HIGH SPEED PLASMA FLOW ABOUT PROBES

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The contents of this thesis, except as described in the Acknowledgements and where credit is indicated by reference, are entirely my own work. My contribution to the Preprint Paper given in Appendix B is the theoretical optimization of the design parameters for the double diaphragm shock tube.

(Greg Allen)
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ABSTRACT

A method for evaluation of the interaction between a uniform magnetic field and a compressible inviscid, electrically conducting ideal gas flow past a blunt body is presented. An experiment using a double diaphragm shock tube to produce a slug of gas with high electrical conductivity is performed to confirm the theoretical results. The theoretical problem is considered in two parts.

Firstly, the flow field around a hemispherical cap in a supersonic flow is evaluated using the method of integral relations. The difficulty associated with the mathematical saddle point singularity in this scheme is seen to be the instability of the integral solution curves to small perturbations. This sensitivity is used to advantage by applying small parameter perturbations to extend the integral curves near the singularity, an innovation due to South [1969]. In this way, the extrapolation error across the sonic line is decreased while at the same time the computational effort expended in determining the integral curve is reduced.

Secondly, the solution for the perturbed magnetic field is obtained using the Biot-Savart relation and solution of the resulting integral equations. The advantage of this approach, for problems in which the boundary conditions are asymptotic to the undisturbed values at infinity, lies in the simple way in which these conditions are satisfied. The problem is reduced to the solution of a Fredholm integral equation and this is solved numerically.

The large magnetic Reynolds number limit to the application of this method is determined.

The analysis is applied to the determination of the induced magnetic field near the nose of a magnetic probe in a supersonic stream with an applied uniform magnetic field. The magnetic flux density measured by the probe is significantly reduced when the magnetic Reynolds number is greater than unity.

The experimental results show reasonable agreement with the predicted values and differences are explained in terms of the short test gas duration and the assumption of ideal gas behaviour for the theoretical computations.
CHAPTER 1
INTRODUCTION

The measurement of the magnetic field in a plasma is usually achieved by insertion of a small probe. This can perturb the plasma by alteration of the plasma conductivity, modification of the electric currents and introduction of an induced magnetic field. The restrictions on probe design to avoid these perturbations are discussed by Lovberg [1965] and Botticher [1968]. When there is relative velocity between the probe and plasma the deflection of the plasma about the probe body produces a velocity-magnetic field interaction different from that in the undisturbed stream. Depending on the strength of this interaction the magnetic field may be perturbed by induced electric currents. For experiments in which it is impossible to keep this interaction weak, the probe signals may be seriously in error. We consider this problem, evaluating the induced magnetic field near the nose of the probe where the search coil or other sensor* is usually located, and take account of the magnetic field perturbation.

Attention is restricted to problems amenable to both theoretical analysis and experimental confirmation while remaining applicable to magnetic field measurements in hypervelocity flows in shock tunnels or behind strong shock waves produced in shock tubes. The flow is supersonic relative to the probe and a bow shock wave forms at the probe nose. The applied magnetic field is assumed to be uniform and aligned with the undisturbed gas stream. With this configuration there are no induced electric currents except in the vicinity of the probe where the velocity is disturbed.

* This may be the element of a galvanomagnetic (Hall probe) or magnetoresistance probe [Botticher, 1968].
The mathematical problem of determining the magnetic field perturbation is complicated by the nature of the governing magneto-hydrodynamic relations. These form a mixed hyperbolic-elliptic set, the precise nature being determined by the local Alfvén and Mach numbers.

If we consider the supersonic flow past a blunt body with no applied magnetic field (Alfvén number infinity) we find that there is a strong fore-and-aft asymmetry in the flow pattern, and that the flow field is disturbed upstream of the body to within a short distance of the nose. The boundary of the disturbance is a shock wave which extends downstream with increasing region of influence. The region between the shock and body is usually termed the shock layer.

The problem of determining the detached shock wave and the flow field behind it for a given body has not been solved analytically. The difficulty lies in the non-linear mixed subsonic-supersonic nature of the problem and that the boundaries between the different regions, the shock wave and the limiting characteristic,* are not known in advance.

Most of the existing techniques for the solution of the blunt body problem are discussed by Hayes and Proebstein [1966]. Three related techniques result in the Newtonian, constant-density and thin layer solutions. The first of these assumes an infinitesimally thin shock layer coincident with the body surface and the latter two make use of assumptions of constant density or thin shock layer to simplify the gas dynamic equations. These solutions are only approximate representations of the physical situation, although they can provide insight into the flow processes. Another approximation method is the

* This is the boundary of the influence of the downstream supersonic flow on the subsonic region.
method of integral relations [Belotserkovskii, 1957] which, in principle, may be extended to any desired degree of accuracy. There are also solutions which may be regarded as "exact" in the sense that no approximations to the flow equations are made and the accuracy is limited only by the numerical technique used. These are the relaxation, unsteady and streamtube-continuity techniques which are, in general, indirect methods by which the flow equations are integrated from a prescribed shock wave to determine the body shape supporting this shock.

Now if we examine the interaction between an applied magnetic field and an electrically conducting flow (which may be either subsonic or supersonic) we find that the induced electric currents may act to modify the magnetic field or the flow field, or both. Associated with these currents is a magnetic body force $J \times B$ which opposes the fluid velocity. The ratio of this force to the inertia force in the stream is $S = \sigma B^2 \ell / \rho u$, where $\ell$ is a characteristic length scale and $\sigma$ is the electrical conductivity, giving a measure of the perturbation of the velocity field. The magnetic field is perturbed when the magnetic Reynolds number, $R_m = \omega \ell / \nu$, is large since $R_m$ is a measure of the efficiency of the magnetic field induction. Theoretical treatments of this type of problem which evaluate the magnetic field perturbation for a specified incompressible flow have been given by Tamada [1964] and Bishop [1968]. These studies are pertinent to flows of liquids and incompressible gas flow but can not be accurately applied to the problems with supersonic free stream owing to the essential differences introduced by the compressibility of the flow, the existence of the bow shock wave in supersonic flow and the fore-and-aft asymmetry.

* These include the methods of Gravalos [1957], Van Dyke [1958], Garabedian and Lieberstein [1958] and Moretti [1967].
The steady flow of a compressible electrically conducting fluid past a slender body in a circular wind tunnel in the presence of an applied aligned magnetic field has been evaluated by Kusukuwa and Suwa [1968]. The problem is dealt with by the slender body approximation in which the fundamental equations are linearized by neglecting higher order terms of the perturbation quantities. This restricts the analysis to low values of $R_m$ and $S$ and therefore it is not applicable to our problem.

Problems of supersonic magnetohydrodynamic flow past a blunt body containing a dipole magnetic field have been considered by Bush [1958], Kemp [1958] and Porter and Cambel [1967]. The shock layer is assumed incompressible and the magnetic field and velocity components are given by similarity forms. From the approximations, these authors are able to compute the perturbation of both the velocity and magnetic field with the velocity field valid in the vicinity of the stagnation point. Ericson et al. [1965] have also performed an order of magnitude analysis for a magnetohydrodynamic flight control problem assuming an average shock layer velocity and simple geometric form for the magnetic field. The induced magnetic field is computed using the Biot-Savart relation and results comparable with those of Bush, Kemp and Porter and Cambel are obtained. However, none of these methods permits evaluation of the precise form of the magnetic field since the form is specified a priori. A more serious error is seen when the region of strongest electric currents, shown in section 2.5 to be near the sonic line, is considered. The assumptions by these authors of incompressible flow restricts the accuracy of the velocity field to the stagnation region, while the region where accuracy in the velocity solution is required extends further around the body.

For an accurate description of the gas dynamic properties of
the supersonic flow past a blunt body, a computational method with accuracy compatible with the accuracy of the magnetic field determination is essential. The method of integral relations was chosen for this purpose since it is relatively simple to implement and there is the possibility of extending the accuracy by increasing the order of approximation. In addition, the analysis of Chushkin [1963] provides the means of extending the simple fluid dynamic calculation to take account of the velocity perturbation caused by the magnetic field-velocity interaction.

It is difficult to incorporate Maxwell's equations into the integral relations technique, for although these equations may be cast in divergence form, the boundary conditions for the magnetic field are not applied at the same locations as those for the supersonic flow. To overcome this difficulty, the problem of evaluating the magnetic field perturbation has been considered on its own assuming that the relevant velocity and electrical conductivity information has been evaluated. This approach is applicable to problems in which the magnetic Reynolds number is of the order of unity while the body force interaction parameter S is small and consequently, the Alfvén number \( \nu/(B^2/\mu_0)^{1/2} \) is large.

The boundary conditions for the magnetic field are not easily specified. For this reason, the solution involving integration of Maxwell's equations has not been employed, but rather a new approach using the Biot-Savart relation and solution of the resulting integral equations. This method is developed in Chapter 2. The range of applicability is derived determining confidence limits for accuracy of the solution. This restriction specifies a maximum value of magnetic

* This is the form of the equations required for application of the method of integral relations.
Reynolds number for which acceptable solutions may be obtained.

The numerical solution for the induced magnetic field is performed for a range of values of $R_m$ and Mach number with the assumptions that:

(i) the problem is time independent;

(ii) the Alfvén number in the flow is large;

(iii) there is no external electric field;

(iv) the electron pressure gradient, ion slip and Hall effect contributions to the electric current density are negligible;

(v) the fluid is an inviscid ideal gas with constant ratio of specific heats $\gamma = 5/3$;

(vi) the applied magnetic field is weak and is uniform and aligned with the undisturbed free stream velocity.

The results for the perturbed magnetic field are expressed in terms of the relative measurable magnetic probe signal. Plots of lines of constant magnetic flux in the vicinity of the probe present an overall picture of the field perturbation.

To confirm these theoretical predictions, a shock tube experiment involving magnetic probes and an applied aligned magnetic field was performed. The measured perturbation of the magnetic probe signal is about 50% of the predicted value. This is reasonable agreement considering that the calculations for the flow field are based on ideal gas assumptions and the shock tube is an ultra-high performance double-diaphragm machine for which operating characteristics are still under investigation.

Another experiment concerned with experimental errors in magnetic probe measurements has been described by Botticher et al. [1967]. The magnetic field associated with cylindrically convergent shock waves produced in a linear z-pinch with extreme diameter to length ratio
(44:10) is measured using probes with different body shapes. The result is that probes with upstream body extension change the probe signal amplitude to a far greater extent than those situated near the nose of quite blunt obstacles. In some cases the original amplitude is reduced by about 66% and is delayed by about 0.2 microsec compared with the probe signal taken without the probe extension. The mechanism of magnetic field reduction is through the replacement of a tube of current carrying plasma from which the magnetic field is derived by the nonconducting probe extension, and therefore is quite different from our experiment.
CHAPTER 2
EVALUATION OF THE INDUCED MAGNETIC FIELD

The relations determining the electric current density, magnetic field and electric field are derived from the general Ohm's law [Sutton and Sherman, 1965, section 5.8] and the Maxwell relations. The requirement that the magnetic field is weak and that the contributions to the conduction current due to the electron pressure gradient, ion slip and Hall effect are negligible allows simplification of Ohm's law to:

\[ J = \sigma (E + u \times B) \]

(2.1)

where \( \sigma \) is the scalar conductivity. For the time independent problem the electric field is assumed negligible and the only contribution to the electric current density is due to the induced electric field \( u \times B \).

The relations between electric field, magnetic field and electric current density are completed by Maxwell's relations:

\[ \nabla \times B = \mu J \]

(2.2)

and

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

(2.3)

For the steady problem equation (2.3) is trivial.

2.1 ESTIMATE OF THE TIME DEPENDENT SOLUTION

Before embarking on the solution of the steady state problem, the initial change in the magnetic field due to the onset of conducting flow is investigated. This starting process indicates the trend in the induced field. If it is compatible with the results from the
steady state solution there is some confidence in the latter solution. In addition, an estimate of the time required to reach steady conditions may be obtained.

We manipulate Maxwell's equations and Ohm's law to obtain the characteristic times. From equation (2.1),

$$E = \frac{1}{\sigma} J - u \times B.$$  

Substituting for \(J\) from equation (2.2) and replacing \(E\) in equation (2.3) gives:

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) - \eta \nabla \times (\nabla \times B), \quad (2.1.1)$$

where \(\eta = 1/\sigma u\). Scaling this equation according to characteristic fluid velocity \(u\), length \(L\), and electrical conductivity \(\sigma_0\),

$$\frac{\partial B}{\partial t} = \frac{u}{L} \left\{ \nabla \times (u \times B) - \frac{1}{R_m} \nabla \times (\nabla \times B) \right\}, \quad (2.1.2)$$

where \(R_m = uL/\eta\) is the magnetic Reynolds number. The first term in the brackets in equation (2.1.2) represents the rate of magnetic field change owing to the induced current, and the second represents that associated with diffusion of the magnetic field. Using a dimensional argument,

$$\Delta B = \Delta t \left( \frac{u}{L} \nabla \times (u \times B) - \Delta t \frac{u}{LR_m} \nabla \times (\nabla \times B) \right), \quad (2.1.3)$$

and the characteristic times \(\tau_I\) and \(\tau_D\) for modification of the field by induced current and diffusion are \(\tau_I = L/u\) and \(\tau_D = R_m L/u = R_m \tau_I\).

It is obvious that for low magnetic Reynolds number the mag-
netic field diffuses faster than it is induced and the characteristic
time to reach a steady state is $L/u$. For large $R_m$, diffusion effects
are slower and the steady state characteristic time is $R_m (L/u)$. These
results determine an order of magnitude estimate of the time required
to reach a steady state and will be considered in Chapter 5 with
regard to the experiment.

Now we consider the nonconducting flow past a body with an
aligned magnetic field. There is no induced field. Let the conduc-
tivity of the fluid be increased instantaneously at time $t = 0$ so that
the electric current density is large enough to induce a non-negligible
magnetic field. For an order of magnitude solution to this time
dependent problem, the electrical conductivity is chosen to be constant
and the flow is assumed to be described by the incompressible flow
about a solid nonconducting sphere. Consequently there is a velocity
potential [Batchelor, 1967, p.452], written in spherical polar
coopinates $(r, \theta, \chi)$ as:

$$\psi = \frac{\nu a^3}{2r} \sin^2 \theta .$$

From this relation, the velocity components $u_R, u_Z$ in cylindrical polar
coopinates $(R, Z, \phi)$ are:

$$u_R = \frac{3}{2} u \frac{a^3}{r^3} \sin \theta \cos \theta ,$$

and

$$u_Z = \frac{u}{2} \left[ 2 + \frac{a^3}{r^3} (\sin^2 \theta - 2 \cos^2 \theta) \right] .$$

The velocity is written:

$$\mathbf{u} = u_z \mathbf{a}_z + u_R \mathbf{a}_R ,$$
where $\mathbf{a}_Z$ and $\mathbf{a}_R$ are unit vectors in the cylindrical polar system.

The applied uniform aligned magnetic field is $\mathbf{B} = B_0 \mathbf{a}_Z$,

where $B_0$ is constant. Accordingly, when $t = 0$, $\nabla \times \mathbf{B} = 0$, and equation (2.1.2) becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{u}{L} \nabla \times (\mathbf{u} \times \mathbf{B}) . \quad (2.1.5)$$

The $Z$, $R$ and $\phi$ components of equation (2.1.5) are:

$$\frac{\partial B_z}{\partial t} = - \frac{B_0}{R} \frac{\partial}{\partial R} R u_R ,$$

$$\frac{\partial B_R}{\partial t} = B_0 \frac{\partial}{\partial Z} u_R ,$$

and

$$\frac{\partial B_\phi}{\partial t} = 0 .$$

Substituting the value of $u_R$ from equation (2.1.4) into these equations gives:

$$\frac{1}{B_0} \frac{\partial B_z}{\partial t} = \frac{3}{2} \frac{u a^3}{r^7} Z(2Z^2 - 3R^2) , \quad (2.1.6)$$

and

$$\frac{1}{B_0} \frac{\partial B_R}{\partial t} = - \frac{3}{2} \frac{u a^3}{r^7} R(R^2 - 4Z^2) . \quad (2.1.7)$$

It can be seen from equation (2.1.6) that if $\frac{\partial B_z}{\partial t} = 0$ then $Z = 0$ or $2Z^2 = 3R^2$. Similarly, from equation (2.1.7), if $\frac{\partial B_R}{\partial t} = 0$ then $R = 0$ or $R^2 = 4Z^2$. Figure 2.1.1 represents the induced magnetic field after the conductivity is switched on and before the diffusion term in equation (2.1.2) is significant. The effect on the total magnetic field is shown in figure 2.1.2. It should be noted that equations (2.1.6) and
Figure 2.1.1: The initial induced magnetic field from the interaction between incompressible conducting flow and uniform magnetic field.
Figure 2.1.2: The initial perturbed magnetic field from the interaction between incompressible conducting flow and uniform magnetic field.
(2.1.7) are applicable only in the conducting flow and consequently do not apply inside the sphere.

The induced field appears to arise from two separate azimuthal current loops giving rise to opposing induced field. For a body more representative of a probe, for example a hemispherical nose cylinder, the velocity downstream of the hemispherical nose becomes parallel to the axis and the corresponding current loops and induced fields are weaker. The induced magnetic field then appears to arise from just the upstream current loops.

The total magnetic field is shown to be deflected from the axis upstream of the body where the induced field opposes that applied. It is pinched back toward the axis downstream of the body. The asymptotic behaviour of the field lines at great distances from the body has not been determined here.

2.2 APPROACH TO THE PROBLEM

Under the conditions already imposed, the steady state problem, formulated in terms of Maxwell's equations, is reduced to the solution of the equations:

\[ \nabla \cdot \mathbf{B} = 0 , \]  
(2.2.1)

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} , \]  
(2.2.2)

and

\[ \mathbf{J} = \sigma (\mathbf{u} \times \mathbf{B}) , \]  
(2.2.3)

with the velocity field specified in a numerical form. The boundary conditions applied to the magnetic field are imposed at infinity, where the field is uniform and in the direction of the undisturbed fluid velocity.

The numerical solution can be obtained in a number of ways.
It is worth considering different methods which might be applied to find which is most appropriate. We look briefly at the numerical difference schemes, the variational methods, particularly the Galerkin technique, and a reformulation of the equations in terms of integral equations. The latter scheme (not to be confused with the method of integral relations) was ultimately selected because of the natural way in which the solution evolves.

A system of numerical difference equations equivalent to equations (2.2.1) and (2.2.2) can be formed [see for example Vitasek, 1969, p.1109]. The set may be solved over a region large enough so that the boundary conditions for the problem are approximately satisfied at the boundaries of the region of integration. For a finite mesh size these boundaries would be at a finite distance from the nose region of the blunt body. With the imposition of such artificial boundary conditions at "infinity" there is no guarantee that the new boundaries will not influence the solution and hence invalidate the computations. It seems likely that the physical system might be unstable since, for very large values of magnetic Reynolds number, the induced magnetic field bears little resemblance to the applied field.

Numerical difference schemes are known to be unstable under some circumstances (e.g. use of high order formula with small local truncation error, Fox, 1962, chapters 4 and 8) and the use of such schemes for the solution of a problem with inherent instabilities could be imprudent.

The variational methods of Ritz, Kantorovitch, Trefftz, Galerkin and the least squares method can be used for the solution of partial differential equations or integral equations [Prager, 1969, p.1045]. In these schemes, a sequence of functions $\{\phi_n\}$, sometimes called trial functions, is chosen subject to certain constraints. The
solution to equations of the form:

\[ Au = w , \] (2.2.4)

where \( A \) is a functional operator, \( u \) and \( w \) are functions and \( u \) is unknown, is approximated by series:

\[ u_n = \sum_{i=1}^{n} a_i \phi_i . \] (2.2.5)

Evaluation of the coefficients \( a_i \) yields the approximate solution to the unknown function.

Consider the Galerkin method for which the restrictions on the sequence of trial functions are:

(i) the \( \phi_i \) are linearly independent,
(ii) they satisfy the boundary conditions, and
(iii) the sequence is complete in the sense that all solutions of equation (2.2.4) are accurately described by equation (2.2.5).

The last restriction requires that the form of the solution be known before the problem can be solved. The method is one of the many methods of weighted residuals [see for example Crandall, 1956; Finlayson and Scriven, 1966].

The residual, \( R \), is formed when the approximation (2.2.5) replaces the function \( u \) in equation (2.2.4), and

\[ R = A \sum_{i=1}^{n} a_i \phi_i - w . \] (2.2.6)

This residual is multiplied in turn by each of the functions \( \phi_i \) in the sequence. These products are integrated over the region of interest.
and the weighted residuals are thus formed. For an exact solution the residual would be zero everywhere but for the approximation this might not be so.

The coefficients \( a_i \) are determined when the weighted residuals are put equal to zero. Let the integration be denoted by the functional scalar product:

\[
(Au_n - w, \phi_i) \equiv \int (Au_n - w) \phi_i \, dr .
\]  

(2.2.7)

The weighted residual is \( R_i = (Au_n - w, \phi_i) \). Substituting for \( u_n \) from equation (2.2.5) gives:

\[
R_i = (A \sum_j a_j \phi_j - w, \phi_i), \quad i = 1, 2, \ldots, n .
\]  

(2.2.8)

When the operator \( A \) is linear, the system (2.2.8) is also linear, and with \( R_i = 0 \),

\[
\sum_j a_j (A \phi_j, \phi_i) = (w, \phi_i), \quad i = 1, 2, \ldots, n .
\]  

(2.2.9)

The coefficients \( a_j \) are found from the solution of the system of simultaneous linear equations (2.2.9), and the approximate solution is determined.

This method has been used by Bishop [1968] to evaluate the induced magnetic field near a perfectly conducting sphere in an incompressible conducting flow with applied dipole fields. The trial functions \( \{\phi_n\} \) are derived from the Legendre polynomials. Bishop presents results with magnetic Reynolds number up to \( R_m = 10.0 \), noting that a limit to the maximum value of \( R_m \) is determined by the number of trial functions used. This is a physical restriction imposed by the
The Galerkin technique is ideally suited to the problem of incompressible inviscid flow about a sphere. The boundary conditions at the sphere can be imposed by choosing suitable forms for the trial functions. The trial functions which should be associated with magnetic field components induced by distributed currents are the Legendre polynomials and associated Legendre polynomials [Jackson, 1962, p.144] with argument $\cos \theta$ (e.g. $P_j^k(\cos \theta)$). There is also a radial dependence, written conveniently as inverse powers of the radial variable, $r$, with origin at the centre of the sphere. The trial functions are consequently simple expressions involving the trigonometric functions $\sin \theta$ and $\cos \theta$ multiplied by inverse powers of $r$.

Taking into account the orthogonality properties of the Legendre and associated polynomials, the integrals which determine the coefficients $a_i$ are straightforward. The expressions for the velocity components are also trigonometric functions.

The Galerkin method is not well suited to the problem of supersonic flow past a nonconducting sphere or spherical nose. The velocity components are not simple geometrical functions of position, and the boundary conditions on the sphere surface are unknown. Expansion of the magnetic field components in terms of the inverse of the radial variable $r$ should be avoided for the nonconducting body since a physically meaningful magnetic field would exist everywhere within the body and would be finite. Terms involving $r^{-n}$ at the origin would be infinite and not representative of the magnetic field there.

A solution exterior to the sphere could be approximated using the Galerkin technique, and the corresponding Dirichlet problem for the
Laplace equation [Rektorys, 1969, p.886] solved simultaneously. Such an approach leads to difficulties since a numerical method for the Dirichlet problem and its boundary conditions would need to be formulated in terms of the Galerkin solution. In view of these difficulties, the Galerkin method was not considered further.

A practical and natural approach to the solution of the induced magnetic fields is the solution of a system of integral equations equivalent to the partial differential equations. This is a natural approach since the induced magnetic field originates from electric currents and is specified in terms of the Biot-Savart relation,

\[ B_{\text{ind}}(x) = \mu \int \frac{J(x') \times s}{4\pi s^3} \, d\mathbf{r}(x') , \quad (2.2.10) \]

where integration is taken over the volume containing electric currents, \( d\mathbf{r} \) is an elemental volume and the quantity \( s \) is the magnitude of the vector \( \mathbf{s} = \mathbf{x} - \mathbf{x}' \). \( B_{\text{ind}}(x) \) and \( J(x) \) represent the induced magnetic field and electric current density respectively at the point \( x \).

It is this fundamental expression in association with the specification that the divergence of \( \mathbf{B} \) is zero \( (\nabla \cdot \mathbf{B} = 0) \) from which Maxwell's relation:

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} \]

is derived. Thus it is exactly equivalent to solve the integral equation (2.2.10) or to solve the partial differential equations (2.2.1) and (2.2.2).

2.3 METHOD OF ANALYSIS

We consider the modification of the magnetic field, resulting
from the interaction between the stream of conducting gas and the magnetic field, using the Biot-Savart relation (2.2.10). This relation is simplified by taking advantage of the symmetry of the problem and the resultant surface integral is integrated numerically. The velocity field and electrical conductivity are assumed to be known at all points. The problem may be scaled so that the magnitude of each quantity is referred to a characteristic value. We take the characteristic values of distance, electrical conductivity, velocity and magnetic field to be \( L, \sigma_0, u_0 \) and \( B_0 \) respectively. In terms of these, the scaling factor for electric current density is \( \sigma_0 u_0 B_0 \). With these substitutions, equation (2.2.10) is written:

\[
B_{\text{ind}}(x) = R_m \int \frac{J(x') \times s}{4\pi s^3} \, d\tau(x') ,
\]

(2.3.1)

since by definition, \( R_m = \mu \sigma_0 u_0 L \).

The induced magnetic field is described as a function of the distributed electric current with density \( J(x) \) given by equation (2.2.3) which, when written in terms of the applied and induced magnetic fields, becomes:

\[
J = \sigma(u \times B_{\text{appl}}) + \sigma(u \times B_{\text{ind}}) .
\]

(2.3.2)

Substituting for \( J \) inside the integral equation (2.3.1), we have:

\[
B_{\text{ind}}(x) = R_m \int \frac{\sigma'(u' \times B_{\text{appl}}') \times s}{4\pi s^3} \, d\tau + R_m \int \frac{\sigma'(u' \times B_{\text{ind}}') \times s}{4\pi s^3} \, d\tau ,
\]

(2.5.3)

where the prime symbol ('') denotes evaluation at the point \( x' \). When the values of the magnetic Reynolds number, electrical conductivity, velocity field and applied magnetic field are specified, equation
(2.3.3) becomes an integral equation for the unknown induced magnetic field.

It is easily seen that the important parameter determining the magnitude of the induced magnetic field is $R_m$, when $u$ and $B_{\text{appl}}$ are not parallel.

2.3.1 Evaluation of the Induced Field by Approximation

From equation (2.3.3) it is possible to derive a simple recurrence formula for an $n^{th}$ order approximation to $B_{\text{ind}}(x)$ when $R_m$ is small. The limitations on $R_m$ are discussed in section 2.3.6.

To simplify the manipulations, it is convenient to introduce a functional operator $L$ defined by:

$$L\gamma = \int \frac{\sigma'(u' \times \gamma') \times s}{4\pi s^3} \, dt',$$

where $\gamma$ is an arbitrary vector function. In terms of $L$, equation (2.3.3) is written:

$$B_{\text{ind}} = R_m L B_{\text{appl}} + R_m L B_{\text{ind}}.$$

After rearrangement, equation (2.3.4) is written:

$$(I - R_m L) B_{\text{ind}} = R_m L B_{\text{appl}},$$

where the identity operator $I$ is defined by $I\gamma = \gamma$. Now if an inverse operator $(I - R_m L)^{-1}$ exists such that:

$$(I - R_m L)^{-1} (I - R_m L) = I,$$

and consequently,

$$(I - R_m L)^{-1} (I - R_m L) B_{\text{ind}} = B_{\text{ind}},$$
The multiplication of this inverse with equation (2.3.5) gives the solution for the induced magnetic field as:

\[ B_{\text{ind}} = (I - R_m L)^{-1} R_m L B_{\text{appl}} . \]  

(2.3.6)

The existence of \((I - R_m L)^{-1}\) depends on the nature of the operator \(L\) and the value of \(R_m\), and is discussed in section 2.3.6.

When \(R_m L\) has a suitable form, namely when \(R_m \ll 1\), the inverse operator can be approximated in terms of the power series expansion:

\[(I - R_m L)^{-1} = I + R_m L + (R_m L)^2 + \ldots . \]  

(2.3.7)

Substituting equation (2.3.7) into equation (2.3.6),

\[ B_{\text{ind}} = (I + R_m L + (R_m L)^2 + \ldots ) R_m L B_{\text{appl}} . \]  

(2.3.8)

The first approximation to \(B_{\text{ind}}\) is:

\[ B_{\text{ind}}^{(1)} = R_m L B_{\text{appl}} \]  

(2.3.9)

and a recurrence formula for the \((n+1)\)th successive approximation is:

\[ B_{\text{ind}}^{(n+1)} = B_{\text{ind}}^{(1)} + R_m L B_{\text{ind}}^{(n)} , \quad n = 1, 2, 3, \ldots \]  

(2.3.10)

The total magnetic field \(B_T = B_{\text{appl}} + B_{\text{ind}}\) can be obtained in a similar way. From equation (2.3.5),

\[(I - R_m L) B_T = B_{\text{appl}} . \]  

(2.3.11)

Multiplying by the inverse \((I - R_m L)^{-1}\),

\[ B_T = (I - R_m L)^{-1} B_{\text{appl}} . \]  

(2.3.12)
and substituting the power series expansion,

$$B_T = (I + R_m L + (R_m L)^2 + \ldots) B_{\text{appl}}.$$  

The recurrence relation for the \(n\)th approximation to the total field is:

$$B_T^{(n)} = B_T^{(0)} + R_m L B_T^{(n-1)}, \quad n = 1, 2, 3, \ldots, \quad (2.3.13)$$

where

$$B_T^{(0)} = B_{\text{appl}}.$$  

From the definition of \(L\), these approximations written in full are:

$$B_{\text{ind}}^{(1)}(x) = R_m \int \frac{\sigma'(u' \times B_{\text{appl}}') \times s}{4\pi s^3} \, dt, \quad (2.3.14)$$

$$B_{\text{ind}}^{(n+1)}(x) = B_{\text{ind}}^{(1)}(x) + R_m \int \frac{\sigma'(u' \times B_{\text{ind}}') \times s}{4\pi s^3} \, dt, \quad (2.3.15)$$

$$B_T^{(0)}(x) = B_{\text{appl}}(x) \quad (2.3.16)$$

and

$$B_T^{(n)}(x) = B_T^{(0)}(x) + R_m \int \frac{\sigma'(u' \times B_T^{(n-1)}) \times s}{4\pi s^3} \, dt, \quad (2.3.17)$$

for \(n = 1, 2, 3, \ldots\).  

An alternative to using these approximations is to evaluate the inverse operator \((I - R_m L)^{-1}\) and use equation (2.3.6) for the solution of \(B_{\text{ind}}\) or equation (2.3.12) for \(B_T\). Substitution of numerical integration formulae into the relevant integrals transforms the operator \(L\) and functions \(B_{\text{ind}}\) and \(B_T\) into an approximating system involving matrices (see section 2.3.5), the inverses of which are well known.
Before performing the integrations we will show that the integrals in equations (2.3.14) to (2.3.17) are finite and bounded in spite of the singularity in the integrand when \( s = x - x' = 0 \).

2.3.2 Boundedness of the Volume Integral

\[
\int \frac{\sigma'(u' \times B') \times s}{4\pi s^3} \, \mathrm{d}r
\]

Let \( M \) be the smallest sphere containing both the region of integration and the singular point \( s = 0 \). Such a sphere exists and has a finite radius, say \( R \). Now it can be shown that:

\[
\int_\mathcal{M} \frac{\mathrm{d}r}{4\pi s^2} \leq R
\]

with equality when \( s = 0 \) at the centre of \( M \). Thus:

\[
\left| \int \frac{\sigma'(u' \times B') \times s}{4\pi s^3} \, \mathrm{d}r \right| \leq \max \left| \sigma'(u' \times B') \right| \int_\mathcal{M} \frac{\mathrm{d}r}{4\pi s^2} \leq \sigma_m' u_m' B_m' R,
\]

where \( \sigma_m', u_m' \) and \( B_m' \) are the maximum magnitudes of \( \sigma', u' \) and \( B' \) in the region of integration. Since these values are finite, the integral is finite and bounded in magnitude.

2.3.3 Reduction of the Volume Integral to a Surface Integral

Suppose the problem is described using the cylindrical polar co-ordinate system \((z, r, \phi)\) with the z axis in the direction of the undisturbed free stream and applied magnetic field. The body is symmetric with respect to rotations about this axis and consequently, in the absence of upstream non-uniformities, so is the velocity.

The vector product of the velocity with the applied magnetic field, \( u \times B_{\text{appl}} \), is a vector in the azimuthal \((\phi)\) direction since \( u \) and \( B_{\text{appl}} \) both lie in the \( r-z \) plane. This is the direction of the induced
electric currents which, through equation (2.3.1), can only induce magnetic field components in the r-z plane. Thus the restriction of the velocity and applied magnetic field to the r-z plane confines the induced magnetic field to the same plane while the electric current density vectors are purely azimuthal. This means that all electric currents in the system form closed loops around the axis.

Through the usual formulae, the components of the induced magnetic field at the point \( \mathbf{r} \) due to the circular loop of current through a neighbourhood of area \( \delta A \) about the point \( \mathbf{r}' \), \( J(\mathbf{r}') \cdot \delta A(\mathbf{r}') \) [see for example Weber, 1950, p.141], are written:

\[
\delta B_r(\mathbf{r}, \mathbf{r}') = \frac{R_m}{2\pi} J' \cdot \delta A' \left( \frac{z - z'}{r} \right) \left\{ \frac{(r + r')^2 + (z - z')^2}{r^2 + (z - z')^2} \right\}^{1/2} \times \left[ -K(k) + \frac{r'^2 + r^2 + (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right], \tag{2.3.18}
\]

\[
\delta B_z(\mathbf{r}, \mathbf{r}') = \frac{R_m}{2\pi} J' \cdot \delta A' \left( (r + r')^2 + (z - z')^2 \right)^{-1/2} \times \left[ K(k) + \frac{r'^2 - r^2 - (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right], \tag{2.3.19}
\]

and

\[
\delta B_\phi(\mathbf{r}, \mathbf{r}') = 0 ,
\]

where the quantity \( k \) is given by:

\[
k^2 = 4\pi r' / \left( (r' + r)^2 + (z - z')^2 \right) , \tag{2.3.20}
\]

and \( K(k) \) and \( E(k) \) are the complete elliptic integrals of first and second kinds respectively [Jahnke and Emde, 1943].

The components of the induced field at \( \mathbf{r} \) are found by adding together all the contributions from points \( \mathbf{r}' \) in the r-z plane through which currents flow. Consequently,
\[ B_{\text{induced}} (x) = \frac{R}{2\pi} \iint J' \left( \frac{z - z'}{x} \right) \left\{ (r + r')^2 + (z - z')^2 \right\}^{-\frac{1}{2}} \]
\[ \times \left[ -K(k) + \frac{r'^2 + r^2 + (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right] \, dr' \, dz' , \quad (2.3.21) \]
\[ B_{\text{induced}} (x) = \frac{R}{2\pi} \iint J' \left\{ (r + r')^2 + (z - z')^2 \right\}^{-\frac{1}{2}} \]
\[ \times \left[ K(k) + \frac{r'^2 - r^2 - (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right] \, dr' \, dz' , \quad (2.3.22) \]
and
\[ B_{\Phi} \text{induced} (x) = 0 . \]

Since, from equation (2.3.2),
\[ J' = \sigma' \left[ u_l^0(B_{\text{appl}}^r + B_{\text{ind}}^r) - u_l^0(B_{\text{appl}}^z + B_{\text{ind}}^z) \right] , \quad (2.3.23) \]
the equations (2.3.21) and (2.3.22) are coupled integral equations each involving the unknowns \( B_{\text{ind}}^r \) and \( B_{\text{ind}}^z \).

2.3.4 The Fredholm Integral Equation

For the analysis it is convenient to amalgamate equations (2.3.21) and (2.3.22) into a single equation and at the same time introduce a new variable, denoted by the symbol \( B \), to replace the \( r \) and \( z \) components of the magnetic field. This is done in the following way.

When \( R \) represents the region where currents flow (region of integration) and \( x \) is an arbitrary point within \( R \), a constant vector \( \mathbf{L} \) is chosen such that the point \( \mathbf{L} + x \) lies outside \( R \). A duplicate of the region of integration, denoted by \( R' \), is constructed consisting of all points described by \( \mathbf{L} + x \). From our choice of \( \mathbf{L} \), \( R \) and \( R' \) do not
intersect and \( \mathbb{R}_L \) consists of all points in \( \mathbb{R} \) displaced by the vector \( \mathbf{L} \).

The new variable \( \mathbf{B} \) can now be defined by \( \mathbf{B}(x) = \mathbf{B}_z(x) \) and \( \mathbf{B}(x + \mathbf{L}) = \mathbf{B}_T(x) \), where the domain of \( \mathbf{B} \) is \( \mathbb{R} \) and \( \mathbb{R}_L \). From equations (2.3.21), (2.3.22) and (2.3.23), and with subscripts denoting induced and applied components as before,

\[
B_{\text{ind}}(\mathbf{y}) = \int \int \ Y(\mathbf{y}, \mathbf{y}') \left[ B_{\text{ind}}(\mathbf{y}') + B_{\text{appl}}(\mathbf{y}') \right] dy_T dy_z, \quad (2.3.24)
\]

where

(i) \( \mathbf{y}' \) and \( \mathbf{y} \) are points in either \( \mathbb{R} \) or \( \mathbb{R}_L \);

(ii) integration is over the region \( \mathbb{R} \) and its duplicate \( \mathbb{R}_L \);

(iii) \( dy_T' \) and \( dy_z' \) are differentials of \( r \) and \( z \);

and

(iv) \( Y(\mathbf{y}, \mathbf{y}') \) is the function defined by:

\[
Y(\mathbf{x}, \mathbf{x}') = \frac{g'}{2\pi} (-u_T') \left\{ (r + r')^2 - (z - z')^2 \right\}^{-\frac{1}{2}}
\]

\[
\times \left[ K(k) + \frac{r'^2 - r^2 - (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right],
\]

\[
Y(\mathbf{x}, \mathbf{L} + \mathbf{x}') = -Y(\mathbf{x}, \mathbf{x}') u_T'/u_T',
\]

\[
Y(\mathbf{x} + \mathbf{L}, \mathbf{x}') = \frac{g'}{2\pi} \left( \frac{z - z'}{r} \right) (u_T') \left\{ (r + r')^2 + (z - z')^2 \right\}^{-\frac{1}{2}}
\]

\[
\times \left[ -K(k) + \frac{r'^2 + r^2 + (z - z')^2}{(r' - r)^2 + (z - z')^2} E(k) \right],
\]

and

\[
Y(\mathbf{x} + \mathbf{L}, \mathbf{x}' + \mathbf{L}) = -Y(\mathbf{x} + \mathbf{L}, \mathbf{x}') u_T'/u_T',
\]

for \( \mathbf{x} \) a point in \( \mathbb{R} \) having co-ordinates \( (r,z) \). Now equation (2.3.24) represents both equations (2.3.21) and (2.3.22) with the \( z \) component of
the magnetic field associated with B in the region R and the r component with B in the duplicate region.

Writing \( B_T(y) \) to represent the total magnetic field (the domain is again \( R + R_L^r \)), equation (2.3.24) can be rearranged to give:

\[
B_T(y) = B_{\text{app1}}(y) + R \int \int Y(y', y'') B_T(y'') \, dy' \, dy''.
\] (2.3.25)

This equation has the form of the Fredholm integral equation of the second kind [Rektorys, 1969, p.918]. The usual technique for numerical solution of these equations is the replacement of the integral operator by a numerical integral operator so that the integral equation is approximated by a linear system of equations [Atkinson, 1967].

2.3.5 Formulae for Numerical Solution of the Integral Equation

Suppose the planar region R and its duplicate \( R_L \) together are divided into \( n \) area elements characterised by points inside each element, \( t_1, t_2, \ldots, t_n \). It is reasonable to divide them into \( n/2 \) elemental areas each, labelled by the \( t_j \) in such a way that for points \( x \) in the \( i \)th element of \( R \), \( i = 1, 2, \ldots, n/2 \), the \((n/2 + i)\)th element of \( R_L \) is specified by points \( x_L \) where \( x_L = L + x \). In other words, \( R \) and \( R_L \) are subdivided in the same way and the subscripts of corresponding points differ by \( n/2 \). Further, let a suitable numerical integration procedure, denoted by \( P \), using values of \( Y(y, y') \) and \( B_T(y') \) at the chosen points \( t_j \) in each element approximate the exact integration. It is then possible to approximate equation (2.3.25) by the equation:

\[
B_n(y) - R \sum_{j=1}^{n} w_j Y_n(y, t_j) R_n(t_j) = B_{\text{app1}}(y),
\] (2.3.26)

where the subscript \( n \) indicates the division of the region of integra-
tion, $R + R_L$, into $n$ areas, $w_j$ represents a weighting function associated with the integration procedure $P$ and point $t_j$, and $Y_n(\gamma, t_j) = Y(\gamma, \gamma')$ and $B_{\text{appl}}(t_j) = B_{\text{appl}}(\gamma')$ when $\gamma' = t_j$. In equation (2.3.26) the numerical values of $R$, $w_j$, $Y_n(\gamma, t_j)$ and $B_{\text{appl}}(\gamma)$ may be inserted for a given value of $\gamma$. Replacing $\gamma$ by $t_i$ for $i = 1, 2, \ldots, n$ yields $n$ equations of the form:

$$B_n(t_i) - R_m \sum_{j=1}^{n} w_j Y_n(t_i, t_j) B_n(t_j) = B_{\text{appl}}(t_i). \quad (2.3.27)$$

The values $B_n(t_i)$ for $i = 1, 2, \ldots, n$ are found from the simultaneous solution of the linear system (2.3.27). These values are subsequently substituted into equation (2.3.26) which, rearranged, may be written:

$$B_n(\gamma) = B_{\text{appl}}(\gamma) + R_m \sum_{j=1}^{n} w_j Y_n(\gamma, t_j) B_n(t_j). \quad (2.3.28)$$

This is the resulting approximate formula for numerical solution to the exact integral equation (2.3.25).

2.3.5.1 Matrix Notation

Writing equation (2.3.27) in matrix notation [see for example Pease, 1965, p.2, p.13] with matrix $L_n \equiv (L_{n ij})$ and vectors $B_n \equiv (B_n)$, $B_A \equiv (B_A)$, where $L_{n ij} \equiv w_j Y_n(t_i, t_j)$, $B_{n i} \equiv B_n(t_i)$ and $B_{A i} \equiv B_{\text{appl}}(t_i)$, we have:

$$(I - R_m L_n) B_n = B_A, \quad (2.3.29)$$

where $I$ is the identity matrix defined by:

$$I_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$
Equation (2.3.29) has the same symbolic form as equation (2.3.11), but it should be noted that equation (2.3.29) is a matrix equation approximating the solution of equation (2.3.11) which is formulated in terms of functions.

2.3.5.2 The Singularity of the Matrix of Coefficients

The matrix $L_n$ in equation (2.3.29) has elements

$$L_{nj} = w_j Y_n(t_i, t_j).$$

The values of $r, z, \sigma, u_r$ and $u_z$ associated with $t_j$ are identical with those associated with $t_{j+n/2}$ when $j = 1, 2, \ldots, n/2$. Consequently from the definition of $Y_n$ in section 2.3.4,

$$Y_n(t_i, t_j) = -Y_n(t_i, t_{j+n/2}) \frac{u_r}{u_z},$$

and the $j^{th}$ column of the matrix $L_n$ is the same as the $(n/2 + j)^{th}$ column except for a constant multiplicative factor. This implies that the matrix $L_n$ has at most $n/2$ linearly independent columns and is therefore singular.

2.3.6 Range of Applicability

The equation for the numerical solution of the magnetic field components at node points, equation (2.3.29), has the matrix form:

$$(I - R_m L) x = b,$$  \hspace{1cm} (2.3.30)

where $R_m$ is the magnetic Reynolds number (a scalar), $L$ is a singular matrix (section 2.3.5.2) and $x$ and $b$ are vectors.

When $R_m$ is so small that:

$$\lim_{R_m \to \infty} (R_m L)^r = 0$$  \hspace{1cm} (2.3.31)
is satisfied, the matrix inverse of \((I - R_m L)\) is [Wilkinson, 1965, p.59]:

\[
(I - R_m L)^{-1} = I + R_m L + (R_m L)^2 + \ldots \tag{2.3.32}
\]

and by condition (2.3.31) the successive terms in the power series expansion in equation (2.3.32) are decreasing in magnitude.

Multiplying equation (2.3.30) by the inverse of \((I - R_m L)\) yields the expression:

\[
x = (I - R_m L)^{-1} b . \tag{2.3.33}
\]

Substituting for \((I - R_m L)^{-1}\) from equation (2.3.32) gives:

\[
x = \{I + R_m L + (R_m L)^2 + \ldots\} b , \tag{2.3.34}
\]

which, neglecting high order terms in \(R_m L\) becomes:

\[
x = b + R_m L b + O((R_m L)^2 b) . \tag{2.3.35}
\]

Equation (2.3.35) is the first approximation to the solution of equation (2.3.30) for small \(R_m\). This result is analogous to that derived in functional notation in equation (2.3.13). The induced magnetic field components are directly proportional to the magnetic Reynolds number and the matrix \(L\) describes the influence of the applied magnetic field at each node point on the induced magnetic field at each other node point.

As the magnitude of \(R_m\) increases the higher order terms in \(R_m L\) in equation (2.3.34) become increasingly important and the approximation (2.3.35) becomes incorrect.

As \(R_m\) tends to infinity, the components of the matrix \(R_m L\) are large compared with those of the identity matrix \(I\) so that:
\[ \lim_{R_m \to \infty} (I - R_m L) = R_m L, \]

and equation (2.3.30) can be written:

\[ R_m L \mathbf{x} = -\mathbf{b}. \quad (2.3.36) \]

The induced magnetic field, \( \mathbf{x}' = \mathbf{x} - \mathbf{b} \) is determined by:

\[ R_m L \mathbf{x}' = -R_m L \mathbf{b}, \quad (2.3.37) \]

or

\[ L \mathbf{x}' = -L \mathbf{b}, \quad (2.3.38) \]

where the scalar quantity \( R_m \) has been cancelled.

Since \( L \) is singular there exists at least one vector \( \mathbf{y} \neq 0 \) [Pease, 1965, p.73] for which:

\[ L \mathbf{y} = 0. \quad (2.3.39) \]

Hence, although \( \mathbf{x}' = -\mathbf{b} \) is a solution of equation (2.3.38), it is not unique since for any arbitrary scalar \( c \), \( \mathbf{x}' = c \mathbf{y} - \mathbf{b} \) is also a solution.

The conclusion to be drawn from the preceding discussion is that using this method it is impossible to obtain a solution for the induced magnetic field when the magnetic Reynolds number is infinitely large. Indeed, since it has been shown that solutions exist for small \( R_m \), there must be some value of \( R_m \) at which the solution of equation (2.3.30) becomes invalid. An estimate of this limiting value of \( R_m \) will be found by examining some properties of the matrices in the problem.

The eigenvectors of the matrix \( L \) are defined to be those vectors \( \mathbf{y}_j \) for which there exists a scalar, \( \lambda_j \), such that:

\[ L \mathbf{y}_j = \lambda_j \mathbf{y}_j. \quad (2.3.40) \]
The quantity $\lambda_j$ is the eigenvalue associated with $\hat{y}_j$. When the matrix $L$ is multiplied by a scalar $R_m$, the resulting matrix $(R_m \cdot L)$ has the same eigenvectors as $L$ but has eigenvalues $R_m \cdot \lambda_j$. The eigenvalues of the identity matrix are all identical and equal to unity since by definition:

$$I \cdot \hat{y}_j = \hat{y}_j.$$ (2.3.41)

Hence the eigenvalues of the matrix $(I - R_m \cdot L)$ in equation (2.3.30) are 

$$(1 - R_m \cdot \lambda_j).$$

If any of the eigenvalues $(1 - R_m \cdot \lambda_j)$ is zero the matrix $(I - R_m \cdot L)$ is singular [Pease, 1965, p.73] and no unique solution of equation (2.3.30) is possible. This happens for values of $R_m$ for which:

$$R_m = \frac{1}{\lambda_j},$$ (2.3.42)

for each of the eigenvalues $\lambda_j$ of $L$. The least value of $R_m$ for which equation (2.3.42) is satisfied corresponds to:

$$R_m = \frac{1}{\max |\lambda_j|}.$$ (2.3.43)

Examination of the restriction (2.3.31) on the value of $R_m$ considered in relation to the series expansion for small $R_m$ yields [Wilkinson, 1965, p.59]:

$$\max |R_m \cdot \lambda_j| < 1,$$

for eigenvalues $R_m \cdot \lambda_j$ of the matrix $R_m \cdot L$. Rearranging this expression gives:

$$R_m < \frac{1}{\max |\lambda_j|},$$ (2.3.44)
which is compatible with equation (2.3.43). Therefore, solution of the system (2.3.30) by series expansion and successive approximation appears to have a more severe restriction on the maximum value of $R_m$ than the direct solution which uses solution of the linear equations.

The foregoing discussion considers the solvability of $n$ simultaneous equations in $n$ unknowns when the arithmetic can be performed exactly. In practice, when these equations are solved using a digital computer, the arithmetic may only be performed with precision governed by the actual machine used.

The nature of the system of linear equations plays a significant role in the accuracy of the solution. Problems are termed "ill-conditioned" [see for example Forsythe and Moler, 1967, p.22, Wilkinson, 1965, p.86] when the values to be computed are very sensitive to small perturbations to the data. Equation (2.3.30) would be ill-conditioned if the solution $x$ was very sensitive to small changes in the matrix $(I - R_m L)$ or the vector $b$.

An example of a badly conditioned problem is the system:

\[
\begin{align*}
    x + 0.99 \, y &= 1.99 \\
    0.99 \, x + 0.98 \, y &= 1.97
\end{align*}
\]

(2.3.45)

whose solution represents the point of intersection of two practically coincident straight lines. The true solution is $x = 1, y = 1$.

Changes in the right hand sides of these equations by -0.0001 and 0.0001 give solutions $x = 3, y = -1.02$, which shows how sensitive the solution is to a small change in the data.

A measure of the sensitivity of a system of linear equations is the condition number. Consider the linear system:

\[
A x = b
\]

(2.3.46)
where \( A \) is a non-singular matrix of order \( n \) and \( \mathbf{x} \) and \( \mathbf{b} \) are vectors of length \( n \). Since \( A \) is non-singular the determinant of \( A \) is non-zero and all the eigenvalues of \( A \) are non-zero. In this case a matrix inverse, \( A^{-1} \), exists and is unique and the system (2.3.46) has a unique solution \( \mathbf{x} \) written:

\[
\mathbf{x} = A^{-1} \mathbf{b} .
\]  

(2.3.47)

If \( A \) is known exactly and there is some uncertainty, \( \delta \mathbf{b} \), in the elements of \( \mathbf{b} \) it can be shown [Forsythe and Moler, 1967, p.20] that the uncertainty in the solution \( \mathbf{x} \), \( \delta \mathbf{x} \), can be written:

\[
\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\|A\|}{\|A^{-1}\|} \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|} .
\]  

(2.3.48)

where \( \|\mathbf{x}\| \) is the vector norm of \( \mathbf{x} \) and \( \|A\| \) the matrix norm of \( A \) [see for example Wilkinson, 1965, p.55, Simmons, 1962, p.54]. The ratio \( \|\delta \mathbf{x}\|/\|\mathbf{x}\| \) is a measure of the relative uncertainty in the solution vector \( \mathbf{x} \) in the same way as \( \|\delta \mathbf{b}\|/\|\mathbf{b}\| \) is a measure of the relative uncertainty in the data vector \( \mathbf{b} \). The meaning of equation (2.3.48) is that \( \|A\| \|A^{-1}\| \) bounds the relative uncertainty in the solution vector to that of the data vector. The quantity \( \|A\| \|A^{-1}\| \) is defined to be the condition number of the matrix \( A \) and is denoted by \( \text{cond}(A) \).

It can also be shown that when \( \mathbf{b} \) is known exactly and there is some uncertainty in the matrix \( A \) [Forsythe and Moler, 1967, p.23] then:

\[
\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x} + \delta \mathbf{x}\|} \leq \text{cond}(A) \frac{\|\delta A\|}{\|A\|} .
\]  

(2.3.49)

In this case the relative uncertainty in the solution of \( A \mathbf{x} = \mathbf{b} \) is
bounded by the product of the condition number of \( A \) with the relative uncertainty in \( A \).

The condition number of a matrix plays a significant role in determining whether the corresponding linear system is ill-conditioned and is a measure of the accuracy of the numerical solution. The value of the condition number depends on the actual matrix norm used. In most common use is the spectral norm \( \| A \|_2 \), which is subordinate to and therefore compatible with the Euclidean vector norm \( \| x \|_2 \) and these are defined by the relations [Wilkinson, 1965, p.57]:

\[
\| x \|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \quad (2.3.50)
\]

and

\[
\| A \|_2 = \left( \text{maximum eigenvalue of } A^T A \right)^{1/2}, \quad (2.3.51)
\]

where \( A^T \) is the matrix transpose of \( A \) which has elements \( (A^T)_{ij} = (A)_{ji} \). The corresponding condition number, the spectral condition number of \( A \) with respect to inversion, is usually denoted by \( \kappa(A) \) [Wilkinson, 1965, p.87] and:

\[
\kappa(A) = \frac{\| A \|_2 \| A^{-1} \|_2}{\left( \text{maximum eigenvalue of } A^T A \right) \left( \text{maximum eigenvalue of } (A^{-1})^T (A^{-1}) \right)^{1/2}}. \quad (2.3.52)
\]

The representation of the spectral condition number in equation (2.3.52) shows that \( \kappa(A) \geq 1 \) with equality only for matrices for which \( A^T A = I \), where \( I \) is the identity matrix. It is generally accepted that the system (2.3.30) is well-behaved with respect to inversion when \( \kappa(A) \leq 100 \). The spectral condition number of the ill-conditioned \( 2 \times 2 \) system (2.3.45) is \( \kappa \sim 40,000 \), and it is not surprising that its
numerical solution is so sensitive to perturbations.

Using equation (2.3.52), the condition number of \((I - R_m L)\) can be evaluated when the maximum modulus eigenvalues of the matrix \((I - R_m L)^T(I - R_m L)\) and its inverse are computed.

To evaluate the maximum modulus eigenvalue of a matrix, consider the \(n \times n\) matrix \(A\) whose eigenvalues are ordered in a decreasing sequence with regard to their absolute values,

\[
|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|.
\]

A sequence of vectors \(x_0, x_1, x_2, \ldots\) is constructed according to the relation:

\[
x_{k+1} = A x_k \quad k = 0, 1, 2, \ldots
\]

where \(x_0\) is an arbitrary non-zero vector. Provided that \(x_0\) is not an eigenvector of \(A\) other than the eigenvector corresponding to \(\lambda_1\), and \(\lambda_1 > \lambda_2\), the value of the maximum modulus eigenvalue \(\lambda_1\) is found by successive iteration and is given by the formula [Korvasova, 1969, p.1161],

\[
x_{k+1,j} = \lambda_1 x_{k,j} + O \left( \frac{\lambda_2}{\lambda_1} \right)^k \quad \text{for } j = 1, 2, \ldots, n,
\]

where \(x_{k,j}\) is the \(j\)th element of the vector \(x_k\). When \(|\lambda_2| = |\lambda_1|\) the method does not converge but special formulae can be applied which yield \(\lambda_1\) and \(\lambda_2\) [Korvasova, 1969].

In the numerical evaluation of the maximum modulus eigenvalue of \((I - R_m L)^T(I - R_m L)\) only the simple formula given above was employed since the eigenvalues were distinct for moderate values of \(R_m\). The evaluation of the maximum modulus eigenvalue of the inverse of this
matrix took several minutes computing time owing to the dimension of the matrix (100 x 100).

Figure 2.3.1 shows the computed condition number of the matrix \((I - R_m L)\) corresponding to various Mach number flow fields and magnetic Reynolds number. For \(R_m\) greater than twenty, the condition numbers are generally too high for confidence in the solution of the system (2.3.30).

The condition number as a means of evaluating the accuracy of the system of linear equations consequently determines the upper bound on \(R_m\) and thus the range of applicability of the method of solution for the induced magnetic field.

2.4 CHOICE OF ALGORITHM FOR SOLUTION OF THE LINEAR EQUATIONS

The choice of a suitable method for numerical solution of the linear system of equations \(Ax = b\) with \(x\) the unknown vector depends on the nature of the matrix \(A\). If most of the elements of \(A\) are non-zero, \(A\) is said to be dense, otherwise \(A\) is sparse. Methods suitable for sparse matrices are not necessarily suitable for dense ones.

No class of algorithm has been found which is superior either in computing effort or accuracy to the method of Gaussian elimination for the solution of linear systems with a general dense matrix [Forsythe and Moler, 1967, p.27]. For sparse matrices, this method changes many of the zero elements to non-zero values which must be stored. For this type of matrix, the iterative methods are usually superior [Forsythe and Moler, 1967, p.120].

The matrix \((I - R_m L)\) is sparse when \(R_m \ll 1\). In this case, the series expansion (2.3.32) is the basis of an iterative solution for the induced field. However, for increasing values of \(R_m\) the order of
Figure 2.3.1: Condition number of the matrix \((I - R_m L)\) for various Mach number and magnetic Reynolds number.
approximation must increase to retain the accuracy of the solution, and as the matrix becomes more dense it is more appropriate to use the method of Gaussian elimination.

Since the range of $R_m$ in which we are interested is of the order of unity, the method of Gaussian elimination is used.

2.5 THE REGION OF INTEGRATION AND NUMERICAL INTEGRATION PROCEDURE

The region of integration over which the Biot-Savart relation is integrated is chosen to be that in which significant electric currents flow. This region is determined using the value of $J$ from Ohm's law with the velocity substituted from the solution of the blunt body problem and electrical conductivity from the Spitzer Hérm relation. For a uniform magnetic field aligned with the axis of symmetry, the relative strengths of currents over the flow field are given by the product $T^{3/2} V_R$, where the electrical conductivity is assumed to be proportional to the absolute temperature $T$ to the power $3/2$, and $V_R$ is the component of fluid velocity normal to the axis. A comparison of the relative current density along the body surface, the shock wave and a surface half-way between them, for Mach 3 flow is given in figure 2.5.1. Since the applied magnetic field is perturbed by these induced currents, figure 2.5.1 is only a guide to the actual current strengths.

When determining the influence of the current loop at one point on the magnetic field at another, the distance between these points must be taken into account. When the radius of the loop is of the same order of magnitude as the separation between the points, the induced field depends approximately on the inverse of the distance. For greater separation, the induced field is dipole-like and varies as the inverse of the cube of the distance. Consequently the magnetic
Figure 2.5.1: Relative electric current along the body surface, shock wave and an intermediate curve.
induction from each current loop is to some extent localised, and distant points are only feebly affected.

The region of integration chosen for the computations is bounded by the shock wave and body surfaces, the axis of symmetry and a line normal to this axis passing through the centre of the sphere (figure 2.5.2). This corresponds to the same region as that over which the flow field is evaluated using the method of integral relations (Chapter 3).

![Figure 2.5.2: Region of integration.](image)

The induced magnetic field arising from electric currents in the region of integration extends its influence a short distance
upstream of the bow shock wave. Consequently, in this upstream region, the velocity and magnetic field are no longer aligned and electric currents opposing the magnetic field perturbation are induced. The effect of these currents on the magnetic field near the probe is small since the free stream electrical conductivity is significantly less than in the stagnation region.

Downstream of the integration region, the velocity and magnetic field are not aligned and again electric currents result. The influence of these currents is to supplement the magnetic field perturbation. However, the velocity is tending to be more nearly parallel to the magnetic field and axis of symmetry, and the electrical conductivity is decreasing toward the free stream value as we proceed downstream. The electric currents are therefore decreasing and this is indicated by extrapolation of the curves in figure 2.5.1. Consequently the influence from this region is also neglected.

Having defined the region of integration we divide it into elementary areas (figure 2.5.3). A mesh is constructed in the following way:

Ten radii from the centre of the sphere are drawn with equal angular separation, the last of these being normal to the axis of symmetry. To cut these radii, three curves are drawn intermediate between the shock wave and body to divide each radius into four equal segments. The node points of this mesh, including the intercepts of the radii with the shock wave and body, are used as the points $t_1$, $t_2$, ..., $t_{n/2}$ introduced in section 2.3.5 and in the same order represent points $t_{n/2+1}$, ..., $t_n$ in $R_L$. There are fifty node points, so in this case $n = 100$.

The elementary areas characterised by the nodes are formed by subdividing each small mesh area into quarters and allocating the
quarters to the adjacent node point (figure 2.5.3). In this way the node point is roughly in the centre of its associated elemental area and velocity, magnetic field, electrical conductivity and geometrical position for the elemental area are approximated by the values at the node point.

The elemental area for the $j^{th}$ node point is approximately $A_j = \Delta \theta \, \varepsilon(\theta_j)/4d$, where $\Delta \theta$ is the angular separation of the radii, $\varepsilon(\theta_j)$ is the distance between the shock wave and body along the radius inclined at an angle $\theta_j$ to the axis, and $d$ takes the value 1, 2 or 4 depending on whether $t_j$ is an interior, boundary or corner point of the region of integration.
We use a simple integration formula for evaluation of the surface integral in equation (2.3.24). We assume the integral over each elemental area to be the product of the approximate area and the integrand evaluated at the node point. Consequently the weighting function in equation (2.3.26) is \( w_j = \Delta \theta \ \epsilon(\theta_j)/4d \), and \( \gamma_n(x,t_j) \) is evaluated by substituting the co-ordinates of \( t_j \) for \( x' \) into the definition for \( Y(x,x') \) (section 2.3.3).

For \( i = j \), the quantity \( \gamma_n(t_i,t_i) \) is infinite. To avoid this singularity we set \( \gamma_n(t_i,t_i) = 0 \). We justify this in physical terms by considering that the function \( w_j \gamma_n(t_i,t_i) \) represents the numerical coefficient determining the magnetic induction at the point \( t_i \) due to the current loop through the associated elementary area. Provided that the electric current density is essentially constant over this area, this coefficient, in the first approximation, should be zero since the effects of the distributed currents tend to be cancelled near their centroid.

There is a singularity in \( \gamma_n(t_i,t_j) \) for \( j = i \pm n/2 \) also and this is treated in the same way as for \( j = i \). Evaluation of the remaining coefficients \( w_j \gamma_n(t_i,t_j) \) is straightforward.

The computed values of the induced magnetic field in the vicinity of the probe are given in Chapter 5.
CHAPTER 3
SOLUTION FOR THE SUPERSONIC BLUNT BODY FLOW

3.1 THE METHOD OF INTEGRAL RELATIONS

The method of integral relations [Dorodnitsyn, 1958, 1962] has been applied to the solution of the nonlinear partial differential equations governing the supersonic flow past a blunt body [Belotserkovskii, 1957, 1958]. Reviews of the method are given by Belotserkovskii and Chushkin [1965], Hayes and Probstein [1966] and Belotserkovskii [1966]. Xerikos and Anderson [1962] indicate the practical difficulties with the one strip approximation resulting from instability of the solution near the sonic line. The scheme is briefly described in this chapter with emphasis on the difficulties encountered in obtaining a solution near the sonic line and the way in which they were overcome. The analysis based on that given by Chushkin [1963, 1965] is given in Appendix A to reveal errors in the equations obtained by Chushkin.

The main advantage of the scheme is that it is suited to automatic digital computation. It does not require the vast data storage areas required by the successive approximation and finite difference methods. The solution is "direct" in the sense that the flow field is found for a specified body shape. Indirect or "inverse" procedures assume a shock shape and evaluate the corresponding body.

The method reduces the problem of integrating a system of nonlinear equations with partial derivatives and two independent variables to that of solving a corresponding approximating system of ordinary differential equations. The procedure is illustrated by considering the system of n partial differential equations,
with boundaries \( a \leq x \leq b \) and \( 0 \leq y \leq \delta(x) \). Here \( P_i \), \( Q_i \) and \( R_i \) are functions specified in terms of \( x \) and \( y \) and \( n \) unknown functions \( u_k(x,y) \). The \( u_k \) are to be found, subject to suitable boundary conditions.

The region of integration is divided into \( m \) strips with boundaries given as suitable functions of the variable \( x \). For simplicity the boundaries of the strips are usually taken to be equally spaced between \( y = 0 \) and \( y = \delta(x) \). Let us assume that they are given by the functions \( y_0(x) \), \( y_1(x) \), ..., \( y_m(x) \) with \( y_0 \) and \( y_m \) corresponding to \( y = 0 \) and \( y = \delta(x) \) respectively. Each of the partial differential equations may be integrated with respect to the variable \( y \) across successive strips, yielding equations:

\[
\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} = R_i, \tag{3.1.1}
\]

\[
\frac{d}{dx} \int_{y_j}^{y_{j+1}} p_i \, dy - P_i(x,y_{j+1}) \frac{dy_{j+1}}{dx} + P_i(x,y_j) \frac{dy_j}{dx} + Q_i(x,y_{j+1})
- Q_i(x,y_j) = \int_{y_j}^{y_{j+1}} R_i \, dy, \tag{3.1.2}
\]

where \( 0 \leq j \leq m-1 \) and \( 1 \leq i \leq n \). These \( mn \) equations remain in exact form. The integrands \( P_i \) and \( R_i \) are not known explicitly for any given point \((x,y)\) since the functions \( u_k \) are not as yet determined. To proceed further expressions for the integrands in equations (3.1.2) must be obtained.

For integration along the \( y \) direction, \( P_i \) and \( R_i \) are approximated as functions of \( y \) in terms of their values at the strip boundaries. Polynomials are the usual approximating functions, the degree of the polynomial being equal to the number of strips, \( m \).
After substitution with the approximation formulae for $P_i$ and $R_i$ and performing the integrations with respect to $y$, a system of ordinary differential equations results. The solution of these can be done using well developed numerical methods.

3.2 NUMERICAL INTEGRATION OF THE SYSTEM OF EQUATIONS

Numerical methods presently available for integration of ordinary differential equations may be divided into two classes: those involving no memory and those involving some memory of the past behaviour of the solution. The Runge Kutta methods [Milne, 1953] belong to the first class and the Adams methods [Fox, 1962] to the second. Methods with memory are generally superior in accuracy for a given elementary step size and computation effort since a better approximating curve using previous solution behaviour may be fitted over the elementary interval. The major problems are stability, choice of quantities to remember, determination of logic for step size control and automatic starting (at the beginning there is nothing to remember).

The method chosen for the integrations was that of Nordsieck [1962]. This involves memory of the past behaviour of the solution and is equivalent to a reformulation of the Adams method since it uses essentially the same quadrature formula. The original method was designed and analysed for a computer using binary arithmetic. It has been adapted to suit floating point arithmetic and thoroughly tested by R.H. Hudson [CSIRO] and I.R. Simpson [ANU] [unpublished].

The desirable characteristics of the Nordsieck method are that:

(i) the solution is stable with a large margin of safety;
(ii) there is a built-in automatic starting procedure;
(iii) optimum step size is automatically chosen to provide the specified accuracy of the solution;

(iv) adjustment of the step size requires little computation since the form of the solution is carried in terms of coefficients of a polynomial;

(v) derivatives need only be evaluated twice at each step (this is the minimum consistent with stability and results in considerable economy of computer effort when the evaluation of derivatives involves a lot of computation); and

(vi) information is provided during each elementary step to enable interpolation of the solution over the elementary interval.

The approximating system of ordinary differential equations obtained by application of the method using one strip has the form (Appendix A):

\[
\frac{d\varepsilon}{d\theta} = -r_1 \cot(\sigma + \theta) ,
\]

\[
\frac{d\phi_0}{d\theta} = - \frac{(Y - 1)}{\phi_0 Y} F_{\theta_0} ,
\]

\[
\frac{d\sigma}{d\theta} = f(\theta; \varepsilon, \phi_0, \sigma, v_0) ,
\]

and

\[
\frac{dv_0}{d\theta} = \frac{N(\theta; \varepsilon, \phi_0, \sigma, v_0)}{a_s^2 - v_0^2} ,
\]

where \( F_{\theta_0} \), \( f \) and \( N \) are specified functions of \( \theta \) and the unknown functions \( \varepsilon, \phi_0, \sigma \) and \( v_0 \). In the denominator of equation (3.2.4), \( a_s^2 \) represents the speed of sound at the sonic point on the body. At that point, the velocity \( v_0 \) equals the speed of sound and the denominator of equation (3.2.4) is zero.
To ensure regularity of the solution for $v_0$ at the sonic point, the numerator $N$ of equation (3.2.4) must also be zero. Otherwise the derivative of $v_0$ with respect to $\theta$ would become infinite and the corresponding solution meaningless.

The boundary conditions for integration of the equations (3.2.1) to (3.2.4) that can be specified are the values of $\phi_0$, $\sigma$ and $v_0$ when $\theta$ is zero. The standoff distance $\epsilon$ is initially unknown. The problem is thus seen to be a two-point boundary value problem with quantities specified at one point ($\phi_0$, $\sigma$, $v_0$ at $\theta = 0$) of the region of integration and a condition ($N(\theta; \epsilon, \phi_0, \sigma, v_0) = 0$) at another. The latter is a moving saddle point singularity [Belotserkovskii, 1958].

For higher approximations using more than one strip, the problem becomes a multipoint boundary value problem. With regard to computing effort, the most expensive part of the solution is the computation of the relevant multipoint boundary value problem. In fact, few authors other than Belotserkovskii and his colleagues appear to have obtained solutions for approximations higher than the first order.

The usual method for obtaining the solution of this two-point boundary value problem is to treat it as an initial value problem [see for example Xerikos and Anderson, 1962] by completing the boundary conditions at the axis of symmetry with an estimate for the initial standoff distance, $\epsilon$. Numerical integration of equations (3.2.1) to (3.2.4) starting from $\theta = 0$ yields integral curves for the quantities $\epsilon$, $\phi_0$, $\sigma$ and $v_0$. A set of integral curves is associated with each different estimate of $\epsilon$.

Examination of the integral curves for $v_0$ with various estimates of $\epsilon$ (figure 3.2.1) shows that the curves coincide at the start of the integration ($\theta = 0$) but diverge as the value of $v_0$ approaches the sonic velocity. Two characteristic types of behaviour
are seen. One set of curves turns in the direction of increasing velocity with rapidly increasing gradient. For these curves the denominator of equation (3.2.4) approaches zero before the numerator. The curves of the other set attain a velocity maximum less than the sonic velocity, which corresponds to the numerator of equation (3.2.4) decreasing to zero before the velocity becomes sonic.

Figure 3.2.1: Integral curve for $v_0$ vs $\theta$. 
If $e$ is considered to be the initial standoff distance for which the numerator and denominator of equation (3.2.4) approach zero together, the curve corresponding to $e$ would pass smoothly through the sonic point. It is unlikely that such an integral curve would be found owing to the mathematical nature of the solution. Under proper initial conditions, these solutions are not stable with respect to infinitesimal perturbation of the integral curves. Such perturbations, which inevitably arise during the numerical integration, cause large variations in the downstream behaviour.

The rapidly increasing curves are associated with initial trial standoff distances less than $e$. This is seen physically as the region contained between the shock wave and body receiving mass flow through the shock wave at a greater rate than can be disposed through the sonic line without rapidly increasing the velocity. On the other hand, when the initial value of $e$ is greater than $e$, the subsonic region has a greater volume and can accommodate more mass than passes through the shock wave. The numerical solution becomes aware of this as the sonic point is approached. Correspondingly, a sharp decrease in velocity results and the velocity curves exhibit a maximum value.

The true solution for the problem is that corresponding to the initial value $e$. Upper and lower bounds for $e$ are determined when integral curves of each type are identified. A simple procedure is used for reducing the interval between these bounds. If the upper and lower bounds are $e_U$ and $e_L$ respectively, the next trial value, $e_T$, is chosen to fall between them, and usually:

$$e_T = \frac{1}{2}(e_U + e_L).$$

When the integral curve corresponding to $e_T$ exhibits behaviour corresponding to a lower bound ($dv_0/d\theta$ rapidly increasing) or an upper bound
(d\(v_0\)/d\(\theta\) negative or zero), \(\varepsilon_T\) replaces the relevant bound. This process may be continued so that after \(n\) cycles the original interval is divided by the factor \(2^n\). Figure 3.2.2 is the flow chart describing the classification of \(\varepsilon_T\) as an upper or lower bound to \(\varepsilon\).

A flow chart for the refinement of \(\varepsilon\) by iteration is given in figure 3.2.3. The estimated upper and lower bounds were examined and adjusted until true upper and lower bounds were obtained. The difference between them was then reduced by interval halving until it was less than \(10^{-7}\). This is much higher precision for the value of \(\varepsilon\) than the three significant digit precision obtained experimentally.

Integral curves corresponding to the two bounds were almost coincident until the velocity was within two per cent of the sonic value where the actual difference between computed values for the velocity was in the fourth significant figure.

The process of interval halving cannot proceed indefinitely to give infinite precision for \(\varepsilon\). This follows from the fact that the digital computer has finite arithmetical precision. Ultimately, numerical roundoff in the arithmetic and accuracy of the numerical integration algorithm places a limit beyond which further computation leads to results which may not be correct. As an example, consider the stage in the integration at which the accumulated error in the value of \(v_0\) is \(\delta v_0\). The most influential term in the denominator of the expression for \(d\(v_0\)/d\(\theta\) near the sonic point is \((a_*^2 - v_0^2)\). The error in this term is \(-2a_* \delta v_0\). Since the magnitude of \(a_*\) is unity, the error in the denominator is twice the accumulated error in \(v_0\). In this case it is possible that integration errors may cause the integral curve associated with the trial value \(\varepsilon_T\) to assume the wrong characteristics near the sonic point and accordingly \(\varepsilon_T\) would be classified wrongly.
Is \( \frac{d\nu_0}{d\theta} > 0 \) and \( \frac{d\sigma}{d\theta} < 0 \)?

\( \varepsilon_T \) is an upper bound

\( \varepsilon_T \) is a lower bound

Is \( \frac{d\nu_0}{d\theta} \geq 2 \left( \frac{d\nu_0}{d\theta} (\theta = 0) \right) \) or \( \nu_0 > a_0 \)?

Figure 3.2.2: Flow chart for testing for divergence of the integral curve and classification of \( \varepsilon_T \) as an upper or lower bound.
Estimate upper and lower bounds $EU, EL$

- Is $EU$ an upper bound? 
  - NO $EL = EU$ → Increase $EU$
  - YES $EL = EL$ → Decrease $EL$

- Is $EL$ a lower bound?
  - NO $EU = EL$ → Decrease $EL$
  - YES $E = (EU + EL)/2$

- Is $E$ an upper bound?
  - NO $EL = E$
  - YES $EU = E$

- Is $EU - EL \leq 10^{-7}$?

Figure 3.2.3: Flow chart for refinement of initial standoff distance $\varepsilon$. 
The sensitivity of the integral curve to perturbation of $v_0$ can be turned to advantage. Since the computed value of $v_0$ contains an accumulated error, $v_0$ can be adjusted at a late point in the integration to prevent the integral curve diverging from the neutrally stable solution. When this technique [due to South, 1969] is employed it is possible to extend the integral curve closer to the sonic point by adjusting $v_0$ within its estimated accumulated error.

Figure 3.2.4 is a flow chart for the extension of the integral curve using South's method. The ordinary differential equations were integrated twice using $\varepsilon_U$ and $\varepsilon_L$ for the initial values of the standoff distance. When the velocity computed by the two solutions differed by $10^{-4}$ the integration was terminated. At that point the upper and lower bounds of the velocity were adjusted by iteration until they were different by less than $10^{-6}$. The integration was recommenced with the adjusted velocity values and the end function values of the previous iteration until the velocities again differed by $10^{-4}$. The process was repeated until no significant advance was made before the curves diverged. In this way, the integral curves were extended until the velocity was within 0.1% of the sonic velocity.

Once the integral curve had been extended to within a very short distance of the saddle point, an extrapolation was applied to continue the integration beyond the sonic point.

The extrapolation was formed using the function values from the last ten integration steps. The functions $v_0$, $\sigma$, $\phi_0$ and $\varepsilon$ were almost linear over these steps. A linear least squares fit to each was computed and values of $\theta$, $\sigma$, $\phi_0$ and $\varepsilon$ were obtained for $v_0 = 1.005 \times a_\infty$. From these values the integration of the ordinary differential equations was restarted and proceeded to $\theta = \pi/2$. 
Upper and lower bounds to $\varepsilon$ are EU and EL and $EU - EL \leq 10^{-7}$

Initiate two integrations of the O.D.E.s simultaneously with initial conditions $\theta = 0$, $\sigma$, $\phi_0$ and $v_0$ in each, but one has EU, the other EL for initial standoff distance - "double" integration

Perform integration step (double)

Is $|V_{EU} - V_{EL}| \geq 10^{-4}$?

NO

YES

Has a significant step been taken before divergence?

YES

Save current function values of $\theta$, $\sigma$, $\phi_0$, $\varepsilon$ for start of next double integration

$VT = (V_{EU} + V_{EL})/2$

Does $VT$ correspond to integral curve with stationary point ($dv_0/d\theta = 0$), i.e. the upper bound set?

NO

$V_{EL} = VT$

YES

$V_{EU} = VT$

Is $|V_{EU} - V_{EL}| \leq 10^{-6}$?

YES

Figure 3.2.4: Flow chart for extension of the integral curve using South's method.
The numerical data associated with the method of integral relations solution was stored on a magnetic tape. This data was used as input to the program evaluating the perturbation of the applied magnetic field caused by interaction with this flow field.

3.3 SUGGESTED FUTURE IMPROVEMENT OF SOLUTION OF THE BLUNT BODY PROBLEM USING THE METHOD OF INTEGRAL RELATIONS

An avenue of research which could return a valuable reward is the application to this problem of the well known methods for solution of two-point boundary value problems [see for example Keller, 1968]. Although this has not been achieved here it is worth noting some ways in which the investigation might proceed.

The sonic point on the body represents a moving saddle point singularity at which the regularity condition is applied. It is moving since the value of the independent variable at the sonic point is unknown. The problem can be changed to one with fixed boundary points by the appropriate independent variable transformation. Since in the subsonic region $v_0$ is a monotonic function of $\theta$, it is permissible to make $v_0$ the independent variable and the end points of the integration become fixed. The transformed problem retains the saddle point singularity but the saddle point is fixed at $v_0 = a_*$, the sonic velocity.

The particular method for solution of the two-point boundary value problem would need to be able to handle the saddle point singularity. This might involve initiating the integration from the sonic point with estimated values of $\theta$, $\phi_0$, $\sigma$ and $\epsilon$ consistent with the regularity condition. It is possible that the new integral curves would not diverge appreciably as the integration proceeded toward $\theta = 0$ and $v_0 = 0$. In this case, the solution of the problem would follow in a straightforward way.
The accuracy of the extrapolation across the saddle point singularity can be improved by perturbing the extrapolated values (figure 3.3.1). The ordinary differential equations should be integrated from the perturbations of the landing point in the supersonic region back toward the saddle point. Just the same as in the subsonic region, divergent integral curves are obtained owing to the nature of the saddle point. The function values are refined as before to obtain the most suitable non-diverging integral curve. These are then the best estimates for values on the solution curve in the supersonic region.

![Diagram showing perturbation and extrapolation](image)

Figure 3.3.1: Increasing the extrapolation accuracy.

* This method for refinement of the extrapolated values came out of a discussion with R.S. Anderssen of the Computer Centre at the Australian National University.
CHAPTER 4
THE EXPERIMENTAL ARRANGEMENT

The experiment was performed in the test section of a shock tube. The slug of high temperature gas produced by this tube was confined by a glass tube while it travelled through an applied magnetic field. Magnetic probes inside a hemispherical cylinder placed in this flow were used to measure perturbation of the magnetic field arising from the induced currents.

Evaluation of the interaction is complicated by two simultaneous processes. The first is the slow diffusion of the field lines into the slug of conducting gas owing to the high electrical conductivity and velocity of the slug. The second is the interaction under investigation, that of the modification of the magnetic field by the flow past the probe body.

4.1 THE SHOCK TUBE

In the experiment a double diaphragm type shock tube with free-piston compression of the driver gas provides the high velocity, electrically conducting slug of gas. Shock velocities in excess of 18 kilometres per sec driving into 0.1 torr of argon at room temperature are obtained, and the electrical conductivity of the shock heated gas, calculated from the Spitzer Härm formula [Spitzer, 1956], is typically $5 \times 10^4$ mho metre$^{-1}$. The Mach number of the test gas is between $M = 2.5$ and $M = 3.0$. High electrical conductivity is required to obtain a magnetic Reynolds number greater than unity near the magnetic probe.

Diaphragm type tubes produce a reasonably homogeneous sample of test gas with known properties in the region between the shock wave
and the driver interface. The temperature, and hence conductivity, produced in the test gas is limited by the speed of the shock wave. This is determined partly by the initial pressure ratio across the diaphragm, but to a greater extent by the ratio of the sound speeds in these regions [Henshall, 1955]. Performance is improved by use of low molecular weight gases, with attendant high speed of sound, for the driver gas. The speed of sound of the driver gas is also increased by heating. Heating by an arc discharge has yielded Mach numbers of approximately 40 in air [Camm and Rose, 1963] and, using free-piston compression, values about 30 have been obtained [Stalker, 1966]. Since shock waves can cause heating, the performance of a shock tube may be improved if it is preceded by an intermediate tube which produces the driver gas for the final tube. This configuration, having two diaphragms, is termed a double-diaphragm shock tube [Bernstein, 1953; Henshall, 1955]. Free-piston compression in conjunction with double diaphragm tubes has been described by Stalker and Plumb [1968] and Sandeman and Allen [1971]. Figure 4.1.1 is a schematic diagram of this type of device.

Application of free-piston compression to shock tube drivers has been discussed by Stalker [1966]. The piston is driven by high pressure air along a tube containing the driver gas which is initially at low pressure. The compression of this gas is rapid, and consequently nearly adiabatic, so that its temperature is raised to approximately 5000 °K at the end of the compression stroke. Rupture of the diaphragm then allows the hot, high pressure driver gas to escape into the intermediate tube, driving before it a shock wave.

Upon reflection of this shock wave at the second diaphragm, the kinetic energy of the shock heated gas is changed into internal energy as the gas comes to rest. At the same time, this diaphragm
Figure 4.1.1: Schematic diagram of the double diaphragm shock tube.
begins to deform and subsequently ruptures to allow the driver gas, this time at a temperature of approximately 12000 °K, to drive a shock wave with very high velocity through the test gas in the final tube.

4.1.1 The Cookie Cutter

The gases flowing out of the end of the shock tube are directed into a cookie cutter, a sharp lipped tube having smaller diameter than the shock tube (figure 4.1.2). The region of test gas near the walls of the shock tube, including the cooler boundary layer, is forced to flow outside the cookie cutter and is excluded from the test region. In this way, the central core of the test gas is sampled and continues to flow along the cookie cutter to the region of magnetic field.

The recoil of the shock tube owing to the movement of the piston is not transmitted to the cookie cutter tube, but is an important parameter in its design. Before shock tube recoil the exit of the shock tube is placed against the camber of the cookie cutter. During the flow of the test gas the shock tube is at its full recoil displacement, typically 1.9 cm, and hence for the tip of the cookie cutter to remain within the Mach cone, the camber angle can be determined from the relation: $(\cot \theta_{\text{camber}} + \cot \theta_{\text{Mach}})\delta r = \text{recoil displacement}$, where $\theta_{\text{camber}}$ and $\theta_{\text{Mach}}$ are the camber and Mach angles respectively and $\delta r$ is the difference between the cookie cutter and shock tube radii. For $M = 2.5$ and shock tube and cookie cutter diameters equal to 4.92 cm and 4.53 cm respectively, the allowable camber angle is $\theta_{\text{camber}} \leq 7.6$ degrees. The actual camber angle in use is 7.0 degrees.

The cookie cutter (figure 4.1.2) consists of two sections of different materials. The upstream section, comprising the cambered lip
Figure 4.1.2: The cookie cutter.
and bracket, is made of austenitic stainless steel and is fixed to the glass section downstream with Plasti-bond adhesive. Hysteresis and finite electromagnetic diffusion time effects associated with ferromagnetic materials would cause an intolerable perturbation to the applied magnetic field. To minimise these influences, the extension of the cookie cutter downstream, in the vicinity of the magnetic field, consists of glass tubing. This is strengthened with four layers of fibre glass cloth wound around the outside so that it withstands the high pressures during shock tube operation. The glass cloth is potted in Plasti-bond and the outer surface machined in a lathe for a neat finish. The cookie cutter assembly is bolted onto the entrance of the test section (figure 4.1.3).

On the outside of the glass section, a four turn loop of 22 SWG copper wire is wound so that changes in the magnetic flux resulting from the conducting flow over the cross section of the glass duct can be monitored. Leads from this search coil are twisted tightly together and lead to a bracket holding the cookie cutter where they are soldered, one to each of two coaxial cables with grounded outer conductor. The signal is conducted out of the test section through vacuum tight seals with BNC connectors at each side.

4.1.2 Timing and Shock Speed Measurements

The time scales of each stage of the shock tube operation are:

(i) 50 millisec for the free-piston compression;

(ii) 5 millisec magnetic field rise time;

(iii) 1 millisec between rupture of the first diaphragm and arrival of the shock wave at the test section; and

(iv) 5 microsec test time.

Within these time scales, timing signals are generated to initiate
Figure 4.1.3: The cookie cutter mounted in the front plate of the test section.
switching on the magnetic field and triggering of the oscilloscope traces to record the probe signals (figure 4.1.4).

For the magnetic field to be near its maximum magnitude during the test time, the associated switching signal is generated prior to the first diaphragm rupture. A mechanical-electrical device consisting of a system of levers attached to the compression tube and the shaft of a variable resistor in an electrical circuit monitors the recoil of the compression tube. This recoil, caused by the reaction to the forward movement of the piston, takes 54 millisec for the compression stroke until diaphragm rupture and is consistent from shot to shot to within 0.5 millisec. From the start of the electrical signal given by the recoil monitor, a delay of 49 to 50 millisec is used before switching the magnetic field. This delay is provided by a transistor multivibrator delay unit with a 40 V, 20 microsec square pulse output.

Shock speed measurements are made using time-of-flight techniques. In the intermediate shock tube signals from two pressure transducers separated by 0.914 metres are recorded on a Tektronix 585A cathode ray oscilloscope with type 82 plug-in operated in the chopped mode. The time between arrival of the pressure front at each station is measured and thus the shock speed is calculated. The intermediate tube shock speed is a guide to the performance of the final tube driver but is not required for evaluation of the state of the test gas.

Shock speed timing in the final shock tube is achieved by a time-of-flight method using apparatus different from the intermediate tube since the pressures to be observed are lower. The arrival of the shock wave is detected by a photodiode which sees the radiation, normal to the shock tube axis, from the shock heated gas. The photodiode signals are fed to electronic counters to measure the time interval
Figure 4.1.4: Timing sequence.
Time base 10 millisec/div
Trigger from recoil monitor

— delay unit signal

— Rogowski coil signal
  $RC = 10^{-2}$
  Sensitivity = 2.6 Kamp/div

— recoil monitor signal

Intermediate tube pressure transducer signals

upper trace 2nd transducer
lower trace 1st transducer

Time base 20 microsec/div
Trigger from lower trace

reflected shock
primary shock

Figure 4.1.4 (continued): Timing sequence.
between arrival at successive stations along the tube.

4.2 THE MAGNETIC FIELD

A uniform magnetic field aligned with the shock tube axis is desirable, but to only some extent realisable in the experiment. Restrictions on the size of coils and their location are imposed by the geometry of the shock tube test section. In addition, all magnetic fields arising from straight solenoids necessarily have end effects where the radial component of the field is nonzero. One way to overcome the end effects is to immerse the whole shock tube in a very large solenoid so that the electrically conducting gas, when it is produced, is immediately in an essentially uniform magnetic field. This solution is most impractical considering the size of coils required and the effect that the steel shock tube walls would have on this field. A practical solution is to construct field coils which, while fitting inside the test section, do not induce large currents in the surrounding brass test section walls and are not so small as to have only a short uniform region. In view of these circumstances, it is anticipated that the fringing of the field lines will have some effect on the magnetic phenomenon under study.

4.2.1 The Field Coils

The coil for production of the magnetic field is a Helmholtz pair [Bötticher, 1968, p.632]. The deviation of the axial component of fields produced by such coils is less than five per cent over a spherical region centred on the geometrical centre of the coil and with radius equal to half the coil radius. In comparison, a solenoid of the same dimensions has a deviation of ten per cent over the same region.

The Helmholtz pair consists of two identical coils separated
Figure 4.2.1: The magnetic field coil.
along their common axis by a distance equal to their radii. The axial magnetic field near the centre of this configuration is given by:

$$B_z(x,z) = 0.715 \mu_0 \frac{NI}{R} \left[ 1 + 0\left(\left(\frac{x}{R}\right)^4 + \left(\frac{z}{R}\right)^4\right)\right],$$

where $N$ is the number of turns on each coil, $I$ is the current threading them and $R$ is the coil radius.

The coils are wound on a perspex former, 3 inches in diameter, in five layers with ten turns of 16 SWG (0.16 cm) annealed copper wire to each layer. The resistance and inductance of each coil is $R = 120$ milliohm and $L = 178$ microhenry. The wire diameter is chosen to make the argument of the discharge damping factor, $e^{-\frac{R}{2L}t}$, small and to reduce the effects of heating in the coil.

The radius of each coil measured from the centre of the windings to the axis is 4.08 cm. The distance between the coils is measured between the winding centres, or equivalently from the end of one coil to the corresponding end of the other. Perspex blocks are cemented between the coils to prevent them from sliding together under the action of strong attractive forces when the electric current flows.

For $R = 4.08$ cm and $N = 50$ the magnetic field near the centre of the Helmholtz pair is $B_z(0) = 1.1 \times 10^{-3}$ webers metre$^{-2}$. The peak current is typically 700 amps producing peak magnetic fields of the order of 0.75 webers metre$^{-2}$.

4.2.2 The Condenser Bank

The condenser bank consists of $100 \times 850$ microfarad electrolytic capacitors (Ducon EWP 850) rated at 450 V, which are connected in parallel by two 1 metre $\times$ 1 metre copper plates and fed to the shock tube test section by means of parallel plate lines. Conduction into the evacuated test section is through two copper rods, each 6 mm
diameter, insulated from the rest of the test section by perspex blanks which replace part of the vacuum chamber wall. Inside the test section, leads from the field coils are clamped directly onto the rods.

The bank is operated nearly critically damped. The peak current is about 450 amps/100 V and electric current rise time with the coils connected in parallel is 4.4 millisec.

The bank is switched by an EE BK 472 ignitron fitted with a coaxial mounting in the parallel plate transmission line. The basic switching unit* consists of a 2.5 kV 0.25 microfarad discharge into a 4 ohm load and coupled by a 1:1 transformer (designed for 15 kV single pulse insulation) onto the triggering electrode of the ignitron. The switch of this unit is a 4C35 hydrogen thyatron triggered by the signal generated by the transistor multivibrator delay unit fed through a cathode follower circuit.

4.2.3 Measurement of Electrical Current

The electric discharge current is measured using a Rogowski coil [Leonard, 1965, p.9]. This is a solenoid 0.45 metres long wound on a flexible plastic tube 0.25 inches (0.63 cm) in diameter with 23 SWG (0.6 mm diameter) copper wire. It is bent to enclose one of the parallel plate transmission line conductors and is held firmly in place by a perspex mount (figure 4.2.2).

The area enclosed by the major circumference of this coil is reduced by returning one of the end leads through the centre of the tube. This not only reduces the pickup of stray magnetic fields not associated with the current being measured, but also allows the coil to be slipped around the conductor through which the electric current flows, thus making the coil more easily removed and replaced.

* This unit was designed and constructed by S. Lee [1968].
Figure 4.2.2: The Rogowski coil.
Provided that the current path is not too close to the coil and the major circumference is topologically equivalent to a torus encircling the current path, the voltage induced across the coil is written:

\[ V_c = \mu \frac{N}{S} A \frac{dI}{dt}, \]

where \( N \) is the number of minor turns, \( A \) is the area of a minor turn and \( S \) is the circumference of the major loop. The integrated signal appearing at the oscilloscope is:

\[ V_0 = \frac{1}{RC} \frac{NA}{S} I, \]

where \( I \) is the total current enclosed by the Rogowski coil and \( RC \) is the integration time constant. The coil has the following constants: \( N = 735; S = 0.45 \) metres; \( A = 0.374 \times 10^{-4} \) metres\(^2\); and \( RC = 10^{-2} \). The calculated sensitivity is 130 Kamp/volt.

4.3 THE MAGNETIC PROBE

The use of coils for measuring magnetic fields has been discussed in detail by Lovberg [1965] and Botticher [1968]. Briefly, an inductive probe is a small cylindrical coil which is inserted into the magnetic field. The change of magnetic flux \( \phi \) through the coil induces a voltage \( U = -d\phi/dt \). If the magnetic field is uniform over the coil cross section \( A \), \( U = -NA dB_n/dt \) where \( N \) is the number of turns and \( B_n \) is the induction in the direction of the coil axis. When placed in a varying magnetic field the integrated signal at the oscilloscope is:

\[ V_0 = -\frac{NA B_n}{RC}, \]

and the risetime of the coil is \( t_r = L_c/R_0 \) where \( L_c \) is the coil impedance and \( R_0 \) the load impedance.
Two small coils are placed in the magnetic field. For the initial tests one was at the geometric centre of the Helmholtz pair and the other 1 cm downstream from the first. In later tests, the field coils were moved upstream 1.2 cm relative to the search coils. These coils are wound with 41 SWG enamelled copper wire on the same delrin former. Their parameters are \( N = 50 \), radius \( a = 1.5 \text{ mm} \), length \( l = 2.0 \text{ mm} \) and inductance \( L = 11 \text{ microhenry} \). The integration time constant is \( RC = 10^{-4} \text{ sec} \) and characteristic impedance of the cables is \( R_0 = 50 \text{ ohm} \). The sensitivity of the probes is \( 0.25 \text{ weber metre}^{-2} \text{ volt}^{-1} \) and risetime is 0.2 microsec.

The electrical signal from each probe is fed through a differential amplifier* before integration.

### 4.3.1 Protection of the Probe Coils

During the operation of the shock tube, the test section is exposed to damage arising from the aerodynamic forces and high pressures associated with the shock wave and the driver gas, and the impact of metal diaphragm fragments. These fragments are carried along with the driver gas and attain energy sufficient to imbed themselves in soft metals such as aluminium or brass. Such impacts are evident on the cookie cutter, the probe and its holder and other exposed parts of the test section.

To prevent damage to the coils, they are protected by a nylon hemispherical nose cylinder (figure 4.3.1) which also serves to give the characteristic length for evaluating the local magnetic Reynolds number. The probe shield diameter is made as large as possible to give

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* Two different differential amplifiers were available: Tektronix type D plug-in in A-B mode with bandwidth 2 MHz; and Tektronix type CA plug-in in A-B mode. Depending on drift and calibration the common mode rejection ratio varied between 100 and 1000.
Figure 4.3.1: Blunt body configuration and search coils.
a large $R_m$ near the nose. If this diameter is too large however, the interaction of the bow shock with the cookie cutter channel must be taken into account.

To reduce the likely perturbing effects of the brass rod supporting the probe, the probe shield is made as long as possible so that the magnetic probes are removed from the vicinity of this mass of conducting material. The ultimate factor limiting the dimensions of the shield is mechanical strength and this depends on the length to diameter ratio and potential damage from diaphragm fragments. One shield, with diameter 6.3 mm and $l/D$ ratio 3, together with the coils inside, was destroyed in one shot due to the combined influences of the diaphragm fragment impact and aerodynamic forces.

The probe shield in use has diameter 1.62 cm and $l/D$ ratio 1.1. This was damaged from time to time by fragment impact but in each case the coils sustained no damage, the shield casing having been fractured and sections torn off. From these observations it is deduced that the dimensions of the probe shield are near optimum consistent with the mechanical, aerodynamic and magnetic Reynolds number restrictions.

The nose radius is 0.81 cm. When the electrical conductivity and velocity of the flow are typically $10^4$ mho/metre and $2 \times 10^4$ m sec$^{-1}$, the magnetic Reynolds number is $R_m = \frac{\mu_0 l v}{\sigma} < 2$.

4.4 THE TEST SECTION

Figure 4.4.1 shows the complete test section assembly from the shock tube (at the left of the figure) to the end of the test section. The dump tank is separated from the test section by an aluminium plate (one inch thick) which serves to support the field coil assembly (figure 4.2.1) and the magnetic probe. The blunt body containing the probe
(figure 4.3.2) has a thread and O-ring seal and is fitted to the end of a 1½ inch diameter brass rod held by the plate. This rod is also fixed at the rear of the dump tank. The shock tube gases are allowed to escape into the dump tank through holes machined in the aluminium plate.
Figure 4.4.1 Test section experimental assembly. Indicated parts are:

a. shock tube;  b. cookie cutter;  c. field coils;  d. field coil support;
  e. probes;  f. probe support;  g. test section outer wall. Scale: 0.4 full size.
The shock tube tests show that there is an interaction between the applied magnetic field and the conducting flow past the probe. Analysis of the integrated probe signals (figures 5.2.1, 5.2.2, 5.2.3 and table 5.2.2) shows that the probes inside the blunt body experience a decreasing magnetic field between 7\% and 12\% of the applied field when the gas slug flows past the nose of the probe. Prior to this signal a smaller perturbation (2\%) to the applied field is observed and appears to originate from the interaction between the shock tube flow and the "fringing" of the Helmholtz field.

The experimental conditions (section 5.1) give a magnetic Reynolds number 4 \pm 1, and the theoretical magnetic field perturbation (figure 5.3.6) is between 12\% and 22\% of that applied for these values of $R_m$. The shape of the signals indicates that the magnetic field has not reached an equilibrium value during the conducting gas test time (approximately 2 microseconds). The estimated time to reach a steady state field (section 2.1) is 6 microseconds and supports the argument that the magnetic field perturbation does not become steady during the test gas time.

The difference between the predicted and observed values of magnetic field perturbation can be explained by taking account of assumptions made in the theory which are not strictly valid in the experiment. The ionized argon flow has been treated as an ideal gas flow with constant ratio of specific heat capacities where the actual flow entails the complexities of chemically reacting gas flow. In addition, to obtain a measurable signal the applied magnetic field was
0.7 weber/m$^2$ which produced a magnetic interaction parameter with value unity near the nose of the probe. Such a large interaction parameter modifies the velocity field and it is shown in section 5.2 that this might reduce the calculated magnetic field perturbation by a factor 0.6.

The probe signals exhibit some degree of irregularity which may have arisen from inhomogeneities in the shock tube flow, a stronger interaction between the velocity and applied magnetic field than has been estimated, or from an inhomogeneity in the applied magnetic field. One feature of the probe signal, a second negative perturbation, can not be explained. Further experiments which may clarify the problem are outlined.
5.1 SHOCK TUBE OPERATION AND TEST GAS CONDITIONS

The shock tube lengths and gas filling pressures necessary to produce a non-attenuating high velocity shock wave (Table 5.1.1) have been determined experimentally by H. J. Sandeman (Sandeman and Allen, 1971). Under these conditions the volumetric compression ratio in the compression tube is 90. The first diaphragm is 10 gauge mild steel plate (unscored) with static bursting pressure $7.2 \times 10^4 \text{ kN m}^{-2}$ (10,500 psi). The second diaphragm is 10 gauge hard aluminium sheet scored to produce four petals.

<table>
<thead>
<tr>
<th>Region</th>
<th>Gas</th>
<th>Pressure</th>
<th>Tube Length</th>
<th>Internal Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>High pressure reservoir</td>
<td>Air</td>
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<td>1.4 m</td>
<td>180 mm</td>
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<td>Helium</td>
<td>$40 \text{ kN m}^{-2}$ (12&quot; Hg)</td>
<td>5 m</td>
<td>124 mm</td>
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<tr>
<td>Intermediate tube</td>
<td>Helium</td>
<td>$37 \text{ kN m}^{-2}$ (11&quot; Hg)</td>
<td>2.5 m</td>
<td>50 mm</td>
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<tr>
<td>Shock tube</td>
<td>Argon</td>
<td>$0.113 \text{ kN m}^{-2}$ (0.1 torr)</td>
<td>3.5 m</td>
<td>50 mm</td>
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</tbody>
</table>

Table 5.1.1: Shock tube dimensions and initial pressures.

The speed of the shock front in the shock tube is assumed to be the same as that of the luminous front. The velocity between four successive stations along the shock tube can be determined using photodiodes to detect the arrival of the luminosity. Table 5.1.2 gives a summary of measured shock speed. The light stations are designated 0, 1, 2, 3 and time $t_j$ is measured from station 0 to station $j$. Distances between successive stations from 0 to 3 are 91.4 cm, 91.4 cm and 60.9 cm. The quantity $v_{ij}$ is an approximation to the velocity between stations $i$ and $j$ calculated using the measured times and these distances. Shots for which the second diaphragm failed to burst (due to piston pre-launch)
<table>
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<th>$t_3$</th>
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<th>$t_{3-t_2}$</th>
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<tr>
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Estimated error $\pm 0.5\mu s \pm 0.5\mu s \pm 0.5\mu s$ $\pm 1\mu s \pm 1\mu s \pm 1\mu s$ 1% 2% 3% 1%

Table 5.1.2a: Summary of measured shock velocities

(Time measurements from oscillograph records).
### Table 5.1.2b: Summary of measured shock velocities

(Time measurements from electronic counters.)

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or for which light station 0 failed are not included in the table.

If we assume that the acceleration of the shock front is constant then the velocity \( v_{ij} = (x_i - x_j)/(t_i - t_j) \) is simply the mean of the velocities at \( i \) and \( j \), \( (v_i + v_j)/2 \). Using this relation we could estimate the speed of the shock wave in the test section by extrapolation of the measured values in the shock tube. The timing measurements do not have the precision required for this estimation however. The velocity is found to be constant within 3% between stations 1 and 3 while before station 1 the shock appears to have been accelerating. We choose to use the average velocity between stations 1 and 3 for the measured shock velocity in the test section.

Table 5.1.2a is derived from shots for which the photodiode signals from stations 1, 2 and 3 were added electronically and displayed on an oscilloscope trace with time base 20 microseconds/division. These oscilloscope records can be read to an accuracy of \( \pm 0.5 \) microseconds. This timing uncertainty has most effect on the determination of speed between stations 2 and 3 where the time interval is typically 35 microseconds and the error is 3% from this source. The table shows that the velocity of the shock front for each shot is constant between stations 1 and 3 to within this \( \pm 3\% \) uncertainty. Timing measurements could not be obtained when there was no signal from a photodiode, in most cases due to grit obstructing the observation porthole at the particular station.

Table 5.1.2b is derived from later shots when electronic counters were used to determine the time elapsed between the luminosity passing station 0 and each other station. The voltage impulses used to start and stop the electronic counters are derived from fast rise (50 ns) monostable multivibrators triggered by the signal from fast PIN photodiodes. Radiated light from the shock front passes through a perspex
porthole in the shock tube wall and along a fibre optic to the photodiode. The accuracy of this system for shock timing is not limited by the resolution of oscillograph records as before. Another advantage is that the oscilloscope otherwise used for shock timing is available for other tasks. There is however a serious disadvantage with the multivibrator system in this experiment. The trigger level of the multivibrator must be set very close to oscillation so that the initial rise of the photodiode signal can start the timing sequence. Deterioration of the photodiode signal delays the timing signal. This imposes an unknown variation in the timing measurements which would be obvious in the oscilloscope recording system where the actual light signals are monitored. The signals from stations 0, 1 and 3 were more reliable in this respect than those from station 2. At this station the fibre optic severely attenuated the light signal and timing measurements could be obtained only for four shots. Uncertainty in all the timing measurements owing to varying sensitivity of the photodiode to the light in the shock tube (contributed to by losses in the fibre optics or dirty observation ports) in the worst case is the time taken by the luminous front to reach its greatest intensity. This is the time for the shock heated gas to reach local thermodynamic equilibrium and is estimated to be half a microsecond. Table 5.1.2b indicates that for most shots the error is not as great as this since the shock velocities calculated from the observations at stations 0, 1 and 3 are constant to within $+3\%$ for a given shot and these measurements of shock velocity are to the same accuracy as was achieved with the simpler oscillograph recording method.

An improvement of this shock timing scheme for future experiments would be the elimination of the fibre optic lines by mounting the photodiode and multivibrator circuits directly on the shock tube. This would eliminate the attenuation of light intensity due to the fibre
optics and possible damage to fibre optics from shock tube recoil.

The state of the gas in the equilibrium region behind a strong shock wave in argon can be calculated numerically using the equilibrium thermodynamic equations (equation of state, Saha equations) and the shock conservation equations (Allen, 1967). The measured shock velocity and the initial pressure in the shock tube are sufficient to determine the equilibrium state of the shock heated gas. Typical values are shown in Table 5.1.3. The electron density has been measured interferometrically (Sanbeman and Allen, 1971) and the fringe shift confirms the calculated electron density. The electrical conductivity for these conditions can be estimated using the Spitzer-Harm (1953) relation. The quantity $\gamma_{\text{effective}}$ in Table 5.1.3 is defined by the relation

$$(P/P_0)^s = \gamma_{\text{effective}} P/P$$

where the derivative of pressure with respect to density for constant entropy is evaluated from equilibrium thermodynamic relations.

The structure of the shock front and the electron density can be estimated from the photographs in figure 8 of Appendix B. Figure 8a shows a shock slug uniform for two microseconds with relaxation regions of about half-a-microsecond in the shock front and at the rear of the slug. These results were obtained in a 1.5" diameter shock tube, 4.3 m long. From this figure the shock front thickness for a Mach 50 shock wave into 0.1 torr of argon is about half-a-microsecond.

Shock structure calculations have been performed by many authors (Petschek and Byron (1957), Teare (1963), Chubb (1968)) for argon with shock Mach number up to 40 and for initial shock tube pressures near 1 torr. Experimental determinations of shock front thickness (Morgan and Morrison, 1965) are essentially in agreement with the computations except that the shock thickness parameter, $(P/P_0)^s$, obtained theoretically is consistently three times as great as the value determined
Table 5.1.3: Typical test gas conditions

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<th>Value 3</th>
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**Units**

Shock velocity, Particle velocity, Speed of sound, $\gamma$ effective, Mach number, Density, Pressure, Specific enthalpy, $A^+$ fraction, $A^{++}$ fraction, Temperature, Electrical conductivity.
experimentally (Chubb, 1968). The thickness parameter decreases with increasing Mach number and is asymptotic to the value $p_1 t_1 = 10^{-7}$ mm Hg sec (Chubb, 1968).

For the Mach 50 shock wave into 0.1 mm Hg of our experiment we estimate from Chubb's results that the shock front region duration is two microseconds using the theoretical result or 0.6 microseconds using the experimental result $\left( (p_1 t_1)_{\text{exp}} = \frac{1}{3} (p_1 t_1)_{\text{theor}} \right)$. The extrapolation of the experimental results reported by Chubb tends to support the observation of the shock front relaxation region in the present study. The conflict between theoretical and experimental results indicates that the shock structure computations should be refined to take into account impurity concentrations and other factors, possibly different collision cross-sections and reactions from those used by Chubb, and effects of radiation.

The structure of the shock heated gas slug produced by the double diaphragm shock tube is interpreted as follows. Following the pressure front and onset of ionization the electron density increases to the equilibrium level or a maximum level within half a microsecond (determined from the channel spectra blurs in figure 8, Appendix B). Behind the frontal structure is a region of hot gas uniform for a few microseconds until the arrival of the driver gas or the rarefaction wave reflected from the driver gas interface in the intermediate tube. Occurrence of this latter process is undesirable since small variations of the shock tube running conditions could result in the rarefaction wave exhausting the shock heated slug before its arrival at the end of the shock tube (figure 8b, Appendix B). Running conditions were chosen so that this did not occur.

The short test gas duration (two to five microseconds) was not anticipated in the design of the experiment and was only evident after
the first experimental run. Its effect is to reduce the time available for equilibrium steady flow past the blunt body. Steady fluid dynamic flow does not develop until the shock wave has reached one diameter past the spherical nose of the body. The time taken for the Mach 50 shock wave to reach this point is one microsecond leaving only one microsecond for steady flow in the case of the two microsecond duration slug.

The flow past the blunt body is complicated by the possibility of ionization taking place. This removes kinetic and translational energy from the atoms and ions and consequently varies the effective speed of sound in the gas over the region of flow. Atoms crossing the bow shock wave move into a region of high pressure and low velocity and hence their internal energy (kinetic) is enhanced at the expense of translational energy. If we assume that the velocity on the stagnation streamline behind the shock is three-tenths of the free stream value the atom-ion temperature changes by 150,000 K. The electron density and velocity variations are assumed to follow the charged particle density and velocity changes to conserve space neutrality. The translational energy of the electrons yields only 20 K. Nonequilibrium of the electron and atom-ion temperatures leads to transfer of energy from the atoms and ions to the electrons. The rate of this exchange can be estimated from the relation given by (Shkarofsky, Johnston and Bachynski, 1966)

\[
\frac{dT_e}{dt} = - \frac{8}{3} n + \frac{Y_{ei} m}{\pi^{3/2}} \frac{T_e - T_+}{M} \left[ \frac{2kT_+}{M} + \frac{2kT_e}{m} \right]^{3/2}
\]

where \( Y_{ei} = 4 \pi \left( z e^2 / 4 \pi \epsilon_0 m \right)^2 \ln \left( 3 \left( \epsilon_0 kT_e \right)^{3/2} / 2z e \right) \left( \pi n_e \right)^{1/2} \).

When the free stream temperature density and velocity are \( 3 \times 10^4 \) K, \( 3 \times 10^{-3} \) kg m\(^{-3}\) and \( 1.5 \times 10^4 \) m s\(^{-1}\) respectively and since we have assumed the density ratio across the shock wave near the stagnation.
region is 3 we can use this equation and the rate of change of atom-ion
temperature \( \frac{dT_+}{dt} = -n_e \frac{dT_e}{dt} / \left( n_A + n_{A^+} + n_{A^{++}} \right) \), to evaluate the
rate of increase of electron temperature. In this approximation changes
in the electron density through ionization have been neglected. If
these processes were taken into account, the increase in electron
density would serve to accelerate the transfer process through the terms
\( n_A Y_{ei} \) and \( (T_e - T_+) / T_e^{3/2} \). Thus the relaxation time calculated using
these assumptions is an upper limit to the true relaxation time.

Integrating first the rate of change of ion temperature,
\[
T_+(t) = T_+(0) + (T_e(0) - T_e(t)) n_e / \left( n_A + n_{A^+} + n_{A^{++}} \right).
\]
By substituting
\[
\frac{dT}{dt} = \frac{Q(1 - T)}{T^{3/2}}
\]
where \( T = T_e / T_m \);
\[
T_m = (n_e T_e(0) + (n_A + n_{A^+} + n_{A^{++}}) T_+(0)) / (n_e + n_A + n_{A^+} + n_{A^{++}})
\]
is the asymptotic electron–ion relaxation temperature; and
\[
Q = \frac{8 \pi m^1}{3 M \frac{3}{4} \pi^3} \left( n_{A^+} Y_{ei}^1 + n_{A^{++}} Y_{ei}^1 (A^{++}) \right) \frac{m}{2kT_m}^{3/2}.
\]
Integrating, we find
\[
t = \frac{1}{Q} \left[ \ln \left( \frac{1+T^{1/2}}{1-T^{1/2}} \right) - 2 \left( \frac{3}{2} T^{3/2} + \frac{1}{2} T \right) \right]^{T_1}_{T_0}.
\]
From the conditions given above, \( Q \approx 0.8 \times 10^7 \text{ s}^{-1} \) and \( T_m \approx 60,000 \text{ K} \),
so for the electron temperature to change from \( 30,000 \text{ K} \) (\( T = 0.5 \)) to
\( 54,000 \text{ K} \) (\( T = 0.9 \)) the time required is \( t \approx 0.1 \) microseconds. The rate
of single ionization of argon given by Petschek and Byron (1957) is
Using the free stream electron temperature and particle number densities (to give an underestimate of this rate) and assuming that single and double ionization both proceed through an intermediate excited level of 12 ev with the same cross-section as the former process, this relation gives \( \frac{dn_e}{dt} = 10^{30} \text{ m}^{-3} \text{ s}^{-1} \). The rate of ionization calculated in this way is fast enough for the ionization to keep pace with transfer of energy to electrons from the ions, and consequently the relaxation time should depend on this energy transfer process which has a characteristic time less than 0.1 microsecond. It appears then that the gas behind the bow shock wave is near thermodynamic equilibrium. It is definite that the chemical composition of the gas is not frozen and since ionization takes place, the gas cannot be considered as an ideal gas.

Since the gas in the region behind the bow shock has been shown to be near equilibrium, the flow cannot be described as ideal gas flow with constant ratio of specific heat capacities, 5/3. If the flow is considered as a real gas flow with ionizing reactions playing a significant role the solution of the fluid dynamic problem becomes a study in itself and the features of the present problem become entangled with the real gas dynamics. As a compromise we choose to describe the flow as one with a constant ratio of specific heat capacities defined by the mean estimated thermodynamic conditions. We require of this ratio that it should give reasonable estimates of the enthalpy and the speed of sound in the usual ideal gas definitions.

Analogous to the relation between the speed of sound and the ratio of heat capacities for an ideal gas, we define an effective ratio.
of specific heat capacities $\gamma_{\text{effective}}$ by

$$\left( \frac{\partial P}{\partial \rho} \right)_s = \gamma_{\text{effective}} \frac{P}{\rho}$$

where $\gamma_{\text{effective}}$ is now a function of state. The derivative of pressure with respect to density at constant entropy is confined to equilibrium states. The free stream conditions in the slug of hot gas are temperature, 30,000 K; density, $3 \times 10^{-3}$ kg m$^{-3}$; pressure, $5 \times 10^4$ N m$^{-2}$; and ionization fractions 0.3 and 0.7. Since there are only few ground state atoms the Saha equation for the reaction $A \rightarrow A^+ + e^-$ is replaced by $\alpha_1 + \alpha_2 = 1$ and the result of differentiating the thermodynamic equations with entropy held constant is

$$\left( \frac{\partial P}{\partial \rho} \right)_s = \left[ \frac{5/2}{3/2} \frac{P}{\rho} + \frac{1}{M} \frac{I_2}{G(\alpha_2)} \left( \frac{5/2}{3/2} + \frac{I_2}{kT} \right) \right] \frac{P}{\rho}$$

where $G(\alpha_2) = \frac{1}{1+\alpha_2} + \frac{1}{\alpha_2} + \frac{1}{1-\alpha_2} + \left( \frac{3+I_2}{kT} \right) \frac{1}{2+\alpha_2}$.

We have defined the effective ratio of specific heat capacities to be

$$\gamma_{\text{effective}} = \left[ \frac{5/2}{3/2} \frac{P}{\rho} + \frac{1}{M} \frac{I_2}{G(\alpha_2)} \left( \frac{5/2}{3/2} + \frac{I_2}{kT} \right) \right]$$

$$\left[ \frac{3}{2} \frac{P}{\rho} + \frac{1}{M} \frac{I_2}{G(\alpha_2)} \left( \frac{3}{2} + \frac{I_2}{kT} \right) \right]$$

to satisfy the relation between the speed of sound and the pressure-density ratio. The "effective" enthalpy is given by

$$h_{\text{effective}} = \frac{\gamma_{\text{effective}}}{\gamma_{\text{effective}} - 1} \frac{P}{\rho}$$

and since $I_2 \approx 10$ kT

$$h_{\text{effective}} \approx \frac{5}{2} \frac{P}{\rho} + \frac{1}{M} \frac{I_2}{10} + \frac{I_2}{2} + \frac{I_2}{10}$$
which approximates the real gas enthalpy to within 20% for thermodynamic conditions near the mean values. It is not possible to exactly satisfy the ideal gas relations for both speed of sound and enthalpy with a single $\gamma_{\text{effective}}$, but assumption of an approximate value leads to results which are more or less correct. This is a re-statement of the result given by Cashman (1971) for the study of a singly ionized gas.

We compute the magnetic Reynolds number, $R_m = \mu \sigma_{ST} a_\ast^2$, for the flow about the blunt body using ideal gas relations to determine the stagnation point electrical conductivity, $\sigma_{ST}$, and sonic point velocity, $a_\ast$, from the free stream values. The value of the ratio of specific heat capacities to represent the ionizing gas should be that appropriate to the free stream since in all regions except very close to the stagnation point the thermodynamic condition of the gas is almost the free stream value. Hence we assume the ratio of specific heat capacities is constant and given to be $\gamma = 1.3$.

The stagnation point sound speed $a_{ST}$, and sonic point velocity are given by

$$a_{ST}^2/a_\infty^2 = 1 + (\gamma - 1)m^2/2$$

and

$$a_\ast^2/a_{ST}^2 = 2/(\gamma + 1).$$

The electrical conductivity, assumed proportional to $T^{3/2}$, is computed for the stagnation point using the free stream value and the first of these relations. Table 5.1.4 shows the relevant quantities in the magnetic Reynolds number evaluation. Owing to the constant $\gamma$ assumption the values determined for $R_m$ are only approximate.
Table 5.1.4: Magnetic Reynolds number evaluation.

For magnetic field $0.75 \ \text{weber/m}^2$ and mean electrical conductivity of $2 \times 10^4 \ \text{mho/m}$ the magnetic interaction parameter for the experiment is $S \approx 1$. 

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<th>$\sigma_{ST}$</th>
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Table 5.1.4: Magnetic Reynolds number evaluation.
5.2 ANALYSIS OF THE PROBE SIGNALS

The features on the oscillograph records are correlated with the predicted position of the test gas on distance-time curves (figures 5.2.1 and 5.2.2). The test gas position is known (to ± 5 mm) at the instant that it reaches the photodiode station 16 cm from the end of the shock tube. Uncertainty in exact timing of the shock has been discussed in section 5.1. The measured velocity permits extrapolation of the shock front location from this fixed point in the distance-time diagrams along the shock tube and into the test section. Shock tube recoil and relative positions of the probes and magnetic field coils are taken into account. For different shots with various shock speeds the location of the test slug at the time of each feature in the probe signal is determined. The uncertainty in true position is about 2 cm and arises from uncertainty in the time-of-flight velocity measurement and to a lesser extent from errors in measurement of the shock tube recoil distance.

In the course of reproduction of search coil traces in figure 5.2.2 (continuation) the polarity was inverted, which accounts for the difference between the polarity shown in these traces and those in figures 5.2.1 and 5.2.2.

The signal obtained from the search coil mounted in the probe body is proportional to the perturbation to the magnetic field. The main sources of this field perturbation are the interaction of the flow with the aligned magnetic field in the region behind the bow shock wave from the body and the interaction of the gas slug with the fringing magnetic field. We now derive an expression for the perturbation caused by the interaction with the fringing field to compare it with that arising from the flow past the probe.

The perturbation of the magnetic field at the probe position can be estimated when the conducting gas is assumed to have uniform
Figure 5.2.1: Oscillograph records correlated with predicted position of test gas: first probe and loop on cookie cutter.
Figure 5.2.2: Oscillograph records correlated with predicted position of test gas: second (downstream) probe.
Figure 5.2.2 (continued): Search coil traces.
Figure 5.2.3: Magnetic field perturbation variations with test gas velocity. Data from table 5.2.2.
conductivity $\sigma$ in a cylindrical region of length $L$ and radius $R$ travelling at speed $v$ along the cookie cutter channel. If the conductivity structure of the shock heated slug is known accurately a solution can be obtained using the results of the following analysis applied to each region of uniformly conducting gas.

The electric current density at each point in the gas is given by $j = \sigma v B_r^\tau$ where $B_r^\tau$ is the radial component of the Helmholtz field. There is no simple analytic expression for the components of the magnetic field at the point $(z,r)$ when $z$ and $r$ are of the same order of magnitude as the Helmholtz coil radius since the exact expressions contain complete elliptic integrals in terms similar to those in equation 2.3.18. However, for $r$ less than the coil radius, $B_r^\tau$ can be approximated by assuming the axial component is uniform over a disk of radius $r$ and integrating the expression for continuity of magnetic field lines ($\nabla \cdot B = 0$) over the surface of a cylindrical element of radius $r$ and length $dz$. The expression for the axial component of the magnetic field at the point $z$ on the axis (measured in the direction of flow from the midpoint of the Helmholtz coils with radius and separation $a$) is

$$B_z = \frac{\mu N I}{2a} \left( (1 + (z+a/2)^2)^{-3/2} + (1 + (z-1/2)^2)^{-3/2} \right)$$

where $z$ is measured in units of the coil radius. With these assumptions, $B_r^\tau = -\frac{1}{r^2} \frac{\partial}{\partial z} B_z^\tau$ can be obtained.

Since the applied magnetic field varies as $z^{-3}$ for $z > 1$, the electric currents a few coil diameters away from the probe produce magnetic field perturbations proportional to $z^{-6}$ and these are negligible.

The electric current flowing through an area $dr\,dz$ in a closed circular loop about the axis produces at the probe position (say $z' = z_0$, ...
\[ \mathbf{r}' = 0 \) an axial component of magnetic field

\[ dB_z = \frac{a \mu_0 \tau_2 j}{2} r \, dz \left( r^2 + (z-z_0)^2 \right)^{-3/2} \]

where \( r \) and \( z \) are again measured in units of coil radius \( a \). The contribution to the field perturbation from all elemental loops of current making up a disk of radius \( R \) and thickness \( dz \) is

\[ dB_{\text{disk}}(z) = \frac{3}{8} \mu_0^2 \pi \sigma v \, dz \left[ (z+\frac{1}{2})(1 + (z+\frac{1}{2})^2)^{-5/2} + (z-\frac{1}{2})(1 + (z-\frac{1}{2})^2)^{-5/2} \right] \]

\[ \times \int_0^R r^3 (r^2 + (z-z_0)^2)^{-3/2} \, dr. \]

Performing the integration,

\[ dB_{\text{disk}}(z) = \frac{dz}{8} \frac{3}{8} \mu_0^2 \pi \sigma v \, f(z), \]

where

\[ f(z) = f_1(z) f_2(z), \]

\[ f_1(z) = (z+\frac{1}{2})(1 + (z+\frac{1}{2})^2)^{-5/2} + (z-\frac{1}{2})(1 + (z-\frac{1}{2})^2)^{-5/2} \]

and

\[ f_2(z) = (r^2 + 2(z-z_0)^2)(r^2 + (z-z_0)^2)^{-3/2} - 2((z-z_0)^2)^{1/2}. \]

In all of these expressions the positive square root is intended and the last term in the expression for \( f_2(z) \) is written in the form which ensures that \( f_2(z \to \pm \infty) = 0. \)

The field perturbation caused by a cylindrical slug of gas with length \( l \) can be obtained from the perturbation due to the disks.

When the front of the slug moves a distance \( dz \), the magnetic field
perturbation is increased by $dB_z(z)$ arising from the electric currents
in the disk at $z$ while simultaneously $dB_z(z-L)$ is subtracted owing to
the removal of a disk of conducting fluid at the back of the slug. Thus

$$\frac{dB_z}{dt} = \frac{3}{8} \mu^2 N I \sigma v (v/a)(f(z) - f(z-L)).$$

In terms of the applied field this is

$$\frac{(dB_z}{dt)}{B_z(0)} = -\frac{3}{8} \frac{\sigma v^2}{R} (f(z) - f(z-L))$$

For the experiment $\sigma = 10^4$ mho m$^{-1}$, $v = 1.5 \times 10^4$ m s$^{-1}$, $R = 25$ mm,
a = 41 mm so that

$$\frac{(dB_z}{dt)}{B_z(0)} = -1.5 \times 10^6 (f(z) - f(z-L)) s^{-1}$$

Since the probe is 12 mm downstream from the geometric centre of the
Helmholtz coils, $z_0 = 0.3$ and the function $f(z)$ is not symmetric about
$z = 0$. The function and its integral for $R = 0.6$ and $z_0 = 0.3$ are shown
in figure 5.2.4. These are drawn from the numerical data in table 5.2.1.

In units of the Helmholtz coil radius ($a = 41$ mm), $L = 1.5$ to
2.0 since the slug length is between 6 and 8 cm. The function $f(z)$ is
negligible for values of $z$ less than (-1.5) and so as the slug approaches
the probe body $f(z-L)$ may be neglected. Thus the magnetic field pertur­
badation before the shock wave passes the probe is proportional to the
integral of $f(z)$ and has a peak value of

$$B_z/B_z(0) = \int \frac{(dB_t}{dt)}{B_z(0)} dt = -1.5 \times 10^6 \int \frac{f(z)dz}{v} a \approx -0.03.$$

These calculations predict that just before the shock wave reaches the
\[
f(z) = f_1(z)f_2(z)
\]

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Table 5.2.1: Numerical values for \( f_1, f_2 \) and \( f \) when \( R = 0.6 \) and \( z_0 = 0.3 \). These values are plotted in figure 5.2.4. \( f_1 \) and \( f_2 \) are symmetric about \( z = 0 \) and \( z = z_0 \) respectively.
Figure 5.2.4: Sketch of the function $f(z)$ and $\int_{-\infty}^{z} f(z') \, dz'$ for $R = 0.6$ and $z_0 = 0.3$. The theory is not strictly valid for $z > -0.5$. 
probe body there is a peak 3% (negative) perturbation of the magnetic field.

Further complication of the signal measured by the probe arises from the deflection of the flow past the flared part of the probe body and in the Mach cone which forms at the exit of the cookie cutter channel. The position of the probe in the cookie cutter was chosen so that the bow shock wave would not impinge on the cookie cutter wall to avoid reflected shock from this wall.

The displacement of the probe from the centre of the field coils is \( z_0 \). For \( z_0 = 0 \) the function \( f(z) \) is anti-symmetric \( (f(-z) = -f(z)) \) but the "fringing field" perturbation prior to the arrival of the shock wave is greater than if \( z_0 \) was slightly positive as in the experiment.

The preceding argument applied to the "fringing field" perturbation is approximate since the slug is assumed to be uniformly conducting, the slug length is not accurately known and the radial components of the applied magnetic field have been approximated rather crudely at points away from the axis. A decrease in slug length serves to reduce the magnitude of the observed fringing field perturbation while variation of conductivity distribution smooths the time derivative of the perturbing field. It is difficult to determine the effect of the error in the approximation of the off-axis radial components. The field is possibly over-estimated. The approximation for \( B_z \) and hence \( B_r \) is not strictly accurate near \( z = 0 \). Figure 5.2.5 is a sketch of the field from the Helmholtz coil. The pecked line is the locus of points where \( B_r = 0 \) and these are not represented in the approximation.

The signal from the magnetic probe is expected to be as follows. The magnetic field at the probe begins to decrease when the shock front is approximately 6 cm (1\( \frac{1}{2} \) Helmholtz coil radii) away and a negative perturbation (3%) is reached 4 microseconds later when the
Figure 5.2.5: Schematic of Helmholtz field illustrating field divergence near $z = 0$. 
wave has reached the centre of the Helmholtz coil. The perturbation turns sharply negative as the shock front encounters the probe body (8 mm radius nose) and the bow shock wave and the associated flow field begin to form. The subsequent deflection of the fluid induces the negative field perturbation at the probe. The signal at this time is the sum of those contributions from the currents in the fluid upstream from the mid-point of the Helmholtz coil and those in the region behind the bow shock which together tend to give negative contributions. The contribution from the currents in the subsonic region behind the bow shock should dominate since they are closer to the probe and occupy much of the volume of the flow. The arrival of the helium driver gas at the probe initiates the collapse of the conducting flow and magnetic field perturbation.

The front probe records (figure 5.2.1) show this behaviour. However, about one microsecond after the collapse a second negative perturbation is observed. The negative peaks are separated in time by 1.6 to 2.6 microseconds for shots of almost identical velocities (frames 248, 270, 274) while in frame 202 only one peak is discernible. The occurrence of the second peak appears to be irregular.

Table 5.2.2 shows the experimentally determined values of magnetic field perturbation at each of the dominant peaks from the traces of the first coil.

There is a time synchronisation error arising from the uncertainty of the time at which the multivibrator at the last light station triggered. Added to this is a possible 0.2 microsecond error in setting the 16 microsecond delay. The total time synchronisation error is possibly 1 microsecond. The error in velocity measurement (3%) has already been discussed. This error is not entirely independent of the time synchronisation since both depend on the triggering of the
<table>
<thead>
<tr>
<th>Shot No.</th>
<th>Frame</th>
<th>Velocity $v_0$</th>
<th>$V_c$</th>
<th>Measured Field Perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$10^4$ m/s</td>
<td>volts</td>
<td>First negative peak</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>weber/m² %</td>
</tr>
<tr>
<td>* 316</td>
<td>202</td>
<td>1.62</td>
<td>180</td>
<td>0.10 11</td>
</tr>
<tr>
<td>325</td>
<td>230</td>
<td>1.62</td>
<td>160</td>
<td>0.08 10</td>
</tr>
<tr>
<td>326</td>
<td>235</td>
<td>1.58</td>
<td>154</td>
<td>0.08 9</td>
</tr>
<tr>
<td>328</td>
<td>240</td>
<td>1.60</td>
<td>135</td>
<td>0.06 8</td>
</tr>
<tr>
<td>* 330</td>
<td>248</td>
<td>1.62</td>
<td>137</td>
<td>0.07 10</td>
</tr>
<tr>
<td>332</td>
<td>257</td>
<td>1.55</td>
<td>148</td>
<td>0.08 10</td>
</tr>
<tr>
<td>* 337</td>
<td>270</td>
<td>1.61</td>
<td>159</td>
<td>0.07 8</td>
</tr>
<tr>
<td>* 338</td>
<td>274</td>
<td>1.63</td>
<td>158</td>
<td>0.06 7</td>
</tr>
<tr>
<td>346</td>
<td>307</td>
<td>1.61</td>
<td>168</td>
<td>0.06 7</td>
</tr>
<tr>
<td>347</td>
<td>311</td>
<td>1.63</td>
<td>162</td>
<td>0.06 8</td>
</tr>
<tr>
<td>348</td>
<td>314</td>
<td>1.66</td>
<td>160</td>
<td>0.10 12</td>
</tr>
</tbody>
</table>

Table 5.2.2: Experimental results showing magnitude of peaks.

The peak applied field is proportional to capacitor voltage $V_c$ and is 1 weber/m² when $V_c = 190$ V. The field perturbation voltage traces are reduced to units of magnetic field using the probe calibration and a pre-amplifier factor. Traces marked * are reproduced in figure 5.2.2 (continuation). For shot 316 (frame 202) the nose of the probe was at the geometric centre of the Helmholtz coils. For all other shots the probe was 12 mm downstream.
last light station. From these sources the error in determination of the shock front position at a given time in the test section is 2 cm.

An analysis of the features of the traces in frames 202, 248, 270 and 274 is now presented, but no definitive explanation for the occurrence of the second peak can be given. The first objective is to determine from the probe signals the instant of abrupt changes in the signals. Table 5.2.3 shows the time elapsed from the beginning of each trace and the shock displacement from the last photodiode station. The displacements are evaluated using the velocity measured between stations 1 and 3 in the shock tube. The shock position can be related to the probe and magnetic field positions from the measured distances given in figure 4.4.1, making allowance for 2 cm shock tube recoil and 16.2 cm between the light station and the end of the shock tube.

The first negative deflection of the signals occurs when the shock front is 4 cm from the centre of the Helmholtz coil. The signal reaches a maximum value of 2% of the applied field over 2½ microseconds. This corresponds to the fringing field perturbation which has been estimated to begin at 1½ coil radii from the Helmholtz coil and to reach a peak of 3% of the applied field. Each trace displays this behaviour. The perturbation in frame 202 is slightly greater than 2% and is accounted for in the theory of the fringing field perturbation by the different position of the probe. For a probe near the centre of the Helmholtz coil the initial "fringing field" perturbation is greater since the probe is closer to the induced electric currents. Frames 191, 195 and 198 which are not reproduced in the thesis are also records of the front probe signal when the probe is at the centre of the Helmholtz coil. These show the enhanced fringing field perturbation and in addition the secondary negative peak absent from frame 202.
Table 5.2.3: Shock position – time analysis of probe signals.

The letters a–e denote features in figure 5.2.6. The absolute position \( L \) of the shock at the given times is known to within an error \( \pm 2 \) cm. The relative positions contain an error of \( \pm 3 \) mm since the times on the trace can be read to an accuracy of \( \pm 0.2 \) microsecond. Displacements from the light station to the centre of the Helmholtz coil, the nose of the probe body and the front probe are 34.3, 34.8 and 35.6 cm respectively, when recoil is 2 cm.
Figure 5.2.6: Sketch of probe trace (frame 270) with significant features marked a-e.
The signal deflects sharply when the shock front reaches the nose of the blunt body. A peak negative perturbation (0.08 ± 0.03 weber/m²) is attained after 2 microseconds. The duration of the flow is too short to obtain a steady state parallel magnetic field. Figure 5.2.3 shows the measured field perturbations at the first negative peak as a function of measured shock velocity. The error bars denote uncertainty in the values of applied field and measured perturbation. Both arise from the accuracy in reading oscilloscope traces. The spread in velocity has been produced from the same nominal shock tube operating conditions and could be accounted for by variations in the second diaphragm burst pressure or gas composition in the intermediate and final shock tubes. Air contamination of either chamber produces slower shock speeds.

The comparison of the magnetic field perturbations for test gases with different velocities shows that the measured perturbation decreases slightly as the velocity increases (figure 5.2.3). This is consistent with test gas duration decreasing as the shock velocity increases.

The field perturbation depends mainly on the electrical conductivity (hence gas composition), the shock speed, the magnitude of the applied field and the test gas duration. There is not sufficient experimental evidence to separate the effects of these factors.

The origin of the second negative peak in the front probe signal is not explained. It begins to form during the period that the conducting flow is along the parallel and flared sections of the nylon nose. At the same time the flow is emerging from the glass channel and becomes electrically connected to the brass tube holding the probe.

The signal from the rear probe (1 cm behind the first) has similar behaviour to that from the front probe. The initial fringing field perturbation and the first negative peak (which has been associated
with the flow past the nose of the probe body) are both smaller in magnitude than those in the first probe signal. The second negative peak is smaller in magnitude also by the same ratio. This last observation appears to discount any theory which associates the second peak with the flow past the flare. In fact, it appears that the disturbance to the magnetic field causing the second peak has its origin upstream from the front probe.

To check the relative performance of the amplifier-oscilloscope systems recording each probe signal, the leads from the probes were interchanged at the back of the test section. The signals from the subsequent shots do not change the result that the ratio of the magnitudes of the features from the front probe are approximately the same as for the second.

Further experiments to eliminate or otherwise determine the nature of the second negative peak are necessary and involve:

(i) isolation of the probes from the brass support by insertion of a non-conducting cylindrical section between the probe and brass support;

(ii) construction of larger diameter Helmholtz field coils to eliminate the small fringing effect and to determine whether the interval between the negative peaks depends on the scale of the magnetic field;

(iii) construction of identical fast rise (less than 0.1 microsecond) differential amplifier systems to amplify the signals from each probe with sufficient sensitivity (50 x gain) to permit applied fields as small as 0.1 weber/m² to be used; and

(iv) determination of the actual shock slug length and electron density structure using interferometric methods for various shock tube operating conditions.
The influence of the magnetic field on the velocity of the conducting gas can be estimated using the momentum equations and Maxwell's equations. For weak interactions the magnetic field and velocity can be written \( \mathbf{B} = \mathbf{B}_0 + \mathbf{B}' \) and \( \mathbf{v} = \mathbf{v}_0 + \mathbf{v}' \) where \( \mathbf{B}_0 \) is the applied field, \( \mathbf{v}_0 \) satisfies the momentum equation without the magnetic force term and the primed quantities denote changes due to the interaction. Neglecting second order terms in the primed quantities and assuming steady state conditions, \( \mu \rho \mathbf{v}_0 \cdot \nabla \mathbf{v}' = (\text{curl } \mathbf{B}') \times \mathbf{B}_0 \). When \( \mathbf{v}_0 \) and \( \mathbf{B}_0 \) are almost parallel,

\[
\frac{\mathbf{v}'/\mathbf{v}_0}{\mathbf{B}'/\mathbf{B}_0} = \frac{B_0^2/\mu \rho v_0^2}{B'_0/\mathbf{B}_0}
\]

In the free stream, the density and velocity are \( 3 \times 10^{-3} \text{ kg/m}^3 \) and \( 1.5 \times 10^4 \text{ m/s} \), and the applied field is approximately \( B_{\text{max}}/z^3 \) (when \( z \geq 1 \)) and \( z \) is the distance from the Helmholtz coil measured in coil radii. Thus

\[
(v'/v_0) = (B'/B_0) \times 0.7/z^6
\]

when the applied field is 0.75 weber/m\(^2\). From Table 5.2.1 \( \int f(z) \, dz < 0.07 \) so that \( B'/(B_0/z^3) \) is less than 0.3 and for \( z > 2 \), \( v'/v_0 \) is negligible. Within the volume defined by the Helmholtz coil, \( v'/v_0 = 0.7 B'/B_0 \), and the relative perturbations to the magnetic field and velocity are of the same order of magnitude. The perturbation to the magnetic field has been estimated to be 3% at the probe.

The interaction between a fringing applied magnetic field and the conducting free stream flow has been examined by Dolder and Hide (1960). The flow converges like that through a nozzle and shock
waves may reflect from the "walls" of this "magnetic nozzle". For the present experiment, the magnetic interaction in the free stream has been shown to produce velocity and magnetic field perturbations of approximately the same order of magnitude. In the free stream near the probe these perturbations should be slight since the Helmholtz field is designed to produce a large region of uniform magnetic field. However, flow irregularities as observed by Dolder and Hide may have arisen from the magnetic interaction in the far fringing field and could possibly have contributed to irregularities in the probe signals.

The interaction in the flow past the blunt body must be considered separately since in this region the velocity and magnetic field are not parallel. Near the blunt body, $v_0$ is approximately $v_\infty \sin \phi$ parallel to the body surface and $(v_0 \times B_0) \approx v_\infty \sin \phi \cos \phi$ where $\phi$ is the angular distance measured around the nose of the body. The linearised equation for $v'$ is

$$\rho v_0 \cdot \nabla v' = \sigma (v_0 \times B_0) \times B_0,$$

from which

$$\frac{\rho v_0 \sin \phi}{R} \frac{\partial v'}{\partial \phi} = -\sigma v_\infty B_0^2 \sin \phi \cos \phi.$$

The density is approximately constant in the subsonic region but the electrical conductivity varies by a factor 5 between the stagnation point and the sonic line. Assuming constant values of density and conductivity and normalising $v'$ with respect to $v_0$ gives

$$\frac{v'}{v_0} = -\sigma B_0^2 R / \rho v_\infty.$$
The direction of the vector \( v' \) is normal to and toward the axis. From the free stream conditions given above and assuming a mean electrical conductivity \( 2 \times 10^4 \text{ mho/m} \), a shock density ratio 3 and nose radius \( R = 8 \text{ mm} \), we find \( v'/v_0 = 1 \). Thus the assumption that second order terms in \( v' \) can be neglected is invalid in this case. This crude analysis determines the appropriate magnetic interaction parameter for this problem.

The magnetic interaction parameter for the flow past the nose of the blunt body, \( v'/v_0 \), has been estimated to be approximately 1.0 and the fluid velocity is significantly modified. The radial component of velocity is reduced in magnitude since the \( \mathbf{j} \times \mathbf{B} \) force for aligned magnetic field and axisymmetric flow is in the radial direction. The bow shock wave is shifted upstream from the no-field position to enable the gas with reduced radial velocity to flow past the probe body. This scheme is sketched in figure 5.2.7.

For magnetic interaction parameter \( S \), the radial velocity component is reduced from the non-interaction value \( v_r \) to approximately \( v_r/(1 + S) \). The region behind the bow shock is increased and as shown in figure 5.2.7, the increased cross-sectional area is approximately \((1 + S/2)\) times the non-interaction value. Since this added region of conducting gas and electric currents is further from the probe the contribution to the magnetic field perturbation at the probe is less per unit current density by approximately the factor

\[
\left( \frac{(1 + \varepsilon/2)/(1 + (S+2)\varepsilon/2)}{(1 + S)} \right)^3
\]

the cube of the ratio of the mean distances from the search coil. For these approximations the ratio of the magnetic field perturbation with velocity changes taken into account to the non-interaction \( (S = 0) \) value is

\[
(1 + (S/2))( (1+\varepsilon/2)/(1 + (2+S)\varepsilon/2) )^3 ) / (1 + S).
\]

For the shock standoff distance \( \varepsilon \geq 0.3 \) and \( S \geq 1 \), this ratio is 0.6.
Figure 5.2.7: Sketch of distortion of the bow shock wave for moderate ($S = 1$) interaction parameter.
5.3 THEORETICAL RESULTS

The magnetic field perturbation arising from interaction between the field and conducting flow past the nose of a spherical body has been evaluated using the method developed in Chapter 2. Various Mach number flows with magnetic Reynolds number up to $R_m = 20$ have been computed. In all cases the ratio of specific heats $\gamma = 5/3$ has been used and the magnetic interaction parameter has been assumed negligible.

Figures 5.3.1 to 5.3.5 give the lines of constant flux for Mach 3 flow. The magnetic field lines are convected in the direction of the flow and the flux density inside the body is reduced as the magnetic Reynolds number increases.

The magnetic flux which would be measured by a search coil situated at the centre of curvature of the nose and having half the nose radius is shown in figure 5.3.6. The ratio of the flux enclosed by this coil to the undisturbed value is given for various magnetic Reynolds number and Mach number. The reduction in the probe signal for a flow with $M = 3, \gamma = 5/3$ and $R_m = 4$ is 16%. Figure 5.3.7 shows the relative magnetic flux which would be measured by the same coil translated along the axis of symmetry. By placing the coil one radius back from the centre of curvature the field perturbation is reduced by about one half.
Figure 5.3.1: Lines of constant magnetic flux shown relative to the shock wave and spherical nose:

\[ R_m = 1.0 \]
\[ M = 3 \]
\[ \gamma = 5/3. \]
Figure 5.3.2: Lines of constant magnetic flux shown relative to the shock wave and spherical nose:

\[ R_m = 5.0 \]
\[ M = 3 \]
\[ \gamma = \frac{5}{3} \]
Figure 5.3.3: Lines of constant magnetic flux shown relative to the shock wave and spherical nose:

\[
\begin{align*}
R_m &= 10 \\
M &= 3 \\
\gamma &= 5/3.
\end{align*}
\]
Figure 5.3.4: Lines of constant magnetic flux shown relative to the shock wave and spherical nose:

\[ R_m = 15 \]
\[ M = 3 \]
\[ \gamma = 5/3. \]
Figure 5.3.5: Lines of constant magnetic flux shown relative to the shock wave and spherical nose:

\[ R_m = 20 \]
\[ M = 3 \]
\[ \gamma = 5/3. \]
Figure 5.3.6: Theoretical magnetic flux measured by a search coil at the centre of probe nose against magnetic Reynolds number.
Figure 5.3.7: Theoretical magnetic flux threading a disk with radius $r = 0.5$ for positions along the axis of symmetry (z-axis). The tip of the probe is $z = 1.0$. 

Mach $N^\circ = 3$

$\gamma = 5/3$
CHAPTER 6
CONCLUSION AND DISCUSSION

A quantitative description of the magnetic field perturbation near the nose of a blunt body in a supersonic stream of conducting gas has been determined. The result is applicable to magnetic probes inserted into a plasma in the presence of applied aligned magnetic fields. The perturbation is strongly dependent on the magnetic Reynolds number (figure 5.3.6) and the shock layer thickness. For decreasing Mach number the shock layer thickens, the induced electric currents occupy a greater volume and the magnetic field perturbation increases.

The solution for the gas dynamic flow past a hemispherical cap is obtained using the method of integral relations one-strip approximation. The innovation due to South [1969] for extension of the integral curve closer to the saddle-point singularity without significantly increasing the computational effort is successfully implemented. From the vicinity of the singular point the solution curves are extrapolated into the supersonic region and the solution proceeds in the usual manner.

Using the computed flow field, the perturbation of an applied uniform aligned magnetic field is evaluated using a new approach which considers the magnetic field evaluation in terms of the Biot-Savart integral. The advantage of this approach, for problems in which the boundary conditions are asymptotic to the undisturbed values at infinity, lies in the simple way in which these conditions are satisfied. The problem is reduced to the solution of a Fredholm integral equation and this is solved numerically by replacing the integral operator by an approximated numerical operator.

The range of applicability is defined by the region in which
the problem is well posed. For large magnetic Reynolds number, the applied field is convected away by the conducting stream and the resulting magnetic field bears little resemblance to that applied. This is coincident with ill-conditioned behaviour of the numerical solution and the usual "condition number" of the linear system of equations is introduced to indicate the confidence level for the solution. However, for the magnetic Reynolds number based on the sonic point velocity, the nose radius and the stagnation point electrical conductivity, solutions for magnetic Reynolds number of the order of $R_m = 20$ are still valid.

The experimental investigation indicates a magnetic field perturbation of between 5% and 10% of the applied field for $R_m = 4$. These results are lower than the theoretically predicted perturbation (between 12% and 22%). The difference can be explained by considering the modification to the velocity field ($S = 1$). In addition, the duration of the test gas (2 microsecond) is not sufficient to produce a steady perturbed field.

The test gas is ionizing as it traverses the bow shock wave and the effective ratio of specific heats is correspondingly less than $\gamma = 5/3$. For lower values of $\gamma$, the calculated shock layer is thinner and by analogy with the effect of variation of Mach number on the flow, the anticipated magnetic field perturbation is reduced. This may also account for some of the difference between the measured and calculated perturbation.

The theoretical calculations of the magnetic perturbation together with the experimental results show that when $R_m > 1.0$ signals from magnetic probes are significantly reduced and this should be taken into account when interpreting magnetic probe measurements from flowing plasmas.
Extension of this research could be undertaken to include the influence of real gas effects and the influence of the magnetic field on the velocity field. This latter problem couples the magnetic field evaluation to the velocity field and vice versa, so that one is not determined without the other. This could be achieved by performing the integral relations solution including the magnetic body-force interaction [Chushkin, 1963; Appendix A], then evaluating the perturbed magnetic field and repeating the process using the perturbed magnetic and velocity fields in place of the undisturbed fields until consistent results are obtained.

Another extension is the refinement of the numerical integration formula in the solution of the Fredholm equation. The simple formula which has been used is equivalent to making the assumption that the electric currents flowing through each elemental area of the integration region are equal to a composite current, the value of which is determined from the electric current density at a given point in each element. A higher order numerical integration formula would effectively take account of variation in the current density over the elemental area.

Further experimental verification of the theory should be undertaken using a high performance device capable of producing a gas slug with test time much greater than say ten microseconds, the time to reach a steady perturbed magnetic field. A long shock tube, operated with shock velocity of $10^4$ metre/sec produces a test time a few times greater than the minimum acceptable flow duration, but the electrical conductivity is sufficient to yield magnetic Reynolds number only of order unity for a practical shock tube diameter. The shock tunnel [Stalker, 1967] provides a hypersonic flow which may last several milliseconds, but to produce a magnetic Reynolds number of the order of
R_m = 10 on a 10 cm radius body the stagnation enthalpy is required to be $5 \times 10^4$ cal/gm. Such high stagnation enthalpy is not available.

The theory for evaluation of the magnetic field perturbation is not only applicable to the problem investigated but to all problems in which the applied magnetic field is axially symmetric and the velocity-magnetic field interaction induces currents which form closed planar loops normal to the symmetry axis. Magnetohydrodynamic flight control problems involving an applied dipole field are therefore amenable to this analysis and the solution has a greater degree of accuracy than the order of magnitude computations given by Bush [1958] and Ericson et al. [1965].
APPENDIX A
SUPERSONIC FLOW PAST A SPHERE USING THE METHOD OF INTEGRAL RELATIONS

A.1 GOVERNING RELATIONS

The expressions for conservation of mass, momentum and energy for an inviscid, compressible, electrically conducting flow with no externally applied electric field have the form:

$$\nabla \cdot (\rho u) = 0 , \quad (A.1.1)$$

$$\nabla p + \nabla \cdot (\rho uu) = \sigma S (u \times B) \times B = F , \quad (A.1.2)$$

and

$$\nabla \cdot (\rho h_T u) = 0 , \quad (A.1.3)$$

where density, velocity, pressure, magnetic field, electrical conductivity and total enthalpy are denoted by $\rho$, $u$, $p$, $B$, $\sigma$ and $h_T$ respectively. In these equations, each quantity has been scaled according to its characteristic value and $S$ is the collection of these having the form:

$$S = \sigma_0 B_0^2 / \rho_\infty a_s . \quad (A.1.4)$$

Distance is scaled according to the radius of the sphere, $\lambda_0$; density in terms of the undisturbed free stream density, $\rho_\infty$; velocity with respect to the speed of sound at the sonic point on the body, $a_s$; and electrical conductivity and magnetic field in terms of their values at some chosen point.

Expanding equation (A.1.3) and using equation (A.1.1) to eliminate $\nabla \cdot (\rho u)$,

$$\rho u \cdot \nabla h_T = 0 . \quad (A.1.5)$$
For a gas with constant ratio of specific heats, $\gamma$, the total enthalpy is defined to be:

$$h_T = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} + \frac{u^2}{2}$$  \hspace{1cm} (A.1.6)

and the speed of sound is defined by:

$$a^2 = \gamma \frac{P}{\rho} .$$  \hspace{1cm} (A.1.7)

Substitution into equation (A.1.6) for $\gamma \frac{P}{\rho}$ gives:

$$h_T = \frac{a^2}{\gamma - 1} + \frac{u^2}{2} .$$  \hspace{1cm} (A.1.8)

At the sonic point on the body, $u_* = a_* = 1$ since $a_*$ is the scaling factor for velocity. The total enthalpy there is:

$$h_{T*} = \frac{\gamma + 1}{2(\gamma - 1)} .$$  \hspace{1cm} (A.1.9)

As can be seen from equation (A.1.5), the total enthalpy is constant along streamlines. Since all streamlines in the flow originate from the undisturbed stream the total enthalpy is constant throughout the whole field of flow. This result, together with equations (A.1.6) and (A.1.9), gives:

$$\frac{P}{\rho} = \frac{\gamma + 1}{2\gamma} \left( 1 - \frac{\gamma - 1}{\gamma + 1} u^2 \right) .$$  \hspace{1cm} (A.1.10)

This may be used in the place of one of the governing equations and is called the integrated momentum equation, or Bernoulli equation.

The number of partial differential equations governing the system can be reduced by one when the gas has a constant ratio of specific heats. This follows after the introduction of an entropy function defined by:

$$\phi = \frac{P}{\rho} \gamma$$
and some manipulation.

The scalar product of the velocity vector \( \mathbf{u} \) with the momentum equation (A.1.2) yields:

\[
\mathbf{u} \cdot \nabla \mathbf{p} + \mathbf{u} \cdot (\nabla \cdot (\rho \mathbf{u}\mathbf{u})) = \mathbf{u} \cdot \mathbf{F} .
\] (A.1.11)

Using equation (A.1.1) and the vector identities:

\[
\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{w} = 0 \quad \text{for an arbitrary vector } \mathbf{w} ,
\] (A.1.12)

\[
\nabla \cdot (\rho \mathbf{u}\mathbf{u}) = (\nabla \cdot \rho \mathbf{u}) \mathbf{u} + (\rho \mathbf{u} \cdot \nabla) \mathbf{u} ,
\] (A.1.13)

and

\[
\nabla (\mathbf{u}^2/2) = (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}) ,
\] (A.1.14)

the second term in equation (A.1.11) is written as:

\[
\mathbf{u} \cdot \nabla (\rho \mathbf{u}\mathbf{u}) = \rho \mathbf{u} \cdot \nabla (\mathbf{u}^2/2) ,
\] (A.1.15)

and so

\[
\mathbf{u} \cdot \nabla \mathbf{p} + \rho \mathbf{u} \cdot \nabla (\mathbf{u}^2/2) = \mathbf{u} \cdot \mathbf{F} .
\] (A.1.16)

Substituting for \( h^2 \) from equation (A.1.6) into equation (A.1.5) gives:

\[
\rho \mathbf{u} \cdot \nabla \left( \frac{\gamma - 1}{\rho} \mathbf{u} + \frac{\mathbf{u}^2}{2} \right) = 0 .
\] (A.1.17)

Adding equation (A.1.16) and equation (A.1.17) and rearranging terms yields:

\[
\mathbf{u} \cdot \left\{ \nabla \mathbf{p} + \gamma \rho \rho \nabla \left( \frac{1}{\rho} \right) \right\} = -(\gamma - 1) \mathbf{u} \cdot \mathbf{F} .
\] (A.1.18)

The term in brackets \( \{ \) is \( \rho \gamma \nabla \phi \). Dividing equation (A.1.18) by \( \rho \gamma \) and substituting with this term gives:

\[
\mathbf{u} \cdot \mathbf{\phi} = - \frac{(\gamma - 1)}{\rho \gamma} \mathbf{u} \cdot \mathbf{F} .
\] (A.1.19)

For an axially symmetric or two dimensional problem only two co-ordinate
directions are relevant. Equation (A.1.19) in terms of orthogonal curvilinear co-ordinates \( s \) and \( n \) with unity Lamé scale factors is written:

\[
u_s \frac{\partial \phi}{\partial s} + u_n \frac{\partial \phi}{\partial n} = - \frac{\gamma - 1}{\rho \gamma} \left( u_s F_s + u_n F_n \right), \quad (A.1.20)\]

where the subscripts \( s \) and \( n \) refer to components in these directions.

The entropy function is \( \phi = \phi(s,n) \). The total derivative with respect to \( s \) is:

\[
\frac{d\phi}{ds} = \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial n} \frac{u_n}{u_s}, \quad (A.1.21)
\]

since

\[
\frac{dn}{ds} = \frac{dn}{dt} \frac{ds}{dt} = \frac{u_n}{u_s}.
\]

Hence

\[
\frac{d\phi}{ds} = - \frac{\gamma - 1}{\rho \gamma} \left( F_s + \frac{u_n}{u_s} F_n \right). \quad (A.1.22)
\]

This ordinary differential equation is used in place of one of the governing relations. In doing so, the number of approximations to be made when the method of integral relations is applied is decreased.

Let us introduce the function \( \tau \) defined by:

\[
\tau = \rho \phi^{-1}
\]

and transform the continuity equation. The divergence of the product \( \tau u \) is:

\[
\nabla \cdot (\tau u) = \rho u \cdot \nabla \phi^{-1}. \quad (A.1.23)
\]

Expanding the gradient term and substituting for \( u \cdot \nabla \phi \) from equation (A.1.19) results in the expression:

\[
\nabla \cdot (\tau u) = \tau p^{-1} u \cdot F. \quad (A.1.24)
\]
A.2 CO-ORDINATE SYSTEM AND NOMENCLATURE

Since the only body shape considered is spherical, it is natural to use spherical polar co-ordinates \((r, \theta)\) in figure A.2.1. The azimuthal co-ordinate takes no part in the analysis since axial symmetry has been assumed.

![Co-ordinate system diagram](image)

Figure A.2.1: Co-ordinate system.

Let the subscripts 0 and 1 denote quantities evaluated on the body and shock respectively.

The distance along a ray with constant \(\theta\) from the body surface to the shock depends only on \(\theta\) and may be denoted by \(\epsilon(\theta)\). The shock shape can be expressed in terms of \(\epsilon\) by:

\[
r_1(\theta) = 1 + \epsilon(\theta)
\]

(A.2.1)

where the body is assumed to have unit radius.

Let the angle between the shock wave and the axis of symmetry
in the plane including this axis be written as the symbol $\sigma$. From the geometry (figure A.2.2),

$$\frac{d\varepsilon}{d\theta} = -(1 + \varepsilon) \cot(\theta + \sigma). \quad (A.2.2)$$

Figure A.2.2: Determination of $d\varepsilon/d\theta$.

The nomenclature of Chushkin [1963] is adhered to so that the ensuing expressions can be compared. The magnitude of the velocity vector $\mathbf{u}$ is written as $w$ and the velocity components in the $r$ and $\theta$ directions are denoted by $u$ and $v$ respectively.

Groups of quantities are identified with a special symbol to facilitate manipulation of the equations. These are:

$$h = rru,$$
$$q = rrv,$$
$$\mu = \sin \theta,$$
G = -r^2\tau p^{-1} (F \cdot w),
Q = r(p + \rho u^2),
P = r\rho uv,
and 
g = r(2p + \rho v^2 + rF).$

The transformed continuity equation (A.1.24) and the θ component of the momentum equation (A.1.2) in terms of the spherical polar co-ordinates are written:

\[
\frac{\partial}{\partial r} rh + \frac{1}{\mu} \frac{\partial}{\partial \theta} \mu q = G, \tag{A.2.3}
\]

and

\[
\frac{\partial}{\partial r} rQ + \frac{1}{\mu} \frac{\partial}{\partial \theta} \mu P = g. \tag{A.2.4}
\]

The equations (A.1.10), (A.1.22), (A.2.2), (A.2.3) and (A.2.4) form a system for the solution of the five unknown functions u, v, p, φ and σ. The boundary conditions are given by Chushkin [1963] and are repeated here for convenience.

### A.3 BOUNDARY CONDITIONS

On the body surface, r = 1 and u = 0. On the shock wave,

\[
u_1 = w_y \sin \theta - w_x \cos \theta,
\]

\[
v_1 = w_y \cos \theta + w_x \sin \theta,
\]

where

\[
w_x = w_\infty \left[ 1 - \frac{2}{\gamma + 1} \left( \sin^2 \sigma - \frac{1}{M_\infty^2} \right) \right],
\]

\[
w_y = (w_\infty - w_x) \cot \sigma,
\]

and

\[
w_\infty^2 = \frac{(\gamma + 1) M_\infty^2}{2 + (\gamma - 1) M_\infty^2},
\]

\[
p_1 = \frac{2}{\gamma + 1} w_\infty^2 \sin^2 \sigma - \frac{\gamma - 1}{2\gamma} \left( 1 - \frac{\gamma - 1}{\gamma + 1} w_\infty^2 \right),
\]

\[
\rho_1 = \frac{\sin^2 \sigma}{1/w_\infty^2 - (\gamma - 1)/(\gamma + 1) \cos^2 \sigma}.
\]
and \[ \phi_1 = p_1/\rho_1^ \gamma. \]

On the flow axis, \( \theta = 0, \nu = 0, \sigma = \pi/2 \) and

\[ \phi = \phi_x = \frac{\gamma + 1}{2\gamma} w_\infty^{-2\gamma} \left( w_\infty^2 - \frac{\gamma - 1}{\gamma + 1} \right). \]

A.4 SOLUTION USING THE METHOD OF INTEGRAL RELATIONS

The governing relations include ordinary differential equations (A.1.22) and (A.2.2) and partial differential equations (A.2.3) and (A.2.4). By means of the method of integral relations, an approximation is made so that the partial differential equations are replaced by ordinary differential equations.

Integrating equation (A.2.3) with respect to \( r \) between the body contour and the shock wave along the ray \( \theta = \) constant and using the Leibniz rule, the result is:

\[
\frac{1}{\mu} \frac{d}{d\theta} \int_{r_0}^{r_1} \mu q \, dr - q_1 \frac{dr_1}{d\theta} + [r h]_0^1 = \int_{r_0}^{r_1} G \, dr, \tag{A.4.1}
\]

where the subscripts 0 and 1 indicate evaluation of the quantity at the body and shock respectively.

For a one-strip solution linear approximations for the integrands \( q \) and \( G \) are made in terms of their values at the body and shock. These are \( q = q_0 + \frac{r - 1}{r_1 - 1} (q_1 - q_0) \), and \( G = G_0 + \frac{r - 1}{r_1 - 1} (G_1 - G_0) \). The approximation is exact when the first derivatives of \( q \) and \( G \) along rays \( \theta = \) constant are constant.

Substituting into equation (A.4.1) and integrating gives:

\[
\frac{1}{\mu} \frac{d}{d\theta} \left\{ \frac{\mu c}{2} (q_0 + q_1) \right\} - q_1 \frac{dr_1}{d\theta} + r_1 h_1 = \frac{c}{2} (G_0 + G_1), \tag{A.4.2}
\]

which is a total differential equation.
Equation (A.2.4) can be reduced in the same way by making the linear approximations \( P = P_0 + \frac{r - 1}{(r_1 - 1)} (P_1 - P_0) \), and \( g = g_0 + \frac{r - 1}{(r_1 - 1)} (g_1 - g_0) \), with the resulting ordinary differential equation:

\[
\frac{1}{\mu} \frac{d}{d\theta} \left\{ \frac{u}{2} \left( P_0 + P_1 \right) \right\} - P_1 \frac{dr_1}{d\theta} + r_1 Q_1 - Q_0 = \frac{e}{2} (g_0 + g_1). \tag{A.4.3}
\]

Expressions for \( \frac{ds}{d\theta} \) and \( \frac{dv_0}{d\theta} \) are obtained from equations (A.4.2) and (A.4.3) after some manipulation. For completeness, the working is included here.

From the Bernoulli equation (A.1.10) and the boundary conditions at the shock wave,

\[ 
\tau_1 = \left( \frac{\gamma + 1}{2\gamma} \left[ 1 - \frac{\gamma - 1}{\gamma + 1} (u_1^2 + v_1^2) \right] \right)^{\frac{1}{\gamma - 1}},
\]

\[ 
\frac{du_1}{d\theta} = v_1 + u_1 \frac{ds}{d\theta},
\]

and

\[ 
\frac{dv_1}{d\theta} = -u_1 + v_1 \frac{ds}{d\theta},
\]

where the prime symbol (') denotes partial differentiation with respect to \( \sigma \).

Using these relations,

\[ 
\frac{dr_1}{d\theta} = - \frac{\tau_1^{2-\gamma}}{\gamma} (u_1 u_1' + v_1 v_1') \frac{ds}{d\theta}.
\]

From the definition of \( q \),

\[ 
\frac{dq_1}{d\theta} = h_1 + q_1 \frac{de}{d\theta} + \frac{r_1 \tau_1^{2-\gamma}}{2\gamma} \frac{d\sigma}{d\theta} \left\{ (\gamma + 1)(1 - v_1^2)v_1' \right. \\
\left. - 2 u_1 u_1' v_1 - (\gamma - 1)u_1^2 v_1' \right\}
\]
where $h_1$ has been substituted for the product $r_1 \tau_1 u_1$. At the body,
the radial velocity $u_0$ is zero, $\tau_0 = 1$, and

$$\tau_0 = \left( \frac{\gamma + 1}{2\gamma} \left[ 1 - \frac{\gamma - 1}{\gamma + 1} v_0^2 \right] \right)^{1/\gamma - 1}.$$

Hence

$$\frac{d\theta_0}{d\theta} = \frac{d\theta_0}{d\theta} = \tau_0 2^{-\gamma} \frac{(\gamma + 1)}{2\gamma} (1 - v_0^2).$$

From equation (A.2.1),

$$\frac{d\theta_1}{d\theta} = \frac{d\theta_1}{d\theta}.$$

Substituting in equation (A.4.2) for $\frac{d\theta_0}{d\theta}$, $\frac{d\theta_1}{d\theta}$ and $\frac{dr_1}{d\theta}$ and rearranging

gives:

$$\frac{dv_0}{d\theta} = \frac{2\gamma}{(\gamma + 1)} \frac{\tau_0 \gamma^2}{(1 - v_0^2)} \left[ G_0 + G_1 + \left( 1 - \frac{2\tau_1}{\varepsilon} \right) h_1 \right]$$

$$- \left( q_0 - \frac{q_1}{\tau_1} \right) \frac{1}{\varepsilon} \frac{d\varepsilon}{d\theta} - \alpha_0 \frac{d\sigma}{d\theta} - \delta_0,$$

(A.4.4)

where

$$\delta_0 = (q_1 + q_0) \cot \theta,$$

and

$$\alpha_0 = \frac{\tau_1 \tau_1 2^{-\gamma}}{2\gamma} \left[ (\gamma + 1)(1 - v_1^2) v_1 - 2u_1 v_1 u_1 \right]$$

$$- (\gamma - 1) u_1^2 v_1].$$

Now we obtain the expansion of the derivatives in equation (A.4.3) to find $d\sigma/d\theta$. From the definition of $\tau$,

$$-1 \rho_1 = \left( \frac{1}{1 + \gamma} \right),$$

and

$$\frac{d\sigma_1}{d\theta} = -\rho_1 \frac{d\sigma}{d\theta} \left\{ \frac{\tau_1 1^{-\gamma}}{\gamma} (u_1 u_1' + v_1 v_1') + \frac{1}{\gamma - 1} \phi_1 \right\}.$$

Since $P_1 = r_1 \rho_1 u_1 v_1$, the derivative of $P_1$ is:
\[
\frac{dP_1}{d\theta} = \frac{P_1}{r_1} \frac{dr}{d\theta} + r_1 \rho_1 (v_1^2 - u_1^2) + \frac{d\sigma}{d\theta} \frac{\rho_1}{r_1} \left\{ u_1 \left[ (\gamma + 1)(1 - v_1^2) v_1' - 2u_1 u'_1 v_1 - (\gamma - 1) u_1^2 v_1' + \frac{q_1 \phi'_1}{(\gamma - 1)\phi_1} \right] + q_1 u_1' \right\}.
\]

On the body, \( P_0 = 0 \) since \( u_0 = 0 \). Substituting into equation (A.4.3) and rearranging terms gives the expression:

\[
\frac{d\sigma}{d\theta} = \frac{1}{\alpha_1} \left[ \varepsilon_0 + g_1 + (Q_0 - r_1 Q_1) \frac{2}{\varepsilon} - r_1 \rho_1 (v_1^2 - u_1^2) + \frac{P_1}{r_1} \frac{1}{\varepsilon} \frac{dr}{d\theta} - \delta_1 \right],
\]

(A.4.5)

where \( \delta_1 = P_1 \cot \theta \),

and \( \alpha_1 = \frac{\rho_1}{r_1} \left[ u_1 \left( \alpha_0 - \frac{q_1 \phi'_1}{\gamma - 1} \phi_1 \right) + q_1 u_1' \right] \).

The expressions for \( \alpha_0 \) and \( \alpha_1 \) given by Chushkin [1963] are in error. Comparison of the computed results with those given by Chushkin shows that either Chushkin used the correct formulae in his computations or the omission of terms \(- (\gamma - 1) u_1^2 v_1' \) and \( \phi_1 \) does not significantly affect the solution.

The expressions (A.4.4) and (A.4.5) may be evaluated to obtain \( \frac{dv_0}{d\theta} \) and \( \frac{d\sigma}{d\theta} \) except when \( \theta = 0 \) since then the terms \( \delta_0 \) and \( \delta_1 \) become indeterminate. New expressions are obtained for \( \theta = 0 \) using l'Hôpital's rule. Now:

\[
\lim_{\theta \to 0} \delta_0 = \lim_{\theta \to 0} \frac{q_0 + q_1}{\tan \theta} = \lim_{\theta \to 0} \left\{ r_0 \frac{dv_0}{d\theta} + r_1 \frac{d\sigma}{d\theta} \right\}.
\]

Similarly,

\[
\lim_{\theta \to 0} \delta_1 = \lim_{\theta \to 0} \frac{p_1}{\tan \theta} = \lim_{\theta \to 0} \left\{ -r_1 \rho_1 u_1^2 + v_1 \frac{d\sigma}{d\theta} \right\}.
\]
Substituting these values into equations (A.4.4) and (A.4.5) and rearranging gives for $\theta = 0$,

$$\frac{d\sigma}{d\theta} = \frac{1}{2\alpha_1} \left[ \sigma_0 + \sigma_1 + \frac{(Q_0 - r_1 Q_1)}{\varepsilon} r_1 \alpha_1 (v_1^2 - u_1^2) \right.$$

$$+ \frac{P_1}{r_1} \frac{1}{\varepsilon} \frac{d\varepsilon}{d\theta} - (-r_1 \alpha_1 u_1^2) \left. \right] \quad (A.4.6)$$

and

$$\frac{dv_0}{d\theta} = \frac{2\gamma}{\gamma + 1} \frac{\gamma - 2}{2(1 - v_0^2)} \left[ G_0 + G_1 + \left( 1 - \frac{2r_1}{\varepsilon} \right) h_1 \right.$$

$$+ \left( q_0 - \frac{q_1}{r_1} \right) \frac{1}{\varepsilon} \frac{d\varepsilon}{d\theta} - 2\alpha_0 \frac{d\sigma}{d\theta} - (-h_1) \left. \right] \quad (A.4.7)$$

These are in the same form as equations (A.4.4) and (A.4.5) except that $\sigma_0$ and $\sigma_1$ are replaced by $(-h_1)$ and $(-v_1 \alpha_1 u_1^2)$ respectively, and a factor of two appears before each of the quantities $\alpha_0$, $\alpha_1$ and $(1 - v_0^2)$. 
APPENDIX B

The double diaphragm shock tube at ANU was designed in 1968. My contribution to this is a theoretical optimisation of the shock tube diameters, lengths, and throat diameters at the diaphragm stations to obtain high shock Mach number and tolerable test gas duration. A paper describing this shock tube was presented at the Eighth International Shock Tube Symposium in London in July 1971. Since the proceedings are not yet published the paper is included in this thesis for reference.

A DOUBLE DIAPHRAGM SHOCK TUBE FOR THE

10-20 KILOMETRE PER SECOND RANGE

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ABSTRACT

A free piston double diaphragm shock tube is described which is producing shock speeds free of attenuation in excess of 17 kilometres sec\(^{-1}\) into 0.1 torr argon. Performance tests show that the shock speeds and attenuation behaviour can be described in terms of ideal shock tube theory. Time resolved channelled spectra measurements of electron density currently in progress, at present support a test slug corresponding to 90% second ionization in argon for a Mach number 55 shock wave.

INTRODUCTION

Pilot tests conducted by Stalker and Plumb [1968] in this
department in 1967 demonstrated that the concept of the double
diaphragm shock tube using a free piston driver [Stalker, 1966] first
stage would produce shock speeds in excess of Mach 50 in air and argon.
In 1969 construction was begun of a machine designed initially to study
the performance characteristics of the technique and subsequently to
act as a test facility for the above speed range.

This paper presents results of the performance tests to date
and reports on the diagnostic experiments which are just beginning. A
more complete presentation particularly of the diagnostics will be
available at the symposium.

SHOCK TUBE DESIGN

A schematic diagram of the machine is shown in fig. 1. A
pressure reservoir of 180 mm internal diameter and 1.4 metres long is
designed for pressures to $2 \times 10^4$ kN-m$^{-2}$, and stores air to drive the
free piston along the compression tube. The coupling between the
reservoir and compression tube contains a spigot (launcher) which holds
the piston prior to the drive stroke. The compression tube consists of
a 4.5" naval gun bored out to 124 mm diameter with the breech at the
high pressure end. This is 5 metres long and has been compression
tested to $10^5$ kN-m$^{-2}$. The piston weighs 20 Kgm and is fitted with two
sets of molybdenum disulphide impregnated nylon piston rings to act as
a seal to the compression tube walls. The breech is closed by a pres­
sure plate, fitted with nylon buffers to remove excess energy from the
piston without damage to the machine, and also contains the first
diaphragm. For the current tests, 10 gauge mild steel plate (unscored)
and bursting at $7.2 \times 10^4$ kN-m$^{-2}$ has been used.

This diaphragm bursts into the intermediate tube which can be
either 40 mm or 50 mm internal diameter. All the tests to date have
Figure 1: Schematic diagram of shock tube.
used the 50 mm tube however. This tube is manufactured from stainless steel hollow bar in 61 cm and 91 cm long sections. These can be joined with mild steel couplings to provide a variation in length from 1.6 to 3.7 metres. The intermediate tube ends in a breech, containing the second diaphragm, with design similar to the first so that the intermediate tube and test shock tube can be interchanged.

The shock tube can be 40 mm or 50 mm in internal diameter and is of the same material and design as the intermediate tube, its length being variable from 1.6 to 5.0 metres. Each section has a hole drilled in the side for fitting shock timing transducers. Fast response pressure transducers have been used in the intermediate tube and fibre optic light guides leading to P.I.N. photodiodes and amplifiers with a response time better than 0.1 microsecond, have been used in the shock tube. The shock tube ends in a test section which can be fitted with interferometer quality windows and a dump tank of sufficient volume to contain all the gases at a low pressure after each shot.

The entire machine with the exception of the test section and dump tank is free to recoil (24 mm), with the piston movement, by supporting it on low friction bearings. A compressed air spring, returns the machine to the same rest position after each firing. The supporting frame under the intermediate and shock tubes is adjustable in length so that the test section can be placed anywhere between 1.5 metres and 9 metres from the first diaphragm. The machine can therefore be operated as a single or double diaphragm shock tube of varying length.

EXPERIMENTAL METHODS

The experiments to the present have concentrated on optimizing the performance and describing the behaviour in terms of ideal
shock tube theory. Thus shock speed measurements have been made using the pressure and optical transducers. Image converter photographs have been taken of the shock in the test section and also time resolved as it passes a vertical slit. These will be presented subsequently.

Experiments have now begun in which the electron density profile behind the shock front is measured by using a time resolved channelled spectrum technique developed in this department [Sandeman, 1971]. Thus, a Mach Zehnder interferometer is adjusted such that the plane of localization of the fringes and the test section are focussed onto the slit of a spectrometer, a "white" light source is used and the optical path difference between the two beams arriving at the slit is preset to a value which is uniform along the slit and produces a "channelled spectrum" [Ditchburn, 1963] at the image plane. This output spectrum is viewed by an image converter operated in the sweep mode in the direction of the slit.

The light source uses a wire exploded by the discharge of a 5,000 joule capacitor and provides sufficient background intensity to swamp the radiation from the shocked gas and expose HP4 film through the image converter camera. The arrival of the shock, shifts the channels from their preset position. The shift is proportional to the refractivity of the component species and since the presentation is one of wavelength against time, the time profile of species of different dispersion can be separated. Two photographs using this technique are shown in fig. 8.

Fig. 2 presents a photograph of the machine and the channelled spectra instrumentation, taken from the test section end. Portion of the image converter camera, the constant deviation spectrometer and interferometer can be seen in the foreground. The various sections of the machine can be identified by reference to fig. 1.
Figure 2: Photograph of shock tube and instrumentation.
PRINCIPLE OF OPERATION

A simplified wave diagram, fig. 3, shows the main features to be expected in free piston double diaphragm operation. The nomenclature is standard and the regions self explanatory.

Stalker [1969] has shown that in free piston shock tube operation the plateau pressure after shock reflection falls below the theoretically predicted value as the volumetric compression ratio of the piston compressor increases and is accompanied by an increase in the shock reflected temperature at tailoring. This in turn leads to an increase in the contact surface speed after expansion through a given pressure ratio. It is noted that in this application tailored interface operation means regions I₂ and I₁ in fig. 3 are identical since not only the velocities, pressures and densities are the same but also the gases. Thus I₅ and I₆ are also identical and the expansion S₃ does not reflect at the "contact surface". If the shock speed into I₀ is higher than the tailored value (over tailored) I₃ reflects as an expansion wave, and vice versa if the conditions are under tailored.

The requirements of fastest shock speed without attenuation and longest test time in a given shock tube length are not necessarily compatible with tailored interface operation however; especially when one has the attended effects of, the diaphragm opening process, and the shock tube boundary layer.

RESULTS AND DISCUSSION

Fig. 4 shows the Mach numbers measured in helium in the intermediate tube at a station 2 metres from the first diaphragm and those in the test shock tube into argon at 0.1 torr pressure, at 4 metres from the second diaphragm, for both the 50 mm and 40 mm test shock tubes. These results are taken at intermediate tube pressures
Figure 3: Simplified wave diagram of shock tube operation.
Figure 4: Performance of shock tube.
close to that which gave a maximum in the shock tube Mach number.

Theoretical estimates of the shock speeds are also shown. These are based on ideal shock tube theory, with area change at the diaphragm into the intermediate tube, using the known volumetric compression ratio (90) of room temperature helium to the diaphragm burst pressure (i.e. reservoir conditions of 7.2 kN-m⁻² pressure and 6,000 °K). The shock tube results assume reservoir conditions corresponding to shock reflection of the intermediate tube shock at the second diaphragm. The area change here is also taken into account.

The experiments are lower than theory primarily because of attenuation of the shock in the intermediate tube of approximately 9 per cent per metre. The effect of area change at the second diaphragm predicted by the theory is confirmed by the experiments while the fall in the shock Mach number with lower intermediate tube pressures than the "optimum", appears to be due to the onset of shock attenuation upstream of the 4 metre station, caused by a reflected expansion, and hence over-tailored operation.

Fig. 5 shows the variation of shock speed and attenuation with varying initial shock tube pressure (Pst), for constant intermediate tube length (2.2 metres or 44 diameters) and initial pressure (Pit) of 40 kN-m⁻² and with area reduction at the second diaphragm.

The maximum shock speed for each curve is compatible with a reservoir pressure of 2.4 × 10⁴ kN-m⁻², (which was measured as the plateau pressure at the closed end of the intermediate tube) and an effective temperature of 14,000 °K. These values compare with shock reflected values calculated from the measured intermediate shock speed of 1.5 × 10⁴ kN-m⁻² and 14,000 °K. This also suggests over tailored operation in the intermediate tube and consequently that the shock attenuation of the 0.5 and 1.00 torr results is due to a reflected
Figure 5: Dependence of shock speed and attenuation on initial shock tube pressure.
expansion wave.

Fig. 6 shows the effect of varying intermediate tube length only. There is no area change at the second diaphragm and the intermediate tube initial pressure is the same as that for fig. 5. The Mach number of 51 is again consistent with reservoir conditions of $2.4 \times 10^8$ kN-m$^{-2}$ and 14,000 °K, this time with no area change. The decreasing shock speed with intermediate tube length, is consistent with shock attenuation in that tube, while the change from shock acceleration (56 diameters) to attenuation (32 diameters) is again consistent with a reflected expansion from an over-tailored contact surface. The arrival of this wave at the shock will be earlier with the shorter intermediate tube, because of the smaller length of reservoir gas. The acceleration noted for the longer tube is presumably due to the diaphragm opening processes.

In fig. 7 the diaphragm opening time has been varied, for constant intermediate tube pressure and shock tube pressure, (0.1 torr) and with no area change at the diaphragm. In one case however the second diaphragm was omitted and this produced a shock Mach number of 39. This compares with a value of 51, using a 16 gauge aluminium diaphragm which was deeply scored to produce four petals.

An 0.13 mm mylar (melinex) plastic sheet, and a 20 gauge mild steel sheet scored to burst at a pressure of $5.5 \times 10^3$ kN-m$^{-2}$, which is approximately twice the pressure behind the intermediate tube shock wave, gave the same shock Mach number of 47. The mylar diaphragm shears about the retaining edge and vaporises, producing radiation from the carbon then in the driver gas. It is most likely that a reflected shock does not form in this case and that the shock drive is via an unsteady expansion from the moving gas at the intermediate shock conditions, $2.7 \times 10^3$ kN-m$^{-2}$ pressure and 6,000 °K temperature. For the
Figure 6: Dependence of shock speed and attenuation on intermediate tube length.
Figure 7: Dependence of shock speed on diaphragm opening time.
mild steel diaphragm the opening time was approximately 90 microseconds compared with 50 microseconds for the aluminium sheet. These times are inferred from the measured time between the last transducer in the intermediate tube and the first in the shock tube, and then extrapolating from the known shock speeds. Minimising the diaphragm opening time while maintaining shock reflection therefore seems to be essential to high shock speeds. The opening time should become less critical as tailored interface operation is approached and this aspect is being further investigated.

Fig. 8 presents two photographs of time resolved channelled spectra of shock waves in argon at Mach numbers of 52, see fig. 8a and 54, see fig. 8b. The shock tube and intermediate tube pressures were 0.09 torr, with 41 kN-m⁻² and 0.1 torr, with 38 kN-m⁻² respectively. The arrival of the shock can be seen from the shift in the channels. In both cases there is an initial shift to the red end of the spectrum of 0.6, see fig. 8a and 0.9, see fig. 8b, of the channel spacing at the 5461 angstrom mercury line. In fig. 8a this shift remains almost constant for 2 microseconds then returns to close to the original position. In fig. 8b however there is an immediate return to a very small shift which in turn disappears 2.5 microseconds after the shock. Densitometer traces and a full analysis of these photographs have yet to be made so the results presented here are approximate only. The initial channel shift to the red corresponds to $6 \times 10^{16}$ and $9 \times 10^{16}$ electrons-cm⁻³ for the Mach 52 and 54 shocks respectively. These can be compared with equilibrium calculations at these Mach numbers of $7.6 \times 10^{16}$ electrons-cm⁻³, arising from 78% second ionization and 22% first ionization and a temperature of 37,500 °K and $8.6 \times 10^{16}$ electrons-cm⁻³, from 84% second and 16% first ionization at 40,000 °K.

The initial conditions were deliberately chosen for fig. 8a,
Figure 8: Time resolved channelled spectra of shock waves.
in order to delay the arrival of a reflected expansion from the intermediate tube contact surface. Both these photographs can be reproduced by repeating the stated initial conditions. The shift of the channels to the blue end is therefore interpreted, at this stage, as further evidence of a reflected expansion which has caught the shock front in fig. 8b, but has been delayed in fig. 8a. Of particular significance also, is the almost constant shift between the shock and the "expansion" indicating that radiation losses are small and the chance of a uniform equilibrium test slug is very high.

Time resolved emission spectra also show a spectrum lasting for this "uniform" time. The helium driver gas following the shock however is cold and free from any radiation. This makes the shock tube particularly suitable as a spectral source over a wide range of shock speeds.

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Normal conventions and notation are adhered to as closely as possible in the text. An underscored character signifies a vector quantity.

- \( a \) speed of sound
- \( B \) magnetic field
- \( E \) electric field
- \( h_T \) total enthalpy
- \( J \) electrical current density
- \( L \) distance
- \( M \) Mach number
- \( p \) pressure
- \( R_m \) magnetic Reynolds number
- \( t \) time
- \( u \) fluid velocity
- \( x, x' \) position vectors
- \( \gamma \) ratio of specific heats
- \( \varepsilon \) standoff distance
- \( \nu \) permeability of free space
- \( \rho \) density
- \( \sigma \) electrical conductivity
- \( \phi \) entropy function

Subscripts
- \( r \) denotes the component in that direction
- \( \theta \) denotes the component in that direction
- \( 0 \) denotes evaluation at the body
- \( 1 \) denotes evaluation at the shock
- \( \ast \) denotes evaluation at the sonic point on the body


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