_s < 0 \) when \( g(0) > 0 \), and this \( g(0) \) is related to the power by

\[
g(0) = \sum_{m = 1}^{\infty} \frac{c_m^2}{\lambda_m} = \langle \psi_1 \mid L_{11}^{-1}\psi_1 \rangle = \langle \psi_1 \mid \frac{\partial^2}{\partial \gamma^2} \rangle = \frac{1}{2} \frac{dP}{d\gamma^2}
\] (3.15)
where the relation \( L_{11} \frac{\partial \psi_1}{\partial \gamma^2} = \psi_1 \), obtained from differentiating the equation \( L_{01} \psi_1 = 0 \) with respect to \( \gamma^2 \), has been used in the derivation. From Figure 3-2 it is seen that within \( 0 \leq \gamma < 1 \), \( \frac{dP}{d\gamma^2} > 0 \), leading to \( g(0) > 0 \) or \( \lambda_s < 0 \), i.e. the numerator of (3.12) is negative. Therefore no real \( \delta \) exists for a symmetric perturbation, and hence in a saturable nonlinear medium the fundamental bound state \( \psi_1 \) is stable to symmetric perturbations.

![Figure 3-3: Dependence of \( \lambda_1 \) and \( \lambda_2 \) of the \( n = 1 \) bound state of the TE family of self-trapped beams on \( \gamma^2 \).](image)

When \( q = 0 \) in (3.13), \( \lambda = \lambda_m \) and the operator \( L_{11} \) admits asymmetric (\( \phi \) dependent) eigenfunctions which are orthogonal to \( \psi_n \). As is shown in Figure 3-3, the largest eigenvalue \( \lambda_2 = \lambda_{as}(< \lambda_1) \) of the asymmetric eigenfunctions is found to be positive as well. This means that for an asymmetric perturbation, the largest quantity \( \langle u | L_{11} u \rangle \) of (3.12) [or the largest eigenvalue of (3.13)] can be greater than zero, and hence real \( \delta \) exists. In other words, the fundamental bound state is unstable to asymmetric perturbations.
3.3.2 Numerical Linear Stability Analysis

A. The Method

Owing to the particular properties of the operators \( L_{01} \) and \( L_{11} \), the method described in section 3.3.1 is only applicable to the fundamental bound state. In this section we use numerical linear stability analysis as another avenue to investigate stability characteristics of the bound states. In contrast to the above approach, this scheme is applicable to all bound states.

To begin with we assume that perturbation of a general bound state can be written in the form of

\[
\psi(r, \phi, z) = \psi_n(r(z)) + \mu f(r, z) \cos(m\phi)
\]  

(3.16)

where \( \mu \) is small parameter, \( f(r, z) \) is a perturbation function (which, in general, is a complex function), and \( m \), an integer, is the azimuthal index. Substituting (3.16) in (3.7) and linearizing with respect to \( \mu \) leads to

\[
\frac{\partial f}{\partial z} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \left( \gamma^2 + \frac{m^2 + 1}{r^2} \right)f + \frac{2\psi_n^2 f + \psi_n^4 f + \psi_n^2 f^*}{(1 + \psi_n^2)^2} = 0
\]  

(3.17)

which clearly defines an eigenvalue problem. In general, this equation admits many different kinds of solutions. However, we are interested in those solutions whose profiles grow exponentially along the \( z \) direction, i.e.

\[
f(r, z) = f(r) \exp(\delta z).
\]  

(3.18)

The eigenvalues (\( \delta \)) of this problem could be either real or imaginary. The real positive eigenvalues indicate exponential growth of the perturbation which results in instability. On the other hand, imaginary eigenvalues represent stable solutions.

In order to solve (3.17) we have used the method which has been described in the first article of Ref. [61]. A Crank-Nicholson scheme [65] was used to propagate a general non-zero initial condition \( f(r, 0) \) for a given \( \psi_n \) and \( m \). Simultaneously,
the following quantity was calculated at each step of the evolution:

$$\delta_{nm} = \frac{\ln\{\text{Re}[f(r, z + \Delta z)]\} - \ln\{\text{Re}[f(r, z)]\}}{\Delta z}$$

(3.19)

where $\Delta z$ is the numerical propagation step size.

Since $f(r, 0)$ is a general non-zero input there are several perturbation eigenmodes present in the initial stages of the propagation. As a result of the exponential nature of the instability development, for large propagation distances, only the perturbation eigenmode corresponding to the largest growth rate (\(\delta_{nm}\)) will dominate. In other words, this method provides us with a numerical tool to calculate the dominant perturbation eigenmode and its corresponding eigenvalue.

The azimuthal index $m$ is intimately connected with the perturbation eigenmode, in that it represents the resulting symmetry of the perturbation eigenmode. That is to say, if $m = 2$ corresponds to the dominant perturbation eigenmode, we expect the initial cylindrically-symmetric profile to break to a two-fold symmetry.

B. The Results of Numerical Stability Analysis

Applying the above numerical scheme to the $n = 1$ and $n = 2$ modes, we obtain the growth rates and perturbation eigenmodes which are displayed in Figures 3-4, 3-5. As is shown in Figure 3-4(a) over the range $0 < \gamma^2 < 0.84$, $m = 2$ asymmetric perturbation dominates, whereas $m = 1$ ('snake-type') instability prevails when $\gamma^2 > 0.84$. In addition, the numerical calculation showed that the perturbation modes with $m = 0$ (symmetric perturbation or neck-type\(^1\)) have zero growth rate. These results are in complete agreement with the predictions of the analytical approach presented in 3.3.1.

The application of this scheme to higher-order bound states shows that all of

\(^1\)This terminology was used by Zakharov and Rubenchik in Ref. [66]. They referred to $m = 0$ perturbation as 'neck-type' instability and $m = 1$ as 'snake-type' instability. The former is a perturbation which preserves the symmetry of the mode and generates modulations longitudinally. On the other hand, the latter corresponds to instabilities which cause the mode to bend in the transverse direction.
them display transverse instabilities. Figures 3-4(b) and 3-4(c) give the dependence of the growth rates on $\gamma^2$ for the $n = 2$ bound state. For the inner ring, it is found that only perturbation eigenmodes corresponding to $m = 1, 2$ can exist. On the other hand, for the outer ring, the value of $m$ is confined to the range $3 \leq m \leq 7$. It should be noted that the upper bound for growth rates is 0.25. This upper bound can be readily obtained by applying the analytical argument of Ref. [56] to the case under consideration.

Figures 3-5 and 3-6 represent dominant perturbation eigenmodes for $n = 1, 2, 3$ where $\gamma^2 = 0.5$, $\gamma^2 = 0.2$ and $\gamma^2 = 0.35$, respectively. In each case the stationary solution (the solid curve) is also shown for comparison purposes. It is clearly seen from these figures that the perturbation eigenmodes are predominantly concentrated around the peak of the ring(s). This means that modulation around the peak of the ring(s) is the chief contributor to the onset of instability.

### 3.3.3 Numerical Simulation of the Wave Equation

Another channel in pursuing the stability characteristics is direct numerical simulation of (3.4). Not only do the numerical simulations complement the predictions of the other aforementioned methods, but they provide us with a visual means of the instability development and give insight to the intrinsic nonlinear dynamics of the system.

To this end, we have conducted numerical simulations of (3.4) on a Cartesian grid using the split-step Fourier method [67, 68]. In order to simulate the imperfections in the medium or random modulations of the initial beam profile, a small noise term has been added to the initial input, i.e.

$$\psi(x, y, 0) = \psi_n [1 + \Gamma(x, y)] \quad (3.20)$$

where we have used a Gaussian distribution for the random noise term, of which the
Figure 3-4: Dependence of the growth rate $\delta$ on $\gamma^2$ (a) for the fundamental bound state ($n = 1$), (b) for the inner ring of the first higher-order bound state ($n = 2$), and (c) for the outer ring of the first higher-order bound state ($n = 2$).
Figure 3-5: Dominant perturbation eigenmodes (a) for the fundamental bound state, where $\gamma^2 = 0.5$ and $m = 2$, (b) for the $n = 2$ bound state, where $\gamma^2 = 0.2$ and $m = 2$ (the inner ring), and (c) for the $n = 2$ bound state, where $\gamma^2 = 0.2$ and $m = 6$ (the outer ring).
Figure 3-6: Dominant perturbation eigenmodes for the $n = 3$ bound state, where $\gamma^2 = 0.35$ and (a) $m = 1$ (the innermost ring), (b) $m = 5$ (the middle ring), and (c) $m = 9$ (the outer ring).
mean is \( \langle \Gamma \rangle = 0 \) and the variance is \( \langle |\Gamma|^2 \rangle = 0.0001 \). It is plausible that a particular choice of the noise term might influence the final outcome of the numerical results. Therefore, to ascertain that the results of numerical simulation do not depend on a particular choice of noise term we have used different realizations of the noise term. All of the numerical results have been cross-checked in this manner.

Figure 3-7 displays the evolution of the initial profile for \( n = 1 \) and \( \gamma^2 = 0.5 \). Recall that linear stability analysis of 3.3.2 predicts that the perturbation eigenmode corresponding to \( m = 2 \) has the largest growth rate for this value of \( \gamma^2 \). It is clearly seen from Figure 3-7 that indeed the initial cylindrical symmetry of the beam is broken and two filaments emerge as the beam propagates. Next we consider the case corresponding to \( n = 2 \) where \( \gamma^2 = 0.2 \). According to linear stability analysis (see Figures 3-4(b) and 3-4(c)) it is expected that the inner ring develops two-fold symmetry and the outer ring six-fold symmetry. Again the numerical simulation shown in Figure 3-8 agrees completely with this prediction. As a further demonstration of the validity of our analysis, we consider the case of \( n = 3 \), where \( \gamma^2 = 0.35 \). Based on linear stability analysis (see Figure 3-6) the dominant perturbation eigenmode for the innermost ring occurs for \( m = 1 \). For the middle ring it occurs for \( m = 5 \), and for the outer ring for \( m = 9 \). As is shown in Figure 3-9, the numerical simulations confirm this prediction. Note that the inner ring in Figure 3-9 does not break up. Instead, its intensity around the ring becomes nonuniform as the beam propagates which is the characteristic of 'snake-type' instability.

The filaments arise from transverse instability. We believe that each of these filaments is a quasi self-trapped beam pattern with maximum intensity at the center, as discussed in Ref. [55]. In the presence of two or more filaments, the location of filaments will vary with the propagation distance, as a result of the interaction force among filaments (see Figures 3-7, 3-8, 3-9), whereas one filament alone will stay there without changing its location. The greater the number of filaments, the more complicated the resulting interaction. Take the simplest case of the presence of two filaments in Figure 3-7. The interaction force between the filaments coupled with the vortex nature of the stationary solutions [see Equation (3.6)] causes the
Figure 3-7: Dynamical evolution of the fundamental bound state $n = 1$ for $\gamma^2 = 0.5$
(a) at $z = 0$, (b) at $z = 40$, and (c) at $z = 50$. 
Figure 3-8: Dynamical evolution of the first higher-order bound state $n = 2$ for $\gamma^2 = 0.2$ (a) at $z = 0$, (b) at $z = 25$, and (c) at $z = 50$. 
Figure 3-9: Dynamical evolution of the second higher-order bound state $n = 3$ for $\gamma^2 = 0.35$ (a) at $z = 0$, (b) at $z = 25$, and (c) at $z = 50$. 
filaments to spiral away (i.e., simultaneous rotation and separation of the filaments as the beam propagates further in the medium) from the center of the bound state. They distance themselves until they reach the equilibrium state of the individual filament (when they are far separated and the interaction between the filaments becomes negligible). A similar development occurs for higher-order states (Figures 3-8 and 3-9).
Chapter 4

Modes of Nonlinear Waveguides

In the preceding chapters we discussed the confinement of an electromagnetic field distribution in a homogeneous nonlinear medium without any intervening structures. In this chapter we specialize our attention to the study of the modes of symmetric nonlinear planar (slab) waveguides, where guiding is provided as a result of the difference between the refractive indices of core and cladding, as well as nonlinear effects. In particular, we shall consider the case where both core and cladding are self-defocusing which gives rise to the dark modes.

Since the early work in this field concentrated on the bright modes, in section 4.1 a brief review of the subject is presented. In addition, as an example, the fundamental TE modes of a layered structure, where a linear film is bounded by semi-infinite self-focusing Kerr media, is discussed. This will serve as background material for this chapter and also will be referred to in the next chapter when the question of the stability of asymmetric modes of this structure is considered.

Section 4.2 contains new material and is devoted to the study of the dark modes of a symmetric structure consisting of a thin self-defocusing film bounded by identical semi-infinite self-defocusing media with a different refractive index profile than that of the film. The general characteristics of the solutions (both symmetric and asymmetric solutions) are discussed and regions of existence are presented. Finally, in section 4.3 it is shown that such a structure admits a novel class of bright modes whose tail asymptotically approaches a non-zero intensity background value.
4.1 Introduction

In 1968 Litvak and Mironov [69] found the exact analytical solutions of the nonlinear wave equations for TE surface waves on the interface between two semi-infinite media, of which one was linear and the other nonlinear. This was an important discovery since such waves do not exist in the linear limit. This result remained unnoticed until the early 1980’s when Tomlinson [70] and Maradudin [71] independently reported exact results for the TE waves. Also the pioneering work by Kaplan [72,73], which dealt with the reflection and refraction of incident light onto the interface of the linear and the nonlinear media, aroused much interest which led to more theoretical work in this area. Later, theoretical studies were extended to layered structures which include both surface and guided modes [28,74–80]. In addition to the theoretical studies, the seminal experimental demonstration of nonlinear guided waves by Vach et al. [81] has greatly encouraged researchers in this area.

4.1.1 Bright Modes of a Nonlinear Waveguide

Consider a layered structure consisting of a thin linear film bounded by a semi-infinite self-focusing Kerr medium as shown in Figure 4-1

\[ n^2(x, |E|^2) = \begin{cases} 
  n_f^2 & |x| < d \\
  n_0^2 + n_2 |E|^2 & |x| > d 
\end{cases} \]  

(4.1)

Figure 4-1: Schematic representation of a nonlinear slab waveguide with a self-focusing cladding and a linear core.

where the refractive index is defined by:
The electric field obeys the wave equation:

$$\nabla^2 E + k^2 n^2 \left( x, |E|^2 \right) E = 0 \tag{4.2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. Since we are considering only the TE waves, the electromagnetic field is assumed to be $E = (0, E_y, 0)$, $H = (H_x, 0, H_z)$ where $H$ is the magnetic field. The electric field is written as:

$$E_y = \frac{1}{2} \psi(x) e^{i(\omega t - n \beta z)} + c.c. \tag{4.3}$$

where $\omega$ is the frequency of the wave, $k = \frac{\omega}{c}$, and $n_e = \frac{\beta}{k}$ is the effective index and $\beta$ is the propagation constant. Substituting (4.3) into (4.2) leads to the following differential equation:

$$\frac{d^2 \psi}{dx^2} - k^2 (n_e^2 - n^2) \psi = 0 \tag{4.4}$$

Integrating equation (4.4) once yields:

$$\left( \frac{d\psi}{dx} \right)^2 - k^2 (n_e^2 - n_f^2) \psi^2 = c_1 \tag{4.5a}$$

$$\left( \frac{d\psi}{dx} \right)^2 - k^2 (n_e^2 - n_c^2 - n_{2o}^2) \psi^2 = c_2 \tag{4.5b}$$

for the film and the cladding respectively. The constants $c_1$ and $c_2$ can be found using the boundary conditions at $x = \pm \infty$ and $x = \pm d$ (see for example [82]). The solution of this equation is found to be [72,73,83–86]:

$$\psi_{cl}(x) = \sqrt{\frac{q}{n_{2o} \cosh[kq(x_1 - x - d)]}} \quad x < -d \tag{4.6a}$$

$$\psi_f(x) = A \cosh(k \gamma(x + d)) + B \sinh(k \gamma(x + d)) \quad |x| < d \tag{4.6b}$$

$$\psi_{cr}(x) = \sqrt{\frac{q}{n_{2o} \cosh[kq(x_2 + x - d)]}} \quad x > d \tag{4.6c}$$

where \(q^2 = n_e^2 - n_{2o}^2\), \(\gamma^2 = |n_f^2 - n_e^2|\), A and B are constants.
Continuity of the field and its derivative at the left interface allow us to evaluate the constants A and B and equation (4.6b) can then be expressed in terms of the field in the cladding [85]:

$$
\psi_f(x) = \psi_d(-d)[\cos(k\gamma(x + d)) + \frac{q}{\gamma} \tanh(kqz_1) \sin(k\gamma(x + d))] \\
(4.7a)
$$

$$
\psi_f(x) = \psi_d(-d)[\cosh(k\gamma(x + d)) + \frac{q}{\gamma} \tanh(kqz_1) \sinh(k\gamma(x + d))] \\
(4.7b)
$$

for $n_e^2 < n_f^2$ and $n_e^2 > n_f^2$ respectively. Using (4.7) and matching the tangential electric and magnetic fields across the right interface the following dispersion relations are obtained:

$$
\tan(2k\gamma d) = \frac{\gamma q(\tanh(kqz_1) + \tanh(kqz_2))}{\gamma^2 - q^2 \tanh(kqz_1) \tanh(kqz_2)} \\
(4.8a)
$$

$$
\tan(2k\gamma d) = \frac{\gamma q(\tanh(kqz_1) + \tanh(kqz_2))}{-\gamma^2 - q^2 \tanh(kqz_1) \tanh(kqz_2)} \\
(4.8b)
$$

The guided wave power per unit length along the z-axis in each region can be calculated by integrating the Poynting’s vector:

$$
P = \frac{1}{2} \text{Re} \left( \int (\mathbf{E} \times \mathbf{H}^*) \cdot \hat{z} \, dx \right) . \\
(4.9)
$$

where the limits of integration are defined by the boundaries of each region. For the left and right claddings (4.9) gives

$$
P_{cl} = \frac{n_e q}{kn_{2o}} (1 - \tanh(kqz_1)) \\
(4.10)
$$

$$
P_{cr} = \frac{n_e q}{kn_{2o}} (1 - \tanh(kqz_2)) \\
(4.11)
$$
respectively. For the cases of \( n_f > n_e \) and \( n_f < n_e \) the power in the film is given by

\[
P_f = \frac{n_e q^2}{2 n_{2o} \cosh^2(k q x_1)} \left[ 2d \left( 1 + \frac{q^2}{\gamma^2} \tanh^2(k q x_1) \right) \right.
\]

\[
+ \frac{\sin(4k \gamma d)}{2k \gamma} \left( 1 - \frac{q^2}{\gamma^2} \tanh^2(k q x_1) \right)
\]

\[
+ \frac{q}{\gamma^2 k} \tanh(k q x_1) \left( 1 - \cos(4k \gamma d) \right) \tan^2(\gamma k x_1)
\]

and,

\[
P_f = \frac{n_e q^2}{2 n_{2o} \cosh^2(k q x_1)} \left[ 2d \left( 1 - \frac{q^2}{\gamma^2} \tanh^2(k q x_1) \right) \right.
\]

\[
+ \frac{\sinh(4k \gamma d)}{2k \gamma} \left( 1 + \frac{q^2}{\gamma^2} \tanh^2(k q x_1) \right)
\]

\[
+ \frac{q}{\gamma^2 k} \tanh(k q x_1) \left( \cosh(4k \gamma d) - 1 \right)
\]  \(4.12\)

respectively.

Using the continuity of the fields and their derivatives across the boundaries, \( x_1 \) and \( x_2 \) can be determined. These values together with the above power formulae can be used to construct a dispersion diagram displaying total power versus the effective index \( n_e \). Figure 4-2 shows the symmetric and asymmetric branches of the TE\(_0\) (fundamental) family of the guided waves for a given set of parameters.

The solid curve in Figure 4-2 represents field distributions which are symmetric with respect to the center of the film. Increasing power creates a minimum at the center of the film and the field maxima are forced into the cladding. The dashed curve, which was reported first in [84], exists only above a certain power threshold.
and represents two degenerate solutions, each of which has a single field maximum. The stability of these asymmetric modes will be discussed in the next chapter.

Figure 4-2: Dependence of TE₀ total power on the effective index for a structure with parameters n₀ = 1.5, n₁ = 2.0 and 2d/λ = 0.4, where the solid line denotes the symmetric modes and the dashed line represents the asymmetric modes.

4.2 Dark Modes in a Nonlinear Waveguide

Although the bright modes have been investigated extensively in recent years, little research has been directed to the study of the dark modes [87]. In the following, the trapping of dark solitary waves in a nonlinear structure composed of a thin self-defocusing film bounded by semi-infinite self-defocusing media will be examined. Due to the complexity of the solutions the symmetric and asymmetric solutions will be discussed separately. Also, the analysis will be restricted to TE light patterns.
4.2.1 Symmetric Dark and Gray Solitary Waves

Consider a layered structure as shown in Figure 4-3

![Figure 4-3: Schematic representation of a nonlinear slab waveguide with a self-defocusing cladding and a self-defocusing core.]

where the refractive index is described by

\[
n^2 = \begin{cases} 
  n_f^2 - n_2 |E|^2 & |x| < d \\
  n_0^2 - n_2 |E|^2 & |x| > d 
\end{cases}
\] (4.14)

The electric field \(E\) takes the form of (4.3) and upon substitution into (4.2) leads to (4.4). After integrating the resulting equation once we obtain:

\[
\left( \frac{d\psi}{dx} \right)^2 - k^2(n_e^2 - n_f^2 + n_2 \frac{\psi^2}{2})\psi^2 = c_1 
\] (4.15a)

\[
\left( \frac{d\psi}{dx} \right)^2 - k^2(n_c^2 - n_0^2 + n_2 \frac{\psi^2}{2})\psi^2 = c_2 
\] (4.15b)

for the film and the cladding respectively. In this case, the boundary conditions at ±\(\infty\) require \(\psi\) to approach a non-zero constant (as opposed to decaying to zero in the case of bright modes of the previous section).

It should be also noted that when the linear refractive index \(n_f\) of the film is smaller than the linear refractive index \(n_o\) of the cladding, the structure is not guiding at low power although it may support one or more linear modes (bright waves) when \(n_f > n_o\). The nonlinearity taking effect at high power, however, changes the guiding characteristics completely. The guiding channels for the dark and gray waves can be created, whereas the mode patterns of the bright waves are
then modified and possibly suppressed (especially for high-order modes), depending on the power.

First consider the case of $n_f < n_o$. The boundary conditions outside the film give rise to a hyperbolic tangent function, and inside the film the solution in general is a Jacobi elliptic function [82,88,89]. Therefore, the possible solitary wave solutions to (4.15), with the refractive index profile of (4.14), are

$$\psi_{cl} = \pm \left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x + x_1)}{\sqrt{2}} \right) \quad x < -d \quad (4.16a)$$

$$\psi_f = \left\{ \begin{array}{l}
\sqrt{\frac{2(1 - m^2)}{n_{2f}}} \frac{\eta f_+(x)}{\cn(k\eta x|m)} \quad |x| < d
\end{array} \right. \quad (4.16b)$$

$$\psi_{cr} = \left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x - x_1)}{\sqrt{2}} \right) \quad x > d \quad (4.16c)$$

where $f_+(x) = \sn(k\eta x|m)$ and $f_-(x) = 1$, $\cn$ and $\sn$ are Jacobi elliptic functions [88], $\gamma = \sqrt{n_o^2 - n_e^2}$ and $m$ is the modulus of Jacobi elliptic function (see Appendix B). The solution associated with the upper sign is referred to as the odd solution (the dark solitary wave) with $\eta = \frac{\nu}{\sqrt{2 - m^2}}$ and $\nu = \sqrt{n_e^2 - n_f^2}$. That corresponding to the lower sign is referred to as the even solution (the gray solitary wave) with $\eta = \frac{\nu}{\sqrt{2m^2 - 1}}$. These solutions exist in the range of $n_f < n_e < n_o$. The continuity of the fields and their derivatives across the boundaries gives rise to the dispersion relations

$$\frac{\gamma^2}{\nu^2} = 2 \sqrt{n_{2o} \sqrt{1 - m^2}} \times \sqrt{\frac{n_{2o}}{n_{2f}} \frac{(1 - m^2) \sn^2 \left( \frac{k\nu}{\sqrt{2 - m^2}} |m| \right)}{\cn^2 \left( \frac{k\nu}{\sqrt{2 - m^2}} |m| \right)}}$$

$$\frac{\gamma^2}{\nu^2} = 2 \sqrt{n_{2o} \sqrt{1 - m^2}} \times \sqrt{\frac{n_{2o}}{n_{2f}} \frac{(1 - m^2)}{2m^2 - 1} \times \left( \sn \left( \frac{k\nu}{\sqrt{2m^2 - 1}} |m| \right) \times \dn \left( \frac{k\nu}{\sqrt{2m^2 - 1}} |m| \right) \right)}$$

for the odd and even solutions respectively.
$x_1$ can be determined by equating the fields at $x = d$ and using the identity \( \tanh^{-1}(y) = \frac{1}{2} \ln \frac{1+y}{1-y} \) to obtain

$$x_1 = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{1 - \frac{(2(1 - m^2)\frac{n_{2o}}{n_{2f}})^{\frac{1}{2}}}{\gamma} \frac{f_\pm(d)}{\sqrt{1 - m^2}}}{\frac{(2(1 - m^2)\frac{n_{2o}}{n_{2f}})^{\frac{1}{2}}}{\gamma} \frac{f_\pm(d)}{\sqrt{1 - m^2}}} \frac{\sqrt{\frac{n_{2f}}{n_{2o}}}}{\sqrt{\frac{n_{2o}}{n_{2f}}}} \frac{\sqrt{n_{2f}}}{\sqrt{n_{2o}}} \frac{\sqrt{1 - m^2}}{\sqrt{1 - m^2}}}.$$

It is important to note that (4.19) by itself does not guarantee the continuity of the fields and their derivatives at the interfaces since both $n_e$ and $x_1$ depend on $m$. For each $m$ within $0 < m < 1$ the dispersion relation (4.17) yields a $n_e$ within $n_f < n_e < n_o$ provided that $0 < \frac{kd\nu}{\sqrt{2 - m^2}} < K(m)$ [In this range both $sn$ and $cn$ functions are positive. See Appendix B for the plots of these functions.] where $K(m)$ is the complete elliptic integral of the first kind. The $m$ and $n_e$ obtained in this manner can then be substituted into (4.19) to obtain $x_1$. As is shown in Figure 4-4(a) $m$ and $n_e$ form a monotonic relationship; i.e. the effective index $n_e$ increases with increasing $m$ whereas the corresponding $x_1$ diminishes with rising $m$ as is illustrated in Figure 4-4(b)].

![Figure 4-4](image_url)

Figure 4-4: Dependence of the effective index $n_e$ and $x_1$ on $m$ for the dark solitary wave trapped in the structure with $\frac{n_o}{n_f} = 1.5$ and $\frac{kd\nu}{\sqrt{2 - m^2}} = 0.25\pi$. 

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This result indicates that the odd solution, or the dark solitary wave (characterized by zero minimum intensity), can exist for any \( m \) within \( 0 < m < 1 \). With increased \( m \) or \( n_e \), the corresponding beam width expands. This relation is shown in Figure 4-5.

![Figure 4-5: Field and intensity profiles of the dark solitary wave for the parameters of the Figure 4-4 and \( \frac{n_{2o}}{n_{2f}} = 1.5 \). The solid curves are for \( m = 0.1 \) and the dashed curves for \( m = 0.95 \), where \( \bar{E} = \psi n_{2o}/\gamma \).

The even solution, or the gray solitary wave [Equation (4.18)], however, can exist only within a limited range of \( m \), i.e., \( \frac{1}{\sqrt{2}} < m < 1 \), corresponding to \( n_f < n_e < n_o \). Similarly, \( n_e \) increases [Figure 4-6(a)] and the beam expands [Figure 4-6(b)] with increasing \( m \) together with the diminishing value of \( x_1 \). The grayness of the gray solitary wave here is measured by

\[
|\bar{E}|^2 = |\psi f_1(0)|^2 n_{2o}/\gamma^2 = \frac{2(1 - m^2)\nu^2 n_{2o}}{(2m^2 - 1)\gamma^2 n_{2f}},
\]

(4.20)

\(^1\)The range of \( m \) is obtained by finding the values of \( m \) for which the fraction \( \sqrt{\frac{1 - m^2}{2m^2 - 1}} \) in (4.18) is a real positive number.
Figure 4-6: (a) Dependence of $n_e$ and $x_1$ on $m$ for the gray solitary wave trapped in a structure with $\frac{n_o}{n_f} = 1.5$ where $\frac{kdv}{\sqrt{2m^2 - 1}} = 0.25\pi$. (b) The corresponding field ($\overline{E} = \psi\sqrt{n_{2o}/\gamma}$) and intensity profiles with $\frac{n_{2o}}{n_{2f}} = 1.5$: the solid lines are for $m = 0.75$ and the dashed lines for $m = 0.95$. 
As is shown in Figure 4-7 the grayness decreases with increasing \( n_e \) for a fixed \( kdn_f \) or decreases with increasing \( kdn_f \) for a given \( n_e \). In the extreme case of \( m \rightarrow 1 \), grayness \( \rightarrow 0 \) and the gray solitary wave reduces to a dark solitary wave.

From the discussion of previous chapters we know that self-guidance in a uniform self-defocusing nonlinear medium is possible for a dark solitary wave and the wave propagates in a self induced waveguide with a convex index profile. Here, in contrast, both gray and dark solitary waves propagate in a self induced waveguide with a concave index profile as shown in the insets of Figures 4-5 and 4-6(b). That is to say, A nonlinear layer sandwiched in a uniform self-defocusing nonlinear medium completely alters the trapping features. On the other hand, similar to the dark solitary wave trapped in a uniform self-defocusing medium, here both gray and dark solitary waves operate at the cut off [see the insets of Figures 4-5 and 4-6(b)], i.e., the effective index \( n_e \) is equal to the minimum value of the induced index profile.

\[ \text{Figure 4-7: Grayness } = |\mathcal{E}|^2 = |\psi_f(0)|^2 n_{2o} / \gamma^2 \text{ of the gray solitary wave (a) versus } n_e \text{ for fixed film widths } kdn_f \text{ and (b) versus the film width } kdn_f \text{ for fixed } n_e, \text{ where } \frac{n_0}{n_f} = 1.5 \text{ and } \frac{n_{2o}}{n_{2f}} = 1.5. \]
4.2.2 Dark Oscillatory Solitary Waves

The gray and dark solitary waves can exist in a hollow self-defocusing structure with \( n_f < n_o \) when the effective wave index \( n_e \) is limited in the range of \( n_f < n_e < n_o \). However, when \( n_e < n_f \) (or \( n_e < n_o \) if \( n_f \) and \( n_o \) are such that \( n_f > n_o \)), so that the structure becomes convex in the absence of nonlinearity, trapping of dark beams can still occur. In this case, the solution of (4.15) outside the film (\(|x| > d\)) has the same form as in (4.16a) and (4.16c) for the dark and gray solitary waves. But inside the film, it reads

\[
\psi_f = (-1)^{(n_e-1)/2} \sqrt{\frac{2}{n_f}} \left[ \frac{\kappa m}{\sqrt{1 + m^2}} \right] \text{sn} \left[ \frac{k \kappa x}{\sqrt{1 + m^2}} \right] m 
\]

(4.21a)

\[
\psi_f = (-1)^{\frac{j-1}{2}} \sqrt{\frac{2}{n_f}} \left[ \frac{\kappa m}{\sqrt{1 + m^2}} \right] \text{cn} \left[ \frac{k \kappa x}{\sqrt{1 + m^2}} \right] m 
\]

(4.21b)

for the odd and even solutions respectively, where \( \kappa = \sqrt{n_f^2 - n_e^2} \). The odd solution corresponds to the upper sign in (4.16a) with \( j = 1, 3, 5, \ldots \) and the even solution corresponds to the lower sign in (4.16a) with \( j = 2, 4, 6, \ldots \). In contrast to the previous case of \( n_f < n_o \), here for \( n_e < \min\{n_o, n_f\} \) an infinite set of trapped mode patterns \( (j=1, 2, 3, \ldots) \) can exist, depending on the value of \( \frac{k \kappa d}{\sqrt{1 + m^2}} \). They differ from one another in the number of intensity peaks \( (j-1) \) that appear within the film as is illustrated in Figure (4-8) for the first four lowest order modes. The relation between \( m \) and \( n_e \) is determined by the following dispersion relations:
\[
\frac{\gamma^2}{\kappa^2} = 2 \sqrt{\frac{n_{20}}{n_{2f}}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_{20}}{n_{2f}}} \text{sn}^2 [\Omega|m] + (-1)^{\frac{\mu + 1}{2}} \text{cn} [\Omega|m] \text{dn} [\Omega|m] \right\}
\] (4.22)

\[
\frac{\gamma^2}{\kappa^2} = 2 \sqrt{\frac{n_{20}}{n_{2f}}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_{20}}{n_{2f}}} \text{cn}^2 [\Omega|m] - (-1)^{\frac{\mu}{2}} (1 - m^2) \text{sn} [\Omega|m] \right\} / \text{dn}^2 [\Omega|m].
\] (4.23)

Using the same method as the previous section, \( x_1 \) is found to be

\[
x_1 = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{1 - (-1)^{\frac{\mu + 1}{2}} \sqrt{\frac{n_{20}}{n_{2f}}} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{sn} [\Omega|m]}{1 + (-1)^{\frac{\mu + 1}{2}} \sqrt{\frac{n_{20}}{n_{2f}}} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{sn} [\Omega|m]}
\] (4.24)

\[
x_1 = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{1 - (-1)^{\frac{\mu}{2}} \sqrt{\frac{n_{20}}{n_{2f}}} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{cn} [\Omega|m]}{1 + (-1)^{\frac{\mu}{2}} \sqrt{\frac{n_{20}}{n_{2f}}} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{cn} [\Omega|m]}
\] (4.25)

for the odd and even solutions respectively, where \( \Omega = \frac{kd}{\sqrt{1 + m^2}} \).

Figure 4-8 displays the first four lowest order modes. The lowest order mode \( j = 1 \) can be trapped when \( 0 < \Omega < K(m) \) with the intensity profile of Figure 4-8(a), which is similar to the case of \( n_f < n_0 \) considered earlier. However, the difference is that here the dark solitary wave propagates in a self-induced waveguide with a convex index profile [see the inset of Figure 4-8(a)] rather than in a hollow waveguide [see the inset of Figure 4-5].
Figure 4-8: Characteristics of dark oscillatory waves (a) for the $j = 1$ mode and (b) for the $j = 2, 3, 4$ modes, where $\bar{E} = \psi \sqrt{n_{2o}/\gamma}$, $\frac{n_o}{n_f} = 0.8$, $\frac{n_{2o}}{n_{2f}} = 0.8$. $m = 0.45$, and $\Omega = 1$ for the $j = 1, 2$ and $\Omega = 4.24$ for $j = 3, 4$ modes.
In general the $j^{th}$ mode (where $j = 2, 3, 4, \ldots$) can be trapped in the range $(j-2)K(m) < Q < jK(m)$. But for a fixed $\Omega$ the trapped mode can be either an even or an odd pattern. For example for a fixed $\Omega$ within $0 < \Omega < K(m)$, either the $j = 1$ mode or $j = 2$ mode can be trapped; within $K(m) < \Omega < 2K(m)$ either the $j = 2$ or the $j = 3$ mode can be trapped and so on. This is illustrated in Figure 4-8.

![Figure 4-9: Dependence of $n_e$ on $m$ (a) for the $j = 1$ mode and (b) for the $j = 2$ mode, where $\Omega = 0.25\pi$. $n_{2o}/n_{2f} = 0.5$ is identified by the solid curves and $n_{2o}/n_{2f} = 1.5$ by the dashed curves.](image)

The dispersion relations (4.22) and (4.23) stipulate the relation between the effective index $n_e$ and the modulus of the Jacobi elliptic functions $m$. As is illustrated in Figure 4-9(a) for the $j = 1$ mode, $n_e$ decreases with an increasing $m$, contrary to the previous case of $n_f < n_e < n_o$. But for the $j = 2$ mode $n_e$ can decrease or increase with an increasing $m$, depending on the ratios of $n_o/n_f$ and $n_{2o}/n_{2f}$ [4-9(b)]. In particular, for small $n_{2o}/n_{2f}$ with $n_o/n_f$ approaching unity [see the case of $n_{2o}/n_{2f} = 0.5$ and $n_o/n_f = 0.8$ in Figure 4-9(b)], $n_e$ decreases and then increases as $m$ varies from 0 to 1; i.e., within a certain range of $m$, one particular $n_e$ can be associated with two degenerate $j = 2$
mode patterns (two unequal $m$) corresponding to two different field profiles. This case is shown in Figure 4-10(a) where $\frac{n_e}{n_f} = 0.51$ and $m = 0.95$, identified by the solid curve, and $m = 0.55$ by the dashed curve. The degenerate characteristic of the $j = 2$ mode is shared by other higher order modes. For example, Figure 4-10(b) displays the field profiles of the degenerate $j = 3$ mode for $n_e = 0.56 n_f$, with $m = 0.41$ marked by the dashed curve and $m = 0.81$ by the solid curve.

Another distinctive feature revealed in Figure 4-9 is the limited range of the value of $m$, $\Delta m = \max\{m\} - \min\{m\}$, within which dark oscillatory solitary waves can exist. The range varies with the nonlinear coefficient ratio $\frac{n_{2o}}{n_{2f}}$ and the linear refractive index ratio $\frac{n_o}{n_f}$. As $\frac{n_{2o}}{n_{2f}}$ increases $\Delta m$ narrows, but $\Delta m$ expands as $\frac{n_o}{n_f}$ increases. For example, when $j = 1$, $\frac{n_o}{n_f} = 0.8$ and $\frac{n_{2o}}{n_{2f}} = 0.5$, dark solitary waves can exist for any $m$ within $0 < m < 1$. However, increasing the ratio $\frac{n_{2o}}{n_{2f}}$ to 1.5 decreases the range to $0 < m < 0.4$. This is in contrast to the previous case (see Figure 4-4(a)) where the mode existed for any $m$ within $0 < m < 1$ regardless of the ratio $\frac{n_{2o}}{n_{2f}}$.

Although the parameters in Figure 4-9 are chosen for $j = 1, 2$ modes, other high-order modes exhibit the same qualitative features. Indeed, changing the value of $\Omega$ alters the value of $\Delta m$ in Figure 4-9 as well, but the qualitative features remain the same. Figure 4-11 outlines the regions for the existence of the first six lowest order modes in terms of $m$ versus $\Omega$ for $\frac{n_{2o}}{n_{2f}} = 0.5$ and $\frac{n_o}{n_f} = 0.8$. 

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Figure 4-10: Demonstration of the field profiles of degenerate mode patterns with $\frac{n_{2o}}{n_{2f}} = 0.5$ and $\frac{n_{2o}}{n_f} = 0.8$. The insets are the corresponding relations between $n_e$ and $m$. (a) $j = 2$ mode ($kdn_f = 0.8$) for $\frac{n_e}{n_f} = 0.51$, and $m = 0.55$ identified by the dashed curve and $m = 0.95$ by the solid curve; (b) $j = 3$ mode ($kdn_f = 2.4$) for $\frac{n_e}{n_f} = 0.56$, with $m = 0.41$ identified by the dashed curve and $m = 0.81$ by the solid curve.
Figure 4-11: Regions for the existence of (a) $j = 1, 3, 5$ modes and (b) $j = 2, 4, 6$ modes in terms of $m$ versus $\Omega$, where $\frac{n_{2o}}{n_{2f}} = 0.5$ and $\frac{n_o}{n_f} = 0.8$. 
4.2.3 Asymmetric Dark Modes

In 4.1.1 it was mentioned that a symmetric nonlinear waveguide, composed of a linear film bounded by identical self-focusing media, can support asymmetric modes above a certain power [84]. At high powers the field is forced out of the film into one of the bounding media, which means that the symmetry of the waveguide can be broken by the optical field. Another consideration is that such asymmetric modes do not have any counterpart in bulk media. This indicates that the asymmetric modes are the product of the interplay between nonlinear effects and the intervening structure.

In comparison with a self-defocusing interface structure [90-92] the self-defocusing waveguide also demonstrates its distinctive features - the fundamental asymmetric dark mode trapped in the self-defocusing waveguide structure has a uniform background intensity (see Figure 4-12) whereas the background intensity of a dark surface wave trapped between two semi-infinite self-defocusing media is different in either medium [90-92]. In addition, the self-defocusing waveguide can admit higher-order asymmetric modes with more than one intensity minimum (see Figure 4-12) whereas “higher-order dark surface waves” do not exist in a self-defocusing interface structure [90-92].

In this section we will investigate the existence of the asymmetric dark modes in symmetric nonlinear waveguides (as shown in Figure 4-3) whose refractive index profile is described by (4.14).

Solutions of the Wave Equation

The procedure for finding the solutions is identical to the previous sections which, in this case, also leads to equations (4.15). Asymmetric solutions do not exist in the case of \( n_f < n_e < n_0 \). This can be shown by the method of contradiction, i.e. by first assuming that asymmetric solutions exist one can simplify the corresponding dispersion relations using the identities in Appendix B which leads to an impossible equation. However, it is found that asymmetric solutions exist when \( n_e < \min\{n_f, n_e\} \). The general expressions for the asymmetric dark modes are given
by:

\[
\psi_{cl} = \left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x + x_1)}{\sqrt{2}} \right) \quad x < -d \quad (4.26a)
\]

\[
\psi_f = (-1)^{\left(\frac{j-1}{2}\right)} \sqrt{\frac{2}{n_{2f}}} \left( \frac{\kappa m}{\sqrt{1 + m^2}} \right) \text{sn} \left[ \frac{k\kappa(x - x_f)}{\sqrt{1 + m^2}} |m| \right] \quad |x| < d \quad (4.26b)
\]

\[
\psi_{cr} = \left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x - x_2)}{\sqrt{2}} \right) \quad x > d \quad (4.26c)
\]

for the odd solution with \( j = 1, 3, 5, \ldots \) and

\[
\psi_{cl} = -\left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x + x_1)}{\sqrt{2}} \right) \quad x < -d \quad (4.27a)
\]

\[
\psi_f = (-1)^{\frac{j}{2}} \sqrt{\frac{2}{n_{2f}}} \left( \frac{\kappa m}{\sqrt{1 + m^2}} \right) \text{cn} \left[ \frac{k\kappa(x - x_f)}{\sqrt{1 + m^2}} |m| \right] \quad |x| < d \quad (4.27b)
\]

\[
\psi_{cr} = \left( \frac{\gamma}{\sqrt{n_{2o}}} \right) \tanh \left( \frac{k\gamma(x - x_2)}{\sqrt{2}} \right) \quad x > d \quad (4.27c)
\]

for the even solution with \( j = 2, 4, 6, \ldots \) \( x_f \) is a measure of asymmetry of the modes. It can be readily seen that when \( x_f = 0 \) the above solutions reduce to the symmetric case. Also, it should be noted that for a fixed value of \( n_e \) and \( m \) there exist two asymmetric solutions. These solutions are mirror images of one another: one corresponding to the intensity minimum at \( x = x_f \) and the other at \( x = -x_f \).
Dispersion Relations

The relationship between the the parameters $x_f$ and $m$ and $n_e$ is determined by the dispersion relations:

\[
\frac{\gamma^2}{k^2} = 2 \sqrt{\frac{n_2}{n_2 f}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_2}{n_2 f}} \text{sn}^2 \left[ \frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}} |m| \right] \right. \\
+ (-1)^{\frac{j-1}{2}} \text{cn} \left[ \frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}} |m| \right] \text{dn} \left[ \frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}} |m| \right] \left. \right\} 
\]

(4.28)

\[
\frac{\gamma^2}{k^2} = 2 \sqrt{\frac{n_2}{n_2 f}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_2}{n_2 f}} \text{cn}^2 \left[ \frac{k \kappa (d + x_f)}{\sqrt{1 + m^2}} |m| \right] \right. \\
- (-1)^{\frac{j-1}{2}} (1 - m^2) \text{sn} \left[ \frac{k \kappa (d + x_f)}{\sqrt{1 + m^2}} |m| \right] \right/ \text{dn}^2 \left[ \frac{k \kappa (d + x_f)}{\sqrt{1 + m^2}} |m| \right]. 
\]

(4.29)

for the odd and even modes respectively. Note that in (4.28) and (4.29) $\pm x_f$ arises due to the degeneracy of the asymmetric solutions.

Using the boundary conditions at $x = \pm d$, $x_1$ and $x_2$ are found to be:

\[
x_{1,2} = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{1 - (-1)^{\frac{j-1}{2}} \sqrt{2n_2} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{sn} \left[ \frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}} |m| \right]}{1 + (-1)^{\frac{j-1}{2}} \sqrt{2n_2} \left( \frac{\kappa m}{\gamma \sqrt{1 + m^2}} \right) \text{sn} \left[ \frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}} |m| \right]} 
\]

(4.30)

for the odd solution and
In the above equations \( j = 1, 2, 3, \ldots \) identifies the mode numbers. These modes differ from one another by the number of intensity peaks that appear within the film. The first four lowest order modes are shown in Figure 4-12.

**Range of Existence**

Similar to the symmetric case, the value of \( \Omega \) is an important factor in determining the range of existence of the \( j^{th} \) mode. For example, the fundamental \((j = 1)\) mode can exist in the range of \( 0 < \Omega < K(m) \). Also, for a fixed value of \( \Omega \) the trapped mode can either be an even or odd pattern as described previously.

\[ n_e \text{ and } \lambda_f \text{ characterize the main properties of the asymmetric modes. Therefore, to investigate the range of existence of such modes we will have to analyze how they vary with } m \text{ and other structural parameters. This added parameter } (\lambda_f) \text{ imposes an additional restriction (compared to the symmetric modes) on the existence of the } j^{th} \text{ mode.} \]

Figures 4-14 and 4-15 show the dependence of \( n_e \) and \( \lambda_f \) on \( m \) for the given structural parameters \( \Omega, \frac{n_o}{n_f}, \text{ and } \frac{n_{2o}}{n_{2f}} \). In order to investigate the effect of structural parameters in (a) and (b) \( \frac{n_{2o}}{n_{2f}} \) is varied and \( \Omega \) and \( \frac{n_o}{n_f} \) are kept constant; in (c) and (d) \( \frac{n_o}{n_f} \) is varied and \( \Omega \) and \( \frac{n_{2o}}{n_{2f}} \) are kept constant; in (e) and (f) \( \Omega \) is varied and \( \frac{n_o}{n_f} \) and \( \frac{n_{2o}}{n_{2f}} \) are kept constant.
Figure 4-12: Demonstration of asymmetric dark modes with $\frac{n_{2o}}{n_{2f}} = 2$ and $\frac{n_o}{n_f} = 1.01$, and $m = 0.6$ for (a) $j = 1$ mode at $\Omega = 0.25\pi$ and $n_e = 0.95936 n_f$, and (b) $j = 2$ mode at $\Omega = 0.5\pi$ and $n_e = 0.96857 n_f$ identified by the dotted curve; $j = 3$ mode at $\Omega = \pi$ and $n_e = 0.96579 n_f$ marked by the dashed curve; $j = 4$ mode at $\Omega = 1.5\pi$ and $n_e = 0.9615 n_f$ shown by the solid curve, where $\overline{E} = \psi \sqrt{n_{2o}/\gamma}$.

It is seen from Figures 4-14(a), 4-14(c), 4-14(e) and 4-15(a), 4-15(c), 4-15(e) that the effective index of the asymmetric modes increases with increasing $m$ for $j = 1$ and $j = 2$ irrespective of the structural parameters. Figures 4-14(b), 4-14(d), 4-14(f) and 4-15(b), 4-15(d), 4-15(f) also show that the value of $x_f$ increases with increasing $m$.

For the fundamental mode ($j = 1$) $x_f$ can be larger than the waveguide half-width ($d$). Figure 4-14(f) shows that $x_f$ can be as large as 20 times the waveguide half-width. An example of an asymmetric mode where $|x_f| > d$ is shown in Figure 4-13. It should be noted that when $|x_f| < d$ the zero intensity occurs at $x = x_f$ but when $|x_f| > d$, the zero intensity occurs at $x = x_2$ (or $x = x_1$) with $x_f \approx x_2$ (or $x_f \approx x_1$). In the case of higher-order modes $x_f$ is always less than or equal to $d$.

Another feature which is displayed in Figures 4-14 and 4-15 is the limited range
of $\Delta m = \max\{m\} - \min\{m\}$ within which the dark modes exist. The range of $\Delta m$ changes with the ratio of $\frac{n_{2o}}{n_{2f}}$, $\frac{n_o}{n_f}$, and the value of $\Omega$. The maximum $\Delta m$ occurs at a finite value of $\frac{n_{2o}}{n_{2f}}$, which is $\frac{n_{2o}}{n_{2f}} = 1.6$ for the $j = 1$ mode at $\frac{n_o}{n_f} = 1.01$ and $\Omega = 0.25\pi$. For $j = 2$ the largest $\Delta m$ occurs at $\frac{n_{2o}}{n_{2f}} = 10$, $\frac{n_o}{n_f} = 1.05$ and $\Omega = 0.5\pi$. A decrease or increase in $\frac{n_{2o}}{n_{2f}}$ from this finite ratio leads to a narrowing in $\Delta m$ (Figures 4-14(a), 4-15(a)). On the other hand, for a fixed $\frac{n_{2o}}{n_{2f}}$, $\Delta m$ increases with decreasing $\frac{n_o}{n_f}$ (> 1) and/or with decreasing $\Omega$ (Figures 4-14(c) and 4-14(e)); it reaches a maximum at $\frac{n_o}{n_f} \rightarrow 1$ and/or $\Omega \rightarrow 0$. For the $j = 2$ mode, similar characteristics are observed in Figure 4-15 except that here the maximum $\Delta m$ occurs at $\Omega = (j - 1)\frac{\pi}{2} = \frac{\pi}{2}$ and it decreases with increasing or decreasing value of $\Omega$ when shifting away from the value of $\frac{\pi}{2}$.

![Figure 4-13: Field and intensity profiles of the fundamental asymmetric dark mode with its center located outside the film where $\frac{n_{2o}}{n_{2f}} = 2$, $\frac{n_o}{n_f} = 1.01$, and $\Omega = 0.1 \pi$, $m = 0.6$, and $n_e = 0.96668 n_f$.](image)
Figure 4.14: Dispersion diagrams; (a), (c) and (e) show the dependence of the effective index $n_e$ on the modulus $m$ and (b), (d) and (f) display the dependence of $x_f$ on $m$ of the fundamental $(j = 1)$ asymmetric dark mode. The parameters are shown on the diagrams.
Figure 4-15: Dispersion diagrams; (a), (c) and (e) show the dependence of the effective index $n_e$ on the modulus $m$ and (b), (d) and (f) display the dependence of $x_f$ on $m$ of the $j = 2$ asymmetric dark mode. The parameters are shown on the diagrams.
Dispersion Relations in Terms of Power

Due to the existence of the infinite nonzero background the power is infinite for the modes discussed above, which prevents us from expressing the dispersion relations in terms of power versus \( n_e \) and \( m \). However, if we subtract the background from the intensity profile and then integrate we obtain a finite value which represents the power in the dark region. This quantity, here termed as complementary power, can be used to characterize the dark modes. The dispersion relations in this case can be presented either in terms of complementary power \( P \) versus the modulus \( m \) or \( n_e \) versus \( P \), where the complementary power is defined as

\[
P = \int_{-\infty}^{\infty} |\psi(x)|^2 - |\psi(\pm \infty)|^2 \, dx. \tag{4.32}
\]

Substituting (4.27) into (4.32) and integrating yields:

\[
P_{\text{odd}} = \frac{\gamma^2 d}{n_{2o}} \left\{ \frac{\sqrt{2}}{kd\gamma} \left[ 2 - \tanh \left( k\gamma \frac{d - x_1}{\sqrt{2}} \right) - \tanh \left( k\gamma \frac{d - x_2}{\sqrt{2}} \right) \right] \right. + 2 - \frac{n_{2o} - 2\gamma}{n_{2f} \gamma^2 \sqrt{1 + m^2}} \left. \left[ 2\Omega - E \left( \text{am} \frac{k\gamma d - x_f}{\sqrt{1 + m^2}} \right|m \right) - E \left( \text{am} \frac{k\gamma d + x_f}{\sqrt{1 + m^2}} \right|m \right] \right\} \tag{4.33}
\]

for the odd solutions and

\[
P_{\text{even}} = P_{\text{odd}} - \frac{2m^2 k}{kn_{2f} \sqrt{1 + m^2}} \left[ \text{sn} \left( \frac{k\gamma d - x_f}{\sqrt{1 + m^2}} |m \right) \cd \left( \frac{k\gamma d - x_f}{\sqrt{1 + m^2}} |m \right) \right.
\]

\[
+ \text{sn} \left( \frac{k\gamma d + x_f}{\sqrt{1 + m^2}} |m \right) \cd \left( \frac{k\gamma d + x_f}{\sqrt{1 + m^2}} |m \right) \tag{4.34}
\]

for the even solutions, where \( E(\text{am} u|m) \) is the normal elliptic integral of the second kind [88]. The relationships between power and \( m \), and power and \( n_e \) are shown in Figures 4-16(a) and 4-16(b) respectively. The power is inversely related to the modulus \( m \) and \( n_e \).
Trapping of the asymmetric dark modes occurs for the hollow waveguides, i.e. \( \frac{n_o}{n_f} > 1 \). Also, no trapping of the asymmetric dark modes is possible when \( \frac{n_{2o}}{n_{2f}} < 1 \). Physically this means that in a hollow waveguide the nonlinearity in the cladding is required to be greater than that in the film so that the high intensity in the cladding induces a refractive index change which converts the hollow waveguide structure into a guiding structure with convex index profile to support the dark waves.

This result can be derived from the dispersion relations (4.28). Equation (4.28) consists of two equations, one with \( +x_f \) and the other with \( -x_f \). Dividing the former by the latter and using the expansion formulae for the the Jacobi elliptic functions (see Appendix B for the expansion formulae) and rearranging, we obtain

---

*Figure 4-16: The relation between power and modulus m and (b) the relation between \( n_e \) and power for the \( j = 1 \) asymmetric mode, where the structural parameters are the same as Figure 4-14(c).*
$$4m \sqrt{\frac{n_{2o}}{n_{2f}}} \sin(a|m) \sin(\alpha x_f|m) \cos(a|m) \cos(\alpha x_f|m) \cos(a|m) \cos(\alpha x_f|m)$$

$$= (-1)^{\frac{2m+1}{2}} [2m^2 \cos^2(a|m) \cos^2(\alpha x_f|m) \sin(a|m) \sin(\alpha x_f|m)$$

$$+ 2 \sin(\alpha x_f|m) \cos(\alpha x_f|m) \cos^2(a|m) \cos^2(\alpha x_f|m)]$$  (4.35)

where \( \alpha = \frac{k \kappa}{\sqrt{1 + m^2}} \). Dividing both sides of (4.35) by

$$2 \sin(\alpha x_f|m) \cos(\alpha x_f|m) \cos^2(a|m) \cos^2(\alpha x_f|m)$$

and using the identity \( \frac{\cos(u|m)}{\sin(u|m)} = \tan(u|m) \) we obtain

$$2m \sqrt{\frac{n_{2o}}{n_{2f}}} \cos(a|m) \cos(\alpha x_f|m) = (-1)^{\frac{2m+1}{2}} (m^2 \cos^2(a|m) \cos^2(\alpha x_f|m) + 1)$$  (4.36)

(4.36) is quadratic in \( \cos(a|m) \cos(\alpha x_f|m) \) which can be treated using the quadratic formula to yield

$$m \cos(a|m) \cos(\alpha x_f|m) = (-1)^{\frac{2m+1}{2}} \left( \sqrt{\frac{n_{2o}}{n_{2f}}} - \sqrt{\frac{n_{2o}}{n_{2f}} - 1} \right).$$  (4.37)

From (4.37) it is obvious that no physically acceptable solution exists when \( \frac{n_{2o}}{n_{2f}} < 1 \). This result also applies to the even modes.

### 4.3 Bright Modes with Non-Zero Background

In this section we demonstrate the existence of a novel class of solutions to (4.15). These solutions are nonlinear bright modes whose characteristics are similar to the dark modes in that they have a non-zero intensity background. The expressions for these solutions are given by
\[
\psi_{cl} = -\frac{\gamma}{\sqrt{n_{20}}} \tanh \left( \frac{k\gamma(x + x_1)}{\sqrt{2}} \right) \quad x < -d \quad (4.38a)
\]

\[
\psi_f = (-1)^{\frac{j-1}{2}} \frac{2}{n_{2f}} \left[ \frac{k_m}{\sqrt{1 + m^2}} \right] \text{cn} \left[ \frac{k\kappa x}{\sqrt{1 + m^2}} |m| \right] \quad |x| < d \quad (4.38b)
\]

\[
\psi_{cr} = \frac{\gamma}{\sqrt{n_{20}}} \tanh \left( \frac{k\gamma(x - x_1)}{\sqrt{2}} \right) \quad x > d \quad (4.38c)
\]

for the symmetric modes with \( j = 1, 3, 5, \ldots \) and

\[
\psi_{cl} = \frac{\gamma}{\sqrt{n_{20}}} \tanh \left( \frac{k\gamma(x + x_1)}{\sqrt{2}} \right) \quad x < -d \quad (4.39a)
\]

\[
\psi_f = (-1)^{\frac{j-1}{2}} \frac{2}{n_{2f}} \left[ \frac{k_m}{\sqrt{1 + m^2}} \right] \text{sn} \left[ \frac{k\kappa x}{\sqrt{1 + m^2}} |m| \right] \quad |x| < d \quad (4.39b)
\]

\[
\psi_{cr} = \frac{\gamma}{\sqrt{n_{20}}} \tanh \left( \frac{k\gamma(x - x_1)}{\sqrt{2}} \right) \quad x > d \quad (4.39c)
\]

for the anti-symmetric modes with \( j = 2, 4, 6, \ldots \) where the \( j \)th mode is characterized by \( j \) intensity peaks as shown in Figure 4-17, where (a) and (c) represent the field profiles and (b) and (d) display the corresponding intensity profiles.
Figure 4-17: Demonstration of field [in (a) and (c)] and intensity [in (b) and (d)] profiles of bright nonlinear modes with non-zero intensity background with \( \frac{n_{20}}{n_{2f}} = 10 \) and \( \frac{n_o}{n_f} = 1.1 \); (a) and (b) show the field and intensity profiles of the \( j = 1 \) mode at \( \Omega = 0.25\pi \), \( m = 0.6 \) and \( n_e = 0.87128n_f \) (identified by the solid curve) and of the \( j = 2 \) mode at \( \Omega = 0.75\pi \), \( m = 0.5 \) and \( n_e = 0.74804n_f \) (marked by the dashed curve); (c) and (d) display the field and intensity profiles of the \( j = 3 \) mode at \( \Omega = 1.25\pi \), \( m = 0.6 \) and \( n_e = 0.96331n_f \) (identified by the solid curve) and of the \( j = 4 \) mode at \( \Omega = 1.75\pi \), \( m = 0.5 \) and \( n_e = 0.93190n_f \) (indicated by the dashed curve).
4.3.1 Dispersion Relations and Range of Existence

The relationship between \( n_e \) and \( m \) in (4.38) and (4.39) is determined by the following dispersion relations

\[
\frac{\gamma^2}{\kappa^2} = 2 \sqrt{\frac{n_{20}}{n_{2f}}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_{20}}{n_{2f}}} \frac{\cn^2 [\Omega|m]}{\dn^2 [\Omega|m]} - (-1)^{\frac{j+1}{2}} (1 - m^2) \sn [\Omega|m] \right\} / \dn^2 [\Omega|m] \tag{4.40}
\]

for the symmetric modes and

\[
\frac{\gamma^2}{\kappa^2} = 2 \sqrt{\frac{n_{20}}{n_{2f}}} \frac{m}{1 + m^2} \left\{ m \sqrt{\frac{n_{20}}{n_{2f}}} \frac{\sn^2 [\Omega|m]}{\dn [\Omega|m]} + (-1)^{\frac{j-1}{2}} \cn [\Omega|m] \dn [\Omega|m] \right\} \tag{4.41}
\]

for the anti-symmetric modes. The corresponding \( x_1 \) is

\[
x_1 = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{(-1)^{\frac{j-1}{2}} \sqrt{2 \frac{n_{20}}{n_{2f}}} \left( \frac{km}{\gamma \sqrt{1 + m^2}} \right) \frac{\cn [\Omega|m]}{\dn [\Omega|m]} - 1}{(-1)^{\frac{j+1}{2}} \sqrt{2 \frac{n_{20}}{n_{2f}}} \left( \frac{km}{\gamma \sqrt{1 + m^2}} \right) \frac{\cn [\Omega|m]}{\dn [\Omega|m]} + 1} \tag{4.42}
\]

and

\[
x_1 = d + \frac{1}{\sqrt{2k\gamma}} \ln \frac{(-1)^{\frac{j-1}{2}} \sqrt{2 \frac{n_{20}}{n_{2f}}} \left( \frac{km}{\gamma \sqrt{1 + m^2}} \right) \frac{\sn [\Omega|m]}{\dn [\Omega|m]} - 1}{(-1)^{\frac{j-1}{2}} \sqrt{2 \frac{n_{20}}{n_{2f}}} \left( \frac{km}{\gamma \sqrt{1 + m^2}} \right) \frac{\sn [\Omega|m]}{\dn [\Omega|m]} + 1} \tag{4.43}
\]

for the symmetric and anti-symmetric modes, respectively. The existence of the \( j^{th} \) mode is limited to the range

\[
(j - 1) K(m) < \Omega < j K(m). \tag{4.44}
\]
Note that in contrast to the asymmetric dark modes discussed previously, there can exist only one bright mode with non-zero intensity background (either symmetric or antisymmetric) at a fixed value of $\Omega$.

Figures 4-18 and 4-19 show the dispersion curves which describe the relationship between $n_e$ and $m$. Here, $n_e$ increases with increasing $m$ when $\frac{n_o}{n_f} > 1$ whereas it decreases with increasing $m$ when $\frac{n_o}{n_f} < 1$ (Figures 4-18(b) and 4-19(b)). Large ratios of nonlinear coefficients lead to a wider trapping range $\Delta m = \max\{m\} - \min\{m\}$ (Figures 4-18(a) and 4-19(a)). With regard to the variation in the ratio of $\frac{n_o}{n_f}$, the maximum $\Delta m$ occurs when $\frac{n_o}{n_f} \approx 1$, and it decreases with increasing $\frac{n_o}{n_f}$ when $\frac{n_o}{n_f} > 1$, and with decreasing $\frac{n_o}{n_f}$ when $\frac{n_o}{n_f} < 1$ (Figures 4-18(b) and 4-19(b)).

Another factor in determining the range of these modes is the value of $\Omega$. Consider, for example, the $j = 1$ symmetric mode. When the value of $\Omega$ is small the trapping range is large (Figure 4-18(c)). On the other hand, the maximum trapping range for $j = 2$ antisymmetric mode occurs when $\Omega \approx 0.75 \pi$. Higher order modes exhibit similar characteristics.

Finally, it should be noted that there are no asymmetric bright modes with non-zero intensity background. This result can be proven by contradiction. First, suppose that there exists such an asymmetric mode. Dispersion relations for such a mode can be obtained by replacing $\Omega = \frac{k \kappa d}{\sqrt{1 + m^2}}$ with $\frac{k \kappa (d \pm x_f)}{\sqrt{1 + m^2}}$ in (4.40) and (4.41). The resulting (4.41) can be further simplified using the same method for deriving (4.37) to give

$$m \frac{\cd(\alpha d|m)}{\cd(\alpha x_f|m)} = (-1)^{j-1} \left( \frac{n_{2o}}{n_{2f}} - \frac{n_{2o}}{n_{2f}} - 1 \right)$$

(4.45)

where $\alpha = \frac{k \kappa}{\sqrt{1 + m^2}}$ and $j = 2, 4, 6, \ldots$. Consider the case of $j = 2$. Based on (4.44) the solution exists when $K(m) < \Omega = \alpha d < 2 K(m)$. However, in this range the left hand side of the equation (4.45) is negative [in this range $\frac{\cd(\alpha d|m)}{\cd(\alpha x_f|m)} < 0$ and since $0 < \alpha x_f < K(m)$, we have $\frac{\cd(\alpha x_f|m)}{\cd(\alpha x_f|m)} > 0$] whereas the right hand side, obviously, is positive. Similar contradiction occurs for higher orders. Hence such an
Figure 4-18: Dispersion relation of the fundamental bright nonlinear mode with non-zero intensity background: (a) for different nonlinear coefficient ratios \( \frac{n_{20}}{n_{2f}} \) with \( \frac{n_0}{n_f} = 1.1 \) and \( \Omega = 0.25\pi \); (b) for different linear refractive index ratios \( \frac{n_0}{n_f} \) with \( \frac{n_{20}}{n_{2f}} = 50 \) and \( \Omega = 0.25\pi \); and (c) for different values of \( \Omega \) with \( \frac{n_{20}}{n_{2f}} = 50 \) and \( \frac{n_0}{n_f} = 1.1 \).
Figure 4-19: Dispersion relation of the $j = 2$ bright nonlinear mode with non-zero intensity background: (a) for different nonlinear coefficient ratios $\frac{n_{20}}{n_{2f}}$ with $\frac{n_o}{n_f} = 1.1$ and $\Omega = 0.75\pi$; (b) for different linear refractive index ratios $\frac{n_o}{n_f}$ with $\frac{n_{20}}{n_{2f}} = 50$ and $\Omega = 0.75\pi$; and (c) for different values of $\Omega$ with $\frac{n_{20}}{n_{2f}} = 50$ and $\frac{n_o}{n_f} = 1.1$. 
asymmetric solution cannot exist.

The dispersion relations can be described in terms of the complementary power versus the modulus $m$ or $n_e$. Substituting the field expressions (4.38) and (4.39) into (4.32) and integrating yields:

\[ P_{\text{sym}} = P_{\text{anti}} + \frac{4m^2\kappa}{kn_2f\sqrt{1 + m^2}} \text{sn}(\Omega|m) \text{cd}(\Omega|m) \quad (4.46) \]

\[ P_{\text{anti}} = \frac{\gamma^2d}{n_2o} \left\{ \frac{\sqrt{2}}{kd\gamma} \left[ \frac{2}{\tanh \left( k\gamma \frac{d - x_1}{\sqrt{2}} \right)} - 2 \right] \right\} \]

\[ -2 + \frac{n_2o}{n_2f} \frac{2\kappa}{kd\gamma^2\sqrt{1 + m^2}} \left[ \frac{2k\kappa d}{\sqrt{1 + m^2}} - 2E(\pm \Omega|m) \right] \quad (4.47) \]

for the symmetric and anti-symmetric bright modes respectively. Figure 4-20 illustrates the dependence of the power on modulus $m$ and the dependence of $n_e$ on the power for these modes. Similar to the case of asymmetric dark modes, power in this case is inversely related to the modulus $m$ and the effective index decreases with increasing the power.

\[ \frac{n_2o}{n_2f} = 50 \]

\[ \frac{n_o}{n_f} = 1.1 \]

\[ \frac{2\Omega}{\pi} = 0.9, 0.99 \]

\[ n_e/m_e \]

\[ 0.3, 0.5 \]

\[ 0.5, 0.7, 0.9 \]

\[ 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 \]

\[ P_{n_2o}/dn_f^2 \]

\[ 0, 0.0001, 0.001 \]

\[ 0, 0.3, 0.5, 1.0 \]

\[ 0, 20, 40, 60 \]

\[ 0, 20, 40, 60 \]

\[ 0.99 = \frac{2\Omega}{\pi} \]

\[ 0.5, 0.7, 0.9 \]

\[ 0.3, 0.5, 0.7, 0.9 \]

\[ 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 \]

\[ Figure 4-20: (a) The relation between power and the modulus $m$ and (b) the relation between $n_e$ and power for the $j = 1$ bright mode where the structural parameters are the same as Figure 4-18(c). \]
Chapter 5

Stability of Nonlinear Modes

5.1 Motivations

Linear waveguides have found their uses in many practical applications since their modes are always stable. The stability of linear modes arises from the fact that guidance is provided as a result of the difference between refractive indices of the core and cladding. In contrast to linear waveguides, stationary nonlinear guided waves (NGW) are manifestations of an overall equilibrium between three interacting mechanisms. These mechanisms are diffraction, the linear guiding effect of the film and cladding, and trapping effect due to the nonlinear refractive index (self-focusing or self-defocusing). It is this third nonlinear mechanism which can cause the overall equilibrium to be broken and which leads to instability. The instability may arise from the imperfections in the waveguide or from stochastic and/or deterministic changes in the field amplitude which can lead to non-stationary behavior of the stationary initial excitation. In this case the field propagating down the waveguide experiences dramatic changes in its field profile. Therefore, it is imperative to investigate the stability characteristics of NGWs.

As was foreshadowed in Chapter 4, this Chapter is devoted to the study of the stability of asymmetric modes of the waveguides discussed in section 4.1.1. Although this problem has been investigated previously both numerically and analytically, there has been a long standing discrepancy between the numerical and
analytical approaches reported in the literature. This controversy will be addressed and resolved in section 5.2. In section 5.3 it will be shown that perturbation of the asymmetric modes gives rise to a class of quasi-periodic solutions. Finally, in section 5.4 the effect of loss on the propagation characteristics of the nonlinear mode is investigated.

5.2 Stability Analysis

5.2.1 Analytical and Numerical Stability Analyses

Consider the structure shown in the Figure 4-1. The propagation of the NGWs in this structure is governed by the nonlinear Schrödinger equation,

\[
\frac{\partial E}{\partial Z} + \frac{k}{2\beta} \frac{\partial^2 E}{\partial X^2} + \left[ \frac{n^2 - (\beta/k)^2}{2\beta/k} - i\gamma(X) \right] E = 0
\]  

(5.1)

where \( n^2 \) is defined by (4.1), \( \beta = n_e k \) is the propagation constant, \( k \) is the wavenumber. \( X = kx, Z = kz, \) and \( \gamma \) is the loss coefficient. To investigate stability characteristics of the NGWs we assume that the structure is lossless (\( \gamma = 0 \)).

Ideally, one would like to solve (5.1) exactly to obtain the \( z \)-dependent behavior of the NGWs. However, due to the inhomogeneity of the refractive index along the \( x \)-axis which results in the lack of translational symmetry, this equation can not be integrated using the elegant inverse scattering transform [11,24]. Consequently, one has to resort to approximate analytical methods and/or numerical techniques. For brevity, the detailed description of these methods is not included here. Nevertheless, the fundamental elements of each method are indicated.

The most widely used analytical (or semi-analytical) approach is linear stability analysis whose application has already been shown in Chapter 3. Essentially, this method is a perturbational analysis in that it assumes the initial stationary solution deviates from the exact solution by a small perturbation. Based on this assumption (5.1) can be linearized leading to a linear eigenvalue equation. This equation can then be treated either using the operator-theoretic approach [93,94] or by solving...
it directly [95–97]. The former utilizes the Sturm-Liouville theory and topological arguments to show that an unstable mode necessitates the existence of a positive eigenvalue. Notwithstanding its elegance, it is only valid for the fundamental modes (TE₀). The latter, however, can be applied to fundamental and higher-order NGWs and attempts to find the perturbational modes (eigenfunctions) as well as the growth rates (eigenvalues) corresponding to these eigenfunctions. The inherent assumption is that initially the instability grows exponentially. Therefore, a positive growth rate is the signature of unstable NGWs. The knowledge of dominant perturbational modes and their eigenvalues allows the identification of the stable and unstable regions of NGW dispersion curves.

Owing to the approximation (linearization) and assumptions in linear stability analysis, it is necessary to test its predictions using numerical techniques. There are two major numerical methods that have been successfully used to solve (5.1). The first method, which is known as the ‘beam propagation method’ (BPM) or split-step Fourier method [98] utilizes Fast Fourier transforms. The other method is characterized by a difference approximation of (5.1) where the Crank-Nicholson Scheme [99] is used to perform the differencing. This results in a system of algebraic equations which then can be solved using the Newton-Picard Algorithm [65].

Associated with any numerical algorithm is numerical noise. However small this may be, it acts as a perturbation to the field profiles under consideration. Hence the numerical techniques provide an automatic test of stability. Additionally, numerical stability analysis can demonstrate the character of non-stationary propagation of NGWs. This is in contrast to the analytical approach which can only predict whether the modes are stable or not. Apart from complementing the existing predictions of the analytical methods, numerical stability analysis is flexible in that it can easily be modified to accommodate effects such as material loss or saturation in nonlinearity.

5.2.2 Stability Prediction and the Controversy

The first analytical prediction of the stability of NGWs of the structure described in Figure 4-1 was reported by Jones and Moloney [93]. For the asymmetric modes
of the structure it was predicted that if the mode is located in the positively-sloped region of the nonlinear dispersion curve, then it is stable (i.e. stable when $\frac{dP}{d\beta} > 0$). On the other hand, if the mode is in the negatively-sloped region, then it is unstable (i.e. unstable when $\frac{dP}{d\beta} < 0$). This result was in agreement with the prediction of numerical stability analysis performed for a particular set of parameters [100, 101] (see Figure 5-1(a)). However, by reducing the structural parameters of $V = kd\sqrt{n_f^2 - n_0^2}$ from $V = 3.3$ in Figure 5-1(a) to $V = 0.97$ in Figure 5-1(b), the minimum, which demarcates the stable and unstable sections of the branch and is marked by the filled triangle in Figure 5-1(a), can disappear, leading to the entire branch of the dispersion curve to be positively sloped. In this case, Leine et al. [102] demonstrated numerically that the asymmetric modes confined within the filled circle (called the bifurcation point) and the filled square (termed as the transition point\(^1\)) are unstable.

\[ V = kd\sqrt{n_f^2 - n_0^2} \]

\( d \) and \( n_f = 2 \), \( d = 0.2 \) and \( n_f = 1.55 \), \( d = 0.4 \), where \( \lambda \) is the wavelength in free space, the filled circles mark the bifurcation point \( \beta = \beta_b \), the filled square identifies the transition point \( \beta = \beta_t \), and the filled triangle represents the minimum \( \beta = \beta_m \). The solid line represents the stable region and the dashed line and dashed-dotted lines stand for the unstable modes.

\( ^1 \) At this point the field maximum coincides with the boundaries between the linear and nonlinear media.

\[ \beta/k \]

\[ \beta/k \]

\[ 0 \]

\[ 0.8 \]

\[ 0 \]

\[ 0.4 \]

\[ 0.2 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1.8 \]

\[ 1.9 \]

\[ 2 \]

\[ 2.1 \]

\[ 2.2 \]

\[ 2.3 \]

\[ 2.4 \]

\[ 2.5 \]

\[ 1.52 \]

\[ 1.53 \]

\[ 1.54 \]

\[ 1.55 \]

\[ \text{Power} \]

\[ \beta/k \]

\( \beta/k \)

\[ \text{Figure 5-1: Dispersion curves for the slab waveguide illustrated in Figure 4-1. Structural parameters are } n_0 = 1.5, \text{ and (a) } n_f = 2, \frac{d}{\lambda} = 0.2 \text{ and (b) } n_f = 1.55, \frac{d}{\lambda} = 0.4, \text{ where } \lambda \text{ is the wavelength in free space, the filled circles mark the bifurcation point } \beta = \beta_b, \text{ the filled square identifies the transition point } \beta = \beta_t, \text{ and the filled triangle represents the minimum } \beta = \beta_m. \text{ The solid line represents the stable region and the dashed line and dashed-dotted lines stand for the unstable modes.} \]
Clearly, this contradiction challenges the $\frac{dP}{d\beta}$ criterion which was advanced by both numerical and analytical stability analyses. Naturally, one wonders where this discrepancy originates from. Is it, for example, due to the violation of the approximations involved in the analytical approach? Or is it because of the breakdown of the numerical scheme? More importantly, which point (the minimum point and the transition point) does correctly demarcate the stable and unstable sections of the branch?

### 5.2.3 Resolution of the Controversy

In general, for a given set of structural parameters both transition and minimum points can coexist in the dispersion diagram. Hence, the first task is to identify how the location of these points varies with changes in structural parameters. As is shown in Figure 5-2 for constant $n_0$ and $n_f$ the relative location of these points varies with the film width $2d$, i.e. $\beta_t$ can be greater or smaller than $\beta_m$, depending on the value of $d$.

![Figure 5-2: Same as Figure 5-1 but $n_f = 1.6$ where (a) $\frac{2d}{\lambda} = 0.6$ and (b) $\frac{2d}{\lambda} = 0.7$.](image)

The overall dependence of the transition point $\beta_t$, the minimum $\beta_m$, and the
bifurcation point $\beta_b$ on $d$ is shown in Figure 5-3. For the parameters $n_f = 1.6$ and $n_o = 1.5$ of Figure 5-3(a), when $\frac{2d}{\lambda} < 0.624$, $\beta_t$ is greater than $\beta_m$, and when $\frac{2d}{\lambda} > 0.624$, $\beta_t$ is smaller than $\beta_m$. Also, for the parameters $n_f = 1.55$ and $n_o = 1.5$ of Figure 5-3(b), when $\frac{2d}{\lambda} < 0.885$, $\beta_t$ is greater than $\beta_m$, and when $\frac{2d}{\lambda} > 0.885$, $\beta_t$ is smaller than $\beta_m$. The shaded regions (I) and (II) which are confined between the lines $\beta = \beta_t$ and $\beta = \beta_m$ represent the controversial regions. According to the criterion proposed in [93, 100, 101] the modes within region (I) are stable and in region (II) are unstable. However, on the basis of the criterion proposed in [102] the modes in region (I) are deemed to be unstable and in region (II) may be stable.

In order to resolve this controversy we conducted a numerical stability analysis using the aforementioned Fourier method (BPM). An example of propagation characteristics is shown in Figure 5-4 where the width of the simulation window is taken to be $40d$ and the total number of sampling points is 1024. Figure 5-4(a) demonstrates the propagation of the asymmetric mode with $\frac{\beta}{k} = 1.5745$ and $\frac{2d}{\lambda} = 0.6$.
(1.573 = \frac{\beta_m}{k} < \frac{\beta}{k} < \frac{\beta_l}{k} = 1.578) which is located in region (I). In Figure 5-4(b) the mode at \( \frac{\beta}{k} = 1.585 \) with \( \frac{2d}{\lambda} = 0.7 \), which is located in region (II), is initially excited. Both modes exhibit oscillatory unstable evolution. Here, Figure 5-4(a) seemingly contradicts the results of the linear stability analysis. However, By increasing the number of sampling points, we find that the propagation starting from the stationary solutions in shaded region (I) is stable, whereas that starting from those in shaded region (II) always demonstrates oscillatory unstable evolution. This result is shown in Figure 5-5 where the initial excitations are identical to Figure 5-4 but with total sampling points of 4096. Similarly, the asymmetric modal solutions within the dashed curve in Figure 5-1(b) demonstrate unstable oscillatory propagation when the sampling points are too small, such as that in Figure 5-6(a) for 128 sampling points [102]. However, stable propagation results when the sampling points are increased, such as to 4096 as shown in Figure 5-6(b). Therefore, we conclude that the numerical and analytical methods do indeed agree, namely, the minimum point demarcates the stable and unstable sections of the asymmetric branch of the dispersion diagram where the stable section is characterized by \( \frac{dP}{d\beta} > 0 \) and the unstable section by \( \frac{dP}{d\beta} < 0 \).

To a large extent, the numerical noise in this case arises from the difference between the refractive indices of the core and cladding. As a result of the discretization of the field, this difference manifests itself in the form of a sudden change in the phase during propagation. The fewer the number of sampling points are the bigger this phase jump is. Therefore, the effect of increasing the number of sampling points is equivalent to decreasing the level of the numerical noise. In addition, it must be noted that the degree of stability of the modes in the stable section is not the same. That is to say as one approaches the minimum point the modes become less and less stable\(^1\) so that at the minimum point the stability changes. Indeed, if one were to propagate a mode in the proximity of the minimum point in the stable section of

\(^1\)Here by less stable we mean the modes are stable to smaller perturbations.
Figure 5-4: Demonstration of the beam propagation of the asymmetric nonlinear mode in the shaded regions of Figure 5-3 with 1024 sampling points. (a) The parameters are of Figure 5-2(a) at $\frac{\beta}{k} = 1.5745$ ($> \frac{\beta_m}{k}$ but $< \frac{\beta_l}{k}$) where $\frac{dP}{d\beta} > 0$; (b) the parameters are of Figure 5-2(b) at $\frac{\beta}{k} = 1.585$ ($< \frac{\beta_m}{k}$ but $> \frac{\beta_l}{k}$).
Figure 5-5: Same as Figure 5-4 but with sampling points of 4096.

Figure 5-6: Demonstration of the beam propagation of the asymmetric nonlinear mode in the dashed section of the 5-1(b) at $\frac{\beta}{k} = 1.5355$ with sampling points of (a) 128 and (b) 4096.
the asymmetric branch of say Figure 5-1(a) using say 128 (or even 512) sampling points, one would observe seemingly unstable behavior. Similar situation occurs when the branch is positively-sloped, in which case the minimum and the bifurcation points coincide.

5.3 Quasi-Periodic Solutions

One of the striking features shown in Figure 5-4 is the oscillatory behavior of the asymmetric mode. This behavior occurs both in the unstable section and weakly stable section. The initial excitation of the asymmetric mode with its peak in the vicinity of the right boundary of the waveguide (termed as the mode on the right) gradually evolves to the corresponding degenerate one with its peak occurring in the vicinity of the left boundary of the waveguide (called the mode on the left) as the instability develops. Upon further evolution the mode returns back to the initially excited mode on the right. This characteristic is repeated at a distance approximately equal to $Z = 480$ for the excitation of Figure 5-4(a) and $Z = 420$ for the excitation of Figure 5-4(b).

In order to compare the field on the right (left) with the evolved field resulting from an initial excitation of the left (right) field we use the following correlation relations:

\[ C_R(Z) = \frac{\int_{-\infty}^{+\infty} |E(X, Z)||E_R(X, 0)|dX}{\int_{-\infty}^{+\infty} |E_R(X, 0)|^2dX} \]

\[ C_L(Z) = \frac{\int_{-\infty}^{+\infty} |E(X, Z)||E_L(X, 0)|dX}{\int_{-\infty}^{+\infty} |E_L(X, 0)|^2dX} \]
where $E(X, Z)$ stands for the evolved field and $E_R$ and $E_L$, one of which is initially excited, denote the modes on the right and the left respectively. Hence a correlation coefficient of 1 represents a perfect matching of the fields under consideration. As is shown in Figure 5-4, $C_R$ and $C_L$ are almost unity ($C_{R,L} > 0.99$), indicating that the propagation starting from the excitations of the weakly stable (Figure 5-4(a)) and unstable (Figure 5-4(b)) modes indeed evolve toward the stationary solution of the paired degenerate modes. Excitations from other unstable or weakly stable points lead to similar results. In general, the closer the excitation to the bifurcation point $\beta_b$ or to $\beta = \max\{\beta, \beta_m\}$, the longer the oscillation period is. We thus conclude that the structure shown in Figure 4-1 may also support a class of quasi-periodic solutions as a result of the instability development of the asymmetric mode.

It should be noted that the quasi-periodic solutions described here are not the same as those discussed in Refs. [100, 101] where the authors demonstrated the aperiodical beam propagation (or aperiodical non-stationary solutions) developed from the excitation of the unstable stationary solutions.

**5.4 Effects of Loss**

Absorptive and dispersive properties of highly nonlinear materials are intimately connected. Hence, an essential step towards understanding and describing the practical situations is to consider material losses. Also, since the existence of loss deteriorates the characteristics of NGWs, it is necessary to estimate the extent of the endurable absorption effects.

Contrary to linear waveguide theory, the presence of loss in nonlinear waveguides results in an interaction of power dissipation and redistribution effects along the nonlinear waveguide. This complex interaction gives rise to non-stationary behavior of NGWs, i.e. in a lossy medium, the stationary propagation of the NGWs is no longer possible and the propagation characteristics of the initial modal excitation are modified as was demonstrated in Refs. [103, 104].

Here we examine the effect of loss on the evolution of the nonlinear mode for the
structure characteristic of Figure 5-2 as a complement to the earlier work [103,104].
We solve (5.1) for the loss profile of $\gamma = 0$ in $|x| < d$ (i.e. no loss in linear film)
and $\gamma = 0.01$ when $|x| < d$ (lossy nonlinear cladding). Our study shows that the
excitation of the mode with $\beta > \beta_m$ follows the dispersion curve down towards
the point $\beta_m$ as is shown in Figures 5-7(a) and 5-8(a). On the other hand, the
excitation below or the evolution passing down through the point $\beta = \beta_m$, leads
to the oscillation of the beam and it wanders away from the dispersion curve; but
eventually it evolves back to the dispersion curve on the symmetric mode branch
below the bifurcation point. This is shown in Figures 5-7 and 5-8 where we have
chosen 1024 sampling points. Increasing the number of sampling points here does
not change the general propagation characteristics.
Figure 5-7: Propagation of asymmetric mode in the waveguides with lossy cladding. The structural parameters are those of Figure 5-2(a). (a) $\frac{\beta}{k} = 1.8$ and (b) $\frac{\beta}{k} = \frac{\beta_m}{k} = 1.5779$.

Figure 5-8: Propagation of asymmetric mode in the waveguides with lossy cladding. The structural parameters are those of Figure 5-2(b). (a) $\frac{\beta}{k} = 1.8$ and (b) $\frac{\beta}{k} = \frac{\beta_m}{k} = 1.5934$.  

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Chapter 6

Saturable Nonlinear Couplers

This chapter is concerned with the study of nonlinear directional coupler (NLDC). In particular, we shall concentrate on couplers with saturating nonlinearity. Section 6.1 provides a brief description of NLDC. In section 6.2, nonlinear saturation models will be discussed. In Section 6.3 passive saturable NLDCs will be considered and their qualitative behavior using bifurcation diagrams and phase space portraits will be presented and their use in device applications will be shown. Finally, in section 6.4 the effect of gain on saturable NLDC will be investigated.

6.1 Introduction

A nonlinear directional coupler is composed of two parallel, closely spaced, single-moded waveguides in a material with intensity-dependent refractive index. Figure 6-1 displays such a device schematically. When the intensity of light is low the NLDC behaves like a linear coupler, namely, evanescent coupling causes that light introduced into guide (1) to completely transfer to guide (2) in one coupling length $L_c$. As the input intensity increases the refractive index changes, and accordingly some light remains in guide (1). At sufficiently high intensities coupling is inhibited. Therefore, this device can act as a power dependent switch. As a result, considerable attention has been given to this device due to its potential applications in optical computing, switching, amplification and optical signal processing [27, 105–136].
The possibility of a twin core nonlinear directional coupler functioning as a power dependent switch was predicted in the early 1980’s by Jensen [27]. He considered a coupler composed of two identical cores with Kerr nonlinearity. In quantitative terms he showed that the amount of light emerging from guide (1) is given by

\[ P_1(L) = P_1(0) \frac{1 + \text{cn} \left( \frac{\pi L}{L_c} \left[ \frac{P_1(0)}{P_c} \right]^2 \right)}{2} \]  

where \( \text{cn} \) is the Jacobi elliptic function, \( L \) is the length of the coupler, \( P_1(0) \) is the input power introduced into guide (1) and \( P_c \) is the critical power at which 50% of the light emerges from each waveguide, and is given by \( P_c = \frac{A \lambda}{L_c n_2} \) where \( A \) is the effective mode area, \( \lambda \) is the vacuum wavelength, and \( n_2 \) is the nonlinear index.

Recently, it has been demonstrated experimentally that for a device of length 2 meters the critical power \( P_c \) is 850 watts [113]. For practical applications, such power levels are too high. Ideally, nonlinear couplers are desired to operate at low powers. This then prompts a search for materials with high nonlinearities [133]. However, high nonlinearities introduce the problem of nonlinear saturation [133,135,136] which gives rise to the limitation of the operational power of the devices using nonlinear couplers [114–117,121,130].
The studies reported so far have mainly concentrated on couplers composed either of two identical cores in the presence of nonlinear saturation [114–117, 127–130] or of two nonidentical nonlinear cores in the absence of nonlinear saturation [108, 110, 131]. Although the special case of the saturable nonlinear couplers with nonlinear coefficients of opposite sign has previously been examined [129], a general analysis has not been carried out. Such a study is necessary as in practice a coupler may be made of two cores with different nonlinear saturation. For example, using erbium as dopant, a difference between doping concentration of the two cores could lead to mismatches in both nonlinearities and in the linear propagation constants of each core. Furthermore, as the nonlinear coefficient of erbium-doped fibers is approximately one million times greater than that of fused quartz [133], an erbium-doped coupler normally operates in saturation region even when the input power is of the order of several milliwatts. This accounts for incomplete power switching from one core to the other even for a coupler of two coupling lengths long [135].

Also, with the advent of newly developed doping technologies, active (e.g. erbium-doped) fibers are now readily available for fabricating active nonlinear couplers [133]. An example of such a device is an erbium-doped coupler operating at a signal frequency (different from the pump wavelength but within the gain spectrum). An active coupler with identical cores has been shown to possess the potential for lower switching power, but unfortunately at the expense of degraded switching characteristics [134]. Thus, there is a need for a comprehensive study of passive and active nonlinear couplers composed of two different saturable nonlinear cores to fully exploit their potential applications.

6.2 Theoretical Framework

6.2.1 Modelling of Saturating Nonlinearity

Due to nonlinearity, the refractive index change of a material increases or decreases with increasing intensity when the light intensity is low but it saturates at a certain level when the intensity is high. This nonlinear index saturation can be described
by various mathematical models - the exponential [114], the two level model [116],
or the polynomial [133]. Although quantitatively different, these models give the
same qualitative picture of nonlinear saturation effect on the device performance.
For instance, the nonlinear saturation index of an erbium-doped fiber [133] outlined
by the polynomial model (the dotted line in Figure 6-2) can well be delineated by
the exponential model and the two level model as shown with the solid line and the
dashed line in Figure 6-2.

\[ y^A_n = 0.27 \times 10^6 [1 - \exp(-1.6 P)] \]
\[ A_n = 0.48 \times 10^6 P/(1 + 1.4 P) \]

**Figure 6-2:** Nonlinear refractive index variation \( \Delta n \) of erbium-doped fibers where
the dotted line describes the experimental measurements characterized
by \( \Delta n = n_2 I + n_3 I^2 + n_4 I^3 \) with \( n_2 = 3.3 \times 10^{-11} \text{cm}^2/\text{W}, \ n_3 = -1.44 \times 10^{-15} \text{cm}^6/\text{W}^2, \) and \( n_3 = 2.09 \times 10^{-20} \text{cm}^6/\text{W}^4. \) The solid line is the exponential
fitting, the dashed line is the two-level fitting, and the power \( P \) (in Watts)
is related to the intensity by \( P = A_{eff} I, \) where \( A_{eff} = 9 \times 10^{-8} \text{cm}^2. \)

**6.2.2 Coupled Mode Theory**

Our analysis is based on the coupled mode theory which has been proven to be
a simple and reliable means for describing the operation of a nonlinear coupler
with weak nonlinear perturbations and sufficiently well separated cores. When gain
and loss are included, the theory can still be employed by simply introducing the imaginary part of the modal propagation constants into the coupled mode equations, which for the modal amplitudes $a_{1,2}$ of individual waveguides, read

$$-i\frac{da_1}{dz} = [\beta_1 + \Delta\beta_1(|a_1|^2) + i\alpha_1(|a_1|^2)]a_1 + Ca_2$$  \hspace{1cm} (6.2a)

$$-i\frac{da_2}{dz} = [\beta_2 + \Delta\beta_2(|a_2|^2) + i\alpha_2(|a_2|^2)]a_2 + Ca_1$$  \hspace{1cm} (6.2b)

where $\beta_j$ is the modal propagation constant of the $j^{th}$ waveguide ($j = 1, 2$), $\alpha_j(|a_j|^2)$ is the saturable gain or loss coefficient with $\alpha_j > 0$ or $\alpha_j < 0$ denoting loss or gain respectively. $C$ refers to the linear coupling coefficient $\text{[124]}$ and $\Delta\beta_j(|a_j|^2)$, and is the power dependent refractive index change for arbitrary nonlinearity in guide $j$. For an exponential saturable nonlinearity, $\Delta\beta_j$ may take the form $\text{[118]}$

$$\Delta\beta_j(|a_j|^2) = \eta_j 2C w_j [1 - \exp(-2\frac{|a_j|^2}{w_j P_{cj}})],$$  \hspace{1cm} (6.3)

and for the two level model

$$\Delta\beta_j(|a_j|^2) = \eta_j 4C \frac{|a_j|^2}{P_{cj}(1 + 2\frac{|a_j|^2}{w_j P_{cj}})},$$  \hspace{1cm} (6.4)

where $P_{cj}$ is the critical power defined for the conventional nonlinear coupler $\text{[27]}$. $\eta_j = \pm 1$ where + and -- correspond to self-focusing and self-defocusing nonlinearities respectively, and $w_j$ refers to the normalized saturation parameter related to the real saturation value $\Delta\beta_{j,\text{sat}}$ by $w_j = \frac{\Delta\beta_{j,\text{sat}}}{2C}$. Physically speaking, $w_j$ is the maximum mismatch that nonlinearity can induce in core $j$ as the power approaches infinity. It should be noted that it is the mismatch, not the refractive index difference, between the two cores that governs the coupler’s behavior $\text{[27,133,136]}$. Clearly, as $w_j \to \infty$ the saturation index ($\Delta\beta_j$) reduces to Kerr nonlinearity. As it was shown in the subsection 6.2.1 the three models of nonlinear saturation are qualitatively equivalent. Hence without loss of generality we are going to employ the exponential model for the analysis.
The system of equations (6.2) can be expressed in terms of real quantities as follows [107]:

\[
\begin{align*}
\frac{dS_1}{dz} &= -(\alpha_1 + \alpha_2)S_1 - (\alpha_1 - \alpha_2)P + 2CS_3 \\
\frac{dS_2}{dz} &= -(\alpha_1 + \alpha_2)S_2 - (\beta_1 - \beta_2 + \Delta\beta_1 - \Delta\beta_2)S_3 \\
\frac{dS_3}{dz} &= -(\alpha_1 + \alpha_2)S_3 - 2CS_1 + (\beta_1 - \beta_2 + \Delta\beta_1 - \Delta\beta_2)S_2 \\
\frac{dP}{dz} &= -(\alpha_1 + \alpha_2)P - (\alpha_1 - \alpha_2)S_1
\end{align*}
\]

(6.5a)  (6.5b)  (6.5c)  (6.5d)

where \(S_1 = |a_1|^2 - |a_2|^2\), \(S_2 = a_1a_2^* + a_1^*a_2\), \(S_3 = ia_1a_2^* - ia_1^*a_2\), and the total power \(P = P_1 + P_2 = (S_1^2 + S_2^2 + S_3^2)^{\frac{1}{2}}\).

In general the above system is not solvable analytically except for certain special cases. One such case is in the presence of nonlinearities, but with \(\alpha_j = 0\). In this case, the total power of the system is conserved, i.e. \(P = \text{constant}\), and the solutions to the above system can be expressed as an integral form [130]

\[
z = \frac{\pm 1}{2C} \int_{S_1(0)}^{S_1(z)} \frac{dS_1}{\{P^2 - S_1^2 - [\Gamma - \delta S_1 - \gamma(S_1)]^2\}^{\frac{1}{2}}},
\]

(6.6)

where the linear mismatch is defined as

\[
\delta = \frac{\beta_1 - \beta_2}{2C},
\]

(6.7)

and \(\Gamma\), which is an invariant of the system, is given by

\[
\Gamma = S_2 + \delta S_1 + \gamma(S_1),
\]

(6.8)

with
\[
\gamma(S_1) = (w_1 - w_2)(S_1 - S_1^0) + P_{c1} w_1^2 \exp\left(-\frac{P}{w_1 p_{c1}}\right)
\]

\[
\cdot \left[\exp\left(-\frac{S_1}{w_1 p_{c1}}\right) - \exp\left(-\frac{S_1^0}{w_1 p_{c1}}\right)\right]
\]

\[
+ P_{c2} w_2^2 \exp\left(-\frac{P}{w_2 p_{c2}}\right) \left[\exp\left(-\frac{S_2}{w_2 p_{c2}}\right) - \exp\left(-\frac{S_2^0}{w_2 p_{c2}}\right)\right],
\]

(6.9)

where \(S_1^0 = S_1(0)\).

### 6.3 Passive Saturable Nonlinear Couplers

We now proceed to analyze the behavior of saturable nonlinear couplers. Equations (6.5) will be the starting point throughout the analysis.

Given the definitions of \(S = P_1 - P_2\) and \(P = P_1 + P_2\), the power in each guide is found to be:

\[
P_{1,2} = \frac{1}{2}(P \pm S_1),
\]

(6.10)

with \(S_1\) being described by (6.6). Evaluation of such an integral is inevitable if one wants to know the details (e.g. period of power transfer) for a given initial excitation. This means that in order to understand the general characteristics of the system it is necessary to do exhaustive numerical calculations. In a conservative system it is possible to avoid such calculations by considering phase space portraits. For the case of passive couplers (no loss and/or gain) since \(P = (S_1^2 + S_2^2 + S_3^2)^{\frac{1}{2}} = constant\), the solutions to (6.5) can be interpreted geometrically, i.e. they form a set of curves on the surface of a sphere of radius \(P\). This, in turn, enables the construction of phase space portraits by projecting the curves onto a plane, such as the \(S_1 - S_3\) plane. Phase space portrait imparts the overall features of the system. Although it gives a qualitative picture, it renders a crucial aid for understanding the properties of couplers and exploiting them for device applications.

The characteristics of the phase space portrait are generally dictated by the
stable and unstable singular points of the phase space which are associated with the 
initial excitation conditions for launching the uniform normal modes of a nonlinear 
coupler. These modes are determined by the simple algebraic equations [130]

\[
\pm \frac{P_1(0) - P_2(0)}{\sqrt{P_1(0)P_2(0)}} = 2\delta + \frac{\Delta \beta_1(P_1(0)) - \Delta \beta_2(P_2(0))}{C},
\]

(6.11)

which may be derived by setting the polynomial in the denominator of the integrand 
of (6.6) to zero and using (6.10) to simplify the resulting equation. In (6.11) the 
upper and lower signs correspond to in-phase and antiphase modes, respectively.

6.3.1 Mode (Bifurcation) Diagrams

The stable and unstable singular points presented on the coordinate system, with 
total power \(P\) as the vertical axis and \(S_1\) as the transverse axis, gives the mode 
(bifurcation) diagram [137] or the power flow portrait [131]. Typical bifurcation 
diagrams for nonsaturable nonlinear couplers and saturable nonlinear couplers com­
posed of two linearly mismatched cores \((\beta_1 - \beta_2 \neq 0)\) have been given in [130,131].

Figure 6-3 presents the mode diagrams of saturable nonlinear couplers with iden­
tical cores where \(w_1 = w_2 = 3 > 1.63\) in (a), and \(1.36 < w_1 = w_2 = 1.6 < 1.63\) in
(b). The dashed curves denote the unstable modes and the solid curves represent 
the stable modes. The mode diagram for \(w_1 = w_2 < 1.36\) is a straight line located 
at \(S_1 = 0\), i.e. all the unstable modes disappear and the nonlinear coupler behaves 
like a linear coupler. As illustrated, irrespective of the values of \(w_1 = w_2\), the mode 
diagrams are symmetric, reflecting the symmetry of the system. However once a 
mismatch (either nonlinear \((w_1 - w_2 \neq 0)\) or linear \((\delta \neq 0)\)) is introduced, the sym­
metry of the mode diagrams is broken and their formation depends on both \(w_1\) and 
the mismatch \((w_1 - w_2)\) as shown in Figures 6-4, 6-5 and 6-6 for different nonlinear 
mismatches. Although the critical \(w_j\), above which one or two unstable modes may 
occur at a fixed power, increases with increasing \(w_1 - w_2\), there can still be three 
typical sets of mode diagrams.

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Figure 6-3: Mode diagrams of nonlinear couplers composed of two identical saturable nonlinear cores ($\delta = 0$ and $w_1 = w_2$). (a) For $w_1 = 3$ (> 1.63) and (b) $w_1 = 1.6$ (1.36 < $w_1$ < 1.63).

Figure 6-4: Mode diagrams showing the location of the in-phase modes on the $S_1$ axis versus power for various values of $w_1$ with small mismatch $w_1 - w_2 = 0.1$ and $\delta = 0$; (a) large $w_1 = 4$ ($w_1 > 2.87$), (b) moderate $w_1 = 2.5$ (1.70 < $w_1$ < 2.87), (c) small $w_1 = 1.65$ ($w_1 < 1.70$). $P_{c2}$ is decided by the relation $w_1 P_{c1} = w_2 P_{c2}$.

Two unstable modes may appear within certain power levels as occurring in Figures 6-4(a) and 6-5(a) for small or moderate mismatches and a large $w_j$. But
for a moderate \( w_j \) and a small or moderate mismatch, or for a large \( w_j \) and a large mismatch, only one unstable mode can possibly survive within certain power levels, such as those shown in Figures 6-4(b), 6-5(b) and 6-6(a). The third of the mode diagrams is for the case of small \( w_j \) as shown in Figures 6-4(c), 6-5(c) and 6-6(b) where only one stable mode can exist. It should be noted that all the modes start at \( S_1 = 0 \) when \( P = 0 \), irrespective of the magnitude of \( w_j \), i.e. the structures are matched as \( P \rightarrow 0 \). This is in contrast to the case when only linear mismatch exists, for which the mode diagrams start at \( S_1 \neq 0 \) when \( P = 0 \). Apart from the case of \( P \rightarrow 0 \) the mode diagrams corresponding to linear mismatch alone [130] and the above results have the same qualitative appearance.

![Figure 6-5](image)

*Figure 6-5:* Mode diagrams showing the location of the in-phase modes on the \( S_1 \) axis versus power for various values of \( w_1 \) with moderate mismatch \( w_1 - w_2 = 0.6 \) and \( \delta = 0 \); (a) large \( w_1 = 8 \) \( (w_1 > 5.61) \), (b) moderate \( w_1 = 5.5 \) \( (2.61 < w_1 < 5.61) \), (c) small \( w_1 = 2.5 \) \( (w_1 < 2.61) \).
The next question to address is what happens when both linear mismatch and mismatch in nonlinear saturation are present. From simple superposition one may deduce that a linearly mismatched nonlinear coupler composed of two different saturable cores may qualitatively be characterized by mode diagrams in Figures 6-4, 6-5, and 6-6, or the ones in Ref. [130], provided that \( \delta(w_1 - w_2) > 0 \) or \( |\delta| > |w_1 - w_2|(|\delta| < |w_1 - w_2|) \), where the linear (nonlinear) mismatch dominates. It is found that this is indeed the case. Nevertheless, when \( \delta(w_1 - w_2) < 0 \) and the magnitudes of \( \delta \) and \( w_1 - w_2 \) are comparable, new types of mode diagrams emerge which are illustrated in Figure 6-7. As shown, a small amount of linear mismatch \( \delta \) has the effect of merging the two branches of the mode diagram in Figure 6-4(a) (with \( \delta = 0 \)) into one branch mode diagram of Figure 6-7(a). With \( \delta \) increasing to \( \delta_c = -0.096 \) (Figure 6-7(b)) a degenerate mode diagram results, similar to that occurring in a symmetric structure of Figure 6-3(a). The difference between Figures 6-3(a) and 6-7(a) is that the bifurcation of the former occurs at two power values but only once for the latter, and that for the former the stable mode at high power \( (P \to \infty) \) is located at \( S_1 = 0 \), but for the latter at \( S_1 \neq 0 \) (at \( S_1 = -0.004P \) in this case). When \( \delta \) increases further, the mode diagram then transits to a typical
Figure 6-7: Mode diagrams for the in-phase modes of mismatched nonlinear couplers with a nonlinear mismatch \( w_1 - w_2 = 0.1 \) and \( w_1 = 4 \) for different linear mismatches; (a) \( \delta = -0.02 \), (b) \( \delta = \delta_c = -0.096 \), and (c) \( \delta = -0.1 \).

one of a linearly mismatched saturable nonlinear coupler (Figure 6-7(c)). Here in Figure 6-7(c) the coupler is synchronized at high power \( (\delta + w_1 - w_2 = 0) \), although mismatched at low power, explaining why only one stable mode is present at \( S_1 = 0 \) when \( P \to \infty \). Figure 6-8 illustrates the characteristics of the bifurcation diagrams for a large mismatch \( w_1 - w_2 \) but with different linear mismatches \( \delta \).

Figure 6-8: Mode diagrams for the in-phase modes of mismatched nonlinear couplers with a nonlinear mismatch \( w_1 - w_2 = 4 \) and \( w_1 = 4 \) for different linear mismatches; (a) \( \delta = -1 \), (b) \( \delta = -4 \), and (c) \( \delta = -5 \).
6.3.2 Application of Mode Diagrams to Switching

A. Switching in the Presence of Nonlinear Mismatch

It has been shown that saturation of nonlinearity degrades switching characteristics of nonlinear couplers (see Figure 6-9(a)) [129]. This results from the fact that at high powers a saturable nonlinear coupler with identical cores will behave like a matched coupler. In this section a method is presented, utilizing the mode diagrams, to relax this limitation.

Considering Figure 6-7(a) \( (w_1 - w_2 = 4, w_1 = 4) \) it is observed that at high powers the coupler is highly mismatched. Also, from Figure 6-4 we know that in the absence of linear mismatch \( (\delta = 0) \) the coupler will be matched at low powers. Hence, it should be possible to rectify this problem by combining two cores with different nonlinear saturation but no linear mismatch. The numerical calculations confirm this prediction. Figure 6-9(a) displays the performance of a saturable nonlinear coupler with \( w_1 = w_2 \). As shown, when \( w_j < 3 \) full power transfer is no longer possible. However, by creating a mismatch in nonlinear saturation the critical saturation value (below which full power transfer is not possible) decreases to \( w_1 = w_{1c} = 2.1 \) for \( w_2 = 0.5 \) (Figure 6-9(b)). In the Figure the critical powers are determined by the relation \( P_{c1}w_1 = P_{c2}w_2 \), arguing that higher nonlinearity has lower saturation and vice versa. Other choices of the ratio of \( \frac{P_{c1}}{P_{c2}} \) lead to similar switching response although with 50% switching point shifted slightly.

The improvement in the tolerance to nonlinear saturation here arises from the mismatch at high power reflected through \( w_1 - w_2 \). Obviously the critical saturation value \( w_{1c} \) will vary with the value \( w_2 \). The optimum (for a nonlinear coupler composed of two self-focusing cores) occurs as \( w_2 \rightarrow 0 \), corresponding to the coupler composed of one linear core and one nonlinear core as shown in Figure 6-9(c). The critical saturation value for this case is \( w_{1c} = 1.7 \) which is nearly half of that for a conventional coupler of Figure 6-9(a). This finding has practical implications. For instance, consider an erbium-doped nonlinear coupler [133,135].

\[^1\](6.5) is integrated using a fourth order Runge-Kutta scheme.
Figure 6-9: Straight-through transmissivity of saturable nonlinear couplers with the length $z = L_c = \frac{\pi}{2\gamma}$. SF, and SDF denote self-focusing, and self-defocusing respectively, and $P_c = P_{c1}$. (a) Conventional saturable nonlinear coupler with $w_1 = w_2$ and $P_{c1} = P_{c2}$; (b) two self-focusing cores with $w_2 = 0.5$ for different $w_1$; (c) one linear core and one self-focusing core with different $w_1$; (d) one self-focusing core with different $w_1$ and one self-defocusing core with $w_2 = 0.7$. In cases (b) and (d), $\frac{P_{c1}}{P_{c2}} = \frac{w_2}{w_1}$ except for the case of $w_1 = \infty$ for which $\frac{P_{c1}}{P_{c2}}$ is set to be 1.
The saturation value of the nonlinear index of an erbium-doped fiber is $\Delta \beta_{1_{\text{sat}}} = 0.25 \times 10^{-6} k n_0$. The critical coupling coefficient for the conventional nonlinear coupler is then decided by

$$w_c = 3 = \frac{\Delta \beta_{1_{\text{sat}}}}{2C} = \frac{0.25 \times 10^{-6} k n_0}{2C},$$

which gives $C = \frac{0.25 \times 10^{-6} k n_0}{6}$. This corresponds to the critical coupler length $L_c = \frac{\pi}{2C} = 2.06 \text{ m}$ for the wavelength $\lambda = 0.514 \mu \text{m}$ and $n_0 = 1.5$, i.e., the length of the coupler composed of two identical erbium-doped fibers must be longer than 2.06 meters so that it can operate as a switch effectively. But an erbium-doped fiber has a typical loss of $8 \frac{dB}{m}$. The two-meter long coupler with a loss of $8 \frac{dB}{m}$ leads to the output power dropped to 2% of the incident power. Loss at this level severely deteriorates switching characteristics [135]. On the other hand, for a coupler composed of an erbium-doped (nonlinear) core and a linear core, the critical saturation value is $w_{1c} = 1.7^1$. This yields the critical coupling coefficient $C = 0.25 \times 10^{-6} k n_0 \frac{3}{4}$ corresponding to the critical coupler length $L_c = 1.17$ meters. This length results in the output power to be 12% of the incident power, which is 6 times more intense than the previous case. With such an amount of loss in one core, the nonlinear coupler may function as a switch, although not as good as in the lossless case, as demonstrated below.

A mismatch at high power can improve the performance of a nonlinear coupler composed of two self focusing cores. A further improvement can be achieved if a nonlinear coupler is constructed by combining one self-focusing core with one self-defocusing core (both saturable). This is due to the fact that the refractive index changes induced by nonlinearity in this case are of opposite sign in the two cores. This is shown in Figure 6-9(d) where the critical saturation value is $w_{1c} = 1$

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1Erbium doping may produce a difference in the linear refractive indices between the cores. An appropriate choice of the ratio of core radii can compensate this difference, leading to the matching of two cores when $P \rightarrow 0$. Suppose for example the core-cladding index difference of the undoped core is 0.003 and that of the doped core is 0.009. Then a ratio of 1.7 of the two core radii will lead to $\delta = 0$. 

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for \( w_2 = 0.7 \), three to four times smaller than that of the conventional nonlinear coupler. This design is in fact a generalization of the earlier reported scheme where the two cores are assumed to have the same nonlinear saturation value [129]. Note that the critical mismatches \( \eta_1 w_{1c} - \eta_2 w_2 \) in Figures 6-9(b)-6-9(d) are nearly all the same although nonlinearities range from self-focusing to self-defocusing.

**Effect of Loss**

As illustrated, nonlinear saturation leads to deteriorated performance of nonlinear couplers. By combining one self-focusing core with another core of different nonlinearity (linear or self-defocusing), detrimental effect of nonlinear saturation on the coupler is greatly reduced compared with that on the conventional coupler composed of two identical saturable self focusing cores. For the sake of simplicity in demonstrating the point, the above analysis has ignored intensity dependent loss which often accompanies high nonlinearity (such as that in erbium-doped fibers) [133,135]. With the introduction of this intensity dependent loss, Equation (5.2) still holds [134] provided that \( \Delta \beta_j(|a_j|^2) + ia_j(|a_j|^2) \) takes the place of \( \Delta \beta_j(|a_j|^2) \), with

\[
\alpha_j(|a_j|^2) = \alpha_j(\infty)(1 - \exp\left(-2\frac{|a_j|^2}{w_j P_{cj}}\right)).
\]  

(6.13)

The presence of this intensity dependent loss contributes further to the deteriorated performance of nonlinear couplers, in addition to the nonlinear saturation effect. This is shown in Figure 6-10 for the conventional saturable nonlinear coupler identified by the dashed curve (which differs from the corresponding one in Figure 6-9(a) only by an additional amount of intensity dependent loss introduced). At the critical value \( w_1 = w_2 = w_c = 3 \) only 68% of the total power at the output can be switched from one core to the another at \( \alpha_j(\infty) = 8 \frac{dB}{m} \). The switching characteristic is also severely affected by the presence of loss.

Naturally, one would wonder the impact of this intensity dependent loss on the nonlinear couplers composed of two different saturable nonlinear cores. Consider a nonlinear coupler composed of one nonlinear core and one linear core. At the
Figure 6-10: Straight-through transmissivity of erbium-doped nonlinear couplers with the length $z = L_c = \frac{\pi}{2c}$ and $\alpha_{ij}(\infty) = 8 \frac{dB}{m}$. The dashed line denotes the conventional nonlinear coupler with $w_1 = w_2 = 3$ and the solid line represents a coupler composed of one nonlinear core and one linear core with $w_1 = w_c = 1.7$.

critical saturation value $w_1 = w_c = 1.7$, and with the loss of $\alpha_{ij}(\infty) = 8 \frac{dB}{m}$, nearly all the power (92%) can be switched from one core to another as shown in Figure 6-10 (identified by the solid curve), demonstrating the switching performance to be much better than that of the conventional nonlinear coupler (the dashed curve), in addition to the relaxed tolerance to nonlinear saturation (or reduced saturation value $w_j$). We thus conclude that the proposed practically realizable couplers composed of one linear core and one nonlinear core indeed offers a way of relaxing nonlinear saturation over the conventional one, irrespective of the presence or absence of loss.

B. Switching Using Both Linear and Nonlinear Mismatch

An interesting aspect of Figure 6-8(b) is that it reflects the characteristics of a highly mismatched coupler at low power and those of a matched coupler at high power. This motivates us to devise a power dependent switch using mismatched (linearly and nonlinearly) saturable nonlinear couplers with length $L_c(= \frac{\pi}{2c})$. The switching characteristics of the proposed device is shown by the solid curve in Figure 6-11 (where $w_1 - w_2 = 1.7$ and $\delta = -1.73$).
Figure 6-11: Switching characteristics of one coupling length long mismatched saturable nonlinear coupler with \( w_1 - w_2 = 1.73 \) and \( w_1 = 1.73 \) where the solid curve indicates the response of the structure with a linear mismatch \( \delta = -1.73 \) \((V \neq 0)\), and the dashed curve denotes the response of the structure without a linear mismatch \( \delta = 0 \) \((V = 0)\).

Note that switching state at the output of core (1) is unity at low power and zero at high power, contrary to that in a conventional nonlinear coupler where the switching state at the output of core (1) is zero at low power and unity at high power. In practice, such a power dependent switch can be realized by combining a linear (undoped silica) core with a saturable nonlinear (erbium-doped operating at the pump wavelength) core. A linear mismatch can be produced by doping concentration, by selection of different core sizes and/or by applying a voltage on electrodes as shown in the inset of Figure 6-11. The interesting case is the linear mismatch induced by a voltage alone. This can be achieved by varying the difference in the core radii so that a linear mismatch introduced from doping in one core is counteracted by the linear mismatch resulting from different core radii. In the absence of an applied voltage this linearly matched coupler functions as a power dependent switch, as demonstrated above. In this case, however, the switching states shown by the dashed curve in Figure 6-11 are opposite to those of the linearly
mismatched coupler (identified by the solid curve in the Figure). Recall that electro-optical switching (at low power limit) operates by switching on and off the voltage applied on the electrodes [139] (see Figure 6-11 where the solid curve is switched to the dashed curve by changing the voltage from $V \neq 0$ to $V = 0$). Using a conventional coupler, optical switching by electrical means may be deteriorated at high power because of a nonlinearity induced mismatch. Nevertheless, here we show that a proper combination of electrically induced (linear) mismatch with a mismatch in nonlinear saturation can lead to ideal optical switching effectively both at low power and high power levels.

For an ideal switching, a mismatch or nonlinear saturation are deemed unwanted since they lead to incomplete or deteriorated power switching. As shown in Figure 6-11, complete power switching in a mismatched saturable nonlinear coupler is made possible by a proper combination of a linear mismatch and a nonlinear mismatch. An adverse effect is converted to a useful one.

It should also be noted that in practice, loss and saturable absorption may be unavoidable. Their presence will affect the power swapping characteristics. When loss and saturable absorption are small, adiabatic power swapping between the two cores results. The coupler behavior in this case is hardly affected by the presence of loss and absorption and the conclusions drawn earlier can be extended to these structures. However, for large loss an analysis involving interaction between nonlinear saturation and loss needs to be developed.

### 6.3.3 Phase Space Portrait

Bifurcation diagrams outline the features of phase space portraits. For the case of a symmetric (identical cores) coupler, the mode diagram uniquely determines the phase space portrait, as is illustrated in Figure 6-12(a) for one stable mode, Figure 6-12(c) for one unstable mode accompanied by two stable modes, and in Figure 6-12(e) for two unstable modes accompanied by three stable modes. Furthermore, for an asymmetric (mismatched cores) the phase space portrait is still solely governed by the mode diagram when the number of unstable modes is less than two.
Figure 6-12: Schematic representation of phase space trajectories (on the $S_1 - S_3$ plane) of nonlinear couplers in the presence of saturation, corresponding to (a) one stable mode of symmetric structure, (b) one stable mode of an asymmetric structure, (c) one unstable mode of a symmetric structure, (d) one unstable mode of an asymmetric structure, (e) two unstable modes of a symmetric structure, (f) and (g) two unstable modes of an asymmetric structure.
This is illustrated in Figure 6-12(b) for one stable mode and in Figure 6-12(d) for one unstable mode accompanied by two stable modes. However, when two unstable modes are involved in an asymmetric structure in the cases of Figures 6-4(a), 6-5(a) and 6-7, within certain power levels, there can be two possible phase portraits being associated with the mode diagrams which are displayed in Figures 6-12(f) and 6-12(g). The trajectory pattern in Figure 6-12(f) is qualitatively the same as 6-12(e). On the other hand, the pattern shown in Figure 6-12(g) (one separatrix nested within another) has no resemblance to Figure 6-12(e). The numerical calculations reveal that the trajectory pattern of Figure 6-12(g) is the very one portraying the behavior of an asymmetric saturable nonlinear coupler operating at power levels where two unstable modes are involved. This means that the trajectory pattern of a nonlinear coupler experiences a qualitative change when a nonlinear mismatch \( w_1 - w_2 \) is introduced into a symmetric structure. A similar situation occurs when only linear mismatch is present [130].

Application to Amplification

The trajectory patterns in Figures 6-12(g) and 6-13(a) show that there are two distinct unstable eigenstates. At the eigenstates, the power splitting between the guides remains unchanged along the length of the coupler after a transient length as shown in Figure 6-13(b) (the dashed curves). These two eigenstates, as indicated, are unstable. Several different periodic power swappings can result once the launching conditions deviate slightly from those leading to eigenstates (Figure 6-13). The magnitude and period of power swapping depend on the strength and direction of a perturbation. In Figure 6-13(b) the two eigenstates occur at \( P_1 = 1.4P_c \) and \( P_2 = 11.1P_c \) (\( S_1 = -0.78P \)) called bias state (I), and \( P_1 = 10.2P_c \) and \( P_2 = 2.3P_c \) (\( S_1 = 0.63P \)) referred to as bias state (II). A small perturbation, which can be a small signal input, to the bias state leads to a large change at the output. The magnitude of the amplification depends on the bias state and the nature of the perturbation, giving rise to four possible different amplification coefficients. Consider the bias state (I). When a perturbing signal is added to guide (1), the signal extracted at the
output of that guide gets a large amplification, while the perturbing signal initially injected into guide (2) leads to a small amplification. Likewise, the bias state (II) may result in a large or small amplification of a signal.

Figure 6-13: Phase space trajectory (on the $S_1 - S_3$ plane) of a nonlinear coupler composed of two mismatched saturable nonlinear cores at power $P = 12.5P_{c1}$, saturation value $w_1 = 3.2$, nonlinear mismatch $w_1 - w_2 = 0.08$ and $\delta = 0$, (b) the corresponding power swapping behavior in core (1) for in-phase excitation of the two guides, where $L_c = \frac{\pi}{2C}$ and $w_1P_{c1} = w_2P_{c2}$.

### 6.4 Active Saturable Nonlinear Couplers

#### 6.4.1 Introduction

Although twin core nonlinear couplers have been investigated theoretically and experimentally over the years, their practical applications are still some distance away, partly because of the operation at high power required to overcome low nonlinearity of material such as silica. This then has prompted a search for materials with high nonlinearities such as the erbium-doped silica whose nonlinear coefficient is $10^6$ times larger than that of silica [133]. The possibility of operation at milliwatt pow-
ers is very attractive, and would be a significant step towards practical nonlinear couplers [135]. Unfortunately, an erbium-doped nonlinear coupler functioning as a switch at the pump wavelength has the disadvantage of slow response in addition to loss and nonlinear saturation, compared with the undoped silica core nonlinear coupler. By shifting the operation wavelength, the problem of slow response can be overcome, as the pump becomes a constant source of injected power into the signal. A signal then experiences gain instead of loss, i.e., by operating at a signal wavelength, the erbium-doped coupler can function as an active nonlinear coupler. In the following analysis we use a two level saturation model [138], namely

\[ \alpha_j(\log_{10}|a_j|^2) = \frac{\alpha_j(\infty)}{(1 + P_{sj}^2)} \]

\[ \text{where } P_{sj} \text{ denotes saturable gain power.} \]

6.4.2 Switching Characteristics of Active Couplers

It has been shown that an active nonlinear coupler (composed of two identical cores) acting as a switch may lead to switching at power levels much lower than a corresponding one without gain [134]. Unfortunately, as illustrated in Figure 6-14(a), its switching characteristics are unacceptable due to the appearance of ripples around the switching power [see the solid and dashed curves around \( \frac{P_1}{P_1 + P_2} = 0.5 \) in Figure 6-14(a)]. The larger the gain the worse the switching response. Furthermore, nonlinear saturation, which is inevitable when high nonlinearity is involved, such as in an erbium-doped nonlinear coupler [135], plays a role more detrimental to switching in an active nonlinear coupler than in a passive nonlinear coupler. For example, at \( w_1 = w_2 = 3 \), the maximum total power that can be switched in an active nonlinear coupler is 65% (see the dashed-dotted curve in Figure 6-14(b)), whereas in the corresponding passive nonlinear coupler full power can still be switched (see the dotted curve in Figure 6-14(b)). Surprisingly, however, the ripples at the edge of the high power state (which are partly responsible for the deteriorated switching of an active Kerr-law nonlinear coupler of Figure 6-14(a)) are gradually suppressed by reducing \( w_j \). Reduction in \( w_j \) can also improve switching response. This motivates us to investigate the effect of a variation of \( w_j \) on the ripples.

Assume that power is introduced into guide (1) initially. Our study shows that
the height of the ripple at the edge of the low power state is proportional to the magnitude of \( w_2 \) and is completely suppressed as \( w_2 \to \infty \), whereas \( w_1 - w_2 \) affects the ripples at the edge of the high power state, fading as \( w_1 - w_2 \) decreases. Figure 6-15 displays the improved switching characteristics of active nonlinear couplers composed of one linear core \( w_2 = 0 \) (so that the ripple at the edge of the low power state is suppressed) and a nonlinear core with different \( w_1 \). Even with the saturation value \( w_1 = 1.7 \) [see the solid curve in Figure 6-15], full power can be switched, in contrast to the matched counterpart of Figure 6-14(b) [see the solid and dashed-dotted curves in Figure 6-14(b)]. This critical saturation value, \( w_1 = 1.7 \), above which complete power can be switched, is the same as that in the corresponding passive nonlinear coupler [140]. This indicates that a mismatch at high power \( (w_1 - w_2) \) can greatly improve the tolerance of an active nonlinear coupler to nonlinear saturation, in addition to ameliorated switching characteristics. Although a coupler with gain of \( \frac{\alpha_1}{2C} = \frac{\alpha_2}{2C} = -0.45 \) and gain saturation powers of \( P_{s_j} = \infty \) (Figure 6-15(a)) and \( P_{s_j} = 0.25P_c \) (Figure 6-15(b)) here are given as examples to demonstrate the effect, other values of gain \( \frac{\alpha_j}{2C} \) and \( P_{s_j} \) result in the same conclusion.

The physical reasons of the above-demonstrated beneficial influence of asymmetric nonlinear saturations on an active nonlinear coupler is due to the mismatch created at high power [140] and can be appreciated partly by comparing the power swapping behavior between the cores versus the coupler length of a mismatched coupler (shown in Figure 6-16(a) with parameters being those of the solid curve in Figure 6-15(a)) with that of a matched coupler (shown in Figure 6-16(b) with parameters being those the solid curve in Figure 6-14(a)) at different power levels. For the matched active coupler of Figure 6-16(b), the point at which power remaining in guide (1) is minimum, moves toward the origin initially, then moves away from the origin, and eventually moves toward the origin again with increasing power in oscillating form. However, for the mismatched coupler of Figure 6-16(a), the corresponding point moves monotonically toward the origin with increasing power.
Figure 6-14: Transmission characteristics of half beat length active nonlinear coupler composed of two identical active self-focusing cores with core (1) initially excited with total power $P$. (a) $w_j = \infty$ (Kerr nonlinearity) with the dotted curves for $\frac{\alpha_1(\infty)}{2C} = 0$, the solid curve for $\frac{\alpha_1(\infty)}{2C} = -0.45$, the dashed curve for $\frac{\alpha_1(\infty)}{2C} = -0.7$; and (b) $w_j = 3$ and $\frac{\alpha_1(\infty)}{2C} = 0$ identified by the dotted curve, $w_j = 3$ and $\frac{\alpha_1(\infty)}{2C} = -0.45$ identified by dashed-dotted curve, $w_j = 1.7$ and $\frac{\alpha_1(\infty)}{2C} = 0$ identified by dashed curve, and $w_j = 1.7$ and $\frac{\alpha_1(\infty)}{2C} = -0.45$ identified by the solid curve. The length of coupler is $L = L_c = \frac{\pi}{2C}$ and $P_{sj} \gg P_{c1}$.

Figure 6-15: Transmission characteristics of half beat length active nonlinear coupler composed of one linear core $w_2 = 0$ and one saturable nonlinear core (which is initially excited with total power $P$) for different $w_1$ where $\frac{\alpha_1(\infty)}{2C} = -0.45$. $w_1 = 4$ is identified by the dashed curve, $w_1 = 1.7$ is identified by the solid curve, and $w_1 = 1$ is identified by the dotted curve. (a) $P_{s_j} = \infty$ and (b) $P_{s_j} = 0.25P_{c1}$. 

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Figure 6-16: Fraction of power remaining in core (1) versus coupler length $Z$ for different input power $P$. The dashed-dotted-dotted curves stand for $\frac{P}{P_{c1}} = 0.2$; the solid curves represent $\frac{P}{P_{c1}} = 0.4$; the dashed curves represent $\frac{P}{P_{c1}} = 0.45$; and the dashed-dotted-dotted-dotted curves represent $\frac{P}{P_{c1}} = 1.5$. (a) $\frac{\alpha_j(\infty)}{2C} = -0.45$ and $w_1 = 1.7$, $w_2 = 0$ and $P_{sj} = \infty$ (b) $w_j = \infty$ and $\frac{\alpha_j(\infty)}{2C} = -0.45$ and $P_{sj} \gg P_{c1}$. 
This monotonic decrease in the period of the active couplers of Figure 6-16(a), which gives rise to the well behaved switching characteristics of Figure 6-15, arises from the presence of the nonlinear mismatch at high power. The effect of this mismatch is gradually exaggerated with increasing power, leading to the shortened period of power swapping, and playing a role equivalent to a linear mismatch in a linear coupler [141]. For a matched active coupler of Figure 6-16(b), no preference on either core is created and thus power swapping between the cores is more or less like a passive nonlinear coupler with the first minimum shifting leftward at low power and rightward at high power, leading to the oscillating feature of Figure 6-14(a).

In the above discussions, attention has been given to a coupler with the same amount of gain in two cores. Had a coupler with unequal gains in the two cores been chosen, the same conclusion would result. For example, a nonlinear coupler composed of one active core and one passive core exhibits unacceptable switching characteristics (Figure 6-17(a)), as well as a deteriorated tolerance to nonlinear saturation (see the dashed curve in Figure 6-17(b) compared with the dotted curve in Figure 6-14(b)), in spite of its switching at a power much lower than $P_c$ of the passive nonlinear coupler. But the introduction of a nonlinear mismatch at high power greatly ameliorates switching characteristics as well as improving tolerance to nonlinear saturation. This is shown in Figure 6-18 where the couplers are formed by combining an active saturable nonlinear core with a passive linear core. This scheme can be realized in practice by combining an erbium-doped fiber with an undoped silica fiber, thus achieving ultra low power switching which may not be possible by using an active nonlinear coupler composed of two identical active cores (see Figure 6-14).
Figure 6-17: Transmission characteristics of a nonlinear coupler composed of one active core and one passive core ($\alpha_2(\infty) = 0$) with the active core initially excited with the total power $P$. (a) $w_1 = w_2 = \infty$, with the solid curve identifying $\frac{\alpha_1(\infty)}{2C} = -0.5$ and the dashed curve identifying $\frac{\alpha_1(\infty)}{2C} = -0.6$, (b) $\frac{\alpha_1(\infty)}{2C} = -0.5$ with the dashed curve for $w_1 = w_2 = 3$ and the solid curve for $w_1 = w_2 = 1.7$. The length of the coupler is $L = L_c \frac{1 + \frac{3}{2} \sin^{-1}(\frac{\alpha_1(\infty)}{2C})}{1 - (\frac{\alpha_1(\infty)}{2C})^2}$ so that complete power is transferred at $P \to 0$.

Figure 6-18: Same as Figure 6-17(b), i.e. $\frac{\alpha_1(\infty)}{2C} = -0.5$ and $\alpha_2(\infty) = 0$ but $w_2 = 0$ with the dashed curve corresponding to $w_1 = 3$ and the solid line to $w_1 = 1.7$ and the dotted curve to $w_1 = 1$. (a) $P_{s1} = \infty$ and (b) $P_{s1} = 2.5P_{c1}$.
To make contact with experiments, consider an example involving typical parameters for which the new predicted effect could be observed. Let $w_1 = 1.7$, $w_2 = 0$, $rac{\alpha_1}{2C} = -0.5$ and $\frac{\alpha_2}{2C} = 0$ for the parameters of the solid curve in Figure 6-18. Assume the refractive index change saturates to the level of $\Delta \beta_1 = 2C w_1 = 0.25 \times 10^{-6} \beta_1$ [133]. The improved switching of the active coupler displayed in Figure 6-18 can then be achieved by a coupler composed of one erbium-doped core and one silica core with its length $L_c = 1.17$ meters and gain of $11.7 \frac{dB}{m}$ in the doped core.
Chapter 7

Conclusions and Summary

In this thesis various nonlinear phenomena have been investigated. Firstly, it has been shown that dark solitons in (1+1) and (2+1)-dimensions evolve adiabatically in the presence of Raman gain and two-photon absorption. In the case of (1+1)-dimensional dark solitons it has been shown that they are more stable to perturbations such as gain or two-photon absorption than their bright counterparts. In addition, it has been shown that in both cases Raman gain and two-photon absorption can completely counterbalance each other, resulting in stationary solutions. The characteristics of these stationary solutions have also been discussed.

The stability characteristics of bright vortex solitary waves (i.e., self-trapped beams with a dark spot surrounded by rings of varying intensity) in saturable self-focusing media have been examined analytically and numerically. It has been shown that the fundamental bound state of the family is stable to symmetric perturbation but unstable to an asymmetric perturbation. The higher-order states are also found to display transverse (modulation) instabilities. The onset of transverse instability is shown to lead to emission of filaments which spiral away from the dark spot and move toward the equilibrium state of the three dimensional self-trapped beam, with maximum intensity at the center.

On the subject of nonlinear guided waves, light trapping in a self-defocusing nonlinear film bounded by an infinite self-defocusing medium of different nonlinearity is examined. Both gray and dark solitary waves are found possible to be trapped in
the structure provided that the linear refractive index $n_f$ in the film is smaller than that in the cladding, $n_o$, and when $n_f < n_e < n_o$. On the other hand, a series of dark oscillatory beams results when $n_e < \min\{n_f, n_o\}$. This contrasts with a uniform self-defocusing medium where only the dark solitary wave can be trapped. It is also shown that this structure (symmetric nonlinear waveguide) admits asymmetric dark modes. In addition, the existence of a new class of bright nonlinear modes with non-zero intensity background is demonstrated. It should be noted that in the analysis it has been assumed that cladding extends to infinity. In practice, cladding layers are finite in extent. In this case, gray and dark solitary waves can still be trapped but they differ quantitatively from those in the structure with cladding layers of infinite extent. However, when cladding layers are much thicker than the film width, as is often the case in practice, the difference in the solutions between finite cladding and infinite cladding is negligible.

The question of the stability of the bright modes of waveguides composed of self-focusing cladding and linear core is discussed. In particular, a reported discrepancy between the numerical and analytical stability analyses is addressed. The reliability of the numerical method in determining the stability of the asymmetric nonlinear mode trapped in this structure is investigated, and thus the controversy on the stability of the mode trapped between the minimum point and transition point on the dispersion curve is resolved. The existence of a class of quasi-periodic solutions is also demonstrated. In addition, the effect of loss in propagation of the asymmetric modes is examined. Small loss is shown to lead to adiabatic evolution on the stable branch of the dispersion curve, whereas initial excitation on the unstable branch results in evolution oscillating around and wandering away from the dispersion curve.

Finally, nonlinear couplers composed of two saturable nonlinear cores are examined. It is found that a mismatch at high power can relax the limitation of saturation effect on a nonlinear coupler functioning as a switch. The coupler made of one self-focusing core and one self-defocusing core is shown to lead to the greatest improvement. The physical mechanism behind the betterment is also given, i.e., due to the mismatch between two cores at high power. In view of practicality, an
example of the coupler composed of a linear core and a nonlinear core is examined in
detail to demonstrate the advantage of the proposed scheme. Insofar as the nonlin-
ear dynamics of this system is concerned, it is found that a mismatch in nonlinearity
results in a dramatic change in the behavior of the coupler. Most interestingly, it is
found that the simultaneous presence of nonlinear and linear mismatches gives rise
to novel bifurcation diagrams leading to possible novel switching devices. It is also
found that a mismatch in nonlinearity can result in improved performance of an
active nonlinear coupler, thus opening the way for ultra-low-power switching using
active saturable nonlinear couplers.

In the final analysis, nonlinear optical phenomena are diverse and rich in charac-
ter. The research presented in this thesis opens a vista into some other intellectually
challenging issues which deserve careful attention. Firstly, Chapter 3 focused on the
propagation of stationary solutions. Although this is an important step in under-
standing the general characteristics of the system, it lacks practicality insofar as
the experimental observation of these solitary waves are concerned. Since in ex-
periments a beam profile can be approximated as a Gaussian, the next step is to
investigate the propagation dynamics of a Gaussian beam with a $2\pi$ phase ramp
in a saturable self-focusing medium. Secondly, the discussion of nonlinear guided
waves in Chapter 4 has centered on the characterization of the stationary modes
of nonlinear waveguides with self-defocusing cores and claddings. The question of
stability of these modes still needs to be addressed. Both of the above problems are
currently being investigated.

It is hoped that this thesis has been able to impart some of the excitement and
joy that the author has experienced in learning the field and being able to contribute
to its advancement.
Appendix A

Two-Photon Absorption and Raman Gain

Note: the material presented in this appendix has been obtained from the following references:


Two-photon absorption is a nonlinear optical process where two quanta are simultaneously absorbed in an electronic transition by a material system that is transparent to either quantum alone. Such an absorption requires that both photons be present simultaneously in the region of the absorbing atom or molecule. Hence, large photon fluence — $10^{20}$ to $10^{30}/cm^2$-sec — is the general requirement for the observation of two-photon absorption.

Phenomenologically, the attenuation of intensity of a light wave as a function of distance inside a material exhibiting one- and two-photon absorption is described by

$$\frac{dI(z)}{dz} = -[\kappa_1 + \kappa_2 I(z)]I(z)$$ \hspace{1cm} (A.1)

where $\kappa_1$ and $\kappa_2$ are the one- and two-photon absorption coefficients, with units of
It is also possible to discuss the two-photon absorption strength in terms of a cross-section per molecule and per photon fluence, $\sigma_2$, related to the two-photon coefficient, $\kappa_2$, by:

$$\sigma_2 = \frac{h\omega}{N}\kappa_2$$

(A.2)

where $h\omega$ is the energy of a single photon of the incident laser wave and $N$ is the number per unit volume of the absorbing molecule. The units of $\sigma_2$ are $\text{cm}^{-1} \text{sec} \text{photon}^{-1} \text{molecule}$, and typical magnitudes are in the range $10^{-51}$ and $10^{-49}$.

By means of the formalism of nonlinear optical susceptibilities one can find an expression for $\kappa_2$. As a particular example, for a two-photon absorption experiment involving a single beam frequency, $\omega$, linearly polarized along the [100] direction of a cubic crystal, $\kappa_2$, is related to the imaginary part of the third order nonlinear susceptibility by:

$$\kappa_2 = \frac{32\pi^2\omega}{n^2c^2} \Im \left( \chi^{(3)}_{xxx}(-\omega,\omega,\omega,\omega) \right)$$

(A.3)

where the subscripts “x” denote the polarization direction of the involved waves (all along the [100] direction in this example), and $n$ is the refractive index of the material at the frequency, $\omega$.

<table>
<thead>
<tr>
<th>Substance</th>
<th>TPA Coefficient $\kappa_2 \times 10^{-9}$ (cm/W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GaAs</td>
<td>60</td>
</tr>
<tr>
<td>GaP</td>
<td>1.7</td>
</tr>
<tr>
<td>InSb</td>
<td>220</td>
</tr>
<tr>
<td>CS₂</td>
<td>0.175</td>
</tr>
</tbody>
</table>

Table A.1: Two-photon absorption coefficients of some materials.

Raman gain arises from stimulated Raman scattering process. Stimulated scattering processes are nonlinear interactions where an incident wave at frequency $\omega_{\text{inc}}$ is converted to a scattered wave at a different frequency $\omega_{\text{scat}}$. The difference in photon energy between the incident and scattered frequencies is taken up or supplied by the nonlinear medium, which undergoes a transition between two of its internal
energy levels. The incident (laser) and scattered (Stokes) frequencies are related by
\[ \omega_s = \omega_l - \omega_0 \] (A.4)
where \( \omega_l \) and \( \omega_s \) are the frequencies of the laser and Stokes waves and \( \omega_0 \) is the frequency of the internal energy level. In this case the third-order nonlinear susceptibility has a negative imaginary part which gives rise to exponential growth of the Stokes wave (assuming that pump depletion is negligible), namely:
\[ I_s = I_s(0) \exp(g I_l L) \] (A.5)
where \( g \) is the Raman gain coefficient, \( I_l \) is the incident pump (laser) intensity, and \( L \) is the length of the nonlinear material. Also, the amplitude of the Stokes wave, \( A_s \), propagating in the forward direction is described by:
\[ \frac{dA_s}{dz} = \frac{3\omega_s}{4n_sc} \text{Im}\left(\chi^{(3)}\right) |A_l|^2 A_s \] (A.6)
where \( A_l \) is the amplitude of the pump and \( n_s \) is the refractive index of the medium at the Stokes frequency. It readily follows from (A.5) and (A.6) that
\[ g = \frac{3\omega_s}{n_s n_l c^2 \varepsilon_0} \text{Im}\left(\chi^{(3)}\right) \] (A.7)

<table>
<thead>
<tr>
<th>Substance</th>
<th>Raman Gain Coefficient ( g \times 10^{-9} ) (cm/W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oxygen</td>
<td>16</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>16</td>
</tr>
<tr>
<td>Methanol</td>
<td>0.4</td>
</tr>
<tr>
<td>CS(_2)</td>
<td>24</td>
</tr>
</tbody>
</table>

*Table A.2: Raman gain coefficients of some materials measured at 694.3 nm.*
Appendix B

The Elliptic Integrals and Functions

This appendix contains extracts from the *Handbook of Elliptic Integrals for Engineers and Physicists* by Paul F. Byrd and Morris D. Friedman.

B.1 Definition of Elliptic Integrals

The integral

\[ I = \int R \left[ t, \sqrt{a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4} \right] dt \]  

is called an **elliptic integral** if the equation

\[ a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4 = 0, \]  

where \( a_0 \) and \( a_1 \) are not both zero, has no multiple roots and if \( R \) is a rational function of \( t \) and of \( \sqrt{a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4} \).

B.1.1 The Canonical Forms of the Elliptic Integrals

Expressing Equation (B.2) in terms of elementary functions the following three fundamental integrals are obtained:
(1) The normal elliptic integral of the first kind:

$$\int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - m^2 t^2)}} = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}$$

$$= \int_0^{u_1} du = u_1 \equiv F(\varphi, m) \quad (B.3)$$

(2) The normal elliptic integral of the second kind:

$$\int_0^y \frac{1 - m^2 t^2}{1 - t^2} \frac{dt}{\sqrt{1 - m^2 \sin^2 \varphi}} \varphi = E(u_1) = E(\varphi, m) \quad (B.4)$$

(3) The normal elliptic integral of the third kind:

$$\int_0^y \frac{dt}{(1 - \alpha^2 t^2)\sqrt{(1 - t^2)(1 - m^2 t^2)}} = \int_0^\varphi \frac{d\varphi}{(1 - \alpha^2 \sin^2 \varphi)\sqrt{(1 - m^2 \sin^2 \varphi)}}$$

$$\equiv \Pi(\varphi, \alpha^2, m) - \infty < \alpha^2 < \infty \quad (B.5)$$

The number $m$ is called the modulus. This number may take real and imaginary values. However, for physical or engineering applications $m$ is within $0 < m < 1$. The variable limit $y$ or $\varphi$ in the above equations is the argument of the normal elliptic integrals. In general the argument may be either real or complex, but it is usually understood that $0 < y < 1$ or $0 < \varphi \leq \frac{\pi}{2}$. For the special case of $y = 1$ the above integrals are termed as complete and they read as:

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}} = F\left(\frac{\pi}{2}, m\right) \equiv K(m) \equiv K \quad (B.6)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - m^2 \sin^2 \varphi} \ d\varphi = E\left(\frac{\pi}{2}, m\right) \equiv E(m) \equiv E \quad (B.7)$$

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1 - \alpha^2 \sin^2 \varphi)\sqrt{(1 - m^2 \sin^2 \varphi)}} = \Pi\left(\frac{\pi}{2}, \alpha^2, m\right) \equiv \Pi\left(\alpha^2, m\right), \quad (B.8)$$

where $\alpha^2 \neq 1$.  

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B.1.2 Special Values

\[
\begin{align*}
E(0, m) &= 0, \\
F(0, m) &= 0, \\
\Pi(0, \alpha^2, m) &= 0.
\end{align*}
\]

\[
\begin{align*}
E(\varphi, 0) &= \varphi, \\
F(\varphi, 0) &= \varphi, \\
\Pi(\varphi, \alpha^2, 0) &= \varphi, \\
&= \tan^{-1}\left(\sqrt{1 - \alpha^2 \tan \varphi} \right), \quad \text{if} \quad \alpha^2 < 1 \\
&= \frac{\tanh^{-1}\left(\sqrt{\alpha^2 - 1} \tan \varphi \right)}{\sqrt{\alpha^2 - 1}}, \quad \text{if} \quad \alpha^2 > 1.
\end{align*}
\]

\[
\begin{align*}
K(0) &= \frac{\pi}{2}, \\
E(0) &= \frac{\pi}{2}, \\
E(\varphi, 1) &= \sin(\varphi), \\
F(\varphi, 1) &= \ln(\tan \varphi + \sec \varphi).
\end{align*}
\]

B.2 Jacobi Elliptic Functions

B.2.1 Definition

Every Jacobi elliptic function is the inverse of an elliptic integral. Instead of considering

\[
u(y_1, m) = \int_0^{y_1} \frac{dt}{\sqrt{(1 - t^2)(1 - m^2 t^2)}} = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m^2 \sin^2 \vartheta}} \equiv F(\varphi, m) \quad (B.9)
\]

one may define inverse functions as \(y_1 = \sin \varphi = \text{sn}(u, m), \varphi = \text{am}(u, m)\); these may be read sine amplitude \(u\) and amplitude \(u\). Two other functions can then be defined by \(\text{cn}(u, m) = \sqrt{1 - y_1^2} = \cos \varphi, \text{dn}(u, m) = \sqrt{1 - m^2 y_1^2}\), requiring that \(\text{sn}(0, m) = 0, \text{cn}(0, m) = 1 \) and \(\text{dn}(0, m) = 1\). The functions \(\text{sn} u, \text{cn} u, \text{dn} u\) are called Jacobi elliptic functions and are one-valued functions of the argument \(u\).
These functions like the trigonometric functions have a real period, and, like the hyperbolic functions, have an imaginary period.

The quotients and reciprocals of $\text{sn}u$, $\text{cn}u$, $\text{dn}u$ are defined by

\[
\begin{align*}
\text{ns}u & = \frac{1}{\text{sn}u}; \quad \text{sc}u = \frac{\text{sn}u}{\text{cn}u}; \quad \text{sd}u = \frac{\text{sn}u}{\text{dn}u}; \\
\text{nc}u & = \frac{1}{\text{cn}u}; \quad \text{cs}u = \frac{\text{cn}u}{\text{sn}u}; \quad \text{cd}u = \frac{\text{cn}u}{\text{dn}u}; \\
\text{nd}u & = \frac{1}{\text{dn}u}; \quad \text{ds}u = \frac{\text{dn}u}{\text{sn}u}; \quad \text{dc}u = \frac{\text{dn}u}{\text{cn}u};
\end{align*}
\]

**B.2.2 Fundamental Relations and Special Values**

\[
\begin{align*}
\text{sn}^2u + \text{cn}^2u & = 1, \\
\text{m}^2\text{sn}^2u + \text{dn}^2u & = 1, \\
\text{dn}^2u - \text{m}^2\text{cn}^2u & = 1 - \text{m}^2, \\
(1 - \text{m}^2)\text{sn}^2u + \text{cn}^2u & = \text{dn}^2u.
\end{align*}
\]

\[
\begin{align*}
\text{sn}(-u) & = -\text{sn}u; & \text{sn}(0) & = 0; & \text{sn}(K) & = 1, \\
\text{cn}(-u) & = \text{cn}u; & \text{cn}(0) & = 1; & \text{cn}(K) & = 0, \\
\text{dn}(-u) & = \text{dn}u; & \text{dn}(0) & = 1; & \text{dn}(K) & = \sqrt{1-\text{m}^2}. \\
\end{align*}
\]

\[
\begin{align*}
\text{sn}(u, 0) & = \sin u; & \text{sn}(u, 1) & = \tanh u; \\
\text{cn}(u, 0) & = \cos u; & \text{cn}(u, 1) & = \text{sech} u; \\
\text{dn}(u, 0) & = 1; & \text{dn}(u, 1) & = \text{sech} u;
\end{align*}
\]

**Addition Formulas**

\[
\begin{align*}
\text{sn}(u \pm v) & = \frac{\text{sn}u \text{cn}v \text{dn}v \pm \text{sn}v \text{cn}u \text{dn}u}{1 - \text{m}^2 \text{sn}^2u \text{sn}^2v}, \\
\text{cn}(u \pm v) & = \frac{\text{cn}u \text{cn}v \mp \text{sn}u \text{sn}v \text{dn}u \text{dn}v}{1 - \text{m}^2 \text{sn}^2u \text{sn}^2v}, \\
\text{dn}(u \pm v) & = \frac{\text{dn}u \text{dn}v \mp \text{m}^2 \text{sn}u \text{sn}v \text{cn}u \text{cn}v}{1 - \text{m}^2 \text{sn}^2u \text{sn}^2v}.
\end{align*}
\]

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B.2.3 Differentiation with Respect to the Argument

\[
\begin{align*}
\frac{\partial}{\partial u} (\text{sn } u) &= \text{cn } u \text{ dn } u, \\
\frac{\partial}{\partial u} (\text{cn } u) &= -\text{sn } u \text{ dn } u, \\
\frac{\partial}{\partial u} (\text{dn } u) &= -m^2 \text{sn } u \text{ cn } u.
\end{align*}
\]

B.2.4 Sketches of Jacobi Elliptic Functions

\[\text{Figure B-1: Representation of Jacobi elliptic functions for one period } (4K) \text{ where } m = 0.5.\]
Bibliography


[38] B. Luther-Davies and Y. Xiaoping, "Waveguides and Y junctions formed in bulk media by using dark spatial solitons," *Opt. Lett.* 17, 496 (1992)


151


(1987)


dependent switching in a coherent nonlinear directional coupler in the presence

[118] M. Romagnoli and G. I. Stegeman, "Saturation of guided wave index with

(1988)


[121] E. Caglioti, S. Trillo, S. Wabnitz, and G. I. Stegeman, "Limitations to all­
optical switching using nonlinear couplers in the presence of linear and non­

directional coupler with a diffusive Kerr-type nonlinearity," *Opt. Lett.* **13**, 419
(1988)


