Polarisation Effects
in the Theory of Optical Solitons

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Declaration

This dissertation describes research undertaken by me in the Optical Sciences Centre, within the Research School of Physical Sciences and Engineering at the Australian National University, during the period March 1992 to October 1995.

Unless specifically disowned, the work contained herein is my own although sections 3.2, 3.4, 4.1, 4.3 & 4.5 were conducted in collaboration with Marc Haelterman.

[Signature]
The production of a thesis is a massive and demoralising task. For me the pain has been greatly reduced by the support of many people. The list is so long that it is difficult to know where to start!

My year spent sharing an office with Marc was one of the ‘growth periods’ of my life. Being able to share in your enthusiasm for the subject and your physical insight was a real privilege. The bulk of the work in this thesis comes from a most productive 12 months of work together. It was very nice to meet you again to continue the friendship and the collaboration in Belgium; thank you for your generous hospitality. I won’t forget our many enjoyable times together on rock faces, either!

Thank you also to Allan who helped get my research career off the ground and who maintains the active and vigorous research community that is the Optical Sciences Centre. It is amazing that after so many years in this field Allan can make the propagation of light beams so exciting. Your disdain for fancy mathematical techniques is a lesson for all of us tempted to respect research merely because it is hard to understand.

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When I came to the dreaded time of transposing the research from neurons into words, Les (my father) and John sacrificed hours of their time being bored witless battling with my turgid prose. If this document makes any sense at all, it will be a consequence of your diligence!

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Abstract

This thesis examines the properties of light beams that are able to trap themselves and propagate without diffraction in nonlinear Kerr materials. Beams of this type, called spatial solitary waves, have potential applications to all-optical switching devices and optical computing. We also study them because they are governed by versatile mathematical models that, in addition to guided waves, describe a variety of physical systems. One may therefore understand more than one physical process by investigating a single mathematical model.

The new results obtained in this thesis can be broken into two main categories. First is the discovery and characterisation of new physical phenomena, in particular of several classes of vector solitary wave. Second is the application of a fascinating mathematical technique, the Hirota method, in the analysis of the integrable $U(n)$ nonlinear Schrödinger equation.

When accounting for polarisation, the simplest model for lightwave propagation is the vector nonlinear Schrödinger equation, one of the integrable soliton equations. More general models exhibit nonlinearly induced birefringence that breaks the symmetry required for integrability.

We make an important generalisation of the concept of a solitary wave by permitting the polarisation components to propagate at different speeds. We thereby locate multi-peaked bright soliton families in a range of media, that we call dynamic solitary waves on account of the beating between the polarisation components. All the non-fundamental forms of these waves appear to be unstable, so we transfer our attention to defocussing media and kink solitary waves. Here we find a bevy of stable bright-dark waves, and a new kind of wave, the polarisation domain wall, that only exists in the presence of nonlinear birefringence. This is a localised structure where the polarisation of the field switches state. It can be naturally extended into three dimensional models where we explore solitary waves and propagation dynamics. We also demonstrate the fundamental nature of the domain wall by linking it with the polarisation modulational instability of plane waves.

Such are the new solitary waves discovered in this work. Our more mathematical results follow similar paths. Investigating the Manakov model for focussing Kerr media we write down the general 2-soliton solution, to our knowledge for the first time. We extract from this solution, as a special case, stationary states that are also special cases of dynamic solitary waves. In defocussing media we find a new type of soliton: the bright-dark soliton, for which we write down general $N$-soliton solutions.

Our final results come from exploring the waveguide X-junctions that are formed by the collision of two solitons. In certain scenarios, these junctions have remarkable properties that can be fully characterised using the multi-solitons of the $U(2)$ and $U(3)$ nonlinear Schrödinger equations. In other cases we characterise these devices using an approximation scheme borrowed from linear waveguide theory.
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Chapter 1

Introduction – Nonlinear Optics

As there are physical and mathematical aspects to this thesis, the introduction is divided into two separate parts. This chapter will introduce nonlinear optics and derive, from first principles, the equations governing nonlinear guided wave propagation in certain idealised media. Chapter 2 introduces a little of the theory of solitons and integrable systems and uses two different methods to find soliton solutions for the nonlinear Schrödinger equation (NSE).

1.1 Electromagnetic fields in dielectrics

1.1.1 The Maxwell equations

Many of the important features of the theory of nonlinear guided waves can be derived, albeit in a phenomenological way, quite painlessly. Such a phenomenological approach reveals many interesting facets of the theory without becoming embroiled in microscopic, material specific models. In the following we shall highlight some of the idealisations necessary to describe the interaction of light and matter in a simple way. A natural starting point is the famous Maxwell equations:

\[ \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \]  
\[ \nabla \cdot \mathbf{B} = 0 \]  
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \]

where \( \rho \) is the charge density and \( \mathbf{J} \) the current density. \( \mathbf{E} \) is the electric field strength and \( \mathbf{B} \) the magnetic field strength. \( \varepsilon_0 \) and \( \mu_0 \) are, respectively, the permittivity and the permeability of free space.

We consider electro-magnetic (EM) fields in nonmagnetic, insulating materials known as dielectrics, that are electrically neutral media in which the electric charges are tightly confined. It is clear that a full quantum mechanical formalism must be invoked if we wish to truly understand the interaction between the fields and the matter. We emphasise that no approach is completely classical, since the response of electrons to an external field depends on their quantum wave-functions. Neither is any approach purely quantum mechanical, since the mathematical tools are not yet available for analysing complex, real world systems using quantum electrodynamics. We are only interested in the macroscopic properties of the medium and so, after making a simple semi-classical model of the atom, we turn our attention solely to the macroscopic effects. Quantitative behaviour and material response close to any quantum mechanical resonances demand a more sophisticated approach.
We assume that the electrons in the material are confined within atoms or molecules by some potential well. Under the influence of an external electric field, the electrons move and therefore create a dipole moment within the material. The electron response, as dictated by the potential well, determines the optical properties of the material. To mathematically describe this effect we define the polarisation vector $P$ as the dipole moment per unit volume. In general one must expand the current density in electric and magnetic multipoles, but for dielectric materials we can safely ignore the magnetic effects and higher order electric multipoles.

When dealing with dielectrics, it is convenient to separate the charges and currents that are free from those that are confined. The only manifestation of the confined charges and currents is in the polarisation field. The Maxwell equations then appear as

$$\nabla \cdot E = (\rho_f - \nabla \cdot P)/\varepsilon_0$$  \hspace{1cm} (1.2a)

$$\nabla \cdot B = 0$$  \hspace{1cm} (1.2b)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$  \hspace{1cm} (1.2c)

$$\nabla \times B = \mu_0 J_t + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} + \mu_0 \frac{\partial P}{\partial t}$$  \hspace{1cm} (1.2d)

It is common, particularly in the engineering literature, to simplify the notation by defining the electric induction field $D = \varepsilon_0 E + P$, but this would not assist with this presentation.

1.1.2 The wave equation

From this point on we consider only the propagation of EM fields far from the influence of sources and currents and therefore set $\rho = 0$ and $J = 0$. The vector wave equation can now be defined \(^1\) by taking the curl of (1.2c) and using (1.2d)

$$\nabla \times \nabla \times E = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$  \hspace{1cm} (1.3)

Clearly $P$ is a function of $E$. $B$ does not enter the nonlinear equations and can be disregarded except as a participant in the wave propagation. $P$ and $E$ are related through the equation

$$P = \varepsilon_0 \chi E.$$  \hspace{1cm} (1.4)

Only in linear, isotropic materials will $P$ be parallel to $B$, so in general $\chi$ is a tensor quantity, called the susceptibility tensor. In general $\chi$ is an integral operator that incorporates spatial and temporal nonlocality and the whole gamut of nonlinear effects. The relationship, eq. (1.4) between $P$ and $E$ is commonly referred to as the constitutive equation. The resulting integro-differential wave equation that governs the propagation of $E$ can only be solved in certain special cases. Systems for which the initial value problem can be solved to give closed form analytical solutions are called integrable. It is problems on the ‘fringe’ of integrability that will be examined in this thesis.

1.2 Nonlinear effects

Until 35 years ago, optics experiments indicated that $P$ was linearly dependent on $E$, so that in the frequency domain $P(\omega) = \varepsilon_0 \chi(\omega)E(\omega)$, where $\chi$ is a constant tensor dependent only on the type of material and the light frequency. There were some nonlinear...

\(^1\)It is a remarkable piece of science history that when this wave equation was first derived by James Clerk Maxwell, he thought he was investigating electrical and magnetic effects alone. He realised that (1.3) describes a wave travelling at the same speed that light does, and that therefore light must be an electromagnetic wave, obeying his equation. A genuinely great moment in science!
1.2 Nonlinear effects

Effects (namely electro-optic, piezo-electric and Kerr effects) involving steady state EM fields, but at optical frequencies everything was linear. This is because all the available optical sources were incoherent which prevented tight focussing of beams and which cancelled higher order effects. Linear systems, while not always exactly soluble, are characterised by predictable and unremarkable behaviour. For this reason optics, prior to 1960, was considered an old science, and little studied by the great minds of physics.

All this changed with the advent of the laser. Schawlow and Townes (1958) predicted that a ruby rod could be used to produce a maser that would operate in the near infrared, and Maiman (1960) and Collins, Nelson, Schawlow, Bond, Garrett, and Kaiser (1960) used this principle to make the first lasers. The authors of the first two papers received the Nobel prize in Physics for their efforts. The coherent illumination of the laser can be focussed onto a tiny spot, generating enormous intensities. Additionally, all the photons of a coherent beam act in concert, reinforcing the effects of one another. Within 12 months, the field of coherent nonlinear optics was born when Franken, Hill, Peters, and Weinreich (1961) observed UV light at 347nm emitted from a quartz crystal being illuminated with a ruby laser producing red light at 694nm. In a very short time scientists had characterised a remarkable variety of new optical effects. Theory and experiment progressed hand in hand in these early years, and many of the experimental results were pre-empted by theoretical studies. Under intense light, materials heat up, ionise, expand, contract, melt or even vaporise while their atoms bend, stretch, reorient or redistribute. The movement of the electrons themselves becomes far more complicated, with quantum resonances often playing a significant role. In this introduction we do not consider the myriad physical processes concerned, but rather develop a phenomenological approach, with the aim of examining light induced changes in refractive index. We expand the susceptibility in a power series in the electric field

\[ P = \varepsilon_0 (\chi^{(1)} E + \chi^{(2)} EE + \chi^{(3)} EEE + \ldots), \]

or, spelling out the tensor multiplication explicitly, one can write this as

\[ P_i = \varepsilon_0 (\chi^{(1)}_{ij} E_j + \chi^{(2)}_{ijkl} E_j E_k + \chi^{(3)}_{ijkl} E_j E_k E_l + \ldots), \]

with summation over the repeated indices. The validity of this expansion clearly depends on the magnitude of the electric field relative to some material dependent value. For non-resonant interactions, one expects that when the external electric field and the atomic electric field become comparable, this expansion will break down. More detailed calculations verify this hypothesis, and the effective expansion parameter is the ratio between the applied field and the atomic field. The atomic electric fields are of the order of $3 \times 10^8 \text{V/cm}$, less than that obtained by many lasers. It is clear that when we enter this regime we shall observe some truly violent behaviour, commonly referred to as optical breakdown, in which avalanche ionisation occurs, forming a plasma (Maker, Terhune, and Savage 1964a).

The second order susceptibility $\chi^{(2)}$ leads to the generation of second harmonics, and sum and difference frequencies, as well as to other effects like parametric amplification. It turns out that, under the electric dipole approximation, $\chi^{(2)}$ is zero in materials with inversion symmetry, i.e., symmetric under the operation $\mathbf{r} \rightarrow -\mathbf{r}$. Examples of such media are glasses like silica and solids composed of centro-symmetric crystals. The lowest order nonlinear effects are then governed by the $\chi^{(3)}$ 4-tensor. The optical Kerr effect and the DC Kerr effect are examples of third order processes and consequently materials whose optical properties are dominated by $\chi^{(3)}$ are called Kerr materials. Other examples are third harmonic generation, stimulated Raman emission, four wave mixing and nonlinear refraction. This thesis shall concentrate on the last effect. The earliest nonlinear optics research concentrated on the generation of different frequencies. Initial studies were therefore looking at the Raman effect (Woodbury and Ng 1962) and
third harmonic generation (Terhune, Maker, and Savage 1962). Askaryan (1962) was the first to recognise the potential for change in the refractive index that intense light provides, and shortly afterwards Hercher (1964) saw the manifestation of this effect when he focussed intense light from a ruby laser onto glass. The laser beam continued to focus inside the material, forming long threads of damage. Maker and Terhune (1965) used self-focussing, as it was called by then, to determine many of the components of the $\chi^{(3)}$ tensor. The self-focussing effect proved to be very important in observing stimulated Raman scattering (Shen and Shaman 1965; Lallemand and Bloembergen 1965), and it was realised that the nonlinear refraction could balance the natural tendency of light to diffract, resulting in a self-trapped beam that travelled in its own waveguide.

1.2.1 The $\chi^{(3)}$ tensor

The 4-tensor $\chi^{(3)}$ has $3^4 = 81$ components that depend on the material and interacting frequencies. In most materials the tensor has many symmetries and a greatly reduced number of independent components. This is because, at the very least, any symmetries at the crystal level are reflected in the macroscopic properties. This principle, formally stated, is called Neumann's principle. The exact relationship between the components for particular crystal symmetry groups can be very difficult to determine. There is a substantial body of literature discussing the physical properties of crystals [see for example, Nye (1957)] while several nonlinear optics texts, most notably Boyd (1992) and Shen (1984), also consider crystal properties.

In particular in cubic media, which are not isotropic — there are still preferred axes of symmetry — we find there to be 21 nonzero components. The nonzero terms take only four different values, and can be grouped accordingly into terms of the form $\chi^{(3)}_{iii}$, $\chi^{(3)}_{iij}$, $\chi^{(3)}_{iij}$, $\chi^{(3)}_{ijj}$. All other terms are zero. In isotropic media there is an additional constraint, namely

$$\chi^{(3)}_{iii} = \chi^{(3)}_{iij} + \chi^{(3)}_{ijj} + \chi^{(3)}_{ijj},$$

leaving a mere 3 material parameters to completely describe processes up to the third order. Other crystal symmetry groups will, of course, require many more parameters to be characterised. For example the tensor for a tetragonal crystal has the same 21 non-zero elements, but there are 11 independent values. A general analysis of third order effects in birefringent materials therefore leads to very complicated mathematical models. Theoretical estimates for the components of $\chi^{(3)}$ are notoriously hard to derive, and even for nonresonant processes involve determination of the ground state electronic wavefunctions. Resonant effects are yet more complex. Consequently computer simulations are now used almost exclusively for this problem.

Referring to eq. (1.5), we expand the polarisation field into components produced by each power in the electric field. $P^{(3)}$ is thus the contribution to the electric polarisation from third order processes. The many different types of nonlinearity respond over a huge variety of time scales, but since no response is instantaneous, the susceptibility is wavelength dependent, to some extent, in all materials. The proximity of a resonance exacerbates the wavelength dependence. Additionally, the response will occur over a finite length scale, resulting in some spatial nonlocality. In general, then, one must write $\chi^{(3)}$ as an integral operator, in which the polarisation at a point $\mathbf{r}$ depends on the electric

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2In the 1970s and 1980s a number of authors successively corrected each other's errors in evaluating the $\chi^{(3)}$ tensor for different crystal symmetries. The final corrections were made by Shang and Hsu (1987)
1.2 Nonlinear effects

field in a neighbourhood of \( \mathbf{r} \)

\[
P^{(3)}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \chi^{(3)}_{ijkl}(\mathbf{r} - \mathbf{r}_1, \mathbf{r} - \mathbf{r}_2, \mathbf{r} - \mathbf{r}_3, t - \tau_1, t - \tau_2, t - \tau_3) \nonumber \
E_j(\mathbf{r}_1, \tau_1)E_k(\mathbf{r}_2, \tau_2)E_l(\mathbf{r}_3, \tau_3)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\tau_1d\tau_2d\tau_3. \tag{1.8}
\]

Optical susceptibilities can be defined by Fourier analysis of the electric field with respect to both time and space

\[
\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\mathbf{k} \mathbf{E}(\mathbf{k}, \omega)e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}. \tag{1.9}
\]

The nonlinear optical susceptibilities therefore depend in general on both the frequency and the wavenumber of the contributing fields. Nonlocality of the nonlinear response is responsible for wavenumber dependence and is therefore referred to as spatial dispersion (Ginsburg 1958). The most common nonlinearities associated with large length scales are thermal processes, electrostrictive processes and molecular redistribution. Electrostriction refers to an increase in density that occurs in many materials when a strong electric field is applied and is always associated with self-focussing. In general it is only when investigating microscopic effects that one need consider spatial dispersion. We concentrate on media whose nonlinear response is dominated by fast processes at the electronic or molecular level. Therefore we neglect any nonlocality in the optical response, which is equivalent to assuming \( \chi^{(3)} \) is independent of the wave-vectors of the fields producing it. \( \chi^{(3)} \) tells us about the interaction of four waves, and depends on the frequency of each contributing field. Suppose that the field can be written as a sum of \( N \) discrete frequencies \( \omega_j \), so that

\[
\mathbf{E}(\mathbf{r}, t) = \sum_{\mu=1}^{N} \mathbf{E}(\mathbf{r}; \omega_\mu)e^{-i\omega_\mu t} \tag{1.10}
\]

For simplicity we consider a transverse wave and assume that the \( z \) component of the field is uniformly zero. Then

\[
P_i^{(3)}(\omega) = \sum_{\mu, \nu, \rho=1}^{N} \sum_{j,k,l=x,y} \chi^{(3)}_{ijkl}(\mathbf{r} = \omega_\mu + \omega_\nu - \omega_\rho)E_j(\omega_\mu)E_k(\omega_\nu)E_l(-\omega_\rho) \tag{1.11}
\]

Since all physical fields must be real, the existence of a forward propagating wave \( e^{-i\omega t} \) always implies the existence of a corresponding backwards propagating wave \( e^{i\omega t} \). The contributing frequencies \( \omega_\mu, \omega_\nu \) and \( \omega_\rho \) can be degenerate, or include acoustic waves, or constant fields (i.e. zero frequency), or backward propagating waves. In this way the single equation eq. (1.11) can be used in the description of processes with diverse physical origins, such as third harmonic generation, four wave mixing, stimulated Raman scattering, stimulated Brillouin scattering, two photon absorption and nonlinear refraction.

1.2.2 Weak guidance approximation

It is now desirable to find simplifications for the vector wave equation (1.3), whose solution in exact form is a tedious and difficult process even when using a modern computer. Let us write eq. (1.3) in another form:

\[
\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla(\nabla \cdot \mathbf{E}) + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{1.12}
\]

The terms on the left hand side constitute a standard linear wave operator in which the Cartesian polarisation components of the field are decoupled. The term involving
the polarisation field $P$ contains any nonlinear effects and will couple the polarisation components. The other term $\nabla(\nabla \cdot E)$ also couples between the components. This $E$ term represents the difference between the vector wave operator $\nabla \times \nabla \times$ and the scalar operator $\nabla^2$. For transverse EM waves and for linear isotropic media this $E$ term will be identically zero. It has been shown that it can also be neglected in linear materials where the total refractive index change is small, i.e. $\Delta n \ll n_0$ (Snyder and Love 1983).

This is called the weak guidance approximation, and also uses the fact that a waveguide structure with small index changes can only guide light rays that are almost parallel.

The nonlinear generalisation of the weak guidance approximation involves some technical asymptotic analysis. This thesis investigates nonlinear refraction, and by demanding that the nonlinear index change be small we can appeal to the result of linear physics, and ignore the term $\nabla(\nabla \cdot E)$. The wave propagation is then described by a set of scalar wave equations, Helmholtz equations, coupled only through the nonlinearity.

$$\nabla^2 E - \frac{1 + \chi^{(1)}}{c^2} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P^{(3)}}{\partial t^2}$$

(1.13)

It is always possible to choose a basis for the polarisation components of $E$ that will diagonalise the 2-tensor $\chi^{(1)}$. Ignoring nonlinear effects, one finds that each polarisation in this basis obeys the scalar equation but propagates with a different speed. These basis vectors are called the principal axes of the crystal. The diagonal elements represent the three independent components of $\chi^{(1)}$, and one usually writes

$$I + \chi^{(1)} = \text{diag}(n_1, n_2, n_3),$$

(1.14)

Light not polarised along one of the principle axes will be composed of different components travelling at different velocities. The polarisation state of this light will change as it propagates through the material. Optical activity is an example of this phenomenon.

1.3 The propagation equations

We illuminate our nonlinear crystal with light of a single frequency, but arbitrary polarisation, such that the beam is aligned with the $z$-axis. Consider the beam to be transversely polarised,

$$E = e^{-i\omega t} [E_x(r)i + E_y(r)j] + c.c.,$$

(1.15)

where c.c. stands for complex conjugate and $i$ and $j$ are unit vectors oriented along the transverse directions $x$ and $y$ respectively. We assume that we can align $i$ and $j$ up with the principle axes of the medium, so the the respective refractive indices are $n_x$ and $n_y$. The nonlinear effect of interest here is the nonlinear refractive index change. Unless special phase matching conditions are satisfied, waves created at other frequencies destructively interfere. We consider media in which phase matching conditions are not met and therefore disregard wave components propagating at other frequencies. Mathematically this means we discard all terms contributing to the third order polarisation that are not oscillating at $e^{-i\omega t}$. We assume that the medium is isotropic with respect to third order processes. This assumption will only hold for materials that are nearly linearly isotropic, i.e. $n_x \approx n_y$. Substituting into eq. (1.11), making use of the intrinsic permutation symmetry and suppressing the frequency dependences, yields

$$\frac{1}{3} P^{(3)}_{zz} = \chi_{zzzz} E_x E_x E_x + \chi_{zyzyy} E_y E_z E_y + \chi_{yyzy} E_y E_y E_y$$

(1.16)

The last term on the RHS of eq. (1.16) is a coherent coupling term in the sense that the phase of the $y$-component of the field affects the propagation of the $x$-component and
1.3 The propagation equations

vice-versa. The wave equation can now be re-written, making use of eq. (1.7)

\[ \nabla^2 E_x - k^2 n_x^2 E_x = 3k^2 \chi_{xxxx}^{(3)} \left[ |E_x|^2 E_x + (1 - B)|E_y|^2 E_x + B E_y^2 E_x^* \right] \]  

(1.17)

\[ \nabla^2 E_y - k^2 n_y^2 E_y = 3k^2 \chi_{xxxx}^{(3)} \left[ |E_y|^2 E_y + (1 - B)|E_x|^2 E_y + B E_x^2 E_y^* \right] \]  

(1.18)

where the wavenumber \( k = \omega / c \) and \( B \) is defined as \( B = \chi_{xyyx} / \chi_{xxxx} \). This elliptic partial differential equation (PDE) can be reduced to a parabolic one using the paraxial approximation: we assume that the light consists of a monochromatic bundle of rays travelling almost parallel to the \( z \)-axis. In fact this is not really an additional assumption. The weak guidance approximation demands that the refractive index change be small, and there are two consequences of this. Firstly small index changes do not couple between the polarisation components, which allows us to discard the coupling term \( \nabla (\nabla \cdot \mathbf{E}) \) in the vector wave equation eq. (1.12). Secondly the small changes cannot reflect or refract a wave through a large angle, and therefore a wave that is initially paraxial will remain that way. These concepts are discussed by Snyder, Mitchell, and Kivshar (1995), where it is mentioned that resonant gratings represent singular points in the perturbation expansion. The field can then be written as

\[ \mathbf{E} = \tilde{E}(\mathbf{r}) e^{i k n_0 z}, \]

(1.19)

where the fast variation in the field is contained in the exponential term. \( n_0 \) is defined as the average of \( n_x \) and \( n_y \). Neglecting the higher \( z \) derivatives in \( \tilde{E} \), eq. (1.18) simplifies to the coupled parabolic equations

\[ \frac{1}{2} \nabla^2 \tilde{E}_x + i k n_0 \partial_z \tilde{E}_x + \delta \tilde{E}_x = \frac{3}{2} k^2 \chi_{xxxx}^{(3)} \left[ (|\tilde{E}_x|^2 + (1 - B)|\tilde{E}_y|^2) \tilde{E}_x + B \tilde{E}_y^2 \tilde{E}_x^* \right] \]  

(1.20)

\[ \frac{1}{2} \nabla^2 \tilde{E}_y + i k n_0 \partial_z \tilde{E}_y - \delta \tilde{E}_y = \frac{3}{2} k^2 \chi_{xxxx}^{(3)} \left[ (|\tilde{E}_y|^2 + (1 - B)|\tilde{E}_x|^2) \tilde{E}_y + B \tilde{E}_x^2 \tilde{E}_y^* \right] \]  

(1.21)

where \( \delta = \frac{1}{4} k^2 (n_x^2 - n_y^2) \) is a measure of the linear birefringence and \( \nabla^2 \) is the transverse laplacian \( \nabla^2 = \partial_x^2 + \partial_y^2 \). In order to remove the coherent coupling terms (i.e those coupling terms in which the phase of the field is significant; here they are of the form \( \tilde{E}_x^2 \tilde{E}_y^* \)), we write the field in terms of circular polarisation components, and rescale to dimensionless units

\[ i \partial_z \tilde{E}_\pm + \frac{1}{2} \nabla^2 \tilde{E}_\pm + \delta \tilde{E}_\mp + \sigma \left[ (1 - B)|\tilde{E}_\pm|^2 + (1 + B)|\tilde{E}_\mp|^2 \right] \tilde{E}_\pm = 0, \]

(1.22)

where we have introduced the dimensionless circularly polarised field components

\[ \tilde{E}_\pm = \left[ 3k^2 \chi_{xxxx}^{(3)} \right]^{-\frac{1}{2}} \left( \tilde{E}_x \pm i \tilde{E}_y \right). \]

(1.23)

and \( z = k n_0 Z \). \( \sigma = \pm 1 \) characterises the sign of the nonlinearity. Note that when expressed in this basis the birefringence appears as a linear coupling; waves of pure circular polarisation are not eigenstates of birefringent media. From this point only slowly varying envelopes will concern us, and we will not make explicit use of the the 'tilde'. All fields can be assumed to be slowly varying envelopes. This is a system of \((2+1)\)-dimensional coupled nonlinear Schrödinger equations (CNSE). The nonlinear terms are of the form \(|\tilde{E}_\pm|^2 \tilde{E}_\pm\) and \(|\tilde{E}_\mp|^2 \tilde{E}_\pm\), and they can both be thought of as a nonlinear refractive index change. The first is called self phase modulation (SPM) and is manifested as self-focussing if \( \sigma = 1 \) and self-defocussing if \( \sigma = -1 \). The second nonlinear
term is called cross phase modulation (XPM), and represents the coupling between the field components. The relative strength of these two effects, a quantity determined by \( B \), depends on the exact nature of the nonlinearity. For many materials whose nonlinearity has a purely electronic origin, \( \chi_{xxy} = \chi_{xyx} = \chi_{xyy} \) and therefore \( B = \frac{1}{3} \). Silica is an example of this sort of medium. Other materials exhibit a wide variety of effects. Wang (1966), Maker, Terhune, and Savage (1964b) and Maker and Terhune (1965) as well as Bloembergen and Lallemand (1966) looked at the nonlinear birefringence effects that are observable in a variety of media. These studies combined theoretical and experimental work and characterised a variety of materials. Often there are several nonlinear processes contributing to the nonlinear refraction effect and therefore the material parameters can take a wide range of values. Zel’dovich and Raizer (1966) showed that if the nonlinearity comes about purely through molecular re-orientation, \( B = \frac{3}{4} \). Some gases fall into this category, and a many liquids are not far off. The so-called electrostriction nonlinearity, arising through changes in the density of the medium does not induce any anisotropy; in this case \( B = 0 \). Electrostrictive nonlinearities invariably exhibit slow response since density changes can only propagate through a material at the local speed of sound. Many nonlinear liquids, such as carbon disulfide, nitrobenzene and acetophenone display a mixture of the two effects, so that \( B \) takes a range of values between 0 and \( \frac{3}{4} \). A slow response, even in a molecular re-orientation model, works against the formation of any induced anisotropy. One can also vary the parameter \( B \) by changing the temperature. This thesis therefore examines a range of values of \( B \), covering a range of potential materials.

Much of the work in this thesis will consider isotropic media for which \( \delta = 0 \). Chapters 2 through 4 examine the \((1+1)\)-dimensional form of this equation in which we set \( \partial_y E = 0 \). This equation is relevant in slab waveguides, and when the beam has quasi-infinite extent in one dimension.

The CNSE (1.22) may be derived to describe a wide variety of physical systems, including several different optical systems. We have chosen to present the details for arbitrarily polarised light beams here; a parallel derivation shows that the CNSE also describes a uniformly polarised paraxial light beam made up of several different frequency components. Some of the details can be found in Agrawal (1991). In this case, \( B \) varies between 0 and \( \frac{1}{3} \) depending on the speed of the response of the material. Bloembergen (1965) showed, in fact, that the nonlinear refractive index can be complex valued, meaning that power is transferred between the wavelength components, for media with finite response times. For a very slow response, one may assume \( B = 0 \) and for a medium with instantaneous electronic nonlinearity, \( B = \frac{1}{3} \).

### 1.3.1 Diffraction, nonlinearity and linear equivalence

One can consider the propagation of a uniformly polarised electric field propagating in an isotropic medium by studying the nonlinear Schrödinger equation, one of the fundamental equations of mathematical physics.

\[
i\partial_z E + \frac{1}{2} \partial^2_x E + \sigma |E|^2 E = 0.
\]  

It is often helpful to view the nonlinear part of this equation as an effective linear refractive index change \(^4\) by letting \( n^2 - n_0^2 = |E|^2 \). If we ignore the nonlinear part of this equation, then a Gaussian beam will always diffract and become broader during propagation, as a consequence of the second spatial derivative. Ignoring this second,

\(^3\)We will use this notation quite a bit in this thesis. \((1+1)\) dimensional means one transverse dimension \((x)\) and one evolutionary or longitudinal dimension \((z)\).

\(^4\)In this thesis we consider the two different effective index profiles \( n_+ \) and \( n_- \) a great deal. We shall talk of \( E_+ \) ‘seeing’ the waveguide corresponding to the refractive index change \( n_+^2 - n_0^2 \). Likewise \( E_- \) sees the refractive index change \( n_-^2 - n_0^2 \).
1.3 The propagation equations

diffractive term results in waves that steepen and form discontinuities within a finite
time. Chiao, Garmire, and Townes (1964) proposed that it might be possible to balance
these effects and within one year Pilipetskii and Rustamov (1965) and Garmire, Chiao,
and Townes (1966) experimentally observed the self-trapping of a light beam in carbon
disulfide liquid. It is often valuable to think of the refractive index change induced
by the passage of the light beam as forming a waveguide. A self-trapped beam, in
which diffraction and nonlinearity are perfectly balanced, must be an exact mode of the
waveguide it has induced into the medium.

A self-trapped beam can be regarded as a solitary wave, and in certain circumstances
also as a soliton. A solitary wave is a wave solution of a nonlinear equation with two
properties: it is solitary, i.e. localised in a particular region of space, and stationary, i.e.
it propagates without any change as a consequence of some balance between different
physical processes. The self-trapped beams are therefore examples of solitary waves.
The definition of a soliton considers the robustness of the wave and is mathematically
based. Solitons shall be considered in Chapter 2.

1.3.2 Solitons in optical fibres

The same equations that we have just derived for the propagation of paraxial beams
of light can also be used to describe the propagation of short optical pulses in optical
fibres. The physical interpretation of the phenomena is quite different, and in many
cases less transparent. In fact, one of the problems facing the designers of optical fibre
transmission systems is dispersion. Dispersion is the wavelength dependence of the
propagation speed. A pulse of light cannot be truly monochromatic, unless it is of
infinite duration, and therefore over a long propagation distance, the different wavelength
consituents will become separated in time. Hasegawa and Tappert (1973a) suggested that
nonlinearity could be used to compensate for this effect, and temporal optical solitons
were thereby conceived. After a long gestation, the temporal soliton was born at Bell
Laboratories and delivered by Mollenauer, Stolen, and Gordon (1980). Hasegawa and
Tappert (1973b) also predicted the possibility of the propagation of dark solitons in fibres
in the normal dispersion regime, but the observation of these did not occur until Emplit,
Hamaide, Reynaud, Froehly, and Barthelemy (1987) and Krökel, Halas, Guiliani, and
Grischkowsky (1988).

The derivation of the equations governing wavepacket propagation in optical fibres
is far more complex than that presented above, and can be found in numerous texts [see
Newell and Moloney (1992) for perhaps the most rigorous derivation]. One considers a
wavepacket comprising a spread of wavelengths, and multiple scale asymptotics are used
to extract the propagation equation. Transverse confinement is provided by the linear
structure of the optical fibre, and the problem is reduced to a (1+1) dimensional nonlin­
er Schrödinger equation. Communications fibres are made from silica, that has a weak
self-focussing electronic nonlinearity. The weakness of the nonlinearity is compensated
for by the fact that the propagation distances are measured in kilometres. The sign of
the dispersion (i.e. whether longer wavelengths travel more or less quickly) takes over
the role played by the sign of the nonlinearity in spatial problems. Fibre designers have
some control over this parameter since the refractive index profile of the fibre lends a
certain dispersion of its own.

Because of the relatively low intensities and the regular electronic nonlinearity of
silica, nonlinear propagation in optical fibres is well described by the perturbative phe-
nomenological models explained in section 1.2 that assumed an instantaneous, local
nonlinearity. When the field is composed of more than one wavelength component or
is elliptically polarised, then coupled equations like eq. (1.22) describe the light prop-
agation. Birefringence effects are more varied than in bulk materials since they are
produced by a loss of cylindrical symmetry in the fibre. Additional terms enter the prop-
agation equations for birefringent optical fibres on account of the fact that the different propagation equations are travelling with different group velocities.

The practical application of solitons to fibre communications has been held back by the additional complications associated with propagation over long distances in the nonlinear regime. A smorgasbord of effects such as two photon absorption, Raman and Brillouin scattering and harmonic generation all have the potential to cause problems. Nonlinear waves do not superpose and therefore there is an interaction between neighbouring pulses in a fibre. More significantly, there is also an interaction between soliton pulses and background noise that disturbs the pulse propagation (Gordon and Haus 1986). Nevertheless recent experimental results have shown that these problems can be overcome with clever use of filters, and that optical solitons have the potential to significantly increase the bandwidth of fibre transmission systems (Mollenauer, Gordon, and Evangelides 1993). Soliton based systems are therefore likely to enter service in the next generation of transoceanic cables.

1.4 Summary of this document

This thesis examines the propagation of light in nonlinear optical materials, where the nonlinearity is dominated by third order processes. Chapter 2 discusses some of the methods that have been developed in the last 25 years for finding solutions to the integrable equations of mathematical physics. We introduce the inverse scattering transform (IST) using the formalism developed by Zakharov and Shabat (1974), Hirota’s method and finally the Darboux transformation method. We illustrate the IST and the Darboux transformation by deriving the two-soliton solution for the NSE.

Chapter 3 investigates solitary waves in Kerr-type self focussing media governed by systems of coupled NSEs. Only when there is no nonlinearly induced birefringence are these systems integrable. We present general soliton solutions for the integrable case, and place results of previous studies in a broader framework, then proceed to numerical methods to find several new categories of solitary waves for non-integrable models. A key feature of this chapter is the dynamic solitary wave, comprising two polarisation components propagating together with different propagation constants. We also examine the stability of these new structures and find that generally they are unstable as a consequence of the instability of a bright soliton to perturbations affecting its direction of propagation.

In Chapter 4 our attention is turned to defocussing materials, and therefore to dark type solitary waves. Once again we use analytical tools (Hirota’s method, to be precise) to investigate the relevant integrable equations, then proceed to numerical analysis. Our analytical work yields a new class of solitons that appear as a dark soliton in one polarisation and a bright one in the other, and we generate the N-soliton solutions for these bright-dark solitons. For the nonintegrable systems the behaviour is much more interesting, and we find a new type of solitary wave, polarisation domain walls (PDWs) and link them to the polarisation modulational instability exhibited by the nonintegrable CNSE. We also investigate the stability of the dark solitary waves, and find them to be universally stable.

Chapter 5 extends these results to include beams of light that vary in two transverse directions. In self-focussing media light beams are unstable and suffer filamentation. In defocussing media we generalise the PDW and the fundamental vortex soliton to bright-dark solitary waves and PDWs of cylindrical symmetry. We examine also the propagation dynamics of these structures, and extend the study to three-component systems and birefringent systems.

Chapter 6 looks at a more concrete problem: what are the coupling properties of the waveguide X-junctions that solitons induce when colliding with one another. A class of these soliton written couplers are described by integrable equations, at least when
the signal and the writing solitons share the same wavelength. This class of couplers has many remarkable properties that are fully characterised by subsets of the algebraic solutions derived in earlier chapters using Hirota's method. Outside the domain of integrability we derive approximate expressions for the transmission properties of these junctions to signals of all wavelengths.

### 1.4.1 A word on the word 'soliton'

There has been much debate in the optics and mathematics communities regarding appropriate use of the word soliton. The word was initially coined by Zabusky and Kruskal (1965) to describe solitary waves that, like particles, would retain their form after interacting with other solitary waves. Thus, according to the original definition, solitons are the localised, robust eigenmodes of integrable systems. Within the field of optics, stationary solitary waves have found such wide application in a variety of integrable, nearly integrable and nonintegrable systems that the word soliton has come to describe any system in which the nonlinearity enables the stationary propagation of localised states. Many purists, needless to say, are unhappy with this development and Martin Kruskal himself, in Calogero (1978), had this to say:

> In recent years the field theorists in mathematical physics have taken to using the term to refer to any solitary wave, not merely those in special systems such as KdV ... This is an encroachment and ambiguity which may be deplored and discountenanced but probably cannot be prevented.

In this thesis I deal with both mathematical solitons, the eigenfunctions of spectral problems, and physical solitary waves. Therefore, in order to differentiate the two, I will restrict the use of the word 'soliton' to the first case. I will make use of terms such as 'self-trapped beam', 'solitary wave', 'localised stationary state' and contractions thereof to describe solutions of nonintegrable equations.
Chapter 2

Introduction – Solitons and Integrability

This chapter reviews and discusses the other, mathematical side of nonlinear guided wave optics: solitons and integrability theory. In particular I will outline the basic elements of the solution of an evolution equation using the inverse scattering transform (IST), and the Darboux transformation method, and use them to derive the two-soliton solution for the nonlinear Schrödinger equation.

2.1 The inverse scattering transform

2.1.1 History

The last thirty years have seen great advances in our physical and mathematical understanding of the role that nonlinearity plays in physical systems. The advent of digital computers has been a major influence on the progress in this field. On the one hand, incredible complexity and unpredictability has been found within a cross-section of relatively simple systems, opening the new science of chaos theory. At the same time, many of the fundamental equations describing nonlinear wave propagation have been shown to exhibit strikingly ordered and regular behaviour.

Fermi, Pasta, and Ulam (1955) were trying to understand the flaws in models that predicted an infinite heat capacity for solids, and they used one of the rare digital computers to examine the evolution of a simple lattice system. They were shocked to find that their initial state would keep re-appearing at a rate far too high to be explained by Poincaré recurrence. This phenomenon is called Fermi-Pasta-Ulam recurrence. Zabusky and Kruskal (1965) investigated this problem more deeply and found that one continuum approximation for the lattice problem investigated by Fermi et al. (1955) was the KdV equation, famous from water wave studies and possessed of solitary wave solutions. Their numerical studies showed that solitary waves of the KdV equation could interpenetrate each other and emerge from their nonlinear interaction intact. Lax (1968) first demonstrated this result analytically and soon Gardner, Greene, Kruskal, and Miura (1967) had developed the inverse scattering method for solving the initial value problem for the KdV equation. It was five years before another nonlinear evolution equation was solved using similar tools. Zakharov and Shabat (1972) found an inverse method for solving the initial value problem for the nonlinear Schrödinger equation. This opened the floodgates and in the next few years whole hierarchies of equations were shown to be integrable. Notable advances were made by Hirota (1973a, 1973b) and Ablowitz, Kaup, Newell, and Segur (1974), so that by 1975 a large class of equations had been found that could be solved using the new methods.

The most notable equations that fall into the integrable basket are the Korteweg-de Vries (KdV) equation (shallow water waves), the Sine-Gordon (SG) equation (periodic
dislocations in crystals) and, most notably, the nonlinear Schrödinger equation (NSE) in scalar and vector form (optics and plasmas and more). The NSE and the KdV are both fundamental equations of mathematical physics that have universal relevance.

### 2.1.2 Solution methods

Since the early days the field has consolidated a great deal, but there are still many significant unanswered questions. Figure 2.1 is a diagrammatical attempt to place the various elements of the spectral transform in context. At the heart of the issue is the nonlinear PDE itself, an evolution equation for the variable $u(x,t)$ given by $u_t = N(u)$ where $N$ is some nonlinear operator. We assume that the equation can be written in the form

$$L_t + [L, M] = 0,$$  \hspace{1cm} (2.1)

where $L$ and $M$ are linear operators, in which $u$ appears as a coefficient, acting on some Hilbert space $\mathcal{H}$, defined with a complete inner product. All the equations mentioned in the previous section can be written in this form. Wider classes of equations have now been shown to be integrable but the discussion is clearer when restricted to equations of the form (2.1). Consider the eigenvalue problem on an eigenvector $\psi$ in $\mathcal{H}$

$$L\psi = \lambda \psi.$$  \hspace{1cm} (2.2)

What about the time evolution of the eigenvalue $\lambda$? Consider

$$L_t \psi + L \psi_t = \lambda_t \psi + \lambda \psi_t.$$  \hspace{1cm} (2.3)

Rearranging and using eqs. (2.1) and (2.2) we get

$$(L - \lambda)(\psi_t - M \psi) = \lambda_t \psi.$$  \hspace{1cm} (2.4)

Now, taking the inner product of $\psi$ with both sides and using the self-adjointness of the operator $L$:

$$((L - \lambda)\psi, \psi_t - M \psi) = \lambda_t (\psi, \psi),$$  \hspace{1cm} (2.5)

showing that $\lambda_t = 0$ and, with eq. (2.4), that the evolution of the eigenvectors is given by

$$\psi_t = M \psi.$$  \hspace{1cm} (2.6)

Equations (2.2) and (2.6) form the Lax pair for the original equation. All integrable evolution equations can be represented as the compatibility condition of an associated Lax pair of equations, which plays a pivotal role in the theory. The dependent variable from the nonlinear PDE enters the Lax equations as a (nonlinear) coefficient, and is frequently considered as the ‘potential’ for the scattering problem. Usually the PDE is recovered by setting $\psi_{xt} = \psi_{tx}$. One can relegate the independent variables $(x,t)$ to the status of mere parameters, and promote the functional dependence on the eigenvalue $\lambda$ to centre stage by considering the spectrum of the linear operator $L$. It is possible to solve the initial value problem (IVP) for the nonlinear PDE using the method of the inverse scattering transform (IST). One first solves the direct scattering problem eq. (2.2), using the initial conditions $u(x,0)$ as the potential. Looking at the asymptotic behaviour, the other Lax equation (2.6) gives the time evolution of the scattering data. It
remains to invert the scattering problem and reconstruct the potential function \(u(x,t)\) from the scattering data. Solution of this inverse problem is very difficult and the mathematical formalism did not exist until work by Gel'fand and Levitan (1951) and Marchenko (1955). In general, one can only find closed form solutions to the inverse scattering problem when the initial conditions correspond to a reflectionless potential. Soliton solutions emerge from this analysis as eigenvectors corresponding to the discrete spectrum of eigenvalues of the scattering operator \(\hat{L}\). The discrete spectrum corresponds in turn to the reflectionless potentials. Since the eigenvalues are constants of the motion then it is clear that certain properties of the solitons remain constant.

The analytic structure of the \(\lambda\) plane also plays a central role in the algebro-geometric methods that were developed by Krichever (1977) and others. One can uniquely define an analytic function (called a Baker-Akhiezer function) on a given Riemann surface of genus \(g\) by fixing the locations of \(g\) simple poles and specifying its behaviour as \(\lambda \to \infty\). Specifying the asymptotics defines a set of differential equations that are obeyed by the Baker-Akhiezer functions. This set of equations is invariably the Lax pair for some nonlinear PDE. One can recover, from the Baker-Akhiezer functions, solutions of the associated nonlinear PDE by considering the asymptotic behaviour at the first order. This method does not solve an IVP, rather it generates a class of solutions. The form of the solutions obtained depends on the genus of the Riemann surface upon which the Baker-Akhiezer functions live. Finite genus surfaces correspond to quasi-periodic multiphase or finite gap solutions that are wave analogs of multiperiodic motions in mechanical systems. Given one solution, we can add poles to the spectrum of the scattering operator to derive another solution. A mapping between different solutions of the PDE is called a Bäcklund or Darboux transformation (BT or DT). The BT that results from adding poles to the spectrum produces combined multiphase/soliton type solutions. If the initial solution was a trivial solution, then the BT will produce pure multisolitons.

Another elegant technique based on algebro-geometric ideas was introduced by Manin (1979) and developed by Date (1982), that we shall call the Krichever-Manin-Date (KMD) scheme. This is a direct method based on an ansatz with defined behaviour (the 'dispersion relation') at \(\lambda \to \infty\) and a component that is polynomial in the eigenvalue \(\lambda\). By choosing an appropriate number of conditions, one can fix uniquely the polynomial and thereby define a function satisfying some Lax pair. Multisoliton and multiphase solutions to the associated nonlinear PDE are obtained easily from this function. One of the elegant qualities of the KMD scheme is that it concentrates on analytical functions and algebraic structures; the solution of a differential equation seems almost a side effect. We stress that Lax pairs are special equation sets that are uniquely defined by their linear dispersion relation. It is possible to derive many integrable systems by commencing with a linear dispersion relation, finding the associated Lax pair, then finding the integrable PDE that is its compatibility condition. It is no surprise that, as discovered by Fermi and co-workers 40 years ago, solutions to integrable equations are quite different in character to solutions of 'normal' equations.

Another method that makes use of a polynomial ansatz and which reduces the problem to a linear dispersion relation is the Hirota method. This method will be used extensively in this thesis. By transforming \(u(x,t)\) to the \(\tau\)-function \(\tau(x,t)\), one can express multi-soliton solutions via simple formulae. Also, in terms of \(\tau\)-functions, integrable equations can be expressed in bilinear form, and solved using a simple polynomial ansatz. Considering the hierarchies of equations that have similar bilinear forms is very interesting and shows up non-integrable equations for which two soliton solutions, but no higher, can be derived. Despite the highly suggestive similarities between the Hirota method and the KMD scheme, unification of these two approaches remains an open problem.

Trying to answer the fundamental question "what is integrability?" leads to further
**Figure 2.1:** Nonlinear evolution equations and solitons: a brief sketch map.
open problems. Last century, Kowalewski (1889, 1890), suggested that one could find closed form analytical solutions if and only if any essential singularities or branch points in the solutions were independent of initial conditions. One can determine whether a given ordinary differential equation (ODE) has this property, and Painlevé rigorously studied all second order ODEs and found exactly 50 with the (now known as) Painlevé property. The solutions to all except 5 were known; the 5 Painlevé transcendents are defined as the solutions to the remainder. Moving to the 1970s, it was conjectured that one could use the same test to determine the integrability of PDEs by applying the Painlevé test to ODEs resulting from symmetry group reductions of the PDE. Weiss (1983) introduced a Painlevé test that could be applied directly to PDEs. One can use the Painlevé test to derive the Lax equations and a Bäcklund transformation for equations with the Painlevé property. Applying the Painlevé test also tells us the transformation needed to obtain the $\tau$-function needed for the Hirota method. It is not clear whether the existence of a Lax pair implies that an equation must possess the Painlevé property.

2.1.3 The Zakharov-Shabat scheme

Two independent groups, one at Potsdam College in the USA (Ablowitz, Kaup, Newell, and Segur 1974), the other in Siberia (Zakharov and Shabat 1974) developed formulations for solving quite general classes of nonlinear evolution equations. The Russian approach leaned heavily on functional analysis and, in its final form, the ZS scheme (Zakharov and Shabat 1974) couched the whole process purely in terms of operators. This scheme is thus more mathematically pleasing but more physically removed than the AKNS (Ablowitz et al. 1974) scheme that retained the notion of scattering and inverse scattering.

We shall present briefly the essential features of the ZS scheme. The scheme uses differential operators in defining the associated scattering (spectral) problem, and then integral operators with special commutators are used to solve the inverse scattering problem. Consider the integral operator

\[ \hat{F} \psi = \int_{-\infty}^{\infty} F(x, z; t) \psi(z, t) \, dz, \quad (2.7) \]

where $\hat{F}(x, z)$ is a matrix valued function. The semicolon indicates that the evolution variable $t$ enters the equations as a parameter only. We assume that $\hat{F}$ can be factorised with the operators $\hat{K}_\pm$ in the following way

\[ 1 + \hat{F} = (1 + \hat{K}_+)^{-1}(1 + \hat{K}_-), \quad (2.8) \]

where $\hat{K}_\pm$ are integral operators such that their kernels $K_+(x, z; t) = 0$ for $z < x$ and $K_-(x, z; t) = 0$ for $z > x$. One can determine the kernels $K_\pm(x, z; t)$ of the operator $\hat{K}_\pm$ from $F(x, z; t)$ by writing the integral operator equation (2.8) explicitly

\[ \pm K_\pm(x, z; t) + F(x, z; t) + \int_{-\infty}^{\infty} K(x, y; t) F(y, z; t) \, dy = 0. \quad (2.9) \]

This is a matrix Gel'fand-Levitan-Marchenko (GLM) equation; a linear Fredholm equation that can be solved formally for $K_\pm(x, z; t)$, using a Neumann expansion, for all well behaved $F(x, z; t)$. Unfortunately simple closed form solutions are available only for certain $F$. One encounters a GLM equation of one form or another in all the spectral transform methods for solving PDEs.

We now define the differential operators that act on the vector functions $\psi(x, t)$ as

\[ \hat{\Delta}^{(1)}_0 = \frac{\partial}{\partial t} - \hat{L}_0 \quad \hat{\Delta}^{(2)}_0 = \hat{M}_0, \quad (2.10) \]

\[ \hat{L}_0 = \sum_{k=1}^{n} l_k \frac{\partial^k}{\partial x^k} \quad \hat{M}_0 = \sum_{k=1}^{n} m_k \frac{\partial^k}{\partial x^k} \quad (2.11) \]
such that \([\hat{\Delta}^{(1)}_0, \hat{\Delta}^{(2)}_0] = 0\). We ‘dress’ the \(\hat{\Delta}^{(i)}_0\) to form new operators \(\hat{\Delta}^{(i)}\)

\[
\hat{\Delta}^{(i)} = (1 + \hat{K}_+^{(i)})\hat{\Delta}^{(i)}_0(1 + \hat{K}_+)^{-1}
\]  

Substitution of eq. (2.11) into this equation and equating to zero the differential parts of the operator relation yields

\[
\hat{\Delta}^{(1)} = \frac{\partial}{\partial t} - \hat{L} \quad \hat{\Delta}^{(2)} = \hat{M},
\]  

\[
\hat{L} = l_n \frac{\partial^n}{\partial z^n} + \sum_{k=0}^{n-1} u_k(x,t) \frac{\partial^k}{\partial x^k} \quad \hat{M} = m_n \frac{\partial^n}{\partial z^n} + \sum_{k=0}^{n-1} v_k(x,t) \frac{\partial^k}{\partial x^k}
\]  

for which \(u_1, u_2, \text{etc}\) can be defined recursively in terms of the \(l_k\) and \(K_+(x,z,t)\), and \(v_1, v_2\) in terms of \(m_k\) and \(K_+(x,z,t)\).

For reasons that shall become clear shortly, we impose the conditions

\[
[\hat{\Delta}^{(1)}_0, \hat{F}] = 0 \quad \text{and} \quad [\hat{\Delta}^{(2)}_0, \hat{F}] = 0.
\]  

In practice, these will be two PDEs with constant coefficients for the kernel function \(F(x,z;t)\), one of which will determine the evolution of \(F\) with time. Consider the operator

\[
[\hat{\Delta}^{(1)}, \hat{\Delta}^{(2)}](1 + \hat{K}_+) = \hat{\Delta}_1(1 + \hat{K}_+)\hat{\Delta}^{(2)}_0 - \hat{\Delta}^{(2)}_0(1 + \hat{K}_+)\hat{\Delta}^{(1)}_0 = (1 + \hat{K}_+)\hat{\Delta}^{(1)}_0, \hat{\Delta}^{(2)}_0 = 0.
\]

Therefore, using the invertibility of the operator \((1 + \hat{K}_+)\),

\[
[\hat{\Delta}^{(1)}, \hat{\Delta}^{(2)}] = \partial_t \hat{L} + [\hat{L}, \hat{M}] = 0,
\]  

which is the Lax \(^1\) equation (2.1), equivalent to a nonlinear evolution equation expressed in terms of the functions \(u(x,t)\) and \(v(x,t)\). One can write the KdV hierarchy in this form, where \(\hat{L}\) and \(\hat{M}\) are scalar operators. Similarly the NSE, and vector forms such as the Manakov model, and the Sine-Gordon equation can be expressed in this form, although one must use matrix differential operators. In the next section we will outline the solution of the NSE using this method. It is possible to develop a formalism for using the Zakharov and Shabat scheme to solve the Cauchy initial value problem. From initial conditions that define \(\hat{L}\) and \(\hat{M}\) at \(t = 0\) one can reconstruct \(K_+(x,z;0)\) from the dressing identity (2.12), as long as we assume \(K_+(x \to \infty, z; t) \to 0\). The integral GLM equation (29) is then solved for \(F(x,z;0)\). Solving the linear PDEs (2.15) gives \(F(x,z;t)\) and we then treat the GLM equation as a Marchenko equation and solve it for \(K_+(x,z; t)\), from which we recover the solution to the initial value problem. Closed form analytical solutions will only be obtained when the initial conditions are exact multisolitons. Initial conditions of this form correspond to reflectionless potentials of the spectral problem. It is frequently more convenient to skip the first, scattering part of the problem and consider instead simple functions \(F(x,z;t)\) that satisfy the pair of linear PDEs (2.15). We hope that these simple solutions will allow the determination of \(K_+(x,z; t)\) from the Marchenko equation (2.9), and thereby yield closed form solutions to the original problem (2.18).

Having developed this formalism, Zakharov and Shabat (1974) systematically considered the simplest scalar and matrix differential operators for \(\hat{L}\) and \(\hat{M}\), to uncover a wide range of evolution equations that could be exactly integrated.

\(^1\)In fact the original Zakharov and Shabat (1974) published form of this scheme presented the solution method for the more general equation \(\alpha \partial_t \hat{L} - \beta \partial_y \hat{M} + [\hat{L}, \hat{M}] = 0\). This extends the Lax equation to systems with two evolution variables \(t\) and \(y\) and is called the Zakharov-Shabat representation. All the equations considered in this thesis can be written in the form (2.18), so for pedagogical reasons our presentation is limited to the case \(\beta = 0\).
2.1.4 The nonlinear Schrödinger equation

The nonlinear Schrödinger equation (NSE)

\[ i \partial_t u + \partial_x^2 u + \sigma |u|^2 u = 0 \]  

(2.19)

can be written in the form of eq. (2.18) where

\[ \hat{L} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + 3 \begin{pmatrix} \pm u^* \\ 0 \end{pmatrix} \]  

(2.20)

\[ i \hat{M} = 3i \partial_x^2 + 2 \begin{pmatrix} \frac{1}{2} \sigma |u|^2 \\ -\sigma \partial_x u^* \\ \sigma |u|^2 \end{pmatrix} \]  

(2.21)

Therefore the NSE can be written as the compatibility condition for the Lax equations

\[ \hat{L} \psi = \lambda \psi \]  

(2.22)

\[ \hat{M} \psi = \partial_t \psi \]  

(2.23)

The NSE can be recovered by considering the identity \( \partial_x \partial_t \psi = \partial_t \partial_x \psi \). Searching for multisolitons we commence with the differential equations that result from the conditions (2.15)

\[ \hat{\Delta}^{(1)}_0 \hat{F} = 0 \]  

(2.24)

\[ \Rightarrow i \partial_t \hat{F}(x,z;t) + 3 \partial_x^2 F - 3 \partial_z^2 F = 0 \]  

(2.25)

\[ \hat{\Delta}^{(2)}_0 \hat{F} = 0 \]  

(2.26)

\[ \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \partial_x F + \partial_z F \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = 0. \]  

(2.27)

The solution method consists of the following: find a solution \( F(x,z;t) \) to the equations (2.27), then find \( K_+(x,z;t) \) satisfying the Marchenko equation (2.9). Finding the solution \( u(x,t) \) to the NSE from \( K_+(x,z;t) \) requires an identity connecting these two quantities. Such an identity can be found by considering the second of the equations (2.12):

\[ \hat{\Delta}^{(2)}(1 + \hat{K}_+) - (1 + \hat{K}_+) \hat{\Delta}^{(2)}_0 = 0. \]  

(2.28)

Writing out the differential operators \( \hat{\Delta}^{(2)} \) and \( \hat{\Delta}^{(2)}_0 \) and the integral operator \( \hat{K}_+ \), then integrating by parts and equating coefficients yields

\[ \begin{pmatrix} 0 & u \\ \sigma u^* & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \hat{K}_+ \]  

(2.29)

\[ \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \partial_x K_+ + \partial_t \hat{K}_+ \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & u \\ \sigma u^* & 0 \end{pmatrix} \hat{K}_+ = 0. \]  

(2.30)

The second of these equations defines the scattering problem that must be solved when dealing with the Cauchy initial value problem. In that case one must find \( K_+(x,z;0) \) in terms of the specified initial conditions \( u(x,0) \). For finding multisolitons it is only necessary to find \( u \) in terms of the kernel \( K_+ \). Assuming the matrix \( K_+(x,z;t) \) to be of the form

\[ K_+(x,z;t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]  

(2.31)
the first of the equations (2.30) gives
\[
\begin{pmatrix}
0 & u \\
\sigma u^* & 0
\end{pmatrix} = \begin{pmatrix}
0 & 3b \\
-3c & 0
\end{pmatrix}
\] (2.32)

This direct correspondence between the off-diagonal elements of \(K_+(x,x;t)\) and \(u(x,t)\) is the final link in the chain and we now proceed to the construction of a multisoliton of the NSE.

2.1.5 Multi-soliton solutions

The linear system of equations (2.27) can easily be satisfied by choosing an exponential form for \(F(x,z;t)\); we obtain the simple solution
\[
F(x,z;t) = \sum_{i=1}^{N} \begin{pmatrix}
0 & r_i e^{\mu_i (x+2z)+9i \nu_i^2 t} \\
S_i e^{\nu_i (x+2z)-9i \nu_i^2 t} & 0
\end{pmatrix}.
\] (2.33)

The linearity of the equations (2.27) means that we can construct a solution, having taken \(F(x,z;t)\) as a sum of any number of exponential terms. It is a feature common to many (if not all) of the methods for deriving soliton solutions that they begin with a linear system whose solutions can be superposed, resulting in solutions to the nonlinear evolution equation that reflect a nonlinear superposition principle. Choosing \(F(x,z;t)\) to be a single exponential results in a single soliton solution, and one can superpose these to construct the multisolitons. For brevity we move straight to the two soliton solution, and therefore take \(N = 2\) in eq. (2.33). Suppressing the parametric time dependence the Marchenko equation (2.9) reads
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} + \begin{pmatrix}
0 & R_1 + R_2 \\
S_1 + S_2 & 0
\end{pmatrix} +
\int_{-\infty}^{\infty} \begin{pmatrix}
b(x,y)\{S_1(y,z) + S_2(y,z)\} & a(x,y)\{R_1(y,z) + R_2(y,z)\} \\
d(x,y)\{S_1(y,z) + S_2(y,z)\} & c(x,y)\{R_1(y,z) + R_2(y,z)\}
\end{pmatrix} dy = 0,
\] (2.34)

where we have set
\[
F(x,z;t) = \begin{pmatrix}
0 & R_1(x,z;t) + R_2(x,z;t) \\
S_1(x,z;t) + S_2(x,z;t) & 0
\end{pmatrix}.
\] (2.35)

The integral equations (2.34) are separable on account of the fact that \(F(x,z) = A(x)B(z)\). By selecting an appropriate form for the kernel \(K_+(x,z)\), factoring out the \(z\) dependence and doing the integrals, the problem is reduced to a set of linear algebraic equations. These can be solved, although for the higher order solutions things get rather messy at this point. The consequence of these tedious calculations is an expression for the kernel \(K_+(x,z;t)\) from which eq. (2.32) gives the 2-soliton for the NSE
\[
u(x,t) =
\frac{r_1 e^{G_1} + r_2 e^{G_2} - \sigma \frac{r_1 r_2 r_1}{2(\mu_1+\mu_2)^2} \left( \frac{\mu_1-\mu_2}{\mu_1+\mu_2} \right)^2 e^{G_1+G_2+G_1^*} - \sigma \frac{r_2^2 r_1}{2(\mu_2+\mu_1)^2} \left( \frac{\mu_2-\mu_1}{\mu_2+\mu_1} \right)^2 e^{G_1+G_2+G_2^*}}{1 + \sigma \sum_{i,j=1}^{2} \frac{r_1 r_2 r_1}{2(\mu_1+\mu_2)^2} e^{G_i+G_j^*} + \frac{r_1^2 r_2^2}{|\mu_1+\mu_1|^2 |\mu_2+\mu_2|^2} \left( \frac{\mu_1-\mu_2}{\mu_1+\mu_2} \right)^2 e^{G_1+G_2+G_1^*+G_2^*}}
\] (2.36)

where
\[
G_j(x,t) = \mu_j x + \frac{1}{2} i \mu_j^2 t
\] (2.37)

One observes that the solution is singular if \(\sigma = -1\). This is for the defocusing NSE that also has bounded kink solutions with finite boundary conditions. The kink solutions
of the NSE will be described in some detail in Chapter 4. The solution presented here with zero boundary conditions is a 2-soliton describing the interaction of two solitons with a cosech profile. We concentrate on the \( \sigma = 1 \) case, in which the fundamental soliton is a sech-profile wave that can be characterised by considering the limiting forms of eq. (2.36). We postpone a detailed discussion of the multisolitons of the NSE until section 2.3 in which the first couple of multisolitons are derived using the method of the Darboux transformation, and described in some detail.

### 2.2 Hirota’s method

In Japan, parallel studies were taking place and Hirota (1971, 1973a, 1973b) developed a surprising method that could generate multisoliton solutions in a purely algebraic way. The Hirota method appears to succeed on all the equations that can be solved using the IST. It is no coincidence that many solution methods for integrable systems revolve around simple solutions to linear equations. In applying the Hirota method we form a polynomial ansatz and obtain linear differential equations for the polynomial coefficients. In the previous section we saw the fundamental role played by exponential solutions \( F(x,z;t) \) to the linear equations (2.27). A nonlinear superposition principle results when these solutions are superposed, creating the multisolitons. What sets the Hirota method apart is its beguiling simplicity and ease of use. This simplicity is due to the fact that the \( \tau \)-function on which the method is based is related to the original problem through an algebraic identity, rather than a differential equation. It is therefore a simple matter to transform the \( \tau \)-function solutions into the desired multisolitons. This is in stark contrast to the spectral methods in which an inverse scattering problem must be solved to recover the dynamical variable. The Hirota method is used extensively in this thesis and we defer a detailed description of its application until section 3.3.

### 2.3 Darboux transformations

#### 2.3.1 An alternative approach to the Lax pair

The set of equations making up the Lax pair for a wide variety of systems has a further property that permits one to apply yet another technique. We use the eigenvalue problem associated with the KdV equation, the linear Schrödinger operator, as an illustrative example

\[
\partial_x^2 \psi + [\lambda - u(x)] \psi = 0. \tag{2.38}
\]

Darboux discovered last century (Darboux 1882) that if one makes the transformation

\[
\psi \rightarrow \tilde{\psi} = \partial_x \psi - \sigma \psi ; \quad u \rightarrow \tilde{u} = u - 2 \partial_x \sigma, \tag{2.39}
\]

where both \( \psi \) and \( \sigma \) satisfy (2.38), then \( \tilde{\psi} \) satisfies the equation

\[
\partial_x^2 \tilde{\psi} + [\lambda - \tilde{u}] \tilde{\psi} = 0, \tag{2.40}
\]

where \( \sigma = \varphi_x \varphi^{-1} \) and \( \psi \) and \( \varphi \) are solutions of eq. (2.38) with eigenvalues \( \lambda \) and \( \lambda_1 \) respectively. Thus it is apparent that by combining two solutions of eq. (2.38) we are able to generate another solution to a problem with a modified potential, but the same eigenvalue.

Now the linear Schrödinger equation is one half of the Lax pair for the KdV equation. Therefore solutions to the KdV equation are potentials of the Schrödinger operator whose temporal evolution leaves the eigenvalue unchanged, and the pure multisolitons are reflectionless potentials. This is most suggestive. In the 1970’s several authors
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(Wahlquist 1976; Matveev 1979) realised the applicability of this sort of transformation to the nonlinear evolution equations. The term “Darboux transformation” (DT) was coined to describe transformations of this kind. It is a special case of a Bäcklund transformation.

In order to apply this transformation to nonlinear wave equations, rather than just to the scattering problem, we must deal in some way with the evolution equation part of the Lax pair. In fact it is necessary to show that the same transformation will also leave the solutions of this equation unchanged. When the utility of Darboux’ method was realised, this question was studied in some detail and some quite general results have been obtained.

In particular, it has been shown that the Darboux invariance is a general property of the linear equations:

\[ \psi_t = \sum_{m=0}^{N} u_m \partial_x^m \psi, \quad (2.41) \]

where \( u_i \) are \( N \times N \) matrices and \( \psi \) are vector valued functions. To be specific, if \( \psi \) and \( \varphi \) are solutions of \((2.41)\) then \( \tilde{\psi} = \partial_x \psi - \sigma \psi \) with \( \sigma = (\partial_x \varphi) \varphi^{-1} \) is another solution of \((2.41)\) with transformed \( u_m \).

This generalisation allows one to solve a hierarchy of equations including the KdV and Kadomtsev-Petviashvili (KP). If both the equations in the Lax pair of an evolution equation admit the same transformation, then the DT will yield new solutions to both the linear problems and to the nonlinear problem. The reasons why each equation in the Lax pair will admit the same DT is rooted in the conserved quantities that form the basis for the IST.

Darboux symmetries have been found for a wide variety of equations, including difference and differential-difference equations (Matveev and Salle 1991). Symmetries of differential equations under transformations of this kind are now well understood with the help of group theoretic approaches. A broad spectrum of nonlinear problems, from \((2+1)\) dimensional systems like the KP equation to lattice type models like the Toda chain, have now been dealt with using the DT approach.

It is worth making a brief mention of the relationship between the DT and its better known cousin, the Bäcklund transformation. The latter refers to any transformation linking two solutions of a differential equation. The DT, a special case of a Bäcklund transformation, is centred around the Lax equations and therefore pertains particularly to the soliton equations. It is particularly useful as non-trivial solutions are often obtained even when the ‘seed’ solution is the zero function.

The major difficulty in applying Bäcklund transformations directly is that in working directly with the evolution equation we can usually only derive differential equations relating different solutions. These are less straightforward than the purely algebraic relations arising from the Darboux transformation. The Darboux transformation also returns transformed values of both the potential and the scattering functions, so that repeated application of the transformation will always generate a hierarchy of solutions.

2.3.2 The nonlinear Schrödinger equation

We will now apply the method of Darboux transformations to the NSE to find a formula for the multisoliton that describes the interaction between two bright solitons. In order to develop the method, we write the Lax equations for the NSE in a convenient form

\[ R_x = J R \lambda + U R \]

\[ R_t = J R \lambda^2 + U R \lambda + J (U^2 - U_x) R. \]
2.3 Darboux transformations

where

\[ R = \begin{bmatrix} r \\ s \end{bmatrix}; \quad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}; \quad U = \begin{bmatrix} 0 & iu \\ iu^* & 0 \end{bmatrix}. \] (2.44)

If we demand compatibility between the two equations by setting \( R_{xt} = R_{tx} \) and eliminate all derivatives of \( r \) and \( s \) then we recover the NSE (2.19). Finding the Lax pair for a given equation is no trivial matter; historically the Lax pair was usually found by serendipity, or working backwards by substituting various linear operators into the Lax equations (2.1).

The Lax pair (2.43) is covariant with respect to the transformation

\[ \tilde{R} = R\lambda - \sigma R \] (2.45)
\[ \tilde{U} = U + [J, \sigma], \] (2.46)

where square brackets refer to the commutator \([a, b] = ab - ba\), \(\sigma = Q\Lambda Q^{-1}\) and \(Q\) is a matrix solution of the Lax pair with eigenvalues given by \(\Lambda\), i.e.

\[ Q = \begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \] (2.47)

This can also be checked – the calculation is straightforward if a little tedious – by transforming \( R \) and then showing that the equations satisfied by the \( \tilde{R} \) are just eqs. (2.43) with \( U \rightarrow \tilde{U} \). It should be noted that one can extend this to a 3x3 system in a trivial manner to derive the Lax pair for the Manakov system. The exact nature of the DT for the Manakov system remains an open question, however.

Let us now write out the DT for the NSE in detail. Setting \( \Delta = r_1 s_2 - r_2 s_1 \) we find

\[ \sigma = \frac{1}{\Delta} \begin{bmatrix} \lambda_1 r_1 s_2 - \lambda_2 r_2 s_1 & - (\lambda_1 - \lambda_2) r_1 r_2 \\ (\lambda_1 - \lambda_2) s_1 s_2 & - \lambda_1 r_2 s_1 + \lambda_2 r_1 s_2 \end{bmatrix}, \] (2.48)

and therefore the transformation for the components of \( U \) reads

\[ \begin{bmatrix} 0 & i\tilde{u} \\ i\tilde{u}^* & 0 \end{bmatrix} = \frac{2i}{\Delta} \begin{bmatrix} 0 & (\lambda_1 - \lambda_2) r_1 r_2 \\ 0 & (\lambda_1 - \lambda_2) r_1 r_2 \end{bmatrix} + \begin{bmatrix} 0 & iu \\ iu^* & 0 \end{bmatrix}. \] (2.49)

In order for this to be consistent for both the field and its conjugate, we require an additional condition on the two eigenvalues \(\lambda_j\) and an additional relation between the two solutions that make up \(Q\)

\[ \lambda_2^* = \pm\lambda_1 \quad r_1^* = s_2 \quad r_2^* = \mp s_1. \] (2.50)

These can be called the reality conditions; without them we solve a complexified form of the NSE that has no physical significance. We are now in a position to write down the Darboux transformation for \(u\)

\[ \tilde{u} = u - 2 \frac{(\lambda_1 \mp \lambda_1^*) r_1 s_2^*}{|r_1|^2 \pm |s_1|^2}. \] (2.51)

Similarly one derives, using the identities (2.46) and (2.48), the DT for the eigenfunctions:

\[ \begin{bmatrix} \tilde{r} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} (\lambda - \lambda_1)|r_1|^2 r \pm (\lambda - \lambda_1^*)|s_1|^2 r \pm (\lambda_1 - \lambda_1^*) r_1 s_2^* s \\ \pm (\lambda - \lambda_1)|s_1|^2 s \pm (\lambda - \lambda_1^*)|r_1|^2 s \pm (\lambda_1 - \lambda_1^*) r_1 s_2^* r \end{bmatrix}. \] (2.52)
2.3.3 The two-soliton solution

We now know that a complete hierarchy of solutions to the NSE can be derived using algebraic methods alone. It is our goal in this section to investigate the hierarchy that arises when the initial solution \( u_0 \) is taken to be identically zero. If we substitute \( u_0 = 0 \) into the Lax pair, eqs. (2.43), then we get

\[
R_x = J R \lambda_1 \quad \text{(2.53a)}
\]

\[
R_t = J R \lambda_1^2, \quad \text{(2.53b)}
\]

which yields the solutions

\[
\begin{bmatrix}
  r \\
  s
\end{bmatrix} =
\begin{bmatrix}
  e^{i \lambda_1 x + i \lambda_1^2 t + i \phi} \\
  e^{-i \lambda_1 x - i \lambda_1^2 t - i \phi}
\end{bmatrix}.
\]

Writing \( \lambda_1 = a_1 + ib_1 \) the second solution in the hierarchy is found using eq. (2.51)

\[
u_1(x,t) = -2i b_1 \text{sech}(2b_1 x + 4a_1 b_1 t) e^{-2ia_1 x - 2i(a_1^2 - b_1^2) t - 2i \phi}.
\]

This is the 1-soliton of the NSE, with a sech profile. It is the eigenfunction of the Lax scattering problem (2.53a) with eigenvalue \( \lambda \). We can view the Darboux transformation process in terms of the spectrum of the linear operator of the eigenvalue equation. Our original selection, \( u_0 = 0 \) corresponded to a scattering problem with no discrete spectrum. In solving the Lax pair we introduced a complex parameter \( \lambda \) that becomes the eigenvalue of the spectral problem corresponding to \( u_1 \). The solution constructed from this is the bright soliton characterised by the complex eigenvalue. One may therefore think of the DT as a process by which we add eigenvalues to the spectrum of the associated scattering problem.

Note that by introducing a single eigenvalue to the spectrum we are constructing the reflectionless potentials. Therefore when we advance to the next stage in the hierarchy we will add another eigenvalue to the operator, and will find a two-soliton solution. First we must use eq. (2.52) to form the eigenfunctions at the second level, that will solve the Lax pair with \( u = u_1 \). We call these functions \( r_{12} \) and \( s_{12} \), as they depend on an additional eigenvalue, which we call \( \lambda_2 \) and write as \( a_2 + i b_2 \).

In order to simplify the calculations we introduce the parameters

\[
\begin{align*}
A_1 &= 2a_1 x + 2(a_1^2 - b_1^2) t - \phi \\
A_2 &= 2a_2 x + 2(a_2^2 - b_2^2) t \\
C_m &= 2b_m x + 4a_m b_m t + \Delta x_m ; \quad m = 1, 2
\end{align*}
\]

and therefore write the initial eigenfunctions as

\[
\begin{bmatrix}
  r_m \\
  s_m
\end{bmatrix} = \begin{bmatrix}
  e^{i \frac{1}{2} A_m - i \frac{1}{2} C_m} \\
  e^{-i \frac{1}{2} A_m - i \frac{1}{2} C_m}
\end{bmatrix}.
\]

Using eq. (2.52) yields

\[
r_{12} = -2ibe^{iA_1 - \frac{i}{2}iA_2 + \frac{i}{2}C_2} + (\lambda_2 - \lambda_1)e^{-C_1 - \frac{i}{2}iA_2 - \frac{1}{2}C_2} + (\lambda_2 - \lambda_1)^*e^{C_1 + \frac{i}{2}iA_2 - \frac{1}{2}C_2}, \quad \text{(2.58)}
\]

\[
s_{12} = -2ibe^{-iA_1 + \frac{i}{2}iA_2 - \frac{i}{2}C_2} + (\lambda_2 - \lambda_1)e^{C_1 - \frac{i}{2}iA_2 + \frac{1}{2}C_2} + (\lambda_2 - \lambda_1)^*e^{-C_1 + \frac{i}{2}iA_2 + \frac{1}{2}C_2}. \quad \text{(2.59)}
\]
2.3 Darboux transformations

The calculation rapidly becomes more complicated as we move up the hierarchy. Some tedious calculation ensues before one finally emerges with the third member, the general two-soliton solution, that reads

\[ u_2(x,t) = u_1 + \frac{2(\lambda_2^* - \lambda_2)r_{12}^s s_{12}}{|r_{12}|^2 + |s_{12}|^2} \]

\[ = \frac{b_1 K e^{-i\Omega_1} + b_2 L e^{-i\Omega_2}}{M} \]

where

\[ K = [(a_2 - a_1)^2 + b_1^2 - b_2^2] \cosh C_2 + 2ib_2(a_2 - a_1) \sinh C_2 \]

\[ L = [(a_2 - a_1)^2 + b_2^2 - b_1^2] \cosh C_1 + 2ib_1(a_2 - a_1) \sinh C_1 \]

\[ M = [(a_2 - a_1)^2 + b_1^2 + b_2^2] \cosh C_1 \cosh C_2 + 2b_1 b_2 \sinh C_1 \sinh C_2 - 2b_1 b_2 \cos(\Omega_1 - \Omega_2) \]

It can be verified that this expression possesses the necessary properties of a solution, and one can use a symbolic computation package to verify that it does indeed satisfy eq. (2.19). This is the same 2-soliton function written in eq. (2.36), expressed in hyperbolic functions rather than exponentials. The asymptotic behaviour of the two soliton solution was first derived using the IST by Zakharov and Shabat (1972) in a classic paper. More than a decade later Gordon (1983) presented this solution, but in an implicit form only. Desem and Chu (1987) also looked at this solution with the aim of determining soliton interaction forces.

Now that we have derived this general expression, it is worthwhile to spend some time investigating its behaviour. From the single soliton solution it is clear that \( a_1 \) refers to the soliton amplitude and width, while \( b_1 \) is the angle relative to the \( t \) axis that each soliton travels at.

First let us briefly consider the case of identical soliton amplitudes. The explicit expression for this case was first presented by Akhmediev, Korneev, and Mitzkevich (1988), so we merely present figure 2.2 as an example of the interaction behaviour of two identical NSE solitons. In the left hand picture we show two in-phase solitons, while in the second we have introduced a \( \pi \) phase shift into the right hand beam so that the solitons are out of phase. Although the solitons in (a) appear to attract and those in (b) appear to repel, the final outcome (the asymptotic behaviour) of the interaction is completely unchanged by the phase change. One can view the pattern at the centre of the interaction as the result of the linear interference between the two solitary waves propagating in different directions.

When the solitons are allowed to have a different amplitude, they no longer maintain a constant phase relationship and therefore the parameter \( \phi \) becomes less important.
Although it will change the intensity quantitatively, it ceases to affect the soliton trajectories in a consistent manner. Figure 2.3 gives two examples of non-degenerate solitons colliding. Figure 2.3(b) shows the interaction of two solitons with a small propagation angle between them. The effect of beating between two beams travelling with different propagation constants is apparent. Figure 2.3(a) shows the same solitons colliding, but with a much greater angle of incidence. This reduces greatly the time scale over which the solitons interact and therefore reduces the effect of that interaction. The phase shift and the position shift are accordingly reduced. The interaction length is less than one beat length, and the beating phenomenon is therefore not all that clear.

Finally we show the behaviour when we allow the angle of incidence between the beams to be zero. In this case the evolution is dominated by the beating effect, whose frequency is proportional to the difference in the soliton amplitudes. Figure 2.4(a) demonstrates this effect. This type of solution (in fact for the case where $b_1 = b_2 = 0$) was presented for the first time by Satsuma and Yajima (1974) in a study of the initial value problem for the NSE. Figure 2.4(b) shows a special case of these solutions that are periodic in $t$, in which the transverse profile periodically collapse to a sech profile with twice the amplitude of a soliton of the same width. This is called a higher order soliton, or more accurately it is the second order soliton, and its period is called the soliton period, and is used as the standard normalised length scale for numerical investigations. It gives a measure of the length scale over which the first order soliton would diffract if the nonlinearity were not present.
Figure 2.4: (a) A bound state formed between two parallel solitons of amplitudes $a_1 = 0.9, a_2 = 1.1$, with position shifted by $\Delta x_1 = 0, \Delta x_2 = 1.0$. (b) A higher order soliton. Notice that the profile periodically collapses to a single sech profile. $a_1 = 0.5, a_2 = 1.5, b_1 = b_2 = 0, \Delta x_1 = \Delta x_2 = 0$. 
Chapter 3

Multiple Soliton Bound States

In this chapter we consider the propagation of arbitrarily polarised beams of light, and particularly the formation of 'vector' solitary waves. A limited family of the equations describing vector solitary waves is integrable; we derive multisoliton solutions for this family. For the nonintegrable systems we use approximation methods and numerical techniques to find new classes of solitary waves. Allowing the polarisation state to vary across the transverse profile unearths a rich family of solitons of varying complexity. By expressing the elliptical polarisation state in terms of orthogonal basis vectors, such as linear polarisation components, we view vector waves as multi-component objects comprising orthogonally polarised beams mutually guiding each other. The novel feature of these states is that the two components have different propagation constants and therefore the polarisation ellipse rotates during propagation while the total intensity remains unchanged.

3.1 Background

Until recently, the bulk of research into optical solitons has considered the light to be composed of a single uniform polarisation. The spatial evolution of light beams in a planar waveguide and the temporal evolution of light pulses in an optical fibre in a Kerr medium can be described by the Nonlinear Schrödinger equation (NSE) when the light is uniformly polarised. Methods for finding rather general solutions for the NSE are outlined in Chapter 2. The power of this equation is confirmed by the many theoretical predictions, based on this equation, that have subsequently been experimentally verified. It is particularly striking within silica optical fibres where the nonlinear response is both very rapid and dominated by third order Kerr effects.

The transverse profiles of uniformly polarised ('scalar') solitons are limited to the simplest cases, as shown by Zakharov and Shabat in 1972 and 1973: fundamental bright solitons in self focussing media and first order dark solitons in defocussing media. By allowing the polarisation of the electric field to vary at different points in space we open up many new possibilities for interesting transverse effects. One finds that a variety of material parameters enter the picture and, in many cases, have a qualitative effect on the solitary wave forms that are present. As discussed in the introduction, when we account for the polarisation of light in Kerr media we find that a coupled set of nonlinear parabolic equations (known from now on as the coupled NSE or the CNSE) governs the propagation of monochromatic paraxial light beams and of light pulses in fibres. Sahadevan, Tamizhmani, and Lakshmanan (1986) showed that in general these equations fail the Painlevé integrability test, meaning that they cannot describe true mathematical solitons. However for the special case of an electrostriction induced nonlinearity (Shen 1966), or for materials of (relatively) slow response time, the field propagation is governed by the Manakov (1974) model, the simplest vector extension of the NSE. The Manakov equation is integrable, but the associated spectral problem is $3 \times 3$ and
analytical investigation tends to be frustrated by algebraic complexity exacerbated by the non-commutativity of the underlying symmetry group \( U(2) \). Despite the efforts of many researchers, there was, until recently, very little addition to the seminal Manakov (1974) investigation into vector solitons.

3.2 Leaping from scalar to vector solitary waves

3.2.1 Haus’ method: the mechanical analogue

We commence with a simple scheme for finding stationary states that leads us to some analytical solutions and that allows qualitative characterisation of many others. The relevant CNSE, from eqn (1.22) appear in dimensionless form as:

\[
\frac{1}{i} \frac{\partial E_\pm}{\partial z} + \frac{1}{2} \frac{\partial^2 E_\pm}{\partial x^2} + \left[ (1 - B)|E_\pm|^2 + (1 + B)|E_\mp|^2 \right] E_\pm = 0 \tag{3.1}
\]

In this equation \( E_+ \) and \( E_- \) represent the counter-rotating circularly polarised (CP) components of the field while \( B = \frac{x_{1231}/x_{1111}}{1} \) is a measure of the nonlinear coupling between polarisations known as cross phase modulation, or XPM. One can view the nonlinear terms in this equation as effecting a change in the refractive index, so that each polarisation ‘sees’ an index change written into the material. Each polarisation sees a different index profile; e.g. \( E_+ \), sees the refractive index \( n_+^2 - n_0^2 \simeq (1 - B)|E_+|^2 + (1 + B)|E_-|^2 \). Only when \( B = 0 \) do \( E_+ \) and \( E_- \) see the same index profile. For non-zero \( B \), then, there is a nonlinearly induced circular birefringence. Constructing a linear model mimicing the nonlinear effects often allows one to utilise the extensive literature of optical waveguide theory. Note that if the field is expressed in terms of the linear polarisation (LP) basis set \( [E_x = \frac{1}{\sqrt{2}}(E_+ + E_-) \text{ and } E_y = \frac{1}{\sqrt{2}}(E_+ - E_-)] \) then the propagation equations have coherent coupling terms of the form \( E_+^2 E_-^* \). These terms are a reflection of the fact that light induces circular birefringence into the material. When using a linearly polarised basis set the linear equivalent is less obvious and less useful. In addition, the new solitary waves presented in this chapter can only be simply expressed using circular polarisation.

The term solitary wave has two connotations; firstly it describes a localised, isolated wave and secondly a stationary wave whose form does not change during propagation. In the context of optics a solitary wave denotes a transverse optical structure that, as a result of some interplay between linear propagation effects and the nonlinearity, will propagate without change in its intensity profile. Therefore we are looking for stationary solutions, those that will satisfy \( \partial_z |E_\pm|^2 = 0 \). Therefore

\[
E_\pm(x, z) = e_\pm(x)e^{i\beta_\pm z}. \tag{3.2}
\]

Borrowing from the linear waveguide literature we call \( \beta_\pm \) the propagation constant of the stationary field. Substituting the ansatz (3.2) into the original system (3.1) reduces it to the ordinary differential equations (ODEs)

\[
-2\beta_\pm e_\pm + \ddot{e}_\pm + 2 \left[ (1 - B)e_\pm^2 + (1 + B)e_\mp^2 \right] e_\pm = 0, \tag{3.3}
\]

where the dot denote derivation with respect to \( x \). These equations also describe the trajectory of a particle moving through a certain potential. We shall spell out this connection to mechanics by presenting these equations in a Hamiltonian form. The above system can be considered to be the equations of motion (i.e. the Euler-Lagrange equations) for a system with Hamiltonian given by

\[
\mathcal{H} = T(e_+, e_-) + V(e_+, e_-)
\]

\[
= \left[ \frac{1}{2} \dot{e}_+^2 + \frac{1}{2} \dot{e}_-^2 \right] + \left[ -\beta_+ e_+^2 - \beta_- e_-^2 + \frac{1}{2} (1 - B) (e_+^4 + e_-^4) + (1 + B) e_+^2 e_-^2 \right], \tag{3.4}
\]
3.2 Leaping from scalar to vector solitary waves

Figure 3.1: Surface representations of the equivalent potential function \( V \) of eq. (3.1), cut out of a block defined by \(-4 < e_+ < 4, -4 < e_- < 4\) and \(-4 < V < 2\). The origin \( r = 0 \) is at the peak of the central hump. One can imagine a ball rolling on these surfaces, its position \((e_+(x), e_-(x))\) tracing out the transverse profile. (a) \( B = 0 \) (Manakov system) (b) \( B = \frac{1}{3} \) (silica fibres) (c) \( B = 0.75 \) (d) \( B = 0.9 \).

which describes the motion, in \((e_+, e_-)\) space, of a frictionless ball moving, under the influence of gravity, around terrain whose topography is defined by the potential function \( V \). Specifically, \( V \) represents the elevation at any point in the terrain. In this analogy, \( x \) assumes the role of time. Figure 3.1 illustrates the potential for \( \beta_+ = \beta_- = 1 \) and several different values of \( B \). This very convenient physical interpretation for the mathematical problem was first presented by Haus (1966) and is often referred to as the mechanical analogue. Using this picture we can, in a very straightforward manner, develop an intuition for the kind of solitary wave solutions that our equations might possess. This is a very simple approach but one that can be an invaluable precursor before embarking on other methods that may be more powerful but are invariably less intuitive.

3.2.2 Scalar solitons

As a beginning, consider the case of uniform circular polarisation and set \( e_-(x) = 0 \), which will constrain the particle to motion in one dimension along the \( e_+ \) axis. Solutions to this constrained problem will also be stationary solutions to the NSE. The ODE for this problem can be completely integrated in terms of Jacobi elliptic functions. In order to keep the presentation as intuitive as possible we first present the solutions to this problem graphically. The different classes of solitary wave solutions to the NSE can therefore be foreseen before presenting their algebraic forms. The equation of motion for scalar fields reads

\[
\dot{e}_+^2 = 2\beta_+e_+^2 - (1 - B)e_+^4 + 2H_0.
\] (3.5)
Figure 3.2: Uniformly polarised cnoidal wave solutions. At the top we indicate the motion inside the potential $V(e_+)$, at the centre the corresponding trajectories in phase space $(e_+, \dot{e}_+)$, and at the bottom the solution profiles $e_+(x)$. Parameter values are $\beta_+ = 1$, $B = \frac{1}{3}$, and (a) $\mathcal{H}_0 = -0.9$, (b) $\mathcal{H}_0 = -0.02$, (c) $\mathcal{H}_0 = 0.0$, (d) $\mathcal{H}_0 = 0.02$, (e) $\mathcal{H}_0 = 4.0$.

This equation describes the motion of a ball rolling around in a structure given by the quartic $V = \frac{1}{4}(1 - B)e_+^4 - \beta_+ e_+^2$, as illustrated in figure 3.1. The function has zeroes at $e_+ = \pm \sqrt{4\beta_+/(1 - B)}$, a local maxima of zero at $e_+ = 0$ and minima of $V = -\beta_+^2/(1 - B)$ at $e_+ = \pm \sqrt{2\beta_+/(1 - B)}$. The total energy given to the ball is given by $\mathcal{H}_0$. More precisely, we choose $\mathcal{H}_0$ to represent the energy of the ball when it is at 'sea level', i.e. where $V = 0$. The different solution classes can be isolated by considering different values of $\mathcal{H}_0$. Figure 3.2 illustrates the different types of solution to this problem. There will be no solution at all if $\mathcal{H}_0$ is less than $-2\beta_+/(1 - B)$. At $\mathcal{H}_0 = -\beta_+^2/\sigma$ there are constant solutions $e_+ = \pm \sqrt{2\beta_+/(1 - B)}$. If $\mathcal{H}_0 > -2\beta_+^2/(1 - B)$ but still negative then the ball can never surmount the hump at the centre of the potential and consequently will be constrained to oscillations within one of the two troughs either side of it. Examples with $\mathcal{H}_0 = -0.9$ and $-0.02$ are shown in figure 3.2(a) and (b). For all values of $\mathcal{H}_0 > 0$ the ball will oscillate between the troughs, slowing as it crosses the central hump, as can be seen in figure 3.2(d) and (e). For $\mathcal{H}_0 = 0$ there will be a constant solution $e_+$. There is, however, another trajectory corresponding to $\mathcal{H}_0 = 0$, separating the two different types of periodic trajectories, called the separatrix, shown in figure 3.2(c). The ball can be thought of as starting with the minutest perturbation from the central massif, going down one side and then asymptotically approaching that point again as $x \to \infty$. Solitary waves correspond to the separatrices of nonlinear systems. Now we present the analysis leading to an algebraic form of the solutions to eq. (3.4). For $-2\beta_+^2/(1 - B) \leq \mathcal{H}_0 \leq 0$ we can rewrite the equation (3.4) in the standard elliptic form

$$\left( \frac{df}{d\zeta} \right)^2 = \left( 1 - f^2 \right) \left( f^2 - \frac{c_-}{c_+} \right), \quad (3.6)$$
3.2 Leaping from scalar to vector solitary waves

where

\( e_+ = \sqrt{c_+ f} \), \( \zeta = \sqrt{\frac{1}{2}(1-B)x} \), and

\( c_\pm = \beta_\pm \pm \sqrt{\beta_\pm^2 - (1-B)x^2} \) \( \text{where } \mathcal{H}_0 = -\lambda^2. \) (3.8)

Note that \( 0 \leq c_- \leq c_+ \). Solutions to this differential equation are the Jacobi elliptic functions of the second kind \( f = dn(\zeta|m) \) [see e.g Whittaker and Watson (1927)], where \( m = \sqrt{1-c_+/c_+} \). Undoing the scaling transformations above gives the solution to eq. (3.4) as

\[ e_+ = \sqrt{c_+} \text{dn} \left( \sqrt{\frac{1}{2}(1-B)} c_+ x \bigg| m \right). \]

The limiting behaviour of the elliptic functions, which can be found in Abramowitz and Stegun (1964) shows that as \( \mathcal{H}_0 \to -\frac{2}{3}+/\sqrt{1-B} \), then

\[ c_+ \to \frac{2\beta_+}{1-B}, \quad \text{and} \quad e_+ \to \sqrt{2\beta_+/(1-B)} \left[ 1 - \frac{m^2}{2} \sin^2 \left( \sqrt{\frac{1}{2}(1-B)c_+ x} \right) \right], \]

implying that the ball undergoes small harmonic oscillations about the minimum in the potential, as in figure 3.2(e). In the opposite limit, \( \mathcal{H}_0 \to 0^- \) then we find \( c_+ \to 4\beta_+/(1-B) \) and \( c_- \to 0 \), and subsequently \( e_+ \to \sqrt{2\beta_+/(1-B)} \text{sech} \left( \sqrt{\beta_+ x} \right) \). This is the classic sech profile solitary wave. The convergence of the solutions upon this form is seen in figure 3.2(b) and (c).

An analogous analysis can be performed to find the solutions for \( \mathcal{H}_0 > 0 \); one derives

\[ \left( \frac{df}{d\xi} \right)^2 = (1-f^2) \left( m' + mf^2 \right), \]

where \( f \) and \( c_+ \) are defined as before, \( \xi = x\sqrt{\frac{1}{2}(1-B)(c_+ + c_-)} \), \( m = (1+c_-/c_+)^{-1}, \) \( m' + m = 1 \). A solution for this equation is \( f = cn(\xi|m) \) and therefore

\[ e_+ = \sqrt{c_+} \text{cn} \left( \sqrt{\frac{1}{2}(1-B)(c_+ + c_-)} x \bigg| m \right) \]

is a solution of (3.4) for \( \mathcal{H}_0 > 0 \). In the limit \( \mathcal{H}_0 \to 0 \) this again tends to the hyperbolic secant profile of the separatrix trajectory as can be seen from figure 3.2(c) and (d). Since these solutions are the Jacobi elliptic functions cn, dn and sn they are called cnoidal waves. Our analysis has limited us to stationary waves; Krichever (1977) has developed algebro-geometric methods for finding a more general class of solutions, called multiphase or quasiperiodic solutions. The quasiperiodic solutions represent the nonlinear superposition of several cnoidal waves, and are thus neither periodic in the transverse or longitudinal co-ordinates. Deriving the multiphase solutions is more complex than finding multisoliton solutions since the eigenfunctions of the associated Lax pair inhabit non-trivial Riemann surfaces.

Solitary waves are distinguished from other stationary states by having trajectories in phase space that are the homoclinic separatrix trajectories. Considering the full mechanical problem of eq. (3.4) one finds the two-dimensional potential \( V(e_+, e_-) \) that is shown for several values of the material parameter \( B \) in figure 3.1. The phase space will be four dimensional and we expect there to be whole families of separatrices, all with \( \mathcal{H}_0 = 0 \) and starting and finishing on the central peak. To find the desirable solutions, then, let us place our imaginary ball on top of the central pinnacle and give it the smallest possible of nudges and allow it to roam freely within \( (e_+, e_-) \) space.
One can extend the applicability of the solutions obtained for the scalar problem to some trivial vector cases of non-zero $\epsilon_\pm$, when the electric field is of a uniform, if elliptic polarisation. General solutions with uniform polarisation can be found if we take the counter-rotating components $\epsilon_\pm$ to be be related by a constant. Therefore one finds them by setting $\epsilon_+ = \gamma \epsilon_+, \beta_+ = \beta_-$, with $\gamma$ a real constant. For scalar solutions we require eq. (3.3) to reduce to a single differential equation

$$1/2 \epsilon_+ - \beta_+ \epsilon_+ + \epsilon_+^3 = 0,$$

which occurs under the condition $\gamma = \pm 1$, unless $B = 0$ when $\gamma$ may be any complex constant and all uniformly polarised states are equivalent under the $U(2)$ group. For $B$ non-zero we must have $\epsilon_+ = \pm \epsilon_-$, describing linearly polarised light. Therefore the only uniformly polarised solutions to the nonintegrable CNSE are either circularly or linearly polarised. One can characterise these solutions by letting $\gamma = 1$ and $\epsilon_x = (\epsilon_+ + \epsilon_-)\sqrt{2}$, $\epsilon_y = 0$. One recovers the same equation as for the circularly polarised profile by making the scaling transformation $\epsilon_x = \sqrt{1-B} \epsilon_+$. Therefore the same family of solutions exist for circularly polarised light and linearly polarised light in isotropic media. On account of the stronger self-focussing suffered by linearly polarised light (Wang 1966), linearly polarised solitons have their intensity reduced by the factor $(1-B)$.

Every solitary wave presented above is a soliton of the NSE and therefore can be built into multisoliton solutions using the methods of Chapter 2. In addition, one can generalise the cnoidal wave solutions, obtainable using the algebro-geometric methods mentioned in chapter 2.

### 3.2.3 Bifurcation points: origins of vector solitary waves

We have now characterised all the stationary waves of uniform polarisation. What about more general solutions? Unfortunately the general CNSE (3.3) is not integrable, forcing approximate and numerical methods to be used. Rather than plunging directly into numerics, let us first build upon things we know and understand: uniformly polarised solitary waves. Because we are looking for solutions that are vector generalisations of the LP and CP states, we expect that the new vector waves will collapse to uniformly polarised waves in certain limits. In these limits the new vector solitary waves should be describable as perturbations of LP or CP solitons. This section considers potential solutions that asymptotically approach circularly polarised solitons.

Planning to add small amounts of another polarisation to a CP soliton, we commence by investigating their stability to perturbations of arbitrary polarisation by means of a standard linear stability analysis. To this end we introduce the ansatz

$$E_+(x, z) = \sqrt{2\beta_+} \text{sech} \left( \sqrt{\frac{2\beta_+}{1-B}} x \right) e^{iz} + \epsilon_+(x, z)$$

$$E_-(x, z) = \epsilon_-(x, z),$$

where $\epsilon_+$ and $\epsilon_-$ are the infinitesimal perturbations in both polarisation components. Substituting eq. (3.16) into (3.1) leads to two decoupled equations in $\epsilon_+$ and $\epsilon_-$.\]

$$i\partial_x \epsilon_+ + \frac{1}{2} \partial_x^2 \epsilon_+ - \beta_+ \epsilon_+ + 2\beta_+ \text{sech}^2 \left( \sqrt{2\beta_+} x \right) (2\epsilon_+ + \epsilon_+^3) = 0 \quad (3.17a)$$

$$i\partial_x \epsilon_- + \frac{1}{2} \partial_x^2 \epsilon_- + 2\kappa \beta_+ \text{sech}^2 \left( \sqrt{2\beta_+} x \right) \epsilon_- = 0, \quad (3.17b)$$

where $\kappa = (1+B)/(1-B)$. The independence of $\epsilon_+$ and $\epsilon_-$ makes the stability analysis very simple. Equation (3.17b) is nothing but the equation of linear stability analysis for a CP soliton. Kuznetsov and Turitsyn (1988) performed this analysis and the result is well known – it merely shows that the soliton is unconditionally stable. Consequently,
only eq. (3.17b) could lead to an instability. We investigate stability in the standard manner by introducing the usual ansatz $e_\pm = y(x)e^{\pm \lambda}$ where $\lambda$ is the growth rate of the arbitrary linear perturbation $y(x)$. This leads to the equation

$$\frac{1}{2}\ddot{y} - 2\kappa \beta_+ \sech^2 \left(\sqrt{2}\beta_+ x\right) y = i\lambda y,$$

(3.18)

where the dots denote derivative with respect to $x$. This eigenvalue equation is well known in linear optical waveguide theory; it is the modal equation of the sech$^2$ index profile planar waveguide (Snyder and Love 1983). If we choose $\beta_- = -i\lambda$, then $\beta_-$ represents the propagation constant of a given mode. The propagation constant associated with any bound function $y(x)$ (i.e. either guided or radiation mode) is real and positive. This means that all the eigenvalues of eq. (3.17b) have a zero real part, $\Re(\lambda) = 0$. In other words, the linear perturbation growth rate is always zero. Such a result indicates that the circularly polarised soliton is neutrally stable with respect to a perturbation of its polarisation state. A perturbation in $E_-$ does not grow or decay and propagates with the soliton in $E_+$. This behaviour is a direct consequence of the energy conservation law of eq. (3.1), that is $\partial_z \left(\int_{-\infty}^{\infty} |E|^2 dx\right) = 0$.

The conclusion of this linear stability analysis can be easily interpreted by means of optical waveguide theory. A spatial soliton in the field $E_+$ induces a waveguide for the field $E_-$ (through cross phase modulation, XPM) and any initial distribution of the field $E_-$ of arbitrarily small amplitude can be decomposed into the guided and the radiation modes of this waveguide. These modes propagate without loss along $z$, as indicated by the neutral stability of the scalar soliton. Of course, only the guided modes in $E_-$ propagate along the axis of the soliton in $E_+$ and are liable to form a bound state of both polarisations. If the initial perturbation is such that only one guided mode is excited then the resulting polarisation bound state is stationary. Silberberg and Sfez (1988) first proposed devices based on XPM-induced guiding while De la Fuente, Barthelemy, and Froehly (1991) observed the guidance of a probe beam by a soliton-created waveguide. In the latter case the two beams were of different frequency and the same polarisation.

From this reasoning we anticipate that when the power in the guided mode $E_-$ is increased into the nonlinear regime, the bound state endures at the expense of some reshaping of the profile of the soliton in $E_+$. The resulting bound states would therefore constitute a new family of vector solitary waves branching from the CP soliton. The branching point corresponds to an infinitesimal CP signal propagating down a counter-rotating CP-soliton induced waveguide. For each guided mode of this waveguide we expect a branching bifurcation from the circularly polarised NSE soliton. These bifurcations can be located by linearising about a CP soliton. First, form a CP soliton with propagation constant $\beta_+$ in one polarisation;

$$e_+ = \sqrt{2\beta_+/1 - B} \sech(\sqrt{2\beta_+ x}),$$

(3.19)

then perturb it with a very small amount of $e_-$. Without loss of generality we take $\beta_+ = \frac{1}{2}$ as we can recover the solutions for any other value by appropriate scaling of $e_+, e_-, \beta_-$ and $x$ (in particular, $e_+ \rightarrow \sqrt{\beta_+} e_+, e_- \rightarrow \sqrt{\beta_-} e_-, \beta_+ \rightarrow \beta_+ + \beta_-, \text{ and } x \rightarrow x/\sqrt{\beta_+}$).

By ignoring all the higher order terms in $e_-$, we linearise and thereby imagine that $e_-$ is travelling through a linear, inhomogeneous medium, with the inhomogeneities given by $e_+$. The propagation of $e_+$ is not affected by $e_-$ while the propagation of $e_-$ is governed by the linear Schrödinger equation

$$\frac{1}{2}\dot{e}_- - \left(\beta_- + \kappa \sech^2 x\right) e_- = 0.$$  

(3.20)

This can be written in the standard form for a Legendre equation. Let $\eta = \tanh x$, then eq. (3.20) is re-expressed as

$$\frac{1}{2} \frac{\partial}{\partial \eta} \left[\left(1 - \eta^2\right) \frac{\partial e_+}{\partial \eta}\right] + \left(\kappa - \frac{\beta_-}{1 - \eta^2}\right) e_+ = 0.$$  

(3.21)
Solutions to this equation are the associated Legendre functions \( P_{\nu}^{\mu}(\eta) \), where \( \nu(\nu + 1) = 2\kappa \) and \( \mu^2 = 2\beta_- \). For \( \beta_- < 0 \), \( \mu \) is complex and there is a continuous spectrum of unbounded solutions whose form is given by the confluent hypergeometric functions. The continuous spectrum represents radiation modes, with negative propagation constant. They are of no further interest here, since they don’t lead to solitary wave type solutions. If \( \beta_- > 0 \), there is a discrete spectrum of solutions bounded on \( \eta \in [-1,1] \), given by \( P_{\nu}^{\nu-\nu}(\eta) \), at values of \( \beta_- = \frac{1}{2}(\nu - n)^2 \), with \( n \) an integer in \([0,\nu]\). It is this discrete set of bounded solutions that are of interest for this work. We anticipate that each of these solutions indicates the point at which a more general elliptically polarised wave branches away. So, for a fixed value of one propagation constant \( (\beta_+) \) we find that there will be a certain number (the integer part of \( \nu \) plus one, in fact) of values for \( \beta_- \) at which there exist linearised solutions to eq. (3.1). Now the waveguide seen by the second beam will depend on the material parameter \( B \), and therefore the number of bound modes that it supports will also depend on \( B \). For \( B = 0 \), the integrable Manakov case, the two polarisations see an identical waveguide. For \( B > 0 \), i.e. for which SPM < XPM, the second field sees a deeper waveguide. The situation is reversed when \( B < 0 \). As \( B > 0 \) increases and approaches unity, there will be more and more linear modes of the waveguide, each corresponding to a bifurcation point. To quantify this, the parameter \( \nu \) defining the number of modes and the material parameter \( B \) are related by the following expression
\[
\frac{1}{2}\nu(\nu + 1) = \frac{1+\beta}{1-B}.
\]
If we wish we can solve this for the positive value \( \nu = \sqrt{2\kappa + \frac{1}{4} - \frac{1}{2}} \). For \( B < 0 \) the waveguide has one bound mode, for \( 0 < B < \frac{1}{2} \) two modes, for \( \frac{1}{2} < B < \frac{5}{6} \) three, with additional higher order modes introduced at \( B = \frac{9}{11}, \frac{7}{8}, \frac{10}{11}, \frac{27}{25} \), etc so that as \( B \) approaches unity the number of modes tends to infinity. A mode at cutoff has infinite tails of constant value, and therefore at the cutoff points themselves one does not expect to find bifurcation points. In fact, modes at cutoff can correspond to bifurcation points, as we shall see in the next section. We write down the algebraic form of the first few modes of these sech profile waveguides, using expressions from Abramowitz and Stegun (1964). The \( n \)th mode is given by \( P_{\nu}^{\nu-\nu}(\tanh x) \) and has propagation constant \( \beta_- = \frac{1}{2}(\nu - n)^2 \).

\[
P_{\nu}^{\nu-\nu} = \frac{2^{\nu}}{\Gamma(\nu + 1)} \text{sech}^\nu x
\]

\[
P_{\nu}^{1-\nu} = \frac{2^{1-\nu}}{\Gamma(\nu)} \tanh x \text{sech}^{-1} x
\]

\[
P_{\nu}^{2-\nu} = \frac{2^{1-\nu}}{\Gamma(\nu)} \text{sech}^{-2} x \left[ (2\nu - 1) \text{tanh}^2 x - 1 \right]
\]

\[
P_{\nu}^{3-\nu} = \frac{2^{2-\nu}}{\Gamma(\nu - 1)} \tanh x \text{sech}^{-3} x \left[ (2\nu - 1) \text{tanh}^2 x - 3 \right]
\]

\[
P_{\nu}^{4-\nu} = \frac{2^{2-\nu}}{\Gamma(\nu - 1)} \text{sech}^{-4} x \left[ (2\nu - 1)(2\nu - 3) \text{tanh}^4 x - 6(2\nu - 3) \text{tanh}^2 x + 3 \right].
\]

Unfortunately no algebraic formula exists, in general, for the \( n \)th mode of the sech\(^2\) waveguide; one must utilise the recursion relations presented in, for example Abramowitz and Stegun (1964), that allow the iterative calculation of the relevant Legendre functions. We illustrate the behaviour for \( B = 0, \frac{1}{3}, \frac{5}{6} \), remembering that the most experimentally important values for it are 0 and \( \frac{5}{6} \). \( B = \frac{1}{3} \) is included as an illustrative example of the higher values of \( B \). Figure 3.3 shows the waveguides and their modes for...
two different values of $B$: $B = \frac{1}{3}$, chosen because it corresponds to many materials with electronic nonlinearities, $B = \frac{5}{6}$, chosen because in media with this characteristic, a soliton induced waveguide supports five modes.

The soliton is the fundamental mode of the waveguide that it induces for itself through self phase modulation (SPM). We have made use of the mechanism of XPM to form a waveguide for light of the second polarisation. We anticipate the existence of vector solitary waves that in the limit $e_- \ll e_+$, will take the forms presented above. We think of one CP wave component as a signal beam being guided by the pump beam of the counter rotating component. Note that once again we have made use of the equivalent linear refractive index change to help motivate our work. This theme will recur throughout the thesis.

We wish to present another argument, motivated by considering induced linear waveguides, suggesting the existence of vector solitary waves. The total bandwidth that is available to the designers of transmission systems, based on temporal solitons, is limited by many factors. One of these is the interaction force between neighbouring solitons that cause in-phase solitons to attract and $\pi$ out-of-phase solitons to repel. The same interaction forces affect the propagation of spatial solitons. If we consider the possibility of forming elliptically polarised solitons, then it would seem plausible to be able to combine different polarisation components in such a way as to cancel the interaction force between two solitary waves.

We propose, then, a stationary solitary wave with a transverse intensity profile describing two parallel solitons of equal amplitude, as shown in figure 3.4(a). Any arguments about attraction and repulsion are only sensible for distinct beams, and we confine our thinking to well separated beams. How can the polarisation state change across this profile, and perhaps along the beams, so as to cancel the nonlinear dynam-
ics? Consider the field to be composed of two superimposed counter-rotating circular polarisation states. Each of these states experiences a slightly different equivalent linear refractive index change or, in other words, sees a slightly different waveguide. The effects of $B$ are not substantive for the purposes of this discussion and will be considered no further. It is clear that for stationary propagation the two channels forming the waveguide closely resemble solitons, therefore the total waveguide will always support two modes. One circular polarisation state occupies the fundamental mode, the counter-rotating state the first mode. What do these modes look like? They are like the modes of a graded index planar waveguide coupler, and are shown in 3.4(b). The dynamics of this system are rather interesting since one component is composed of two in-phase beams, the other two out-of-phase beams. As we are able to tune the relative intensities in each mode without qualitatively changing the waveguides, we can cancel completely the interaction forces and form a stationary state. Since the two lowest order modes of a planar coupler are not degenerate, there will always be a finite difference between the propagation constants of each mode.

To best visualise the polarisation state of the field at any point, think instead in terms of linear polarisation states $e_x = (e_+ + e_-)/\sqrt{2}$ and $e_y = (e_+ - e_-)/i\sqrt{2}$. Figure 3.4(c) shows the transverse profile in terms of the linear polarisation. The two adjacent beams are orthogonally polarised, and are in pure linear polarisation states. Figure 3.4(c) does not show the relative phase; $e_-$ is $90^\circ$ out of phase. There is a $90^\circ$ phase difference and a $90^\circ$ polarisation difference between the interacting beams. As the beams propagate, counter-rotating CP components $e_{\pm}$ beat against each other. This beating will cause the polarisation state in the beams to rotate together, always remaining $90^\circ$ out of phase. This effect is easily seen by considering the intensity at the peak of the right hand beam, expressed in terms of $e_x$ and $e_y$.

$$|e_x(x_0,z)|^2 = e^{\beta_+ \cos(2\pi z/C)}$$
$$|e_y|^2 = \sin(2\pi z/C),$$  

(3.28)

where $C = 4\pi/(\beta_+ - \beta_-)$ is a length scale that measures the beat length of the induced waveguide coupler. $\beta$ is the average of the two propagation constants $\beta_{\pm}$. In this example $C$ gives the length over which the polarisation makes one complete rotation.\footnote{The nonlinearity has induced circular birefringence into the material. Since all isotropic Kerr materials with non-zero $B$ exhibit this effect, we will work using the counter-rotating polarisaton basis set as much as possible in this thesis} In the limit where the two beams become well separated, the propagation constants will merge and $C$ will tend to infinity.

The above reasoning suggests that vector stationary states combining first and second modes can form very different than those proposed by the bifurcation analysis of §3.2.3. Our analytical tools have been able to show the guidance of one beam by another. By conjecture we have considered the possibilily of mutual guidance, and shall now lay the foundations for this investigation.

### 3.3 Vector solitons of the Manakov model

In the 1980s several researchers examined systems of CNSEs, looking for solutions of the type we are interested in here. Invariably they looked at the integrable systems, as these yielded algebraic solutions that are far more convenient to work with than outputs of thousands of computer simulations.

It was shown by Tratnik and Sipe (1988) that Hirota's method could be used also to derive soliton solutions for an ODE analogous to eq. (3.4). Slightly before this work, Christodoulides and Joseph (1988) had applied the inverse scattering method to the same problem. They were considering birefringent systems, and by choosing an appropriate ansatz could reduce the CNSE to ODEs. This was really a mathematical trick to reduce
3.3 Vector solitons of the Manakov model

![Graphical representation](image)

**Figure 3.4:** Cancelling soliton interaction by combining even and odd supermodes. (a) the total intensity profile $I(x)$ of two adjacent beams. (b) the two modes of the waveguide represented by (a). (c) The amplitude in each linear polarisation state.

the problem to a soluble form, and was based upon shaky physical assumptions trying to match the phases of the wave components. Using our ansatz of eq. (3.2) we can derive the same equations as a reduction of the Manakov equations. Since they considered a reduced form of the integrable Manakov model, it is no surprise that the Hirota method could be applied. The authors believed that the birefringence was necessary for the formation of such bound states. However by starting with the integrable $B = 0$ case of (3.1) (that has wide applicability, in any case) we bypass any unphysical assumptions, and extend their work by deriving multi-soliton solutions for the Manakov system. Radhakrishnan and Lakshmanan (1995) claim to have derived this multisoliton; however they have chosen the wrong ansatz and only succeed in finding solutions of uniform polarisation. The Manakov system reads

\[
i\partial_z E_\pm + \frac{1}{2} \partial^2_x E_\pm + \left(|E_+|^2 + |E_-|^2\right) E_\pm = 0 \quad (3.29)
\]

Note that this system of equations is invariant with respect to the $U(2)$ group of transformations; that is, fields resulting from any linear transformation of the fields $E_\pm$ that preserves the $L_2$ norm $|E_+|^2 + |E_-|^2$ are also solutions of eqs. (3.29). It is therefore clear that linear and circular polarisation is equivalent in this sort of material, and that any solution of eqs. (3.29) with constant polarisation is equivalent to a solution of the NSE. This symmetry emphasises the vector nature of the fields governed by this equation (in contrast to nonintegrable cases), and we will use bold symbols to indicate two-component vector quantities. In this form the Manakov equation reads

\[
i\partial_z E + \frac{1}{2} \partial^2_x E + \left(E^\dagger E\right) E = 0 \quad (3.30)
\]
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where

\[ E = \begin{bmatrix} E_+ \\ E_- \end{bmatrix} \quad \text{and} \quad E^\dagger = (E_+, E_-^*) \] (3.31)

This notation also liberates us from the constraints of the \( U(2) \) problem and we can now consider the \( U(n) \) NSE by taking \( E \) to be an \( n \)-vector. In the context of polarisation structures this may not appear to be a physically worthwhile problem. Nevertheless, if we consider the electric field to be made up of components not of different polarisation but of different wavelength, then many different components may interact and the \( U(n) \) equation must be used. Additionally, Chapter 6 will make use of this result for \( n = 3 \).

A further reason for investigating the general case is the novelty factor: previous studies have looked at the higher dimensional cases in passing only.

### 3.3.1 Hirota’s Bilinear operator

The two soliton solution to the NSE presented in §2.3 is a rational function of sums of exponentials. The simple form taken by the numerator and the denominator suggests that transforming to new variables might lead to an easy way to find the multisolitons of the NSE. This proves to be a valuable idea that is applicable to all the integrable equations, and the so called \( \tau \) functions that result prove to have many interesting properties (Newell 1985). We convert to \( \tau \)-type Hirota functions by making the substitution

\[ E = g/f \] (3.32)

where \( f \) is a scalar, real function. Making this substitution produces a more complicated equation

\[ if^2g_z - ifgf_z + f^2g_{xx} - fg_xf_z - fg_xg_z + gf_z^2 - (g^\dagger g)g = 0 \] (3.33)

Hirota (1973a) introduced a new differential operator, called the bilinear operator, that allowed equations of this type to be written more simply. This unconventional operator is defined as

\[ D_x^m D_z^n (f(x,z) \cdot g(x,z)) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^n (f(x,z)g(x',z')) \bigg|_{x=x',z=z} \] (3.34)

Note that by bilinear we mean that the operator is linear in each argument independently. The bilinear operator works just like the Liebnitz rule, except for a difference in sign that is manifested in every second term. We present two rather pointed examples:

\[ D_x (g \cdot f) = g_x \cdot f - g \cdot f_x \] (3.35)

\[ D_x^2 (g \cdot f) = g_{xx} \cdot f - 2g_xf_x + g \cdot f_{xx} \] (3.36)

From which it is clear that one can write equation (3.33) in Hirota’s bilinear form

\[ f \left[ iD_z + \frac{1}{2} D_x^2 \right] (g \cdot f) + g \left( g^\dagger g - D_x^2 (f \cdot f) \right) = 0 \] (3.37)

For bright solitons, we impose the boundary conditions \( u(\pm \infty, z), e_-(\pm \infty, z) = 0 \), for all \( z \), and since the components of \( g \) are independent functions, eq. (3.37) can be separated into

\[ B_1 (g \cdot f) = 0, \quad B_2 (f \cdot f) = g^\dagger g \] (3.38)

where we have defined the operators \( B_1 \) and \( B_2 \) as

\[ B_1 = iD_z + \frac{1}{2} D_x^2, \quad B_2 = \frac{1}{2} D_x^2 \] (3.39)
3.3 Vector solitons of the Manakov model

Table 3.1: The equations for $f_i$ and $g_i$ at each power in the parameter $\lambda$ that correspond to the two-soliton solution. The solution is defined by the equations up to $\lambda^4$; a valid solution will satisfy the remaining equations.

### 3.3.2 Solution of the Hirota equations

The fact that the $U(n)$ NSE system passes the Painlevé test for integrability gives us confidence in pursuing Hirota’s method to derive the multisolitons. The solution method is straightforward: commencing with a polynomial ansatz in some parameter $\lambda$, we treat $\lambda$ as an infinitesimal and equate coefficients at each order in it. The equations at each order are differential equations that can be iteratively solved to yield $r$-function solutions.

A polynomial ansatz suitable for obtaining the 2-soliton solution is

$$f = 1 + \lambda^2 f_2 + \lambda^4 f_4, \quad g = \lambda g_1 + \lambda^3 g_3$$

At each power of $\lambda$ we get an equation from each of eqs. (3.38). Table 3.1 shows the equations for $g_i$ and $h_i$ at each order in $\lambda$.

**First order:** we obtain (from Table 3.1)

$$\left(i\partial_x + \frac{1}{2}\partial_x^2\right) g_1 = 0$$

The solutions to these linear equations are exponential functions. We place ourselves inexorably on the path toward a two-soliton solution by taking a two-exponential form for $g_1$

$$g_1 = c_1 e^{\eta_1} + c_2 e^{\eta_2}$$

where $\eta_j = \kappa_j x + \frac{1}{2} \kappa_j^2 t$ and the $c_j$ are complex constants of integration, unit vectors normalised so that $c_j^* c_j = 1$. The significance of the $c_j$ will become apparent. At this stage we are locked into pursuit of the 2-soliton. No further parameters need be introduced. When evaluating $f_2, g_3, \ldots, f_4$ from the other equations in Table 3.1 one must apply Occam’s razor and choose any constants of integration to be zero, but no further assumptions need be made. Any other approach will lead to incompatibilities at higher orders in $\lambda$.

**Second order:**

$$\partial_x^2 f_2 = \sum_{j,k=1}^2 c_k^* c_j e^{\eta_j} + \eta_k^*.$$
Note that as we have already introduced all the independent parameters necessary for a two-soliton solution, we set all integration constants to zero. This leads to

$$f_2 = \sum_{j,k=1}^{2} \frac{c_j^* c_j}{(\kappa_j + \kappa_j^*)} e^{\eta_j + \eta_j^*}. \quad (3.44)$$

To simplify the expressions that quickly grow more complicated we introduce the complex scalar, matrix

$$\Gamma_{jk} = \frac{c_j^* c_j}{\kappa_j + \kappa_k^*}. \quad (3.45)$$

**Third order:** we simplify and obtain

$$B_1(g_3 \cdot 1) = (\partial_x g_1)(\partial_x f_2) - g_1 B_1(1 \cdot f_2) = \sum_{j,k,l=1}^{2} (\kappa_k - \kappa_j) c_j^* c_j e^{\eta_j + \eta_k + \eta_l^*}. \quad (3.46)$$

Using the permutation symmetry in $j$ and $k$ in the right hand side, we find $g_3$ to be

$$g_3 = \sum_{j=1}^{2} \frac{(\kappa_2 - \kappa_1)}{(\kappa_1 + \kappa_1^*)(\kappa_2 + \kappa_2^*)} [\Gamma_{1j} c_2 - \Gamma_{2j} c_1] e^{\eta_1 + \eta_2 + \eta_j^*}. \quad (3.47)$$

**Fourth order:** after some algebra, we find that the fourth order term is a single exponential, given by

$$f_4 = \frac{|\kappa_2 - \kappa_1|^2}{(\kappa_1 + \kappa_1^*)(\kappa_2 + \kappa_2^*)} (\Gamma_{11} \Gamma_{22} - \Gamma_{21} \Gamma_{12}) e^{\eta_1 + \eta_2 + \eta_j^*}. \quad (3.48)$$

**Higher orders:** it can be shown by substitution that the calculated expressions satisfy the remaining equations of Table 3.1 without further conditions. Now the effect of the polynomial parameter $\lambda$ can be removed from the expansion by transforming $\eta_j^0 \rightarrow \eta_j^0 - \ln \lambda$, and hereafter we merely take $\lambda = 1$. Therefore the solution for $E$ reads

$$E = \frac{g_1 + g_3}{1 + f_2 + f_4}. \quad (3.50)$$

This is the two soliton solution describing the interaction of arbitrarily polarised solitons of the Manakov system (3.29). It is instructive to spend some time examining the form of this solution, and its behaviour in certain special cases and asymptotic limits. The parameter space of the two soliton solution can be divided into two classes: the periodic solutions in which the two solitons propagate in parallel [for which $\Im(\kappa_1) = \Im(\kappa_2)$], and the aperiodic solutions describing soliton collisions.\(^2\) We begin with the latter case that was analysed asymptotically by Manakov (1974) who first integrated eq. (3.29) using the inverse scattering transform. It is instructive to consider first the profiles of single, isolated solitons far from any interaction regions. To facilitate investigation of the asymptotic forms we choose one of $R(\kappa_j)$ to be negative. We therefore let $\kappa_1 = -a_1 + i b_1$ and $\kappa_2 = a_2 + i b_2$ where $a_j \geq 0$ for $j = 1, 2$, and $b_1 \geq b_2$, a choice that breaks the symmetry of the system. This symmetry breaking choice enables one to consider the reference frame of the $j$-th incoming soliton, where $\eta_j \approx 0$, and find that $\eta_k \ll 0$ for $k \neq j$ in this reference frame. Before the collision, then, the $j$-th soliton takes the form:

$$E(x, z \rightarrow -\infty) \approx \sum_{j=1}^{2} c_j a_j \text{sech} \left[ \pm a_j (x - b_j z) + \Re(\eta_j^0) + \delta_j^0 \right] e^{i(b_j x + \frac{1}{2}(a_j^2 - b_j^2) z + \Im(\eta_j^0))}. \quad (3.51)$$

\(^2\)The strange symbol $\Im$ comes from typing $\Im$ in $\LaTeX$, and will be used in this document to denote the imaginary part of a complex expression.
The physical roles played by the various parameters can now be clarified. The $a_j$ describe the soliton amplitude (and dictate its width also), $\arctan b_j$ the angle the $j$th soliton makes with the $z$-axis. The real part of $\eta_j^0$ shifts the position of the centre of the soliton, while the imaginary part changes its shape. The unit vectors $c_j$ give the polarisation unit vector for each incoming soliton. In the $U(2)$ model, if one component of $c_j$ is zero then the $j$-th soliton is initially circularly polarised, while if the components are equal the $j$-th soliton is of pure linear polarisation. $\delta_j^0 = -\ln(2a_j)$ gives the position of the centre of the soliton. Note that the solitons in isolation are nothing more than single NSE solutions; transformations under the symmetry group $U(n)$ connect the soliton of eq. (3.51) to a scalar NSE soliton. The $U(n)$ transformations can sweep one of the $c_j$ through its complete parameter space.

After the interaction the soliton eigenvalue (indicated here by $\kappa$) is, of course, unchanged, and so each soliton will be travelling in the same direction with the same amplitude. However the position of the centre of the soliton, its phase and ‘polarisation’ can be changed. The fact that these changes are permitted is a reflection of the underlying $U(n)$ symmetry of the Manakov equations. Following our soliton through the interaction region and out the other side, where $\eta_j \sim 0, \eta_k \rightarrow \infty$ for $j = 1, 2$ and $k \neq j$, we find

$$e' = \sum_{j=1}^{2} c_j' a_j \mathrm{sech} \left[ \pm a_j (x - b_j z) + \Re(\eta_j^0) + \delta_j' \right] e^{i[b_j x + \frac{1}{2}(a_j^2 - b_j^2)z + \Im(\eta_j^0) + \phi_j']} ,$$

where the polarisation rotation is defined by

$$c_j' = \frac{(\kappa_k + \kappa_j^*)(c_k^t c_j) c_k - [\kappa_j + \kappa_j^*] c_j}{\sqrt{\kappa_1 + \kappa_2^* |^2 - (\kappa_1 + \kappa_1^*)(\kappa_2 + \kappa_2^*)| c_1^t c_2^* |^2}}$$

and the position shift

$$\delta_j' - \delta_j = \pm \frac{1}{2} \ln \left[ \frac{|\kappa_1 - \kappa_2|}{|\kappa_1 + \kappa_2^*|} \left( 1 - \frac{(\kappa_1 + \kappa_1^*)(\kappa_2 + \kappa_2^*)}{|\kappa_1 + \kappa_2^*|^2} | c_1^t c_2^* |^2 \right) \right].$$

The change in phase is embodied by $\phi_j'$, we won't write out the rather complicated form of this function as it doesn’t play a significant role in the interaction. The position shift, $\delta_j' - \delta_j$, was presented in Manakov (1974) and it is worth noting only that the shift is minimised when the solitons are initially orthogonally polarised ($c_1^t c_2 = 0$), and maximised when they are parallel, corresponding to the scalar problem, when ($c_1^t c_2 = 1$). From here we consider the $U(2)$ NSE and consider the vector $E$ to be an arbitrarily polarised field. The major novelty uncovered by analysing the $U(2)$ Manakov model is polarisation rotation which warrants further investigation. Prior to doing this we address the question: how big is the parameter space spanned by the two soliton solution? Without loss of generality we can investigate the case of $\kappa_1 = 1, c_1 = (1, 0)^T$, and $\eta_j = 0$. This leaves five real parameters to investigate: the amplitude, direction, polarisation (requiring two numbers), and phase of the second soliton (defined in $a_1, b_1$ and $c_2$).

The significance of the phase demands a little explanation as its importance is not immediately apparent. Since the CNSE only exhibits incoherent coupling, it is clear that the total phase in either $E_+$ or $E_-$ has no bearing on the outcome. Changing the overall phase between the two solitons, equivalent to changing the relative phase of $c_1$ and $c_2$ but

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3The choice of absolute polarisation is arbitrary in the Manakov model on account of its symmetry under the $U(2)$ group of transformations. Under $U(2)$ we can transform to any polarisation basis set $(E_+, E_-) \rightarrow (u, v)$ that preserves the total intensity $|u|^2 + |v|^2$. We stick to the CP basis set since the symmetry is broken in the non-integrable CNSE models in which the CP basis is far more convenient to work with.
not altering their component parts, will affect the transient behaviour in the interaction
region, but does not change the asymptotic values. However a shift in the relative phase
between the polarisation components of each soliton does play an important role. This
is seen if we take $c_1 = \frac{1}{\sqrt{2}}(1,1)^T$, and then let $c_1 = \frac{1}{\sqrt{2}}(1,\sigma)^T$. If $\sigma = 1$ then
the solitons are like polarised, while if $\sigma = -1$ then they are orthogonally polarised.
Nonetheless, a little thought will show that for our choice of a circular polarisation
state, $c_2 = (1,0)^T$, only the magnitude of the components of $c_2$ affect the final collision
outcome. Therefore we are free to take $c_2$ to be real, and define a ‘relative polarisation
angle’ $\cos \psi = c_1^Tc_2$. Figure 3.5 shows how the level of polarisation rotation depends
on the relative amplitude $a_2$, the angle of incidence $\theta_2 = \arctan b_2$ and the polarisation
angle $\psi$.

Note that when $a_2 \gg a_1$, the polarisation shift is maximal at $\psi = 45^\circ$ and the
polarisation state of a soliton can be rotated through almost $90^\circ$. A polarisation shift of
this magnitude may be useful for all-optical switching applications.

There is another important feature of these results that was pointed out in Manakov
(1974): since the $c_j$ change upon collision by an amount dependent on the relative
polarisation, the total position and polarisation shift in an N-soliton interaction will
depend on the order in which collisions take place, unlike the NSE where the total shift
is merely the sum of shifts from pairwise collisions. The non-commutative nature of the
collisions reflects the non-commutability of the symmetry group of the Manakov model.
The parameters $\eta_j$ therefore enter the N-soliton formulae in a non-trivial way. For this
reason the general N-soliton result for the Manakov system takes a very complex form
and has never been written down, despite claims by Radhakrishnan and Lakshmanan
(1995) to have done so.
Figure 3.6: The two soliton solution describing the collision between two arbitrarily polarised solitons, as given in eqs. (3.53). At left is \(|E_+|\), in the centre \(|E_-|\) and at right \(I = |E_+|^2 + |E_-|^2\). In particular note that the beating apparent in the first example is not apparent in the second, and that in the third interaction one polarisation appears to cause repulsion while the other attracts.

Figure 3.6 shows three different types of collision between bright solitons of the Manakov system. Beyond polarisation rotation there is little qualitative novelty in the collisions of vector solitons as compared to scalar solitons; nevertheless we present three cases of particular interest. At the top we show a general interaction between solitons of different amplitude and polarisation. One can observe that both beams are rotated, with the effect greater on the weaker beam. In the middle is shown an interaction between two orthogonally polarised beams, in which we observe none of the beating characteristic of bright soliton collisions. At the bottom is the collision between two like-polarised beams of equal amplitude in which one component is 'in phase' and the other \(\pi\) out of phase [i.e \(c_1 = \frac{1}{\sqrt{2}}(1,1)^T\) and \(c_2 = \frac{1}{\sqrt{2}}(1,-1)^T\)], a disguised collision between orthogonal solitons. Unsurprisingly one does not observe beating phenomena in this collision either.

Let us now investigate the other portion of the solution space, for which both solitons propagate in parallel. These periodic solutions are a small part of the parameter space but are of particular interest in the context of this thesis. In addition, there has never been a study of the bound state solutions of the Manakov system since the original work avoided much of the algebraic morass by considering the asymptotic forms alone. Note, first of all, that these solutions, like the higher order solitons of the scalar NSE, are not,
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Figure 3.7: Bound 2-solitons of the Manakov model. Amplitudes are given by \( a_1 = 1.0, a_2 = 1.05 \). In (a) we show two coincident solitons with a \( \Delta \psi = 90^\circ \); (b) shows the same soliton eigenvalues, but a position shift of \( \eta_2 = 2.5 \), while (c) shows coincident solitons with \( \Delta \psi = 90^\circ \). This last example shows a stationary state in which the beating between the modes has been cancelled.

in general, stable to small perturbations of the initial conditions. This fact is easy to see since a small perturbation will (in general) shift the values of the soliton eigenvalues so that the soliton directions no longer coincide. This will, of course, cause the solitons to diverge at large \( z \) and break up the bound state. The bound states can be said, then, to possess zero binding energy.

To simplify the expressions, we commence by shifting the solitons by \( \eta_j^0 - \eta_j^0 - \frac{1}{2}(\delta_1 + \delta_1^T) \), so that when the shifts are set to zero, the solitons coincide. Again we choose \( a_1 = 1.0, b_1 = 0 \) and \( c_1 = (1,0)^T \). To give the solutions reflection symmetry about \( z = 0 \), we also take \( \eta_2^0/a_2 = -\eta_1^0/a_1 = \Delta x \). Figure 3.7 gives several examples of the bound modes. At top is shown a mode with \( \eta_2^0 = 0, \psi_2 = 75^\circ, a_2 = 1.05 \). At the centre we show the same mode with the second soliton shifted with \( \Delta x = 10 \), and then at the bottom we present the same mode again with \( \psi = 90^\circ \). Remarkably, this last case is a stationary bound mode, first discovered in pioneering works of Christodoulides and Joseph (1988) (in a limited form) and Tratnik and Sipe (1988). By analysing the 2-soliton of the vector NSE in full generality we can understand the algebraic framework within which the stationary solution fits. In fact, whenever \( b_1 = b_2 \) and \( \psi = 90^\circ \), eq. (3.53) describes two stationary copropagating solitons. We restrict our investigation to \( c_1 = (1,0)^T \) (and
therefore \( c_2 = (0, 1)^T \); a \( U(2) \) transformation can be used to generate further solutions (that are really equivalent) for which the beams are orthogonally polarised but not pure CP states. In these solutions there will be a beating in the components that does not appear in the total intensity. For simplicity (and without losing generality) we take \( b_1 = b_2 = 0 \) and \( a_1 > a_2 \), and reduce the solution to the following form:

\[
E_+ = \frac{1}{\chi} 2a_1 \sqrt{(\Omega)} e^{i/2 a_1^2 x^2} \cosh[a_2(x - \Delta x_2)]
\]

\[
E_- = \frac{1}{\chi} 2a_2 \sqrt{(\Omega)} e^{i/2 a_2^2 x^2} \sinh[a_1(x - \Delta x_1)]
\]

where

\[
\chi = \Omega \cosh[a_1(x - \Delta x_1) + a_2(x - \Delta x_2)] + \cosh[a_1(x - \Delta x_1) - a_2(x - \Delta x_2)]
\]

This is the same expression obtained by Tratnik and Sipe (1988) from their analysis of eq. (3.14) for \( B = 0 \). It is clear from the 2-soliton formulae that it can only exhibit a stationary state when \( c_1^T c_2 = 0 \), since the elimination of the beating terms demands that each component must only contain a single soliton part. Three component waves, described by three coupled equations, will feature both two and three peaked stationary states, when each soliton is orthogonal to all the others, i.e. \( c_j^T c_k = \delta_{jk} \). Figure 3.8 illustrates some cases in which \( \Delta x_1 = \Delta x_2 = 0 \). These solutions are symmetric in \( E_+ \) and antisymmetric in \( E_- \). This case was not considered explicitly by Tratnik and Sipe (1988), yet we shall find that this symmetric form alone exists outside the domain of integrability. We plot these stationary states in the \((E_+, E_-)\) plane at left, then at right we show the transverse intensity profiles \( E_+(x) \), \( E_-(x) \) and \( I(x) \). For \( \epsilon < a_1 \) eqs. (3.56) and (3.56) read:

\[
E_+ = a_1 e^{i/2 a_1^2 x^2} \sech(a_1 x)
\]

\[
E_- = a_2 e^{i/2 a_2^2 x^2} \tanh(a_1 x)
\]

This is exactly the result that we obtained in the bifurcation analysis of the previous section. It’s a solution comprising a bright soliton in one polarisation and the second mode, propagating at cutoff, in the the second polarisation, and is shown in figure 3.13(a). If, on the other hand, we let \( a = a_1 - \epsilon = a_2 + \epsilon, \epsilon \ll a, \) we find:

\[
E_+ = \frac{a + \epsilon}{\sqrt{2}} e^{i/2(a + 2\epsilon) x^2} \left\{ \sech[a(x + \Delta)] + \sech[a(x - \Delta)] \right\}
\]

\[
E_- = \frac{a - \epsilon}{\sqrt{2}} e^{i/2(a - 2\epsilon) x^2} \left\{ \sech[a(x + \Delta)] - \sech[a(x - \Delta)] \right\}
\]

where \( \Delta = \frac{1}{2} \ln(a/2\epsilon) \). In one polarisation we have the sum of two half-solitons, while in the other the difference. This solution is of the form proposed by heuristic reasoning in §3.2.3, validating the essential idea behind that argument. As emphasised in that section, the difference in the propagation constants \( 1/2 a_1^2 - 1/2 a_2^2 \) causes a beating that manifests itself in a rotation of the polarisation ellipse. The term ‘dynamic soliton’ has been coined by Snyder, Hewlett, and Mitchell (1994) to describe localised waves like these whose polarisation changes but whose intensity remains constant during propagation.

Figure 3.9 shows the transverse profiles of the stationary states when \( \Delta x_j \neq 0 \), for which the symmetry properties are lost. Tratnik and Sipe (1988) examined the stationary states in the limit \( \Delta x_1 - \Delta x_2 \to \infty \). Simple asymptotics show that eq. (3.56) reduces to two orthogonally polarised, well separated, sech-profile solitons, with amplitudes \( a_j \), centred at \( x = \Delta x_j \). In Figure 3.9 we increase the soliton separation towards the bottom of the figure, and it is clear that the solution resembles two separate solitons of different amplitudes. Solutions of this form do not carry over into the non-integrable models.
Figure 3.8: Symmetric stationary bound modes of the Manakov system. In all cases $a_1 = 1$. (a) $a_2 = 0.1$, (b) $a_2 = 0.8$, (c) $a_2 = 0.999$. (a) shows the solution branch bear the bifurcation point. (b) is an intermediate bound state while in the wave in (c) resembles two distinct beams $90^\circ$ out of phase and orthogonally polarised.

### 3.4 Vector solitary waves in nonintegrable systems

#### 3.4.1 Shooting for solutions

In the preceding sections we have established the existence of dynamic solitary waves in two opposite limits, and have found analytic forms for them in the Manakov system. The only remaining link in the chain is to find the complete solution branches for the general CNSE, and characterise the different classes of solitary wave that we have found. This task falls to the numerical shooting scheme, a standard method for integrating boundary value problems for ODEs.

In terms of the mechanical analogy, the shooting method consists of placing the ball just off to the side of the central peak, and giving it just the right amount of a kick to simulate being rolled asymptotically off the peak at $x \to -\infty$, remembering that $x$ plays the role of time. By varying the angle at which the ball rolls off, we can search for solutions that return asymptotically to the peak as $x \to \infty$. These are the separatrix trajectories corresponding to solitary waves. We formalise the initial conditions for this method by linearising eq. (3.3) around the point $(e_+, e_-) = 0$. The resultant decoupled linear equations are satisfied by

$$e_+ = \cos \theta \ e^{\sqrt{2}\beta_+ x} \quad \text{and} \quad e_- = \sin \theta \ e^{\sqrt{2}\beta_- x}. \quad (3.61)$$

Where one propagation constant, $\beta_+$, will be set to $\frac{1}{2}$ without loss of generality. For the numerical solution, then, we set $\beta_-$ and use eq. (3.61) as the initial conditions for large negative $x$. and $\theta$ becomes the shooting parameter. It is apparent that $\theta = 0$ and $45^\circ$ will lead to the scalar CP and LP solitons respectively.
Asymmetric stationary bound modes of the Manakov system. These solutions are the same as (b) from the previous figure, but with position shifts $\Delta x_1 = -\Delta x_2 \neq 0$. (a) $\Delta x_1 = 0.1$, (b) $\Delta x_1 = 0.8$, (c) $\Delta x_1 = 2.5$. Between (a) and (c) one observes the decomposition of the bound state into separated solitons propagating in parallel.

In order to present concrete examples we commence by fixing the material parameter $B$ to $\frac{1}{2}$. For this value there are two values of $\beta_-$ of special interest, as they correspond to the linear modes of the $e_+^{-}$-induced waveguide. They are $\beta_+ = \beta_0 = 1.22$ for the fundamental mode and $\beta_+ = \beta_2 = 0.15$ for the second mode. Figure 3.10 shows some of the solutions for $\beta_+ > \beta_- > \frac{1}{2}$. It is apparent that $\beta_+ = \beta_0$ does indeed correspond to a bifurcation point at the end of a branch of new solutions. These are elliptically polarised fundamental mutually guiding modes whose polarisation will vary during propagation as the two modes beat against one another. We call these zeroth order since there are no nodes in either polarisation component. Note that the corresponding fundamental vector solitons of the Manakov system are much less interesting. The Manakov soliton is a uniformly polarised soliton that is only distinguishable from the scalar soliton when it encounters a differently polarised soliton. One can easily see why the Manakov soliton is less interesting: since both polarisation components form the fundamental mode of the same waveguide they must be proportional. The varying ellipticity for $B \neq 0$ is a consequence of the fact that one polarisation sees a 'deeper' waveguide than the other. Figure 3.13 gives an indication of the relative widths of each component for different $B$ when one component has a very low intensity.

It has traditionally been convenient to represent the solutions on an energy bifurcation diagram $Q(\beta_-)$, where $Q$ is the total power defined as

$$Q = \int_{-\infty}^{+\infty} (e_+^2 + e_-^2) \, dx. \quad (3.62)$$

This representation is particularly natural for the scalar and vector NSE where the total power plays a special role as the induced refractive index change. This is particularly true
for the NSE where the slope of the curve $Q(\beta_-)$ is a stability criterion for solitary waves. While for the CNSE with $B \neq 0$ $Q$ has no such direct relevance, it proves convenient to represent the solutions graphically this way, and figure 3.11 is just such a diagram, in which $\beta_+$ is fixed. The lower curve on this figure demonstrates the evolution of the fundamental vector solitary wave from a circular CP soliton of propagation constant $\beta_+ = \frac{1}{2}$ through an LP soliton to a counter-rotating CP soliton with $\beta_- = 1/\beta_0$. The points $d_1, d_2$ and $d_3$ indicate the points at which the separatrices and profiles were shown in figure 3.10.

Figure 3.12 shows the results of the investigation into the solution branch originating in the second mode, and therefore considers the region $1 > \beta_- > \beta_1$. We again show the separatix trajectories and the solitary wave profiles. We have verified the appearance along the $e_+$ axis of a single loop (one-node) trajectory near $\beta_+ = \beta_1$. As $\beta_-$ increases towards unity the separatix loop divides into two lobes that narrow progressively and approach the lines $e_+ = \pm e_-$ as $\beta_- \to 1$. In this limit, the solitary wave is composed of two distinct pulses of almost linear polarisation for sufficiently large $\beta_-$. In the limit $\beta_- \to 1$ the pulses become infinitely separated LP NSE solitons, whose plane of polarisation rotates at a frequency $(1 - \beta_-)/2$. This evolution along the solution branch exactly mirrors that of the combined symmetric/antisymmetric stationary states shown in figure 3.8. There appears to be no analogue of the asymmetric states shown in figure 3.9.

Figure 3.13 shows the energy bifurcation diagram $Q(\beta_-)$ for those solution branches that lie entirely in the region $\beta_- < \beta_+$ for $B = \frac{1}{3}$. The points labelled $a_1, a_2$ and $a_3$ correspond to the separatrices and profiles shown in figure 3.12. The line joining them shows the solution branch of the one node solitary wave, beginning at $\beta_- = \beta_1$ with the energy of a CP soliton and tending towards two LP solitons as $\beta_- \to \beta_+$. 

**Figure 3.10**: Separatrices in $(e_+, e_-)$ space (at left), the field profiles $e_+(x), e_-(x)$ (centre) and the ellipticity profile $q = (e_+ - e_-)/(e_+ + e_-)$ (right) of the fundamental dynamic soliton for $B = \frac{1}{3}$ and $\beta_+ = 0.5$. In all figures of this type we use a solid line for $e_+$ and a dotted one for $e_-$. $f_3$: $\beta_- = 0.7$, $f_2$: $\beta_- = 0.9$, $f_1$: $\beta_- = 1.1$. 

---

**Diagram Description**

- **Separatrices**: The separatrices are shown in the $(e_+, e_-)$ space, indicating the boundaries between different regions of behavior for the system.
- **Field Profiles**: The field profiles $e_+(x), e_-(x)$ are depicted, showing the behavior of the positive and negative components of the field, respectively.
- **Ellipticity Profile**: The ellipticity profile $q = (e_+ - e_-)/(e_+ + e_-)$ is shown, which quantifies the degree of ellipticity in the field components.

These visualizations provide a comprehensive view of the fundamental dynamic soliton's properties for specific parameter values, facilitating a deeper understanding of its behavior under given conditions.
Figure 3.11: Energy bifurcation diagram showing the fundamental branch and the 'blended' soliton (which is described in the next section). Note that the blended state is symmetric, and that for $\beta_- = 0.5$ both polarisation modes have the same propagation constant.

Figure 3.12: The one node dynamic soliton: separatrices and profiles. $B = 1/3$ and $\beta_+ = 0.5$. $a_1$: $\beta_- = 0.18$, $a_2$: $\beta_- = 0.39$ and $a_3$: $\beta_- = 0.49$. 
The advantage of the numerical technique is that it is general and can find solutions that do not associate with the known soliton solutions in such a regular manner. Having found the solutions we expected, we are now free to investigate further. A second type of solitary wave was identified that bifurcates from the CP solution at $\beta_- = 0$, and is indicated by the dotted line labelled $b$ in figure 3.13. These solitary waves are symmetric functions of $x$ and possess two nodes in $e_-$. They originate from a more complicated bifurcation which cannot be studied by means of a linear perturbation analysis because the linear equation (3.20) does not admit two node solutions that decay at infinity.

The symmetry in the profile of these solitary waves corresponds to separatrix trajectories that are folded upon themselves at a turning point $(e^+_0, e^-_0)$ where the velocities $\dot{e}_+, \dot{e}_-$ are zero. We therefore minimise the computation involved by using this midpoint as the initial state for the shooting method. If we use the polar co-ordinates $e_+ = r \cos \phi$ and $e_- = r \sin \phi$ then the Hamiltonian relation (3.4) reads

$$r^2 = \frac{1 - (1 - \beta_) \sin^2 \phi}{1 - B \cos^2 2\phi} \quad (3.63)$$

We then fix the initial value of $r$ and use $\phi$ as the shooting parameter, and start with the ball at rest. $\phi$ indicates the polarisation state of the field at the centre point of the soliton. Figure 3.14 shows the 2-node separatrix and solitary wave profiles for $\beta_-$ corresponding to the points $b_1, b_2$ and $b_3$ on figure 3.13. As $\beta_-$ tends to 1 the solitary wave transforms itself into a bound state of three LP NSE solitons (infinitely well separated
at $\beta_-=1$). Note that a similar type of composite soliton has recently been described by Akhmediev and Ankiewicz (1993) in a study of nonlinear fibre couplers.

Yet another type of solitary wave has also been identified using the shooting method. These states, with three, five, seven and more nodes, provide compelling evidence for the existence of an infinite hierarchy. These additional branches correspond to higher order separatrix trajectories that merge at the bifurcation point with the first order separatrices studied above. Near the bifurcation the $2n-1$ node solution represents bound states of $n$ CP solitons (at $\beta_- = \beta_{jt}$ the distance between them is infinite). The solid curve connecting points labelled $e_{1-3}$ represents the solution branch on the bifurcation diagram 3.13 while figure 3.14 shows the corresponding separatrices and profiles. It is clear that as $\beta_-$ tends to unity the solitary wave becomes a bound state of four equidistant LP solitons. In both limiting regions the total energy carried by this solitary wave is therefore twice the energy of the first order wave.

We also found higher order separatrices with even numbers of nodes in $e_-$ that are higher terms in a hierarchy based on the two-node solutions. The dotted line at the top of figure 3.13 is a six node branch that for sufficiently large $\beta_-$ represents bound states of seven LP solitons. We found no apparent limitation on this soliton multiplication process, but were limited by the growing numerical accuracy that is required to calculate the correspondingly more complex trajectories. As illustrated in 3.13 the higher order branches exhibit a more complex behaviour related to a non-uniqueness of the solutions in a certain range of the parameter $\beta_-$. Unlike similar bifurcation diagrams of scalar systems, one cannot draw any conclusions about stability from these diagrams.

Returning to the case of $B = 5/6$ then, of course, we find a far more complex situation. We consider primarily the solutions characterised by the bifurcation analysis of §3.2.3. We expect to find five of these bound solitary waves, the fundamental one branching at $\beta_- = 8.89$, the higher orders at $5.17, 2.46, 0.74, 0.0235$. Figures 3.16 and
Figure 3.15: Illustration of the existence of the higher order solitary waves as the concatenation of lower order states: the three node solitary waves $c_1$: $\beta_1 = 0.18$, $c_2$ $\beta_2 = 0.35$ and $c_3$ $\beta_3 = 0.49$.

3.17 show the form of all five solution branches, both near to and far from the bifurcation points.

The energy versus propagation constant bifurcation diagram, that shows these solutions, is not a good representation of this system as the modes of different orders intersect one another. Nevertheless the energy becomes a multi valued function of the propagation constant for several of these solutions. We shall investigate the third order wave more closely. First note that we expect to find two different solution branches with 3 zeroes in $e_x$, that will converge at $\beta_1 = \beta_4$. One wave family will originate from a single soliton state at $\beta_1 = 0.74$, as shown in figure 3.16, while the other will branch from two CP solitons at $\beta_1 = 5.17$, like the solitary wave of figure 3.15. Figure 3.17 shows the bifurcation diagram for this solution branch. Note that in fact there are four solutions for each propagation constant for a certain parameter range.

We have also verified the existence of solitary wave hierarchies of which these states are the building blocks. There is a great number and profusion of these hierarchies for $B = 5/6$ and we don’t consider them except to note their existence.

One implication from the results of this section is that soliton trains can be built up that do not interact and therefore will not disturb each other’s propagation. This is a potentially useful feature both for soliton based communication systems and all optical switching devices. However if this is ever going to be observed, let alone used, these solutions must be at least stable, and preferably robust.

3.4.2 Degenerate solutions

The existence of the solitary waves hierarchies, ascertained in the previous section, shows that it is possible to concatenate the lower order solutions to obtain higher order states. There are many different types of solutions that one may find using the shooting method
Figure 3.16: $B = 5/6$. The three lowest order solution branches that bifurcate from the circularly polarised soliton state. In all cases $\beta_+ = 0.5$. (a) fundamental soliton. $\beta_- = 6.0$ (b) first order, $\beta_- = 0.52$ and 3.0 (c) second order $\beta_- = 0.52$ and 1.8.
and we illustrate one further variety of special interest. Figure 3.11 shows the bifurcation diagram for the simplest example of the new solution type, as well as that of the fundamental vector solitary wave. The solution branch of this solitary wave is a loop beginning and ending at the same state. This limiting state is a bound state between infinitely separated CP and LP solitons. Figure 3.19 shows the profiles of this new type of solitary wave. Note that as traverses the lower branch, the solution resembles a fundamental vector solitary wave (see figure 3.10) alongside a CP state. There is one point in the solution branch of particular interest here: $e_1$ corresponds to a degenerate state in which both polarisation modes propagate with the same propagation constant.

The solutions shown in figure 3.19 can exist for equal propagation constants. This 'static solitary wave' is a qualitatively different phenomenon that warrants further investigation. At the point $b_1$ the two polarisation components are mirror images of one another. This means one polarisation will see an asymmetric waveguide, the other a waveguide that is the mirror image. It is clear that the mirror image modes are degenerate. In order to find general degenerate solutions of this form we commence by setting $\beta_- = \beta_+ = \beta = \frac{1}{3}$, and investigate the solution branch as a function of $B$. Having made the assumption that the fields stay in-phase at all times, we choose to work in the linear
polarisation basis set, $e_x$ and $e_y$. In this basis eq. (1.22) reads
\begin{align*}
-\beta e_x + \frac{1}{2} \partial^2_x e_x + e_x^2 + (1 - 2B)e_y^2 e_x &= 0 \quad (3.64) \\
-\beta e_y + \frac{1}{2} \partial^2_y e_y + e_y^3 + (1 - 2B)e_x^2 e_y &= 0 \quad (3.65)
\end{align*}

The major difference in using this basis set is that $XPM < SPM$, and indeed that $XPM = 0$ at $B = \frac{1}{2}$ and therefore equations (3.65) decouple completely. Any functions $e_x$ and $e_y$ that are individually solutions of the scalar NSE can be combined to form a solution of eq. (3.65). In practice this means that any two identical NSE solitons, with orthogonal linear polarisation and 90° phase difference, will not affect each other at all. Each polarisation is a separate soliton of the scalar NSE, totally decoupled from the other polarisation. The birefringent system analysed by Christodoulides and Joseph (1988) exhibited the same peculiarity at $B = \frac{1}{2}$. At values of $B$ greater than $\frac{1}{2}$, $XPM$ and $SPM$ are of opposite signs.

The problem can be expressed in terms of finding the trajectories of a ball in the potential
\begin{equation}
\mathcal{V} = -\beta r^2 + \frac{r^4}{4} \left(1 - B \sin^2 2\theta\right) \quad (3.66)
\end{equation}

Once again we are only interested in the bound solitary wave solutions corresponding to the separatrix trajectories for which $(e_+, e_-) = (\dot{e}_+, \dot{e}_-) = 0$ for $\tau \to \pm \infty$.

We next examine how these solitary waves are affected by variation of the parameter $B$. As $B \to 1$ the self phase modulation becomes insignificant and the nonlinear effect is limited to the cross coupling terms, i.e. one polarisation is guided completely by the other. At $B = 1$ the fields of each circular polarisation of the solitary wave must therefore coincide exactly and hence the only possible solutions are linearly polarised.
Figure 3.19: The separatrix trajectories and the $e_+$ and $e_-$ profiles for the fundamental vector solitary wave and the one node blended soliton. Note that at both ends of the solution branch the solution tends to one CP soliton and one $n = 0$ vector solitary wave. $e_1$: $\beta_+ = 0.5$, $e_2$: $\beta_+ = 1.0$, $e_3$: $\beta_+ = 0.6$. Note also the symmetry that is present in $c_1$: both polarisation components have the same propagation constant.

NSE solitons. This reasoning suggests that as $B \to 1$ the solutions break apart into separate pulses, each of pure linear polarisation. The separation between these pulses tends to infinity as $B \to 1$.

In other words, branching bifurcations exist at $B = 0$ and 1 where the new solutions branch from sets of infinitely separated NSE solitons. At $B = 0$ the solutions bifurcate from circularly polarised solitons; at $B = 1$ the new solutions merge with linearly polarised solitons. Similar branching bifurcations have been observed in studies of vector solitary waves in birefringent media (Eleonskii, Korolev, Kulagin, and Shil'nikov 1991; Close, Giuliano, Hellwarth, Hess, McClung, and Wagner 1966; Blow and Doran 1985).

Figure 3.20 illustrates the influence of $B$ on the single node solitary waves. (a) considers $B = 3/4$, materials in which the nonlinearity is due to molecular re-orientation. The solitary wave resembles three separate linearly polarised pulses, each $90^\circ$ phase shifted and orthogonally polarised with respect to its neighbours. The case of $B = 0.4$ is shown in (b) where the three pulses are heavily overlapping, and the solitary wave has varying elliptic polarisation. In (c) ($B = 0.25$) the field is best described as two counter-rotating, circularly polarised pulses while in (d) ($B = 0.03$) these pulses become well separated, and approach NSE solitons. At $B = 1/2$ the only possible solutions are those where each linear polarisation obeys a separate NSE.

Next we explore the heirarchy of higher order separatrix trajectories of the potential of eq. (3.67). We consider only $B = 1/3$, corresponding to the nonlinearity of silica fibres and other media with electronic nonlinearities. Figure 3.21 shows the first four higher order solitary waves that are obtained for this value of $B$. Two kinds of trajectories can be identified in figure 3.21: the closed loops in which there are an odd number of zeroes
Figure 3.20: The transformation of the single node solutions wrought by variation of the parameter $B$, showing the branching behaviour from various sets of polarised solitons. (a) $B = 0.75$, (b) $B = 0.4$, (c) $B = 0.25$ and (d) $B = 0.03$. Trajectories in $(e_+, e_-)$ space are shown at left, in the middle are shown field amplitudes of $e_+$ (solid line) and $e_-$ (dotted line) as a function of the time $\tau$, while on the right is the total field intensity.
Multiple Soliton Bound States

Figure 3.21: Higher order degenerate solitary waves. (a) two node solution (b) three node (c) four node. The profiles resemble three-, four- and five-soliton bound states.

in $e_-$ [in (b)], and the trajectories that fold back upon themselves and correspond to even numbers of zeroes [(a) and (c)]. The corresponding intensity profiles appearing in figure 3.21 show that the higher order solitary waves constitute trains of distinct pulses. One can infer that the whole hierarchy exhibits the same variation with the material parameter $B$ that was observed in figure 3.20.

It is possible to extend the analysis of degenerate modes to birefringence systems. We use equation (1.22) with non-zero birefringence $\delta \neq 0$, and then derive the equivalent potential in the form

$$ V = -\beta r^2 + \frac{1}{2} r^4 [1 + B \cos^2(2\theta)] - \delta r^2 \sin(2\theta) \quad (3.67) $$

On the left of figure 3.22 are contour plots of the birefringent potential. The minima of the potential that lay on the $e_+$ and $e_-$ axes are shifted in pairs towards fast axis, which for $\delta > 0$ is the line $e_y = 0$. At $\delta = \beta B/(1 - B)$ these minima coincide at the points $e_+ = e_- = \beta$. The saddle points of the potential can be found on the LP axes at the points $e_x = \pm \sqrt{\beta + \delta}$ and $e_y = \pm \sqrt{\beta - \delta}$. For levels of birefringence $\delta > \beta$ the saddle points on the slow axis disappear altogether.

Clearly these changes will affect the simple solitary waves of the system. Firstly, the LP states will have their amplitude affected. In fact, the only effect of birefringence on LP solitons is to scale the effective $\beta$ by an amount $\delta$. An LP soliton on the slow axis will have its amplitude decreased,

$$ e_x = \sqrt{\beta - \delta} \text{sech} \left( \sqrt{\beta - \delta} x \right), \quad (3.68) $$

while one on the fast axis will be taller and narrower

$$ e_y = \sqrt{\beta + \delta} \text{sech} \left( \sqrt{\beta + \delta} x \right), \quad (3.69) $$
The changes to the CP states are more profound. It is no longer possible to have pure CP states – the fact that the LP components propagate with different speeds has the effect of coupling power between the CP components. The CP solitary waves of isotropic media will therefore become solitary waves of nonuniform elliptic polarisation. It is necessary to use the shooting methods to ascertain their profiles and three examples are shown in figure 3.22. As the birefringence increases, and the positions of the minima approach the fast axis, so too the fundamental elliptically polarised soliton states become almost linearly polarised. For $\delta > |\beta|/2$, the solitary wave is polarised purely along the fast axis. Note that the Haus method can only be used to find the degenerate solitary waves of birefringent materials. This is a serious limitation; there is little doubt that the solutions we present here represent a cross section through the full solution family. For this reason it is not possible to identify many of the bifurcation points. Nevertheless, we can extend the results for the degenerate modes, and infer the existence of wider classes of solutions. We expect that the one node degenerate solution of the previous section will carry through into birefringent materials. Figure 3.23 shows the trajectories in $(e_+, e_-)$ space and the transverse profiles of this solution. In order to characterise the solution, we consider the domains of positive and negative birefringence separately. The solution in isotropic media is symmetric about the $e_x$ axis. This is the fast axis for positive $\delta$. As shown in figure 3.23(a), for negative $\delta$, the solitary wave appears as the bound state of two elliptically polarised solitons that converge on opposite sides of the origin as $\delta \to -\beta/2$. As the medium becomes more isotropic, the trajectory fills out and makes the transition to the bound state of 3 LP solitary waves, as shown in figure 3.23(b) and (c). There is another degenerate solitary wave of this type, although its solution branch does not explicitly connect with the first variety. Shown in figure 3.23(d), it forms a bound state between two fundamental vector solitary waves that are converging on the same LP state. Since it is not possible to locate the branching points
Figure 3.23: Some of the higher order degenerate bound modes that exist in birefringent media. (a) The one-node state from figure 3.19 at $\delta = 0.05$ and (b) $0.3$. (c) The two-soliton degenerate bound state, comprising two elliptically polarised solitary waves from 3.22. (d) the two-node solution.

between the various solutions, we do not present an energy bifurcation diagram showing these solutions.

As an indication of yet higher order solutions, we have examined the effect of birefringence on the two node bound modes. As $\delta \to \beta$, the solution tends to linearly polarised states, three 'slow' solitons interleaved with two 'fast' solitons. The method that we have used to find vector solitary waves in birefringent systems is very limited by restricting our search to degenerate modes. A complete characterisation of the solitary wave solutions to this system would require an investigation of the three parameters $\beta_-, \delta$, and $B$. This is simply not possible using these techniques.

3.5 Are these solutions stable?

It is clearly of crucial importance for potential applications to determine the stability of these new solutions. Several hints relating to their stability can be obtained using physical reasoning. It is currently the goal of several researchers to develop general stability criteria for solitary wave solutions of CNSE systems. These criteria will revolve
around the functional relationships between the two propagation constants and the power in each polarisation component. The current incompleteness of this research leaves us with but one option: full numerical solution of the CNSE, a system of coupled parabolic PDEs.

It is not meaningful to ask whether the soliton solutions for integrable problems presented in section 3.3 are stable. Integrable differential equations are called ‘evolution’ equations in a rather loose sense of the word because the infinity of conservation laws means that nothing really evolves at all. Any parameter of significance is a conserved quantity, and one can decompose initial data into various conserved parts. An initial condition corresponding to the perturbation to a soliton state will, in general, result in a slight modification of the soliton parameters, and the appearance of a small quantity of ‘radiation’. The soliton and the nonsoliton component will not interact in any substantive way and will therefore be spatially separated after a long enough time. In terms of stability of the soliton state, the relevant question is the long time effect of the perturbation of the soliton parameters. For a single soliton solution, the perturbation may change the position, the speed or the amplitude of the soliton by a small amount. Nothing significant changes. However for bound states of parallel solitons, the small perturbation in the speed is critical. If the relative speed of the two components of the bound state are jilted out of coincidence by the perturbation, then the two components will, ever so slowly, drift apart. There is no binding energy between them. At very long times each component will comprise a separate soliton. Thus a higher order bound state of any integrable system will be unstable, but the instability has an arbitrarily small linear growth rate. Nothing further need be said concerning the ‘stability’ of the solitons of section 3.3.

To numerically integrate eq. (3.1) we have used a Crank-Nicholson finite differencing scheme (Press 1971) and, for confirmation, a split step Fourier method (Feit and Fleck 1978). Algorithms used in implementing these two techniques are outlined in numerous textbooks [see, e.g. Agrawal (1991), Marcuse (1990) and Newell and Moloney (1992)]. A nice discussion of the various methods as applied to the scalar NSE may be found in Taha and Ablowitz (1984). We used a variety of perturbations in testing stability, usually a small addition of Gaussian random noise across the pulse profile. In all simulations the type of perturbation had no qualitative effect as long as it was small enough. Tiny, unavoidable inaccuracies arising from the discreteness of the numerical grid excited enough unstable modes to dominate the evolution.

The fundamental soliton appears to be stable. Figure 3.24 shows the initial and final state where a large perturbation was imposed. The initial condition is a uniformly polarised stationary state of the $B = 0$ Manakov system. During propagation the beam radiates a little (see figure 3.25) and the pulse in each polarisation adjusts its width so that the final state is a fundamental vector solitary wave.

Unfortunately this is a pyrrhic victory since this soliton is not particularly interesting and, as we shall see, the higher order solitons are all unstable. The one node solitary wave is more stable when it is closer to the CP soliton state, i.e. when its two linearly polarised components overlap more. This is not surprising since this solution is formally stable in the infinitesimal region next to the bifurcation point, when the power in the second polarisation component is arbitrarily small. It is not clear whether there is a distinct point on the solution branch at which the solitary wave becomes unstable. Numerically it was only possible to determine that as the power in the second mode decreased that the time until breakdown was increasing, apparently without bound.

The time, measured in beat lengths $C = 2\pi/\Delta\beta$, until breakdown is a monotonically decreasing function of $\beta_-$. However the beat length is an exponential function of the soliton separation and so for larger separations the actual time until collapse increases. At the limit $\beta_- = \beta_+$, when the solution appears as well separated LP solitons, the breakdown appears to be caused by the same sort of gradual drift instability that affects
Figure 3.24: Numerical integration of eq. (3.1), showing the stability of the fundamental dynamic soliton to large perturbations. (a) Initial conditions are a uniform but elliptically polarised soliton, i.e. $e_+ = A \text{sech}(bx)$, $e_- = B \text{sech}(bx)$, chosen so that $A^2 + B^2 = b^2$. (b) After propagation, the field has evolved to the fundamental vector solitary wave.

soliton states in integrable systems. However in the nonintegrable system there is no graceful, gradual parting of company. Once there is a small difference in velocities between the components, the collapse will become catastrophic.

Due to its practical importance – solitons are likely to be used in some of the next generation of transoceanic fibre links – the problem of soliton interaction in optical fibres has attracted much attention in recent years. Several techniques for the reduction of soliton interaction have been suggested such as amplitude or phase shifting (Desem and Chu 1987), the use of third order dispersion, active temporal modulation schemes, as well as passive filtering methods. Rather than dealing with the question of the mathematical stability of these states, we might consider instead the possibility that, if the onset of the drift instability was slow enough, these states might still improve the state of the art.

Since this work was done, the state of the art has advanced considerably. This section presents results that are somewhat dated. Sliding-frequency guiding filters, developed at Bell labs by Mollenauer, Gordon, and Evangelides (1993) on the one hand, and phase dependent amplifiers (Kutz, Kath, Liow, and Kumar 1993) on the other, have effectively solved the problems that soliton interaction posed for transmission system designers. In the following, we compare the evolution of higher order solitary waves with in-phase bright solitons to quantify the gains of our method relative to older, simpler systems. These simulations model an idealised, lossless fibre and do not consider effects such as
random birefringence or amplified spontaneous emission noise arising from the use of amplifiers. Were the solitary waves stable, these simulations would predict an almost infinite potential bandwidth.

Figure 3.26 examines the evolution of an experimentally realisable situation. The initial conditions for (a) and (c) are a superposition of LP solitons, with each soliton polarised at 90° and 90° out of phase with respect to its neighbours. Separation between solitons is 2 FWHM. For (b) and (d) we launch in phase solitons that are superposed, with a separation twice as large. One can gain a factor of two improvement in bandwidth by using polarisation and phase multiplexing. This is inferior to quite spectacular results obtained recently by Mollenauer, Mamyshev, and Neubelt (1994) using filtering methods.
Figure 3.26: Contour plot showing (a) the evolution of the total intensity profile of the two-soliton bound state obtained by approximation with orthogonally polarised and 90° out of phase LP NSE solitons of the exact solution labelled $a_3$ in figure 3.12. The separation between the solitons is $X = 2.3$ times their FWHM. (b) The evolution of the two in-phase, likewise polarised LP NSE solitons separated by $2X = 4.6$ FWHM. (c) the evolution of the total intensity of the approximated $\beta_\perp = 0.95$ two node solution. The pulse separation is 2 FWHM (d) the same system with the central pulse removed. We observe a drift instability in the soliton bound state whose length scale is of the same order as the collision length of the in-phase solitons.
Chapter 4

Polarisation Domain Walls

Having characterised the solitary waves that occur in a variety of self-focussing materials, it is appropriate to turn our attention to self-defocussing media and the corresponding kink solitary waves. On account of the theoretically infinite energy contained in their tails, kink solitons have traditionally been shaded by their bright counterparts, and are frequently considered irrelevant to the applications. It is interesting to note, then, that the new vector waves found in this chapter are both stable and robust, and therefore may be more relevant to experiment than those of Chapter 3.

4.1 Bifurcations of the black soliton

4.1.1 Kink solitons of the NSE

This section will briefly repeat the analysis of §3.2.2, using the self-adjoint defocussing NSE instead. This NSE describes also the propagation of light in the normal dispersion regime in optical fibres. In dimensionless form it reads:

\[ i \partial_t E + \frac{1}{2} \partial_x^2 E - |E|^2 E = 0, \tag{4.1} \]

which we reduce, by using the ansatz \( E(x, z) = u(x) e^{-i \beta z} \), to the equation of mechanics

\[ \mathcal{H}_0 = \frac{1}{2} \dot{e}^2 + \beta e^2 - \frac{1}{2} e^4, \tag{4.2} \]

Consider the motion of a particle inside this potential that has a trough at the centre while there are finite peaks placed symmetrically on either side. The potential is exactly the negative of the potential for the focussing problem, eq. (3.5). What are the different categories of solution to this problem? Every trajectory with \( \mathcal{H}_0 > \sqrt{2} \beta \) will diverge as \( x \to \infty \), slowing down briefly to traverse the origin and the peaks surrounding it. For \( \mathcal{H}_0 = \beta^2 \) there are stationary solutions \( e = \pm \sqrt{\beta} \), and a kink solitary wave that traverses from one peak to the other. This is the separatrix trajectory. For \( 0 < \mathcal{H}_0 < \beta^2 \) there are two types of solution: one that oscillates forever within the central trough, and one that climbs up the outer wall from \( e = \infty \) only to fall back after reaching a high point of \( \mathcal{H}_0 \) at \( e^2 = \beta + \sqrt{\beta^2 + \mathcal{H}_0} \). This second type of solution exists for any \( \mathcal{H}_0 \), while the oscillatory solution collapses to a point for \( \mathcal{H}_0 = 0 \) and disappears altogether for negative energies.

Only solutions that are bounded for all time are of physical significance; therefore we consider the oscillatory type of solution alone. As in chapter 3, we write the equation in the standard form for elliptic functions, as

\[ \left( \frac{dv}{d\zeta} \right)^2 = \left( 1 - v^2 \right) \left( 1 - k^2 v^2 \right), \tag{4.3} \]
where \( e = \sqrt{a} \nu \), \( k^2 = a/b \) and \( \zeta = \sqrt{a/b} \). \( a < b \) are the two critical points of the equation (4.2). This equation leads to the simple solution for \( e \) of

\[
e = \sqrt{a} \sin \left( \sqrt{vb} \cdot \frac{x}{\sqrt{a/b}} \right).
\]

The Jacobi function \( \text{sn} \) has all the correct properties: at small (but positive) energies, which corresponds to the limit \( a \to 0 \), we get the reduction \( e \approx \sqrt{a} \sin \left( \sqrt{vb} \cdot \frac{x}{\sqrt{a/b}} \right) \). At the opposite limit, which corresponds to a trajectory that has exactly the right energy to start and finish at rest on top of one of the peaks, we get \( a \to b \to \beta_-/v \) and the localised kink solution reads

\[
e = \sqrt{a} \tanh \left( \sqrt{a} x \right).
\]

This is the black soliton solution to the scalar NSE. It has been observed as a dark stripe on a laser beam by Swartzlander, Anderson, and Regan (1991) and as a pulse in an optical fibre by Krökel, Halas, Guiliani, and Grischkowsky (1998).

The close parallel between the Haus-type analysis for bright and dark solitons suggests that there may be no qualitative difference between. However there is more to the black soliton than meets the eye. The distinguishing features arise on account of the grey soliton family, of which it is a member.

It is clear that it would not be possible to consider multi-soliton solutions to the black soliton alone, since it is carrying the baggage of infinite tails along with it. No, in order to collide these kinks, we demand a soliton that can travel across the background carrier wave. To this end we must try a new kind of ansatz, in which we allow the soliton this new freedom. If all goes well, we expect to recover the black soliton as a special case.

There is no simple way to arrive at the general dark soliton solution (i.e. a solution incorporating the family of grey solitons) in an \textit{a priori} way, and in fact the existence of this sort of soliton was not known until Zakharov and Shabat (1973) integrated the system completely and presented the single dark soliton and the family of N-soliton solutions. For brevity we shall utilise again the Hirota method to derive the dark soliton solution. Analogous to eq. (3.33), the NSE is equivalent to the bilinear equation:

\[
f \left( iD_x + \frac{1}{2} D_x^2 \right)(g \cdot f) - g \left( \frac{1}{2} D_x^2 + |g|^2 \right)(f \cdot f) = 0.
\]

One cannot break apart eq. (4.6) as we did in Chapter 2 with eq. (3.37) since eq. (3.38) cannot be satisfied by functions with finite boundary conditions. Instead, following Radhakrishnan and Lakshmanan (1995) we introduce a parameter \( \chi \) (whose significance will become apparent shortly), and break eq. (4.6) into

\[
B_1(g \cdot f) = 0 \quad B_2(f \cdot f) = -|g|^2,
\]

where

\[
B_1 = iD_x + \frac{1}{2} D_x^2 - \chi \quad \text{and} \quad B_2 = \frac{1}{2} D_x^2 - \chi.
\]

We choose the ansatz

\[
f = 1 + \lambda^2 f_2 \quad \text{and} \quad g = g_0(1 + \lambda^2 g_2),
\]

motivated by the fact that, give or take an overall phase factor \( g_0 \), \( f \) and \( g \) must have the same asymptotic behaviour. At the zeroth order we find

\[
|g_0|^2 = \chi \quad \text{and} \quad B_1(g_0 \cdot 1) = 0,
\]
4.1 Bifurcations of the black soliton

for which we choose the solution

\[ g_0 = r_0 e^{ip}, \quad \text{where} \quad p = a_0 x + (\frac{1}{2}a_0^2 - \chi)z. \]  

(4.11)

At the second order we find

\[ C_1(g_2 \cdot 1) + C_1(1 \cdot f_2) = 0 \quad \text{and} \quad B_2(f_2 \cdot 1 + 1 \cdot f_2) = -|g_0|^2(g_2 + g_2^*), \]

(4.12)

where \( C_1 = iD_x + ia_0D_x + \frac{1}{2}D_x^2 \). Unlike the bright soliton analysis, we must determine \( f_2 \) and \( g_2 \) simultaneously. Fortunately this is not arduous, and one can easily verify the solution

\[ f_2 = e^{ix} \quad \text{and} \quad g_2 = -\frac{\rho^*}{\rho} e^{ix}, \]

(4.13)

where

\[ q = 2a_1(x + b_1z) + q_0 \quad \text{and} \quad \rho = a_1 + i(a_0 + b_1). \]  

(4.14)

There is an additional constraint here, imposed from the second equation above, that \( |ho|^2 = |r_0|^2 \). The general form of the dark soliton can now be written out as

\[ e = r_0 \left\{ i \sin \phi + \cos \phi \tanh[a_1(x + b_1)z + q_0] \right\} e^{i[a_0 x + (\frac{1}{2}a_0^2 - |r_0|^2)z - \phi]}, \]

(4.15)

and the intensity

\[ |u|^2 = |r_0|^2 \left\{ 1 - \cos^2 \phi \text{sech}^2[a_1(x + b_1)z + q_0] \right\}, \]

(4.16)

where \( \phi = \arctan \left( \frac{a_0 + b_1}{a_1} \right) \) and the soliton parameters are connected by the constraint \( a_1^2 + (a_0 + b_1)^2 = |r_0|^2 \). This is a localised kink structure on a background plane wave; \( r_0 \) gives the amplitude and overall phase of the background while \( a_0 \) gives its direction. The soliton half width is \( a_1^{-1} \) and the angle it makes with the z axis is \( \theta = \arctan b_1 \).

At \( z = 0 \) the soliton is centred at \( -q_0 \). Without loss of generality we take \( \Im(r_0) = 0 \), \( a_0 = \phi \) and \( q_0 = 0 \), simplifying our constraint to \( a_1^2 + b_1^2 = |r_0|^2 \). Starting with \( b_1 = 0 \) we recover the black soliton solution of eq. (4.5), with width \( a_1 = r_0 \) and \( \phi = 0 \). For non-zero \( b \), the intensity at the centre of the soliton is given by \( r_0^2 \sin^2 \phi \), while the phase in the background differs on each side of the soliton by \( \pi - 2\phi \). \( b_1 \) is bounded above by \( r_0 \), and as \( b_1 \to r_0 \) then \( a_1 \to 0 \) (and the width \( \to \infty \)) and the contrast, defined as \( (I_{\text{max}} - I_{\text{min}})/I_{\text{max}} = \cos^2 \phi \), tends to zero. Figure 4.1 shows the form of the dark soliton, for varying levels of contrast. Unlike the bright solitons, the grey solitons form a genuine family of solutions. Note also that once the background wave is fixed, there is only one free parameter that determines the propagation direction, the half width and the contrast. One could continue this analysis and derive the multisoliton solutions; interested readers may consult Blow and Doran (1985) for the exact two-soliton form, derived using the IST.

Dark solitons, despite the baggage of infinite tails, have some interesting properties that suit them well to many applications. Being a form of kink wave, they are topologically trapped and therefore are very robust in a variety of situations. They are also far more useful in bulk materials as they propagate in self-defocusing media. In the self-focusing realm of bright solitons, light beams suffer from filamentation, only ceasing to collapse when the medium is damaged or saturation is reached. Dark soliton research therefore continues, albeit on a smaller scale than the work on bright solitons. Vector dark solitary waves have recently been discovered (Kivshar and Turitsyn 1993b), in which the intensities of the two components are balanced to ensure equal group velocity.
Figure 4.1: The dark soliton of the defocussing NSE. On the left are the real and imaginary parts of the profile $e(x)$, at centre the total intensity profile, while at right we show an image of the real part of $e(x,z)$, so that phase variation can be observed. $\tau_0 = 1$, (a) a black soliton. Notice that the contrast is unity, and the phase change is $\pi$ radians. (b) $b_1 = 0.2$ (c) $b_1 = 0.95$. Note both the increase in width and reduced phase change in the latter case.

4.1.2 Bifurcation points

We have shown in chapter 3 that it is possible to form solitary waves that are bound states of two orthogonal polarisation components of the electric field. In the case of bright solitons, these states appear as trains of bright pulses that travel in parallel, and whose polarisation state changes during propagation. It was fundamental to that analysis that the new solutions originate from bifurcations of scalar NSE solitons.

In this section we apply the same reasoning to derive new solitary waves that are instead based on dark solitons. We locate the bifurcation points and trace the solution branches of the new solitary waves. The vector solitary waves of (Kivshar and Turitsyn 1993b) are shown to be a subset of the new solutions. The limit, far from the bifurcation, of the new solutions, shows a new, fundamental kind of solitary wave. This is a localised structure that separates adjacent regions of orthogonal polarisation, and is therefore named a polarisation domain wall (PDW). This object also exists in birefringent materials, and we discuss this PDW further in the next section.

We wish, initially, to consider the propagation of light in defocussing, isotropic, Kerr type media and therefore consider eq. 1.22 with $\delta = 0$:

$$i\partial_x E_\pm + \frac{1}{2} \partial^2_x E_\pm - [(1 - B)|E_\pm|^2 + (1 + B)|E_\mp|^2] E_\pm = 0,$$

that reduces to a scalar NSE when we set $E_+$ or $E_- = 0$, or $E_+ = \pm E_-$. These reductions will lead, respectively, to the circularly polarised (CP) and linearly polarised (LP) tanh-profile soliton states encountered already. The CP states will be neutrally
4.1 Bifurcations of the black soliton

Figure 4.2: Surface and contour plots of the potential function of eq. (4.18), with the trajectories of the CP (blue and aqua lines) and LP (purple and yellow) solitons inscribed on it. $B = \frac{1}{3}$ and $\beta_+ = \beta_- = \frac{1}{2}$.

stable against small perturbations in the other polarisation component, as the bright solitons were shown to be in §3.2.3. In a parallel analysis we will try to locate the bifurcation points by forming a CP black soliton in one polarisation and considering perturbations to this soliton in the other polarisation.

It is instructive firstly to show separatrix trajectories in $(e_+, e_-)$ space that correspond to known solutions. Once again we make the simple, stationary, ansatz $E_{\pm}(x, z) = e_{\pm}(x)e^{i\theta_{\pm}z}$, which leads to the same mechanical model of a ball rolling around inside the topography whose elevation is given by

$$V(e_+, e_-) = \beta_+ e_+^4 + \beta_- e_-^4 - \frac{1}{4}(1 - B) (e_+^4 + e_-^4) - \frac{1}{2}(1 + B)e_+^2 e_-^2.$$ \hspace{1cm} (4.18)

This is clearly the complete inverse of the previous potential and in figure 4.2 we show the shape of the potential for $B = \frac{1}{3}$. Figure 4.2 also shows the potential as a contour plot, and superposed upon it are the uniformly polarised soliton states. CP solitons connect the four peaks (i.e. local maxima) at

$$(e_+, e_-) = \left( \pm \frac{\sqrt{2\beta_+}}{1 - B}, 0 \right) \quad \text{and} \quad (e_+, e_-) = \left( 0, \pm \frac{\sqrt{2\beta_-}}{1 - B} \right), \hspace{1cm} (4.19)$$

that surround the central basin. The LP solitons join the saddle points that are found at

$$(e_+, e_-) = \left( \pm \sqrt{\beta_+ + \beta_- + \frac{\beta_- - \beta_+}{2B}}, \pm \sqrt{\beta_+ + \beta_- + \frac{\beta_+ - \beta_-}{2B}} \right). \hspace{1cm} (4.20)$$

It is apparent from this that LP solitons do not qualify as candidates for this sort of bifurcation: they start and finish at saddle points of the potential. A saddle does not allow the same trajectory freedom as does a maxima. A trajectory asymptotically approaching a saddle can only come along a single line. This is a consequence of nonlinear circular birefringence that exists in many isotropic media.

Having examined known kink solitons of the CNSE, let us consider new solutions that may bifurcate from them. The transverse profiles of black and bright solitons are modes of the same induced waveguide. In particular, the bright soliton represents the fundamental mode, while the black soliton takes the second mode exactly at the point where it is ‘cut-off’ – i.e. where it is on the very edge of being a bound mode. In chapter 3 (figure 3.2), we showed the modes of the soliton induced waveguide for the other polarisation component. For $B = 0$ both components see the same waveguide, and
we can see that the second mode is at the very point of cut-off. Since the black soliton induces the same waveguide, we can fix $B$ and find the same bifurcations that were present for the bright soliton. The values of $\beta_-/\beta_+$ corresponding to these bifurcations will be different from the bright case since $\beta_+$ dark and bright solitons, despite inducing the same waveguide, have different propagation constants.

Taking $e_+(x,z) = Ae^{i\beta_+ z} \tanh bx$ (where $\beta_+ = b^2$ and $A = b/\sqrt{1-B}$), and $e_- = y(x)e^{i\beta_- z} \ll e_+$ we derive, to the first order, a linear Schrödinger equation for $e_-.$

$$\frac{1}{2} \dot{y} - \left[ (\beta_- - \kappa) + 2\kappa \beta_+ \text{sech}^2(\sqrt{2 \beta_+} x) \right] y = 0. \quad (4.21)$$

This is a direct analogue of the modal equation (3.20). The only difference is that the propagation constant $\beta_-$ has been shifted by an amount $\kappa = (1+B)/(1-B).$ Without loss of generality we take $\beta_+ = 1/2.$ The separatrix trajectories must begin and end at the maximum points of $V;$ therefore the solutions of the eigenvalue problem are restricted to those functions $y(x)$ that decay at $x \to \pm \infty,$ i.e. we only consider the bound modes of the waveguide. Being the same waveguide induced by the bright soliton, the results all carry over. The waveguide supports the integer part of $\nu$ modes, where $\nu(\nu+1) = K.$ The profile of the $n$-th mode is given by $P_n^\nu$ where $P_n^\nu$ are the associated Legendre functions. See section 3.2.3 for more details. The propagation constant of the $n$-th mode is given by $\beta_- = \kappa - \frac{1}{2}(\nu - n)^2.$ For $B = \frac{1}{3},$ the only case that we shall investigate here, there are two modes

$$y_0 = \text{sech} x \quad \text{at} \quad \beta_- = \beta_0 = 0.39, \quad \text{and} \quad (4.22)$$
$$y_1 = \text{sech} x \tanh x \quad \text{at} \quad \beta_- = \beta_1 = 0.92. \quad (4.23)$$

One expects that these two points $\beta_- = \beta_{0,1}$ correspond to branching bifurcations that will lead to a new form of solitary wave.

Note that the preceding analysis can be generalised to grey solitons. Grey solitons induce waveguides that are wider and shallower but that always support the same number of bound modes. In the parlance of linear waveguide theory, we say that the waveguide parameter $V$ is the same in both cases. The grey soliton has a different propagation constant, and the profiles of the modes are changed quantitatively, but the essence of the results remains the same.

### 4.2 Bright-dark solitons of the Manakov system

#### 4.2.1 The single bright-dark soliton

Continuing to follow paths taken in Chapter 3, we now investigate the integrable CNSE that comes from assuming the defocussing Manakov model, i.e. setting $B = 0$ in eq. (4.17). One anticipates the existence of solutions originating in the bifurcation points found in the previous section. These solutions are a kink wave in one polarisation but a bound mode in the other. Once again we shall use the Hirota method, having proven itself already in this thesis. Since the fields $E_+$ and $E_-$ take qualitatively different forms, we introduce a new, scalar, function $h$ and set $e_+ = g/f, \ e_- = h/f.$ The uncoupled Hirota equations take the form

$$B_1(g \cdot f) = 0, \quad B_1(h \cdot f) = 0 \quad (4.24)$$
$$B_2(f \cdot f) = -|g|^2 - |h|^2, \quad (4.25)$$

where

$$B_1 = iD_z + \frac{1}{2}D_x^2 - \chi \quad \text{and} \quad B_2 = \frac{1}{2}D_x^2 - \chi. \quad (4.26)$$
Table 4.1: Equations resulting from the quadratic ansatz, eq. (4.27), used to find the single bright-dark soliton. After casting eq. (4.17) with $B = 0$ into bilinear form and forming a quadratic ansatz in the parameter $\lambda$ one equates the coefficients at each power of $\lambda$ to obtain these equations for $f_1, g_x$ and $h_x$.

We introduce, for $h$, an ansatz suitable for bright solitons, and one for $g$ appropriate for kinks. This means that $f$ and $g$ will have terms only in even powers of the polynomial parameter while $h$ only has terms in odd powers. Looking for single solitons,

$$f = 1 + \lambda^2 f_2, \quad g = g_0(1 + \lambda^2 g_2) \quad \text{and} \quad h = \lambda h_1. \quad (4.27)$$

Once again we take individual powers of $\lambda$, and the resultant equations are shown in Table 4.1.

There are no additional difficulties encountered when ‘cranking the handle’ to generate this type of solution. One rapidly arrives at the following solution form:

$$g_0 = \tau_0 e^{ip}, \quad p = a_0 x - \left(\frac{1}{2} a_0^2 + |\tau_0|^2\right) z$$

$$h_1 = e^{\eta_1}, \quad \eta_1 = \kappa_1 x + i \left(\frac{1}{2} \kappa_2^2 - |\tau_0|^2\right) z + \eta_1^0$$

$$f_2 = f_2^0 e^{\eta_1 + \eta_1^0}, \quad f_2^0 = \left[(\kappa_1 + \kappa_2^* \left|\frac{|\tau_0|^2}{|\rho_1|^2} - 1\right|)^{-1} \rho_1 = \kappa_1 - i a_0 \right.$$

$$g_2 = -\frac{\rho_1}{\rho_1^*} f_2^0 e^{\eta_1 + \eta_1^0}. \quad (4.28)$$

After some algebraic manipulation the expression for the fields can be written in the more transparent form

$$E_+ = \tau_0 \left\{ \cos \phi + \cos \phi \tanh \left[ a_1 (x - b_1 z) + \mathcal{R}(\eta_1^0) \right] \right\} e^{ia_0 x - \left(\frac{1}{2} a_0^2 + |\tau_0|^2\right) z}$$

$$E_- = a_1 \sqrt{\frac{|\tau_0|^2}{|\rho_1|^2} - 1} \sech \left[ a_1 (x - b_1 z) + \mathcal{R}(\eta_1^0) \right] e^{i \left\{ b_1 x + \left[ \frac{1}{2} (a_1^2 - b_1^2) - y^2 \right] z + \mathcal{R}(\eta_1^0) \right\}}, \quad (4.29)$$

where $\phi = \arctan \left( \frac{a_0 + b_1}{a_1} \right)$ and the solution will become singular unless the constraint $|\rho_1|^2 \leq |\tau_0|^2$ is satisfied. It is readily seen that this is a combination of a kink wave and a sech profile bound mode. This is a straightforward generalisation of the scalar grey soliton, eq. (4.6). The $E_+$ component of the field takes the grey soliton profile, but in this case the contrast at a given propagation angle is reduced by the influence of the bound component $E_-$. The total intensity of the bright-dark soliton takes the deceptively simple form:

$$I = |e_+|^2 + |e_-|^2 = |\tau_0|^2 - a_0^2 \sech^2 \left[ a_1 (x - b_1 z) + \mathcal{R}(\eta_1^0) \right]. \quad (4.31)$$
Figure 4.3: The bright-dark soliton of the Manakov system. On the left we show the intensity profiles in each polarisation component, $|e_+(x)|^2$ and $|e_-(x)|^2$, then the total intensity profile $I(x) = |e_+|^2 + |e_-|^2$. Finally we show images of the real parts of $e_+(x,z)$ and $e_-(x,z)$ respectively, in which the phase variation is clear. $\tau_0 = 1$, and (a) $a_1 = 0.8$, (b) $a_1 = 0.5$ (c) $a_1 = 0.2$ show the progressive 'greying out' of the $b_1 = 0.4$ grey soliton presented in figure 4.1.

Without losing generality we take $a_0$ and $\Im(\tau_0)$ to be zero. The parameter constraint now reads $a_1^2 + b_1^2 < \tau_0^2$. As $a_1 \to \tau_0$ the intensity in $e_-$ tends to zero and the soliton becomes the scalar grey soliton (4.6). The contrast of the soliton goes to zero as $a_1 \to 0$, which can occur at any propagation angle. Figure 4.3 shows the form of a single BD soliton.

4.2.2 Bright-dark soliton interactions

Having determined the form of the single BD solitons using the Hirota method, one would expect it to be a reasonably straightforward matter to extend this calculation to include the interaction between BD solitons. Happily this turns out to be true. We therefore choose the 2-soliton polynomial ansatz

$$f = 1 + \lambda^2 f_2 + \lambda^4 f_4 \quad g = g_0(1 + \lambda^2 g_2 + \lambda^4 g_4) \quad h = \lambda h_1 + \lambda^3 h_3,$$

which leads to the equations presented in Table 4.2. The solution to these equations can be found by proceeding down the table, beginning with the lowest order. At the first
Table 4.2: The equations for \( f_i, g_i \) and \( h_i \) at each power in the parameter \( \lambda \), after making the quartic ansatz, eq. (4.32). The solution is defined by the equations up to \( \lambda^4 \); a valid solution will satisfy the remaining equations.

| \( \lambda^0 \) | \( B_1(g_0 \cdot 1) = 0 \) | \( B_2(1 \cdot 1) = -|g_0|^2 \) |
| --- | --- | --- |
| \( \lambda^1 \) | \( B_1(h_1 \cdot 1) = 0 \) |  |
| \( \lambda^2 \) | \( B_1(g_0g_2 \cdot 1 + g_0 \cdot f_2) = 0 \) | \( B_2(f_2 \cdot 1 + f_2 \cdot f_2) = -|g_0|^2(g_2 + g_2^*) - |h_1|^2 \) |
| \( \lambda^3 \) | \( B_1(h_3 \cdot 1 + h_1 \cdot f_2) = 0 \) |  |
| \( \lambda^4 \) | \( B_1(g_0g_4 \cdot 1 + g_0 \cdot f_2 + g_0 \cdot f_4) = 0 \) | \( B_2(f_4 \cdot 1 + f_2 \cdot f_2 + 1 \cdot f_4) = -|g_0|^2(g_4 + |g_2|^2 + g_4^*) - h_1h_3^* - h_1^*h_3 \) |
| \( \lambda^5 \) | \( B_1(h_3 \cdot f_2 + h_1 \cdot f_4) = 0 \) |  |
| \( \lambda^6 \) | \( B_1(g_0g_4 \cdot f_2 + g_0g_2 \cdot f_4) = 0 \) | \( B_2(f_4 \cdot f_2 + f_2 \cdot f_4) = -|g_0|^2(g_4g_2^* + g_4^*g_2) - |h_3|^2 \) |
| \( \lambda^7 \) | \( B_1(h_3 \cdot f_4) = 0 \) |  |
| \( \lambda^8 \) | \( B_1(g_0g_4 \cdot f_4) = 0 \) | \( B_2(f_4 \cdot f_4) = |g_0|^2|g_4|^2 \) |

In order to extract the 'essence' from this expression we investigate its behaviour when the two solitons are well separated. One should note first that when the soliton directions coincide, we can form a bound state, in which the solitons never separate. We will return to this bound state later on. Let us take \( b_1 > b_2 \) so that soliton 1 crosses soliton 2 from left to right as \( z \) increases. Using the techniques already plumbed in some depth in this
thesis we obtain the position shifts from the formula to be

\[ \Delta x_j = \frac{1}{a_j} \ln |\mu_{1,2}\mu_{1,2^*}| = \frac{1}{a_j} \left| \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \left( \frac{\tau_0^2 + \kappa_1 \kappa_2}{\tau_0^2 - \kappa_1 \kappa_2} \right) \right|. \] (4.34)

There is not much of interest in this expression; like most multisolitons, the position shift tends to infinity when the soliton eigenvalues coincide. The main difference between this solution and the bright-bright collision is that there is no equivalent ‘polarisation rotation’ upon collision.

Let us now explore some facets of the solutions graphically. Figure 4.4(a) shows the collision between two BD solitons of different directions and contrast. Longitudinal and transverse beating, invariably absent in collisions between pure kink waves, is clearly visible. Note that the amount of energy carried in the bound mode is an invariant of the equations and therefore there can be no swapping of power in an interaction. Figure 4.4(b) and (c) show bound states, in which \( b_1 = b_2 \). Unlike the bright case, there are no bound states here that are stationary, although three-component bright-bright-dark solitons satisfying the U(3) NSE will exhibit such stationary solutions. One should also note that all such bound states are, as discussed in section 3.5, unstable to arbitrary perturbations of the initial conditions.

4.2.3 N-soliton bright-dark solutions

Unlike the focussing solitons of the Manakov system, the initial position of the BD solitons does not play a significant role. Therefore there are no obstructions to writing down the complete \( N \)-soliton solution. In fact this solution is a trivial generalisation of the two soliton case. We write it in a compact notation similar to that used in Radhakrishnan and Lakshmanan (1995).

\[ f = \sum_{a=0,1} M_1(a) \exp \left( \sum_{j=1}^{2N} a_j \eta_j + \sum_{1 \leq j < k}^{2N} a_j a_k A_{j,k} \right) \]

\[ g = \tau_0 e^{ip} \left[ \sum_{a=0,1} M_1(a) \exp \left( \sum_{j=1}^{2N} a_j (\eta_j + \xi_j) + \sum_{1 \leq j < k}^{2N} a_j a_k A_{j,k} \right) \right] \]

\[ h = \sum_{a=0,1} M_1(a) \exp \left( \sum_{j=1}^{2N} a_j \eta_j + \sum_{1 \leq j < k}^{2N} a_j a_k A_{j,k} \right), \] (4.35)

where the first sum is over all possible permutations of the vector \( a = (a_1, a_2, \ldots, a_N) \), and

\[ \eta_j = \kappa_j x + \frac{1}{2} \kappa_j^2 z + \eta_j^0 \]

\[ \xi_j = -\rho_j \]

\[ e^{A_{j,k+N}} = e^{A_{k+N,j}} = \mu_{j,k}^* \]

\[ e^{A_{j,k}} = \mu_{j,k} \]

\[ M_1 = \begin{cases} 1 & \text{if } \sum_{j=1}^N a_j = \sum_{j=N+1}^{2N} a_j \\ 0 & \text{otherwise} \end{cases} \]

\[ M_2 = \begin{cases} 1 & \text{if } \sum_{j=1}^N a_j = \sum_{j=N+1}^{2N} a_j + 1 \\ 0 & \text{otherwise} \end{cases} \]

Asymptotics reveal the total position shift suffered by the \( j \)-th beam to be
4.2 Bright-dark solitons of the Manakov system

Figure 4.4: Four examples of bright-dark soliton interactions. At left we show the Amplitude $|e_+|$, in the centre $|e_-|$ and at right the total intensity $I$. In (a) we show a general collision in which $\kappa_1 = 0.7 - 0.01i, \kappa_2 = 0.97 + 0.03i$, showing the beating phenomena. In (b) the solitons are mirror images of one another $\kappa_1 = 0.7 + 0.3i, \kappa_2 = 0.7 - 0.3i$. The bottom two pictures illustrate bound modes: in (c) we show a bound mode where the two solitons are truly coincident and almost equal in contrast, giving rise to very strong beating effects; $\kappa_1 = 0.7, \kappa_2 = 0.77$. In (d) we show the same two solitons, this time spatially separated by 2 half-widths and the interaction is reduced to small fluctuations in position and amplitude; $\kappa_1 = 0.7, \kappa_2 = 0.77$. 
Figure 4.5: Two examples of three soliton bright-dark dynamic soliton solutions to the integrable U(2) CNSE. (a): a generic sort of collision, with $\kappa_1 = 0.4 - 0.4i$, $\kappa_2 = 0.8 - 0.05i$ and $\kappa_3 = 0.7 - 0.6i$. (b): a quasiperiodic solution consisting of coincident solitons with different contrast, defined by $b_2 = 0.6, 0.4$ and $0.2$.

$$\Delta x_j = \frac{1}{2} \sum_{1 \leq k < j} (A_{j,k} + A_{j,k+N} + A_{j+N,k+N} + A_{j+N,k})$$
$$- \frac{1}{2} \sum_{j < k \leq N} (A_{j,k} + A_{j,k+N} + A_{j+N,k+N} + A_{j+N,k}). \quad (4.36)$$

This is a simple sum of the phase shifts that would be suffered in paired collisions with each soliton in turn. The interactions therefore commute, like the collisions of scalar solitons, and unlike vector bright soliton interactions. Figure 4.5 shows two examples of three soliton interactions. At the top we show a standard, non-degenerate interaction in which all three beams have different contrast and direction. Below this we show a case in which all the eigenvalues have the same imaginary part and are therefore coincident. Note that there are now three separate beat frequencies involved and that the solution is therefore quasiperiodic in $z$.

4.3 Bright-Dark solitons in nonintegrable CNSEs

4.3.1 The solution branches associated with bifurcations

We have established the existence of bright-dark solitons for CNSEs with $B = 0$ and for the asymptotic limit $\epsilon_\infty \to 0$. We now wish to extend this class to include general bright-dark solitary waves of the nonintegrable CNSEs. We will use numerical methods, commencing with investigations for $B = \frac{1}{3}$ near the bifurcation points found in §4.1.2. Associated bifurcation branches will be found for values of $\beta_\infty$ corresponding to an
increase in the power of $e_-$. These solutions can easily be calculated from eq. (4.18) using the faithful numerical shooting scheme.

Given the symmetries of the solution corresponding to the first bifurcation (i.e. $e_+$ antisymmetric, $e_-$ symmetric), we can search for it using initial conditions $e_+(0) = 0$ and $e_-(0) = 0$. It is therefore convenient to choose $e_-(0)$ as the shooting parameter, and the value of $e_+(0)$ is calculated from it so as to obtain a trajectory with the correct energy. Figure 4.6 displays four examples to indicate the evolution of the separatrices as $\beta_-$ tends from $\beta_0$ to 1. The corresponding bifurcation diagram appears in figure 4.7. We see that as $\beta_-$ increases from $\beta_0$ a pulse, centred around the origin, grows in the $e_-$ field component; in the same region, the slope of the wave $e_+$ decreases. This is a much more interesting effect than is exhibited by the BD solitons of the Manakov system. In that case, the bright and dark components were the same width, that width increasing as the soliton contrast diminished. Here the solution no longer appears as a single entity, but forms two separate structures. Note that this only occurs for $B > 0$; cross phase modulation is required to be greater than self phase modulation to observe the breakup of the bright-dark state into two separate parts. Analogously with the bright soliton case, the solution as $\beta_+ \rightarrow 1$ tends to two independent, localised structures, in this case bound kink waves of a new type. They constitute vector solitary waves of the dark type (i.e. non-vanishing boundary conditions) separating domains of orthogonal polarisation, and therefore we christen them polarisation domain walls. The existence of such coupled kink waves can be easily understood from the analysis of the potential $V$.

The second bifurcation produced by the solution (4.23) is more complex, although in many waves qualitatively identical. It occurs at $\beta_- = \beta_1 = 0.78$ and since the corresponding guided mode $y_1$ is antisymmetric it does not involve a change in symmetry of the total field. The energy bifurcation diagram of this second bright-dark wave is shown in figure 4.7 while figure 4.8 shows two examples of separatrix trajectories and the corresponding envelopes. Like the first bifurcation, the profile of $e_+$ flattens around the origin. Here this process is accompanied by the growth of two $\pi$ out of phase pulses in $e_-$. As a result, when $\beta_-$ approaches $\beta_+$ the field distribution is composed of one CP NSE dark soliton surrounded by two identical polarisation domain walls. We have established that the BD soliton is the $B = 0$ limit of the first bifurcation branch. To what solution of the Manakov system does the second bifurcation correspond? In the limit $B \rightarrow 0$ this complicated, compound solitary wave degenerates into a simple vector dark soliton. Note that the second mode of the waveguide is no longer bound when $B = 0$ and therefore the solution is a dark-dark form. The vector dark solution was recently derived for the first time by Kivshar and Turitsyn (1993b).

As in the previous chapter, the solitary waves that we have presented in this section can be thought of as multiple-PDW bound states. In the limit $\beta_- \rightarrow \beta_+$, the trajectories of the first bifurcation tend to the separatrices of the first and second quadrant domain walls combined, while those of second bifurcation tend to combinations of first and third quadrant PDWs. As the difference between the propagation constants increases, the separation between the PDWs decreases. At any separation then, one can form a stationary bound state between PDWs, suggesting that there is no interaction force between them, analogous to two bright solitons orthogonally polarised and $90^\circ$ out of phase.

One more comment needs to be made regarding separatrix trajectories on the potential corresponding to the defocussing problem. Unlike the focussing problem, there is not a great menagerie of solutions. One can concatenate the PDWs into trains of any length, and interpose black solitons at certain points, but there are no further classes of solution.
Figure 4.6: Trajectories and profiles of the bound solitary waves $e_+$ and $e_-$ corresponding to the zeroth bifurcation branch of figure 4.7. Curves are given for different values of $\epsilon = \beta_+ - \beta_-$, (a) $\epsilon = 0.2$, (b) $\epsilon = 0.05$ (c) $\epsilon = 10^{-3}$, (d) $\epsilon = 10^{-5}$.
4.3 Bright-Dark solitons in nonintegrable CNSEs

4.3.2 Polarisation domain walls

Let us now study the polarisation domain walls (PDW) in isolation, rather than as the limiting cases of the bifurcation branches as $\beta_- \rightarrow \beta_+$. Their existence can be easily explained by means of an analysis of the potential $V(e_+,e_-)$. Separatrices are trajectories that connect either saddle points or maxima. In figure 4.9 we show surface and contour illustrations of $V(e_+,e_-)$ in the particular case where $\beta_- = \beta_+$ and $B = \frac{1}{3}$, with the separatrix trajectories corresponding to PDWs inscribed upon them.

The solitary wave envelopes $e_+(x)$ and $e_-(x)$ corresponding to the separatrix of the first quadrant ($e_+, e_- > 0$) are shown in figure 4.9. The envelopes of the separatrices of the other quadrants only differ by changes of the signs of $e_+$ and $e_-$. We recognise the kink shape of the waves of figure 4.6(d). The total intensity profile, $I = e_+^2 + e_-^2$, consists of a dark pulse inscribed onto a constant background. The values $q = \pm 1$ correspond to circular polarisation states of opposite handedness while $q = 0$ describes a linear polarisation state. We see that the bound kink solitary waves $e_+$ and $e_-$ form a localised structure consisting of a dip in the constant background accompanied by a progressive inversion of the ellipticity of the field. Since this localised structure forms the boundary between two regions of orthogonal, stable eigenpolarisations of the Kerr medium (Maker, Terhune, and Savage 1964b), it can be called a polarisation domain wall. Let us note, finally, that the existence of such a mode is closely tied to the tensor character of the Kerr nonlinearity. It is, in fact, easy to see from eq. (4.18) that the potential $V$ possesses maxima on the $e_+$ and $e_-$ axes only if $B > 0$ (i.e. that $\chi_{1221}^{(3)} \neq 0$). The $B$ dependence of $V$ indicates that, for a given intensity, the smaller the $B$ the broader the domain walls. In the limit $B \rightarrow 0$ their width becomes infinite. The influence of $B$ on the PDW profile is indicated in figure 4.10 where we plot $e_+$ and $e_-$ envelopes for several values of $B$. Note that a trivial analytical solution exists for the case $B = \frac{1}{2}$. For this value, the two linearly polarised components decouple if they have the same propagation constant, quite unphysical behaviour. At $B = \frac{1}{2}$ the PDW is a black soliton.

![Diagram](image)

**Figure 4.7:** The bifurcation diagram showing both bifurcations of the dark soliton at $B = \frac{1}{3}$. An energy bifurcation diagram is not suitable for kink type solitary waves; instead the dependent parameter is the peak amplitude in the $e_-$ field component.
Figure 4.8: (a) Two separatrix trajectories of the second bifurcation branch; the values of $\beta_-$ are (a) $\beta_- = 1.6$ and (b) $\beta_- = 1.002$. In (b) the wave resembles a bound state of a PDW, a black soliton and another PDW.

Figure 4.9: The polarisation domain wall of the first quadrant. We show the trajectory superposed on surface and contour representations of the potential $V(e_+, e_-)$, and at right the profiles $e_{\pm}(x)$. 
in one LP component, and a plane wave in the other.

To a certain extent these solitary waves are qualitatively similar to those of Zakharov and Mikhailov (1987). In that case, however, the localised structures result from the nonlinear interaction between two counterpropagating beams of orthogonal polarisation in a dispersion-less Kerr material. In fact, the mathematical solutions presented by Zakharov and Mikhailov (1987) are best physically interpreted as localised structures separating adjacent eigenstates of counterpropagating waves (Kaplan 1983). In our case the localised structures are separating orthogonal eigenpolarisations of single wave systems. Since we have restricted the analysis to isotropic Kerr media, these eigenpolarisations are pure LP and CP states. To be more precise, and to differentiate the vector kink solitary waves that we have discovered from the solitary waves of Zakharov and Mikhailov (1987), one should perhaps call them 'diffractive' or 'dispersive' polarisation domain walls depending on whether we consider the spatial or the temporal problem.

An interesting parallel can be drawn from a topological point of view between the polarisation domain walls considered here and a new class of localised lattice structures in discrete models recently reported theoretically in the literature by Kivshar and Turitsyn (1993a) and Denardo, Galvin, Greenfield, Larrazza, Puttermann, and Wright (1992) and observed experimentally by Denardo, Larrazza, Puttermann, and Roberts (1992) in simple systems of coupled pendula. In the latter case the localised structures separate domains of symmetric lattice vibration eigenmodes.

4.3.3 Another bright-dark wave

The two bright-dark waves that we have characterised so far as bifurcations of the black soliton are also bound states of PDW solitary waves. In particular, the first type is a bound state of PDWs of the first and second quadrants. We now confirm numerically, using the shooting scheme, the existence of a bound state of two PDWs of the first quadrant. Figure 4.11(a) illustrates this bright-dark solitary wave, and figure 4.12 shows the corresponding bifurcation diagram. One naturally asks the question: since these solutions don't bifurcate from black solitons, what form do they take when the power in $e_-$ is small? In answering this, we resort once more to the mechanical analogue. For $\beta_- = \beta_+$ there are saddle points on the axes $e_+ = e_-$, corresponding to the linearly polarised solutions. The trajectory corresponding to the new solution branch starts at the maxima on the $e_+$ axis, traverses the saddle and reaches a high point on the other side before retracing its path. When $\beta_-$ increases, these saddle points move toward the $e_+$ axis, shortening the distance before the trajectory folds back upon itself. As $\beta_- \to \kappa \beta_+$ the saddle points converge on the local maxima at $(e_+, e_-) = (\pm \sqrt{2\beta/(1 - B)}, 0)$, which are replaced by saddle points for $\beta_- > \kappa \beta_+$. These trajectories need to begin and end on a maximum and therefore they are cut-off for $\beta_- > \kappa \beta_+$. Figure 4.12 shows the energy bifurcation diagram for this type of solution. The existence of a bifurcation at $\beta_- = \kappa = 2$ is apparent. Note that the solid line on the horizontal axis denotes a CP plane wave and not a CP kink soliton as before. The form of this bright-dark solitary wave near the bifurcation is shown in figure 4.11(b). The profile spreads out and the contrast goes to zero like the profile of the integrable bright dark solitons of section 4.2. Note that the domain of existence of this BD wave goes to zero as $B \to 0$ and therefore it is endemic to nonintegrable CNSEs.

4.3.4 Stability of the new solutions

As in chapter 3 it is crucially important whether or not the new solutions are stable. First note that the bifurcation should not affect the stability of the branch of the circularly polarised NSE soliton. As we have seen earlier, the CP soliton is always neutrally stable with respect to perturbations of its polarisation state. Since we do not have analytic expressions for the new vector solitary waves, the stability of the new branch
Figure 4.10: Trajectories and profiles of the first quadrant PDW for two values of the material parameter $B$: (a) $B = 0.25$, (b) $B = 0.75$.

Figure 4.11: Two examples of the 'odd one out' solution branch, that does not originate as a linear mode of a black soliton. In one limit, it describes well separated PDWs of the first quadrant (a); in the other, a broad and flat solitary wave approaching a circularly polarised plane wave (b).
must be investigated numerically. We have performed numerical simulations of the propagation equation (4.17) with the solitary waves of figure 4.6 as the initial conditions. We verified their stability with respect to several different types of perturbations. Figure 4.13 illustrates the stability with respect to large amplitude perturbations. The initial conditions here are not exact profiles of the solitary waves, but rather the kind of profiles that one might expect to be generated in the laboratory: Gaussian functions $e_+(x) = \pm \left(1 - e^{-ax^2}\right)$ for $x \neq 0$ and $e_-(x) = e^{-ax^2}$, where the parameter $a$ is adjusted to match the width of the solitary wave of figure 4.6(b); its value is $a = 0.36$. We see in figure 4.13(a) that after oscillation and the emission of radiation (the transient behaviour) the field settles down to a stationary steady state. This state is identified as being a solitary wave solution whose parameter $\beta_-$ is close to that of figure 4.6(b) [due to radiation loss the final intensity in $E_-$ is slightly lower and therefore $\beta_\text{final} \approx 0.47$]. Figure 4.13(b) shows the initial (dotted lines) and final (solid lines) profiles of both polarisation components. This result illustrates the robust nature of the vector solitary waves which originate from the dark NSE soliton bifurcation associated with the fundamental mode of the XPM induced waveguide.

To illustrate the importance of the bifurcation of the circularly polarised dark NSE soliton we have numerically simulated the propagation of the dark soliton through an idealised amplifier. The situation is idealised since we consider a selective amplification of one circular polarisation component only. We do not discuss here the different possible mechanisms for achieving such a feature in practice. We simply assume that the component $E_+$ of the field is not affected by the amplifier and that the component $E_-$ undergoes adiabatic amplification. The initial condition corresponds to a dark NSE soliton for the field $E_+$ and a small Gaussian envelope centred on the soliton for the polarisation component $E_-$. The intensity of the initial Gaussian beam (or pulse) is one-tenth of the dark soliton background intensity. Figure 4.14 shows the evolution of the intensity profiles of both fields at various stages during the propagation. We see that while the intensity in $E_-$ increases, the dark soliton broadens and exhibits a flat region of zero intensity around the origin $x = 0$. The intensity in $E_-$ saturates when it
Figure 4.13: Numerical simulation of the propagation of a perturbed solitary wave of the new solution branch. The initial field envelopes are given in the text. (a) Contour plot of the intensity $|E_+(x, z)|^2$ (note that damping techniques have been used to reduce reflection on the edge of the numerical window) (b) initial (dotted lines) and final (solid lines) profiles of both field components.

Approaches unity and after $z \simeq 250$ the effect of the amplification is only to broaden the field profile further. As a result, the intensity profiles in both polarisation components acquire a square shape and two distinct pairs of kink waves are formed. In fact, this scenario reproduces the evolution of the solitary wave solutions along the bifurcation branch studied above (compare figure 4.14 with figure 4.6). This simulation provides further confirmation of the stability of the whole solution branch.

Solitary waves of the second solution branch have also been shown to be stable in propagation. In figure 4.15 we show the example of a perturbed solitary wave close to the bifurcation. The initial condition for the field $E_+$ is the dark NSE soliton whereas the initial envelope in $E_-$ is given by $e_-(x) = axe^{-bx^2}$ where $a$ and $b$ are chosen roughly to approach the amplitude and width of a solitary wave solution ($a = 0.1, b = 0.5$). As in the case of figure 4.13 we observe a reshaping of the dark soliton accompanied by radiation and damped oscillations. Both fields $E_+$ and $E_-$ rapidly settle down to a steady state identified as being the solitary wave solution corresponding to the parameter $\beta_\sim \simeq 0.92$.

We have also checked the propagation stability of the PDWs themselves by numerical integration of the CNSE. As for the compound solitary waves studied above, we have verified their stability with respect to large perturbations. Figure 4.16 shows a rather drastic stability test. It shows, in the form of a pixel image of the intensity of the field $E_+$, the collision of a polarisation domain wall (located at $x = 0$) with a grey soliton. The grey soliton has the form $e_+ = e_0 \{\cos \phi \tanh [\eta(x - \omega z)] - i \sin \phi \}$ where $\omega = |e_0| \sin \phi$. In order to avoid any discontinuities in the uniform background, we must choose $e_0 = e^{i\phi}$ (the amplitude of the domain wall is real and equal to unity). For the example of figure 4.16(a) the transverse velocity $\omega = \sin \phi$ of the grey soliton is determined by the value $\phi = 0.1$, which corresponds to a total phase change of $\pi - 2\phi = 0.8\pi$ across the profile of the grey soliton. This represents a perturbation of almost twice the amplitude of the domain wall itself. We see, however, that this large perturbation does not affect the domain wall. The grey soliton bounces back from the collision and the shape and trajectory of the domain wall remain almost totally unchanged. Figure 4.16(b) shows that at higher angles of incidence the non-soliton behaviour is more pronounced. The behaviour of the PDW is dictated by the fact that the total energy in each polarisation component is conserved. As the grey soliton passes
4.3 Bright-Dark solitons in nonintegrable CNSEs

Figure 4.14: Evolution of the intensity profiles $|E_+|^2$ (solid lines) and $|E_-|^2$ (dotted lines), obtained when the polarisation component $E_-$ is adiabatically amplified. The initial conditions are described in the text. We clearly verify the solitary waves of the new solution branch of Figures 4.7 and 4.6.

Figure 4.15: Numerical simulation of the propagation of a perturbed solitary wave of the second solution branch. The initial field profiles are as given in the text. (a) contour plot of the intensity $|E_+|^2$ (b) Initial (dotted lines) and final (solid lines) intensity profiles of the polarisation component $E_-$. 
through into the other polarisation component, the PDW shifts away to compensate for the fact that a ‘hole’ has traversed from one polarisation to the other. The effect on the grey solitary wave is more interesting. The phase change across the solitary wave remains constant, but it picks up a bright component in $E_-$ as well, and forms a grey-bright solitary wave. This bright component reduces the propagation angle. The formation of this state suggests that the results obtained for bright-dark solitons of the Manakov model carry over to the nonintegrable model. These simulations show the domain walls to be very robust and is most encouraging for their eventual observation.

4.4 Domain walls in birefringent materials

4.4.1 Stationary states

The approach of this section is less motivated by physical reasoning than by the simple desire to generalise. We use results from the previous chapter and transpose to media in which there is some birefringence. For this reason we postulate directly the form of the solutions, and utilise the equivalent potential method of Haus (1966) without introduction. These equivalent potentials show us that a number of hitherto unrecognised vector solitary waves exist, and in particular show the form of the PDWs for different levels of birefringence.

In this section we deal with light in a birefringent medium. Because, in this sort of material, light of different linear polarisations travels at different velocity, it is no longer convenient to consider the circularly polarised basis set. In normalised units the propagation of the linear polarisation components of an electric field in Kerr media is described by the CNSE (1.22)

$$i\partial_z E_\pm + \frac{1}{2} \partial_z^2 E_\pm - [(1 - B)|E_\pm|^2 + (1 + B)|E_\mp|^2] E_\pm + \delta E_\mp = 0,$$

(4.37)

where all the symbols take their usual meanings. For the temporal problem we are neglecting the difference in the group velocity between the modes, a somewhat contentious action described in Wabnitz, Wright, and Stegeman (1990). For the spatial problem we are ignoring the higher order effects of birefringence and assuming an isotropic form for the $\chi^{(3)}$ tensor.

We look for bound solitary wave solutions in which both polarisation components have the same propagation constant and are $\pi/2$ out of phase: $E_+ = e_+e^{-i\beta z}$ and $E_- = e_-e^{-i\beta z}$ where the envelopes $e_+$ and $e_-$ and the propagation constant $\beta$ are real. Substitution of these forms of $E_+$ and $E_-$ into eq. (4.37) leads (as always!) to a coupled set of real ordinary differential equations, that describe motion inside a potential that is given by

$$V(e_+, e_-) = \beta(e_+^2 + e_-^2) - \frac{1}{2}(e_+^2 + e_-^2)^2 + 2Be_+^2e_-^2 + \delta(e_+^2 - e_-^2).$$

(4.38)

This potential possesses maxima and saddle points that correspond to uniform (i.e plane wave or continuous wave) stationary solutions to eq. (4.37). The polarisation states of these waves constitute the eigenpolarisations of the birefringent Kerr material. Setting $\nabla V = 0$ in eq (4.38) we can easily calculate these points. We find four saddle points on the $e_+$ and $e_-$ axes: $(e_+ = \pm \sqrt{\beta + \delta}, e_- = 0)$ and $(e_+ = 0, e_- = \pm \sqrt{\beta - \delta})$. They represent the LP eigenstates, parallel to the fast and slow axes of the birefringent material. The potential $V$ also possesses four maxima that are of much greater interest to us

$$e_+ = \pm \left[\frac{1}{2} \left(\frac{\beta}{1 - B} + \frac{\delta}{B}\right)\right]^\frac{1}{2}, \quad e_- = \pm \left[\frac{1}{2} \left(\frac{\beta}{1 - B} - \frac{\delta}{B}\right)\right]^\frac{1}{2}.$$
Figure 4.16: Image plots of numerical simulations of collisions between a grey soliton and a polarisation domain wall. The intensity of green in the image corresponds to $|E_+|^2$ while red corresponds to $|E_-|^2$. The grey soliton is introduced in the field $E_+$ and it is adjusted to match the background of the domain wall. (a) The grey soliton parameter $\phi = 0.1$, and the grey soliton is repulsed by the PDW that propagates without change. (b) At higher angles of incidence ($\phi = 0.3$) the solitary waves pass through one another.

Figure 4.17: The linearly polarised solitons of birefringent ($\delta = 0.4$) Kerr media, shown superposed on the contour and surface plots of the potential $V$. 
These are the elliptical eigenpolarisations of the material. In the absence of birefringence ($\delta = 0$) they are pure CP states (i.e. $e_+ = \pm e_-\cdot$ The solitary wave solutions of (4.37) correspond to the separatix trajectories connecting either saddle points or maxima. The separatrices that connect the pairs of opposite saddle points correspond to the LP NSE dark solitons. Setting $e_- = 0$ and $e_+ = 0$ in eq. (4.37), we find the dark soliton solutions of the NSE appearing as $e_x, e_y = \gamma \tanh(\gamma x)$ where $\gamma = \sqrt{\beta \pm \delta}$ for $e_x$ and $e_y$. Figure 4.17 shows the profiles and the trajectories of this potential. In isotropic media, ($\delta = 0$) the separatrices that connect opposite maxima lie on the bisecting lines $e_+$ or $e_- = 0$ and correspond to the circularly polarised NSE dark solitons. Setting $e_x = 0$ in eq (4.37) we obtain $e_\pm = \gamma \tanh(\gamma x)$ where $\gamma' = \beta\pm/[2(1-B)]$.

As shown in eq. (4.39) the eigenpolarisations of birefringent Kerr materials are no longer simple CP states. Figure 4.18 shows, at left, contour plots of the potential $V(e_+, e_-)$ for $\delta = 0.2$ and $B = \frac{1}{3}$. We see that the birefringence breaks the four-fold symmetry of the isotropic potential into a twofold symmetry with respect to the $e_+$ and $e_-$ axes. As a result, the saddle points are still on those axes, and the corresponding separatrices give the linearly polarised NSE solitons mentioned above. The maxima, however, are now closer to the $e_+$ axis. In fact, from eq. (4.39) it is apparent that the maxima merge with the saddle points of this axis if $\delta$ becomes larger than $\beta B/(1 - B)$. Note that the normalised intensity $I = e_+^2 + e_-^2$ at the maxima is given by $I = \beta/(1 - B)$. This shows that the intensity of the waves and the birefringence of the material must be such that $\delta < BI$ in order to have polarisation domain walls in isotropic media. The trajectories that connect opposite maxima (and that correspond to CP NSE solitons in the small birefringence limit) no longer lie on the bisecting lines $e_+ = \pm e_-$ and must be calculated numerically. They are represented by the red lines in figure 4.18. We see that the solitary wave consists of a transition between two uniform waves of the same amplitude and eigenpolarisation but of opposite signs, (i.e as with the NSE dark soliton there is an abrupt $\pi$ phase shift in both $e_+$ and $e_-$). This soliton constitutes a generalisation of the circularly polarised NSE black solitons to birefringent materials.

In the presence of birefringence there are two types of domain wall trajectory that connect adjacent maxima. They are represented by the green and red lines of figure 4.18 and constitute a domain wall between two uniform waves of counter-rotating elliptical eigenpolarisations. The two types can be distinguished by the nodes in the CP components. The PDWs of type I are confined completely to a single quadrant in the $(e_+, e_-)$ plane and do not have any nodes. Type II PDWs (as we christen them) have a single node in each polarisation component. One may think of type I PDWs as connecting those maxima that are converging, while the type II PDWs connect those that move apart as the birefringence increases. As the birefringence increases the contrast of the type I PDW goes to zero and for $\delta > B$ it has collapsed to a plane wave. The type II and the vector black soliton converge to a linearly polarised black soliton as the birefringence increases.

4.4.2 Dynamics of domain walls in birefringent media

In isotropic materials the dynamics of the PDW are very simple: it is extremely robust and cannot be deviated from its course. This is because the PDW is intimately tied to conserved quantities of the CNSE, rather like the soliton eigenvalues are conserved quantities of integrable equations. In this case the conserved quantities are the total integrated power in each polarisation component. The conservation of this quantity means that the PDW is constrained — any PDW movement in the $x$ direction would violate the conservation law. In birefringent materials the power in each polarisation is no longer conserved independently. Therefore we expect much more interesting propagation dynamics. And we are not disappointed!

There are three classes of objects whose interactions are to be characterised. The two
Figure 4.18: the potential surfaces and the soliton trajectories for (a) $\delta = 0.1$ (b) $\delta = 0.2$ (c) $\delta = 0.4$. 
types of PDW (those with and those without nodes) and the black soliton generalisation of the PDW. We emphasise that the coupled equations (4.37) are not well studied, and the only known solitary waves are those presented above. Notably absent from the list of known solitary waves are any kind of grey kink soliton that moves at a different velocity to the background wave. In addition, we anticipate the existence of a more general class of PDW that does the same.

With this in mind, we have conducted several simple simulations designed to give some insight into the behaviour of solitary waves in this system, and to give some clue to any other solitary waves that might exist. This is not an exhaustive study that will characterise in a rigorous manner the influence of all the relevant parameters. For simplicity we refer to the PDW without any nodes in the CP profiles as a type I PDW. This is the PDW whose contrast tends to zero as the birefringence increases. The type II PDW has a greater contrast, and possesses zeroes. Figure 4.19 shows the interactions of identical solitary waves. The PDWs of type I attract one another and eventually annihilate themselves. Those of type II are repelled, with a smaller repulsive force than the black solitary waves that also repel. Note that in the first two cases the interaction created a PDW solitary wave that was travelling across the background. The beam propagation method is not a suitable tool for characterising new solutions; nevertheless we note the existence of this sort of PDW in birefringent systems. Similarly the vector black soliton evolves into a vector grey soliton moving at a different velocity than the background.

The interactions between different solitary waves are not so easily characterised in terms of attraction and repulsion. The introduction of the birefringence term changes the concept of conserved momentum which widens the interaction possibilities. Figure 4.20 shows the evolution of initially overlapping solitary waves. Type I and type II PDWs both move in the direction of the type I PDW, with steadily increasing separation. This can be interpreted as a type II being attracted to a type I, but the type I being repelled by a type II. A type I PDW and a black solitary wave form a bound state with unchanging separation, but moving relative to the background in the direction of the black solitary wave. The type I PDW is attracted to the black solitary wave but repels it.

The interaction force between type II PDWs and the black solitary wave depends on their separation. From close range, there is repulsion, but from further away the they are attracted to each other. Figure 4.20(c) shows how, from close range, the two solitary waves break apart. Note the beating phenomena that is apparent in the grey vector soliton on the right of this picture. It is possible that a more sophisticated ansatz approach could find these solutions. Figure 4.21 shows that when the initially overlapping solitary waves are further apart, they form a bound state.

4.5 Polarisation modulational instability and domain walls

4.5.1 Introduction

Modulational instability (MI) is a general characteristic of wave propagation in non-linear dispersive media and is a common occurrence in such diverse fields as plasma physics, fluid dynamics and nonlinear optics. It refers to the physical process in which a weak periodic perturbation of a uniform intense carrier wave grows exponentially as a result of the interplay between dispersion and nonlinearity. In the optics context special attention has been paid to MI in Kerr media in the scalar approximation where light propagation is described by the NSE. In this approximation MI is only observed for self-focussing (or anomalous GVD, in the case of optical fibres) media. This condition is also necessary for the existence of bright solitons. Using direct substitution methods, Akhmediev and Korneev (1986) and Akhmediev, Eleonskii, and Kulagin (1987) have
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Figure 4.19: Interactions between identical solitary waves of birefringent systems. $\delta = 0.2$. The initial conditions consist of (a) two overlapping type I PDWs, (b) two type II PDWs and (c) two vector black solitary waves.

Figure 4.20: The interactions between the three different types of solitary waves that exist in birefringent systems. $\delta = 0.2$ (a) a type I (on the right) and a type II PDW, (b) a type I (right) and vector black solitary wave, (c) a type II PDW (left) and a black solitary wave, of small initial separation.

Figure 4.21: The evolution of a type II PDW and a vector black solitary wave, when the initial separation is large. The two solitary waves form a bound state.
developed an analytical approach to MI for the NSE. A general three parameter family of first order solutions to the NSE was given that ranged from the solitary waves through stationary states to space and time periodic solutions. In this way, a direct and explicit link between MI and the fundamental bright NSE soliton was established for the first time.

It has been known since the early study of Berkhoer and Zakharov (1970) in the context of plasma physics that incoherent coupling (brought about through XPM) in the CNSE leads to an extension of the frequency domain of MI to defocussing systems, or the normal dispersion case. XPM is a general phenomenon characteristic of the simultaneous nonlinear propagation of several waves belonging to different optical modes. It has been shown, in the context of fibre optics, that MI in the normal dispersion can also occur through XPM between waves of different frequencies. Agrawal (1987) foresaw the fundamental importance of this phenomenon when he conjectured that a soliton must exist that is associated with PMI in the same way as the bright soliton is associated with scalar MI. Analogous to the work of Akhmediev et al. (1987) we establish a direct link between XPM induced PMI and the associated soliton. The incoherently coupled NSE being non-integrable, most of our developments are based on numerical simulations, although an approximate analytical approach to the problem is used as a guide in the search for new solutions.

4.5.2 Modulational instability and the CNSE

We consider the propagation of light of arbitrary polarisation in isotropic Kerr media and once again find ourselves dealing with the CNSE of eq. (4.17). Since the pioneering work of Berkhoer and Zakharov (1970), MI of the CNSE and generalised forms has been extensively studied in the literature. For the sake of clarity, we only give here the main steps of the development of the linear stability analysis of the problem of interest to us, namely, the stability of the linearly polarised plane wave. The linearly polarised solution to the CNSE reads

\[ E_+ = E_0 e^{i\Omega t} = E_0 e^{i\Theta_0 z}. \]

The stability of this solution against the growth of a periodic perturbation is examined by introducing to the CNSE the ansatz

\[ E_{\pm} = (a_\pm + \epsilon_{\pm}) e^{-i\omega t} e^{i\Theta_0 z} \]

where \[ \epsilon_{\pm} = \epsilon_{\pm} e^{i\lambda_0 \cos(\Omega t)} \]

Linearising with respect to \[ \epsilon_{\pm} \]

leads to a characteristic polynomial of the fourth degree in \( \lambda \). Among the corresponding four eigenvalues \( \lambda \), only two are potentially unstable. They are

\[ \lambda_1 = \Omega \sqrt{-\sigma P_0 - \frac{1}{4} \Omega^2} \]

\[ \lambda_2 = \Omega \sqrt{\sigma B P_0 - \frac{1}{4} \Omega^2}, \]

where \( P_0 \) is the normalised power of the initial plane wave polarisation components \( P_0 = |E_0|^2 \). The first eigenvalue \( \lambda_1 \) corresponds to MI in the anomalous dispersion regime, \( \sigma = -1 \). The gain maximum \( \lambda_{1(max)} \) is obtained at the frequency \( \Omega_{1m} = \sqrt{-2\sigma P_0} \). Note that this instability is independent of the coefficient \( B \), i.e., it does not involve the tensor character of the Kerr nonlinearity. Moreover, the eigenvector associated with \( \lambda_1 \) at the optimal frequency \( \Omega_{1m} \) has the form

\[ v_1 = (\Re(\epsilon_+), \Im(\epsilon_+), \Re(\epsilon_-), \Im(\epsilon_-))_{\lambda=\lambda_1} = (1, 1, 1, 1), \]

which shows that, under the effect of the instability, both polarisation components exhibit the same evolution. In other words, as is well known, MI of the CNSE in the anomalous dispersion regime does not involve any changes in the linear polarisation state of the field and does not differ from MI in the scalar NSE. The theory of Akhmediev and Korneev (1986) can then be applied, and, naturally, the soliton associated with
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Figure 4.22: Evolution of the intensity profiles of the two linearly polarisation components $E_\pm(x,z)$ of an initially linearly polarised plane wave slightly perturbed by a periodic perturbation at the optimal frequency $\Omega_{2m}$, for $B = \frac{1}{3}$. One can verify the formation of two identical but out of phase periodic structures in $E_\pm$.

This instability is the usual bright NSE soliton which represents here a pulse of constant and uniform linear polarisation.

The situation is fundamentally different for the instability governed by the second eigenvalue $\lambda_2$. We see from eq. (4.40b) that, contrary to MI of the scalar NSE, this instability only occurs in the normal dispersion regime, $\sigma = +1$. Its maximum gain is $\lambda_{2m} = BP_0$, which corresponds to the optimal frequency $\Omega_{2m} = \sqrt{2\sigma BP_0}$. The role of the coefficient $B$ shows that MI, in this case, depends intrinsically on the tensor character of the Kerr nonlinearity. The smaller the value of $B$, the smaller the PMI gain at its optimal frequency. The instability disappears in the absence of nonlinear birefringence – when $B = 0$. Another important difference with respect to scalar MI is revealed by analysis of the eigenvector associated with $\lambda_2$. At the optimal frequency $\Omega_{2m}$, this eigenvector has the form $\nu_2 = (1, 1, -1, -1)$, which indicates that PMI induces the exponential growth of periodic perturbations of opposite sign in the two circular polarisation components of the initial linearly polarised wave. As a consequence, one may expect that, up to the nonlinear stage in the evolution of the instability, the envelopes of the two circularly polarised fields exhibit two identical but $\pi$ out of phase periodic structures. Since this instability involves a change of the state of polarisation of the field, it it called polarisation modulational instability, or PMI. Figure 4.22 shows the onset of PMI, obtained by numerical simulation of eq. (4.17) for $B = \frac{1}{3}$. The initial conditions correspond to a linearly polarised plane wave slightly perturbed by a periodic signal at the optimal frequency $E_\pm = 1 \pm \epsilon \cos(\Omega_{2m} x)$, where $\epsilon = 10^{-3}$. Periodic boundary conditions are considered. As predicted from the linear stability analysis, we verify in figure 4.22 the formation of two $\pi$ out of phase identical periodic structures in both polarisation components. Note that this result is general and does not depend on the type of initial perturbations. For instance, when seeding PMI with a white noise, the spectral components of the noise at the optimal frequency are selected in both polarisations and, in agreement with the form of the eigenvector $\nu_2$, they grow with a phase of $\pi/4$ and $5\pi/4$ with respect to the the initial signal, giving rise to $\pi$ phase-shifted structures.

Let us note finally that the linear stability analysis applied to continuous waves of pure circular polarisation shows that such waves obey the dynamics of the scalar NSE and therefore are modulationally stable in the normal dispersion regime.
4.5.3 Truncated three wave model

Complete solution families for the NSE were found by Akhmediev, Eleonskii, and Kulagin (1987) for the scalar NSE that include both PMI and soliton solutions. Thanks to this analytic solution, we can now formulate the link between the bright soliton and MI as follows. The dynamics of MI is described by the space and time periodic solutions of the family in a given range of the parameters. Varying the family parameters shows that stationary periodic solutions constitute limiting states of these complex solutions. Since the stationary periodic solutions have the form of the elliptic function of the first kind \( cn \), they have been called cnoidal waves. As discussed in §3.2.2, the elliptic cosine tends to a periodic train of pulses whose envelopes become the hyperbolic secant functions corresponding to bright NSE solitons in the limit \( T \to \infty \).

Ideally, in order to find the general soliton associated with PMI we should look for a similar general family of solutions of the CNSE (4.17). Seeing the complexity of the these equations as compared to the single NSE, this task appears rather difficult, especially given their nonintegrability. Even a numerical approach to the problem, similar to the one originally applied to the NSE (Akhmediev and Korneev 1986), would be difficult to develop without further information on the dynamical features of eq. (4.17). This is the reason we propose here an approximate model for the dynamical features of eq. (4.17). This model provides a simple qualitative picture of the dynamics of periodic solutions of the CNSE. In particular, it reveals the existence of space and time periodic solutions as well as stationary periodic solutions. The predictions of this model are used then as a guide in the numerical search for the exact stationary periodic solutions.

Since the early work of Infeld (1981) on the theory of FPU (Fermi, Pasta, and Ulam 1955) recurrence in the NSE, it is known that much physical insight into the complex dynamics of MI can be obtained by means of the Galerkin approximation. For the NSE this approximation leads to a truncated three wave model which reduces the complexity of the dynamics to that of the conservative motion of a particle in a one-dimensional potential (Trillo and Wabnitz 1991a; Trillo and Wabnitz 1991c; Capellini and Trillo 1991). Three wave truncated models were also proven by Haelterman, Trillo, and Wabnitz (1992, 1993) to be efficient in the study of MI in optical cavities described by the driven and damped nonlinear Schrödinger equations. The three wave truncation approach to the problem of modulational instability in the CNSE was first proposed by Trillo and Wabnitz (1991b) for the study of nonlinear modulation of coupled waves in birefringent optical fibres. We show in the following that this approach is suitable to the study of periodic solutions of eq. (4.17).

The stability analysis of §4.5.2 and more particularly the form of the eigenvector \( v_2 \) as well as the results of the numerical simulation illustrated in figure 4.22, suggest that it should be possible to describe PMI of eq. (4.17) by means of a three mode Fourier truncation of the form:

\[
E_{\pm}(x, z) = E_0(z) \pm \sqrt{2} E_1(z) \cos(\Omega x),
\]

(4.42)

where \( \Omega \) is the frequency of the temporal patterns. Naturally this truncation is only valid as long as higher harmonics do not significantly influence the system dynamics. In order to ensure that higher harmonics are not linearly unstable we impose the condition \( \Omega > \Omega_c/2 \) where, according to eq. (4.40b), \( \Omega_c = 2\sqrt{BP_0} \) is the total PMI bandwidth or the PMI cut-off frequency (we set \( \sigma = +1 \) from now on). Note that in eq. (4.42) we have assumed equal amplitudes for the two sideband waves [or Fourier components \( e^{i\pi x} \) and \( e^{-i\pi x} \)]. In the following we call \( E_0 \) and \( E_1 \) the amplitudes of the pump and sideband waves respectively. With the notation of eq. (4.42) the power carried by each polarisation component is given by \( P = |E_0|^2 + |E_1|^2 \). It is important to note that the ansatz (4.42) is consistent with eq. (4.17) in the sense that the substitution of \( E_+ \) and \( E_- \) as given by eq. (4.42) into the first equation eq. (4.17) leads to the same set of ODEs.
as their substitution into the second. This unique set of ODEs appears as

\[ i \partial_t E_0 + |E_0|^2 E_0 + (1 - B)|E_1|^2 E_0 - B E_1^* E_0^* = 0 \]  
\[ i \partial_t E_1 + \frac{1}{2} \Omega^2 E_1 + (1 - B)|E_0|^2 E_1 - \frac{3}{2}|E_1|^2 E_1 - B E_0 E_1^* = 0. \]

Consequently, although the original model consists of two coupled NSEs, the truncated model has a form analogous to that derived for the single NSE. This result is important because it means that PMI can be described in terms of the dynamics of only two coupled Fourier modes. Moreover, it is easy to see from eq. (4.44) that the total power \( P \) is a conserved quantity and that the power flow between the pump and sidebands only depends on their relative phase. Introducing the powers \( P_0 \) and \( P_1 \) and the phases \( \phi_0 \) and \( \phi_1 \) of the pump and sideband waves through the relations \( E_k = \sqrt{P_k} e^{i \phi_k} \) for \( k = 0, 1 \), we can then reduce the model to a set of two differential relations of the real variable \( P_1 \) and \( \phi = \phi_0 - \phi_1 \), namely

\[ \frac{dP_1}{dz} = 2B(P - P_1)P - 1 \sin(2\phi) \]
\[ \frac{d\phi}{dz} = -\frac{1}{2} \Omega^2 + BP - 2 \left( B + \frac{1}{4} \right) P_1 + B(P - 2P_1) \cos(2\phi). \]

These two self-consistent coupled real equations describe the physical content of the approximation model eq. (4.44) [i.e. the only information lost when deriving eq. (4.45) is the absolute phase of the field]. In order to investigate the existence of eigensolutions (as fixed points) and the dynamics of eq. (4.45), it is convenient to formulate the problem as a Hamiltonian system first done in Trillo and Wabnitz (1991b). Introducing the normalised sideband power \( \eta = P_1/P \) it is easy to see from eq (4.45) that the variables \( \eta \) and \( \phi \) are conjugate through the Hamiltonian

\[ H = (\kappa - B) \eta + B(\eta - 1) \eta \cos(2\phi) + (B + \frac{1}{4}) \eta^2, \]

where \( \kappa = \Omega^2/(2P) \), and thus obey the equations

\[ \frac{d\eta}{d\xi} = \frac{dH}{d\phi} \quad \text{and} \quad \frac{d\phi}{d\xi} = \frac{dH}{d\eta}, \]

where we have introduced the scaled longitudinal coordinate \( \xi = zP \). The dynamical features of PMI and, more generally, of the periodic solutions of the CNSE are qualitatively described by this simple one dimensional Hamiltonian model. This model is remarkably similar to that derived in Haelterman, Trillo, and Wabnitz (1992) for the study of MI and three wave mixing in Kerr media in the scalar approximation. In a given Kerr material, i.e. for a given value of the Hamiltonian (4.46) there is one degree of freedom determined by the parameter \( \kappa \). In the following we consider the value \( B = \frac{1}{3} \) which corresponds to the nonlinearity of optical silica fibres.

Following the methodology of Haelterman, Trillo, and Wabnitz (1992) and Trillo and Wabnitz (1991b) and related works, it proves convenient to characterise the dynamics by means of a phase space representation. We consider the phase space of polar coordinates \((\eta, \phi)\) in such a way that the sideband power \( \eta \) and the relative phase \( \phi \) are simply represented by the modulus and angle of the phase point, respectively. The origin of this plane therefore represents a linearly polarised plane wave [i.e. \( E_1 = 0 \) in (4.42)]. The trajectories of eq. (4.46) are the contour lines of \( H \) in the phase space. They are plotted in figure 4.23 for the particular case of the optimal PMI frequency i.e. \( \kappa = \Omega_{\text{opt}}/(2P) = B \) [note that in eq. (4.40b) the continuous wave power \( P_0 \) is equal to the total wave power \( P \) since the sideband amplitude is assumed to be arbitrarily small in the stability analysis]. A simple glance at figure (4.40b) allows us to see that the dynamical behaviour of PMI is similar to that of MI in the scalar NSE, in particular, the instability of the plane wave (represented by the origin) corresponds to the presence of a hyperbolic unstable
Figure 4.23: Phase space portrait of the dynamics of the space and time periodic solitons given by the approximate truncated Hamiltonian model for the optimal frequency. The orbits are the contour lines of the Hamiltonian $H$ in polar coordinates $(\eta, \phi)$. We verify the existence of the homoclinic orbit corresponding to PMI of the linearly polarised plane wave. The arrows indicate the stable and unstable directions of the origin as obtained from the linear stability analysis of §4.5.2.

fixed point of the Hamiltonian system (Trillo and Wabnitz 1991b). In other words, the system exhibits a homoclinic orbit characteristic of the FPU recurrence in the NSE, first found by Infeld (1981). This orbit is a separatrix that divides the phase plane into two domains. The orbits of these domains represent the two types of space and time periodic solutions of the CNSE (4.17) around the optimal frequency of PMI. Note that, in accordance with the ansatz (4.42), if a point $(\eta, \phi)$ running on an orbit represents the periodic evolution of one circular polarisation component, say $E_+$, then the point $(\eta, \phi + \pi)$ running on the corresponding symmetric orbit represents the evolution of the component $E_-$. The arrows on the separatrix orbit indicate the unstable directions of the origin. The angles $\phi = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ correspond naturally to the expression of the eigenvector $v_2$ obtained from the linear stability analysis.

We have checked the validity of the truncated Hamiltonian model (4.46) by comparing its predictions with the results of direct numerical integration of the CNSE. For the sake of simplicity we set the total power $P$ to unity in such a way that $\xi = z$ and the comparison between both models is straightforward. Figure 4.24 shows the evolution in $z$ of the intensity at $z = 0$ of both polarisation components in the case of a continuous wave slightly perturbed by an amplitude modulation, i.e. with the initial conditions $E_\pm(x, 0) = 1 \pm \epsilon \cos \Omega x$ where $\epsilon = 10^{-2}$, and $\Omega = 1$ ($\kappa = 0.5$). The initial phase $\phi$ is the zero which means that the evolution is described by an orbit inside the homoclinic separatrix. The solid curves in figure 4.24 give the intensities at $x = 0$ obtained from the Hamiltonian model. Using the ansatz (4.42) and expressing it in terms of $\eta$ and $\phi$ [whose evolution is given by eq. (4.47)], we find

$$|E_\pm(x = 0, z)|^2 = 1 + \eta(z) \pm 2\sqrt{2[1 - \eta(z)]}\eta(z) \cos[\phi(z)].$$

(4.48)

The dotted curves are the result of numerical integration of the CNSE (4.17). We verify here the validity of the predictions of the truncated model. In particular we observe that the CNSE exhibits a recurrent behaviour analogous to that of the scalar NSE. Figure 4.25 illustrates the recurrence for the two types of orbits shown in figure 4.23. It
shows the evolution of the intensity profiles of both polarisation components obtained by numerical simulation. Figure 4.25(a) corresponds to the parameters of figure 4.24, whereas figure 4.25(b) corresponds to an outer orbit obtained with an initial phase \( \phi(z = 0) = \frac{\pi}{2} \) i.e. the initial conditions are \( E_{\pm}(x, z = 0) = 1 \pm \epsilon(\Omega x) \) (with the same values of \( \epsilon \) and \( \Omega \) as before). A good qualitative agreement between the CNSE and the truncated model is observed as long as the frequency \( \Omega \) is higher than half the PMI cut-off frequency, \( \frac{1}{2}\Omega_c \). Let us note that the similarity with the scalar NSE dynamics is only qualitative. In fact, due to the coupling between the two out of phase periodic waves, the field envelopes contain less harmonics than in the case of a single nonlinear wave and the truncated three wave model is more accurate for the CNSE than for the scalar NSE. For instance, at the optimal frequency of MI, the energy content of the higher harmonics is of the order of 30% of the total field energy in the case of the NSE while in the present case the discrepancy is only a few percent (see figure 4.24).

The Hamiltonian formulation of the problems as given by eq. (4.46) and (4.47) reveals an important feature of the CNSE, namely, the existence of elliptic fixed points corresponding to steady state solutions of these equations. These fixed points appear in figure 4.23 as the limiting state of the inner orbits for low values of \( H \). They can easily be calculated from eq. (4.46) and (4.47). We find

\[
\eta_e = \frac{4B - 2\kappa}{8B + 1}, \quad \phi_e = 0, \pi.
\]

(4.49)

Naturally the two values of the phase correspond each to circular polarisation components of opposite handedness. The stationary periodic solutions thus consist of the
superposition of two $\pi$ out of phase periodic structures of counter-rotating polarisations. From eq. (4.44), (4.45) and (4.47) it is easy to see that the fixed point solutions $E_+$ and $E_-$ have the form

$$E_\pm = \left[ \sqrt{1 - \eta_e} \pm \sqrt{2\eta_e \cos(\Omega x)} \right] \sqrt{P} e^{i\beta z},$$

(4.50)

where $\beta = P(1 - 2B\eta_e)$ is the propagation constant of the stationary periodic solution. The calculations of the absolute phase ($e^{i\beta z}$) of these solutions is necessary for the comparison with the exact stationary periodic solutions of the CNSE, which will be studied in the next section. Note that the $\pi$ out of phase envelopes of $E_+$ and $E_-$ exhibit no phase variation – they are described by real functions. Figure 4.26 shows the envelopes of $|E_+|$ and $|E_-|$ of the stationary periodic solution (4.50) corresponding to the elliptic fixed points of figure 4.23. (i.e. $\kappa = B = \frac{1}{3}$).

These stationary periodic solutions, which appear as limiting states of the more general space and time periodic solutions, are equivalent to the cnoidal waves of the NSE. The NSE cnoidal waves can be calculated analytically and may be expressed in terms of Jacobian elliptic functions (Infeld 1981). This closed analytic form permitted Infeld (1981) to establish an explicit link between MI and the bright NSE soliton, since, on the one hand, the cnoidal waves appear as limiting states of space and time periodic solutions, and on the other hand they tend to the one soliton solution as their period is increased to infinity. For the CNSE the situation is more difficult because no analytical solutions are known to exist. Moreover, the approximate function derived above only

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**Figure 4.25:** Evolution of the intensity profiles of the two polarisation components obtained by numerical simulation. (a) the initial conditions are those of figure 4.24. i.e. it corresponds to an inner orbit of the phase space portrait. (b) the initial condition corresponds to an outer orbit. We observe in this case the regular alternations between $\pi$ out of phase periodic patterns predicted by the Hamiltonian model.
4.5 Polarisation modulational instability and domain walls

Figure 4.26: Stationary periodic solutions of the CNSE. Solid lines: envelopes of the stationary periodic solutions given of (4.50) for the optimal frequency $\Omega_{2m}$ and $B = 13$; the period is then $T = 2\pi/\Omega_{2m} = 7.7$. Dotted lines: stationary periodic solutions of the same frequency obtained by numerical integration of eq. (4.18) with the value of propagation constant supplied by the truncated model, i.e. $\beta_\perp = 0.88$.

provides a good qualitative description of the periodic solutions of the CNSE in a limited frequency range. Ideally in order to find the soliton associated with PMI we should look for the limiting states of the stationary periodic solutions as their frequency tends to zero. Clearly, this cannot be done with the the truncated three wave model since a decrease of the frequency would lead to the appearance of the dreaded higher harmonics. We then have to study numerically the stationary periodic solutions of the CNSE. This is the purpose of the next section. As we shall see, the information brought by the Hamiltonian model greatly simplifies this task.

To wrap up this section devoted to the approximation of the truncated three wave model we have shown that, in the Hamiltonian representation, PMI corresponds to the existence of a hyperbolic fixed point which is itself associated with two elliptic points representing stationary periodic solutions of the CNSE. The limit in which the period of these solutions tends to infinity provides the soliton associated with PMI.

4.5.4 Stationary periodic solutions and domain walls

We now consider the exact stationary periodic solutions of the CNSE. The approximation method of §4.5.3 reveals that the stationary periodic solutions can be described as the superposition of two out of phase periodic envelopes of counter-rotating circular polarisation. It also shows that, in a limited frequency range, these envelopes are well approximated by in-phase ($\phi = 0$) or $\pi$ out of phase ($\phi = \pi$) pump and sideband Fourier modes. This result is very important because it suggests that the envelopes of the stationary periodic functions can be expressed in terms of real functions. This leads us to make the standard “stationary” ansatz, and arrive at the reduced mechanical problem already met several times before.

Now the linearly polarised plane wave is represented in the potential by the saddle points on the lines $e_+ = \pm e_-$. There are two maxima either side of each saddle point that correspond to the circularly polarised eigenstates. The stationary periodic solutions
Figure 4.27: Periodic trajectories in the plane \((e_+, e_-)\) of the unit mass in the potential \(V\) obtained from the shooting method with \(B = \frac{1}{3}\) and \(\beta_+ = \beta_- = 1\). The shooting parameters are (a) \(a_{\text{max}} = 1.17, a_{\text{min}} = 0.82\); (b) \(a_{\text{max}} = 1.43, a_{\text{min}} = 0.52\); (c) \(a_{\text{max}} = 1.62, a_{\text{min}} = 0.23\); (d) \(a_{\text{max}} = 1.73, a_{\text{min}} = 0.03\); which correspond, respectively, to \(T(a) = 5.6, T(a) = 6.2, T(a) = 7.8, T(d) = 13\).
correspond to the oscillations of our friendly ball around the saddle and up toward the peaks. This is illustrated in figure 4.27 where we plot the trajectories in the plane of \((e_+, e_-)\) of four stationary periodic solutions of different periods for \(B = \frac{1}{3}\) [the contour lines of the potential \(V\) are included in the plots]. They were obtained from the shooting method. For values of the ratio \(a_{\text{max}}/a_{\text{min}}\) close to unity, i.e. for trajectories which lie in a small region around the saddle point, the frequency \(\Omega\) of the oscillations is high, whereas the trajectories obtained with larger values of \(a_{\text{max}}/a_{\text{min}}\) correspond to the oscillations of lower frequency. For very large periods, these trajectories span an angle of almost 90° in the first quadrant of the \((e_+, e_-)\) plane and their extremities approach the maxima of \(V\).

Figure 4.28 shows examples of stationary periodic solutions corresponding to such trajectories at different frequencies. As the period increases, the amplitude \(a_{\text{max}}\) increases while \(a_{\text{min}}\) decreases and the envelopes \(e_+(x)\) and \(e_-(x)\) flatten around these extrema, indicating the appearance of higher harmonics in the solutions. In figure 4.29 we plot the modulation depth \(\Delta = (a_{\text{max}} - a_{\text{min}})/a_{\text{max}}\) of the envelopes as a function of their period. We see that \(\Delta\) tends to unity as the period tends to infinity. Note that under the threshold period, \(T_c = 5.44\), no periodic solutions are found. The existence of this threshold can be foreseen from the linear stability analysis and the Hamiltonian model developed above. When the frequency \(\Omega\) tends to the PMI cut-off frequency \(\Omega_c = 2\sqrt{BP_0}\) of the PMI gain given in eq. (4.40b), the homoclinic orbit of the Hamiltonian becomes arbitrarily small and vanishes when \(\Omega = \Omega_c\). Naturally, the corresponding fixed points tend to the origin or, in other words, the stationary periodic solutions tend to a stable plane wave solution and \(\Delta \to 0\). It is easy to verify that the threshold period corresponds, in agreement with this reasoning, to the cut-off frequency, that is \(T_c = 2\pi/\Omega_c\). In fact, with the choice \(\beta_+ = 1\) and \(B = \frac{1}{3}\) the threshold period is calculated to be \(T_c = 5.44\) in agreement with the results of figure 4.29.
Figure 4.29: Modulation depth $\Delta$ of the stationary periodic solutions as a function of their period $T$. A period threshold $T_c$ is observed under which the stationary periodic solutions no longer exist. From the linear stability analysis we find $T_c = 2\pi/\Omega_c$, which, for $B = \frac{1}{3}$ and $\beta_- = 1$ considered here, gives $T_c = 5.44$.

Figure 4.28 shows that in the other limit, at very low frequencies, the stationary periodic solutions consist of an alternation between quasi-plane wave domains of counter-rotating circular polarisations. It is clear that in the limit of infinite period, that this will become the polarisation domain wall of section 4.4. It is therefore clear that the PDW is related to PMI, at opposite ends of some solution family, for the CNSE in the same way that the bright soliton is linked to MI for the scalar NSE. In this sense we might say that the PDW is a more fundamental soliton (at least for the CNSE) than is the standard black soliton. That, of course, is a matter for conjecture.

This section highlights, in a very direct way, the limitations of the analysis as it currently stands. Since the NSE falls into the integrable basket, we are able to generate wonderful closed form solutions in terms of transcendental functions of one form or another. Thus one can find the soliton and the modulational instability at opposite ends of a single solution family. However since the CNSE falls outside this camp, we are forced to utilise approximations and computer simulations to deduce that there is a solution family of as yet unknown form linking the polarisation domain wall with polarisation modulational instability.
Chapter 5

Kink Waves in (2+1) Dimensions

In this chapter we extend much of analysis conducted in the last two chapters to systems with two transverse dimensions. The equations representing these systems are not integrable, and therefore we cannot find analytical solutions at all. On account of the robustness of the kink-type solitary waves\(^1\) our attention is restricted to defocussing media, in which the simplest scalar solitary wave is an optical vortex. We consider systems with several copropagating components and find new, cylindrically symmetric solution families and determine the bifurcation points at which they originate. Investigating the propagation dynamics of these solitary wave structures, we find them to be stable. Nevertheless, far from being robust, they evolve in fascinating ways.

5.1 Kink waves of cylindrical symmetry

5.1.1 Scalar states: vortices

Askaryan (1962) and, later, Chiao, Garmire, and Townes (1964) predicted that a light beam of a particular power propagating in certain nonlinear materials may trap itself. This self trapping was experimentally observed in CS\(_2\) liquid by Garmire, Chiao, and Townes (1966). However, Kelley (1965) found that cylindrical light beams propagating in self-focussing materials suffer from collapse which is only halted by higher order processes that are frequently a consequence of material damage. The experimental observation of self-trapped beams was therefore relying on some sort of saturation of the nonlinearity. Catastrophic self-focussing, called filamentation since a multi-moded laser beam breaks into many tiny filaments, was first observed by Hercher (1964) who saw long threads of damage after focussing pulsed laser beams onto a variety of materials. The fundamental solitary wave in a focussing material collapses to a single filament and therefore is not stable in a Kerr material. No such effects are observed in defocussing media, whose most fundamental self trapped beam is a stable kink wave called a vortex solitary wave. The nature of this kink solitary wave was first realised by Snyder, Poladian, and Mitchell (1992) who considered the modes of circular fibres as idealised threshold law nonlinearities. Almost simultaneously, Swartzlander and Law (1992) succeeded in observing the vortex and, with supercomputer simulations, uncovered fascinating propagation dynamics, showing that optical vortices behave in many ways like fluid mechanical vortices. The optical vortex appears to be related to the transverse modulational instability (MI) of a dark soliton stripe in the same way that bright solitons are related to MI of the plane wave (see section 4.5). This relationship has been clarified somewhat by Pelinovsky, Stepanyants, and Kivshar (1995) in an analytical and numerical study of the (2+1) dimensional defocussing NSE.

\(^1\)Most of the bright vector solitary waves found in chapter 3 are unstable, while the kink solutions are invariably quite robust. It therefore seems prudent to concentrate on kink solitons for this study.
It is usual to consider the optical vortex solitary wave in bulk Kerr media in which light propagation is described by the \((2+1)\) dimensional NSE. We write this equation here in cylindrical polar co-ordinates:

\[
\frac{i}{\partial z} \frac{\partial E}{\partial z} + \frac{1}{2} \left( \frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E}{\partial \varphi^2} \right) - |E|^2 E = 0. \tag{5.1}
\]

This is a simplified form of eq. (1.22) and can be obtained from it by setting \(E_- = \delta = B = 0\). We look for a stationary, cylindrically symmetric solution to this equation by investigating solutions of the form

\[
E(r, \varphi, z) = e(r) e^{i\varphi} e^{i\beta z}. \tag{5.2}
\]

\[\text{Figure 5.1: The optical vortex solitary wave. At the left we show the radial intensity profile } |e(r)|^2 \text{ while on the right we visualise the complex valued function } E(r, \varphi) \text{ with a flow type diagram. The size of the arrows indicate the magnitude of the field, while their direction gives the phase. An arrow pointing vertically represents a real value, one horizontally an imaginary value.}\]

It is not possible to find analytical solutions to the ensuing problem; Swartzlander and Law (1992) and subsequent works have solved it numerically although one could use a variational method to find approximating functions. Figure 5.1 shows, at left, the radial profile and at right, the transverse form of the field, shown as a flow protrait to emphasise the vortex-like appearance. While this solution lacks any explicit vorticity\(^2\), anyone who has observed a numerical simulation showing these structures will be convinced that the vorticity is more than skin deep. A planar \((1+1)\) dark soliton stripe embedded in \((2+1)\) dimensions breaks into numerous optical vortices like a pressure wave in gas dynamics succumbs to turbulence (Swartzlander and Law 1992). An optical vortex solitary wave induces a waveguide for itself, and the field profile is the second mode of this waveguide at the point of cut-off. There is no known grey soliton family associated with the vortex soliton; no scalar kink wave in three dimensions can move at a different speed than its carrier wave.

5.1.2 Vector states and bifurcations of vortices

In previous chapters we have started by considering the known soliton solutions of the NSE and generalised them to vector solitary waves. In particular we located bifurcation

\(^2\)Typically a physical process must exhibit swirling motion around a singular point to earn the title 'vortex'. The optical vortex, by contrast, possesses nothing but the form that any complex function of two real variables takes at a simple zero. As will be seen in future figures, one can represent an optical vortex in several different ways using a flow diagram and only one of them looks like a vortex.
points that signalled the existence of new solution families. These new solitary waves arise from the mutual trapping between orthogonal polarisations [an early discussion of this concept can be found in Menyuk (1987)]. Solitary waves, of all shapes and sizes, in which self phase and cross phase modulation conspire to allow stationary propagation have now been found in a variety of media, including $\chi^{(2)}$ materials (DeSalvo, Hagen, Shiek-Bahae, Stegeman, and Stryland 1992; Hayata and Koshiba 1993). Examples of these states are bright higher order bound states (Tratnik and Sipe 1988; Haelterman and Sheppard 1994), mixtures of bright and dark modes (Haelterman and Sheppard 1994; Buryak and Kivshar 1995) and dark-dark states (Kivshar and Turitsyn 1993b).

Of particular interest in self-defocussing media are polarisation domain walls (PDWs), introduced in the last chapter. The polarisation state switches in the region of the PDW. Spatial PDWs were first described analytically (Zakharov and Mikhailov 1987) for an integrable system that describes counterpropagating beams in which the Kerr self effect has been ignored. In the last chapter we found PDW solitary waves that exist in a general class of nonintegrable CNSE that describe multi component wave propagation in many physical systems.

We wish to extend the concept of mutual trapping between different components of a light wave to (2+1)-dimensional systems and find solitary waves of the PDW type that bifurcate from optical vortices. Two novel varieties of circularly symmetric solitary waves will be identified. One type is found by examining limits in which one polarisation is a vortex soliton and the intensity in the other polarisation is arbitrarily small. By the same simple physical reasoning used in chapters 3 and 4 the existence of branching points is foreseen. New solitary waves bifurcate from the vortex solitons at these points, and one can then trace the full length of the solution branches associated with the new solitary waves.

Far from the bifurcation the solutions are composed of two polarisation domains of orthogonal polarisation separated by a circular PDW. Analysis of these limiting states leads us to deduce the existence of the second variety of solitary wave. Following this, we consider colour domain walls that arise in light beams that are uniformly polarised but not monochromatic. Finally we present the results of numerical investigations into the dynamics and stability of these new structures.

Looking for cylindrically symmetric solitary waves in bulk, isotropic defocussing Kerr type materials, we take $\delta = 0, \sigma = -1$ in equation (1.22) to give

\begin{equation}
\begin{aligned}
  i \partial_z E_\pm + \frac{1}{2} \left( \partial_r^2 E_\pm + \frac{1}{r} \partial_r E_\pm + \frac{1}{r^2} \partial_\varphi E_\pm \right) - [ (1 - B) |E_\pm|^2 + (1 + B) |E_\mp|^2 ] E_\pm &= 0
\end{aligned}
\end{equation}

Note that normalised units are used; $E_\pm(r, \varphi, z)$ are the counter-rotatating CP field components, $z$ is the space coordinate along the propagation axis, $r$ and $\varphi$ are cylindrical transverse co-ordinates and the birefringence is zero. Although numerical methods are necessary to find solitary wave solutions to these nonintegrable equations, some insight into the kind of solutions we can expect will shed light on the numerical analysis that must be undertaken. An optical vortex is the second mode, at cut-off, of the cylindrical waveguide that it induces through self phase modulation (SPM) (Snyder, Poladian, and Mitchell 1992). Proceeding in the same manner as in previous chapters, we use this waveguide, via the mechanism of cross phase modulation (XPM), to guide an arbitrarily weak probe beam of the other polarisation component. In this way a bound state that comprises both field components is formed. One anticipates that when the power in the probe is increased into the nonlinear regime, this bound state can still exist, at the expense of a reshaping of the profile of the pump beam. It will then describe a new type of solitary wave, in which a beam of finite extent (the bright part) is guided by an orthogonally polarized beam of effectively infinite extent (the dark part). These new solutions can be said to bifurcate from the vortex soliton solution branch.
This reasoning leads us to propose the existence of cylindrically symmetric stationary solutions of the form

$$E_{\pm}(r, \varphi, z) = e_{\pm}(r)e^{i\ell_{\pm}\varphi} e^{-i\beta_{\pm}z}$$  \hspace{1cm} (5.4)

where the envelopes $e_{\pm}$ are real functions, $\beta_{\pm}$ their respective propagation constants and $l_{\pm}$ the azimuthal numbers. Substitution into eqs. (5.3) yields the real ODEs

$$2\beta_{\pm}e_{\pm} + e''_{\pm} + \frac{1}{r} \partial_{r} e_{\pm} - \frac{l^2}{r^2} e_{\pm} - 2 \left[ (1 - B)e_{\pm}^2 + (1 + B)e_{\mp}^2 \right] e_{\pm} = 0$$ \hspace{1cm} (5.5)

We cannot express this problem in the Hamiltonian formulation familiar from previous chapters. There does exist an equivalent potential that is exactly the same as for the (1+1) problem, but the motion of a ball within this potential is no longer conservative. The term with coefficient $1/r$ represents a friction term decaying with time, while the $l^2/r^2$ term represents a time decaying repulsion from the origin $(e_{+}, e_{-}) = (0, 0)$. For this reason one cannot derive much value from the mechanical picture and we do not use it.

Consider an optical vortex, with azimuthal dependence $e^{i\varphi}$, to be formed in the field component $e_{+}$. Without loss of generality we choose $\beta_{+} = 1$. Since XPM is greater than SPM (for $B > 0$), the soliton induced waveguide (as seen through XPM by the other arbitrarily weak polarisation) supports at least two linear guided modes and we expect to find a new solution branch corresponding to each of these modes. The fundamental mode has no azimuthal variation (i.e. $l_{-} = 0$) while the second has an azimuthal dependence of $e^{i\varphi}$ ($l_{-} = 1$) and an intensity node at the origin. We consider only $B = 1/3$ (XPM/SPM = 2) as a representative example. For larger values of $B$ we must expect other solutions (corresponding to higher order modes), whose properties will not differ in any qualitative way from the solutions described here.

For $B = 1/3$ the fundamental ($l_{-} = 0$) mode of the soliton induced waveguide propagates with $\beta_{-} = 1.21$ and the second ($l_{-} = 1$) mode with $\beta_{-} = 1.92$. These values of $\beta_{-}$ should yield the points at which the new solutions branch from the vortex soliton solution. Numerical investigation of eqs. (5.5) using a shooting technique confirms the existence of bifurcation points at these values of $\beta_{-}$ and enables one to follow the new solution branches away from these points. This process, that parallels exactly that presented for (1+1) dimensions in §4.1.2, is demonstrated graphically for the first bifurcation in figure 5.2.

### 5.1.3 Polarisation domain wall solitary waves

Figure 5.2 shows the field profiles of the first solution branch both near to and far from the point where it branches from the scalar vortex. The profile near the bifurcation point (i.e. at low signal power) is shown in (a). As expected, the profile of the field $e_{+}$ closely resembles a vortex soliton, while $e_{-}$ takes the profile of the first or second mode of the induced waveguide. The case of $\beta_{-} \rightarrow \beta_{+}$, i.e. when the intensity of the signal $e_{-}$ tends to that of the pump $e_{+}$, is shown in 5.2(c). The signal beam $(e_{-})$ is a wide, flat region contained at its edge by the 'pump' field. The dividing line between these regions is an annular PDW. Therefore it is reasonable to consider these bright dark solitons as circular polarisation domains. Figure 5.3(a) and (b) give two examples of the solitary wave associated with the second mode of the vortex-induced waveguide. In figure 5.3(a), near the bifurcation, one observes a simple central zero in $E_{\pm}$, associated with the second order mode. In (b), as $\beta_{-} \rightarrow \beta_{+}$ this node has itself become a scalar vortex, guided by its own waveguide.

Both the solutions studied so far originate from vortex solitons and so the surrounding field is limited to a phase dependence of $e^{i\varphi}$ (i.e. $l_{+} = 1$). Further solutions, that could not be characterised as bifurcations of scalar kink waves, were found for the (1+1) system...
5.1 Kink waves of cylindrical symmetry

Figure 5.2: The bright dark vector solitary wave originating in the bifurcation of the vortex soliton associated with the fundamental mode. The dark component of the field is illustrated in black, the bright component in grey. In (a) the amplitude in the bright component \( E_- \) is small and the dark component \( E_+ \) still closely resembles a scalar vortex. In (b) the amplitude in \( E_- \) is greater so the profile of \( E_+ \) is reshaped. In (c) \( E_- \) has plateaued, forming a circular domain. The interface between the components is a ring shaped PDW.

in Chapter 4. We are justified in asking ourselves what other solitary wave type might exist here. Since the coupling between the two polarisations is incoherent, the profile of the PDWs is independent of the relative phase of the overlapping fields. Therefore the form of the large radius PDW localised structures, for which \( \beta_- \rightarrow \beta_+ \), will not change if the azimuthal dependence of the surrounding field \( e_+ \) is suppressed. This reasoning leads us to investigate, via numerical solution of eqs. (5.5), solutions that have no azimuthal variation in \( e_+ \) (i.e. \( l_+ = 0 \)).

The radial field profiles of these solutions are shown in figure 5.3. Figure 5.3(c) and (d) show examples of the \((l_+, l_-) = (0, 0)\) solution; these are circular polarisation domains without azimuthal phase dependence. This is the first time, to our knowledge, that \((2+1)\)-D kink solitary waves with flat phase fronts have been described. Solutions in this form show that one-dimensional domain walls can bend into rings to form two-dimensional solitary waves that are comprised of circular domains of any radius. This \((0,0)\) solution has a direct analogue in Chapter 4, but the cylindrically symmetric problem possesses another solution branch. The \((0,1)\) solution appears in (e) and (f) and differs in possessing a vortex at its center. Instead of branching from self guiding vortices, these solutions find their origins in plane waves. There is therefore no linear guided wave picture to provide a physical insight into the bifurcation, as there was for the first type of solution. Both pump and signal fields become progressively broader and flatter near the branching point, at \( \beta_- = \frac{1+\beta_+}{2} \beta_+ = 2.0 \) for both solution branches. The form of these solitary waves approaching these points is shown in figure 5.3(c) and (e).

The analysis of the reduced system of eqs. (5.5) gives no indication of the stability of these solutions. The full eqs. (5.3) have been solved numerically and several propagations
Figure 5.3: The three other types of cylindrical bright dark solitary wave. (a) and (b): bright dark solution originating in the second mode of the vortex, $l_1 = 1, l_2 = 1$; (c) and (d): $l_1 = 0, l_2 = 0$; (e) and (f) $l_1 = 0, l_2 = 1$. 
5.1 Kink waves of cylindrical symmetry

Figure 5.4: Snapshots showing the linearly polarised field components $E_x$ and $E_y$ during the propagation of the $l_1,l_2 = (0,1)$ solitary wave. Saturation of blue in the images is proportional to $|E_x|^2$, that in yellow proportional to $|E_y|^2$. Circularly polarised portions of the beam appear grey. We show one complete rotation of the linearly polarised lobes.

Figure 5.5: Evolution of an elliptical domain. The initial conditions consist of a $(1+1)$ PDW bent into a n elliptical ring, with no azimuthal phase dependence. Green colour saturation corresponds to $|E_+|^2$, red to $|E_-|^2$. In light regions the field is predominantly one wavelength component, in dark regions the other. The final box shows the field to have evolved to a circularly symmetric $l_+ = 0,l_- = 0$ solitary wave.

carried out. For brevity, we only present simulations of the $l_+ = 0$ solutions that have no azimuthal dependence in the surrounding field. The topologically trapped $l_+ = 1$ solitary waves appear to be at least as stable as the solutions whose simulations are presented here.

Note that structures with $l_- \neq l_+$ will display interesting propagation dynamics arising from beating between the two polarisation components that have different propagation constants. Consider the line (really a ring, or cylinder) running down the middle of the domain wall [where $e_+(r) = e_-(r) = u_0$] and consider the linearly polarised field component given by $E_x = (E_+ + E_-)/\sqrt{2}$. Assuming the structure to be stable, we can determine the evolution of the intensity of this component:

$$|E_x(\varphi,z)|^2 = \epsilon_+^2 \left[ 1 + \cos((l_+ - l_-)\varphi - (\beta_+ - \beta_-)z) \right]$$

(5.6)

At any instant the intensity of the field $E_x$ in the middle of the domain wall varies sinusoidally with $\varphi$, resulting in lobes. During propagation the position of the lobes is proportional to $z$; therefore they trace out a helical path whose ‘pitch’ depends on the difference between the propagation constants. In figure 5.4 we illustrate this effect using coloured images. One complete rotation of the linearly polarised lobes lying on the domain boundary is shown. Other simulations have demonstrated that large perturbations imposed upon this solution (up to 10% of the total power) decay until the radiative components are vanishingly small, leaving the pure solitary wave shown in figure 5.4.

Figure 5.5 shows the evolution of the field profile from the initial condition of an elliptically shaped polarisation domain with no azimuthal dependence. One can think of this as a large perturbation to the $(0,0)$ solution. The domain oscillates about the circular solution many times before settling down finally to the $(e_+,e_-) = (0,0)$ circular polarisation domain solitary wave. These field dynamics indicate the stable nature of this solitary wave. Note that according to eqs. (5.3), the power in each polarisation component is a constant of the motion.

In conclusion, a new class of two-dimensional dark vector solitary wave has been found in bulk media with self-defocussing Kerr nonlinearity. The existence and location
of branching bifurcations from which the new solutions originate is foreseen from simple physical arguments and confirmed by numerical analysis. Simulations suggest that the new solitary waves are stable and that they play a central role in the rich polarisation dynamics exhibited by these systems. By allowing a single beam to be divided into many separate polarization domains (and thereby carry many bits of information) these new solutions may prove useful for all-optical signal processing.

5.1.4 Colour domain walls

Until this point we have considered the evolution of monochromatic beams of light of arbitrary elliptical polarisation. The evolution of a beam of light of uniform polarisation but composed of several distinct wavelengths can be described by a related set of coupled nonlinear Schrödinger equations. In fact the evolution of a field comprising $N$ different frequencies is governed by the following system (De la Fuente, Barthelemy, and Froehly 1991)

$$ik_jn_0\partial_z E_j + \frac{1}{2}\nabla_j^2 E_j - k_j^2n_0\alpha_j\sum_{j=1}^{N} \sigma_{jk}|E_k|^2E_j = 0 \quad j = 1, \ldots, N$$

(5.7)

The Laplacian is taken only over the polar transverse co-ordinates $r = (r, \varphi)$. $z$ is the longitudinal coordinate. $E_j(r, z)$ is the envelope of the field with wavelength $k_j$ that sees the refractive index $n_j = n_0 + \alpha_j\sum_{k=1}^{N} \sigma_{jk}|E_k|^2$. The tensor $\sigma_{jk}$ is wavelength dependent and related to the speed of response of the materials. Its diagonal elements are always scaled to unity. In slow materials $\sigma_{jk} = 1$ for all $j, k$ as all wavelengths see the same, slow refractive index change. The system is therefore described by the (2+1) U(n) NSE. In fast materials $\sigma_{jk} = 2$ for $j \neq k$, a consequence of the fact that more nonlinear terms contribute to XPM than to SPM. Hereafter we consider only those cases where the cross terms are all equal and define $\sigma = \sigma_{jk}$ for $j \neq k$.

We consider first the copropagation of two wavelengths. By using a monochromatic vortex as a waveguide for a signal field of different wavelength we anticipate the existence of bound states between wavelength components that bifurcate from vortex solitons. Since cross phase modulation is greater than self phase modulation we also expect to find domain wall solitary waves. It remains only for us to clarify the differences between wavelength domains and polarisation domains.

As before we propose the existence of circularly symmetric stationary solutions of the form $E_j(r, \varphi, z) = e_j(r)e^{i\ell_j}e^{-i\beta_jz}$ where the envelopes $e_1$ and $e_2$ are real functions; $\beta_1$ and $\beta_2$ their respective propagation constants and $l_1$ and $l_2$ are the azimuthal numbers. As before this ansatz reduces eqs. 5.7 to a pair of real ODEs

$$\beta_j e_j + \frac{1}{2} \left(e_j'' + \frac{1}{\rho_j} e'_j - \frac{\beta_j^2}{\rho_j^2} e_j\right) - \rho_j (e_j^2 + \sigma e_k^2) e_j = 0 \quad j = 1, 2; \quad k \neq j$$

(5.8)

where $\rho_j = k_j^2\alpha_j n_0$. Hamiltonian variables for this problem are $f = \sqrt{\rho_1} e_1$ and $g = \sqrt{\rho_2} e_2$, and the mechanical potential is

$$V = \beta_1 f^2 + \beta_2 g^2 - \frac{1}{2}(\rho_1 f^4 + \rho_2 g^4) - \sigma \rho_1 \rho_2 f^2 g^2$$

(5.9)

Note that the energy of the ball moving through this potential is not conserved when its trajectory is described by eq. (5.8). One finds the same four wave types, corresponding to permutations of $l_1, l_2 = 0, 1$, that were shown in figures 5.2 and 5.3. These solutions tend to circular domains only when the wavelengths are close enough together and the material response is fast enough to satisfy $\sigma^{-1} > \rho_2/\rho_1 > \sigma$. Within this constraint the bright-dark solitary waves of different frequencies are qualitatively identical to those monochromatic waves of elliptical polarisation. The exact profiles of the colour domain walls (CDWs) themselves are, however, a little different to PDWs since the permutation
5.2 Propagation dynamics in 3D

5.2.1 Steerable PDWs

In order for these new structures to be of practical use in switching devices, we need to be able to direct the domains in some way or another. In particular we wish to discover if there are domain solitary waves analogous to grey solitons: solitary waves that propagate unchanged at a different velocity to the background wave. There are several reasons why this sort of solitary wave is not easy to characterise. Firstly the direction of travel will break the cylindrical symmetry. Secondly, like bright-dark states in (1+1) dimensions, a travelling solution has a more complex phase dependence. Therefore an ansatz involving real functions needs to be quite sophisticated. Nevertheless we expect that in systems without coherent cross-coupling one might be able to impose a phase ‘chirp’ on the bright component without destroying the integrity of the solution. If the solitary wave is robust then the chirped state would radiate a little and propagate at an angle to the background. In this way we could find approximate numeric forms for travelling waves.

We have conducted preliminary numerical investigations into these objects by launching chirped forms of stationary solutions where the wavefront of the secondary field is oriented at an angle to the primary wavefront. After a transient radiatory period, a stable travelling domain is formed. This result strongly suggests that the stationary wavelength domain solitary waves are part of a larger family of robust, steerable solitary waves.

Figure 5.6: Transverse profiles $e_j(x)$ of a (1+1) dimensional CDW. $k_1/k_2 = 2$. Note the asymmetry in the amplitudes: the field component with the shorter wavelength forms a sharper domain wall than the other.

Symmetry has been lost. Figure 5.6 shows the profile of a (1+1) dimensional CDW, in which the asymmetry is apparent.

The introduction of further wavelength components widens the spectrum of potential solitary waves. We expect to find at least one family of three component waves bifurcating from each two component solitary wave. As an example, consider the $(l_1, l_2) = (1, 1)$ solitary wave far from the bifurcation: a vortex surrounded by a ring PDW. To a third frequency, the vortex induces a cylindrical waveguide while the ring PDW induces a ring shaped waveguide. The compound waveguide therefore supports at least two modes (and probably three) each of which corresponds to a new solution branch that can be explored by increasing the power in the third wavelength component. We do not present any details of these further bifurcations as no new physical principles are involved. In addition, there is reason to believe that solitary waves of this form would be unstable, as will be seen in the next section.
waves, in the same way that the black soliton is only a part of the steerable grey soliton family. Fig 5.7 shows a collision between two small domains that leads to their eventual coalescence, while figure 5.8 shows a small domain annihilating itself against an infinite domain wall. These two results clearly show that despite their robustness to certain perturbations, simple travelling domains are easily consumed in collisions. Higher order domains, containing topologically trapped ‘vortex’ nodes may prove to be more robust.

5.2.2 Interfacial energy

Like many interfaces in physics, the dynamics of these structures are driven by an energy cost per unit length that is associated with the domain walls. The systems therefore evolve to minimise the length of the boundaries, within the constraint that the total power in each component is always conserved. The cylindrical symmetry of the stationary states thus represents a state of minimum energy. Figure 5.5 demonstrates the stability of the $l_1 = 0, l_2 = 0$ solitary wave, and shows the system acting to minimise the domain wall length. Because of this, we expect a general trend in which the PDW solitons tend to attract one another and coalesce, as bubbles in a liquid. An excellent example of this kind of behaviour is shown in figure 5.8 where a travelling domain is incident upon a PDW stripe. The domain strikes the interface and disappears, leaving ‘ripples’ propagating outwards along the interface.

Three component systems behave in much the same way, although one must remember that the power in each mode is independently conserved. Figure 5.9 shows an example of the kind of behaviour to be expected; two incident travelling domain waves trap one another, forming a state that minimises the total domain length. This implies that ‘embedded cylinder’ solitary waves comprising several coaxial annular domains would not be stable. This is confirmed by simulations which show that the cylindrical symmetry is broken as the domains re-organise themselves to minimise the interface length.

There is one additional question to be addressed in considering more than two components. Is there a greater interfacial energy associated with certain interfaces than with others? One might expect that the interfaces that separate more disparate wavelengths may invoke a greater energy cost. Figure 5.10 shows that this is indeed the case. Here blue corresponds to the shortest wavelength, green a little longer and red the longest. The system evolves so that the length of the blue-red interface is minimised in preference to the green-blue or green-red interfaces.
5.2 Propagation dynamics in 3D

Figure 5.7: A collision between two travelling domains. The domains attract, spiral about each other, then coalesce and eventually settle to form a single stationary domain (not shown).

Figure 5.8: The collision of a circular travelling domain with a quasi-infinite domain wall. The behaviour is strikingly reminiscent of a bubble at the surface of a liquid.

Figure 5.9: Collision between two wavelength domains where the bright component of each domain is a different wavelength.

Figure 5.10: Evolution where the wavelengths are very different. Note that the interface separating the blue and the red components appears to be associated with a higher energy cost.
Kink Waves in (2+1) Dimensions
Chapter 6

Soliton Written X-junctions

Previous chapters of this thesis have used extensively the idea that a self trapped light beam induces ('writes') a waveguide as physical motivation in our search for new solitary waves. This chapter makes concrete use of this property by considering the fabrication of waveguide devices with spatial soliton beams. One would expect such waveguide devices to have special properties as a consequence of the exotic fabrication technique. In particular, we investigate the transmission properties of the waveguide X-junctions that are written by colliding spatial solitons. Fortunately a portion of the parameter space of interest is described by the integrable Manakov system and leads to exact results. When we consider more general cases, we use approximation methods based on linear waveguide theory.

6.1 Introduction

Chiao, Garmire, and Townes (1964) proposed that spatial solitary waves may exist in self focussing media. The brighter centre of the beam increases the refractive index more than the edges and so travels more slowly, permitting the wavefront to remain flat during propagation.Shortly afterwards Garmire, Chiao, and Townes (1966) observed the self-trapping of a light beam in CS$_2$ liquid. Later Zakharov and Shabat (1973) extended the concept of the optical spatial solitons into defocussing media, presenting the kink soliton solutions to the nonlinear Schrödinger equation (NSE). Recently Snyder, Mitchell, and Poladian (1991) considered the refractive index change in terms of writing a waveguide into the material for which the soliton profile represents the fundamental mode. At the same time, De la Fuente, Barthelemy, and Froehly (1991) successfully used a soliton as a waveguide for a weak probe beam of different wavelength. Subsequently Luther-Davies and Yang (1992) have done the same for dark solitons, and Swartzlander and Law (1992) for vortex solitons. Silberberg and Sfez (1988) pre-empted some of this by suggesting the use of spatial soliton collisions for all-optical switching devices.

Chapter 2 derives, in two different ways, an expression describing the collision between two arbitrary solitons in a Kerr type material. Therefore we have complete and exact analytical descriptions for the waveguide X-junctions that we consider in this section. Unfortunately these solutions, as can be seen from examining figures 2.2–2.4, are in general very complicated, and exact analytical forms cannot be obtained using traditional linear waveguide theory. Our answer to this dilemma is two pronged: for cases where the wavelengths are identical, and soliton and signal see the same waveguide, the transmission properties of the junction can be found exactly by solving the integrable Manakov system with infinitesimal intensity in one component. This case is of special interest because the integrability of the underlying equations means that the junctions possess striking properties. Our second approach to the problem is to use coupled mode theory, which assumes that the waveguides remain well separated and vary slowly. We find analytical expressions characterising the X-junctions for those cases in which the
colliding solitons fulfill these criteria.

6.2 Exact ‘modes’ of some X-junctions

6.2.1 An integrable system

This chapter considers two quite different kinds of device. In materials with a reasonably rapid response time, the soliton waveguide exists only when the soliton itself is present. The devices relevant to this sort of material involve the control of light by light and the ability to switch a probe beam rapidly by altering the pump beams. In this case the pump and the probe are always propagating together and if they are of the same wavelength then they must be orthogonally polarised to be distinguishable. The propagation of two beams will therefore be well described by the Manakov equations if cross phase and self phase modulation are equal.

The other kind of device that we consider here is one that is written into a particular material that preserves the refractive index change permanently, or at least for a long time. The medium must either have a very long response time, exhibit memory effects, or be permanently changed by intense light (like photosensitive materials). With this permanent kind of device, the pump and signal will always see the same waveguide and when they are of the same wavelength, will be described by the integrable equations.

In this section we are only interested in the problems that are integrable. They are described, in dimensionless form, by equations that are the linear limit of the $U(3)$ NSE

$$i\partial_z E_p + \frac{1}{2} \partial_x^2 E_p + \sigma \left( E_p^\dagger E_p \right) E_p = 0$$  

(6.1a)

$$i\partial_z E_{\text{sig}} + \frac{1}{2} \partial_x^2 E_{\text{sig}} + \sigma \left( E_p^\dagger E_p \right) E_{\text{sig}} = 0.$$  

(6.1b)

The first equation, governing the pump field $E_p$, is decoupled from the second that governs the signal field $E_{\text{sig}}$. In the scalar case it is the NSE, while for a junction written with elliptically polarised beams it is the $U(2)$ Manakov system. Either way the problem is integrable. The second equation is a linear Schrödinger equation describing scattering from the potential well $V(x,z) = -\sigma E_p(x,z)^\dagger E_p(x,z)$. The bound solutions to this scattering problem represent the $z$-dependent ‘modes’ of the X-junction.

6.2.2 Bright soliton X-junctions

In the self-focussing case ($\sigma = 1$) the solution to this equation can be found by taking the linear limit $|E_{\text{sig}}|^2 \ll E^\dagger E$ of the collision of two bright solitons of the $U(3)$ Manakov system. The solution describing this collision was derived using Hirota’s method in §3.3 and presented in eqs. (3.42) et seq. In the following we use the parameters explained in §3.3. In order to explore the coupling properties of the X-junction we excite one input port with the probe beam. The initial polarisation states of the two solitons take the form

$$c_1 = \begin{bmatrix} \sqrt{1 - \epsilon^2} \\ 0 \\ \epsilon \end{bmatrix}, \quad c_2 = \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \end{bmatrix}$$  

(6.2)

Where $\epsilon$ is an infinitesimal giving the amplitude of the signal field. Ignoring the higher order terms in $\epsilon$, $E_p$ represents the collision of two vector solitons with an initial polarisation difference of $\psi$ radians, while $E_{\text{sig}}$ becomes the linear $z$-dependent mode of the waveguide X-junction. We do not write out the form of this mode here; it can be
obtained from the results of section 3.3, where all the hard work has been done. It is a trivial matter to obtain, from eq. (3.53), the power that emerges in each port

\[ P'_1 = P_1 \frac{(a_1 - a_2)^2 + (b_1 - b_2)^2}{(a_1 - a_2)^2 + (b_1 - b_2)^2 + 4a_1a_2 \cos^2 \psi} \]  
(6.3)

\[ P'_2 = P_1 \frac{4a_1a_2 \cos \psi}{(a_1 - a_2)^2 + (b_1 - b_2)^2 + 4a_1a_2 \cos^2 \psi} \]  
(6.4)

where a signal beam of power \( P_1 \) was directed down port 1, and \( P'_j \) indicates the amount of power that is transmitted by the junction. One should remember that the amplitude of the \( j \)-th soliton is given by \( a_j \) and that they are travelling at angles \( \arctan \beta_j \) to the \( z \)-axis. The peak power \( P_{j(\text{peak})} \) emerging in the \( j \)-th guide is related to the integrated power \( P'_j \) by \( P_{j(\text{peak})} = a_j P'_j \). Note that \( P'_1 + P'_2 = P_1 \), as must be the case since there is no loss in the junction. Consider first of all the junction written by colliding scalar solitons, i.e. \( \psi = 0 \). Figure 6.1 is a surface plot showing how the reflection coefficient \( R = P'_2/P_1 \) varies according to the parameters \( a_1/a_2 \) and \( \Delta b = b_1 - b_2 \). For high angles of incidence, and for a large amplitude difference, the reflection coefficient drops to zero. Note that with nearly identical solitons we can achieve almost 100% reflection (as the solitons coincide). Figure 6.2 shows the waveguides \( |E_{\text{sig}}|^2 \) and their ‘modes’ \( |E_{\text{sig}}|^2 \) for some scalar collisions. If the colliding solitons consist of different field components (e.g. non identical polarisation) then reflection into the other soliton is reduced by the factor \( \cos \psi \). When the solitons are in orthogonal states then the junction always transmits 100%. This fascinating property was first realised by Luther-Davies and Yang (1993). While this behaviour does not appear immediately promising for switching applications, the reduction of cross talk to zero could prove useful.
Figure 6.2: Bright soliton induced junctions: exact solutions. The refractive index profile and the form of the signal field for three different angles of incidence. At the bottom we show a bound state, in which the angle of incidence is zero. It is clear that nothing is transmitted in this case. (a) $a_1 = 1$, $a_2 = 1.4$, $\Delta b = 1.0$ (b) $a_1 = 1$, $a_2 = 1.4$, $\Delta b = 4.0$ (c) $a_1 = 0.99$, $a_2 = 1.01$, $\Delta b = 0.0$

6.2.3 Dark soliton X-junctions

For defocussing materials we take $\sigma = -1$ in eq. (6.1), and then to explore the waveguide properties of dark soliton induced X-junctions, we use the bright-dark soliton solution that was derived in §4.2. Note that for this case we also have the N-soliton solution. However the very reasons that made the derivation of the N-soliton case straightforward make the transmission properties of the dark soliton junctions simple to the point of being uninteresting. Since the soliton eigenvalue is a conserved quantity of the Manakov equations, the amount of ‘bright’ in a bright-dark soliton cannot change when it interacts with other solitons. Therefore all dark soliton induced X-junctions will transmit 100% of the signal. This remarkable property, that dark solitons junctions share with orthogonally polarised bright solitons, was first pointed out by Luther-Davies and Yang (1992) after extensive numerical simulations.

Figure 6.3 shows the index profiles $|E|^2$ and the modes $|E_{\text{sig}}|^2$ of two different soliton induced X-junctions. Observe how the black soliton collision is tuned so that exactly 100% of the power couples through, remaining in the original guide.

In summary, the junctions whose writing and operation is described by integrable
systems [in particular, the $U(n)$ NSE] possess many special properties. The most important single property of these junctions is that they are always lossless. This is very important as it permits the construction of lossless waveguide X-junctions with much shorter overall length than can be fabricated using conventional designs. The other striking property is the fact that junctions written by dark solitons or orthogonally polarised bright solitons always transmit 100%. The experimental observation of these remarkable consequences of integrability is made more difficult by the requirement that the induced waveguide must be almost perfect. Imperfections will prevent the fine phase cancellation that their special properties rest on.

6.3 Different wavelengths: approximation scheme

In the limit of small incident angles interacting solitons remain well separated throughout the interaction if the force between them is repulsive (see figure 6.4). It is possible to apply coupled mode theory [as explained in Chapter 19, Snyder and Love (1983)] when this condition is satisfied. The force between bright solitons is repulsive if they are out of phase, while dark solitons always repel. Bright solitons can only maintain a constant phase relationship if they are propagating with the same propagation constant, demanding equal amplitude. Thus there are two regimes under which a collision can be thought of in this way: dark solitons at small incident angles, and identical bright solitons launched with a 180° phase shift, also at small incident angles. We assume that the shape of the solitons does not change, and that the collision appears as two identical waveguides whose separation changes slowly. First we obtain expressions describing the separation of these waveguides. Local mode coupling theory (McIntyre and Snyder 1973) is used to determine the coupling between the soliton induced waveguide cores at every point in the collision. The integral of this expression allows the determination of the power that emerges in each core. The illumination may be of the same or a different wavelength to that which wrote the device.
Figure 6.4: Schematic of the system under consideration. The interacting solitons induce waveguide cores whose separation varies with $z$. The lower core (core 1) is illuminated by a small signal of power $P_{in}$ and the power that emerges in each of the two cores is given by $P_1$ and $P_2$.

6.3.1 The trajectories of interacting solitons

Dark solitons

We consider dark solitons in self-defocussing Kerr media governed by the NSE (4.1). The NSE is integrable, and one class of the analytic solutions that can be derived for it are the multisolitons that describe interactions between any number of solitons. The two soliton solution for two identical dark Kerr solitons yields the following field intensity (Blow and Doran 1985)

$$|\psi|^2 = b^2 - \frac{4(b^2 - a^2)(a^2 + ab \cosh(2a\sqrt{b^2 - a^2} Z) \cosh(2\sqrt{b^2 - a^2} X))}{b \cosh(2a\sqrt{b^2 - a^2} Z) + a \cosh(2\sqrt{b^2 - a^2} X)}^2,$$

where $b$ determines the amplitude of the background field (and the soliton width) and $a$ the ‘darkness’ of each soliton ($a$ also determines the angle of collision). The standard dark soliton eigenvalue for each soliton is given by $a/b$. Very little information can be read instinctively from (6.5); the information it carries is hidden by the unfortunate functional form. However we know that for small input angles the interaction can be represented by two separate solitons, whose separation varies as the beams propagate (see Fig 6.4). It is in this limit (where $a \ll b$) that we shall operate. We therefore expect the solution to take the form of two separated negative hyperbolic secant pulses on a constant background.

$$|\psi_{guess}|^2 = b^2 - b^2[\text{sech}^2(bX + \alpha) + \text{sech}^2(bX - \alpha)]$$

The separation of the beam centres is given throughout the collision by $2\alpha/b$, i.e. $\alpha$ represents the cores’ half-separation measured in terms of the beam half-width $b$. For beams remaining well separated at all times eq. (6.5) reduces to eq. (6.6) if the separation $2\alpha$ is given by

$$\cosh \alpha \simeq \sqrt{\frac{b}{2a}} \cosh(2ab Z)$$

(6.7)
The two soliton solution has been studied with similar aims previously: Zhao and Bourkoff (1989) used beam propagation methods with initial conditions of two distinct, parallel solitons. They then fitted a curve to their numerical trajectories and obtained the expression

\[ e^{2(a-a_0)} = \frac{1}{2} [1 + e^{2b^2 Z \exp(-2a_0)}] \]  

(6.8)

where \( a_0 \) is the separation at \( Z = 0 \). This expression was designed to show the long distance behaviour and is not capable of describing accurately the solitons around the point of closest approach (near \( Z = 0 \)). As such it is of little value to us. Figure 6.5 compares this curve with our approximation and the exact solution. The region displayed in this figure corresponds to the upper right hand quadrant of figure 6.4. The result obtained directly from the exact solution provides a demonstrably better fit in the domain of interest to us (i.e. near the point of closest approach).

**Figure 6.5:** Comparison of the approximation of eq. (6.7) derived here (solid line) with the exact result (eq. (6.5); dashed line) and the approximation of Zhao and Bourkoff (1989) (eq. (6.8); dots and dashes). \( b = 1, a = 0.01 \) and the axes correspond to the \( x \) and \( z \) axes from figure 6.4

**Bright solitons**

The interaction of bright solitons is algebraically more complicated since one observes pseudo-linear interference fringes that are absent in dark soliton interactions. There is also an additional degree of freedom in the interaction: the relative phase of the colliding beams. Fortunately several approximate schemes for determining the soliton trajectory exist. Mitchell, Snyder, and Poladian (1991) examined the interaction of two well separated, almost parallel solitary waves and used an integral invariant of the generalised NSE to obtain a differential equation for the soliton motion. Their result is valid for all self-focussing nonlinearities with Kerr media as a special case only. Earlier, Gordon (1983) derived the same result in Kerr media directly from the exact two-soliton
solution. We only require the approximation of the former result. The separation of
distinct, bright solitons in any nonlinear self-focussing media is given by Mitchell et al.
(1991):

\[ \frac{1}{2} \left( \frac{dr}{dZ} \right)^2 + V(r) = E_0 \]  

(6.9)

where \( E_0 \) is the initial ‘energy’ imparted to the system, \( 2r \) the soliton separation and
\( V(r) \) is given by

\[ V(r) = \frac{\int \left[ n_1^2(r) - n_0^2 \right] E_1(r)E_2(r) \, dx}{2n_0^2 \int e_1^2 \, dx} \]  

(6.10)

e_1 and \( e_2 \) are the scalar field amplitudes and \( n_1(r) \) is the refractive index profile induced
by the first soliton. In the dimensionless units and Kerr nonlinearity of eq. (6.1) this
potential can be written as

\[ V(r) = \frac{\int |\psi_1|^2 \psi_1(r) \psi_2(r) \, dX}{2 \int \psi_1^2 \, dX} \]  

(6.11)

For two solitons in Kerr media \( \psi_i = b \text{sech}[b(x \pm r)] \) and the interaction potential evaluates
in the first order to

\[ V(r) = b^2 \text{sech}(2br) \]  

(6.12)

Approximating \( V \) by a decaying exponential allows the integration of (6.9). The trajectory so derived is written here in terms of \( \alpha = br \) and \( a = \sqrt{E_0/2} \), analogously to the
dark soliton trajectory

\[ \cosh \alpha \simeq \frac{b}{2a} \cosh(2abZ) \]  

(6.13)

Fig 6.6 compares the exact result [from Zakharov and Shabat (1972) and Gordon
(1983)] with our analytical expression in the region where our expression is expected
to break down. While the approximation deteriorates for higher angles, we find these
results highly encouraging since at these angles the order parameter in our asymptotic
expansion is approaching unity.

6.3.2 General properties of induced X-junctions

Equipped with the soliton trajectories, we can calculate the coupling properties of the
X-junctions induced by soliton collisions. A junction is formed in this manner by (1)
simultaneously copropagating a small signal beam with one of the solitons, or by (2) writ­
ing the device into a material in which the refractive index change is quasi-permanent.

In the first instance even small signal beams affect the waveguide in which they are
travelling when the signal wavelength (\( \lambda_s \)) and the pump wavelength (\( \lambda_p \)) are equal.
We can circumvent this problem in two ways: changing \( \lambda_s \) and/or using orthogonal
polarisations. When we copropagate a signal of similar polarisation, it will see a different
induced waveguide than does the soliton (Agrawal 1991) when the medium is of rapid
response. One can account for the difference in self-phase and cross-phase modulation
using the parameter \( \kappa \) introduced in §3.2.3. \( \kappa \) gives the ratio XPM/SPM. A discussion of
the effective value of XPM/SPM for media with different temporal response can be found
in Bloembergen and Lallemmand (1966). The essential conclusion is that for all but the
fastest electronic nonlinearities, XPM=SPM (i.e. \( \kappa = 1 \)). For an assumed instantaneous
response, XPM/SPM = 2, since there are twice as many terms contributing to XPM.
6.3 Different wavelengths: approximation scheme

Figure 6.6: Comparison of the approximate separation for bright solitons of eq. (6.13) (dashed line) with the exact result [from Gordon (1983); solid line] in the region of breakdown. The values of the parameters ($b = 1$ while $a = 0.2$ and 0.6) were chosen to demonstrate the breakdown of the approximation.

When the signal is polarised orthogonally to the pump things change a lot, although a nonlocal response in either space or time still leads to the Manakov model with $\kappa = 1$. For non-degenerate wavelengths the coherent coupling terms in eq. (1.18) no longer oscillate at the same frequency and will therefore average to zero. Therefore XPM will invariably be less than (or equal to) SPM for non-degenerate orthogonally polarised waves. This corresponds to $\kappa < 1$. The practical ramification of $\kappa$ is to change the modulation depth of the waveguide junction seen by the signal. For $\kappa > 1$ the signal sees a deeper waveguide than the soliton, while for $\kappa < 1$ the situation is reversed. Note also that the primary effect of shifting the wavelength is to alter the waveguide parameters.

The following theory is best presented in dimensional form and so we turn to dimensional units for the remainder of the chapter.

**Coupled mode theory**

The coupling coefficient between the local modes $i$ and $j$ of a slowly varying, weakly guiding waveguide structure is (Snyder and Love 1983)

$$C = \frac{k_s}{2n_0} \frac{\int \left[ n_i^2(x) - n_0^2 \right] e_i(x)e_j(x) \, dx}{\int e_i^2 \, dx}$$  

(6.14)

where $n_i(x)$ is the refractive index variation of core $i$ in isolation. In applying (6.14) one assumes that the second (antisymmetric) mode of the structure is not cut-off at any point in the device. If this mode is cut-off there will be large signal loss and the output will be evenly divided between the cores (Snyder and Rühl 1985; Love and Ankiewicz 1985). We analyse the onset of this effect in the next section. Both bright and dark
Soliton Written X-junctions

Solitons induce the same waveguide: the effective induced refractive index profile is given by

\[ n_f(x) - n^2_0 = \frac{2b^2 \kappa}{k_p^2} \text{sech}^2(bx \pm \alpha) \]  \hspace{1cm} (6.15)

where the parameter \( \kappa \) gives the ratio of cross phase to self phase modulation. \( \kappa \) depends on the third order susceptibility tensor and therefore varies with \( k_s, k_p \) and the type of material being used. The linear signal that is the fundamental mode of these waveguides is

\[ e(x) = N \text{sech}^2(bx \pm \alpha) \]  \hspace{1cm} (6.16)

where

\[ s = \frac{1}{2} \left( \sqrt{4V^2 + 1} - 1 \right) \]  \hspace{1cm} (6.17)

\( V = k_s^2 \rho^2(n_{\text{max}}^2 - n_0^2) \) is the waveguide parameter as seen by the signal and \( N \) is an arbitrary complex number. It should be noted that we assume throughout that the induced devices are lossless; numerical studies indicate that this condition is only met exactly for the integrable case considered in section 6.2. Nevertheless, for slowly varying couplers (frequently called adiabatic couplers) the lossless assumption is valid.

Substitution of (6.15) and (6.16) into (6.14) gives

\[ C \approx \frac{2^{s+1} b^2 \kappa \Gamma(s+1/2)}{(s+1) k_s n_0 \Gamma(1/2) \Gamma(s)} \int_{-\infty}^{\infty} \text{sech}^{2+s}(bx + \alpha) \text{sech}^s(bx - \alpha) \, dx \]  \hspace{1cm} (6.18)

For general \( s \) we must assume well separated beams (i.e. \( \alpha \gg 1 \)) in order to perform the integral. This approximation leads to

\[ C \approx \frac{2^{s+1} b^2 \kappa \Gamma(s+1/2)}{(s+1) k_s n_0 \Gamma(1/2) \Gamma(s)} \text{sech}^s(2\alpha) \]  \hspace{1cm} (6.19)

Equation (6.19) furnishes us with the coupling coefficient between the fundamental modes of two sech-profile waveguides with separation of \( 2\alpha \). We can now make use of equations (6.7) and (6.13) to determine how \( C \) varies through the interaction.

**Antisymmetric mode cut-off**

A coupler consisting of two infinitely well separated waveguides will always support two modes since the fundamental mode of each guide is never cut-off. However when low \( V \)-valued waveguides approach each other, a point is reached when the second (antisymmetric) mode is cut off. When this happens in junctions, half the power is lost to radiation and the field is divided symmetrically between the two output ports. This effect will invalidate the coupled mode theory approach at large signal wavelengths when the \( V \)-value of the induced waveguide drops substantially.

A coupler comprising two identical waveguides each supporting a single mode of propagation constant \( \beta_1 = \beta_2 = \beta \) will support symmetric and antisymmetric modes whose propagation constants are given by \( \beta_{\pm} = \beta^2 \pm C(r) \). From this \( \beta_+ > \beta \) and the symmetric fundamental mode is never cut-off. The second mode will be cut-off when \( \beta_- = k n_0 \). This condition is satisfied when

\[ W^2 = C(r) \]  \hspace{1cm} (6.20)

where \( W = \sqrt{\beta^2 - k^2 n_0^2} \) is the modal parameter of the fundamental mode of each core in isolation. For the fields considered here, \( W = s \) [this may be seen from eq. (6.17)]. Substituting eq. (6.19) into this and approximating the hyperbolic secant by an exponential
6.3 Different wavelengths: approximation scheme

\[
\alpha_{\text{cutoff}} = \frac{1}{2s} \log \left[ \frac{b^2}{k_s n_0} \frac{2^{s+2}}{s^2(s+1) \Gamma(1/2) \Gamma(s)} \right] \tag{6.21}
\]

To locate the onset of this effect we must determine the point of closest approach of the beams. This can be determined by taking eqs. (6.7) and (6.13) at \( z = 0 \).

\[
\alpha_{\text{dark}}(0) \simeq 1/2 \log(b/a) \quad \text{and} \quad \alpha_{\text{bright}}(0) \simeq \log(b/a) \tag{6.22}
\]

For the same angle of incidence the dark solitons will get twice as close because the force of repulsion is much less. The interaction of dark beams lacks the strong phase dependent 'superposition force' that dominates bright soliton collisions. Figure 6.7 combines (6.22) with (6.21) and shows the region in which the coupled mode theory should be valid. Only at high angles and long signal wavelengths is cut-off a problem. As the single-moded region becomes more extensive we expect the radiation loss to approach 50% and the remainder of the signal to be split symmetrically between the output ports.

![Figure 6.7: Onset of antisymmetric mode cut-off for bright soliton couplers (solid line) and dark soliton couplers (dotted line) for \( b=1 \), plotted as a function of the ratio between the signal and the pump wavelengths \( (\lambda_s/\lambda_p) \). Cut-off occurs at the closest approach of the beams when \( a_c \) lies above the relevant line.](image)

**Bright soliton couplers**

For bright solitons the combination of eq. (6.13) and (6.19) yields:

\[
C(z) = \frac{1}{k_s n_0} \frac{2^{s+1} a^{2s} \kappa}{(s+1) b^{2s-2} \Gamma(s) \Gamma(1/2)} \sech^{2s} \left( \frac{2ab}{k_p n_0} z \right) \tag{6.23}
\]
If we now consider the case where one of the induced waveguides is illuminated with the signal mode, then the power that couples into the other core has been shown to be (Snyder and Love 1983)

\[ P_2 = \sin^2 \left[ \int_{-\infty}^{\infty} C(z) \, dz \right] \]  

(6.24)

For the coupling coefficient given above, this yields (for bright solitons)

\[ P_2 = \sin^2 \left[ \frac{2\kappa}{(s+1) \lambda_p} \left( \frac{2a}{b} \right)^{2s-1} \right] \]  

(6.25)

**Figure 6.8:** Transmission coefficient \( \frac{P_2}{P_{in}} \) of bright soliton induced junctions, as a function of \( \frac{\lambda_s}{\lambda_p} \). For the solid curve \( a/b = 0.02 \); for the dashed curve \( a/b = 0.1 \).

Figure 6.8 shows the variation of transmission coefficient with wavelength for \( a/b = 0.02 \) and \( a/b = 0.1 \). Eq. (6.25) shows that the only effect of the material parameter \( \kappa \) is to scale the effective wavelength ratio. We set it to unity in all future calculations. The wavelength sensitivity decreases with increasing incident angle, however at larger incident angles the junction will lose signal to radiation when operated at disparate wavelengths. Cut-off also occurs earlier for larger angles. The solid curve (representing smaller angles of incidence) will therefore be far more accurate.

When \( s = 1 \) the transmitted power can be written as

\[ P_2 = \sin^2 \left( \sqrt{2} \frac{\theta}{\theta_c} \right) \]  

(6.26)

\( \theta_c \) is the critical angle of total internal reflection and \( \theta \) is the incident angle. This result agrees with the small angle limit of previous numerical deductions (Akhmediev and Ankiewicz 1993) and the analytical work of section 6.2.
6.3 Different wavelengths: approximation scheme

Dark soliton couplers

We expect dark soliton collisions to be very interesting since our analytical studies indicate 100% coupling regardless of the incident angle. This behaviour is very useful for designing dark soliton induced devices because signal pathways will not be disturbed by cross talk arising during collisions.

Following the same method as for bright beams, we substitute eq. (6.7) into eq. (6.19) and make use of eq. (6.24), which yields

$$P_2 = \sin^2 \left[ \frac{2s \lambda_s}{(s+1) \lambda_p} \left( \frac{a}{b} \right)^{s-1} \frac{\Gamma(s+1/2) \Gamma(s/2)}{\Gamma(s) \Gamma(s/2+1/2)} \right]$$

(6.27)

![Figure 6.9: Transmission coefficient of dark soliton induced junctions, using the same parameters as figure 6.8. Note that there is always 100% transmission at \((\lambda_s/\lambda_p) = 1\)

as expected, for identical wavelengths (in fact for \(\lambda_s = \kappa \lambda_p\)) the induced coupler is a half beat length device independent of incident angle (in complete agreement with exact results). The variation of transmitted power with signal wavelength is shown in figure 6.9 for two values of \(a/b\). Although our approximation is exact for \(\lambda_s = \lambda_p\) we do not expect such accuracy at other wavelengths; as in figure 6.8 the solid line representing a smaller incident angle will be more accurate. Once again the high-angle device shows less wavelength dependence, a consequence of a short interaction length.

We find that both bright and dark solitons induce couplers whose 'length' (expressed in terms of the coupling length) increases as the signal wavelength is increased relative to the soliton (pump) wavelength. At short signal wavelengths little light is coupled across while at long wavelengths the signal will swap from core to core. For \(\lambda_s = \lambda_p\) numerical simulations (Akhmediev and Ankiewicz 1993; Yang, Luther-Davies, and Krolikowski 1993) have been carried out. For bright solitons our result is consistent with the numerical studies, showing the exchanged power to increase with incident angle.
6.4 Summary

The use of spatial solitons to write waveguide devices enables the switching and steering of light by light itself, without intervening fabricated structures. In this chapter we have examined the behaviour of these devices in Kerr media: in doing so we have made contact with the limited numerical studies examining these systems. We first considered devices that could be modelled using integrable $U(n)$ nonlinear Schrödinger equations. We had already derived complete expressions describing the propagation of light through these devices in Chapters 3 and 4. Characterising their behaviour was therefore a simple matter. The integrability of the underlying equations lends a touch of the miraculous to these X-junctions: they are always lossless and when fabricated using dark solitons or orthogonally polarised solitons, there is zero cross talk.

Sadly the previous study is only relevant for a limited class of materials and a limited range of devices. We therefore have also conducted a more general investigation using coupled mode theory and asymptotics of the exact solutions. These results are only valid for cases where the soliton channels change slowly and remain well separated throughout the device. Practical devices must, in any case, be fabricated within these constraints. Expressions describing both bright and dark soliton trajectories were derived and then used to calculate (for the first time) the transmission properties of the couplers induced in these collisions. The results obtained here agree with our own analytical formulae, and with other numerical studies.
Chapter 7

Conclusions and Further Work

This thesis has been directed towards novel optical phenomena that occur in materials exhibiting the optical Kerr effect. In particular, we have investigated the different types of solitary wave that one might find when the field is composed of several independent components, interacting through the nonlinearity. Optical solitary waves manifest themselves in two distinct physical systems. They can be temporal pulses in optical fibres, where they may soon be used in commercial transmission systems. Additionally they appear as self-trapped spatial beams propagating through a bulk material, having potential application to all optical switching. An investigation into solitary waves therefore has potential relevance to two different disciplines. In addition to this, true solitons, as solutions of integrable equations, are of mathematical interest in themselves.

This thesis traverses a broad spectrum of interests. Included are studies of the integrable Manakov system, the discovery and investigation of many qualitatively new types of solitary wave, and the characterisation of soliton written X-junction devices. There is work of interest to both mathematician and experimental physicist.

The focussing $U(2)$ CNSE was first integrated by Manakov (1974), but since then has seen little interest. We have used Hirota's method to derive a number of new solutions to focussing and defocussing integrable CNSE. In particular we have found, for the first time, bright-dark solitons of the Manakov equations, and derived the bright-dark $N$-soliton formulae. These solutions help the characterisation of the equation and also tell us how multi-component solitons interact. The wide applicability of the Manakov system make any solutions particularly relevant. However the integrable systems are only a part of the story, and when the $U(2)$ symmetry of the Manakov system is broken we have used numerical and approximation methods to locate new solitary waves. The interest in bright soliton bound states is lessened by their instability, and it is the polarisation domain wall that is the most interesting of the new solitary waves found in this thesis. The robustness and lack of interaction forces that characterise domain wall propagation make it potentially suitable for transmission and switching applications.

The prospect of using light as a fabricating tool for waveguide devices has a certain appeal, and we therefore examine the properties that this type of junction is likely to possess. Like previous chapters, the integrability of the Manakov system again proves useful and we are able to completely characterise the X-junctions for certain cases when the signal and pump are of the same frequency. The miraculous consequences of integrability are physically realised so that waveguide X-junctions formed from solitons exhibit remarkable properties. Total elimination of loss to radiation and through cross talk would be feasible if only these devices could be fabricated.

The most clearly incomplete work in this thesis relates to solitary waves with two transverse dimensions. While we have conducted some numerical simulations to characterise their propagation dynamics, the behaviour of our solitary waves and of optical vortices is not well understood. Future devices incorporating all optical switching are likely to utilise bulk Kerr materials; understanding the propagation of light in such
materials is therefore important. There is a striking visual resemblance between the propagation of optical vortices and domain walls, governed by a wave equation, and the dynamics of vortices and interfaces in fluids, governed by a diffusion equation. One expects there to be a unifying mathematical structure that links these apparently disparate physical phenomena.

The obvious next step from any theoretical study like this is the experimental observation of the phenomena that we have predicted. Since the polarisation and wavelength domains have been shown to be stable to a wide variety of perturbations, it seems logical to search for experimental verification of these solitary waves. One might expect to find our solitary waves in optical fibres or in bulk Kerr materials like polymers, liquids and atomic vapours. At the Université Libre de Bruxelles in Belgium work is currently being undertaken to try and observe PDW solitary waves in optical fibres.
References


REFERENCES


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