DIFFERENTIABLE MANIFOLDS
MODELLED ON LOCALLY CONVEX SPACES

by

Truong Công Nghê

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Kính dâng Cậu Mạ

STATEMENT

The results in this thesis are the product of my own research, except where specifically stated otherwise.

Truong Công Nghê

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ABSTRACT

We construct differentiable manifolds modelled on locally convex spaces using Yamamuro's theory of Γ-differentiation [81], [82], manifolds which we term as Γ-manifolds.

Then corresponding to the strong notion of $B\Gamma$-differentiability in Yamamuro's theory [82] we obtain the subclass of $B\Gamma$-manifolds. We show how to extend to these $B\Gamma$-manifolds the standard properties of Banach manifolds: The Smale Density Theorem [4] as well as the Transversality Theory [4], [31].

As first applications, we give several simple results about genericity of smooth maps using our Γ-technique instead of the usual standard Banach techniques.

CONTENTS

INTRODUCTION

During the last two decades, there has been considerable development of the theory of Banach manifolds starting with J. Eells [20] in 1958. He constructed a smooth manifold from the set of continuous maps between two manifolds. This was the first example of a non-trivial Banach manifold. Since then, many authors have contributed to the theory: R. Abraham [2], F.E. Browder [11], [12], R. Bonic and J. Frampton [8], H.I. Eliasson [26], K.D. Elworthy and A.J. Tromba [28], [29], N.H. Kuiper and D. Burghelea [41], J. McAlpin [25], R.S. Palais [58], [59], [60], S. Smale [67], [68], [29], A. Weinstein [77], [78], and many others. Eells' paper [21] is a good survey of development in this area.

However, it is quickly apparent that, for many purposes, Banach manifolds are inadequate. In particular, they are not suitable for applications to mechanics, for instance, to the Cauchy problem of an equation of evolution, and the space of C^{∞} diffeomorphisms on a compact manifold. Marsden's note [48] is a good survey of these and other related matters.

Thus there is an urgent need for a suitable theory of manifolds modelled on locally convex spaces, or at least on Fréchet spaces. In fact, several attempts have been made in this direction. We mention the work of A. Bastiani [7], W.D. Curtis and F.R. Miller [15], H.R. Fischer [30], J. Kijowski, W. Szczyrba and J. Komorowski [37], [38], [39], N. Krikorian [40], J. Leslie [45], H. Omori [54], [55], [56] and F. Sergeraert [70].

As is well-known, there is a previledged notion of differentiation in the normed case, the so-called Fréchet differentiation, or more up to date, the bounded differentiation. However, in the case of locally convex spaces, there is a variety of possible differentiations (see [6], [36], [76], [80]). For instance, Leslie's work [45] is based on Bastiani's differentiation, Sergeraert's [70] on Hyers', Curtis-Miller's [15] on strong differentiation, and so on. Unfortunately, all these differentiations share a common weakness: the lack of the Inverse Mapping Theorem, the essential tool for the investigation of differential results.

Recently, in 1975, in an attempt to overcome this weakness, S. Yamamuro [81] has given a new differentiation for which the Inverse Mapping Theorem and all of its equivalent forms hold. However, since this differentiation is too strong for some purposes, he has found it necessary to define a weaker notion (see [52], [82]). He terms this differentiation Γ-differentiation whereas the previous notion is referred to as $B\Gamma$-differentiation. These matters will be taken up in sections 1 to 3 of Chapter 1 of this thesis. It should be noted that, in the same year of 1975, H.R. Fischer [30] has independently proposed a differentiation which is almost equivalent to the Γ-differentiation.

In this thesis, we shall use Yamamuro's Γ-differentiation to construct manifolds modelled on locally convex spaces. These manifolds are called Γ-manifolds. Then corresponding to the strong notion of $B\Gamma$-differentiation we have the subclass of $B\Gamma$-manifolds. We will show how to extend the standard properties of Banach manifolds to this class of $B\Gamma$-manifolds. For example, we have been able to define $B\Gamma$-transversality, a generalisation of the standard notion of transversality ([4], [31], [75]) and to prove all the standard theorems for this generalised notion.

This thesis is divided into five chapters. In the first chapter, we prove two local results on Γ-differentiation, namely the Γ-omega lemma and the $B\Gamma$-differentiability of the evaluation map. These results will be needed later in the text.

Chapter 2 is devoted to definitions and examples of Γ- and $B\Gamma$-manifolds as well as Γ- and $B\Gamma$-bundles. We shall prove that the space $C^{\infty}(X, Y)$ of C^{∞} maps from a compact C^{∞} manifold X into a (finite-dimensional) C^{∞} manifold Y is a Γ-manifold of class C^{∞}_{Γ}. Hence the space $\mathrm{Diff}^{\infty}(X)$

and $\text{Emb}^{\infty}(X, Y)$ introduced in [31] are Γ-manifolds. We will also give several simple examples of $B\Gamma$-manifolds.

From Chapter 3 onward, we will restrict our attention to the subclass of $B\Gamma$-manifolds and will use the full strength of the Inverse Mapping Theorem. Chapter 3 contains a generalisation of the Smale Density Theorem [68] to $B\Gamma$-manifolds followed by a brief discussion of the notion of $B\Gamma$-maps between Γ-manifolds. This notion cannot be defined in any natural fashion, however, it yields many interesting results. For example, using this notion, we can get the results in Chapter III of Omori's Lecture Notes [54]. With this notion, we have also been able to give a yet more general Γ-version of the Smale Density Theorem.

The standard transversality theory ([4], [31], [33]) is generalised to the $B\Gamma$-context in Chapter 4 under the name of $B\Gamma$-transversality. We show that all the standard transversal theorems remain valid: the $B\Gamma$-Transversal Density Theorem and the $B\Gamma$-Transversal Isotopy Theorem.

Some applications of our $B\Gamma$-Transversal Density Theorem appear in Chapter 5 where we give simple "generic" results for local smooth maps which parallel the usual ones. They are local versions of the more general global results in [31], [33], [43]. The only difference is that here we follow the Γ-technique instead of the standard Banach techniques.

Two papers based on the contents of Chapter 3 and Chapter 4 have been accepted for publication [51], [52].

For the reader's convenience, we include at the end of the thesis a list of notation as well as an index of terminology.

After this thesis had been completely typed, we discovered two recent works of H.R. Fischer and J. Gutkecht which are closely related to it. They are added to the bibliography as additional references [AR1] and [AR2].

CHAPTER 1

Γ-DIFFERENTIATION

The purpose of this chapter is to give two local results about Γ-differentiation which shall be needed later: the Γ-omega lemma and the *B*Γ-differentiability of the evaluation map.

For the sake of completeness, we include in the first three sections, §1-§3, the Γ-differentiation theory of Yamamuro. The main results which shall be needed are stated without proof. For more details we refer to [52], [81], [82]. In §4 we give a criterion for Γ-differentiability in case Γ consists of a family of norms. In §5, we combine the work of Irwin [35] with the criterion in §4 to prove the Γ-omega lemma, the main step for proving that the space $C^{\infty}(X, Y)$ is a Γ-manifold (see Chapter 2).

The last section, §6, is devoted to the study of *B*Γ-differentiability of the evaluation map; the results are needed for Chapter 5.

1. Calibration

A *calibration* for a locally convex space (LCS) is a set of continuous semi-norms which induces the topology. For a LCS E , the set $P(E)$ of all continuous semi-norms on E is obviously the biggest calibration for E .

Let $\mathcal{E}$ be a family of LCS's. A map p defined on $\mathcal{E}$ is called a *semi-norm map* if, for each $E \in \mathcal{E}$, the value p_E of p at E belongs to $P(E)$. We call a set Γ of semi-norm maps on $\mathcal{E}$ a *calibration for* $\mathcal{E}$ if, for each $E \in \mathcal{E}$, the set

$$\Gamma_E = \{p_E \mid p \in \Gamma\} \tag{1}$$

is a calibration for E .

Let $\mathcal{E}$ be a family of LCS's and let Γ be a calibration for $\mathcal{E}$. Let $E, F \in \mathcal{E}$ and U be an open subset of E . Then a map $f : U \subseteq E \to F$ is said to be Γ-*continuous at* $a \in U$ if, for any $\varepsilon > 0$ and $p \in \Gamma$, there exists $\delta > 0$ such that the following condition holds:

$$\big(p_E(x) < \delta \text{ and } a+x \in U\big) \Rightarrow \big(p_F[f(a+x)-f(a)] < \varepsilon\big) . \tag{2}$$

In other words, f is Γ-continuous at $a \in U$ if, for each $p \in \Gamma$, we have $p_F[f(a+x_n)-f(a)] \to 0$ whenever $p_E(x_n) \to 0$ and $a + x_n \in U$. Note that the fact that U is open is not used in this definition; but under this condition, we can say that a map which is Γ-continuous at one point is continuous there.

As usual, we say that $f : U \subseteq E \to F$ is Γ-*continuous on* U if it is Γ-continuous at every point of U.

(1.1) PROPOSITION. *A linear map* $u : E \to F$ *is* Γ-*continuous at one point (hence on* E*) iff, for each* $p \in \Gamma$,

$$p_{(E,F)}(u) = \sup\{p_F[u(x)] \mid p_E(x) \le 1\} < +\infty . \tag{3}$$

We denote the set of all Γ-continuous linear maps of E into F by $L_\Gamma(E, F)$. It is obvious that $L_\Gamma(E, F)$ is a linear space and $p_{(E,F)}$ (defined in (3)) is a semi-norm on $L_\Gamma(E, F)$ for each $p \in \Gamma$. We put

$$\Gamma_{(E,F)} = \{p_{(E,F)} \mid p \in \Gamma\} \tag{4}$$

and regard $L_\Gamma(E, F)$ as a locally convex space calibrated by $\Gamma_{(E,F)}$.

As to the composition, we have the following usual results which also imply its continuity.

(1.2) PROPOSITION. *Let* $E, F, G \in \mathcal{E}$. *If* $u \in L_\Gamma(E, F)$ *and* $v \in L_\Gamma(F, G)$ *then* $v \circ u \in L_\Gamma(E, G)$ *and*

$$p_{(E,G)}(v \circ u) \le p_{(E,F)}(u)\, p_{(F,G)}(v) \quad \textit{for all} \quad p \in \Gamma . \tag{5}$$

Note that Proposition (1.2) does not imply that $L_\Gamma(E, E)$ is an algebra because the first E and the second E may have different calibrations. If they have the identical calibration then we denote it by $L_\Gamma(E)$ which is then an algebra with jointly continuous multiplication.

In the sequel, we shall sometimes drop (E, F) from $p_{(E,F)}$; for instance, (3) shall permit us to write

$$p_F[u(x)] \le p(u)p_E(x) \quad \text{if} \quad x \in E \quad \text{and} \quad u \in L_\Gamma(E, F) \ . \tag{6}$$

We shall say that a map $f : U \subseteq E \to F$ (where $E, F \in \mathcal{E}$, Γ is a calibration for $\mathcal{E}$, and $U \subseteq E$ open) is *strongly* Γ*-continuous* at $a \in U$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following condition holds:

$$\big(p \in \Gamma,\ p_E(x) < \delta \text{ and } a+x \in U\big) \Rightarrow \big(p_F[f(a+x)-f(a)] < \varepsilon\big) \ . \tag{7}$$

In this case, unlike the case of the Γ-continuity, δ *does not depend on* p . It is easy to see that f is strongly Γ-continuous at $a \in U$ iff the following condition is satisfied: if $(p_n)_E(x_n) \to 0$ for sequences $p_n \in \Gamma$ and $x_n \in E$ such that $a + x_n \in U$, we have $(p_n)_F[f(a+x_n)-f(a)] \to 0$. $f : U \subseteq E \to F$ is *strongly* Γ*-continuous on* U if it is strongly Γ-continuous at every point $x \in U$.

(1.3) PROPOSITION. *A linear map* $u : E \to F$ *is strongly* Γ*-continuous at one point (hence on* E *) iff*

$$\|u\|_\Gamma = \sup_{p\in\Gamma} \sup\{p_F[u(x)] \mid p_E(x) \le 1\} < +\infty \ . \tag{8}$$

Such a map will be called a $B\Gamma$*-bounded linear map* as is explained right now.

Let $E \in \mathcal{E}$. An element $x \in E$ is said to be Γ*-bounded* if

$$\|x\|_\Gamma = \sup\{p_E(x) \mid p \in \Gamma\} < +\infty \ . \tag{9}$$

The set of all Γ-bounded elements of E will be denoted by $B\Gamma(E)$, which will always be regarded as a normed space equipped with the norm defined by (9).

(1.4) PROPOSITION. *If* E *is sequentially complete, then* $B\Gamma(E)$ *is a Banach space.*

In general, $B\Gamma(E)$ is small as a subset of E.

(1.5) PROPOSITION. *Let E be Baire. If $B\Gamma(E)$ contains an interior point with respect to the relative topology from E, then E is normable.*

Now let $\mathcal{E}$ be a family of LCS's and Γ be a calibration for $\mathcal{E}$. For $E, F \in \mathcal{E}$, we have seen that $L_\Gamma(E, F)$ is equipped with the calibration $\Gamma_{(E,F)}$ (defined in (4)). Therefore, by extending each $p \in \Gamma$ over $L_\Gamma(E, F)$ we may suppose that $L_\Gamma(E, F) \in \mathcal{E}$. Then, we denote the space $B\Gamma[L_\Gamma(E, F)]$ by $L_{B\Gamma}(E, F)$.

Thus $u \in L_{B\Gamma}(E, F)$ iff $\|u\|_\Gamma$, defined by (8), is finite. In other words, the set of $B\Gamma$-bounded linear maps $E \to F$ coincides with the set of strongly Γ-continuous linear maps $E \to F$. Hence, if $u \in L_{B\Gamma}(E, F)$, we have

$$p_F[u(x)] \leq \|u\|_\Gamma p_E(x) \quad \text{if } x \in E \text{ and } p \in \Gamma. \tag{10}$$

We shall always regard $L_{B\Gamma}(E, F)$, the space of all $B\Gamma$-bounded linear maps $E \to F$, as a normed space equipped with the above norm.

(1.6) PROPOSITION. *If F is sequentially complete, $L_\Gamma(E, F)$ is sequentially complete and $L_{B\Gamma}(E, F)$ is a Banach space.*

If we denote by $L(E, F)$ the space of all continuous linear maps $E \to F$ equipped with the topology of uniform convergence on bounded sets, then we have the relation

$$L_{B\Gamma}(E, F) \subseteq L_\Gamma(E, F) \subseteq L(E, F). \tag{11}$$

$L_{B\Gamma}(E, F)$ is, in general, a small subset of $L(E, F)$. However, the following fact shows that $L(E, F)$ is covered by $L_{B\Gamma}(E, F)$.

(1.7) PROPOSITION. *Let E, F be LCS's and $u \in L(E, F)$. Then there exists a calibration Γ for $E \times F$ such that $u \in L_{B\Gamma}(E, F)$.*

Note that in Proposition (1.7) the calibration Γ for which

$u \in L_{B\Gamma}(E, F)$ depends heavily on the given map u. If we specify a particular calibration on E or F, then $L_{B\Gamma}(E, F)$ does not cover $L(E, F)$. Yamamuro has given several examples of such a character (see [82]).

2. Γ-Family

Let $\mathcal{E}$ be a family of LCS's and Γ be a calibration for $\mathcal{E}$. As we have seen in §1, the family $\mathcal{N}$ of all normed spaces is equipped with a single calibration λ (called the *norm calibration*). Hence, it is possible to extend each $p \in \Gamma$ to a semi-norm map $\hat{p}$ on $\mathcal{N} \cup \mathcal{E}$ by putting

$$\hat{p}_E = \begin{cases} p_E & \text{if } E \in \mathcal{E}, \\ \lambda_E & \text{if } E \in \mathcal{N}. \end{cases} \tag{1}$$

Next, let $E, F \in \mathcal{E}$. Then as we have seen in §1, each space $L_\Gamma(E, F)$ is equipped with the calibration $\Gamma_{(E,F)}$ which is uniquely determined by Γ. Hence, each $p \in \Gamma$ can be extended over $L_\Gamma(E, F)$.

Now let $E \in \mathcal{E}$ and F be a linear subspace of E. Then, for each $p \in \Gamma$, the restriction $p_E|_F$ of p_E on F is a semi-norm on F. Then the pair $(F, \Gamma_E|_F)$ where

$$\Gamma_E|_F = \{p_E|_F \mid p \in \Gamma\} \tag{2}$$

is uniquely determined.

From the above remarks, we can now give a convenient definition:

A family $\mathcal{E}$ of LCS's is called a Γ-*family* if Γ is a calibration for $\mathcal{E}$ (see §1) and the following conditions are satisfied:

(i) $\mathcal{N} \subset \mathcal{E}$ and $p_E = \lambda_E$ for every $E \in \mathcal{N}$ and $p \in \Gamma$;

(ii) if $E, F \in \mathcal{E}$, then $G = L_\Gamma(E, F) \in \mathcal{E}$ and $p_G = p_{(E,F)}$;

(iii) if $E \in \mathcal{E}$ and F is a linear subspace of E, then the space F calibrated by $\Gamma_E|_F$ (defined by (2)) belongs to $\mathcal{E}$ and $p_F = p_E|_F$.

The members of a Γ-family $\mathcal{E}$ are thus the pairs (E, Γ_E) consisting of $E \in \mathcal{E}$ and the E-component Γ_E of Γ. We often call these members *objects* of $\mathcal{E}$. If (F, Γ_F) is another object of $\mathcal{E}$ then we define the *morphisms*, which we shall call Γ-*morphisms*, as Γ-continuous linear maps $E \to F$. $B\Gamma$-bounded linear maps $E \to F$ will be called $B\Gamma$-*morphisms*. The Γ-*isomorphisms* and $B\Gamma$-*isomorphisms* are then naturally defined.

When $\mathcal{E}$ is a Γ-family we shall frequently write $E \in \mathcal{E}$ to denote that (E, Γ_E) is an object of $\mathcal{E}$.

We define the Γ-*products* of members of $\mathcal{E}$ as follows. Let $E, F \in \mathcal{E}$. Then the product $E \times F$ may or may not belong to $\mathcal{E}$. If it does, and moreover, the projections

$$\pi_E : E \times F \to E \quad \text{and} \quad \pi_F : E \times F \to F \tag{3}$$

and the embeddings:

$$i_E : E \to E \times F \quad \text{and} \quad i_F : F \to E \times F \tag{4}$$

are Γ-morphisms, then we call $E \times F$ a Γ-*product* and denote it by $E \times_\Gamma F$. This definition can be generalised in an obvious way to the Γ-products of more than two spaces.

When the projections (3) and embeddings (4) are $B\Gamma$-morphisms, the Γ-product is called a $B\Gamma$-*product* and is denoted by $E \times_{B\Gamma} F$. We note that if $E, F \in N$ then the product $E \times F$ is always a $B\Gamma$-product.

A Γ-family $\mathcal{E}$ is said to be *a* Γ-*family with* Γ-*product* iff for all $E, F \in \mathcal{E}$, the Γ-product $E \times_\Gamma F$ exists and belongs to $\mathcal{E}$. A similar definition holds for a Γ-*family with* $B\Gamma$-*product*.

Now let E_1 and E_2 be linear subspaces of $E \in \mathcal{E}$ and E be a direct sum of E_1 and E_2. By the assumption, $(E_i, \Gamma_E|_{E_i})$, $i = 1, 2$, belong to $\mathcal{E}$ and hence the embedding maps

$$E_i \to E \quad (i = 1, 2)$$

are always $B\Gamma$-morphisms. However, the projections

$$E \to E_i \quad (i = 1, 2)$$

are not necessarily Γ-morphisms. If they are, we shall call the direct sum a Γ-*direct sum* and denote it by $E_1 \oplus_\Gamma E_2$. The $B\Gamma$-*direct sum* $E_1 \oplus_{B\Gamma} E_2$ is defined similarly.

Let $\mathcal{E}$ be a Γ-family and let $E, F, G \in \mathcal{E}$.

(2.1) PROPOSITION. *The evaluation map*

$$L_\Gamma(E, F) \times_\Gamma E \to F : (u, x) \mapsto u(x)$$

(respectively $L_{B\Gamma}(E, F) \times_{B\Gamma} E \to F : (u, x) \mapsto u(x)$ *) is* Γ-*continuous (respectively* $B\Gamma$-*continuous).*

(2.2) PROPOSITION. *Let* $E = E_1 \times_\Gamma E_2$. *Then a bilinear map* $u : E \to F$ *is* Γ-*continuous at one point (hence everywhere) iff for any* $p \in \Gamma$, *there exists a positive constant* γ_p *such that*

$$p_F[u(x_1, x_2)] \leq \gamma_p p_{E_1}(x_1) p_{E_2}(x_2) \qquad (5)$$

for all $(x_1, x_2) \in E$ *and all* $p \in \Gamma$.

If the product is a $B\Gamma$-*product (i.e.* $E = E_1 \times_{B\Gamma} E_2$ *), then the bilinear map* u *is* $B\Gamma$-*continuous iff the same inequality holds with* $\gamma > 0$ *independent of* p:

$$p_F[u(x_1, x_2)] \leq \gamma p_{E_1}(x_1) p_{E_2}(x_2), \quad \forall (x_1, x_2), \quad \forall p \in \Gamma. \qquad (6)$$

We shall denote by $L^2_\Gamma(E_1 \times_\Gamma E_2, F)$ (respectively $L^2_{B\Gamma}(E_1 \times_{B\Gamma} E_2, F)$) the space of all Γ-continuous (respectively $B\Gamma$-continuous) bilinear maps of $E_1 \times_\Gamma E_2$ (respectively $E_1 \times_{B\Gamma} E_2$) into F. We shall regard

$L_\Gamma^2(E_1 \times_\Gamma E_2, F)$ as a LCS whose calibration consists of seminorms p defined by:

$$p(u) = \sup \{p_F[u(x_1, x_2)] \mid p_{E_i}(x_i) \le 1,\ i = 1, 2\} \tag{7}$$

for all $u \in L_\Gamma^2(E_1 \times_\Gamma E_2, F)$ and $p \in \Gamma$.

We regard $L_{B\Gamma}^2(E_1 \times_{B\Gamma} E_2, F)$ as a normed space with the norm:

$$\|u\|_\Gamma = \sup_{p \in \Gamma} p(u) \quad \text{for all} \quad u \in L_{B\Gamma}^2(E_1 \times_{B\Gamma} E_2, F) . \tag{8}$$

(2.3) PROPOSITION. *$L_\Gamma^2(E \times_\Gamma F, G)$ is Γ-isomorphic to $L_\Gamma(E, L_\Gamma(F, G))$ by the correspondence*

$$L_\Gamma^2(E \times_\Gamma F, G) \ni u \longmapsto (x \longmapsto u_x) \in L_\Gamma(E, L_\Gamma(F, G)) . \tag{9}$$

Similarly, $L_{B\Gamma}^2(E \times_{B\Gamma} F, G)$ is $B\Gamma$-isomorphic to $L_{B\Gamma}(E, L_{B\Gamma}(F, G))$ by the same correspondence (9).

3. Γ-Differentiation

Let $\mathcal{E}$ be a Γ-family and $E, F \in \mathcal{E}$. Let U be an open subset of E.

For maps $f : U \to F$ and $u : E \to F$, we put

$$r_u(f, a, x) = f(a+x) - f(a) - u(x) \tag{1}$$

when $a, a+x \in U$.

A map $f : U \to F$ is said to be Γ-*differentiable at* $a \in U$ if there exists $u \in L_\Gamma(E, F)$ such that the following condition is satisfied: for any $p \in \Gamma$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$p_F[r_u(f, a, x)] < \varepsilon p_E(x) \quad \text{whenever} \quad p_E(x) < \delta \quad \text{and} \quad a+x \in U . \tag{2}$$

If this is the case, the Γ-morphism u is uniquely determined; we shall call it the *Γ-derivative of* f *at* a and denote it by $f'(a)$.

If f is Γ-differentiable at every point of U , we say that f is *Γ-differentiable on* U , and then we have a map

$$f' : U \subseteq E \to L_{\Gamma}(E, F) \tag{3}$$

which again maps an open subset of an object $E \in \mathcal{E}$ into another object $L_{\Gamma}(E, F) \in \mathcal{E}$. If f' is Γ-continuous at $a \in U$, then f is said to be *continuously Γ-differentiable at* a or C^1_{Γ} *at* a . If f is C^1_{Γ} at every point of U , it is called a C^1_{Γ} *map* of U into F . Similarly, we can define C^k_{Γ} *maps of* U *into* F and the set of all such maps will be denoted by $C^k_{\Gamma}(U, F)$. The set $C^{\infty}_{\Gamma}(U, F)$ is the intersection of all $C^k_{\Gamma}(U, F)$ with respect to k , where as $C^0_{\Gamma}(U, F)$ consists of all Γ-continuous maps of U into F .

There is a corresponding notion of differentiability when we replace the LCS $L_{\Gamma}(E, F)$ by the normed space $L_{B\Gamma}(E, F)$ throughout the above definitions. It is the "$B\Gamma$-bounded" version of the Γ-differentiability.

A map $f : U \subseteq E \to F$ is said to be *BΓ-differentiable at* $a \in U$ if there exists $u \in L_{B\Gamma}(E, F)$ such that the same condition for the Γ-differentiability holds. f is called *continuously BΓ-differentiable at* $a \in U$ iff furthermore, the map

$$f' : U \subseteq E \to L_{B\Gamma}(E, F) \tag{4}$$

is Γ-continuous at a . Such a map is also called $C^1_{B\Gamma}$ *at* a . The set of all $C^1_{B\Gamma}$ maps of U into F will be denoted by $C^1_{B\Gamma}(U, F)$.

Repeating this process, we obtain $C^k_{B\Gamma}(U, F)$ and we put

$$C^{\infty}_{B\Gamma}(U, F) = \bigcap_{k=0}^{\infty} C^{k}_{B\Gamma}(U, F) . \tag{5}$$

Thus, whenever we deal with the $B\Gamma$-differentiability, the derivatives are $B\Gamma$-bounded linear maps (called the *$B\Gamma$-derivatives* to be distinguished from the previous Γ-derivatives) and the continuity of $B\Gamma$-derivatives is as maps into the normed space $L_{B\Gamma}(E, F)$.

The following three propositions are obvious.

(3.1) PROPOSITION. *If* $f : U \to F$ *is* Γ*-differentiable at* $a \in U$ *, it is* Γ*-continuous at* a .

(3.2) PROPOSITION. $L_{\Gamma}(E, F) \subseteq C^{\infty}_{\Gamma}(E, F)$ *and* $L_{B\Gamma}(E, F) \subseteq C^{\infty}_{B\Gamma}(E, F)$.

For $u \in L_{\Gamma}(E, F)$ *, we have*

$$u'(x) = u \text{ for all } x \in E \tag{6}$$

and

$$u^{(k)}(x) = 0 \text{ for all } x \in E \text{ and } k \geq 2 . \tag{7}$$

(3.3) PROPOSITION. $L^{2}_{\Gamma}\left(E_{1} \times_{\Gamma} E_{2}, F\right) \subseteq C^{\infty}_{\Gamma}\left(E_{1} \times_{\Gamma} E_{2}, F\right)$ *and* $L^{2}_{B\Gamma}\left(E_{1} \times_{B\Gamma} E_{2}, F\right) \subseteq C^{\infty}_{B\Gamma}\left(E_{1} \times_{B\Gamma} E_{2}, F\right)$.

For $u \in L^{2}_{\Gamma}\left(E_{1} \times_{\Gamma} E_{2}, F\right)$ *we have:*

$$u'(a, b) = u_{a} + u_{b} \tag{8}$$

where $u_{a} : y \longmapsto u(a, y)$ *and* $u_{b} : x \longmapsto u(x, b)$.

Let us recall some definitions given in [80], [81]: let U, E, F be as above. Then $f : U \subseteq E \to F$ is said to be *Fréchet differentiable at* $a \in U$ (or better *boundedly differentiable at* $a \in U$) if there exists $u \in L(E, F)$ (see §1) such that

$$\varepsilon^{-1} r_{u}(f, a, \varepsilon x) \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly on each bounded set;} \tag{9}$$

that is,

$$\lim_{\varepsilon \to 0} \sup_{x \in B} p\left[\varepsilon^{-1} r_u(f, a, \varepsilon x)\right] = 0 \tag{10}$$

for any bounded subset B and for any $p \in P(F)$. The properties of this differentiation have been investigated in [80] in detail. $f : U \to F$ is said to be *Gâteaux differentiable at* $a \in U$ if there exists $u \in L(E, F)$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} r_u(f, a, \varepsilon x) = 0 \tag{11}$$

for each $x \in E$. In both cases, we denote u by $f'(a)$ which stands for the *bounded derivative* (or *Fréchet derivative*) *at* a for the first case and the *Gâteaux derivative at* a for the second case.

It is immediate that bounded differentiability implies Gâteaux differentiability. The following result gives us a relationship between the Γ-differentiability and the bounded differentiability:

(3.4) PROPOSITION. *Let* $k \geq 0$ *be an integer. Then if* $f : U \subseteq E \to F$ *is* k*-times* Γ*-differentiable at* $a \in U$, *it is* k*-times boundedly differentiable at* a *with the same derivative.*

If f *is* C^k_Γ *at* $a \in U$, *it is* k*-times continuously boundedly differentiable at* a.

One of the immediate consequences of Proposition (3.4) is the following

(3.5) THEOREM (The Mean Value Theorem). *Let* $f : U \subseteq E \to F$ *be* Γ*-differentiable on* U. *Then, for each* $p \in \Gamma$ *and* $x \in E$ *such that* $a+\xi x \in U$ *for all* $\xi \in [0, 1]$, *there exists* $\theta \in (0, 1]$ *such that*

$$p_F[f(a+x)-f(a)] \leq p[f'(a+\theta x)]p_E(x) . \tag{12}$$

If f *is* BΓ*-differentiable on* U, *then*

$$p_F[f(a+x)-f(a)] \leq \|f'(a+\theta x)\|_\Gamma p_E(x) . \tag{13}$$

Using this Mean Value Theorem, one can prove the following fundamental fact.

(3.6) THEOREM. *Assume that* U *is convex, open in* E , $f : U \to F$ *is Gâteaux differentiable on* U *and the Gâteaux derivative* $f'(x)$ *belongs to* $L_\Gamma(E, F)$ *for each* $x \in U$. *If the map* $f' : U \to L_\Gamma(E, F)$ *is* Γ*-continuous at* $a \in U$, *then* f *is* Γ*-differentiable at* a *with the same derivative.*

If, furthermore, $f'(x) \in L_{B\Gamma}(E, F)$ *for each* $x \in U$ *and* $f' : U \to L_{B\Gamma}(E, F)$ *is* Γ*-continuous at* a , f *is* $B\Gamma$*-differentiable at* a.

The following proposition is the main tool for the proof of the chain rules.

(3.7) PROPOSITION. *The composition map*

$$\text{comp} : L_\Gamma(E, F) \times_\Gamma L_\Gamma(F, G) \to L_\Gamma(E, G)$$

(respectively $\text{comp} : L_{B\Gamma}(E, F) \times L_{B\Gamma}(F, G) \to L_{B\Gamma}(E, G)$ *) is a* C^∞_Γ*-map (respectively a* $C^\infty_{B\Gamma}$*-map).*

(3.8) PROPOSITION (Chain Rules). *Let* $\mathcal{E}$ *be a* Γ*-family,* $E, F, G \in \mathcal{E}$ *and* $U \subseteq E$, $V \subseteq F$ *be open. Let* $k \geq 0$ *be an integer. Then if* $f : U \to V \subseteq F$ *is* C^k_Γ *(respectively* $C^k_{B\Gamma}$ *) at* $a \in U$, $g : V \to G$ *is* C^k_Γ *(respectively* $C^k_{B\Gamma}$ *) at* $f(a) \in V$, *the composite* $g \circ f : U \to G$ *is* C^k_Γ *(respectively* $C^k_{B\Gamma}$ *) at* $a \in U$.

If we denote by $GL_{B\Gamma}(E, F)$, $(E, F \in \mathcal{E})$, the set of all $B\Gamma$-isomorphisms of E onto F , then the following fact is well-known because $GL_{B\Gamma}(E, F)$ and $L_{B\Gamma}(E, F)$ are normed spaces.

(3.9) PROPOSITION. *If* F *is sequentially complete, then* $GL_{B\Gamma}(E, F)$ *is open in* $L_{B\Gamma}(E, F)$ *and the inverse operation* $u \mapsto u^{-1}$ *is* $C^\infty_{B\Gamma}$ *on* $GL_{B\Gamma}(E, F)$.

We now add basic properties of the partial derivatives. Let $\mathcal{E}$ be a

Γ-family; $E_1, E_2, F \in \mathcal{E}$ and suppose $E = E_1 \times_\Gamma E_2 \in \mathcal{E}$. Let $U = U_1 \times U_2$ where U_i is an open subset of E_i for $i = 1, 2$. Then a map $f : U \to F$ is said to be *partially* Γ-*differentiable at* $(a_1, a_2) \in U$ *with respect to the first variable* if the partial map $f_{a_2} : U_1 \to F$ defined by

$$f_{a_2}(x) = f(x, a_2) \tag{14}$$

of U_1 into F is Γ-differentiable at a_1. The derivative will be denoted by $\partial_1 f(a_1, a_2)$, which is an element of $L_\Gamma(E_1, F)$. In the same way the partial Γ-derivative $\partial_2 f(a_1, a_2)$ of f at (a_1, a_2) can be defined. By repeating the process we can also define higher partial Γ-derivatives.

If the LCS E is a $B\Gamma$-product, $E = E_1 \times_{B\Gamma} E_2$, then we can define *partial* $B\Gamma$-*derivatives* $\partial_1 f(a_1, a_2)$ and $\partial_2 f(a_1, a_2)$. They belong respectively to $L_{B\Gamma}(E_1, F)$ and $L_{B\Gamma}(E_2, F)$.

(3.10) PROPOSITION. *Let* $E = E_1 \times_\Gamma E_2$ *and* $U = U_1 \times U_2$ *for open subsets* U_i *of* E_i $(i = 1, 2)$. *Then, if* f *is* Γ-*differentiable at* $a = (a_1, a_2) \in U$, *then* $\partial_1 f$ *and* $\partial_2 f$ *exist at* a *and*

$$f'(a)(x) = \partial_1 f(a)(x_1) + \partial_2 f(a)(x_2) \tag{15}$$

for all $x = (x_1, x_2) \in E$.

If, moreover, U *is convex, then* f *is* C^1_Γ *at* a *iff* $\partial_1 f$ *and* $\partial_2 f$ *are* Γ-*continuous at* a.

The $B\Gamma$-*versions are valid if* E *is a* $B\Gamma$-*product.*

We now consider the case of a mapping in a product. Let $F = F_1 \times_\Gamma F_2$ and let $f : U \subseteq E \to F$, where U is open in $E \in \mathcal{E}$. Then f can be written in the partial maps as follows:

$$f(x) = \big(f_1(x), f_2(x)\big) \ , \quad \forall x \in U \ , \tag{16}$$

where $f_i : U \to F_i$ for $i = 1, 2$.

(3.11) PROPOSITION. *Let* $f : U \to F = F_1 \times_\Gamma F_2$ *as above and let* $k \geq 0$ *be an integer. Then* f *is* k*-times* Γ*-differentiable or* C^k_Γ *at* a *iff* f_i $(i = 1, 2)$ *are* k*-times* Γ*-differentiable or* C^k_Γ *respectively at* a .

The $B\Gamma$*-version is valid if* F *is a* $B\Gamma$*-product.*

The following two theorems are the most important in $B\Gamma$-differentiation theory and shall be used later.

(3.12) THEOREM (Inverse Mapping Theorem). *Let* $f : U \subseteq E \to F$ *be as usual. Assume that* E *is sequentially complete,* $f \in C^k_{B\Gamma}(U, F)$ *and* $f'(a)$ *is a* $B\Gamma$*-isomorphism for some* $a \in U$. *Then,* f *is a local* $C^k_{B\Gamma}$*-diffeomorphism at* a .

(3.13) THEOREM (Implicit Function Theorem). *Let* $U = U_1 \times U_2$ *with* U_i *open in* E_i ; $f : U \to F$. *Suppose that* F *is sequentially complete and* $E = E_1 \times E_2$ *be a* $B\Gamma$*-product. Let* $f \in C^k_{B\Gamma}(U, F)$, $f(a_1, a_2) = 0$ *and* $\partial_2 f(a_1, a_2)$ *be a* $B\Gamma$*-isomorphism of* E_2 *onto* F . *Then, there is an open neighbourhood* $\Omega_1 \times \Omega_2$ *of* (a_1, a_2) *and also a map* $g \in C^k_{B\Gamma}(\Omega_1, \Omega_2)$ *such that* $g(a_1) = a_2$ *and*

$$f^{-1}(0) \cap (\Omega_1 \times \Omega_2) = \{(x, g(x)) \mid x \in \Omega_1\} \ . \tag{17}$$

If this is the case,

$$g'(x) = -\left[\partial_2 f(a_1, a_2)^{-1}\right] \circ \partial_1 f(a_1, a_2) \ . \tag{18}$$

4. A Criterion for Γ-Differentiability

In this section we give an useful criterion for the Γ-differentiability of mappings $f : U \subseteq E \to F$ where the calibrations Γ_E and Γ_F consist of families of *norms* $\{p_E \mid p \in \Gamma\}$ and $\{p_F \mid p \in \Gamma\}$ respectively. The criterion simplifies very much when the calibration Γ_E is an increasing sequence of norms: $\Gamma_E = \{\|\cdot\|_r\}_{r=0,1,2,\ldots}$ and the calibration Γ_F is a single norm. In this case, we have even $B\Gamma$-differentiability. The results shall be used in the next two sections.

Let $\mathcal{E}$ be a Γ-family. Let $E, F \in \mathcal{E}$ and suppose that for each $p \in \Gamma$ the semi-norm $p_E \in \Gamma_E$ (respectively $p_F \in \Gamma_F$) is a *norm*. Thus we have:

$$\Gamma_E = \{p_E \mid p \in \Gamma\} = \text{family of norms}, \tag{1}$$

$$\Gamma_F = \{p_F \mid p \in \Gamma\} = \text{family of norms}. \tag{2}$$

Now let $U \subseteq E$ be open and let $f : U \subseteq E \to F$ be a map. For each $p \in \Gamma$, consider the normed spaces

$$E_p = (E, p_E) \text{ and } F_p = (F, p_F) . \tag{3}$$

Then we say that f is *p-differentiable at* $a \in U$ iff $f : U \subseteq E_p \to F_p$ is differentiable in the usual sense of mapping between normed spaces. Similarly, the notion of C_p^k for f is defined for any integer $k \geq 1$ or ∞. We shall denote by $f_p^{(r)}(a)$ the rth p-derivative of f at $a \in U$ and by

$$f_p^{(r)} : U \subseteq E_p \to L_s^r(E_p, F_p) \tag{4}$$

the rth p-derivative of f, for $0 \leq r \leq k$.

If for each $p \in \Gamma$, $L(E_p, F_p)$ denotes the space of linear continuous maps from the normed space E_p into the normed space F_p, then we have immediately:

$$L_\Gamma(E, F) = \bigcap_{p\in\Gamma} L\left(E_p, F_p\right) . \tag{5}$$

The following theorem, suggested to me by Dr Yamamuro, is an useful criterion for Γ-differentiability.

(4.1) THEOREM. *Let* r *be an integer greater than or equal to* 1 *or* $+\infty$ *and let* $E, F \in \mathcal{E}$ *with the condition that* $\Gamma_E = \{p_E \mid p \in \Gamma\}$ *and* $\Gamma_F = \{p_F \mid p \in \Gamma\}$ *are families of norms. Let* $U \subseteq E$ *be open and consider a map* $f : U \subseteq E \to F$.

Then f *is* C^r_Γ *iff the following conditions are satisfied:*

(a) for every $p \in \Gamma$, f *is* C^r_p ;

(b) for $k = 1, \ldots, r$, *we have* $f_p^{(k)} = f_q^{(k)}$ *for all* $p, q \in \Gamma$.

Condition (b) can be dropped if F *is sequentially complete.*

Proof. NECESSITY: We first prove the case $r = 1$ and then use induction on r .

(i) Case $r = 1$: Suppose $f : U \subseteq E \to F$ is C^1_Γ. Then by definition, f is Γ-differentiable at every $a \in U$ and the Γ-derivative $f' : U \subseteq E \to L_\Gamma(E, F)$ is Γ-continuous. Hence for each $a \in U$, there is $f'(a) \in L_\Gamma(E, F)$ such that for all $\varepsilon > 0$ and $p \in \Gamma$ we can find a $\delta > 0$ such that

$$\left(p_E(x) < \delta \text{ and } a+x \in U\right) \Rightarrow \left(p_F[f(a+x)-f(a)-f'(a)x] < \varepsilon p_E(x)\right) . \tag{6}$$

We want to prove *(a)* and *(b)* for the case $r = 1$.

To do this, let $p \in \Gamma$. Then by (5), we have $f'(a) \in L\left(E_p, F_p\right)$, and for all $\varepsilon > 0$, there is a $\delta > 0$ (given above) such that (6) holds. This means $f' : U \subseteq E_p \to F_p$ is p-differentiable at $a \in U$, and the

p-derivative is

$$f_p'(a) = f'(a) \ . \tag{7}$$

Since p_E, p_F are norms, the p-derivative is unique. Thus (7) is meaningful and we also have

$$f_p' = f_q' \quad \text{for all} \quad p, q \in \Gamma \ . \tag{8}$$

It remains to prove that $f_p' : U \subseteq E_p \to L(E_p, F_p)$ is p-continuous. But since $f_p' = f'$ and since f' is Γ-continuous this follows quickly.

(ii) General Case: Suppose that the necessary condition is true for $r \geq 1$, we want to prove it for $r + 1$, that is, we want to prove if $f \in C_\Gamma^{r+1}$ then

(a) for every $p \in \Gamma$, f is C_p^{r+1},

(b) for every $k = 1, \ldots, r+1$, $f_p^{(k)} = f_q^{(k)}$ for all $p, q \in \Gamma$.

Indeed, by hypothesis, the Γ-derivative $f' : U \subseteq E \to L_\Gamma(E, F)$ is C_Γ^r, where the calibration of $L_\Gamma(E, F) = \tilde{F} \in \mathcal{E}$ is:

$$\Gamma_{\tilde{F}} = \{p_{(E,F)} \mid p \in \Gamma\} = \text{family of norms.} \tag{9}$$

Thus, by induction hypothesis we have:

$$\text{for all} \quad p \in \Gamma \ , \quad f' : U \subseteq E_p \to \left(L_\Gamma(E, F), p_{(E,F)}\right) \ \text{is} \ C_p^r ; \tag{10}$$

$$f_p'^{(k)} = f_q'^{(k)} \quad \text{for all} \quad p, q \in \Gamma \ \text{and all} \ k = 1, 2, \ldots, r \ . \tag{11}$$

Furthermore, by the above part (i), the case $r = 1$ is true, we also have:

$$f : U \subseteq E_p \to F_p \ \text{is} \ C_p^1 ; \tag{12}$$

thus (10) and (12) give (a); (11) gives (b) as desired.

SUFFICIENCY: We first prove the case $r = 1$, and then proceed by induction on r.

(i) Case $r = 1$: Suppose we have *(a)*, *(b)* of the theorem for the case $r = 1$. We want to prove that $f : U \subseteq E \to F$ is C^1_Γ .

Let $a \in U$. Then by (a), for all $p \in \Gamma$, there is a $f'_p(a) \in L(E_p, E_p)$ such that for all $\varepsilon > 0$, we can find $\delta > 0$ verifying:

$$\left(p_E(x) < \delta,\ a+x \in U\right) \Rightarrow \left(p_F[f(a+x)-f(a)-f'_p(a).x] < \varepsilon p_E(x)\right) . \tag{13}$$

Then by (b) we have $f'_p(a) = f'_q(a)$ for all $p, q \in \Gamma$. Thus we can define a linear map $f'(a) \in L_\Gamma(E, F)$ by putting:

$$f'(a).x = f'_p(a).x \quad \text{for all} \quad x \in U \tag{14}$$

where p is any norm in Γ . The fact that $f'(a) \in L_\Gamma(E, F)$ follows from (5).

Thus for all $\varepsilon > 0$ and $p \in \Gamma$ we can find $\delta > 0$ (given above) such that (13) holds. This means that f is Γ-differentiable at $a \in U$ and has Γ-derivative $f'(a)$ equal to the p-derivative $f'_p(a)$ for all $p \in \Gamma$.

It remains to prove that $f' : U \subseteq E \to L_\Gamma(E, F)$ is Γ-continuous. Since $f' = f'_p$ for all $p \in \Gamma$ and since f'_p is p-continuous for all $p \in \Gamma$, this follows quickly.

(ii) General Case. Suppose the sufficient condition is true for $r > 1$. We want to prove it for $r + 1$. That is, we want to prove that conditions (a), (b) for $r + 1$ imply that f is C^{r+1}_Γ .

First note that the proof for $r = 1$ gives us f is C^1_Γ . Now consider the Γ-derivative $f' : U \subseteq E \to L_\Gamma(E, F)$. Then conditions (a), (b) give:

$$\text{for all } p \in \Gamma ,\ f'_p : U \subseteq E_p \to L\left(E_p, F_p\right) \text{ is } C^r_p ; \tag{15}$$

$$f'^{(k)}_p = f'^{(k)}_q \quad \text{for all} \quad k = 1, 2, \ldots, r \text{ and all } p, q \in \Gamma . \tag{16}$$

By induction hypothesis, (15) and (16) give:

$$f' : U \subseteq E \to L_\Gamma(E, F) \quad \text{is} \quad C^r_\Gamma . \tag{17}$$

Thus, since f is already C^1_Γ, (17) implies that f is C^{r+1}_Γ as desired.

Now suppose that F is sequentially complete. We want to prove that condition (a) in the theorem implies (b).

First prove the case $r = 1$: suppose $f : U \subseteq E \to F$ is C^1_p for all $p \in \Gamma$ and let $a \in U$. Then if $(\alpha_m > 0)$ is a sequence of positive numbers converging to 0, we have for each $x \in E$ and each $p \in \Gamma$:

$$p_F\left[\frac{f(a+\alpha_m x)-f(a)}{\alpha_m} - f'_p(a).x\right] \to 0 \quad \text{when} \quad m \to +\infty . \tag{18}$$

From this, it follows quickly that:

$$f'_p(a).x = \lim_{m\to\infty} \frac{f(a+\alpha_m x)-f(a)}{\alpha_m} \quad (\text{since } p_F \text{ is a norm}) \tag{19}$$

and

$$\left\{\frac{f(a+\alpha_m x)-f(a)}{\alpha_m}\right\} \quad \text{is a Cauchy sequence in } F . \tag{20}$$

Thus, since F is sequentially complete this Cauchy sequence converges to an unique element in F, which proves:

$$f'_p(a).x = f'_q(a).x \quad \text{for all} \quad x \in E . \tag{21}$$

Since a is arbitrary we have the desired result.

Now suppose that the case $r \geq 1$ is true and let us prove the case $r + 1$. First note that by Proposition (1.6), for each integer j, $L^j_\Gamma(E, F)$ is sequentially complete. Then by the induction hypothesis we have

$$f^{(r)}_p = f^{(r)}_q = f^{(r)} \in L^r(E, F) \quad \text{for all} \quad p, q \in \Gamma \tag{22}$$

and

$$f^{(r)} : U \subseteq E \to L^r_\Gamma(E, F) \equiv \tilde{F} \text{ is } C^1_p \text{ for all } p \in \Gamma . \quad (23)$$

Fix $a \in U$. Then for each $x \in E$ and each $p \in \Gamma$, we have again

$$p_{\tilde{F}}\left[\frac{f^{(r)}\left(a+\alpha_m x\right)-f^{(r)}(a)}{\alpha_m} - f_p^{(r+1)}(a).x\right] \to 0 \text{ as } m \to +\infty \quad (24)$$

where $p_{\tilde{F}}$ is the norm on $L^r_\Gamma(E, F)$ induced by $p \in \Gamma$:

$$p_{\tilde{F}}(u) = \sup\{u\left(x_1, \ldots, x_r\right) \mid p_E\left(x_1\right) \le 1, \ldots, p_E\left(x_r\right) \le 1\} . \quad (25)$$

Thus the same argument as above gives us:

$$f_p^{(r+1)}(a).x = f_q^{(r+1)}(a).x \text{ for all } x \in E \text{ , all } p, q \in \Gamma . \quad // \quad (26)$$

Later we shall have occasion to investigate the $B\Gamma$-differentiability of a map $f : U \subseteq E \to F$ where the calibration Γ_E of E is an increasing sequence of norms $\|\cdot\|_n$ $(n = 0, 1, 2, \ldots)$ and the calibration Γ_F of F is just a single norm $\|\cdot\|_F$.

In this particular case, we have the following criterion for $B\Gamma$-differentiability.

(4.2) COROLLARY. *Let $f : U \subseteq E \to F$ be a map, U being open in E. Suppose that E is calibrated by $\Gamma = \{\|\cdot\|_n\}$, an increasing sequence of norms $(n = 0, 1, 2, \ldots)$ and F is calibrated by the norm-calibration $\|\cdot\|_F$. Suppose furthermore that F is complete with respect to $\|\cdot\|_F$ and for each $n = 0, 1, 2, \ldots$, denote by $E_n = \left(E, \|\cdot\|_n\right)$ the corresponding normed space.*

Let r be an integer greater than or equal to 1 or $+\infty$. Then $f : U \subseteq E \to F$ is $C^r_{B\Gamma}$ iff for all $n = 0, 1, 2, \ldots$, $f : U \subseteq E_n \to F$ is C^r in the usual sense as map between normed spaces.

Proof. Since F is complete, Theorem (4.1) ensures that f is C^r_Γ iff $f : U \subseteq E_n \to F$ is C^r_Γ for all $n = 0, 1, 2, \ldots$.

We claim that for all integers k :

$$L^k_{B\Gamma}(E;\ F) = L^k_{\Gamma}(E;\ F)\ . \tag{27}$$

Indeed, let $u \in L^k_{\Gamma}(E;\ F)$. Then by definition there is a constant $\alpha_0 > 0$ such that for all $x_1, \ldots, x_k \in E$, we have:

$$\|u(x_1,\ \ldots,\ x_k)\|_F \le \alpha_0 \|x_1\|_0\ \ldots\ \|x_k\|_0\ . \tag{28}$$

Since the sequence of norms $\|\cdot\|_n$ is increasing, (28) implies

$$\|u(x_1,\ \ldots,\ x_k)\|_F \le \alpha_0 \|x_1\|_n\ \ldots\ \|x_k\|_n \tag{29}$$

for all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_k \in E$; which proves (27) as claimed.

Thus, since f is C^r_{Γ} and since for all k , $1 \le k \le r$ and all $a \in U$, $D^k f(a) \in L^k_{B\Gamma}(E;\ F)$, the corollary is proved. //

5. The Γ-Omega Lemma

In this section we prove the Γ-version of the ω-lemma in [1] (Corollary 3.8, p. 9). This shall be globalised later in Chapter 2 and shall be used to prove that the space $C^\infty(X, Y)$ of C^∞ maps $X \to Y$ (where X is compact) is a Γ-manifold (see Chapter 2).

Let E, F, G be Banach spaces, $X \subseteq E$ be *compact* and $Y \subseteq F$ be open. Let $C^\infty(X, F)$ be the space of C^∞ maps of X into F . Then, for an integer $i \ge 0$, we have, for each $f \in C^\infty(X, F)$,

$$\sup_{x \in X} \|D^i f(x)\|_{L^i(E;F)} < +\infty\ . \tag{1}$$

For $n = 0, 1, 2, \ldots$ and for $f \in C^\infty(X, F)$, define:

$$\|f\|_n = \sup_{x \in X} \{\|f(x)\| + \|Df(x)\| + \ldots + \|D^n f(x)\|\} < +\infty \tag{2}$$

and let $C^n(X; F)$ denote the Banach space of all C^n maps $X \to F$.

Then define

$$C^\infty(X, F) = \bigcap_{n=0}^{\infty} C^n(X, F) \tag{3}$$

which is regarded as a LCS calibrated by the sequence of increasing norms

$$\Gamma = \{\|\cdot\|_n\}_{n=0,1,2,\ldots}. \tag{4}$$

Let $C^n(X, Y)$ [respectively $C^\infty(X, Y)$] be the subset of all $f \in C^n(X, Y)$ [respectively $f \in C^\infty(X, Y)$] such that $f(X) \subseteq Y$. Then it is clear that $C^n(X, Y)$ is open in $C^n(X, F)$ for each $n = 0, 1, 2, \ldots$ and $C^\infty(X, Y)$ is open in $C^\infty(X, F)$ calibrated by (4).

Let $C^n(X, G)$ and $C^\infty(X, G)$ be the similar spaces, where $C^\infty(X, G)$ is calibrated by a similar sequence of increasing norms

$$\Gamma' = \{\|\cdot\|_n\}_{n=0,1,2,\ldots}. \tag{5}$$

Then we have the following Γ-version of the ω-lemma given in [1], p. 9:

(5.1) PROPOSITION (Γ-omega lemma). *Let E, F, G be Banach spaces, $X \subseteq E$ compact and $Y \subseteq F$ open. Then, for a fixed $g \in C^\infty(Y, G)$, the map*

$$\omega_g \equiv g_* : C^\infty(X, Y) \subseteq C^\infty(X, F) \to C^\infty(X, G) : f \mapsto g_*(f) = g \circ f \tag{6}$$

is C^∞_Γ with respect to the above calibrations (4) *and* (5) *for $C^\infty(X, F)$ and $C^\infty(X, G)$ respectively.*

Proof. We apply Theorem (4.1). Since $C^\infty(X, G)$ is a Fréchet space (see e.g. [34], [50]) it suffices to verify condition (a) of the theorem. To do this, let us put

$$\tilde{U} = C^\infty(X, Y) \; ; \; \tilde{E} = C^\infty(X, F) \; ; \; \tilde{F} = C^\infty(X, G) \; . \tag{7}$$

Then $\tilde{U}$ is open in $\tilde{E}$ and we have $g_* : \tilde{U} \subseteq \tilde{E} \to \tilde{F}$.

For each $n = 0, 1, 2, \ldots$, we put

$$\tilde{E}_n = (\tilde{E}, \|\cdot\|_n) \text{ and } \tilde{F}_n = (\tilde{F}, \|\cdot\|_n) . \tag{8}$$

Then condition (a) means that

$$g_* : \tilde{U} \subseteq \tilde{E}_n \to \tilde{F}_n \tag{9}$$

is C^∞ as map between normed spaces $\tilde{E}_n$ and $\tilde{F}_n$. This in turn follows quickly from Theorem 6 in [35], p. 117. //

(5.2) Remark. In the proof of the Proposition (5.1) we do not need the explicit form of the kth derivatives of g_* $(k = 0, 1, 2, \ldots)$. Actually, using the results in [35], it is not hard to see the following formula for the derivative $D^k g_*(f)$ of g_* at $f \in C^\infty(X, Y)$:

$$\left[D^k g_*(f).\eta_1 \ldots \eta_k\right](x) = D^k g\big(f(x)\big).\eta_1(x) \ldots \eta_k(x) \tag{10}$$

for $\eta_1, \ldots, \eta_k \in C^\infty(X, F)$ and $x \in X$.

(5.3) Remark. Proposition (5.1) still holds if we replace the norm (2) by the following norm

$$\|f\|_n = \max_{0 \le i \le n} \left\{\sup_{x \in X} \|D^i f(x)\|\right\} \tag{11}$$

for each $f \in C^\infty(X, F)$ and each $n = 0, 1, 2, \ldots$.

6. The Evaluation Map

In this section, we prove the $B\Gamma$-differentiability of a kind of evaluation map, the result of which shall be used later in some applications of the $B\Gamma$-Transversal Density Theorem (see Chapter 5).

Let E, F be Banach spaces, $U \subseteq E$ open, convex. Recall that for a nonnegative integer r, $P^r(E, F)$ is the Banach space of polynomials $E \to F$ of degree less than or equal to r (see, e.g. [4], [13]):

$$P^r(E, F) = F \times L(E, F) \times L_s^2(E, F) \times \ldots \times L_s^r(E, F) \tag{1}$$

where $L_s^i(E, F)$ $(2 \le i \le r)$ denotes the space of symmetric i-linear maps $E \to F$.

For each $\xi \in C^r(U, F)$ and each $x \in U$, $P^r\xi(x)$ is the point of $P^r(E, F)$ given by

$$P^r\xi(x) = \left(\xi(x),\ D\xi(x),\ D^2\xi(x),\ \ldots,\ D^r\xi(x)\right) . \tag{2}$$

Now choose the following norm on $P^r(E, F)$:

$$\|(a_0, a_1, \ldots, a_r)\| = \|a_0\| + \|a_1\| + \ldots + \|a_r\| \tag{3}$$

for all $(a_0, a_1, \ldots, a_r) \in P^r(E, F)$, and for each $\xi \in C^r(U, F)$, define

$$\|\xi\|_r = \sup_{x \in U} \{\|P^r\xi(x)\|\} . \tag{4}$$

Let $B^r(U, F)$ denote the space of all $\xi \in C^r(U, F)$ such that $\|\xi\|_r < +\infty$, and put

$$B^\infty(U, F) = \bigcap_{r=0}^{\infty} B^r(U, F) . \tag{5}$$

Now consider the product $B^\infty(U, F) \times E$ and for each $r = 0, 1, 2, \ldots$ define the following norm p_r on $B^\infty(U, F) \times E$,

$$p_r(\xi, x) = \|\xi\|_r + \|x\|_E \quad \text{for all } \xi \in B^\infty(U, F) ,\ x \in E . \tag{6}$$

Fix an integer $r \geq 1$ and regard $B^\infty(U, F) \times E$ as a LCS calibrated by the following sequence of increasing norms:

$$\Gamma = \{p_{r+i}\}_{i=0,1,2,\ldots} \tag{7}$$

and consider the norm-calibration $\|\cdot\|_F$ on F . Then we have the following

(6.1) PROPOSITION. *Let r be an integer greater than or equal to 1 , E, F be Banach spaces and $U \subseteq E$ open. Then the evaluation map*

$$ev : B^\infty(U, F) \times U \subseteq B^\infty(U, F) \times E \to F \tag{8}$$

given by $ev(\xi, x) = \xi(x)$ for $\xi \in B^\infty(U, F)$, $x \in F$, is $C^r_{B\Gamma}$ with respect to the calibration (7) *on $B^\infty(U, F) \times E$ and the norm-calibration on F . Furthermore, for each $k \leq r$, the $B\Gamma$-derivative $D^k ev(\xi, x)$ is given by*

$$D^k ev(\xi, x).(\eta_1, h_1) \dots (\eta_k, h_k)$$

$$= D^k\xi(x).h_1 \dots h_k + \sum_{l=1}^{k} D^{k-1}\eta_l(x).h_1 \dots \hat{h}_l \dots h_k \quad (9)$$

(where $\hat{h}_l$ *means the factor* h_l *is deleted); for* $\xi \in B^\infty(U, F)$ *,* $\eta_i \in B^\infty(U, F)$ $(1 \le i \le k)$ *,* $x \in U$ *and* $h_i \in F$ $(1 \le i \le k)$ *.*

Proof. To prove the $B\Gamma$-differentiability of ev , we apply Corollary (4.2). First, for all $i = 0, 1, 2, \dots$ the mapping

$$\Phi : (B^\infty(U, F) \times U, p_{r+i}) \to \left(B^{r+i}(U, F) \times U, p_{r+i}\right) \quad (10)$$

$$(\xi, x) \longmapsto \Phi(\xi, x) = (\xi, x)$$

considered as map between normed spaces, is the restriction of a linear continuous map and hence of class C^∞ . On the other hand, by Theorem (10.3) in [4], p. 25, the map

$$ev_{r+i} : \left(B^{r+i}(U, F) \times U, p_{r+i}\right) \to (F, \|\cdot\|_F) , \quad (11)$$

$$(\xi, x) \longmapsto ev_{r+i}(\xi, x) = \xi(x)$$

is of class C^{r+i} , *a fortiori* C^r , for all $i = 0, 1, 2, \dots$. Since the composite map $ev_{r+i} \circ \Phi$ is exactly the map ev in (8), we have the first part of the proposition.

For the formula (9), we use the proof of Theorem (10.3) in [4], p. 25: if for each $k = 1, 2, \dots, r$ we denote by $D^k ev_{r+i}(\xi, x)$ the kth derivative of the map (11) at $(\xi, x) \in B^\infty(U, F) \times U$, then

$$D^k ev_{r+i}(\xi, x).(\eta_1, h_1) \dots (\eta_k, h_k)$$

$$= D^k\xi(x).h_1 \dots h_k + \sum_{l=1}^{k} D^{k-1}\eta_l(x).h_1 \dots \hat{h}_l \dots h_k \quad (12)$$

where $\hat{h}_l$ means the factor h_l is deleted, and where $\eta_i \in B^\infty(U, F)$ and $h_i \in F$ for $1 \le i \le k$. Thus (9) follows. //

More generally, we have the following:

(6.2) PROPOSITION. *Let* E, F *be Banach spaces,* $U \subseteq E$ *be open, convex and bounded,* r *be an integer greater than or equal to* 1 *and* k *be an integer* $0 \le k \le r$. *Then the map*

$$ev_k : B^\infty(U, F) \times U \to U \times P^k(E, F) : (\xi, x) \longmapsto \big(x, P^k\xi(x)\big) \tag{13}$$

is $C^r_{B\Gamma}$ *with respect to the calibration* $\Gamma = \{p_{r+k+i}\}_{i=0,1,2,\ldots}$ *on* $B^\infty(U, F) \times E$ *and the norm-calibration on* $E \times P^k(E, F)$.

For each $(\xi, x) \in B^\infty(U, F) \times U$, *the* $B\Gamma$*-derivative*

$$Dev_k(\xi, x) : B^\infty(U, F) \times E \to E \times P^k(E, F) \tag{14}$$

is given by

$$Dev_k(\xi, x).(\zeta, h) = \big(h, \zeta(x)+D\xi(x).h, \ldots, D^k\zeta(x)+D^{k+1}\xi(x).h\big) \tag{15}$$

for $(\zeta, h) \in B^\infty(U, F) \times E$, *and is onto.*

Proof. We can write ev_k as a composite of the following maps:

$$B^\infty(U, F) \times U \xrightarrow{\ \Psi\ } B^\infty\big(U, P^k(E, F)\big) \times U \xrightarrow{\ \Phi\ } U \times P^k(E, F) \tag{16}$$

$$(\xi, x) \longmapsto \big(P^k\xi, x\big) \longmapsto \big(x, P^k\xi(x)\big) .$$

We choose as the calibration for $B^\infty\big(U, P^k(E, F)\big) \times E$ the following sequence of increasing norms:

$$\tilde{\Gamma} = \{\tilde{p}_{r+i}\}_{i=0,1,2,\ldots} \tag{17}$$

defined by

$$\tilde{p}_{r+i}(\zeta, x) = \|\zeta\|_{r+i} + \|x\|_E \quad \text{for all} \quad (\zeta, x) \in B^\infty\big(U, P^k(E, F)\big) \times E \tag{18}$$

where $\|\cdot\|_{r+i}$ is the norm in $B^\infty\big(U, P^k(E, F)\big)$ with respect to the norm (3) in $P^r(E, F)$.

Then the map

$$P^k : B^\infty(U, F) \to B^\infty\big(U, P^k(E, F)\big) : \xi \longmapsto P^k\xi \tag{19}$$

is obviously linear. Furthermore, it can be seen that

$$\|P^k\xi\|_{r+i} \leq (k+1)\|\xi\|_{r+k+i} \quad \text{for all } i = 0, 1, 2, \ldots . \tag{20}$$

Since k is a constant, this proves that P^k is a linear $B\Gamma$-continuous map, hence of class $C^\infty_{B\Gamma}$.

Thus the above composite is $C^r_{B\Gamma}$ since the map

$$ev : B^\infty\big(U, P^k(E, F)\big) \times U \to P^k(E, F) : (\zeta, x) \to \zeta(x) \tag{21}$$

is $C^r_{B\tilde{\Gamma}}$ (with respect to $\tilde{\Gamma}$ defined in (17) and $\|\cdot\|_{P^k(E,F)}$ defined in (3)) by Proposition (6.1).

For each $(\xi, x) \in B^\infty(U, F) \times U$, we have by (16),

$Dev_k(\xi, x).(\zeta, h)$

$= D(\Phi \circ \Psi)(\xi, x).(\zeta, h)$

$= D\Phi\big(P^k\xi, x\big) \circ D\Psi(\xi, x).(\zeta, h)$

$= D\Phi\big(P^k\xi, x\big)\big(P^k\zeta, h\big)$

$= D\big(P^k\xi\big)(x).h + P^k\zeta(x)$ (by (9))

$= \big(D\xi(x).h, D^2\xi(x).h, \ldots, D^{k+1}\xi(x).h\big) + \big(\zeta(x), D\zeta(x), \ldots, D^k\zeta(x)\big)$

$$= \big(\zeta(x)+D\xi(x).h, D\zeta(x)+D^2\xi(x).h, \ldots, D^k\zeta(x)+D^{k+1}\xi(x).h\big) . \tag{22}$$

Thus we have (15) as desired.

We now prove that $Dev_k(\xi, x)$ is onto. Let $\big(h_0, a_0, a_1, \ldots, a_k\big)$ be an arbitrary element in $E \times P^k(E, F)$. We want to find a $(\zeta, h) \in B^\infty(U, F) \times E$ such that

$$\big(h, \zeta(x)+D\xi(x).h, \ldots, D^k\zeta(x)+D^{k+1}\xi(x).h\big) = \big(h_0, a_0, \ldots, a_k\big) . \tag{23}$$

Taking $h = h_0$, then (23) gives

$$\left.\begin{aligned} \zeta(x) + D\xi(x).h_0 &= a_0 \\ D\zeta(x) + D^2\xi(x).h_0 &= a_1 \\ \ldots\ldots& \\ D^k\zeta(x) + D^{k+1}\xi(x).h_0 &= a_k \end{aligned}\right\} \; . \tag{24}$$

That is, we must find $\zeta \in B^\infty(U, F)$ such that

$$\zeta(x) = a_0 - D\xi(x).h_0 = b_0, \ldots, D^k\zeta(x) = a_k - D^{k+1}\xi(x).h_0 = b_k \tag{25}$$

where $b_i \in L_s^i(E, F)$ $(i = 0, 1, \ldots, k)$ are given.

That condition is satisfied if we take ζ defined by

$$\zeta : U \subseteq E \to F : y \mapsto \zeta(y) = b_0 + \frac{b_1}{1!}(y-s) + \ldots + \frac{b_k}{k!}(y-x)^{(k)} \tag{26}$$

Since U is open, convex and bounded, it is easy to see that $\zeta \in B^\infty(U, F)$ and $D^k\zeta(x) = b_i$ for $i = 0, 1, \ldots, k$. //

CHAPTER 2

Γ-MANIFOLDS AND Γ-BUNDLES

In this chapter, we construct Γ-manifolds and Γ-bundles modelled on locally convex spaces using the Γ-differentiation of Yamamuro. The models of a Γ-manifold are open subsets of the members of a Γ-family and the transition maps are supposed to be Γ-differentiable (see Chapter 1, §3). We shall prove that the space of C^∞ maps from a compact manifold X into a (finite-dimensional) manifold Y is a Γ-manifold of class C_Γ^∞. Hence the space $\mathrm{Diff}^\infty(X)$ of C^∞-diffeomorphisms of a compact manifold X, and the space $\mathrm{Emb}^\infty(K, X)$ of C^∞-embeddings of a compact manifold K into a manifold X are both Γ-manifolds.

Corresponding to the notion of $B\Gamma$-differentiability, we have the $B\Gamma$-manifolds (or Γ-manifolds of bounded type). More precisely, $B\Gamma$-manifolds are Γ-manifolds with the requirement that the transition maps are $B\Gamma$-differentiable. We shall give some examples of simple $B\Gamma$-manifolds.

In the last section of this chapter, we shall give a brief exposition of Γ- and $B\Gamma$-bundles and an useful example of a $B\Gamma$-bundle, the $B\Gamma$-bundle $L_{B\Gamma}(\tau_X, \tau_Y)$ of $B\Gamma$-linear maps (see [4] for the Banach case).

1. Γ-manifolds

We follow the treatment of [4].

Let E be a Γ-family (see Chapter 1, §2). Then a *local* Γ-*manifold* is an open subset of a member $E \in E$. A C_Γ^r-*local manifold morphism* is a C_Γ^r map between local Γ-manifolds. These form a category whose isomorphisms are just C_Γ^r-diffeomorphism (see Chapter 1).

Let X be a Hausdorff space. A *Γ-manifold chart* (or simply a *Γ-chart*) on X is a pair (U, α) where U is an open subset of X and α is a homeomorphism from U onto a local Γ-manifold. Two Γ-charts (U, α) and (V, β) are *C^r_Γ-compatible* iff the composition (also called *transition map*)

$$\beta \circ \alpha^{-1} : \alpha(U \cap V) \to \beta(U \cap V)$$

is a local C^r_Γ-manifold isomorphism [i.e. a C^r_Γ-diffeomorphism in the sense of Chapter 1]. A *Γ-atlas of class* C^r_Γ (or a *C^r_Γ-atlas*) on X is a collection of Γ-charts $\{(U, \alpha)\}$ any two of which are C^r_Γ-compatible and such that the U's cover X. A Γ-atlas is *maximal* iff it contains each Γ-chart which is Γ-compatible with all of its members. Clearly, every Γ-atlas extends uniquely to a maximal Γ-atlas.

A *Γ-manifold of class* C^r_Γ (or simply a *C^r_Γ-manifold*) is a Hausdorff topological space X together with a maximal C^r_Γ-atlas on X. As usual, we often suppress notation for the maximal C^r_Γ-atlas on X but simply let X refer ambigously to both the underlying topological space and the maximal Γ-atlas. Instead of saying that a Γ-chart (U, α) is a member of the maximal Γ-atlas, we say that (U, α) is an *admissible* *Γ-chart* on X.

If all the models E_α coincide to a fixed member $E \in \mathcal{E}$ then we have a *pure* *Γ-manifold* (*modelled on* E).

Let X be a C^r_Γ-manifold $(r \geq 1)$ and let x be a point of X. We consider triples (U, α, v) where (U, α) is a Γ-chart at x and v is an element of $E \in \mathcal{E}$ [in which $\alpha(U)$ lies]. Following the standard way (see Lang [44]) we say that to such triples (U, α, v) and (V, β, w) are *equivalent* if the Γ-derivative of $\beta \circ \alpha^{-1}$ at $\alpha(x)$ maps v onto w.

The formula reads

$$\left(\beta \circ \alpha^{-1}\right)'\left(\alpha(x)\right).v = w \; . \tag{1}$$

An equivalence class of such triples forms an entity called a Γ-*tangent vector* of X at x. The set of all these tangent vectors is called the Γ-*tangent space* of X at x, and is denoted by $T_x X$.

Each admissible Γ-chart (U, α) at x determines a bijection of $T_x X$ onto the LCS $E \in \mathcal{E}$ (in which $\alpha(U)$ lies) namely:

$$\Phi_{(U,\alpha)} : T_x X \to E$$

$$\dot{x} = \overline{(U, \alpha, v)} \longmapsto \Phi_{(U,\alpha)}(\dot{x}) = v$$

where $\overline{(U, \alpha, v)}$ denotes the equivalence class of (U, α, v). Furthermore, if (U, α) and (V, β) are two C^r_Γ-compatible Γ-charts at x, then it follows quickly from (1) that for each $p \in \Gamma$, there exist $\gamma_p > 0$ and $\delta_p > 0$ such that if $w = \left(\beta \circ \alpha^{-1}\right)'\left(\alpha(x)\right).v$ and $v = \left(\alpha \circ \beta^{-1}\right)'\left(\beta(x)\right).w$, then

$$\delta_p p_F(w) \le p_E(v) \le \gamma_p p_F(w) \; . \tag{2}$$

Thus, by means of the bijection $\Phi_{(U,\alpha)}$, we can transport to $T_x X$ the LCS structure of E as well as the calibration of E. More precisely, we define the calibration for $T_x X$ by

$$\Gamma_{T_x X} = \{p_{T_x X} \mid p \in \Gamma\} \tag{3}$$

with

$$p_{T_x X}(\dot{x}) = p_E(v) \quad \text{if} \quad \dot{x} = \overline{(U, \alpha, v)} \; . \tag{4}$$

By (2) it follows quickly that all the Γ-notions considered on $T_x X$ remain the same if we define the LCS structure on $T_x X$ via $\Phi_{(U,\beta)}$ when (V, β) is C^r_Γ-compatible with (U, α).

We can define the tangent space at a point $x \in X$ of a Γ-manifold X by another equivalent approach as follows (see [4]).

Let $\mathbb{R}$ be the real line endowed with the standard norm (i.e. the absolute value $|\cdot|$). Then $\mathbb{R} \in E$ by our definition of a Γ-family (see Chapter 1, §2). If X is a Γ-manifold of class C^r_Γ $(r \geq 1)$, then a map $c : I \subseteq \mathbb{R} \to X$ where I is an open interval in $\mathbb{R}$ is said to be of class C^1_Γ if for every $t \in \mathbb{R}$, there is a Γ-chart (U, α) at $c(t) \in X$ such that the map $\alpha \circ c : I \to \alpha(U) \subseteq E$ is C^1_Γ (in the sense of Chapter 1). A C^1_Γ-*curve* in X is a C^1_Γ map from an open interval in $\mathbb{R}$ containing 0 to X. Curves c_1 and c_2 are *tangent at a point* $x \in X$ iff $c_1(0) = c_2(0) = x$ and for some (and hence every) admissible Γ-chart (U, α) at x, we have

$$\left(\alpha \circ c_1\right)'(0).1 = \left(\alpha \circ c_2\right)'(0).1 \tag{5}$$

where $(\alpha \circ c)'(0)$ is the Γ-derivative of $\alpha \circ c$ at 0 .

A C^1_Γ-curve c is called a *curve at* x iff $c(0) = x$. Among the curves c at x , tangency (at x) is an equivalence relation. If we denote by X_x the set of all equivalence classes, then it is easy to see that for each Γ-chart (U, α) , there exists a bijection of X_x onto E (the member of E in which $\alpha(U)$ lies), namely,

$$\Phi_{(U,\alpha)} : X_x \to E \tag{6}$$

$$[c]_x \mapsto (\alpha \circ c)'(0).1$$

and we can identify X_x to T_xX defined in the previous paragraph.

Now let X be a C^r_Γ-manifold and Ω be an open subset of X . Then it is possible, in the obvious way, to induce a C^r_Γ-manifold structure on

Ω , by taking as Γ-charts for Ω the intersections

$$(U \cap \Omega, \alpha|U \cap \Omega) . \tag{7}$$

The open subset Ω with this C^r_Γ-manifold structure is called an *open Γ-submanifold* of X .

More generally, we define the Γ-submanifolds as follows:

Let $\mathcal{E}$ be a Γ-family and let X be a C^r_Γ-manifold. Let $A = \{(U, \alpha)\}$ be the Γ-atlas of X . Let W be a subset of X . Then we say that an admissible Γ-chart $(U, \alpha) \in A$ *has the* Γ-*submanifold property for* W *in* X *at* $x \in W$ if the following conditions are satisfied:

(i) the LCS $E \in \mathcal{E}$ (in which $\alpha(U)$ lies) admits a direct Γ-decomposition $E = E_1 \oplus_\Gamma E_2$ (see Chapter 1) into two closed Γ-splitting subspaces E_1 and E_2 ;

(ii) $\alpha(U) = U_1 + U_2$ where U_1 and U_2 are open neighbourhoods of 0 in E_1 and E_2 respectively;

(iii) $\alpha(x) = 0$ and $\alpha(W \cap U) = U_1 \subseteq E_1$.

It is not hard to see that if every point $x \in W$ has a Γ-chart (U, α) with the above property, then the family

$$A_W = \{(W \cap U, \alpha|W \cap U) \mid (U, \alpha) \in A\} \tag{8}$$

is a Γ-atlas for W. Note that if (V, β) is another such Γ-chart at x , then the transition map

$$\beta \circ \alpha^{-1}|_{\alpha(W \cap U \cap V)} : \alpha(W \cap U \cap V) \subseteq E_1 \to \beta(W \cap U \cap V) \subseteq F_1$$

is C^r_Γ and $E_1, F_1 \in \mathcal{E}$ (since $\mathcal{E}$ is a Γ-family).

The subset W with the above C^r_Γ-manifold structure is called a *Γ-submanifold* of X. Note that for $x \in W$, the tangent space $T_x W$ is a Γ-splitting subspace of $T_x X$.

We now define the C^r_Γ maps between C^r_Γ-manifolds. Let X, Y be C^r_Γ-manifolds and let $f : X \to Y$ be a map. Then we define the *local representative of* f (with respect to the Γ-charts (U, α) and (V, β)) to be the map

$$f_{\alpha\beta} = \beta \circ f \circ \alpha^{-1} : \alpha(U) \subseteq E \to \beta(V) \subseteq F \tag{9}$$

where $E, F \in \mathcal{E}$ are respectively the spaces containing $\alpha(U)$ and $\beta(V)$.

A map $f : X \to Y$ is *of class* C^r_Γ (or more categorically a C^r_Γ-*manifold morphism*) iff for every $x \in X$ and every admissible Γ-chart (V, β) on Y with $f(x) \in V$, there exists an admissible Γ-chart (U, α) on X such that $x \in U$, $f(U) \subseteq V$ and the local representative $f_{\alpha\beta}$ is a local C^r_Γ-manifold morphism (i.e. C^r_Γ in the sense of Chapter 1, §3).

If $f : X \to Y$ is a C^r_Γ map $(r \geq 1)$ then, as usual, it induces a linear map

$$T_x f : T_x X \to T_{f(x)} Y \tag{10}$$

called the Γ-*tangent map* of f at x. In the Γ-charts (U, α) and (V, β), this tangent map is represented by the Γ-derivative

$$f'_{\alpha\beta}(\alpha(x)) : E \to F \tag{11}$$

(see [4] for the Banach case).

2. Examples of Γ-manifolds

In this section we give some examples of Γ-manifolds. Let X be a *compact* C^∞-manifold and let Y be a finite-dimensional C^∞ manifold.

We denote by $C^\infty(X, Y)$ the space of all C^∞ maps from X to Y. As a first example of Γ-manifold, we shall prove that $C^\infty(X, Y)$ is a C^∞_Γ-manifold in the sense of §1.

To do so, we first prove the global version of the Γ-omega lemma in Chapter 1, §5.

Let X be a compact C^∞ manifold and let

$$\pi : E \to X \ , \ \rho : F \to X \tag{1}$$

be two C^∞ (Banach) vector bundles having the same compact base space X. Then a mapping $f : E \to F$ is *fibre-preserving* (see [1]) iff $\rho \circ f = \pi$.

We denote by $S^\infty(\pi)$ and $S^\infty(\rho)$ the spaces of C^∞ sections of π and ρ respectively.

We endow $S^\infty(\pi)$ and $S^\infty(\rho)$ the following calibrations: cover π and ρ by a finite number of pseudo compact *VB* charts $\left(U_i, \alpha_i^0, \alpha_i\right)$ and $\left(U_i, \alpha_i^0, \beta_i\right)$ $(1 \le i \le n)$ where $\left\{\left(U_i, \alpha_i^0\right)\right\}_{1\le i\le n}$ is an atlas of X (see [1], p. 15). Then each $\gamma \in S^\infty(\pi)$ has the following principal part with respect to the *VB*-chart $\left(U_i, \alpha_i^0, \alpha_i\right)$:

$$\tilde{\gamma}_{\alpha_i} : \alpha_i^0\left(\overline{U}_i\right) \to E_{\alpha_i} \quad (1 \le i \le n) \tag{2}$$

with $\tilde{\gamma}_{\alpha_i} \in C^\infty\left(\alpha_i^0\left(\overline{U}_i\right), E_{\alpha_i}\right)$ and $\alpha_i^0\left(\overline{U}_i\right)$ is compact.

For $r = 0, 1, 2, \ldots$, define

$$\|\tilde{\gamma}_{\alpha_i}\|_r = \sup_{x\in\alpha_i^0\left(\overline{U}_i\right)} \left\{\|\tilde{\gamma}_{\alpha_i}(x)\| + \|D\tilde{\gamma}_{\alpha_i}(x)\| + \ldots + \left\|D^r\tilde{\gamma}_{\alpha_i}(x)\right\|\right\} < +\infty \tag{3}$$

and

$$\|\gamma\|_r = \sum_{i=1}^{n} \|\tilde{\gamma}_{\alpha_i}\|_r \quad \text{for} \quad \gamma \in S^\infty(\pi) . \tag{4}$$

Then the set

$$\Gamma = \{\|\cdot\|_r\}_{r=0,1,2,\ldots} \tag{5}$$

is a calibration for $S^\infty(\pi)$.

Similarly, we have the calibration

$$\Gamma' = \{\|\cdot\|_r\}_{r=0,1,2,\ldots} \tag{6}$$

for $S^\infty(\rho)$.

Now if $\Omega \subseteq E$ is an open set such that $\pi|\Omega : \Omega \to X$ is surjective, let $S^\infty(\Omega) \subseteq S^\infty(\pi)$ denote the open set of sections with image in Ω .

If $f : \Omega \subseteq E \to F$ is a C^∞ fibre-preserving map, let

$$f_* : S^\infty(\Omega) \subseteq S^\infty(\pi) \to S^\infty(\rho) \tag{7}$$

denote the composition mapping induced by f :

$$f_*(\gamma) = f \circ \gamma \quad \text{for all } \gamma \in S^\infty(\Omega) . \tag{8}$$

Then the local Γ-omega lemma in Chapter 1 may be globalised as follows (see [1] for the Banach case).

(2.1) LEMMA. *Let* X *be a compact* C^∞ *manifold and let* $\pi : E \to X$, $\rho : F \to X$ *be two* C^∞ *(Banach) vector bundles having the same base space* X . *Let* $f : E \to F$ *be a* C^∞ *fibre-preserving map as above. Then*

$$f_* : S^\infty(\Omega) \subseteq S^\infty(\pi) \to S^\infty(\rho)$$

defined by $f_*(\gamma) = f \circ \gamma$ *for all* $\gamma \in S^\infty(\Omega)$ *is* C^∞_Γ *with respect to the calibrations* (5) *and* (6) *for* $S^\infty(\pi)$ *and* $S^\infty(\rho)$.

Proof. We first prove that f_* is C^1_Γ . Cover π and ρ by a finite number of the pseudocompact VB charts $\left\{\left(U_i, \alpha^0_i, \alpha_i\right)\right\}_{1\le i\le n}$ and $\left\{\left(U_i, \alpha^0_i, \beta_i\right)\right\}_{1\le i\le n}$ as above.

Now for each such pair of VB charts (U, α_0, α) and (U, α_0, β) with $\overline{U} \subseteq V$ and (V, α_0, α), (V, α_0, β) VB-charts (see the definition of pseudocompact charts [4]) we have

$$(f \circ \gamma)_\beta = f_{\alpha\beta} \circ \gamma_\alpha \quad \text{for all } \gamma \in S^\infty(\Omega) \tag{9}$$

where

$$\gamma_\alpha = \alpha \circ \gamma \circ \alpha_0^{-1} : \alpha_0(V) \subseteq G \to E_\alpha , \tag{10}$$

$$(f \circ \gamma)_\beta = \beta \circ (f \circ \gamma) \circ \alpha_0^{-1} : \alpha_0(V) \subseteq G \to F_\beta \tag{11}$$

are local representatives of γ and $f \circ \gamma$, and $f_{\alpha\beta}$ is the local representative of f (G being the member of E in which $\alpha_0(V)$ lies).

Hence

$$(f_*(\gamma))_\beta = f_{\alpha\beta_*}(\gamma_\alpha) \tag{12}$$

with

$$f_{\alpha\beta} : \alpha_0(V) \times E_\alpha \to \alpha_0(V) \times F_\beta$$

and

$$f_{\alpha\beta_*} : C^\infty(\alpha_0(\bar{U}), \alpha_0(V) \times E_\alpha) \to C^\infty(\alpha_0(\bar{U}), \alpha_0(V) \times F_\beta) . \tag{13}$$

Note that $\alpha_0(\bar{U})$ is compact, $\alpha_0(V) \times E_\alpha$ is open in $G \times E_\alpha$ and $\alpha_0(V) \times F_\beta$ is open in $G \times F_\beta$.

Consider the map

$$\Phi : C^\infty(\alpha_0(\bar{U}), E_\alpha) \to C^\infty(\alpha_0(\bar{U}), \alpha_0(V) \times E_\alpha) \subseteq C^\infty(\alpha_0(\bar{U}), G \times E_\alpha)$$

defined by

$$\Phi(\tilde{\gamma}_\alpha) = \gamma_\alpha \quad \text{for all} \quad \gamma_\alpha \in C^\infty(\alpha_0(\bar{U}), E_\alpha) \tag{14}$$

where

$$\gamma_\alpha(x) = (x, \tilde{\gamma}_\alpha(x)) \quad \text{for all} \quad x \in \alpha_0(\bar{U}) . \tag{15}$$

Then it is easy to see that Φ is C_Γ^∞ with respect to the natural calibrations on $C^\infty(\alpha_0(\bar{U}), E_\alpha)$ and $C^\infty(\alpha_0(\bar{U}), G \times E_\alpha)$ (see Chapter 1, §3).

Indeed, we have

$$\begin{aligned}[\Phi(\tilde{\gamma}_\alpha+\tilde{\eta}_\alpha)-\Phi(\tilde{\gamma}_\alpha)](x) &= (x, \tilde{\gamma}_\alpha(x)+\tilde{\eta}_\alpha(x)) - (x, \tilde{\gamma}_\alpha(x)) \\ &= (0, \tilde{\eta}_\alpha(x)) \quad \text{for all} \quad x \in \alpha_0(\bar{U})\end{aligned}$$

which gives

$$\Phi'(\tilde{\gamma}_\alpha) \in L_\Gamma(C^\infty(\alpha_0(\bar{U}), E_\alpha), C^\infty(\alpha_0(\bar{U}), G \times E_\alpha))$$

defined by

$$\Phi'(\tilde{\gamma}_\alpha) : \tilde{\eta}_\alpha \mapsto 0 \times \tilde{\eta}_\alpha \,. \tag{16}$$

The map Φ' is Γ-continuous because it is a constant map, and we have

$$\Phi^{(k)} = 0 \quad \text{for} \quad k \geq 2 \,. \tag{17}$$

Now consider the composite $\mathrm{pr}_2 \circ f_{\alpha\beta} : \alpha_0(V) \times E_\alpha \to F_\beta$. Then $\mathrm{pr}_2 \circ f_{\alpha\beta} \in C^\infty$ and induces

$$\left(\mathrm{pr}_2 \circ f_{\alpha\beta}\right)_* : C^\infty\left(\alpha_0(\overline{U}),\, \alpha_0(V) \times E_\alpha\right) \to C^\infty\left(\alpha_0(\overline{U}),\, F_\beta\right) \tag{18}$$

which is C^∞_Γ by the local Γ-omega lemma (5.1).

Thus the composite $\left(\mathrm{pr}_2 \circ f_{\alpha\beta}\right)_* \circ \Phi$:

$$C^\infty\left(\alpha_0(\overline{U}),\, E_\alpha\right) \xrightarrow{\;\Phi\;} C^\infty\left(\alpha_0(\overline{U}),\, \alpha_0(V) \times E_\alpha\right) \xrightarrow{\;\left(\mathrm{pr}_2 \circ f_{\alpha\beta}\right)_*\;} C^\infty\left(\alpha_0(\overline{U}),\, F_\beta\right)$$

is C^∞_Γ by Chapter 1.

If $\gamma \in S^\infty(\Omega)$ and $\eta \in S^\infty(\pi)$, we can define the following C^∞ section of ρ :

$$\zeta = f'_*(\gamma) \,.\, \eta \in S^\infty(\rho) \tag{19}$$

by requiring that the principal part of ζ with respect to the *VB*-chart $(U,\, \alpha_0,\, \beta)$ be:

$$\zeta_\beta : \alpha_0(\overline{U}) \to F_\beta$$

given by

$$\begin{aligned} \zeta_\beta\left(\alpha_0(x)\right) &= \left(f'_*(\gamma).\eta\right)_\beta\left(\alpha_0(x)\right) \\ &= \mathrm{pr}_2 \circ \partial_2 f_{\alpha\beta}\left(\alpha_0(x),\, \tilde{\gamma}_\alpha\left(\alpha_0(x)\right)\right).\tilde{\eta}_\alpha\left(\alpha_0(x)\right) \,. \end{aligned} \tag{20}$$

Note that formula (20) gives us a well-defined section ζ (independent of the *VB*-chart chosen).

Now we have

$$\{[(\mathrm{pr}_2 \circ f_{\alpha\beta})_* \circ \Phi]'(\tilde{\gamma}_\alpha).\tilde{\eta}_\alpha\}(\alpha_0(x))$$

$$= \{\mathrm{pr}_2 \circ f'_{\alpha\beta}(\gamma_\alpha) \circ \Phi'(\tilde{\gamma}_\alpha).\tilde{\eta}_\alpha\}(\alpha_0(x))$$

$$= \mathrm{pr}_2 \circ f'_{\alpha\beta}(\alpha_0(x), \tilde{\gamma}_\alpha(\alpha_0(x))).(0, \tilde{\eta}_\alpha(\alpha_0(x)))$$

$$= \mathrm{pr}_2 \circ [\partial_1 f_{\alpha\beta}(\alpha_0(x), \tilde{\gamma}_\alpha(\alpha_0(x))).0 + \partial_2 f_{\alpha\beta}(\alpha_0(x), \tilde{\gamma}_\alpha(\alpha_0(x))).\tilde{\eta}_\alpha(\alpha_0(x))]$$

$$= \mathrm{pr}_2 \circ \partial_2 f_{\alpha\beta}(\alpha_0(x), \tilde{\gamma}_\alpha(\alpha_0(x))).\tilde{\eta}_\alpha(\alpha_0(x)).$$

Hence by (20), we have

$$[(\mathrm{pr}_2 \circ f_{\alpha\beta})_* \circ \Phi]'(\tilde{\gamma}_\alpha).\tilde{\eta}_\alpha = \widetilde{(f'_*(\gamma).\eta)}_\beta$$

$$= \text{principal part of } f'_*(\gamma).\eta . \qquad (21)$$

Since $(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi$ is C^1_Γ for each i $(1 \le i \le n)$ we have:

For all $\varepsilon > 0$, all r, there is $\delta_i(\varepsilon, r) > 0$ $(1 \le i \le n)$ such that

$$\|(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i} + \tilde{\eta}_{\alpha_i}) - (\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i}) - [(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi]'(\tilde{\gamma}_{\alpha_i}).\tilde{\eta}_{\alpha_i}\|_r \le \varepsilon\|\tilde{\eta}_{\alpha_i}\|$$

whenever $\|\tilde{\eta}_{\alpha_i}\|_r < \delta_i$ $(1 \le i \le n)$. Thus, since the principal parts of the local representatives of $f_*(\gamma)$ and $f_*(\gamma+\eta)$ are:

$$\widetilde{(f_*(\gamma))}_{\beta_i} = \mathrm{pr}_2 \circ f_{\alpha_i\beta_{i*}}(\gamma_{\alpha_i}) = (\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i}) ,$$

$$\widetilde{(f_*(\gamma+\eta))}_{\beta_i} = (\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i} + \tilde{\eta}_{\alpha_i})$$

we have by definition

$$\|f_*(\gamma+\eta) - f_*(\gamma) - f'_*(\gamma).\eta\|_r$$

$$= \sum_{i=1}^n \|\widetilde{(f_*(\gamma+\eta))}_{\beta_i} - \widetilde{(f_*(\gamma))}_{\beta_i} - \widetilde{(f'_*(\gamma).\eta)}_{\beta_i}\|_r$$

$$= \sum_{i=1}^n \|(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i} + \tilde{\eta}_{\alpha_i}) - (\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi(\tilde{\gamma}_{\alpha_i}) - [(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi]'(\tilde{\gamma}_{\alpha_i}).\tilde{\eta}_{\alpha_i}\|_r .$$

Hence, taking $\delta = \min(\delta_1, \ldots, \delta_n) > 0$, we have

$$\|f_*(\gamma+\eta)-f_*(\gamma)-f'_*(\gamma).\eta\|_r \leq \varepsilon\|\eta\|_r \quad \text{whenever } \|\eta\|_r < \delta .$$

Furthermore, from the fact that

$$[(\mathrm{pr}_2 \circ f_{\alpha_i\beta_i})_* \circ \Phi]'(\tilde{\gamma}_{\alpha_i}) \in L_\Gamma\left[C^\infty\left(\overset{0}{\alpha}_i(\overline{U}_i), E_{\alpha_i}\right), C^\infty\left(\overset{0}{\alpha}_i(\overline{U}_i), F_{\beta_i}\right)\right]$$

for all $1 \leq i \leq n$, it follows quickly that for the above defined map

$$f'_*(\gamma) : \eta \longmapsto f'_*(\gamma).\eta$$

we have $f'_*(\gamma) \in L_\Gamma(S^\infty(\pi), S^\infty(\rho))$ and $f'_* : S^\infty(\Omega) \to L_\Gamma(S^\infty(\pi), S^\infty(\rho))$ is Γ-continuous.

Hence f_* is of class C^1_Γ as desired.

The proof for C^r_Γ for any $r \geq 2$ is analogous with the use of the following facts:

For $\gamma \in S^\infty(\Omega)$, $\eta^1, \ldots, \eta^r \in S^\infty(\pi)$, the rth Γ-derivative $f^{(r)}_*(\gamma)$ is defined by the analogue of (20),

$$(f^{(r)}_*(\gamma).\eta^1 \ldots \eta^r)_\beta = \mathrm{pr}_2 \circ \partial^k_2 f_{\alpha\beta}(\gamma_\alpha).\tilde{\eta}^1_\alpha \ldots \tilde{\eta}^k_\alpha .$$

and we have the following analogous formula of (21):

$$[(\mathrm{pr}_2 \circ f_{\alpha\beta})_* \circ \Phi]^{(k)}(\tilde{\gamma}_\alpha).\tilde{\eta}^1_\alpha \ldots \tilde{\eta}^k_\alpha = \mathrm{pr}_2 \circ \partial^k_2 f_{\alpha\beta}(\gamma_\gamma).\tilde{\eta}^1_\alpha \ldots \tilde{\eta}^k_\alpha . \quad // \ (22)$$

Now let X be a compact C^∞-manifold, Y be a finite-dimensional C^∞ manifold and let $C^\infty(X, Y)$ denote the space of all C^∞ maps from X to Y .

If $s : TY \to T^2Y$ is a spray on Y (see [1]) then there is a neighbourhood $\mathcal{D}_s \subseteq TY$ of the zero-section and a neighbourhood $F_s \subseteq Y \times Y$ of the diagonal such that $\mathrm{Exp}^s : \mathcal{D}_s \to F_s$ is a C^∞ diffeomorphism (see [1], p. 31).

If $f \in C^\infty(X, Y)$, we have the diffeomorphism

$$s_f \equiv f^*\mathrm{Exp}^s : f^*\mathcal{D}_s \to \mathcal{D}_{f,s}$$

where $\mathcal{D}_{f,s} \subseteq X \times Y$ is a neighbourhood of the graph of f.

If $U_{f,s} \subseteq C^\infty(X, Y)$ consists of maps g such that $\mathrm{graph}(g) \subseteq \mathcal{D}_{f,s}$, then the map

$$\varphi_{f,s} : U_{f,s} \to C_f^\infty(X, TY) \equiv S^\infty(f^*TY) \tag{23}$$

defined by $g \mapsto s_f^{-1} \circ \mathrm{graph}(g)$ is a homeomorphism of $U_{f,s}$ onto an open subset of $C_f^\infty(X, TY)$ (where $C_f^\infty(X, TY)$ is the space of C^∞ vector fields along f, i.e. the space $S^\infty(f^*TY)$). We shall call the pair $(U_{f,s}, \varphi_{f,s})$ a natural chart.

(2.2) THEOREM. *Let X be a compact C^∞ manifold and Y be a finite-dimensional C^∞ manifold. Then the family $\{(U_{f,s}, \varphi_{f,s})\}$ of natural charts is a Γ-atlas of class C_Γ^∞ on $C^\infty(X, Y)$ if we take as calibration for $S^\infty(f^*TY)$ the one defined by* (5). *Hence $C^\infty(X, Y)$ is a C_Γ^∞ manifold.*

Proof. We follow the proof in Abraham [1], p. 32. Let $(U_{f,s}, \varphi_{f,s})$ and $(U_{f',s'}, \varphi_{f',s'})$ be natural charts, and suppose $U_{f,s} \equiv U_{f',s'}$. It suffices to show that $\varphi_{f',s'} \circ \varphi_{f,s}^{-1}$ is a C_Γ^∞-diffeomorphism. But it is clear that

$$\varphi_{f',s'} \circ \varphi_{f,s}^{-1}(\gamma) = F_*(\gamma) \equiv F \circ \gamma$$

where

$$F = [f'^*\mathrm{Exp}^{s'}]^{-1} \circ [f^*\mathrm{Exp}^s] .$$

But s and s' are C^∞ sprays and f, f' are of class C^∞, so it is evident that F is a fibre-preserving map of class C^∞. By Lemma (2.1), F_* is of class C_Γ^∞. Clearly

$$F_*^{-1} = (F^{-1})_*$$

so F_* is a C_Γ^∞-diffeomorphism. //

As an immediate consequence of (2.2) we have the following two examples of Γ-manifolds.

(2.3) COROLLARY. *Let* X *be a compact* C^∞ *manifold and let* $\mathrm{Diff}^\infty(X)$ *denote the space of all* C^∞*-diffeomorphisms of* X *onto itself. Then* $\mathrm{Diff}^\infty(X)$ *is a* C_Γ^∞*-manifold.*

Proof. By Proposition 1.10 in [31], p. 75, $\mathrm{Diff}^\infty(X)$ is open in $C^\infty(X, X)$. Thus, it is an open Γ-submanifold of $C^\infty(X, X)$, i.e. a C_Γ^∞-manifold. //

(2.4) COROLLARY. *Let* X *be a compact* C^∞ *manifold and* Y *be a finite-dimensional* C^∞ *manifold. Let* $\mathrm{Emb}^\infty(X, Y)$ *denote the space of* C^∞*-embeddings of* X *into* Y. *Then* $\mathrm{Emb}^\infty(X, Y)$ *is a* C_Γ^∞*-manifold.*

Proof. Note that $\mathrm{Emb}^\infty(X, Y)$ is open in $C^\infty(X, Y)$ (see [31]). Then the proof is similar to the one of (2.3). //

3. *B*Γ-manifolds

There is a special kind of Γ-manifolds which are useful in application since we have the Inverse Mapping Theorem only for *B*Γ-differentiability (see Chapter 1, §3). In this section we shall define these *B*Γ-manifolds, and their corresponding *B*Γ-submanifolds. In the next section we shall give several simple examples of *B*Γ-manifolds.

If in the definition of Γ-manifold (see §2) we require that the transition maps

$$\beta \circ \alpha^{-1} : \alpha(U \cap V) \subseteq E \to \beta(U \cap V) \subseteq F$$

are $C_{B\Gamma}^r$ for all compatible Γ-charts (U, α) and (V, β) (see Chapter 1, §3) then the corresponding Γ-manifold X will be called a *B*Γ*-manifold of class* $C_{B\Gamma}^r$. The Γ-charts (U, α) and (V, β) are then called *B*Γ*-charts* *of class* $C^r_{B\Gamma}$ (or $C^r_{B\Gamma}$*-charts*).

Note that the only difference between a Γ-manifold and a $B\Gamma$-manifold is about the extra condition on transition maps. Thus $B\Gamma$-manifolds may be called Γ*-manifolds of bounded type* in the sense that the coefficients γ_p and δ_p in formula (2), §1, are bounded:

$$\sup_{p\in\Gamma} \gamma_p = \gamma < +\infty , \tag{1}$$

$$\sup_{p\in\Gamma} \delta_p = \delta < +\infty , \tag{2}$$

and we have the following double inequalities

$$\delta p_F(w) \le p_E(v) \le \gamma p_F(w) \tag{3}$$

for v and w satisfying

$$w = \left(\beta \circ \alpha^{-1}\right)'(\alpha x)v \quad \text{and} \quad v = \left(\alpha \circ \beta^{-1}\right)'(\beta x).w . \tag{4}$$

From (3) we see immediately that if X is a $B\Gamma$-manifold of class $C^r_{B\Gamma}$ $(r \ge 1)$ then the definition of the tangent space T_xX at a point $x \in X$ does not change if we take

$$T_xX = \left\{ \overline{(U, \alpha, v)} \;\middle|\; \begin{array}{l} (U, \alpha) \text{ is a } B\Gamma\text{-chart at } x \\ v \in E \in \mathcal{E} , \ \alpha(U) \subseteq E \end{array} \right\} . \tag{5}$$

Similarly, the definition of tangent space X_x via C^1_Γ-curves as in §1 does not change as well:

$$X_x = \left\{ [c]_x \mid c : I \to X, \ C^1_\Gamma\text{-curve at } x \right\} \tag{6}$$

and we have the equivalence between the two definitions.

For two $B\Gamma$-manifolds X and Y of class $C^r_{B\Gamma}$ $(r \ge 1)$ we can define $C^r_{B\Gamma}$ maps $f : X \to Y$ as well as C^r_Γ-maps.

More precisely, a map $f : X \to Y$ is of class $C^r_{B\Gamma}$ iff for every $x \in X$ and every $C^r_{B\Gamma}$ admissible chart (V, β) on Y with $f(x) \in V$, there exists a $C^r_{B\Gamma}$-admissible chart (U, α) on X such that $x \in U$, $f(U) \subseteq V$ and the local representative $f_{\alpha\beta}$ is a $C^r_{B\Gamma}$ map from $\alpha(U) \subseteq E$ to $\beta(V) \subseteq F$. It is then easily seen that the definition does not depend on $C^r_{B\Gamma}$-charts (U, α) and (V, β).

If $f : X \to Y$ is a $C^r_{B\Gamma}$ map $(r \geq 1)$ then we can define the *BΓ-tangent map* at $x \in X$:

$$T_x f : T_x X \to T_{f(x)} Y \tag{7}$$

as usual. The difference between this $B\Gamma$-tangent map and the Γ-tangent map defined in §1 is that, in local $C^r_{B\Gamma}$-charts (U, α) and (V, β), the $B\Gamma$-tangent map is represented by the $B\Gamma$-derivative $f'_{\alpha\beta}\big(\alpha(x)\big) \in L_{B\Gamma}(E, F)$ (unlike the case of Γ-tangent map where $f'_{\alpha\beta}\big(\alpha(x)\big) \in L_{\Gamma}(E, F)$).

We now define the $B\Gamma$-submanifolds of a $B\Gamma$-manifold. Let X be a $B\Gamma$-manifold of class $C^r_{B\Gamma}$. If in the definition of Γ-submanifold (as in §1) we require that the Γ-decomposition $E = E_1 \oplus_{\Gamma} E_2$ in condition (i) be a *BΓ-decomposition* (i.e. $E = E_1 \oplus_{B\Gamma} E_2$) then A_W is a $C^r_{B\Gamma}$-atlas for a $B\Gamma$-manifold structure on W. W is then called a *BΓ-submanifold of class* $C^r_{B\Gamma}$ of the $C^r_{B\Gamma}$-manifold X and (U, α) is said to *have the BΓ-submanifold property for W in X at x*. Note that the $B\Gamma$-tangent space $T_x W$ of the $B\Gamma$-submanifold W at $x \in W$ is a $B\Gamma$-splitting subspace of the $B\Gamma$-tangent space $T_x X$ (see Chapter 3, §1).

4. Examples of $B\Gamma$-manifolds

We give three simple examples of $B\Gamma$-manifolds.

EXAMPLE 1. Let $Z = \left\{\big(x_1, x_2, x_3\big) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - 1 = 0\right\}$ be the cylinder in $\mathbb{R}^3$ defined in [32], p. 115. Then Z is a Riemannian submanifold of dimension 2.

If $(q_0^1, q_0^2) \in \mathbb{R}^2$ is a point, consider the mapping

$$\iota : \Omega \subseteq \mathbb{R}^2 \to Z \tag{1}$$

defined in a neighbourhood Ω of (q_0^1, q_0^2) by

$$\iota(q^1, q^2) = (\cos q^1, \sin q^1, q^2) \quad (q^1, q^2) \in \Omega . \tag{2}$$

Then ι is a local isometry mapping Ω onto a neighbourhood V of $\iota(q_0^1, q_0^2) = q_0 \in Z$. Furthermore, ι induces

$$\iota_* \; \Omega \times \mathbb{R}^2 \to TV \subseteq TZ \tag{3}$$

which maps the canonical basis $\{D_1|_{(q^1,q^2)}, D_2|_{(q^1,q^2)}\}$ of $\mathbb{R}^2$ at (q^1, q^2) onto an orthonormal basis

$$\{i_1(q) = \iota_*(D_1|_{(q^1,q^2)}); i_2(q) = \iota_*(D_2|_{(q^1,q^2)})\} \tag{4}$$

of the tangent space T_qZ at each point $q = \iota(q^1, q^2) \in V$ (see [32], p. 115).

Let $\beta = \iota^{-1} : V \to \Omega$ be the inverse map of ι, then (V, β, β_*) (where $\beta_* = \iota_*^{-1}$) can be taken as a VB-chart for the tangent bundle TZ.

Now let $I \subseteq \mathbb{R}^m$ be a compact subset and consider the space $C^\infty(I, Z)$ of all C^∞ maps from I to Z (i.e. maps which are C^∞ in a neighbourhood of I).

For each $a \in C^\infty(I, Z)$, we denote by $C_a^\infty(I, TZ)$ the space of C^∞-vector fields along a:

$$C_a^\infty(I, TZ) = \{v : I \to TZ \mid v \in C^\infty \text{ and } \pi \circ v = a\} \tag{5}$$

where $\pi : TZ \to Z$ is the natural map.

Then $C_a^\infty(I, TZ)$ is obviously a vector space. We define a calibration on $C_a^\infty(I, TZ)$ as follows:

Fix a VB-atlas $\{(V_j, \beta_j, \beta_{j*})\}$ for TZ with the VB-charts $(V_j, \beta_j, \beta_{j*})$ defined as above. Then for each $x \in I$, there exists a $(V_j, \beta_j, \beta_{j*})$ such that $a(x) \in V_j$. We can thus find a neighbourhood $\tilde{U}(x)$ of x such that $a(\tilde{U}(x)) \subseteq V_j$ and $a|_{\tilde{U}(x)} : \tilde{U}(x) \to V_j$ is C^∞. We can furthermore find an open relatively compact neighbourhood $U(x)$ of x such that:

$$U(x) \subseteq \overline{U}(x) \subseteq \tilde{U}(x) . \tag{6}$$

Since I is compact, we can cover I by a finite number of such U's : $U_1^a, U_2^a, \ldots, U_M^a$ (the number M may depend on a). Thus $\left\{U_1^a, U_2^a, \ldots, U_M^a\right\}$ is an open covering of I with the property that $\overline{U}_i^a$ is compact $(1 \le i \le M)$ and each U_i^a is mapped into a V_j.

Now, if $v \in C_a^\infty(I, TZ)$ then on each U_i^a $(1 \le i \le M)$ we have a C^∞ map

$$v_i = v|_{U_i^a} : U_i^a \to TV_j \subseteq TZ \tag{7}$$

and for each $x \in U_i^a$,

$$v_i(x) = v_i^1(x) i_1(x) + v_i^2(x) i_2(x) \in T_{a(x)}Z \tag{8}$$

where $\{i_1(x), i_2(x)\}$ is the orthonormal basis of $T_{a(x)}Z$ defined as above. Obviously the components $v_i^1 : U_i^a \to \mathbb{R}$ and $v_i^2 : U_i^a \to \mathbb{R}$ are C^∞.

For each integer $r = 0, 1, 2, \ldots$, define

$$\|v_i\|_r^a = \sup_{x \in \overline{U}_i^a} \left\{\left\|D^r v_i^1(x)\right\| + \left\|D^r v_i^2(x)\right\|\right\} < +\infty \tag{9}$$

and

$$\|v\|_r^a = \sum_{i=1}^{M} \|v_i\|_r^a . \tag{10}$$

Then it is easy to see that $\|\cdot\|_r^a$ are semi-norms on $C_a^\infty(I, TZ)$ for each $a \in C^\infty(I, Z)$.

Note that if $b \in C^\infty(I, Z)$ is another map, then we have another open covering $\{U_1^b, U_2^b, \ldots, U_N^b\}$ with the above property and for $w \in C_b^\infty(I, TZ)$,

$$w(x) = w_k^1(x)j_1(x) + w_k^2(x)j_2(x) \quad \forall x \in I \tag{11}$$

where $\{j_1(x), j_2(x)\}$ is a basis for $T_{b(x)}Z$, and

$$\|w_k\|_r^b = \sup_{x \in \overline{U}_k^b} \left\{ \left\| D^r w_k^1(x) \right\| + \left\| D^r w_k^2(x) \right\| \right\}, \tag{12}$$

$$\|w\|_r^b = \sum_{k=1}^{N} \|w_k\|_r^b . \tag{13}$$

Thus we have a family of LCS's $C_a^\infty(I, TZ)$, $a \in C^\infty(I, Z)$, calibrated by the calibrations

$$\Gamma^a = \left\{ \|\cdot\|_r^a \right\}_{r=0,1,2,\ldots} . \tag{14}$$

(4.1) THEOREM. *Let* $I \subseteq \mathbb{R}^m$ *be a compact subset and let* Z *be the above cylinder. Then* $C^\infty(I, Z)$ *is a* $B\Gamma$*-manifold of class* $C_{B\Gamma}^\infty$ *if we take as local models for it the family of LCS's* $C_a^\infty(I, TZ)$ $\left(a \in C^\infty(I, Z)\right)$ *calibrated by* Γ^a *in* (14).

Proof. First, we know that $\exp : TZ \to Z$ is C^∞ and there exists an open neighbourhood S of the zero-section of TZ such that

$$\pi \times \exp : S \to Z \times Z \tag{15}$$

maps S C^∞-diffeomorphically onto a neighbourhood 0 of the diagonal $\Delta \subseteq Z \times Z$ ($\pi : TZ \to Z$ being the natural map). Its inverse

$$\varphi : 0 \to S \tag{16}$$

is given by ([45], p. 268)

$$\varphi(v, m) = \left(v, \exp_v^{-1}(m)\right) \text{ for all } (v, m) \in 0 . \tag{17}$$

For each $a \in C^\infty(I, Z)$ let

$$\Omega_a = \{l \in C^\infty(I, Z) \mid (a(x), l(x)) \in 0 \text{ for all } x \in I\} . \tag{18}$$

Then Ω_a is a neighbourhood of a in $C^\infty(I, TZ)$, and the map

$$\psi_a : \Omega_a \to C_a^\infty(I, TZ) \tag{19}$$

defined by $l \mapsto \psi_a(l)$ with

$$\psi_a(l)(x) = \exp_{a(x)}^{-1}\left(l(x)\right) \text{ for all } x \in I , \tag{20}$$

maps Ω_a onto the open subset $\psi_a(\Omega_a) \subseteq C_a^\infty(I, TZ)$.

Take (Ω_a, ψ_a) , $a \in C^\infty(I, Z)$, as a Γ-chart for $C^\infty(I, Z)$ at a .

We need to prove that if $\Omega_a \cap \Omega_b \neq \emptyset$ then

$$\psi = \psi_b \circ \psi_a^{-1} : \psi_a(\Omega_a \cap \Omega_b) \to \psi_b(\Omega_a \cap \Omega_b) \tag{21}$$

is $C_{B\Gamma}^\infty$ with respect to the calibrations Γ^a and Γ^b on $C_a^\infty(I, TZ)$ and $C_b^\infty(I, TZ)$.

It is obvious that, for $v \in \psi_a(\Omega_a \cap \Omega_b) \subseteq C_a^\infty(I, TZ)$ we have $w = \psi(v) \in C_b^\infty(I, TZ)$ given by (see [45])

$$w(x) = \exp_{b(x)}^{-1}\left(\exp_{a(x)}\left(v(x)\right)\right) \text{ for all } x \in I . \tag{22}$$

Now, with respect to the VB-chart $(V_j, \beta_j, \beta_{j*})$ constructed above, for each $q \in Z$, the exponential map

$$\exp_q : T_qZ \to Z$$

is given by (see [32], p.116)

$$v = v^1 i_1 + v^2 i_2 \mapsto \exp_q(v) = \left(\cos(v^1+q^1), \sin(v^1+q^1), v^2+q^2\right) . \tag{23}$$

Thus, for all $x \in I$, $a(x) \in Z$ and $\beta\left(a(x)\right) = \left(a^1(x), a^2(x)\right) \in \mathbb{R}^2$, then

$v(x) \in T_{a(x)}Z$ is given by

$$v(x) = v^1(x)I_1(x) + v^2(x)I_2(x) \tag{24}$$

where $I_1(x) = i_1\big(a(x)\big)$, $I_2(x) = i_2\big(a(x)\big)$ are unit vectors of the basis $\{I_1(x), I_2(x)\}$ of $T_{a(x)}Z$, and

$$\exp_{a(x)}\big(v(x)\big) = \big(\cos\big(v^1(x)+a^1(x)\big), \sin\big(v^1(x)+a^1(x)\big), v^2(x)+a^2(x)\big) . \tag{25}$$

Similarly, for all $x \in I$, $\beta\big(b(x)\big) = \big(b^1(x), b^2(x)\big) \in \mathbb{R}^2$ and $w(x) \in T_{b(x)}Z$ with

$$w(x) = w^1(x)J_1(x) + w^2(x)J_2(x) \tag{26}$$

where $\{J_1(x), J_2(x)\}$ is the basis $\{j_1\big(b(x)\big), j_2\big(b(x)\big)\}$ for $T_{b(x)}Z$, and

$$\exp_{b(x)}\big(w(x)\big) = \big(\cos\big(w^1(x)+b^1(x)\big), \sin\big(w^1(x)+b^1(x)\big), w^2(x)+b^2(x)\big) . \tag{27}$$

From (22), (25), (27), we have for all $x \in I$,

$$\left.\begin{aligned} w^1(x) &= v^1(x) + a^1(x) - b^1(x) + k2\pi \\ & \qquad\qquad (k \text{ is a constant integer}) \\ w^2(x) &= v^2(x) + a^2(x) - b^2(x) \end{aligned}\right\} . \tag{28}$$

That is, for $v(x) = v^1(x)I_1(x) + v^2(x)I_2(x)$, we have

$$\psi(v)(x) = \big(v^1(x)+a^1(x)-b^1(x)+k2\pi\big)J_1(x) + \big(v^2(x)+a^2(x)-b^2(x)\big)J_2(x) . \tag{29}$$

We now prove that ψ is $C^\infty_{B\Gamma}$. Indeed, consider the coverings $\left\{U^a_1, U^a_2, \ldots, U^a_M\right\}$ and $\left\{U^b_1, U^b_2, \ldots, U^b_N\right\}$, then for each $j = 1, 2, \ldots, N$, we have by (29),

$$\big(\psi(v+h)-\psi(v)\big)_j(x) = h^1(x)J_1(x) + h^2(x)J_2(x) \ , \quad x \in I \ , \tag{30}$$

if $v \in \psi_a\big(\Omega_a \cap \Omega_b\big)$, $h \in C^\infty_a(I, TZ)$ with $h(x) = h^1(x)I_1(x) + h^2(x)I_2(x)$ for all $x \in I$.

Define

$$L : C_a^\infty(I,\ TZ) \to C_b^\infty(I,\ TZ)$$

$$h = h^1 I_1 + h^2 I_2 \longmapsto L(h) = h^1 J_1 + h^2 J_2$$

(i.e. for all $x \in I$, $L(h)(x) = h^1(x)J_1(x) + h^2(x)J_2(x)$). Then obviously L is linear. We claim that L is linear $B\Gamma$-continuous. Indeed, by definition

$$\|L(h)\|_r^b = \sum_{j=1}^{N} \|L(h)_j\|_r^b$$

with

$$\|L(h)_j\|_r^b = \sup_{x \in \overline{U}_j^b} \left\{ \left\| D^r \left(L(h)_j^1(x) \right) \right\| + \left\| D^r \left(L(h)_j^2(x) \right) \right\| \right\}$$

$$= \sup_{x \in \overline{U}_j^b} \{ \|D^r h^1(x)\| + \|D^r h^2(x)\| \} .$$

But since $\overline{U}_j^b$ is covered by $\left\{ \overline{U}_1^a, \overline{U}_2^a, \ldots, \overline{U}_M^a \right\}$ we have

$$\|L(h)_j\|_r^b \le \max_{1 \le i \le M} \left\{ \sup_{x \in \overline{U}_i^a} \left[\|D^r h^1(x)\| + \|D^r h^2(x)\| \right] \right\}$$

$$\le \max_{1 \le i \le M} \left\{ \|h_i\|_r^a \right\} \le \sum_{i=1}^{M} \|h_i\|_r^a$$

$$\le \|h\|_r^a \quad \text{for all } j = 1, 2, \ldots, N .$$

Hence

$$\|L(h)\|_r^b = \sum_{j=1}^{N} \|L(h)_j\|_r^b \le N\|h\|_r^a . \tag{31}$$

Since N is independent of r , L is linear $B\Gamma$-continuous:

$$L \in L_{B\Gamma}\left(C_a^\infty(I,\ TZ),\ C_b^\infty(I,\ TZ) \right) . \tag{32}$$

Furthermore, by (30),

$$[\psi(v+h)-\psi(v)-L(h)](x) = oJ_1(x) + oJ_2(x) \quad \forall x \in I$$

which implies

$$\|(\psi(v+h)-\psi(v)-L(h))_j\|_r^b = 0 \quad \text{for all} \quad j = 1, 2, \ldots, N \ ;$$

that is,

$$\|\psi(v+h)-\psi(v)-L(h)\|_r^b = 0 \ . \tag{33}$$

Thus ψ is $B\Gamma$-differentiable at each $v \in \psi_a(\Omega_a \cap \Omega_b)$ and the $B\Gamma$-derivative of ψ at v is given by

$$D\psi(v).h = L(h) \quad \text{for all} \quad h \in C_a^\infty(I, TZ) \ . \tag{34}$$

Hence $D\psi : \psi_a(\Omega_a \cap \Omega_b) \to L_{B\Gamma}(C_a^\infty(I, TZ), C_b^\infty(I, TZ))$ is the constant map $v \mapsto L$ which proves that ψ is $C_{B\Gamma}^\infty$. //

(4.2) COROLLARY. *Let M be a compact manifold and let Z be the above cylinder. Then $C^\infty(M, Z)$ is a $B\Gamma$-manifold of class $C_{B\Gamma}^\infty$ if we take as local models for it the family of LCS's $C_a^\infty(M, TZ)$ $(a \in C^\infty(M, Z))$.*

Proof. First recall a terminology. A *pseudo-compact chart* (U, φ) of a manifold M is a chart with $\overline{U}$ compact and satisfying the following condition: there exists another chart (V, ψ) of M such that $\overline{U} \subseteq V$ and $\psi|_U = \varphi$. In other words, pseudo-compact charts are charts with relatively compact domain and which can be extended over a neighbourhood of the closure of the domain.

Cover M by a finite number of pseudo-compact charts $\{(W_\alpha, \chi_\alpha)\}_{1\le\alpha\le n}$. Then each $\overline{W}_\alpha$ $(1 \le \alpha \le n)$ is mapped by χ_α onto a compact subset $I_\alpha = \chi_\alpha(\overline{W}_\alpha) \subseteq \mathbb{R}^m$ $(m = \dim M)$.

For each $a \in C^\infty(M, Z)$, consider the space $C_a^\infty(M, TZ)$ of C^∞-vector fields along a , and for each integer $r = 0, 1, 2, \ldots$, define a semi-norm $\|\cdot\|_r^a$ on $C_a^\infty(M, TZ)$ as follows:

Let $v \in C_a^\infty(M, TZ)$, then for each α $(1 \le \alpha \le n)$ put

$$\left.\begin{aligned} a_\alpha &= a \circ \chi_\alpha^{-1} \in C^\infty(I_\alpha, Z) \\ v_\alpha &= v \circ \chi_\alpha^{-1} \in C^\infty_{a_\alpha}(I_\alpha, TZ) \end{aligned}\right\} . \tag{35}$$

Thus, since $I_\alpha \subseteq \mathbb{R}^m$ is compact, $\|v_\alpha\|_r^{a_\alpha}$ can be defined by (10).

Put

$$\|v\|_r^a = \sum_{\alpha=1}^{n} \|v_\alpha\|_r^{a_\alpha} \quad \text{for} \quad v \in C_a^\infty(M, TZ) . \tag{36}$$

Then it is not hard to see that $C_a^\infty(M, TZ)$ is a LCS calibrated by

$$\Gamma^a = \left\{\|\cdot\|_r^a\right\}_{r=0,1,2,\ldots} . \tag{37}$$

Now following the construction in the proof of Theorem (4.1), let (Ω_a, ψ_a) be defined as above with I replaced by M and $x \in I$ replaced by $m \in M$.

We must prove that

$$\psi = \psi_b \circ \psi_a^{-1} : \psi_a(\Omega_a \cap \Omega_b) \subseteq C_a^\infty(M, TZ) \to \psi_b(\Omega_a \cap \Omega_b) \subseteq C_b^\infty(M, TZ)$$

is $C^\infty_{B\Gamma}$ with respect to the calibrations Γ^a, Γ^b defined by (37).

For each $v \in \psi_a(\Omega_a \cap \Omega_b) \subseteq C_a^\infty(M, TZ)$ we have $\psi(v) \in C_b^\infty(M, TZ)$, with

$$\psi(v)(m) = \exp_{b(m)}^{-1}\left(\exp_{a(m)}(v(x))\right) \quad \text{for all} \quad m \in M . \tag{38}$$

Hence, if we denote by

$$\psi(v)_\alpha = \psi(v) \circ \chi_\alpha^{-1} \quad (1 \le \alpha \le n) \tag{39}$$

we have for each $1 \le \alpha \le n$:

$$\psi(v)_\alpha(x) = \exp_{b_\alpha(x)}^{-1}\left(\exp_{a_\alpha(x)}(v_\alpha(x))\right) \quad \forall x \in I_\alpha \tag{40}$$

where a_α, b_α and v_α are defined in (35).

By the proof of Theorem (4.1), for each $1 \le \alpha \le n$, there is a $B\Gamma$-continuous linear map

$$L_\alpha : C^\infty_{a_\alpha}\left(I_\alpha, TZ\right) \to C^\infty_{b_\alpha}\left(I_\alpha, TZ\right)$$

defined by

$$h_\alpha = h^1_\alpha I_1 + h^2_\alpha I_2 \mapsto L_\alpha\left(h_\alpha\right) = h^1_\alpha J_1 + h^2_\alpha J_2$$

with the property

$$\|\psi(v+h)_\alpha - \psi(v)_\alpha - L_\alpha\left(h_\alpha\right)\|^{b_\alpha}_r = 0 \quad (1 \le \alpha \le n) . \tag{41}$$

Now define the linear map

$$L : C^\infty_a(M, TZ) \to C^\infty_b(M, TZ)$$

by

$$h = h^1 I_1 + h^2 I_2 \mapsto L(h) = h^1 J_1 + h^2 J_2$$

where $\{I_1(m), I_2(m)\}$ and $\{J_1(m), J_2(m)\}$ are, for each $m \in M$, orthonormal base for $T_{a(m)}Z$ and $T_{b(m)}Z$ respectively.

Note that if $x \in I_\alpha$ and $\chi^{-1}_\alpha(x) = m$, then $T_{a_\alpha(x)}Z \equiv T_{a(m)}Z$ and the basis $\{I_1(x), I_2(x)\}$ of $T_{a_\alpha(x)}Z$ coincide to the basis $\{I_1(m), I_2(m)\}$. Similar results hold for $\{J_1(m), J_2(m)\}$ and $T_{b(m)}Z$.

Thus we have

$$L(h)_\alpha = L_\alpha\left(h_\alpha\right) \quad \text{for} \quad 1 \le \alpha \le n . \tag{42}$$

From (42), it follows that

$$\|L(h)\|^b_r = \sum_{\alpha=1}^n \|L(h)_\alpha\|^{b_\alpha}_r$$

$$= \sum_{\alpha=1}^n \|L_\alpha\left(h_\alpha\right)\|^{b_\alpha}_r \le \sum_{\alpha=1}^n \gamma_\alpha \|h_\alpha\|^{a_\alpha}_r .$$

Putting $\gamma = \max\{\gamma_1, \ldots, \gamma_n\}$ we have

$$\|L(h)\|^b_r \le \gamma\left(\sum_{\alpha=1}^n \|h_\alpha\|^{a_\alpha}_r\right) ;$$

that is,

$$\|L(h)\|_r^b \le \gamma \|h\|_r^a$$

and $L \in L_{B\Gamma}\left(C_a^\infty(M, TZ), C_b^\infty(M, TZ)\right)$.

Furthermore, by (41), we have

$$\|\psi(v+h)-\psi(v)-L(h)\|_r^b = \sum_{\alpha=1}^n \|\psi(v+h)_\alpha-\psi(v)_\alpha-L(h)_\alpha\|_r^{b_\alpha}$$

$$= \sum_{\alpha=1}^n \|\psi(v+h)_\alpha-\psi(v)_\alpha-L_\alpha(h_\alpha)\|_r^{b_\alpha} = 0$$

which proves that ψ is $B\Gamma$-differentiable at $v \in \psi_a(\Omega_a \quad \Omega_b)$ and $D\psi(v) = L$ for all v . Hence $D\psi : v \mapsto L$ is a constant map and ψ is $C_{B\Gamma}^\infty$ as desired. //

EXAMPLE 2. Let $K = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2+x_2^2-(x_3^2)/8 = 0,\ x_3 > 0\right\}$ be the half-cone defined in [32], p. 116. Then K is a Riemannian submanifold of $\mathbb{R}^3$ of dimension 2 .

If $\left(q_0^1, q_0^2\right) \in \mathbb{R}^2\setminus\{0\}$ is a point, and $\Omega \subseteq \mathbb{R}^2\setminus\{0\}$ is an open neighbourhood of $\left(q_0^1, q_0^2\right)$, consider the map

$$\iota : \Omega \subseteq \mathbb{R}^2\setminus\{0\} \to K \tag{43}$$

defined by

$$\iota(q^1, q^2) = \left(-q^1 + \frac{4}{3}\frac{(q^1)^3}{\left((q^1)^2+(q^2)^2\right)^2},\ q^2 - \frac{4}{3}\frac{(q^2)^2}{(q^1)^2+(q^2)^2},\ \frac{2\sqrt{2}}{3}\sqrt{(q^1)^2+(q^2)^2}\right) \tag{44}$$

for $(q^1, q^2) \in \Omega$.

Then by [32], ι is a local isometry mapping Ω onto an open neighbourhood V of $\iota(q^1, q^2) \in K$, and for each $(q^1, q^2) \in \Omega$ the vectors

$$i_1(q) = \iota_*\left(D_1|_{(q^1,q^2)}\right), \quad i_2(q) = \iota_*\left(D_2|_{(q^1,q^2)}\right)$$

form an orthonormal basis for T_qK where $q = \iota(q^1, q^2) \in K$. Furthermore, the exponential mapping

$$\exp_q : T_q K \to K$$

is given in [32], p. 116, as follows: for $v = v^1 i_1 + v^2 i_2 \neq \lambda\left(-q^1 i_1 - q^2 i_2\right)$, $\lambda \geq 1$, we have

$$\exp_q(v) = \left(-\left(v^1+q^1\right) + \frac{4}{3}\frac{\left(v^1+q^1\right)^3}{\left(v^1+q^1\right)^2+\left(v^2+q^2\right)^2},\ \left(v^2+q^2\right) - \frac{4}{3}\frac{\left(v^2+q^2\right)^3}{\left(v^1+q^1\right)^2+\left(v^2+q^2\right)^2},\ \frac{2\sqrt{2}}{3}\sqrt{\left(v^1+q^1\right)^2+\left(v^2+q^2\right)^2} \right) . \quad (45)$$

Let $\beta = \iota^{-1} : V \to \mathbb{R}^2$ be the inverse of ι and β_* the tangent map. Then (V, β, β_*) is a *VB*-chart for TK.

If $I \subseteq \mathbb{R}^m$ is a compact subset and $a \in C^\infty(I, K)$ we can endow the space $C_a^\infty(I, TK)$ with a calibration Γ^a defined as in Example 1 (formulae (9) and (10)) and if $b \in C^\infty(I, K)$, $C_b^\infty(I, TK)$ has a similar calibration Γ^b (defined by (12), (13)).

(4.3) THEOREM. *Let $I \subseteq \mathbb{R}^m$ be a compact subset and let K be the above half-cone. Then $C^\infty(I, K)$ is a $B\Gamma$-manifold of class $C^\infty_{B\Gamma}$ if we take as local models the family of LCS's $C_a^\infty(I, TK)$ calibrated by Γ^a defined as above.*

Proof. We follow the construction in the proof of Theorem (4.1). All that we need is to prove that

$$\psi = \psi_b \circ \psi_a^{-1} : \psi_a\left(\Omega_a \cap \Omega_b\right) \subseteq C_a^\infty(I, TK) \to \psi_b\left(\Omega_a \cap \Omega_b\right) \subseteq C_b^\infty(I, TK)$$

is $C^\infty_{B\Gamma}$ where

$$\psi(v)(x) = \exp^{-1}_{b(x)}\left(\exp_{a(x)}\left(v(x)\right)\right) \text{ for all } x \in I . \quad (46)$$

Using the formula (45) for the exponential map $\exp_q$, it can be seen that if $\beta\left(a(x)\right) = \left(a^1(x), a^2(x)\right)$, $\beta\left(b(x)\right) = \left(b^1(x), b^2(x)\right)$, for $x \in I$, then

$$v(x) = v^1(x)I_1(x) + v^2(x)I_2(x)$$

and

$$w(x) = w^1(x)J_1(x) + w^2(x)J_2(x)$$

where $w = \psi(x)$ and $\{I_1(x), I_2(x)\}$, $\{J_1(x), J_2(x)\}$ are the orthonormal base of $T_{a(x)}Z$ and $T_{b(x)}Z$.

Then from (46), we see that there are three possibilities.

$$\text{(I)} \begin{cases} w^1(x) = v^1(x) + a^1(x) - b^1(x) , \\ \qquad\qquad\qquad\qquad (x \in I) , \\ w^2(x) = v^2(x) + a^2(x) - b^2(x) , \end{cases}$$

$$\text{(II)} \begin{cases} w^1(x) = (-\tfrac{1}{2})v^1(x) + \frac{\sqrt{3}}{2} v^2(x) + \frac{-a^1(x)+a^2(x)\sqrt{3}-2b^1(x)}{2} , \\ \qquad\qquad\qquad\qquad (x \in I) , \\ w^2(x) = \left(-\frac{\sqrt{3}}{2}\right)v^1(x) + (-\tfrac{1}{2})v^2(x) + \frac{-a^1(x)\sqrt{3}-a^2(x)-2b^2(x)}{2} , \end{cases}$$

$$\text{(III)} \begin{cases} w^1(x) = (-\tfrac{1}{2})v^1(x) + \left(-\frac{\sqrt{3}}{2}\right)v^2(x) + \frac{-a^1(x)-a^2(x)\sqrt{3}-2b^1(x)}{2} , \\ \qquad\qquad\qquad\qquad (x \in I) . \\ w^2(x) = \left(\frac{\sqrt{3}}{2}\right)v^1(x) + (-\tfrac{1}{2})v^2(x) + \frac{a^1(x)-a^2(x)-2b^2(x)}{2} , \end{cases}$$

In any case, the components $w^1(x), w^2(x)$ of $w(x)$ are affine functions of the components $v^1(x), v^2(x)$ of $v(x)$:

$$\left.\begin{aligned} w^1(x) &= c_1v^1(x) + c_2v^2(x) + \gamma_1a^1(x) + \gamma_2a^2(x) + \delta_1b^1(x) + \delta_2b^2(x) \\ w^2(x) &= d_1v^1(x) + d_2v^2(x) + \eta_1a^1(x) + \eta_2a^2(x) + \lambda_1b^1(x) + \lambda_2b^2(x) \end{aligned}\right\} \qquad (47)$$

where the c's, d's, γ's, η's, δ's and λ's are constant.

From (47) it follows quickly that ψ is $B\Gamma$-differentiable at every

$v \in \psi_a(\Omega_a \cap \Omega_b)$ and has $B\Gamma$-derivative at v given by

$$D\psi(v) = L : C_a^\infty(I, TK) \to C_b^\infty(I, TK) \tag{48}$$

$$h = h^1 I_1 + h^2 I_2 \mapsto L(h) = \left(c_1 h^1 + c_2 h^2\right) J_1 + \left(d_1 h^1 + d_2 h^2\right) J_2 .$$

From that ψ is $C_{B\Gamma}^\infty$. //

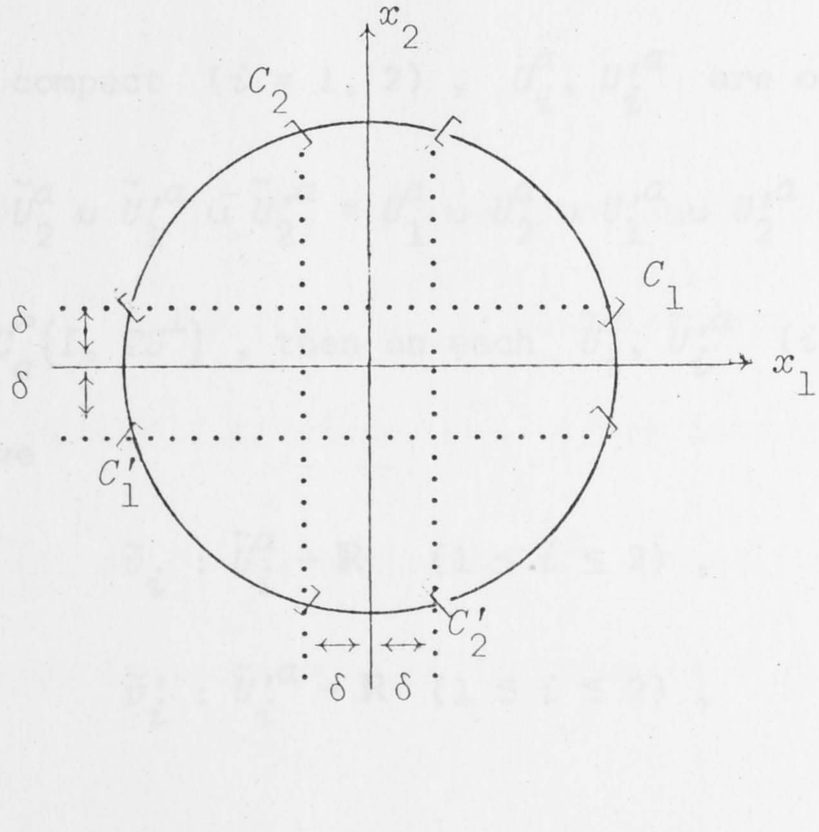
(4.4) COROLLARY. *Let M be a compact manifold and let K be the above half-cone. Then $C^\infty(M, K)$ is a $B\Gamma$-manifold of class $C_{B\Gamma}^\infty$ if we take as local models the family of LCS's $C_a^\infty(M, TK)$ $\left(a \in C^\infty(M, K)\right)$.*

Proof. The construction of calibrations Γ^a on $C_a^\infty(M, TK)$ is similar to the one of Corollary (4.2) and the proof is omitted. //

EXAMPLE 3. I am indebted to Dr Yamamuro for giving me this example of a $B\Gamma$-manifold.

Let I be a compact subset of $\mathbb{R}^m$ and let S^1 be the 1-sphere (defined, e.g., in [32], p. 2). Consider the space $C^\infty\left(I, S^1\right)$ of all C^∞ maps $I \to S^1$.

On S^1 we have a standard atlas defined by the four charts $\left(C_1, \varphi_1\right)$, $\left(C_2, \varphi_2\right)$, $\left(C_1', \varphi_1'\right)$ and $\left(C_2', \varphi_2'\right)$:



$$\varphi_1 : C_1 = \left\{(x_1, x_2) \in S^1 \mid x_1 > 0\right\} \to \mathbb{R} : (x_1, x_2) \mapsto \varphi_1(x_1, x_2) = x_1 ,$$

$$\varphi_2 : C_2 = \left\{(x_1, x_2) \in S^1 \mid x_2 > 0\right\} \to \mathbb{R} : (x_1, x_2) \mapsto \varphi_2(x_1, x_2) = x_2 ,$$

$$\varphi_1' : C_1' = \left\{(x_1, x_2) \in S^1 \mid x_1 < 0\right\} \to \mathbb{R} : (x_1, x_2) \mapsto \varphi_1'(x_1, x_2) = x_2 ,$$

$$\varphi_2' : C_2' = \left\{(x_1, x_2) \in S^1 \mid x_2 < 0\right\} \to \mathbb{R} : (x_1, x_2) \mapsto \varphi_2'(x_1, x_2) = x_1 .$$

Let $\delta > 0$ be a small constant $(0 < \delta < 0.001)$ and consider the subarcs $\tilde{C}_i, \tilde{C}_i'$ $(i = 1, 2)$ given by (see the figure above)

$$\tilde{C}_1 = \left\{(x_1, x_2) \in S^1 \mid x_1 \geq \delta\right\},$$

$$\tilde{C}_2 = \left\{(x_1, x_2) \in S^1 \mid x_2 \geq \delta\right\},$$

$$\tilde{C}_1' = \left\{(x_1, x_2) \in S^1 \mid x_1 \leq -\delta\right\},$$

$$\tilde{C}_2' = \left\{(x_1, x_2) \in S^1 \mid x_2 \leq -\delta\right\}.$$

If $\alpha \in C^\infty(I, S^1)$, we define a calibration on $C_\alpha^\infty(I, TS^1)$ as follows:

For $i = 1, 2$, we put

$$U_i^\alpha = \alpha^{-1}(C_i), \quad U_i'^\alpha = \alpha^{-1}(C_i'), \tag{49}$$

$$\tilde{U}_i^\alpha = \alpha^{-1}(\tilde{C}_i), \quad \tilde{U}_i'^\alpha = \alpha^{-1}(\tilde{C}_i'). \tag{50}$$

Then $\tilde{U}_i^\alpha, \tilde{U}_i'^\alpha$ are compact $(i = 1, 2)$, $U_i^\alpha, U_i'^\alpha$ are open in I and

$$\tilde{U}_1^\alpha \cup \tilde{U}_2^\alpha \cup \tilde{U}_1'^\alpha \cup \tilde{U}_2'^\alpha = U_1^\alpha \cup U_2^\alpha \cup U_1'^\alpha \cup U_2'^\alpha = I . \tag{51}$$

Now, if $v \in C_\alpha^\infty(I, TS^1)$, then on each $\tilde{U}_i^\alpha, \tilde{U}_i'^\alpha$ $(i = 1, 2)$ v has local representative

$$\tilde{v}_i : \tilde{U}_i^\alpha \to \mathbb{R} \quad (1 \leq i \leq 2),$$

$$\tilde{v}_i' : \tilde{U}_i'^\alpha \to \mathbb{R} \quad (1 \leq i \leq 2),$$

which are C^∞.

For each $s \in I$, we have $a(s) = \left(a_1(s),\, a_2(s)\right) \in S^1 \subseteq \mathbb{R}^2$. Then, for every integer $k = 0, 1, 2, \ldots$, we define

$$\left.\begin{aligned}
\|\tilde{v}_1\|_k^a &= \sup_{s \in \tilde{U}_1^a} \left\{\left|\frac{d^k}{ds^k}\left(\frac{\tilde{v}_1(s)}{a_1(s)}\right)\right|\right\} \\
\|\tilde{v}_2\|_k^a &= \sup_{s \in \tilde{U}_2^a} \left\{\left|\frac{d^k}{ds^k}\left(\frac{\tilde{v}_2(s)}{a_2(s)}\right)\right|\right\} \\
\|\tilde{v}'_1\|_k^a &= \sup_{s \in \tilde{U}'^a_1} \left\{\left|\frac{d^k}{ds^k}\left(\frac{\tilde{v}'_1(s)}{a_1(s)}\right)\right|\right\} \\
\|\tilde{v}'_2\|_k^a &= \sup_{s \in \tilde{U}'^a_2} \left\{\left|\frac{d^k}{ds^k}\left(\frac{\tilde{v}'_2(s)}{a_2(s)}\right)\right|\right\}
\end{aligned}\right\} \tag{52}$$

and for all $v \in C_a^\infty(I, TS^1)$, we put

$$\|v\|_k^a = \|\tilde{v}_1\|_k^a + \|\tilde{v}_2\|_k^a + \|\tilde{v}'_1\|_k^a + \|\tilde{v}'_2\|_k^a . \tag{53}$$

Then it is easy to see that $\|\cdot\|_k^a$ is a norm on $C_a^\infty(I, TS^1)$ because of (51). Thus for each $a \in C^\infty(I, S^1)$ we have a calibration

$$\Gamma^a = \left\{\|\cdot\|_k^a\right\}_{k=0,1,2,\ldots} \tag{54}$$

for $C_a^\infty(I, TS^1)$.

(4.5) THEOREM. *Let $I \subseteq \mathbb{R}^m$ be a compact subset and let $C^\infty(I, S^1)$ be the space of all C^∞ maps from I to the 1-sphere S^1. Then $C^\infty(I, S^1)$ is a $B\Gamma$-manifold of class $C^\infty_{B\Gamma}$ if we take as models the LCS's $C_a^\infty(I, TS^1)$ $\left(a \in C^\infty(I, S^1)\right)$ calibrated by Γ^a given in* (54).

Proof. We follow the construction in Examples 1 and 2. The only thing to prove is that

$$\psi = \psi_b \circ \psi_a^{-1} : \psi_a(\Omega_a \cap \Omega_b) \subseteq C_a^\infty(I, TS^1) \to \psi_b(\Omega_a \cap \Omega_b) \subseteq C_b^\infty(I, TS^1)$$

is $C^\infty_{B\Gamma}$ with respect to the calibrations Γ^a, Γ^b defined by (54).

But, for $v \in \psi_a(\Omega_a \cap \Omega_b)$, $\psi(v) \in C_b^\infty(I, TS^1)$ is given by

$$\psi(v)(s) = \exp_{b(s)}^{-1}\left(\exp_{a(s)}(v(s))\right) \quad \forall s \in I . \tag{55}$$

To simplify the notation we put

$$\Phi(t) = \exp_{b(s)}^{-1}\left(\exp_{a(s)}(t)\right) \quad \text{for} \quad t \in T_{a(s)}S^1 . \tag{56}$$

Note that for each $s \in I$, $a(s) \in S^1$ and $T_{a(s)}S^1$ is 1-dimensional (which is identified to the real line $\mathbb{R}$ with basis 1). Then for all $s \in I$,

$$v(s) = t(s) \in \mathbb{R} .$$

The same argument holds for $\psi(v)(s) \in T_{b(s)}S^1$:

$$\psi(v)(s) = \Phi\left(t(s)\right) \in \mathbb{R} .$$

Then, using the formula for $\exp_e : T_eS^1 \to S^1$ given in [32] at the north pole

$$\exp_e(t) = (\sin t, \cos t) \quad \text{for} \quad t \in T_eS^1$$

and the fact that at arbitrary points $a(s) \in S^1$,

$$\exp_{a(s)} = \alpha \circ \exp_e \circ \alpha_*^{-1}$$

where α is an isometry $\alpha \in O(2)$ such that $\alpha(e) = a$, we can see: if

$$a(s) = \left(a_1(s), a_2(s)\right) \in C_i \quad \text{or} \quad C_i' \quad (i = 1, 2) ,$$

$$b(s) = \left(b_1(s), b_2(s)\right) \in C_j \quad \text{or} \quad C_j' \quad (j = 1, 2) ,$$

then

$$\Phi(t) = \pm \frac{b_j}{a_i} t + c \tag{57}$$

where c is a constant and the sign $+$ or $-$ depends on the mutual C_i, C_j, C'_i, C'_j.

We define a map $L : C^\infty_a(I, TS^1) \to C^\infty_b(I, TS^1)$ by $h \mapsto L(h)$ as follows:

For each $s \in I$, if $s \in U^a_i \cap U^b_j$ (or $s \in U'^a_i \cap U^b_j$ or $s \in U'^a_i \cap U'^b_j$ or $s \in U^a_i \cap U'^b_j$) $(i, j = 1, 2)$ then $h(s) = h_i(s) \in \mathbb{R}$ and we put

$$L(h)(s) = \pm \frac{b_j(s)}{a_i(s)} h_i(s) \in \mathbb{R} . \tag{58}$$

(The sign $+$ or $-$ is determined by (57).)

Then it can be checked that $L(h)$ is well-defined and $L(h) \in C^\infty_b(I, TS^1)$ for each $h \in C^\infty_a(I, TS^1)$. Furthermore,

$$\|L(h)\|^b_k = \|\widetilde{L(h)_1}\|^b_k + \|\widetilde{L(h)_2}\|^b_k + \|\widetilde{L(h)'_1}\|^b_k + \|\widetilde{L(h)'_2}\|^b_k$$

with

$$\|\widetilde{L(h)_i}\|^b_k = \sup_{s \in \tilde{U}^b_i} \left\{ \left| \frac{d^k}{ds^k} \left(\frac{\widetilde{L(h)_i(s)}}{b_i(s)} \right) \right| \right\}$$

$$= \sup_{s \in \tilde{U}^b_i} \left\{ \left| \frac{d^k}{ds^k} \left(\frac{h_i(s)}{a_i(s)} \right) \right| \right\}$$

$$\leq \|h\|^a_k \quad \text{for } i = 1, 2$$

and similarly

$$\|\widetilde{L(h)'_i}\|^b_k \leq \|h\|^a_k \quad \text{for } i = 1, 2 .$$

Thus

$$\|L(h)\|^b_k \leq 4\|h\|^a_k \quad \text{for all } h \in C^\infty_a(I, TS^1)$$

which proves that $L \in L_{BT}\left(C^\infty_a(I, TS^1), C^\infty_b(I, TS^1)\right)$.

Furthermore, by construction we have

$$\|\widetilde{\left(\psi(v+h)-\psi(v)-L(h)\right)_i}\|_k^b = 0 \quad (i = 1,\ 2)\ ,$$

$$\|\widetilde{\left(\psi(v+h)-\psi(v)-L(h)\right)_i'}\|_k^b = 0 \quad (i = 1,\ 2)\ .$$

Hence

$$\|\psi(v+h)-\psi(v)-L(h)\|_k^b = 0$$

which proves that ψ is $B\Gamma$-differentiable with $B\Gamma$-derivative at v given by $D\psi(v) = L$.

Hence $D\psi : v \mapsto L$ is a constant map, and ψ is $C^\infty_{B\Gamma}$. //

(4.6) COROLLARY. *Let M be a compact manifold and let S^1 be the 1-sphere. Then $C^\infty(M, S^1)$ is a $B\Gamma$-manifold of class $C^\infty_{B\Gamma}$ if we take as models the LCS's $C^\infty_a(M, TS^1)$ $(a \in C^\infty(M, S^1))$.*

Proof. The proof is analogous to the proof of Corollary (4.4). //

Remarks. (1) Dr Yamamuro has kindly informed me that the construction of calibration Γ^a on $C^\infty_a(M, TS^1)$ can be given via the covariant derivative along a as follows.

For each $v \in C^\infty_a(M, TS^1)$, we define

$$\|v\|_k^a = \sup_{x \in M} \left\{ \left| \left(D_a^n v\right)(x) \right|_{a(x)} \right\}$$

where D_a^n denotes the nth covariant derivative along a and $|\cdot|_{a(x)}$ denotes the Riemannian norm in the tangent space $T_{a(x)}S^1$.

A simple calculation will show that in local charts, we refind the previous formulae.

(2) Let M be a compact manifold. We denote by C the family of all Riemannian manifolds X such that $C^\infty(M, X)$ can be given a $B\Gamma$-manifold structure. Then, as we see, by Theorems (4.2), (4.4) and (4.6), C contains the cylinder Z , the cone K and the 1-sphere S^1 .

An important question is to find which space X belongs to C .

Dr Yamamuro has announced that:

(i) every Euclidean space $\mathbb{R}^n$ belongs to C ;

(ii) if $X \in C$ and $Y \in C$ then $X \times Y \in C$;

(iii) if $X \in C$ and X is isometric to Y then $Y \in C$.

In particular, all flat manifolds (i.e. Riemannian manifolds with curvature identically zero) belong to C .

The problem to see whether C contains a non flat manifold is still not known.

5. Γ-Bundles

In this section, we give a short exposition about Γ-bundles and $B\Gamma$-bundles to see that the usual Banach bundles can be generalised to the Γ-theory.

For simplicity, we only consider the $B\Gamma$-case which shall be needed later. The Γ-case is similar. Our treatment follows [4].

Let $\mathcal{E}$ be a Γ-family with $B\Gamma$-product (Chapter 1, §2). We suppose that all $E \in \mathcal{E}$ are sequentially complete.

A *local* $B\Gamma$*-bundle* is a map $\pi : U \times F \to U$ where U is an open subset of an element $E \in \mathcal{E}$, F is another element of $\mathcal{E}$ and π is the first projection. (Note that we consider $U \times F$ as a subset of the space $E \times_{B\Gamma} F \in \mathcal{E}$.)

Let $\pi : U \times F \to U$ and $\pi' : U' \times F' \to U'$ be two local $B\Gamma$-bundles. Then a pair (f, f_0) with $f : U \times F \to U' \times F'$ and $f_0 : U \to U'$ is a C^r *local* $B\Gamma$*-bundle morphism* if the following conditions are satisfied:

$LB\Gamma BM$ 1: The diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{f} & U' \times F' \\ \pi \downarrow & & \downarrow \pi' \\ U & \xrightarrow{f_0} & U' \end{array}$$

is commutative.

LBΓBM 2: For all $x \in U$, the mapping $f^{\#}(x) : F \to F'$ defined by

$$f(x, v) = \left(f_0(x),\ f^{\#}(x)v\right)$$

is linear $B\Gamma$-continuous $\left(\text{i.e. } f^{\#}(x) \in L_{B\Gamma}(F, F')\right)$.

LBΓBM 3: The mappings $f_0 : U \subseteq E \to U' \subseteq E'$ and $f^{\#} : U \subseteq E \to L_{B\Gamma}(F, F')$ are $C^r_{B\Gamma}$.

The local $B\Gamma$-bundles and C^r local $B\Gamma$-bundle morphisms form a category with $(f, f_0) \circ (g, g_0)$ defined by $(f \circ g, f_0 \circ g_0)$. Hence we have a notion of isomorphism. More precisely, (f, f_0) is a C^r *local BΓ-bundle isomorphism* iff it is a C^r local $B\Gamma$-bundle morphism and there is a C^r local $B\Gamma$-bundle morphism (g, g_0) such that $f \circ g$, $g \circ f$, $f_0 \circ g_0$, and $g_0 \circ f_0$ are the identities.

For C^r local $B\Gamma$-bundle morphisms we have the following property (see [4] for the Banach case):

(5.1) PROPOSITION. *Let* E *be a* Γ*-family with* $B\Gamma$*-product and suppose that all members of* E *are sequentially complete.*

(I) *For* C^r *local* $B\Gamma$*-bundle morphisms, the condition*

LBΓBM 4: $f : U \times F \to U' \times F'$ *is* $C^r_{B\Gamma}$

holds. Furthermore, if every $E \in \mathrm{E}$ *is finite-dimensional, condition LBΓBM*4 *may replace condition LBΓBM* 3 *in the definition of* C^r *local BΓ-bundle morphisms.*

(II) *A C^r local $B\Gamma$-bundle morphism (f, f_0) is a C^r local $B\Gamma$-bundle isomorphism iff f_0 is a $C^r_{B\Gamma}$-diffeomorphism and for each $x \in U$, $f^{\#}(x)$ is a $B\Gamma$-isomorphism [i.e. $f^{\#}(x) \in \mathrm{GL}_{B\Gamma}(F, F')$ for all $x \in U$].*

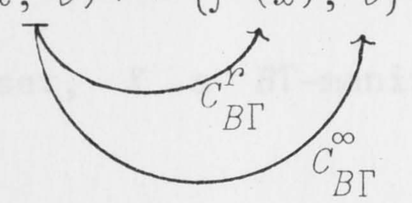
Proof. (I) We have $f : U \times F \subseteq E \times_{B\Gamma} F \to U' \times F' \subseteq E' \times_{B\Gamma} F'$ given by

$$(x, v) \longmapsto \left[f_0(x), f^{\#}(x)v\right] . \tag{1}$$

The first partial map f_0 is $C^r_{B\Gamma}$ by $LB\Gamma BM$ 3 . As for the second, $(x, v) \longmapsto f^{\#}(x)v$, we can write

$$U \times F \to L_{B\Gamma}(F, F') \times U \xrightarrow{ev} F'$$

$$(x, v) \longmapsto \left(f^{\#}(x), v\right) \longmapsto f^{\#}(x)v .$$



Since $(x, v) \longmapsto f^{\#}(x)$ is $C^r_{B\Gamma}$ by $LB\Gamma BM$ 3 and the evaluation map ev is $C^\infty_{B\Gamma}$ (by Chapter 1, Proposition (2.1)) f is $C^r_{B\Gamma}$. Thus we have the first part of (I).

The second part of (I) follows quickly from Proposition 1 in [44], p. 35.

(II) The only if part is obvious. We prove the if part. Let g_0 be the inverse of f_0 , then $g_0 : U' \to U$ is $C^r_{B\Gamma}$. We define the inverse g of f as follows:

$$g : U' \times F' \to U \times F \tag{2}$$

$$(x', v') \longmapsto g(x', v') = \left[g_0(x'), f^{\#}\left(g_0(x')\right)^{-1}v'\right] .$$

Since $f^{\#}\big(g_0(x')\big) \in \mathrm{GL}_{B\Gamma}(F, F')$, for all $x' \in U'$, the inverse $f^{\#}\big(g_0(x')\big)^{-1}$ is defined. Then for each $x' \in U'$,

$$g^{\#}(x') = f^{\#}\big(g_0(x')\big)^{-1} \in \mathrm{GL}_{B\Gamma}(F', F) . \tag{3}$$

Furthermore, $g^{\#}$ can be considered as a composite

$$U' \xrightarrow{g_0} U \xrightarrow{f^{\#}} \mathrm{GL}_{B\Gamma}(F, F') \subseteq L_{B\Gamma}(F, F') \xrightarrow{\mathrm{inv}} \mathrm{GL}_{B\Gamma}(F', F) \subseteq L_{B\Gamma}(F', F) ,$$

$$x' \mapsto g_0(x') \mapsto f^{\#}\big(g_0(x')\big) \mapsto f^{\#}\big(g_0(x')\big)^{-1} = g^{\#}(x') .$$

Since $f^{\#} : U \to L_{B\Gamma}(F, F')$ is $C^r_{B\Gamma}$ (by *LBΓBM* 3), and the map

$$\underline{\mathrm{inv}} : \mathrm{GL}_{B\Gamma}(F, F') \to \mathrm{GL}_{B\Gamma}(F', F) \tag{4}$$

$$u \longmapsto u^{-1}$$

is $C^{\infty}_{B\Gamma}$ (see Chapter 1, §3) $g^{\#}$ is $C^r_{B\Gamma}$ as desired. //

Now let E be a set, X a $B\Gamma$-manifold of class $C^r_{B\Gamma}$ and $\pi : E \to X$ a surjective map.

A C^r *BΓ-bundle chart* on π is a triple (α, α_0, U) where (U, α_0) is an admissible C^r $B\Gamma$-chart on X , $\alpha : \pi^{-1}(U) \to \alpha_0(U) \times F_\alpha$ (where $F_\alpha \in E$ and is not necessarily the ambient space of $\alpha_0(U)$) is a bijection, and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\alpha} & \alpha_0(U) \times F_\alpha \\ \downarrow & & \downarrow \\ U & \xrightarrow{\alpha_0} & \alpha_0(U) \end{array}$$

commutes. (The right-hand map is the natural projection.)

Two C^r $B\Gamma$-bundle charts (α, α_0, U) and (β, β_0, V) are $C^r_{B\Gamma}$ *compatible* iff the pair $\left(\beta \circ \alpha^{-1}, \beta_0 \circ \alpha_0^{-1}\right)$ is a C^r local $B\Gamma$-bundle isomorphism from the local $B\Gamma$-bundle $\alpha(U \cap V) \times F_\alpha \to \alpha(U \cap V)$ to the local $B\Gamma$-bundle $\beta(U \cap V) \times F_\beta \to \beta(U \cap V)$.

A C^r *$B\Gamma$-bundle atlas* on π is a collection $\{(\alpha, \alpha_0, U)\}$ of C^r $B\Gamma$-bundle charts on π , any two of which are $C^r_{B\Gamma}$-compatible and such that the U's cover X . A C^r $B\Gamma$-bundle atlas is *maximal* iff it contains each C^r $B\Gamma$-bundle chart which is $C^r_{B\Gamma}$-compatible with all of its members. As before, every C^r $B\Gamma$-bundle atlas extends uniquely to a maximal C^r $B\Gamma$-bundle atlas. A C^r *$B\Gamma$-bundle* is a surjective map $\pi : E \to X$ where E is a set and X is a $B\Gamma$-manifold of class $C^r_{B\Gamma}$ together with a maximal C^r $B\Gamma$-bundle atlas. Elements of the maximal $B\Gamma$-bundle atlas will be called *admissible* C^r *$B\Gamma$-bundle charts* on π .

Before giving some examples of $B\Gamma$-bundles, we define the morphisms between them.

Let $\pi : E \to X$ and $\pi' : E' \to X'$ be C^r $B\Gamma$-bundles, $f : E \to E'$ and $f_0 : X \to X'$ a pair of maps, and (α, α_0, U) and (β, β_0, V) admissible C^r $B\Gamma$-bundle charts on π and π' respectively with $f_0(U) \subseteq V$ and $f(\pi^{-1}(U)) \subseteq \pi'^{-1}(V)$. Then we define the *local representative* of (f, f_0) (with respect to the $B\Gamma$-bundle charts) by

$$(f, f_0)_{\alpha\beta} = (f_{\alpha\beta}, f_{\alpha\beta_0}) \tag{5}$$

where

$$f_{\alpha\beta} = \beta \circ f \circ \alpha^{-1} \quad \text{and} \quad f_{\alpha\beta_0} = \beta_0 \circ f_0 \circ \alpha_0^{-1} . \tag{6}$$

We say that (f, f_0) is a C^r *$B\Gamma$-bundle morphism from* π *to* π' , for short a C^r *$B\Gamma B$-morphism*, iff for every $x \in X$ and every admissible

$B\Gamma B$-chart (β, β_0, V) on π' with $f_0(x) \in V$, there is an admissible $B\Gamma B$-chart (α, α_0, U) on π with $x \in U$, $f_0(U) \subseteq V$, $f(\pi^{-1}(U)) \subseteq \pi'^{-1}(V)$ and such that $(f, f_0)_{\alpha\beta}$ is a C^r local $B\Gamma$-bundle morphism.

We now give some simple examples of $B\Gamma$-bundles. The first one is the standard bundle, the tangent bundle of a $B\Gamma$-manifold.

(5.2) PROPOSITION. *Let X be a $B\Gamma$-manifold of class $C^{r+1}_{B\Gamma}$ and let $TX = \bigcup_{x \in X} T_x X$ be the union of all tangent spaces. We define a map*

$$\tau_X : TX \to X$$

by

$$\tau_X(\dot{x}) = x \quad for \quad \dot{x} \in T_x X .$$

Then $\tau_X : TX \to X$ is a C^r $B\Gamma$-bundle.

Proof. Let (U, α) be an admissible $B\Gamma$-chart on X with $\alpha(U)$ an open subset of $E_\alpha \in E$. We define

$$T\alpha : \tau_X^{-1}(U) \to \alpha(U) \times E_\alpha$$

by setting

$$T\alpha(\dot{x}) = \left(\alpha\left(\tau_X(\dot{x})\right), v\right)$$

where $\dot{x} = \overline{(U, \alpha, v)}$ (see §1).

Then it is easy to see that $(T\alpha, \alpha, U)$ is a C^r local $B\Gamma$-bundle chart on τ_X (it is usually called the *natural* $B\Gamma$-*bundle chart* associated with the $B\Gamma$-chart (U, α)).

We take as a C^r $B\Gamma$-bundle atlas on τ_X the set of all such natural $B\Gamma$-bundle charts (as (U, α) ranges over the admissible $B\Gamma$-charts on X).

The only thing left is to prove that two such charts $(T\alpha, \alpha, U)$ and $(T\beta, \beta, V)$ are $C^r_{B\Gamma}$ compatible. This follows quickly from the following formula for the transition map

$$(T\beta) \circ T\alpha^{-1} : \alpha(U \cap V) \times E_\alpha \to \beta(U \cap V) \times F_\beta$$

and Proposition (5.2) (II):

$$(T\beta) \circ T\alpha^{-1}(x, v) = \left(\beta \circ \alpha^{-1}(x), \left(\beta \circ \alpha^{-1}\right)'(x)v\right) . \qquad //$$

The second example of $B\Gamma$-bundle is an useful one, the $B\Gamma$-bundle $L_{B\Gamma}(\pi, \pi')$, which shall be used later in the proof of the $B\Gamma$-transversal isotopy theorem (see Chapter 4, §4).

Our treatment follows from [4], p. 21.

Let $\pi : E \to X$ and $\pi' : E' \to X'$ be two C^r $B\Gamma$-bundles (where X, X' are two $B\Gamma$-manifolds of class $C^r_{B\Gamma}$). For each $x \in X$ (respectively $x' \in X'$) denote by E_x (respectively $E'_{x'}$) the fibre over x (respectively over x').

Define the set $L_{B\Gamma}(E, E')$ by

$$L_{B\Gamma}(E, E') = \bigcup_{(x,x') \in X \times_{B\Gamma} X'} L_{B\Gamma}\left(E_x, E'_{x'}\right) . \qquad (7)$$

One has a natural projection

$$L_{B\Gamma}(\pi, \pi') : L_{B\Gamma}(E, E') \to X \times_{B\Gamma} X'$$

given by

$$L_{B\Gamma}(\pi, \pi')(\lambda) = (x, x') \qquad (8)$$

where

$$\lambda \in L_{B\Gamma}\left(E_x, E'_{x'}\right) .$$

Given $B\Gamma$-bundle charts (α, α_0, U) and (α', α'_0, U') on π and π' respectively, one defines a *natural* $B\Gamma$*-bundle chart* on $L(\pi, \pi')$ as follows.

Let E_α and $E'_{\alpha'}$ be the ambient spaces of $\alpha_0(U)$ and $\alpha'_0(U')$ respectively; i.e. $\alpha_0(U) \subseteq E_\alpha \in \mathcal{E}$, $\alpha'_0(U') \subseteq E'_{\alpha'} \in \mathcal{E}$. Suppose $\alpha\left(\pi^{-1}(U)\right) = \alpha_0(U) \times F_\alpha$ and $\alpha'\left(\pi'^{-1}(U')\right) = \alpha'_0(U') \times F'_{\alpha'}$. Then for $x \in U$,

$x' \in U'$ and $\lambda \in L_{B\Gamma}(E_x, E'_{x'})$ define

$$L_{\alpha\alpha'}(\lambda) \in \alpha_0(U) \times \alpha'_0(U') \times L_{B\Gamma}(F_\alpha, F'_{\alpha'})$$

by

$$L_{\alpha\alpha'}(\lambda) = \left(\alpha_0(x), \alpha'_0(x'), \lambda_{\alpha\alpha'}(x, x')\right) \tag{9}$$

where $\lambda_{\alpha\alpha'}(x, x') \in L_{B\Gamma}(F_\alpha, F'_{\alpha'})$ is defined by

$$\lambda_{\alpha\alpha'}(x, x')v = \mathrm{pr}_2 \circ \alpha' \circ \lambda \circ \alpha^{-1}(\alpha_0(x), v) \tag{10}$$

for $v \in F_\alpha$ and where $\mathrm{pr}_2 : \alpha'_0(U') \times F'_{\alpha'} \to F'_{\alpha'}$ is the natural projection on the second factor. The set of all such natural $B\Gamma$-bundle charts (as (α, α_0, U) and (α', α'_0, U') range over the admissible $B\Gamma$-bundle charts on π and π' respectively) is called the *natural* $B\Gamma$*-bundle atlas* for $L_{B\Gamma}(\pi, \pi')$.

(5.3) PROPOSITION. *The natural* $B\Gamma$*-bundle atlas on* $L_{B\Gamma}(\pi, \pi')$ *is a* C^r $B\Gamma$*-bundle atlas; hence* $L_{B\Gamma}(\pi, \pi')$ *(together with the maximal* $B\Gamma$*-bundle atlas which extends the natural* $B\Gamma$*-bundle atlas) is a* C^r $B\Gamma$*-bundle called the* $B\Gamma$*-bundle of linear* $B\Gamma$*-continuous maps of* π *and* π'.

Proof. Let (β, β_0, V) and (β', β'_0, V) be $B\Gamma$-bundle charts on π, π' at $\alpha_0^{-1}(x)$ and $\alpha_0'^{-1}(x')$ respectively. Then the transition map

$$\left(L_{\beta\beta'} \circ L^{-1}_{\alpha\alpha'}\right)^{\#}(x, x') : L_{B\Gamma}(F_\alpha, F'_{\alpha'}) \to L_{B\Gamma}(F_\beta, F'_{\beta'})$$

is given by

$$\left(L_{\beta\beta'} \circ L^{-1}_{\alpha\alpha'}\right)^{\#}(x, x')\lambda = \mu' \circ \lambda \circ \mu^{-1} \tag{11}$$

for $\lambda \in L_{B\Gamma}(F_\alpha, F'_{\alpha'})$ and where

$$\mu = (\beta \circ \alpha^{-1})^{\#}(x) \quad \text{and} \quad \mu' = (\beta' \circ \alpha'^{-1})^{\#}(x') . \tag{12}$$

From this Proposition (5.3) follows. //

CHAPTER 3

THE SMALE DENSITY THEOREM

From now on, except for the last section, §4, of this chapter, we shall always restrict our interest to the $B\Gamma$-manifolds defined in Chapter 2, §3.

The purpose of this chapter is to generalise the Smale Density Theorem (see [4], [68]) from the Banach case to the $B\Gamma$-context.

In the first section we give a brief exposition of $B\Gamma$-splitting maps; in §2 we define the $B\Gamma$-Fredholm maps, a generalisation of the standard Fredholm maps in the Banach case (see e.g. [4]). The $B\Gamma$-version of the Smale Density Theorem will be stated and proved in §3.

In the last section, §4, we include the work of S. Yamamuro about a possibility of defining $C^r_{B\Gamma}$ maps from one C^r_Γ manifold to another. We shall see that, with this notion of SC^r_Γ maps, we can state and prove a yet more general "Γ-version" of the Smale Density Theorem.

1. $B\Gamma$-Splitting Maps

Let E be a Γ-family and $E \in E$ a member. Recall that ([82], Chapter V, §1) a closed subspace E_1 of E is said to be *$B\Gamma$-splitting in* E if we can find a closed complement E_2 for E_1 (in E) such that the canonical projections $P_i : E = E_1 \oplus E_2 \to E_i$ $(i = 1, 2)$ are $B\Gamma$-continuous. The closed subspace E_2 is called a *$B\Gamma$-complement of* E_1 (in E), and a decomposition $E = E_1 \oplus E_2$ with the above property is denoted by $E = E_1 \oplus_{B\Gamma} E_2$.

(1.1) EXAMPLE. Let $\Omega \subseteq \mathbb{R}^m$ be an open, convex and bounded subset,

$\mathbb{R}^m$ be another Euclidean space. Then for each integer $k = 0, 1, 2, \ldots$ we denote by $B^k(\Omega, \mathbb{R}^m)$ the space of all C^k maps $\Omega \to \mathbb{R}^m$ such that the norm

$$\|f\|_k = \sup_{x \in \Omega} \{\|f(x)\| + \|Df(x)\| + \ldots + \|D^k f(x)\|\} \tag{1}$$

is *finite*.

We denote by $E = B^\infty(\Omega, \mathbb{R}^m)$ the intersection of all $B^k(\Omega, \mathbb{R}^m)$:

$$B^\infty(\Omega, \mathbb{R}^m) = \bigcap_{k=0}^{\infty} B^k(\Omega, \mathbb{R}^m) = \Big\{f : \Omega \to \mathbb{R}^m \mid f \in C^\infty \text{ and } \|f\|_k < +\infty \text{ for all } k\Big\} . \tag{2}$$

Then we shall see in Lemma (1.1), Chapter 5, that E is a separable Fréchet space if we equip E with the sequence of increasing norms $\{\|\cdot\|_k\}_{k=0,1,2,\ldots}$ defined by (1).

Now let $x \in \Omega$ and $k \geq 0$ be fixed and define (see [4])

$$K_1 = \{\zeta \in B^\infty(\Omega, \mathbb{R}^m) \mid D^i\zeta(x) = 0 \text{ for } i = 0, 1, 2, \ldots, k\} . \tag{3}$$

Then, as shall be seen in the proof of Lemma (1.3), Chapter 5, there always exists an integer i_0 such that if we give the following calibration for E,

$$\Gamma = \{\|\cdot\|_{r+k+i}\}_{i \geq i_0} \tag{4}$$

then K_1 is $B\Gamma$*-splitting* in E .

Now let E, F be two members of $\mathcal{E}$. Then a $B\Gamma$-continuous linear map $S : E \to F$ is called *double* $B\Gamma$*-splitting* if both Ker S and Im S are $B\Gamma$-splitting and such that there exists a $B\Gamma$-complement of Ker S in E such that the restriction of S to this $B\Gamma$-complement is a $B\Gamma$-isomorphism. In other words, S is double $B\Gamma$-splitting iff:

(i) E_1 = Ker S is a closed subspace of E ; there exists a closed subspace E_2 of E such that $E = E_1 \oplus_{B\Gamma} E_2$;

(ii) $F_2 = \operatorname{Im} S$ is a closed subspace of F ; there exists a closed subspace F_1 of F such that $F = F_1 \oplus_{B\Gamma} F_2$;

(iii) $S|_{E_2} : E_2 \to F_2$ is a $B\Gamma$-isomorphism.

A $B\Gamma$-continuous linear map $S : E \to F$ is called a *$B\Gamma$-splitting surjection* if it is a surjection, its kernel Ker S is $B\Gamma$-splitting in E , and there exists a $B\Gamma$-complement of Ker S in E such that the restriction of S to this $B\Gamma$-complement is a $B\Gamma$-isomorphism onto F . In other words, S is a $B\Gamma$-splitting surjection iff S is double $B\Gamma$-splitting and onto.

We shall denote by $\mathrm{SL}_{B\Gamma}(E, F)$ the space of all $B\Gamma$-splitting surjections of E onto F .

The following two results are due to S. Yamamuro ([82]).

(1.2) PROPOSITION. *The set* $\mathrm{SL}_{B\Gamma}(E, F)$ *of $B\Gamma$-splitting surjections of E onto F is open in the space $L_{B\Gamma}(E, F)$ of $B\Gamma$-continuous linear maps $E \to F$.*

Proof (see [82]). Let $u \in \mathrm{SL}_{B\Gamma}(E, F)$ and denote by $P_i : E = E_1 \oplus_{B\Gamma} E_2 \to E_i$ $(i = 1, 2)$ the projections corresponding to the $B\Gamma$-decomposition of E into $B\Gamma$-direct sum with $E_1 = u^{-1}(0)$. Denote by

$$u_2 = u|_{E_2} : E_2 \to F \tag{5}$$

the $B\Gamma$-isomorphism given in the definition. (Here $F = F_2 = \operatorname{Im} u$.) Then it is obvious that

$$P_1 + u_2^{-1} u = 1_E . \tag{6}$$

Now, if $v \in L_{B\Gamma}(E, F)$ is such that

$$\|u-v\|_\Gamma < \frac{1}{\|u_2^{-1}\|_\Gamma} \tag{7}$$

then

$$w = 1_E - u_2^{-1}(u-v)$$

is a $B\Gamma$-isomorphism of E onto E, and

$$uw = uP_1 + uu_2^{-1}v = uu_2^{-1}v = v \, . \qquad (8)$$

Since u and w are surjective, v is also a surjection.

To see that v is double $B\Gamma$-splitting, we put

$$S_1 = w^{-1}P_1 w \quad \text{and} \quad S_2 = w^{-1}P_2 w \, . \qquad (9)$$

Then these are projections, and

$$S_1 S_2 = S_2 S_1 = 1 \quad \text{and} \quad S_1 + S_2 = 1 \, . \qquad (10)$$

Moreover, we have $S_1(E) = v^{-1}(0)$ because

$$S_1(E) = w^{-1}P_1(E) = w^{-1}u^{-1}(0) = v^{-1}(0) \, .$$

Thus $v \in \mathrm{SL}_{B\Gamma}(E, F)$ as desired. //

(1.3) PROPOSITION. *If* $u \in \mathrm{SL}_{B\Gamma}(E, F)$ *and* $v \in \mathrm{GL}_{B\Gamma}(E, E)$ *, then* $uv \in \mathrm{SL}_{B\Gamma}(E, F)$.

Proof (see [82]). Since v is an isomorphism of E, we have

$$E = v^{-1}(E) = v^{-1}(E_1) + v^{-1}(E_2) \qquad (11)$$

(where E_1 and E_2 are as in the proof of Proposition (1.2)) and

$$(uv)^{-1}(0) = v^{-1}(E_1) \, . \qquad (12)$$

Furthermore,

$$v^{-1}(E_2) \xrightarrow{v} E_2 \xrightarrow{u} F \qquad (13)$$

is a $B\Gamma$-isomorphism and the projections

$$E \to v^{-1}(E_1) \quad \text{and} \quad E \to v^{-1}(E_2) \qquad (14)$$

are given by $v^{-1}P_1 v$ and $v^{-1}P_2 v$ respectively, which are $B\Gamma$-continuous.

Hence $uv \in SL_{B\Gamma}(E, F)$. //

2. $B\Gamma$-Fredholm Maps

Let $\mathcal{E}$ be a Γ-family as usual. Then a $B\Gamma$-continuous linear map $S : E \to F$ (E, F being members of $\mathcal{E}$) is called a *$B\Gamma$-Fredhom operator* iff

(a) S is double $B\Gamma$-splitting (see §1),

(b) Ker S is finite-dimensional,

(c) Im S is finite-codimensional.

In this case, if n = dim Ker S and p = codim Im S , the integer (possibly positive, negative or zero) $n - p$ is called *the index of* S , in symbol ind(S) . Thus

$$\text{ind}(S) = \dim \text{Ker } S - \text{codim Im } S . \tag{1}$$

Note that when $\mathcal{E}$ is the category of Banach spaces with the norm-calibration Γ , then $B\Gamma$-Fredholm operators are exactly the usual standard Fredholm operators (see e.g. [4], [68]).

Now let X, Y be $C^r_{B\Gamma}$-manifolds ($r \geq 1$) and $f : X \to Y$ be a $C^r_{B\Gamma}$ mapping. Then f is a *$B\Gamma$-Freholm map* iff for every $x \in X$ the $B\Gamma$-tangent map $T_x f : T_x X \to T_{f(x)} Y$ is a $B\Gamma$-Fredholm operator.

(2.1) EXAMPLE. Let Ω, $\mathbb{R}^n$, $\mathbb{R}^m$ and $B^\infty(\Omega, \mathbb{R}^m)$ be as in Example (1.1), §1. For each integer $k = 0, 1, 2, \ldots$ denote by $P^k(n, m)$ the space of polynomials $\mathbb{R}^n \to \mathbb{R}^m$ of degree less than or equal to k :

$$P^k(n, m) = P^k(\mathbb{R}^n; \mathbb{R}^m) = \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m) \times \ldots \times L^k_s(\mathbb{R}^n, \mathbb{R}^m) \tag{2}$$

defined in §6, Chapter 1.

Let r be an integer greater than or equal to 1 and k another integer such that $0 \leq k \leq r$, and define the mapping

$$ev_k : B^\infty(\Omega, \mathbb{R}^m) \times \Omega \to \Omega \times P^k(n, m) \tag{3}$$

by

$$ev_k(\xi, x) = \left(x, P^k\xi(x)\right) \quad \text{for all } x \in \Omega\ , \quad \xi \in B^\infty\left(\Omega, \mathbb{R}^m\right)\ . \tag{4}$$

We regard Ω and $\Omega \times P^k(n, m)$ as (finite-dimensional) Banach manifolds with the corresponding canonical norm-calibrations and $B^\infty\left(\Omega, \mathbb{R}^m\right)$ as a $C^\infty_{B\Gamma}$ manifold (modelled on the Fréchet space $B^\infty\left(\Omega, \mathbb{R}^m\right)$) with the calibration

$$\Gamma = \left\{\|\cdot\|_{r+k+i}\right\}_{i \geq i_0} \tag{5}$$

where i_0 is an integer given by Lemma (1.3), Chapter 5. Then $B^\infty\left(\Omega, \mathbb{R}^m\right) \times \Omega$ is a $C^\infty_{B\Gamma}$-manifold (the $B\Gamma$-product of $B\Gamma$-manifolds) and the map ev_k is a $B\Gamma$-Fredholm map of class $C^r_{B\Gamma}$ (see Lemma (1.3), Chapter 5).

The following proposition gives us a convenient form for the local representative of a $B\Gamma$-Fredholm map (see [4] for the Banach case).

(2.2) PROPOSITION. *Let X and Y be $C^r_{B\Gamma}$-manifolds* $(r \geq 1)$ *modelled respectively on $E, F \in \mathcal{E}$, E sequentially complete, and let $f : X \to Y$ be a $B\Gamma$-Fredholm map of class $C^r_{B\Gamma}$* $(r \geq 1)$. *Let $x \in X$. Then we can find admissible $B\Gamma$-charts (U, α) on X centred at x , and (V, β) on Y , centred at $f(x)$, with the following properties:*

(i) $E = E_1 \oplus_{B\Gamma} E_2$ where E_1 and E_2 are closed subspaces of E , $\dim E_1 = n < +\infty$, $U \subseteq$ domain of f ; α maps U $C^r_{B\Gamma}$-diffeomorphically onto $B_1 + B_2$ with B_i closed convex circled neighbourhoods of 0 *in B_i* $(i = 1, 2)$;

(ii) $F = F_1 \oplus_{B\Gamma} F_2$, where F_1 and F_2 are closed subspaces of F , $\dim F_1 = p < +\infty$; *$f(U) \subseteq V$; β maps V $C^r_{B\Gamma}$-diffeomorphically onto an open subset of $F_1 \oplus_{B\Gamma} F_2$;*

(iii) the local representative $f_{\alpha\beta} : \alpha(U) \subseteq E \to \beta(V) \subseteq F$ *has the form*

$$f_{\alpha\beta} = \eta + \Phi \circ P_2 \tag{6}$$

where $\eta : \alpha(U) \subseteq E \to F_1$ *is* $C^r_{B\Gamma}$ *with* $\eta'(0) = 0$; Φ *is a* $B\Gamma$*-isomorphism of* E_2 *onto* F_2 *and* P_2 *is the second projection of* $E = E_1 \oplus_{B\Gamma} E_2$ *onto* E_2 .

Proof. We may find admissible $B\Gamma$-charts $(\tilde{U}, \varphi)$ at x and $(\tilde{V}, \tilde{\psi})$ at $f(x)$ such that the local representative $f_{\varphi\psi} : \varphi(\tilde{U}) \subseteq E \to \psi(\tilde{V}) \subseteq F$ is of class $C^r_{B\Gamma}$ $(r \geq 1)$ and the $B\Gamma$-derivative $f'_{\varphi\psi}\big(\varphi(x)\big) : E \to F$ is a $B\Gamma$-Fredholm operator. By suitable translations (which are $C^\infty_{B\Gamma}$-diffeomorphisms) we may suppose that $\varphi(x) = 0$ and $\psi\big(f(x)\big) = 0$. Thus $f_{\varphi\psi}(0) = 0$ and the $B\Gamma$-derivative $f'_{\varphi\psi}(0) : E \to F$ is double $B\Gamma$-splitting.

Let $E_1 = \operatorname{Ker} f'_{\varphi\psi}(0)$ and $F_2 = \operatorname{Im} f'_{\varphi\psi}(0)$. Then $\dim E_1 = n < +\infty$; $\operatorname{codim} F_2 = p < +\infty$ and there exist closed subspaces E_2 of E and F_1 of F such that $E = E_1 \oplus_{B\Gamma} E_2$, $F = F_1 \oplus_{B\Gamma} F_2$ and such that $\Phi = f'_{\varphi\psi}(0)|_{E_2} : E_2 \to F_2$ is a $B\Gamma$-isomorphism.

Let $P_i : E_1 \oplus_{B\Gamma} E_2 \to E_i$ $(i = 1, 2)$ and $Q_i : F_1 \oplus_{B\Gamma} F_2 \to F_i$ $(i = 1, 2)$ be corresponding projections. Then P_i and Q_i are $B\Gamma$-continuous.

We can write

$$f'_{\varphi\psi}(0) = Q_1 \circ f'_{\varphi\psi}(0) + Q_2 \circ f'_{\varphi\psi}(0) . \tag{7}$$

But since $\operatorname{Im} f'_{\varphi\psi}(0) = F_2$, we have

$$Q_1 \circ f'_{\varphi\psi}(0) = 0 . \tag{8}$$

Define $k : E \to E$ in a neighbourhood of $0 \in E$ by

$$k = P_1 + \Phi^{-1} \circ Q_2 \circ f_{\varphi\psi} . \tag{9}$$

Then it is not hard to see that k is a $C^r_{B\Gamma}$-map with $k(0) = 0$, $k'(0) = 1_E \in \mathrm{GL}_{B\Gamma}(E, E)$.

By the Inverse Mapping Theorem (3.12), §3, Chapter 1, we can find a neighbourhood Ω' of 0 contained in $\varphi(\tilde{U})$ and a $C^r_{B\Gamma}$-diffeomorphism $\lambda : \Omega' \to \Omega$ so that $k \circ \lambda^{-1}$ is the identity on Ω (where Ω is a neighbourhood of 0 in E):

$$k \circ \lambda^{-1} = \mathrm{id}_\Omega . \tag{10}$$

Let Λ be any neighbourhood of 0 in $F = F_1 \oplus_{B\Gamma} F_2$ such that $f_{\varphi\psi}(\Omega') \subseteq \Lambda \subseteq \psi(\tilde{V})$ and let $\mu = \mathrm{id}_\Lambda : \Lambda \to \Lambda$. We put

$$U = \varphi^{-1}(\Omega') , \quad V = \psi^{-1}(\Lambda) , \tag{11}$$

$$\alpha = \lambda \circ \varphi : U \to \alpha(U) \subseteq E ; \quad \beta = \mu \circ \psi : V \to \beta(V) \subseteq F . \tag{12}$$

Then it is clear that (U, α) and (V, β) are $B\Gamma$-charts on X and Y and the local representative of f with respect to these $B\Gamma$-charts is

$$f_{\alpha\beta} = f_{\varphi\psi} \circ \lambda^{-1} = Q_1 \circ f_{\varphi\psi} \circ \lambda^{-1} + Q_2 \circ f_{\varphi\psi} \circ \lambda^{-1} . \tag{13}$$

We put $\eta = Q_1 \circ f_{\varphi\psi} \circ \lambda^{-1}$, then $\eta \in C^r_{B\Gamma}$ and from (8) we see that

$$\eta'(0) = 0 . \tag{14}$$

Furthermore, from (9) and (10), it is not hard to see that

$$Q_2 \circ f_{\varphi\psi} \circ \lambda^{-1} = \Phi \circ P_2 . \tag{15}$$

By making routine adjustments in α, β, U and V we can satisfy the conditions in the proposition. //

Recall that a map g between topological spaces is *locally closed* iff every point in the domain of the definition of g has an open neighbourhood Ω such that $g|\overline{\Omega}$ is a closed map (i.e. maps closed sets to closed sets).

(2.3) PROPOSITION. *A $B\Gamma$-Fredholm map $f : X \to Y$ is locally closed.*

Proof. Let x be an arbitrary point in X contained in the domain of f. We want to prove that there exists an open neighbourhood Ω of x (in X) such that $f|\overline{\Omega}$ is a closed map.

By Proposition (2.2) we can find $B\Gamma$-charts (U, α) on X and (V, β) on Y such that $x \in U$, $f(x) \in V$ and the properties (i), (ii), (iii) in Proposition (2.2) hold.

Let D_1 be an open (convex circled) bounded neighbourhood of 0 in E_1 such that $\overline{D}_1 \subseteq B_1$ and let D_2 be an open (convex circled) neighbourhood of 0 in E_2 such that $\overline{D}_2 \subseteq B_2$. Since $\dim E_1 = n < +\infty$, $\overline{D}_1$ is compact.

Let $\Lambda = D_1 + D_2$. Then $\overline{\Lambda} = \overline{D}_1 + \overline{D}_2 \subseteq B_1 + B_2 = \alpha(U)$. We claim that $f_{\alpha\beta}|\overline{\Lambda}$ is closed. Indeed, if $A \subseteq \overline{\Lambda}$ is closed, we see as follows that $f_{\alpha\beta}(A)$ is closed:

Choose a net $\left\{y_1^i+y_2^i\right\} \subseteq f_{\alpha\beta}(A)$ such that $y_1^i + y_2^i \to y_1 + y_2$, say,

$$y_1^i + y_2^i = f_{\alpha\beta}\left(x_1^i+x_2^i\right) \tag{16}$$

where $x_1^i + x_2^i \in A$ for all i.

Since $A \subseteq \overline{\Lambda} = \overline{D}_1 + \overline{D}_2$, for all i, we have

$$x_1^i \in \overline{D}_1 \ ; \ x_2^i \in \overline{D}_2 \ ; \ y_1^i \in F_1 \ \text{and} \ y_2^i \in \Phi\left(\overline{D}_2\right) \subseteq F_2 \ . \tag{17}$$

Since $\overline{D}_1$ is compact, and $\left\{x_1^i\right\} \subseteq \overline{D}_1$, we may assume (replacing $\left\{x_1^i\right\}$ by a subnet if necessary) that

$$x_1^i \to x_1 \in \overline{D}_1 \ . \tag{18}$$

Then by (iii) in Proposition (2.2) we have

$$y_1^i + y_2^i = f_{\alpha\beta}\left(x_1^i+x_2^i\right) = \eta\left(x_1^i+x_2^i\right) + \Phi\left(x_2^i\right) \ . \tag{19}$$

Since $\Phi\left(x_2^i\right) \to y_2 \in F_2$, and Φ is a homeomorphism, we have

$$\left.\begin{aligned} x_2^i &\to \Phi^{-1}\left(y_2\right) \in \overline{D}_2 \\ x_1^i &\to x_1 \in \overline{D}_1 \end{aligned}\right\} . \tag{20}$$

Hence $x_1^i + x_2^i \to x_1 + \Phi^{-1}\left(y_2\right)$. Since $\left\{x_1^i + x_2^i\right\} \subseteq A$ and A is closed, we must have

$$x_1 + \Phi^{-1}\left(y_2\right) \in A . \tag{21}$$

Since η is continuous,

$$\eta\left(x_1^i + x_2^i\right) \to \eta\left(x_1 + \Phi^{-1}\left(y_2\right)\right) \tag{22}$$

and

$$f_{\alpha\beta}\left(x_1^i + x_2^i\right) = \eta\left(x_1^i + x_2^i\right) + \Phi\left(x_2^i\right) \to \eta\left(x_1 + \Phi^{-1}\left(y_2\right)\right) + y_2 . \tag{23}$$

But, by (21), we have

$$y_1^i + y_2^i \to f_{\alpha\beta}\left(x_1 + \Phi^{-1}\left(y_2\right)\right) \in f_{\alpha\beta}(A) \tag{24}$$

Thus, we have $y_1 + y_2 \in f_{\alpha\beta}(A)$ and $f_{\alpha\beta}|\overline{\Lambda}$ is closed.

Take $\Omega = \alpha^{-1}(\Lambda)$, then it is clear that $f|\overline{\Omega}$ is closed. //

3. $B\Gamma$-Version of Smale Theorem

Let E be a Γ-family, $E, F \in E$ and let X, Y be $C_{B\Gamma}^r$-manifolds modelled on E, F respectively $(r \geq 1)$.

Let $f : X \to Y$ be a $C_{B\Gamma}^r$-map $(r \geq 1)$. Following [4], we say that a point $x \in X$ is a *regular point* of f iff the $B\Gamma$-tangent map $T_x f : T_x X \to T_{f(x)} Y$ is surjective; x is a *critical point* of f iff it is not regular.

If $C \subseteq X$ is the set of critical points of f, then $f(C) \subseteq Y$ is the set of *critical values* of f and $Y\backslash f(C)$ is the set of *regular values* of f.

The set of regular values of f is denoted by R_f or $R(f)$. In addition, for an arbitrary set $A \subseteq X$, we follow [4] to define

$$R_f|A = Y\backslash f(A \cap C) . \tag{1}$$

In particular, if $U \subseteq X$ is open, then $R_f|U = R(f|U)$.

(3.1) REMARK. S. Yamamuro has given the following definitions for regular point and regular value in the linear case (see [82], p. 62): let $\mathcal{E}$ be a Γ-family, $E, F \in \mathcal{E}$ and $U \subseteq E$ be open. Let $f : U \subseteq E \to F$ be a $C^r_{B\Gamma}$ map $(r \geq 1)$. Then a point $x \in U$ is a *regular point* of f iff the $B\Gamma$-derivative $f'(x)$ is a $B\Gamma$-splitting surjection of E onto F (i.e. $f'(x) \in \mathrm{SL}_{B\Gamma}(E, F)$). x is a *critical point* of f iff $f'(x) \notin \mathrm{SL}_{B\Gamma}(E, F)$. Obviously his notion of regular point is stronger than ours and the set of critical points of f in his sense is *bigger* than ours, and both coincide in the case of finite-dimentional spaces.

For a $B\Gamma$-Fredholm map, we have the following property.

(3.2) PROPOSITION. *Let $f : X \to Y$ be a $B\Gamma$-Fredholm map of class $C^r_{B\Gamma}$ $(r \geq 1)$, where X, Y are $C^r_{B\Gamma}$-manifolds modelled on $E, F \in \mathcal{E}$ with F sequentially complete. Then the set of regular points of f is open in X, hence the set C of critical points of f is closed in X.*

Proof. Let $x_0 \in X$ be a regular point of f. We want to prove that there exists a neighbourhood Ω of x_0 such that every $x \in \Omega$ is a regular point of f.

By definition of regular points, the $B\Gamma$-tangent map $T_{x_0}f : T_{x_0}X \to T_{f(x_0)}Y$ is onto. Furthermore, since f is a $B\Gamma$-Fredholm map, $T_{x_0}f$ maps $T_{x_0}X$ onto $T_{f(x_0)}Y$, $\mathrm{Ker}\, T_{x_0}f$ is $B\Gamma$-splitting in $T_{x_0}X$ and there exists a $B\Gamma$-complement of $\mathrm{Ker}\, T_{x_0}f$ such that the restriction of $T_{x_0}f$ to this

$B\Gamma$-complement is a $B\Gamma$-isomorphism.

Let (U, φ) and (V, ψ) be admissible $B\Gamma$-charts at x_0 and $f(x_0)$ respectively. Then $f'_{\varphi\psi}(\varphi(x_0)) \in SL_{B\Gamma}(E, F)$ (by the above discussion). Consider the composite map

$$U \subseteq X \xrightarrow{\varphi} \varphi(U) \subseteq E \xrightarrow{f'_{\varphi\psi}} L_{B\Gamma}(E, F) \tag{2}$$

then $f'_{\varphi\psi} \circ \varphi$ is continuous, and $f'_{\varphi\psi} \circ \varphi(x_0) \in SL_{B\Gamma}(E, F)$.

Since $SL_{B\Gamma}(E, F)$ is open in $L_{B\Gamma}(E, F)$ by Proposition (1.2), we can find an open neighbourhood Ω of x_0 with $\Omega \subseteq U$ such that

$$\forall x \in \Omega \Rightarrow f'_{\varphi\psi}(\varphi(x)) \in SL_{B\Gamma}(E, F) \tag{3}$$

and, *a fortiori*, $f'_{\varphi\psi}(\varphi(x)) : E \to F$ is onto for every $x \in \Omega$. //

(3.3) REMARK. The Proposition (3.2) still holds if we define critical points of f as in Remark (3.1), that is, x is a critical point of $f : X \to Y$ iff $T_x f \notin SL_{B\Gamma}(T_x X, T_{f(x)} Y)$ where we identify $T_x X$ (respectively $T_{f(x)} Y$) to E (respectively F) in the Γ-family $\mathcal{E}$.

Before stating the Smale Density Theorem, we restate the Sard Density Theorem ([4], [68]) in a convenient form as follows:

A subset of a topological space is *residual* iff it is the countable intersection of open dense sets. Recall that if the topological space is Baire then a residual subset is dense.

(3.4) THEOREM (Sard Density Theorem). *Let X and Y be finite-dimensional C^r manifolds with* $\dim X = n$, $\dim Y = p$ *and with X Lindelöf. Let $f : X \to Y$ be a C^r map. Then if* $r > \max(0, n-p)$, R_f *is residual (and hence dense) in Y.*

Proof. See, for example, [4].

Now recall that if X, Y are $C^r_{B\Gamma}$-manifolds modelled on $E, F \in \mathcal{E}$ and if $f : X \to Y$ is $C^r_{B\Gamma}$, then if E and F are normable then by Proposition (3.4), Chapter 1, f is C^r with respect to any admissible norms on E and F.

We now state and prove the main theorem of this chapter whose Banach version is due to Smale ([68]).

(3.5) THEOREM ($B\Gamma$-version of Smale Density Theorem). *Let $\mathcal{E}$ be a Γ-family, $E, F \in \mathcal{E}$ sequentially complete. Let X, Y be $C^r_{B\Gamma}$-manifolds modelled on E, F respectively with X Lindelöf and let $f : X \to Y$ be a $B\Gamma$-Fredholm map of class $C^r_{B\Gamma}$. Suppose that $r > \max(0, \text{ind } T_x f)$ for every $x \in X$. Then R_f is residual in Y (hence dense in Y if the model F of Y is Baire).*

Proof. We follow the method in [4]. Let x_0 be an arbitrary point in X. We shall construct a neighbourhood Z of x_0 so that $R_f | \overline{Z}$ is open dense in Y.

First, we may choose admissible $B\Gamma$-charts (U, α) centred at x_0 and (V, β) centred at $f(x_0)$ verifying the properites (i), (ii), and (iii) in Proposition (2.2). We may suppose that

$$\alpha(U) = B_1 + B_2 \subseteq E = E_1 \oplus_{B\Gamma} E_2 \; ; \quad \beta(V) = D_1 + D_2 \subseteq F = F_1 \oplus_{B\Gamma} F_2 \qquad (4)$$

where B_i, D_i $(i = 1, 2)$ are closed neighbourhoods of 0 in E_i, F_i $(i = 1, 2)$ respectively. Furthermore, we can suppose

$$\Phi(B_2) = D_2 \qquad (5)$$

where $\Phi : E_2 \to F_2$ is the $B\Gamma$-isomorphism in Proposition (2.2).

Since $f_{\alpha\beta}(\alpha(U)) \subseteq \beta(V) = D_1 + D_2$, it follows immediately that

$$R(f_{\alpha\beta}) \supseteq \beta(V)^c = (D_1 + D_2)^c \qquad (6)$$

where A^c denotes the complementary of A in F $(A \subseteq F)$.

Thus we have $f_{\alpha\beta} : \alpha(U) = B_1 + B_2 \to \beta(V) = D_1 + D_2$ with

$$f_{\alpha\beta} = \eta + \Phi \circ P_2 \tag{7}$$

where $\eta : \alpha(U) \to F_1$ is $C^r_{B\Gamma}$ and $\eta'(0) = 0$.

By hypotheses, we have

$$\dim E_1 = n \ , \ \dim F_1 = p \ , \ \text{ind}\ T_{x_0} f = n - p \tag{8}$$

and

$$r > \max(0, n-p) \ . \tag{9}$$

We now show that $R(f|U)$ is dense in Y. Indeed, it suffices to show that $R(f_{\alpha\beta})$ is dense in F.

Let e' be an arbitrary point in F_2. Two cases are possible.

(1) $e' \notin D_2 = \Phi(B_2)$: Then for every $y \in F_1$, we have

$$y + e' \notin \beta(V) = D_1 + D_2 \ .$$

That is

$$F_1 + e' \subseteq \beta(V)^c \subseteq R(f_{\alpha\beta}) \quad \text{(by (6))}.$$

Hence

$$R(f_{\alpha\beta}) \cap (F_1 + e') = F_1 + e' \tag{10}$$

which implies that

$$R(f_{\alpha\beta}) \cap (F_1 + e') \text{ is dense in } F_1 + e' \ . \tag{11}$$

(2) $e' \in D_2 = \Phi(B_2)$: Since Φ is a homeomorphism, there exists one and only one $e \in B_2$ such that $\Phi(e) = e'$.

Define a map $\eta_e : B_1 \subseteq E_1 \to F_1$ by

$$\eta_e(x) = \eta(x+e) \quad (x \in B_1) \ . \tag{12}$$

Then it is clear that η is a C^r map from the open neighbourhood B_1 of 0 in E_1 $(\dim E_1 = n)$ into a finite-dimensional space F_1 $(\dim F_1 = p)$ with $r > \max(0, n-p)$.

By Sard Density Theorem (3.4), $R(\eta_e)$ is dense in F_1. Furthermore, it is not hard to see that

$$R(f_{\alpha\beta}) \cap (F_1+e') = R(\eta_e) + e' . \tag{13}$$

Thus, since $R(\eta_e)$ is dense in F_1, we see that

$$R(f_{\alpha\beta}) \cap (F_1+e') \text{ is dense in } F_1 + e' . \tag{14}$$

So, by (11) and (14), we have proved that for each $e' \in F_2$, $R(f_{\alpha\beta}) \cap (F_1+e')$ is dense in $F_1 + e'$. Thus $R(f_{\alpha\beta})$ is dense in F and we have $R(f|U)$ dense in Y as desired.

By Proposition (2.3) we can choose an open neighbourhood Z of x_0 such that $\overline{Z} \subseteq U$ and $f|\overline{Z}$ is closed. By Proposition (3.2), the set C of critical points of f is closed in X. Hence

$$R_f|\overline{Z} = Y \backslash f(\overline{Z} \cap C) \text{ is open in } Y . \tag{15}$$

Since $R(f|U) \subseteq R_f|\overline{Z}$, $R_f|\overline{Z}$ is also dense.

Now, since X is Lindelöf, we can find a countable cover $\{Z_i\}$ of X such that $R_f|\overline{Z}_i$ is open dense. Since

$$R_f = \bigcap_i R_f|\overline{Z}_i \tag{16}$$

it follows that R_f is residual and we have proved the theorem. //

(3.6) REMARK. If we follow Yamamuro's definition of regular points as in Remark (3.1) then Theorem (3.5) is still true by analogous proof. Actually the set R_f in this case is smaller than the normal one, but is still residual.

The following proposition is a standard corollary of Theorem (3.5) (see [68] for the Banach case).

(3.7) COROLLARY. *Let $E, F \in$ E be sequentially complete with F Baire and let X, Y be $C^r_{B\Gamma}$-manifolds $(r \geq 1)$ modelled on E, F respectively with X Lindelöf. If $f : X \to Y$ is a BΓ-Fredholm map of* *class* $C^r_{B\Gamma}$ $(r \geq 1)$ *and* $\operatorname{ind} T_x f < 0$ *for all* $x \in X$ *, then its image contains no interior points.*

Proof. Since for all $x \in X$, $\operatorname{ind} T_x f < 0$, the condition $r > \max(0, \operatorname{ind} T_x f)$ for all $x \in X$ is trivially verified.

Furthermore, the condition $\operatorname{ind} T_x f < 0$ for all $x \in X$ also implies that all $x \in X$ is a critical point of f . Hence the image $f(X)$ is exactly the set of critical values of f and thus has no interior points as indicated by the last part of the Smale Density Theorem (3.5). //

If we follow Yamamuro's definition for regular points (see Remark (3.1)), then the Smale Density Theorem has another consequence as follows.

(3.8) COROLLARY. *Let* X, Y *be as in Corollary* (3.7). *If* $f : X \to Y$ *is a* $B\Gamma$*-Fredholm map of class* $C^r_{B\Gamma}$ *with* $r > \max(0, \operatorname{ind} T_x f)$ *for all* $x \in X$. *Then for almost all* $y \in Y$, $f^{-1}(y)$ *is either empty or a* $B\Gamma$*-submanifold of class* $C^r_{B\Gamma}$ *of* X .

Proof. By Smale Density Theorem given in Remark (3.6), almost all $y \in Y$ is a regular value of f (in the sense of Yamamuro's definition as in Remark (3.1)), that is $y = f(x)$ for a $x \in X$ with $T_x f \in SL_{B\Gamma}(T_x X, T_{f(x)} Y)$. Hence we have either $f^{-1}(y)$ is empty or $f^{-1}(y) \neq \emptyset$ and is a closed $B\Gamma$-submanifold of X . The fact that $f^{-1}(y)$ is a $B\Gamma$-submanifold can be proved directly from the definitions or by just noting that $\{y\}$ is a $B\Gamma$-submanifold of Y of dimension 0 , and y is a regular value of f (in Yamamuro's sense) iff $f \pitchfork_{B\Gamma} \{y\}$, and apply Corollary (1.2), Chapter 4. //

4. A Possibility of Generalisation

The results in this section are due to S. Yamamuro. I thank him for permitting me to include them here.

The purpose of this section is to discuss a possibility of defining $C^r_{B\Gamma}$ maps between C^r_Γ-manifolds, a non-standard notion. The reason for this shall be given now.

As we have seen in Chapter 2, the class of $B\Gamma$-manifolds is rather small while the class of Γ-manifolds is considerably larger. And the reason for introducing these $B\Gamma$-manifolds is to get the Inverse Mapping Theorem (3.12) in Chapter 1. Thus we have encountered a difficult choice:

(i) either we restrict ourselves to the small class of $B\Gamma$-manifolds (with the Inverse Mapping Theorem available) but which is not suitable for many applications, or

(ii) we have to force ourselves to define the notion of *$B\Gamma$-maps between Γ-manifolds*, so that whenever we need, we can use the Inverse Mapping Theorem or its equivalent forms.

Dr S. Yamamuro has proposed a solution for (ii) and we shall show in this section that with this general notion of SC^r_Γ maps the Smale Density Theorem (3.5) can be stated and proved. Dr Yamamuro has also informed me that with the use of SC^r_Γ maps he has been able to refind several results in [54], §III.

Let E be a Γ-family, X, Y be Γ-manifolds of class C^r_Γ $(r \geq 1)$ modelled on $E, F \in E$ respectively. Let $f : X \to Y$ be a mapping and $x \in X$ be a point. Then a pair of C^r_Γ-charts (U, α) and (V, β) of X and Y respectively is said to be a *pair of strong C^r_Γ-charts for f at x* (or for short, *a pair of SC^r_Γ-charts for f at x*) iff $x \in U$, $f(x) \in V$, $f(U) \subseteq V$ and the local representative $f_{\alpha\beta} : \alpha(U) \subseteq E \to \beta(V) \subseteq F$ is a $C^r_{B\Gamma}$ map (see Chapter 1, §3).

We say that $f : X \to Y$ is *strongly C^r_Γ at x* (for short *of class*

SC^r_Γ *at* x) iff f is C^r_Γ at x, and in addition, there is a pair $\{(U, \alpha), (V, \beta)\}$ of strong C^r_Γ-charts for f at x. In other words, f is strongly C^r_Γ at x iff we can find a pair of admissible Γ-charts (U, α) and (V, β) at x and $y = f(x)$ respectively such that, with respect to these Γ-charts, the local representative of f is of class $C^r_{B\Gamma}$.

$f : X \to Y$ is a *strongly* C^r_Γ *map* (or *of class* SC^r_Γ) iff it is SC^r_Γ at every point $x \in X$. Note that when X and Y are $B\Gamma$-manifolds of class $C^r_{B\Gamma}$ then SC^r_Γ maps $X \to Y$ coincide with $C^r_{B\Gamma}$ maps $X \to Y$.

Now consider a SC^r_Γ map $f : X \to Y$ between Γ-manifolds X, Y of class C^r_Γ $(r \geq 1)$. We say that f *has the* $B\Gamma$*-Fredholm property at* $x \in X$ if, with respect to a pair of SC^r_Γ-charts $\{(U, \alpha), (V, \beta)\}$, the Γ-derivative $f'_{\alpha\beta}(\alpha(x)) : E \to F$ is a $B\Gamma$-Fredhom operator (see §2). In this case, we define the *index of* f *at* x *with respect to the pair of* SC^r_Γ*-charts* $\{(U, \alpha), (V, \beta)\}$ as follows:

$$\begin{aligned}\operatorname{ind}\big(f, x, (U, \alpha), (V, \beta)\big) &= \operatorname{ind} f'_{\alpha\beta}(\alpha(x)) \\ &= \dim \operatorname{Ker} f'_{\alpha\beta}(\alpha(x)) - \operatorname{codim} \operatorname{Im} f'_{\alpha\beta}(\alpha(x)) \,. \quad (1)\end{aligned}$$

We say that $f : X \to Y$ has *the* $B\Gamma$*-Fredholm property* iff it has the $B\Gamma$-Fredholm property at every $x \in X$.

With this general definition, we have the following version of Proposition (2.2).

(4.1) PROPOSITION. *Let* X, Y *be* C^r_Γ*-manifolds* $(r \geq 1)$ *modelled on* $E, F \in E$, E *be sequentially complete, and let* $f : X \to Y$ *be a* SC^r_Γ *map* *having the $B\Gamma$-Fredholm property at a point* $x \in X$. *Then we can find admissible* Γ*-charts* (U, α) *on* X *centred at* x, *and* (V, β) *on* Y *centred at* $f(x)$ *with the following properties:*

(i) $E = E_1 \oplus_{B\Gamma} E_2$ *where* E_1 *and* E_2 *are closed subspaces of* E, $\dim E_1 = n < +\infty$, $U \subseteq$ *domain of* f ; α *maps* U C^r_Γ*-diffeomorphically onto* $B_1 + B_2$ *with* B_i *closed convex circled neighbourhoods of* 0 *in* E_i $(i = 1, 2)$;

(ii) $F = F_1 \oplus_{B\Gamma} F_2$ *where* F_1 *and* F_2 *are closed subspaces of* F, $\dim F_1 = p < +\infty$, $f(U) \subseteq V$; β *maps* V C^r_Γ*-diffeomorphically onto an open subset of* $F_1 \oplus_{B\Gamma} F_2$;

(iii) the local representative $f_{\alpha\beta} : \alpha(U) \subseteq E \to \beta(V) \subseteq F$ *has the form*

$$f_{\alpha\beta} = \eta + \Phi \circ P_2$$

where $\eta : \alpha(U) \subseteq E \to F_1$ *is* $C^r_{B\Gamma}$ *with* $\eta'(0) = 0$; Φ *is a* $B\Gamma$*-isomorphism of* E_2 *onto* F_2 *and* P_2 *is the second projection of* $E = E_1 \oplus_{B\Gamma} E_2$ *onto* E_2.

Proof. We start from a pair of SC^r_Γ-charts $\{(\tilde{U}, \varphi), (\tilde{V}, \psi)\}$ for f at x and proceed exactly as in the proof of Proposition (2.2). //

(4.2) PROPOSITION. *Let* $f : X \to Y$ *be a* SC^r_Γ *map* $(r \geq 1)$ *having the* $B\Gamma$*-Fredholm property. Then* f *is locally closed.*

Proof. Exactly as the proof of Proposition (2.3) with the use of Proposition (4.1). //

Now let $f : X \to Y$ be a C^r_Γ-map between C^r_Γ-manifolds X and Y $(r \geq 1)$. Then the notion of regular point, critical point, regular value and critical value of f can be defined exactly as in the $B\Gamma$-case of §3.

For instance, a point $x \in X$ is a *regular point* of f iff the Γ-tangent map $T_x f : T_x X \to T_{f(x)} Y$ is onto. Note that this means that for every pair of Γ-charts (U, α) and (V, β) at x and $f(x)$ respectively with $f(U) \subseteq V$, the Γ-derivative $f'_{\alpha\beta}(\alpha(x))$ of the local representative $f_{\alpha\beta} : \alpha(U) \subseteq E \to \beta(V) \subseteq F$ is onto.

As before, we denote by R_f the set of regular values of $f : X \to Y$. Then we have the following analogue of Proposition (3.2).

(4.3) PROPOSITION. *Let $f : X \to Y$ be a SC^r_Γ map having the $B\Gamma$-Fredholm property where X, Y are C^r_Γ-manifolds modelled on $E, F \in \mathcal{E}$; F sequentially complete. Then the set of regular points of f is open in X, hence the set C of critical points of f is closed in X.*

Proof. Let $x_0 \in X$ be a regular point of f. Then by definition of regular points, the Γ-tangent map $T_{x_0} f : T_{x_0} X \to T_{f(x_0)} Y$ is onto. Furthermore, since f has the $B\Gamma$-Fredholm property at x_0, we can find a pair of SC^r_Γ-charts $\{(U, \varphi), (V, \psi)\}$ for f at x_0, such that

$$f'_{\varphi\psi}(\varphi(x_0)) : E \to F \tag{2}$$

is a $B\Gamma$-Fredholm operator. In other words, $f'_{\varphi\psi}(\varphi(x_0)) \in SL_{B\Gamma}(E, F)$. Then by an argument analogous to the one in the proof of Proposition (3.2) we can see that there is an open neighbourhood Ω of x_0 such that every $x \in \Omega$ is a regular point of f. //

With all these results we can now state and prove the "Γ-version" of the Smale Density Theorem (3.5).

(4.4) THEOREM. *Let $\mathcal{E}$ be a Γ-family, $E, F \in \mathcal{E}$ sequentially complete. Let X, Y be C^r_Γ-manifolds modelled on E, F respectively with X Lindelöf. Let $f : X \to Y$ be a SC^r_Γ map $(r \geq 1)$ having the $B\Gamma$-Fredholm* *property. Suppose that for each* $x \in X$ *we can find a pair of* SC^r_Γ*-charts* $\{(U, \alpha), (V, \beta)\}$ *at* x *such that*

$$r > \max\{0, \operatorname{ind}(f; x, (U, \alpha), (V, \beta))\} . \tag{3}$$

Then the set R_f *of regular values of* f *is residual in* Y *(hence dense in* Y *if the model* F *of* Y *is Baire).*

Proof. Exactly as the proof for the Smale Density Theorem (3.5) using Propositions (4.1), (4.2) and (4.3). //

CHAPTER 4

$B\Gamma$-TRANSVERSALITY

In this chapter we generalise the usual notion of transversality ([4], [31], [75]) to maps between $B\Gamma$-manifolds.

In the first section we give the definition of $B\Gamma$-transversality and prove some standard properties. In §2, we consider representations of $B\Gamma$-manifolds (following the treatment of [4]) and in §3 we prove the $B\Gamma$-Transversal Density Theorem, a generalisation of the one in [4]. In the last section, §4, we prove the $B\Gamma$-Transversal Isotopy Theorem ([4], [75]).

1. $B\Gamma$-Transversality

Let E be a Γ-family (see Chapter 1, §2) and let $E, F \in E$. We say that a $B\Gamma$-continuous linear map $u : E \to F$ is *a* $B\Gamma$*-splitting surjection* iff u is surjective, $\ker u$ is $B\Gamma$-splitting in E and there exists a $B\Gamma$-complement of $\ker u$ such that the restriction of u to this $B\Gamma$-complement is a $B\Gamma$-isomorphism onto F .

Now, let X, Y be two $C^1_{B\Gamma}$-manifolds modelled on E, F respectively. Let $f : X \to Y$ be a $C^1_{B\Gamma}$-map and $W \subseteq Y$ be a $C^1_{B\Gamma}$-submanifold (Chapter 2, §3).

We say that f is *$B\Gamma$-transversal to* W *at a point* $x \in X$, in symbols $f \pitchfork_{B\Gamma} xW$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and the following condition is satisfied:

(*) there exist a chart (U, α) centred at x , a chart (V, β) verified the $B\Gamma$-submanifold property for W in Y at y such that, if $T_y W$ is represented by F_1 in F with

$F_1 \oplus_{B\Gamma} F_2 = F$) and $Q_2 : F = F_1 \oplus_{B\Gamma} F_2 \to F_2$ is the second projection, then the composite

$$E \xrightarrow{f'_{\alpha\beta}(0)} F \xrightarrow{Q_2} F_2 \tag{1}$$

is a $B\Gamma$-splitting surjection.

The definition is independent of the charts chosen.

We say that f is *$B\Gamma$-transversal to* W, in symbols $f \pitchfork_{B\Gamma} W$, iff $f \pitchfork_{B\Gamma} xW$ for every $x \in X$.

Note that if $E = B$, the category of Banach spaces, (with the norm-calibration) then by the Banach theorem (see [13]) our definition of $B\Gamma$-transversality reduces to the usual one as defined in Lang [44].

Furthermore, the condition (*) can be formulated equivalently in global form as follows:

(**) There exists a $B\Gamma$-complement Z of T_yW in T_yY such that if we denote by $Q : T_yY = T_yW \oplus_{B\Gamma} Z \to Z$, the second projection, then the composite

$$T_xX \xrightarrow{T_xf} T_yY \xrightarrow{Q} Z \tag{2}$$

is a $B\Gamma$-splitting surjection.

The following theorem for the local representative of $B\Gamma$-transversality is an analogue of the one in [4].

(1.1) THEOREM (Local Representative of $B\Gamma$-Transversality). *Let X, Y be $C^r_{B\Gamma}$-manifolds $(r \geq 1)$ modelled on E, F respectively with E sequentially complete. Let $f : X \to Y$ be a $C^r_{B\Gamma}$-map, $W \subseteq Y$ a $C^r_{B\Gamma}$-submanifold, and $x \in X$ such that $y = f(x) \in W$.*

Then a necessary and sufficient condition for $f \pitchfork_{B\Gamma} xW$ is that there exist admissible $B\Gamma$-charts (U, α) and (V, β) at x and $f(x)$ respectively, $B\Gamma$-decompositions $E = E_1 \oplus_{B\Gamma} E_2$, $F = F_1 \oplus_{B\Gamma} F_2$ for E

and F *such that:*

(a) $\alpha(U) = B_1 + B_2 \subseteq E = E_1 \oplus_{B\Gamma} E_2$; $\beta(V) = D_1 + D_2 \subseteq F = F_1 \oplus_{B\Gamma} F_2$,

$\alpha(x) = 0$, $\beta(y) = 0$, $\beta(W \cap V) = D_1$, *where* B_i *and* D_i

$(i = 1, 2)$ *are open neighbourhoods of* 0 *in* E_i, F_i

$(i = 1, 2)$ *respectively;*

(b) *the local representative* $f_{\alpha\beta} : B_1 + B_2 \to D_1 + D_2$ *of* f *has the form*

$$f_{\alpha\beta} = \eta + \Phi \circ P_2$$

where $\eta : B_1 + B_2 \subseteq E \to D_1 \subseteq F_1$ *is a* $C^r_{B\Gamma}$*-map,* Φ *is a* $B\Gamma$*-isomorphism of* E_2 *onto* F_2 *and* $P_2 : E = E_1 \oplus_{B\Gamma} E_2 \to E_2$ *is the second projection.*

Proof. SUFFICIENCY. Suppose that there exist admissible $B\Gamma$-charts (U, α) and (V, β) verifying (a) and (b) in the theorem. We want to prove that $f \pitchfork_{B\Gamma} xW$.

Indeed, with respect to these $B\Gamma$-charts, T_xX, T_yY and $T_xf : T_xX \to T_yY$ are represented respectively by

$$T_xX = E \ , \quad T_yY = F \ , \quad T_xf = f'_{\alpha\beta}(0) : E \to F \ . \tag{3}$$

Since, by (a), (V, β) has the $B\Gamma$-submanifold property for W at y with

$$\beta(V) = D_1 + D_2 \subseteq F_1 \oplus_{B\Gamma} F_2 = F$$

$$\beta(W \cap V) = D_1 \subseteq F_1 \ ,$$

T_yW is represented by

$$T_yW = F_1 \ .$$

We claim that the composite $Q_2 \circ f'_{\alpha\beta}(0)$,

$$E \xrightarrow{f'_{\alpha\beta}(0)} F \xrightarrow{Q_2} F$$

is a $B\Gamma$-splitting surjection.

Indeed, by (b), we have

$$f'_{\alpha\beta}(0) = \eta'(0) + \Phi \circ P_2 \quad \text{with} \quad \eta'(0) : E \to F_1 . \tag{4}$$

Then for all $x \in E$, $x = x_1 + x_2 \in E_1 \oplus_{B\Gamma} E_2$ and

$$f'_{\alpha\beta}(0)x = \underbrace{\eta'(0)x}_{\in F_1} + \underbrace{\Phi \circ P_2(x)}_{\in F_2} .$$

Hence $Q_2 \circ f'_{\alpha\beta}(0)x = Q_2 \circ \Phi \circ P_2(x)$. That is,

$$Q_2 \circ f'_{\alpha\beta}(0) = Q_2 \circ \Phi \circ P_2 \tag{5}$$

which is a surjective map of E onto F_2 (by (b)).

Furthermore,

$$\begin{aligned}\operatorname{Ker}\left(Q_2 \circ f'_{\alpha\beta}(0)\right) &= f'_{\alpha\beta}(0)^{-1}\left(Q_2^{-1}(0)\right) = f'_{\alpha\beta}(0)^{-1}\left(F_1\right) \\ &= \left\{x = x_1 + x_2 \in E_1 \oplus_{B\Gamma} E_2 \mid \eta'(0)(x) + \Phi\left(x_2\right) \in F_1\right\} \\ &= \left\{x = x_1 + x_2 \in E_1 \oplus_{B\Gamma} E_2 \mid \Phi\left(x_2\right) = 0\right\} \quad \left(\text{because } F_1 \cap F_2 = \{0\}\right) \\ &= \left\{x = x_1 + x_2 \in E_1 \oplus_{B\Gamma} E_2 \mid x_2 = 0\right\} \quad (\text{because } \Phi \text{ is a } B\Gamma\text{-isomorphism}) \\ &= E_1\end{aligned}$$

which is $B\Gamma$-splitting in E (with a $B\Gamma$-complement E_2).

Now it follows quickly from (5) that

$$Q_2 \circ f'_{\alpha\beta}(0)\big|_{E_2} = \Phi : E_2 \to F_2 \tag{6}$$

which is a $B\Gamma$-isomorphism by (b).

NECESSITY. Suppose that $f \pitchfork_{B\Gamma} xW$. We want to prove that there exist $B\Gamma$-charts (U, α) and (V, β) with the properties (a) and (b) in the theorem.

First, since W is a $B\Gamma$-submanifold of Y and $y = f(x) \in W$, we can find a $B\Gamma$-chart $\left(V_1, \beta_1\right)$ at y having the $B\Gamma$-submanifold property for W in Y at y ; that is,

$$\beta_1(V_1) = C_1 + C_2 \subseteq F_1 \oplus_{B\Gamma} F_2 = F ,$$

$$\beta_1(W \cap V_1) = C_1 \subseteq F_1 ,$$

$$\beta_1(y) = 0 .$$

Moreover, we can find a chart (U_1, α_1) at x with $\alpha_1(x) = 0$, $f(U_1) \subseteq V_1$ and

$$f_{\alpha_1\beta_1} : \alpha_1(U_1) \subseteq E \to \beta_1(V_1) \subseteq F$$

is $C^r_{B\Gamma}$.

Since $f \pitchfork_{B\Gamma} xW$, the composite map

$$E \xrightarrow{f'_{\alpha_1\beta_1}(0)} F \xrightarrow{Q_2} F_2$$

is a $B\Gamma$-splitting surjection. That is, $Q_2 \circ f'_{\alpha_1\beta_1}(0)$ is surjective and, if we denote by $E_1 = f'_{\alpha_1\beta_1}(0)^{-1}(F_1)$, then E_1 is $B\Gamma$-splitting in E , and there exists a $B\Gamma$-complement E_2 of E_1 (in E) such that

$$Q_2 \circ f'_{\alpha_1\beta_1}(0)\Big|_{E_2} : E_2 \to F_2 \text{ is a } B\Gamma\text{-isomorphism.} \tag{7}$$

Denote by

$$\Phi = Q_2 \circ f'_{\alpha_1\beta_1}(0)\Big|_{E_2} : E_2 \xrightarrow[\approx]{B\Gamma} F_2 . \tag{8}$$

Now, consider the map $Q_2 \circ f_{\alpha_1\beta_1} : \alpha_1(U_1) \subseteq E \to F$, then obviously

$$Q_2 \circ f_{\alpha_1\beta_1}(0) = 0 . \tag{9}$$

Define $k : \alpha_1(U_1) \subseteq E = E_1 \oplus_{B\Gamma} E_2 \to E = E_1 \oplus_{B\Gamma} E_2$ by

$$k = P_1 + \Phi^{-1} \circ Q_2 \circ f_{\alpha_1\beta_1} \tag{10}$$

(i.e. for all $x = x_1 + x_2 \in \alpha_1(U_1) \subseteq E_1 \oplus_{B\Gamma} E_2$, put

$$k(x) = x_1 + \Phi^{-1} \circ Q_2 \circ f_{\alpha_1\beta_1}(x)\,) \; .$$

Then k is $C^r_{B\Gamma}$ and $k(0) = 0$. Furthermore

$$k'(0) = P_1 + \Phi^{-1} \circ Q_2 \circ f'_{\alpha_1\beta_1}(0) = P_1 + P_2 = \mathrm{id}_E \; . \tag{11}$$

(The fact that $\Phi^{-1} \circ Q_2 \circ f'_{\alpha_1\beta_1}(0) = P_2$ is proved as follows:

$$\forall x = x_1 + x_2 \in E_1 \oplus_{B\Gamma} E_2 \Rightarrow f'_{\alpha_1\beta_1}(0)x = f'_{\alpha_1\beta_1}(0).x_1 + f'_{\alpha_1\beta_1}(0).x_2$$

$$\Rightarrow Q_2 \circ f'_{\alpha_1\beta_1}(0).x = Q_2 \circ f'_{\alpha_1\beta_1}(0).x_2 = \Phi(x_2)$$

$$\Rightarrow \Phi^{-1} \circ Q_2 \circ f'_{\alpha_1\beta_1}(0).x = \Phi^{-1}(\Phi(x_2)) = x_2 = P_2(x) \;.)$$

By the Inverse Mapping Theorem (3.12), Chapter 1, k is a local $C^r_{B\Gamma}$-diffeomorphism. Let U_2 be a sufficiently small neighbourhood of 0 in $E = E_1 \oplus_{B\Gamma} E_2$ such that $U_2 \subseteq \alpha_1(U_1)$ and let α_2 be a $C^r_{B\Gamma}$-diffeomorphism of U_2 onto an open neighbourhood of 0 in $E_1 \oplus_{B\Gamma} E_2$ such that

$$k \circ \alpha_2^{-1} = \mathrm{id}_E \; . \tag{12}$$

Then

$$Q_2 \circ f_{\alpha_1\beta_1} \circ \alpha_2^{-1} = \Phi \circ P_2 \; . \tag{13}$$

Indeed, if $x = x_1 + x_2 \in \alpha_2(U_2) \subseteq E_1 \oplus_{B\Gamma} E_2$ and

$\alpha_2^{-1}(x) = x_1' + x_2' \in E_1 \oplus_{B\Gamma} E_2$ then by (12) and (13),

$$k \circ \alpha_2^{-1}(x) = k(x_1'+x_2')$$

$$= x_1' + \Phi^{-1} \circ Q_2 \circ f_{\alpha_1\beta_1}(x_1'+x_2') = x_1 + x_2$$

which implies

$$\left.\begin{aligned} x_1' &= x_1 \\ \Phi^{-1} \circ Q_2 \circ f_{\alpha_1\beta_1}\left(x_1'+x_2'\right) &= x_2 \end{aligned}\right\} . \tag{14}$$

Hence

$$\begin{aligned} Q_2 \circ f_{\alpha_1\beta_1}\left(x_1'+x_2'\right) &= \Phi\left(x_2\right) \\ &= \Phi \circ P_2\left(x_1+x_2\right) \ ; \end{aligned}$$

that is,

$$Q_2 \circ f_{\alpha_1\beta_1} \circ \alpha_2^{-1}(x) = \Phi \circ P_2(x) \quad \text{for all} \quad x \in \alpha_2\left(U_2\right) . \tag{15}$$

We put

$$\left.\begin{aligned} &\alpha = \alpha_2 \circ \alpha_1 \ , \quad U = \alpha_1^{-1}\left(U_1\right) \\ &\beta = \beta_1 \ , \quad y \in V \text{ small neighbourhood} \subseteq V_1 \end{aligned}\right\} . \tag{16}$$

Then (U, α) and (V, β) are the $B\Gamma$-charts desired if we make routine adjustments in U and V, because

$$f_{\alpha\beta} = \beta_1 \circ f \circ \left(\alpha_2 \circ \alpha_1\right)^{-1} = \beta_1 \circ f \circ \alpha_1^{-1} \circ \alpha_2^{-1} = f_{\alpha_1\beta_1} \circ \alpha_2^{-1} . \tag{17}$$

That is,

$$\begin{aligned} f_{\alpha\beta} &= Q_1 \circ f_{\alpha_1\beta_1} \circ \alpha_2^{-1} + Q_2 \circ f_{\alpha_1\beta_1} \circ \alpha_2^{-1} \\ &= \eta + \Phi \circ P_2 \quad \text{(by (13))} \end{aligned} \tag{18}$$

if we put $\eta = Q_1 \circ f_{\alpha_1\beta_1} \circ \alpha_2^{-1}$ which is a $C^r_{B\Gamma}$-map. //

(1.2) COROLLARY. *Let* X, Y *be as in Theorem* (1.1), $f : X \to Y$ *be a* $C^r_{B\Gamma}$*-map and* $W \subseteq Y$ *a* $C^r_{B\Gamma}$*-submanifold* $(r \geq 1)$. *Suppose that* $f \pitchfork_{B\Gamma} W$. *Then* $f^{-1}(W)$ *is either empty or a* $C^r_{B\Gamma}$*-submanifold of* X. *In the later case, we have* *(a)* *for* $x \in f^{-1}(W)$ *and* $y = f(x)$,

$$\left(T_x f\right)^{-1}\left(T_y W\right) = T_x\left(f^{-1}(W)\right) ;$$

(b) W *and* $f^{-1}(W)$ *have the same codimension: more precisely, for* $x \in f^{-1}(W)$, $y = f(x)$, *any* $B\Gamma$*-complement to* $T_x\left(f^{-1}(W)\right)$ *in* $T_x X$ *is* $B\Gamma$*-isomorphic to any* $B\Gamma$*-complement to* $T_y W$ *in* $T_y Y$.

Proof. Let $x \in f^{-1}(W)$, then $y = f(x) \in W$ and $f \pitchfork_{B\Gamma} xW$. By Theorem (1.1) we can find $B\Gamma$-charts (U, α) and (V, β) centred at x and y respectively, with

$$\alpha(U) = B_1 + B_2 \subseteq E_1 \oplus_{B\Gamma} E_2 = E ,$$

$$\beta(V) = D_1 + D_2 \subseteq F_1 \oplus_{B\Gamma} F_2 = F ,$$

and such that

$$f_{\alpha\beta} = \eta + \Phi \circ P_2 : B_1 + B_2 \to D_1 + D_2 \tag{19}$$

where $\eta : B_1 + B_2 \subseteq E \to F_1$ is $C^r_{B\Gamma}$ and $\Phi : E_2 \to F_2$ is a $B\Gamma$-isomorphism.

From this, we claim that $\alpha\left(f^{-1}(W) \cap U\right) = B_1$. Indeed, let $x' \in f^{-1}(W) \cap U$, then $y' = f(x') \in W \cap V$ and $\beta\left(f(x')\right) \in D_1 \subseteq F_1$. Furthermore, by (19), if $\alpha(x') = b_1 + b_2 \in B_1 + B_2$ we have

$$f_{\alpha\beta}\left(\alpha(x')\right) = f_{\alpha\beta}\left(b_1+b_2\right) = \eta\left(b_1+b_2\right) + \Phi\left(b_2\right) \in D_1 \subseteq F_1 . \tag{20}$$

Since $\Phi\left(b_2\right) \in F_2$ and $F_1 \cap F_2 = \{0\}$, this implies

$$\Phi\left(b_2\right) = 0 \text{ , i.e. } b_2 = 0 . \tag{21}$$

Hence

$$\alpha(x') \in B_1 \text{ for all } x' \in f^{-1}(W) \cap U .$$

That is,

$$\alpha\left(f^{-1}(W) \cap U\right) \subseteq B_1 \, . \tag{22}$$

Conversely,

$$B_1 \subseteq \alpha\left(f^{-1}(W) \cap U\right) \tag{23}$$

since for every $b_1 \in B_1$,

$$f_{\alpha\beta}\left(b_1\right) = \eta\left(b_1\right) + \Phi \circ P_2\left(b_1\right) = \eta\left(b_1\right) \in D_1 \, .$$

Thus

$$\beta\left[f\left(\alpha^{-1}\left(b_1\right)\right)\right] \in D_1 \Rightarrow f\left[\alpha^{-1}\left(b_1\right)\right] \in \beta^{-1}\left(D_1\right) = W \cap V$$

$$\Rightarrow f\left[\alpha^{-1}\left(b_1\right)\right] \in W$$

$$\Rightarrow \alpha^{-1}\left(b_1\right) \in f^{-1}(W) \cap V$$

$$\Rightarrow b_1 \in \alpha\left(f^{-1}(W) \cap V\right) \, .$$

Thus, for all $x \in f^{-1}(W)$, there exists a $B\Gamma$-chart (U, α) with $\alpha(U) = B_1 + B_2 \subseteq E_1 \oplus_{B\Gamma} E_2$ and $\alpha(x) = 0$, $\alpha\left(f^{-1}(W) \cap U\right) = B_1$. That is $f^{-1}(W)$ is a $C^r_{B\Gamma}$-submanifold of X .

(a) In the $B\Gamma$-charts (U, α) and (V, β) , we have

$$T_x X = E \, , \quad T_y Y = F \, ,$$

$$T_x\left(f^{-1}(W)\right) = E_1 \, , \quad T_y W = F_1$$

and

$$T_x f = f'_{\alpha\beta}(0) : E = E_1 \oplus_{B\Gamma} E_2 \to F = F_1 \oplus_{B\Gamma} F_2 \, .$$

Furthermore, by the proof of Theorem (1.1), we have

$$f'_{\alpha\beta}(0)^{-1}\left(F_1\right) = E_1 \, . \tag{24}$$

Hence

$$\left(T_x f\right)^{-1}\left(T_y W\right) = T_x\left(f^{-1}(W)\right) \, . \tag{25}$$

(b) Since in the $B\Gamma$-charts (U, α) and (V, β), E_2 is a $B\Gamma$-complement to $E_1 = T_x\left(f^{-1}(W)\right)$ in $E = T_x X$ and F_2 is a $B\Gamma$-complement to $F_1 = T_y W$ in $F = T_y W$, this part follows immediately from the fact that $\Phi : E_2 \to F_2$ is a $B\Gamma$-isomorphism. //

2. Representations of $B\Gamma$-Manifolds

In this section we generalise the treatement in [4] to the $B\Gamma$-manifolds.

Let E be a Γ-family and let $E, F, G \in E$ such that the $B\Gamma$-product (see Chapter 1) $E \times_{B\Gamma} F \in E$. Let A, X, Y be $C^r_{B\Gamma}$-manifolds modelled on E, F, G respectively. We shall denote by $A \times_{B\Gamma} X$ the $C^r_{B\Gamma}$-manifold product of A and X (which is modelled on $E \times_{B\Gamma} F \in E$), and by $C^r_{B\Gamma}(X, Y)$ the space of $C^r_{B\Gamma}$-maps from X to Y.

Let $\rho : A \to C^r_{B\Gamma}(X, Y)$ be a map. For $a \in A$, we follow [4] to write ρ_a instead of $\rho(a)$, i.e. $\rho_a : X \to Y$ is a $C^r_{B\Gamma}$-map. Following [4], we say that ρ is a $C^r_{B\Gamma}$-*representation* iff the evaluation map

$$\mathrm{ev}_\rho : A \times_{B\Gamma} X \to Y \tag{1}$$

given by

$$\mathrm{ev}_\rho(a, x) = \rho_a(x) \quad \text{for } a \in A,\ x \in X, \tag{2}$$

is a $C^r_{B\Gamma}$-map from $A \times_{B\Gamma} X$ to Y.

Now let (A, α) be an admissible $B\Gamma$-chart for A at a, (U, β) an admissible $B\Gamma$-chart for X at x and (V, γ) an admissible $B\Gamma$-chart for Y at $\mathrm{ev}_\rho(a, x) = \rho_a(x) = y$. Then $(A \times U, \alpha \times \beta)$ is a $B\Gamma$-chart for

$A \times_{B\Gamma} X$ at (a, x), and we have the following representative for ev_ρ :

$$(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma} : \alpha(A) \times \beta(U) \subseteq E \times_{B\Gamma} F \to \gamma(V) \subseteq G \tag{3}$$

given by:

$$\begin{aligned}(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}(e, f) &= \gamma \circ (\mathrm{ev}_\rho) \circ (\alpha \times \beta)^{-1}(e, f) \quad \forall(e, f) \in \alpha(A) \times \beta(U) \subseteq E \times_{B\Gamma} F \\ &= \gamma \circ (\mathrm{ev}_\rho)(\alpha^{-1}(e), \beta^{-1}(f)) \\ &= \gamma\left[\rho_{\alpha^{-1}(e)}(\beta^{-1}(f))\right] \\ &= \left(\gamma \circ \rho_{\alpha^{-1}(e)} \circ \beta^{-1}\right)(f) .\end{aligned}$$

That is

$$(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}(e, f) = (\rho_{\alpha^{-1}(e)})_{\beta,\gamma}(f) \tag{4}$$

where $(\rho_{\alpha^{-1}(e)})_{\beta,\gamma}$ is the local representative of $\rho_{\alpha^{-1}(e)} : X \to Y$ in the $B\Gamma$-charts (U, β) and (V, γ) at x and $y = \rho_a(x)$ respectively.

If $(a, x) \in A \times_{B\Gamma} X$, then we have

$$T_a A \cong E , \quad T_x X \cong F , \quad T_y Y \cong G$$

and

$$T_{(a,x)}(A \times_{B\Gamma} X) \cong T_a A \times_{B\Gamma} T_x X = E \times_{B\Gamma} F .$$

We want to calculate the tangent map

$$T_{(a,x)}\mathrm{ev}_\rho : T_{(a,x)}(A \times_{B\Gamma} X) \to T_{\rho_a(x)}Y = T_y Y .$$

In the above $B\Gamma$-charts, we can write

$$T_{(a,x)}(A \times_{B\Gamma} X) = \{(\alpha(a), \beta(x))\} \times E \times_{B\Gamma} F ; \quad T_{\rho_a(x)}Y = \{\gamma(y)\} \times G \tag{5}$$

and $T_{(a,x)}\mathrm{ev}_\rho$ is given by

$$T(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}(\alpha(a), \beta(x)) : (\alpha(a), \beta(x)) \times E \times_{B\Gamma} F \to \gamma(y) \times G$$

with

$$\big(\alpha(a),\ \beta(x),\ e,\ f\big) \longmapsto \big(\gamma(y),\ (\mathrm{ev}_\rho)'_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \beta(x)\big).(e,\ f)\big)$$

where

$$\gamma(y) = (\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \beta(x)\big)$$

and

$$(\mathrm{ev}_\rho)'_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \beta(x)\big) : E \times_{B\Gamma} F \to G \quad \text{is linear } B\Gamma\text{-continuous.}$$

By the Γ-differentiation theory applied to

$$(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma} : \alpha(A) \times \beta(U) \subseteq E \times_{B\Gamma} F \to \gamma(V) \subseteq G$$

we can write

$$(\mathrm{ev}_\rho)'_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \beta(x)\big) = \partial_1(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\alpha(a)\big) + \partial_2(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\beta(x)\big)\ . \tag{6}$$

Hence, for each $(\dot{a},\ \dot{x}) \in T_aA \times_{B\Gamma} T_xX$, identified to $(e,\ f) \in E \times_{B\Gamma} F$,

we have

$$\big(T_{(a,x)}\mathrm{ev}_\rho\big)(\dot{a},\ \dot{x}) = \big(T_{1(a,x)}\mathrm{ev}_\rho\big)\dot{a} + \big(T_{2(a,x)}\mathrm{ev}_\rho\big)\dot{x}\ . \tag{7}$$

Furthermore, we claim that

$$T_{2(a,x)}\mathrm{ev}_\rho = T_x\rho_a\ . \tag{8}$$

Indeed, $\rho_a : X \to Y$ is represented in the $B\Gamma$-charts $(U,\ \beta)$ and $(V,\ \gamma)$

by

$$(\rho_a)_{\beta,\gamma} : \beta(U) \subseteq F \to \gamma(V) \subseteq G$$

and $T_{2(a,x)}\mathrm{ev}_\rho$ is represented by

$$\partial_2(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\beta(x)\big) = (\mathrm{ev}_\rho)'_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \cdot\big)\big(\beta(x)\big) \tag{9}$$

where $(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\alpha(a),\ \cdot\big) : \beta(U) \subseteq F \to \gamma(V) \subseteq G$ denotes the partial map

obtained by *fixing* $\alpha(a) \in \alpha(A)$. But when we fix $\alpha(a) \in \alpha(A)$, we have,

for all $f \in F$:

$$\begin{aligned}(\mathrm{ev}_\rho)_{\alpha\times\beta,\gamma}\big(\alpha(a),\ f\big) &= \gamma \circ \rho_a \circ \beta^{-1}(f) \\ &= (\rho_a)_{\beta,\gamma}(f)\ .\end{aligned} \tag{10}$$

That is,

$$\left(\mathrm{ev}_\rho\right)'_{\alpha\times\beta,\gamma}\left(\alpha(a),\ \cdot\right) = \left(\rho_a\right)'_{\beta,\gamma}\left(\beta(x)\right) . \tag{11}$$

Since $T_x\rho_a$ is represented by $\left(\rho_a\right)'_{\beta,\gamma}\left(\beta(x)\right)$, (8) follows quickly.

The above discussion gives:

(2.1) PROPOSITION. *Let* A, X, Y *be* $C^r_{B\Gamma}$*-manifolds modelled on* E, F, $G \in E$ *respectively, with* $r \geq 1$, *and let* $\rho : A \to C^r_{B\Gamma}(X, Y)$ *be a* $C^r_{B\Gamma}$ *representation. Let* $\mathrm{ev}_\rho : A \times_{B\Gamma} X \to Y$ *be the evaluation map.*

Then for each $(a, x) \in A \times_{B\Gamma} X$, *we have*

$$T_{(a,x)}\mathrm{ev}_\rho = T_{1(a,x)}\mathrm{ev}_\rho + T_x\rho_a \tag{12}$$

which means that for every $(\dot{a}, \dot{x}) \in T_{(a,x)}\left(A \times_{B\Gamma} X\right)$, *we have*

$$\left(T_{(a,x)}\mathrm{ev}_\rho\right)(\dot{a}, \dot{x}) = \left(T_{1(a,x)}\mathrm{ev}_\rho\right)\dot{a} + \left(T_x\rho_a\right)\dot{x} . \tag{13}$$

3. The $B\Gamma$-Transversal Density Theorem

In this section, we apply the Smale Density Theorem in Chapter 3 to prove the $B\Gamma$-Transversal Density Theorem, a generalisation of the one in [4] to the $B\Gamma$-manifolds and $B\Gamma$-transversality.

First, we prove a technical lemma.

Let E be a Γ-family and let $F, G \in E$ such that $F \times_{B\Gamma} G \in E$. (For the definition of $F \times_{B\Gamma} G$ see Chapter 1.) Denote by $\pi_1 : F \times_{B\Gamma} G \to F$ the first projection. We say that a $B\Gamma$-splitting subspace E of $F \times_{B\Gamma} G$ *$B\Gamma$-adapts* π_1 if there exists a $B\Gamma$-complement H of E in $F \times_{B\Gamma} G$ such that

$$\pi_1\left(E \oplus_{B\Gamma} H\right) = \pi_1(E) \oplus_{B\Gamma} \pi_1(H)\ (= F) . \tag{1}$$

Note that in the category of Banach spaces with the norm-calibration Γ, every finite-codimensional subspace of $F \times G$ with $\dim G < +\infty$, $B\Gamma$-adapts the first projection $F \times G \to F$. (See the proof of Lemma 19.2 in [4], p. 49.)

The following lemma is a generalisation of Lemma 19.2 in [4] to the $B\Gamma$-context.

(3.1) LEMMA. *Let* $F, G \in \mathcal{E}$ *with* $F \times_{B\Gamma} G \in \mathcal{E}$, $\dim G = n < +\infty$, $\pi_1 : F \times_{B\Gamma} G \to F$ *be the projection onto the first factor and let* $E \subseteq F \times_{B\Gamma} G$ *be a* $B\Gamma$*-splitting subspace of codimension* q . *Suppose that* E $B\Gamma$*-adapts* π_1 . *Then the restriction* $\pi = \pi_1|E$ *is a* $B\Gamma$*-Fredholm operator with index* $n - q$.

Proof. We can find a $B\Gamma$-complement H for E in $F \times_{B\Gamma} G$ such that (1) holds. We have $\dim H = \operatorname{codim} E = q$. Let P and Q be the first and second projections corresponding to the $B\Gamma$-decomposition $F \times_{B\Gamma} G = E \oplus_{B\Gamma} H$. Then P and Q are $B\Gamma$-continuous.

We put $L = P\pi(E)$ and $K = E \cap (0 \times G)$. We shall prove

$$K \oplus_{B\Gamma} L = E , \tag{2}$$

$$L \underset{B\Gamma}{\approx} \pi(E) . \tag{3}$$

Indeed, first consider F as $F \times 0 \subseteq F \times_{B\Gamma} G$, then, for every $e \in E$, $\pi(e) \in F$ can be written in a unique decomposition

$$\pi(e) = P\pi(e) + Q\pi(e)$$

with $P\pi(e) \in E$, $Q\pi(e) \in H$.

Then

$$\pi^2(e) = \pi(e) = \pi P\pi(e) + \pi Q\pi(e)$$

which implies

$$\pi[e-P\pi(e)] + \pi(h) = 0 \quad \text{where} \quad h = -Q\pi(e) \in H . \tag{4}$$

Since $e - P\pi(e) \in E$ and $\pi(E) \oplus \pi(H) = F$, (4) implies

$$\pi[e-P\pi(e)] = 0 \quad \text{for every} \quad e \in E . \tag{5}$$

Now, for every $e \in E$, we can write

$$e = [e-P\pi(e)] + P\pi(e) \in K + L . \tag{6}$$

The fact that $e - P\pi(e) \in K$ follows from (5) and from the obvious relation

$K = \ker \pi = E \cap (0 \times G)$.

But we also have $K \cap L = \{0\}$ (by (5)) hence $E = K \oplus L$. Since the projection on L corresponding to the decomposition $K \oplus L$ is nothing but the composite $P\pi$ which is $B\Gamma$-continuous, we have (2).

As for (3), we note that

$$\pi|_L : L \to \pi(E)$$

is one-to-one, onto and $B\Gamma$-continuous, and by (5), we can see that the inverse map is $P|_{\pi(E)}$ which is also $B\Gamma$-continuous. Hence we have (3).

From the above discussion, it follows easily that π is a $B\Gamma$-Fredholm operator and

$$\text{ind } \pi = n - q \text{ .} \quad // \tag{7}$$

We now state and prove the main theorem (see [4]).

(3.2) THEOREM ($B\Gamma$-Transversal Density Theorem). *Let* $\mathcal{E}$ *be a* Γ*-family,* $F, G, J \in \mathcal{E}$ *with* F *sequentially complete and* $F \times_{B\Gamma} G \in \mathcal{E}$. *Let* A, X, Y *be* $C^r_{B\Gamma}$*-manifolds modelled respectively on* F, G, J ; $\rho : \mathrm{A} \to C^r_{B\Gamma}(X, Y)$ *a* $C^r_{B\Gamma}$*-representation;* $W \subseteq Y$ *a* $C^r_{B\Gamma}$*-submanifold (not necessarily closed) and* $\mathrm{ev}_\rho : \mathrm{A} \times_{B\Gamma} X \to Y$ *the evaluation map.*

Define $\mathrm{A}_W \subseteq \mathrm{A}$ *by*

$$\mathrm{A}_W = \{a \in \mathrm{A} : \rho_a \pitchfork_{B\Gamma} W\} \text{ .} \tag{8}$$

Assume that

(a) X *has finite dimension* n *and* W *has finite codimension* q ,

(b) A *and* X *are second countable,*

(c) $r > \max(0, n-q)$,

(d) $\mathrm{ev}_\rho \pitchfork_{B\Gamma} W$,

(e) *for every* $(a, x) \in \mathrm{A} \times_{B\Gamma} X$ *such that* $y = \rho_a(x) \in W$, *the* $B\Gamma$*-splitting subspace* $\left(T_{(a,x)}\mathrm{ev}_\rho\right)^{-1}\left(T_y W\right)$ *of* *assertion* $T_{(a,x)}(A \times_{B\Gamma} X) = T_a A \times_{B\Gamma} T_x X$ *$B\Gamma$-adapts the first projection*

$T_a A \times_{B\Gamma} T_x X \to T_a A$.

Then A_W *is residual in* A . *Moreover, if the model* F *of the manifold* A *is a Baire LCS, then* A_W *is dense in* A .

Proof (see [4]). Before proving the theorem, we note that condition (e) in the theorem is well-defined (independent of the $B\Gamma$-charts chosen).

Define $B = ev_\rho^{-1}(W) \subseteq A \times_{B\Gamma} X$. Then by Corollary (1.2), B is a $B\Gamma$-submanifold of $A \times_{B\Gamma} X$ of codimension q . Furthermore, for every $(a, x) \in B$ we have

$$T_{(a,x)}B = \left(T_{(a,x)}ev_\rho\right)^{-1}\left(T_y W\right) \tag{9}$$

and is a $B\Gamma$-splitting subspace of $T_a A \times_{B\Gamma} T_x X = T_{(a,x)}\left(A \times_{B\Gamma} X\right)$.

Let $P_1 : A \times_{B\Gamma} X \to A$ be the projection on the first factor and let $P : B \to A$ be given by $P = P_1 | B$.

Clearly P is a $C^r_{B\Gamma}$-map. Let R_P be the set of regular values of P . We shall prove that

$$P \text{ is a } B\Gamma\text{-Fredholm map of constant index } n - q \text{ ,} \tag{10}$$

$$A_W = R_P \text{ .} \tag{11}$$

The theorem then follows from the Smale Density Theorem of Chapter 3.

(*) $P : B \to A$ is a $B\Gamma$-Fredholm map of index $n - q$

Choose $(a, x) \in B$, we must show that the tangent map $T_{(a,x)}P : T_{(a,x)}B \to T_a A$ is a $B\Gamma$-Fredholm operator with index $n - q$.

If in Lemma (3.1) we read $T_x X$ for G , $T_a A$ for F , $T_a A \times_{B\Gamma} T_x X = T_{(a,x)}\left(A \times_{B\Gamma} X\right)$ for $F \times_{B\Gamma} G$, $T_{(a,x)}B$ for E , $T_{(a,x)}P_1$ for π_1 and $T_{(a,x)}P$ for π , then Lemma (3.1) gives us the desired assertion if we note that condition (e) means E $B\Gamma$-adapts π_1 .

(**) $A_W = R_P$

The inclusion $A_W \subseteq R_P$ is easy (see [4], p. 50). We only need to prove $R_P \subseteq A_W$.

Choose $a \in R_P$. We want to prove that $a \in A_W$, that is $\rho_a \pitchfork_{B\Gamma} W$.

Two cases can occur.

(a) $q \leq n$: Let $x \in X$ so that $y = \rho_a(x) \in W$. From the hypothesis (d), $ev_\rho \pitchfork_{B\Gamma} xW$, we can find $B\Gamma$-charts (A, α_1) centred at a , (U, β_1) centred at x and (V, γ) satisfies

$$\gamma(V) = D_1 + D_2 \subseteq J_1 \oplus_{B\Gamma} J_2 = J ,$$

$$\gamma(y) = 0 ,$$

$$\gamma(W \cap V) = D_1 \subseteq J_1 .$$

Then $(A \times U, \alpha_1 \times \beta_1)$ is a $B\Gamma$-chart for $A \times_{B\Gamma} X$ and with respect to these charts, the composite

$$F \times_{B\Gamma} G \xrightarrow{(ev_\rho)'_{\alpha_1\times\beta_1,\gamma}(0)} J \xrightarrow{Q_2} J_2$$

is a $B\Gamma$-splitting surjection.

That means

$$\operatorname{Ker}\left(Q_2 \circ (ev_\rho)'_{\alpha_1\times\beta_1,\gamma}(0)\right) = (ev_\rho)'_{\alpha_1\times\beta_1,\gamma}(0)^{-1}(J_1) = E \tag{12}$$

is $B\Gamma$-splitting in $F \times_{B\Gamma} G$;

there exists a $B\Gamma$-complement H_1 of E in $F \times_{B\Gamma} G$ such that

$$Q_2 \circ (ev_\rho)'_{\alpha_1\times\beta_1,\gamma}(0)\Big|_{H_1} : H_1 \overset{\approx}{\underset{B\Gamma}{\longrightarrow}} J_2 . \tag{13}$$

Now, by condition (e), we have a $B\Gamma$-complement H of E in $F \times_{B\Gamma} G$ such that

$$\pi_1\left(E \oplus_{B\Gamma} H\right) = \pi_1(E) \oplus_{B\Gamma} \pi_1(H) = (F) . \tag{14}$$

Since H and H_1 are $B\Gamma$-complements of the same subspace E in $F \times_{B\Gamma} G$; we can find a $B\Gamma$-isomorphism

$$\Phi : F \times_{B\Gamma} G = E \oplus_{B\Gamma} H_1 \to E \oplus_{B\Gamma} H = F \times_{B\Gamma} G$$

such that Φ maps H_1 onto H and E onto itself.

Put $\alpha \times \beta = \Phi \circ \left(\alpha_1 \times \beta_1\right)$: then $(A \times U, \alpha \times \beta)$ is a $B\Gamma$-chart for $A \times_{B\Gamma} X$ and with respect to $(A \times U, \alpha \times \beta)$ and the above chart (V, γ) we have the following properties:

$$E = \left(\mathrm{ev}_\rho\right)'_{\alpha\times\beta,\gamma}(0)^{-1}\left(J_1\right) \text{ is } B\Gamma\text{-splitting in } F \times_{B\Gamma} G ; \tag{15}$$

there exists a $B\Gamma$-complement H of E in $F \times_{B\Gamma} G$ such that

$$\pi_1\left(E \oplus_{B\Gamma} H\right) = \pi_1(E) \oplus_{B\Gamma} \pi_1(H) ; \tag{16}$$

the composite $Q_2 \circ \left(\mathrm{ev}_\rho\right)'_{\alpha\times\beta,\gamma}(0)$ restricted to H is a

$B\Gamma$-isomorphism of H onto J_2 . (17)

Now, since $a \in \mathbb{R}_P$, $T_{(a,x)}P : T_{(a,x)}\mathcal{B} \to T_a\mathcal{A}$ is onto. Thus $\pi : E \to F$ is onto, where $\pi = \pi_1|E$ and $\pi_1 : F \times_{B\Gamma} G \to F$ is the first projection.

We thus have

$$\pi(E) = \pi_1(E) = F .$$

Since $\pi_1(E) \oplus_{B\Gamma} \pi_1(H) = F$ and $\pi_1(H) \subseteq F$, we have

$$\pi_1(H) = \pi_1(H) \cap F = \pi_1(H) \cap \pi_1(E) = \{0\} .$$

Hence

$$H \subseteq 0 \times G$$

and we can write

$$H = \{0\} \times \tilde{H} \text{ with } \tilde{H} \subset G . \tag{18}$$

Now consider $\rho_a : X \to Y$ and $T_x\rho_a : T_xX \to T_yY$. Then in the $B\Gamma$-charts (U, β) and (V, γ) above, $T_x\rho_a$ is represented by the $B\Gamma$-derivative

$$\left(\rho_\alpha\right)'_{\beta,\gamma}(0) : G \to J .$$

If we denote by

$$K = E \cap (0 \times G) = \{0\} \times \tilde{K}$$

then, from the discussion in §2 (see Proposition (2.1)), it is not hard to see

$$\tilde{K} = \left(\rho_\alpha\right)'_{\beta,\gamma}(0)^{-1}\left(J_1\right) \subseteq G \tag{19}$$

and

$$G = \tilde{K} \oplus_{B\Gamma} \tilde{H} , \tag{20}$$

$$\left(\mathrm{ev}_\rho\right)'_{\alpha\times\beta,\gamma}(0)\big|_H = 0 \times \left(\left(\rho_\alpha\right)'_{\beta,\gamma}(0)\big|_{\tilde{H}}\right) . \tag{21}$$

From (21) and (17) it follows quickly that

$$Q_2 \circ \left(\rho_\alpha\right)'_{\beta,\gamma}(0)\big|_{\tilde{H}} : \tilde{H} \xrightarrow[B\Gamma]{\approx} J_2$$

and the composite

$$G \xrightarrow{\left(\rho_\alpha\right)'_{\beta,\gamma}(0)} G \xrightarrow{Q_2} J_2$$

is a $B\Gamma$-splitting surjection, which proves $\rho_\alpha \pitchfork_{B\Gamma} xW$.

(b) $q > n$: From the above discussion, we have

$$\rho_\alpha(X) \cap W = \emptyset ;$$

that is,

$$\rho_\alpha \pitchfork_{B\Gamma} W .$$

So, in any case, $\alpha \in R_P \Rightarrow \alpha \in A_W$ and the proof is completed. //

(3.3) REMARK. We note that Theorem (3.2) is a generalisation of the corresponding one in [4], since in the Banach case (with the norm-calibration), condition (e) in Theorem (3.2) is automatically satisfied, and our $B\Gamma$-transversality reduces to the usual transversality.

4. The $B\Gamma$-Transversal Isotopy Theorem

In this section we generalise the important Transversal Isotopy Theorem in [4] to our context of $B\Gamma$-manifolds and $B\Gamma$-transversality.

We suppose that the Γ-family E is a Γ-family with $B\Gamma$-product (see Chapter 1, §2) and let A, X, Y be $C^r_{B\Gamma}$-manifolds $(r \geq 1)$ modelled respectively on $E, F, G \in E$.

Let $\rho : A \to C^r_{B\Gamma}(X, Y)$ be a $C^r_{B\Gamma}$-representation. Then the following theorem is a generalisation of Theorem 18.2 in [4], p. 47.

(4.1) THEOREM (Openness of $B\Gamma$-transversality). *Let* A, X, Y *be* $C^1_{B\Gamma}$*-manifolds modelled respectively on* $E, F, G \in E$ *, where* E *is a* Γ*-family with* $B\Gamma$*-product. Let* $W \subseteq Y$ *be a closed* $C^1_{B\Gamma}$*-submanifold,* $K \subseteq X$ *a compact subset of* X *and* $\rho : A \to C^1_{B\Gamma}(X, Y)$ *a* $C^1_{B\Gamma}$*-representation.*

Then the subset A_{KW} *of* A *defined by*

$$A_{KW} = \{a \in A \mid \rho_a \pitchfork_{B\Gamma} xW \ \text{ for } \ x \in K\}$$

is open in A .

Proof (see [4]). Consider the $B\Gamma$-bundle of $B\Gamma$-continuous linear maps (see Chapter 2, §5):

$$L_{B\Gamma}(\tau_X, \tau_Y) : L_{B\Gamma}(TX, TY) \to X \times_{B\Gamma} Y$$

whose fibre over a point $(x, y) \in X \times_{B\Gamma} Y$ is the space $L_{B\Gamma}(T_xX, T_yY)$ of $B\Gamma$-continuous linear maps from T_xX to T_yY . Define the following subset Ω of $L_{B\Gamma}(TX, TY)$. An element $A \in L_{B\Gamma}(TX, TY)$ is in Ω iff the following condition is satisfied: if $x \in X$, $y \in Y$ and $A \in L_{B\Gamma}(T_xX, T_yY)$ then

(i) either $y \notin W$, or

(ii) $y \in W$, and there exists a $B\Gamma$-complement Z of T_yW in

T_yY such that the composite

$$T_xX \xrightarrow{A} T_yY \xrightarrow{Q_2} Z$$

$\left(\text{where } Q_2 : T_yY = T_yW \oplus_{B\Gamma} Z \to Z \text{ is the second projection}\right)$

is a $B\Gamma$-splitting surjection.

Then Ω is open in $L_{B\Gamma}(TX, TY)$. Indeed we have

$$\Omega = \Omega_1 \cup \Omega_2 \tag{1}$$

with

$$\Omega_1 = \{A \in L_{B\Gamma}(TX, TY) \mid \text{if } A \in L_{B\Gamma}\left(T_xX, T_yY\right) \text{ then } y \notin W\} \tag{2}$$

which is open in $L_{B\Gamma}(TX, TY)$ because W is closed, and

$$\Omega_2 = \{A \in L_{B\Gamma}(TX, TY) \mid \text{if } A \in L_{B\Gamma}\left(T_xX, T_yY\right) \text{ then } y \in W \text{ and (ii) is satisfied}\} \tag{3}$$

which is open in $L_{B\Gamma}(TX, TY)$ by Proposition (1.2), Chapter 3.

Now consider $W' = L_{B\Gamma}(TX, TY)\backslash\Omega$, then W' is closed in $L_{B\Gamma}(TX, TY)$. Consider the map

$$\rho' : \mathrm{A} \to C^0_{B\Gamma}\left(X, L_{B\Gamma}(TX, TY)\right) \tag{4}$$

defined by $a \longmapsto \rho'_a : x \longmapsto \rho'_a(x) = T_x\rho_a$. Then, since ρ is a $C^1_{B\Gamma}$-representation, ρ' is a $C^0_{B\Gamma}$-representation.

By construction, $\rho_a \pitchfork_{B\Gamma} xW$ if and only if $\rho'_a(x) \notin W'$. Hence

$$\mathrm{A}_{KW} = \{a \in \mathrm{A} \mid \rho'_a(K) \cap W' = \emptyset\} \tag{5}$$

which is open in A by Theorem 18.1 in [4], p. 46, about openness of non-intersection. //

We now prove the most important property of $B\Gamma$-transversality: the stability of $B\Gamma$-transversal intersection. This is a generalisation of the one in [4] to the $B\Gamma$-context.

First recall briefly what is meant by a C^r-isotopy. Let X be a Banach manifold of class C^r. A C^r-diffeomorphism $F : X \to X$ is *C^r-isotopic to the identity* iff there is a C^r-map

$$\varphi : X \times I \to X \tag{6}$$

(where I is an open interval in $\mathbb{R}$ containg [0, 1]) such that $\varphi_t = \varphi|_{X\times\{t\}}$ is a C^r-diffeomorphism for all $t \in [0, 1]$ and $\varphi_0 = id_X$, $\varphi_1 = F$.

Now, if W_0 and W are two submanifolds of X, we say that *W is C^r-istopic to W_0* iff there is a C^r-diffeomorphism

$$F : X \to X$$

such that $F(W_0) = W$ and F is C^r-isotopic to the identity.

Now let E be a Γ-family with $B\Gamma$-product and A, X, Y be $C^{r+1}_{B\Gamma}$-manifolds $(r \geq 1)$ modelled on $E, F, G \in E$. Suppose that X is *compact* (hence F is a finite-dimensional space).

Let $\rho : A \to C^{r+1}_{B\Gamma}(X, Y)$ be a $C^{r+1}_{B\Gamma}$-representation, $W \subseteq Y$ a closed $C^{r+1}_{B\Gamma}$-submanifold, and $a_0 \in A$ a point such that $\rho_{a_0} \pitchfork_{B\Gamma} W$.

For each $a \in A$, let $W_a = \rho_a^{-1}(W) \subseteq X$. Then by Theorem (4.1), $\rho_a \pitchfork_{B\Gamma} W$ for $a \in A$ sufficiently near a_0. Hence, by Corollary (1.2), W_a is a $C^{r+1}_{B\Gamma}$-submanifold of X (i.e. a C^{r+1}-submanifold of X in the Banach sense since X is a (Banach) finite-dimensional manifold). One might expect that for a near a_0, the submanifolds W_a and W_{a_0} are close. The $B\Gamma$-Transversal Isotopy Theorem makes this precise. It says that W_a and W_{a_0} are isotopic.

(4.2) THEOREM (*$B\Gamma$-Transversal Isotopy Theorem*, [4]). *Let E be a Γ-family with $B\Gamma$-product. Let X be a compact C^{r+3}-manifold $(r \geq 1)$ modelled on a (finite-dimensional) space $F \in E$, A and Y be $C^{r+1}_{B\Gamma}$-manifolds $(r \geq 1)$ modelled on $E, G \in E$. Let $\rho : A \to C^{r+1}_{B\Gamma}(X, Y)$ be a $C^{r+1}_{B\Gamma}$-representation and $W \subseteq Y$ be a closed $C^{r+1}_{B\Gamma}$-submanifold. Let $a_0 \in A$ be a point, and for each $a \in A$, let $W_a = \rho_a^{-1}(W) \subseteq X$.*

Then if $\rho_{a_0} \pitchfork_{B\Gamma} W$, there is an open neighbourhood N of a_0 in A such that, for $a \in N$, W_a is C^r-isotopic to W_{a_0}.

Proof (see [4]). This proof is exactly the one in [4] rewritten in our language of $B\Gamma$-manifolds and $B\Gamma$-transversality.

Since X is a finite-dimensional manifold of class C^{r+3} $(r \geq 1)$ and W_{a_0} is a closed submanifold of X, we can find a C^{r+1}-total tubular neighbourhood of W_{a_0} in X (see [31] or [44]), that is, we can find an open neighbourhood Ω of W_{a_0} in X, a surjective map

$$\pi : \Omega \to W_{a_0}$$

and a C^{r+1}-vector bundle structure on π which makes Ω an open submanifold of X.

Take a Riemannian metric on π and a reduction to the Hilbert group (Lang [44], Chapter VII); and let $\|\cdot\| : \Omega \to \mathbb{R}$ be the Finsler associated with this Riemannian metric in the usual fashion; that is,

$$\|\omega\|^2 = \langle \omega, \omega \rangle \quad \text{for} \quad \omega \in \Omega \tag{7}$$

where $\langle\ ,\ \rangle$ is the Riemannian metric.

Thus we have an admissible covering of π by VB charts (α, α_0, U)

where U is an open subset of W_{a_0} , $\alpha : \pi^{-1}(U) \to \alpha_0(U) \times F_\alpha$, and for each such chart a norm $\|\cdot\|_\alpha$ on F_α such that

$$\|\omega\| = \|\alpha(\omega)\|_\alpha \quad \text{for} \quad \omega \in \pi^{-1}(U) . \tag{8}$$

This covering is called the *reduced atlas for* π (see [4]).

To prove Theorem (4.2) we first prove a technical lemma.

(4.3) LEMMA ([4]). *There is an open neighbourhood* N *of* a_0 *in* A *such that, for* $a \in N$, $W_a = \rho_a^{-1}(W)$ *is the image of a* C^{r+1}*-section of* π ; *that is, for* $a \in N$, *there is a* $\xi_a \in S^{r+1}(\pi)$ *such that* $W_a = \xi_a\left(W_{a_0}\right)$.

Proof (see [4]). For each real number $t > 0$ we define

$$B_t = \{\omega \in \Omega \mid \|\omega\| < t\} \tag{9}$$

and for an open subset U of W_{a_0} we define $B_t(U) \subseteq \Omega$ by

$$B_t(U) = \{\omega \in \pi^{-1}(U) \mid \|\omega\| < t\} . \tag{10}$$

If (α, α_0, U) is a member of the reduced atlas with
$\alpha(\pi^{-1}(U)) = \alpha_0(U) \times F_\alpha$, then

$$\alpha(B_t(U)) = \alpha_0(U) \times B_{\alpha t} \tag{11}$$

where $B_{\alpha t}$ is the open ball of radius t centred at the origin of F_α .

For each $x \in X$, we define an open neighbourhood N_x of a_0 in A and an open neighbourhood Z_x of x in X as follows.

(i) If $x \notin W_{a_0}$, then $\rho_{a_0}(x) \notin W$. Since $W \subseteq Y$ is closed, and the evaluation map $\text{ev}_\rho : A \times_{B\Gamma} X \to Y$ is continuous, we may take N_x and Z_x satisfying the condition

$$Z_x \cap W_a = \emptyset \quad \text{for} \quad a \in N_x . \tag{12}$$

(ii) Suppose $x \in W_{a_0}$. Then $\rho_{a_0}(x) \in W$. Choose an admissible $B\Gamma$-chart (V, β) in Y at $\rho_{a_0}(x)$ having the $B\Gamma$-submanifold property for W; that is,

$$\beta(V) = V_1 + V_2 \subseteq G_1 \oplus_{B\Gamma} G_2 = G ,$$

$$\beta\left(\rho_{a_0}(x)\right) = 0 ,$$

$$\beta(W \cap V) = V_1 \subseteq G_1 .$$

Because the evaluation map $\mathrm{ev}_\rho : A \times_{B\Gamma} X \to Y$ is continuous, we may choose an open neighbourhood N_x of a_0 in A, a VB-chart (α, α_0, U) on π at x from the reduced atlas, and a real number $t > 0$ such that

$$\rho_a\left(B_t(U)\right) \subseteq V \quad \text{for all} \quad a \in N_x . \tag{13}$$

Let H_α be the model space of the manifold W_{a_0} (then $\alpha_0(U) \subseteq H_\alpha$) and F_α be the model of the fibre of π in the chart (α, α_0, U), that is,

$$\alpha\left(\pi^{-1}(U)\right) = \alpha_0(U) \times F_\alpha \subseteq H_\alpha \times F_\alpha = (F) . \tag{14}$$

Then as we see above

$$\alpha\left(B_t(U)\right) = \alpha_0(U) \times B_{\alpha t} .$$

Let $Q_2 : G_1 \oplus_{B\Gamma} G_2 = G \to G_2$ be the second projection and consider the map

$$f : N_x \times \alpha_0(U) \times B_{\alpha t} \subseteq A \times_{B\Gamma} H_\alpha \times F_\alpha \to V_2 \subseteq G_2 \tag{15}$$

defined by

$$(a, u, v) \longmapsto f(a, u, v) = Q_2 \circ \beta \circ \rho_a \circ \alpha^{-1}(u, v) \tag{16}$$

for $a \in N_x$, $u \in \alpha_0(U) \subseteq H_\alpha$ and $v \in B_{\alpha t} \subseteq F_\alpha$.

Assume for simplicity that $\alpha(x) = (0, 0)$ and $\beta\left(\rho_{a_0}(x)\right) = 0$.

Then f is obviously $C^{r+1}_{B\Gamma}$. Furthermore, since $\rho_{a_0} \pitchfork_{B\Gamma} W$, we can see that the partial $B\Gamma$-derivative

$$\partial_3 f(a_0, 0, 0) : F_3 \to G_2$$

is a $B\Gamma$-isomorphism of F_3 onto G_2.

By the Implicit Function Theorem (3.13), Chapter 1, (making N_x, $\alpha_0(U)$ and $B_{\alpha t}$ smaller if necessary) we have a $C^{r+1}_{B\Gamma}$-map

$$h : N_x \times \alpha_0(U) \subseteq A \times_{B\Gamma} H_\alpha \to B_{\alpha t} \subseteq F_\alpha \tag{17}$$

so that

$$f^{-1}(0) \cap (N_x \times \alpha_0(U) \times B_{\alpha t}) = \{(a, x, h(a, x)) \mid (a, x) \in N_x \times \alpha_0(U)\} . \tag{18}$$

We take

$$Z_x = \alpha^{-1}(\alpha_0(U) \times B_{\alpha t}) = B_t(U) . \tag{19}$$

Then

$$W_a \cap Z_x = \alpha^{-1}\left[f_a^{-1}(0) \cap (\alpha_0(U) \times B_{\alpha t})\right] \tag{20}$$

for $a \in N_x$; here $f_a : \alpha_0(U) \times B_{\alpha t} \to V_2$ is given by

$$f_a(u, v) = f(a, u, v) \quad \text{for } (u, v) \in \alpha_0(U) \times B_{\alpha t} . \tag{21}$$

This completes the definition of Z_x and N_x in case (ii).

By the compactness of X, finitely many of the Z_x's cover X, say

$$X = Z_{x_1} \cup \ldots \cup Z_{x_n} . \tag{22}$$

Then define N by

$$N = N_{x_1} \cap \ldots \cap N_{x_n} . \tag{23}$$

We claim that this neighbourhood N satisfies the conclusion of the lemma: for each fixed $a \in N$, $W_a \subseteq \Omega$ and W_a intersects each fibre $\Omega_x = \pi^{-1}(x)$, $(x \in W_{a_0})$, in exactly one point.

To prove this, fix $a \in N$. If $x_i \notin W_{a_0}$, then $Z_{x_i} \cap W_a = \emptyset$ by

(12); for $x_i \in W_{a_0}$, $Z_{x_i} \subseteq \Omega$ by (19). Thus $W_a \subseteq \Omega$. Choose $x \in W_{a_0}$. We must show that $W_a \cap \Omega_x$ consists of a single point. By (18) and (20) and since the Z_{x_i}'s cover X , $W_a \cap Z_{x_i} \cap \Omega_x$ consists of exactly one point for some $i = 1, \ldots, n$; indeed, this is true for each $i = 1, \ldots, n$ such that $x \in Z_{x_i}$. Suppose that $W_a \cap \Omega_x$ contains two points and let these two points be in Z_{x_i} and Z_{x_j} , respectively. By (12) we must have $x_i, x_j \in W_{a_0}$. Then by (19),

$$Z_{x_i} = B_{t_i}(U_i) \quad \text{and} \quad Z_{x_j} = B_{t_j}(U_j) \tag{24}$$

where $x \in U_i \cap U_j$. But either $t_i \leq t_j$ or $t_j \leq t_i$; hence either $\Omega_x \cap Z_{x_i} \subseteq \Omega_x \cap Z_{x_j}$ or $\Omega_x \cap Z_j \subseteq \Omega_x \cap Z_{x_i}$. Thus, in either case, $W_a \cap \Omega_x \cap Z_{x_k}$ ($k = i$ or j) consists of two points contradicting our previous conclusion. This proves that $W_a \cap \Omega_x$ contains exactly one point.

Hence W_a is the image of a section ξ_a of π for each $a \in N$. Furthermore, for $a \in N$ and each sufficiently small VB chart (α, α_0, U) of the reduced atlas, the map $h_a : \alpha_0(U) \subseteq H_\alpha \to B_{\alpha t} \subseteq F_\alpha$ given by $h_a(x) = h(a, x)$ for $x \in \alpha_0(U)$ (h being the map constructed above) is the principal part of a local representative of ξ_a . As h was $C^{r+1}_{B\Gamma}$, ξ_a is $C^{r+1}_{B\Gamma}$. Since H_α and F_α are finite-dimensional, ξ_a is also C^{r+1} , which ends the proof of Lemma (4.3).

The proof of Theorem (4.2) is straightforward, as follows: by Lemma (4.3), we can find an open neighbourhood N of a_0 in A such that for each $a \in N$, there is a C^{r+1} section of π , ξ_a , such that

$\xi_a\left(W_{a_0}\right) = W_a$. And for this $\xi_a \in S^{r+1}(\pi)$, $\xi_a\left(W_{a_0}\right)$ is C^r-isotopic to W_{a_0} by Lemma 20.4 in [4], p. 53. //

CHAPTER 5

APPLICATIONS

In this chapter we give several applications of the $B\Gamma$-Transversal Density Theorem of Chapter 4. They are simple local results similar to the global results given in [31], [33], [43]. Our method is also the one in [31], [43]. But there are differences between the results obtained in this chapter and the previous ones in [31], [33], [43]. The first difference is that the spaces in our results have different topologies from the one in their results. The second one, and probably the most remarkable one, is that we follow the $B\Gamma$-technique instead of the usual standard techniques as in [31], [33], [43].

In the first section we fix the notations and prove two useful lemmas. The next two sections, §§2, 3, are devoted to two simple applications: the Morse functions defined on an open convex and bounded neighbourhood $\Omega \subseteq \mathbb{R}^n$, and the 0-transversal vector fields on Ω. In §4, for the sake of completeness, we include the Infinite Codimension Lemma from the paper [43] of Kurland and Robbin which will be used in all later applications. The remaining sections, §5 to §7, are for other applications which range from critical points of C^∞ local vector fields to the fixed points of C^∞ maps.

1. Preparatory Lemmas

Throughout this chapter, Ω shall always stand for an *open*, *convex and bounded* subset of an Euclidean space $\mathbb{R}^n$, and for each integer $k = 0, 1, 2, \ldots$, $P^k(n, m)$ shall stand for the space of polynomials of degree less than or equal to k from $\mathbb{R}^n$ to another Euclidean space $\mathbb{R}^m$:

$$P^k(n, m) = \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m) \times L_s^2(\mathbb{R}^n, \mathbb{R}^m) \times \ldots \times L_s^k(\mathbb{R}^n, \mathbb{R}^m) , \qquad (1)$$

defined in Example (2.1), Chapter 3.

We consider the space $B^\infty(\Omega, \mathbb{R}^m)$ defined in Example (1.1), Chapter 3, calibrated by the following sequence of increasing norms

$$\Gamma = \{\|\cdot\|_k\}_{k=0,1,2,\ldots} \tag{2}$$

where each $\|\cdot\|_k$ is defined by (1) in §1, Chapter 3.

Then we have the following:

(1.1) LEMMA. *The space* $B^\infty(\Omega, \mathbb{R}^m)$ *equipped with the sequence of increasing norms in* (2) *is a separable Fréchet space.*

Proof. The fact that $B^\infty(\Omega, \mathbb{R}^m)$ is Fréchet is well-known. We need only prove separability. For each integer $k = 0, 1, 2, \ldots$ define the space

$$UB^k(\Omega, \mathbb{R}^m) = \{f : \Omega \to \mathbb{R}^m \mid f \in B^k(\Omega, \mathbb{R}^m) \text{ and } f, Df, \ldots, D^k f \text{ are uniformly continuous on } \Omega\} .$$

Then it is easily seen that $UB^k(\Omega, \mathbb{R}^m)$ is a closed subspace of $B^k(\Omega, \mathbb{R}^m)$. Thus since $B^k(\Omega, \mathbb{R}^m)$ is a Banach space, $UB^k(\Omega, \mathbb{R}^m)$ is a Banach space for each $k = 0, 1, 2, \ldots$.

We now prove that the Banach spaces $UB^k(\Omega, \mathbb{R}^m)$ are all separable.

First prove the result for $k = 0$: we have

$$UB^0(\Omega, \mathbb{R}^m) = \{f : \Omega \to \mathbb{R}^m \mid f \in B^0(\Omega, \mathbb{R}^m) \text{ and } f \text{ is uniformly continuous on } \Omega\} .$$

By Theorem (3.15.6) in [16], p. 55, f can be extended to a uniformly continuous map $\bar{f} : \bar{\Omega} \to \mathbb{R}^m$ with $\bar{\Omega}$ compact.

Denote by $C^0(\bar{\Omega}, \mathbb{R}^m)$ the space of continuous maps $\bar{\Omega} \to \mathbb{R}^m$ with the norm $\|f\|_0 = \sup_{x \in \bar{\Omega}} \|f(x)\|$. Then since $\bar{\Omega}$ is compact, we know that $C^0(\bar{\Omega}, \mathbb{R}^m)$ is separable (applying the Stone-Weierstrass Theorem).

Consider the mapping $\Phi : UB^0(\Omega, \mathbb{R}^m) \to C^0(\overline{\Omega}, \mathbb{R}^m)$ defined by $\Phi(f) = \overline{f}$. Then it is easy to see that Φ is linear bijective with the inverse Φ^{-1} given by

$$\Phi^{-1}(g) = g|\Omega \quad \text{for} \quad g \in C^0(\overline{\Omega}, \mathbb{R}^m) .$$

Furthermore if $\overline{f} = \Phi(f)$, then we have

$$\|\overline{f}\|_0 = \sup_{x\in\overline{\Omega}} \|\overline{f}(x)\| = \sup_{x\in\Omega} \|\overline{f}(x)\| = \|f\|_0 . \tag{3}$$

Thus Φ is a toplinear isomorphism. Hence $UB^0(\Omega, \mathbb{R}^m)$ is a separable Banach space.

Now for each $k = 1, 2, 3, \ldots$, define the mapping

$$P^k : UB^k(\Omega, \mathbb{R}^m) \to UB^0(\Omega, P^k(n, m))$$

$$f \longmapsto P^k f$$

where $P^k f : \Omega \to P^k(n, m)$ is given by $P^k f(x) = \left(f(x), Df(x), \ldots, D^k f(x)\right)$ for all $x \in \Omega$.

Since each $f \in UB^k(\Omega, \mathbb{R}^m)$ implies immediately that $P^k f \in UB^0(\Omega, \mathbb{R}^m)$, the map P^k is well defined.

P^k is obviously linear and one-to-one. Furthermore

$$\|P^k f\|_0 = \sup_{x\in\Omega} \{\|f(x)\| + \|Df(x)\| + \ldots + \|D^k f(x)\|\} = \|f\|_k . \tag{4}$$

Hence P^k is continuous.

We claim that the image $P^k[UB^k(\Omega, \mathbb{R}^m)]$ is closed in the separable Banach space $UB^0(\Omega, P^k(n, m))$.

Indeed, let $\left\{P^k f_l\right\} \subseteq P^k[UB^k(\Omega, \mathbb{R}^m)]$ be a sequence converging to ζ in $UB^0(\Omega, P^k(n, m))$; that is,

$$\left\|P^k f_l - \zeta\right\|_0 \underset{l\to\infty}{\to} 0 . \tag{5}$$

Then we have by (5),

$$\left\|P^k f_l - P^k f_h\right\|_0 = \|f_l - f_h\|_k \to 0 \quad \text{as} \quad l,\ h \to \infty .$$

Hence $\{f_l\}$ is a Cauchy sequence in the Banach space $UB^k(\Omega, \mathbb{R}^m)$. Thus

$f_l \to f$ for $f \in UB^k(\Omega, \mathbb{R}^m)$, i.e.

$$\|f_l - f\|_k \to 0 \quad \text{as} \quad l \to \infty .$$

Thus we have

$$\left\|P^k f_l - P^k f\right\|_0 \to 0 \quad \text{as} \quad l \to \infty . \tag{6}$$

Let $\varepsilon > 0$ be given: then (5) and (6) show that

$$\|P^k f - \zeta\|_0 < 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we must have

$$\|P^k f - \zeta\|_0 = 0 .$$

Hence

$$\zeta = P^k f$$

and $P^k[UB^k(\Omega, \mathbb{R}^m)]$ is closed in $UB^0(\Omega, P^k(n, m))$. Thus $P^k[UB^k(\Omega, \mathbb{R}^m)]$ is a separable Banach space and P^k is a toplinear isomorphism. Hence

$UB^k(\Omega, \mathbb{R}^m)$ is a separable Banach space for each $k = 0, 1, 2, \ldots$. (7)

Now if $f \in B^\infty(\Omega, \mathbb{R}^m)$, the mean value theorem proves that $D^k f : \Omega \to L^k_s(\mathbb{R}^n, \mathbb{R}^m)$ is uniformly continuous for all $k = 0, 1, 2, \ldots$.

Thus we have

$$B^\infty(\Omega, \mathbb{R}^m) = \bigcap_{k=0}^{\infty} [UB^k(\Omega, \mathbb{R}^m)] . \tag{8}$$

Since for all $k = 0, 1, 2, \ldots$, we have

$$B^\infty(\Omega, \mathbb{R}^m) \subseteq UB^k(\Omega, \mathbb{R}^m) \subseteq B^k(\Omega, \mathbb{R}^m) \tag{9}$$

the topology induced on $B^\infty(\Omega, \mathbb{R}^m)$ from the one of $B^k(\Omega, \mathbb{R}^m)$ coincides with the one induced by the topology of $UB^k(\Omega, \mathbb{R}^m)$. Thus by (7), each

$\|\cdot\|_k$ on $UB^k(\Omega, \mathbb{R}^m)$ induces a countable basis for the induced topology on $B^\infty(\Omega, \mathbb{R}^m)$. Since $k = 0, 1, 2, \ldots$ is countable and since the $\{\|\cdot\|_k\}$ is increasing, $B^\infty(\Omega, \mathbb{R}^m)$ equipped with $\Gamma = \{\|\cdot\|_k\}_{k=0,1,2,\ldots}$ has countable basis. Since $B^\infty(\Omega, \mathbb{R}^m)$ is metrisable, this is equivalent to the fact that $B^\infty(\Omega, \mathbb{R}^m)$ is separable. //

(1.2) REMARK. Since the sequence $\{\|\cdot\|_k\}_{k=0,1,2,\ldots}$ is increasing, if we endow $B^\infty(\Omega, \mathbb{R}^m)$ by

$$\tilde{\Gamma} = \{\|\cdot\|_k\}_{k\geq i_0}$$

where i_0 is an integer, then $B^\infty(\Omega, \mathbb{R}^m)$ is still a separable Fréchet space because Γ and $\tilde{\Gamma}$ are equivalent.

Note that each $\|\cdot\|_j$ on $B^\infty(\Omega, \mathbb{R}^m)$ induces a norm p_j on $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$ as in §5, Chapter 1:

$$p_j(\xi, x) = \|\xi\|_j + \|x\| \quad \text{for all} \quad (\xi, x) \in B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n . \tag{10}$$

The following lemma shall be used in all the proofs of the applications of this chapter.

(1.3) LEMMA. *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open convex bounded subset,* r *be an integer greater than of equal to* 1 *and* k *be an integer such that* $0 \leq k \leq r$. *Then we can always find an integer* i_0 *such that the following assertions are true:*

(a) the map $\mathrm{ev}_k : B^\infty(\Omega, \mathbb{R}^m) \times \Omega \to \Omega \times P^k(n, m)$ *defined by*

$$\mathrm{ev}_k(\xi, x) = \left(x, P^k\xi(x)\right) \quad \text{for } x \in \Omega,\ \xi \in B^\infty(\Omega, \mathbb{R}^m) \text{ is } C^r_{B\Gamma}$$

with respect to the calibration $\Gamma = \{p_{r+k+i}\}_{i\geq i_0}$ *on* $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$ *and the norm-calibration on* $\mathbb{R}^n \times P^k(n, m)$ *;*

(b) the $B\Gamma$-derivative $\mathrm{Dev}_k(\xi, x) : B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^n \times P^k(n, m)$

(for each $(\xi, x) \in B^\infty(\Omega, \mathbb{R}^m) \times \Omega$ *) is onto and has kernel $B\Gamma$-splitting in* $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$ *;*

(c) for any ($B\Gamma$-splitting) subspace F_1 *of* $\mathbb{R}^n \times P^k(n, m)$ *the inverse image* $E_1 = \mathrm{Dev}_k(\xi, x)^{-1}(F_1)$ *is $B\Gamma$-splitting in* $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$ *and has a $B\Gamma$-complement* E_2 *such that the restriction of* $\mathrm{Dev}_k(\xi, x)$ *to* E_2 *is a $B\Gamma$-isomorphism onto a $B\Gamma$-complement* F_2 *of* F_1 *in* $\mathbb{R}^n \times P^k(n, m)$ *;*

(d) the subspace $E_1 = \mathrm{Dev}_k(\xi, x)^{-1}(F_1)$ *$B\Gamma$-adapts the first projection* $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n \to B^\infty(\Omega, \mathbb{R}^m)$.

Proof. (a) By the first part of Proposition (6.2), Chapter 1, the map ev_k is $C^r_{B\Gamma}$ with respect to the calibration $\Gamma = \{p_{r+k+i}\}_{i\geq 0}$. Then for the integer i_0 (determined in (b)) ev_k is still $C^r_{B\Gamma}$ with respect to the calibration $\Gamma = \{p_{r+k+i}\}_{i\geq i_0}$ and still has the same $B\Gamma$-derivative $\mathrm{Dev}_k(\xi, x)$ at $(\xi, x) \in B^\infty(\Omega, \mathbb{R}^m) \times \Omega$.

(b) The ontoness of $\mathrm{Dev}_k(\xi, x)$ is the second part of Proposition (6.2), Chapter 1.

For a fixed $(\xi, x) \in B^\infty(\Omega, \mathbb{R}^m) \times \Omega$, define (see [4]).

$$K_1 = \{\zeta \in B^\infty(\Omega, \mathbb{R}^m) \mid D^i\zeta(x) = 0 \quad \text{for} \quad i = 0, 1, \ldots, k\} ,$$

$$K_2 = \{\zeta \in B^\infty(\Omega, \mathbb{R}^m) \mid D^i\zeta \equiv 0 \quad \text{for} \quad i \geq k+1\} .$$

Then K_1 and K_2 are closed subspaces of $B^\infty(\Omega, \mathbb{R}^m)$ (equipped with the family of norms $\|\cdot\|_{r+k+i}$, $i \geq 0$).

We now prove that $B^{\infty}(\Omega, \mathbb{R}^m) = K_1 \oplus_{B\Gamma} K_2$. Indeed, first prove that $B^{\infty}(\Omega, \mathbb{R}^m) = K_1 \oplus K_2$. Let $\zeta \in K_1 \cap K_2$ then since $D^i\zeta \equiv 0$ for $i \geq k+1$, Taylor's Formula (see [4], p. 4) gives us

$$\zeta(y) = \zeta(x) + \frac{D\zeta(x)}{1!}(y-x) + \ldots + \frac{D^k\zeta(x)}{k!}(y-x)^{(k)} \quad \text{for } y \in \Omega . \tag{11}$$

Since $\zeta \in K_1$, this implies $\zeta(y) = 0$, $\forall y \in \Omega$; that is, $\zeta = 0$. Hence $K_1 \cap K_2 = \{0\}$.

Furthermore $K_1 + K_2 = B^{\infty}(\Omega, \mathbb{R}^m)$ as seen by the following argument. Let $\zeta \in B^{\infty}(\Omega, \mathbb{R}^m)$. Define the mappings

$$\zeta_1 : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m , \quad \zeta_2 : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

by

$$\zeta_2(y) = \zeta(x) + \frac{D\zeta(x)}{1!}(y-x) + \ldots + \frac{D^k\zeta(x)}{k!}(y-x)^{(k)} \quad (\forall y \in \Omega) \tag{12}$$

and

$$\zeta_1(y) = \zeta(y) - \zeta_2(y) \quad \text{for all } y \in \Omega . \tag{13}$$

Then since Ω is open, convex and bounded, it can be seen that $\zeta_1, \zeta_2 \in B^{\infty}(\Omega, \mathbb{R}^m)$.

Furthermore by (13) we have $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in K_1$ (because $D^j\zeta_1(x) = D^j\zeta(x) - D^j\zeta_2(x) = 0$ for all $j = 0, 1, 2, \ldots, k$ by (12)) and $\zeta_2 \in K_2$ (because $D^i\zeta_2 \equiv 0$ for $i \geq k+1$).

Note that

$$K_2 = \{\zeta \in B^{\infty}(\Omega, \mathbb{R}^m) \mid D^i\zeta \equiv 0, i \geq k+1\}$$

$$= P^k(n, m) = \text{space of polynomials } \mathbb{R}^n \to \mathbb{R}^m$$

of degree less than or equal to k . Thus $\dim K_2 < +\infty$.

Since $\dim K_2 < +\infty$ and $K_1 \oplus K_2 = B^\infty(\Omega, \mathbb{R}^m)$, and since $B^\infty(\Omega, \mathbb{R}^m)$ is a Fréchet space (by Lemma (1.1)), this is a topological sum.

Thus the mapping

$$B^\infty(\Omega, \mathbb{R}^m) = K_1 \oplus K_2 \to K_2 : \zeta = \zeta_1 + \zeta_2 \mapsto \zeta_2 \tag{14}$$

is continuous.

Furthermore, since $D^i\zeta \equiv 0$ for $\zeta \in K_2$, $i \geq k+1$, we have on K_2,

$$\|\zeta\|_j = \|\zeta\|_k \quad \text{for all} \quad j \geq k \text{ , all } \zeta \in K_2 \text{ .}$$

Hence $\|\zeta\|_{r+k+i} = \|\zeta\|_k$ for all $i = 0, 1, 2, \ldots$ and all $\zeta \in K_2$.

The continuity of the mapping (14) implies there are a positive number α and an integer i_0 such that

$$\|\zeta_2\|_k < \alpha\|\zeta\|_{r+k+i_0} \quad \text{for all} \quad \zeta \in B^\infty(\Omega, \mathbb{R}^m) \text{ .}$$

Hence

$$\|\zeta_2\|_{r+k+i} = \|\zeta_2\|_k < \alpha\|\zeta\|_{r+k+i_0} \leq \alpha\|\zeta\|_{r+k+i} \tag{15}$$

for all $i \geq i_0$ and all $\zeta \in B^\infty(\Omega, \mathbb{R}^m)$, which proves that

$$B^\infty(\Omega, \mathbb{R}^m) = K_1 \oplus_{B\Gamma} K_2$$

where $\Gamma = \{p_{r+k+i}\}_{i \geq i_0}$.

By a simple calculation we have

$$\operatorname{Ker} Dev_k(\xi, x) = K_1 \times \{0\} \text{ .} \tag{16}$$

Furthermore

$$B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n = (K_1 \times \{0\}) \oplus_{B\Gamma} \left(K_2 \times \mathbb{R}^n\right) \text{ .} \tag{17}$$

Indeed, for an arbitrary $(\zeta, h) \in B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$, we can write $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in K_1$, $\zeta_2 \in K_2$. Hence

$$(\zeta, h) = (\zeta_1, 0) + (\zeta_2, h) \in (K_1 \times \{0\}) + \left(K_2 \times \mathbb{R}^n\right) \text{ .}$$

On the other hand, if $(\zeta, h) \in (K_1 \times \{0\}) \cap \left(K_2 \times \mathbb{R}^n\right)$ then

$$(\zeta, h) \in K_1 \times \{0\} \Rightarrow h = 0 \ , \ \zeta \in K_1 \ ,$$

$$(\zeta, h) \in K_2 \times \mathbb{R}^n \Rightarrow \zeta \in K_2 \ .$$

Since $K_1 \cap K_2 = \{0\}$, this implies $\zeta = 0$, i.e. $(\zeta, h) = (0, 0)$ and

$$(K_1 \times \{0\}) \cap \left(K_2 \times \mathbb{R}^n\right) = \{(0, 0)\} \ .$$

Thus we have $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n = (K_1 \times \{0\}) \oplus \left(K_2 \times \mathbb{R}^n\right)$. Furthermore, the mapping

$$(\zeta, h) = (\zeta_1, 0) + (\zeta_2, h) \longmapsto (\zeta_2, h) \tag{18}$$

is continuous (because $K_2 \times \mathbb{R}^n$ is finite dimensional) and for all $i \geq i_0$ we have

$$\begin{aligned} p_{r+k+i}(\zeta_2, h) &= \|\zeta_2\|_{r+k+i} + \|h\| \\ &= \|\zeta_2\|_k + \|h\| \leq \alpha\|\zeta\|_{r+k+i} + \|h\| \\ &\leq \beta\left(\|\zeta\|_{r+k+i} + \|h\|\right) = \beta p_{r+k+i}(\zeta, h) \end{aligned}$$

where $\beta \geq \max(\alpha, 1)$. Hence we have (17).

Since Ker $Dev_k(\xi, x) = K_1 \times \{0\}$, (17) proves that Ker $Dev_k(\xi, x)$ is $B\Gamma$-splitting in $B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n$.

(c) If F_1 is a ($B\Gamma$-splitting) subspace of $\mathbb{R}^n \times P^k(n, m)$ then $\dim F_1 < +\infty$, $\operatorname{codim} F_1 = l < +\infty$ because $\mathbb{R}^n \times P^k(n, m)$ is finite-dimensional. We also have

$$K_1 \times \{0\} = \text{Ker } Dev_k(\xi, x) \subseteq E_1 = Dev_k(\xi, x)^{-1}(F_1) \ . \tag{19}$$

Let us denote by $L = E_1 \cap \left(K_2 \times \mathbb{R}^n\right) \subseteq K_2 \times \mathbb{R}^n$, then by (19) we have

$$E_1 = L \oplus_{B\Gamma} (K_1 \times \{0\}) \ . \tag{20}$$

Since $\operatorname{codim} L < +\infty$ and since $K_2 \times \mathbb{R}^n$ is a normed space with norm $\|(\zeta_2, h)\| = \|\zeta_2\|_k + \|h\|$, we can find a complement E_2 of L in $K_2 \times \mathbb{R}^n$ such that

$$\pi(L) \oplus_{B\Gamma} \pi(E_2) = K_2 \quad \text{(see Chapter 4, §3),} \tag{21}$$

where $\pi : B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n \to B^\infty(\Omega, \mathbb{R}^m)$ is the first projection.

Furthermore, since $Dev_k(\xi, x)$ is onto by part (b), we have

$$\operatorname{codim} E_1 = \dim E_2 = l \tag{22}$$

and $Dev_k(\xi, x)|_{E_2}$ is one-to-one.

Let $F_2 = Dev_k(\xi, x)(E_2) \subseteq \mathbb{R}^n \times P^k(n, m)$; then

$$\dim F_2 = \operatorname{codim} F_1 = l . \tag{23}$$

Furthermore, it can be seen easily that

$$F_2 \cap F_1 = \{0\} . \tag{24}$$

Thus (23) and (24) imply

$$F_1 \oplus F_2 = \mathbb{R}^n \times P^k(n, m) . \tag{25}$$

Since on $\mathbb{R}^n \times P^k(n, m)$ we have the norm calibration, we also have

$$F_1 \oplus_{B\Gamma} F_2 = \mathbb{R}^n \times P^k(n, m) . \tag{26}$$

Thus, $Dev_k(\xi, x)|_{E_2} : E_2 \to F_2$ is one-to-one, bijective, $B\Gamma$-continuous with E_2 equal to a $B\Gamma$-complement of E_1, F_2 equal to a $B\Gamma$-complement of F_1. Since $E_2 \subseteq K_2 \times \mathbb{R}^n$ where the induced calibration reduces to the norm $\|\cdot\|_k + \|\cdot\|$, the Banach theorem implies

$$Dev_k(\xi, x) : E_2 \to F_2 \quad \text{is a } B\Gamma\text{-isomorphism.}$$

(d) By the proof in (c), E_2 is a $B\Gamma$-complement of L in $K_2 \times \mathbb{R}^n$. We also have

$$B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n = (K_1 \times \{0\}) \oplus_{B\Gamma} (K_2 \times \mathbb{R}^n)$$
$$= (K_1 \times \{0\}) \oplus_{B\Gamma} (L \oplus_{B\Gamma} E_2)$$
$$= [(K_1 \times \{0\}) \oplus_{B\Gamma} L] \oplus_{B\Gamma} E_2 .$$

Thus, by (20),

$$B^\infty(\Omega, \mathbb{R}^m) \times \mathbb{R}^n = E_1 \oplus_{B\Gamma} E_2 . \tag{27}$$

Furthermore, we have

$$\pi(E_1) = \pi((K_1 \times \{0\})+L) = \pi(K_1 \times \{0\}) + \pi(L) ,$$

$$\pi(E_1) = K_1 \oplus \pi(L) \quad \text{since} \quad \pi(L) \subseteq K_2 \quad \text{complementary to} \quad K_1 . \tag{28}$$

Thus

$$\pi(E_1) + \pi(E_2) = K_1 + (\pi(L)+\pi(E_2))$$
$$= K_1 + K_2 \quad \text{(by (21))}.$$

That is,

$$\pi(E_1) + \pi(E_2) = B^\infty(\Omega, \mathbb{R}^m) . \tag{29}$$

Now if $z \in \pi(E_1) \cap \pi(E_2)$ then $z = \pi(e_1)$ with $e_1 \in E_1$. By (20) we can write $e_1 = l + (k_1, 0)$ with $l \in L$, $k_1 \in K_1$. Hence

$$\pi(e_1) = \pi(l) + k_1 \quad \text{with} \quad \pi(l) \in \pi(L) \subseteq K_2 .$$

This implies $k_1 = 0$ and $\pi(e_1) \in \pi(L)$.

Thus $z = \pi(e_1) \in \pi(L) \cap \pi(E_2) = \{0\}$ and we have

$$\pi(E_1) \oplus \pi(E_2) = B^\infty(\Omega, \mathbb{R}^m) . \tag{30}$$

We need to prove that (30) is actually a $B\Gamma$-sum. Indeed, we have

$$\pi(E_1) \oplus \pi(E_2) = [K_1 \oplus \pi(L)] \oplus \pi(E_2) ,$$

$$\pi(E_1) \oplus \pi(E_2) = K_1 \oplus [\pi(L) \oplus \pi(E_2)] = K_1 \oplus_{B\Gamma} K_2 = B^\infty(\Omega, \mathbb{R}^m) . \tag{31}$$

We want to prove that the mapping $\pi(E_1) \oplus \pi(E_2) \to \pi(E_2)$,

$$z = \pi(e_1) + \pi(e_2) \longmapsto \pi(e_2) \tag{32}$$

is $B\Gamma$-continuous.

But we can write, by (31),

$$z = \pi(e_1) + \pi(e_2) = k_1 + \pi(l) + \pi(e_2) \in K_1 \oplus_{B\Gamma} K_2 .$$

Hence there is a constant $\alpha > 0$ such that

$$p_{r+k+i}\big(\pi(l)+\pi(e_2)\big) \le \alpha p_{r+k+i}(z) \quad \text{for all} \quad i = 0, 1, 2, \ldots . \tag{33}$$

Since $\pi(L) \oplus_{B\Gamma} \pi(E_2) = K_2$, we can find $\beta > 0$ such that

$$p_{r+k+i}\big(\pi(e_2)\big) \le \beta p_{r+k+i}\big(\pi(l)+\pi(e_2)\big) \quad \text{for all} \quad i = 0, 1, 2, \ldots . \tag{34}$$

Thus (33) and (34) give

$$p_{r+k+i}\big(\pi(e_2)\big) \le \beta\alpha p_{r+k+i}(z) \quad \text{for all} \quad i = 0, 1, 2, \ldots \tag{35}$$

and we have

$$\pi(E_1) \oplus_{B\Gamma} \pi(E_2) = B^\infty(\Omega, \mathbb{R}^m) \tag{36}$$

which proves that E_1 $B\Gamma$-adapts π . //

2. First Application: Morse Functions

Let $\Omega \subseteq \mathbb{R}^n$ be open convex and bounded, and consider the space $B^\infty(\Omega, \mathbb{R})$ of all C^∞ functions $\Omega \to \mathbb{R}$ with all derivatives bounded on Ω. Then $B^\infty(\Omega, \mathbb{R})$ is a separable Fréchet space by Lemma (1.1).

Recall that a point $x \in \Omega$ is a *critical point* of $f \in B^\infty(\Omega, \mathbb{R})$ iff $Df(x) = 0$. x is a *non-degenerate critical point* of f iff the Hessian $\mathrm{Hess}(f)_x = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$ is non singular (see [31], [33]).

A function $f \in B^\infty(\Omega, \mathbb{R})$ is a *Morse function* if all the critical points of f are non-degenerate. We shall denote by $M^\infty(\Omega, \mathbb{R})$ the subset of Morse functions in $B^\infty(\Omega, \mathbb{R})$.

Morse functions can be characterised by condition on 1-jets as follows. Let $J^1(\Omega, \mathbb{R}) = \Omega \times \mathbb{R} \times L(\mathbb{R}^n, \mathbb{R})$ be the space of 1-jets $\Omega \to \mathbb{R}$, and let

$$S_1 = \{\sigma \in J^1(\Omega, \mathbb{R}) \mid \text{corank}\ \sigma = 1\}$$

be the submanifold of $J^1(\Omega, \mathbb{R})$ defined in [31], p. 60. Then we have (see [31], Definition 6.1 and Proposition 6.4),

$$M^\infty(\Omega, \mathbb{R}) = \left\{f \in B^\infty(\Omega, \mathbb{R}) \mid j^1 f \pitchfork S_1\right\}. \qquad (1)$$

Note that the submanifold S_1 has codimension n in $J^1(\Omega, \mathbb{R})$ (see [31], Theorem 5.4).

(2.1) PROPOSITION. *The set* $\bar{M}^\infty(\Omega, \mathbb{R})$ *of Morse functions is dense in* $B^\infty(\Omega, \mathbb{R})$. *In other words, every function in* $B^\infty(\Omega, \mathbb{R})$ *can be approximated by Morse functions.*

Proof. We apply the $B\Gamma$-Transversal Density Theorem. Put $A = B^\infty(\Omega, \mathbb{R})$ considered as a $B\Gamma$-manifold and in the $B\Gamma$-Transversal Density Theorem read $X = \Omega \subseteq \mathbb{R}^n$, $Y = J^1(\Omega, \mathbb{R}) = \Omega \times \mathbb{R} \times L(\mathbb{R}^n, \mathbb{R})$, and $W = S_1 \subseteq J^1(\Omega, \mathbb{R})$.

Define the representation $\rho : A \to C^1_{B\Gamma}(\Omega, J^1(\Omega, \mathbb{R}))$ by $\rho(f) = j^1 f$, where $f \in A = B^\infty(\Omega, \mathbb{R})$ and $j^1 f : \Omega \to J^1(\Omega, \mathbb{R})$ is the 1-jet prolongation of f (see [31], [33]) (see also §4).

Then we have all the conditions of the $B\Gamma$-Transversal Density Theorem: $\text{ev}_\rho : B^\infty(\Omega, \mathbb{R}) \times \Omega \to J^1(\Omega, \mathbb{R}) = \Omega \times P^1(n, 1)$ is $C^1_{B\Gamma}$ if we take the calibration $\Gamma = \{p_{2+i}\}_{i \geq i_0}$ given by Lemma (1.3).

(a) $\Omega \subseteq \mathbb{R}^n$ is open, thus considered as a manifold of dimension n; S_1 has codimension n.

(b) $A = B^\infty(\Omega, \mathbb{R})$ is second countable by Lemma (1.1); Ω is second countable.

(c) $1 > \max(0, n-n) = 0$.

(d) $\mathrm{ev}_\rho \pitchfork_{B\Gamma} S_1$ follows from part (c) of Lemma (1.3).

(e) For all $(f, x) \in A \times_{B\Gamma} \Omega$ such that

$$y = j^1 f(x) = \left(x, P^1 f(x)\right) \in S_1 ,$$

the $B\Gamma$-splitting subspace $Dev_\rho(f, x)^{-1}\left(T_y S_1\right)$ of $B^\infty(\Omega, \mathbb{R}) \times \mathbb{R}^n$ $B\Gamma$-adapts the first projection $B^\infty(\Omega, \mathbb{R}) \times \mathbb{R}^n \to B^\infty(\Omega, \mathbb{R})$ by part (d), Lemma (1.3).

Thus

$$A_{S_1} = \{f \in B^\infty(\Omega, \mathbb{R}) \mid \rho_f \pitchfork S_1\} \tag{2}$$

is dense in A.

But this (2) is exactly (1). //

(2.2) REMARK. According to the general result given in [33], p. 147 (or [31], p. 63) for any manifold X, Morse functions $X \to \mathbb{R}$ form a dense (and open) set in $C^\infty_S(M, \mathbb{R})$.

If we take $X = \Omega$ the open submanifold of $\mathbb{R}^n$, then Morse functions in $C^\infty(\Omega, \mathbb{R})$ is dense in $C^\infty_S(\Omega, \mathbb{R})$, the space $C^\infty(\Omega, \mathbb{R})$ equipped with the (strong) Whitney topology. But the induced topology on $B^\infty(\Omega, \mathbb{R})$ is not the same as the topology defined by the sequence of increasing norms $\{\|\cdot\|_k\}_{k\geq 0}$. Hence our result, for this particular case, seems to be new.

3. Second Application: 0-Transversal Vector Fields (see [4], p. 62)

Let $\Omega \subseteq \mathbb{R}^n$ be open convex and bounded as before. Consider the tangent bundle $T\Omega = \Omega \times \mathbb{R}^n$ and the space of all C^∞ sections of $T\Omega$; that is, the space of all maps $\xi : \Omega \to T\Omega$ such that $\pi \circ \xi = \mathrm{id}_\Omega$ where $\pi : T\Omega \to \Omega$ is the natural projection.

Thus each $\xi : \Omega \to T\Omega$ has the following form:

$$\xi(x) = \left(x, \tilde{\xi}(x)\right) \quad \text{for all } x \in \Omega \tag{1}$$

where $\tilde{\xi} : \Omega \to \mathbb{R}^n$ is C^∞.

Denote by $S^\infty(T\Omega)$ the space of all such ξ with $\tilde{\xi} \in B^\infty(\Omega, \mathbb{R}^n)$ defined in §1:

$$S^\infty(T\Omega) = \{\xi : \Omega \to T\Omega \mid \xi(x) = (x, \tilde{\xi}(x)) \text{ for all } x \in \Omega \text{ and } \tilde{\xi} \in B^\infty(\Omega, \mathbb{R}^n)\} . \quad (2)$$

Consider the usual topology on $B^\infty(\Omega, \mathbb{R}^n)$ defined by the sequence of increasing norms $\|\cdot\|_k$, and for each $k = 0, 1, 2, \ldots$ define

$$\|\xi\|_k = \|\tilde{\xi}\|_k \quad \text{for all } \xi \in S^\infty(T\Omega) . \quad (3)$$

Then $S^\infty(T\Omega)$ equipped with $\{\|\cdot\|_k\}_{k\geq 0}$ is a separable Fréchet space isomorphic to $B^\infty(\Omega, \mathbb{R}^n)$ via the toplinear isomorphism

$$\Phi : S^\infty(T\Omega) \to B^\infty(\Omega, \mathbb{R}^n) \quad (4)$$

defined by $\Phi(\xi) = \tilde{\xi}$.

Now a point $x \in \Omega$ is a *critical point* of $\xi \in S^\infty(T\Omega)$ iff $\tilde{\xi}(x) = 0 \in \mathbb{R}^n$; that is, $\xi(x) = (x, 0)$. Then x is called a *nondegenerate critical point* of ξ iff $D\tilde{\xi}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective. This is equivalent to the requirement that $D\tilde{\xi}(x)$ is a toplinear isomorphism.

Denote by $(T\Omega)_0$ the image of the zero section in $T\Omega$; that is,

$$(T\Omega)_0 = \{0_x \in T\Omega \mid x \in \Omega\} = \Omega \times \{0\} \subseteq \Omega \times \mathbb{R}^n . \quad (5)$$

Then $(T\Omega)_0$ is a closed submanifold of $T\Omega$ and has codimension equals n.

Thus x is a critical point of ξ iff $\xi(x) \in (T\Omega)_0$ and it is a nondegenerate critical point of ξ iff

$$\xi \pitchfork_x (T\Omega)_0 . \quad (6)$$

Indeed we have $D\xi(x) = (\mathrm{Id}, D\tilde{\xi}(x)) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, thus

$$D\xi(x)(\mathbb{R}^n) + T_{\xi(x)}((T\Omega)_0) = T_{\xi(x)}((T\Omega)) ; \quad (7)$$

that is,

$$\mathbb{R}^n \times \mathbb{R}^n + \left(\mathbb{R}^n \times \{0\}\right) = \mathbb{R}^n \times \mathbb{R}^n \tag{8}$$

if and only if $D\tilde{\xi}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is onto.

We say that $\xi \in S^\infty(T\Omega)$ is a 0-*transversal vector field* (*on* Ω) (see [4], p. 62) iff every critical point of ξ is nondegenerate. Let $G_0^\infty(\Omega)$ denote the set of all 0-transversal vector fields in $S^\infty(T\Omega)$:

$$G_0^\infty(\Omega) = \{\xi \in S^\infty(T\Omega) \mid \xi \pitchfork (T\Omega)_0\} . \tag{9}$$

(3.1) PROPOSITION. *The set* $G_0^\infty(\Omega)$ *of* 0-*transversal vector fields on* Ω *is dense in* $S^\infty(T\Omega)$.

Proof. In the $B\Gamma$-Transversal Density Theorem read for A, X, Y , W respectively $S^\infty(T\Omega)$, Ω, $T\Omega = \Omega \times \mathbb{R}^n$, $(T\Omega)_0 = \Omega \times \{0\}$ and consider

$$\rho : S^\infty(T\Omega) \to C^1_{B\Gamma}(\Omega, T\Omega)$$

defined by $\rho(\xi) = \xi$ where the right-hand side ξ is considered as a C^∞ map $\Omega \to T\Omega = \Omega \times \mathbb{R}^n$.

Then $\mathrm{ev}_\rho : S^\infty(T\Omega) \times \Omega \to T\Omega = \Omega \times \mathbb{R}^n$ is the composite:

$$S^\infty(T\Omega) \times \Omega \xrightarrow{\Phi\times \mathrm{id}_\Omega} B^\infty\left(\Omega, \mathbb{R}^n\right) \times \Omega \xrightarrow{\mathrm{ev}_0} \Omega \times P^0(n, n) \tag{10}$$

$$(\xi, x) \longmapsto (\tilde{\xi}, x) \longmapsto \left(x, \tilde{\xi}(x)\right)$$

where the map $\Phi : \xi \longmapsto \tilde{\xi}$ is the $B\Gamma$-isomorphism (4), and ev_0 is defined in Lemma (1.3).

We consider the calibration $\{p_i\}_{i \geq i_0}$ defined in Lemma (1.3). Then ev_ρ is $C^1_{B\Gamma}$ and we have:

(a) $\Omega \subseteq \mathbb{R}^n$ has dim $= n$; $W \equiv (T\Omega)_0$ has codim $= n$;

(b) $A = S^\infty(T\Omega)$ and Ω are second countable;

(c) $1 > \max(0, n-n) = 0$;

(d) $\mathrm{ev}_\rho \pitchfork_{B\Gamma} W$ follows from part (c) of Lemma (1.3) and (10);

(e) for each $(\xi, x) \in A \times_{B\Gamma} \Omega$ such that $\rho_\xi(x) = \xi(x) \in W = (T\Omega)_0$

we have

$$\mathcal{D}ev_\rho(\xi, x) = \mathcal{D}ev_0(\xi, x) \circ \left(\Phi \times \mathrm{id}_{\mathbb{R}^n}\right) \tag{11}$$

and condition (e) follows quickly.

Thus the $B\Gamma$-Transversal Density Theorem gives

$$A_W = \{\xi \in S^\infty(T\Omega) \mid \xi \pitchfork (T\Omega)_0\} \tag{12}$$

is dense in $S^\infty(T\Omega)$.

Since A_W is exactly $G_0^\infty(\Omega)$ we have proved the proposition. //

(3.2) REMARK. Every $\xi \in G_0^\infty(\Omega)$ has isolated critical points.

Indeed, if $C_\xi(0)$ denotes the set of all critical points of ξ , then since $\xi \pitchfork (T\Omega)_0$, we have by Corollary (1.2), Chapter 4,

$$C_\xi(0) = \xi^{-1}\left((T\Omega)_0\right) = \text{submanifold of dimension zero.} \tag{13}$$

Thus, Proposition (3.1) also proves that there is a dense subset $G_0^\infty(\Omega) \subseteq S^\infty(T\Omega)$ such that every $\xi \in G_0^\infty(\Omega)$ has only isolated critical points. In the third application in §5 we shall prove this result directly using the Infinite Codimension Lemma of Kurland and Robin ([43]).

4. The Infinite Codimension Lemma

For the sake of completeness, we include in this section the first two sections of [43] about the Infinite Codimension Lemma.

We denote by E_n or simply E the ring of germs at $0 \in \mathbb{R}^n$ of real valued C^∞ functions of n real variables and by M_n or simply M the maximal ideal in E . Thus M consists of those germs which vanish at 0 . From the formula

$$f(x) = f(0) + \sum_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dtx_i$$

it follows that M is generated by coordinate functions

$$M = \langle x_1, \ldots, x_n \rangle . \tag{1}$$

We frequently use:

(4.1) LEMMA (Nakayama). *If* I *and* I' *are ideals in* E *and if* I *is finitely generated and*

$$I \subseteq I' + MI$$

then $I \subseteq I'$.

Proof. Let $g_1, \ldots, g_m$ generate I. By hypothesis

$$g_i = h_i + \sum_j a_{ij} g_j$$

where $h_i \in I'$ and $a_{ij} \in M$. Thus

$$\sum_j (\delta_{ij} - a_{ij}) g_j = h_i . \tag{2}$$

The matrix on the left hand side of (2) is invertible as it is the identity matrix when $x = 0$. Thus each generator g_i of I is a combination of the elements h_i of I' and hence in I'. //

As an application of Nakayama's lemma, we prove:

(4.2) PROPOSITION. *Let* I *be an ideal in* E *of codimension less than or equal to* k. *Then* $M^k \subseteq I$.

By "codimension" we always mean "codimension as a real vector subspace of a real vector space".

(4.3) COROLLARY. *An ideal in* E *has finite codimension if and only if it contains some power of the maximal ideal.*

Proof. Consider the sequence

$$I \subseteq I + M^{k+1} \subseteq I + M^k \subseteq \ldots \subseteq I + M \subseteq I + E = E .$$

There are $k + 2$ inclusion signs and if I has codimension less than or equal to k at least two inclusions must be equality. Thus

$$I + M^{j+1} = I + M^j$$

for some $j = 0, 1, \ldots, k$ so $M^j \subseteq I + MM^j$ so $M^j \subseteq I$ by Nakayama's lemma. This proves the proposition as $M^k \subseteq M^j$ since $j \leq k$. The corollary follows immediately from the fact that M^k has finite codimension in E (in fact the monomials of order less than k form a basis for E/M^k). //

The main importance for us of ideals of finite codimension lies in the following.

(4.4) PROPOSITION. *Let $I \subseteq E$ be an ideal. If I has finite codimension, then the origin is at most an isolated zero of I; that is, there are elements $h_1, \ldots, h_m$ of I such that the only $x \in \mathbb{R}^n$ for which $h_1(x) = \ldots = h_m(x) = 0$ is $x = 0$.*

Proof. If I has finite codimension then $M^k \subseteq I$ for some k (by Proposition (4.2)) and we may take $h_1, \ldots, h_m$ to be the monomials of order k. //

Now let $J^k(n, 1)$ denote the vector space of k-jets of germs at 0 of maps $f : \mathbb{R}^n \to \mathbb{R}$. In other words, $J^k(n, 1)$ is nothing but the vector space $P^k(n, 1)$ (defined in §1) of all real polynomials in n-variables of order less than or equal to k. This is a quotient of E :

$$J^k(n, 1) \simeq E/M^{k+1} \tag{3}$$

and is hence an algebra. (The multiplication is performed by multiplying polynomials in the usual fashion and then dropping the terms of order greater than k.)

The projection of E onto $J^k(n, 1)$ is denoted by

$$E \to J^k(n, 1) : f \mapsto j^k f(0) . \tag{4}$$

Of course, $j^k f(0)$ is nothing more than the Taylor polynomial of order k

of f at 0. A polynomial is a real valued function (among other things) so we have an inclusion

$$J^k(n, 1) \subseteq E \tag{5}$$

but this inclusion (unlike the projection (4)) must be used with caution as it does not behave well under changes of coordinates (i.e., is not invariantly defined).

We denote by $E_{n,p}$ the set of germs at zero of C^∞ maps $g : \mathbb{R}^n \to \mathbb{R}^p$. Thus $E = E_n = E_{n,1}$ and $E_{n,p}$ is a free E-module on p generators. Similarly, $J^k(n, p)$ denotes the space of k-jets of maps $g \in E_{n,p}$ (the space $P^k(n, p)$ defined in §1). This is a free $J^k(n, 1)$-module on p generators and a quotient of $E_{n,p}$:

$$J^k(n, p) = E_{n,p}/M_n E_{n,p} . \tag{6}$$

We denote the projection by

$$E_{n,p} \to J^k(n, p) : g \mapsto j^k g(0) \tag{7}$$

and also use the non-invariant inclusion

$$J^k(n, p) \subseteq E_{n,p} . \tag{8}$$

An element $g \in E_{n,p}$ consists of p functions $g_1, \ldots, g_p \in E_{n,1}$ and we denote by $\langle g \rangle$ the ideal in $E = E_{n,1}$ generated by $g_1, \ldots, g_p$:

$$\langle g \rangle = \langle g_1, \ldots, g_p \rangle . \tag{9}$$

The inclusion (8) means that every $u \in J^k(n, p)$ determines an ideal $\langle u \rangle$ in E. It also determines the ideal in $J^k(n, 1)$ generated by its coordinates $u_1, \ldots, u_p \in J^k(n, 1)$. We denote the latter ideal by $\langle u \rangle_k$:

$$\langle u \rangle_k = (\langle u \rangle + M^{k+1})/M^{k+1} \subseteq J^k(n, 1) . \tag{10}$$

Now let V be a finite dimensional vector space. Then an *algebraic* *variety* in V is the zero set of a finite set of functions $p : V \to \mathbb{R}$ where each $p(x)$ is a polynomial in the coefficients of $x \in V$ relative to some (and hence any) basis of V. According to a theorem of Whitney [79], an algebraic variety is a *finite union of submanifolds*. The codimension of the variety is the codimension of a submanifold of largest dimension from this finite union.

The following theorem called the Infinite Codimension Lemma is given by H. Kurland and J. Robin ([43]) and shall be used in all the later applications.

(4.5) THEOREM (Infinite Codimension Lemma, [43]). *There are subsets* $W^k \subseteq J^k(n, n)$, $k = 1, 2, \ldots$ *such that:*

(a) *if* $g \in E_{n,n}$ *and* $j^k g(0) \notin W^k$ *then either* $g(0) \neq 0$ *or* g *has an isolated zero at* 0 ;

(b) W^k *is an algebraic variety;*

(c) *the codimension of* W^k *in* $J^k(n, n)$ *tends to infinity with* k .

Proof. For each integer $k = 1, 2, \ldots$, we let W^k be the set of all $u \in J^k(n, n)$ such that the codimension of $\langle u \rangle_k$ in $J^k(n, 1)$ is greater than k :

$$W^k = \left\{ u \in J^k(n, n) \mid \operatorname{codim}_{J^k(n,1)} (\langle u \rangle_k) > k \right\} . \tag{11}$$

We shall prove the properties (a), (b), (c) of Theorem (4.5) by two lemmas.

(4.6) LEMMA. *Let* $g \in E_{n,n}$. *Then* $j^k g(0) \in W^k$ *if and only if the codimension of* $\langle u \rangle_k$ *in* $J^k(n, 1)$ *is greater than* k .

Proof of lemma. Suppose $j^k g(0) \notin W^k$. Then $\langle j^k g(0) \rangle_k$ has codimension less than or equal to k in $J^k(n, 1)$. By the second isomorphism theorem $\langle j^k g(0)\rangle + M^{k+1}$ has codimension less than or equal to k in E. As g and $j^k g(0)$ differ by terms of order $k+1$ this last ideal is $\langle g\rangle + M^{k+1}$. By Proposition (4.2) we have $M^k \subseteq \langle g\rangle + M^{k+1}$; so by Nakayama's lemma $M^k \subseteq \langle g\rangle$, so $\langle g\rangle = \langle g\rangle + M^{k+1}$ and has codimension less than or equal to k. Conversely if $\langle g\rangle$ has codimension less than or equal to k, then $M^{k+1} \subseteq M^k \subseteq \langle g\rangle$, so the Noether isomorphism theorem shows that $j^k g(0) \notin \bar{W}^k$ as required. //

Proof of (a). (a) then follows immediately from Lemma (4.6) and Proposition (4.4).

Proof of (b). Let $d = \dim\left(J^k(n, 1)\right) - k$. Then W^k is the set of all $u \in J^k(n, n)$ such that the vector space $\langle u\rangle_k$ has dimension less than d. The set of all elements $x^\alpha u_i \in J^k(n, 1)$ (where x^α ranges over the monomials of order less than or equal to k in the coordinates $x_1, \ldots, x_n$ and $u_1, \ldots, u_n$ are the coordinates of u) span the vector space $\langle u\rangle_k$. Think of elements of $J^k(n, 1)$ as column vectors and let $M(u)$ be the rectangular matrix whose columns are the k-jets of the $x^\alpha u_i$. Then $\langle u\rangle_k$ has dimension less than d if and only if every $d \times d$ minor of $M(u)$ vanishes. This expresses the condition $u \in W^k$ as a system of algebraic equations (each of degree d) in the coefficients of u proving that W^k is algebraic as required.

Proof of (c). To prove (c) we need another lemma.

(4.7) LEMMA. *Let $l > k$ and let $\pi : J^l(n, n) \to J^k(n, n)$ be the projection*

$$\pi\left(j^l g(0)\right) = j^k g(0)$$

for $g \in E_{n,n}$. Then

$$W^l \subseteq \pi^{-1}(W^k) \text{ .}$$

Proof of Lemma. Let $g \in E_{n,n}$ satisfy $j^k g(0) \notin W^k$. By Lemma (4.6), $\langle g \rangle$ has codimension less than or equal to k in E. Then by $k \le l$ and Lemma (4.6) again, $j^l g(0) \notin W^l$ as required. //

Now π is a linear surjection so the codimension of $\pi^{-1}(W^k)$ in $J^l(n, n)$ is just the codimension of W^k in $J^k(n, n)$. Hence to prove (c) it suffices to show that

for every k, there is an $l > k$ such that no point of W^l is an interior point of $\pi^{-1}(W^k)$. (12)

To prove this choose k and let $h \in E_{n,n}$ be the germ whose coordinates are given by

$$h_i(x) = x_i^{k+1} \text{ ,}$$

$i = 1, \ldots, n$. Clearly $M^{nk+1} \subseteq \langle h \rangle$ so that $\langle h \rangle$ has finite codimension. Let l be the codimension of $\langle h \rangle$ in E. Then by Lemma (4.6), $j^l h(0) \notin W^l$.

Now suppose $g \in E_{n,n}$ satisfies $j^l g(0) \in W^l$, and for $t \in \mathbb{R}$, let

$$g_t = (1-t)g + th \text{ .}$$

As $j^k g_t(0) = (1-t)j^k g(0)$, it follows that $j^l g_t(0) \in \pi^{-1}(W^k)$ for all t. By (a) the condition

$$j^l g_t(0) \in W^l$$

is algebraic in t. It holds for $t = 0$ but fails for $t = 1$; thus it can hold for at most finitely many t. In particular it fails for t arbitrarily close to zero showing that $j^l g(0)$ is not an interior point of $\pi^{-1}(W^k)$ as required. //

(4.8) REMARK. Equation (11) for the definition of W^k is rather explicit and one could presumably compute the codimension of W^k by counting the number of independent equations from this list. This gives another method to prove (c). However that looks rather tedious.

5. Third Application: Zeros of C^∞ Vector Fields

Let $\Omega \subseteq \mathbb{R}^n$ be open convex bounded as always and let $S^\infty(T\Omega)$ be the space of C^∞ vector fields on Ω whose derivatives of all order are bounded (see §3). Recall that each $\xi \in S^\infty(T\Omega)$ has the form $\xi(x) = \big(x, \tilde{\xi}(x)\big)$ for $x \in \Omega$, where $\tilde{\xi} \in B^\infty\big(\Omega, \mathbb{R}^n\big)$.

We endow $S^\infty(T\Omega)$ with the family of increasing norms

$$\|\xi\|_k = \|\tilde{\xi}\|_k = \sup_{x \in \Omega} \{\|\tilde{\xi}(x)\| + \ldots + \|D^k\tilde{\xi}(x)\|\} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

defined by (3) in §3. Then $S^\infty(T\Omega)$ is a separable Fréchet space isomorphic to $B^\infty\big(\Omega, \mathbb{R}^n\big)$ by the isomorphism Φ defined in (4), §3.

Recall that a point $x \in \Omega$ is a *zero* (or *critical point*) of ξ iff $\tilde{\xi}(x) = 0 \in \mathbb{R}^n$.

(5.1) PROPOSITION. *There is a dense subset* $G \subseteq S^\infty(T\Omega)$ *such that every* $\xi \in G$ *has the property that* ξ *has only isolated zeros.*

Proof. We apply the Infinite Codimension Lemma (4.5) to find an integer k so large that

$$q = \operatorname{codim}\big(W^k \text{ in } J^k(n, n)\big) > n \tag{1}$$

where W^k is the algebraic subset constructed by (11) in §4.

Let $J^k(T\Omega) \to \Omega$ be the vector bundle of k-jets of vector fields on Ω (see [4], p. 19):

$$J^k(T\Omega) = \Omega \times J^k(n, n) \ . \tag{2}$$

Define

$$\tilde{W} = \Omega \times W^k \subseteq \Omega \times J^k(n, n) = J^k(T\Omega) . \tag{3}$$

Then $\tilde{W}$ is a finite union of submanifolds of $J^k(T\Omega)$ of codimension greater than n :

$$\tilde{W} = \bigcup_{j=1}^{N} \tilde{W}_j \text{ where for } 1 \leq j \leq N , \ \tilde{W}_j = \text{submanifold of } J^k(T\Omega)$$

$$\text{and } q_j = \operatorname{codim}\left[\tilde{W}_j \text{ in } J^k(T\Omega)\right] > n . \tag{4}$$

Moreover, if $\xi : \Omega \to T\Omega = \Omega \times \mathbb{R}^n$ is any vector field such that $j^k\xi(\Omega) \cap \tilde{W} = \emptyset$, then ξ has only isolated zeros in Ω by part (c) of the Infinite Codimension Lemma (4.5). Here $j^k\xi : \Omega \to J^k(T\Omega)$ denotes the k-jet extension of $\xi : \Omega \to T\Omega$.

We now apply the $B\Gamma$-Transversal Density Theorem. Let $A = S^\infty(T\Omega)$ and consider the map

$$\rho : A \to C^\infty_{B\Gamma}\left(\Omega, J^k(T\Omega)\right)$$

defined by $\rho(\xi) = j^k\xi$ for each $\xi \in A$; $j^k\xi$ is considered as a C^∞ map $\Omega \to J^k(T\Omega)$. Then

$$\operatorname{ev}_\rho : A \times_{B\Gamma} \Omega \to J^k(T\Omega) = \Omega \times J^k(n, n) \equiv \Omega \times P^k(n, n)$$

$$(\xi, x) \longmapsto j^k\xi(x) = \left(x, P^k\tilde{\xi}(x)\right)$$

is of class $C^r_{B\Gamma}$ by Lemma (1.3) where r is an integer

$$r > \max(q, k) \tag{5}$$

and $\Gamma = \{p_{r+k+i}\}_{i \geq i_0}$ given in Lemma (1.3).

Define for each $1 \leq j \leq N$,

$$G_j = \left\{\xi \in A \mid j^k\xi(\Omega) \cap \tilde{W}_j = \emptyset\right\} \tag{6}$$

and consider

$$G = \{\xi \in A \mid j^k\xi(\Omega) \cap \tilde{W} = \emptyset\} . \tag{7}$$

Then it follows quickly that

$$G = \bigcap_{j=1}^{N} G_j = \{\xi \in S^{\infty}(T\Omega) \mid \xi \text{ has only isolated zeros}\}. \qquad (8)$$

For each j $(1 \le j \le N)$, ev_{ρ} and $\tilde{W}_j$ verify all the conditions of the $B\Gamma$-Transversal Density Theorem:

(a) Ω has $\dim = n$; $\tilde{W}_j$ has codimension $q_j > n$;

(b) $A = S^{\infty}(T\Omega)$ and Ω are second countable;

(c) $r > \max(0, n-q_j)$ since $n - q_j < 0$;

(d) $ev_{\rho} \pitchfork_{B\Gamma} \tilde{W}_j$ by part (c) of Lemma (1.3);

(e) follows exactly as in the proof of Proposition (3.1).

Thus for each $1 \le j \le N$, G_j is residual in $S^{\infty}(T\Omega) = A$ since we have

$$\xi \in G_j \text{ iff } j^k\xi \pitchfork \tilde{W}_j \qquad (9)$$

since $\operatorname{codim}\left(\tilde{W}_j \text{ in } J^k(n, n)\right) = q_j > n = \dim \Omega$.

Hence G is residual by (8), and thus is dense in $S^{\infty}(T\Omega)$ because $S^{\infty}(T\Omega)$ is Baire. //

(5.2) REMARK. Proposition (5.1) proves the existence of a dense subset $G \subseteq S^{\infty}(T\Omega)$ with the property that each $\xi \in G$ has only isolated zeros; while Proposition (3.1) exhibits explicitly such a subset, the set $G_0^{\infty}(\Omega)$ of 0-transversal vector fields on Ω (see Proposition (3.1) and Remark (3.2)).

6. Fourth Application: Finite-to-One Maps

Let $\Omega \subseteq \mathbb{R}^n$ be open convex and bounded as always and let $f \in B^{\infty}(\Omega, \mathbb{R}^n)$. We say that f is *locally finite-to-one* (see [43]) iff every point $x \in \Omega$ has a neighbourhood $U \subseteq \Omega$ such that $f^{-1}(y) \cap U$ is finite for all $y \in \mathbb{R}^n$.

Recall that $B^\infty(\Omega, \mathbb{R}^m)$ is a separable Fréchet space calibrated by the sequence of increasing norms

$$\|f\|_k = \sup_{x\in\Omega} \{\|f(x)\| + \|Df(x)\| + \dots + \|D^k f(x)\|\} \tag{1}$$

defined in §1.

(6.1) PROPOSITION. *There is a dense subset* $G \subseteq B^\infty(\Omega, \mathbb{R}^n)$ *such that each* $f \in G$ *has the property that* $f : \Omega \to \mathbb{R}^n$ *is locally finite-to-one.*

Proof. We apply the Infinite Codimension Lemma (4.5) to find an integer k so large that

$$q = \operatorname{codim}\left(W^k \text{ in } J^k(n, n)\right) > n \tag{2}$$

where W^k is the algebraic subset constructed by (11) in §4.

Consider the k-jets bundle $J^k(\Omega, \mathbb{R}^n) = \Omega \times J^k(n, n) \equiv \Omega \times P^k(n, n)$ and define

$$\tilde{W} = \Omega \times W^k \subseteq \Omega \times J^k(n, n) = J^k(\Omega; \mathbb{R}^n) . \tag{3}$$

Then $\tilde{W}$ is a finite union of submanifolds of $J^k(\Omega; \mathbb{R}^n)$ of codimension greater than n .

Consider the map $\rho : A \equiv B^\infty(\Omega; \mathbb{R}^n) \to C^r_{B\Gamma}(\Omega; J^k(\Omega; \mathbb{R}^n))$ defined by $\rho(f) = j^k f$ for each $f \in A$, where r is an integer sufficiently large, say $r > \max(q, k)$.

Then, as usual, the map

$$\mathrm{ev}_\rho : A \times_{B\Gamma} \Omega \to J^k(\Omega; \mathbb{R}^n) = \Omega \times P^k(n, n) \tag{4}$$

$$(f, x) \mapsto j^k f(x) = (x, f(x), \dots, D^k f(x))$$

is $C^r_{B\Gamma}$ with respect to the calibration $\Gamma = \{p_{r+k+i}\}_{i\geq i_0}$ defined in the Lemma (1.3).

Let $G = \{f \in A \mid j^k f(\Omega) \cap \tilde{W} = \emptyset\}$ then G is dense in $A = B^\infty(\Omega; \mathbb{R}^n)$ as usual.

It remains to be seen that if $f \in G$ then f is locally finite-to-one; that is, that any $f : \Omega \to \mathbb{R}^n$ such that $j^k f(\Omega) \cap \tilde{W} = \emptyset$ is locally finite-to-one. Fix such an f, and let $x \in \Omega$ be an arbitrary point. Then if y is a point in $\mathbb{R}^n$, then by part (b) of the Infinite Codimension Lemma (4.5), since $j^k f(x) \notin W^k$, we have: there is a neighbourhood U of x such that $f^{-1}(y) \cap U = \emptyset$ or is finite. Indeed, by suitable translations we can suppose $x = 0 \in \Omega$ and $y = 0 \in \mathbb{R}^n$, $f = \psi \circ g \circ \Phi$. Then either $g(0) \neq 0$ (which means $f(x) \neq y$) or $g(0) = 0$ and 0 is an isolated zero for g (which means there is a neighbourhood U of x such that $\forall x' \in U$, $x' \neq x \Rightarrow f(x') \neq y$). //

7. Fifth Application: Fixed Points of C^∞ Maps

Let $\Omega \subseteq \mathbb{R}^n$ be open, convex and bounded as usual, and consider the separable Fréchet space $B^\infty(\Omega; \mathbb{R}^n)$ defined in §1. Then we have (see also [43]),

(7.1) PROPOSITION. *There is a dense subset* $G \subseteq B^\infty(\Omega; \mathbb{R}^n)$ *such that every* $f \in G$ *has only isolated fixed points.*

Proof. As usual, we choose k so large that

$$\operatorname{codim}\left(W^k \text{ in } J^k(n, n)\right) > n \tag{1}$$

where W^k is the algebraic set defined in the Infinite Codimension Lemma (4.5).

Fix k and consider the bundle $J^k(\Omega; \mathbb{R}^n)$ (see [4]),

$$J^k(\Omega; \mathbb{R}^n) = \Omega \times J^k(n, n) = \Omega \times \mathbb{R}^n \times J_0^k(n, n) \tag{2}$$

where $J_0^k(n, n)$ denotes those k-jets with no constant term so that $\Omega \times \Omega \times J_0^k(n, n)$ is open in $\Omega \times \mathbb{R}^n \times J_0^k(n, n) \equiv J^k(\Omega; \mathbb{R}^n)$ since Ω is open.

Note that $W^k \subseteq J_0^k(n, n)$ and define

$$\tilde{W} = \Delta \times (1+W^k) \tag{3}$$

where Δ is the diagonal of $\Omega \times \Omega$ and $1 + W^k$ denotes the translate of W^k by the k-jet of the identity map $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$:

$$1 + W^k = j^k(\mathrm{id}) + W^k = \{j^k(\mathrm{id})+u \mid u \in W^k\} . \tag{4}$$

Then $\tilde{W}$ is finite union of submanifolds of $J^k(\Omega; \mathbb{R}^n)$ of codimension greater than n.

Define $G = \{f \in B^\infty(\Omega; \mathbb{R}^n) \mid j^k f(\Omega) \cap \tilde{W} = \emptyset\}$ then G is dense as usual.

It remains to be seen that any $f : \Omega \to \mathbb{R}^n$ such that $j^k f(\Omega) \cap \tilde{W} = \emptyset$ has only isolated fixed points. But if $x \in \Omega$ such that $j^k f(x) = (x, f(x), Df(x), \ldots, D^k f(x)) \notin \tilde{W} = \Delta \times (1+W^k)$ then either $x \neq f(x)$, that is, x is not a fixed point, or $x = f(x)$ and $(Df(x), \ldots, D^k f(x)) \notin 1 + W^k$. That is, by putting $g = f - \mathrm{id}$, $g(x) = 0$ and $D^k g(x) \notin W^k$; which implies, by the Infinite Codimension Lemma, that x is an isolated zero of g (i.e. fixed point of f). //

NOTATION

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