NUMERICAL PROCEDURES

FOR

VOLTERRA INTEGRAL EQUATIONS

BY

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PREFACE

Several of the results of this thesis have been established during collaboration with R.S. Anderssen and F.R. de Hoog. In particular, the work of chapters 2 and 3 is based on results established jointly with F.R. de Hoog while some of the results in chapter 4 are based on work with R.S. Anderssen. The publication details regarding this work are: Anderssen, de Hoog and Weiss (1972), de Hoog and Weiss (1972, a, b, c), Weiss (1972) and Weiss and Anderssen (1972). Often, within the thesis, the text of these papers has been closely followed.

Elsewhere in this thesis, unless another source is acknowledged, the work described is my own.

Richard Weiss

ABSTRACT

This thesis investigates new finite difference methods for the numerical solution of Volterra integral equations.

After a brief discussion of the relevant literature in chapter 1, implicit Runge-Kutta methods, based on interpolatory quadrature formulae, are derived in chapter 2 for Volterra integral equations of the second kind,

$$y(t) = g(t) + \int_0^t K\big(t, s, y(s)\big)ds , \quad t \geq 0$$

with continuous kernels $K(t, s, y)$. Convergence of the schemes is examined. The order of convergence is equal to the degree of precision of the related quadrature formula plus one. In addition, the methods are shown to be numerically stable. For certain choices of quadrature formulae they are also A-stable in the sense of Dahlquist.

In chapter 3, the implicit Runge-Kutta methods developed in chapter 2 are applied to Volterra integral equations of the first kind,

$$g(t) = \int_0^t k(t, s)y(s)ds , \quad t \geq 0 ,$$

where $k(t, s)$ satisfies certain smoothness conditions and $k(t, t) \neq 0$. For the schemes obtained, simple necessary and sufficient conditions for convergence and numerical stability are derived. From these conditions, schemes which are convergent of arbitrarily high order and numerically stable can be constructed.

Finite difference schemes for the generalized Abel equation,

$$g(t) = \int_0^t \frac{k(t,s)}{(t-s)^\alpha} y(s)ds , \quad t \geq 0 , \quad 0 < \alpha < 1 ,$$

where $k(t, s)$ satisfies appropriate smoothness conditions and

$k(t, t) \neq 0$, are investigated in chapter 4. Using product integration techniques, the midpoint, Euler and trapezoidal methods for Volterra integral equations of the first kind with continuous kernels and the schemes developed in chapter 3 are extended to this equation. Convergence results for the product integration analogues of the midpoint, Euler and trapezoidal methods are derived.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Volterra integral equations are usually classified into equations of the second kind

$$y(t) = g(t) + \int_0^t K\big(t, s, y(s)\big)ds \ , \quad t \geq 0$$

and equations of the first kind

$$g(t) = \int_0^t K\big(t, s, y(s)\big)ds \ , \quad t \geq 0 \ .$$

In most numerical schemes, the equations are discretized at a number of distinct points and quadrature rules are used to approximate the integral terms. If such methods are convergent, the order of convergence will depend on the accuracy of the quadrature rule used. In the construction of efficient quadrature rules the smoothness properties of the specific kernel $K(t, s, y)$ under consideration are a critical factor. It is therefore useful to distinguish between equations with continuous and singular kernels.

In this thesis, new finite difference methods for Volterra integral equations of the first and second kind with continuous kernels and for equations of the first kind with kernels having an algebraic singularity are examined.

In the following four sections a short survey of finite difference schemes for Volterra integral equations is given. For a comprehensive bibliography on approximate methods for Volterra integral equations, the reader is referred to Noble (1971 a, b).

1.2 Volterra Equations of the Second Kind with Continuous Kernels

The general form of a second kind Volterra integral equation with a continuous kernel is

$$y(t) = g(t) + \int_0^t K\big(t, s, y(s)\big)ds \ , \quad t \geq 0 \ , \tag{1.1}$$

where

(1.2.1) $K(t, s, y)$ is continuous with respect to t and s and uniformly Lipschitz continuous with respect to y for $0 \leq s \leq t \leq T < \infty$ and all finite y ,

(1.2.2) $g(t)$ is continuous on $0 \leq t \leq T$.

Under these conditions, (1.1) has a unique continuous solution on $0 \leq t \leq T$ (see for instance Davis (1960), p. 415).

If the kernel does not depend on t , and $g(t)$ is differentiable, then (1.1) reduces to an ordinary differential equation. This correspondence has made it possible for certain classes of methods for ordinary differential equations to be extended to (1.1).

(i) Runge-Kutta methods. The derivation of standard Runge-Kutta methods for (1.1) has been the subject of extensive investigation (see for instance Pouzet (1960, 1962), Laudet and Oules (1960) and Beltjukov (1965)).

(ii) Linear multistep methods. The extensions of these methods to (1.1) are sometimes referred to as step by step methods (see Linz (1967 a)). Early advocates of these methods include Fox and Goodwin (1953), Jones (1961) and Noble (1964). A general definition of linear multistep methods for (1.1) combined with a rigorous theoretical treatment has been provided by Kobayasi (1966). The concept of the repetition factor of a linear multistep method has been introduced by Linz (1967 a) who shows that schemes with a repetition factor of one are numerically stable. He also gives examples of schemes with a repetition factor greater than one which are only weakly stable. A more elegant proof of Linz's result is given by Noble (1969).

A class of methods derived independently of ordinary differential equations are the block by block methods. The characteristic feature of these schemes is that a "block" of approximations to $y(t)$ is obtained at each step, rather than a single approximation. The use of block by block methods was first suggested by Young (1954) in the context of Volterra integral equations of the second kind with singular kernels. Linz (1967 a) has formulated the methods for equations with continuous kernels and established convergence.

Other finite difference methods not mentioned above include the schemes developed by Hung (1970) which produce spline approximations to $y(t)$.

1.3 Volterra Equations of the Second Kind with Singular Kernels

The general form of second kind Volterra integral equations with singular kernels is

$$y(t) = g(t) + \int_0^t p(t, s)K\big(t, s, y(s)\big)ds \ , \quad t \geq 0 \ , \qquad (1.2)$$

where

(1.3.1) $K(t, s, y)$ and $g(t)$ satisfy the conditions (1.2.1) and (1.2.2) respectively,

(1.3.2) $\displaystyle\int_{t_1}^{t_2} |p(t, s)|ds \to 0$ as $t_2 \to t_1$, uniformly in t and t_1 for $0 \leq t_1 \leq t_2 \leq t \leq T$.

The conditions (1.3.1) and (1.3.2) are sufficient for the existence of a unique continuous solution on $0 \le t \le T$ (see Evans (1910)). Some typical forms of $p(t, s)$ encountered in practice are

$$p(t, s) = (t-s)^{-\frac{1}{2}} \ , \quad p(t, s) = s\left(t^2 - s^2\right)^{-\frac{1}{2}} \ .$$

Young (1954) has suggested the use of product integration for the approximate solution of integral equations with singular kernels. Using this technique, the linear multistep methods and the block by block methods mentioned in §1.2 can be extended to (1.2). Specific examples of methods of this kind have been considered by Noble (1964) and Oules (1964). The convergence of methods based on product integration has been investigated by Linz (1967 a).

1.4 Volterra Equations of the First Kind with Continuous Kernels

To date, only linear equations

$$g(t) = \int_0^t k(t, s)y(s)ds \ , \quad t \ge 0 \ , \tag{1.3}$$

have been considered in the literature. To ensure the existence of a unique continuous solution on $0 \le t \le T < \infty$, the following conditions are required (see for instance Tricomi (1957), pp 15-16):

(1.4.1) $k(t, s)$ and $\partial k(t, s)/\partial t$ are continuous on $0 \le s \le t \le T$,

(1.4.2) $k(t, t) \ne 0$ on $0 \le t \le T$,

(1.4.3) $g(t)$ is continuously differentiable on $0 \le t \le T$ and $g(0) = 0$.

The linear multistep methods and the block by block methods introduced in §1.2 can be formulated for (1.3). However, a general analysis of these schemes has as yet not been given. Only the methods obtained when approximating the integral term in (1.3) by means of the trapezoidal, midpoint and Euler rules, respectively, have been investigated.

The trapezoidal method for the case of a convolution kernel $k(t, s) = k(t-s)$ has been examined by Jones (1961) who established order two convergence. In addition, Jones observed that the error incurred when using the trapezoidal method can be highly oscillatory. This oscillatory behaviour was explained by Kobayasi (1967) who proved that the trapezoidal method is convergent of order two for a general $k(t, s)$ but is only weakly stable. Convergence for a general $k(t, s)$ was established independently by Linz (1967 a).

Noble (1964) suggested that the midpoint method would be a more suitable numerical procedure. This was verified by Linz (1967 a) who established order two convergence and numerical stability for this scheme.

The Euler method has been shown to be convergent of order one and numerically stable by Linz (1967 a).

Hung (1970) considers variants of the trapezoidal and midpoint methods which are based on product integration. The product integration analogue of the midpoint method has also been examined by Squire (1969) and Linz (1971).

1.5 Volterra Equations of the First Kind with Singular Kernels

The first kind equations with singular kernels most frequently encountered take one of three forms:

$$\text{(i)} \qquad g(t) = 2\int_t^T \frac{y(s)s}{(s^2-t^2)^{\frac{1}{2}}}\,ds\ , \quad 0 \le t \le T < \infty . \tag{1.4}$$

An analytic inversion formula for (1.4) is known (see for instance Minerbo and Levy (1969)). Under appropriate conditions it is given by

$$\text{(iii)} \qquad y(t) = -\frac{1}{\pi}\int_t^T \frac{g'(s)}{(s^2-t^2)^{\frac{1}{2}}}\,ds\ , \quad 0 \le t \le T\ . \tag{1.5}$$

Approximate solutions can be obtained by solving (1.4) directly by a numerical procedure, or alternatively, by applying numerical quadrature to (1.5). In both cases the quadrature formulae used are obtained by product integration.

Direct methods which use piecewise constant and piecewise linear approximations to $y(t)$ have been suggested by Schardin (1933) and Winckler (1948) respectively. Recently, Einarsson (1971) has examined methods where $y(t)$ is approximated by a cubic spline.

In the procedures based on (1.5), $g(t)$ is approximated by piecewise polynomials of low order and the integrals are evaluated analytically. Such schemes are given in Olsen (1959), Bockasten (1961), Nestor and Olsen (1962), Edels, Hearne and Young (1962) and Free (1963).

$$\text{(ii)} \qquad g(t) = \int_0^t \frac{k(t,s)y(s)}{(t-s)^{\alpha}}\,ds\ , \quad 0 < \alpha < 1\ , \quad t \ge 0\ , \tag{1.6}$$

where $k(t, s)$ is continuous and $k(t, t) \neq 0$. Eq. (1.6) with $k(t, s) = 1$ and $\alpha = \frac{1}{2}$ is Abel's integral equation. After an appropriate transformation of variables (see for instance Minerbo and Levy (1969)), (1.4) takes the form (1.6) with $k(t, s) = 1$ and $\alpha = \frac{1}{2}$. For the case $k(t, s) = 1$, an analytic inversion formula is known (see for instance Tricomi (1957), p. 40). A class of methods for the evaluation of this inversion formula in the case $\alpha = \frac{1}{2}$ is considered in Edels, Hearne and Young (1962).

In general, inversion formulae are not available and schemes based on (1.6) must be used. Methods of this type were developed by Durbin (1971) for a particular class of equations which arise in the analysis of Brownian motion and diffussion processes.

(iii) $$g(t) = \int_0^t \frac{y(s)}{(t^2-s^2)^{\frac{1}{2}}}\, ds\ , \quad t \geq 0\ . \tag{1.7}$$

This equation was first studied by Linz (1967 b) who examines a direct method and a method which utilizes the known inversion formula. Atkinson (1971 a) has proved convergence of a direct method based on a piecewise linear approximation to $y(t)$.

1.6 Thesis Outline

Chapter 2 deals with the numerical solution of second kind Volterra integral equations with continuous kernels (Eq. (1.1)). Implicit Runge-Kutta methods based on interpolatory quadrature formulae are developed for (1.1) and convergence and stability of the methods are examined. These schemes contain as a subclass the block by block methods given in Linz (1967 a).

It is shown that the methods are convergent and that the order of convergence is equal to the degree of precision of the related quadrature formula plus one. This extends a result obtained by Axelsson (1969).

The asymptotic behaviour of the numerical solution for a "small" gridspacing is investigated and the methods are shown to be numerically stable. For certain choices of quadrature formulae they also have the stronger property of being A-stable in the sense of Dahlquist (1963).

Finally it is concluded that the schemes utilizing Lobatto or Radau quadrature display highest possible orders of convergence combined with favourable stability properties.

In chapter 3, the numerical solution of linear Volterra integral equations of the first kind with continuous kernels (Eq. (1.3)) is considered. After a comparison of the stability properties of the trapezoidal and midpoint methods is made, the implicit Runge-Kutta methods developed in chapter 2 are applied to (1.3). The results of this chapter show that the implicit Runge-Kutta methods contain schemes which are convergent of arbitrarily high order and numerically stable. As far as the author is aware these are the first high order schemes developed for first kind equations.

Finite difference methods for a class of first kind Volterra equations with singular kernels, (Eq. (1.6)), which will be called generalized Abel equations, are examined in chapter 4. The midpoint, Euler and trapezoidal methods used for the numerical solution of first kind equations with continuous kernels are extended to (1.6) by means of product integration. The resulting schemes are examined and shown to be convergent under appropriate conditions on $k(t, s)$ and $y(t)$. Finally the implicit Runge-Kutta methods of chapter 3 are extended to (1.6) by product integration and investigated numerically.

CHAPTER 2

IMPLICIT RUNGE-KUTTA METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

2.1 Introduction

As mentioned in chapter 1, the extension of explicit Runge-Kutta methods for ordinary differential equations to Volterra integral equations of the second kind with continuous kernels,

$$y(t) = g(t) + \int_0^t K\big(t, s, y(s)\big)ds , \tag{2.1}$$

has received considerable attention.

Implicit Runge-Kutta methods for the solution of ordinary differential equations have been studied by Butcher (1964 a, b), Axelsson (1969) and Wright (1970) and have been shown to possess desirable stability properties combined with high order of convergence. Butcher shows that to each Runge-Kutta process there corresponds a numerical quadrature formula.

In this chapter, the idea of constructing implicit Runge-Kutta methods, based on quadrature formulae, is extended to (2.1). The basis of the quadrature formulae under consideration for the interval $[0, 1]$ is Lagrangian interpolation with respect to a set of points $\{u_1, u_2, \ldots, u_n\}$ with $0 \le u_1 < u_2 < \ldots < u_n = 1$. Two different methods are defined for each set and are shown to be convergent of order $p + 1$ where p is the degree of precision of the associated quadrature formula. In addition the methods are shown to be numerically stable in the sense of Kobayasi (1966) and for special choices of points have the stronger property of being A-stable in the sense of Dahlquist (1963).

The conditions on $g(t)$ and $K(t, s, y)$ ensuring the existence of a unique continuous solution of (2.1) have been given in §1.2 and are restated here for convenience:

(2.1.1) $K(t, s, y)$ is continuous with respect to t and s for all finite y on $0 \le s \le t \le T$,

(2.1.2) $K(t, s, y)$ satisfies the Lipschitz condition

$$|K(t, s, y_1)-K(t, s, y_2)| \le L|y_1-y_2| ,$$

$$0 \le s \le t \le T , \quad |y_1|, |y_2| < \infty \tag{2.2}$$

where L is a constant independent of t, s, y_1 and y_2,

(2.1.3) $g(t)$ is continuous on $0 \le t \le T$.

However, in the subsequent convergence analysis and in the analysis of the asymptotic behaviour of the error, additional smoothness conditions will be imposed on $K(t, s, y)$ and $g(t)$.

It should be noted that although the analysis is presented only for the scalar equation (2.1), the generalization to a system of Volterra integral equations of the second kind follows immediately.

2.2 Preliminaries

In this section some notation is introduced and four lemmas and two corollaries which will be required in subsequent analysis are presented.

Let $0 \le u_1 < u_2 < \ldots < u_n = 1$ and

$$\omega(t) = (t-u_1)(t-u_2) \ldots (t-u_n) = \sum_{j=0}^{n} \alpha_j t^{n-j} .$$

Define

$$L_k(t) = \frac{\omega(t)}{(t-u_k)\,\omega'(u_k)}\ , \quad k = 1, \ldots, n\ , \tag{2.3}$$

$$\left.\begin{aligned} a_{jkr} &= \int_0^{u_j} s^r L_k(s)ds\ , \quad j = 1, \ldots, n\ ;\ r = 0, 1, \ldots\ , \\ a_k &= a_{nk0} \\ \text{and} \\ a_{jk} &= a_{jk0}\ . \end{aligned}\right\} \tag{2.4}$$

Denote the relation

$$\int_0^1 \omega(s)ds \neq 0$$

by $\omega(t) \in \mathcal{P}_0$, and the relations

$$\left.\begin{aligned} &\int_0^1 s^r\omega(s)ds = 0\ , \quad r = 0, \ldots, \nu-1,\ \nu > 0 \\ &\text{and} \\ &\int_0^1 s^\nu\omega(s)ds \neq 0 \end{aligned}\right\} \tag{2.5}$$

by $\omega(t) \in \mathcal{P}_\nu$.

Let

$$R_i(f) = \int_0^{u_i} f(s)ds - \sum_{k=1}^{n} a_{ik} f(u_k)\ , \quad i = 1, \ldots, n\ . \tag{2.6}$$

The following lemmas and corollaries are generalizations of results due to Axelsson (1969).

<u>Lemma 2.1.</u>

If $\omega(t) \in \mathcal{P}_\nu$, then

$$R_i(u^{n+q}) = \int_0^{u_i} s^q\omega(s)ds - \sum_{r=1}^{q} \alpha_r R_i(u^{n+q-r})\ , \quad q = 0, \ldots, n\ ,$$

and

$$R_n(u^{n+q}) = 0\ , \quad q \leq \nu - 1\ . \tag{2.7}$$

Proof. Clearly

$$R_i(u^q) = 0 \ , \quad q = 0, \ldots, n-1 \ ,$$

and

$$t^{n+q} = t^q\omega(t) - t^q \sum_{r=1}^{n} \alpha_r t^{n-r} \ .$$

The result follows from the linearity of the operator R_i . #

Remark 2.1. This result includes the well known fact that the degree of precision of the quadrature formula

$$\int_0^1 f(s)ds \simeq \sum_{k=1}^{n} a_k f(u_k)$$

is $n + \nu - 1$.

Lemma 2.2.

If $f_q(t)$ is a polynomial of degree less than or equal to q , and $r + q \le n + \nu - 2$, then

$$\sum_{k=1}^{n} a_k u_k^r \int_0^{u_k} f_q(s)ds = \int_0^1 \frac{1-s^{r+1}}{r+1} f_q(s)ds \ .$$

Proof. Using (2.4), lemma 2.1 and partial integration,

$$\sum_{k=1}^{n} a_k u_k^r \int_0^{u_k} f_q(s)ds = \sum_{k=1}^{n} \int_0^1 L_k(x)dx \, u_k^r \int_0^{u_k} f_q(s)ds$$

$$= \int_0^1 x^r \int_0^x f_q(s)ds \, dx$$

$$= \int_0^1 \frac{1-s^{r+1}}{r+1} f_q(s)ds \ . \quad \#$$

Corollary 2.1.

$$\sum_{k=1}^{n} a_k u_k^r R_k(u^{n+p}) = 0 \ , \quad r + p = 0, \ldots, \nu-2 \ .$$

Proof. Using lemmas 2.1, 2.2 and Eq. (2.5),

$$\sum_{k=1}^{n} a_k R_k\left(u^n\right) = \sum_{k=1}^{n} a_k \int_0^{u_k} \omega(s)ds$$

$$= \int_0^1 (1-s)\omega(s)ds$$

$$= 0 .$$

Hence the result is true for $r + p = 0$.

Suppose the result is true for $r + p = 0, \ldots, \ell-1$; $\ell-1 < \nu-2$. Let $r + p = \ell$. Then, applying lemmas 2.1, 2.2 and (2.5),

$$\sum_{k=1}^{n} a_k u_k^r R_k\left(u^{n+p}\right) = \sum_{k=1}^{n} a_k u_k^r \left(\int_0^{u_k} s^p\omega(s)ds - \sum_{j=1}^{p} \alpha_j R_k\left(u^{n+p-j}\right)\right)$$

$$= \int_0^1 \frac{1-s^{r+1}}{r+1} s^p\omega(s)ds = 0 .$$

The result follows by induction. #

Corollary 2.2.

$$\sum_{k=1}^{n} a_k u_k^r R_k\left(\left(u-u_k\right)^{n+p}\right) = 0 , \quad r + p = 0, \ldots, \nu-2 .$$

Proof. The result follows from corollary 2.1 and the linearity of the operator R_k. #

Lemma 2.3.

If $r \le \nu$, then

$$\sum_{k=1}^{n} a_k u_j^{r+1}\left(u_k-1\right)^r L_m\left(u_j u_k\right) = \sum_{\ell=0}^{r} \binom{r}{\ell}\left(-u_j\right)^{r-\ell} a_{jm\ell} . \qquad (2.8)$$

Proof. Using (2.7),

$$\sum_{k=1}^{n} a_k u_j^{r+1}(u_k-1)^r L_m(u_j u_k) = u_j^{r+1}\int_0^1 (s-1)^r L_m(u_j s)ds$$

$$= \int_0^{u_j} (s-u_j)^r L_m(s)ds$$

$$= \sum_{\ell=0}^{r} \binom{r}{\ell}(-u_j)^{r-\ell}\int_0^{u_j} s^\ell L_m(s)ds$$

$$= \sum_{\ell=0}^{r} \binom{r}{\ell}(-u_j)^{r-\ell} a_{jm\ell} . \qquad \#$$

Remark 2.2. For $r = 0$, Eq. (2.8) reduces to

$$u_j \sum_{k=1}^{n} a_k L_m(u_j u_k) = a_{jm} . \tag{2.9}$$

This identity will be used repeatedly.

The following lemma which provides an estimation of the growth of the solutions of nonhomogeneous difference equations is given in Henrici (1962), p. 313.

Lemma 2.4.

If $|\varepsilon_i| \le A \sum_{k=0}^{i-1} |\varepsilon_k| + B$, $i = 1, 2, \ldots$, $A, B > 0$ and $|\varepsilon_0| \le \eta$, then

$$|\varepsilon_i| \le (B+A\eta)e^{(i-1)A} , \quad i = 1, 2, \ldots .$$

2.3 Numerical Schemes

In this section two implicit Runge-Kutta methods corresponding to a fixed set $\{u_1, u_2, \ldots, u_n\}$ are defined.

The aim is to obtain an approximation to the solution $y(t)$ on the grid

$$t_i = ih , \quad i = 0, \ldots, I ; \quad h = T/I .$$

Let

$$t_{ij} = t_i + u_j h\ , \quad j = 1, \ldots, n\ ; \quad i = 0, \ldots, I-1\ .$$

Discretizing (2.1) yields

$$y\left(t_{ij}\right) = g\left(t_{ij}\right) + \int_0^{t_i} K\left(t_{ij}, s, y(s)\right)ds + \int_{t_i}^{t_{ij}} K\left(t_{ij}, s, y(s)\right)ds\ ,$$

$$j = 1, \ldots, n\ ; \quad i = 0, \ldots, I-1\ . \qquad (2.10)$$

To approximate $\int_0^{t_i} K\left(t_{ij}, s, y(s)\right)ds$, the quadrature formula

$$\int_{t_\ell}^{t_{\ell+1}} f(s)ds \simeq h \sum_{k=1}^{n} f\left(t_{\ell k}\right)\ , \quad \ell = 0, \ldots, i-1\ ,$$

is used. An approximation to $\int_{t_i}^{t_{ij}} K\left(t_{ij}, s, y(s)\right)ds$ is now required. A natural extension of the above is to apply the quadrature formula

$$\int_{t_i}^{t_{ij}} f(s)ds \simeq \sum_{k=1}^{n} h\, a_{jk}\, f\left(t_{ik}\right)\ .$$

This leads to the numerical scheme

$$Y_{ij} = g\left(t_{ij}\right) + \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} h\, a_k K\left(t_{ij}, t_{\ell k}, Y_{\ell k}\right) + \sum_{k=1}^{n} h\, a_{jk} K\left(t_{ij}, t_{ik}, Y_{ik}\right)\ ,$$

$$j = r, \ldots, n\ ; \quad i = 0, \ldots, I-1\ , \qquad (2.11)$$

where

$$r = \begin{cases} 1 & \text{if } u_1 > 0 \\ 2 & \text{if } u_1 = 0 \end{cases}$$

and Y_{ij} denotes the approximation to $y\left(t_{ij}\right)$. If $u_1 = 0$ then

$Y_{01} = g(0)$ and $Y_{i+1,1} = Y_{in}$, $i = 0, \ldots, I-2$.

For each i, (2.11) represents a system of $n + 1 - r$ simultaneous equations in Y_{ij}, $j = r, \ldots, n$. It follows from a contraction mapping argument that this system has a unique solution if h is sufficiently small. Also, it may be seen from (2.11) that values of $K(t, s, y)$ are required outside the region $0 \le s \le t \le T$ and this could cause difficulties in practice if the kernel is badly behaved outside this region.

This problem can be overcome by using a different approximation for $\int_{t_i}^{t_{ij}} K\left(t_{ij}, s, y(s)\right)ds$. First approximate

$$\int_{t_i}^{t_{ij}} K\left(t_{ij}, s, y(s)\right)ds \quad \text{by} \quad \int_{t_i}^{t_{ij}} K\left(t_{ij}, s, \sum_{k=1}^{n} L_k\left(\frac{s-t_i}{h}\right)y\left(t_{ik}\right)\right)ds$$

and then apply the quadrature formula

$$\int_{t_i}^{t_{ij}} f(s)ds \simeq \sum_{k=1}^{n} hu_j a_k f\left(t_i+u_j u_k h\right) .$$

This yields the numerical scheme

$$Y_{ij} = g\left(t_{ij}\right) + \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k K\left(t_{ij}, t_{\ell k}, Y_{\ell k}\right)$$

$$+ \sum_{k=1}^{n} hu_j a_k K\left(t_{ij}, t_i+u_j u_k h, \sum_{r=1}^{n} L_r\left(u_j u_k\right)Y_{ir}\right),$$

$$j = r, \ldots, n ; \quad i = 0, \ldots, I-1 . \quad (2.12)$$

Clearly, in (2.12), $K(t, s, y)$ is not required outside the region $0 \le s \le t \le T$. For the special case

$$K(t, s, y) = \lambda y , \quad \lambda = \text{const},$$

the schemes (2.11) and (2.12) are the same. This is a consequence of (2.9).

It should be noted that the scheme (2.11) can also be considered as arising from a piecewise continuous spline interpolating to $K(t, s, y(s))$ at $s = t_{ij}$, $j = 1, \ldots, n$; $i = 0, \ldots, I-1$.

Methods based on splines with full continuity have been considered by Hung (1970) and have been shown to be divergent for splines with degree greater than two.

Remark 2.3. In the sequel only the schemes with $u_1 > 0$ will be considered. However with slight notational modifications the analysis applies also to the schemes with $u_1 = 0$.

2.4 Convergence of the Numerical Schemes

Let $\omega(t) \in P_\nu$. In the subsequent analysis it will be assumed that

(2.4.1) $K(t, s, y)$ is $n + \nu + 1$ times continuously differentiable with respect to t, s and y respectively on $0 \le s \le t+\delta$, $0 \le t \le T$, $|y| \le \bar{y}$, where δ is a fixed positive number for the scheme (2.11), $\delta = 0$ for the scheme (2.12) and $\bar{y} = \max\limits_{0 \le t \le T} |y(t)|$,

(2.4.2) $\frac{\partial K}{\partial y}(t, s, y)$ is Lipschitz continuous with respect to y on $0 \le s \le t+\delta$, $0 \le t \le T$, $|y| \le \bar{y}+\tau$, $\tau > 0$.

(2.4.3) $g(t)$ is $n + \nu + 1$ times continuously differentiable on $0 \le t \le T$.

The conditions (2.4.1) with $\delta = 0$ and (2.4.3) ensure that $y(t)$ is $n + \nu + 1$ times continuously differentiable on $0 \le t \le T$. However, condition (2.4.3) is not used explicitly in the subsequent analysis.

Define

$$y_{ij} = y(t_{ij}) , \quad j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 ,$$

and

$$\varepsilon_{ij} = y_{ij} - Y_{ij} \ , \quad j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ .$$

Firstly, the method (2.11) with $h < \delta$ will be considered. Subtraction of (2.11) from (2.10) yields

$$\varepsilon_{ij} = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k\{K(t_{ij}, t_{\ell k}, y_{\ell k}) - K(t_{ij}, t_{\ell k}, Y_{\ell k})\}$$

$$+ \sum_{k=1}^{n} ha_{jk}\{K(t_{ij}, t_{ik}, y_{ik}) - K(t_{ij}, t_{ik}, Y_{ik})\} + R_{ij} \ ,$$

$$j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ , \tag{2.13}$$

where

$$R_{ij} = P_{ij} + Q_{ij}$$

and

$$P_{ij} = \sum_{\ell=0}^{i-1} \left\{ \int_{t_\ell}^{t_{\ell+1}} K(t_{ij}, s, y(s))ds - \sum_{k=1}^{n} ha_k \ K(t_{ij}, t_{\ell k}, y_{\ell k}) \right\} ,$$

$$Q_{ij} = \int_{t_i}^{t_{ij}} K(t_{ij}, s, y(s))ds - \sum_{k=1}^{n} ha_{jk} \ K(t_{ij}, t_{ik}, y_{ik}) \ .$$

The following lemma provides an asymptotic expansion for R_{ij} .

<u>Lemma 2.5.</u>

Let

$$K(t, s) = K(t, s, y(s))$$

and

$$K^{(m)}(t, s) = \frac{\partial^m}{\partial \eta^m} K(t, \eta)\Big|_{\eta=s} \ .$$

Then

$$R_{ij} = \sum_{p=0}^{\nu} h^{n+p} \phi_{pj}(t_{ij}) + O(h^{n+\nu+1}) \ ,$$

$$j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ , \tag{2.14}$$

where

$$\phi_{pj}(t) = K^{(n+p-1)}(t,\, t)\, \frac{R_j\left[(u-u_j)^{n+p-1}\right]}{(n+p-1)!}\,,$$

$$p = 0, \ldots, \nu-1;\; j = 1, \ldots, n\,,$$

$$\phi_{\nu j}(t) = \frac{R_n\left(u^{n+\nu}\right)}{(n+\nu)!} \int_0^t K^{(n+\nu)}(t,\, s)ds + \frac{K^{(n+\nu-1)}(t,t)}{(n+\nu-1)!}\, R_j\left[(u-u_j)^{n+\nu-1}\right]\,,$$

$$j = 1, \ldots, n\,.$$

<u>Proof.</u> By Taylor series expansion,

$$\int_{t_\ell}^{t_{\ell+1}} K\left(t_{ij},\, s\right)ds - h \sum_{k=1}^{n} a_k\, K\left(t_{ij},\, t_{\ell k}\right)$$

$$= h^{n+\nu+1}\, \frac{K^{(n+\nu)}\left(t_{ij}, t_\ell\right)}{(n+\nu)!}\, R_n\left(u^{n+\nu}\right) + O\left(h^{n+\nu+2}\right)\,.$$

Thus

$$P_{ij} = h^{n+\nu}\, \frac{R_n\left(u^{n+\nu}\right)}{(n+\nu)!} \int_0^{t_{ij}} K^{(n+\nu)}\left(t_{ij},\, s\right)ds + O\left(h^{n+\nu+1}\right)\,.$$

Again from Taylor series expansion,

$$Q_{ij} = h \int_0^{u_j} K\left(t_{ij},\, t_{ij} + \left(s-u_j\right)h\right)ds - \sum_{k=1}^{n} ha_{jk}\, K\left(t_{ij},\, t_{ij} + \left(u_k - u_j\right)h\right)$$

$$= \sum_{r=0}^{n+\nu-1} \frac{h^{r+1}}{r!}\, K^{(r)}\left(t_{ij},\, t_{ij}\right) \left\{\int_0^{u_j} \left(s-u_j\right)^r ds - \sum_{k=1}^{n} a_{jk}\left(u_k - u_j\right)^r\right\} + O\left(h^{n+\nu+1}\right)$$

$$= \sum_{r=n}^{n+\nu-1} \frac{h^{r+1}}{r!}\, K^{(r)}\left(t_{ij},\, t_{ij}\right) R_j\left[(u-u_j)^r\right] + O\left(h^{n+\nu+1}\right)\,.$$

The result follows. #

<u>Corollary 2.3.</u>

If $\nu > 1$, then

$$\sum_{j=1}^{n} a_j u_j^r \phi_{pj}(t) = 0 \ , \quad r = 0, \ldots, \nu-p-1 \ ; \quad p = 0, \ldots, \nu-1 \ .$$

Proof. The result follows from corollary 2.1. #

In the subsequent analysis use will be made of

Lemma 2.6.

There exists a constant K such that

$$e_i = \max_{1\le j\le n} |\varepsilon_{ij}| \le Kh^n \ , \quad i = 0, \ldots, I-1 \ , \text{ if } \nu = 0 \ ,$$

or

$$e_i \le Kh^{n+1} \ , \quad i = 0, \ldots, I-1 \ , \text{ if } \nu > 0 \ .$$

Proof. Taking absolute values in (2.13) and applying the Lipschitz condition (2.2) yields

$$|\varepsilon_{ij}| \le \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} hL\,|a_k|\,|\varepsilon_{\ell k}| + \sum_{k=1}^{n} hL\,|a_{jk}|\,|\varepsilon_{ik}| + |R_{ij}| \ ,$$

$$j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ .$$

From lemmas 2.1 and 2.5, there exists a constant C such that

$$|R_{ij}| \le Ch^n \ , \quad j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ , \text{ if } \nu = 0 \ ,$$

or

$$|R_{ij}| \le Ch^{n+1} \ , \quad j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ , \text{ if } \nu > 0 \ .$$

Hence

$$e_i \le nhL\,A \sum_{\ell=0}^{i} e_\ell + Ch^n \ , \quad i = 0, \ldots, I-1 \ , \text{ if } \nu = 0 \ ,$$

or

$$e_i \le nhL\,A \sum_{\ell=0}^{i} e_\ell + Ch^{n+1} \ , \quad i = 0, \ldots, I-1 \ , \text{ if } \nu > 0 \ ,$$

where

$$A = \max_{\substack{1\le j\le n \\ 1\le k\le n}} |a_{jk}| \ .$$

The result follows from lemma 2.4. #

The above lemma gives a convergence result for the scheme (2.11). Generally however, this result is not the best possible and in particular a more accurate estimate can be obtained for ε_{in}, $i = 0, \ldots, I-1$. For this analysis four additional lemmas are required.

Lemma 2.7.

Let the functions $\theta_j(t)$, $j = 1, \ldots, n$, satisfy

$$\sum_{j=1}^{n} a_j u_j^r \theta_j(t) = 0 \ , \quad r = 0, \ldots, q \ , \text{ where } 0 < q < \nu-1 \ .$$

Then

$$\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jkm} u_k^\ell \theta_k(t) = 0 \ , \quad p + \ell + m = 0, \ldots, q-1 \ .$$

Proof. Using lemma 2.2,

$$\begin{aligned}\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jkm} u_k^\ell \theta_k(t) &= \sum_{k=1}^{n} u_k^\ell \theta_k(t) \sum_{j=1}^{n} a_j u_j^p \int_0^{u_j} s^m L_k(s)ds \\ &= \sum_{k=1}^{n} u_k^\ell \theta_k(t) \int_0^1 s^m \frac{\left(1-s^{p+1}\right)}{p+1} L_k(s)ds \ ,\end{aligned}$$

$$p + m = 0, \ldots, \nu-1 \ . \quad (2.15)$$

By binomial expansion,

$$\begin{aligned} s^r &= \left[\left(s-u_k\right)+u_k\right]^r \\ &= \sum_{q=0}^{r} \binom{r}{q} \left(s-u_k\right)^q u_k^{r-q} \ ,\end{aligned}$$

and hence

$$s^r L_k(s) = u_k^r L_k(s) + \sum_{q=1}^{r} \binom{r}{q} \left(s-u_k\right)^{q-1} u_k^{r-q} \omega(s)/\omega'\left(u_k\right) \ .$$

Thus, from (2.5),

$$\int_0^1 s^r L_k(s)ds = u_k^r a_k \ , \quad r = 0, \ldots, \nu \ . \qquad (2.16)$$

On substitution of (2.16) into (2.15) it follows that

$$\sum_{j=1}^{n} a_j u_j^p \sum_{k=1}^{n} a_{jkm} u_k^{\ell} \theta(t) = \frac{1}{p+1} \sum_{k=1}^{n} a_k u_k^{\ell+m} \left(1-u_k^{p+1}\right) \theta_k(t)$$

$$= 0 \ , \quad m + \ell + p = 0, \ldots, q-1 \ . \ \#$$

<u>Lemma 2.8.</u>

Let $f(t, s)$ be $M + 1$ times continuously differentiable in the region $0 \le s \le t \le T$ and denote

$$f^{(r)}(t, s) = \frac{\partial^r}{\partial \eta^r} f(t, \eta)\Big|_{\eta=s} \ .$$

Then

$$h \sum_{\ell=0}^{i-1} f\left(t_{ij}, t_{\ell k}\right) = \int_0^{t_{ij}} f\left(t_{ij}, s\right)ds + \sum_{m=0}^{M-1} h^{m+1} \psi_{jkm}\left(t_{ij}\right) + O\left(h^{M+1}\right) \ ,$$

$$M \ge 0 \ ,$$

where

$$\psi_{jkm}(t) = f^{(m)}(t, 0) \sum_{r=0}^{m+1} C_{mr} u_k^r + f^{(m)}(t, t) \sum_{r=0}^{m+1} D_{mr}\left(u_k - u_j - 1\right)^r$$

and

$$C_{mr} \ , \ D_{mr} \ , \ r = 0, \ldots, m+1 \ ; \ m = 0, \ldots, M-1 \ ,$$

are constants.

<u>Proof.</u> By the Euler-Maclaurin sum formula, (see for example Ralston (1965), p. 133),

$$h \sum_{\ell=0}^{i-1} f\left(t_{ij}, t_{\ell k}\right) = \int_{u_k h}^{t_{i-1}+u_k h} f\left(t_{ij}, s\right) ds + \frac{h}{2}\left(f\left(t_{ij}, u_k h\right) + f\left(t_{ij}, t_{i-1}+u_k h\right)\right) + \sum_{r=1}^{[M/2]} \frac{h^{2r} B_{2r}}{(2r)!} \left\{f^{(2r-1)}\left(t_{ij}, t_{i-1}+u_k h\right) - f^{(2r-1)}\left(t_{ij}, u_k h\right)\right\} + O\left(h^{M+1}\right) , \qquad (2.17)$$

where B_r , $r = 1, 2, \ldots$, are the Bernoulli numbers. By Taylor series expansion,

$$\int_0^{u_k h} f\left(t_{ij}, s\right) ds = \sum_{m=0}^{M-1} h^{m+1} \frac{f^{(m)}\left(t_{ij}, 0\right)}{(m+1)!} u_k^{m+1} + O\left(h^{M+1}\right) , \qquad (2.18)$$

$$\int_{t_{i-1}+u_k h}^{t_{ij}} f\left(t_{ij}, s\right) ds = - \sum_{m=0}^{M-1} h^{m+1} \frac{f^{(m)}\left(t_{ij}, t_{ij}\right)}{(m+1)!} \left(u_k - u_j - 1\right)^{m+1} + O\left(h^{M+1}\right) , \qquad (2.19)$$

$$f^{(2r-1)}\left(t_{ij}, u_k h\right) = \sum_{m=0}^{M-2r} h^m \frac{f^{(2r-1+m)}\left(t_{ij}, 0\right)}{m!} u_k^m + O\left(h^{M-2r+1}\right) , \quad (2r \le M) \qquad (2.20)$$

and

$$f^{(2r-1)}\left(t_{ij}, t_{i-1}+u_k h\right) = \sum_{m=0}^{M-2r} h^m \frac{f^{(2r-1+m)}\left(t_{ij}, t_{ij}\right)}{m!} \left(u_k - u_j - 1\right)^m + O\left(h^{M-2r+1}\right) , \quad (2r \le M) . \qquad (2.21)$$

The result follows by substitution of (2.18), (2.19), (2.20) and (2.21) into (2.17). #

Lemma 2.9.

The scheme (2.11) is 0-stable in the sense of Stetter (1965).

Proof. The result follows in the same way as lemma 2.6. #

Since from lemma 2.5, R_{ij} has an asymptotic expansion in integral powers of h, it follows from the 0-stability of the scheme (2.11) and Stetter (1965), theorem 1, p. 21, that ε_{ij} also possesses such an expansion, viz. there exists a unique set of functions

$$\left\{e_{pj}(t) \in C^{\nu-p}[0, T] , \quad j = 1, \ldots, n ; \quad p = 0, \ldots, \nu\right\}$$

such that

$$\varepsilon_{ij} = \sum_{p=0}^{\nu} h^{n+p} e_{pj}\left(t_{ij}\right) + O\left(h^{n+\nu+1}\right) ,$$

$$j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 . \quad (2.22)$$

In the following lemma, a recurrence relation for the functions $e_{pj}(t)$, $j = 1, \ldots, n$; $p = 0, \ldots, \nu$, is derived. Using this relation, it will be possible to obtain estimates for ε_{ij} , $j = 1, \ldots, n$; $i = 0, \ldots, I-1$, which are sharper than the bound given in lemma 2.6.

Lemma 2.10.

If $\nu > 1$, then the functions $e_{pj}(t)$, $j = 1, \ldots, n$; $p = 0, \ldots, \nu-1$, satisfy the relations

$$e_{0j}(t) = 0 , \quad j = 1, \ldots, n ,$$

$$e_{pj}(t) = \phi_{pj}(t) + \sum_{m=0}^{p-1} \sum_{q=0}^{p-m-1} \binom{p-m-1}{q} \frac{k^{(p-m-q-1)}(t,t)}{(p-m-1)!} \sum_{k=1}^{n} a_{jk} e_{mk}^{(q)}(t) \left(u_k-u_j\right)^{p-m-1} ,$$

$$j = 1, \ldots, n ; \quad p = 1, \ldots, \nu-1 , \quad (2.23)$$

and

$$\sum_{j=1}^{n} a_j u_j^r e_{pj}(t) = 0 \ , \quad r = 0, \ldots, \nu-p-1 \ ; \quad p = 0, \ldots, \nu-1 \ , \quad (2.24)$$

where

$$k(t, s) = \frac{\partial}{\partial \eta} K(t, s, \eta)\Big|_{\eta=y(s)} \ ,$$

$$k^{(r)}(t, s) = \frac{\partial^r}{\partial \eta^r} k(t, \eta)\Big|_{\eta=s} \ ,$$

$$e_{mk}^{(r)}(t) = \frac{d^r}{dt^r} e_{mk}(t) \ .$$

Proof. From Taylor's theorem,

$$K(t, s, y) - K(t, s, Y) = k(t, s)(y-Y) + O\left((y-Y)^2\right) .$$

Hence (2.13) becomes

$$\varepsilon_{ij} = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k \, k\left(t_{ij}, t_{\ell k}\right)\varepsilon_{\ell k} + \sum_{k=1}^{n} ha_{jk} \, k\left(t_{ij}, t_{ik}\right)\varepsilon_{ik} + R_{ij} + O\left(h^{2n}\right) .$$

Substitution of (2.14) and (2.22) and division by h^n yields

$$\sum_{p=0}^{\nu} h^p e_{pj}\left(t_{ij}\right) = \sum_{p=0}^{\nu} h^p \sum_{k=1}^{n} \left\{a_k h \sum_{\ell=0}^{i-1} k\left(t_{ij}, t_{\ell k}\right) e_{pk}\left(t_{\ell k}\right) + a_{jk} h \, k\left(t_{ij}, t_{ik}\right) e_{pk}\left(t_{ik}\right)\right\} + \sum_{p=0}^{\nu} h^p \phi_{pj}\left(t_{ij}\right) + O\left(h^{\nu+1}\right) ,$$

$$j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ . \quad (2.25)$$

From lemma 2.8, with $f(t, s) = k(t, s)e_{pk}(s)$, there exists a set of functions $\{\Phi_{jkpm}(t) \ , \ m = 0, \ldots, \nu-p\}$ such that

$$h \sum_{\ell=0}^{i-1} k\left(t_{ij}, t_{\ell k}\right) e_{pk}\left(t_{\ell k}\right) = \int_0^{t_{ij}} k\left(t_{ij}, s\right) e_{pk}(s)ds + \sum_{m=0}^{\nu-p-1} h^{m+1} \Phi_{jkpm}\left(t_{ij}\right) + O\left(h^{\nu-p+1}\right) ,$$

$$j = 1, \ldots, n \ ; \quad k = 1, \ldots, n \ ; \quad p = 0, \ldots, \nu \ . \quad (2.26)$$

Also, from Taylor series expansion,

$$k\left(t_{ij}, t_{ik}\right)e_{pk}\left(t_{ik}\right)$$

$$= \sum_{m=0}^{\nu-p-1} h^m \frac{\left(u_k-u_j\right)^m}{m!} \frac{\partial^m}{\partial\eta^m}\left(k(t, \eta)e_{pk}(\eta)\right)\Big|_{\eta=t_{ij}} + O\left(h^{\nu-p}\right) ,$$

$$k = 1, \ldots, n \; ; \; p = 0, \ldots, \nu . \quad (2.27)$$

Substituting (2.26) and (2.27) into (2.25) and applying Leibnitz's rule gives

$$\sum_{p=0}^{\nu} h^p e_{pj}\left(t_{ij}\right) = \sum_{p=0}^{\nu} h^p \left\{ \sum_{k=1}^{n} a_k \int_0^{t_{ij}} k\left(t_{ij}, s\right) e_{pk}(s)ds + \phi_{pj}\left(t_{ij}\right) \right\}$$

$$+ \sum_{p=1}^{\nu} h^p \sum_{r=0}^{p-1} \sum_{k=1}^{n} \left\{ a_k \, \Phi_{jkr,p-r-1}\left(t_{ij}\right) \right.$$

$$+ a_{jk} \sum_{q=0}^{p-r-1} \binom{p-r-1}{q} k^{(p-r-q-1)}\left(t_{ij}, t_{ij}\right)$$

$$\left. e_{rk}^{(q)}\left(t_{ij}\right) \frac{\left(u_k-u_j\right)^{p-r-1}}{(p-r-1)!} \right\} + O\left(h^{\nu+1}\right) ,$$

$$j = 1, \ldots, n \; ; \; i = 0, \ldots, I-1 \quad (2.28)$$

Clearly, from lemma 2.6,

$$e_{0j}(t) = 0 , \quad j = 1, \ldots, n .$$

Now consider the case $p = 1$. From lemma 2.8,

$$\Phi_{jk00}(t) = 0 , \quad j = 1, \ldots, n \; ; \; k = 1, \ldots, n .$$

Hence, equating coefficients of h in (2.28) yields

$$e_{1j}(t) = \int_0^t k(t, s) \sum_{k=1}^{n} a_k e_{1k}(s)ds + \phi_{1j}(t) , \quad j = 1, \ldots, n , \quad (2.29)$$

and since

$$\sum_{j=1}^{n} a_j = 1 ,$$

it follows that

$$\sum_{j=1}^{n} a_j e_{1j}(t) = \int_0^t k(t, s) \sum_{j=1}^{n} a_j e_{1j}(s)ds + \sum_{j=1}^{n} a_j \phi_{1j}(t) . \quad (2.30)$$

From corollary 2.3,

$$\sum_{j=1}^{n} a_j \phi_{1j}(t) = 0 .$$

Thus (2.30) is a homogeneous Volterra integral equation of the second kind and so

$$\sum_{j=1}^{n} a_j e_{1j}(t) = 0 .$$

It follows from (2.29) that

$$e_{1j}(t) = \phi_{1j}(t) , \quad j = 1, \ldots, n ,$$

and so from corollary 2.3,

$$\sum_{j=1}^{n} a_j u_j^r e_{1j}(t) = 0 , \quad r = 0, \ldots, \nu-2 .$$

The proof proceeds by induction. Assume the lemma is true for $p = 1, \ldots, \ell-1 < \nu-1$. Then from (2.24), (2.26) and lemma 2.8,

$$\sum_{k=1}^{n} a_k \, \phi_{jkr,\ell-r-1}(t) = 0 , \quad j = 1, \ldots, n , \quad r = 0, \ldots, \ell-1 .$$

Hence, equating coefficients of h^ℓ in (2.28), yields

$$e_{\ell j}(t) = \int_0^t k(t, s) \sum_{k=1}^{n} a_k \, e_{\ell k}(s)ds$$

$$+ \sum_{r=0}^{\ell-1} \sum_{q=0}^{\ell-r-1} \binom{\ell-r-1}{q} \frac{k^{(\ell-r-q-1)}(t,t)}{(\ell-r-1)!} \sum_{k=1}^{n} a_{jk} \, e_{rk}^{(q)}(t)\left(u_k-u_j\right)^{\ell-r-1}$$

$$+ \phi_{\ell j}(t) , \quad j = 1, \ldots, n , \quad (2.31)$$

and so,

$$\sum_{j=1}^{n} a_j e_{\ell j}(t) = \int_0^t k(t, s) \sum_{j=1}^{n} a_j e_{\ell j}(s)ds + \sum_{r=0}^{\ell-1} \sum_{q=0}^{\ell-r-1} \binom{\ell-r-1}{q}$$

$$\frac{k^{(\ell-r-q-1)}(t,t)}{(\ell-r-1)!} \sum_{j=1}^{n} a_j \sum_{k=1}^{n} a_{jk} e_{rk}^{(q)}(t)\left(u_k-u_j\right)^{\ell-r-1}$$

$$+ \sum_{j=1}^{n} a_j \, \phi_{\ell j}(t) \ . \qquad (2.32)$$

From corollary 2.3,

$$\sum_{j=1}^{n} a_j \, \phi_{\ell j}(t) = 0 \ ,$$

and from (2.24) and lemma 2.7,

$$\sum_{j=1}^{n} a_j \sum_{k=1}^{n} a_{jk} e_{rk}^{(q)}(t)\left(u_k-u_j\right)^{\ell-r-1} = 0 \ ,$$

$$q = 0, \ldots, \ell-r-1 \ , \quad r = 0, \ldots, \ell-1 \ .$$

It follows that (2.32) is a homogeneous Volterra integral equation of the second kind and therefore

$$\sum_{j=1}^{n} a_j e_{\ell j}(t) = 0 \ .$$

Thus, from (2.31),

$$e_{\ell j}(t) = \phi_{\ell j}(t)$$

$$+ \sum_{r=0}^{\ell-1} \sum_{q=0}^{\ell-r-1} \binom{\ell-r-1}{q} \frac{k^{(\ell-r-q-1)}(t,t)}{(\ell-r-1)!} \sum_{k=1}^{n} a_{jk} e_{rk}^{(q)}(t)\left(u_k-u_j\right)^{\ell-r-1} \ ,$$

$$j = 1, \ldots, n \ ,$$

and from (2.24) and lemma 2.7,

$$\sum_{j=1}^{n} a_j u_j^r \, e_{\ell j}(t) = 0 \ , \quad 0 \le r \le \nu-\ell-1 \ .$$

Hence the result is true for $p = \ell$ and so the lemma follows. #

In the following theorem it will be shown that

$\varepsilon_{in} = O\left(h^{n+\nu}\right)$. This is the main convergence result for the scheme

(2.11).

Theorem 2.1.

If $\omega(t) \in P_\nu$, then

$$\varepsilon_{in} = h^{n+\nu} e_{\nu n}\left(t_{i+1}\right) + O\left(h^{n+\nu+1}\right) , \quad i = 0, \ldots, I-1 ,$$

where $e_{\nu n}(t)$ satisfies an equation of the form

$$e_{\nu n}(t) = \zeta_{\nu n}(t) + \int_0^t k(t, s) e_{\nu n}(s) ds .$$

Proof. From (4.22),

$$\varepsilon_{in} = \sum_{p=0}^{\nu} h^{n+p} e_{pn}\left(t_{i+1}\right) + O\left(h^{n+\nu+1}\right) .$$

If $\nu = 1$, then, from lemma 2.6,

$$e_{0n}(t) = 0 ,$$

and if $\nu > 1$, then from lemma 2.5 and lemma 2.10,

$$e_{pn}(t) = 0 , \quad p = 0, \ldots, \nu-1 .$$

This proves the first part of the theorem.

Equating powers of h^ν in (2.28) yields equations of the form

$$e_{\nu j}(t) = \xi_{\nu j}(t) + \int_0^t k(t, s) \sum_{k=1}^{n} a_k e_{\nu k}(s) ds , \quad j = 1, \ldots, n ,$$

and hence

$$\sum_{j=1}^{n} a_j e_{\nu j}(t) = \sum_{j=1}^{n} a_j \xi_{\nu j}(t) + \int_0^t k(t, s) \sum_{j=1}^{n} a_j e_{\nu j}(s) ds$$

$$= \sum_{j=1}^{n} a_j \xi_{\nu j}(t) - \xi_{\nu n}(t) + e_{\nu n}(t) .$$

It follows that

$$e_{\nu n}(t) = \xi_{\nu n}(t) + \int_0^t k(t, s) \sum_{j=1}^{n} a_j\left(\xi_{\nu j}(s)-\xi_{\nu n}(s)\right)ds + \int_0^t k(t, s)e_{\nu n}(s)ds$$

$$= \zeta_{\nu n}(t) + \int_0^t k(t, s)e_{\nu n}(s)ds . \qquad \#$$

Now the convergence of the scheme (2.12) will be investigated. The analysis proceeds as for the scheme (2.11). Subtracting (2.12) from (2.10) yields

$$\begin{aligned}\varepsilon_{ij} = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k\{K(t_{ij}, t_{\ell k}, y_{\ell k})-K(t_{ij}, t_{\ell k}, Y_{\ell k})\} \\ + \sum_{k=1}^{n} hu_j a_k\left\{K\left(t_{ij}, t_i+u_j u_k h, \sum_{m=1}^{n} L_m(u_j u_k) y_{im}\right)\right. \\ \left. - K\left(t_{ij}, t_i+u_j u_k h, \sum_{m=1}^{n} L_m(u_j u_k) Y_{im}\right)\right\} + R_{ij} , \\ j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 , \quad (2.33)\end{aligned}$$

where

$$R_{ij} = P_{ij} + Q_{ij} + S_{ij} ,$$

with

$$P_{ij} = \sum_{\ell=0}^{i-1} \left\{\int_{t_\ell}^{t_{\ell+1}} K(t_{ij}, s, y(s))ds - \sum_{k=1}^{n} ha_k K(t_{ij}, t_{\ell k}, y_{\ell k})\right\} ,$$

$$Q_{ij} = \int_{t_i}^{t_{ij}} \{K(t_{ij}, s, y(s))-K(t_{ij}, s, \tilde{y}(s))\}ds ,$$

$$S_{ij} = \int_{t_i}^{t_{ij}} K(t_{ij}, s, \tilde{y}(s))ds - \sum_{k=1}^{n} hu_j a_k K\left(t_{ij}, t_i+u_j u_k h, \tilde{y}(t_i+u_j u_k h)\right)$$

and

$$\tilde{y}(s) = \sum_{m=1}^{n} L_m\left(\frac{s-t_i}{h}\right) y_{im} , \quad t_i < s \le t_{i+1} .$$

Lemma 2.11.

Let $K^{(r)}(t, s)$ and $k^{(r)}(t, s)$ be defined as previously.

Then

$$R_{ij} = \sum_{p=0}^{\nu} h^{n+p}\, \phi_{pj}\left(t_{ij}\right) + O\left(h^{n+\nu+1}\right) ,$$

$$j = 1, \ldots, n \; ; \; i = 0, \ldots, I-1 ,$$

where

$$\phi_{pj}(t) = \sum_{r=0}^{p-1} \frac{k^{(p-r-1)}(t,t)y^{(n+r)}(t)}{(p-r-1)!\,(n+r)!} \sum_{q=n}^{n+r} \sum_{\ell=0}^{p-r-1} \binom{n+r}{q}\binom{p-r-1}{\ell}$$

$$\left(-u_j\right)^{n+p-q-\ell-1} \int_0^{u_j} s^{\ell}\, \mu_q(s)\omega(s)ds \; , \; p = 0, \ldots, \nu-1 ,$$

$$\phi_{\nu j}(t) = \frac{R_n\left(u^{n+\nu}\right)}{(n+\nu)!} \int_0^t K^{(n+\nu)}(t, s)ds + \sum_{r=0}^{\nu-1} \frac{k^{(\nu-r-1)}(t,t)y^{(n+r)}(t)}{(\nu-r-1)!\,(n+r)!}$$

$$\sum_{q=n}^{n+r} \sum_{\ell=0}^{\nu-r-1} \binom{n+r}{q}\binom{\nu-r-1}{\ell} \left(-u_j\right)^{n+\nu-q-\ell-1} \int_0^{u_j} s^{\ell}\, \mu_q(s)\omega(s)ds$$

and

$$\mu_q(t)\omega(t) = t^q - \sum_{k=1}^{n} L_k(t)u_k^q .$$

Proof. In the same way as in lemma 2.5,

$$P_{ij} = h^{n+\nu} \frac{R_n\left(u^{n+\nu}\right)}{(n+\nu)!} \int_0^{t_{ij}} K^{(n+\nu)}\left(t_{ij}, s\right)ds + O\left(h^{n+\nu+1}\right) .$$

From Taylor's theorem,

$$Q_{ij} = h \int_0^{u_j} k\left(t_{ij}, t_i+hs\right)\{y\left(t_i+hs\right)-\tilde{y}\left(t_i+hs\right)\}ds + O\left(h^{2n+1}\right) .$$

Now

$$y(t_i+hs) - \tilde{y}(t_i+hs) = \sum_{r=0}^{\nu-1} h^{n+r} \frac{y^{(n+r)}(t_{ij})}{(n+r)!} \left\{ (s-u_j)^{n+r} - \sum_{k=1}^{n} L_k(s)(u_k-u_j)^{n+r} \right\} + O(h^{n+\nu})$$

$$= \sum_{r=0}^{\nu-1} h^{n+r} \frac{y^{(n+r)}(t_{ij})}{(n+r)!} \sum_{q=n}^{n+r} \binom{n+r}{q} (-u_j)^{n+r-q} \left\{ s^q - \sum_{k=1}^{n} L_k(s) u_k^q \right\} + O(h^{n+\nu}) ,$$

and

$$k(t_{ij}, t_i+hs) = \sum_{r=0}^{\nu-1} h^r \frac{k^{(r)}(t_{ij},t_{ij})}{r!} (s-u_j)^r + O(h^\nu)$$

$$= \sum_{r=0}^{\nu-1} h^r \frac{k^{(r)}(t_{ij},t_{ij})}{r!} \sum_{q=0}^{r} \binom{r}{q} (-u_j)^{r-q} s^q + O(h^\nu) .$$

The result follows, since

$$S_{ij} = O(h^{n+\nu+1}) . \qquad \#$$

Corollary 2.4.

If $\nu > 1$, then

$$\sum_{j=1}^{n} a_j u_j^r \phi_{pj}(t) = 0 , \quad r = 0, \ldots, \nu-p-1 ; \quad p = 0, \ldots, \nu-1 .$$

Proof. The result follows from lemmas 2.11, 2.2 and Eq. (2.5). #

The following two lemmas can be proved in the same way as lemmas 2.6 and 2.9.

Lemma 2.12.

There exists a constant K such that

$$e_i = \max_{1\le j\le n} |\varepsilon_{ij}| \le Kh^n , \quad i = 0, \ldots, I-1 , \text{ if } \nu = 0 ,$$

or

$$e_i \le Kh^{n+1} \quad , \quad i = 0, \ldots, I-1 \; , \text{ if } \; \nu > 0 \; .$$

Lemma 2.13.

The scheme (2.12) is 0-stable in the sense of Stetter (1965).

From lemmas 2.11 and 2.13 and Stetter (1965) there exists a unique set of functions

$$\left\{ e_{pj}(t) \in C^{\nu-p}[0, T] \; , \; j = 1, \ldots, n \; ; \; p = 0, \ldots, \nu \right\}$$

such that

$$\varepsilon_{ij} = \sum_{p=0}^{\nu} h^{n+p} \, e_{pj}\left(t_{ij}\right) + O\left(h^{n+\nu+1}\right) \; . \qquad (2.34)$$

A recurrence relation for $e_{pj}(t)$, analogous to (2.23), is obtained in

Lemma 2.14.

If $\nu > 1$, then the functions $e_{pj}(t)$, $j = 1, \ldots, n$; $p = 0, \ldots, \nu-1$, satisfy

$$e_{0j}(t) = 0 \; , \qquad\qquad j = 1, \ldots, n \; ,$$

$$e_{pj}(t) = \phi_{pj}(t) + \sum_{\ell=1}^{p-1} \sum_{r=0}^{\ell} \frac{k^{(r)}(t,t)}{r! \; (\ell-r)!} \sum_{k=1}^{n} e^{(\ell-r)}_{p-\ell-1,k}(t) \left(u_k - u_j\right)^{\ell-r} b_{jkr} \; ,$$

$$j = 1, \ldots, n \; ; \; p = 0, \ldots, \nu-1 \; ,$$

and

$$\sum_{j=1}^{n} a_j \, u_j^r \, e_{pj}(t) = 0 \; , \; r = 0, \ldots, \nu-p-1 \; ; \; p = 0, \ldots, \nu-1 \; ,$$

where

$$b_{jkr} = \sum_{q=0}^{r} \binom{r}{q} \left(-u_j\right)^{r-q} a_{jkq} \; .$$

Proof. The application of Taylor's theorem and lemma 2.12 to (2.33) yields

$$\varepsilon_{ij} = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k\, k\left(t_{ij}, t_{\ell k}\right)\varepsilon_{\ell k}$$

$$+ \sum_{m=1}^{n} \varepsilon_{im} \sum_{k=1}^{n} hu_j a_k\, k\left(t_{ij}, t_i+u_j u_k h\right) L_m\left(u_j u_k\right) + R_{ij} + O\left(h^{2n}\right),$$

$$j = 1, \ldots, n\ ;\ i = 0, \ldots, I-1\ . \quad (2.35)$$

Also by Taylor series expansion and lemma 2.3,

$$\sum_{k=1}^{n} u_j a_k\, k\left(t_{ij}, t_i+u_j u_k h\right) L_m\left(u_j u_k\right)$$

$$= \sum_{r=0}^{\nu-1} h^r \frac{1}{r!} k^{(r)}\left(t_{ij}, t_{ij}\right) b_{jmr} + O\left(h^{\nu}\right).$$

Substituting (2.34) into (2.35) and dividing by h^n yields

$$\sum_{p=0}^{\nu} h^p e_{pj}\left(t_{ij}\right) = \sum_{p=0}^{\nu} h^p \left\{ \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k\, k\left(t_{ij}, t_{\ell k}\right) e_{pk}\left(t_{\ell k}\right) + \phi_{pj}\left(t_{ij}\right) \right\}$$

$$+ \sum_{p=0}^{\nu} h^{p+1} \sum_{r=0}^{\nu-p-1} h^r \frac{k^{(r)}\left(t_{ij}, t_{ij}\right)}{r!} \sum_{k=1}^{n} e_{pk}\left(t_{ik}\right) b_{jkr} + O\left(h^{\nu+1}\right),$$

$$j = 1, \ldots, n\ ;\ i = 0, \ldots, I-1\ .$$

Proceeding as in lemma 2.10, it follows that

$$\sum_{p=0}^{\nu} h^p e_{pj}\left(t_{ij}\right) = \sum_{p=0}^{\nu} h^p \left\{ \sum_{r=0}^{p-1} \sum_{k=1}^{n} a_k\, \phi_{jkr,p-r-1}\left(t_{ij}\right) \right.$$

$$\left. + \sum_{k=1}^{n} a_k \int_0^{t_{ij}} k\left(t_{ij}, s\right) e_{pk}(s)\,ds + \phi_{pj}\left(t_{ij}\right) \right\}$$

$$+ \sum_{p=1}^{\nu} h^p \left\{ \sum_{\ell=0}^{p-1} \sum_{r=0}^{\ell} \frac{k^{(r)}\left(t_{ij}, t_{ij}\right)}{r!\,(\ell-r)!} \sum_{k=1}^{n} \left(u_k - u_j\right)^{\ell-r} \right.$$

$$\left. e^{(\ell-r)}_{p-\ell-1,k}\left(t_{ij}\right) b_{jkr} \right\} + O\left(h^{\nu+1}\right), \quad j = 1, \ldots, n\ .$$

The lemma follows by induction in a similar way to lemma 2.10. #

The following theorem is proved in a similar manner to theorem 2.1.

Theorem 2.2.

If $\omega(t) \in P_\nu$, then

$$\varepsilon_{in} = h^{n+\nu} e_{\nu n}(t_{i+1}) + O(h^{n+\nu+1}) , \quad i = 0, \ldots, I-1 ,$$

where $e_{\nu n}(t)$ satisfies an equation of the form

$$e_{\nu n}(t) = \zeta_{\nu n}(t) + \int_0^t k(t, s) e_{\nu n}(s) ds .$$

2.5 Numerical Stability

Consider the integral equation (2.1) and suppose that a perturbation $\delta g(t)$ to $g(t)$ causes a change $\delta y(t)$ to the solution $y(t)$. Then,

$$y(t) + \delta y(t) = g(t) + \delta g(t) + \int_0^t K\big(t, s, y(s)+\delta y(s)\big) ds .$$

Expanding $K(t, s, y+\delta y)$ by a Taylor series and neglecting the $O(\delta y^2)$ term yields

$$\delta y(t) = \delta g(t) + \int_0^t k(t, s)\delta y(s) ds , \tag{2.36}$$

where

$$k(t, s) = \frac{\partial}{\partial \eta} K(t, s, \eta)\Big|_{\eta = y(s)} .$$

The linearized equation (2.36) characterizes the sensitivity of $y(t)$ with respect to a small perturbation in $g(t)$. It is clear that this sensitivity must be reflected in the growth of the discretization error and the propagation of rounding errors. Hence, the best that can be expected for a finite difference method, is that the leading term in the asymptotic expansion of the error satisfies an equation of the same form as (2.36). In this case, the method is called <u>numerically stable</u>. This concept was first introduced by Kobayasi (1966) and has been further developed by Linz (1967 a) and Noble (1969).

From theorems 2.1 and 2.2 it is clear that the pure discretization error of the schemes (2.11) and (2.12) will grow in a stable manner. However, to investigate the numerical stability fully, it is necessary to consider the propagation of rounding errors which can be characterized by the propagation of perturbations in Y_{0j} , $j = 1, \ldots, n$.

First consider the scheme (2.11) and suppose that in the first step, approximations $\overline{Y}_{0j}$, $j = 1, \ldots, n$, which satisfy

$$\overline{Y}_{0j} = Y_{0j} - \delta_j \ , \quad j = 1, \ldots, n \ ,$$

have been calculated instead of Y_{0j} , $j = 1, \ldots, n$. Denote

$$\delta = \max_{j=1,\ldots,n} |\delta_j| \ .$$

Using the values $\tilde{Y}_{0j}$, $j = 1, \ldots, n$, (2.11) will generate a new sequence of approximations $\overline{Y}_{ij}$, $j = 1, \ldots, n$, $i = 1, \ldots, I-1$, given by

$$\overline{Y}_{ij} = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k \, K\left(t_{ij}, t_{\ell k}, \overline{Y}_{\ell k}\right) + \sum_{k=1}^{n} ha_{jk} \, K\left(t_{ij}, t_{ik}, \overline{Y}_{ik}\right) ,$$

$$j = 1, \ldots, n \ ; \quad i = 1, \ldots, I-1 \ . \qquad (2.37)$$

Let

$$\overline{\varepsilon}_{ij} = y_{ij} - \overline{Y}_{ij} \ , \quad j = 1, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ .$$

Then,

$$\overline{\varepsilon}_{0j} = \varepsilon_{0j} + \delta_j \ , \quad j = 1, \ldots, n \ , \qquad (2.38)$$

where ε_{0j} , $j = 1, \ldots, n$, are given by (2.13). By an argument similar to lemma 2.6, it follows that

$$\bar{\varepsilon}_{ij} = O(h^n) + O(h\delta) , \quad j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 . \quad (2.39)$$

Subtraction of (2.37) from (2.10) and the use of Taylor's theorem, (2.38) and (2.39) yield

$$\bar{\varepsilon}_{ij} = \sum_{\ell=1}^{i-1} \sum_{k=1}^{n} ha_k \, k(t_{ij}, t_{\ell k})\bar{\varepsilon}_{\ell k} + \sum_{k=1}^{n} ha_{jk} \, k(t_{ij}, t_{ik})\bar{\varepsilon}_{ik}$$

$$+ \sum_{k=1}^{n} ha_k \, k(t_{ij}, t_{0k})(\varepsilon_{0j}+\delta_j) + R_{ij} + O(h^{2n}) + O(h^2\delta) ,$$

$$j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 .$$

Hence, by superposition,

$$\bar{\varepsilon}_{ij} = \varepsilon_{ij} + \hat{\varepsilon}_{ij} , \quad j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 ,$$

where

$$\hat{\varepsilon}_{0j} = \delta_j , \quad j = 1, \ldots, n ,$$

$$\hat{\varepsilon}_{ij} = \sum_{\ell=1}^{i-1} \sum_{k=1}^{n} ha_k \, k(t_{ij}, t_{\ell k})\hat{\varepsilon}_{\ell k} + \sum_{k=1}^{n} ha_{jk} \, k(t_{ij}, t_{ik})\hat{\varepsilon}_{ik}$$

$$+ \sum_{k=1}^{n} ha_k \, k(t_{ij}, t_{0k})\delta_k + O(h^{2n}) + O(h^2\delta) ,$$

$$j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 , \quad (2.40)$$

and ε_{ij} , $j = 0, \ldots, n$; $i = 0, \ldots, I-1$, is the pure discretization error given by (2.13). By lemma 2.4,

$$\hat{\varepsilon}_{ij} = O(h\delta) + O(h^{2n}) , \quad j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 .$$

Let

$$e_{ij} = \frac{\hat{\varepsilon}_{ij}}{h} , \quad j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 .$$

Then, from (2.40),

$$e_{ij} = \sum_{\ell=1}^{i-1} \sum_{k=1}^{n} ha_k\, k\left(t_{ij}, t_{\ell k}\right) e_{\ell k} + \sum_{k=1}^{n} ha_{jk}\, k\left(t_{ij}, t_{ik}\right) e_{ik}$$

$$+ \sum_{k=1}^{n} a_k\, k\left(t_{ij}, t_{0k}\right)\delta_k + O\left(h^{2n-1}\right) + O(h\delta) ,$$

$$j = 1, \ldots, n \; ; \; i = 1, \ldots, I-1 .$$

This equation can be interpreted as a finite difference method applied to the system of integral equations

$$e_j(t) = \int_0^t k(t, s) \sum_{k=1}^{n} a_k e_k(s)ds + k(t, 0) \sum_{k=1}^{n} a_k \delta_k ,$$

$$j = 1, \ldots, n . \quad (2.41)$$

Using lemma 2.4, this method can easily be shown to be convergent of order one, and hence

$$e_{ij} = e_j\left(t_{ij}\right) + O\left(h^{2n-1}\right) + O(h\delta) , \quad j = 1, \ldots, n \; ; \quad i = 1, \ldots, I-1 .$$

From (2.41),

$$e_j(t) = e(t) , \quad j = 1, \ldots, n ,$$

where

$$e(t) = \int_0^t k(t, s)e(s)ds + k(t, 0) \sum_{k=1}^{n} a_k \delta_k$$

and it follows that

$$\hat{\varepsilon}_{ij} = he\left(t_{ij}\right) + O\left(h^{2n}\right) + O\left(h^2\delta\right) , \quad j = 1, \ldots, n \; ; \quad i = 1, \ldots, I-1 ,$$

which implies that the scheme (2.11) is numerically stable.

In the same way, numerical stability can be established for the scheme (2.12).

2.6 A-Stability and Stiff A-Stability

So far in the analysis of the numerical schemes it has been assumed that the product of the step size h and the Lipschitz constant L is "small". However it is well known from ordinary differential equations, which are a special case of (2.1), that such a choice of h is unsatisfactory if for instance the solution consists of a slowly varying main component over which there are superimposed some rapidly decaying components. This is due to the fact that when the contribution from the decaying components has died out, it is desirable to choose the step size h with respect to the rate of variation of the main component rather than the rate of variation of the negligible components. Equations of this type are called stiff equations and several special stability properties for numerical methods, in particular A-stability in the sense of Dahlquist (1963) and stiff A-stability in the sense of Axelsson (1969) have been introduced to cope with this difficulty.

A numerical method is called A-stable if, when applied to the problem

$$y(t) = 1 + \lambda \int_0^t y(s)ds \ , \quad Re(\lambda) < 0 \ ,$$

with an arbitrary step size h, then

$$\lim_{\substack{i\to\infty \\ h \text{ fixed}}} Y_i = 0 \ ,$$

where Y_i denotes the numerical approximation to $y(t_i)$. If in addition

$$\lim_{\substack{h\to\infty \\ i \text{ fixed}}} Y_i = 0 \quad \text{for all } i \ ,$$

then the method is called stiffly A-stable. Clearly the same difficulties can be expected for the more general problem (2.1) and so A-stability or stiff A-stability is essential for an efficient general numerical method.

In §2.7 various choices of points $\{u_1, \ldots, u_n\}$ which lead to A-stable or stiffly A-stable methods are given.

Hung (1970) has shown A-stability for a Hermitian spline method which uses the differentiated form of (2.1) and so requires $\partial K(t, s, y)/\partial t$ analytically.

2.7 The Choice of Points

From the convergence results of theorems 2.1 and 2.2 it is clear that the choice of points $\{u_1, u_2, \ldots, u_n\}$ is important.

A natural choice of points would appear to be the equally spaced points $u_i = \frac{i-1}{n-1}$, $i = 1, \ldots, n$, $n \geq 2$. For $n = 2$ these methods correspond to the well known trapezoidal method and for $n = 3$ to the block by block methods considered by Linz (1967 a) who observed their stability in his numerical examples. Also for $n = 5$ the methods correspond to a slight modification of the block by block methods considered by Cambell and Day (1971). For these points, $\omega(t) \in P_0$ for $n = 2r$ and $\omega(t) \in P_1$ for $n = 2r + 1$, $r \geq 1$, and so from theorems 2.1 and 2.2 the methods with $n = 2r + 1$ and $n = 2r + 2$ have order $2r + 2$ convergence. Hence in order to obtain order r convergence it is necessary to solve at least $r - 2$ simultaneous equations at each step. Also it has been shown numerically by Wright (1970) that the methods are A-stable for $n \leq 9$.

This situation however is not the best possible. A more suitable choice of points are those considered by Axelsson (1969) for ordinary differential equations, i.e., let u_i , $i = 1, \ldots, n$, be the zeros of $P_n(t) + a\,P_{n-1}(t) + b\,P_{n-2}(t)$, a, b real, $b \leq 0$, where $P_n(t)$ is the n-th degree Legendre polynomial defined on

[0, 1] and a, b are chosen so that $u_1 \geq 0$ and $u_n = 1$. In this case $\omega(t) \in P_\nu$ where,

$$\nu = \begin{cases} n-2 \ , & b \neq 0 \\ n-1 \ , & a \neq 0 \ , \ b = 0 \ . \end{cases}$$

In particular let u_i be the zeros of $P_n(t) - P_{n-1}(t)$, $(u_1 > 0 \ , \ u_n = 1)$, and $P_n(t) - P_{n-2}(t)$, $(u_1 = 0 \ , \ u_n = 1)$, so that the methods correspond to Radau and Lobatto quadrature respectively. Axelsson has shown that these methods are stiffly A-stable and A-stable respectively. In addition the methods have the advantage that a_i, $i = 1, \ldots, n$, are positive and that for Lobatto quadrature only r simultaneous equations have to be solved at each step to obtain order $2r$ convergence. This is best possible.

2.8 Numerical Results

In this section the numerical solutions of two simple examples are given to illustrate some of the features of the methods. The schemes (2.11) and (2.12) considered were based on Radau quadrature with degree of precision four $(\nu = 2)$

$$\left(\text{i.e.}\ \ u_1 = \frac{1-\frac{1}{5}(1+\sqrt{6})}{2} \ , \ \ u_2 = \frac{1+\frac{1}{5}(\sqrt{6}-1)}{2} \ , \ \ u_3 = 1\right) .$$

The fifth order convergence of the methods is illustrated by the application to

$$y(t) = 1 + t - \cos(t) - \int_0^t \cos(t-s)y(s)ds \ , \quad 0 \leq t \leq 2 \ ,$$

which has the solution $y(t) = t$.

The errors are tabulated in tables 2.1 and 2.2. It should be noted that the errors for method (2.11) are appreciably larger than the errors for method (2.12). Numerical computations show that this is the case for many examples.

Table 2.1.

Method (2.11)			
t	h = 0.4	h = 0.2	h = 0.1
0.4	-2.396 E-6	-7.446 E-8	-2.316 E-9
0.8	-4.608 E-6	-1.446 E-7	-4.519 E-9
1.2	-6.391 E-6	-2.022 E-7	-6.349 E-9
1.6	-7.638 E-6	-2.437 E-7	-7.682 E-9
2.0	-8.347 E-6	-2.685 E-7	-8.498 E-9

Table 2.2.

Method (2.12)			
t	h = 0.4	h = 0.2	h = 0.1
0.4	2.647 E-7	7.704 E-9	2.323 E-10
0.8	4.165 E-7	1.212 E-8	3.655 E-10
1.2	4.803 E-7	1.398 E-8	4.220 E-10
1.6	4.843 E-7	1.413 E-8	4.271 E-10
2.0	4.550 E-7	1.334 E-8	4.043 E-10

The advantage of stiff A-stability is illustrated by the application of the methods to

$$y(t) = \left((1+t)\exp(-10t)+1\right)^{\frac{1}{2}} + (1+t)\left(\left(1-\exp(-10t)\right)+10\log(1+t)\right)$$

$$- 10\int_0^t \frac{1+t}{1+s}\, y(s)^2 ds \ , \quad 0 \le t \le 19 \ ,$$

which has the solution

$$y(t) = \left((1+t)\exp(-10t)+1\right)^{\frac{1}{2}} .$$

The Lipschitz constant for this example is effectively 20 and so from the remarks of §2.6, a conventional multistep method will not work well for a large step size. The methods were applied with $h = 0.1$ on the interval $[0, 1]$ and then the step size was increased on $(1, 19]$. The resulting systems of nonlinear equations were solved by Newton-Raphson iteration. The errors are given in tables 2.3 and 2.4.

Table 2.3.

Method (2.11)		
t	h = 1.5	h = 3.0
4.0	-1.780 E-3	-2.843 E-2
7.0	-3.058 E-4	-5.376 E-3
10.0	-1.015 E-4	-1.392 E-3
13.0	-4.577 E-5	-5.430 E-4
16.0	-2.446 E-5	-2.716 E-4
19.0	-1.458 E-5	-1.558 E-4

Table 2.4.

Method (2.12)		
t	h = 1.5	h = 3.0
4.0	-3.271 E-5	-2.302 E-3
7.0	-1.717 E-6	-1.834 E-4
10.0	-2.699 E-7	-1.952 E-5
13.0	-7.191 E-8	-3.757 E-6
16.0	-2.452 E-8	-1.150 E-6
19.0	-1.077 E-8	-4.554 E-7

2.9 Conclusion

Since the methods given by (2.11) require values of the kernel $K(t, s, y)$ outside the region $0 \le s \le t \le T$, difficulties can be expected if the kernel is badly behaved there. As the schemes (2.12) avoid this problem, they should be used in these cases.

As illustrated by a numerical example, a large stepsize h is feasible in certain cases. The choice of a large stepsize is a particularly desirable feature for finite difference methods for Volterra integral equations since the amount of computation required is proportional to $(T/h)^2$. Hence the implicit Runge-Kutta methods are superior to the conventional multistep methods in certain situations. In addition, they have the advantage of being self starting.

If the integral equation (2.1) is nonlinear, the schemes require the solution of a nonlinear system of equations at each step. This can be done by standard methods such as the Newton-Raphson iteration.

CHAPTER 3

HIGH ORDER METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

3.1 Introduction

In this chapter, the implicit Runge-Kutta methods developed in chapter 2 are extended to Volterra integral equations of the first kind with continuous kernels,

$$g(t) = \int_0^t k(t, s)y(s)ds \ , \quad 0 \le t \le T \ . \tag{3.1}$$

For convenience, the conditions ensuring the existence of a unique continuous solution are repeated below.

(3.1.1) $k(t, s)$ and $\partial k(t, s)/\partial t$ are continuous on $0 \le s \le t \le T$,

(3.1.2) $k(t, t) \ne 0$ on $0 \le t \le T$,

(3.1.3) $g(t)$ is continuously differentiable on $0 \le t \le T$ and $g(0) = 0$.

Under these conditions, (3.1) is equivalent to the second kind Volterra integral equation

$$g'(t) = k(t, t)y(t) + \int_0^t \frac{\partial k}{\partial t}(t, s)y(s)ds \ , \quad 0 \le t \le T \ , \tag{3.2}$$

which is obtained by differentiating (3.1) (see for instance Tricomi (1957), pp 15-16).

Since (3.2) is a Volterra integral equation of the second kind, it follows that $y(t)$ depends continuously on $g'(t)$. This implies that the problem (3.1) is improperly posed. The effect of this will be briefly discussed in §3.11.

Numerical approximations to $y(t)$ can be obtained via (3.2) by methods for Volterra integral equations of the second kind (cp. §1.2 and chapter 2). However, in some situations (for example, if $g(t)$ is given in tabulated form) it is desirable to determine the solution directly from (3.1). The derivation and analysis of such methods form the basis of this chapter.

To illustrate some of the features of different numerical schemes for (3.1), a comparison between the trapezoidal and midpoint methods is made in §3.2. In §3.3, implicit Runge-Kutta methods for (3.1) are constructed. It will be shown that the schemes with $u_1 > 0$ (methods I) and the schemes with $u_1 = 0$ (methods II) are generalizations of the midpoint and trapezoidal methods respectively. The remaining sections of the chapter investigate the convergence and stability properties of methods I and II.

3.2 A Comparison Between the Trapezoidal and Midpoint Methods

Let $t_i = ih$, $i = 0, \ldots, I$; $h = T/I$, and discretize (3.1) at t_i, $i = 1, \ldots, I$, to obtain

$$g(t_i) = \int_0^{t_i} k(t_i, s)y(s)ds \, , \quad i = 1, \ldots, I \, . \tag{3.3}$$

Approximating the integrals in (3.3) by the trapezoidal rule yields the <u>trapezoidal method</u>,

$$g(t_i) = h\left\{\frac{k(t_i,0)}{2}Y_0 + \sum_{\ell=1}^{i-1} k(t_i, t_\ell)Y_\ell + \frac{k(t_i,t_i)}{2}Y_i\right\} , \quad i = 1, \ldots, I \, , \tag{3.4}$$

where Y_i is the numerical approximation to $y(t_i)$. The trapezoidal method requires a starting value Y_0 which must be determined independently (for example from (3.2), $Y_0 = g'(0)/k(0, 0)$).

The application of the midpoint rule to (3.3) leads to the midpoint method,

$$g(t_i) = h \sum_{\ell=0}^{i-1} k(t_i,\ t_\ell + h/2) Y_{\ell+\frac{1}{2}} \ , \quad i = 1, \ldots, I \ , \tag{3.5}$$

where $Y_{i+\frac{1}{2}}$ is the approximation to $y(t_i + h/2)$.

As suggested by Noble (1964), pp 254-258, it is instructive to apply methods for first kind equations to the problem of differentiation, i.e.

$$g(t) = \int_0^t y(s)ds \ ; \ y(t) = g'(t) \ .$$

For the trapezoidal method, subtraction of (3.4) from (3.4) with i replaced by $i + 1$ yields

$$Y_{i+1} = 2\left(\frac{g(t_{i+1}) - g(t_i)}{h}\right) - Y_i \ , \quad i = 0, \ldots, I-1 \ . \tag{3.6}$$

Assuming an exact starting value, it is easy to verify that this difference equation has the solution

$$Y_i = \left\{2g(t_i) + 4 \sum_{\ell=1}^{i-1} (-1)^\ell g(t_{i-\ell})\right\}/h + (-1)^i g'(0) \ , \quad i = 1, \ldots, I \ .$$

Clearly, this is not a suitable differentiation formula since it does not preserve the local behaviour of the operator $\frac{d}{dt}$.

On the other hand, the same differencing applied to the midpoint method yields

$$Y_{i+\frac{1}{2}} = \frac{g(t_{i+1}) - g(t_i)}{h}, \quad i = 0, \ldots, I-1 \ ,$$

which is the standard central difference approximation to $g'(t_i + h/2)$.

This indicates that the midpoint method will be superior to the trapezoidal method. The following results due to Kobayasi (1967) and Linz (1967 a) confirm this. For convenience it will be assumed that $k(t, s)$ and $g(t)$ are sufficiently smooth.

Theorem 3.1 (Trapezoidal method, Kobayasi (1967)).

Let $e(t)$ be the solution of

$$k(t, t)e(t) + \int_0^t \frac{\partial k}{\partial t}(t, s)e(s) = \phi'(t)$$

where

$$\phi(t) = -\frac{1}{12}\left[\left(\frac{\partial}{\partial s}\right)\left(k(t, s)y(s)\right)\right]_0^t$$

and let $q_r(t)$, $r = 1, 2$ be the solutions of

$$k(t, t)q_r'(t) + \frac{\partial k}{\partial s}(t, t)q_r(t) = 0 \ , \quad r = 1, 2 \ , \tag{3.7}$$

with the initial conditions

$$q_1(0) = -e(0) \ ,$$

$$q_2(0) = -1 \ .$$

Then

$$Y_i = y(t_i) + h^2\left(e(t_i) + (-1)^i q_1(t_i)\right)$$
$$+ (-1)^i \delta q_2(t_i) + O(h^3) + O(h\delta) \ , \quad i = 0, \ldots, I \ ,$$

where δ is the starting error $y(0) - Y_0$.

From this result the trapezoidal method is convergent of order two. To illustrate its stability properties, consider the equation

$$-1 + t + e^{-t} = \int_0^t (1+t-s)y(s)ds$$

which has the solution

$$y(t) = te^{-t} .$$

It can be seen that $e(t)$ satisfies an equation of the same form as (3.2) and hence grows in a similar way to $y(t)$. However, from (3.7),

$$q_1(t) = -e(0)\exp(t),$$

$$q_2(t) = -\exp(t),$$

and clearly $q_1(t_i)$ and $q_2(t_i)$ will dominate the numerical solution for large t_i. Hence the trapezoidal method is not numerically stable.

Although the midpoint method does not require a starting value it is instructive to examine the propagation of an initial perturbation in the numerical solution since this illustrates the effect of rounding error. It is therefore convenient to proceed as in §2.5 and assume that in the first step an approximation $\overline{Y}_{\frac{1}{2}} = Y_{\frac{1}{2}} - \delta$ has been computed. Then (3.5) will generate a new sequence of approximations $\overline{Y}_{i+\frac{1}{2}}$, $i = 1, \ldots, I-1$, given by

$$g(t_i) = h \sum_{\ell=0}^{i-1} k(t_i, t_\ell + h/2)\overline{Y}_{\ell+\frac{1}{2}}, \quad i = 2, \ldots, I.$$

<u>Theorem 3.2</u> (Midpoint method).

Let

$$\psi_1(t) = \frac{1}{24}\left[\left(\frac{\partial}{\partial s}\right)\left(k(t, s)y(s)\right)\right]\Big|_0^t,$$

$$\psi_2(t) = \frac{1}{k(0,0)} \frac{\partial k}{\partial t}(t, 0) \frac{\partial k}{\partial s}(0, 0) - \frac{\partial^2 k}{\partial s \partial t}(t, 0)$$

and let $e(t)$ and $d(t)$ be the solutions of

$$k(t, t)e(t) + \int_0^t \frac{\partial k}{\partial t}(t, s)e(s)ds = \psi_1'(t)$$

and

$$k(t, t)d(t) + \int_0^t \frac{\partial k}{\partial t}(t, s)d(s)ds = \psi_2(t).$$

Then

$$\overline{Y}_{i+\frac{1}{2}} = y\left(t_i+h/2\right) + h^2 e\left(t_i+h/2\right) + h^2\delta d\left(t_i+h/2\right) + O\left(h^3\right) + O\left(h^3\delta\right) ,$$

$$i = 2, \ldots, I-1 .$$

<u>Proof</u>. This theorem with $\delta = 0$ is given in Linz (1967 a). The result for $\delta \neq 0$ is established in the same way as the corresponding result for the implicit Runge-Kutta schemes with $u_1 > 0$ which will be derived in §3.5. #

Theorem 3.2 implies that the midpoint method is convergent of order two and numerically stable since $e(t)$ and $d(t)$ satisfy equations of the same form as (3.2). In addition, a perturbation δ propagates as $O\left(h^2\delta\right)$. This is rather surprising since a perturbation δ in finite difference methods for (3.2) propagates as $O(h\delta)$ (cp. §2.5). It implies that the midpoint method is a very robust scheme.

3.3 Numerical Schemes

For convenience, some notation introduced in §2.2 and §2.3 will firstly be reiterated. Let

$$0 \leq u_1 < u_2 < \ldots < u_n = 1 ,$$

$$\omega(t) = \prod_{k=1}^{n} \left(t-u_k\right) ,$$

$$L_k(t) = \frac{\omega(t)}{\left(t-u_k\right)\omega'\left(u_k\right)} , \quad k = 1, \ldots, n ,$$

$$a_{jk} = \int_0^{u_j} L_k(s)ds , \quad k = 1, \ldots, n ; \quad j = 1, \ldots, n ,$$

$$a_k = a_{nk} , \quad k = 1, \ldots, n ,$$

$$t_i = ih , \quad i = 0, \ldots, I ; \quad h = T/I$$

and

$$t_{ij} = t_i + u_j h , \quad j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 .$$

The discretized form of (3.1) is

$$g(t_{ij}) = \int_0^{t_{ij}} k(t_{ij}, s) y(s) ds ,$$

$$j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 . \quad (3.8)$$

Approximating $\int_0^{t_{ij}} k(t_{ij}, s) y(s) ds$ as in the derivation of method (2.11) yields the numerical scheme

$$g(t_{ij}) = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k \, k(t_{ij}, t_{\ell k}) Y_{\ell k} + \sum_{k=1}^{n} ha_{jk} \, k(t_{ij}, t_{ik}) Y_{ik} ,$$

$$j = r, \ldots, n ; \quad i = 0, \ldots, I-1 , \quad (3.9)$$

where

$$r = \begin{cases} 1 & \text{if } u_1 > 0 \\ 2 & \text{if } u_1 = 0 \end{cases}$$

and Y_{ij} is the approximation to $y(t_{ij})$. As in (2.11), if $u_1 = 0$, then $Y_{i+1,1} = Y_{in}$, $i = 0, \ldots, I-2$, and a starting value Y_{01} is required.

For each i , $i = 0, \ldots, I-1$, (3.9) represents a system of linear equations for Y_{ij} , $j = r, \ldots, n$. From conditions (3.1.1) and (3.1.2) it follows that these systems are nonsingular if h is sufficiently small.

In (3.9) values of $k(t, s)$ are required in the region $t \le s \le t+h(1-u_1)$, $0 \le t \le T$. Although this does not cause theoretical difficulties since h can be taken to be arbitrarily small, computational difficulties may arise (see §3.10, example (3.63)).

This difficulty can be avoided if the scheme (3.9) is modified in the same way as (2.12), i.e.

$$g(t_{ij}) = \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k\, k(t_{ij}, t_{\ell k}) Y_{\ell k}$$

$$+ \sum_{k=1}^{n} hu_j \sum_{m=1}^{n} a_m\, k(t_{ij}, t_i + u_j u_m h) L_k(u_j u_m) Y_{ik} ,$$

$$j = r, \ldots, n \; ; \quad i = 0, \ldots, I-1 . \qquad (3.10)$$

For $k(t, s) = \text{const}$ the methods (3.9) and (3.10) are identical. This is a consequence of (2.9).

Remark 3.1. By subtracting (3.9) with $j = n$ and i replaced by $i - 1$ from (3.9), it can easily be verified that for $k(t, s) = 1$ and $u_1 > 0$,

$$Y_{ij} = p_i'(t_{ij}) , \quad j = 1, \ldots, n \; ; \quad i = 0, \ldots, I-1 , \qquad (3.11)$$

where $p_i(t)$, $t_i < t \le t_{i+1}$, is the polynomial of degree n interpolating to $g(t)$ at the points $t_i, t_{i1}, \ldots, t_{in}$. Eq. (3.11) is a standard formula for numerical differentiation.

On the other hand, for $u_1 = 0$, the same differencing yields a relation which is similar to (3.6) and cannot be interpreted as a local differentiation formula.

Hence intuitively it can be expected that schemes with $u_1 > 0$ will be superior to those with $u_1 = 0$.

In the sequel, the schemes with $u_1 > 0$ and $u_1 = 0$ will be denoted by methods I and II respectively.

Remark 3.2. Clearly, methods corresponding to (3.9) and (3.10) can be constructed for $u_n < 1$. These schemes can be treated by an analysis similar to that presented here. Also, it should be noted that all the analysis extends easily to systems of first kind Volterra equations.

3.4 Methods I: Convergence

For the analysis of methods I the following conditions on $k(t, s)$ and $g(t)$ are required.

(3.4.1) $k(t, s)$ is $n + 2$ times continuously differentiable on $0 \le s \le t+\delta$, $0 \le t \le T$, where δ is a fixed positive number for the scheme (3.9) and is equal to zero for the scheme (3.10),

(3.4.2) $\partial^{n+3} k(t, s)/\partial t^{n+3}$ is continuous on $0 \le s \le t \le T$,

(3.4.3) $g(t)$ is $n + 3$ times continuously differentiable on $0 \le t \le T$.

The conditions (3.4.1) with $\delta = 0$, (3.4.2) and (3.4.3) imply that $y(t)$ is $n + 2$ times continuously differentiable on $0 \le t \le T$.

The main result of this section is

Theorem 3.3.

The schemes (3.9) and (3.10) are convergent of order n.

Subtracting (3.9) with $h < \delta$ and (3.10) from (3.8) yields

$$0 = h \sum_{k=1}^{n} a_{jk}\, k(t_{ij}, t_{ik})\varepsilon_{ik} + h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})\varepsilon_{\ell k} + R_{ij} ,$$

$$j = 1, \ldots, n \; ; \; i = 0, \ldots, I-1 , \quad (3.12)$$

and

$$0 = h \sum_{k=1}^{n} u_j \sum_{r=1}^{n} a_r\, k(t_{ij}, t_i + u_j u_r h) L_k(u_j u_r)\varepsilon_{ik}$$

$$+ h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})\varepsilon_{\ell k} + \tilde{R}_{ij} ,$$

$$j = 1, \ldots, n \; ; \; i = 0, \ldots, I-1 , \quad (3.13)$$

where

$$\varepsilon_{ij} = y(t_{ij}) - Y_{ij} ,$$

$$R_{ij} = \int_0^{t_{ij}} k(t_{ij}, s)y(s)ds - h \sum_{k=1}^{n} a_{jk}\, k(t_{ij}, t_{ik})y(t_{ik})$$

$$- h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})y(t_{\ell k}) \qquad (3.14)$$

and

$$\tilde{R}_{ij} = \int_0^{t_{ij}} k(t_{ij}, s)y(s)ds$$

$$- h \sum_{k=1}^{n} u_j \sum_{r=1}^{n} a_r\, k(t_{ij}, t_i + u_j u_r h) L_k(u_j u_r) y(t_{ik})$$

$$- h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})y(t_{\ell k}) . \qquad (3.15)$$

<u>Lemma 3.1.</u>

There exist continuously differentiable functions $\phi(t)$, $\tilde{\phi}(t)$ with $\phi(0) = 0$, $\tilde{\phi}(0) = 0$ and continuous functions $\psi_j(t)$, $\tilde{\psi}_j(t)$, $j = 1, \ldots, n$, such that

$$R_{ij} = -h^n \phi(t_{ij}) - h^{n+1} \psi_j(t_{ij}) + O(h^{n+2}) ,$$

$$\tilde{R}_{ij} = -h^n \tilde{\phi}(t_{ij}) - h^{n+1} \tilde{\psi}_j(t_{ij}) + O(h^{n+2}) ,$$

$$j = 1, \ldots, n ; \quad i = 0, \ldots, I-1 .$$

<u>Proof</u>. The result follows from the Euler Maclaurin sum formula. #

<u>Proof of theorem 3.3</u>. First consider the scheme (3.9).

Subtraction of (3.12) with i replaced by $i - 1$ and $j = n$ from (3.12) and division by $hk(t_i, t_i)$ yields

$$\sum_{k=1}^{n} a_{jk} \frac{k(t_{ij}, t_{ik})}{k(t_i, t_i)} \varepsilon_{ik} = -h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k \frac{k(t_{ij}, t_{\ell k}) - k(t_i, t_{\ell k})}{hk(t_i, t_i)} \varepsilon_{\ell k}$$

$$- \frac{R_{ij} - R_{i-1,n}}{hk(t_i, t_i)} , \quad j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 . \qquad (3.16)$$

Let

$$M = \min_{0 \le t \le T} |k(t, t)| > 0 ,$$

$$K = \max_{\substack{0 \le s \le t \\ 0 \le t \le T}} \left| \frac{\partial k}{\partial t}(t, s) \right|$$

and

$$a = \max_{1 \le j \le n} |a_j| .$$

From Taylor's theorem and lemma 3.1, there exists a constant C such that

$$\left| \frac{R_{ij} - R_{i-1,n}}{hk(t_i, t_i)} \right| \le Ch^n , \quad j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 ,$$

for a sufficiently small h. Taking absolute values in (3.16) and applying the above estimates yields

$$\left| \sum_{k=1}^{n} a_{jk} \frac{k(t_{ij}, t_{ik})}{k(t_i, t_i)} \varepsilon_{ik} \right| \le \frac{hKa}{M} \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} |\varepsilon_{\ell k}| + Ch^n ,$$

$$j = 1, \ldots, n ; \quad i = 1, \ldots, I-1 . \quad (3.17)$$

Introduce the n dimensional vectors

$$\underline{\varepsilon}_i = (\varepsilon_{i1}, \ldots, \varepsilon_{in})^T , \quad i = 0, \ldots, I-1 ,$$

and the $n \times n$ matrices

$$\overline{A}^{(i)} = \left(\overline{a}_{jk}^{(i)} \right) , \quad i = 1, \ldots, I-1 ,$$

where

$$\overline{a}_{jk}^{(i)} = a_{jk} \frac{k(t_{ij}, t_{ik})}{k(t_i, t_i)} .$$

Clearly, $\overline{A}^{(i)}$ is nonsingular for sufficiently small h.

Let

$$L = \max_{1 \le i \le I-1} \left\| \overline{A}^{(i)^{-1}} \right\|$$

and

$$e_i = \| \underline{\varepsilon}_i \| , \quad i = 0, \ldots, I-1 ,$$

where $\|\cdot\|$ is the usual maximum norm. It follows from (3.17) that

$$\frac{e_i}{L} \le \frac{hKan}{M} \sum_{\ell=0}^{i-1} e_\ell + Ch^n , \quad i = 1, \ldots, I-1 . \tag{3.18}$$

Also, from Taylor's theorem there exists a constant C_1 such that

$$|R_{0j}| \le C_1 h^{n+1} , \quad j = 1, \ldots, n ,$$

and hence from (3.12),

$$e_0 \le C_2 h^n , \quad C_2 = \text{const} .$$

The application of lemma 2.4 to (3.18) yields

$$e_i \le C_3 h^n , \quad i = 0, \ldots, I-1 , \quad C_3 = \text{const} .$$

Hence the theorem holds for the scheme (3.9).

Using (2.9), a similar argument establishes the analogous result for the scheme (3.10). #

3.5 Methods I: Numerical Stability

From the remark in §3.2, Eq. (3.1) can be solved via (3.2). If a numerically stable method is used to solve this second kind equation, it is known (cp. §2.5) that the leading term in the asymptotic expansion of the error satisfies an equation of the form

$$k(t, t)e(t) + \int_0^t \frac{\partial k}{\partial t}(t, s)e(s)ds = \xi(t) . \tag{3.19}$$

If a direct method is to be comparable, the leading term in the asymptotic expansion of the error must behave in a similar manner. In this case, the method will be called numerically stable.

The leading term in the asymptotic expansion of methods I will now be investigated. As in §2.5, it will be assumed that in the first step, approximations

$$\overline{Y}_{0j} = Y_{0j} - \delta_j , \quad j = 1, \ldots, n ,$$

have been computed.

First consider the scheme (3.9). Using the values $\overline{Y}_{0j}$, $j = 1, \ldots, n$, (3.9) will generate a new sequence of approximations $\overline{Y}_{ij}$, $j = 1, \ldots, n$; $i = 1, \ldots, I-1$, given by

$$g(t_{ij}) = h \sum_{k=1}^{n} a_{jk}\, k(t_{ij}, t_{ik})\overline{Y}_{ik} + h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})\overline{Y}_{\ell k} ,$$

$$j = 1, \ldots, n \; ; \; i = 1, \ldots, I-1 . \quad (3.20)$$

Let

$$\overline{\varepsilon}_{ij} = y(t_{ij}) - \overline{Y}_{ij} , \; j = 1, \ldots, n \; ; \; i = 0, \ldots, I-1 .$$

Subtracting as previously yields

$$0 = h \sum_{k=1}^{n} a_{jk}\, k(t_{ij}, t_{ik})\overline{\varepsilon}_{ik} + h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k(t_{ij}, t_{\ell k})\overline{\varepsilon}_{\ell k} + R_{ij} ,$$

$$j = 1, \ldots, n \; ; \; i = 1, \ldots, I-1 , \quad (3.21)$$

and therefore,

$$\sum_{k=1}^{n} a_{jk}\, k(t_{1j}, t_{1k})\overline{\varepsilon}_{1k} = - \sum_{k=1}^{n} a_k\, k(t_{1j}, t_{0k})\overline{\varepsilon}_{0k} - \frac{R_{1j}}{h} ,$$

$$j = 1, \ldots, n . \quad (3.22)$$

Differencing (3.21) as before and dividing by h yields

$$\sum_{k=1}^{n} a_{jk}\, k(t_{ij}, t_{ik})\overline{\varepsilon}_{ik} + h \sum_{\ell=2}^{i-1} \sum_{k=1}^{n} a_k \frac{k(t_{ij}, t_{\ell k}) - k(t_i, t_{\ell k})}{h} \overline{\varepsilon}_{\ell k}$$

$$= -h \sum_{k=1}^{n} a_k \left\{ \frac{k(t_{ij}, t_{1k}) - k(t_i, t_{1k})}{h} \overline{\varepsilon}_{1k} + \frac{k(t_{ij}, t_{0k}) - k(t_i, t_{0k})}{h} \overline{\varepsilon}_{0k} \right\}$$

$$- \frac{R_{ij} - R_{i-1,n}}{h} , \; j = 1, \ldots, n \; ; \; i = 2, \ldots, I-1 . \quad (3.23)$$

Define the $n \times n$ matrices

$$A = (a_{ij}) ,$$

$$B = (b_{ij}) = (a_j) ,$$

$$D = (d_{ij}) = (u_j \delta_{ij}) ,$$

where δ_{ij} is the Kronecker delta, and the n dimensional vectors

$$\underline{\delta} = \left(\delta_1, \ldots, \delta_n\right)^T$$

and

$$\underline{\theta}(t) = \left(\theta_1(t), \ldots, \theta_n(t)\right)^T$$

where

$$\underline{\theta}(t) = \left\{\frac{\partial^2 k}{\partial t \partial s}(t, 0) DB\left(D-(I+D)A^{-1}B\right)\right.$$

$$\left. + \frac{\frac{\partial k}{\partial t}(t,0)\frac{\partial k}{\partial s}(0,0)}{k(0,0)} DB\left((I+D)A^{-1}B-D\right)\right\}\underline{\delta} . \quad (3.24)$$

In addition, let $d_j(t)$, $j = 1, \ldots, n$, be the solution of the system of integral equations

$$k(t, t) \sum_{k=1}^{n} a_{jk} d_k(t) + u_j \int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=1}^{n} a_k d_k(s) ds = -\theta_j(t) ,$$

$$j = 1, \ldots, n , \quad (3.25)$$

and let $e_j(t)$, $j = 1, \ldots, n$, be the solution of

$$k(t, t) \sum_{k=1}^{n} a_{jk} e_k(t) + u_j \int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=1}^{n} a_k e_k(s) ds$$

$$= u_j \phi'(t) + \psi_j(t) - \psi_n(t) , \quad j = 1, \ldots, n . \quad (3.26)$$

Define

$$z_{ij} = \overline{\varepsilon}_{ij} - h^2 d_j\left(t_{ij}\right) - h^n e_j\left(t_{ij}\right) ,$$

$$j = 1, \ldots, n ; \quad i = 2, \ldots, I-1 .$$

Then from (3.23),

$$\sum_{k=1}^{n} a_{jk}\, k\left(t_{ij}, t_{ik}\right) z_{ik} + h \sum_{\ell=2}^{i-1} \sum_{k=1}^{n} a_k \frac{k\left(t_{ij}, t_{\ell k}\right) - k\left(t_i, t_{\ell k}\right)}{h} z_{\ell k}$$

$$= -h \sum_{k=1}^{n} a_k \left\{ \frac{k\left(t_{ij}, t_{1k}\right) - k\left(t_i, t_{1k}\right)}{h} \bar{\varepsilon}_{1k} + \frac{k\left(t_{ij}, t_{0k}\right) - k\left(t_i, t_{0k}\right)}{h} \bar{\varepsilon}_{0k} \right\}$$

$$+ h^2 \theta_j\left(t_i\right) + O\left(h^{n+1}\right) + O\left(h^3 \|\underline{\delta}\|\right) ,$$

$$j = 1, \ldots, n \; ; \quad i = 2, \ldots, I-1 . \quad (3.27)$$

Using (3.22), (3.24) and the identity

$$BA^{-1}B = B ,$$

a tedious but straightforward computation yields

$$h^2 \theta_j\left(t_i\right) - h \sum_{k=1}^{n} a_k \left\{ \frac{k\left(t_{ij}, t_{1k}\right) - k\left(t_i, t_{1k}\right)}{h} \bar{\varepsilon}_{1k} + \frac{k\left(t_{ij}, t_{0k}\right) - k\left(t_i, t_{0k}\right)}{h} \bar{\varepsilon}_{0k} \right\}$$

$$= O\left(h^{n+1}\right) + O\left(h^3 \|\underline{\delta}\|\right) .$$

The application of lemma 2.4 to (3.27) therefore yields:

<u>Theorem 3.4.</u>

$$\bar{\varepsilon}_{ij} = h^n e_j\left(t_{ij}\right) + h^2 d_j\left(t_{ij}\right) + O\left(h^{n+1}\right) + O\left(h^3 \|\underline{\delta}\|\right) ,$$

$j = 1, \ldots, n$; $i = 2, \ldots, I-1$, where $d_j(t)$ and $e_j(t)$ are given by (3.25) and (3.26) respectively.

To establish the corresponding result for the scheme (3.10), define the $n \times n$ matrix

$$C = \left(c_{ij}\right) = \left[u_i \sum_{k=1}^{n} \left(1 + u_i u_k\right) a_k L_j\left(u_i u_k\right)\right]$$

and the n dimensional vector

$$\underline{\tilde{\theta}}(t) = \left(\tilde{\theta}_1(t), \ldots, \tilde{\theta}_n(t)\right)^T ,$$

where

$$\underline{\tilde{\theta}}(t) = \left\{\frac{\partial^2 k}{\partial t \partial s}(t, 0)DB\left(D-(D+I)A^{-1}B\right)\right.$$

$$\left. + \frac{\frac{\partial k}{\partial t}(t,0)\frac{\partial k}{\partial s}(0,0)}{k(0,0)} DBA^{-1}\left(CA^{-1}B-BD\right)\right\}\underline{\delta} .$$

Let the functions $\tilde{d}_j(t)$, $j = 1, \ldots, n$, satisfy

$$k(t, t) \sum_{k=1}^{n} a_{jk} \tilde{d}_k(t) + u_j \int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=1}^{n} a_k \tilde{d}_k(s)ds = -\tilde{\theta}_j(t) ,$$

$$j = 1, \ldots, n , \quad (3.28)$$

and $\tilde{e}_j(t)$, $j = 1, \ldots, n$, satisfy

$$k(t, t) \sum_{k=1}^{n} a_{jk} \tilde{e}_k(t) + u_j \int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=1}^{n} a_k \tilde{e}_k(s)ds$$

$$= u_j\tilde{\phi}'(t) + \tilde{\psi}_j(t) - \tilde{\psi}_n(t) , \quad j = 1, \ldots, n . \quad (3.29)$$

The following result on the asymptotic behaviour of the error, for the scheme (3.10), is obtained in the same way as theorem 3.4.

Theorem 3.5.

$$\overline{\varepsilon}_{ij} = h^n\tilde{e}_j(t_{ij}) + h^2\tilde{d}_j(t_{ij}) + O(h^{n+1}) + O(h^3\|\underline{\delta}\|) ,$$

$j = 1, \ldots, n$; $i = 2, \ldots, I-1$, where $\tilde{d}_j(t)$ and $\tilde{e}_j(t)$ are given by (3.28) and (3.29) respectively.

Remark 3.3. It is clear that the solutions of the systems (3.25), (3.26), (3.28) and (3.29) are characterized by the n th equation which is of the form (3.19) and therefore methods I are numerically stable. Also, perturbations of magnitude $\|\underline{\delta}\|$ are propagated with magnitude $h^2\|\underline{\delta}\|$. This is the same behaviour as observed in the midpoint method (cp. §3.2).

3.6 Methods I: Numerical Results

In order to illustrate convergence and numerical stability, the schemes (3.9) and (3.10) with $u_1 = \frac{1}{3}$, $u_2 = \frac{2}{3}$ and $u_3 = 1$ are applied to the equation

$$-1 + t + e^{-t} = \int_0^t (1+t-s)y(s)ds \ , \quad 0 \le t \le 20 \ ,$$

where

$$y(t) = te^{-t} \ .$$

The errors for various stepsizes h are tabulated in tables 3.1 and 3.2.

Table 3.1.

Method (3.9)			
t	h = 0.4	h = 0.2	h = 0.1
4.0	-1.086 E-4	-1.125 E-5	-1.290 E-6
8.0	-1.563 E-6	-7.202 E-8	-2.613 E-9
12.0	7.715 E-8	1.257 E-8	1.730 E-9
16.0	5.137 E-9	6.940 E-10	8.964 E-11
20.0	1.949 E-10	2.498 E-11	2.932 E-12

Table 3.2.

Method (3.10)			
t	h = 0.4	h = 0.2	h = 0.1
4.0	-1.572 E-4	-1.817 E-5	-2.181 E-6
8.0	-4.016 E-6	-4.343 E-7	-5.033 E-8
12.0	1.111 E-8	2.194 E-9	3.194 E-10
16.0	3.679 E-9	4.460 E-10	5.468 E-11
20.0	1.661 E-10	1.976 E-11	2.094 E-12

3.7 Methods II: Preliminaries

Let

$$\omega(t) = \sum_{k=0}^{n-1} v_k t^{k+1} ,$$

$$\omega(t+1) = \sum_{k=0}^{n-1} w_k t^{k+1}$$

and define the $(n-1) \times (n-1)$ matrices

$$M = (m_{ij}) = (a_{i+1,j+1}) ,$$

$$B = (b_{ij}) = (-a_{i+1,1}\, \delta_{j,n-1}) ,$$

where δ_{ij} is the Kronecker delta,

$$D = (d_{ij}) = (u_{j+1}\, \delta_{ij}) ,$$

$$R = (r_{ij}) = (\delta_{j,n-1}) ,$$

$$C = (c_{ij}) = DR(A-B) ,$$

$$W = (w_{ij}) = (\delta_{i,n-1}\, \delta_{j,n-1}) ,$$

$$X = (x_{ij}) = (\delta_{i,n-1}\, \delta_{j,n-1} + \delta_{i,n-2}\, \delta_{j,n-2}) .$$

The following lemmas examine some properties of the above matrices.

Lemma 3.2.

The nonzero eigenvalue of $M^{-1}B$ is

$$\eta = \frac{w_0}{v_0} = \frac{\prod_{k=1}^{n-1}(1-u_k)}{\prod_{k=2}^{n}(-u_k)} .$$

Proof. Consider the differential equation

$$y' = \frac{y}{\lambda} , \quad y(0) = 1 , \quad 0 \le t \le 1 , \quad \lambda > 0$$

and approximate the solution $y(t)$ by

$$q(t) = \sum_{k=0}^{n} c_k t^k$$

where

$$c_0 = 1$$

and c_k , $k = 1, \ldots, n$, are determined by collocation at the points u_k , $k = 1, \ldots, n$.

Let

$$\underline{q} = \left(q(u_2), q(u_3), \ldots, q(u_n)\right)^T .$$

Wright (1970) shows that

$$q(u_n) = \frac{\int_0^\infty e^{-s/\lambda}\omega(s+1)ds}{\int_0^\infty e^{-s/\lambda}\omega(s)ds} = \frac{\sum_{k=0}^{n-1}(k+1)!w_k\lambda^k}{\sum_{k=0}^{n-1}(k+1)!v_k\lambda^k} \qquad (3.30)$$

and remarks that this collocation is equivalent to the implicit Runge-Kutta scheme

$$(\lambda I-M)\underline{q} = \underline{x}$$

where

$$\underline{x} = (\lambda+a_{21}, \lambda+a_{31}, \ldots, \lambda+a_{n1})^T .$$

Clearly, since M is nonsingular, $\lambda I - M$ is nonsingular if λ is sufficiently small, and hence by Cramer's rule,

$$q(u_n) = \frac{\det(\lambda S-N)}{\det(\lambda I-M)} , \tag{3.31}$$

where

$$S = (s_{ij}) = (\delta_{ij}+\delta_{n-1,j}-\delta_{i,n-1}\delta_{j,n-1}) ,$$

$$N = (n_{ij}) = (m_{ij}(1-\delta_{j,n-1})-a_{i+1,1}\delta_{j,n-1}) .$$

Equating (3.30) and (3.31) and taking the limit as λ tends to zero, yields

$$\frac{\det N}{\det M} = \frac{w_0}{v_0} .$$

The result follows since the characteristic equation is

$$\det(\eta M-B) = \eta^{n-2}(\eta \det M - \det N) = 0 . \qquad \#$$

<u>Lemma 3.3.</u>

Let E and Y be the block matrices

$$E = \left[\begin{array}{c|c} I+M^{-1}B & -M^{-1}B \\ \hline I & 0 \end{array}\right] , \quad Y = \left[\begin{array}{c|c} \eta W & 0 \\ \hline 0 & I \end{array}\right] .$$

Then, if $\eta \neq 1$, there exists a nonsingular matrix H such that

$$E = H^{-1}YH .$$

<u>Proof</u>. Since

$$E - \lambda I = \left[\begin{array}{c|c} I & 0 \\ \hline I & I \end{array}\right] \left[\begin{array}{c|c} (1-\lambda)I & -M^{-1}B \\ \hline 0 & M^{-1}B-\lambda I \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline -I & I \end{array}\right] ,$$

the characteristic equation is

$$\det(E-\lambda I) = (1-\lambda)^{n-1}(-\lambda)^{n-2}(\eta-\lambda) = 0 .$$

Since it is easily shown that E has a complete set of eigenvectors, the result follows. #

Lemma 3.4.

If $\eta = 1$, then

(i) $CM^{-1}B = 0$,

(ii) $\left(M^{-1}C\right)^2 = M^{-1}C$,

(iii) there exists a nonsingular matrix G such that

$$M^{-1}(B+C) = G^{-1}XG .$$

Proof.

(i) $CM^{-1}B = D\,R\,(M-B)M^{-1}B$

$$= D\,R\,B\,\left(I-M^{-1}B\right)$$

$$= b_{n-1,n-1}D\,R\,\left(I-M^{-1}B\right)$$

$= 0$, since the $(n-1)$th row of $\left(I-M^{-1}B\right)$ is zero.

(ii) Consider the vector

$$\underline{v} = \left(I-M^{-1}B\right)\underline{e}$$

where

$$\underline{e} = (1, 1, \ldots, 1)^T .$$

Then, from (i),

$$M^{-1}D\,R\,B\,\left(I-M^{-1}B\right)\underline{e} = \underline{0} .$$

Also, since

$$\int_0^{u_r} \sum_{k=1}^{n} L_k(s)ds = u_r ,$$

it follows that

$$(M-B)\underline{e} = \underline{u}$$

where

$$\underline{u} = \left(u_2, u_3, \ldots, u_n\right)^T .$$

The use of this relation yields

$$M^{-1}D\,R\,M\,(I-M^{-1}B)\underline{e} = M^{-1}D\,R\,\underline{u}$$

$$= M^{-1}D\,\underline{e}$$

$$= M^{-1}\,\underline{u}$$

$$= (I-M^{-1}B)\,\underline{e}\,.$$

Hence,

$$M^{-1}C\,\underline{v} = M^{-1}D\,R\,(M-B)\,\underline{v} = \underline{v}\,.$$

The result follows since $M^{-1}C$ is of rank 1 and hence has $n-2$ linearly independent eigenvectors corresponding to the zero eigenvalue.

(iii) Since $\eta = 1$,

$$M^{-1}B\,\underline{v} = \underline{0}\,,$$

and it follows from (ii) that

$$M^{-1}(B+C)\,\underline{v} = \underline{v}\,.$$

Let

$$\underline{\omega} = M^{-1}B\,\underline{e}\,.$$

Then, from (i),

$$M^{-1}C\,\underline{\omega} = \underline{0}\,,$$

and since $\omega_{n-1} = \eta = 1$, therefore

$$M^{-1}B\,\underline{\omega} = \underline{\omega}\,.$$

Hence,

$$M^{-1}(B+C)\,\underline{\omega} = \underline{\omega}\,.$$

Since $M^{-1}(B+C)$ has rank two and two independent eigenvectors corresponding to the eigenvalue 1 , it is diagonalizable and so the result follows. #

In addition use will be made of the following two lemmas.

Lemma 3.5. (Jones (1961)).

Let q_i , $i = 1, \ldots, I$, satisfy

$$q_{i+1} = (1+2hL)q_i + h^2 L \sum_{\ell=0}^{i} q_\ell + Lh^{m+1} \ , \quad L = \text{const} > 0 \ ,$$

and let

$$0 \le q_0 \ .$$

Then, there exists a constant K such that

$$0 \le q_i \le K\left(q_0 + h^m\right) \ , \quad i = 0, \ldots, I \ .$$

Lemma 3.6.

Let $-1 \le \eta < 1$ and $f(t)$, $0 \le t \le T$, be a continuously differentiable function. Then

$$\sum_{\ell=0}^{i-1} \eta^\ell f\left(t_\ell\right) = \frac{f(0) - \eta^i f\left(t_i\right)}{1-\eta} + O(h) \ , \quad i = 1, \ldots, I \ .$$

Proof. For $\eta = -1$ the result follows from the Euler-Maclaurin sum formula. If $-1 < \eta < 1$, partial summation yields

$$\sum_{\ell=0}^{i-1} \eta^\ell f\left(t_\ell\right) = f(0) \frac{1-\eta^i}{1-\eta} + O(h)$$

and since

$$\eta^i \left(f\left(t_i\right) - f(0)\right) = O(h) \ ,$$

the result follows. #

3.8 Methods II: Convergence

In the analysis of methods II it will be assumed that conditions (3.4.1), (3.4.2) and (3.4.3), with n replaced by $n + 2$, are satisfied.

The results of this section can be summarized in

Theorem 3.6.

The schemes (3.9) and (3.10) are convergent if and only if

$$-1 \le \eta = \frac{\prod_{k=1}^{n-1} \left(1-u_k\right)}{\prod_{k=2}^{n} \left(-u_k\right)} \le 1$$

and the order of convergence is $n - 1$ if $\eta = 1$ and n otherwise.

Before a proof of this theorem can be given, some preliminary results are required. Subtraction of (3.9) with $h < \delta$ and (3.10) from (3.8) yields

$$0 = h \sum_{k=1}^{n} a_{jk}\, k\left(t_{ij}, t_{ik}\right)\varepsilon_{ik} + h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k\left(t_{ij}, t_{\ell k}\right)\varepsilon_{\ell k} + R_{ij} ,$$

$$j = 2, \ldots, n \; ; \; i = 0, \ldots, I-1 , \quad (3.32)$$

and

$$0 = h \sum_{k=1}^{n} u_j \sum_{r=1}^{n} a_r\, k\left(t_{ij}, t_i + u_j u_r h\right) L_k\left(u_j u_r\right)\varepsilon_{ik}$$

$$+ h \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} a_k\, k\left(t_{ij}, t_{\ell k}\right)\varepsilon_{\ell k} + \tilde{R}_{ij} ,$$

$$j = 2, \ldots, n \; ; \; i = 0, \ldots, I-1 , \quad (3.33)$$

respectively, where

$$\varepsilon_{ij} = y\left(t_{ij}\right) - Y_{ij}$$

and R_{ij} and $\tilde{R}_{ij}$ have the same form as (3.14) and (3.15) respectively.

Lemma 3.7.

There exist unique functions $\phi(t)$, $\tilde{\phi}(t)$, $\psi_j(t)$, $\tilde{\psi}_j(t)$, $\Theta_j(t)$, $\tilde{\Theta}_j(t)$, $j = 2, \ldots, n$, $0 \le t \le T$, satisfying

(i) $\phi(t), \tilde{\phi}(t)$ are three times continuously differentiable and $\phi(0) = 0$, $\tilde{\phi}(0) = 0$,

(ii) $\psi_j(t), \tilde{\psi}_j(t)$, $j = 2, \ldots, n$, are two times continuously differentiable,

(iii) $\Theta_j(t), \tilde{\Theta}_j(t)$, $j = 2, \ldots, n$, are continuously differentiable,

such that

$$R_{ij} = -h^n \phi(t_{ij}) - h^{n+1} \psi_j(t_{ij}) - h^{n+2} \Theta_j(t_{ij}) + O(h^{n+3}) ,$$

$$\tilde{R}_{ij} = -h^n \tilde{\phi}(t_{ij}) - h^{n+1} \tilde{\psi}_j(t_{ij}) - h^{n+2} \tilde{\Theta}_j(t_{ij}) + O(h^{n+3}) ,$$

$$j = 2, \ldots, n \; ; \; i = 0, \ldots, I-1 .$$

Proof. The result follows from the Euler-Maclaurin sum formula. #

Firstly the scheme (3.9) will be considered. To simplify the notation, define the $(n-1) \times (n-1)$ matrices

$$M_{i\ell} = \left(m_{jk}^{(i\ell)}\right) , \quad \ell = 0, \ldots, i \; ; \; i = 0, \ldots, I-1 ,$$

where

$$m_{jk}^{(ii)} = m_{jk} \, k\left(t_{i,j+1}, t_{i,k+1}\right) ,$$

$$m_{jk}^{(i,i-1)} = a_{j+1,1} \, k\left(t_{i,j+1}, t_i\right)\delta_{n-1,j} + a_{k+1} \, k\left(t_{i,j+1}, t_{i-1,k+1}\right) ,$$

$$m_{jk}^{(i\ell)} = a_1 \, k\left(t_{i,j+1}, t_{\ell+1}\right)\delta_{n-1,j} + a_{k+1} \, k\left(t_{i,j+1}, t_{\ell,k+1}\right) ,$$

$$\ell = 0, \ldots, I-2 ,$$

and the $n - 1$ dimensional vectors

$$\left.\begin{aligned} \underline{\varepsilon}_i &= \left(\varepsilon_{i2}, \ldots, \varepsilon_{in}\right)^T \\ \underline{R}_i &= \left(R_{i2}, \ldots, R_{in}\right)^T \\ \underline{b}_i &= \left(b_{i2}, \ldots, b_{in}\right)^T \end{aligned}\right\} \; i = 0, \ldots, I-1 ,$$

where

$$b_{0j} = a_{j1} \, k\left(t_{0j}, 0\right) , \quad j = 2, \ldots, n ;$$

$$b_{ij} = a_1 \, k\left(t_{ij}, 0\right) , \quad j = 2, \ldots, n \; ; \; i = 1, \ldots, I-1 .$$

Then (3.32) can be rewritten in vector form as

$$h \sum_{\ell=0}^{i} M_{i\ell}\underline{\varepsilon}_\ell + h\varepsilon_{01}\underline{b}_i + \underline{R}_i = 0 , \quad i = 0, \ldots, I-1 . \tag{3.34}$$

To establish convergence, bounds on the solutions of difference equations of the form (3.34) are required. To obtain these, it is necessary to distinguish between two cases.

Case 1. $-1 \le \eta < 1$.

Lemma 3.8.

Let the $n-1$ dimensional vectors $\underline{z}_i$, $i = 0, \ldots, I-1$, satisfy

$$h \sum_{\ell=0}^{i} M_{i\ell}\underline{z}_\ell + h\bar{z}\,\underline{b}_i = h^m\underline{\alpha}_i + h^{m+1}\underline{\beta}_i + O\left(h^{m+2}\right) ,$$

$$i = 0, \ldots, I-1 , \tag{3.35}$$

where

$$\underline{\alpha}_i = \left(\alpha(t_{i2}), \ldots, \alpha(t_{in})\right)^T ,$$

$$\underline{\beta}_i = \left(\beta_2(t_{i2}), \ldots, \beta_n(t_{in})\right)^T ,$$

and

(i) $\alpha(t)$ has a Lipschitz continuous first derivative,

(ii) $\beta_j(t)$, $j = 2, \ldots, n$, are Lipschitz continuous.

Then

$$\|\underline{z}_i\| = O(\bar{z}h) + O\left(\|\underline{z}_1\|\right) + O\left(h^m\right) , \quad i = 1, \ldots, I-1 .$$

Proof. From (3.35) it follows that

$$M_{i+1,i+1}\underline{z}_{i+1} + \left(M_{i+1,i} - (I+R)M_{ii}\right)\underline{z}_i + \sum_{\ell=0}^{i-1}\left(M_{i+1,\ell} - (I+R)M_{i\ell} + RM_{i-1,\ell}\right)\underline{z}_\ell$$

$$+ \bar{z}\left(\underline{b}_{i+1} - (I+R)\underline{b}_i + R\underline{b}_{i-1}\right)$$

$$= h^{m-1}\left(\underline{\alpha}_{i+1} - (I+R)\underline{\alpha}_i + R\underline{\alpha}_{i-1}\right) + h^m\left(\underline{\beta}_{i+1} - (I+R)\underline{\beta}_i + R\underline{\beta}_{i-1}\right) + O\left(h^{m+1}\right) ,$$

$$i = 1, \ldots, I-2 . \tag{3.36}$$

Division of (3.36) by $k(t_{i+1}, t_{i+1})$ and the applications of Taylor's theorem yield

$$(M+hS_i)\underline{z}_{i+1} - (M+B+hT_i)\underline{z}_i + (B+hU_i)\underline{z}_{i-1} - h^2 \sum_{\ell=0}^{i-1} W_{i\ell}\underline{z}_\ell - h^2\overline{z}\,\underline{x}_i$$

$$= h^{m+1}\underline{r}_i , \qquad i = 1, \ldots, I-2 , \qquad (3.37)$$

where S_i, T_i, U_i, $W_{i\ell}$ and $\underline{x}_i$, $\underline{r}_i$ are matrices and vectors, respectively, the norms of which are bounded by a constant independent of h, i and ℓ. Since M is nonsingular, $(M+hS_i)$ is nonsingular if h is sufficiently small and therefore

$$(M+hS_i)^{-1} = M^{-1} + hN_i ,$$

where

$$\|N_i\| \le D , \quad D = \text{const} , \quad i = 1, \ldots, I-2 .$$

Multiplication of (3.37) by $M^{-1} + hN_i$ yields

$$\underline{z}_{i+1} = \left(I+M^{-1}B+h\overline{T}_i\right)\underline{z}_i - \left(M^{-1}B+h\overline{U}_i\right)\underline{z}_{i-1}$$

$$+ h^2 \sum_{\ell=0}^{i-1} \overline{W}_{i\ell}\underline{z}_\ell + h^2\overline{z}\,\overline{\underline{x}}_i + h^{m+1}\overline{\underline{r}}_i , \quad i = 1, \ldots, I-2 , \qquad (3.38)$$

where $\overline{T}_i$, $\overline{U}_i$, $\overline{W}_{i\ell}$ and $\overline{\underline{x}}_i$, $\overline{\underline{r}}_i$ are matrices and vectors, respectively, which are bounded as above. Define the $2n - 2$ dimensional vectors

$$\underline{Z}_0 = (\underline{z}_0, \underline{0})^T , \quad \underline{Z}_i = (\underline{z}_i, \underline{z}_{i-1})^T , \quad i = 1, \ldots, I-1 ,$$

$$\underline{q}_i = (\overline{\underline{r}}_i, \underline{0})^T , \quad i = 1, \ldots, I-2 ,$$

$$\underline{y}_i = (\overline{\underline{x}}_i, \underline{0})^T , \quad i = 1, \ldots, I-2 ,$$

and the block matrices

$$F_i = \left[\begin{array}{c|c} \bar{T}_i & -\bar{U}_i \\ \hline 0 & 0 \end{array}\right] , \quad i = 1, \ldots, I-2 ,$$

and

$$G_{i\ell} = \left[\begin{array}{c|c} \bar{W}_{i\ell} & 0 \\ \hline 0 & 0 \end{array}\right] , \quad \ell = 0, \ldots, i-1 ; \quad i = 1, \ldots, I-2 .$$

Then (3.38) can be rewritten as

$$\underline{Z}_{i+1} = \left(E+hF_i\right)\underline{Z}_i + h^2 \sum_{\ell=0}^{i-1} G_{i\ell}\underline{Z}_\ell + h^2\bar{z}\,\underline{y}_i + h^{m+1}\underline{q}_i ,$$

$$i = 1, \ldots, I-2 . \quad (3.39)$$

From lemma 3.3,

$$E = H^{-1}YH ,$$

and denoting

$$\hat{\underline{Z}}_i = H\underline{Z}_i ,$$

it follows on multiplying (3.39) by H that

$$\hat{\underline{Z}}_{i+1} = \left(Y+hHF_iH^{-1}\right)\hat{\underline{Z}}_i + h^2 \sum_{\ell=2}^{i-1} HG_{i\ell}H^{-1}\hat{\underline{Z}}_\ell$$

$$+ h^2H\left(G_{i0}\underline{Z}_0+G_{i1}\underline{Z}_1+\bar{z}\underline{y}_i\right) + h^{m+1}H\underline{q}_i , \quad i = 2, \ldots, I-2 .$$

By induction, it can be shown that

$$\|\hat{\underline{Z}}_i\| \le P_{i-2} , \quad i = 2, \ldots, I-1 ,$$

where

$$P_0 = \|\hat{\underline{Z}}_2\| ,$$

$$P_{i+1} = (1+2hL)P_i + h^2L \sum_{\ell=0}^{i} P_\ell + L\{h^2|\bar{z}|+h^{m+1}\} , \quad i = 0, \ldots, I-4 ,$$

and L is an appropriate constant. The result follows from lemma 3.5 since from (3.35),

$$\|\underline{z}_2\| = O(h\bar{z}) + O\left(\|\underline{z}_1\|\right) + O\left(h^m\right) . \qquad \#$$

Case 2. $\eta = 1$.

Lemma 3.9.

Let $\underline{z}_i$, $i = 0, \ldots, I-1$, satisfy

$$h \sum_{\ell=0}^{i} M_{i\ell}\underline{z}_\ell + h\bar{z}\underline{b}_i = h^{m+1}\underline{\alpha}_i + O\left(h^{m+2}\right) , \quad i = 0, \ldots, I-1 , \qquad (3.40)$$

where

$$\underline{\alpha}_i = \left(\alpha\left(t_{i2}\right), \ldots, \alpha\left(t_{in}\right)\right)^T$$

and $\alpha(t)$ has a Lipschitz continuous first derivative. Then

$$\|\underline{z}_i\| = O(\bar{z}) + O\left(h^m\right) , \quad i = 0, \ldots, I-1 .$$

Proof. From (3.40) it follows that

$$M_{ii}\underline{z}_i + \sum_{\ell=0}^{i-1} \left(M_{i\ell}-RM_{i-1,\ell}\right)\underline{z}_\ell + \bar{z}\left(\underline{b}_i-R\underline{b}_{i-1}\right) = h^m\left(\underline{\alpha}_i-R\underline{\alpha}_{i-1}\right) + O\left(h^{m+1}\right) ,$$

$$i = 1, \ldots, I-1 .$$

Division by $k\left(t_i, t_i\right)$ and the application of Taylor's theorem yield

$$\left(M+hS_i\right)\underline{z}_i - \left(B+hT_i\right)\underline{z}_{i-1} + hC \sum_{\ell=0}^{i-1} \frac{\frac{\partial k}{\partial t}\left(t_i,t_\ell\right)}{k\left(t_i,t_i\right)} \underline{z}_\ell + h^2 \sum_{\ell=0}^{i-1} U_{i\ell}\underline{z}_\ell + h\bar{z}\underline{x}_i$$

$$= h^{m+1}\underline{r}_i , \qquad i = 1, \ldots, I-1 , \qquad (3.41)$$

where S_i, T_i, $U_{i\ell}$ and $\underline{x}_i$, $\underline{r}_i$ are matrices and vectors, respectively, the norms of which are bounded by a constant independent of h, i and ℓ .

Multiplication of (3.41) by $M^{-1}CM^{-1}$ and the application of lemma 3.4, (i) and (ii), yield

$$hM^{-1}C\sum_{\ell=0}^{i-1}\frac{\frac{\partial k}{\partial t}(t_i,t_\ell)}{k(t_i,t_i)}\,\underline{z}_\ell = -\left(M^{-1}C+hM^{-1}CM^{-1}S_i\right)\underline{z}_i$$

$$+\,hM^{-1}CM^{-1}T_i\underline{z}_{i-1} - h^2\sum_{\ell=0}^{i-1} M^{-1}CM^{-1}U_{i\ell}\underline{z}_\ell$$

$$-\,M^{-1}CM^{-1}\left(h\bar{z}\underline{x}_i - h^{m+2}\underline{r}_i\right)\ , \quad i = 1, \ldots, I-1\ . \qquad (3.42)$$

From Taylor's theorem,

$$\frac{\frac{\partial k}{\partial t}(t_i,t_\ell)}{k(t_i,t_i)} = \frac{\frac{\partial k}{\partial t}(t_{i+1},t_\ell)}{k(t_{i+1},t_{i+1})} + O(h)\ . \qquad (3.43)$$

Substitution of (3.43) and (3.42) into (3.41) with i replaced by $i+1$, yields an equation of the form

$$\left(M+h\bar{S}_i\right)\underline{z}_{i+1} = \left(B+C+h\bar{T}_i\right)\underline{z}_i + h\bar{V}_i\underline{z}_{i-1} + h^2\sum_{\ell=0}^{i}\bar{U}_{i\ell}\underline{z}_\ell + h\bar{z}\,\bar{\underline{x}}_i + h^{m+1}\bar{\underline{r}}_i\ ,$$

$$i = 1, \ldots, I-2\ .$$

The result follows in a similar way to lemma 3.8, using lemma 3.4, (iii), instead of lemma 3.3. #

Remark 3.4. It is clear that (3.33) can be written in the form (3.34) if the $M_{i\ell}$ are redefined appropriately. Using (2.9) it can be shown that lemmas 3.8 and 3.9 hold with these definitions of $M_{i\ell}$.

Proof of theorem 3.6. From (3.34) and lemma 3.7,

$$\|\underline{\varepsilon}_0\| = O(\varepsilon_{01}) + O(h^n)\ , \quad \|\underline{\varepsilon}_1\| = O(\varepsilon_{01}) + O(h^n)\ .$$

Hence it follows from lemma 3.7 and lemmas 3.8 and 3.9, respectively that

$$\|\underline{\varepsilon}_i\| = O(\varepsilon_{01}) + O(h^{n-1})\ , \quad i = 0, \ldots, I-1\ ,$$

if $\eta = 1$, and

$$\|\underline{\varepsilon}_i\| = O(\varepsilon_{01}) + O(h^n)\ , \quad i = 0, \ldots, I-1\ ,$$

if $-1 \le \eta < 1$. This establishes sufficiency and order of convergence for the scheme (3.9). The extension to the scheme (3.10) follows from remark 3.4.

If $k(t, s) = 1$, then $\underline{\varepsilon}_i$ can be shown to satisfy an equation of the form

$$\underline{\varepsilon}_i = M^{-1}B\underline{\varepsilon}_{i-1} + h^n \underline{r}_i \ , \quad i = 1, \ldots, I-1 \ .$$

Necessity follows, since the nonzero eigenvalue of $M^{-1}B$ is η . #

Remark 3.5. The fact that for $\eta = 1$ the schemes are convergent or order $n - 1$ instead of order n is similar to a phenomenon associated with weakly stable finite difference operators for initial value problems in partial differential equations (see Richtmyer and Morton (1967), p. 95), since in the equation corresponding to (3.39) for $\eta = 1$,

$$E^i = \left[\begin{array}{c|c} I+M^{-1}B & -M^{-1}B \\ \hline I & 0 \end{array}\right]^i = \left[\begin{array}{c|c} I+M^{-1}B & -M^{-1}B \\ \hline I & 0 \end{array}\right] + (i-1)\left[\begin{array}{c|c} M^{-1}B & -M^{-1}B \\ \hline M^{-1}B & -M^{-1}B \end{array}\right] ,$$

and hence,

$$0 < iC_1 \le \|E^i\| \le iC_2 \ , \quad i = 1, 2, \ldots \ , \quad C_1, C_2 = \text{const} \ .$$

3.9 Methods II: Numerical Stability

In this section, the leading term in the asymptotic error expansion for the schemes (3.9) and (3.10) will be investigated. As in the convergence proof it is necessary to distinguish between the cases $-1 \le \eta < 1$ and $\eta = 1$.

Case 1. $-1 \le \eta < 1$.

First consider the scheme (3.9).

Theorem 3.7.

Let $e_j(t)$, $j = 2, \ldots, n$, be the solution of the system of second kind Volterra integral equations

$$k(t, t)\left\{\sum_{k=2}^{n} a_{jk}e_k(t)+a_{j1}e_n(t)\right\}+ u_j\int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=2}^{n} \alpha_k e_k(s)ds$$

$$= u_j\phi'(t) + \psi_j(t) - \psi_n(t) , \quad j = 2, \ldots, n , \tag{3.44}$$

where

$$\alpha_k = a_k + \delta_{nk} a_1$$

and $\phi(t)$, $\psi_j(t)$, $j = 2, \ldots, n$ are defined by lemma 3.7. In addition, let $q_r(t)$, $r = 1, 2$, satisfy

$$k(t, t)q_r'(t) + \frac{\partial k}{\partial s}(t, t)q_r(t) = 0 , \quad r = 1, 2 , \tag{3.45}$$

with the initial conditions

$$q_1(0) = \frac{\psi_n(0)}{\eta k(0,0)} \sum_{k=1}^{n-1} \tilde{m}_{n-1,k} - e_n(0) , \tag{3.46}$$

where $\tilde{m}_{jk}$ are the elements of M^{-1} , and

$$q_2(0) = 1 .$$

Also let

$$\underline{\omega} = \left(\omega_1, \omega_2, \ldots, \omega_{n-1}\right)^T = M^{-1}B\underline{e}$$

where

$$\underline{e} = (1, 1, \ldots, 1)^T .$$

Then

$$\varepsilon_{ij} = h^n\left(e_j(t_{ij})+\eta^i\omega_{j-1}q_1(t_{ij})\right) + \varepsilon_{01}\eta^i\omega_{j-1}q_2(t_{ij})$$

$$+ O\left(h^{n+1}\right) + O\left(h\varepsilon_{01}\right) , \quad j = 2, \ldots, n ; \quad i = 1, \ldots, I-1 .$$

Proof. From the Euler Maclaurin sum formula, Taylor's theorem and (3.44),

$$\sum_{k=2}^{n} ha_{jk}\, k(t_{ij}, t_{ik})e_k(t_{ij}) + ha_{j1}k(t_{ij}, t_i)e_n(t_i)$$

$$- ha_1\{k(t_{ij}, t_i)e_n(t_i)-k(t_{ij}, 0)e_n(0)\}$$

$$+ \sum_{\ell=0}^{i-1}\sum_{k=2}^{n} h\alpha_k\, k(t_{ij}, t_{\ell k})e_k(t_{\ell k})$$

$$= \phi(t_{ij}) + h\{\psi_j(t_{ij})-\psi_n(t_{ij})-\gamma(t_{ij})\} - h^2\rho_j(t_{ij}) + O(h^3) ,$$

$$j = 2, \ldots, n ;\quad i = 0, \ldots, I-1 , \tag{3.47}$$

where $\gamma(t)$ is twice continuously differentiable, $\gamma(0) = 0$, and $\rho_j(t)$, $j = 2, \ldots, n$, are continuously differentiable. Noting that

$$\omega_{n-1} = \eta$$

and

$$\sum_{k=2}^{n} a_{jk}\omega_{k-1} + a_{j1} = 0 ,\quad j = 2, \ldots, n \tag{3.48}$$

it follows from Taylor's theorem and (3.45) that

$$\sum_{k=2}^{n} ha_{jk}\, k(t_{ij}, t_{ik})\omega_{k-1}q_r(t_{ik}) + ha_{j1}\, k(t_{ij}, t_i)q_r(t_i) = O(h^3) ,$$

$$r = 1, 2 ;\quad j = 2, \ldots, n ;\quad i = 0, \ldots, I-1 . \tag{3.49}$$

By Taylor's theorem, (3.48), lemma 3.6 and (3.45),

$$\sum_{\ell=0}^{i-1} \eta^{\ell}\left\{\sum_{k=2}^{n} ha_k\, k(t_{ij}, t_{\ell k})\omega_{k-1}q_r(t_{\ell k})+ha_1\, k(t_{ij}, t_{\ell})q_r(t_{\ell})\right\}$$

$$= h^2\sum_{\ell=0}^{i-1}\eta^{\ell}\left\{\sum_{k=2}^{n} u_k a_k\omega_{k-1}\left[\frac{\partial k}{\partial s}(t_i, t_{\ell})q_r(t_{\ell})+k(t_i, t_{\ell})q_r'(t_{\ell})\right]\right\} + O(h^3)$$

$$= -h^2\mu_r(t_{ij}) + O(h^3) ,$$

$$r = 1, 2 ;\quad j = 2, \ldots, n ;\quad i = 0, \ldots, I-1 , \tag{3.50}$$

where $\mu_r(t)$, $r = 1, 2$ are continuously differentiable. Define

$$z_{ij} = \varepsilon_{ij} - h^n\left\{e_j\left(t_{ij}\right)+\eta^i\omega_{j-1}q_1\left(t_{ij}\right)\right\} - \varepsilon_{01}\eta^i\omega_{j-1}q_2\left(t_{ij}\right) ,$$

$$j = 2, \ldots, n \; ; \; i = 0, \ldots, I-1 ,$$

$$z_{i+1,1} = z_{in} , \; i = 0, \ldots, I-2 ,$$

and

$$z_{01} = - h^n\left(e_n(0)+q_1(0)\right) .$$

Then, from (3.32), lemma 3.7, (3.47), (3.49) and (3.50),

$$\sum_{k=1}^{n} ha_{jk}\, k\left(t_{ij},\, t_{ik}\right)z_{ik} + \sum_{\ell=0}^{i-1}\sum_{k=1}^{n} ha_k\, k\left(t_{ij},\, t_{\ell k}\right)z_{\ell k}$$

$$= h^{n+1}\{\psi_n\left(t_{ij}\right)+\gamma\left(t_{ij}\right)\} + h^{n+2}\{\theta_j\left(t_{ij}\right)+\rho_j\left(t_{ij}\right)+\mu_1\left(t_{ij}\right)\}$$

$$+ h^2\varepsilon_{01}\mu_2\left(t_{ij}\right) + O\left(h^{n+3}\right) + O\left(h^3\varepsilon_{01}\right) ,$$

$$j = 2, \ldots, n \; ; \; i = 0, \ldots, I-1 . \quad (3.51)$$

Also from (3.51) and (3.46),

$$z_{01} = O\left(h^n\right)$$

and

$$z_{1j} = O\left(h^{n+1}\right) + O\left(h\varepsilon_{01}\right) , \; j = 2, \ldots, n .$$

The application of lemma 3.8 to (3.51) yields

$$z_{ij} = O\left(h^{n+1}\right) + O\left(h\varepsilon_{01}\right) , \; j = 2, \ldots, n \; ; \; i = 1, \ldots, I-1 ,$$

and the result follows. #

A corresponding result will now be derived for the scheme (3.10). For notational convenience, introduce the $n - 1$ dimensional vectors

$$\underline{\sigma} = \left(\sigma_2, \sigma_3, \ldots, \sigma_n\right)^T$$

and

$$\underline{\tau} = \left(\tau_2, \tau_3, \ldots, \tau_n\right)^T$$

where

$$\sigma_j = \sum_{k=2}^{n} u_j^2\omega_{k-1} \sum_{r=2}^{n} a_r u_r L_k(u_j u_r) + u_j^2 \sum_{r=2}^{n} a_r u_r L_1(u_j u_r)$$

$$- \frac{1}{1-\eta} \sum_{k=2}^{n} u_k a_k \omega_{k-1} \ , \quad j = 2, \ldots, n \ ,$$

and

$$\tau_j = \sum_{k=2}^{n} u_j u_k \omega_{k-1} \sum_{r=1}^{n} a_r L_k(u_j u_r) - \frac{1}{1-\eta} \sum_{k=2}^{n} u_k a_k \omega_{k-1} \ , \ j = 2, \ldots, n \ ,$$

and the $(n-1) \times (n-1)$ matrix

$$P = (p_{jk}) = M^{-1}\left(I - \frac{R}{1-\eta}\right)^{-1} .$$

<u>Theorem 3.8.</u>

Let $\tilde{e}_j(t)$, $j = 2, \ldots, n$, be the solution of the system of second kind Volterra integral equations

$$k(t, t)\left\{\sum_{k=2}^{n} a_{jk}\tilde{e}_k(t) + a_{j1}\tilde{e}_n(t)\right\} + u_j \int_0^t \frac{\partial k}{\partial t}(t, s) \sum_{k=2}^{n} \alpha_k \tilde{e}_k(s) ds$$

$$= u_j\tilde{\phi}'(t) + \tilde{\psi}_j(t) - \tilde{\psi}_n(t) \ , \quad j = 2, \ldots, n \ , \qquad (3.52)$$

where $\tilde{\phi}(t)$ and $\tilde{\psi}_j(t)$, $j = 2, \ldots, n$, are defined by lemma 3.7.

In addition, let $\tilde{q}_r(t)$, $r = 1, 2$, satisfy

$$k(t, t)\tilde{q}_r'(t) \sum_{k=1}^{n-1} p_{n-1,k}\tau_{k+1} + \frac{\partial k}{\partial s}(t, t)\tilde{q}_r(t) \sum_{k=1}^{n-1} p_{n-1,k}\sigma_{k+1} = 0 \ ,$$

$$r = 1, 2 \ , \qquad (3.53)$$

with the initial conditions

$$\tilde{q}_1(0) = \frac{\tilde{\psi}_n(0)}{\eta k(0,0)} \sum_{k=1}^{n-1} \tilde{m}_{n-1,k} - \tilde{e}_n(0) \qquad (3.54)$$

and

$$\tilde{q}_2(0) = 1 \ .$$

Then

$$\varepsilon_{ij} = h^n\left\{\tilde{e}_j(t_{ij})+\eta^i\omega_{j-1}\tilde{q}_1(t_{ij})\right\} + \varepsilon_{01}\eta^i\omega_{j-1}\tilde{q}_2(t_{ij})$$

$$+ O(h^{n+1}) + O(h\varepsilon_{01}) \ , \quad j = 2, \ldots, n \ ; \quad i = 1, \ldots, I-1 \ .$$

Proof. Using (2.9) it follows as previously that

$$\sum_{k=2}^{n} hu_j \sum_{m=1}^{n} a_m k(t_{ij}, t_i+u_ju_mh)L_k(u_ju_m)\tilde{e}_k(t_{ik})$$

$$+ hu_j \sum_{m=1}^{n} a_m k(t_{ij}, t_i+u_ju_mh)L_1(u_ju_m)\tilde{e}_n(t_i)$$

$$- ha_1\{k(t_{ij}, t_i)\tilde{e}_n(t_i)-k(t_{ij}, 0)\tilde{e}_n(0)\}$$

$$+ h \sum_{\ell=0}^{i-1} \sum_{k=2}^{n} \alpha_k k(t_{ij}, t_{\ell k})\tilde{e}_k(t_{\ell k})$$

$$= \tilde{\phi}(t_{ij}) + h\{\psi_j(t_{ij})-\psi_n(t_{ij})-\tilde{\gamma}(t_{ij})\} - h^2\tilde{\rho}_j(t_{ij}) + O(h^3) \ ,$$

$$j = 2, \ldots, n \ ; \quad i = 0, \ldots, I-1 \ , \qquad (3.55)$$

where $\tilde{\gamma}(t)$ is twice continuously differentiable, $\tilde{\gamma}(0) = 0$ and $\tilde{\rho}_j(t)$, $j = 2, \ldots, n$, are continuously differentiable.

Introduce the functions $x_{rj}(t)$, $r = 1, 2$; $j = 2, \ldots, n$, where

$k(t, t)x_{rj}(t)$

$$= -k(t, t)\tilde{q}_r'(t) \sum_{k=1}^{n-1} p_{j-1,k}\sigma_{k+1} - \frac{\partial k}{\partial s}(t, t)\tilde{q}_r(t) \sum_{k=1}^{n-1} p_{j-1,k}\tau_{k+1} \ ,$$

$$r = 1, 2 \ ; \quad j = 2, \ldots, n-1 \ , \qquad (3.56)$$

and

$$x_{rn}(t) = 0 \ , \quad r = 1, 2 \ .$$

From Taylor's theorem, (2.9) and (3.48) it follows that

$$\sum_{k=2}^{n} hu_j \sum_{m=1}^{n} a_m k\left(t_{ij}, t_i+u_ju_mh\right)L_k\left(u_ju_m\right)\left(\omega_{k-1}\tilde{q}_r\left(t_{ik}\right)+hx_{rk}\left(t_{ik}\right)\right)$$

$$+ hu_j \sum_{m=1}^{n} a_m k\left(t_{ij}, t_i+u_ju_mh\right)L_1\left(u_ju_m\right)\tilde{q}_r\left(t_i\right)$$

$$= h^2\left\{\frac{\partial k}{\partial s}\left(t_i, t_i\right)\tilde{q}_r\left(t_i\right)\left(\sum_{k=2}^{n} u_j^2\omega_{k-1} \sum_{m=2}^{n} a_mu_mL_k\left(u_ju_m\right)\right.\right.$$

$$\left.+ u_j^2 \sum_{m=2}^{n} a_mu_mL_1\left(u_ju_m\right)\right) + k\left(t_i, t_i\right)\tilde{q}_r'\left(t_i\right) \times$$

$$\left.\left(\sum_{k=2}^{n} u_ju_k\omega_{k-1} \sum_{m=1}^{n} a_mL_k\left(u_ju_m\right)\right) + k\left(t_i, t_i\right) \sum_{k=2}^{n} a_{jk}x_{rk}\left(t_i\right)\right\} + O\left(h^3\right),$$

$$r = 1, 2 \;;\; j = 2, \ldots, n \;;\; i = 0, \ldots, I-1 \;. \quad (3.57)$$

In the same way as in (3.50) it can be shown that

$$\sum_{\ell=0}^{i-1} \eta^{\ell}\left\{\sum_{k=2}^{n} ha_k k\left(t_{ij}, t_{\ell k}\right)\left(\omega_{k-1}\tilde{q}_r\left(t_{\ell k}\right)+hx_{rk}\left(t_{\ell k}\right)\right)\right.$$

$$\left.+ ha_1 k\left(t_{ij}, t_{\ell}\right)\tilde{q}_r\left(t_{\ell}\right)\right\}$$

$$= -h^2\left\{\frac{\eta^i}{1-\eta}\left(\sum_{k=2}^{n} a_k\omega_{k-1}u_k\left(\frac{\partial k}{\partial s}\left(t_i, t_i\right)\tilde{q}_r\left(t_i\right)+k\left(t_i, t_i\right)\tilde{q}_r'\left(t_i\right)\right)\right.\right.$$

$$\left.\left.+ k\left(t_i, t_i\right) \sum_{k=2}^{n} a_kx_{rk}\left(t_i\right)\right) + \tilde{\mu}_r\left(t_i\right)\right\} + O\left(h^3\right) ,$$

$$r = 1, 2 \;;\; j = 2, \ldots, n \;;\; i = 1, \ldots, I-1 \;, \quad (3.58)$$

where $\tilde{\mu}_r(t)$, $r = 1, 2$, are continuously differentiable. It follows from (3.56), (3.57) and (3.58) that

$$\eta^i\left\{\sum_{k=2}^{n} hu_j \sum_{m=1}^{n} a_m k\left(t_{ij}, t_i+u_ju_mh\right)L_k\left(u_ju_m\right)\left(\omega_{k-1}\tilde{q}_r\left(t_{ik}\right)+hx_{rk}\left(t_{ik}\right)\right)\right.$$

$$\left.+ hu_j \sum_{m=1}^{n} a_m k\left(t_{ij}, t_i+u_ju_mh\right)L_1\left(u_ju_m\right)\tilde{q}_r\left(t_i\right)\right\}$$

$$+ \sum_{\ell=0}^{i-1} \eta^{\ell}\left\{\sum_{k=2}^{n} ha_k k\left(t_{ij}, t_{\ell k}\right)\left(\omega_{k-1}\tilde{q}_r\left(t_{\ell k}\right)+hx_{rk}\left(t_{\ell k}\right)\right)\right.$$

$$\left.+ ha_1 k\left(t_{ij}, t_{\ell}\right)\tilde{q}_r\left(t_{\ell}\right)\right\}$$

$$= -h^2\tilde{\mu}_r(t_{ij}) + O(h^3) ,$$

$$r = 1, 2 ; \quad j = 2, \ldots, n ; \quad i = 0, \ldots, I-1 . \qquad (3.59)$$

Let

$$z_{ij} = \varepsilon_{ij} - h^n\left\{\tilde{e}_j(t_{ij})+\eta^i\omega_{j-1}\tilde{q}_1(t_{ij})+h\eta^i x_{1j}(t_{ij})\right\}$$

$$- \varepsilon_{01}\left\{\eta^i\omega_{j-1}\tilde{q}_2(t_{ij})+h\eta^i x_{2j}(t_{ij})\right\} ,$$

$$j = 2, \ldots, n ; \quad i = 0, \ldots, I-1 ,$$

$$z_{i+1,1} = z_{in} , \quad i = 0, \ldots, I-2 ,$$

$$z_{01} = -h^n\{\tilde{e}_n(0)+\tilde{q}_1(0)\} .$$

It follows from (3.33), lemma 3.7, (3.55) and (3.59) that

$$\sum_{k=1}^{n} hu_j \sum_{m=1}^{n} a_m k(t_{ij}, t_i+u_j u_m h) L_k(u_j u_m) z_{ik}$$

$$+ \sum_{\ell=0}^{i-1} \sum_{k=1}^{n} ha_k k(t_{ij}, t_{\ell k}) z_{\ell k}$$

$$= h^{n+1}\left\{\tilde{\psi}_n(t_{ij})+\tilde{\gamma}(t_{ij})\right\} + h^{n+2}\left\{\tilde{\theta}_j(t_{ij})+\tilde{\rho}_j(t_{ij})+\tilde{\mu}_1(t_{ij})\right\}$$

$$+ h^2\varepsilon_{01}\tilde{\mu}_2(t_{ij}) + O(h^{n+3}) + O\left(h^3\varepsilon_{01}\right) ,$$

$$j = 2, \ldots, n ; \quad i = 0, \ldots, I-1 . \qquad (3.60)$$

Using (3.54) and (3.60) it follows that

$$z_{01} = O(h^n)$$

and

$$z_{1j} = O(h^{n+1}) + O(h\varepsilon_{01}) , \quad j = 2, \ldots, n .$$

The result follows from remark 3.4. #

<u>Remark 3.6</u>. It can be seen that $e_j(t)$, $j = 2, \ldots, n$, and $\tilde{e}_j(t)$, $j = 2, \ldots, n$, are characterized by the $(n-1)$th equations of (3.44) and (3.52) respectively, which are of the form (3.19). Also, $\eta^i q_r(t_i)$, $\eta^i\tilde{q}_r(t_i)$, $r = 1, 2$, decrease rapidly

with increasing i if $|\eta| < 1$ and hence schemes with $|\eta| < 1$ are numerically stable. If $\eta = -1$, the schemes are only weakly stable since equations (3.45) and (3.53) can have exponentially increasing solutions for decreasing $y(t)$ (see for example (3.64)). These methods however can be salvaged to a certain extent by the application of smoothing procedures. The generalization of a smoothing procedure suggested by Jones (1961) for the trapezoidal method can be used, i.e.

$$\tilde{Y}_{ij} = \frac{1}{4}\left(Y_{i-1,j}+2Y_{ij}+Y_{i+1,j}\right) , \quad j = 2, \ldots, n \; ; \quad i = 2, \ldots, I-2 .$$

From theorems 3.7 and 3.8, $\tilde{Y}_{ij}$ gives a 'smooth' approximation to $y\left(t_{ij}\right)$ for the schemes (3.9) and (3.10). Kobayasi (1967) has considered a more sophisticated procedure for the trapezoidal method and a generalization of this may be more suitable.

Case 2. $\eta = 1$.

Only the case $k(t, s) = 1$ will be briefly discussed. Let

$$\underline{r}_i = \left(r_{i2}, r_{i3}, \ldots, r_{in}\right)^T , \quad i = 0, \ldots, I-1 ,$$

where

$$r_{ij} = \frac{1}{h}\int_{t_i}^{t_{ij}} \left[y(s) - \sum_{k=1}^{n} L_k\left(\frac{s-t_i}{h}\right) y\left(t_{ik}\right)\right] ds$$

$$= h^n y^{(n)}\left(t_{ij}\right) \int_0^{u_j} \omega(s) ds + O\left(h^{n+1}\right) , \quad j = 2, \ldots, n .$$

From (3.33),

$$\underline{\varepsilon}_0 = \varepsilon_{01} M^{-1} B \underline{e} + M^{-1} \underline{r}_0 , \tag{3.61}$$

where

$$\underline{e} = (1, 1, \ldots, 1)^T .$$

Also, subtracting (3.33) with $j = n$ and i replaced by $i - 1$ from (3.33) yields

$$\underline{\varepsilon}_i = M^{-1}B\underline{\varepsilon}_{i-1} + M^{-1}\underline{r}_i \,, \quad i = 1, \ldots, I-1 \,. \tag{3.62}$$

Let

$$\underline{\omega} = \left(\omega_1, \omega_2, \ldots, \omega_{n-1}\right)^T = A^{-1}B\underline{e} \,.$$

Since

$$\omega_{n-1} = \eta = 1$$

it follows that the $(n-1)$th equations of (3.61) and (3.62) take the form

$$\varepsilon_{0n} = \varepsilon_{01} + h^n c\, y^{(n)}\left(t_{0n}\right) + O\left(h^{n+1}\right) ,$$

and

$$\varepsilon_{in} = \varepsilon_{i-1,n} + h^n c\, y^{(n)}\left(t_{in}\right) + O\left(h^{n+1}\right), \quad i = 1, \ldots, I-1 ,$$

respectively, where

$$c = \sum_{k=1}^{n-1} \tilde{m}_{n-1,k} \int_0^{u_{k+1}} \omega(s)ds \,.$$

Hence,

$$\varepsilon_{ij} = h^{n-1}\omega_{j-1}\, c\int_0^{t_{ij}} y^{(n)}(s)ds + \omega_{j-1}\varepsilon_{01} + O\left(h^n\right)$$

$$= h^{n-1}\omega_{j-1}\, c\left\{y^{(n-1)}\left(t_{ij}\right) - y^{(n-1)}(0)\right\} + \omega_{j-1}\varepsilon_{01} + O\left(h^n\right) ,$$

$$j = 2, \ldots, n \,; \quad i = 0, \ldots, I-1 \,.$$

The appearance of the term $y^{(n-1)}(0)$ in this expansion implies that the method is not a local differentiation formula. This is also the reason why the ε_{01} term is propagated throughout. Since the leading term in the expansion varies smoothly with i, the approximations cannot be improved by smoothing procedures. These features indicate that methods with $\eta = 1$ are unsuitable for $k(t, s) = 1$ and should therefore not be used in practice.

3.10 Methods II: Numerical Results

The foregoing analysis appears to be contradicted by an example given by Linz (1967 a). Linz considers the scheme (3.9) with $u_1 = 0$, $u_2 = \frac{1}{3}$, $u_3 = \frac{2}{3}$ and $u_4 = 1$ applied to

$$t\left(\frac{1}{2} + \frac{4}{5}t^4\right) = \int_0^t \left(\frac{1}{2} + t^4 - s^4\right)y(s)ds \ , \quad 0 \le t \le T \ , \qquad (3.63)$$

which has the solution

$$y(t) = 1 \ .$$

Linz observes that the method appears to diverge for stepsizes $h = 0.3$ and 0.15 if $T \ge 1.2$. The 'explosion' in the error for $T \ge 1.2$, however is not due to divergence of the method but results from taking the stepsize too large. The effect of this is two fold. Firstly, for $T \ge 1.2$, $k(t, s)$ has a zero in the region $t \le s \le t + \frac{2}{3}h$ and secondly the derivatives of $hk(t, s)$ become too large for the analysis to be valid. It should be noted that the scheme (3.10) avoids the first difficulty. In table 3.3 on p.86 the errors tabulated for a smaller stepsize indicate that the methods (3.9) and (3.10) are convergent.

The order of convergence of various schemes is illustrated by application to

$$1 + t - \sin t - \cos t = \int_0^t (1+t-s)y(s)ds \ , \quad 0 \le t \le 1.2 \ ,$$

which has the solution

$$y(t) = \sin t \ .$$

In tables 3.4 and 3.5 the errors are tabulated for the schemes (3.9) and (3.10) respectively with $u_1 = 0$, $u_2 = \frac{1}{2}$ and $u_3 = 1$ $(\eta = 1)$ and tables 3.6 and 3.7 contain the errors for the schemes (3.9) and (3.10) respectively with $u_1 = 0$, $u_2 = \frac{1}{3}$, $u_3 = \frac{2}{3}$ and $u_4 = 1$ $(\eta = -1)$.

Table 3.3.

	Method (3.9)		Method (3.10)	
t	h = 0.15	h = 0.0375	h = 0.15	h = 0.0375
0.3	5.062 E-7	2.348 E-9	3.692 E-7	1.303 E-9
0.6	9.466 E-6	3.830 E-8	5.022 E-6	1.800 E-8
0.9	6.897 E-5	2.675 E-7	1.769 E-5	6.481 E-8
1.2	2.032 E-3	5.558 E-6	5.823 E-5	2.725 E-7
1.5	1.173 E 1	2.549 E-3	4.889 E-4	4.691 E-6

Table 3.4.

Method (3.9)			
t	h = 0.3	h = 0.15	h = 0.075
0.3	-3.587 E-4	-9.151 E-5	-2.299 E-5
0.6	-2.268 E-4	-6.076 E-5	-1.544 E-5
0.9	5.450 E-4	1.295 E-4	3.194 E-5
1.2	2.105 E-3	5.161 E-4	1.284 E-4

Table 3.5.

Method (3.10)			
t	h = 0.3	h = 0.15	h = 0.075
0.3	-5.759 E-4	-1.439 E-4	-3.597 E-5
0.6	-1.145 E-3	-2.862 E-4	-7.156 E-5
0.9	-1.661 E-3	-4.152 E-4	-1.038 E-4
1.2	-2.080 E-3	-5.201 E-4	-1.300 E-4

Table 3.6.

Method (3.9)			
t	h = 0.3	h = 0.15	h = 0.075
0.3	7.045 E-5	-6.855 E-7	-4.263 E-8
0.6	-2.842 E-5	-1.746 E-6	-1.086 E-7
0.9	9.621 E-5	-3.252 E-6	-2.025 E-7
1.2	-8.581 E-5	-5.289 E-6	-3.294 E-7

Table 3.7.

Method (3.10)			
t	h = 0.3	h = 0.15	h = 0.075
0.3	2.488 E-5	-9.538 E-8	-6.096 E-9
0.6	-3.951 E-6	-2.562 E-7	-1.639 E-8
0.9	2.804 E-5	-4.892 E-7	-3.133 E-8
1.2	-1.223 E-5	-7.960 E-7	-5.099 E-8

To illustrate the weak stability for $\eta = -1$, the scheme (3.10) with $u_1 = 0$, $u_2 = \frac{1}{3}$, $u_3 = \frac{2}{3}$ and $u_4 = 1$ is applied to

$$-1 + t + e^{-t} = \int_0^t (1+t-s)y(s)ds\ , \quad 0 \le t \le 9\ , \tag{3.64}$$

where

$$y(t) = te^{-t}\ .$$

From (3.53), the dominant part of the error behaves like $\text{const}(-1)^i e^{0.52t_i}$, which agrees with the numerical results given in table 3.8.

The numerical stability of schemes with $|\eta| < 1$ is demonstrated by applying the scheme (3.10) with $u_1 = 0$, $u_2 = \frac{1}{2}$, $u_3 = \frac{3}{4}$ and $u_4 = 1$ $\left(\eta = -\frac{1}{3}\right)$ to (3.64). The errors tabulated in table 3.8 decrease exponentially.

Table 3.8.

	Method (3.10), $\eta = -1$	Method (3.10), $\eta = -\frac{1}{3}$
t	h = 0.3	h = 0.3
0.9	-1.098 E-4	-4.909 E-5
1.8	1.495 E-4	-1.461 E-6
2.7	-2.360 E-4	1.583 E-6
3.6	3.717 E-4	1.850 E-6
4.5	-5.809 E-4	1.077 E-6
5.4	9.113 E-4	5.378 E-7
6.3	-1.428 E-3	2.420 E-7
7.2	2.238 E-3	1.014 E-7
8.1	-3.507 E-3	3.973 E-8
9.0	5.497 E-3	1.445 E-8

Table 3.9 contains the errors incurred when applying the schemes (3.9) and (3.10) with $u_1 = 0$, $u_2 = \frac{1}{2}$ and $u_3 = 1$ ($\eta = 1$) to (3.64). For both methods the error appears to grow exponentially. However, the errors for the scheme (3.10) increase less rapidly.

Table 3.9.

	Method (3.9), $\eta = 1$	Method (3.10), $\eta = 1$
t	h = 0.3	h = 0.3
0.9	1.460 E-2	3.316 E-3
1.8	3.908 E-2	5.038 E-3
2.7	9.628 E-2	6.480 E-3
3.6	2.364 E-1	8.106 E-3
4.5	5.812 E-1	1.011 E-2
5.4	1.429 E 0	1.264 E-2
6.3	3.515 E 0	1.580 E-2
7.2	8.648 E 0	1.978 E-2
8.1	2.127 E 1	2.476 E-2
9.0	5.231 E 1	3.099 E-2

3.11 Conclusion

It has been shown that implicit Runge-Kutta methods applied to first kind Volterra equations can lead to high order schemes. If $u_1 = 0$, then the methods are convergent under appropriate conditions on $\{u_1, u_2, \ldots, u_n\}$ and numerically stable under further restrictions. In particular, symmetrically spaced points always lead to weakly stable schemes. On the other hand, the schemes with $u_1 > 0$

are always convergent and numerically stable. These are the schemes which should be used in practice.

In the analysis so far only the case of exact data, i.e., $g(t)$ is given analytically, has been considered. However, since (3.1) is an improperly posed problem, small perturbations in g can cause large perturbations in y. The effect of this in the numerical schemes considered is that a perturbation δg_ℓ in g results in a perturbation δy_ℓ in y which is proportional to $\delta g_\ell/h$. The propagation of perturbations of this type has been discussed in §3.5 and §3.9. A rigorous treatment of the effect of a noise $\{\delta g_{ij}, j = 1, \ldots, n; i = 0, \ldots, I-1\}$ in g can be obtained by considering each perturbation separately and then using superposition.

The above indicates that when g is given as data, h must not be small. On the other hand, h should be small so that the discretization error is kept small. The actual h used when implementing the methods should be chosen with these two facts in mind.

CHAPTER 4

THE NUMERICAL SOLUTION OF THE GENERALIZED ABEL EQUATION

4.1 Introduction

In this chapter, finite difference schemes for the generalized Abel equation,

$$g(t) = \int_0^t \frac{k(t,s)}{(t-s)^\alpha} y(s)ds \ , \ t \geq 0 \ , \ 0 < \alpha < 1 \ , \qquad (4.1)$$

are examined. The motivation for this study is the need for efficient methods for equations of the form

$$t^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\{a(t)\}^2/t\right) = \int_0^t (t-s)^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\{a(t)-a(s)\}^2/(t-s)\right)y(s)ds \quad (4.2)$$

where $a(t)$ can be any differentiable function with $a(0) > 0$. Such equations arise in the analysis of Brownian motion and diffusion processes (see Durbin (1971)).

In §4.2 some results on the analytic solution of (4.1) are presented. A short discussion of product integration is given in §4.3. In the remainder of the chapter, this technique is used to extend the methods for first kind Volterra integral equations with continuous kernels to (4.1). Convergence results for the product integration analogues of the midpoint, Euler and trapezoidal methods are also given.

It should be noted that the methods derived and the analysis given can be extended to systems of equations.

4.2 The Analytic Solution

The question of the existence and uniqueness of a solution of (4.1) is summarized in the following theorem.

Theorem 4.1.

If

(4.2.1) $k(t,t) \neq 0$, $t \in [0, T]$,

(4.2.2) $k(t, s)$ and $\partial k(t, s)/\partial t$ are continuous on $0 \leq s \leq t \leq T$,

(4.2.3) $g(t) \in C[0, T]$, $g(0) = 0$ and

$$G(t) = \frac{d}{dt}\int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds \in C[0, T] ,$$

then (4.1) has a unique solution $y(t) \in C[0, T]$.

A proof can be found in Kowalewski (1930), pp 80-82, where it is shown that under the conditions (4.2.1)-(4.2.3), Eq. (4.1) is equivalent to the second kind Volterra equation

$$y(t) = \frac{G(t)}{L(t,t)} - \int_0^t \left\{\frac{1}{L(t,t)} \partial L(t, s)/\partial t\right\} y(s)ds , \quad 0 \leq s \leq T , \quad (4.3)$$

where

$$L(t, s) = \int_0^1 \frac{k\{s+u(t-s),s\}}{u^{\alpha}(1-u)^{1-\alpha}} du .$$

From (4.2.1) and (4.2.2),

$$L(t, t) = k(t, t) \frac{\pi}{\sin\pi\alpha} \neq 0 , \quad t \in [0, T] ,$$

and $L(t, s)$ and $\partial L(t, s)/\partial t$ are continuous on $0 \leq s \leq t \leq T$. Hence the result follows from the standard theory for second kind Volterra integral equations.

Remark 4.1. It is clear, on differentiating (4.3), that

$y(t) \in C^m[0, T]$, $m \geq 1$, if

(4.2.4) $k(t, s)$ has continuous partial derivatives up to order m and

$$\partial^{m+1}k(t, s)/\partial t^{m+1} \text{ is continuous on } 0 \leq s \leq t \leq T ,$$

and

(4.2.5) $G(t) \in C^m[0, T]$.

If $g(t) \in C^1[0, T]$, then, by partial integration,

$$G(t) = \int_0^t \frac{g'(s)}{(t-s)^{1-\alpha}}\, ds \;. \tag{4.4}$$

If $g(t) \in C^{m+1}[0, T]$, then repeated partial integration of (4.4) yields

$$G(t) = (\alpha-1)! \left\{ \sum_{\ell=0}^{m-1} \frac{1}{(\alpha+\ell)!} g^{(\ell+1)}(0) t^{\alpha+\ell} + \frac{1}{(\alpha+m-1)!} \int_0^t g^{(m+1)}(s)(t-s)^{\alpha+m-1} ds \right\} .$$

Thus, in general, if $g(t)$ is smooth on $[0, T]$, then $G(t)$, and hence $y(t)$, is smooth on $(0, T]$, but not necessarily on $[0, T]$. However, if

$$g^{(\ell)}(0) = 0 \;, \quad \ell = 1, \ldots, m \;, \tag{4.5}$$

then

$$G(t) = \frac{(\alpha-1)!}{(\alpha+m-1)!} \int_0^t g^{(m+1)}(s)(t-s)^{\alpha+m-1} ds \in C^m[0, T] \;.$$

Hence, it follows from theorem 4.1 that for Eq. (4.2), $y(t) \in C^m[0, T]$ if $a(t) \in C^{m+1}[0, T]$.

Remark 4.2. From (4.3) it is clear that $y(t)$ depends continuously on $G(t)$. However, $G(t)$ does not depend continuously on $g(t)$. In fact, $G(t)/(\alpha-1)!$ is the order $1 - \alpha$ fractional derivative of $g(t)$. In §4.7 it will be discussed how this behaviour is reflected in finite difference schemes for (4.1).

4.3 Product Integration

A common procedure for the numerical evaluation of integrals of the form

$$I_w(f) = \int_a^b w(s)f(s)ds$$

where $f(t)$ is "smooth" and $w(t)$ is absolutely integrable on $[a, b]$, is product integration. In this technique, $I_w(f)$ is approximated by $I_w(\tilde{f})$, where $\tilde{f}(t)$ is an approximation to $f(t)$.

The difference between product integration rules and standard quadrature formulae is that only an approximation to $f(t)$ is used. This is desirable for instance if the weight function $w(t)$ is highly oscillatory or has singularities on $[a, b]$.

Examples of product integration include weighted Gaussian quadrature and the Filon formulae for the evaluation of Fourier coefficients (see for instance Squire (1970)).

As well as being a technique for numerical quadrature, product integration is a basis for the derivation of numerical schemes for integral equations with singular kernels. Such methods were first investigated by Young (1954). A survey of the application of product integration to Fredholm integral equations is given in Atkinson (1971 b). The use of this technique for Volterra integral equations has been briefly discussed in §1.3 and §1.5. Recently, de Hoog and Weiss (1972 d) have derived accurate convergence rates of product integration methods for second kind integral equations with kernels having algebraic or logarithmic singularities.

4.4 The Product Integration Analogues of the Midpoint, Euler and Trapezoidal Methods

In this section, the midpoint, Euler and trapezoidal methods for (3.1) are generalized to (4.1).

Firstly introduce the notation

$$t_i = ih \ , \quad i = 0, \ldots, I \ ; \quad h = T/I \ ,$$

$$t_{i+\frac{1}{2}} = (i+\tfrac{1}{2})h \ ,$$

$$y_i = y(t_i)\ , \qquad y_{i+\frac{1}{2}} = y(t_{i+\frac{1}{2}})\ ,$$

$$k_{ij} = k(t_i,\ t_j)\ , \qquad k_{i,j+\frac{1}{2}} = k(t_i,\ t_{j+\frac{1}{2}})\ .$$

As previously, numerical schemes are obtained by approximating the integral terms in the discretized form of (4.1),

$$g(t_i) = \int_0^{t_i} \frac{k(t_i,s)}{(t_i-s)^{\alpha}}\, y(s)ds$$

$$= \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{k(t_i,s)}{(t_i-s)^{\alpha}}\, y(s)ds\ , \quad i = 1,\ \ldots,\ I\ . \tag{4.6}$$

Due to the singularity at $s = t_i$, it is appropriate to use formulae based on product integration, with

$$w_i(t) = (t_i-t)^{-\alpha}$$

as the weight function.

In the first method to be derived, $k(t_i,\ s)y(s)$ is replaced by the piecewise constant function

$$K_h(t_i,\ s) = k_{i,j+\frac{1}{2}}\, y_{j+\frac{1}{2}}\ , \quad t_j \le s < t_{j+1}\ , \quad j = 0,\ \ldots,\ i-1\ .$$

Substitution of this approximation into (4.6) leads to the numerical scheme

$$g(t_i) = \sum_{j=0}^{i-1} Y_{j+\frac{1}{2}}\, k_{i,j+\frac{1}{2}}\, V_{i-j}\ , \quad i = 1,\ \ldots,\ I\ , \tag{4.7}$$

where

$$V_{\ell} = \int_{(\ell-1)h}^{\ell h} s^{-\alpha} ds = \frac{1}{1-\alpha}\, h^{1-\alpha}\left(\ell^{1-\alpha}-(\ell-1)^{1-\alpha}\right)\ , \quad \ell = 1,\ \ldots,\ I\ ,$$

and $Y_{i+\frac{1}{2}}$ is the approximation to $y_{i+\frac{1}{2}}$. This is the analogue of the <u>midpoint method</u>.

The use of the piecewise constant function

$$M_h(t_i,\ s) = k_{ij}\, y_j\ , \quad t_j \le s < t_{j+1}\ , \quad j = 0,\ \ldots,\ i-1\ ,$$

to approximate $k(t_i,s)y(s)$ yields Euler's method,

$$g(t_i) = \sum_{j=0}^{i-1} Y_j\, k_{ij}\, V_{i-j} \ , \quad i = 1, \ldots, I \ , \tag{4.8}$$

where Y_i is the approximation to y_i.

Finally, approximating $k(t_i, s)y(s)$ by the piecewise linear function

$$N_h(t_i, s) = \{(t_{j+1}-s)k_{ij}\, y_j + (s-t_j)k_{i,j+1}\, y_{j+1}\}/h \ ,$$
$$t_j \le s \le t_{j+1} \ , \quad j = 0, \ldots, i-1 \ ,$$

leads to the trapezoidal method,

$$g(t_i) = \sum_{j=1}^{i} Y_j\, k_{ij}\, W_{i-j} + Y_0\, k_{i0}\, \overline{W}_i \ , \quad i = 1, \ldots, I \ , \tag{4.9}$$

where

$$W_0 = \frac{1}{h}\int_0^h \frac{h-s}{s^\alpha}\, ds = \left(\frac{1}{1-\alpha} - \frac{1}{2-\alpha}\right)h^{1-\alpha} \ ,$$

$$W_\ell = \frac{1}{h}\left(\int_{(\ell-1)h}^{\ell h} \frac{s-(\ell-1)h}{s^\alpha}\, ds + \int_{\ell h}^{(\ell+1)h} \frac{(\ell+1)h-s}{s^\alpha}\, ds\right)$$

$$= \left(\frac{1}{1-\alpha} - \frac{1}{2-\alpha}\right)h^{1-\alpha}\left((\ell+1)^{2-\alpha} - 2\ell^{2-\alpha} + (\ell-1)^{2-\alpha}\right) \ , \quad \ell = 1, \ldots, I-1 \ ,$$

$$\overline{W}_\ell = \frac{1}{h}\int_{(\ell-1)h}^{\ell h} \frac{s-(\ell-1)h}{s^\alpha}\, ds$$

$$= h^{1-\alpha}\left\{\frac{1}{2-\alpha}\left(\ell^{2-\alpha} - (\ell-1)^{2-\alpha}\right) - \frac{\ell-1}{1-\alpha}\left(\ell^{1-\alpha} - (\ell-1)^{1-\alpha}\right)\right\} \ , \quad \ell = 1, \ldots, I \ ,$$

and Y_i is the approximation to y_i. The trapezoidal method requires a starting value Y_0 which can be determined from (4.3), i.e.,

$$Y_0 = y_0 = \frac{\sin \pi\alpha}{\pi k(0,0)} \lim_{t\to 0} \frac{d}{dt} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}}\, ds \ . \tag{4.10}$$

If $g'(t)$ is continuous, it follows immediately from (4.4) that

(4.10) reduces to

$$Y_0 = y_0 = 0 .$$

From conditions (4.2.1) and (4.2.2), the triangular systems (4.7), (4.8) and (4.9) are nonsingular if h is sufficiently small.

4.5 Convergence Results

The convergence of the schemes derived in the preceeding section will now be examined. For notational convenience it will be assumed that A_i, B_i, C_i, D_i , $i = 1, 2, \ldots$, are positive constants.

The basic lemma used in the convergence proofs is

Lemma 4.1.

Let x_i , $i = 0, 1, \ldots$ satisfy

$$|x_0| \leq |b_0| ,$$

$$|x_i| \leq \left| \sum_{j=0}^{i-1} a_{ij} x_j + b_i \right| , \quad i = 1, 2, \ldots \tag{4.11}$$

where

$$\rho_i = 1 - \sum_{j=0}^{i-1} |a_{ij}| > 0 , \quad i = 1, 2, \ldots ,$$

and

$$|b_0| \leq K , \quad K = \text{const} ,$$

$$|b_i| \leq K\rho_i , \quad i = 1, 2, \ldots .$$

Then

$$|x_i| \leq K , \quad i = 0, 1, \ldots .$$

The proof follows from induction.

Firstly, the midpoint method (4.7) will be considered. In the analysis, the following lemma concerning the summation of

$$c_\ell = \ell^{1-\alpha} - (\ell-1)^{1-\alpha} , \quad \ell = 1, 2, \ldots , \quad 0 < \alpha < 1 ,$$

will be required.

Lemma 4.2.

(i) $$\sum_{\ell=1}^{i} c_\ell = i^{1-\alpha} ,$$

(ii) $$\sum_{\ell=1}^{i} \left(c_\ell - c_{\ell+1}\right) = 1 - \left((i+1)^{1-\alpha} - i^{1-\alpha}\right)$$

$$\le 1 - (1-\alpha)i^{-\alpha} + \frac{1-\alpha}{2}\alpha i^{-1-\alpha}$$

$$\le 1 - (1-\alpha)\left(1 - \frac{\alpha}{2}\right)i^{-\alpha} ,$$

(iii) $$\sum_{\ell=1}^{\infty} \left(c_\ell - c_{\ell+1}\right) = 1 ,$$

(iv) $$\sum_{\ell=1}^{i} \ell\left(c_\ell - c_{\ell+1}\right) = i^{1-\alpha} - i\left((i+1)^{1-\alpha} - i^{1-\alpha}\right)$$

$$\le \alpha i^{1-\alpha} + \frac{(1-\alpha)}{2}\alpha i^{-\alpha} .$$

Proof. Parts (i), (ii) and (iii) follow immediately from the definition of c_ℓ . Part (iv) follows by partial summation. #

In addition, use will be made of

Lemma 4.3.

Let $\phi(t)$ and $\psi(t)$ be functions satisfying:

(a) $|\phi(t)|$ is integrable over the interval $[a, b]$, and

(b) $\psi(t)$ is Lipschitz continuous on $[a, b]$,

then

$$\left|\int_a^b \phi(s)\psi(s)ds - \psi\left(\frac{a+b}{2}\right)\int_a^b \phi(s)ds\right| \le L\,\frac{b-a}{2}\int_a^b |\phi(s)|ds ,$$

where L is the Lipschitz constant of $\psi(t)$.

Proof. As, for $s \in [a, b]$,

$$\left|\psi(s) - \psi\left(\frac{a+b}{2}\right)\right| \le L\,\frac{b-a}{2} ,$$

the result follows immediately. #

Since most of the arguments for $k(t, s) = 1$ generalize, this case will be considered first.

Theorem 4.2.

If $y'(t)$ is Lipschitz continuous on $[0, T]$, then, for $k(t, s) = 1$, the midpoint method is convergent of order 1.

Proof. The proof consists of three basic steps:

(I) the construction of the error equation (4.11),

(II) the estimation of the $|b_i|$, $(i = 0, 1, \ldots)$, and

(III) the estimation of ρ_i, $(i = 1, 2, \ldots)$, and hence the immediate application of lemma 4.1.

(I) On subtracting

$$g(t_i) = \sum_{j=0}^{i-1} Y_{j+\frac{1}{2}} V_{i-j}$$

from

$$g(t_i) = \sum_{j=0}^{i-1} y_{j+\frac{1}{2}} V_{i-j} + \sum_{j=0}^{i-1} \tau_{ij}$$

where

$$\tau_{ij} = \int_{t_j}^{t_{j+1}} \frac{1}{(t_i-s)^\alpha} (y(s)-y_{j+\frac{1}{2}}) ds$$

it follows that

$$\sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}} V_{i-j} + \sum_{j=0}^{i-1} \tau_{ij} = 0 \ , \quad i = 1, \ldots, I \ , \qquad (4.12)$$

where

$$\varepsilon_{i+\frac{1}{2}} = y_{i+\frac{1}{2}} - Y_{i+\frac{1}{2}} \ .$$

Subtraction of (4.12) from (4.12) with i replaced by $i + 1$ yields

$$\varepsilon_{i+\frac{1}{2}} V_1 + \sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}} (V_{i+1-j} - V_{i-j}) + \beta_i = 0 \ , \quad i = 1, \ldots, I-1 \ , \qquad (4.13)$$

where

$$\beta_i = \tau_{i+1,i} - \sum_{j=0}^{i-1} \left(\tau_{ij} - \tau_{i+1,j}\right) .$$

Division of (4.13) by V_1 and substitution of

$$a_\ell = \left(V_\ell - V_{\ell+1}\right)/V_1$$

$$= \ell^{1-\alpha} - (\ell-1)^{1-\alpha} - \left((\ell+1)^{1-\alpha} - \ell^{1-\alpha}\right)$$

yield

$$\varepsilon_{i+\frac{1}{2}} = \sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}}\, a_{i-j} - \frac{1-\alpha}{h^{1-\alpha}} \beta_i \ , \quad i = 1, \ldots, I-1 . \tag{4.14}$$

(II) From the Taylor series expansion for $y(s)$ at $t_{j+\frac{1}{2}}$,

$$y(s) - y_{j+\frac{1}{2}} = y'\left(t_{j+\frac{1}{2}}\right)\left(s - t_{j+\frac{1}{2}}\right) + r_j(s) \ ,$$

$$t_j \le s \le t_{j+1} \ , \quad j = 0, \ldots, I-1 \ ,$$

where

$$\left|r_j(s)\right| \le \frac{h^2}{4} L$$

and L is the Lipschitz constant of $y'(s)$. Consequently,

$$\beta_i = h^{2-\alpha}\left(y'\left(t_{i+\frac{1}{2}}\right)\gamma - \sum_{j=0}^{i-1} y'\left(t_{j+\frac{1}{2}}\right)\Theta_{i-j}\right) + \delta_i \ , \tag{4.15}$$

where

$$\gamma = \frac{1}{h^{2-\alpha}} \int_{t_i}^{t_{i+1}} \frac{1}{\left(t_{i+1}-s\right)^\alpha} \left(s - t_{i+\frac{1}{2}}\right) ds$$

$$= \frac{1}{h^{2-\alpha}} \int_0^h \frac{1}{(h-s)^\alpha} \left(s - \frac{h}{2}\right) ds \ ,$$

$$\Theta_{i-j} = \frac{1}{h^{2-\alpha}} \int_{t_j}^{t_{j+1}} \left(\frac{1}{(t_i - s)^{\alpha}} - \frac{1}{(t_{i+1} - s)^{\alpha}} \right) (s - t_{j+\frac{1}{2}}) ds$$

$$= \frac{1}{h^{2-\alpha}} \int_0^h \left(\frac{1}{((i-j)h-s)^{\alpha}} - \frac{1}{((i+1-j)h-s)^{\alpha}} \right) \left(s - \frac{h}{2}\right) ds$$

$$= \frac{1}{2-\alpha} \left((i-j+1)^{2-\alpha} - 2(i-j)^{2-\alpha} + (i-j-1)^{2-\alpha} \right)$$

$$- \frac{1}{2} \frac{1}{1-\alpha} \left((i-j+1)^{1-\alpha} - (i-j-1)^{1-\alpha} \right)$$

$$+ \frac{1}{1-\alpha} (i-j) \left(2(i-j)^{1-\alpha} - (i-j+1)^{1-\alpha} - (i-j-1)^{1-\alpha} \right) \quad (4.16)$$

and

$$\delta_i = \int_{t_i}^{t_{i+1}} \frac{1}{(t_{i+1} - s)^{\alpha}} r_i(s) ds$$

$$- \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(\frac{1}{(t_i - s)^{\alpha}} - \frac{1}{(t_{i+1} - s)^{\alpha}} \right) r_j(s) ds .$$

From lemma 4.2 (iii)

$$|\delta_i| \le \frac{1}{1-\alpha} h^{1-\alpha} \frac{L}{2} h^2 . \quad (4.17)$$

Next, by Taylor series expansion,

$$(\ell+1)^{2-\alpha} - 2\ell^{2-\alpha} + (\ell-1)^{2-\alpha} = (2-\alpha)(1-\alpha)\ell^{-\alpha} + O\left(\ell^{-2-\alpha}\right) ,$$

$$(\ell+1)^{1-\alpha} - (\ell-1)^{1-\alpha} = 2(1-\alpha)\ell^{-\alpha} + O\left(\ell^{-2-\alpha}\right) ,$$

$$\ell\left(2\ell^{1-\alpha} - (\ell+1)^{1-\alpha} - (\ell-1)^{1-\alpha}\right) = \alpha(1-\alpha)\ell^{-\alpha} + O\left(\ell^{-2-\alpha}\right) .$$

Hence,

$$0 < \Theta_\ell < A_1 \ell^{-2-\alpha} , \quad (4.18)$$

where A_1 depends on α . Furthermore, it can easily be verified that

$$\left|\gamma - \sum_{\ell=1}^{i} \theta_\ell\right| = \frac{1}{h^{2-\alpha}} \left|\int_0^h \frac{1}{\left((i+1)h-s\right)^\alpha}\left(s - \frac{h}{2}\right) ds\right|$$

$$\leq A_2 i^{-1-\alpha} . \qquad (4.19)$$

By partial summation,

$$\sum_{j=0}^{i-1} y'\left(t_{j+\frac{1}{2}}\right)\Theta_{i-j} = \sum_{\ell=1}^{i} y'\left(t_{i-\ell+\frac{1}{2}}\right)\Theta_\ell$$

$$= y'\left(t_{i-\frac{1}{2}}\right) \sum_{\ell=1}^{i} \Theta_\ell$$

$$+ \sum_{\nu=2}^{i} \left\{y'\left(t_{i-\nu+\frac{1}{2}}\right) - y'\left(t_{i-(\nu-1)+\frac{1}{2}}\right)\right\} \sum_{\ell=\nu}^{i} \Theta_\ell .$$

On noting that

$$\left|y'\left(t_{i-\nu+\frac{1}{2}}\right) - y'\left(t_{i-(\nu-1)+\frac{1}{2}}\right)\right| \leq L\, h ,$$

it follows from (4.18) that

$$\left|\sum_{\nu=2}^{i} \left\{ y'\left(t_{i-\nu+\frac{1}{2}}\right) - y'\left(t_{i-(\nu-1)+\frac{1}{2}}\right)\right\} \sum_{\ell=\nu}^{i} \Theta_\ell\right| \leq A_3\, h .$$

Hence, using (4.18) and (4.19),

$$\left|y'\left(t_{i+\frac{1}{2}}\right)\gamma - \sum_{j=0}^{i-1} y'\left(t_{j+\frac{1}{2}}\right)\Theta_{i-j}\right|$$

$$\leq \left|y'\left(t_{i+\frac{1}{2}}\right)\left(\gamma - \sum_{\ell=1}^{i} \Theta_\ell\right)\right| + \left|y'\left(t_{i+\frac{1}{2}}\right) - y'\left(t_{i-\frac{1}{2}}\right)\right| \sum_{\ell=1}^{i} \Theta_\ell + A_3 h$$

$$\leq \bar{F} A_2 i^{-1-\alpha} + A_4 h , \qquad (4.20)$$

where $\bar{F} = \max_{s \in [0,T]} |y'(s)|$. Since

$$h = \frac{T}{I} \leq \frac{T}{i} , \quad i = 1, \ldots, I ,$$

it follows from (4.17) and (4.20) that

$$|\beta_i| \leq A_5 h^{2-\alpha}\, i^{-1} . \qquad (4.21)$$

(III) By (4.12) and lemma 4.3,

$$|\varepsilon_{\frac{1}{2}}| \leq \frac{L}{2} h . \qquad (4.22)$$

Also, using (4.22), (4.14) and (4.21),

$$\left.\begin{aligned} &|\varepsilon_{\frac{1}{2}}| \le A_6 h \ , \\ &|\varepsilon_{i+\frac{1}{2}}| \le \sum_{j=1}^{i-1} |\varepsilon_{j+\frac{1}{2}}| a_{i-j} + A_6 h i^{-1} \ , \quad i = 1, \ldots, I-1 \ . \end{aligned}\right\} \tag{4.23}$$

Since, by lemma 4.2 (ii),

$$\rho_i = 1 - \sum_{j=0}^{i-1} a_{i-j} \ge (1-\alpha)\left(1 - \frac{\alpha}{2}\right) i^{-\alpha} \ ,$$

the application of lemma 4.1 yields the required result

$$|\varepsilon_{i+\frac{1}{2}}| \le \frac{A_6}{(1-\alpha)\left(1-\frac{\alpha}{2}\right)} h \ , \quad i = 0, \ldots, I-1 \ . \qquad \#$$

Corollary 4.1.

In general the order one convergence of theorem 4.2 is best possible.

Proof. If $y(t) = t$, $t \in [0, T]$, then

$$\varepsilon_{\frac{1}{2}} = h\left(\frac{1}{2} - \frac{1-\alpha}{2-\alpha}\right) \ . \qquad \#$$

The result of theorem 4.2 will now be extended to the case when $k(t, s)$ satisfies

(4.5.1) $|k(t, t)| \ge k_0 > 0$, $t \in [0, T]$, and

(4.5.2) $k(t, s)$ and $\partial k(t, s)/\partial s$ are Lipschitz continuous with respect to t and s on $0 \le s \le t \le T$.

Theorem 4.3.

If $y'(t)$ is Lipschitz continuous on $[0, T]$ and $k(t, s)$ satisfies (4.5.1) and (4.5.2), then the midpoint method is convergent of order 1 .

Proof. The proof proceeds as for theorem 4.2.

(I) The error equations corresponding to (4.13) are

$$\varepsilon_{i+\frac{1}{2}} k_{i+1,i+\frac{1}{2}} V_1 + \sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}}\left(k_{i+1,j+\frac{1}{2}} V_{i+1-j} - k_{i,j+\frac{1}{2}} V_{i-j}\right) + \beta_i = 0 \ ,$$
$$i = 1, \ldots, I-1 \ , \tag{4.24}$$

where

$$\beta_i = \tau_{i+1,i} - \sum_{j=0}^{i-1} \left(\tau_{ij} - \tau_{i+1,j}\right)$$

and

$$\tau_{ij} = \int_{t_j}^{t_{j+1}} \frac{1}{\left(t_i - s\right)^{\alpha}} \{y(s)k\left(t_i, s\right) - y_{j+\frac{1}{2}} k_{i,j+\frac{1}{2}}\} ds .$$

Now, (4.24) can be rewritten as

$$\varepsilon_{i+\frac{1}{2}} k_{i+1,i+\frac{1}{2}} V_1 + \sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}} \{k_{i+1,j+\frac{1}{2}} \left(V_{i+1-j} - V_{i-j}\right)$$

$$+ \left(k_{i+1,j+\frac{1}{2}} - k_{i,j+\frac{1}{2}}\right) V_{i-j}\} + \beta_i = 0 , \quad i = 1, \ldots, I-1 . \quad (4.25)$$

Division of (4.25) by $k_{i+1,i+\frac{1}{2}} V_1$ yields

$$\varepsilon_{i+\frac{1}{2}} = \sum_{j=0}^{i-1} \varepsilon_{j+\frac{1}{2}} \left\{a_{i-j} \frac{k_{i+1,i+\frac{1}{2}} + \left(k_{i+1,j+\frac{1}{2}} - k_{i+1,i+\frac{1}{2}}\right)}{k_{i+1,i+\frac{1}{2}}}\right.$$

$$\left. + \left((i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}\right) \frac{k_{i,j+\frac{1}{2}} - k_{i+1,j+\frac{1}{2}}}{k_{i+1,i+\frac{1}{2}}}\right\} - \phi_i ,$$

$$i = 1, \ldots, I-1 , \quad (4.26)$$

where

$$\phi_i = \frac{1-\alpha}{k_{i+1,i+\frac{1}{2}} h^{1-\alpha}} \beta_i .$$

(II) If

$$y_i^1(s) = y(s)k\left(t_{i+1}, s\right)$$

and

$$y_i^2(s) = y(s)\left\{k\left(t_{i+1}, s\right) - k\left(t_i, s\right)\right\} ,$$

then

$$\phi_i = \left[\frac{1-\alpha}{k_{i+1,i+\frac{1}{2}}h^{1-\alpha}}\left[\int_{t_i}^{t_{i+1}} \frac{1}{(t_{i+1}-s)^{\alpha}}\left(y_i^1(s)-y_i^1(t_{i+\frac{1}{2}})\right)ds\right.\right.$$

$$\left.\left. - \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\left(\frac{1}{(t_i-s)^{\alpha}}-\frac{1}{(t_{i+1}-s)^{\alpha}}\right)\left(y_i^1(s)-y_i^1(t_{j+\frac{1}{2}})\right)ds\right]\right]$$

$$+\left\{\frac{1-\alpha}{k_{i+1,i+\frac{1}{2}}h^{1-\alpha}}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\frac{1}{(t_i-s)^{\alpha}}\left(y_i^2(s)-y_i^2(t_{j+\frac{1}{2}})\right)ds\right\} .$$

The expression in square brackets can be estimated by the arguments of theorem 4.2, (II), and the expression in curly brackets by the application of Taylor's theorem and lemma 4.3. These arguments yield

$$|\phi_i| \le \left[B_1 hi^{-1}\right] + \left\{B_2 h^{1+\alpha}\right\} \le B_3 hi^{-\alpha} , \quad i = 1, \ldots, I-1 . \tag{4.27}$$

(III) From the Lipschitz continuity of $k(t, s)$, (4.26) and (4.27) it follows that for sufficiently small h ,

$$|\varepsilon_{i+\frac{1}{2}}| \le \sum_{j=0}^{i-1} |\varepsilon_{j+\frac{1}{2}}|\tilde{a}_{i-j} + B_3 hi^{-\alpha} , \quad i = 1, \ldots, I-1 . \tag{4.28}$$

where

$$\tilde{a}_\ell = a_\ell + \delta a_\ell = a_\ell + \frac{h}{k_0/2} L\left\{a_\ell \ell + \left(\ell^{1-\alpha}-(\ell-1)^{1-\alpha}\right)\right\} ,$$

and L is the Lipschitz constant of $k(t, s)$. In general, lemma 4.1 cannot be used directly for the estimation of $|\varepsilon_{i+\frac{1}{2}}|$ as previously. However, if the interval $[0, T]$ is replaced with the finite covering

$$[0, \Delta T), [\Delta T, 2\Delta T), \ldots, [(m-1)\Delta T, m\Delta T] , \quad m\Delta T = T ,$$

such that

$$\Delta T\left(\frac{1}{k_0/2} L(1+\alpha)\right) \le \frac{1-\alpha}{2} ,$$

then lemma 4.1 can be applied to each subinterval.

Let $r = \left[\frac{I}{m}\right]$. Using the equation corresponding to (4.12) and lemma 4.3 for the estimation of $|\varepsilon_{\frac{1}{2}}|$, (4.28) yields for the subinterval $[0, \Delta T)$,

$$\left.\begin{aligned} |\varepsilon_{\frac{1}{2}}| &\le B_3 h \\ |\varepsilon_{i+\frac{1}{2}}| &\le \sum_{j=0}^{i-1} |\varepsilon_{j+\frac{1}{2}}| \tilde{a}_{i-j} + B_3 h i^{-\alpha}, \quad i = 1, \dots, r_1 - 1, \end{aligned}\right\} \quad (4.29)$$

where $r_1 = r + 1$ if $(r+\frac{1}{2})h < \Delta T$, and $r_1 = r$ otherwise. By lemma 4.2 (ii), (iv) and (i),

$$\sum_{j=0}^{i-1} \tilde{a}_{i-j} = \sum_{\ell=1}^{i} a_\ell + \sum_{\ell=1}^{i} \delta a_\ell \le 1 - (1-\alpha) i^{-\alpha} + \frac{1-\alpha}{2} \alpha i^{-1-\alpha}$$

$$+ \frac{h}{\Delta T}\left\{\frac{\Delta T}{k_0/2} L(1+\alpha)\right\}\left\{i^{1-\alpha} + \frac{\alpha(1-\alpha) i^{-\alpha}}{2(1+\alpha)}\right\}.$$

Since

$$h = \frac{T}{I} = \frac{m\Delta T}{I} \le \frac{\Delta T}{r} \le \frac{\Delta T}{i}, \quad i = 1, \dots, r,$$

it follows that

$$\sum_{j=0}^{i-1} \tilde{a}_{i-j} \le 1 - \frac{(1-\alpha)^2}{2}\left(1 - \frac{\alpha}{2(1+\alpha)}\right) i^{-\alpha}, \quad i = 1, \dots, r_1 - 1.$$

Therefore

$$\rho_i = 1 - \sum_{j=0}^{i-1} \tilde{a}_{i-j} \ge B_4 i^{-\alpha} > 0, \quad i = 1, \dots, r_1 - 1.$$

The application of lemma 4.1 to (4.29) immediately yields

$$|\varepsilon_{i+\frac{1}{2}}| \le \frac{B_3}{B_4} h, \quad i = 0, \dots, r_1 - 1,$$

which implies order one convergence for the subinterval $[0, \Delta T)$.

For the subinterval $[\Delta T, 2\Delta T)$, (4.28) yields

$$\left.\begin{aligned} |\varepsilon_{r_1+\frac{1}{2}}| &\le \mu_0 + B_3 h \\ |\varepsilon_{r_1+i+\frac{1}{2}}| &\le \sum_{j=0}^{i-1} |\varepsilon_{r_1+j+\frac{1}{2}}| \tilde{a}_{i-j} + \mu_i + B_3 h i^{-\alpha}, \quad i = 1, \dots, r_2 - 1 \end{aligned}\right\} \quad (4.30)$$

where $r_2 = r + 1$ if $\left(r_1+r+\frac{1}{2}\right)h < 2\Delta T$, and $r_2 = r$ otherwise, and

$$\mu_i = \sum_{j=1}^{r_1} |\varepsilon_{r_1-j+\frac{1}{2}}| \tilde{a}_{i+j} .$$

By lemma 4.3 (iii), (iv) and (i),

$$|\mu_i| \le \frac{B_3}{B_4} h \sum_{j=1}^{r_1} \tilde{a}_{i+j} \le B_5 h i^{-\alpha} , \quad i = 1, \ldots, r_2-1 ,$$

and

$$|\mu_0| \le B_5 h .$$

In the same way as for (4.29), the application of lemma 4.1 to (4.30) yields

$$|\varepsilon_{r_1+i+\frac{1}{2}}| \le \frac{B_3+B_5}{B_4} h , \quad i = 0, \ldots, r_2-1 ,$$

which implies order one convergence for the subinterval $[\Delta T, 2\Delta T)$.

Clearly, the process continues inductively, and theorem 4.3 is an immediate consequence of the finiteness of the covering. #

Although in general the midpoint method is convergent of order 1 , it is possible under certain assumptions on $y(t)$ and $k(t, s)$, to obtain higher order convergence.

Theorem 4.4.

If

(4.5.3) $y'(t)$ is Lipschitz continuous on $[0, T]$,

(4.5.4) $y(0) = 0$, $y'(0) = 0$, and

(4.5.5) $k(t, s)$, $\partial k(t, s)/\partial t$, $\partial k(t, s)/\partial s$, $\partial^2 k(t, s)/\partial s^2$ and $\partial^2 k(t, s)/\partial s \partial t$ are Lipschitz continuous with respect to t and s on $0 \le s \le t \le T$,

then the midpoint method is convergent of order $2 - \alpha$.

Proof. The result can be obtained by a slight modification of the proof of theorem 4.3. The only difference is that

$$|\varepsilon_{\frac{1}{2}}| \le C_1 h^2 \tag{4.31}$$

and that (4.27) can be replaced by the estimate

$$|\phi_i| \le C_2 h^{2-\alpha} i^{-\alpha} \ , \quad i = 1, \ldots, I-1 \ . \tag{4.32}$$

The estimate (4.31) is obtained by Taylor series expansion of $k(h, s)y(s)$ at $s = t_{\frac{1}{2}}$.

To derive (4.32), note that from theorem 4.3, (II),

$$\phi_i = \frac{1-\alpha}{k_{i+1,i+\frac{1}{2}} h^{1-\alpha}} \left(\phi_i^1 + \phi_i^2\right)$$

where

$$\phi_i^1 = \int_{t_i}^{t_{i+1}} \frac{1}{(t_{i+1}-s)^\alpha} \left(y_i^1(s) - y_i^1(t_{i+\frac{1}{2}})\right) ds$$

$$- \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(\frac{1}{(t_i - s)^\alpha} - \frac{1}{(t_{i+1}-s)^\alpha}\right) \left(y_i^1(s) - y_i^1(t_{j+\frac{1}{2}})\right) ds$$

and

$$\phi_i^2 = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i - s)^\alpha} \left(y_i^2(s) - y_i^2(t_{j+\frac{1}{2}})\right) ds \ .$$

Firstly, ϕ_i^1 will be considered. The Taylor series expansion of $y_i^1(s)$ at $t_{j+\frac{1}{2}}$ yields an equation of the same form as (4.15), viz.

$$\phi_i^1 = h^{2-\alpha} \left(\frac{dy_i^1}{ds}(t_{i+\frac{1}{2}})\gamma - \sum_{j=0}^{i-1} \frac{dy_i^1}{ds}(t_{j+\frac{1}{2}})\Theta_{i-j}\right) + \tilde{\delta}_i \ , \quad i = 1, \ldots, I-1 \ ,$$

where

$$|\tilde{\delta}_i| \le C_3 h^{3-\alpha} \ .$$

Consequently, the arguments of theorem 4.2, (II) can be used to obtain a result corresponding to (4.20), viz.

$$\left|\frac{dy_i^1}{ds}\left(t_{i+\frac{1}{2}}\right)\gamma - \sum_{j=0}^{i-1}\frac{dy_i^1}{ds}\left(t_{j+\frac{1}{2}}\right)\Theta_{i-j}\right| \le \left|\frac{dy_i^1}{ds}\left(t_{i+\frac{1}{2}}\right)\left(\gamma - \sum_{\ell=1}^{i}\theta_\ell\right)\right| + C_4 h \ .$$

By (4.19) and conditions (4.5.3), (4.5.4) and (4.5.5),

$$\left|\frac{dy_i^1}{ds}\left(t_{i+\frac{1}{2}}\right)\left(\gamma - \sum_{\ell=1}^{i}\theta_\ell\right)\right| \le C_5 ihi^{-1-\alpha} \le C_5 h$$

and hence

$$\left|\phi_i^1\right| \le C_6 h^{3-\alpha} \ . \qquad (4.33)$$

To estimate ϕ_i^2, write

$$y_i^2(s) - y_i^2\left(t_{j+\frac{1}{2}}\right) = \left(y(s)-y_{j+\frac{1}{2}}\right)\left(k\left(t_{i+1}, s\right)-k\left(t_i, s\right)\right)$$

$$- y_{j+\frac{1}{2}}\left\{\left(k_{i+1,j+\frac{1}{2}}-k\left(t_{i+1}, s\right)\right) - \left(k_{i,j+\frac{1}{2}}-k\left(t_i, s\right)\right)\right\} .$$

By Taylor series expansion,

$$\left(y(s)-y_{j+\frac{1}{2}}\right)\left(k\left(t_{i+1}, s\right)-k\left(t_i, s\right)\right)$$

$$= y'\left(t_{j+\frac{1}{2}}\right)\frac{\partial k}{\partial t}\left(t_i, t_{j+\frac{1}{2}}\right)\left(s-t_{j+\frac{1}{2}}\right)h + O\left(h^3\right) ,$$

and

$$\left(k_{i+1,j+\frac{1}{2}}-k\left(t_{i+1}, s\right)\right) - \left(k_{i,j+\frac{1}{2}}-k\left(t_i, s\right)\right)$$

$$= -\frac{\partial^2 k}{\partial t\partial s}\left(t_i, t_{j+\frac{1}{2}}\right)\left(s-t_{j+\frac{1}{2}}\right)h + O\left(h^3\right) .$$

Hence,

$$\phi_i^2 = h^{3-\alpha}\sum_{\ell=1}^{i}\left\{y'\left(t_{i-\ell+\frac{1}{2}}\right)\frac{\partial k}{\partial t}\left(t_i, t_{i-\ell+\frac{1}{2}}\right)\right.$$

$$\left. + y\left(t_{i-\ell+\frac{1}{2}}\right)\frac{\partial^2 k}{\partial t\partial s}\left(t_i, t_{i-\ell+\frac{1}{2}}\right)\right\} \times \left\{\frac{1}{h^{2-\alpha}}\int_0^h \frac{1}{(\ell h-s)^\alpha}\left(s - \frac{h}{2}\right)ds\right\} + O\left(h^3\right)$$

and since it can easily be verified that

$$\frac{1}{h^{2-\alpha}}\int_0^h \frac{1}{(\ell h-s)^\alpha}\left(s - \frac{h}{2}\right)ds = O\left(\ell^{-1-\alpha}\right) ,$$

it follows that

$$|\phi_i^2| \le C_7 h^{3-\alpha} \quad , \quad i = 1, \ldots, I-1 \; . \tag{4.34}$$

Combining (4.33) and (4.34) yields

$$|\phi_i| \le C_8 h^2 \le C_2 h^{2-\alpha} i^{-\alpha} \quad , \quad i = 1, \ldots, I-1 \; .$$

Using (4.31) and (4.32) the theorem can be established in the same way as theorem 3.4. #

Remark 4.3. As α tends to zero, this result corresponds to the order two convergence of the midpoint scheme obtained in §3.2.

Remark 4.4. The condition (4.5.4) is always satisfied for Eq. (4.2).

This completes the discussion of the midpoint method.

The convergence of Euler's method, (4.8), will now be considered. Since the weights V_ℓ , $\ell = 1, 2, \ldots$, for Euler's method are the same as for the midpoint method, a similar analysis applies. In fact, only the arguments used in the derivation of the estimate corresponding to (4.27) differ slightly. Consequently, the following theorem will be stated without proof.

Theorem 4.5.

If $y'(t)$ is Lipschitz continuous on $[0, T]$ and the conditions (4.5.1) and (4.5.2) hold, then Euler's method is convergent of order 1 .

In the remainder of this section, the convergence properties of the trapezoidal method (4.9) will be investigated. As in the midpoint method, the case $k(t, s) = 1$ will be considered first.

Theorem 4.6.

Let $k(t, s) = 1$ and $y''(t)$ be Lipschitz continuous on $[0, T]$. Then, for $\alpha \in [\alpha_0, 1)$, where $\alpha_0 \doteqdot 0.4150$, the trapezoidal method is convergent of order 2 .

Proof. The proof consists of the same basic steps as the proof of theorem 4.2.

(I) Let $y_h(s)$ be the piecewise linear function interpolating to $y(s)$ at the gridpoints,

$$y_h(s) = \{(t_{j+1}-s)y_j+(s-t_j)y_{j+1}\}/h ,$$

$$t_j \le s \le t_{j+1} , \quad j = 0, \ldots, I-1 ,$$

and

$$e_h(s) = y_h(s) - y(s) .$$

Then (4.6) can be rewritten as

$$g(t_i) = \int_0^{t_i} \frac{y_h(s)}{(t_i-s)^\alpha} ds - \int_0^{t_i} \frac{e_h(s)}{(t_i-s)^\alpha} ds . \tag{4.35}$$

Subtraction of (4.9) from (4.35) yields

$$\sum_{j=1}^{i} \varepsilon_j W_{i-j} + \varepsilon_0 \overline{W}_i = \int_0^{t_i} \frac{e_h(s)}{(t_i-s)^\alpha} ds , \quad i = 1, \ldots, I , \tag{4.36}$$

where $\varepsilon_i = y_i - Y_i$. Subtraction of (4.36) from (4.36) with i replaced by $i + 1$ and division by W_0 yield the basic error equation

$$\varepsilon_{i+1} = \sum_{j=1}^{i} \varepsilon_j a_{i-j} + b_i , \quad i = 1, \ldots, I-1 , \tag{4.37}$$

where

$$a_0 = (W_0-W_1)/W_0 = 3 - 2^{2-\alpha} ,$$

$$a_\ell = (W_\ell-W_{\ell+1})/W_0 ,$$

$$b_i = (\beta_i^1+\beta_i^2)/W_0 ,$$

$$\beta_i^1 = \int_{t_i}^{t_{i+1}} \frac{e_h(s)}{(t_{i+1}-s)^\alpha} ds$$

$$- \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left[\frac{1}{(t_i-s)^\alpha} - \frac{1}{(t_{i+1}-s)^\alpha}\right] e_h(s) ds \quad (4.38)$$

and

$$\beta_i^2 = \varepsilon_0(\overline{W}_i - \overline{W}_{i+1}) .$$

(II) Firstly β_i^1 will be considered. By a standard theorem on Lagrangian interpolation,

$$e_h(s) = \frac{(s-t_j)(t_{j+1}-s)}{2} y''(t_j) + r_j(s) ,$$

$$t_j \le s \le t_{j+1} , \quad j = 0, \ldots, I-1 , \quad (4.39)$$

where

$$|r_j(s)| \le \frac{Lh^3}{8} \quad (4.40)$$

and L is the Lipschitz constant of $y''(s)$. Substitution of (4.39) into (4.38) yields

$$\beta_i^1 = h^{3-\alpha}\left[y''(t_i)\gamma - \sum_{j=0}^{i-1} y''(t_j)\Theta_{i-j}\right] + \delta_i , \quad i = 1, \ldots, I-1 ,$$

where

$$\gamma = \frac{1}{2h^{3-\alpha}} \int_0^h s(h-s)^{1-\alpha} ds ,$$

$$\Theta_\ell = \frac{1}{2h^{3-\alpha}} \left(\int_0^h s(h-s)\left[\frac{1}{(\ell h-s)^\alpha} - \frac{1}{((\ell+1)h-s)^\alpha}\right] ds\right.$$

and

$$\delta_i = \int_{t_i}^{t_{i+1}} \frac{r_i(s)}{(t_{i+1}-s)^\alpha} ds - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left[\frac{1}{(t_i-s)^\alpha} - \frac{1}{(t_{i+1}-s)^\alpha}\right] r_j(s) ds .$$

It follows from (4.40) that

$$|\delta_i| \le D_1 h^{4-\alpha} \ , \quad i = 1, \ldots, I-1 \ . \tag{4.41}$$

Also, it is easy to verify that

$$0 < \Theta_\ell < D_2 \ell^{-1-\alpha} \ , \quad \ell = 1, 2, \ldots \ , \tag{4.42}$$

and

$$|\gamma - \sum_{\ell=1}^{i} \Theta_\ell| \le D_3 i^{-\alpha} \ , \quad i = 1, 2, \ldots \ . \tag{4.43}$$

By partial summation,

$$\sum_{j=0}^{i-1} y''(t_j)\Theta_{i-j} = y''(t_{i-1}) \sum_{\ell=1}^{i} \Theta_\ell$$

$$+ \sum_{\nu=2}^{i} \{y''(t_{i-\nu}) - y''(t_{i+1-\nu})\} \sum_{\ell=\nu}^{i} \Theta_\ell \ . \tag{4.44}$$

From the Lipschitz continuity of $y''(t)$ and (4.42) it follows that

$$\left|\sum_{\nu=2}^{i} \{y''(t_{i-\nu}) - y''(t_{i+1-\nu})\} \sum_{\ell=\nu}^{i} \Theta_\ell\right| \le D_4 h i^{1-\alpha} \ ,$$

$$i = 2, \ldots, I-1 \ . \tag{4.45}$$

The use of (4.43), (4.44) and (4.45) yields

$$\left|y''(t_i)\gamma - \sum_{j=0}^{i-1} y''(t_j)\Theta_{i-j}\right|$$

$$\le \left|y''(t_i)\left(\gamma - \sum_{\ell=1}^{i} \Theta_\ell\right)\right| + |y''(t_i) - y''(t_{i-1})| \sum_{\ell=1}^{i} \Theta_\ell + D_4 h i^{1-\alpha}$$

$$\le F_2 D_3 i^{-\alpha} + \gamma L h + D_4 h i^{1-\alpha} \ , \qquad i = 1, \ldots, I-1 \ , \tag{4.46}$$

where $F_2 = \max_{0 \le s \le T} |y''(s)|$. Since

$$h \le \frac{T}{i} \ , \quad i = 1, \ldots, I \ ,$$

it follows from (4.41) and (4.46) that

$$|\beta_i^1| \le D_5 h^{3-\alpha} i^{-\alpha} \ . \tag{4.47}$$

From the asymptotic expansion of $(\overline{W}_i - \overline{W}_{i+1})$ it is clear that

$$|\beta_i^2| \le |\varepsilon_0| D_6 h^{1-\alpha} i^{-\alpha} . \qquad (4.48)$$

Hence, from (4.47) and (4.48)

$$|b_i| \le D_7 \left(h^2 + |\varepsilon_0|\right) i^{-\alpha} , \quad i = 1, \ldots, I-1 . \qquad (4.49)$$

(III) From (4.36),

$$|\varepsilon_1| \le D_8 \left(h^2 + |\varepsilon_0|\right) \qquad (4.50)$$

and from (4.37) and (4.49),

$$|\varepsilon_{i+1}| \le \sum_{j=1}^{i} |\varepsilon_j| |a_{i-j}| + D_7 \left(h^2 + |\varepsilon_0|\right) i^{-\alpha} , \quad i = 1, \ldots, I-1 . \qquad (4.51)$$

By the definition of a_ℓ ,

$$\sum_{\ell=0}^{i-1} a_\ell = 1 - W_i , \quad i = 1, 2, \ldots , \qquad (4.52)$$

$$a_\ell > 0 , \quad \ell = 1, 2, \ldots , \qquad (4.53)$$

$$\left.\begin{array}{l} 0 \le a_0 < 1 , \quad \alpha \in \left[\alpha_0, 1\right) , \\ -1 < a_0 < 0 , \quad \alpha \in \left(0, \alpha_0\right) , \quad \alpha_0 = 2 - \dfrac{\ell n 3}{\ell n 2} \doteqdot 0.4150 \end{array}\right\} . \qquad (4.54)$$

It can easily be verified that

$$W_i \ge D_9 i^{-\alpha} , \quad i = 1, 2, \ldots .$$

Hence, from (4.52),

$$\sum_{\ell=0}^{i-1} a_\ell \le 1 - D_9 i^{-\alpha} , \quad i = 1, 2, \ldots . \qquad (4.55)$$

It is now possible to apply lemma 4.1 to (4.51). From (4.50), (4.53), (4.54) and (4.55) the hypotheses of the lemma are satisfied for $\alpha \in \left[\alpha_0, 1\right)$ and hence

$$|\varepsilon_i| \le D_{10} \left(h^2 + |\varepsilon_0|\right) , \quad i = 1, \ldots, I . \qquad \#$$

Remark 4.5. The restriction $\alpha \in [\alpha_0, 1)$ in theorem 4.6 is only introduced for the analysis of the proof and has no natural meaning in connection with the trapezoidal method. In fact, convergence of order 2 has been established for the slightly larger interval $[\alpha_1, 1)$ where $\alpha_1 \doteq 0.2117$, (cp. Weiss (1972)). This is obtained by substituting (4.37) with i replaced by $i - 1$ into (4.37). Extensive numerical computations indicate that the trapezoidal method is convergent of order 2 for all $\alpha \in (0, 1)$, (cp. §4.6).

Since the extension of theorem 4.6 to a general $k(t, s)$ follows in a similar way to theorem 4.3, the following result is stated without proof.

Theorem 4.7.

If $k(t, s)$ satisfies (4.5.1) and

(4.5.6) $y''(t)$ is Lipschitz continuous on $[0, T]$, and

(4.5.7) $k(t, s)$, $\partial k(t, s)/\partial t$, $\partial k(t, s)/\partial s$ and $\partial^2 k(t, s)/\partial s^2$ are Lipschitz continuous with respect to t and s on $0 \le s \le t \le T$,

then, for $\alpha \in [\alpha_0, 1)$, the trapezoidal method is convergent of order 2.

4.6 Numerical Results

The methods in §4.4 have been applied to the following equations:

$$\text{(i)}\quad \frac{t^{1-\alpha}}{(1-\alpha)2i}\{{}_1F_1(1; 2-\alpha; it) - {}_1F_1(1; 2-\alpha; -it)\} = \int_0^t \frac{y(s)}{(t-s)^\alpha}\,ds\,, \quad 0 \le t \le \pi\,, \tag{4.56}$$

where ${}_1F_1$ is Kummer's hypergeometric function and $i = \sqrt{-1}$. The solution is $y(t) = \sin t$ (see Erdélyi (1954), p. 189).

$$\text{(ii)} \quad t^{-\frac{1}{2}}\exp\left(-\frac{1}{2t}\{1+t\}^2\right) = \int_0^t (t-s)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}(t-s))y(s)ds , \qquad 0 \le t \le 1 , \tag{4.57}$$

where

$$y(t) = \frac{(2\pi t)^{-\frac{1}{2}}}{2}\left\{\exp\left(-\frac{1}{2t}\{1+t\}^2\right)\left(\frac{1}{t}-1\right) + \exp(-2)\exp\left(-\frac{1}{2t}\{1-t\}^2\right)\left(\frac{1}{t}+1\right)\right\} .$$

This is an equation of the form (4.2), (cp. Durbin (1971)).

The errors incurred when applying the midpoint method to (4.56) with $\alpha = 0.5$ and (4.57) are listed in tables 4.1 and 4.2 respectively. In the row headed "max" in these tables the value of $\max_{0 \le i \le I-1} |\varepsilon_{i+\frac{1}{2}}|$ is given.

In table 4.1 the errors for fixed t decrease by approximately 5.2 when h is divided by three, which would correspond to convergence of order 1.5. However, the maximum error, which occurs at the first gridpoint, is of order one, as predicted by corollary 4.1. In table 4.2 it is clear that convergence of order 1.5 is obtained. This agrees with the order predicted by theorem 4.4.

Table 4.1.

t	$h = \pi/10$	$h = \pi/30$	$h = \pi/90$	$h = \pi/270$
$3\pi/20$	-2.631 E-2	-4.992 E-3	-9.292 E-4	-1.755 E-4
$7\pi/20$	4.303 E-3	1.211 E-3	2.770 E-4	5.784 E-5
$11\pi/20$	2.634 E-2	5.534 E-3	1.112 E-3	2.186 E-4
$15\pi/20$	3.630 E-2	7.366 E-3	1.451 E-3	2.824 E-4
$19\pi/20$	3.149 E-2	6.219 E-3	1.204 E-3	2.323 E-4
max	5.065 E-2	1.739 E-2	5.815 E-3	1.939 E-3

Table 4.2.

t	$h = 1/10$	$h = 1/30$	$h = 1/90$	$h = 1/270$
3/20	-1.068 E-2	-2.377 E-3	-4.465 E-4	-8.234 E-5
7/20	5.037 E-3	1.135 E-3	2.282 E-4	4.466 E-5
11/20	2.381 E-3	4.695 E-4	8.998 E-5	1.726 E-5
15/20	9.689 E-4	1.813 E-4	3.409 E-5	6.486 E-6
19/20	4.066 E-4	7.379 E-5	1.369 E-5	2.586 E-6
max	1.151 E-2	5.228 E-3	1.184 E-3	2.387 E-4

In tables 4.3 and 4.4 the errors obtained when using Euler's method for the solution of (4.56) with $\alpha = 0.5$ and (4.57), respectively, are given. These results verify the order 1 convergence predicted by theorem 4.5.

Table 4.3.

t	h = π/20	h = π/40	h = π/80	h = π/160
π/5	-7.610 E-2	-3.539 E-2	-1.696 E-2	-8.275 E-3
2π/5	-2.970 E-2	-1.285 E-2	-6.031 E-3	-2.951 E-3
3π/5	2.594 E-2	1.392 E-2	6.983 E-3	3.426 E-3
4π/5	7.105 E-2	3.518 E-2	1.726 E-2	8.470 E-3
π	8.874 E-2	4.290 E-2	2.091 E-2	1.027 E-2

Table 4.4.

t	h = 0.1	h = 0.05	h = 0.025	h = 0.0125
0.2	-6.693 E-2	-2.496 E-2	-9.753 E-3	-4.262 E-3
0.4	4.164 E-3	3.057 E-3	1.516 E-3	7.132 E-4
0.6	8.474 E-3	3.725 E-3	1.701 E-3	8.016 E-4
0.8	5.691 E-3	2.482 E-3	1.146 E-3	5.465 E-4
1.0	3.595 E-3	1.590 E-3	7.433 E-4	3.579 E-4

Finally, the trapezoidal method was applied to (4.56) ($\alpha = 0.5$, $\alpha = 0.05$) and (4.57). The errors tabulated in tables 4.5, 4.6 and 4.7 respectively, demonstrate the order 2 convergence. In particular, the result for (4.56) with $\alpha = 0.05$ illustrates the order 2 convergence for $\alpha \notin [\alpha_0, 1)$.

Table 4.5.

t	h = π/20	h = π/40	h = π/80	h = π/160
π/5	-1.056 E-3	-2.753 E-4	-7.081 E-5	-1.805 E-5
2π/5	-1.814 E-3	-4.634 E-4	-1.177 E-4	-2.975 E-5
3π/5	-1.894 E-3	-4.771 E-4	-1.201 E-4	-3.017 E-5
4π/5	-1.259 E-3	-3.100 E-4	-7.684 E-5	-1.911 E-5
π	-1.476 E-4	-2.541 E-5	-4.436 E-6	-7.794 E-7

Table 4.6.

t	h = π/20	h = π/40	h = π/80	h = π/160
π/5	-1.213 E-3	-3.023 E-4	-7.552 E-5	-1.888 E-5
2π/5	-1.963 E-3	-4.892 E-4	-1.222 E-4	-3.055 E-5
3π/5	-1.964 E-3	-4.894 E-4	-1.223 E-4	-3.056 E-5
4π/5	-1.216 E-3	-3.028 E-4	-7.560 E-5	-1.889 E-5
π	-4.883 E-6	-5.015 E-7	-6.097 E-8	-7.815 E-9

Table 4.7.

t	h = 0.1	h = 0.05	h = 0.025	h = 0.0125
0.2	-9.098 E-3	-1.333 E-3	-3.671 E-4	-9.285 E-5
0.4	-1.032 E-3	-1.909 E-4	-4.679 E-5	-1.134 E-5
0.6	-2.146 E-5	5.912 E-6	2.026 E-6	6.468 E-7
0.8	6.771 E-5	2.065 E-5	5.353 E-6	1.389 E-6
1.0	5.447 E-5	1.550 E-5	3.939 E-6	1.004 E-6

Remark 4.6. Numerical instability has not been observed in any of the above examples.

4.7 Error Propagation in the Midpoint Method

In the application to practical problems it is necessary to examine the case when values

$$g_i = g(t_i) + \delta g_i \tag{4.58}$$

are used in the numerical schemes instead of the exact values $g(t_i)$. Only the midpoint method will be considered. However, similar results can be obtained for the Euler and trapezoidal methods.

Using g_i given by (4.58), the midpoint method will generate approximations $\overline{Y}_{i+\frac{1}{2}}$, $i = 0, \ldots, I-1$, satisfying

$$g(t_i) + \delta g_i = \sum_{j=0}^{i-1} \overline{Y}_{j+\frac{1}{2}} k_{i,j+\frac{1}{2}} V_{i-j} , \quad i = 1, \ldots, I .$$

Let

$$\delta Y_{i+\frac{1}{2}} = \overline{Y}_{i+\frac{1}{2}} - Y_{i+\frac{1}{2}} , \quad i = 0, \ldots, I-1 .$$

Then

$$\delta g_i = \sum_{j=0}^{i-1} \delta Y_{j+\frac{1}{2}} k_{i,j+\frac{1}{2}} V_{i-j} , \quad i = 1, \ldots, I-1 . \tag{4.59}$$

Assume that $k(t, s) = 1$ and first consider the effect of a single perturbation δg_ℓ. From (4.59),

$$\delta Y_{\ell+j-\frac{1}{2}} = \delta g_\ell \frac{1-\alpha}{h^{1-\alpha}} \chi_j , \quad j = 0, \ldots, I-\ell ,$$

where χ_j, $j = 0, 1, \ldots$, are given by

$$\chi_0 = 1 ,$$

$$\sum_{j=0}^{i} \chi_j V_{i+1-j} = 0 \ , \quad i = 1, 2, \ldots \ .$$

For the general case, superposition yields

$$\delta Y_{i+\frac{1}{2}} = \frac{1-\alpha}{h^{1-\alpha}} \sum_{j=1}^{i+1} \delta g_j \chi_{i+1-j} \ , \quad i = 0, \ldots, I-1 \ . \tag{4.60}$$

An accurate estimate for χ_j could not be obtained analytically and so its behaviour was investigated numerically. From table 4.8 it can be seen that $\chi_j = O(j^{-2+\alpha})$. On the basis of this numerical result it follows from (4.60) that

$$|\delta Y_{i+\frac{1}{2}}| \le \frac{C}{h^{1-\alpha}} \max_{j=1,\ldots,i+1} |\delta g_j| \ ,$$

$$i = 0, \ldots, I-1 \ , \quad C = \text{const.} \tag{4.61}$$

Numerical computation indicates that this bound remains valid for a general $k(t, s)$. It should be noted that (4.61) has an amplification factor of $1/h^{1-\alpha}$ compared with an amplification factor of $1/h$ in the numerical schemes for (3.1) (cp. §3.11). For the important case $\alpha = \frac{1}{2}$ this represents a considerable improvement over $\alpha = 0$ and indicates that these equations are more amenable to a numerical solution.

Table 4.8.

	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
χ_{10}/χ_{100}	106.62	39.39	13.64
χ_{100}/χ_{1000}	87.53	32.26	12.71
$10^{2-\alpha}$	79.43	31.62	12.59

4.8 High Order Methods

High order schemes for (4.1) can be constructed by extending the methods of §3.3. Discretizing (4.1) at the points t_{ij} , $j = 1, \ldots, n$; $i = 0, \ldots, I-1$, yields

$$g(t_{ij}) = \int_0^{t_{ij}} \frac{k(t_{ij},s)}{(t_{ij}-s)^\alpha} y(s)ds$$

$$= \sum_{\ell=0}^{i-1} \int_{t_\ell}^{t_{\ell+1}} \frac{k(t_{ij},s)}{(t_{ij}-s)^\alpha} y(s)ds + \int_{t_i}^{t_{ij}} \frac{k(t_{ij},s)}{(t_{ij}-s)^\alpha} y(s)ds \ ,$$

$$j = 1, \ldots, n \ ; \ i = 0, \ldots, I-1 \qquad (4.62)$$

The application of product integration based on the approximation

$$k(t_{ij}, s)y(s) \simeq \sum_{k=1}^{n} k(t_{ij}, t_{\ell k})y(t_{\ell k})L_k\left(\frac{s-t_\ell}{h}\right) , \quad j = 1, \ldots, n ,$$

$$t_\ell \le s < t_{\ell+1} \ ; \ \ell = 0, \ldots, i-1 \ ; \ i = 0, \ldots, I-1 ,$$

to (4.62) leads to the numerical scheme

$$g(t_{ij}) = \sum_{\ell=0}^{i} \sum_{k=1}^{n} Y_{\ell k} k(t_{ij}, t_{\ell k}) w_{jk}^{(i-\ell)} ,$$

$$j = r, \ldots, n \ ; \ i = 0, \ldots, I-1 \qquad (4.63)$$

where

$$w_{jk}^{(0)} = h^{1-\alpha} \int_0^{u_j} \frac{L_k(s)}{(u_j-s)^\alpha} ds , \quad j = r, \ldots, n \ ; \ k = 1, \ldots, n ,$$

$$w_{jk}^{(m)} = h^{1-\alpha} \int_0^1 \frac{L_k(s)}{(m+u_j-s)^\alpha} ds ,$$

$$j = r, \ldots, n , \quad k = 1, \ldots, n \ ; \quad m = 1, \ldots, I-1 ,$$

and Y_{ij} is the approximation to $y(t_{ij})$. The scheme (4.63) is the product integration analogue of (3.9). For each i , $i = 0, \ldots, I-1$, (4.63) is a linear system in the unknowns Y_{ij} , $j = r, \ldots, n$.

Since $k(t, s)$ is continuous and $k(t, t) \neq 0$, these systems are nonsingular if h is sufficiently small. In (4.63), values of $k(t, s)$ are required in the region $t \leq s \leq t+(1-u_1)h$, $0 \leq t \leq T$. This can be avoided if (4.63) is modified in a similar way to (3.10).

While convergence results for (4.63) have not been obtained, numerical computations carried out and the results of chapter 3 indicate that

(i) if $u_1 > 0$, then the scheme is convergent of order n, and

(ii) if $u_1 = 0$, a <u>sufficient</u> condition for convergence of order n is

$$-1 \leq \eta = \frac{\prod_{k=1}^{n-1}(1-u_k)}{\prod_{k=2}^{n}(-u_k)} \leq 1 .$$

Some numerical results supporting this conjecture will now be given. The schemes (4.63) with $u_1 = \frac{1}{3}$, $u_2 = \frac{2}{3}$ and $u_3 = 1$, $u_1 = 0$, $u_2 = \frac{1}{2}$ and $u_3 = 1$ $(\eta = 1)$ and $u_1 = 0$, $u_2 = \frac{1}{3}$, $u_3 = \frac{2}{3}$ and $u_4 = 1$ $(\eta = -1)$ have been applied to Eq. (4.57). The errors $y(t_{ij}) - Y_{ij}$ listed in tables 4.9, 4.10 and 4.11 respectively, indicate convergence of order 3, 3 and 4. Although the errors are only given for the interval $0 \leq t \leq 1$, the methods have been used to calculate approximations on $0 \leq t \leq 10$. Numerical instability has not been observed.

Table 4.9.

t	h = 0.1	h = 0.05	h = 0.025	h = 0.0125
0.2	-1.948 E-4	-2.455 E-6	1.040 E-6	1.990 E-7
0.4	7.064 E-4	6.423 E-6	7.723 E-7	9.326 E-8
0.6	1.085 E-5	5.842 E-7	7.642 E-8	8.903 E-9
0.8	2.554 E-6	-5.810 E-8	-1.632 E-9	-3.443 E-10
1.0	1.070 E-6	-9.572 E-8	-7.615 E-9	-9.871 E-10

Table 4.10.

t	h = 0.1	h = 0.05	h = 0.025	h = 0.0125
0.2	-4.727 E-4	-1.026 E-4	-1.107 E-6	5.345 E-7
0.4	2.520 E-4	2.562 E-5	2.646 E-6	2.972 E-7
0.6	4.316 E-5	2.524 E-6	2.438 E-7	2.655 E-8
0.8	4.789 E-6	-5.242 E-8	-9.088 E-8	-1.584 E-9
1.0	-4.697 E-8	-2.348 E-7	-2.562 E-8	-3.229 E-9

Table 4.11

t	h = 0.1	h = 0.05	h = 0.025	h = 0.0125
0.2	5.862 E-4	1.854 E-6	3.810 E-7	2.056 E-8
0.4	2.308 E-6	-6.011 E-7	-3.312 E-8	-2.020 E-9
0.6	-5.651 E-7	-7.394 E-8	-3.482 E-9	-2.185 E-10
0.8	3.715 E-8	-1.261 E-8	-2.653 E-10	-2.112 E-11
1.0	9.113 E-8	-3.813 E-9	7.390 E-11	1.232 E-12

4.9 Conclusion

In the preceeding sections a variety of finite difference schemes for the generalized Abel equation has been constructed. As a general purpose method, the scheme (4.63) with $u_1 = 0$, $u_2 = \frac{1}{2}$ and $u_3 = 1$ appears to be suitable since it combines high order accuracy with simplicity of implementation.

In this chapter, a number of problems remain unsolved. In particular, the convergence proof for the trapezoidal method for all $\alpha \in (0, 1)$, the convergence of the high order methods and the derivation of asymptotic error estimates still require examination. Such problems could provide a fruitful avenue for further research.

REFERENCES

Anderssen, R.S., de Hoog, F.R. and Weiss, R. (1972). On the numerical solution of Brownian motion processes. J. Appl. Prob., to appear.

Atkinson, K.E. (1971 a). The numerical solution of an Abel integral equation by a product trapezoidal method. Report, Computer Centre, The Australian National University.

Atkinson, K.E. (1971 b). A survey of numerical methods for the solution of Fredholm integral equations of the second kind. To appear in the proceedings of the symposium "Numerical Solution of Integral Equations with Physical Applications", 1971 Fall National Meeting of SIAM.

Axelsson, O. (1969). A class of A-stable methods. BIT, 9, pp 185-199.

Beltjukov, B.A. (1965). An analogue of the Runge-Kutta method for the solution of a nonlinear integral equation of the Volterra type. Differential Equations, 1, pp 417-433.

Bockasten, K. (1961). Transformation of observed radiances into radial distribution of the emission of a plasma. J. Opt. Soc. Amer., 51, pp 943-947.

Butcher, J.C. (1964 a). Implicit Runge-Kutta processes. Math. Comp., 18, pp 50-64.

Butcher, J.C. (1964 b). Integration processes based on Radau quadrature formulas. Math. Comp., 18, pp 233-244.

Cambell, G.M. and Day, J.T. (1971). A block by block method for the numerical solution of Volterra integral equations. BIT, 11, pp 120-124.

Dahlquist, G. (1963). A special stability problem for linear multistep methods. BIT, 3, pp 27-43.

Davis, H.T. (1960). Introduction to Nonlinear Differential and Integral Equations. United States Atomic Energy Commission.

de Hoog, F.R. and Weiss, R. (1972 a). Implicit Runge-Kutta methods for second kind Volterra integral equations. Submitted for publication.

de Hoog, F.R. and Weiss, R. (1972 b). On the solution of Volterra integral equations of the first kind. Submitted for publication.

de Hoog, F.R. and Weiss, R. (1972 c). High order methods for Volterra integral equations of the first kind. SIAM J. Numer. Anal., to appear.

de Hoog, F.R. and Weiss, R. (1972 d). Asymptotic expansions for product integration. Math. Comp., to appear.

Durbin, J. (1971). Boundary-crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of the Kolmogorov-Smirnov test. J. Appl. Prob., 8, pp 431-453.

Edels, H., Hearne, K. and Young, A. (1962). Numerical solutions of the Abel integral equation. J. Math. and Phys., 41, pp 62-75.

Einarsson, B. (1971). Numerical solution of Abel's integral equation with spline functions. FOA 2 Rapport, C2455-11(25). Försvarets Forskningsanstalt, Avdelning 2, Stockholm.

Erdélyi, A. (Ed.) (1954). Tables of Integral Transforms, Vol. II. McGraw-Hill.

Evans, G.C. (1910). Volterra's integral equation of the second kind, with discontinuous kernel. Trans. Amer. Math. Soc., 11, pp 393-413.

Fox, L. and Goodwin, E.T. (1953). The numerical solution of nonsingular linear integral equations. Phil. Trans. Roy. Soc., 245, pp 501-534.

Frie, W. (1963). Zur Auswertung der Abelschen Integralgleichung. Ann. Physik., 10, pp 332-339.

Henrici, P. (1962). Discrete Variable Methods in Ordinary Differential equations. John Wiley, New York.

Hung, H.S. (1970). The numerical solution of differential and integral equations by spline functions. MRC Technical Summary Report #1053, University of Wisconsin, Madison.

Jones, J.G. (1961). On the numerical solution of convolution integral equations and systems of such equations. Math. Comp., 15, pp 131-142.

Kobayasi, M. (1966). On numerical solution of the Volterra integral equations of the second kind by linear multistep methods. Rep. Stat. Appl. Res., JUSE, 13, pp 1-21.

Kobayasi, M. (1967). On numerical solution of the Volterra integral equations of the first kind by trapezoidal rule. Rep. Stat. Appl. Res., JUSE, 14, pp 1-14.

Kowalewski, G. (1930). Integralgleichungen. Walter de Gruyter & Co., Berlin W und Leipzig.

Linz, P. (1967 a). The numerical solution of Volterra integral equations by finite difference methods. MRC Technical Summary Report #825. The University of Wisconsin, Madison.

Linz, P. (1967 b). Applications of Abel transforms to the numerical solution of problems in electrostatics and elasticity. MRC Technical Summary Report #826, The University of Wisconsin, Madison.

Linz, P. (1971). Product integration methods for Volterra integral equations of the first kind. BIT, 11, pp 413-421.

Loudet, M. and Oules, H. (1960). Sur l'integration numerique des equations integrales du type de Volterra; in "Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-differential Equations". Birkhauser Verlag, Basel, pp 117-121.

Minerbo, G.N. and Levy, M.E. (1969). Inversion of Abel's integral equation by means of orthogonal polynomials. SIAM J. Numer. Anal., 6, pp 598-616.

Nestor, O.H. and Olsen, H.N. (1960). Numerical methods for reducing line and surface probe data. SIAM Rev., 2, pp 200-207.

Noble, B. (1964). The numerical solution of nonlinear integral equations and related topics; in "Nonlinear Integral Equations", ed. P.M. Anselone. University of Wisconsin Press, Madison, pp 215-318.

Noble, B. (1969). Instability when solving Volterra integral equations of the second kind by multistep methods; in "Lecture Notes in Mathematics", 109. Springer Verlag, pp 23-39.

Noble, B. (1971 a). A bibliography on: "Methods for solving integral equations" - Author listing. MRC Technical Summary Report #1176, The University of Wisconsin, Madison.

Noble, B. (1971 b). A bibliography on: "Methods for solving integral equations" - Subject listing. MRC Technical Summary Report #1177, The University of Wisconsin, Madison.

Olsen, H.N. (1959). Thermal and electrical properties of an argon plasma. Physics of Fluids, 2, pp 614-623.

Oules, H. (1964). Resolution numerique d'une equation integrale singuliere. Rev. Franc. de Traitm. de l'Inf., 7, pp 117-124.

Pouzet, P. (1960). Methode d'integration numerique des equations integrales et integro-differentielles du type de Volterra de seconde espece, Formules de Runge-Kutta; in "Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-differential Equations". Birkhauser Verlag, Basel, pp 362-368.

Pouzet, P. (1962). Etude, en vue de leur traitment numerique d'equations integrales et integro-differentielles du type de Volterra pour des problemes de conditions initiales. Thesis, University of Strassbourg.

Ralston, A. (1965). A First Course in Numerical Analysis. McGraw Hill, New York.

Richtmyer, R.D. and Morton, K.W. (1967). Difference Methods for Initial Value Problems; Second Edition. Interscience Publishers.

Schardin, H. (1933). Theorie und Anwendungen des Mach-Zehnderschen Interferenz-Refractrometers. Zeits. f. Instrumentenk., 53, pp 396-403.

Squire, W. (1969). Numerical solution of linear Volterra equations of the first kind. Aerospace Engineering TR-15. West Virginia University.

Squire, W. (1970). Integration for Engineers and Scientists. American Elsevier, New York.

Stetter, H.J. (1965). Asymptotic expansions for the error of discretization algorithms for non-linear functional equations. Num. Math., 7, pp 18-31.

Tricomi, F.G. (1957). Integral Equations. Interscience Publishers, Inc., New York.

Weiss, R. (1972). Product integration for the generalized Abel equation. Math. Comp., 26, pp 177-190.

Weiss, R. and Anderssen, R.S. (1972). A product integration method for a class of singular first kind Volterra equations. Num. Math., 18, pp 442-456.

Winckler, J. (1948). The Mach interferometer applied to studying an axially symmetric supersonic air jet. Rev. Sci. Instr., 19, pp 307-322.

Wright, K. (1970). Some relationships between implicit Runge-Kutta, collocation and Lanczos τ methods, and their stability properties. BIT, 10, pp 217-227.

Young, A. (1954). The application of approximate product integration to the numerical solution of integral equations. Proc. Roy. Soc. London (A), 224, pp 561-573.