THE ANALYTIC PROPERTIES
OF THE
TRIANGLE AND BOX DIAGRAM AMPLITUDES

by

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To my Parents

and Lorraine

PREFACE

Apart from the first two introductory chapters, and except where acknowledged by appropriate references, this thesis describes original work carried out under the supervision of Dr W.S. Woolcock. The work presented in chapters 3 and 4 is based on the following papers co-authored with Dr Woolcock:

"The Analytic Properties of the Scalar Vertex Function",

Nuclear Physics (1971), B28, 605.

"The Analytic Properties of the Box Diagram Amplitude I",

submitted to *Annals of Physics*.

The work in chapter 5 has been carried out by the candidate alone.

No part of this thesis has been presented for a degree at any other university.

J. J. Frederiksen

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ABSTRACT

An investigation is made of the analytic properties of two Feynman amplitudes, the triangle diagram amplitude and the box diagram amplitude, by using two different methods. After the first two introductory chapters, the scalar triangle diagram amplitude is studied in chapter 3. The Feynman parametrisation of this amplitude is transformed directly into a single spectral representation in the Mandelstam variable t . In this way, the different forms of the spectral representation, the weight functions and the thresholds are obtained directly for all possible mass configurations involving stable external particles. Then, by starting with a particular normal threshold spectral representation obtained by the method of direct transformation (or by a heuristic method which is also discussed in this chapter), the different forms of the spectral representation, including cases for which the threshold is anomalous, are obtained by continuing in the external masses squared.

In chapter 4, the method of direct transformation is extended to apply to the scalar box diagram amplitude. In this way, a double spectral representation in the Mandelstam variables s and t is established, and necessary and sufficient conditions for its validity are found. Further, by the same method, a number of different spectral representations for the box diagram amplitude are obtained for cases when the double spectral representation is no longer valid. The method of analytic continuation is then used in chapter 5 to establish spectral representations for the box diagram amplitude for both real and complex s and t . In particular, for cases when the external particles are stable, spectral representations are obtained for all s in the upper half complex plane and for almost all $-\infty < t < +\infty$. These spectral representations are established by

starting with a particular normal threshold spectral representation, obtained in chapter 4 by the method of direct transformation (or by a heuristic method which is also discussed in chapter 4), and continuing in the external masses, and then in s and t .

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1. HISTORICAL INTRODUCTION

1-1. Field theory

In 1935 Yukawa (1935) suggested that the nuclear forces are due to the exchange of heavy quanta, just as the electromagnetic interaction is due to the exchange of photons; the short-range nature of the nuclear forces is then explained by the fact that the quanta are massive. This idea led to the development of the Lagrangian field theory of the strong interactions. In the simplest form of the theory the strong interactions are transmitted by the pion fields; more generally, a field theory can be set up which involves the fields of all the strongly interacting particles.

The application of Lagrangian field theory to the strong interactions met with two major obstacles. Firstly, there is no classical limit to the forces, so there is no clue to the form of the interaction. Secondly, the coupling, in contrast to the electromagnetic coupling, is so strong that perturbation calculations cannot be trusted to give even a rough indication of the correctness of a particular trial form of the interaction. Two of the simplest commonly used interaction densities for describing the pion-nucleon interaction are the pseudoscalar coupling, which is renormalizable (Dyson 1949a, b), and the pseudovector coupling, which is unrenormalizable; both are linear in the pseudoscalar pion field.

In the earliest comparisons of pion-nucleon theory with experiments, weak coupling or perturbation theory was used. It was found that, in the Born approximation, the pseudovector coupling gives reasonable agreement with experiment for the total cross section for low energy scattering of pions by protons, while the pseudoscalar theory disagrees with experiment. However both theories predict very large s -wave scattering of pions whereas the experiments showed strong p -wave

scattering and weak s -wave scattering. (See, for example, Marshak (1952) or Bethe and de Hoffmann (1955) where detailed reviews of the early theories and experiments can be found.)

In 1952 Drell and Henley (1952) proposed a solution to the s -wave problem. They used a transformation discovered by Dyson (1948) to convert the linear renormalizable pseudoscalar coupling into a linear pseudovector coupling plus a large number of correction terms. The most important of these, they showed, has the same effect as a strong repulsive central potential between the pion and the nucleon. The Born approximation for this short range repulsive interaction gives the large s -wave scattering predicted by weak coupling theory. However, by also including some higher order effects they were able to obtain reasonable agreement with experimental data for low energy s -wave scattering. To obtain better agreement with experimental results, and, in particular, to obtain accurate s -wave scattering lengths, it is however, necessary to write a dispersion relation for the s -wave amplitudes. (See, for example, Hamilton and Woolcock (1963).)

Chew (1953, 1954a, b) attacked the other part of the problem, namely p -wave pion-nucleon scattering. He used a crude static model in which the nucleon recoil was neglected and for the interaction Hamiltonian used a pseudovector coupling with a finite energy cut off and a small momentum cut off. Then, by varying two arbitrary parameters, he was able to correlate the theory with the data for a number of low energy experiments, including p -wave pion-nucleon scattering.

A major step in the understanding of p -wave scattering came with the application of an equation for the transition amplitude to the cut off pseudovector interaction between a pion and a static nucleon.

This equation for the transition amplitude was derived independently by Low (1955) and Lehmann, Symanzik and Zimmermann (1955) and the resultant theory is generally known as Chew-Low theory (Chew and Low (1956)). Using this equation, Chew and Low were able to derive dispersion relations and obtain solutions for the p -wave amplitude for low energy pion-nucleon scattering. The solutions chosen were not unique, as pointed out by Castillejo, Dalitz and Dyson (1956), but they did lead to results which were in good agreement with experiment. In particular, for an appropriate energy cut off, the solutions lead to the observed $N^*(3, 3)$ resonance which dominates low energy pion-nucleon scattering.

The Chew-Low theory only applies to low energy p -wave scattering and the dispersion relations derived are only approximate. It is, however, possible to deduce exact dispersion relations and improve on the results of Chew-Low theory. Heuristic derivations of exact dispersion relations were given by a number of authors, for example, Salam (1956) and Capps and Takeda (1956), and these dispersion relations were first used to improve the static equations of Chew-Low theory by Chew, Goldberger, Low and Nambu (1957a). Rigorous proofs were subsequently given by Bogoliubov, Medvedev and Polivanov (1959), Bremermann, Oehme and Taylor (1958) and Lehmann (1958). These proofs were based not on Lagrangian field theory but on the Lehmann-Symanzik-Zimmermann formulation of field theory (Lehmann, Symanzik and Zimmermann (1955)).

Lehmann, Symanzik and Zimmermann found Lagrangian field theory unsatisfactory from a theoretical as well as a practical point of view. Besides the problem of choosing an appropriate Lagrangian density for describing a particular interaction, there is the problem

of renormalization which is encountered when one attempts to use the theory. When the S -matrix for the interaction is expanded in a perturbation series in the coupling constant, divergent terms appear which must be removed by a prescription called renormalization (Dyson (1949a, b)). Although this prescription gives finite results for the so-called renormalizable theories, divergent renormalization constants appear both in the field equations and in the commutation relations of the renormalized operators. Finally, for strong interactions, such as the pion-nucleon interaction, perturbation theory cannot be trusted since the coupling constant squared is greater than unity in the renormalizable theories.

In their new formulation of field theory, Lehmann, Symanzik and Zimmermann used only those properties of quantum field theory which they believed to be model independent, and completely avoided using unrenormalized fields and renormalization constants. They assumed, for example, that the field of spin zero particles of mass m can be described by a linear Hermitian operator $\phi(x)$ which operates in a Hilbert space of state vectors, and further limited the properties of $\phi(x)$ by imposing the following requirements.

1. *Invariance principle.* The theory should be invariant under arbitrary Poincaré transformations.
2. *Microcausality or local commutativity.* The space-like commutators of field operators should vanish:

$$[\phi(x), \phi(y)] = 0$$

when $(x-y)$ is a space-like vector.

3. *Asymptotic condition.* The field operator $\phi(x)$ should, in the limits as the time $x_0 \rightarrow -\infty$ and $x_0 \rightarrow +\infty$, be equivalent to the free field operators $\phi_{in}(x)$ and $\phi_{out}(x)$ respectively.

The limit is taken in the sense of weak convergence.

These requirements can also be generalized to apply to particles other than spin zero particles.

By combining the basic principles of the Lehmann-Symanzik-Zimmermann formulation of field theory with

- (a) *the assumption of the existence of a complete set of eigenstates of the energy-momentum operator corresponding to non-negative values of the energy,*
 - (b) *assumptions about the existence of certain Fourier transforms,*
- and
- (c) *assumptions about the properties of the spectrum of intermediate states for the system in question,*

it is then possible, for some processes, to deduce certain analytic properties of the transition amplitude. From these analyticity properties, dispersion relations connecting measurable quantities can then be obtained (Bremermann, Oehme and Taylor (1958)).

1-2. Dispersion relations

Although there were earlier related works, the real starting point of the theory of dispersion relations came with the papers of Kronig (1926) and Kramers (1927; see also Kramers (1956)). Both were concerned with the classical dispersion of light, and obtained an integral equation, called a dispersion relation, connecting the real and imaginary parts of the refractive index:

$$\operatorname{Re}(n(\omega)-1) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im}n(\omega')d\omega'}{\omega'-\omega},$$

where P denotes Cauchy principal value. Kramers showed that the dispersion relation follows from the requirement that the refractive index, considered as a function of a complex frequency ω , be an

analytic function in the upper half complex ω -plane and approach unity at infinity. He then showed that in a medium described by such a refractive index, electromagnetic signals cannot propagate faster than the velocity of light, and that the dispersion relation is in fact based on the requirement of causality.

In 1946 Kronig (1946) suggested that the causality requirement might be imposed on the S -matrix for elementary particle processes to limit its form. Subsequently dispersion relations were derived for non-relativistic quantum mechanical scattering processes and later, the Kramers-Kronig relation was applied to problems in quantum electrodynamics such as forward scattering of light by the nuclear Coulomb field.

The first attempt at a complete formulation of the causality condition and a derivation of the Kramers-Kronig relation within the framework of quantum field theory was made by Gell-Mann, Goldberger and Thirring (1954). The field theoretical formulation of microscopic causality is that the commutator (resp. anticommutator) of two boson (resp. fermion) Heisenberg field operators at two different space-time points must vanish if the separation of the points is space-like. The proof of the dispersion relation given by Gell-Mann, Goldberger and Thirring for the forward scattering of light was based on the use of perturbation theory, but this deficiency was soon remedied by Goldberger (1955a).

In the following paper, Goldberger (1955b) attacked the problem of obtaining dispersion relations for the forward scattering of massive particles. Using the microcausality condition together with the equation for the scattering amplitude derived independently by Low (1955) and Lehmann, Symanzik and Zimmermann (1955), he obtained

the correct form of the dispersion relations without giving a rigorous proof. Dispersion relations for the forward scattering of pions by nucleons were then obtained in a heuristic way by Goldberger, Miyazawa and Oehme (1955) and the first comparison with experiment was made by Anderson, Davidson and Kruse (1955).

The next step was to generalize the dispersion relations to apply to non-forward scattering. Heuristic derivations were given independently by a number of authors, for example Salam (1956) and Capps and Takeda (1956). The integration in these dispersion relations is over the energy variable with the Lorentz invariant momentum transfer held constant. These fixed momentum transfer dispersion relations were first applied by Chew, Goldberger, Low and Nambu (1957a). They used them to improve the static equations of Chew-Low theory by including first order nucleon recoil effects and calculating the s and d wave phase shifts as well as the p -wave phase shifts. Chew, Goldberger, Low and Nambu (1957b) also wrote down similar relations for the photo production of pions from nucleons and discussed the photo production process for energies up to and including the $N^*(3, 3)$ resonance. Since then there have been numerous applications of dispersion relations for both forward and non-forward scattering. (See, for example, Sreaton (1961), Hamilton and Woolcock (1963), Shirkov, Serebryakov and Meshcheryakov (1969) and Burkhardt (1969)).

The first rigorous proof of dispersion relations for forward scattering of massive particles was given by Symanzik (1957), and, as mentioned in sect. 1-1, general proofs for non-forward scattering were given by Bogoliubov, Medvedev and Polivanov (1959), Bremermann, Oehme and Taylor (1958) and Lehmann (1958). These proofs showed that fixed momentum transfer dispersion relations were valid, provided that

the momentum transfer satisfied certain inequalities and that the masses of the interacting particles also satisfied some inequalities. For pion-nucleon scattering the fixed momentum transfer dispersion relations were proved for physical pion and nucleon masses and for

$$0 \leq -t \leq \frac{32}{3} \cdot \frac{2M+\mu}{2M-\mu} \cdot \mu^2 \approx 12\mu^2.$$

Here M is the nucleon mass, μ the pion mass and

$$t = (p_N - p'_N)^2 = (p_\pi - p'_\pi)^2,$$

where p_N and p'_N are the four-momenta of the incident and outgoing nucleon, p_π and p'_π the four-momenta of the incident and outgoing pion. For nucleon-nucleon scattering and the nucleon electromagnetic form factor or vertex function, both of which involve coupling to the pion field, dispersion relations could only be proved if

$$\mu > (\sqrt{2}-1)M.$$

Unfortunately, the observed masses do not satisfy this inequality and hence it was not known whether dispersion relations were valid for these processes.

Since rigorous proofs of dispersion relations for nucleon-nucleon scattering and for electromagnetic form factors could not be given (in fact no proofs have yet been found (Goldberger, 1970)) many authors began investigating the analytic properties of the Feynman integrals which occur in the perturbation expansion of the scattering matrix (Feynman (1949a, b), Dyson (1949a, b)). Feynman had previously developed a graphical representation of the perturbation series in connection with quantum electrodynamics and had given a set of rules for writing down the integral associated with a particular Feynman diagram. Similar sets of rules had also been developed for the interactions of hadrons and it was hoped that the analytic properties

of the individual Feynman integrals associated with a particular amplitude would indicate the analytic properties of that amplitude itself. This was despite the fact that the magnitude of the individual Feynman integrals was not believed to be significant in the case of the strong interactions.

In chapter 2, we discuss some of the methods that have been used to obtain dispersion relations for Feynman integrals, and to obtain the thresholds and weight functions in these dispersion relations. We also discuss a method of obtaining the weight functions in approximate dispersion relations for scattering amplitudes and vertex functions. In this method, a dispersion relation is obtained, for a particular amplitude, by making assumptions about the analytic properties and behaviour of the amplitude at infinity. The weight function is then obtained by using the unitarity relation (discussed in sects. 2-3, 3-2 and 4-2). Approximations to the amplitudes on the right hand side of the unitarity relation are made, and in this way an approximation to the imaginary part of the amplitude (weight function) is obtained. In the following chapters, we shall see that, for the scalar vertex function and the two-particle $\rightarrow$ two-particle scattering amplitude, this method can, under certain circumstances, yield the same dispersion relations, with the same weight functions, as those obtained from the triangle diagram Feynman amplitude and the box diagram Feynman amplitude respectively. It has, in fact, been argued by a number of authors (for example, Stapp (1962), Polkinghorne (1962) and Taylor (1964)) that, in a theory of scalar particles, the set of singularities that one obtains for a given amplitude by iterating the unitarity relation in a particular way, includes the set of singularities obtained in the perturbation expansion of the amplitude in a simple field theory with a $g\phi^3$ interaction.

2. DISPERSION RELATIONS FROM PERTURBATION THEORY AND THE UNITARITY RELATION

2-1. Introduction

In this chapter, we review some of the investigations of analyticity properties of Feynman amplitudes. We also discuss a method of using the unitarity relation together with certain approximations to obtain the weight functions in approximate dispersion relations for scattering amplitudes and vertex functions. For most of this chapter, and for all of chapters 3, 4 and 5, we neglect spin effects and consider only amplitudes involving scalar particles. In addition, the external particles, though not necessarily the internal particles, are assumed to be stable.

Investigations of Feynman amplitudes involving scalar particles are, of course, simpler than the corresponding investigations involving particles with spin, and hence most investigations of the analytic properties of Feynman amplitudes have involved scalar particles. Most elementary particles are, in fact, not scalar, and therefore it might be thought that such investigations would not be very useful for gaining an insight into the analytic properties of transition amplitudes involving elementary particles. However, simple local field theories of scalar particles, such as the ϕ^3 -type theories, have been very useful models for discovering new effects (such as anomalous thresholds) and for postulating analytic properties of transition amplitudes. The complications introduced by including spin effects occur, for simple local field theories, in the *numerator* of the Feynman integral. Hence, it is expected that a particular Feynman integral involving scalar particles will have some singularities in common with the corresponding Feynman integral involving particles with spin. It should be noted, however, that the

non-Landau or second-type singularities (Cutkosky (1960 a, b), Fairlie, Landshoff, Nuttall and Polkinghorne (1962)) depend on the spin of the particles (Nakanishi (1971), sect. 16-1).

We begin in sect. 2-2 by reviewing the first investigations of the analyticity properties of Feynman integrals, in particular, the Feynman triangle and box diagram amplitudes. In this section, we also discuss Mandelstam's method of approximating the amplitudes on the right hand side of the unitarity relation, obtaining a normal threshold dispersion relation for a certain range of masses and then continuing in the external masses, to obtain a dispersion relation for a wider range of masses. In sect. 2-3, the Mandelstam double spectral representation and investigations of the analytic properties of the box diagram amplitude are discussed. Then, in sect. 2-4, a discussion of the derivation of the Landau-Cutkosky rules and their application to investigations of analyticity properties of Feynman integrals is given. Finally, in sect. 2-5 we conclude this chapter with an account of the work to be carried out in chapters 3, 4 and 5.

2-2. Single dispersion relations from perturbation theory and the unitarity relation

The first attempts at establishing dispersion relations for the triangle diagram vertex function (or amplitude) associated with the Feynman diagram shown in fig. 3-1, were those of Karplus, Sommerfield and Wichmann (1958) and Nambu (1958). Karplus, Sommerfield and Wichmann considered the general mass case when the six particles shown in fig. 3-1 were all different, but restricted their investigation to scalar self-conjugate particles interacting through a $g\phi^3$ -type interaction Hamiltonian density. The resultant theory is renormalizable (Dyson (1949 b)). Further they considered only external particles

which satisfied the *stability criterion*, that the rest mass of all intermediate states to which a single particle is coupled must exceed the rest-mass of that particle. They wrote down the Feynman parametrisation of the triangle diagram vertex function $F_{\Delta}(t)$ and then argued that it can be written as a dispersion relation (or spectral representation) in the momentum transfer:

$$F_{\Delta}(t) = \frac{1}{\pi} \int_{v_{1+}}^{\infty} \frac{\text{Im} F_{\Delta}(v+i\cdot 0) dv}{v-t}.$$

Here $t = (p_A - p_B)^2$ and p_A, p_B are the four-momenta of the particles A, B in fig. 3-1. They did not obtain the explicit form of the spectral function $\text{Im} F_{\Delta}(v+i\cdot 0)$ but they discovered that the threshold v_{1+} is not always the *normal threshold*, $(E+F)^2$. When the inequality

$$F(E^2 + G^2 - A^2) + E(F^2 + G^2 - B^2) < 0$$

is satisfied, v_{1+} becomes an *anomalous threshold* whose value is less than $(E+F)^2$. Here A, B, E, F and G refer to the masses of the particles in fig. 3-1. Normal thresholds, which had been discussed earlier by Eden (1952), refer to the threshold energies at which new physical processes become possible. Anomalous thresholds are additional singularities which do not have a simple physical interpretation. In fact, Karplus, Sommerfield and Wichmann showed that the anomalous threshold occurs for the triangle diagram amplitude when the intermediate particles are on the mass shell, and then energy and momentum can be conserved only for reactions between particles with complex momenta. Nevertheless, as we shall see later, there have been attempts to explain observable effects in terms of anomalous singularities.

In his investigation of the nucleon electromagnetic vertex function, Nambu included spin effects. He decomposed the nucleon electromagnetic form factor into two scalar functions multiplied by irreducible covariants and then, from perturbation theory, wrote dispersion relations for each of the scalar functions, without giving the explicit form of the spectral function. He also studied the hyperon electromagnetic form factor in a simplified model involving hyperons, nucleons and K -mesons and, by examining the triangle diagram contribution, showed that under certain circumstances anomalous thresholds occur.

Symanzik (1958) used a scalar model for the nucleon electromagnetic vertex function and argued, without considering the individual diagrams in detail, that dispersion relations can be written for any Feynman integral which is obtained in the perturbation expansion of the nucleon electromagnetic form factor. A similar heuristic argument had been used earlier by Nambu (1957) to write down dispersion relations for a vertex function in quantum electrodynamics.

At about the same time Källén and Wightman (1958) obtained the domain of analyticity of the scalar vertex function by using the general requirements of Lorentz invariance, a reasonable mass spectrum shape and local commutativity. They also explicitly integrated a certain derivative of the general mass case triangle diagram amplitude and argued that its singularities are the same as those of the triangle diagram amplitude. However the nature of the second-type singularity (Cutkosky (1960 a, b)) is different for the two functions, being a pole for the derivative with respect to the external masses squared considered by Källén and Wightman, and a square root singularity for the triangle diagram amplitude (Oehme (1959), Fairlie,

Landshoff, Nuttall and Polkinghorne (1962)).

In a series of papers, Oehme (1958, 1959, 1960) used the results obtained by Källén and Wightman to write down a dispersion relation in t for the derivative of the triangle diagram vertex function, for a particular case when some of the masses of the particles were equal. He then gave a heuristic derivation of a dispersion relation for the particular equal mass case triangle diagram amplitude and presented an intuitive non-relativistic interpretation of anomalous thresholds. In particular, he examined the effect of the triangle diagram anomalous threshold on a static 'charge' distribution associated with the deuteron electromagnetic vertex function, in a model involving only scalar particles. He took a three dimensional Fourier transform of the triangle diagram vertex function and showed that the resultant static 'charge' distribution is a superposition of Yukawa clouds, and that the longest range component is determined by the threshold of the triangle diagram vertex function. He then argued that, of the triangle diagrams which contribute to the deuteron electromagnetic vertex function, the one with the lowest normal threshold, namely $4\mu^2$, has A and B (in fig. 3-1) each being a deuteron, E and F each a pion, G two nucleons and T a photon. The range corresponding to this normal threshold is approximately $(2\mu)^{-1}$. However, if instead one considers the triangle diagram with A and B each being a deuteron, E , F and G each a nucleon and T a photon, then the resultant triangle diagram amplitude has an anomalous threshold approximately equal to $16M\varepsilon$. Here μ is the pion mass, M the nucleon mass and ε the binding energy of the deuteron. The range corresponding to this anomalous threshold is approximately $(16M\varepsilon)^{-\frac{1}{2}}$.

which is larger than $(2\mu)^{-1}$. If one considers the Schrödinger wave function for a bound state of a proton and a neutron with a small binding energy ϵ , then one finds that the range of the charge distribution is also approximately $(16M\epsilon)^{-\frac{1}{2}}$. Thus, in this rough model, the range obtained by considering only normal thresholds is too small, while that obtained by including an anomalous threshold agrees with Schrödinger theory.

Mandelstam (1960) examined the problem of obtaining approximate dispersion relations for the process $\pi + \pi \rightarrow N + \bar{N}$ and for the nucleon electromagnetic form factor. He treated all particles as being scalar, and by making a single nucleon exchange approximation to one of the amplitudes on the right hand side of the unitarity relation (discussed in sects. 2-3, 3-2 and 4-2) and assuming that the threshold is a normal threshold, he obtained normal threshold dispersion relations in t , the momentum transfer. Then, he supposed that a baryon of mass B , instead of a nucleon, was exchanged and obtained a modified spectral function for the case when $M^2 < B^2 + \mu^2$, where M is the nucleon mass and μ the pion mass. To obtain the form of the dispersion relation when $M^2 > B^2 + \mu^2$, he examined the path taken in the complex plane by the singularity of the spectral function as the masses were changed. In the continuation procedure, he gave M^2 a small positive imaginary part and found that the path of integration used to define the dispersion relation had to be deformed when $M^2 > B^2 + \mu^2$. Further, he found that the threshold for the dispersion relation when $M^2 > B^2 + \mu^2$ was an anomalous threshold, whose value was less than the normal threshold value, $4\mu^2$, and that the spectral function in the anomalous region was different from the one above

$4\mu^2$. Mandelstam's method of approximating the amplitudes on the right hand side of the unitarity relation, obtaining a normal threshold dispersion relation for a certain range of masses and then continuing in the external masses to obtain a dispersion relation for a wider range of masses, will be discussed in detail in chapters 3, 4 and 5.

Blankenbecler and Nambu (1960) studied the deuteron electromagnetic form factor in a simplified model involving scalar spinless deuterons, nucleons and another baryon. Firstly, they discussed the form factor from the point of view of the Schrödinger equation with the aim of giving a physical interpretation of the anomalous thresholds which occur for the triangle diagram amplitude. Then, from perturbation theory, they wrote down the dispersion relation for their particular equal mass case triangle diagram amplitude and gave an interpretation of anomalous thresholds similar to that given earlier by Oehme (1959). Finally, they obtained the same dispersion relation for their particular equal mass case by using Mandelstam's method of approximating the amplitudes on the right hand side of the unitarity relation, obtaining a normal threshold dispersion relation for a certain range of masses and then continuing in the external masses to obtain a dispersion relation for a wider range of masses. Blankenbecler and Cook (1960) also discussed the deuteron electromagnetic form factor from the point of view of dispersion theory, using Mandelstam's method, and compared their results with Schrödinger theory.

Mandelstam's method was also used by Nishijima (1962) and Gross (1964 a) for studying the deuteron electromagnetic form factor in a model consisting of scalar particles, by Liu (1962) for studying a scalar vertex function in a theory containing three different particles

and by Simonov (1962, 1963) for studying a scalar vertex function in a theory where the internal particles were the same. Nishijima also obtained the dispersion relation for his particular equal mass case triangle diagram amplitude by using the Cutkosky rules (discussed in sect. 2-4). Fronsda1 and Norton (1964) also used Mandelstam's method to obtain the spectral representation for the general mass case triangle diagram amplitude, without giving the details of how the method must be modified for this case. In a further paper Gross (1964 b) included spin effects in his study of the deuteron electro-magnetic form factor.

The first attempts at establishing dispersion relations for the box diagram amplitude associated with the Feynman diagram shown in fig. 4-1, were those of Grisaru (1958) and Karplus, Sommerfield and Wichmann (1959). Grisaru examined the nucleon-nucleon scattering amplitude in fourth order perturbation theory and gave a heuristic derivation of single dispersion relations for small momentum transfers. He did not, however, obtain the thresholds nor the form of the spectral functions. Karplus, Sommerfield and Wichmann investigated the analytic properties of the scalar box diagram amplitude for the general mass case, when all the particles shown in fig. 4-1 were different. They wrote down the Feynman parametrisation of the box diagram amplitude and argued that it can be written as a single dispersion relation in the momentum transfer variable t . They did not obtain the explicit form of the spectral function, but showed that under certain conditions anomalous thresholds occur for the box diagram amplitude, as well as for the triangle diagram amplitude. By examining the integrand of the Feynman integral, they showed that the box diagram amplitude is an analytic function of both the energy and the momentum transfer for a

restricted range of the energy and momentum transfer. Wu (1961) also studied the scalar box diagram amplitude and showed that it can be explicitly integrated to give a sum of 192 Spence functions.

Most of the other investigations of analyticity properties that we shall review, involve the Mandelstam representation or the Landau-Cutkosky rules. We therefore treat these separately in the next two sections.

2-3. The Mandelstam representation

In 1958 Mandelstam (1958) considered the problem of finding a relativistic analog of Chew-Low theory for pion-nucleon scattering. To do this, he needed the analytic properties of the scattering amplitude as a function of both the energy and the momentum transfer. Before writing down the Mandelstam representation for the invariant scattering amplitude of two-particle $\rightarrow$ two-particle collision processes, we introduce the Mandelstam variables s , t and u and briefly discuss the unitarity relation and the substitution law or crossing relation. We shall need some of the results given in this section in the later chapters.

We start by considering the scattering process involving stable scalar particles:

$$1: \quad AB \rightarrow CD .$$

This process can be represented diagrammatically as in fig. 4-1, but with the internal lines unspecified. (It is also customary to replace the square by a circle when the total scattering amplitude rather than the Feynman diagram is considered.) The probability that the process $AB \rightarrow CD$ will take place is given by the square of the modulus of the matrix element $\langle p_C p_D | S | p_A p_B \rangle$ where the plane wave states are

normalized so that $\langle p|p' \rangle = \delta^{(3)}(p-p')$, and S is the scattering operator defined, for example, in sect. 7-2 of Roman (1969). The law of conservation of probability is expressed by the unitarity relation for the S -matrix:

$$S^\dagger S = S S^\dagger = 1. \quad (2-1)$$

The S -matrix element (or invariant amplitude) for the process $AB \rightarrow CD$ depends only on two parameters other than the masses of the particles (see, for example, Roman (1969), appendix 2), for example, the total energy and a suitably chosen scattering angle. Sometimes, however, it is convenient to regard it as a function of the three invariants

$$\begin{aligned} s_1 \equiv s &= (p_A + p_B)^2 = (p_C + p_D)^2, \\ t_1 \equiv t &= (p_A - p_C)^2 = (p_B - p_D)^2, \\ u_1 \equiv u &= (p_A - p_D)^2 = (p_B - p_C)^2. \end{aligned} \quad (2-2)$$

These invariants are not independent; in fact, since all the particles are on the mass shell, it follows from energy-momentum conservation that

$$s + t + u = A^2 + B^2 + C^2 + D^2, \quad (2-3)$$

where A, B, C, D are the masses of the particles. As mentioned above, the process may also be characterised by its total energy W_1 and scattering angle θ_1 in the centre-of-momentum system (CMS). If n_1, n'_1 are unit vectors in the directions of p_A, p_C respectively in the CMS, then θ_1 is the angle between n and n' which satisfies $0 \leq \theta_1 \leq \pi$ and $\cos \theta_1 = n_1 \cdot n'_1$. Also, if q_1 is the magnitude of the three-momentum of either A or B and q'_1 is the magnitude of

the three-momentum of either C or D in the CMS, then it can be shown that

$$\begin{aligned}
 W_1 &= \sqrt{A^2 + q_1^2} + \sqrt{B^2 + q_1^2} = \sqrt{C^2 + q_1'^2} + \sqrt{D^2 + q_1'^2}, \\
 s_1 &= W_1^2, \\
 t_1 &= A^2 + C^2 - 2\sqrt{A^2 + q_1^2}\sqrt{C^2 + q_1'^2} + 2q_1 q_1' \cos\theta_1 \\
 &= \frac{1}{2}(A^2 + B^2 + C^2 + D^2) - \frac{1}{2}s_1 - \frac{1}{2}(A^2 - B^2)(C^2 - D^2)s_1^{-1} + 2q_1 q_1' \cos\theta_1, \quad (2-4) \\
 u_1 &= A^2 + D^2 - 2\sqrt{A^2 + q_1^2}\sqrt{D^2 + q_1'^2} - 2q_1 q_1' \cos\theta_1 \\
 &= \frac{1}{2}(A^2 + B^2 + C^2 + D^2) - \frac{1}{2}s_1 + \frac{1}{2}(A^2 - B^2)(C^2 - D^2)s_1^{-1} - 2q_1 q_1' \cos\theta_1, \\
 q_1 &= \frac{[s_1 - (A+B)^2]^{\frac{1}{2}}[s_1 - (A-B)^2]^{\frac{1}{2}}}{2s_1^{\frac{1}{2}}} \equiv q_{AB}(s_1)
 \end{aligned}$$

and

$$q_1' = \frac{[s_1 - (C+D)^2]^{\frac{1}{2}}[s_1 - (C-D)^2]^{\frac{1}{2}}}{2s_1^{\frac{1}{2}}} \equiv q_{CD}(s_1).$$

The physical process $AB \rightarrow CD$ corresponds to

$$s_1 \geq \max\{(A+B)^2, (C+D)^2\}, \quad |\cos\theta_1| \leq 1. \quad (\text{See, for example, Källén (1964), Rasche and Woolcock (1970).})$$

We now define the Lorentz invariant scattering amplitude

$T_1(s_1, t_1)$ in terms of the S -operator by

$$\langle p_C p_D | S - 1 | p_A p_B \rangle = -i(2\pi)^{-2} \delta^{(4)}(p_C + p_D - p_A - p_B) \left(E_A E_B E_C E_D \right)^{-\frac{1}{2}} T_1(s_1, t_1), \quad (2-5)$$

where E_A, E_B, E_C, E_D are the energies of the particles A, B, C, D respectively. The unitarity relation in eq. (2-1) can also be expressed in terms of $T_1(s_1, t_1)$. In particular, suppose that for

the process $AB \rightarrow CD$, there is a two-particle intermediate state (EF) with the same internal quantum numbers as (AB) and (CD) , and that $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$. Then, dropping the subscript 1, the contribution of this state to $\text{Im}T_{AB \rightarrow CD}(s, t(s, n \cdot n'))$ in the unitarity relation is

$$\begin{aligned} & \text{Im}T_{AB \rightarrow CD}^{(EF)}(s, t(s, n \cdot n')) \\ &= \frac{-q_{EF}(s)}{32\pi^2 s^{\frac{1}{2}}} \iint d\bar{n} T_{CD \rightarrow EF}^*(s, t(s, n' \cdot \bar{n})) T_{AB \rightarrow EF}(s, t(s, n \cdot \bar{n})) \quad (2-6) \end{aligned}$$

for $s \geq (E+F)^2$. (See, for example, Kibble (1960), Rasche and Woolcock (1970).) Here $q_{EF}(s)$ is defined in a similar way to $q_{AB}(s)$ and $q_{CD}(s)$, $\bar{n}$ is an arbitrary unit vector and the integration is over the surface of the unit sphere. Time reversal invariance for the strong interactions means that the reverse process $CD \rightarrow AB$ must also have the same invariant scattering amplitude in the same physical region; similarly symmetry under particle-antiparticle conjugation for the strong interactions means that the processes involving the corresponding antiparticles $\overline{AB} \rightarrow \overline{CD}$ and $\overline{CD} \rightarrow \overline{AB}$ are also described by the same invariant scattering amplitude in the same physical region.

Besides regarding fig. 4-1 (with the internal lines unspecified) as representing process 1: $AB \rightarrow CD$, it can also be regarded as representing other processes related to process 1. Process 1 is usually called the s -channel reaction since s is the square of the CMS energy for this reaction. Similarly, reactions for which t and u are the square of the CMS energy are called the t and u -channel reactions respectively. We take the t -channel reaction to be

2:

$$A\bar{C} \rightarrow \bar{B}D$$

and for this process the corresponding Mandelstam variables are

$$s_2 = (p_A + p_{\bar{C}})^2 = (p_A - p_C)^2 = (p_B - p_D)^2 = t$$

$$t_2 = (p_A - p_{\bar{B}})^2 = (p_A + p_B)^2 = (p_C + p_D)^2 = s$$

$$u_2 = (p_A - p_D)^2 = (p_C - p_B)^2 = u.$$

Similarly the u -channel reaction is

3:

$$A\bar{D} \rightarrow C\bar{B}$$

and the Mandelstam variables are

$$s_3 = (p_A + p_{\bar{D}})^2 = (p_A - p_D)^2 = (p_C - p_B)^2 = u$$

$$t_3 = (p_A - p_C)^2 = (p_B - p_D)^2 = t$$

$$u_3 = (p_A - p_{\bar{B}})^2 = (p_A + p_B)^2 = (p_C + p_D)^2 = s.$$

We can define the invariant scattering amplitudes $T_2(s_2, t_2)$, $T_3(s_3, t_3)$ for processes 2 and 3 in a similar way to $T_1(s_1, t_1)$.

For reaction 2 to be a physical process, we must have

$$s_2 \geq \max\{(A+C)^2, (B+D)^2\}, \quad |\cos\theta_2| \leq 1$$

and for reaction 3 to be a physical process, we must have

$$s_3 \geq \max\{(A+D)^2, (C+B)^2\}, \quad |\cos\theta_3| \leq 1.$$

The physical regions for the three reactions (and their time reversed and charge conjugate reactions) can also be expressed in terms of the three invariants s , t and u , as mentioned earlier. To do this, it is convenient to use a trilinear system of coordinates in a plane, so that each of s , t and u is treated on an equal footing, and it is found that the three physical regions do not overlap (Kibble (1960)).

The substitution law, or crossing relation, states that a single

analytic function $F(\zeta_1, \zeta_2, \zeta_3)$ of the three complex variables

$\zeta_1, \zeta_2, \zeta_3$ describes all three reactions 1, 2 and 3, and that the physical invariant amplitude for a particular channel can be obtained by taking an appropriate boundary value of $F(\zeta_1, \zeta_2, \zeta_3)$.

Mandelstam (1958) postulated a particular integral form for

$F(\zeta_1, \zeta_2, \zeta_3)$. The Mandelstam hypothesis states that $F(\zeta_1, \zeta_2, \zeta_3)$ is defined and regular except when any one of the ζ_i is real and satisfies $\Sigma_i \leq \zeta_i < \infty$ and except for isolated simple poles.

Further, in its unsubtracted form, $F(\zeta_1, \zeta_2, \zeta_3)$ has the following representation:

$$\begin{aligned}
 F(\zeta_1, \zeta_2, \zeta_3) = & \text{Pole terms} \\
 & + \int_{\Sigma_1}^{\infty} \int_{\Sigma_2}^{\infty} \frac{\rho_{12}(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2}{(\sigma_1 - \zeta_1)(\sigma_2 - \zeta_2)} \\
 & + \int_{\Sigma_1}^{\infty} \int_{\Sigma_3}^{\infty} \frac{\rho_{13}(\sigma_1, \sigma_3) d\sigma_1 d\sigma_3}{(\sigma_1 - \zeta_1)(\sigma_3 - \zeta_3)} \\
 & + \int_{\Sigma_2}^{\infty} \int_{\Sigma_3}^{\infty} \frac{\rho_{23}(\sigma_2, \sigma_3) d\sigma_2 d\sigma_3}{(\sigma_2 - \zeta_2)(\sigma_3 - \zeta_3)}, \quad (2-7)
 \end{aligned}$$

where each of the pole terms is of the form

$$\frac{g_i g_i'}{\zeta_i - M_{ij}^2}.$$

Here the real numbers g_i, g_i' are rationalized renormalized coupling constants and the real numbers M_{ij} are the masses of single particles with the same internal quantum numbers as the initial and final states of process i .

The $\rho_{ij}(\sigma_i, \sigma_j)$ are the real-valued double spectral functions, and the real numbers Σ_i are the squares of the total masses of the two-particle states, of lowest possible total mass, with the same internal quantum numbers as the initial and final states of process i . The region in which the double spectral function $\rho_{ij}(\sigma_i, \sigma_j)$ is non-zero is not rectangular; unitarity restricts it to a smaller region bounded by curves asymptotic to Σ_i and Σ_j . The unsubtracted form of the Mandelstam representation in eq. (2-7) was postulated on the assumption that the integrals converge. If they do not, then a subtracted form is needed (Mandelstam (1958, 1959 a), Rasche and Woolcock (1970)).

The substitution law now states that the physical invariant amplitudes $T_i(s_i, t_i(s_i, \cos\theta_i))$ are the boundary values of $F(\zeta_1, \zeta_2, \zeta_3)$ according to the following prescription:

$$\begin{aligned} T_1(s_1, t_1(s_1, \cos\theta_1)) &= \lim_{\substack{z \rightarrow 0 \\ \text{Im}z > 0}} F(s_1+z, t_1(s_1, \cos\theta_1), u_1(s_1, \cos\theta_1)) , \\ T_2(s_2, t_2(s_2, \cos\theta_2)) &= \lim_{\substack{z \rightarrow 0 \\ \text{Im}z > 0}} F(t_2(s_2, \cos\theta_2), s_2+z, u_2(s_2, \cos\theta_2)) , \\ T_3(s_3, t_3(s_3, \cos\theta_3)) &= \lim_{\substack{z \rightarrow 0 \\ \text{Im}z > 0}} F(u_3(s_3, \cos\theta_3), t_3(s_3, \cos\theta_3), s_3+z) . \end{aligned}$$

For this prescription to be meaningful, restrictions on the values of $t_1(s_1, \cos\theta_1)$, $u_1(s_1, \cos\theta_1)$ etc., must be made as discussed by Rasche and Woolcock (1970).

The Mandelstam representation for the invariant scattering amplitude has been proved for a class of potentials in the case of non-relativistic potential scattering by Blankenbecler, Goldberger,

Khuri and Treiman (1960). However, no proof of the validity of the Mandelstam representation exists based on either the Lehmann-Symanzik-Zimmermann formulation of field theory or perturbation theory. In fact, in certain circumstances, even the lower order amplitudes in perturbation theory violate it. For the box diagram amplitude associated with the Feynman diagram shown in fig. 4-1, anomalous thresholds and complex singularities occur for sufficiently large external masses, and then it is not possible to write the amplitude as a double dispersion integral over a real domain. However, for sufficiently small external masses, the box diagram amplitude can be written as a double dispersion relation in s and t and, for an appropriate range of values of the external masses, a double dispersion integral can be written even in the presence of anomalous thresholds. (We discuss the various integral forms of the box diagram amplitude in detail in chapters 4 and 5.)

Mandelstam (1959 a) argued that, in the absence of anomalous thresholds, the box diagram amplitude $T_{\square}(s, t)$ can be written as a double dispersion integral over a real domain. He wrote down the Feynman integral for $T_{\square}(s, t)$ and then gave a prescription for calculating the imaginary part of $T_{\square}(s+i\cdot 0, t)$ when $s \geq (E+F)^2$; here E and F are the masses of the particles shown in fig. 4-1. Then, noting that $T_{\square}(s, t)$ can be written as a single normal threshold dispersion relation in s :

$$T(s, t) = \frac{1}{\pi} \int_{(E+F)^2}^{\infty} \frac{\text{Im}T(v+i\cdot 0, t)dv}{v-s} \quad (2-8)$$

he obtained a double dispersion relation in s and t for $T_{\square}(s, t)$,

and gave the explicit form of the double spectral function and the boundary curve. By examining the boundary curve of the double spectral function, he obtained a condition on the masses of the interacting particles, for which the double spectral representation could no longer be written.

In a following paper Mandelstam (1959 b) considered the problem of obtaining an approximate dispersion relation for $T_{AB \rightarrow CD}(s, t)$. He approximated $\text{Im}T_{AB \rightarrow CD}(s, t)$ by the right hand side of eq. (2-6) for $s \geq (E+F)^2$ and by zero for $s < (E+F)^2$, and then made pole approximations for each of the amplitudes on the right hand side of eq. (2-6). Then, by using eq. (2-8), he obtained an approximation to $T_{AB \rightarrow CD}(s, t)$, in the form of a double dispersion relation in s and t . This double dispersion relation was exactly the same as the one he had obtained for the Feynman box diagram amplitude. (We discuss this heuristic method in detail in chapter 4).

Tarski (1960) studied the five branches of the curve where the double spectral function of the box diagram amplitude is singular. In his analysis, he supposed that the internal, as well as the external, masses of the particles shown in fig. 4-1 were stable. By using Hadamard's lemma (discussed in sect. 2-4), he then obtained the singular and regular points of the box diagram amplitude. Fowler, Landshoff and Lardner (1960) studied the branches of the curve where the double spectral function is singular for cases when some of the internal particles were unstable.

The method of analytic continuation proposed earlier by Mandelstam (1960) was applied by Gribov, Terent'ev and Ter-Martirosyan (1961 a, b) to a particular equal mass case of the box diagram

amplitude. They started with the double dispersion relation obtained by Mandelstam (1959 a, b) and, by continuing in the external masses and t , obtained a new representation consisting of a double and a single integral part, for sufficiently large external masses. A number of equal mass cases were also investigated using the method of analytic continuation by Fronsda1, Mahanthappa and Norton (1962), Fronsda1, Norton and Mahanthappa (1963) and Simonov (1963 a, b); they wrote the box diagram amplitude as a double Cauchy integral over a complex domain for sufficiently large external masses. The general mass case for which all the masses of the particles in fig. 4-1 are different was investigated by Gribov, Damilov and Dyatlov (1961, 1962). They again started with the double spectral representation obtained by Mandelstam, continued in the external masses and t , and obtained a new representation consisting of a double and a single integral part, for sufficiently large external masses. (We discuss the work of Gribov, Danilov and Dyatlov on the box diagram amplitude in detail in chapter 5.)

Despite the fact that no proof exists for the validity of the Mandelstam representation in field theory, it has been widely used for analysing the interactions of hadrons. Detailed reviews of some of the applications of the Mandelstam representation are given in the books by Screation (1961), Shirkov, Serebryakov and Meshcheryakov (1969), Burkhardt (1969) and Martin and Spearman (1970).

2-4. The Landau-Cutkosky rules

Soon after the first investigations of the analytic properties of individual Feynman integrals had been made, a number of authors examined the problem of establishing the analytic properties of a general Feynman amplitude in simple renormalizable theories, including

the ϕ^3 , ϕ^4 and $\phi^3 + \phi^4$ type theories. Except for the simpler amplitudes, such as the pole amplitude, a general Feynman amplitude in the momentum space representation is a multiple integral both in the Feynman parametrised form and in the unparametrised form, which is obtained directly from the Feynman rules (Chrisholm (1952), Eden, Landshoff, Olive and Polkinghorne (1966)); we refer to this book as ELOP). One of the main tools in studying the analytic properties of Feynman integrals has been a lemma originally due to Hadamard (1898). This lemma was first used in studying Feynman integrals by Eden (1952) and extended by Polkinghorne and Screatton (1960 a, b). Hadamard's lemma gives *necessary* conditions for a function defined by an integral over a path in the complex plane to be singular. The form of the lemma given in appendix 7 of Barut (1967) is as follows:

"LEMMA. Consider the integral

$$g(w) = \int_C f(z, w) dz$$

over an arc C in the z -plane. Let D be a neighborhood of the arc C , and let G be a domain in the w -plane. Let further $f(z, w)$ be regular in both variables, except for a finite number of isolated singularities or branch points, for any value of the other variable, when $z \in D$ and $w \in G$; D and G may extend over many Riemann sheets of the function f . Under these conditions $g(w)$ can be singular at a point $w_0 \in G$ only if

- (1) $f(z, w_0)$ in z -plane has a singularity coinciding with the end points of the arc C (end point singularity),
- (2) two singularities of f , $z_1(w)$ and $z_2(w)$, approach the arc C from opposite sides and pinch the arc precisely at

$w = w_0$ (coincident or pinching singularities),

- (3) a singularity $z(w)$ tends to infinity as $w \rightarrow w_0$ deforming the contour with itself to infinity; one has to make a change of variables to bring the point ∞ to the finite plane to see what happens.

For a fixed contour C and $w \in G$, the integral defines $g(w)$ to be holomorphic in some domain. If one singularity approaches C we can modify the contour, thereby obtaining an analytic continuation of the function $g(w)$ in w -plane. Clearly this procedure of analytic continuation breaks down if one of the three situations in the lemma occurs. These are, however, necessary conditions, not sufficient. It may be that the integral is still regular in spite of these conditions."

Hadamard's lemma can also be generalized in a plausible but non-rigorous way, to apply to functions defined by multiple integrals (Polkinghorne and Screaton (1960 a, b), ELOP, sect. 2.1).

Landau (1959 a, b, 1960) proposed a set of equations for determining the surfaces of singularity of Feynman integrals in the complex space of the external momentum variables of the Feynman graphs. He also gave a geometrical method of solving these equations, which involves constructing what are known as dual diagrams. This method of associating a dual diagram with a Feynman graph had been used earlier by Karplus, Sommerfield and Wichman (1959) in their study of the triangle and box diagram amplitudes. The Landau equations were also proposed by Bjorken (1959), Nakanishi (1959) and Mathews (1959).

The Landau equations were derived by Polkinghorne and Screaton (1960 a, b), using the generalized form of Hadamard's lemma. They used the Feynman parametrised form of the Feynman integrals and

obtained the form of the Landau equations, which determines the surfaces of singularity of the Feynman amplitudes in the complex space of certain scalar products, which can be formed from the external momenta of the Feynman graphs. They noted that, in their derivation of the Landau equations, no distinction was made between singularities of the integrand which approached the path of integration from the same side of the path and then coincided, and singularities which approached the path from opposite sides and pinched the path. It was, however, argued that when the Feynman amplitude is continued onto all its sheets, then the Landau equations give the possible singular points. This, they argued, is because this continuation may be achieved by considering the totality of contours joining the given end points and thus every coincident singularity pinches some contour. They also noted that, since we are concerned in physics with particular Riemann sheets, it is necessary to supplement the Landau equations with an analysis of whether the coincident singularities pinch the contour. This, in fact, is not simple in practice. The fact that the Landau equations do not determine on which Riemann sheets the singularities lie has been emphasized by several other authors, for example, Cutkosky (1961), Landshoff (1963), Norton (1964), Hwa and Teplitz (1966, p. 3). We also note from Hadamard's lemma that, even if it is known that the path is pinched by the coincident singularities, this does not necessarily mean that the Feynman amplitude is in fact singular at the corresponding point (Fairlie, Landshoff, Nuttall and Polkinghorne (1962), ELOP, p. 121).

Cutkosky (1960 a, b) used the generalized form of Hadamard's lemma to obtain a formula for the discontinuity of a general Feynman integral across a branch cut starting from any Landau singularity.

The Landau singularities are often branch point singularities with cuts attached to them. However, this is not always so; for the pentagonal diagram, for example, the leading singularity (ELOP, p. 54) is a pole. The formula derived by Cutkosky is strictly only valid for physical-region singularities. For singularities out of the physical region, the discontinuity is only determined up to a sign. Then, the only reliable way to interpret the discontinuity formula is to make an analytic continuation in some of the masses so that the singularities come into the physical region, evaluate the discontinuity and continue the result back to the desired values of the masses. This procedure is not simple in practice (ELOP, p. 113).

Cutkosky's formula is a generalization of the prescription given by Mandelstam (1959 a) for obtaining the imaginary part of the box diagram amplitude (in eq. (2-8)). Cutkosky also used his discontinuity formula to obtain the Mandelstam double spectral function, and gave a geometrical method of obtaining this spectral function. Further, he obtained the discontinuity corresponding to the anomalous threshold or leading singularity of the triangle diagram amplitude and noted when the discontinuity is singular. Since the discontinuity is just the difference of the Feynman integral on two of its Riemann sheets, the Feynman integral must have this singularity on at least one of its sheets. However, the singularity did not appear to be given by the Landau equations and hence Cutkosky called it a "non-Landauian" singularity. This non-Landau singularity is in fact an example of a wide class of singularities, also called second-type singularities, corresponding to special solutions of the Landau equations which do not appear among the dual diagram solutions (Fairlie, Landshoff, Nuttall and Polkinghorne (1962), Fowler (1962)). The second-type singularities correspond to pinch configurations in which

some of the components of the loop momenta of the Feynman diagram are infinite. A further class of singularities called mixed singularities has also been found (Fairlie, Landshoff, Nuttall and Polkinghorne (1962); Fowler (1963), Drummond (1963)).

Hadamard's lemma and the Landau-Cutkosky rules have been used extensively for analysing both individual Feynman integrals and whole classes of Feynman integrals. As already mentioned, Cutkosky used his rules to obtain the Mandelstam double spectral function and the discontinuity across the cut from the anomalous threshold singularity of the triangle diagram amplitude. As mentioned in sect. 2-3, Tarski (1960) studied the Landau curve of the box diagram amplitude and obtained the regular and singular points of this amplitude by using Hadamard's lemma. A similar analysis was carried out by Eden, Landshoff, Polkinghorne and Taylor (1961 a). Fowler, Landshoff and Lardner (1960) and Bonnevey, Aitchison and Dowker (1961) studied the Landau surface of the triangle diagram amplitude.

Landshoff and Treiman (1962) used the Landau-Cutkosky rules to obtain the weight functions in an anomalous threshold dispersion relation for the general mass case triangle diagram amplitude. They then gave examples of production amplitudes in which the triangle diagram singularities might, in certain circumstances, produce a sharp peaking. A number of further examples of production amplitudes in which the triangle diagram anomalous threshold and second-type singularities might produce observable effects have been proposed. (See, for example, Aaron (1962), Halpern and Watson (1963), Anisovich and Dakhno (1964), Baldo and Recami (1970).) It has also been proposed that the leading Landau singularity of the box diagram amplitude produces observable effects in certain scattering and production processes. (See, for example, Month (1967), Collas and Norton (1967).)

The Landau surfaces associated with a number of other individual Feynman amplitudes have also been studied. (See, for example, Eden, Landshoff, Polkinghorne and Taylor (1961 b), Fowler (1963), Olive and Taylor (1962).) The dual diagram method (Landau (1959 a, b, 1960); see also Landshoff (1960)) was used by Taylor (1960) to investigate the class of amplitudes obtained in the perturbation expansion of the vertex function. The class of Feynman amplitudes that contribute to the two-particle $\rightarrow$ two-particle scattering amplitude has also been investigated using Hadamard's lemma and the Landau equations (Eden (1960 a, b, c, 1961), Landshoff, Polkinghorne and Taylor (1960, 1961)). Further references and applications of Hadamard's lemma and the Landau-Cutkosky rules are given in ELOP and by Nakanishi (1971).

In 1964, Fotiadi, Froissart, Lascoux and Pham (1964) proposed that homology theory be used as a rigorous way of analysing the analytic structure of individual Feynman integrals. They formulated the problem of determining the structure of the Riemann surface associated with a Feynman integral, in such a way that they could apply some important theorems in algebraic topology. In this way, they converted the problem from being a geometric one to an algebraic one, with the description of the surface given in terms of homology groups. This method of using homology theory in studying the analytic properties of Feynman integrals involves setting up quite a complicated mathematical formalism which is probably unfamiliar to most physicists. Further, the analysis needed to establish the Riemann sheet properties of individual Feynman integrals is, even for the lower order diagrams, quite complicated.

An introduction to homology theory and its application to the study of Feynman integrals is given in the book by Hwa and Teplitz

(1966). This book also contains excerpts from two unpublished papers by Fotiadi and Pham on the application of homology theory to the box and triangle diagram amplitudes.

2-5. Conclusion

We conclude this chapter with an account of the work to be carried out in chapters 3, 4 and 5. In these chapters, dispersion relations for the general mass case triangle and box diagram amplitudes are obtained *by a direct method*, which does not rely on the Landau-Cutkosky rules or the complicated mathematical formalism of homology theory. We also discuss, in detail, the heuristic method of approximating the amplitudes on the right hand side of the unitarity relation to obtain the weight functions in approximate normal threshold dispersion relations for the scalar vertex function and two-particle $\rightarrow$ two-particle scattering amplitude.

Before proceeding with the main methods in chapter 3, the above heuristic method is used to write down the approximate normal threshold dispersion relation for the scalar vertex function for the case when the six particles in fig. 3-1 are different. The Feynman parametrisation of the triangle diagram amplitude is written down in sect. 3-3, and the main part of chapter 3 is then devoted to transforming this amplitude into the form of a dispersion relation in the momentum transfer t . In this way, the explicit forms of the dispersion relation, the weight function and the threshold are obtained *directly* for all possible mass configurations involving externally stable particles. This method, in fact, involves only real analysis, once the Feynman parametrised form of the amplitude has been obtained. Thus, there are no problems of deciding on which Riemann sheets the singularities of the amplitude lie. Further, no assumptions or

approximations are needed to obtain the various forms of the dispersion relation. In sects. 3-7 and 3-8, the details of the method of analytic continuation in the external masses (originally proposed by Mandelstam (1960)) are given for the case when the masses of the six particles in fig. 3-1 are different. We start with a particular normal threshold dispersion relation obtained by the method of direct transformation (or by the heuristic method), and, by continuing in the external masses, obtain the dispersion relation for all other mass configurations involving externally stable particles.

These methods are then extended in chapters 4 and 5 to apply to the box diagram amplitude. Again, we discuss the heuristic method of using the unitarity relation to obtain the weight function in an approximate normal threshold dispersion relation for the two-particle $\rightarrow$ two-particle scattering amplitude, before proceeding with the main method in chapter 4. This method is again direct transformation of the Feynman parametrised form of the box diagram amplitude into the form of a dispersion relation. As for the triangle diagram vertex function, no assumptions or approximations are used and the method, in fact, involves only real analysis, so that again there are no problems of deciding on which Riemann sheets the singularities of the amplitude lie. By using this direct method, we obtain the double spectral representation in s and t , and necessary and sufficient conditions for its validity. Further, by the same method, a number of different spectral representations for the box diagram amplitude are obtained for cases when the double spectral representation is no longer valid. For these cases, the amplitude is written as the sum of a double dispersion integral in s and t and a single dispersion integral in s . A spectral representation is, in fact, obtained for

all possible mass configurations involving externally stable particles and for $t < (G-H)^2$, where G and H are the masses of the internal particles shown in fig. 4-1. Initially, we restrict s to lie on the negative real axis, but since the box diagram amplitude is always written as a dispersion relation in s , continuation in s is immediate.

In chapter 5, the method of analytic continuation is used to obtain a spectral representation for the general mass case box diagram amplitude for both real and complex s and t . In particular, for cases when the external particles in fig. 4-1 are stable, a spectral representation is obtained for all s in the upper half complex plane and for almost all $-\infty < t < +\infty$. To obtain the various forms of the spectral representation, we start with a particular normal threshold double dispersion relation, obtained by the method of direct transformation in chapter 4 (or by the heuristic method), and continue in the external masses, and then in s and t . Finally, we compare our results with those of Gribov, Danilov and Dyatlov (1961, 1962).

3. THE TRIANGLE DIAGRAM AMPLITUDE

3-1. Introduction

In this chapter we investigate the analytic properties of the triangle diagram amplitude $F_{\Delta}(t)$ associated with the Feynman diagram shown in fig. 3-1. This amplitude describes the conversion of a scalar particle A into a scalar particle B via interaction with a third scalar particle T ; the intermediate particles in the process are E , F and G . The main object is to obtain a spectral representation of $F_{\Delta}(t)$ for all masses satisfying the stability conditions $A < (E+G)$, $B < (F+G)$. We also obtain an approximation to the total scalar vertex function $F(t)$ for the conversion of a scalar particle A into a scalar particle B via interaction with a third scalar particle T . We write this approximation in the form of a single spectral representation and then show that this spectral representation is the same as that obtained for $F_{\Delta}(t)$ when $(E+F) \geq (A+B)$.

We begin in sect. 3-2 by writing down the total scalar vertex function $F(t)$. Then we obtain an approximation to $F(t)$ in the form of a single spectral representation by first using unitarity to evaluate the contribution to $\text{Im}F$ from a state $(F\bar{E})$ with the same internal quantum numbers as $(B\bar{A})$ and then making an s -channel single pole approximation to the amplitude for $B\bar{A} \rightarrow F\bar{E}$ and a constant approximation to the vertex function for the vertex EFT .

The rest of this chapter is then devoted to a study of the triangle diagram amplitude $F_{\Delta}(t)$. In sect. 3-3 we write down the Feynman parametrisation of $F_{\Delta}(t)$ and then transform it into a more convenient form. The implications of the stability conditions $A < (E+G)$, $B < (F+G)$ for the vertices of the diagram are also

Fig. 3-1.

Triangle diagram for the vertex function.

discussed. The conditions on the masses of the interacting particles for which normal and anomalous thresholds occur are found in sect. 3-4 by examining the boundary of the region of integration in the double integral form obtained in sect. 3-3.

In sect. 3-5, we reverse the order of integration in this double integral and carry out one of the integrations to obtain the desired single spectral representation of the triangle diagram vertex function. Sect. 3-6 lists the explicit forms of the weight functions for all possible cases and we also discuss the continuity of the weight functions. In particular, we show that when $(E+F) \geq (A+B)$, the weight function $X(\xi)$ obtained from the calculations of sects. 3-3 to 3-6 is the same as that obtained in sect. 3-2.

Finally, in sects. 3-7 and 3-8 we obtain, from the spectral representation for the case $(E+F) \geq (A+B)$, the spectral representation for all other cases detailed in sect. 3-6 by continuing in A^2 and B^2 .

3-2. Approximation to the scalar vertex function

As discussed in chapter 2, the method of approximating the amplitudes on the right hand side of the unitarity relation and then obtaining a dispersion relation, is well known. We now generalize and expand the heuristic argument of Barut (1967) to apply to the case when the six particles shown in fig. 3-1 are different.

With plane wave states normalized so that $\langle p' | p \rangle = \delta^{(3)}(p' - p)$, we define the scalar vertex function F for the scalar vertex (ABT) by

$$F_{ABT}(t) = -(2\pi)^3 (4E_A E_B)^{\frac{1}{2}} \langle p_B | j_T(0) | p_A \rangle, \quad (3-1)$$

where E_A and E_B are the energies of the particles A and B respectively. In eq. (3-1), $j_T(x)$ is the current associated with the particle T , that is,

$$j_T(x) = (\square + T^2) \phi_T(x)$$

where $\phi_T(x)$ is the field operator for the particle T and

$$\square = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial \mathbf{x}^2}.$$

The function F defined in eq. (3-1) is Lorentz invariant. Now, besides A^2 and B^2 , there is only one independent kinematical Lorentz invariant that can be formed from p_A and p_B . We choose this to be

$$t = (p_A - p_B)^2;$$

the function F may thus be written as a function of t . For four-momenta p_A, p_B corresponding to particles on the mass shell,

$$\begin{aligned} t &= \left(\sqrt{A^2 + p_A^2} - \sqrt{B^2 + p_B^2} \right)^2 - (p_A - p_B)^2 \\ &\leq A^2 + B^2 - 2\sqrt{A^2 + p_A^2} \sqrt{B^2 + p_B^2} + 2|p_A||p_B|, \end{aligned}$$

that is,

$$t \leq (A-B)^2.$$

Thus $F(t)$ has been defined by eq. (3-1) for t real and $\leq (A-B)^2$.

We suppose that $F(t)$ has the following properties.

(1) $F(t)$ is a real valued function of t for four-momenta p_A, p_B

corresponding to particles on the mass shell, that is, for

$t \leq (A-B)^2$. That $F(t)$ is real valued for $t \leq (A-B)^2$ can be

seen from the fact that the strong interactions are invariant under particle-antiparticle conjugation and hence, from the *TPC* theorem, under the product of time reversal and space inversion. Also, since t remains the same when the signs of both p_A and p_B are changed, this result also follows from time reversal invariance. (See for example, Barton 1965, chapter 7 or Roman 1969, p. 509.)

- (2) $F(t)$ may be continued to give an analytic function $F(\zeta)$ which is regular in the whole ζ -plane except for a cut along the real axis from a threshold $(\geq (A+B)^2)$ to $+\infty$.
- (3) $F(\zeta) \rightarrow 0$ uniformly as $|\zeta| \rightarrow \infty$ in any direction in the complex plane.

Suppose that $(F\bar{E})$ is a physical two-particle state with the same internal quantum numbers as the state $(B\bar{A})$. We might expect this state to give rise to a branch point of $F(\zeta)$ at $\zeta = (E+F)^2$ and a branch cut in $F(\zeta)$ from $(E+F)^2$ to $+\infty$ along the real axis; if this is the case, we say that the state $(F\bar{E})$ gives rise to a normal threshold. The first approximation we are going to make is to use the unitarity relation to evaluate $\text{Im}F(t+i.0)$, the imaginary part of the boundary value of $F(\zeta)$ as ζ approaches the cut in $F(\zeta)$ from above, and to include on the right hand side of the unitarity relation the contribution from the state $(F\bar{E})$ only. For the case $(E+F) \geq (A+B)$ we suppose that the threshold is a normal threshold. Then the contribution of the state $(F\bar{E})$ to $\text{Im}F(t+i.0)$ will begin to differ from zero at $t = (E+F)^2$ and

$$\text{Im}F_{ABT}^{(F\bar{E})}(t+i.0) = -\frac{q_{EF}(t)}{32\pi^2 t^2} F_{EFT}(t+i.0) \iint d\Omega_{\mathbf{n}} T_{BA \rightarrow FE}^*(t+i.0, \mathbf{n} \cdot \mathbf{n}') \quad (3-2)$$

for $t \geq (E+F)^2$. Here

$$q_{EF}(t) = \frac{[t-(E+F)^2]^{\frac{1}{2}}[t-(E-F)^2]^{\frac{1}{2}}}{2t^{\frac{1}{2}}}, \quad (3-3)$$

while $\mathbf{n}$ is an arbitrary unit vector and the integration is over the surface of the unit sphere, so that the integral is independent of $\mathbf{n}$. Finally, $T_{\overline{BA} \rightarrow \overline{FE}}$ is the invariant scattering amplitude for the process $\overline{BA} \rightarrow \overline{FE}$ (defined in sect. 2-3) while F_{EFT} is defined in the same way as F_{ABT} in eq. (3-1) with $A \rightarrow E$, $B \rightarrow F$.

There are now two more approximations to make. Assuming that the Mandelstam hypothesis is valid for the process $\overline{BA} \rightarrow \overline{FE}$ and its related processes (see sect. 2-3), and that there is a single particle $\overline{G}$ with the same internal quantum numbers as the states $(F\overline{B})$ and $(E\overline{A})$, we approximate $T_{\overline{BA} \rightarrow \overline{FE}}$ by the contribution from the s -channel pole in this amplitude corresponding to $\overline{G}$. This gives

$$T_{\overline{BA} \rightarrow \overline{FE}}(t, \mathbf{n} \cdot \mathbf{n}') = \frac{F_{AEG}(G^2)F_{BFG}(G^2)}{s(t, \mathbf{n} \cdot \mathbf{n}') - G^2} \quad (3-4)$$

where

$$s(t, \mathbf{n} \cdot \mathbf{n}') = \frac{1}{2}(A^2 + B^2 + E^2 + F^2) - \frac{1}{2}t - \frac{1}{2}(A^2 - B^2)(E^2 - F^2)t^{-1} + 2q_{EF}(t)q_{AB}(t)\mathbf{n} \cdot \mathbf{n}' \quad (3-5)$$

and

$$q_{AB}(t) = \frac{[t-(A+B)^2]^{\frac{1}{2}}[t-(A-B)^2]^{\frac{1}{2}}}{2t^{\frac{1}{2}}}. \quad (3-6)$$

The expression for $s(t, \mathbf{n} \cdot \mathbf{n}')$ in eq. (3-5) can be obtained from that for t_1 in equation (2-4) by making the appropriate change of variables and defining $\cos \theta$ by $\mathbf{n} \cdot \mathbf{n}' = \cos \theta$. We also approximate F_{EFT} by a constant and write it as $F_{EFT}(T^2)$. The quantities

$F_{AEG}(G^2)$, $F_{BFG}(G^2)$ and $F_{EFT}(T^2)$ are what are usually called rationalized coupling constants.

Combining the three approximations just made, we obtain, using eqs. (3-2) - (3-6),

$$\begin{aligned} \text{Im}F(t+i.0) &\approx - \frac{q_{EF}(t)F_{AEG}(G^2)F_{BFG}(G^2)F_{EFT}(T^2)}{32\pi^2 t^{\frac{1}{2}}} \\ &\quad \cdot 2\pi \int_{-1}^1 d(\cos \theta) \left[\frac{1}{2}(A^2+B^2+E^2+F^2) - \frac{1}{2}t \right. \\ &\quad \left. - \frac{1}{2}(A^2-B^2)(E^2-F^2)t^{-1} + 2q_{EF}(t)q_{AB}(t)\cos\theta - G^2 \right]^{-1} \\ &= \frac{g}{16\pi[t-(A+B)^2]^{\frac{1}{2}}[t-(A-B)^2]^{\frac{1}{2}}} \ln \left(\frac{\tilde{\Sigma}(t)+1}{\tilde{\Sigma}(t)-1} \right), \end{aligned} \quad (3-7)$$

where

$$g = F_{AEG}(G^2)F_{BFG}(G^2)F_{EFT}(T^2) \quad (3-8)$$

and

$$\tilde{\Sigma}(t) = \frac{t^2+t(2G^2-A^2-B^2-E^2-F^2)+(A^2-B^2)(E^2-F^2)}{\sqrt{t-(E+F)^2}\sqrt{t-(E-F)^2}\sqrt{t-(A+B)^2}\sqrt{t-(A-B)^2}}. \quad (3-9)$$

It can in fact be shown that $\tilde{\Sigma}(t) > 1$ for $t > (E+F)^2$; this is done in sect. 3-5. Inspection of the approximate expression for $\text{Im}F(t+i.0)$ in eq. (3-7) shows that for large t it behaves like $t^{-1} \ln t$.

Next we want to write our approximation to $F(\zeta)$ in the form of an unsubtracted dispersion relation. To do this we use the Cauchy representation (see for example, Copson 1948, p. 66),

$$F(\zeta) = (2\pi i)^{-1} \int_C \frac{F(z)dz}{z-\zeta}$$

where the contour C consists of a large circular arc, of radius R

and centre at the origin of the z -plane, which runs from just above the real positive axis in an anti-clockwise direction to just below the real positive axis, plus a straight line segment running just below the real positive axis from R to $(E+F)^2$, a small clockwise semi-circle around the point $((E+F)^2, 0)$ and a straight line segment running just above the real positive axis from $(E+F)^2$ to R . Thus, the closed path C does not pass through the branch cut from $(E+F)^2$ to $+\infty$ along the real positive axis and $F(z)$ is regular inside and on C .

We now shrink the semi-circle to a point and let $R \rightarrow +\infty$. Since $F(z) \rightarrow 0$ uniformly as $|z| \rightarrow +\infty$ in any direction in the complex plane, it follows that the integral over the large circular arc vanishes and we are left with the integral along the straight line segments. Also, since $F(z)$ is regular except for a cut along the real positive axis from $(E+F)^2$ to $+\infty$ and $\text{Im}F(v) = 0$ for $-\infty < v < (E+F)^2$, it follows from the Schwarz reflection principle that $F(z) = [F(z^*)]^*$ for $z \notin [(E+F)^2, +\infty)$. In particular

$$F(v+i.0) - F(v-i.0) = 2i \text{Im}F(v+i.0)$$

for $(E+F)^2 \leq v < +\infty$ and hence we may write our approximation to $F(\zeta)$ in the form of an unsubtracted dispersion relation:

$$F(\zeta) \approx F_{\Delta}(\zeta) = \frac{g}{8\pi^2} J(\zeta)$$

where

$$J(\zeta) = \int_{(E+F)^2}^{\infty} \frac{dv}{(v-\zeta)} \frac{1}{[v-(A+B)^2]^{\frac{1}{2}} [v-(A-B)^2]^{\frac{1}{2}}} \ln \left(\frac{\tilde{\Sigma}(v)+1}{\tilde{\Sigma}(v)-1} \right). \quad (3-10)$$

Eq. (3-10) has been established for the case when $(E+F)^2 \geq (A+B)^2$.

For other mass configurations, $J(\zeta)$ can be obtained by analytic continuation; this is done in sects. 3-7 and 3-8.

At this stage we introduce the variables ξ, τ_0 and the constants a, b defined by

$$\begin{aligned} 2EF\xi &= v - E^2 - F^2, & 2EF\tau_0 &= \zeta - E^2 - F^2, \\ 2EGa &= E^2 + G^2 - A^2, & 2FGb &= F^2 + G^2 - B^2. \end{aligned} \quad (3-11)$$

In some respects, it is simpler to retain the original variables v and ζ ; this is done in our paper, Frederiksen and Woolcock (1971). However, when we consider the box diagram amplitude in chapters 4 and 5, it will be more convenient to use a set of variables like those on the right hand side of eq. (3-11). Since most of the results that we establish for the triangle diagram amplitude will be needed when we consider the box diagram amplitude, we express our results in terms of the quantities defined in eq. (3-11). It is also convenient to define an auxiliary function

$$v(\xi) = 2EF\xi + E^2 + F^2, \quad (3-12)$$

and to express some of the results in terms of it.

Now,

$$\tilde{\Sigma}(v) \equiv \Sigma(\xi) = \frac{G(\xi)}{\sqrt{\xi^2 - 1} \sqrt{F(\xi)}}, \quad (3-13)$$

where

$$G(\xi) = \xi^2 - 1 + (EF)^{-1} G[(Ea + Fb)\xi + Eb + Fa], \quad (3-14)$$

$$\begin{aligned} F(\xi) &= (4E^2F^2)^{-1} [(v(\xi))^2 - 2v(\xi)(A^2 + B^2) + (A^2 - B^2)^2] \\ &= \xi^2 - 1 + (EF)^{-2} [(2EF\xi + E^2 + F^2)(EGa + FGb - G^2) \\ &\quad + (EGa - FGb)^2 - (EGa - FGb)(E^2 - F^2)] \\ &= (\xi - g_+)(\xi - g_-) \end{aligned} \quad (3-15)$$

and

$$\begin{aligned} g_{\pm} &= (2EF)^{-1} [(A \pm B)^2 - E^2 - F^2] \\ &= (2EF)^{-1} [(\sqrt{E^2 + G^2 - 2EGa} \pm \sqrt{F^2 + G^2 - 2FGb})^2 - E^2 - F^2] . \end{aligned} \quad (3-16)$$

Thus

$$\begin{aligned} 2EFJ(\zeta) &\equiv I(\tau_0) \\ &= \int_1^{\infty} \frac{d\xi X(\xi)}{\xi - \tau_0} \end{aligned} \quad (3-17)$$

where the weight function $X(\xi)$ is given by

$$X(\xi) = \frac{\frac{1}{2}}{\sqrt{F(\xi)}} \ln \left(\frac{\Sigma(\xi) + 1}{\Sigma(\xi) - 1} \right) . \quad (3-18)$$

3-3. Transformation of the triangle diagram amplitude

We now study the triangle diagram vertex function associated with the Feynman diagram of fig. 3-1. The definition of the vertex function in eq. (3-1) corresponds to the following definition in terms of the S -operator:

$$\langle p_B | S | p_T p_A \rangle = -i \delta^{(4)}(p_B - p_A - p_T) (2\pi)^{-\frac{1}{2}} \cdot (8E_T E_A E_B)^{-\frac{1}{2}} F((p_A - p_B)^2) .$$

Eq. (3-1) can be obtained from the above equation by using the reduction formula:

$$\begin{aligned} \langle p_B | S | p_T p_A \rangle &= i \int d^4x (2\pi)^{-3/2} (2E_T)^{-\frac{1}{2}} \exp.(-ip_T x) \cdot \langle p_B | j_T(x) | p_A \rangle \\ &= i (2\pi)^4 \delta^{(4)}(p_B - p_A - p_T) (2\pi)^{-3/2} (2E_T)^{-\frac{1}{2}} \cdot \langle p_B | j_T(0) | p_A \rangle . \end{aligned}$$

(See for example, Lehmann, Symanzik and Zimmermann 1955, Roman 1969.)

Therefore, using the Feynman rules of Schweber (1961) we find that the contribution to the matrix element $\langle p_B | S | p_T p_A \rangle$ corresponding to the triangle diagram in fig. 3-1 is

$$\langle p_B | S | p_T p_A \rangle_\Delta = (-i)^3 (2\pi)^{-9/2} (8E_A E_B E_T)^{-\frac{1}{2}} .$$

$$\begin{aligned} & \cdot \iiint \frac{d^4 p_E d^4 p_F d^4 p_G (i)^3}{[p_E^2 - E^2 + i\epsilon][p_F^2 - F^2 + i\epsilon][p_G^2 - G^2 + i\epsilon]} \cdot \\ & \cdot \delta^{(4)}(p_B - p_F - p_G) \delta^{(4)}(p_A - p_E - p_G) \delta^{(4)}(p_T + p_E - p_F) \cdot \\ & \cdot F_{AEG}(G^2) F_{BFG}(G^2) F_{EFT}(T^2) \end{aligned}$$

Hence, leaving out the $i\epsilon$'s, we find that the triangle diagram vertex function is

$$F_\Delta((p_A - p_B)^2) = i(2\pi)^{-4} g \int \frac{d^4 p_G}{[(p_A - p_G)^2 - E^2][(p_B - p_G)^2 - F^2][p_G^2 - G^2]}$$

where g is defined in eq. (3-8). We then use the Feynman identity

$$\begin{aligned} \frac{1}{A_1 A_2 \dots A_n} &= (n-1)! \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \dots \int_0^1 d\alpha_n \cdot \\ &\cdot \frac{\delta(1 - \alpha_1 - \alpha_2 - \dots - \alpha_n)}{[\alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_n A_n]^n} \end{aligned} \quad (3-19)$$

for the case $n = 3$ and the fact that

$$\int \frac{d^4 k}{(k^2 - \tau)^{n-2}} = \frac{i\pi^2}{(-\tau)^{n-2}} \frac{\Gamma(2)\Gamma(n-2)}{\Gamma(n)} \quad (3-20)$$

for the case $n = 3$. Eq. (3-20) is derived in Jauch and Rohrlich (1955, appendix A) and $\Gamma(n) = (n-1)!$ in eq. (3-20). Finally, using the fact that the particles A and B are on the mass shell together with eq. (3-11) we find on continuing to the complex variable ζ , that

$$F_\Delta(\zeta) \equiv \frac{g}{16\pi^2 EF} I(\tau_0)$$

where

$$I(\tau_0) = EF \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left[E^2\alpha + F^2\beta + G^2(1-\alpha-\beta) + (2EG\alpha - E^2 - G^2)\alpha(1-\alpha-\beta) \right. \\ \left. + (2FG\beta - F^2 - G^2)\beta(1-\alpha-\beta) - (2EF\tau_0 + E^2 + F^2)\alpha\beta \right]^{-1}. \quad (3-21)$$

For most of this chapter, τ_0 can take any real or complex value except $\tau_0 \notin [-(2EF)^{-1}(E^2+F^2), +\infty)$. This restriction on τ_0 ensures that eq. (3-21) is well defined. The notation anticipates the fact (to be proved in due course) that for $(A+B) \leq (E+F)$, the functions $I(\tau_0)$ given in eqs. (3-17) and (3-21) are the same.

Our object now is to transform $I(\tau_0)$ into the form of a single dispersion relation in τ_0 . In the process we show that for certain mass configurations, the threshold is not a normal threshold but an anomalous threshold whose value is less than the normal threshold value. We also obtain the explicit functional forms of the weight functions for all mass configurations allowed by the stability conditions $A < (E+G)$, $B < (F+G)$, which forbid the decays $A \rightarrow E+G$, $B \rightarrow F+G$. In particular, we show that for $(E+F) \geq (A+B)$, the weight function obtained by transformation of eq. (3-21) is exactly that given in eqs. (3-18) and (3-13).

The first transformation is

$$\lambda = (\alpha+\beta)^{-1} - 1, \quad v = (\alpha+\beta)\beta^{-1}$$

with the inverse

$$\alpha = v^{-1}(v-1)(1+\lambda)^{-1}, \quad \beta = v^{-1}(1+\lambda)^{-1},$$

and the Jacobian of the transformation is given by

$$\left| \frac{\partial(\alpha, \beta)}{\partial(\lambda, v)} \right| = (1+\lambda)^{-3} v^{-2}.$$

The transformation changes the triangular region of integration in the

$\alpha\beta$ -plane into the region

$$\{(\lambda, \nu) \mid \lambda \geq 0, \nu \geq 1\}.$$

Thus

$$I(\tau_0) = \frac{1}{2} \int_0^\infty \frac{d\lambda}{\lambda+1} \int_1^\infty \frac{d\nu}{\nu-1} \frac{1}{g(\lambda, \nu) - \tau_0}, \quad (3-22)$$

where

$$2EGg(\lambda, \nu) + E^2 + F^2 = \phi(\lambda)\nu + \psi(\lambda)\nu(\nu-1)^{-1}, \quad (3-23)$$

$$\phi(\lambda) = G^2\lambda^2 + 2EGa\lambda + E^2, \quad (3-24)$$

$$\psi(\lambda) = G^2\lambda^2 + 2FGb\lambda + F^2, \quad (3-25)$$

and a and b are given in eq. (3-11).

We now need a result concerning the stability conditions, which we put in the form of a lemma. Note that $A < E + G$ is equivalent to $a > -1$, $B < F + G$ is equivalent to $b > -1$, $A > 0$ implies that $a < (2EG)^{-1}(E^2 + G^2)$ and $B > 0$ implies that $b < (2FG)^{-1}(F^2 + G^2)$.

LEMMA. $\phi(\lambda) > 0$ (resp. $\psi(\lambda) > 0$) for all $\lambda \geq 0$ if and only if $a > -1$ (resp. $b > -1$).

Proof. The minimum of the quadratic function $\phi(\lambda)$ is at

$$\lambda_m = -EG^{-1}a$$

and its minimum value is

$$\phi(\lambda_m) = E^2(1-a^2). \quad (3-26)$$

Now if $a \geq 0$, ($A \leq \sqrt{E^2 + G^2}$), then $\lambda_m \leq 0$ and since

$\phi(0) = E^2 > 0$, it follows that $\phi(\lambda) > 0$ for all $\lambda \geq 0$. If

$-1 < a < 0$, ($\sqrt{E^2 + G^2} < A < E + G$), then $\lambda_m > 0$, but from eq.

(3-26), $\phi(\lambda_m) > 0$ and again $\phi(\lambda) > 0$ for all $\lambda \geq 0$.

For the converse, note that either $\lambda_m \leq 0$ or $\lambda_m > 0$. If

$\lambda_m \leq 0$, then $\alpha \geq 0$, ($A < \sqrt{E^2 + G^2} < E + G$), while if $\lambda_m > 0$ then $\alpha < 0$, ($A > \sqrt{E^2 + G^2} > |E - G|$). In the latter case $\phi(\lambda) \geq 0$ for all $\lambda \geq 0$ implies that $\phi(\lambda_m) > 0$ and so $\alpha > -1$, ($A < E + G$).

Now take λ fixed, $\lambda \geq 0$, and consider $g(\lambda, v)$ as a function of v for $v > 1$. From the lemma it follows that $g(\lambda, v) \rightarrow +\infty$ as $v \downarrow 1$ and $v \rightarrow +\infty$. Moreover, since

$$2EF \frac{\partial g(\lambda, v)}{\partial v} = \phi(\lambda) - \psi(\lambda)(v-1)^{-2},$$

we see that $g(\lambda, v)$ has just one turning point (a minimum) in $(1, +\infty)$, namely

$$v_m(\lambda) = 1 + \sqrt{\psi(\lambda)} / \sqrt{\phi(\lambda)}. \quad (3-27)$$

From eqs. (3-23) and (3-27) it follows that

$$2EFg(\lambda, v_m(\lambda)) + E^2 + F^2 = (\sqrt{\phi(\lambda)} + \sqrt{\psi(\lambda)})^2 > 0.$$

Consequently $I(\tau_0)$ in eq. (3-22) is well defined provided

$\tau_0 \notin [-(2EF)^{-1}(E^2 + F^2), \infty)$. In the sequel it will be useful to be able to refer to the functions $h(\lambda)$, $k(\lambda)$ defined by

$$\begin{aligned} 2EFh(\lambda) + E^2 + F^2 &= (\sqrt{\phi(\lambda)} + \sqrt{\psi(\lambda)})^2, \\ 2EFk(\lambda) + E^2 + F^2 &= (\sqrt{\phi(\lambda)} - \sqrt{\psi(\lambda)})^2, \end{aligned} \quad (3-28)$$

which by the lemma are defined for $\lambda \geq 0$.

We now replace the integration over v by an integration over the variable $\xi = g(\lambda, v)$. For fixed $\xi \geq h(\lambda)$ there are two values of v (call them $v_{\pm}(\xi, \lambda)$) such that

$$g(\lambda, v_{\pm}(\xi, \lambda)) = \xi.$$

From eq. (3-23), $v_{\pm}(\xi, \lambda)$ are the roots of the quadratic equation

$$\phi(\lambda)v^2 + (\psi(\lambda) - \phi(\lambda) - v(\xi))v + v(\xi) = 0,$$

where $v(\xi) = 2EF\xi + E^2 + F^2$, as given in eq. (3-12). Thus

$$v_{\pm}(\xi, \lambda) = \frac{1}{2}(\phi(\lambda))^{-1} \left[v(\xi) + \phi(\lambda) - \psi(\lambda) \pm 2EF\sqrt{U(\xi, \lambda)} \right],$$

where

$$\begin{aligned} U(\xi, \lambda) &= (4E^2F^2)^{-1} [(v(\xi) + \phi(\lambda) - \psi(\lambda))^2 - 4v(\xi)\phi(\lambda)] \\ &= (\xi - h(\lambda))(\xi - k(\lambda)). \end{aligned} \quad (3-29)$$

We also see that

$$U(\xi, \lambda) \geq 0 \quad \text{for} \quad \xi \geq h(\lambda)$$

and then

$$v_+(\xi, \lambda) \geq v_-(\xi, \lambda).$$

Now

$$\frac{\partial v_{\pm}(\xi, \lambda)}{\partial \xi} = \frac{1}{2\phi(\lambda)} \left[1 \pm \frac{v(\xi) - \phi(\lambda) - \psi(\lambda)}{\{U(\xi, \lambda)\}^{\frac{1}{2}}} \right] = \pm \frac{v_{\pm}(\xi, \lambda) - 1}{[U(\xi, \lambda)]^{\frac{1}{2}}}$$

and so from eq. (3-22),

$$I(\tau_0) = \int_0^{\infty} \frac{d\lambda}{\lambda+1} \int_{h(\lambda)}^{\infty} \frac{d\xi}{(\xi - \tau_0)[U(\xi, \lambda)]^{\frac{1}{2}}}. \quad (3-30)$$

The region of integration is now to the right of the line $\lambda = 0$ and above the curve $\xi = h(\lambda)$.

3-4. Conditions for normal and anomalous thresholds

To reverse the order of integration in eq. (3-30) we need to examine the function $h(\lambda)$ for $\lambda \geq 0$. It is convenient to write $\phi(\lambda)$, $\psi(\lambda)$ as

$$\phi(\lambda) = p\lambda^2 + 2q\lambda + r, \quad \psi(\lambda) = p\lambda^2 + 2q'\lambda + r', \quad (3-31)$$

where, from eqs. (3-24) and (3-25),

$$p = G^2, \quad q = EG\alpha, \quad q' = FGb, \quad r = E^2, \quad r' = F^2. \quad (3-32)$$

Then

$$2EFh_{\lambda}(\lambda) = 2(\sqrt{\phi(\lambda)} + \sqrt{\psi(\lambda)}) \left[\frac{p\lambda+q}{\sqrt{\phi(\lambda)}} + \frac{p\lambda+q'}{\sqrt{\psi(\lambda)}} \right], \quad (3-33)$$

and so $h_\lambda(\lambda)$ exists and is continuous on $[0, \infty)$. Now either $h_\lambda(\lambda)$ never vanishes on $[0, \infty)$ or else it does; we want a criterion which divides the two cases. If $h_\lambda(\lambda)$ does vanish on $[0, \infty)$ there exists a $\lambda_m \geq 0$ such that at least one of the numbers $(p\lambda_m + q)$, $(p\lambda_m + q')$ vanishes. Suppose that $(p\lambda_m + q) = 0$. Then

$$\phi(\lambda_m) = p^{-1}(pr - q^2) > 0$$

and so

$$pr - q^2 > 0.$$

Also,

$$\begin{aligned} (p\lambda + q) / \sqrt{\phi(\lambda)} + (p\lambda + q') / \sqrt{\psi(\lambda)} \\ = p^{\frac{1}{2}} \left[\frac{p\lambda + q}{\{(p\lambda + q)^2 + (pr - q^2)\}^{\frac{1}{2}}} + \frac{p\lambda + q'}{\{(p\lambda + q')^2 + (pr' - q'^2)\}^{\frac{1}{2}}} \right]. \end{aligned}$$

Consequently, $h_\lambda(\lambda_0) = 0$ with $\lambda_0 \geq 0$, if and only if

$$(i) \quad pr - q^2 > 0, \quad pr' - q'^2 > 0, \quad \text{and} \quad (3-34)$$

$$(ii) \quad \frac{p\lambda_0 + q}{(pr - q^2)^{\frac{1}{2}}} = - \frac{p\lambda_0 + q'}{(pr' - q'^2)^{\frac{1}{2}}} \quad (3-35)$$

solves to give $\lambda_0 \geq 0$, that is,

$$\lambda_0 = - \frac{q(pr' - q'^2)^{\frac{1}{2}} + q'(pr - q^2)^{\frac{1}{2}}}{p[(pr - q^2)^{\frac{1}{2}} + (pr' - q'^2)^{\frac{1}{2}}]} \geq 0. \quad (3-36)$$

Writing $\xi_{1+} = h(\lambda_0)$, we have

$$\begin{aligned} 2EF\xi_{1+} + E^2 + F^2 &= \phi(\lambda_0) (1 + \sqrt{\psi(\lambda_0)} / \sqrt{\phi(\lambda_0)})^2 \\ &= \phi(\lambda_0) \frac{[(pr - q^2)^{\frac{1}{2}} + (pr' - q'^2)^{\frac{1}{2}}]^2}{pr - q^2} \\ &= p^{-1} [p(r+r') - 2qq' + 2(pr - q^2)^{\frac{1}{2}}(pr' - q'^2)^{\frac{1}{2}}] \quad (3-37) \end{aligned}$$

Then, from eq. (3-32), we find that

$$\xi_{1\pm} = -ab \pm (1-a^2)^{\frac{1}{2}}(1-b^2)^{\frac{1}{2}}, \quad (3-38)$$

where eq. (3-38) also defines the quantity ξ_{1-} which we shall need

later. Also, when eqs. (3-34) and (3-36) hold,

$$\begin{aligned} 2EFh_{\lambda\lambda}(\lambda_0) &= \frac{2(\sqrt{\phi(\lambda_0)} + \sqrt{\psi(\lambda_0)})^2}{\sqrt{\phi(\lambda_0)}\sqrt{\psi(\lambda_0)}} \left[p - \frac{(p\lambda_0 + q)^2}{\phi(\lambda_0)} \right] \\ &= 2[\phi(\lambda_0)\psi(\lambda_0)]^{-\frac{1}{2}} [(pr - q^2)^{\frac{1}{2}} + (pr' - q'^2)^{\frac{1}{2}}]^2 > 0. \end{aligned} \quad (3-39)$$

We have established that either $h_{\lambda}(\lambda)$ vanishes exactly once on $[0, \infty)$ or it does not vanish at all. Knowing that $h_{\lambda}(\lambda)$ is continuous on $[0, \infty)$, that $h_{\lambda}(\lambda) > 0$ for all sufficiently large λ , and that $h_{\lambda\lambda}(\lambda_0) > 0$ when $h_{\lambda}(\lambda_0) = 0$, we conclude that $h_{\lambda}(\lambda_0) = 0$ with $\lambda_0 \geq 0$, if and only if $h_{\lambda}(0) \leq 0$, or

$$a + b \leq 0. \quad (3-40)$$

The required criterion is therefore the sign of $h_{\lambda}(0)$.

When $h_{\lambda}(0) \geq 0$, $h_{\lambda}(\lambda) > 0$ for all $\lambda > 0$ and $h(\lambda)$ increases strictly from $h(0) = 1$ to $+\infty$ as λ increases from 0 to $+\infty$. In this case the smallest value of $h(\lambda)$ on $[0, \infty)$ is 1, and as we shall see this corresponds to a *normal* threshold for $I(\tau_0)$. If however $h_{\lambda}(0) < 0$, $h(\lambda)$ has a local minimum at $\lambda_0 (> 0)$ and $h(\lambda_0) = \xi_{1+} (< 1)$ is the smallest value of $h(\lambda)$ on $[0, \infty)$. In this case $h(\lambda)$ decreases strictly from 1 to ξ_{1+} on $[0, \lambda_0]$ and then increases strictly from ξ_{1+} to $+\infty$ on $[\lambda_0, +\infty)$, and $I(\tau_0)$ can be shown to have an *anomalous* threshold ξ_{1+} .

Next we examine under what conditions on A, B, E, F , an anomalous

threshold is possible. From the stability conditions $A < E + G$, $B < F + G$ we see that G is bounded below by

$$G > \max\{0, (A-E), (B-F)\} \quad (\equiv G_m) .$$

For an anomalous threshold to occur, we see from eqs. (3-40) and (3-11) that G is bounded above by

$$G^2 < (E+F)^{-1} [FA^2 + EB^2 - EF(E+F)] \quad (\equiv G_o^2) .$$

When $G_o^2 > G_m^2$ there will be a range of values of G for which an anomalous threshold occurs. There are four cases to consider.

(i) When $A \leq E$, $B \leq F$, we have $G_m = 0$ and $G_o^2 \leq 0$ and so an anomalous threshold cannot occur.

(ii) When $B > F$, $A \leq E$, we have $G_m = (B-F)$ and

$$G_o^2 - (B-F)^2 = F(E+F)^{-1} [(B-F)-(E-A)][(A+E)-(B-F)] .$$

Thus $G_o^2 > (B-F)^2$ if and only if

$$E - A < B - F < A + E .$$

(iii) When $A > E$, $B \leq F$, the results of case (ii) apply with

$E \leftrightarrow F$, $A \leftrightarrow B$. Hence $G_o^2 > (A-E)^2$ if and only if

$$F - B < A - E < B + F .$$

(iv) When $A > E$, $B > F$ we use the expression for

$G_o^2 - (B-F)^2$ given in case (ii) above, and the analogous equation

$$G_o^2 - (A-E)^2 = E(E+F)^{-1} [(A-E)+(B-F)][(B+F)-(A-E)] ,$$

to show that $G_o^2 > \max\{(B-F)^2, (A-E)^2\}$ if and only if

one of the three following conditions holds:

$$A - E < B - F < A + E , \text{ or}$$

$$B - F < A - E < B + F, \text{ or}$$

$$A - E = B - F.$$

Summarizing the results of the four cases, we see that $G_c^2 > G_m^2$ if and only if *one* of the three following conditions holds:

$$|A-E| < B - F < A + E, \text{ or}$$

$$|B-F| < A - E < B + F, \text{ or}$$

$$A - E = B - F > 0. \quad (3-41)$$

An equivalent, but more useful and elegant, condition for an anomalous threshold to be possible is

$$|A-B| < E + F < A + B. \quad (3-42)$$

That (3-42) is a necessary condition may be seen as follows. We saw earlier (eq. (3-34)) that, if there is an anomalous threshold,

$pr - q^2 > 0$, $pr' - q'^2 > 0$. Then it follows from eq. (3-26) that $h(\lambda)$ is defined and real for all real λ and it has a local minimum at λ_0 (> 0) where it attains its smallest value. Hence $h(\lambda)$ is strictly decreasing on $(-\infty, \lambda_0]$ and strictly increasing on $[\lambda_0, +\infty)$ and so

$$1 = h(0) < h(-1) = g_+, \text{ or} \quad (3-43)$$

$$(E+F)^2 = v(h(0)) < v(h(-1)) = (A+B)^2,$$

where g_+ is given in eq. (3-16) and $v(\xi)$ is given in eq. (3-12).

Further, from eqs. (3-32) and (3-11) we see that

$$p(A-B)^2 = p[(p-2q+r)^{\frac{1}{2}} - (p-2q'+r')^{\frac{1}{2}}]^2,$$

and on using eqs. (3-38) and (3-37) we have

$$\begin{aligned}
pv(\xi_{1\pm}) - p(A-B)^2 &= \pm 2(pr-q^2)^{\frac{1}{2}}(pr'-q'^2)^{\frac{1}{2}} + 2[(p-q)^2 + (pr-q^2)]^{\frac{1}{2}}[(p-q')^2 + (pr'-q'^2)]^{\frac{1}{2}} \\
&\quad - 2(p-q)(p-q') \\
&= 2(pr-q^2)^{\frac{1}{2}}(pr'-q'^2)^{\frac{1}{2}} [\pm 1 + \cosh(\chi_1 - \chi_2)] .
\end{aligned}$$

Here χ_1 and χ_2 are defined by

$$\sinh \chi_1 = \frac{p-q}{(pr-q^2)^{\frac{1}{2}}} , \quad \sinh \chi_2 = \frac{p-q'}{(pr'-q'^2)^{\frac{1}{2}}} . \quad (3-44)$$

Consequently

$$\begin{aligned}
(A-B)^2 \leq v(\xi_{1-}) < v(\xi_{1+}) = v(h(\lambda_0)) < v(h(0)) = (E+F)^2 , \text{ or} \\
g_- \leq \xi_{1-} < \xi_{1+} < 1 ,
\end{aligned} \quad (3-45)$$

where g_- is given in eq. (3-16). Eqs. (3-43) and (3-45) then give eq. (3-42).

To prove that (3-42) is also a sufficient condition for an anomalous threshold to be possible we need to consider cases (ii), (iii) and (iv) above. For case (ii), (3-42) implies that

$$E - A < B - F , \quad B - F < A + E .$$

For case (iii) the results of case (ii) apply with $E \leftrightarrow F$, $A \leftrightarrow B$.

For case (iv) one of the three following relations holds:

$$0 < A - E < B - F , \text{ or}$$

$$0 < B - F < A - E , \text{ or}$$

$$0 < A - E = B - F .$$

When (3-42) is used, we find that one of the three relations in (3-41) holds and hence an anomalous threshold is possible (for an appropriate range of values of G) whenever (3-42) holds.

Notice that the transition case $h_\lambda(0) = 0$, that is $a + b = 0$ or $G^2 = G_c^2$, can occur only when $G_m^2 < G_c^2$. In the following, it will

be convenient to separate the transition case for special consideration; a normal threshold will therefore mean that $h_\lambda(0) > 0$. We now have the following cases to consider in sects. 3-5 and 3-6.

$$(1) \quad (A+B)^2 \leq (E+F)^2, \quad (g_+ \leq 1) \quad . \quad \text{Normal threshold only.}$$

$$(2) \quad (E+F)^2 \leq (A-B)^2, \quad (1 \leq g_-) \quad . \quad \text{Normal threshold only.}$$

$$(3) \quad (A-B)^2 < (E+F)^2 < (A+B)^2, \quad (g_- < 1 < g_+) \quad .$$

$$(a) \quad \text{Normal threshold case} \quad \left(G^2 > G_c^2 \quad \text{or} \quad a + b > 0 \right) ;$$

$$(b) \quad \text{Anomalous threshold case} \quad \left(G^2 < G_c^2 \quad \text{or} \quad a + b < 0 \right) ;$$

$$(c) \quad \text{Transition case} \quad \left(G^2 = G_c^2 \quad \text{or} \quad a + b = 0 \right) . \quad (3-46)$$

To visualize the situation more clearly we let $x_1 = A^2$, $x_2 = B^2$ and show in fig. 3-2 the regions in the (x_1, x_2) plane for which the various cases occur. The case $E < G$, $F < G$ has been sketched. The transition case occurs for the open segment of the line

$$F(E^2 + G^2 - x_1) + E(F^2 + G^2 - x_2) = 0 \quad (3-47)$$

between the points $((G-E)^2, (G+F)^2)$ and $((G+E)^2, (G-F)^2)$. The anomalous threshold case corresponds to the interior of the triangle whose sides are the transition line (eq. (3-47)) and the lines

$x_1 = (E+G)^2$, $x_2 = (F+G)^2$. The normal threshold case occurs for the remainder of the interior of the rectangle bounded by the axes and the lines $x_1 = (E+G)^2$, $x_2 = (F+G)^2$. Also shown in fig. 3-2 is the parabola

Fig. 3-2.

Normal and anomalous threshold regions, transition line (eq. (3-47)) and regions for which $E + F < |A-B|$, $|A-B| < E + F < A + B$, $A + B < E + F$, for the case $E < G$, $F < G$.

$$(x_1 - x_2)^2 - 2(E+F)^2(x_1 + x_2) + (E+F)^4 = 0, \quad (3-48)$$

which touches the axes at $((E+F)^2, 0)$ and $(0, (E+F)^2)$. Its axis is the line $x_1 = x_2$, its vertex the point $(\frac{1}{4}(E+F)^2, \frac{1}{4}(E+F)^2)$, its focus the point $(\frac{1}{2}(E+F)^2, \frac{1}{2}(E+F)^2)$ and its directrix the line $(x_1 + x_2) = 0$. Points inside the parabola satisfy the inequality $|A-B| < E + F < A + B$. That this is a necessary condition for an anomalous threshold to occur was shown above (eq. (3-46)); it is also evident from fig. 3-2. The interior of the region enclosed by the parabola and the two axes satisfies $A + B < E + F$, as indicated in fig. 3-2. There are also two regions for which $E + F < |A-B|$. When $F > G$ or (resp. and) $E > G$, one (resp. both) of these regions disappears and the region satisfying $A + B < E + F$ is then cut off at the line $x_1 = (E+G)^2$ or (resp. and) the line $x_2 = (F+G)^2$.

3-5. Spectral representation of the triangle diagram amplitude

In this section we reverse the order of integration in eq. (3-30) and carry out the integral over λ to obtain the spectral representation of $I(\tau_0)$. We start by considering the function $U(\xi, \lambda)$ defined in eq. (3-29), which is a quadratic in λ :

$$4E^2F^2U(\xi, \lambda) = a(\xi)\lambda^2 + 2b(\xi)\lambda + c(\xi), \quad (3-49)$$

where

$$\begin{aligned} a(\xi) &= 4[(q-q')^2 - p\nu(\xi)] \\ &= 4G^2[(Ea-Fb)^2 - \nu(\xi)], \end{aligned} \quad (3-50)$$

$$\begin{aligned} b(\xi) &= 2[(q-q')(r-r') - (q+q')\nu(\xi)] \\ &= 2G[(Ea-Fb)(E^2-F^2) - (Ea+Fb)\nu(\xi)], \end{aligned} \quad (3-51)$$

$$\begin{aligned}
 c(\xi) &= (r+r'-v(\xi))^2 - 4rr' \\
 &= 4E^2F^2(\xi^2-1) ,
 \end{aligned} \tag{3-52}$$

and $v(\xi)$ is given by eq. (3-12). The discriminant of this quadratic function is

$$\begin{aligned}
 (b(\xi))^2 - a(\xi)c(\xi) &= 4v(\xi) \left[p(v(\xi))^2 + 2v(\xi)\{2qq' - p(r+r')\} \right. \\
 &\quad \left. + \{p(r-r')^2 + 4(q-q')(qr'-q'r)\} \right] .
 \end{aligned} \tag{3-53}$$

The discriminant of the quadratic function of ξ (or $v(\xi)$) in square brackets can be shown to be a positive constant multiple of

$$\begin{aligned}
 (pr-q^2)(pr'-q'^2) . \text{ Thus, if } (pr-q^2)(pr'-q'^2) < 0 , \text{ or equivalently} \\
 (1-a^2)(1-b^2) < 0 , \text{ then}
 \end{aligned}$$

$$(b(\xi))^2 - a(\xi)c(\xi) > 0 \text{ for } \xi > -(2EF)^{-1}(E^2+F^2) .$$

The inequality $\xi > -(2EF)^{-1}(E^2+F^2)$ is satisfied by all $\xi \geq h(\lambda)$ as can be seen from eq. (3-28).

We now turn to the cases when $(pr-q^2)(pr'-q'^2) \geq 0$ or equivalently $(1-a^2)(1-b^2) \geq 0$, for which

$$\begin{aligned}
 (b(\xi))^2 - a(\xi)c(\xi) &= 4pv(\xi)(v(\xi)-v(\xi_{1+}))(v(\xi)-v(\xi_{1-})) \\
 &= 16E^2F^2G^2(2EF\xi+E^2+F^2)(\xi-\xi_{1+})(\xi-\xi_{1-}) ,
 \end{aligned} \tag{3-54}$$

where

$$\begin{aligned}
 v(\xi_{1\pm}) &= p^{-1} \left[p(r+r') - 2qq' \pm 2\{(pr-q^2)(pr'-q'^2)\}^{\frac{1}{2}} \right] , \\
 \xi_{1\pm} &= -ab \pm \sqrt{(1-a^2)(1-b^2)} .
 \end{aligned} \tag{3-55}$$

Note that ξ_{1+} is just the quantity defined in eq. (3-38) for the

anomalous threshold case. When $(1-a^2)(1-b^2) \geq 0$, $\xi_{1\pm}$ may be

written in the form

$$\xi_{1\pm} = 1 - \frac{1}{2} \left(\sqrt{1+a} \sqrt{1+b} \mp \sqrt{(1-a)(1-b)} \right)^2 . \quad (3-56)$$

It follows from eq. (3-56) that, when $(1-a^2)(1-b^2) \geq 0$,

$$\xi_{1-} \leq \xi_{1+} \leq 1 . \quad (3-57)$$

The two equality signs in eq. (3-57) cannot occur together. Moreover $\xi_{1+} = 1$ is the transition case.

When $(pr-q^2) \geq 0$, $(pr'-q'^2) \geq 0$ or equivalently $(1-a^2) \geq 0$, $(1-b^2) \geq 0$, there is another form for $\xi_{1\pm}$, namely

$$\xi_{1\pm} = -1 + \frac{1}{2} \left(\sqrt{(1+a)(1-b)} \pm \sqrt{(1-a)(1+b)} \right)^2 . \quad (3-58)$$

Thus, for the anomalous threshold and transition cases, for which eq. (3-34) holds, we have

$$-(2EF)^{-1}(E^2+F^2) \leq -1 \leq \xi_{1-} < \xi_{1+} \leq 1 . \quad (3-59)$$

Eqs. (3-59) and (3-54) show that, for the anomalous threshold and transition cases,

$$(b(\xi))^2 - a(\xi)c(\xi) \geq 0 \quad \text{for } \xi \geq \xi_{1+} ,$$

equality occurring if and only if $\xi = \xi_{1+}$. Also, from eqs. (3-57)

and (3-54) it follows that, for a normal threshold case

$$(b(\xi))^2 - a(\xi)c(\xi) > 0 \quad \text{for } \xi \geq 1 .$$

We have now shown that, for each $\xi \geq \min\{h(\lambda) \mid \lambda \geq 0\}$, the quadratic equation in λ

$$U(\xi, \lambda) = 0$$

has two real roots $\lambda_{\pm}(\xi)$ given by

$$\lambda_{\pm}(\xi) = [a(\xi)]^{-1} \left[-b(\xi) \mp \{(b(\xi))^2 - a(\xi)c(\xi)\}^{\frac{1}{2}} \right] , \quad (3-60)$$

except for the one point at most where $a(\xi) = 0$. For a normal threshold, $\lambda_{+}(\xi) \neq \lambda_{-}(\xi)$ for each $\xi \geq 1$. For the anomalous and

transition cases, $\lambda_+(\xi) \neq \lambda_-(\xi)$ for each $\xi > \xi_{1+}$ and

$$\lambda_+(\xi_{1+}) = \lambda_-(\xi_{1+}) .$$

Since

$$b(1) = -4EFG(E+F)(a+b)$$

it follows that for a normal threshold

$$\lambda_+(1) = 0 \neq \lambda_-(1) ,$$

while for an anomalous threshold

$$\lambda_-(1) = 0 \neq \lambda_+(1) .$$

For the transition case, of course,

$$\lambda_+(1) = \lambda_-(1) = 0 .$$

Notice too, that as $\xi \rightarrow +\infty$

$$\lambda_{\pm}(\xi) \sim \pm \frac{1}{\sqrt{2}} G^{-1}(EF)^{\frac{1}{2}} \xi^{\frac{1}{2}} .$$

For the anomalous threshold and transition cases, eqs. (3-50) and (3-37) show that

$$a(\xi_{1+}) = -4[(pr-q^2)^{\frac{1}{2}} + (pr'-q'^2)^{\frac{1}{2}}]^2 < 0$$

and since $a(\xi)$ is linear in ξ ,

$$a(\xi) < 0 \quad \text{for} \quad \xi \geq \xi_{1+} . \quad (3-61)$$

Also

$$b(\xi_{1+}) = -4p^{-1}[q(pr'-q'^2)^{\frac{1}{2}} + q'(pr-q^2)^{\frac{1}{2}}][(pr-q^2)^{\frac{1}{2}} + (pr'-q'^2)^{\frac{1}{2}}] ,$$

so that

$$\lambda_+(\xi_{1+}) = \lambda_-(\xi_{1+}) = \lambda_0$$

where λ_0 is given in eq. (3-36),

For these cases, then, we see that $\lambda_-(\xi)$ is the inverse of the strictly decreasing function $h(\lambda)$ on $(-\infty, \lambda_0]$; the function

$\lambda_-(\xi)$ therefore strictly decreases from λ_0 to $-\infty$ as ξ increases from ξ_{1+} to $+\infty$. Similarly, $\lambda_+(\xi)$ is the inverse of the strictly increasing function $h(\lambda)$ on $[\lambda_0, +\infty)$ and therefore strictly increases from λ_0 to $+\infty$ as ξ increases from ξ_{1+} to $+\infty$. For a normal threshold case, $\lambda_+(1) = 0$, $\lambda_+(\xi) \sim \frac{1}{\sqrt{2}} G^{-1} (EF)^{\frac{1}{2}} \xi^{\frac{1}{2}}$ as $\xi \rightarrow +\infty$, and $\lambda_+(\xi)$ is continuous on $[1, +\infty)$ except for the one point at most where $a(\xi) = 0$. Thus we see that $\lambda_+(\xi)$ is the inverse of the strictly increasing function $h(\lambda)$ on $[0, +\infty)$, and $\lambda_+(\xi)$ strictly increases from 0 to $+\infty$ as ξ increases from 1 to $+\infty$. We deduce further that if $\xi_a \geq 1$, where ξ_a is the solution of $a(\xi) = 0$, then

$$c(\xi_a) \geq 0, \quad b(\xi_a) < 0 \quad \text{and} \quad \lambda_+(\xi_a) = -c(\xi_a)/2b(\xi_a).$$

Finally, we see from eq. (3-60) that

$$\begin{aligned} [a(\xi) - 2b(\xi)c(\xi)] [\lambda_+(\xi) + 1]^{-1} + b(\xi) - a(\xi) \\ = \pm [(b(\xi))^2 - a(\xi)c(\xi)]^{\frac{1}{2}}. \end{aligned} \quad (3-62)$$

Now we reverse the order of integration in eq. (3-30). For the normal threshold and transition cases we have

$$I(\tau_0) = \int_1^\infty \frac{d\xi X(\xi)}{\xi - \tau_0}, \quad (3-63)$$

while for the anomalous threshold case we obtain

$$I(\tau_0) = \int_{\xi_{1+}}^1 \frac{d\xi Y(\xi)}{\xi - \tau_0} + \int_1^\infty \frac{d\xi X(\xi)}{\xi - \tau_0}, \quad (3-64)$$

where

$$X(\xi) = \int_0^{\lambda_+(\xi)} \frac{d\lambda}{(\lambda+1)[U(\xi, \lambda)]^{\frac{1}{2}}}, \quad (3-65)$$

$$Y(\xi) = \int_{\lambda_-(\xi)}^{\lambda_+(\xi)} \frac{d\lambda}{(\lambda+1)[U(\xi, \lambda)]^{\frac{1}{2}}}. \quad (3-66)$$

Eqs. (3-63) and (3-64) give the desired single dispersion relations for $I(\tau_0)$. To obtain the weight functions $X(\xi)$ and $Y(\xi)$, we carry out the integration over λ .

We start by considering $Y(\xi)$. From eqs. (3-43), (3-45), (3-15) and (3-49) we see that when $\xi_{1+} \leq \xi \leq 1$,

$$\begin{aligned} U(\xi, -1) = F(\xi) &= (4E^2F^2)^{-1} (a(\xi) - 2b(\xi) + c(\xi)) \\ &= (\xi - g_+) (\xi - g_-) < 0. \end{aligned} \quad (3-67)$$

Hence, from eq. (3-49),

$$\begin{aligned} Y(\xi) &= 2EF \int_{\lambda_-(\xi)}^{\lambda_+(\xi)} \frac{d\lambda}{(\lambda+1)[a(\xi)\lambda^2 + 2b(\xi)\lambda + c(\xi)]^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{-F(\xi)}} \left[\arcsin \frac{4E^2F^2F(\xi)[\lambda_+(\xi)+1]^{-1} + b(\xi) - a(\xi)}{\{(b(\xi))^2 - a(\xi)c(\xi)\}^{\frac{1}{2}}} \right. \\ &\quad \left. - \arcsin \frac{4E^2F^2F(\xi)[\lambda_-(\xi)+1]^{-1} + b(\xi) - a(\xi)}{\{(b(\xi))^2 - a(\xi)c(\xi)\}^{\frac{1}{2}}} \right] \\ &= \frac{\pi}{\sqrt{-F(\xi)}}, \end{aligned} \quad (3-68)$$

on using eqs. (3-62) and (3-67).

From eqs. (3-63) and (3-64) we see that we need $X(\xi)$ for $\xi \geq 1$ and in this case

$$4E^2F^2U(\xi, 0) = c(\xi) = 4E^2F^2(\xi^2 - 1) \geq 0. \quad (3-69)$$

There are two separate cases to consider:

(a) $(A-B)^2 < v(\xi) < (A+B)^2$ or $g_- < \xi < g_+$. In this case

$F(\xi) < 0$ and

$$\begin{aligned}
 X(\xi) &= 2EF \int_0^{\lambda_+(\xi)} \frac{d\lambda}{(\lambda+1)[a(\xi)\lambda^2+2b(\xi)\lambda+c(\xi)]^{\frac{1}{2}}} \\
 &= \frac{1}{\sqrt{-F(\xi)}} \left[\arcsin \frac{4E^2F^2F(\xi)[\lambda_+(\xi)+1]^{-1}+b(\xi)-a(\xi)}{\{(b(\xi))^2-a(\xi)c(\xi)\}^{\frac{1}{2}}} \right. \\
 &\quad \left. - \arcsin \frac{c(\xi)-b(\xi)}{\{(b(\xi))^2-a(\xi)c(\xi)\}^{\frac{1}{2}}} \right] \\
 &= \frac{1}{\sqrt{-F(\xi)}} \left[\frac{\pi}{2} - \arcsin \frac{c(\xi)-b(\xi)}{\{(b(\xi))^2-a(\xi)c(\xi)\}^{\frac{1}{2}}} \right], \quad (3-70)
 \end{aligned}$$

on using eqs. (3-62) and (3-67). Here

$$(b(\xi))^2 - a(\xi)c(\xi) = (c(\xi)-b(\xi))^2 - c(\xi)[a(\xi)-2b(\xi)+c(\xi)], \quad (3-71)$$

and so, using eqs. (3-69) and (3-67), eq. (3-70) may be rewritten in the form

$$\begin{aligned}
 X(\xi) &= \frac{1}{\sqrt{-F(\xi)}} \left[\frac{\pi}{2} - \arcsin \frac{c(\xi)-b(\xi)}{\{-4E^2F^2F(\xi)\}^{\frac{1}{2}}\{c(\xi)\}^{\frac{1}{2}}} \right] \\
 &= \frac{1}{\sqrt{-F(\xi)}} \left[\frac{\pi}{2} - \arcsin \frac{G(\xi)}{\sqrt{-F(\xi)} \sqrt{\xi^2-1}} \right], \quad (3-72)
 \end{aligned}$$

where

$$\begin{aligned}
 G(\xi) &= (4E^2F^2)^{-1}(c(\xi)-b(\xi)) \\
 &= \xi^2 - 1 + (EF)^{-1}G[(Ea+Eb)\xi+Eb+Fa], \quad (3-73)
 \end{aligned}$$

as given in eq. (3-14).

$$(b) \quad v(\xi) < (A-B)^2, \quad (\xi < g_-) \quad \text{or} \quad v(\xi) > (A+B)^2, \quad (\xi > g_+).$$

In these cases $F(\xi) > 0$ and

$$\begin{aligned}
X(\xi) &= 2EF \int_0^{\lambda_+(\xi)} \frac{d\lambda}{(\lambda+1)[a(\xi)\lambda^2 + 2b(\xi)\lambda + c(\xi)]^{\frac{1}{2}}} \\
&= \frac{1}{\sqrt{F(\xi)}} \cdot \ln \left[\frac{\lambda+1}{4E^2F^2F(\xi) + [b(\xi)-a(\xi)](\lambda+1) + 4E^2F^2\sqrt{F(\xi)}[U(\xi,\lambda)]^{\frac{1}{2}}} \right] \Bigg|_0^{\lambda_+(\xi)} \\
&= \frac{1}{\sqrt{F(\xi)}} \ln \left[\frac{c(\xi)-b(\xi)+2EF\sqrt{F(\xi)}\sqrt{c(\xi)}}{4E^2F^2F(\xi)[\lambda_+(\xi)+1]^{-1} + b(\xi)-a(\xi)} \right], \quad (3-74)
\end{aligned}$$

on using eqs. (3-67), (3-69) and (3-49). Note that

$$4E^2F^2F(\xi) + [b(\xi)-a(\xi)](\lambda+1) > 0 \quad \text{for } 0 \leq \lambda \leq \lambda_+(\xi). \quad (3-75)$$

Since $F(\xi) > 0$, the result follows immediately, if $[b(\xi)-a(\xi)] \geq 0$.

If however, $[b(\xi)-a(\xi)] < 0$, we use the fact that

$$4E^2F^2F(\xi) = a(\xi) - 2b(\xi) + c(\xi)$$

together with eq. (3-62) to deduce that

$$4E^2F^2F(\xi) + [b(\xi)-a(\xi)](\lambda_+(\xi)+1) = (\lambda_+(\xi)+1) [(b(\xi))^2 - a(\xi)c(\xi)]^{\frac{1}{2}} > 0,$$

and eq. (3-75) follows.

Using eqs. (3-62) and (3-71), we see that the argument of the logarithm in eq. (3-74) is

$$\left[\frac{c(\xi)-b(\xi)+2EF\sqrt{F(\xi)}\sqrt{c(\xi)}}{c(\xi)-b(\xi)-2EF\sqrt{F(\xi)}\sqrt{c(\xi)}} \right]^{\frac{1}{2}}$$

and so

$$X(\xi) = \frac{\frac{1}{2}}{\sqrt{F(\xi)}} \ln \left(\frac{\Sigma(\xi)+1}{\Sigma(\xi)-1} \right) \quad (3-76)$$

where

$$\Sigma(\xi) = \frac{G(\xi)}{\sqrt{F(\xi)}\sqrt{\xi^2-1}}, \quad (3-77)$$

as given in eq. (3-13). Here we have used eqs. (3-69) and (3-73). Eqs. (3-71), (3-73) and (3-75) show that $\Sigma(\xi) > 1$ for $\xi > 1$, and consequently $\tilde{\Sigma}(t) > 1$ for $t > (E+F)^2$ where $\tilde{\Sigma}(t)$ is given in eq. (3-9).

3-6. Explicit forms of the weight functions $X(\xi)$ and $Y(\xi)$

In this section we consider the explicit form of $X(\xi)$ and $Y(\xi)$ for all the cases detailed in (3-46).

$$(1) \quad (A+B)^2 \leq (E+F)^2, \quad (g_+ \leq 1) \quad . \quad \text{Normal threshold only.}$$

In this case $X(\xi)$ is given, for $\xi > 1$, by eqs. (3-76) and (3-77). This is exactly the result given in eqs. (3-18) and (3-13) of sect. 3-2. Thus we have shown that the non-rigorous method used in sect. 3-2 gives, for the case $(A+B)^2 \leq (E+F)^2$, the same result as that obtained by transforming the Feynman parametrisation of the triangle diagram amplitude.

From eqs. (3-73), (3-51) and (3-52) we see that

$$4E^2F^2G(1) = -b(1) = 4EFG(E+F)(a+b), \quad (3-78)$$

so that for a normal threshold,

$$G(1) > 0.$$

It then follows from eqs. (3-77) and (3-76) that, as $\xi \downarrow 1$, $\Sigma(\xi) \rightarrow +\infty$ and $X(\xi) \downarrow 0$.

$$(2) \quad (E+F)^2 \leq (A-B)^2, \quad (1 \leq g_-) \quad . \quad \text{Normal threshold only.}$$

Again $X(\xi)$ is given, for $1 < \xi < g_-$ and for $\xi > g_+$ by eqs. (3-76) and (3-77). The argument of case (1) above shows that $\Sigma(\xi) \rightarrow +\infty$ and $X(\xi) \downarrow 0$ as $\xi \downarrow 1$. For $g_- < \xi < g_+$, $X(\xi)$ is given by eq. (3-72). We now establish the continuity of $X(\xi)$ at $\xi = g_-$ and at $\xi = g_+$. Note that $G(\xi)$ is a quadratic function of

ξ which is positive at $\xi = 1$ and for all sufficiently large ξ . We denote the solutions of $G(\xi) = 0$ by $\gamma_{\pm}$ where $\gamma_{\pm}$ are either complex conjugates or real and satisfy $\gamma_+ \geq \gamma_-$. Then $G(\xi) > 0$ for all $\xi > 1$ if $\gamma_{\pm}$ are complex conjugates or if $\gamma_- \leq \gamma_+ < 1$. The only other possibility is $1 < \gamma_- \leq \gamma_+$. In this case we see from eqs. (3-71), (3-54), (3-69) and (3-72) that

$$[G(\xi)]^2 - (\xi^2 - 1)F(\xi) = E^{-2}F^{-2}G^2(2EF\xi + E^2 + F^2)(\xi - \xi_{1+})(\xi - \xi_{1-}), \quad (3-79)$$

so that using the properties of $\xi_{1\pm}$ given in sect. 3-5 we have

$$g_- < \gamma_- \leq \gamma_+ < g_+ \quad (3-80)$$

and

$$G(g_{\pm}) > 0.$$

We can now expand the logarithm in eq. (3-76) and the inverse tangent in eq. (3-72), and we find that the left and right hand limits of $X(\xi)$ at $\xi = g_-$ are equal. Each of these limits is equal to the value of $X(g_-)$ obtained by direct evaluation of the integral in eq. (3-65), namely

$$X(g_-) = \frac{2\sqrt{g_-+1}\sqrt{g_- - 1}}{G(g_-)}, \quad (3-81)$$

and hence $X(\xi)$ is continuous at $\xi = g_-$. Similarly $X(\xi)$ is continuous at $\xi = g_+$, with

$$X(g_+) = \frac{2\sqrt{g_++1}\sqrt{g_+ - 1}}{G(g_+)}. \quad (3-82)$$

$$(3) \quad (A-B)^2 < (E+F)^2 < (A+B)^2, \quad (g_- < 1 < g_+).$$

(a) Normal threshold case.

For $1 < \xi < g_+$, $X(\xi)$ is given by eq. (3-72), while for

$\xi > g_+$, $X(\xi)$ is given by eqs. (3-76) and (3-77). The argument given in case (1) above shows that $X(\xi) \downarrow 0$ as $\xi \downarrow 1$, while the argument in case (2) shows that $X(\xi)$ is continuous at $\xi = g_+$, with $X(g_+)$ given by eq. (3-82).

(b) Anomalous threshold case.

For $\xi_{1+} \leq \xi \leq 1$, $Y(\xi)$ is given by eq. (3-68). For $1 < \xi < g_+$, $X(\xi)$ is given by eq. (3-72) and for $\xi > g_+$, $X(\xi)$ is given by eqs. (3-76) and (3-77). In this case we see from eq. (3-78) that $G(1) < 0$ and consequently $\gamma_- < 1 < \gamma_+$. We then use the fact that $F(\xi)$, as well as $G(\xi)$, is a quadratic function of ξ which is positive for all sufficiently large ξ , together with eq. (3-79), to show that

$$F(\gamma_+) < 0 .$$

Hence

$$1 < \gamma_+ < g_+ ,$$

$$G(g_+) > 0 ,$$

and $X(\xi)$ is again continuous at $\xi = g_+$, with $X(g_+)$ given in eq. (3-82). Since $G(1) < 0$, it follows from eq. (3-72) that $X(\xi)$ has a right hand limit at $\xi = 1$ which is just $Y(1)$ given by eq. (3-68) when $\xi = 1$.

(c) Transition Case

As in the normal threshold case, $X(\xi)$ is given by eq. (3-72) for $1 < \xi < g_+$ and by eqs. (3-76) and (3-77) for $\xi > g_+$. Again $X(g_+)$ is given by eq. (3-82) and $X(\xi)$ is continuous at $\xi = g_+$. Eq. (3-78) shows that, in this case, $G(1) = 0$. It then follows from eq. (3-72) that, as $\xi \downarrow 1$,

$$X(\xi) \rightarrow \frac{\frac{1}{2}\pi}{\sqrt{g_+-1}\sqrt{1-g_-}}.$$

We have now completed the study of the weight functions $X(\xi)$ and $Y(\xi)$ for all the cases given in (3-46).

3-7. Continuation of $I(\tau_0)$: Normal threshold case

As discussed in chapter 2, the method of analytic continuation used originally by Mandelstam (1960) has been very useful in examining the analyticity properties of Feynman amplitudes. We now give the details of the continuation procedure for the case when the masses of the six particles shown in fig. 3-1 are different. We start from the single dispersion relation for $I(\tau_0)$ for the case $(A+B)^2 \leq (E+F)^2$, $(g_+ \leq 1)$, and continue in A^2 and B^2 , the squares of the external masses. The internal masses E, F, G are arbitrary positive numbers which remain constant throughout. The stability conditions confine A^2, B^2 to the intervals

$$0 < A^2 < (E+G)^2, \quad 0 < B^2 < (F+G)^2,$$

or, in terms of a and b given in eq. (3-11),

$$-1 < a < (2EG)^{-1}(E^2+G^2), \quad -1 < b < (2FG)^{-1}(F^2+G^2). \quad (3-83)$$

We recall that the case $(A+B)^2 \leq (E+F)^2$ was the case for which we obtained a single dispersion relation in sect. 3-2 by using a heuristic argument. However it is not easily seen from eqs. (3-17) and (3-18) how to carry out the continuation; so we examine instead the expression for $I(\tau_0)$ given in eq. (3-21). We replace a and b in eq. (3-21) by the complex variables z_1 and z_2 respectively, and

define formally a function

$$I(\tau_0, z_1, z_2) = EF \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left[E^2 \alpha^2 + F^2 \beta^2 + G^2 (1-\alpha-\beta)^2 + 2EGz_1 \alpha(1-\alpha-\beta) + 2FGz_2 \beta(1-\alpha-\beta) - 2EF\tau_0 \alpha\beta \right]^{-1}. \quad (3-84)$$

To simplify the continuation procedure, τ_0 is taken to be real;

$\text{Re}\tau_0 = y_0$, $\text{Im}\tau_0 = 0$ and in addition we restrict y_0 to

$y_0 \leq -(2EF)^{-1}(E^2 + F^2)$. We then see from eq. (3-84) that

$I(y_0, z_1, z_2)$ is defined and is an analytic function of (z_1, z_2) at least in the Cartesian product $S_1 \times S_2$ of the open sets

$$S_1 : -1 < \text{Re}z_1 = a < (2EG)^{-1}(E^2 + G^2), \quad \text{Im}z_1 < 0$$

$$S_2 : -1 < \text{Re}z_2 = b < (2FG)^{-1}(F^2 + G^2), \quad \text{Im}z_2 < 0.$$

(The above choice of imaginary parts is equivalent to Feynman's prescription of adding negative imaginary parts to the internal masses squared. We could of course equally well have chosen $\text{Im}z_1 > 0$, $\text{Im}z_2 > 0$.)

As we have already seen in sect. 3-3, $I(y_0, a, b)$ is defined for $-1 < a < (2EG)^{-1}(E^2 + G^2)$, $-1 < b < (2FG)^{-1}(F^2 + G^2)$, and its value can be obtained from $I(y_0, z_1, z_2)$ by letting $\text{Im}z_1 \uparrow 0$, $\text{Im}z_2 \uparrow 0$, with a, b held fixed. In fact, when $a + b > 0$ so that the threshold is a normal threshold, the continuation can be carried out without going into the complex plane. It is only when we consider the transition and anomalous threshold cases that we need to take z_1 and z_2 into the lower half plane and then back to the real axis.

We begin by writing $I(y_0)$ in full for the case $(A+B)^2 \leq (E+F)^2$, $(g_+ \leq 1)$, which is a normal threshold case, satisfying

$$a + b > 0. \quad (3-85)$$

We found by the heuristic argument of sect. 3-2, and confirmed in sect. 3-6, that

$$I(y_0) = \int_1^\infty \frac{d\xi X(\xi)}{\xi - y_0}, \quad (3-86)$$

where

$$X(\xi) = \frac{\frac{1}{2}}{\sqrt{F(\xi)}} \ln E(\xi), \quad (3-87)$$

$$E(\xi) = \frac{G(\xi) + H(\xi)}{G(\xi) - H(\xi)}, \quad (3-88)$$

$$G(\xi) = \xi^2 + 1 + (EF)^{-1} G[(Ea + Fb)\xi + Eb + Fa], \quad (3-89)$$

$$H(\xi) = \sqrt{\xi^2 - 1} \sqrt{F(\xi)}, \quad (3-90)$$

$$F(\xi) = (\xi - g_+)(\xi - g_-), \quad (3-91)$$

$$g_\pm = (2EF)^{-1} [(\sqrt{E^2 + G^2 - 2EGa} \pm \sqrt{F^2 + G^2 - 2FGb})^2 - E^2 - F^2]. \quad (3-92)$$

The function $G(\xi)$ is also given by eqs. (3-73) and (3-14), $F(\xi)$ by eqs. (3-67) and (3-15) and $g_\pm$ by eq. (3-16). We saw in sects. 3-5 and 3-6 that for this case

$$G(\xi) > H(\xi) \geq 0$$

for $\xi \geq 1$ ($\geq g_+$), equality occurring if and only if $\xi = 1$. Thus $E(\xi) \geq 1$ for $\xi \geq 1$ and $\ln E(\xi) \geq 0$.

To do the continuation in a and b , or equivalently in A^2 and B^2 , we choose $\tau_0 = y_0 < -(2EF)^{-1}(E^2 + F^2)$, so that the denominator in eq. (3-86) does not vanish during the continuation procedure. (In fact $y_0 < -1$ would be sufficient to ensure this.) The functions

defined in eqs. (3-86) to (3-92) are then considered to depend explicitly on the complex variables z_1, z_2 which replace a, b respectively, the logarithm in eq. (3-87) and the square roots in eqs. (3-87), (3-90) and (3-92) being now interpreted in a more general sense. The singularities of $X(\xi)$ must be studied since, as pointed out by Mandelstam (1960) and Eden (1952, 1967, sect. 4.5), the behaviour of these singularities as z_1, z_2 go into the lower half plane and return to the real axis determines how the path for the integration over ξ must be deformed to carry out the continuation.

Considering first only real z_1, z_2 we see that $X(\xi)$ is singular when

$$G(\xi) = \pm H(\xi) ,$$

that is when

$$0 = [G(\xi)]^2 - [H(\xi)]^2 = E^{-2}F^{-2}G^2(2EF\xi + E^2 + F^2)(\xi - \xi_{1+})(\xi - \xi_{1-}) , \quad (3-93)$$

where $\xi_{1\pm}$ are given in eq. (3-55) and of course $-(2EF)^{-1}(E^2 + F^2) \leq -1$.

When $(1-a)(1-b) < 0$, $\xi_{1\pm}$ are complex conjugates, while if $(1-a)(1-b) \geq 0$, $\xi_{1\pm}$ are given by eq. (3-56) and the inequalities in eq. (3-57) hold. Thus $\xi_{1+} = 1$ if and only if

$$a + b = 0 ,$$

which is the condition for the transition case.

We now see that, provided eq. (3-85) continues to hold, a and b , or equivalently A^2 and B^2 , can be changed from the case when $(A+B)^2 \leq (E+F)^2$, $(g_+ \leq 1)$ and the singularities $\xi_{1\pm}$ will not appear on the path of integration $[1, +\infty)$. Further, the arguments leading to eqs. (3-82) and (3-81) show that $X(\xi)$ is not singular at

$\xi = g_+$ (resp. $\xi = g_-$) when $g_+ \geq 1$ (resp. $g_- \geq 1$). Thus, the right hand side of eq. (3-86) continues to define $I(y_0)$, with the logarithm in eq. (3-87) and the square roots in eqs. (3-87) and (3-90) interpreted more generally, as remarked earlier. We now check that this prescription reproduces the expressions for $X(\xi)$ given in sects. 3-5 and 3-6 for the three subcases detailed there.

(1) $(A+B)^2 \leq (E+F)^2$, $\{g_+ \leq 1\}$. This is the case from which we started the continuation and $X(\xi)$ is given by eqs. (3-76) and (3-77) (repeated in eqs. (3-87) to (3-92) and in eqs. (3-18) and (3-13)).

(2) $(E+F)^2 \leq (A-B)^2$, $\{1 \leq g_-\}$. When $1 \leq \xi < g_-$ and when $g_+ < \xi$, we have $F(\xi) > 0$ and $E(\xi)$ defined in eq. (3-88) is real. From eq. (3-93) we see that $|G(\xi)| > |H(\xi)|$, so that $E(\xi) > 0$ and the logarithm in eq. (3-87) is the ordinary natural logarithm. Thus $X(\xi)$ is again given by eqs. (3-76) and (3-77).

When $g_- < \xi < g_+$, we have $F(\xi) < 0$. Defining

$$K(\xi) = \sqrt{\xi^2 - 1} \sqrt{-F(\xi)} > 0,$$

we have

$$\begin{aligned} X(\xi) &= \frac{\frac{1}{2}}{-i\sqrt{-F(\xi)}} \log \left[\frac{G(\xi) - iK(\xi)}{G(\xi) + iK(\xi)} \right] \\ &= \frac{\arg[G(\xi) + iK(\xi)]}{\sqrt{-F(\xi)}} = \frac{1}{\sqrt{-F(\xi)}} \left[\frac{\pi}{2} - \arctan \frac{G(\xi)}{K(\xi)} \right], \end{aligned} \quad (3-94)$$

as given in eq. (3-72). This result for $X(\xi)$ does not depend on which square root of $F(\xi)$ is chosen.

(3) $(A-B)^2 < (E+F)^2 < (A+B)^2$, $\{g_- < 1 < g_+\}$. The arguments of case (2) apply. When $\xi > g_+$, $X(\xi)$ is given by eqs. (3-76) and

(3-77); when $1 < \xi < g_+$, $X(\xi)$ is given by eq. (3-72).

Note that the arguments in sect. 3-6 concerning the right hand limit of $X(\xi)$ at $\xi = 1$ and the continuity of $X(\xi)$ at $\xi = g_+$ (resp. $\xi = g_-$) when $g_+ \geq 1$ (resp. $g_- \geq 1$) may still be used; they are independent of the evaluation of the Feynman integral.

3-8. Continuation of $I(\tau_0)$: Transition and anomalous threshold cases

To carry out the continuation of $I(y_0)$ from the normal threshold cases where $a + b > 0$ to the anomalous threshold cases where $a + b < 0$, we need to use complex z_1, z_2 which replace a, b . This is because for the transition case between the two cases, $a + b = 0$, $\xi_{1+} = 1$ for real a, b and hence there is a singularity of the integrand at an end point of the path of integration in eq. (3-86).

We choose to start the continuation from a normal threshold case for which $|a| < 1$, $|b| < 1$ and in addition $g_- < 1 < g_+$ (or $(A-B)^2 < (E+F)^2 < (A+B)^2$); these requirements are, of course, satisfied for all anomalous threshold cases (eq. (3-46)). The functions defined in eqs. (3-86) - (3-92) and in eq. (3-55) are now considered to depend on z_1, z_2 which replace a, b . Then, for sufficiently small $\text{Im} z_1, \text{Im} z_2$,

$$\text{Reg}_- < 1 < \text{Reg}_+ \quad (3-95)$$

throughout the continuation procedure and an adaptation of the argument leading to eq. (3-82) shows that $X(\xi)$ is not singular at $\xi = g_+$.

In fact, we shall carry out the continuation in such a way that we only need this result for real a, b and hence real g_+ . Further, for sufficiently small $\text{Im}z_1, \text{Im}z_2$, we see from eq. (3-45) that for the anomalous threshold case where $a + b < 0$,

$$\text{Re}g_- < \text{Re}\xi_{1\pm}. \quad (3-96)$$

Eqs. (3-95) and (3-96) ensure that we can carry out the continuation in such a way that g_- never lies on the (possibly deformed) path of integration.

From eq. (3-93) we see that $X(\xi)$ is singular when $\xi = \xi_{1\pm}$ and $\xi = -(2EF)^{-1}(E^2 + F^2) \leq -1$, where $\xi_{1\pm}$ are given in eq. (3-55) with z_1, z_2 replacing a, b , that is,

$$\xi_{1\pm} = -z_1 z_2 \pm \left[\left(1 - z_1^2 \right) \left(1 - z_2^2 \right) \right]^{\frac{1}{2}}.$$

To study the behaviour of $\xi_{1\pm}$ in the complex ξ plane as we take z_1, z_2 into the lower half planes and then back to the real axis, it is convenient to introduce the quantities

$$\alpha = \arccos a, \quad \beta = \arccos b \quad (3-97)$$

where

$$0 < \alpha < \pi, \quad 0 < \beta < \pi$$

since $|a| < 1, |b| < 1$. We also introduce the quantities

$$\theta_1 = \arccos z_1, \quad \theta_2 = \arccos z_2. \quad (3-98)$$

Here θ_1 (resp. θ_2) is defined by continuity for small $\text{Im}z_1$ (resp. $\text{Im}z_2$) so that it reduces to α (resp. β) when $\text{Im}z_1 \uparrow 0$ (resp. $\text{Im}z_2 \uparrow 0$). Then

$$\xi_{1\pm} = -\cos(\theta_1 \pm \theta_2). \quad (3-99)$$

The steps in the continuation are then as follows

- (a) Start with (a, b) such that $a + b > 0$, (that is, start with (α, β) such that $\alpha + \beta < \pi$);
- (b) make $\text{Im}z_1 < 0$, $\text{Im}z_2 < 0$, where $\text{Im}z_1, \text{Im}z_2$ are both small;
- (c) fix $\text{Im}z_1, \text{Im}z_2$ and change (a, b) until $a + b \leq 0$ (that is, change (α, β) until $\alpha + \beta \geq \pi$);
- (d) let $\text{Im}z_1 \uparrow 0$, $\text{Im}z_2 \uparrow 0$.

We need to consider the way in which $\xi_{1\pm}$ move as we go through steps

(a)-(d). If terms up to second order in $\text{Im}z_1, \text{Im}z_2$ are retained then

$$\text{Re}(\theta_{1\pm}\theta_2) \approx (\alpha\pm\beta) + \frac{1}{2}\left\{\cos\alpha(\sin\alpha)^{-3}(\text{Im}z_1)^2 \pm \cos\beta(\sin\beta)^{-3}(\text{Im}z_2)^2\right\},$$

$$\text{Im}(\theta_{1\pm}\theta_2) \approx \left\{\text{Im}z_1(\sin\alpha)^{-1} \pm \text{Im}z_2(\sin\beta)^{-1}\right\}.$$

Using the fact that

$$\cos\theta = \cos(\text{Re}\theta)\cosh(\text{Im}\theta) - i\sin(\text{Re}\theta)\sinh(\text{Im}\theta), \quad (3-100)$$

we then find that

$$-\text{Re}\xi_{1\pm} \approx \cos(\alpha\pm\beta)$$

$$\begin{aligned} & -\frac{1}{2}\sin(\alpha\pm\beta)\left\{\cos\alpha(\sin\alpha)^{-3}(\text{Im}z_1)^2 \pm \cos\beta(\sin\beta)^{-3}(\text{Im}z_2)^2\right\} \\ & + \frac{1}{2}\cos(\alpha\pm\beta)\left\{\text{Im}z_1(\sin\alpha)^{-1} \pm \text{Im}z_2(\sin\beta)^{-1}\right\}^2, \end{aligned} \quad (3-101a)$$

$$-\text{Im}\xi_{1\pm} \approx \sin(\alpha\pm\beta)\left\{\text{Im}z_1(\sin\alpha)^{-1} \pm \text{Im}z_2(\sin\beta)^{-1}\right\}. \quad (3-101b)$$

We see from eq. (3-101a) that for sufficiently small $\text{Im}z_1, \text{Im}z_2$,

$\text{Re}\xi_{1-}$ is always less than 1, so that this singularity does not

force us to deform the path of integration over ξ . Also $\text{Re}\xi_{1-} < \text{Re}\xi_{1+}$.

always so that the (possibly deformed) path does not become pinched between the two singularities. From eq. (3-101), we also see that the path taken by ξ_{1+} as steps (a)-(d) are taken is as shown in fig.

3-3. Fig. 3-3 is drawn for the case when at step (c), $a + b < 0$.

If at step (c), $a + b = 0$, then ξ_{1+} moves as shown in fig. 3-3 until it meets the real axis to the right of $(1, 0)$ and then moves to the left along the real axis to $\xi_{1+} = 1$ at step (d).

For the anomalous threshold case, once ξ_{1+} crosses the real axis to the right of $(1, 0)$ and moves into the lower half-plane, the path of integration used to define $I(y_0)$ must be deformed; the modified path is shown in fig. 3-4. At step (a), Cauchy's theorem assures us that the path C could replace the interval $[1, +\infty)$ of the real axis as the path of the ξ integration in eq. (3-86) without affecting $I(y_0)$. We choose the path C at step (a) such that in the neighbourhood of $\xi = g_+$ it lies in the lower half plane and such that for every point on C , $\text{Re}\xi > g_-$. Then as we take steps (b), (c) and (d), eqs. (3-95) and (3-96) ensure that we can continuously deform the path of integration such that for every point on C , $\text{Re}\xi > \text{Re}g_-$ and $\xi = g_+$ does not lie on the path C until after step (d) has been taken, (see fig. 3-4).

The function $X(\xi)$ will now be required for complex ξ , which involves interpreting the square roots in eqs. (3-87) and (3-90) and the logarithm in eq. (3-87) in a more general sense. The required continuation of $I(y_0)$ for step (c) is then given by

$$I(y_0) = \int_C \frac{d\xi X(\xi)}{\xi - y_0} \quad (3-102)$$

Fig. 3-3.

Path taken by ξ_{1+} in the complex plane in steps (a)-(d) of sect. 3-8.

Fig. 3-4.

Modified path C required when ξ_{1+} moves into the lower half-plane.

where $X(\xi)$ is obtained by the continuation procedure.

The logarithmic singularity at $\xi = \xi_{1+}$ has associated with it a branch cut. As ξ_{1+} traces out the path in fig. 3-3, we take this branch cut to be the line segment joining ξ_{1+} to 1, plus the interval $(-\infty, 1]$ of the real axis. The curve C is drawn in such a way that this branch cut does not pass through it, (except at the point $(1, 0)$). The branch cut from the logarithmic singularity at $\xi = \xi_{1-}$ is taken to be the straight line segment joining ξ_{1-} to the point $\{-(2EF)^{-1}(E^2+F^2), 0\}$, plus the interval $(-\infty, -(2EF)^{-1}(E^2+F^2)]$ of the real axis, while the branch cut from the point $\{-(2EF)^{-1}(E^2+F^2), 0\}$ is taken to be the interval $(-\infty, -(2EF)^{-1}(E^2+F^2)]$ of the real axis. The branch cuts of the square root functions in eqs. (3-87) and (3-90) are taken between g_- and g_+ and between -1 and 1 in such a way that they do not pass through C , (except at the point $(1, 0)$).

At step (d), ξ_{1+} approaches the real axis from below, and the path C is forced into the real axis at $\xi = \xi_{1+}$. It may then be deformed to the segment of the real axis from ξ_{1+} to $+\infty$ with the interval $[\xi_{1+}, 1]$ being traversed *in both directions*. Thus

$$I(y_0) = \int_1^\infty \frac{d\xi X(\xi)}{\xi - y_0} + \int_{\xi_{1+}}^1 \frac{d\xi \text{disc.}[X(\xi)]}{\xi - y_0}, \quad (3-103)$$

where

$$\begin{aligned} \text{disc.}[X(\xi)] = & \left[(\text{value of } X(\xi) \text{ below the cut from } \xi_{1+} \text{ to } 1) \right. \\ & \left. - (\text{value of } X(\xi) \text{ above the cut from } \xi_{1+} \text{ to } 1) \right], \end{aligned}$$

and it remains to evaluate $X(\xi)$ and $\text{disc.}[X(\xi)]$.

For $\xi > 1$, the arguments of sect. 3-7 apply to the evaluation of $X(\xi)$. Given any $\xi > 1$, $\text{Im}z_1, \text{Im}z_2$ can be chosen sufficiently small for ξ_{1+} to cross the real axis to the left of ξ . Hence $X(\xi)$ is given by eqs. (3-76) and (3-77) for $\xi > g_+$ and by eq. (3-72) when $1 < \xi < g_+$. This same argument applies to the transition case. The right hand limit of $X(\xi)$ at $\xi = 1$ is obtained exactly as in sect. 3-6 and as mentioned earlier $X(\xi)$ is continuous at $\xi = g_+$ with $X(g_+)$ given by eq. (3-82).

Next we evaluate $\text{disc.}[X(\xi)]$ for $\xi_{1+} < \xi < 1$. To obtain $[F(\xi)]^{\frac{1}{2}}$ and $H(\xi)$, we note that ξ , on the path C , has approached the real axis from below. Thus taking the cuts in the square root functions in eqs. (3-87) and (3-90) as being the straight line segments between g_- and g_+ and between -1 and 1 at step (d), we see that, with

$$g_- < \xi_{1+} < \xi < 1 < g_+,$$

$$[F(\xi)]^{\frac{1}{2}} = -i\sqrt{-F(\xi)} \quad (3-104)$$

and

$$H(\xi) = -M(\xi),$$

where

$$M(\xi) = \sqrt{g_+ - \xi} \sqrt{\xi - g_-} \sqrt{1 - \xi} \sqrt{\xi + 1} > 0.$$

From eq. (3-93) it follows that

$$[G(\xi_{1+})]^2 - [M(\xi_{1+})]^2 = 0,$$

and using eq. (3-59) as well, we have

$$[G(\xi)]^2 \geq [M(\xi)]^2 > 0 \quad \text{for } \xi_{1+} \leq \xi < 1 ,$$

so that $G(\xi)$ cannot vanish in this interval. Since, from eq. (3-78), $G(1) < 0$, it follows that $G(\xi_{1+}) < 0$ and hence

$$G(\xi_{1+}) + M(\xi_{1+}) = 0 .$$

Thus $E(\xi)$ in eq. (3-88) can be put in the form

$$E(\xi) = \frac{[G(\xi) - M(\xi)]^2}{E^{-2} F^{-2} G^2 (2EF\xi + E^2 + F^2) (\xi - \xi_{1+}) (\xi - \xi_{1-})} , \quad (3-105)$$

so that

$$E(\xi) \sim (\text{Positive valued function of } \xi) (\xi - \xi_{1+})^{-1} .$$

Hence

$$\text{disc.}[\arg E(\xi)] = -\text{disc.}[\arg(\xi - \xi_{1+})] = -2\pi ,$$

and

$$\text{disc.}[\log E(\xi)] = -2\pi i , \quad (3-106)$$

where the discontinuity is again computed as

$$[(\text{value below cut}) - (\text{value above cut})] .$$

Strictly, $(-2\pi i)$ is the limit of the left hand side of eq. (3-106) as ξ_{1+} approaches the real axis from below, and as C is deformed to the segment of the real axis from ξ_{1+} to $+\infty$ with the interval $[\xi_{1+}, 1]$ being traversed in both directions. Finally, from eqs.

(3-87), (3-104) and (3-106) we have

$$\text{disc.}[X(\xi)] = \frac{\pi}{\sqrt{-F(\xi)}} = Y(\xi) , \quad (3-107)$$

where $Y(\xi)$ is given in eq. (3-68).

We now continue $I(y_0)$ to complex τ_0 . For the normal threshold

and transition cases $I(\tau_0)$ is given by eq. (3-63) for all $\tau_0 \notin [1, +\infty)$ and for the anomalous threshold case $I(\tau_0)$ is given by eq. (3-64) for all $\tau_0 \notin [\xi_{1+}, +\infty)$. The explicit forms of the weight functions are as given in sects. 3-5 and 3-6 for the various cases given in eq. (3-46).

In the following two chapters, the methods used in this chapter are extended to apply to the box diagram amplitude. In chapter 4, we extend the method of direct transformation to obtain a number of different spectral representations for the box diagram amplitude. We find that (for the range of s and t considered) the box diagram amplitude can be written either in the form of a double dispersion integral in s and t or as the sum of a double dispersion integral in s and t and a single dispersion integral in s , depending on the values of the masses of the particles in fig. 4-1. In this chapter, we also discuss a heuristic method of obtaining the weight function in an approximate normal threshold dispersion relation for the two-particle $\rightarrow$ two-particle scattering amplitude. Then, in chapter 5, we start with a particular normal threshold double dispersion relation obtained in chapter 4 and continue in the external masses and then in s and t . In this way, we obtain a spectral representation for the box diagram amplitude for both real and complex s and t . In particular, a spectral representation is obtained for all s in the upper half complex plane and for almost all $-\infty < t < +\infty$.

4. THE BOX DIAGRAM AMPLITUDE: DIRECT TRANSFORMATION

4-1. Introduction

In this chapter we start our investigation of the analytic properties of the box diagram amplitude $T_{\square}(s, t)$ associated with the Feynman diagram shown in fig. 4-1. This amplitude describes the collision process $AB \rightarrow CD$ where the intermediate particles in the process are E, F, G and H as shown in fig. 4-1. Our main object is to obtain a spectral representation of $T_{\square}(s, t)$ for all masses satisfying the stability conditions $A < (E+G)$, $B < (F+G)$, $C < (E+H)$, $D < (F+H)$. We also obtain an approximation to the scalar invariant scattering amplitude $T(s, t)$ for the process $AB \rightarrow CD$ and write this approximation in the form of a double dispersion relation. Then we show that when $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$ this double spectral representation is the same as that obtained for $T_{\square}(s, t)$.

We begin in sect. 4-2 by writing down the scalar invariant scattering amplitude $T(s, t)$ for the process $AB \rightarrow CD$. Then we obtain an approximation to $T(s, t)$ in the form of a double spectral representation by using unitarity to evaluate the contribution to $\text{Im}T$ from a two-particle intermediate state (EF) and making t -channel single pole approximations to the amplitudes for the processes $AB \rightarrow EF$, $CD \rightarrow EF$.

The rest of this chapter is then devoted to a study of the box diagram amplitude $T_{\square}(s, t)$. In section 4-3 we transform the box diagram amplitude from its Feynman parametrised form to a more convenient form and discuss the implication of the stability conditions $A < (G+E)$, $B < (G+F)$, $C < (H+E)$, $D < (H+F)$ for the vertices of the diagram. The boundary of the region of integration in the triple

Fig. 4-1.

Box diagram for the scattering process $AB \rightarrow CD$.

integral form obtained in section 4-3 is studied in section 4-4 and in section 4-5 we obtain some results necessary for reversing the order of integration in the triple integral.

The order of integration is then reversed in sections 4-6 and 4-7 and two of the integrations are carried out to obtain a single spectral representation of the box diagram amplitude. We also show that under certain circumstances the single dispersion integral can be written as a double dispersion integral in s and t .

Finally in section 4-8 we study the region in which the double spectral function is non-zero. For a restricted range of s and t (or x and y) we find relations between the masses of the interacting particles for which the double spectral representation is valid and give the explicit form of the spectral representation for all allowed mass configurations, including cases for which the double spectral representation breaks down. In particular, we show that for the case when $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$, the double spectral representation obtained by the heuristic method in sect. 4-2 is the same as that obtained by the direct transformation of the Feynman parametrisation of the box diagram amplitude.

4-2. Approximation to the invariant amplitude

As discussed in chapter 2, the method of approximating the amplitudes on the right hand side of the unitarity relation and then obtaining a dispersion relation is well known (Mandelstam 1959b). We now give a slightly modified version of Mandelstam's heuristic method and then in sects. 4-6 and 4-8 compare the result obtained in this way with that obtained from the transformation of the Feynman parametrisation of the box diagram amplitude.

With plane wave states normalised so that $\langle p' | p \rangle = \delta^{(3)}(p' - p)$

we define the scalar Lorentz invariant scattering amplitude $T(s, t)$ for the process $AB \rightarrow CD$ in terms of the S -operator by

$$\langle p_C p_D | (S-1) | p_A p_B \rangle = -i(2\pi)^{-2} \delta^{(4)}(p_C + p_D - p_A - p_B) \frac{1}{4} (E_A E_B E_C E_D)^{-\frac{1}{2}} T(s, t), \quad (4-1)$$

where

$$s = (p_A + p_B)^2 = (p_C + p_D)^2,$$

$$t = (p_A - p_C)^2 = (p_D - p_B)^2$$

are two independent kinematical invariants.

Suppose that there is a two-particle state (EF) with the same internal quantum numbers as (AB) and (CD) , and that $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$. Then, as discussed in sect. 2-3, (see also Rasche and Woolcock, 1970), the contribution of this state to $\text{Im} T_{AB \rightarrow CD}(s, t(s, n \cdot n'))$ in the unitarity relation is

$$\begin{aligned} & \text{Im} T_{AB \rightarrow CD}^{(EF)}(s, t(s, n \cdot n')) \\ &= - \frac{q_{EF}(s)}{32\pi^2 s^{\frac{1}{2}}} \iint d\Omega \bar{n} T_{CD \rightarrow EF}^*(s, t(s, \bar{n} \cdot n')) T_{AB \rightarrow EF}(s, t(s, \bar{n} \cdot n)) \end{aligned} \quad (4-2)$$

for $s \geq (E+F)^2$. In eq. (4-2)

$$q_{EF}(s) = \frac{[s - (E+F)^2]^{\frac{1}{2}} [s - (E-F)^2]^{\frac{1}{2}}}{2s^{\frac{1}{2}}}, \quad (4-3)$$

n, n' are unit vectors in the directions of p_A, p_C respectively in the centre-of-momentum system, $\bar{n}$ is an arbitrary unit vector and the integration is over the surface of the unit sphere.

Next we make a one-particle exchange approximation to

$$T_{AB \rightarrow EF}(s, t(s, \bar{n} \cdot n)),$$

$$T_{AB \rightarrow EF}(s, t(s, \bar{n} \cdot n)) \approx \frac{-g_{AEG} g_{BFG}}{G^2 - t_{AB \rightarrow EF}(s, \bar{n} \cdot n)} \quad (4-4)$$

and similarly

$$T_{CD \rightarrow EF}(s, t(s, \bar{n} \cdot n')) \approx \frac{-g_{CEH} g_{DFH}}{H^2 - t_{CD \rightarrow EF}(s, \bar{n} \cdot n')} , \quad (4-5)$$

where

$$t_{AB \rightarrow EF}(s, \bar{n} \cdot n) = \frac{1}{2}(A^2 + B^2 + E^2 + F^2) - \frac{1}{2}s - \frac{1}{2}(A^2 - B^2)(E^2 - F^2)s^{-1} + 2q_{AB}(s)q_{EF}(s)\bar{n} \cdot n \quad (4-6)$$

and similarly for $t_{CD \rightarrow EF}(s, \bar{n} \cdot n')$. The functions $q_{AB}(s)$, $q_{CD}(s)$ are defined in a similar way to $q_{EF}(s)$, (eq. (4-3)).

The quantities g_{AEG} , g_{BFG} , g_{CEH} , g_{DFH} are the usual rationalised coupling constants.

Approximating $\text{Im}T_{AB \rightarrow CD}(s, t(s, n \cdot n'))$ by $\text{Im}T_{AB \rightarrow CD}^{(EF)}(s, t(s, n \cdot n'))$ (eq. (4-2)) and making the approximations of eqs. (4-4), (4-5), we obtain

$$\begin{aligned} \text{Im}T_{AB \rightarrow CD}(s, t(s, n \cdot n')) &\approx -\frac{q_{EF}(s)g}{32\pi^2 s^{\frac{1}{2}}} \iint \frac{d\Omega \bar{n}}{[G^2 - t_{AB \rightarrow EF}(s, \bar{n} \cdot n)][H^2 - t_{CD \rightarrow EF}(s, \bar{n} \cdot n')]} \\ &= \frac{-g}{128\pi^2 s^{\frac{1}{2}} q_{AB}(s)q_{CD}(s)q_{EF}(s)} \iint \frac{d\Omega \bar{n}}{(\tau_2 - \bar{n} \cdot n)(\tau'_2 - \bar{n} \cdot n')} , \end{aligned}$$

where

$$g = g_{AEG} g_{BFG} g_{CEH} g_{DFH} , \quad (4-7)$$

$$\tau_2 = [2q_{AB}(s)q_{EF}(s)]^{-1} [G^2 - \frac{1}{2}(A^2 + B^2 + E^2 + F^2) + \frac{1}{2}s + \frac{1}{2}(A^2 - B^2)(E^2 - F^2)s^{-1}] ,$$

$$\tau'_2 = [2q_{CD}(s)q_{EF}(s)]^{-1} [H^2 - \frac{1}{2}(C^2 + D^2 + E^2 + F^2) + \frac{1}{2}s + \frac{1}{2}(C^2 - D^2)(E^2 - F^2)s^{-1}] .$$

Next we use the result, given in the appendix of Rasche and Woolcock (1970);

$$\iint \frac{d\Omega \bar{n}}{(\tau_2 - \bar{n} \cdot n) (\tau'_2 - \bar{n} \cdot n')} = 4\pi \int_{u_0}^{\infty} \frac{du}{(u - n \cdot n') [(u - u_0) (u - u_1)]^{\frac{1}{2}}}$$

where

$$\begin{aligned} u_0 &= \tau_2 \tau'_2 \pm \left\{ \tau_2^2 - 1 \right\}^{\frac{1}{2}} \left\{ \tau'^2_2 - 1 \right\}^{\frac{1}{2}}, \\ u_1 & \end{aligned}$$

and we obtain

$$\text{Im} T_{AB \rightarrow CD}(s, t(s, n \cdot n'))$$

$$\approx \frac{-g}{32\pi s^{\frac{1}{2}} q_{AB}(s) q_{CD}(s) q_{EF}(s)} \int_{u_0}^{\infty} \frac{du}{(u - n \cdot n') [(u - u_0) (u - u_1)]^{\frac{1}{2}}}.$$

We now make the change of variable

$$w = 2q_{AB}(s)q_{CD}(s)u + \frac{1}{2}(A^2 + B^2 + C^2 + D^2) - \frac{1}{2}s - \frac{1}{2}(A^2 - B^2)(C^2 - D^2)s^{-1}$$

to give

$$\text{Im} T_{AB \rightarrow CD}(s, t(s, n \cdot n')) \approx \frac{-g}{8\pi} \int_{\tilde{f}_+(s)}^{\infty} \frac{dw}{(w - t_{AB \rightarrow CD}(s, n \cdot n')) \sqrt{\tilde{F}(s, w)}}, \quad (4-8)$$

where $t_{AB \rightarrow CD}(s, n \cdot n')$ is defined in a similar way to $t_{AB \rightarrow EF}(s, \bar{n} \cdot n)$

(eq. (4-6)) and

$$\tilde{f}_{\pm}(s) = G^2 + H^2 + [s^2 - 2(E^2 + F^2)s + (E^2 - F^2)^2]^{-1}.$$

$$\begin{aligned} & \cdot [\{s - 2(E^2 + F^2)\} \{ (E^2 + G^2 - A^2)(F^2 + H^2 - D^2) + (E^2 + H^2 - C^2)(F^2 + G^2 - B^2) \} \\ & + 2E^2(F^2 + G^2 - B^2)(F^2 + H^2 - D^2) + 2F^2(E^2 + G^2 - A^2)(E^2 + H^2 - C^2) \\ & \pm 2\{ (s^2 - 2(E^2 + F^2)s + (E^2 - F^2)^2)G^2 + (s - 2(E^2 + F^2))(E^2 + G^2 - A^2)(F^2 + G^2 - B^2) \\ & + E^2(F^2 + G^2 - B^2)^2 + F^2(E^2 + G^2 - A^2)^2 \}^{\frac{1}{2}} \cdot \\ & \cdot \{ (s^2 - 2(E^2 + F^2)s + (E^2 - F^2)^2)H^2 + (s - 2(E^2 + F^2))(E^2 + H^2 - C^2)(F^2 + H^2 - D^2) \\ & + E^2(F^2 + H^2 - D^2)^2 + F^2(E^2 + H^2 - C^2)^2 \}^{\frac{1}{2}} \}, \end{aligned}$$

$$\tilde{F}(s, w) = (s - (E+F)^2) (s - (E-F)^2) (w - \tilde{f}_+(s)) (w - \tilde{f}_-(s)) .$$

The integration in eq. (4-8) can be carried out and, dropping the subscript $AB \rightarrow CD$ and writing t for $t(s, n \cdot n')$, we find that our approximation to $\text{Im}T(s, t)$ behaves like $s^{-1} \ln s$ for large s .

We now assume that $T(s, t)$, for fixed t in some real interval, may be continued to give an analytic function $T(\zeta_1, t)$ which is regular in the whole ζ_1 -plane, except for the usual right and left hand cuts and simple poles (see for example Martin and Spearman 1970, chapter 6). Further, we assume that $T(\zeta_1, t) \rightarrow 0$ uniformly as $|\zeta_1| \rightarrow \infty$ in any direction in the complex plane and that $T(\zeta_1, t)$ has the Schwarz reflection property. Next we use an argument similar to the one used in sect. 3-2 to establish a single dispersion relation for our approximation to $F(\zeta)$. In the resulting integral representation of $T(\zeta_1, t)$ we take only the integral over the right hand cut and approximate the weight function $\text{Im}T(s, t)$ by eq. (4-8) for $s \geq (E+F)^2$ and by zero for $s < (E+F)^2$. We then find that our approximation to $T(\zeta_1, t)$ may be written in the form

$$T(\zeta_1, t) \approx - \frac{g}{8\pi^2} J(\zeta_1, t) ,$$

where

$$J(\zeta_1, t) = \iint_{\tilde{R}} \frac{dv dw}{(v - \zeta_1)(w - t) \sqrt{\tilde{F}(v, w)}} , \quad (4-9)$$

$$\tilde{R} = \left\{ (v, w) \mid v \geq (E+F)^2 , w \geq \tilde{f}_+(v) \right\} .$$

From this integral form of $J(\zeta_1, t)$ we see that continuation in t is possible, to give a function $J(\zeta_1, \zeta_2)$ of two complex variables

ζ_1, ζ_2 , whose singular points may be deduced from eq. (4-9).

For further calculations it is convenient to introduce the variables ξ, η, σ, τ and the constants a, b, c, d by

$$\begin{aligned} 2EF\xi &= v - E^2 - F^2, & 2GH\eta &= w - G^2 - H^2, \\ 2EF\sigma &= \zeta_1 - E^2 - F^2, & 2GH\tau &= \zeta_2 - G^2 - H^2, \\ 2EG\alpha &= E^2 + G^2 - A^2, & 2FGb &= F^2 + G^2 - B^2, \\ 2EHc &= E^2 + H^2 - C^2, & 2Fhd &= F^2 + H^2 - D^2. \end{aligned} \quad (4-10)$$

In terms of these quantities,

$$\begin{aligned} (2GH)^{-1} \left[\tilde{f}_{\pm}(v) - G^2 - H^2 \right] &\equiv f_{\pm}(\xi) = (\xi^2 - 1)^{-1} [(\xi - 1)(ad + bc) + (a + b)(c + d) \\ &\quad \pm \{\xi^2 - 1 + 2ab(\xi - 1) + (a + b)^2\}^{\frac{1}{2}} \{\xi^2 - 1 + 2cd(\xi - 1) + (c + d)^2\}^{\frac{1}{2}}], \end{aligned} \quad (4-11)$$

$$\begin{aligned} (16E^2F^2G^2H^2)^{-1} \tilde{F}(v, w) &\equiv F(\xi, \eta) \\ &= (\xi^2 - 1)(\eta - f_+(\xi))(\eta - f_-(\xi)) \\ &= (\xi^2 - 1)(\eta^2 - 1) - 2(\xi - 1)(\eta - 1)(ad + bc) - 2(\xi - 1)(a + c)(b + d) \\ &\quad - 2(\eta - 1)(a + b)(c + d) + (ad - bc)^2 - (a + b + c + d)^2, \end{aligned} \quad (4-12)$$

and

$$\begin{aligned} 4EFGHJ(\zeta_1, \zeta_2) &\equiv I(\sigma, \tau) \\ &= \iint_{R_1} \frac{d\xi d\eta}{(\xi - \sigma)(\eta - \tau)\sqrt{F(\xi, \eta)}}, \end{aligned} \quad (4-13)$$

where

$$R_1 = \{(\xi, \eta) \mid \xi \geq 1, \eta \geq f_+(\xi)\}.$$

4-3. Transformation of the box diagram amplitude

We now study the box diagram amplitude associated with the Feynman diagram of fig. 4-1. Using the Feynman rules of Schweber (1961) and the definition of the invariant amplitude given in eq. (4-1), (with the

left hand side as well as $T(s, t)$ subscripted with $\square$, together with eqs. (3-19) and (3-20), we find, on continuing to complex variables, that the box diagram amplitude takes the form

$$T_{\square}(\zeta_1, \zeta_2) = \frac{-g}{32\pi^2 EFGH} I(\sigma, \tau),$$

where

$$I(\sigma, \tau) = 2EFGH \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_0^{1-\alpha-\beta} d\gamma \cdot \\ \cdot [E^2\alpha + F^2\beta + G^2(1-\alpha-\beta-\gamma) + H^2\gamma - A^2(1-\alpha-\beta-\gamma)\alpha - B^2(1-\alpha-\beta-\gamma)\beta - C^2\alpha\gamma - D^2\beta\gamma \\ - (2EF\sigma + E^2 + F^2)\alpha\beta - (2GH\tau + G^2 + H^2)(1-\alpha-\beta-\gamma)\gamma]^{-2} \quad (4-14)$$

and g is defined in eq. (4-7). The notation anticipates the fact (to be proved in due course) that for $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$ the functions $I(\sigma, \tau)$ given in eqs. (4-13) and (4-14) are the same. For the remainder of this chapter σ and τ are restricted to be real. Further, for most of this chapter we restrict x and y , where $x = \text{Re}\sigma$, $y = \text{Re}\tau$, to the intervals $x \leq -(2EF)^{-1}(E^2 + F^2)$, $y < -1$. In sect. 4-8 and in chapter 5, we discuss how these restrictions may be removed.

Our object now is to transform $I(x, y)$ into the form of a double dispersion integral over a real domain or, when this is not possible, into the form of a double dispersion relation in x and y plus a single dispersion relation in x . We start by generalizing the transformation used in sect. 3-3 for the triangle diagram amplitude. The set of transformations used are

1) $\zeta = (1-\gamma)^{-1}$, with α and β as the other integration variables,

2) $\lambda = (\alpha+\beta)^{-1}\zeta^{-1} - 1$, $\nu = (\alpha+\beta)\beta^{-1}$, with ζ as the other

integration variable,

- 3) $\mu = (\lambda+1)(\zeta-1)$, with λ and v as the other integration variables.

These transformations can be combined into the transformation

$$\lambda = (\alpha+\beta)^{-1}(1-\alpha-\beta-\gamma), \quad \mu = (\alpha+\beta)^{-1}\gamma, \quad v = \beta^{-1}(\alpha+\beta),$$

with the inverse

$$\alpha = v^{-1}(v-1)(1+\lambda+\mu)^{-1}, \quad \beta = v^{-1}(1+\lambda+\mu)^{-1}, \quad \gamma = \mu(1+\lambda+\mu)^{-1}.$$

The Jacobian of the transformation is given by

$$\left| \frac{\partial(\alpha, \beta, \gamma)}{\partial(\lambda, \mu, v)} \right| = (1+\lambda+\mu)^{-4} v^{-2}$$

and the new region of integration is

$$\{(\lambda, \mu, v) \mid \lambda \geq 0, \mu \geq 0, v \geq 1\}.$$

Thus

$$I(x, y) = 2EFGH \int_0^\infty d\lambda \int_0^\infty d\mu \int_1^\infty dv \cdot$$

$$\cdot [(v-1)\phi(\lambda, \mu, y) + \psi(\lambda, \mu, y) - v^{-1}(v-1)v(x)]^{-2}, \quad (4-15)$$

where

$$\phi(\lambda, \mu, y) = G^2\lambda^2 + H^2\mu^2 - 2GH\lambda\mu + 2EGa\lambda + 2EHc\mu + E^2, \quad (4-16)$$

$$\psi(\lambda, \mu, y) = G^2\lambda^2 + H^2\mu^2 - 2GH\lambda\mu + 2FGb\lambda + 2F Hd\mu + F^2, \quad (4-17)$$

$$v(x) = 2EFx + E^2 + F^2. \quad (4-18)$$

The constants A^2, B^2, C^2, D^2 have been expressed in terms of a, b, c, d (eq. (4-10)).

Next we need a result concerning the stability conditions, which we put in the form of a lemma. Note that $A < E + G$ is equivalent to $a > -1$, $B < F + G$ is equivalent to $b > -1$, $C < E + H$ is equivalent to $c > -1$, and $D < F + H$ is equivalent to $d > -1$. Also $A > 0$ implies that $a < (2EG)^{-1}(E^2 + G^2)$, $B > 0$ implies

$b < (2FG)^{-1}(F^2+G^2)$, $C > 0$ implies $c < (2EH)^{-1}(E^2+H^2)$ and $D > 0$ implies $d < (2FH)^{-1}(F^2+H^2)$.

LEMMA. For $y < -1$, $\phi(\lambda, \mu, y) > 0$ (resp. $\psi(\lambda, \mu, y) > 0$) for all $\lambda \geq 0$, $\mu \geq 0$ if and only if $a > -1$ and $c > -1$ (resp. $b > -1$ and $d > -1$) .

Proof. For the necessity of the conditions, we note that $\phi(\lambda, \mu, y) > 0$ for all $\lambda \geq 0$, $\mu \geq 0$, means that, in particular, $\phi(\lambda, 0, y) > 0$ and $\phi(0, \mu, y) > 0$. Hence, from the lemma in sect. 3-3, $a > -1$ and $c > -1$. Note that $\phi(\lambda, 0, y) = \phi(\lambda)$ where $\phi(\lambda)$ is given in eq. (3-24).

For the sufficiency, we note first that if $a > -1$ and $c > -1$, then $\phi(\lambda, \mu, y) > 0$ on the boundary of the first quadrant, that is, $\phi(\lambda, 0, y) > 0$ and $\phi(0, \mu, y) > 0$. For fixed $\mu \geq 0$, the minimum of the quadratic function $\phi(\lambda, \mu, y)$ is at

$$\lambda_m(\mu, y) = G^{-1}(Hy\mu - Ea)$$

and its minimum value is

$$\Phi_1(\mu, y) \equiv \phi(\lambda_m(\mu, y), \mu, y) = -\left\{P_1\mu^2 + 2Q_1\mu + R_1\right\}$$

where

$$P_1 = H^2(y^2-1) , \quad Q_1 = -EH(ya+c) , \quad R_1 = E^2(a^2-1) .$$

Now if $a \geq 0$, then $\lambda_m(\mu, y) \leq 0$ and if $-1 < a < 0$ and

$Ea(Hy)^{-1} \leq \mu < \infty$, then $\lambda_m(\mu, y) \leq 0$. In both cases the minimum

value of $\phi(\lambda, \mu, y)$ for $\lambda \geq 0$, fixed μ occurs when $\lambda = 0$ and then $\phi(0, \mu, y) > 0$, from the lemma in sect. 3-3. If however

$-1 < a < 0$ and $0 \leq \mu < Ea(Hy)^{-1}$, then $\lambda_m(\mu, y) > 0$ and the minimum

value of $\phi(\lambda, \mu, y)$ for fixed μ is $\Phi_1(\mu, y)$. Now

$$\Phi_{1\mu}(\mu, y) = -2(P_1\mu + Q_1) \quad , \quad \Phi_{1\mu\mu}(\mu, y) = -2P_1 < 0 \quad ,$$

so that $\Phi_1(\mu, y)$ has at most one turning point in the region

$0 \leq \mu < Ea(Hy)^{-1}$, and then the turning point is a maximum turning point. Thus the minimum value of $\phi(\lambda, \mu, y)$ occurs when either $\lambda = 0$ or $\mu = 0$ and hence $\phi(\lambda, \mu, y) > 0$ for $\lambda \geq 0$, $\mu \geq 0$ if $a > -1$, $c > -1$.

Now eq. (4-15) may be written in the form

$$\begin{aligned} I(x, y) &= EF \int_0^\infty d\lambda \int_0^\infty d\mu (\lambda\mu)^{-1} \frac{\partial}{\partial y} \int_1^\infty dv \cdot \\ &\quad \cdot [v(v-1)\phi(\lambda, \mu, y) + v\psi(\lambda, \mu, y) - (v-1)v(x)]^{-1} \\ &= \frac{1}{2} \int_0^\infty \frac{d\lambda}{\lambda} \int_0^\infty \frac{d\mu}{\mu} \frac{\partial}{\partial y} \int_1^\infty \frac{dv}{v-1} \frac{1}{g(\lambda, \mu, v, y) - x} \quad , \end{aligned} \quad (4-19)$$

where

$$2EFg(\lambda, \mu, v, y) + E^2 + F^2 = \phi(\lambda, \mu, y)v + \psi(\lambda, \mu, y)v(v-1)^{-1} \quad .$$

The interchange of the order of integration with respect to v and differentiation with respect to y in eq. (4-19) is justified by using the theorem given, for example, in section 1.88 of Titchmarsh (1939). Note that, from the lemma, it follows that

$[v(v-1)\phi(\lambda, \mu, y) + v\psi(\lambda, \mu, y) - (v-1)v(x)]$ is positive for $\lambda \geq 0$,

$\mu \geq 0$, $v \geq 1$, $y < -1$, $x \leq -(2EF)^{-1}(E^2 + F^2)$. To interchange the order of differentiation with respect to y and integration with respect to λ and μ , it is necessary to use limiting procedures because of the factor $(\lambda\mu)^{-1}$ in eq. (4-19).

Next we replace the integration over v by an integration over the variable $\xi = g(\lambda, \mu, v, y)$ and by using the results in sect. 3-3 we find that

$I(x, y)$

$$= \int_0^\infty d\mu \lim_{\delta \downarrow 0} \left[\mu^{-1} \frac{\partial}{\partial y} \left\{ \int_\delta^\infty \frac{d\lambda}{\lambda} \int_{h(\lambda, \mu, y)}^\infty \frac{d\xi}{(\xi-x)(U(\xi, \lambda, \mu, y))^{\frac{1}{2}}} \right\} \right], \quad (4-20a)$$

$I(x, y)$

$$= \lim_{\delta \downarrow 0, \epsilon \downarrow 0} \frac{\partial}{\partial y} \left[\int_\epsilon^\infty \frac{d\mu}{\mu} \int_\delta^\infty \frac{d\lambda}{\lambda} \int_{h(\lambda, \mu, y)}^\infty \frac{d\xi}{(\xi-x)(U(\xi, \lambda, \mu, y))^{\frac{1}{2}}} \right], \quad (4-20b)$$

where

$$U(\xi, \lambda, \mu, y) = (\xi - h(\lambda, \mu, y)) (\xi - k(\lambda, \mu, y)), \quad (4-21)$$

$$\frac{h(\lambda, \mu, y)}{k(\lambda, \mu, y)} = (2EF)^{-1} [\{\sqrt{\phi(\lambda, \mu, y)} \pm \sqrt{\psi(\lambda, \mu, y)}\}^2 - E^2 - F^2], \quad (4-22)$$

and $\phi(\lambda, \mu, y)$, $\psi(\lambda, \mu, y)$ are given by eqs. (4-16) and (4-17).

4-4. Study of $h(\lambda, \mu, y)$

To reverse the order of integration in (4-20a) or (4-20b) we need to examine the function $h(\lambda, \mu, y)$ for $\lambda \geq 0$, $\mu \geq 0$. By analogy with eqs. (3-31) and (3-32) we write $\phi(\lambda, \mu, y)$ and $\psi(\lambda, \mu, y)$ as

$$\phi(\lambda, \mu, y) = p_1 \lambda^2 + 2q_1(\mu, y)\lambda + r_1(\mu), \quad (4-23)$$

$$\psi(\lambda, \mu, y) = p_1 \lambda^2 + 2q'_1(\mu, y)\lambda + r'_1(\mu),$$

where

$$p_1 = G^2, \quad q_1(\mu, y) = G(-Hy\mu + Ea), \quad r_1(\mu) = H^2\mu^2 + 2EHc\mu + E^2, \quad (4-24)$$

$$q'_1(\mu, y) = G(-Hy\mu + Fb), \quad r'_1(\mu) = H^2\mu^2 + 2FHd\mu + F^2.$$

(Note that

$$p_1 = p, \quad q_1(0, y) = q, \quad q'_1(0, y) = q', \quad r_1(0) = r, \quad r'_1(0) = r',$$

where p, q, q', r, r' are given in eq. (3-32).)

We can now apply the argument of sect. 3-4 to show that for fixed $\mu \geq 0$, $h_\lambda(\lambda_0(\mu, y), \mu, y) = 0$ with $\lambda_0(\mu, y) \geq 0$ if and only if

$$(i) \quad \Phi_1(\mu, y) > 0, \quad \Psi_1(\mu, y) > 0, \text{ and}$$

$$(ii) \quad \frac{p_1 \lambda_0(\mu, y) + q_1(\mu, y)}{\sqrt{\Phi_1(\mu, y)}} = - \frac{p_1 \lambda_0(\mu, y) + q_1(\mu, y)}{\sqrt{\Psi_1(\mu, y)}}$$

solves to give $\lambda_0(\mu, y) \geq 0$, that is,

$$\lambda_0(\mu, y) = \frac{-q_1(\mu, y)\sqrt{\Psi_1(\mu, y)} - q'_1(\mu, y)\sqrt{\Phi_1(\mu, y)}}{p_1(\sqrt{\Phi_1(\mu, y)} + \sqrt{\Psi_1(\mu, y)})} \geq 0. \quad (4-25)$$

Here

$$\begin{aligned} \Phi_1(\mu, y) &\equiv p_1^{-1} [p_1 r_1(\mu) - (q_1(\mu, y))^2] = -\{P_1 \mu^2 + 2Q_1 \mu + R_1\}, \\ \Psi_1(\mu, y) &\equiv p_1^{-1} [p_1 r'_1(\mu) - (q'_1(\mu, y))^2] = -\{P_1 \mu^2 + 2Q'_1 \mu + R'_1\} \end{aligned} \quad (4-26)$$

and

$$\begin{aligned} P_1 &= H^2(y^2 - 1), \quad Q_1 = -EH(ya + c), \quad R_1 = E^2(a^2 - 1), \\ Q'_1 &= -FH(yb + d), \quad R'_1 = F^2(b^2 - 1). \end{aligned} \quad (4-27)$$

We also find that when $\Phi_1(\mu, y) > 0$, $\Psi_1(\mu, y) > 0$ and eq. (4-25) holds, then $h_{\lambda\lambda}(\lambda_0(\mu, y), \mu, y) > 0$. This result can be obtained by analogy with eq. (3-39).

It now follows that, for fixed $\mu \geq 0$, $h_\lambda(\lambda_0(\mu, y), \mu, y) = 0$ with $\lambda_0(\mu, y) \geq 0$ if and only if $h_\lambda(0, \mu, y) \leq 0$. Consequently $h_\lambda(0, \mu, y) \leq 0$ implies

$$\Phi_1(\mu, y) > 0, \quad \Psi_1(\mu, y) > 0. \quad (4-28)$$

We also see that $h(\lambda, \mu, y)$ increases strictly from $h(0, \mu, y)$ to $+\infty$ as λ increases from 0 to $+\infty$, if $h_\lambda(0, \mu, y) \geq 0$. If however $h_\lambda(0, \mu, y) < 0$, then $h(\lambda, \mu, y)$ decreases strictly from $h(0, \mu, y)$ to $H_1(\mu, y)$ as λ increases from 0 to $\lambda_0(\mu, y)$ and increases strictly from $H_1(\mu, y)$ to $+\infty$ as λ increases from

$\lambda_0(\mu, y)$ to $+\infty$. By analogy with eq. (3-37) and using eqs. (4-26) and (4-24) as well, we see that $H_1(\mu, y) \equiv h(\lambda_0(\mu, y), \mu, y)$ is given by

$$\begin{aligned}
 & 2EFH_1(\mu, y) + E^2 + F^2 \\
 &= p_1^{-1} \left\{ p_1 \left[r_1(\mu) + r_1'(\mu) \right] - 2q_1(\mu, y)q_1'(\mu, y) \right. \\
 &\quad \left. + 2 \left[p_1 r_1(\mu) - (q_1(\mu, y))^2 \right]^{\frac{1}{2}} \left[p_1 r_1'(\mu) - (q_1'(\mu, y))^2 \right]^{\frac{1}{2}} \right\} \\
 &= \left(\sqrt{\Phi_1(\mu, y)} + \sqrt{\Psi_1(\mu, y)} \right)^2 + (Ea - Fb)^2. \quad (4-29)
 \end{aligned}$$

Note also that

$$h_\lambda(0, \mu, y) = (EF)^{-1} \left(\sqrt{r_1(\mu)} + \sqrt{r_1'(\mu)} l_1(\mu, y) \right), \quad (4-30)$$

where

$$l_1(\mu, y) = \frac{q_1(\mu, y)}{\sqrt{r_1(\mu)}} + \frac{q_1'(\mu, y)}{\sqrt{r_1'(\mu)}}. \quad (4-31)$$

Thus the sign of $h_\lambda(0, \mu, y)$ is the sign of $l_1(\mu, y)$. To determine when $h_\lambda(0, \mu, y)$ is negative we must study the function $l_1(\mu, y)$ in the range of values of μ for which the inequalities in (4-28) hold.

Now if $a \geq 1$ and we choose $-y > (2EH)^{-1}(E^2 + H^2)$ (the maximum value c is permitted to take) we see from eq. (4-27) that $P_1 > 0$, $Q_1 > 0$, $R_1 \geq 0$ and thus, from eq. (4-26), $\Phi_1(\mu, y) \leq 0$ for $\mu \geq 0$. The first inequality of (4-28) is then violated and $h_\lambda(0, \mu, y) > 0$ for all $\mu \geq 0$. The same conclusion holds if $b \geq 1$ and we choose $-y > (2FH)^{-1}(F^2 + H^2)$. Similarly, if $c \geq 1$ and $-y > (2EG)^{-1}(E^2 + G^2)$, or if $d \geq 1$ and $-y > (2FG)^{-1}(F^2 + G^2)$, then $h_\mu(\lambda, 0, y) > 0$ for

all $\lambda \geq 0$. As we shall see, these results enable us to handle any case in which at least one of the quantities a, b, c, d is greater than or equal to 1.

We next consider what happens when $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$. In this case $\Phi_1(\mu, y) > 0$ on $[0, \mu_+]$ and $\Phi_1(\mu, y) < 0$ on $(\mu_+, +\infty)$, where

$$\mu_+ = P_1^{-1} \left\{ -Q_1 + \sqrt{Q_1^2 - P_1 R_1} \right\} > 0,$$

since $P_1 > 0$, $R_1 < 0$. Similarly, $\Psi_1(\mu, y) > 0$ on $[0, \mu'_+]$ and $\Psi_1(\mu, y) < 0$ on $(\mu'_+, +\infty)$, where

$$\mu'_+ = P_1^{-1} \left\{ -Q'_1 + \sqrt{Q_1'^2 - P_1 R'_1} \right\} > 0.$$

Thus $h_\lambda(0, \mu, y) > 0$ for $\mu \geq \min\{\mu_+, \mu'_+\}$.

From eq. (4-24) we see that

$$[r_1(\mu)]^{\frac{3}{2}} \frac{d}{d\mu} \left(\frac{q_1(\mu, y)}{\sqrt{r_1(\mu)}} \right) = -EGH[H(yca+a)\mu + E(y+ac)],$$

the right hand side being linear in μ . Its value at $\mu = 0$ is

$$-E^2 GH(y+ac) > 0$$

while its value at $\mu = \mu_+$ is

$$\begin{aligned} & -E^2 GH(y^2-1)^{-1} \sqrt{(yc+a)^2 + (y^2-1)(1-c^2)} \\ & \cdot [y \sqrt{(yc+a)^2 + (y^2-1)(1-c^2)} + (yc+a)] > 0 \end{aligned}$$

since $y < -1$, $|a| < 1$, $|c| < 1$. Thus the derivative of $q_1(\mu, y)/\sqrt{r_1(\mu)}$ is positive on $[0, \mu_+]$ and similarly the derivative of $q'_1(\mu, y)/\sqrt{r'_1(\mu)}$ is positive on $[0, \mu'_+]$. Using eq. (4-31) we see that $z_1(\mu, y)$ strictly increases in the interval

$0 \leq \mu \leq \min\{\mu_+, \mu'_+\}$. But we have already seen that $h_\lambda(0, \mu, y) > 0$,

and hence from eq. (4-30), $z_1(\mu, y) > 0$, for $\mu \geq \min\{\mu_+, \mu'_+\}$. Thus, if

$$G^{-1}z_1(0, y) = (a+b) \geq 0,$$

then $z_1(\mu, y) > 0$ and $h_\lambda(0, \mu, y) > 0$ for all $\mu > 0$. However, if $(a+b) < 0$, then $z_1(\mu, y)$ and $h_\lambda(0, \mu, y)$ have exactly one zero μ_z in $(0, +\infty)$ and $z_1(\mu, y) < 0$ and $h_\lambda(0, \mu, y) < 0$ on $[0, \mu_z]$, $z_1(\mu, y) > 0$ and $h_\lambda(0, \mu, y) > 0$ on $(\mu_z, +\infty)$, where

$$0 < \mu_z < \min\{\mu_+, \mu'_+\}.$$

Now we study the function $H_1(\mu, y)$ for $0 \leq \mu \leq \mu_z$, in the case $(a+b) < 0$. Note first that

$$H_1(0, y) = \xi_{1+},$$

where

$$\xi_{1\pm} = -ab \pm \sqrt{(1-a^2)(1-b^2)}. \quad (4-32)$$

as given in eq. (3-38). The quantities $\xi_{1\pm}$ are defined and distinct whenever $|a| < 1$, $|b| < 1$ and can be written in the form given in eqs. (3-56) and (3-58). Consequently they satisfy the inequalities in (3-59), that is,

$$-1 \leq \xi_{1-} < \xi_{1+} \leq 1. \quad (4-33)$$

From eqs. (4-29) and (4-26) we see that $H_1(\mu, y)$ is defined and real on the interval $\max\{\mu_-, \mu'_-\} \leq \mu \leq \min\{\mu_+, \mu'_+\}$, where

$$\mu_{\pm} = P_1^{-1} \left[-Q_1 \pm \sqrt{Q_1^2 - P_1 R_1} \right], \quad \mu'_{\pm} = P_1^{-1} \left[-Q'_1 \pm \sqrt{Q'^1_1{}^2 - P_1 R'_1} \right]$$

and $\mu_+ > 0$, $\mu'_+ > 0$, $\mu_- < 0$, $\mu'_- < 0$. Because of the form of

$H_1(\mu, y)$ (eq. (4-29)) its properties are very similar to those of the

function $h(\lambda)$ of chapter 3. The derivative of $H_1(\mu, y)$ vanishes exactly once on its interval of definition, at the point given by

$$\mu(y) = \frac{-Q_1(Q_1'^2 - P_1 R_1')^{\frac{1}{2}} - Q_1'(Q_1^2 - P_1 R_1)^{\frac{1}{2}}}{P_1[(Q_1^2 - P_1 R_1)^{\frac{1}{2}} + (Q_1'^2 - P_1 R_1')^{\frac{1}{2}}]}. \quad (4-34)$$

Note that

$$\mu_{\pm} - \mu(y) = \frac{(Q_1^2 - P_1 R_1)^{\frac{1}{2}}[Q_1' - Q_1 \pm (Q_1^2 - P_1 R_1)^{\frac{1}{2}} \pm (Q_1'^2 - P_1 R_1')^{\frac{1}{2}}]}{P_1[(Q_1^2 - P_1 R_1)^{\frac{1}{2}} + (Q_1'^2 - P_1 R_1')^{\frac{1}{2}}]},$$

and a similar expression holds for $(\mu' - \mu(y))$. Then, since $(a+b) < 0$

and hence $|a| < 1$, $|b| < 1$, it follows from eq. (4-27) that

$$\max\{\mu_-, \mu'_-\} < \mu(y) < \min\{\mu_+, \mu'_+\}.$$

We also find that

$$2EFH_{1\mu\mu}(\mu(y), y)$$

$$= -2[\Phi_1(\mu(y), y)\Psi_1(\mu(y), y)]^{-\frac{1}{2}}\left[\left(Q_1^2 - P_1 R_1\right)^{\frac{1}{2}} + \left(Q_1'^2 - P_1 R_1'\right)^{\frac{1}{2}}\right]^2 < 0,$$

so that $H_1(\mu, y)$ takes its *maximum* value on its interval of definition

at $\mu = \mu(y)$. Then

$$2EFH_1(\mu(y), y) + E^2 + F^2 - (Ea - Fb)^2$$

$$= P_1^{-1}\left\{-P_1(R_1 + R_1') + 2Q_1Q_1' + 2\left(Q_1^2 - P_1 R_1\right)^{\frac{1}{2}}\left(Q_1'^2 - P_1 R_1'\right)^{\frac{1}{2}}\right\}, \quad (4-35)$$

that is, $H_1(\mu(y), y) = g_+(y)$, where quantities $g_{\pm}(y)$ are defined by

$$g_{\pm}(y) = (y^2 - 1)^{-1}\left\{(y-1)(ad+bc) + (a+c)(b+d)\right.$$

$$\left. \pm [y^2 - 1 + 2(y-1)ac + (a+c)^2]^{\frac{1}{2}} [y^2 - 1 + 2(y-1)bd + (b+d)^2]^{\frac{1}{2}}\right\}. \quad (4-36)$$

Further, from eqs. (4-34) and (4-27) it follows that $\mu(y) > 0$ if and only if

$$(ya+c)[(yb+d)^2+(y^2-1)(1-b^2)]^{\frac{1}{2}} + (yb+d)[(ya+c)^2+(y^2-1)(1-a^2)]^{\frac{1}{2}} > 0 .$$

We then use the fact that

$$\tanh \chi_1(y) + \tanh \chi_2(y) > 0$$

if and only if

$$\sinh \chi_1(y) + \sinh \chi_2(y) > 0 ,$$

where $\chi_1(y)$ and $\chi_2(y)$ are defined by eq. (4-A6) of appendix 4A,

to show that $\mu(y) > 0$ if and only if

$$h_1(y) > 0 , \quad (4-37)$$

where

$$h_1(y) = (ya+c) \sqrt{1-b^2} + (yb+d) \sqrt{1-a^2} . \quad (4-38)$$

The above analysis shows that there are three possible subcases of the case where $(a+b) < 0$.

- (1a) $H_1(\mu, y)$ increases strictly on $[0, \mu(y)]$ and decreases strictly on $[\mu(y), \mu_L]$. For this subcase $H_{1\mu}(0, y) > 0$ or, using eqs. (4-29), (4-26), (4-27) and (4-38),

$$h_1(y) > 0 ,$$

$$\text{and } 0 < \mu(y) < \mu_L .$$

- (2a) $H_1(\mu, y)$ decreases strictly on $[0, \mu_L]$. For this subcase $H_{1\mu}(0, y) \leq 0$ or $h_1(y) \leq 0$ and $\mu(y) \leq 0$.

- (3a) $H_1(\mu, y)$ increases strictly on $[0, \mu_L]$. For this subcase $H_{1\mu}(0, y) > 0$ or $h_1(y) > 0$ and $\mu(y) \geq \mu_L$.

We now note that $\phi(\lambda, \mu, y)$, $\psi(\lambda, \mu, y)$ defined in eqs. (4-16), (4-17) can also be written in the form

$$\phi(\lambda, \mu, y) = p_2 \mu^2 + 2q_2(\lambda, y)\mu + r_2(\lambda),$$

$$\psi(\lambda, \mu, y) = p_2 \mu^2 + 2q'_2(\lambda, y)\mu + r'_2(\lambda),$$

where

$$p_2 = H^2, \quad q_2(\lambda, y) = H(-Gy\lambda + Ec), \quad r_2(\lambda) = G^2\lambda^2 + 2EGa\lambda + E^2, \quad (4-39)$$

$$q'_2(\lambda, y) = H(-Gy\lambda + Fd), \quad r'_2(\lambda) = G^2\lambda^2 + 2FGb\lambda + F^2,$$

and we can obtain results similar to those just obtained, by making the interchanges $\lambda \leftrightarrow \mu$, $1 \rightarrow 2$. By making these interchanges in eqs. (4-25) and (4-31) we obtain the functions $\mu_0(\lambda, y)$ and $l_2(\lambda, y)$ respectively. The definition of $\mu_0(\lambda, y)$ will involve the functions $\Phi_2(\lambda, y)$, $\Psi_2(\lambda, y)$ given by

$$\Phi_2(\lambda, y) = p_2^{-1} \left[p_2 r_2(\lambda) - (q_2(\lambda, y))^2 \right] = - \left\{ P_2 \lambda^2 + 2Q_2 \lambda + R_2 \right\},$$

$$\Psi_2(\lambda, y) = p_2^{-1} \left[p_2 r'_2(\lambda) - (q'_2(\lambda, y))^2 \right] = - \left\{ P_2 \lambda^2 + 2Q'_2 \lambda + R'_2 \right\},$$

where

$$P_2 = G^2(y^2 - 1), \quad Q_2 = -EG(ya + a), \quad R_2 = E^2(\sigma^2 - 1), \quad (4-40)$$

$$Q'_2 = -FG(yd + b), \quad R'_2 = F^2(d^2 - 1).$$

We define $H_2(\lambda, y)$ by $H_2(\lambda, y) \equiv h(\lambda, \mu_0(\lambda, y), y)$ and the equation corresponding to (4-29) is

$$2EFH_2(\lambda, y) + E^2 + F^2 = (\sqrt{\Phi_2(\lambda, y)} + \sqrt{\Psi_2(\lambda, y)})^2 + (Ec - Fd)^2, \quad (4-41)$$

If $(\sigma + d) \geq 0$, then $l_2(\lambda, y) > 0$ and $h_\mu(\lambda, 0, y) > 0$ for all $\lambda > 0$ and $h(\lambda, \mu, y)$ increases strictly from $h(\lambda, 0, y)$ to $+\infty$ as μ increases from 0 to $+\infty$, for each fixed $\lambda \geq 0$. If, however, $(\sigma + d) < 0$, then $l_2(\lambda, y)$ and $h_\mu(\lambda, 0, y)$ both have exactly one

zero λ_z on $(0, +\infty)$ and $l_2(\lambda, y) < 0$ and $h_\mu(\lambda, 0, y) < 0$ on $[0, \lambda_z)$, $l_2(\lambda, y) > 0$ and $h_\mu(\lambda, 0, y) > 0$ on $(\lambda_z, +\infty)$. In this case, if $0 \leq \lambda < \lambda_z$, then $h(\lambda, \mu, y)$ decreases strictly from $h(\lambda, 0, y)$ to $H_2(\lambda, y)$ as μ increases from 0 to $\mu_0(\lambda, y)$ and then increases strictly from $H_2(\lambda, y)$ to $+\infty$ as μ increases from $\mu_0(\lambda, y)$ to $+\infty$. If $\lambda_z \leq \lambda < +\infty$, then $h(\lambda, \mu, y)$ increases strictly from $h(\lambda, 0, y)$ to $+\infty$ as μ increases from 0 to $+\infty$.

Next we study the function $H_2(\lambda, y)$ for $0 \leq \lambda \leq \lambda_z$, in the case $(c+d) < 0$. Firstly we note that

$$H_2(0, y) = \xi_{2+}$$

where

$$\xi_{2\pm} = -cd \pm \sqrt{(1-c^2)(1-d^2)}, \quad (4-42)$$

and

$$-1 \leq \xi_{2-} < \xi_{2+} \leq +1 \quad (4-43)$$

when $|c| < 1$, $|d| < 1$. The function $H_2(\lambda, y)$ is defined and real on the interval $\max\{\lambda_-, \lambda'_-\} \leq \lambda \leq \min\{\lambda_+, \lambda'_+\}$, where $0 < \lambda_z < \min\{\lambda_+, \lambda'_+\}$,

$$\lambda_{\pm} = P_2^{-1} \left\{ -Q_2 \pm \sqrt{Q_2^2 - P_2 R_2} \right\}, \quad \lambda'_{\pm} = P_2^{-1} \left\{ -Q'_2 \pm \sqrt{Q'^2_2 - P_2 R'_2} \right\}$$

and $\lambda_+ > 0$, $\lambda'_+ > 0$, $\lambda_- < 0$, $\lambda'_- < 0$. We find that the derivative of $H_2(\lambda, y)$ vanishes exactly once on its interval of definition, at the point

$$\lambda(y) = \frac{-Q_2 (Q_2'^2 - P_2 R_2')^{\frac{1}{2}} - Q'_2 (Q_2^2 - P_2 R_2)^{\frac{1}{2}}}{P_2 [(Q_2^2 - P_2 R_2)^{\frac{1}{2}} + (Q_2'^2 - P_2 R_2')^{\frac{1}{2}}]}. \quad (4-44)$$

We also find that $H_2(\lambda, y)$ takes its *maximum* value on its interval of definition at $\lambda = \lambda(y)$. Then

$$2EFH_2(\lambda(y), y) + E^2 + F^2 - (Ec - Fd)^2 = P_2^{-1} \left\{ -P_2(R_2 + R'_2) + 2Q_2Q'_2 + 2 \left\{ Q_2^2 - P_2R_2 \right\}^{\frac{1}{2}} \left\{ Q_2'^2 - P_2R'_2 \right\}^{\frac{1}{2}} \right\}, \quad (4-45)$$

that is, $H_2(\lambda(y), y) = g_+(y)$, where $g_+(y)$ is defined in eq. (4-36).

By analogy with eq. (4-37) we find that $\lambda(y) > 0$ if and only if

$$h_2(y) > 0 \quad (4-46)$$

where

$$h_2(y) = (yc + a)\sqrt{1-d^2} + (yd + b)\sqrt{1-c^2}. \quad (4-47)$$

Again, there are three possible subcases of the case where $(c+d) < 0$.

(1b) $H_2(\lambda, y)$ increases strictly on $[0, \lambda(y)]$ and decreases strictly on $[\lambda(y), \lambda_l]$. For this subcase $H_{2\lambda}(0, y) > 0$ or using eqs. (4-41), (4-40) and (4-47),

$$h_2(y) > 0,$$

and

$$0 < \lambda(y) < \lambda_l.$$

(2b) $H_2(\lambda, y)$ decreases strictly on $[0, \lambda_l]$. For this subcase $H_{2\lambda}(0, y) \leq 0$ or $h_2(y) \leq 0$ and $\lambda(y) \leq 0$.

(3b) $H_2(\lambda, y)$ increases strictly on $[0, \lambda_l]$. For this subcase $H_{2\lambda}(0, y) > 0$ or $h_2(y) > 0$ and $\lambda(y) \geq \lambda_l$.

Now we consider the case when $(a+b) < 0$ and $(c+d) < 0$. From eqs. (4-35) and (4-45) it follows that

$$h\{\lambda(y), \mu_0(\lambda(y), y), y\} = h\{\lambda_0(\mu(y), y), \mu(y), y\} = g_+(y) .$$

It can also be shown that

$$\lambda_0(\mu(y), y) = \lambda(y) , \quad \mu_0(\lambda(y), y) = \mu(y) . \quad (4-48)$$

To see this, we note firstly that

$$H^2(Q_2^2 - P_2 R_2) = G^2(Q_1^2 - P_1 R_1) , \quad H^2(Q_2'^2 - P_2 R_2') = G^2(Q_1'^2 - P_1 R_1')$$

and

$$\frac{\Phi_1(\mu(y), y)}{Q_1^2 - P_1 R_1} = \frac{\Psi_1(\mu(y), y)}{Q_1'^2 - P_1 R_1'} .$$

Thus

$$\begin{aligned} \lambda_0(\mu(y), y) &= - \frac{q_1(\mu(y), y) \sqrt{Q_1'^2 - P_1 R_1'} + q_1'(\mu(y), y) \sqrt{Q_1^2 - P_1 R_1}}{p_1 [\sqrt{Q_1^2 - P_1 R_1} + \sqrt{Q_1'^2 - P_1 R_1'}]} \\ &= \frac{H^2 [EG(ya+c) \sqrt{Q_1'^2 - P_1 R_1'} + FG(yb+d) \sqrt{Q_1^2 - P_1 R_1}]}{p_1 p_1 [\sqrt{Q_1^2 - P_1 R_1} + \sqrt{Q_1'^2 - P_1 R_1'}]} \\ &= \lambda(y) , \end{aligned}$$

on using eq. (4-40). The other relation in (4-48) can be proved in a similar way. We thus see that the arcs $\lambda = \lambda_0(\mu, y)$ and $\mu = \mu_0(\lambda, y)$, which are defined in the intervals $\max\{\lambda_-, \lambda'_-\} \leq \lambda \leq \min\{\lambda_+, \lambda'_+\}$ and $\max\{\mu_-, \mu'_-\} \leq \mu \leq \min\{\mu_+, \mu'_+\}$ respectively, have exactly one point of intersection, namely $(\lambda(y), \mu(y))$. We have just established that there are three subcases of the case where $(a+b) < 0$ distinguished by the location of $\mu(y)$ and three subcases of the case where $(c+d) < 0$ distinguished by the location of $\lambda(y)$. We now want to find how many of the locations of the point $(\lambda(y), \mu(y))$ are possible.

We saw earlier (eq. (4-28)) that $z_1(\mu, y) > 0$ if $\Phi_1(\mu, y) \leq 0$ or $\Psi_1(\mu, y) \leq 0$. Similarly $z_2(\lambda, y) > 0$ if $\Phi_2(\mu, y) \leq 0$ or $\Psi_2(\lambda, y) \leq 0$. Next, we consider the case where $\max\{\mu_-, \mu'_-\} < \mu < \min\{\mu_+, \mu'_+\}$ and as well $\max\{\lambda_-, \lambda'_-\} < \lambda < \min\{\lambda_+, \lambda'_+\}$. From eqs. (4-31) and (4-26) it follows that

$$z_1(\mu, y) = p_1^{\frac{1}{2}} \{ \sinh \kappa(\mu, y) + \sinh \kappa'(\mu, y) \}$$

where

$$\sinh \kappa(\mu, y) = \frac{q_1(\mu, y)}{\sqrt{p_1 \Phi_1(\mu, y)}}, \quad \sinh \kappa'(\mu, y) = \frac{q'_1(\mu, y)}{\sqrt{p_1 \Psi_1(\mu, y)}}$$

Thus $z_1(\mu, y) < 0$ if and only if

$$\sinh \kappa(\mu, y) + \sinh \kappa'(\mu, y) < 0.$$

Also from eq. (4-25) we see that

$$\lambda_0(\mu, y) = - \frac{p_1^{\frac{1}{2}} \sqrt{\Phi_1(\mu, y) \Psi_1(\mu, y)} (\sinh \kappa(\mu, y) + \sinh \kappa'(\mu, y))}{p_1 [\sqrt{\Phi_1(\mu, y)} + \sqrt{\Psi_1(\mu, y)}]},$$

so that $z_1(\mu, y) < 0$ if and only if $\lambda_0(\mu, y) > 0$. In particular $z_1(\mu(y), y) < 0$ if and only if $\lambda(y) > 0$ and hence from eqs. (4-46) and (4-47), $\mu(y) < \mu_z$ and $\lambda(y) > 0$ if and only if $h_2(y) > 0$.

Similarly $\lambda(y) < \lambda_z$ and $\mu(y) > 0$ if and only if $h_1(y) > 0$.

Thus we are left with just three subcases of the case $(a+b) < 0$, $(c+d) < 0$:

- (i) $h_1(y) > 0$, $h_2(y) > 0$ or $0 < \lambda(y) < \lambda_z$, $0 < \mu(y) < \mu_z$;
- (ii) $h_1(y) \leq 0$, $h_2(y) > 0$ or $0 < \lambda_z \leq \lambda(y)$, $\mu(y) \leq 0$;
- (iii) $h_1(y) > 0$, $h_2(y) \leq 0$ or $\lambda(y) \leq 0$, $0 < \mu_z \leq \mu(y)$.

Different methods of showing that the subcase $h_1(y) \leq 0$, $h_2(y) \leq 0$

is not possible when $y < -1$ are given in our paper, Frederiksen and Woolcock (1972).

Next we want to establish some further results for the subcases (i), (ii) and (iii). Firstly we find a relationship between μ_L and $\mu_0(0, y)$. By analogy with eq. (3-37) we see that

$$r_1(\mu_0(0, y)) = \frac{-R_2 v(\xi_{2+})}{[(-R_2)^{\frac{1}{2}} + (-R'_2)^{\frac{1}{2}}]^2}$$

where from eqs. (4-18) and (4-43), $v(\xi_{2+}) > (E-F)^2$. A similar equation holds for $r'_1(\mu_0(0, y))$ and hence from eq. (4-31)

$$z_1(\mu_0(0, y), y) = \frac{EFG((-R_2)^{\frac{1}{2}} + (-R'_2)^{\frac{1}{2}})h_2(y)}{(v(\xi_{2+}))^{\frac{1}{2}}(-R_2)^{\frac{1}{2}}(-R'_2)^{\frac{1}{2}}}$$

and $\mu_L < \mu_0(0, y)$ if and only if $h_2(y) > 0$. Similarly

$\lambda_L < \lambda_0(0, y)$, if and only if $h_1(y) > 0$. For subcase (i) we see

that $h(0, \mu, y)$ strictly decreases as μ increases from 0 to $\mu_0(0, y)$ and $h(\lambda, \mu(y), y)$ strictly decreases as λ increases from 0 to $\lambda(y)$. Here $0 < \mu(y) < \mu_L < \mu_0(0, y)$ so that

$$1 > h(0, \mu(y), y) > g_+(y),$$

where $g_+(y)$ is given by eq. (4-36). Also, from the properties of $H_1(\mu, y)$ (subcase (1a)) and $H_2(\mu, y)$ (subcase (1b)) we see that

$$\min\{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\} = \min\{\xi_{1+}, \xi_{2+}\},$$

and

$$\max\{\xi_{1+}, \xi_{2+}\} < g_+(y).$$

For subcase (ii) we see from the properties of $H_1(\mu, y)$ (subcase (2a))

and $H_2(\lambda, y)$ (subcase (3b)) that

$$g_+(y) \geq \xi_{1+} = H_1(0, y) > H_1(\mu_L, y) = h(0, \mu_L, y) > \xi_{2+}.$$

Similarly for subcase (iii)

$$g_+(y) \geq \xi_{2+} > \xi_{1+}.$$

In appendix 4-A, some further results involving the quantities $\xi_{1\pm}$, $\xi_{2\pm}$ and $g_{\pm}(y)$ are given.

Finally we summarize our results for the subcases (i), (ii) and (iii) of the case where $(a+b) < 0$, $(c+d) < 0$:

(i) $h_1(y) > 0$, $h_2(y) > 0$. For this subcase

$$0 < \lambda(y) < \lambda_L < \lambda_0(0, y), \quad 0 < \mu(y) < \mu_L < \mu_0(0, y),$$

$$\max\{\xi_{1+}, \xi_{2+}\} < g_+(y) < 1,$$

$$\min\{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\} = \min\{\xi_{1+}, \xi_{2+}\};$$

(ii) $h_1(y) \leq 0$, $h_2(y) > 0$. For this subcase

$$0 < \lambda_0(0, y) \leq \lambda_L \leq \lambda(y), \quad \mu(y) \leq 0, \quad 0 < \mu_L < \mu_0(0, y),$$

$$\xi_{2+} < \xi_{1+} \leq g_+(y),$$

$$\min\{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\} = \xi_{2+};$$

(iii) $h_1(y) > 0$, $h_2(y) \leq 0$. For this subcase

$$\lambda(y) \leq 0, \quad 0 < \lambda_L < \lambda_0(0, y), \quad 0 < \mu_0(0, y) \leq \mu_L \leq \mu(y),$$

$$\xi_{1+} < \xi_{2+} \leq g_+(y),$$

$$\min\{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\} = \xi_{1+}.$$

4-5. Solutions of $U(\xi, \lambda, \mu, y) = 0$

In this section we study the behaviour of the zeros of $U(\xi, \lambda, \mu, y)$, first when ξ, μ and y are held fixed and then when ξ, λ and y are fixed. The approach is very similar to that used in sect. 3-5. From eqs. (4-21), (4-22), (4-16) and (4-17) we have

$$4E^2F^2U(\xi, \lambda, \mu, y) = a_1(\xi)\lambda^2 + 2b_1(\xi, \mu, y)\lambda + c_1(\xi, \mu), \quad (4-49)$$

where

$$\begin{aligned} a_1(\xi) &= 4[(q_1(\mu, y) - q'_1(\mu, y))^2 - p_1 v(\xi)] \\ &= 4G^2[(Ea - Fb)^2 - v(\xi)], \end{aligned} \quad (4-50)$$

$$\begin{aligned} b_1(\xi, \mu, y) &= 2[(q_1(\mu, y) - q'_1(\mu, y))(r_1(\mu) - r'_1(\mu)) - (q_1(\mu, y) + q'_1(\mu, y))v(\xi)] \\ &= 2[2GH\{(Ea - Fb)(Ec - Fd) + yv(\xi)\}\mu \\ &\quad + G\{(Ea - Fb)(E^2 - F^2) - (Ea + Fb)v(\xi)\}], \end{aligned} \quad (4-51)$$

$$\begin{aligned} c_1(\xi, \mu) &= (r_1(\mu) - r'_1(\mu))^2 - 2v(\xi)(r_1(\mu) + r'_1(\mu)) + \{v(\xi)\}^2 \\ &= 4H^2[(Ec - Fd)^2 - v(\xi)]\mu^2 \\ &\quad + 2H[(Ec - Fd)(E^2 - F^2) - (Ec + Fd)v(\xi)]\mu + 4E^2F^2(\xi^2 - 1), \end{aligned} \quad (4-52)$$

and $v(\xi)$ is given in eq. (4-18). By analogy with eq. (3-53) we see that

$$\{b_1(\xi, \mu, y)\}^2 - a_1(\xi)c_1(\xi, \mu) > 0 \quad \text{for } \xi > -(2EF)^{-1}(E^2 + F^2)$$

if $\Phi_1(\mu, y)\Psi_1(\mu, y) < 0$ where $\Phi_1(\mu, y)$ and $\Psi_1(\mu, y)$ are given by

(4-26). For cases where $\Phi_1(\mu, y)\Psi_1(\mu, y) \geq 0$,

$$\{b_1(\xi, \mu, y)\}^2 - a_1(\xi)c_1(\xi, \mu) = 16E^2F^2G^2v(\xi)(\xi - H_1(\mu, y))(\xi - K_1(\mu, y)), \quad (4-53)$$

where

$$\begin{aligned} &\left. \begin{aligned} &2EFH_1(\mu, y) + E^2 + F^2 \\ &2EFK_1(\mu, y) + E^2 + F^2 \end{aligned} \right\} \\ &= p_1^{-1} \left\{ p_1(r_1(\mu) + r'_1(\mu)) - 2q_1(\mu, y)q'_1(\mu, y) \right. \\ &\quad \left. \pm \left[\left(p_1r_1(\mu) - (q_1(\mu, y))^2 \right) \left(p_1r'_1(\mu) - (q'_1(\mu, y))^2 \right) \right]^{\frac{1}{2}} \right\}. \end{aligned} \quad (4-54)$$

Note that $H_1(\mu, y)$ is just the quantity defined in eq. (4-29) for

the case when $\Phi_1(\mu, y) > 0$, $\Psi_1(\mu, y) > 0$. When $\Phi_1(\mu, y)\Psi_1(\mu, y) \geq 0$,

$$\begin{aligned}
 H_1(\mu, y) \\
 K_1(\mu, y) &= h(0, \mu, y) - (2EFp_1)^{-1} \left\{ \sqrt{(q_1(\mu, y) + \sqrt{p_1 r_1(\mu)}) (q_1'(\mu, y) + \sqrt{p_1 r_1'(\mu)})} \right. \\
 &\quad \left. \pm \sqrt{(q_1(\mu, y) - \sqrt{p_1 r_1(\mu)}) (q_1'(\mu, y) - \sqrt{p_1 r_1'(\mu)})} \right\}^2
 \end{aligned}$$

and hence

$$K_1(\mu, y) \leq H_1(\mu, y) \leq h(0, \mu, y). \quad (4-55)$$

The two equality signs in eq. (4-55) cannot occur together. When

$\Phi_1(\mu, y) \geq 0$, $\Psi_1(\mu, y) \geq 0$, there is another form for $H_1(\mu, y)$,

$K_1(\mu, y)$, namely

$$\begin{aligned}
 H_1(\mu, y) \\
 K_1(\mu, y) &= k(0, \mu, y) + (2EFp_1)^{-1} \left\{ \sqrt{(\sqrt{p_1 r_1(\mu)} + q_1(\mu, y)) (\sqrt{p_1 r_1'(\mu)} - q_1'(\mu, y))} \right. \\
 &\quad \left. \pm \sqrt{(\sqrt{p_1 r_1(\mu)} - q_1(\mu, y)) (\sqrt{p_1 r_1'(\mu)} + q_1'(\mu, y))} \right\}^2, \quad (4-56)
 \end{aligned}$$

where $k(0, \mu, y)$ is given in eq. (4-22). Thus, for this case

$$-(2EF)^{-1}(E^2 + F^2) \leq k(0, \mu, y) \leq K_1(\mu, y) \leq H_1(\mu, y) \leq h(0, \mu, y). \quad (4-57)$$

Eqs. (4-57) and (4-53) show that when $(a+b) \leq 0$ and $0 \leq \mu \leq \mu_L$,

$$(b_1(\xi, \mu, y))^2 - a_1(\xi) \sigma_1(\xi, \mu) \geq 0 \quad \text{for } \xi \geq H_1(\mu, y),$$

equality occurring if and only if $\xi = H_1(\mu, y)$. Also, from eqs. (4-55)

and (4-53) it follows that when $(a+b) > 0$ and $\mu \geq 0$ or when

$(a+b) \leq 0$ and $\mu > \mu_L$,

$$(b_1(\xi, \mu, y))^2 - a_1(\xi) \sigma_1(\xi, \mu) > 0 \quad \text{for } \xi \geq h(0, \mu, y).$$

We have now shown that for each

$$\xi \geq \min\{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \text{ fixed and } \geq 0\},$$

the quadratic equation in λ ,

$$U(\xi, \lambda, \mu, y) = 0$$

has two real roots $\lambda_{\pm}(\xi, \mu, y)$ given by

$$\lambda_{\pm}(\xi, \mu, y) = [a_1(\xi)]^{-1} \left[-b_1(\xi, \mu, y) \mp \left\{ (b_1(\xi, \mu, y))^2 - a_1(\xi) a_1(\xi, \mu) \right\}^{\frac{1}{2}} \right] , \quad (4-58)$$

except for the one point at most where $a_1(\xi) = 0$.

From eqs. (4-51), (4-22), (4-16) and (4-17) we see that

$$b_1(h(0, \mu, y), \mu, y) = -4 \{ \sqrt{r_1(\mu)} + \sqrt{r'_1(\mu)} \} \sqrt{r_1(\mu)} \sqrt{r'_1(\mu)} z_1(\mu, y) , \quad (4-59)$$

where $z_1(\mu, y)$ is given in eq. (4-31). Consequently

$$\lambda_+(h(0, \mu, y), \mu, y) = 0 \neq \lambda_-(h(0, \mu, y), \mu, y) \quad \text{when } z_1(\mu, y) > 0 ,$$

$$\lambda_-(h(0, \mu, y), \mu, y) = 0 \neq \lambda_+(h(0, \mu, y), \mu, y) \quad \text{when } z_1(\mu, y) < 0 ,$$

and

$$\lambda_+(h(0, \mu, y), \mu, y) = \lambda_-(h(0, \mu, y), \mu, y) = 0 \quad \text{when } z_1(\mu, y) = 0 .$$

Notice too that as $\xi \rightarrow +\infty$

$$\lambda_{\pm}(\xi, \mu, y) \sim \pm \frac{1}{\sqrt{2}} G^{-1} (EF)^{\frac{1}{2}} \xi^{\frac{1}{2}} .$$

When $(a+b) \leq 0$ and $0 \leq \mu \leq \mu_1$, we see from eqs. (4-50) and (4-29)

that

$$a_1(H_1(\mu, y)) = -4 \left[\left\{ p_1 r_1(\mu) - (q_1(\mu, y))^2 \right\}^{\frac{1}{2}} + \left\{ p_1 r'_1(\mu) - (q'_1(\mu, y))^2 \right\}^{\frac{1}{2}} \right]^2 < 0$$

and since $a_1(\xi)$ is linear in ξ ,

$$a_1(\xi) < 0 \quad \text{for } \xi \geq H_1(\mu, y) . \quad (4-60)$$

Also

$$b_1(H_1(\mu, y), \mu, y) = -4 p_1^{-1} \left[\left\{ p_1 r_1(\mu) - (q_1(\mu, y))^2 \right\}^{\frac{1}{2}} + \left\{ p_1 r'_1(\mu) - (q'_1(\mu, y))^2 \right\}^{\frac{1}{2}} \right] \cdot \\ \cdot \left[q_1(\mu, y) \left\{ p_1 r'_1(\mu) - (q'_1(\mu, y))^2 \right\}^{\frac{1}{2}} + q'_1(\mu, y) \left\{ p_1 r_1(\mu) - (q_1(\mu, y))^2 \right\}^{\frac{1}{2}} \right] ,$$

so that

$$\lambda_+(H_1(\mu, y), \mu, y) = \lambda_-(H_1(\mu, y), \mu, y) = \lambda_0(\mu, y) \quad (4-61)$$

where $\lambda_0(\mu, y)$ is given in eq. (4-25).

Thus, when $l_1(\mu, y) < 0$, $\lambda_-(\xi, \mu, y)$ is the inverse of the strictly decreasing function $h(\lambda, \mu, y)$ on $[0, \lambda_0(\mu, y)]$; the function $\lambda_-(\xi, \mu, y)$ therefore strictly decreases from $\lambda_0(\mu, y)$ to 0 as ξ increases from $H_1(\mu, y)$ ($= h(\lambda_0(\mu, y), \mu, y)$) to $h(0, \mu, y)$. Similarly $\lambda_+(\xi, \mu, y)$ is the inverse of the strictly increasing function $h(\lambda, \mu, y)$ on $[\lambda_0(\mu, y), +\infty)$ and therefore strictly increases from $\lambda_0(\mu, y)$ to $+\infty$ as ξ increases from $H_1(\mu, y)$ to $+\infty$. When $l_1(\mu, y) \geq 0$ we see that $\lambda_+(\xi, \mu, y)$ is a continuous function of ξ on $[h(0, \mu, y), +\infty)$ except for the one point at most where $a_1(\xi) = 0$. Thus $\lambda_+(\xi, \mu, y)$ is the inverse of the strictly increasing function $h(\lambda, \mu, y)$ on $[0, +\infty)$ and $\lambda_+(\xi, \mu, y)$ strictly increases from 0 to $+\infty$ as ξ increases from $h(0, \mu, y)$ to $+\infty$. We deduce further that if $\xi_a \geq h(0, \mu, y)$, where ξ_a is the solution of $a_1(\xi) = 0$, then from eqs. (4-49) and (4-22),

$$a_1(\xi_a, \mu) \geq 0, \quad b_1(\xi_a, \mu, y) < 0 \quad \text{and}$$

$$\lambda_+(\xi_a, \mu, y) = -a_1(\xi_a, \mu)/2b_1(\xi_a, \mu, y).$$

We can also write

$$4E^2 F^2 U(\xi, \lambda, \mu, y) = a_2(\xi) \mu^2 + 2b_2(\xi, \lambda, y) \mu + c_2(\xi, \lambda), \quad (4-62)$$

where

$$a_2(\xi) = 4H^2[(Ec-Fd)^2 - v(\xi)] , \quad (4-63)$$

$$b_2(\xi, \lambda, y) = 2[2GH\{(Ea-Fb)(Ec-Fd) + yv(\xi)\}\mu + H\{(Ec-Fd)(E^2-F^2) - (Ec+Fd)v(\xi)\}] , \quad (4-64)$$

$$c_2(\xi, \lambda) = 4G^2[(Ea-Fb)^2 - v(\xi)]\mu^2 + 2G[(Ea-Fb)(E^2-F^2) - (Ea+Fd)v(\xi)]\mu + 4E^2F^2(\xi^2-1) . \quad (4-65)$$

For each $\xi \geq \min\{h(\lambda, \mu, y) \mid \mu \geq 0, \lambda \text{ fixed and } \geq 0\}$, the quadratic equation in μ ,

$$U(\xi, \lambda, \mu, y) = 0 ,$$

has two real roots $\mu_{\pm}(\xi, \lambda, y)$ given by

$$\mu_{\pm}(\xi, \lambda, y) = [a_2(\xi)]^{-1} \cdot \left[-b_2(\xi, \lambda, y) \pm \{(b_2(\xi, \lambda, y))^2 - a_2(\xi)c_2(\xi, \lambda)\}^{\frac{1}{2}} \right] . \quad (4-66)$$

We can also write statements about the behaviour of $\mu_{\pm}(\xi, \lambda, y)$ analogous to those for $\lambda_{\pm}(\xi, \mu, y)$. These results are sufficient to reverse the order of integration in eq. (4-20) and perform the required integrations for all cases except the case where $(a+b) < 0$, $(c+d) < 0$.

Now we look at the case $(a+b) < 0$ and consider the function $U(\xi, \lambda_0(\mu, y), \mu, y)$ for $0 \leq \mu \leq \mu_1$. From eqs. (4-21), (4-22), (4-25), (4-29) and (4-26), we have

$$\begin{aligned} & 4E^2F^2U(\xi, \lambda_0(\mu, y), \mu, y) \\ &= 4E^2F^2[\xi - h(\lambda_0(\mu, y), \mu, y)][\xi - k(\lambda_0(\mu, y), \mu, y)] \\ &= \left[(\sqrt{\Phi_1(\mu, y)} + \sqrt{\Psi_1(\mu, y)})^2 - v(\xi) + (Ea-Fb)^2 \right] \cdot \\ & \quad \cdot \left[(\sqrt{\Phi_1(\mu, y)} - \sqrt{\Psi_1(\mu, y)})^2 - v(\xi) + (Ea-Fb)^2 \right] \\ &= A_1(\xi, y)\mu^2 + 2B_1(\xi, y)\mu + C_1(\xi) , \end{aligned} \quad (4-67)$$

where

$$A_1(\xi, y) = 4[(Q_1 - Q'_1)^2 - P_1\{(Ea - Fb)^2 - v(\xi)\}] , \quad (4-68)$$

$$B_1(\xi, y) = 2[(Q_1 - Q'_1)(R_1 - R'_1) - (Q_1 + Q'_1)\{(Ea - Fb)^2 - v(\xi)\}] , \quad (4-69)$$

$$\begin{aligned} C_1(\xi) &= (R_1 - R'_1)^2 - 2(R_1 + R'_1)[(Ea - Fb)^2 - v(\xi)] + [(Ea - Fb)^2 - v(\xi)]^2 \\ &= 4E^2F^2(\xi - \xi_{1+})(\xi - \xi_{1-}) , \end{aligned} \quad (4-70)$$

and $P_1, Q_1, Q'_1, R_1, R'_1$ are given by eq. (4-27) and $\xi_{1\pm}$ by eq. (4-32).

The discriminant of the quadratic function of μ in eq. (4-67) is

$$\begin{aligned} [B_1(\xi, y)]^2 - A_1(\xi, y)C_1(\xi) &= 4[(Ea - Fb)^2 - v(\xi)] \cdot \left[P_1\{(Ea - Fb)^2 - v(\xi)\}^2 \right. \\ &\quad \left. + 2\{2Q_1Q'_1 - P_1(R_1 + R'_1)\}\{(Ea - Fb)^2 - v(\xi)\} + P_1(R_1 - R'_1)^2 + 4(Q_1 - Q'_1)(Q_1R'_1 - Q'_1R_1) \right] , \end{aligned}$$

and the second factor vanishes when

$$\begin{aligned} 2EF\xi + E^2 + F^2 - (Ea - Fb)^2 \\ = P_1^{-1} \left[2Q_1Q'_1 - P_1(R_1 + R'_1) \pm 2 \left\{ Q_1^2 - P_1R_1 \right\}^{\frac{1}{2}} \left\{ Q_1'^2 - P_1R'_1 \right\}^{\frac{1}{2}} \right] , \end{aligned}$$

giving $\xi = g_{\pm}(y)$, with $g_{\pm}(y)$ defined in eq. (4-36). Thus

$$\begin{aligned} [B_1(\xi, y)]^2 - A_1(\xi, y)C_1(\xi) \\ = 16E^2F^2H^2[(Ea - Fb)^2 - v(\xi)](y^2 - 1)(\xi - g_+(y))(\xi - g_-(y)) \\ = 4E^2F^2G^{-2}H^2a_1(\xi)F(\xi, y) , \end{aligned} \quad (4-71)$$

using eqs. (4-50) and (4-A3).

Since $U(\xi, \lambda_0(\mu, y), \mu, y) = 0$ when

$$\xi = h(\lambda_0(\mu, y), \mu, y) (\equiv H_1(\mu, y)) ,$$

we see that, for

$$\min\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\} \leq \xi \leq \max\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\} ,$$

the quadratic equation in μ ,

$$U(\xi, \lambda_0(\mu, y), \mu, y) = 0, \quad (4-72)$$

has one real root, satisfying $0 \leq \mu \leq \mu_L$. Thus, for this range of

ξ , the discriminant is non-negative and there is a second real root.

From eqs. (4-A8) and (4-A9) we see that $F(\xi, y) < 0$ for

$\min\{\xi_{1+}, \xi_{2+}\} \leq \xi < g_+(y)$, $y < -1$. Then from eqs. (4-60) and

(4-71) it follows that $[B_1(\xi, y)]^2 - A_1(\xi, y)C_1(\xi)$ is in fact positive

for $\min\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\} < \xi < \max\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\}$. The

two roots of eq. (4-72) are given by

$$\begin{aligned} \mu_{\alpha}(\xi, y) &= [A_1(\xi, y)]^{-1} \left[-B_1(\xi, y) \mp \left\{ (B_1(\xi, y))^2 - A_1(\xi, y)C_1(\xi) \right\}^{\frac{1}{2}} \right] \\ \beta \\ &= [A_1(\xi, y)]^{-1} \left[-B_1(\xi, y) \mp 2EFG^{-1}H(-a_1(\xi))^{\frac{1}{2}} \{-F(\xi, y)\}^{\frac{1}{2}} \right], \quad (4-73) \end{aligned}$$

using eqs. (4-67) and (4-71).

Now from eq. (4-68) we see that

$$A_1(g_{\pm}(y), y) = 4 \left[\left(Q_1^{2-P_1 R_1} \right)^{\frac{1}{2}} \pm \left(Q_1'^{2-P_1 R_1'} \right)^{\frac{1}{2}} \right]^2 > 0$$

and since $A_1(\xi, y)$ is linear in ξ , it follows from eqs. (4-A8) and

(4-A9) that

$$\mu_{\alpha}(\xi, y) \leq \mu_{\beta}(\xi, y)$$

for

$$\min\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\} \leq \xi \leq \max\{H_1(\mu, y) \mid 0 \leq \mu \leq \mu_L\},$$

equality occurring if and only if $\xi = g_+(y)$. Then

$$B_1(g_+(y), y) = 4P_1^{-1} \left[Q_1 \left(Q_1'^{2-P_1 R_1'} \right)^{\frac{1}{2}} + Q_1' \left(Q_1^{2-P_1 R_1} \right)^{\frac{1}{2}} \right] \left[\left(Q_1^{2-P_1 R_1} \right)^{\frac{1}{2}} + \left(Q_1'^{2-P_1 R_1'} \right)^{\frac{1}{2}} \right]$$

and consequently

$$\mu_{\alpha}(g_+(y), y) = \mu_{\beta}(g_+(y), y) = \mu(y) \quad (4-74)$$

where $\mu(y)$ is given in eq. (4-34). Also, from eqs. (4-69), (4-70) and (4-32) we see that

$$C_1(\xi_{1+}) = 0, \quad B_1(\xi_{1+}, y) = -4EFH(E\sqrt{1-a^2} + F\sqrt{1-b^2})h_1(y).$$

Thus if $h_1(y) > 0$ then $\mu_\alpha(\xi_{1+}, y) = 0$, if $h_1(y) < 0$ then

$\mu_\beta(\xi_{1+}, y) = 0$ and if $h_1(y) = 0$ then

$$\mu_\alpha(\xi_{1+}, y) = \mu_\beta(\xi_{1+}, y) = 0.$$

In a similar way we may write

$$4E^2F^2U(\xi, \lambda, \mu_0(\lambda, y), y) = A_2(\xi, y)\lambda^2 + 2B_2(\xi, y)\lambda + C_2(\xi),$$

with $A_2(\xi, y)$, $B_2(\xi, y)$, $C_2(\xi)$ obtained from the right hand sides of eqs. (4-68), (4-69), (4-70) by the replacements $1 \rightarrow 2$, $a \rightarrow c$, $b \rightarrow d$ where these quantities appear explicitly. The roots of the quadratic equation in λ ,

$$U(\xi, \lambda, \mu_0(\lambda, y), y) = 0,$$

will be denoted by $\lambda_{\alpha\beta}(\xi, y)$, for $\xi \in \{H_2(\lambda, y) \mid 0 \leq \lambda \leq \lambda_l\}$.

They are obtained from the right hand side of eq. (4-73) by replacing 1 by 2 and interchanging G and H where these quantities appear explicitly.

We now give the behaviour of $\lambda_{\alpha\beta}(\xi, y)$ and $\mu_{\alpha\beta}(\xi, y)$ for the three subcases (i)-(iii) of the case $(a+b) < 0$, $(c+d) < 0$ detailed in the last part of section 4-4.

(i) The function $\lambda_\alpha(\xi, y)$ strictly increases from 0 to $\lambda(y)$ as ξ increases from ξ_{2+} to $g_+(y)$, while $\lambda_\beta(\xi, y)$ strictly decreases from λ_l to $\lambda(y)$ as ξ increases from $H_2(\lambda_l, y)$ to $g_+(y)$. Similarly the function $\mu_\alpha(\xi, y)$ strictly increases from 0

to $\mu(y)$ as ξ increases from ξ_{1+} to $g_+(y)$, while $\mu_\beta(\xi, y)$ strictly decreases from μ_λ to $\mu(y)$ as ξ increases from $H_1(\mu_\lambda, y)$ to $g_+(y)$.

(ii) The function $\lambda_\alpha(\xi, y)$ strictly increases from 0 to λ_λ as ξ increases from ξ_{2+} to $H_2(\lambda_\lambda, y)$, while $\mu_\beta(\xi, y)$ strictly decreases from μ_λ to 0 as ξ increases from $H_1(\mu_\lambda, y)$ to ξ_{1+} .

(iii) The function $\mu_\alpha(\xi, y)$ strictly increases from 0 to μ_λ as ξ increases from ξ_{1+} to $H_1(\mu_\lambda, y)$, while $\lambda_\beta(\xi, y)$ strictly decreases from λ_λ to 0 as ξ increases from $H_2(\lambda_\lambda, y)$ to ξ_{2+} .

We also note from eqs. (4-58), (4-53) and (4-61) that $\lambda_+(\xi, \mu, y) = \lambda_-(\xi, \mu, y)$, in the region $\mu \geq 0$, $\xi \in \{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\}$, if and only if $0 \leq \mu \leq \mu_\lambda$, $\xi = h(\lambda_0(\mu, y), \mu, y) \equiv H_1(\mu, y)$. Similarly $\mu_+(\xi, \lambda, y) = \mu_-(\xi, \lambda, y)$ in the region $\lambda \geq 0$, $\xi \in \{h(\lambda, \mu, y) \mid \lambda \geq 0, \mu \geq 0\}$ if and only if $0 \leq \lambda \leq \lambda_\lambda$, $\xi = h(\lambda, \mu_0(\lambda, y), y) \equiv H_2(\lambda, y)$. Thus, using eq. (4-67) and the statements made in the preceding paragraph, we may make the following assertions concerning subcase (i) of the case $(a+b) < 0$, $(c+d) < 0$:

$$\lambda_+(\xi, \mu_\alpha(\xi, y), y) = \lambda_-(\xi, \mu_\alpha(\xi, y), y) \quad \text{for } \xi_{1+} \leq \xi \leq g_+(y),$$

$$\lambda_+(\xi, \mu_\beta(\xi, y), y) = \lambda_-(\xi, \mu_\beta(\xi, y), y) \quad \text{for } H_1(\mu_\lambda, y) \leq \xi \leq g_+(y),$$

$$\lambda_+(\xi, \mu, y) \neq \lambda_-(\xi, \mu, y) \quad \text{for } 0 \leq \mu \leq \mu_\lambda, \quad \xi > H_1(\mu, y) \quad \text{and}$$

$$\text{for } \mu > \mu_\lambda, \quad \xi \geq h(0, \mu, y),$$

$$\mu_+(\xi, \lambda_\alpha(\xi, y), y) = \mu_-(\xi, \lambda_\alpha(\xi, y), y) \quad \text{for } \xi_{2+} \leq \xi \leq g_+(y),$$

$$\begin{aligned} \mu_+(\xi, \lambda_\beta(\xi, y), y) &= \mu_-(\xi, \lambda_\beta(\xi, y), y) \quad \text{for } H_2(\lambda_z, y) \leq \xi \leq g_+(y), \\ \mu_+(\xi, \lambda, y) &\neq \mu_-(\xi, \lambda, y) \quad \text{for } 0 \leq \lambda \leq \lambda_z, \quad \xi > H_2(\lambda, y) \quad \text{and} \\ &\text{for } \lambda > \lambda_z, \quad \xi \geq h(\lambda, 0, y). \end{aligned}$$

In particular, from eqs. (4-74), (4-35), (4-61) and (4-48) it follows that

$$\lambda_\pm(g_+(y), \mu_\alpha(g_+(y), y), y) = \lambda_\pm(g_+(y), \mu_\beta(g_+(y), y), y) = \lambda(y)$$

and similarly

$$\mu_\pm(g_+(y), \lambda_\alpha(g_+(y), y), y) = \mu_\pm(g_+(y), \lambda_\beta(g_+(y), y), y) = \mu(y).$$

For subcase (ii) of the case $(a+b) < 0$, $(c+d) < 0$ we find that

$$\begin{aligned} \lambda_+(\xi, \mu_\beta(\xi, y), y) &= \lambda_-(\xi, \mu_\beta(\xi, y), y) \quad \text{for } H_1(\mu_z, y) \leq \xi \leq \xi_{1+}, \\ \lambda_+(\xi, \mu, y) &\neq \lambda_-(\xi, \mu, y) \quad \text{for } 0 \leq \mu \leq \mu_z, \quad \xi > H_1(\mu, y) \quad \text{and} \\ &\text{for } \mu > \mu_z, \quad \xi \geq h(0, \mu, y), \\ \mu_+(\xi, \lambda_\alpha(\xi, y), y) &= \mu_-(\xi, \lambda_\alpha(\xi, y), y) \quad \text{for } \xi_{2+} \leq \xi \leq H_2(\lambda_z, y), \\ \mu_+(\xi, \lambda, y) &\neq \mu_-(\xi, \lambda, y) \quad \text{for } 0 \leq \lambda \leq \lambda_z, \quad \xi > H_2(\lambda, y) \quad \text{and} \\ &\text{for } \lambda > \lambda_z, \quad \xi \geq h(\lambda, 0, y). \end{aligned}$$

Similar relations can be deduced for subcase (iii). The results for subcases (i) and (ii) discussed in this section and in sect. 4-4 are summarised in figs. 4-2 and 4-3 respectively. These figures show the contour map of the surface defined by $\xi = h(\lambda, \mu, y)$ for $\lambda \geq 0$, $\mu \geq 0$.

In the next two sections we start with eq. (4-20a) or eq. (4-20b) and manipulate $I(x, y)$ into the form of a spectral representation. The various cases to be considered will be denoted as follows:

Fig. 4-2.

Contour map of the surface defined by $\xi = h(\lambda, \mu, y)$ for $\lambda \geq 0$, $\mu \geq 0$ in case III (i).

Fig. 4-3.

Contour map of the surface defined by $\xi = h(\lambda, \mu, y)$ for $\lambda \geq 0$, $\mu \geq 0$ in case III (ii).

- I $(a+b) \geq 0$, $(c+d) \geq 0$;
- II (i) $(a+b) \geq 0$, $(c+d) < 0$;
- II (ii) $(a+b) < 0$, $(c+d) \geq 0$;
- III (i) $(a+b) < 0$, $(c+d) < 0$, subcase (i) ;
- III (ii) $(a+b) < 0$, $(c+d) < 0$, subcase (ii) ;
- III (iii) $(a+b) < 0$, $(c+d) < 0$, subcase (iii) .

4-6. Evaluation of $I(x, y)$: Cases I and II

We now reverse the order of integration in eq. (4-20a) and obtain spectral representations of $I(x, y)$ for cases I and II defined at the end of sect. 4-5. When $(a+b) \geq 0$, we see from sect. 4-4 that $h_\lambda(0, \mu, y) > 0$ (or $z_\perp(\mu, y) > 0$) for all $\mu > 0$ and so from sect. 4-5, $\lambda_+(\xi, \mu, y)$ is the inverse of the strictly increasing function $h(\lambda, \mu, y)$ on $0 \leq \lambda < \infty$, for each $\mu \geq 0$. Thus eq. (4-20a) can be written as

$$I(x, y) = 2EF \int_0^\infty d\mu \lim_{\delta \downarrow 0} \left[\mu^{-1} \frac{\partial}{\partial y} \int_{h(\delta, \mu, y)}^\infty \frac{d\xi}{(\xi-x)} \Lambda(\xi, \delta, \mu, y) \right] , \quad (4-75)$$

where

$$\Lambda(\xi, \delta, \mu, y) = \int_\delta^{\lambda_+(\xi, \mu, y)} \frac{d\lambda}{\lambda [a_1(\xi)\lambda^2 + 2b_1(\xi, \mu, y)\lambda + c_1(\xi, \mu)]^{\frac{1}{2}}} \quad (4-76)$$

and $a_1(\xi)$, $b_1(\xi, \mu, y)$, $c_1(\xi, \mu)$ are defined in eqs. (4-50), (4-51) and (4-52). We now use the approach described in sect. 3-5 to evaluate $\Lambda(\xi, \delta, \mu, y)$. From eqs. (4-21) and (4-49),

$$c_1(\xi, \mu) = 4E^2F^2 \{ \xi - h(0, \mu, y) \} \{ \xi - k(0, \mu, y) \}$$

and so, since $h(\lambda, \mu, y)$ is strictly increasing on $0 \leq \lambda < \infty$, we have

$$c_1(\xi, \mu) > 0 \quad \text{for } \mu \geq 0, \quad \xi > h(0, \mu, y).$$

Also, from eq. (4-58),

$$\begin{aligned} c_1(\xi, \mu) [\lambda_{\pm}(\xi, \mu, y)]^{-1} + b_1(\xi, \mu, y) \\ = \pm \left[(b_1(\xi, \mu, y))^2 - c_1(\xi) c_1(\xi, \mu) \right]^{\frac{1}{2}}. \end{aligned} \quad (4-77)$$

Thus

$$\begin{aligned} \Lambda(\xi, \delta, \mu, y) &= \frac{1}{[c_1(\xi, \mu)]^{\frac{1}{2}}} \cdot \\ &\cdot \ln \left[\frac{c_1(\xi, \mu) + \delta b_1(\xi, \mu, y) + \{c_1(\xi, \mu)\}^{\frac{1}{2}} \{4E^2 F^2 U(\xi, \delta, \mu, y)\}^{\frac{1}{2}}}{\delta \{ (b_1(\xi, \mu, y))^2 - c_1(\xi) c_1(\xi, \mu) \}^{\frac{1}{2}}} \right], \end{aligned} \quad (4-78)$$

using eq. (4-77) and the fact that $U(\xi, \lambda_+(\xi, \mu, y), \mu, y) = 0$. We

see from eq. (4-77) that $c_1(\xi, \mu) + \delta b_1(\xi, \mu, y) > 0$ for

$0 \leq \delta \leq \lambda_+(\xi, \mu, y)$ so that the argument of the logarithm is positive.

Now

$$\begin{aligned} \frac{\partial}{\partial y} \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{(\xi-x)} \Lambda(\xi, \delta, \mu, y) \\ = - \frac{\partial h(\delta, \mu, y)}{\partial y} \frac{1}{(h(\delta, \mu, y)-x)} \Lambda(h(\delta, \mu, y), \delta, \mu, y) \\ + \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{(\xi-x)} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y). \end{aligned} \quad (4-79)$$

From eq. (4-21), $U(h(\delta, \mu, y), \delta, \mu, y) = 0$ and since

$\lambda_+(h(\delta, \mu, y), \mu, y) = \delta$, eqs. (4-77) and (4-78) show that

$\Lambda(h(\delta, \mu, y), \delta, \mu, y) = 0$. Also

$$\frac{\partial h(\delta, \mu, y)}{\partial y} = - \frac{GH}{EF} \cdot \frac{\delta \mu (\sqrt{\phi(\delta, \mu, y)} + \sqrt{\psi(\delta, \mu, y)})^2}{[\phi(\delta, \mu, y) \psi(\delta, \mu, y)]^{\frac{1}{2}}}$$

is finite and hence the first term on the right hand side of eq. (4-79)

vanishes. Differentiating the right hand side of eq. (4-78) with

respect to y , we have

$$\mu^{-1} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) = - \frac{4GHv(\xi) [\delta a_1(\xi) + b_1(\xi, \mu, y)]}{[(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu)] [4E^2 F^2 U(\xi, \delta, \mu, y)]^{\frac{1}{2}}} . \quad (4-80)$$

To take the limit $\delta \downarrow 0$ in eq. (4-75), we treat separately the terms involving $\delta a_1(\xi)$ and $b_1(\xi, \mu, y)$ in the numerator of eq. (4-80).

First we recall from sect. 4-5 that the function $(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu)$ cannot vanish for $\xi \geq h(0, \mu, y)$, $\mu \geq 0$. Also, for fixed $\mu \geq 0$,

$$\left| v(\xi) a_1(\xi) \left[(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu) \right]^{-1} \right|$$

is bounded by a constant as $\xi \rightarrow +\infty$, and hence it has a finite maximum value for $h(0, \mu, y) \leq \xi < \infty$. Thus, we can find a finite $M (> 0)$ such that

$$\left| \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi (-4GHv(\xi) \delta a_1(\xi))}{(\xi - x) [(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu)] [4E^2 F^2 U(\xi, \delta, \mu, y)]^{\frac{1}{2}}} \right| \leq M\delta \int_{h(\delta, \mu, y)}^{\infty} \frac{(v(\xi))^{-1} d\xi}{[(\xi - h(\delta, \mu, y))(\xi - k(\delta, \mu, y))]^{\frac{1}{2}}} . \quad (4-81)$$

The last integration can be carried out, (see for example eq. (3-74) or eqs. (4-91) and (4-92)), and we find that the above expression vanishes as $\delta \downarrow 0$. Similarly, for fixed $\mu \geq 0$,

$$\left| v(\xi) b_1(\xi, \mu, y) \left[(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu) \right]^{-1} \right|$$

has a finite maximum value for $h(0, \mu, y) \leq \xi < \infty$ and we can find a finite $M' > 0$ such that

$$\begin{aligned}
& \left| \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi 4GHv(\xi) b_1(\xi, \mu, y)}{(\xi-x) [(b_1(\xi, \mu, y))^2 - a_1(\xi) c_1(\xi, \mu)]} \right. \\
& \quad \cdot \left. \{ [4E^2 F^2 U(\xi, 0, \mu, y)]^{-\frac{1}{2}} - [4E^2 F^2 U(\xi, \delta, \mu, y)]^{-\frac{1}{2}} \} \right| \\
& \leq M' \left| \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{v(\xi)} \{ [U(\xi, 0, \mu, y)]^{-\frac{1}{2}} - [U(\xi, \delta, \mu, y)]^{-\frac{1}{2}} \} \right|. \quad (4-82)
\end{aligned}$$

The last integration can be carried out and we find that the above expression vanishes as $\delta \downarrow 0$. Using these two results and eqs. (4-80), (4-79) and (4-75) we have

$$I(x, y) = -8EFGH \cdot$$

$$\cdot \int_0^{\infty} d\mu \int_{h(0, \mu, y)}^{\infty} \frac{d\xi v(\xi) b_1(\xi, \mu, y)}{(\xi-x) [c_1(\xi, \mu)]^{\frac{1}{2}} [(b_1(\xi, \mu, y))^2 - a_1(\xi) c_1(\xi, \mu)]}. \quad (4-83)$$

Note that, if $|a| < 1$, $|b| < 1$, $(a+b) \geq 0$, eq. (4-83) has been shown to hold for $y < -1$. If, however, $\max\{a, b\} \geq 1$, we have imposed a stricter restriction on y , as discussed in section 4-4.

We now consider case I, where $(a+d) \geq 0$ in addition to $(a+b) \geq 0$. Then $h(0, \mu, y)$ (which is independent of y) strictly increases from 1 to $+\infty$ as μ increases from 0 to $+\infty$. From section 4-5 it follows that $\mu_+(\xi, 0, y)$ is the inverse of the strictly increasing function $h(0, \mu, y)$ and so

$$I(x, y) = \int_1^{\infty} \frac{d\xi}{(\xi-x)} X(\xi, y), \quad (4-84)$$

where

$$X(\xi, y) = -8EFGH \int_0^{\mu_+(\xi, 0, y)} \frac{d\mu b_1(\xi, \mu, y) v(\xi)}{[c_1(\xi, \mu)]^{\frac{1}{2}} [(b_1(\xi, \mu, y))^2 - a_1(\xi) c_1(\xi, \mu)]}. \quad (4-85)$$

On making the change of variable

$$\alpha = \mu^{-1},$$

we find that

$$X(\xi, y) = -8EFGH\nu(\xi) \int_{\alpha_+(\xi)}^{\infty} \frac{d\alpha [\beta(\xi, y) + b_1(\xi, 0, y)\alpha]}{[a_2(\xi) + 2b_2(\xi, 0, y)\alpha + c_2(\xi, 0)\alpha^2]^{\frac{1}{2}}} \cdot \\ \cdot \left[(\beta(\xi, y) + b_1(\xi, 0, y)\alpha)^2 - a_1(\xi) \{a_2(\xi) + 2b_2(\xi, 0, y)\alpha + c_2(\xi, 0)\alpha^2\} \right]^{-1},$$

where

$$\alpha_{\pm}(\xi) = [\mu_{\pm}(\xi, 0, y)]^{-1} \\ = [c_2(\xi, 0)]^{-1} \left[-b_2(\xi, 0, y) \pm \left\{ (b_2(\xi, 0, y))^2 - a_2(\xi)c_2(\xi, 0) \right\}^{\frac{1}{2}} \right], \quad (4-86)$$

and the quantity $\beta(\xi, y)$ is the coefficient of μ in $b_1(\xi, \mu, y)$

(or λ in $b_2(\xi, \lambda, y)$), that is

$$\beta(\xi, y) = 4GH[(Ea-Fb)(Ec-Fd) + yv(\xi)]. \quad (4-87)$$

Next we make the change of variable

$$u^2 = [\alpha - \alpha_+(\xi)] [\alpha - \alpha_-(\xi)]^{-1}$$

with the inverse

$$\alpha = \alpha_-(\xi) + (\alpha_+(\xi) - \alpha_-(\xi)) [1 - u^2]^{-1},$$

and

$$\frac{d\alpha}{du} = 2(\alpha_+(\xi) - \alpha_-(\xi))u [1 - u^2]^{-2}.$$

Then

$$X(\xi, y) = \frac{16EFGH\nu(\xi)}{[c_2(\xi, 0)]^{\frac{1}{2}}} \int_0^1 \frac{du (lu^2 - m)}{(lu^2 - m)^2 + 4mu^2}, \quad (4-88)$$

where

$$l = \beta(\xi, y) + b_1(\xi, 0, y)\alpha_-(\xi) = (\mu_-(\xi, 0, y))^{-1} b_1(\xi, \mu_-(\xi, 0, y), y),$$

$$m = \beta(\xi, y) + b_1(\xi, 0, y)\alpha_+(\xi) = (\mu_+(\xi, 0, y))^{-1}b_1(\xi, \mu_+(\xi, 0, y), y) , \quad (4-89)$$

$$n = -a_1(\xi)[c_2(\xi, 0)]^{-1}\left[(b_2(\xi, 0, y))^2 - a_2(\xi)c_2(\xi, 0)\right] .$$

To carry out the integration in eq. (4-88), we need to know the sign of $lm - n$ where

$$lm - n = [c_2(\xi, 0)]^{-2}\left[(\beta(\xi, y)c_2(\xi, 0) - b_1(\xi, 0, y)b_2(\xi, 0, y))^2 - \left\{(b_1(\xi, 0, y))^2 - a_1(\xi)c_1(\xi, 0)\right\}\left\{(b_2(\xi, 0, y))^2 - a_2(\xi)c_2(\xi, 0)\right\}\right] .$$

In deriving this expression we use eq. (4-86) and the fact that

$$c_1(\xi, 0) = c_2(\xi, 0) . \quad \text{We also find that}$$

$$\beta(\xi, y)c_2(\xi, 0) - b_1(\xi, 0, y)b_2(\xi, 0, y) = -16E^2F^2GH\nu(\xi)G(\xi, y)$$

where $G(\xi, y)$ is given in eq. (4-A5) and from eqs. (4-53), (4-54) and (4-32), (or eq. (3-54)), we see that

$$(b_1(\xi, 0, y))^2 - a_1(\xi)c_1(\xi, 0) = 16E^2F^2G^2\nu(\xi)(\xi - \xi_{1+})(\xi - \xi_{1-}) .$$

Similarly

$$(b_2(\xi, 0, y))^2 - a_2(\xi)c_2(\xi, 0) = 16E^2F^2H^2\nu(\xi)(\xi - \xi_{2+})(\xi - \xi_{2-})$$

where $\xi_{2\pm}$ are given in eq. (4-42). Using the above three equations

together with eq. (4-65) we find that

$$lm - n = 16G^2H^2(\xi^2 - 1)^{-2}[\nu(\xi)]^2 .$$

$$\begin{aligned} & \cdot [\{G(\xi, y)\}^2 - (\xi^2 + 2ab\xi + a^2 + b^2 - 1)(\xi^2 + 2cd\xi + c^2 + d^2 - 1)] \\ & = 16G^2H^2(\xi^2 - 1)^{-1}[\nu(\xi)]^2 F(\xi, y) , \end{aligned} \quad (4-90)$$

where $F(\xi, y)$ is given in eq. (4-12). In deriving eq. (4-90), eq. (4-A4) was also used.

If $F(\xi, y) > 0$, then since $\xi > 1$, $(lm - n) > 0$ and in this case

$$\frac{(lu^2-m)}{(lu^2-m)^2+4mu^2} = \frac{1}{2pl} \left[\frac{u-p}{(u-p)^2+q} - \frac{u+p}{(u+p)^2+q} \right],$$

where

$$p = l^{-1}(lm-n)^{\frac{1}{2}}, \quad q = nl^{-2}.$$

The integration in eq. (4-88) can now be carried out to give

$$X(\xi, y) = \frac{4EFGH\nu(\xi)}{[c_2(\xi, 0)]^{\frac{1}{2}} pl} \ln \left| \frac{(1-p)^2+q}{(1+p)^2+q} \right|$$

where $\ln$ refers to the natural logarithm and

$$l(1+p^2+q) = l + m = -8GH\nu(\xi)(\xi^2-1)^{-1}G(\xi, y).$$

Then using eq. (4-90) we find that

$$X(\xi, y) = \frac{1}{2\sqrt{F(\xi, y)}} \ln \left[\frac{G(\xi, y) + \sqrt{(\xi^2-1)F(\xi, y)}}{G(\xi, y) - \sqrt{(\xi^2-1)F(\xi, y)}} \right], \quad (4-91)$$

irrespective of the sign of l . Since $\xi_{1\pm}$ are either complex

conjugates or satisfy (3-57), and similar results hold for $\xi_{2\pm}$, it

follows from eq. (4-A4) that the argument of the logarithm in eq. (4-91)

is positive for $\xi > 1$.

Next, we show that

$$X(\xi, y) = \int_{f_+(\xi)}^{\infty} \frac{d\eta}{(\eta-y)\sqrt{F(\xi, \eta)}}, \quad (4-92)$$

when $y < f_-(\xi)$, where $f_{\pm}(\xi)$ are given in eq. (4-11). In fact,

since $y < -1$ and, as we show in sect. 4-8, $f_+(\xi) > -1$ for $\xi > 1$,

eq. (4-92) holds for $F(\xi, y) > 0$. The right hand side of eq. (4-92)

is equal to

$$\frac{1}{\sqrt{F(\xi, y)}} \ln \left| \frac{F(\xi, y) - (n-y)G(\xi, y) - \sqrt{F(\xi, y)F(\xi, n)}}{(n-y)} \right| \Bigg|_{\eta=f_+(\xi)}^{\eta \rightarrow \infty}$$

$$= \frac{1}{\sqrt{F(\xi, y)}} \ln \left| \frac{G(\xi, y) + \sqrt{(\xi^2 - 1)F(\xi, y)}}{F(\xi, y)(y - f_+(\xi))^{-1} + G(\xi, y)} \right| ,$$

where, from eqs. (4-A5), (4-11) and (4-12),

$$F(\xi, y) + G(\xi, y)(y - f_+(\xi))$$

$$= (y - f_+(\xi)) [(\xi^2 - 1)(y - f_-(\xi)) + (\xi^2 - 1)(-y + \frac{1}{2}f_+(\xi) + \frac{1}{2}f_-(\xi))]$$

$$= (y - f_+(\xi)) [\xi^2 - 1 + 2ab(\xi - 1) + (a+b)^2]^{\frac{1}{2}} \cdot [\xi^2 - 1 + 2cd(\xi - 1) + (c+d)^2]^{\frac{1}{2}} .$$

Finally, using eq. (4-A4) we see that the right hand side of eq. (4-92) is equal to the right hand side of eq. (4-91).

When $F(\xi, y) < 0$, we use different methods to carry out the integration in eq. (4-88). First we recall from sect. 4-4 that $l_1(\mu, y) > 0$ for all $\mu > 0$ if $(a+b) \geq 0$. Then we use eqs. (4-59) and (4-89) together with the fact that $\mu_+(\xi, 0, y) > 0$ for $\xi > 1$ to show that $m < 0$ for $\xi > 1$. If in addition $n < 0$, then it follows from eq. (4-90) that $l > 0$. In this case we find that

$$\frac{lu^2 - m}{(lu^2 - m)^2 + 4nu^2} = \frac{\frac{1}{2}}{lu^2 - m + 2(-n)^{\frac{1}{2}}u} + \frac{\frac{1}{2}}{lu^2 - m - 2(-n)^{\frac{1}{2}}u} .$$

Thus

$$X(\xi, y) = \frac{8EFGH\nu(\xi)}{[c_2(\xi, 0)]^{\frac{1}{2}}\sqrt{-(lm-n)}} \left[\arctan \frac{l+(-n)^{\frac{1}{2}}}{\sqrt{-(lm-n)}} + \arctan \frac{l-(-n)^{\frac{1}{2}}}{\sqrt{-(lm-n)}} \right]$$

$$= \frac{1}{\sqrt{-F(\xi, y)}} \left[\frac{\pi}{2} - \arctan \frac{-(l+m)}{2\sqrt{-(lm-n)}} \right]$$

$$= \frac{1}{\sqrt{-F(\xi, y)}} \left[\frac{\pi}{2} - \arctan \frac{G(\xi, y)}{\sqrt{\xi^2 - 1}\sqrt{-F(\xi, y)}} \right] . \quad (4-93)$$

If $n > 0$ we have

$$\frac{l u^2 - m}{(l u^2 - m)^2 + 4 n u^2} = \frac{1}{l(P+Q)} \left\{ \frac{P}{u^2 + P^2} + \frac{Q}{u^2 + Q^2} \right\}$$

where

$$\frac{P}{Q} = \sqrt{-(lm-n)l^{-2}} \pm \sqrt{nl^{-2}}.$$

Thus

$$X(\xi, y) = \frac{8EFGH\nu(\xi)}{[c_2(\xi, 0)]^{\frac{1}{2}} l \sqrt{-(lm-n)l^{-2}}} \cdot \left[\arctan \frac{|l|}{\sqrt{-(lm-n)} + \sqrt{n}} + \arctan \frac{|l|}{\sqrt{-(lm-n)} - \sqrt{n}} \right]$$

and again $X(\xi, y)$ is given by eq. (4-93) irrespective of the sign of l . By using an argument similar to the one that led from eq. (3-65) to eq. (3-72) we can show that $X(\xi, y)$ given in eq. (4-93) can also be written in the integral form in eq. (4-92).

Combining eqs. (4-84) and (4-92), we have shown that, for case I $((a+b) \geq 0, (c+d) \geq 0)$,

$$I(x, y) = \int_1^\infty \frac{d\xi}{(\xi-x)} \int_{f_+}^\infty \frac{d\eta}{(\eta-y)\sqrt{F(\xi, \eta)}}. \quad (4-94)$$

Eq. (4-94) has been established for $x \leq -(2EF)^{-1}(E^2 + F^2)$ and $y < -1$ if $|a| < 1$, $|b| < 1$. If $\max\{a, b\} \geq 1$, a further restriction on y is required, as discussed in section 4-4.

For case II (i), $(a+b) \geq 0$ but $(c+d) < 0$. From section 4-4 (or section 3-4) it follows that $h(0, \mu, y)$ decreases strictly from 1 to ξ_{2+} on $[0, \mu_0(0, y)]$ and then increases strictly from ξ_{2+} to $+\infty$ on $[\mu_0(0, y), +\infty)$. From section 4-5, $\mu_-(\xi, 0, y)$ is the inverse of $h(0, \mu, y)$ on $[0, \mu_0(0, y)]$ and $\mu_+(\xi, 0, y)$ is the inverse of $h(0, \mu, y)$ on $[\mu_0(0, y), +\infty)$. Thus

$$I(x, y) = \int_1^{\infty} \frac{d\xi}{(\xi-x)} X(\xi, y) + \int_{\xi_{2+}}^1 \frac{d\xi}{(\xi-x)} Y(\xi, y), \quad (4-95)$$

where $X(\xi, y)$ is again given by eq. (4-85) and hence by eq. (4-92) as well, and

$$Y(\xi, y) = -8EFGH \int_{\mu_-(\xi, 0, y)}^{\mu_+(\xi, 0, y)} \frac{d\mu b_1(\xi, \mu, y) v(\xi)}{[c_1(\xi, \mu)]^{\frac{1}{2}} [b_1(\xi, \mu, y)]^2 - a_1(\xi) c_1(\xi, \mu)}. \quad (4-96)$$

Now we make the change of variable

$$\alpha = \mu^{-1},$$

and then the change of variable

$$u^2 = [\alpha - \alpha_+(\xi)] [\alpha_-(\xi) - \alpha]^{-1}$$

with the inverse

$$\alpha = \alpha_-(\xi) - (\alpha_-(\xi) - \alpha_+(\xi)) [1+u^2]^{-1},$$

and

$$\frac{d\alpha}{du} = 2u (\alpha_-(\xi) - \alpha_+(\xi)) [1+u^2]^{-2}.$$

Here $\alpha_{\pm}(\xi)$ are given in eq. (4-86). Then

$$Y(\xi, y) = - \frac{16EFGH v(\xi)}{[-a_2(\xi, 0)]^{\frac{1}{2}}} \int_0^{\infty} \frac{du (lu^2+m)}{(lu^2+m)^2 - 4mu^2}, \quad (4-97)$$

where l, m, n are defined in eq. (4-89).

From eqs. (4-43) and (4-90) it follows that $(lm-n)$ has the opposite sign to $F(\xi, y)$ when $\xi_{2+} \leq \xi < 1$, while eqs. (4-A3),

(4-A9) and (4-A10) show that $F(\xi, y) < 0$. Thus $(lm-n) > 0$ for $\xi_{2+} \leq \xi < 1$, $y < -1$. Next we recall from sect. 4-4 that

$l_1(\mu, y) > 0$ for all $\mu > 0$ if $(a+b) \geq 0$, and then we use eqs.

(4-59) and (4-89) together with the fact that $\mu_{\pm}(\xi, 0, y) > 0$ for

$\xi_{2+} \leq \xi < 1$ to show that

$$l < 0, \quad m < 0 \quad \text{for} \quad \xi_{2+} \leq \xi < 1.$$

The integration in eq. (4-97) can now be carried out. If $n > 0$, we use the fact that

$$\frac{lu^2+m}{(lu^2+m)^2-4nu^2} = \frac{\frac{1}{2}}{lu^2+2n^{\frac{1}{2}}u+m} + \frac{\frac{1}{2}}{lu^2-2n^{\frac{1}{2}}u+m}$$

to show that

$$Y(\xi, y) = \frac{\pi}{\sqrt{-F(\xi, y)}}, \quad (4-98)$$

using eq. (4-90). If $n < 0$ we have

$$\frac{lu^2+m}{(lu^2+m)^2-4nu^2} = \frac{1}{l(P'+Q')} \left[\frac{P'}{u^2+P'^2} + \frac{Q'}{u^2+Q'^2} \right],$$

where

$$\begin{matrix} P' \\ Q' \end{matrix} = (\sqrt{ml^{-1}-nl^{-2}} \pm \sqrt{-nl^{-2}}).$$

and again $Y(\xi, y)$ is given by eq. (4-98).

Combining eqs. (4-95), (4-92) and (4-98) we have, for case II (i) $\{(a+b) \geq 0, (c+d) < 0\}$,

$$I(x, y) = \int_1^\infty \frac{d\xi}{(\xi-x)} \int_{f_+(\xi)}^\infty \frac{d\eta}{(\eta-y)\sqrt{F(\xi, \eta)}} + \int_{\xi_{2+}}^1 \frac{d\xi}{(\xi-x)} \cdot \frac{\pi}{\sqrt{-F(\xi, y)}}. \quad (4-99)$$

Eq. (4-99) has been established for exactly the same values of x and y as those given after eq. (4-94). In a similar way, for case II (ii) where $(a+b) < 0, (c+d) \geq 0$, $I(x, y)$ is given by eq. (4-99) with ξ_{2+} replaced by ξ_{1+} .

4-7 Evaluation of $I(x, y)$: Case III

In this section we obtain spectral representations of $I(x, y)$ for case III for which $(a+b) < 0$ and $(c+d) < 0$. Firstly we consider

case III (i) for which, in addition $h_1(y) > 0$ and $h_2(y) > 0$ where $h_1(y)$ and $h_2(y)$ are defined in eqs. (4-38) and (4-47). The results of sects. 4-4 and 4-5, summarised in fig. 4-2, enable us to reverse the order of integration in eq. (4-20b), giving

$$\begin{aligned}
 I(x, y) = 2EF \lim_{\delta \rightarrow 0, \epsilon \rightarrow 0} \frac{\partial}{\partial y} \left[\int_{h(\delta, \epsilon, y)}^{\infty} \frac{d\xi}{\xi - x} \int_{\epsilon}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \Lambda(\xi, \delta, \mu, y) \right. \\
 + \int_{g_+(y)}^{h(\delta, \epsilon, y)} \frac{d\xi}{\xi - x} \left\{ \int_{\epsilon}^{\mu_-(\xi, \delta, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) + \int_{\mu_-(\xi, \delta, y)}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \Lambda(\xi, \delta, \mu, y) \right\} \\
 + \left. \int_{H_1(\epsilon, y)}^{g_+(y)} \frac{d\xi}{\xi - x} \int_{\epsilon}^{\mu_{\alpha}(\xi, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) + \int_{H_2(\delta, y)}^{g_+(y)} \frac{d\xi}{\xi - x} \int_{\delta}^{\lambda_{\alpha}(\xi, y)} \frac{d\lambda}{\lambda} \Delta_2(\xi, \lambda) \right], \quad (4-100)
 \end{aligned}$$

where

$$\Delta_1(\xi, \mu) = \int_{\lambda_-(\xi, \mu, y)}^{\lambda_+(\xi, \mu, y)} \frac{d\lambda}{\lambda [\alpha_1(\xi) \lambda^2 + 2b_1(\xi, \mu, y) \lambda + c_1(\xi, \mu)]^{\frac{1}{2}}} = \frac{\pi}{[-c_1(\xi, \mu)]^{\frac{1}{2}}}, \quad (4-101)$$

$$\Delta_2(\xi, \lambda) = \int_{\mu_-(\xi, \lambda, y)}^{\mu_+(\xi, \lambda, y)} \frac{d\mu}{\mu [\alpha_2(\xi) \mu^2 + 2b_2(\xi, \lambda, y) \mu + c_2(\xi, \lambda)]^{\frac{1}{2}}} = \frac{\pi}{[-c_2(\xi, \lambda)]^{\frac{1}{2}}},$$

and $\Lambda(\xi, \delta, \mu, y)$ is defined in eq. (4-76). Applying Leibnitz's theorem to eq. (4-100), we obtain

$$\begin{aligned}
 I(x, y) = 2EF \lim_{\delta \rightarrow 0, \epsilon \rightarrow 0} \left[\int_{h(\delta, \epsilon, y)}^{\infty} \frac{d\xi}{\xi - x} \int_{\epsilon}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \right. \\
 + \int_{g_+(y)}^{h(\delta, \epsilon, y)} \frac{d\xi}{\xi - x} \int_{\mu_-(\xi, \delta, y)}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\
 + \int_{H_1(\epsilon, y)}^{g_+(y)} \frac{d\xi}{\xi - x} \frac{1}{\mu_{\alpha}(\xi, y)} \frac{\partial \mu_{\alpha}(\xi, y)}{\partial y} \Delta_1(\xi, \mu_{\alpha}(\xi, y)) \\
 + \left. \int_{H_2(\delta, y)}^{g_+(y)} \frac{d\xi}{\xi - x} \frac{1}{\lambda_{\alpha}(\xi, y)} \frac{\partial \lambda_{\alpha}(\xi, y)}{\partial y} \Delta_2(\xi, \lambda_{\alpha}(\xi, y)) \right] \\
 + J(x, y) + K(x, y), \quad (4-102)
 \end{aligned}$$

where $J(x, y)$ and $K(x, y)$ are defined and shown to be zero in appendix 4-B. The details of the limiting procedure are also given in appendix 4-B and the final expression for $I(x, y)$ is

$$\begin{aligned}
 I(x, y) = & \int_1^{\infty} \frac{d\xi}{\xi-x} X(\xi, y) + \int_{g_+(y)}^1 \frac{d\xi}{\xi-x} Y(\xi, y) \\
 & + 2EF \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{\xi-x} \frac{1}{\mu_{\alpha}(\xi, y)} \frac{\partial \mu_{\alpha}(\xi, y)}{\partial y} \frac{\pi}{[-c_1(\xi, \mu_{\alpha}(\xi, y))]^{\frac{1}{2}}} \\
 & + 2EF \int_{\xi_{2+}}^{g_+(y)} \frac{d\xi}{\xi-x} \frac{1}{\lambda_{\alpha}(\xi, y)} \frac{\partial \lambda_{\alpha}(\xi, y)}{\partial y} \frac{\pi}{[-c_2(\xi, \lambda_{\alpha}(\xi, y))]^{\frac{1}{2}}}, \quad (4-103)
 \end{aligned}$$

where $X(\xi, y)$ is defined in eq. (4-85) and $Y(\xi, y)$ in eq. (4-96). The argument of section 4-6 shows that $X(\xi, y)$ is again given by eq. (4-92).

The function $Y(\xi, y)$ may also be put in the form given in eq. (4-97). To carry out the integration we need the signs of l, m, n (eq. (4-89)). Now

$$0 < \mu_-(\xi, 0, y) < \mu_l < \mu_+(\xi, 0, y) \quad \text{for } g_+(y) \leq \xi < 1$$

and so $l_1(\mu_-(\xi, 0, y), y) < 0$ and $l_1(\mu_+(\xi, 0, y), y) > 0$, where $l_1(\mu, y)$ is given in eq. (4-31). From eq. (4-59) it then follows that

$$l > 0 \quad \text{and} \quad m < 0 \quad \text{for } g_+(y) \leq \xi < 1.$$

Next note that, from eqs. (4-60) and (4-32), $a_1(\xi) < 0$ for $\xi \geq \xi_{1+}$ when $(a+b) < 0$. Also

$$c_2(\xi, 0) < 0 \quad \text{for } |\xi| < 1 \quad \text{and} \quad (b_2(\xi, 0, y))^2 - a_2(\xi)c_2(\xi, 0) > 0$$

for $\xi > \xi_{2+}$. Remembering the characteristics of case III (i) given at the end of section 4-4, it follows from eq. (4-89) that

$$n < 0 \quad \text{for } g_+(y) \leq \xi < 1.$$

With the definitions $r = l^{-\frac{1}{2}}(-m)^{\frac{1}{2}}(>0)$, $q = nl^{-2}(<0)$, we see that

$$\int_0^\infty \frac{du(u^2-r^2)}{(u^2-r^2)^2-4qu^2} = \int_0^r \frac{dv(v^2-r^2)}{(v^2-r^2)^2-4qv^2} + \int_r^\infty \frac{du(u^2-r^2)}{(u^2-r^2)^2-4qu^2} = 0,$$

on putting $u = r^2 v^{-1}$ in the integral over $[0, r]$. Thus

$$Y(\xi, y) = 0 \quad \text{for } g_+(y) \leq \xi < 1.$$

Next we evaluate the third term on the right hand side of eq.

(4-103). From eq. (4-73) we have

$$\frac{1}{\mu_\alpha(\xi, y)} \frac{\partial \mu_\alpha(\xi, y)}{\partial y} = \frac{\frac{1}{2}\mu_\alpha(\xi, y) \frac{\partial A_1(\xi, y)}{\partial y} + \frac{\partial B_1(\xi, y)}{\partial y}}{[B_1(\xi, y)^2 - A_1(\xi, y)C_1(\xi)]^{\frac{1}{2}}},$$

and it follows from eqs. (4-68), (4-69), (4-27) and (4-51) that

$$\frac{1}{2}\mu \frac{\partial A_1(\xi, y)}{\partial y} + \frac{\partial B_1(\xi, y)}{\partial y} = HG^{-1}b_1(\xi, \mu, y).$$

Thus, using eq. (4-71), we have

$$\begin{aligned} \frac{2EF}{\mu_\alpha(\xi, y)} \frac{\partial \mu_\alpha(\xi, y)}{\partial y} \frac{1}{[-c_1(\xi, \mu_\alpha(\xi, y))]^{\frac{1}{2}}} \\ = \frac{b_1(\xi, \mu_\alpha(\xi, y), y)}{[-a_1(\xi)]^{\frac{1}{2}}[-c_1(\xi, \mu_\alpha(\xi, y))]^{\frac{1}{2}}[-F(\xi, y)]^{\frac{1}{2}}}, \end{aligned}$$

since, from eqs. (4-60) and (4-32), $a_1(\xi) < 0$ for $\xi \geq \xi_{1+}$ and

from eq. (4-A8), $F(\xi, y) < 0$ for $\xi_{1+} \leq \xi < g_+(y)$. By using eqs.

(4-67) and (4-53) we established at the end of sect. 4-5 that for case

III (i)

$$\left\{ b_1(\xi, \mu_\alpha(\xi, y), y) \right\}^2 = a_1(\xi)c_1(\xi, \mu_\alpha(\xi, y)),$$

and

$$\lambda_+(\xi, \mu_\alpha(\xi, y), y) = \lambda_-(\xi, \mu_\alpha(\xi, y), y) > 0 \quad \text{for } \xi_{1+} < \xi \leq g_+(y).$$

Thus

$$\lambda_{\pm}(\xi, \mu_{\alpha}(\xi, y), y) = -[a_1(\xi)]^{-1} b_1(\xi, \mu_{\alpha}(\xi, y), y) > 0$$

and hence

$$b_1(\xi, \mu_{\alpha}(\xi, y), y) = [-a_1(\xi)]^{\frac{1}{2}} [-c_1(\xi, \mu_{\alpha}(\xi, y))]^{\frac{1}{2}} > 0.$$

This completes the evaluation of the third term on the right side of eq. (4-103). The fourth term may be evaluated in the same way, so that

$$\begin{aligned} I(x, y) = & \int_1^{\infty} \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \\ & + \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{\xi-x} \frac{\pi}{\sqrt{-F(\xi, y)}} + \int_{\xi_{2+}}^{g_+(y)} \frac{d\xi}{\xi-x} \frac{\pi}{\sqrt{-F(\xi, y)}}. \quad (4-104) \end{aligned}$$

Eq. (4-104) has been established for case III (i), for which $(a+b) < 0$, $(c+d) < 0$,

$$y < \min \left\{ -1, \frac{a\sqrt{1-b^2} + d\sqrt{1-a^2}}{-a\sqrt{1-b^2} - b\sqrt{1-a^2}}, \frac{a\sqrt{1-d^2} + b\sqrt{1-c^2}}{-a\sqrt{1-b^2} - b\sqrt{1-a^2}} \right\}. \quad (4-105)$$

We use eqs. (4-38) and (4-47) together with the fact that

$$(a\sqrt{1-b^2} + b\sqrt{1-a^2}) < 0 \text{ when } (a+b) < 0 \text{ to obtain eq. (4-105). As}$$

$$\text{usual } x \leq -(2EF)^{-1}(E^2 + F^2).$$

For case III (ii), $h_1(y) \leq 0$ and $h_2(y) > 0$ in addition to

$(a+b) < 0$, $(c+d) < 0$. Using the results of sections 4-4 and 4-5, summarized in fig. 4-3, one can show, in a similar way to the above, that

$$\begin{aligned} I(x, y) = & \int_1^{\infty} \frac{d\xi}{\xi-x} X(\xi, y) + \int_{\xi_{1+}}^1 \frac{d\xi}{\xi-x} Y(\xi, y) \\ & + 2EF \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-x} \frac{\partial \lambda_{\alpha}(\xi, y)}{\partial y} \frac{1}{\lambda_{\alpha}(\xi, y)} \frac{\pi}{\sqrt{-c_2(\xi, \lambda_{\alpha}(\xi, y))}}. \end{aligned}$$

We find that $Y(\xi, y) = 0$ for $\xi_{1+} \leq \xi < 1$ and that

$$I(x, y) = \int_1^\infty \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^\infty \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-x} \frac{\pi}{\sqrt{-F(\xi, y)}}. \quad (4-106)$$

We showed in sect. 4-4 that when $(a+b) < 0$, $(c+d) < 0$, the case $h_1(y) \leq 0$, $h_2(y) \leq 0$ is not possible for any $y < -1$. Thus, for case III (ii) where $h_1(y) \leq 0$, $h_2(y) > 0$, it follows that

$$-1 < \frac{a\sqrt{1-d^2} + b\sqrt{1-c^2}}{-c\sqrt{1-d^2} - d\sqrt{1-c^2}}.$$

Thus the function $I(x, y)$ is given by eq. (4-106) for

$$x \leq -(2EF)^{-1}(E^2 + F^2) \quad \text{and}$$

$$\frac{c\sqrt{1-b^2} + d\sqrt{1-a^2}}{-a\sqrt{1-b^2} - b\sqrt{1-a^2}} \leq y < -1. \quad (4-107)$$

Finally, for case III (iii) where $h_1(y) > 0$ and $h_2(y) \leq 0$ in addition to $(a+b) < 0$, $(c+d) < 0$, $I(x, y)$ is given by eq. (4-106) with

$$\xi_{1+} \leftrightarrow \xi_{2+}, \quad \text{for } x \leq -(2EF)^{-1}(E^2 + F^2) \quad \text{and}$$

$$\frac{a\sqrt{1-d^2} + b\sqrt{1-c^2}}{-c\sqrt{1-d^2} - d\sqrt{1-c^2}} \leq y < -1.$$

4-8. Spectral representation of the box diagram amplitude

In this section we study the set of points $\{(\xi, \eta) \mid F(\xi, \eta) = 0\}$, which we call the curve Γ , and then give the explicit form of the spectral representation of $I(x, y)$ for all mass configurations allowed by the stability conditions. The function $F(\xi, \eta)$ is defined in eq. (4-12); it is a quadratic polynomial in ξ and η . One factorisation of $F(\xi, \eta)$ is given in eq. (4-12), the functions $f_{\pm}(\xi)$ being given by eq. (4-11) or, alternatively, by

$$f_{\pm}(\xi) = (\xi^2 - 1)^{-1} \left\{ (\xi - 1)(ad + bc) + (a + b)(c + d) \right. \\ \left. \pm [(\xi - \xi_{1+})(\xi - \xi_{1-})(\xi - \xi_{2+})(\xi - \xi_{2-})]^{\frac{1}{2}} \right\}.$$

The quantities $\xi_{1\pm}$, defined in eq. (4-32), are complex conjugate numbers when $(1-a)(1-b) < 0$; when $(1-a)(1-b) \geq 0$ they are real and

$$\xi_{1-} \leq \xi_{1+} \leq 1,$$

as given in eq. (3-57). A similar statement (with $a \rightarrow c$, $b \rightarrow d$) holds for $\xi_{2\pm}$ defined in eq. (4-42). Thus the functions $f_{\pm}(\xi)$ are real for all ξ if $\xi_{1\pm}$ are complex conjugates and $\xi_{2\pm}$ are complex conjugates, while if at least one pair is real then $f_{\pm}(\xi)$ are real for $\xi \geq \xi_{\alpha}$, where

$$\xi_{\alpha} = \begin{cases} \max\{\xi_{1+}, \xi_{2+}\} & \text{if both } \xi_{1+}, \xi_{2+} \text{ are real,} \\ \xi_{1+} & \text{if } \xi_{1+} \text{ is real and } \xi_{2+} \text{ is complex,} \\ \xi_{2+} & \text{if } \xi_{2+} \text{ is real and } \xi_{1+} \text{ is complex.} \end{cases} \quad (4-108)$$

Note that one of the numbers $f_{\pm}(1)$ is indeterminate.

From eq. (4-A3) we have an alternative factorisation of $F(\xi, \eta)$, the functions $g_{\pm}(\eta)$ being given by eq. (4-36) or, alternatively, by

$$g_{\pm}(\eta) = (\eta^2 - 1)^{-1} \left\{ (\eta - 1)(ad + bc) + (a + c)(b + d) \right. \\ \left. \pm [(\eta - \eta_{1+})(\eta - \eta_{1-})(\eta - \eta_{2+})(\eta - \eta_{2-})]^{\frac{1}{2}} \right\}.$$

The quantities $\eta_{1\pm}$, defined in eq. (4-A2), are either complex conjugate numbers or are both real, with

$$\eta_{1-} \leq \eta_{1+} \leq 1,$$

and similarly for $\eta_{2\pm}$ (also defined in eq. (4-A2)). Thus the functions $g_{\pm}(\eta)$ are real for all η if $\eta_{1\pm}$ are complex conjugates and $\eta_{2\pm}$ are

complex conjugates, while if at least one pair is real then $g_{\pm}(\eta)$ are real for $\eta \geq \eta_{\alpha}$, where η_{α} is defined in a similar way to ξ_{α} (eq. (4-108) with $\xi \rightarrow \eta$),

The curve Γ has several branches (Tarski 1960), but for our purposes it is sufficient to examine only one, namely the branch Γ_1 for which $\eta \rightarrow +\infty$ either as $\xi \downarrow 1$ or as $\xi \uparrow 1$ and for which $\xi \rightarrow +\infty$ either as $\eta \downarrow 1$ or as $\eta \uparrow 1$. Using eqs. (4-11) and (4-36) we obtain the following properties of $f_{\pm}(\xi)$ and $g_{\pm}(\eta)$ which we need to determine the behaviour of Γ_1 .

When $(a+b)(c+d) > 0$,

$$f_{+}(\xi) \sim (\xi-1)^{-1}(a+b)(c+d) \text{ for } \xi \downarrow 1,$$

so that $f_{+}(\xi) \rightarrow +\infty$ as $\xi \downarrow 1$. Also,

$$g_{+}(\eta) \sim 1 + \eta^{-1}(a+b)(c+d) \text{ for } \eta \rightarrow +\infty,$$

so that $g_{+}(\eta) \downarrow 1$ as $\eta \rightarrow +\infty$. Similarly, when $(a+c)(b+d) > 0$,

$f_{+}(\xi) \downarrow 1$ as $\xi \rightarrow +\infty$ and $g_{+}(\eta) \rightarrow +\infty$ as $\eta \downarrow 1$.

When $(a+b)(c+d) < 0$, $g_{+}(\eta) \uparrow 1$ as $\eta \rightarrow +\infty$, $f_{-}(\xi) \rightarrow +\infty$ as $\xi \uparrow 1$, and

$$f_{+}(1) = 1 + [2(a+b)(c+d)]^{-1}[(ad-bc)^2 - (a+b+c+d)^2].$$

We also note that, in this case, either $(a+b) < 0$, and then

$|a| < 1$, $|b| < 1$ and

$$f_{\pm}(\xi_{1+}) = \frac{c\sqrt{1-b^2} + d\sqrt{1-a^2}}{-a\sqrt{1-b^2} - b\sqrt{1-a^2}}, \quad (4-109)$$

or $(c+d) < 0$, in which case $|c| < 1$, $|d| < 1$ and

$$f_{\pm}(\xi_{2+}) = \frac{a\sqrt{1-d^2} + b\sqrt{1-c^2}}{-c\sqrt{1-d^2} - d\sqrt{1-c^2}}. \quad (4-110)$$

Similarly, when $(a+c)(b+d) < 0$, $f_{+}(\xi) \uparrow 1$ as $\xi \rightarrow +\infty$, $g_{-}(\eta) \rightarrow +\infty$ as $\eta \uparrow 1$, and

$$g_{+}(1) = 1 + [2(a+c)(b+d)]^{-1} [(ad-bc)^2 - (a+b+c+d)^2].$$

We also note that in this case, either $(a+c) < 0$, and then $|a| < 1$, $|c| < 1$ and

$$g_{\pm}(\eta_{1+}) = \frac{b\sqrt{1-c^2} + d\sqrt{1-a^2}}{-a\sqrt{1-c^2} - c\sqrt{1-a^2}},$$

or $(b+d) < 0$, in which case $|b| < 1$, $|d| < 1$ and

$$g_{\pm}(\eta_{2+}) = \frac{a\sqrt{1-d^2} + c\sqrt{1-b^2}}{-b\sqrt{1-d^2} - d\sqrt{1-b^2}}.$$

When $(a+b)(c+d) > 0$ and $(a+c)(b+d) < 0$, we find that

$$g_{+}(1) > 1.$$

For, if $(a+c) > 0$, $(b+d) < 0$, then $|b| < 1$, $|d| < 1$, and

$$\begin{aligned} (a+b+c+d)^2 - (ad-bc)^2 \\ = [(a+b)(1+d) + (c+d)(1-b)][(a+b)(1-d) + (c+d)(1+b)] > 0, \end{aligned}$$

and similarly if $(a+c) < 0$, $(b+d) > 0$. In the same way, if

$(a+b)(c+d) < 0$ and $(a+c)(b+d) > 0$, then

$$f_{+}(1) > 1.$$

When $(a+b)(c+d) < 0$ and $(a+c)(b+d) < 0$, $f_{+}(1)$ and $g_{+}(1)$ are either both greater than 1, both equal to 1, or both less than 1.

We are now in a position to sketch the required branch Γ_1 and the region R above and to the right of it for which $F(\xi, \eta) > 0$. We do not consider the degenerate cases for which at least one of the quantities $(a+b)$, $(c+d)$, $(a+c)$, $(b+d)$ is zero; these can be

obtained by modification of the above analysis. There are then four cases:

$$(1a) \quad (a+b)(c+d) > 0, \quad (a+c)(b+d) > 0;$$

$$(1b) \quad (a+b)(c+d) > 0, \quad (a+c)(b+d) < 0;$$

$$(2a) \quad (a+b)(c+d) < 0, \quad (a+c)(b+d) > 0;$$

$$(2b) \quad (a+b)(c+d) < 0, \quad (a+c)(b+d) < 0.$$

The branch Γ_1 and the region R (subscripted according to the case under consideration) are sketched in fig. 4-4 for these four cases. In case (2b) the intersections of Γ_1 with the lines $\xi = 1$, $\eta = 1$

depend on the sign of $(a+b+c+d)^2 - (ad-bc)^2$. Regions R_{1a} and R_{1b} are subcases of the region

$$R_1 = \{(\xi, \eta) \mid \xi > 1, \eta > f_+(\xi)\}, \quad (4-111)$$

while regions R_{2a} and R_{2b} are subcases of the region

$$R_2 = \{(\xi, \eta) \mid \xi_\alpha < \xi < 1, f_+(\xi) < \eta < f_-(\xi)\} \\ \cup \{(\xi, \eta) \mid \xi \geq 1, \eta > f_+(\xi)\}. \quad (4-112)$$

We now give the arguments used to establish the behaviour of Γ_1 .

For case (1a), we note that the functions $f_\pm(\xi)$ are defined and differentiable for $\xi > 1$, and $f_-(\xi) < f_+(\xi)$; a similar statement holds for the functions $g_\pm(\eta)$. Knowing the asymptotic behaviour of $f_+(\xi)$, we deduce that $f_+(\xi) > 1$ for $\xi > 1$; for otherwise there would be two real values of ξ for which $F(\xi, 1) = 0$, which is impossible. Next, $f_+(\xi)$ cannot take the same value for two *different* values of ξ . For if $f_+(\xi_A) = f_+(\xi_B)$, with $\xi_A < \xi_B$, then $f'_+(\xi)$ vanishes at least once in the interval (ξ_A, ξ_B) , at ξ_m say; but

Fig. 4-4.

The branch Γ_1 of the curve Γ for which $F(\xi, \eta) = 0$, and the corresponding region R for which $F(\xi, \eta) > 0$, in the four cases (1a), (1b), (2a), (2b) of sect. 4-8.

then $g_+(f_+(\xi_m)) = g_-(f_+(\xi_m))$, which is impossible. Thus $f_+(\xi)$ is monotonic decreasing on $(1, +\infty)$, and similarly $g_+(\eta)$ is monotonic decreasing on $(1, +\infty)$ and they are inverse functions. The arguments used to establish the behaviour of Γ_1 for the other cases are similar but slightly more complicated. For case (1b), we use the additional facts that $g_+(1) > 1$, $g_+(\eta_\alpha) = g_-(\eta_\alpha)$ together with the asymptotic behaviour of $g_+(\eta)$ and the behaviour of $g_-(\eta)$ as $\eta \uparrow 1$. Further, we use the fact that $(f_+(\xi) - \eta_\alpha) \geq 0$ for $\xi > 1$, which follows by analogy with eq. (4-A7). The arguments leading to the behaviour of Γ_1 for case (2a) are similar to those for case (1b). For case (2b) we use the additional facts that $(g_+(\eta) - \xi_\alpha) \geq 0$ for $\eta \geq 1$ and $(g_\pm(\eta) - \xi_\alpha) \geq 0$ for $\eta_\alpha \leq \eta < 1$. These relations follow from eqs. (4-A7) and (4-A12). Further, we use the relations $(f_+(\xi) - \eta_\alpha) \geq 0$ for $\xi \geq 1$ and $(f_\pm(\xi) - \eta_\alpha) \geq 0$ for $\xi_\alpha \leq \xi < 1$ which follow by analogy with eqs. (4-A7) and (4-A12). In this case the exact ordering of the points 1 , $g_+(1)$ and $g_\pm(\eta_\alpha)$ and also of 1 , $f_+(1)$ and $f_\pm(\eta_\alpha)$ varies depending on the values of a , b , c and d .

Our object now is to express $I(x, y)$ as a double integral over the region R_1 or R_2 or, when this is not possible, as a double integral over the region R_1 or R_2 plus a single integral. To do this for the case when $(a+b)(c+d) < 0$ requires some additional results. When $(c+d) < 0$, $(a+b) > 0$ and, in addition, $a \geq 1$, $b \geq 1$, we have

$$\xi_{2+} = -1 + \frac{1}{2} \left[(1+c)^{\frac{1}{2}}(1-d)^{\frac{1}{2}} + (1+d)^{\frac{1}{2}}(1-c)^{\frac{1}{2}} \right]^2 > -1 ,$$

$$\xi_{1+} = -1 - \frac{1}{2} \left[(1+a)^{\frac{1}{2}}(b-1)^{\frac{1}{2}} - (1+b)^{\frac{1}{2}}(a-1)^{\frac{1}{2}} \right]^2 \leq -1 ,$$

and thus $\xi_{2+} > \xi_{1+}$. Hence, if $(c+d) < 0$ and $\max\{a, b\} > 1$, it

follows from eq. (4-108) that $\xi_{\alpha} = \xi_{2+}$. If $(c+d) < 0$ and

$(a+b) > 0$, but $\max\{a, b\} \leq 1$, it is convenient to define angles

$\alpha, \beta, \gamma, \delta$ by

$$\alpha = \arccos a, \quad \beta = \arccos b, \quad \gamma = \arccos c, \quad \delta = \arccos d, \quad (4-113)$$

where

$$0 \leq \alpha < \pi, \quad 0 \leq \beta < \pi, \quad 0 < \gamma < \pi, \quad 0 < \delta < \pi.$$

(These angles are similar to those used by Karplus, Sommerfield and Wichmann 1959.) Then

$$\xi_{1\pm} = -\cos(\alpha \pm \beta), \quad \xi_{2\pm} = -\cos(\gamma \pm \delta),$$

$$\eta_{1\pm} = -\cos(\alpha \pm \gamma), \quad \eta_{2\pm} = -\cos(\beta \pm \delta).$$

The condition $(c+d) < 0$ is equivalent to $\pi < (\gamma+\delta) < 2\pi$, while

$(a+b) > 0$ is equivalent to $0 \leq (\alpha+\beta) < \pi$. If, in addition,

$(\alpha+\beta+\gamma+\delta) \leq 2\pi$, then $0 < (\gamma+\delta-\pi) \leq \pi - (\alpha+\beta) \leq \pi$, and so

$\cos(\gamma+\delta) \leq \cos(\alpha+\beta)$ and $\xi_{1+} \leq \xi_{2+}$. If, however, $(\alpha+\beta+\gamma+\delta) > 2\pi$,

then $\xi_{2+} < \xi_{1+}$. In particular, when $\alpha + \beta + \gamma + \delta = 2\pi$, then

$$\xi_{1+} = \xi_{2+} \quad \text{and} \quad f_{\pm}(\xi_{1+}) = f_{\pm}(\xi_{2+}) = \eta_{1+} = \eta_{2+}.$$

We have now shown that, when $(c+d) < 0$ and $(a+b) > 0$,

$\xi_{\alpha} = \xi_{2+}$ if $\max\{a, b\} > 1$ or if $-1 < a \leq 1$, $-1 < b \leq 1$ and

$(\alpha+\beta+\gamma+\delta) \leq 2\pi$. However, $\xi_{\alpha} = \xi_{1+}$ if $-1 < a \leq 1$, $-1 < b \leq 1$ and

$(\alpha+\beta+\gamma+\delta) > 2\pi$. Also, when $(c+d) < 0$, $(a+b) > 0$ and $y < -1$,

$$\int_{f_{+}(\xi)}^{f_{-}(\xi)} \frac{d\eta}{(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}} = \frac{\pi}{\sqrt{-F(\xi, y)}}, \quad (4-114)$$

for $\xi_\alpha \leq \xi < 1$. To prove eq. (4-114), note that $f_\pm(\xi) > -1$ for $\xi_\alpha \leq \xi < 1$, so that, from eq. (4-12), $F(\xi, y) < 0$ for $\xi_\alpha \leq \xi < 1$ and $y < -1$. That $f_\pm(\xi) > -1$ for $\xi_\alpha \leq \xi < 1$ may be seen as follows. The argument above shows that $\xi_\alpha > -1$. Similarly for case (2b), $\eta_\alpha > -1$, which shows that for this case $f_\pm(\xi) > -1$ for $\xi_\alpha \leq \xi < 1$. For case (2a) this result follows directly from the properties of $f_+(\xi)$ summarized in fig. 4-4.

We are now in a position to give the explicit form of $I(x, y)$ for the various mass configurations.

I: $(a+b) > 0$, $(c+d) > 0$. From eq. (4-94) it follows that

$$I(x, y) = \iint_{R_1} \frac{d\xi d\eta}{(\xi-x)(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}}, \quad (4-115)$$

where $F(\xi, \eta)$ is given in eq. (4-12) and R_1 in eq. (4-111). Eq. (4-115) has been shown to be valid for $x \leq -(2EF)^{-1}(E^2 + F^2)$, $y < -1$ if $|a| < 1$, $|b| < 1$ or if $|c| < 1$, $|d| < 1$. In other cases we have had to restrict y to a smaller range (section 4-4). Once the double spectral representation (4-115) has been obtained, however, we see that $I(x, y)$ is defined by it for $x < 1$, $y < 1$ in case (1a), and for $x < 1$, $y < \eta_\alpha$ in case (1b). Note, too, that the representation of $I(x, y)$ given by eq. (4-115) is exactly that obtained heuristically in section 4-2 (eq. (4-13)).

II. (i): $(a+b) > 0$, $(c+d) < 0$. From eqs. (4-99) and (4-114) and from the discussion of the preceding paragraph it follows that

$$I(x, y) = \iint_{R_2} \frac{d\xi d\eta}{(\xi-x)(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}} \quad (4-116)$$

if $\max\{a, b\} > 1$ or if $-1 < a \leq 1$, $-1 < b \leq 1$ and $(\alpha + \beta + \gamma + \delta) \leq 2\pi$.

However, if $-1 < a \leq 1$, $-1 < b \leq 1$ and $(\alpha + \beta + \gamma + \delta) > 2\pi$, then

$$I(x, y) = \iint_{R_2} \frac{d\xi d\eta}{(\xi-x)(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{(\xi-x)} \frac{\pi}{\sqrt{-F(\xi, y)}}, \quad (4-117)$$

and $I(x, y)$ no longer has a double spectral representation. In eqs. (4-116) and (4-117), R_2 is given in eq. (4-112), ξ_{1+} in eq.

(4-32), ξ_{2+} in eq. (4-42) and $\alpha, \beta, \gamma, \delta$ in eq. (4-113). Eq.

(4-116) has been shown to be valid for $x \leq -(2EF)^{-1}(E^2 + F^2)$,

$y < -1$ if $|a| < 1$, $|b| < 1$. In other cases we have had to restrict

y to a smaller range as discussed in sect. 4-4. However, once the

double spectral representation in eq. (4-116) has been obtained, we

see that $I(x, y)$ is defined by it for $x < \xi_\alpha$, $y < 1$ in case (2a),

and for $x < \xi_\alpha$, $y < \eta_\alpha$ in case (2b). Eq. (4-117) has been shown to

be valid for $x \leq -(2EF)^{-1}(E^2 + F^2)$, $y < -1$. In fact we see from eq.

(4-117) that $I(x, y)$ is defined by it at least for $x < \xi_{2+}$,

$y < -1$. A detailed discussion of $I(x, y)$ as a function of x and

y is given in chapter 5.

II (ii): $(a+b) < 0$, $(c+d) > 0$. The results for case II (i) apply, with the replacements $a \rightarrow c$, $b \rightarrow d$, $\xi_{1+} \leftrightarrow \xi_{2+}$.

III: $(a+b) < 0$, $(c+d) < 0$. Thus $(\alpha + \beta) > \pi$, $(\gamma + \delta) > \pi$ and $(\alpha + \beta + \gamma + \delta) > 2\pi$. There are three subcases to consider.

(i) $h_1(y) > 0$, $h_2(y) > 0$. It follows from eq. (4-104) that

$$I(x, y) = \iint_{R_1} \frac{d\xi d\eta}{(\xi-x)(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{(\xi-x)} \frac{\pi}{\sqrt{-F(\xi, y)}} + \int_{\xi_{2+}}^{g_+(y)} \frac{d\xi}{(\xi-x)} \frac{\pi}{\sqrt{-F(\xi, y)}}. \quad (4-118)$$

The functions $h_1(y)$ and $h_2(y)$ are given in eqs. (4-38) and (4-47), $F(\xi, \eta)$ in eq. (4-12), R_1 in eq. (4-111), $g_+(y)$ in eq. (4-36), ξ_{1+} in eq. (4-32) and ξ_{2+} in eq. (4-42). Eq. (4-118) defines $I(x, y)$ for $x < \min\{\xi_{1+}, \xi_{2+}\}$ and for

$$y < \min\{f_{\pm}(\xi_{1+}), f_{\pm}(\xi_{2+}), -1\}, \quad (4-119)$$

where $f_{\pm}(\xi_{1+}), f_{\pm}(\xi_{2+})$ are given in eqs. (4-109), (4-110) respectively.

The restriction on y comes directly from eq. (4-105).

(ii) $h_1(y) \leq 0$, $h_2(y) > 0$. From eqs. (4-106) and (4-107) we have

$$I(x, y) = \iint_{R_1} \frac{d\xi d\eta}{(\xi-x)(\eta-y)} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{(\xi-x)} \frac{1}{\sqrt{-F(\xi, y)}}, \quad (4-120)$$

for $x < \xi_{2+}$ and for

$$f_{\pm}(\xi_{1+}) \leq y < -1. \quad (4-121)$$

(iii) $h_1(y) > 0$, $h_2(y) \leq 0$. The result in (ii) applies, with the interchange $\xi_{1+} \leftrightarrow \xi_{2+}$.

We thus see that for the range of x and y considered above, $I(x, y)$ cannot be given as a double dispersion integral over the real domain R_1 or R_2 if and only if $-1 < a \leq 1$, $-1 < b \leq 1$, $-1 < c \leq 1$, $-1 < d \leq 1$ and $\alpha + \beta + \gamma + \delta > 2\pi$.

In conclusion, we see that our method of establishing dispersion relations for the box diagram amplitude is very similar to that used in sects. 3-3 to 3-6 to establish dispersion relations for the triangle diagram amplitude. This is particularly so for the cases where $a + b \geq 0$ (or $(c+d) \geq 0$); for then $h(\lambda, \mu, y)$ is strictly increasing with λ for all fixed $\mu > 0$, and $\lambda_+(\xi, \mu, y)$ is the

inverse of $h(\lambda, \mu, y)$ on $[0, +\infty)$. The order of integration in eq. (4-20a) can then be reversed to obtain eq. (4-75) and (4-76), in a way similar to that leading from eq. (3-30) to eqs. (3-63) and (3-65). Further, the integration in eq. (4-76) can then be carried out in the same way as that in eq. (3-65). Once the limiting procedure has been carried out, $I(x, y)$ takes the form in eq. (4-83) with the region of integration the same as that used to define $I(\tau_0)$ in eq. (3-30), with the replacements $\lambda \rightarrow \mu$, $a \rightarrow c$, $b \rightarrow d$, $G \rightarrow H$. Thus, the order of integration in eq. (4-83) can be reversed in exactly the same way as for eq. (3-30), and $I(x, y)$ is given by eqs. (4-84) and (4-85) when $(c+d) \geq 0$ (similar to eqs. (3-63) and (3-65)) and by eqs. (4-95), (4-85) and (4-86) when $(c+d) < 0$ (similar to eqs. (3-64), (3-65) and (3-66)). The integral representations of the weight functions $X(\xi, y)$ and $Y(\xi, y)$ in eqs. (4-85) and (4-96) are, of course more complicated than those for $X(\xi)$ and $Y(\xi)$ in eqs. (3-64) and (3-66). However, we find that the integrated forms of $X(\xi, y)$ given in eqs. (4-91) and (4-93) are similar to those for $X(\xi)$ in eqs. (3-76) and (3-72). Also, the form of $Y(\xi, y)$ given in eq. (4-98) is similar to that of $Y(\xi)$ given in eq. (3-68).

For case III, where $(a+b) < 0$, $(c+d) < 0$, the similarity between the method used for the box diagram amplitude and that used for the triangle diagram is less apparent. Now the properties of the functions $H_1(\mu, y)$ (eq. (4-29)) and $H_2(\lambda, y)$ (eq. (4-41)), and of the inverses $\mu_\alpha(\xi, y)$, $\mu_\beta(\xi, y)$ (eq. (4-73)) and $\lambda_\alpha(\xi, y)$, $\lambda_\beta(\xi, y)$, are needed. However, the transformations leading to eq. (4-15) have been chosen in such a way that the properties of $H_1(\mu, y)$ and $H_2(\mu, y)$ can be established in a similar way to the properties

of $h(\lambda)$ in eq. (3-28). Further, $\mu_\alpha(\xi, y)$, $\mu_\beta(\xi, y)$ and $\lambda_\alpha(\xi, y)$, $\lambda_\beta(\xi, y)$ can be written in a form similar to that of $\lambda_\pm(\xi)$ in eq.

(3-60). The method of obtaining the spectral representations for case III, once the properties of the surfaces in figs. 4-2 and 4-3 have been established, is, of course, more complicated than that leading to the spectral representations for the triangle diagram amplitude.

In the next chapter, the other method used in chapter 3, namely analytic continuation, is applied to the box diagram amplitude.

APPENDIX 4A

We collect here a number of results involving the quantities $\xi_{1\pm}$, $\xi_{2\pm}$ and $g_{\pm}(y)$ introduced in eqs. (4-32), (4-42) and (4-36).

From eq. (4-36) we see that $g_{\pm}(y)$ can be written in the form

$$g_{\pm}(y) = (y^2 - 1)^{-1} \left[(y-1)(ad+bc) + (a+c)(b+d) \right. \\ \left. \pm \sqrt{(y-\eta_{1+})(y-\eta_{1-})(y-\eta_{2+})(y-\eta_{2-})} \right] \quad (4-A1)$$

where

$$\eta_{1\pm} = -ac \pm \sqrt{(1-a^2)(1-c^2)}, \quad \eta_{2\pm} = -bd \pm \sqrt{(1-b^2)(1-d^2)}. \quad (4-A2)$$

The properties of $\eta_{1\pm}$, $\eta_{2\pm}$ are similar to those of $\xi_{1\pm}$, $\xi_{2\pm}$.

The quantities $\eta_{1\pm}$ are complex conjugate numbers when $(1-a)(1-c) < 0$

and when $(1-a)(1-c) \geq 0$ they are real and

$$\eta_{1-} \leq \eta_{1+} \leq 1.$$

The two equality signs in the above equation cannot occur together.

In particular, when $|a| < 1$, $|c| < 1$,

$$-1 \leq \eta_{1-} < \eta_{1+} \leq 1.$$

Similar statements hold for $\eta_{2\pm}$ (with $a \rightarrow b$, $c \rightarrow d$). Thus, the

functions $g_{\pm}(y)$ are real for all y if $\eta_{1\pm}$ are complex conjugates

and $\eta_{2\pm}$ are complex conjugates, while if at least one pair is real,

then $g_{\pm}(y)$ are real for $y \geq \eta_{\alpha}$ where η_{α} is defined in a similar

way to ξ_{α} (eq. (4-108) with $\xi \rightarrow \eta$). Also, when $|a| < 1$,

$|b| < 1$, $|c| < 1$ and $|d| < 1$, $g_{\pm}(y)$ are real for

$y \leq \min\{\eta_{1-}, \eta_{2-}\}$ and in particular for $y < -1$.

Using eq. (4-12) we find that

$$\begin{aligned}
& (y^2-1) \{ \xi - g_+(y) \} \{ \xi - g_-(y) \} \\
&= (y^2-1) \{ \xi^2-1 \} + 2(1-y)(\xi-1)(ad+bc) + 2(1-y)(a+b)(c+d) \\
&\quad - 2(\xi-1)(a+c)(b+d) + (ad-bc)^2 - (a+b+c+d)^2 \\
&= F(\xi, y) .
\end{aligned} \tag{4-A3}$$

Also,

$$\begin{aligned}
(\xi^2-1)F(\xi, y) &= [G(\xi, y)]^2 - \{ \xi^2+2ab\xi+a^2+b^2-1 \} \{ \xi^2+2cd\xi+c^2+d^2-1 \} \\
&= [G(\xi, y)]^2 - (\xi-\xi_{1-})(\xi-\xi_{1+})(\xi-\xi_{2-})(\xi-\xi_{2+}) ,
\end{aligned} \tag{4-A4}$$

where

$$G(\xi, y) = -y(\xi^2-1) + (\xi-1)(ad+bc) + (a+b)(c+d) . \tag{4-A5}$$

From eqs. (4-32), (4-36) and (4-A1) we see that when $|a| < 1$, $|b| < 1$ and $y > 1$ and also when $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$ and $y < -1$,

$$\begin{aligned}
g_{\pm}(y) - \xi_{1\pm} &= \sqrt{(1-a^2)(1-b^2)} \left[-1 + \frac{(ya+c)(yb+d)}{(y^2-1)\sqrt{(1-a^2)(1-b^2)}} \right. \\
&\quad \left. \pm \left\{ 1 + \frac{(ya+c)^2}{(y^2-1)(1-a^2)} \right\}^{\frac{1}{2}} \left\{ 1 + \frac{(yb+d)^2}{(y^2-1)(1-b^2)} \right\}^{\frac{1}{2}} \right] .
\end{aligned}$$

We now define the quantities $\chi_1(y)$ and $\chi_2(y)$ by

$$\sinh \chi_1(y) = \frac{ya+c}{\sqrt{(y^2-1)(1-a^2)}} , \quad \sinh \chi_2(y) = \frac{yb+d}{\sqrt{(y^2-1)(1-b^2)}} . \tag{4-A6}$$

Then

$$g_{\pm}(y) - \xi_{1\pm} = \sqrt{(1-a^2)(1-b^2)} [-1 \pm \cosh \{ \chi_1(y) \pm \chi_2(y) \}] \tag{4-A7}$$

and $(g_+(y) - \xi_{1+}) \geq 0$, equality holding if and only if

$$(\chi_1(y) + \chi_2(y)) = 0 \quad \text{or} \quad h_1(y) = 0 , \quad \text{where } h_1(y) \text{ is defined in eq.}$$

(4-38). Similarly

$$g_-(y) - \xi_{1-} = \sqrt{(1-a^2)(1-b^2)} [1 - \cosh \{ \chi_1(y) - \chi_2(y) \}] \leq 0 ,$$

so that, combining this relation with eqs. (4-A7) and (4-33), we have

$$g_-(y) \leq \xi_{1-} < \xi_{1+} \leq g_+(y) \quad (4-A8)$$

when $|a| < 1$, $|b| < 1$ and $y > 1$ and also when $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$ and $y < -1$. In the same way

$$g_-(y) \leq \xi_{2-} < \xi_{2+} \leq g_+(y) \quad (4-A9)$$

when $|c| < 1$, $|d| < 1$ and $y > 1$ and also when $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$ and $y < -1$.

From eq. (4-12),

$$F(1, y) = 2(-y+1)(a+b)(c+d) + (ad-bc)^2 - (a+b+c+d)^2.$$

If, then, $(a+b)(c+d) < 0$ and $y < -1$, we have

$$\begin{aligned} F(1, y) &< (ad-bc)^2 - (a+b-c-d)^2 \\ &= -[(a+b)(1+d) - (c+d)(1+b)][(a+b)(1+c) - (c+d)(1+a)] < 0. \end{aligned}$$

But from eq. (4-A3)

$$F(1, y) = (y^2 - 1)(1 - g_+(y))(1 - g_-(y)),$$

so that, since $g_-(y) < g_+(y)$, we have

$$g_-(y) < 1 < g_+(y) \text{ when } (a+b)(c+d) < 0, \quad y < -1 \quad (4-A10)$$

Next, we establish some results needed for case (2b) of sect. 4-8 for which $(a+b)(c+d) < 0$, $(a+c)(b+d) < 0$. We suppose that

$(a+b) < 0$, $(c+d) > 0$, $(a+c) > 0$, $(b+d) < 0$; then $|a| < 1$, $|b| < 1$, $|d| < 1$. For $\eta_\alpha \leq y < 1$, where η_α is defined in eq.

(4-108) with $\xi \rightarrow \eta$, we see from eqs. (4-32) and (4-36) that

$$\begin{aligned} g_\pm(y) - \xi_{1\pm} &= \sqrt{(1-a^2)(1-b^2)} \left[-1 - \frac{(ya+c)(yb+d)}{(1-y^2)\sqrt{1-a^2}\sqrt{1-b^2}} \right. \\ &\quad \left. \mp \sqrt{\frac{(ya+c)^2}{(1-y^2)(1-a^2)} - 1} \sqrt{\frac{(yb+d)^2}{(1-y^2)(1-b^2)} - 1} \right]. \end{aligned}$$

Now, $(a+c) > 0$, $(b+d) < 0$ and since

$$(ya+c)^2 - (1-y^2)(1-a^2) > 0, \quad (yb+d)^2 - (1-y^2)(1-b^2) > 0$$

for $\eta_\alpha < y \leq 1$, it follows that

$$(ya+c) > 0, \quad (yb+d) < 0 \quad \text{for } \eta_\alpha \leq y \leq 1.$$

We then define the quantities $\kappa_1(y)$ and $\kappa_2(y)$ by

$$\cosh \kappa_1(y) = \frac{ya+c}{\sqrt{1-y^2}\sqrt{1-a^2}}, \quad \cosh \kappa_2(y) = -\frac{yb+d}{\sqrt{1-y^2}\sqrt{1-b^2}} \quad (4-A11)$$

and thus

$$g_\pm(y) - \xi_{1+} = \sqrt{(1-a^2)(1-b^2)} [-1 + \cosh(\kappa_1(y) \mp \kappa_2(y))] \quad (4-A12)$$

APPENDIX 4B

In this appendix we give the mathematical details of the limiting procedures carried out in section 4-7. The functions $J(x, y)$ and $K(x, y)$ appearing in eq. (4-102) are given by

$$J(x, y) = 2EF \lim_{\delta \downarrow 0, \epsilon \downarrow 0} \cdot$$

$$\begin{aligned} & \cdot \left\{ \int_{h(\delta, \epsilon, y)}^{\infty} \frac{d\xi}{\xi-x} \frac{1}{\mu_+(\xi, \delta, y)} \frac{\partial \mu_+(\xi, \delta, y)}{\partial y} \Lambda(\xi, \delta, \mu_+(\xi, \delta, y), y) \right. \\ & + \int_{g_+(y)}^{h(\delta, \epsilon, y)} \frac{d\xi}{\xi-x} \left[\frac{1}{\mu_+(\xi, \delta, y)} \frac{\partial \mu_+(\xi, \delta, y)}{\partial y} \Lambda(\xi, \delta, \mu_+(\xi, \delta, y), y) \right. \\ & \left. \left. - \frac{1}{\mu_-(\xi, \delta, y)} \frac{\partial \mu_-(\xi, \delta, y)}{\partial y} \left(\Lambda(\xi, \delta, \mu_-(\xi, \delta, y), y) - \Delta_1(\xi, \mu_-(\xi, \delta, y)) \right) \right] \right\}, \end{aligned} \quad (4-B1)$$

$$K(x, y) = 2EF \lim_{\delta \downarrow 0, \epsilon \downarrow 0} \cdot$$

$$\begin{aligned} & \cdot \left\{ \frac{1}{h(\delta, \epsilon, y) - x} \frac{\partial h(\delta, \epsilon, y)}{\partial y} \left[- \int_{\epsilon}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \Lambda(\xi, \delta, \mu, y) \right. \right. \\ & + \left. \int_{\mu_-(\xi, \delta, y)}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \Lambda(\xi, \delta, \mu, y) + \int_{\epsilon}^{\mu_-(\xi, \delta, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) \right] \Bigg|_{\xi=h(\delta, \epsilon, y)} \\ & + \frac{1}{g_+(y) - x} \frac{dg_+(y)}{dy} \left[- \int_{\epsilon}^{\mu_-(\xi, \delta, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) - \int_{\mu_-(\xi, \delta, y)}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \Lambda(\xi, \delta, \mu, y) \right. \\ & + \left. \int_{\epsilon}^{\mu_{\alpha}(\xi, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) + \int_{\delta}^{\lambda_{\alpha}(\xi, y)} \frac{d\lambda}{\lambda} \Delta_2(\xi, y) \right] \Bigg|_{\xi=g_+(y)} \\ & - \frac{1}{H_1(\epsilon, y) - x} \frac{\partial H_1(\epsilon, y)}{\partial y} \left[\int_{\epsilon}^{\mu_{\alpha}(\xi, y)} \frac{d\mu}{\mu} \Delta_1(\xi, \mu) \right] \Bigg|_{\xi=H_1(\epsilon, y)} \\ & - \frac{1}{H_2(\delta, y) - x} \frac{\partial H_2(\delta, y)}{\partial y} \left[\int_{\delta}^{\lambda_{\alpha}(\xi, y)} \frac{d\lambda}{\lambda} \Delta_2(\xi, \lambda) \right] \Bigg|_{\xi=H_2(\delta, y)} \Bigg\} . \quad (4-B2) \end{aligned}$$

We show first that $J(x, y)$ vanishes. From eq. (4-66) it follows that

$$\frac{1}{\mu_{\pm}(\xi, \delta, y)} \frac{\partial \mu_{\pm}(\xi, \delta, y)}{\partial y} \text{ are finite for } \xi \geq g_+(y) > H_2(\delta, y) .$$

From section 4-5 we see that, for sufficiently small δ ,

$$\lambda_-(\xi, \mu_-(\xi, \delta, y), y) = \delta \text{ for } g_+(y) \leq \xi \leq h(\delta, \epsilon, y) ,$$

$$\lambda_+(\xi, \mu_+(\xi, \delta, y), y) = \delta \text{ for } g_+(y) \leq \xi < +\infty .$$

Also, from the definition of $\mu_{\pm}(\xi, \lambda, y)$ we have

$$U(\xi, \delta, \mu_{\pm}(\xi, \delta, y), y) = 0 .$$

Thus, using eqs. (4-77) and (4-78), it follows that

$$\Lambda(\xi, \delta, \mu_+(\xi, \delta, y), y) = 0 \quad \text{for } \xi \geq g_+(y), \quad c_1(\xi, \mu_+(\xi, \delta, y)) > 0.$$

When $c_1(\xi, \mu) < 0$,

$$\Lambda(\xi, \delta, \mu, y) =$$

$$\frac{1}{[-c_1(\xi, \mu)]^{\frac{1}{2}}} \left\{ \frac{\pi}{2} - \arcsin \frac{b_1(\xi, \mu, y) + \delta^{-1} c_1(\xi, \mu)}{[(b_1(\xi, \mu, y))^2 - c_1(\xi) c_1(\xi, \mu)]^{\frac{1}{2}}} \right\} \quad (4-B3)$$

and again $\Lambda(\xi, \delta, \mu_+(\xi, \delta, y), y) = 0$ for $\xi \geq g_+(y)$. Also, from

eqs. (4-76) and (4-101) we see that

$$\Lambda(\xi, \delta, \mu_-(\xi, \delta, y), y) = \Delta_1(\xi, \mu_-(\xi, \delta, y))$$

$$\text{for } g_+(y) \leq \xi \leq h(\delta, \epsilon, y)$$

and so $J(x, y) = 0$.

Next we show that $K(x, y)$ vanishes. From eq. (4-22) and the lemma of section 4-3 we see that $\partial h(\delta, \epsilon, y)/\partial y$ is finite for $y < -1$. Then, since $\mu_-(h(\delta, \epsilon, y), \delta, y) = \epsilon$ for sufficiently small δ, ϵ , and $\Delta_1(h(\delta, \epsilon, y), \epsilon)$ is finite, it follows that the first term in eq. (4-B2) is zero. From eq. (4-A1), $dg_+(y)/dy$ is finite for $y < -1$ and reference to fig. 4-2 shows that the term in square brackets multiplying $dg_+(y)/dy$ is zero, being the difference of two repeated integrals obtained from the same double integral. Thus the second term in eq. (4-B2) vanishes. From eq. (4-29) we see that $\partial H_1(\epsilon, y)/\partial y$ is finite and, since $\mu = \mu_\alpha(\xi, y)$, and $\xi = H_1(\mu, y)$ are inverse functions when $0 \leq \mu \leq \mu(y)$, $\xi_{1+} \leq \xi \leq g_+(y)$,

$$\mu_\alpha(H_1(\epsilon, y), y) = \epsilon.$$

Next we note from eq. (4-57) that

$$k(0, \mu, y) < H_1(\mu, y) \leq h(0, \mu, y)$$

for $0 \leq \mu \leq \mu_L$, the equality occurring if and only if $\mu = \mu_L$. Also from eqs. (4-21) and (4-49),

$$c_1(\xi, \mu) = 4E^2F^2(\xi - k(0, \mu, y))(\xi - h(0, \mu, y)). \quad (4-B4)$$

Hence

$$c_1(H_1(\epsilon, y), \epsilon) < 0$$

and from eq. (4-101), $\Delta_1(H_1(\epsilon, y), \epsilon)$ is finite. Thus the third term on the right hand side of eq. (4-B2) is zero. Similarly the fourth term is zero and thus $K(x, y) = 0$.

We now take the limits $\delta \downarrow 0$, $\epsilon \downarrow 0$ in eq. (4-102). In the third and fourth terms in square brackets we may directly take the limits $\epsilon \downarrow 0$ and $\delta \downarrow 0$ respectively, to give the third and fourth terms on the right side of eq. (4-103). Next we reverse the order of integration in the first two terms inside the square brackets in eq. (4-102) to give

$$\begin{aligned} & \int_{\epsilon}^{\mu_-(g_+(y), \delta, y)} \frac{d\mu}{\mu} \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{\xi - x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\ & + \int_{\mu_-(g_+(y), \delta, y)}^{\mu_+(g_+(y), \delta, y)} \frac{d\mu}{\mu} \int_{g_+(y)}^{\infty} \frac{d\xi}{\xi - x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\ & + \int_{\mu_+(g_+(y), \delta, y)}^{\infty} \frac{d\mu}{\mu} \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{\xi - x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \end{aligned}$$

(see fig. 4-5). The infinitesimal ϵ now appears only in the first term and the limit $\epsilon \downarrow 0$ may be taken without difficulty.

The region of integration in the (μ, ξ) plane divides into two parts, according to the sign of $c_1(\xi, \mu)$. From eqs. (4-57) and

(4-B4),

Fig. 4-5.

The curves $\xi = h(0, \mu, y)$ and $\xi = h(\delta, \mu, y)$, with δ small and positive, for $\mu \geq 0$.

$$c_1(\xi, \mu) \geq 0 \text{ if and only if } \xi \geq h(0, \mu, y).$$

the function $c_1(\xi, \mu)$ being defined on the region $\mu \geq 0$,

$\xi \geq \min\{h(\lambda, \mu, y) \mid \lambda \geq 0\}$. Thus we see from fig. 4-5 that the integral over the region in which $c_1(\xi, \mu) \leq 0$ is

$$\begin{aligned} & \int_0^{\mu_-(g_+(y), \delta, y)} \frac{d\mu}{\mu} \int_{h(\delta, \mu, y)}^{h(0, \mu, y)} \frac{d\xi}{\xi - x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\ & + \int_{\mu_-(g_+(y), \delta, y)}^{\mu_-(g_+(y), 0, y)} \frac{d\mu}{\mu} \int_{g_+(y)}^{h(0, \mu, y)} \frac{d\xi}{\xi - x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y). \end{aligned}$$

Now note that eq. (4-80) holds regardless of the sign of $c_1(\xi, \mu)$;

for $c_1(\xi, \mu) < 0$, differentiation of eq. (4-B3) also gives eq.

(4-80). The function $(b_1(\xi, \mu, y))^2 - a_1(\xi)c_1(\xi, \mu)$ cannot vanish

for $\xi > g_+(y)$, while for $\xi = g_+(y)$ it vanishes only for $\mu = \mu(y)$,

and $\mu(y) < \mu_-(g_+(y), \delta, y)$. Thus the function has a positive

minimum on the region of integration and the modulus of the integral under consideration may be majorised by

$$M'' \int_0^{\mu_-(g_+(y), 0, y)} d\mu \int_{h(\delta, \mu, y)}^{h(0, \mu, y)} \frac{d\xi}{(\xi - h(\delta, \mu, y))^{\frac{1}{2}}} \rightarrow 0 \text{ as } \delta \downarrow 0.$$

The integral over the region in which $c_1(\xi, \mu) \geq 0$ is

$$\begin{aligned}
& \int_0^{\mu_-(g_+(y), 0, y)} \frac{d\mu}{\mu} \int_{h(0, \mu, y)}^{\infty} \frac{d\xi}{\xi-x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\
& + \int_{\mu_-(g_+(y), 0, y)}^{\mu_+(g_+(y), \delta, y)} \frac{d\mu}{\mu} \int_{g_+(y)}^{\infty} \frac{d\xi}{\xi-x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\
& + \int_{\mu_+(g_+(y), \delta, y)}^{\infty} \frac{d\mu}{\mu} \int_{h(\delta, \mu, y)}^{\infty} \frac{d\xi}{\xi-x} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\
& = \int_1^{\infty} \frac{d\xi}{\xi-x} \int_0^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) \\
& + \int_{g_+(y)}^1 \frac{d\xi}{\xi-x} \int_{\mu_-(\xi, 0, y)}^{\mu_+(\xi, \delta, y)} \frac{d\mu}{\mu} \frac{\partial}{\partial y} \Lambda(\xi, \delta, \mu, y) .
\end{aligned}$$

When the limit $\delta \downarrow 0$ is taken in the above equation, we find that the first term on the right hand side gives, when it is multiplied by $2EF$, the first term on the right hand side of eq. (4-103). Similarly, the second term on the right hand side of the above equation gives, when multiplied by $2EF$, the second term on the right hand side of eq. (4-103). The method of taking the limit $\delta \downarrow 0$ is similar to that used in sect. 4-6, with the terms involving $\delta a_1(\xi)$ and $b_1(\xi, \mu, y)$ in the numerator of eq. (4-80) again being treated separately. In this case, we note that $a_1(\xi)$ and $b_1(\xi, \mu, y)$ are both linear in ξ . Further, from eqs. (4-53), (4-67) and (4-73) we see that

$$\{b_1(\xi, \mu, y)\}^2 - a_1(\xi)a_1(\xi, \mu) = 4G^2v(\xi)A_1(\xi, y)(\mu-\mu_\alpha(\xi, y))(\mu-\mu_\beta(\xi, y))$$

where $A_1(\xi, y)$ is linear in ξ and positive for $\xi \geq g_+(y)$ and

$(\mu-\mu_\alpha(\xi, y))(\mu-\mu_\beta(\xi, y))$ cannot vanish in the region of integration.

We also note that $U(\xi, \delta, \mu, y)$ can be written as in eq. (4-62)

(with $\lambda \rightarrow \delta$) where $a_2(\xi)$ is linear in ξ and negative for $\xi \geq g_+(y)$. By using these results, the limit $\delta \downarrow 0$ can then be taken, when the theorem given, for example, in sect. 225 of Hobson (1957) is used.

5. THE BOX DIAGRAM AMPLITUDE: ANALYTIC CONTINUATION

5.1. Introduction

In this chapter, we start with the double spectral representation in eq.(4-13) or (4-115) for the case $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$, $x < -1$, $y < -1$ and continue in the external masses and in x and y . In this way, we obtain the spectral representation of $I(x, y)$ for all masses satisfying the stability conditions $A < (E+G)$, $B < (F+G)$, $C < (E+H)$, $D < (F+H)$ and for wider ranges of x and y . In particular, the spectral representation of $I(\sigma, y+i\cdot 0)$ is obtained for all σ in the upper half complex plane and for almost all $-\infty < y < +\infty$. This method of analytic continuation was originally proposed by Mandelstam (1960) and subsequently used by Gribov, Danilov and Dyatlov (1961, 1962) (referred to as GDD) in their investigation of the general mass case box diagram amplitude. We find, however, that some of the expressions for $I(x, y)$, that we obtain by using this method, disagree with those given in GDD. Further, many of the details of the method, in particular details of the movement of singularities at the critical points and the precautions that need to be taken to ensure that the path of integration is not pinched between the singularities, are not given in GDD.

In sects. 5-2, 5-3 and 5-4, the spectral representations of $I(x, y)$ for cases I, II, and III (ii) and (iii), detailed at the end of sect. 4-5, are obtained by starting with the spectral representation in eq. (4-13) or (4-115) for the case when $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$, $x < -1$, $y < -1$ and continuing in the external masses squared. Then in sect. 5-5, we start with the spectral representation for case I, given in eq. (4-115) for $x < -1$, $y < -1$, and continue in x and y to obtain the spectral representation for wider ranges of x and y , including both real and complex x and

y . In particular, we obtain $I(\sigma, y+i\cdot 0)$ for all σ in the upper half plane and for almost all $-\infty < y < +\infty$. Similarly, in sect. 5-6 we start with the double spectral representation given in eq. (4-116), for $x < -1$, $y < -1$, and continue in x and y to obtain the spectral representation for wider ranges of x and y . The subcase of case II, for which eq. (4-116) holds when $x < -1$, $y < -1$, we call case II D. The other subcase of case II, for which the double spectral representation is not valid even when $x < -1$, $y < -1$, we call case II S.

To obtain the spectral representation for cases II S and III, for wider ranges of x and y than those given in sect. 4-8, the properties of the curve Γ , for which $F(\xi, y) = 0$, are needed. Thus, in sect. 5-7 we study the five branches of the curve Γ in the case $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$. In sect. 5-8, the spectral representation for case II S when $x < -1$, $y < -1$, is written as the sum of a double spectral representation over the region R_2 (eq. (4-112)) and a single dispersion relation in x . The double integral part can be continued in x and y in exactly the same way as the spectral representation of $I(x, y)$ in case II D. The continuation in x and y of the single dispersion relation is then carried out to obtain a single spectral representation for all σ in the upper half plane and for almost all $-\infty < y < +\infty$. We, in fact, find a single formula which summarizes all the possible subcases. The above separation into a double and single integral part when $x < -1$, $y < -1$ is different from the one used in GDD. However, we find it more convenient, since it allows us to treat the single integral parts in cases II S and III in a unified way. For case III (ii) (or (iii)), $I(x, y)$ can, for $x < -1$, $y < -1$, be written as the sum of a

double spectral representation over the region R_1 (eq. (4-111)) and a single dispersion relation in x . The double integral part can then be continued in x and y in exactly the same way as the spectral representation of $I(x, y)$ in case I. In sect. 5-9, the continuation of the single integral part is then carried out to give a single spectral representation for all σ in the upper half plane and for almost all $-\infty < y < +\infty$. It is found that the possible subcases can again be summarized by the same formula as that given in sect. 5-8. In particular, the continuation in x and y gives the spectral representation for case III (i) (eq. (4-118)) when the inequality in eq. (4-119) holds and when $x < \min\{\xi_{1+}, \xi_{2+}\}$.

Finally, in sect. 5-10, we compare our results with those given in GDD.

5-2. Continuation in external masses: Case I

In this section and in the following two sections we obtain a spectral representation for cases I, II and III (ii) and III (iii) detailed at the end of sect. 4-5, by starting with the spectral representation in eq. (4-13) for the case when $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$, $y < -1$, $x < -1$ and continuing in the external masses squared. We noted after eq. (4-115) that the double spectral representation given there or in eq. (4-13) was valid for wider ranges of x and y than $x \leq -(2EF)^{-1}(E^2+F^2)$, $y < -1$. In this section and in sects. 5-3 and 5-4, it will, however, be convenient to restrict x and y to the intervals $x < -1$, $y < -1$. The internal masses E, F, G and H are arbitrary positive numbers which remain constant throughout the continuation procedure. We see from eq. (3-46) of sect. 3-4 that when $(A+B) \leq (E+F)$,

$$F(E^2+G^2-A^2) + E(F^2+G^2-B^2) > 0 ,$$

that is,

$$a + b > 0 \quad (5-1)$$

and when, $(C+D) \leq (E+F)$,

$$c + d > 0 . \quad (5-2)$$

Consequently, as noted after eq. (4-115), the case from which we start the continuation is a subcase of case I. The stability conditions confine A^2 , B^2 , C^2 and D^2 to the intervals

$$0 < A^2 < (E+G)^2 , \quad 0 < B^2 < (F+G)^2 ,$$

$$0 < C^2 < (E+H)^2 , \quad 0 < D^2 < (F+H)^2 ,$$

or

$$\begin{aligned} -1 < a < (2EG)^{-1}(E^2+G^2) , \quad -1 < b < (2FG)^{-1}(F^2+G^2) , \\ -1 < c < (2EH)^{-1}(E^2+H^2) , \quad -1 < d < (2FH)^{-1}(F^2+H^2) . \end{aligned} \quad (5-3)$$

The method of continuation that we use is very similar to the one used in sects. 3-7 and 3-8. From eq. (4-10) we see that A^2 , B^2 , C^2 and D^2 can be expressed in terms of the quantities a , b , c and d respectively, and hence $I(\sigma, \tau)$, given in eq. (4-14), is a function of a , b , c and d . We begin by replacing a , b , c and d by the complex variables z_1 , z_2 , z_3 and z_4 respectively and define formally a function

$$I(\sigma, \tau, z_1, z_2, z_3, z_4)$$

$$\begin{aligned} &= 2EFGH \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_0^{1-\alpha-\beta} d\gamma [E^2\alpha^2 + F^2\beta^2 + G^2(1-\alpha-\beta-\gamma)^2 \\ &\quad + H^2\gamma^2 + 2EGz_1(1-\alpha-\beta-\gamma)\alpha + 2FGz_2(1-\alpha-\beta-\gamma)\beta \\ &\quad + 2EH z_3 \alpha \gamma + 2FH z_4 \beta \gamma - 2EF\sigma\alpha\beta - 2GH\tau(1-\alpha-\beta-\gamma)\gamma]^{-2} . \end{aligned} \quad (5-4)$$

From eq. (5-4) we see that $I(\sigma, \tau, z_1, z_2, z_3, z_4)$ is an analytic function of $(\sigma, \tau, z_1, z_2, z_3, z_4)$ at least in the Cartesian product

$$S_\sigma \times S_\tau \times S_1 \times S_2 \times S_3 \times S_4$$

of the open sets

$$S_\sigma : -\infty < \operatorname{Re}\sigma = x < +\infty, \quad \operatorname{Im}\sigma > 0,$$

$$S_\tau : -\infty < \operatorname{Re}\tau = y < +\infty, \quad \operatorname{Im}\tau > 0,$$

$$S_1 : -1 < \operatorname{Re}z_1 = a < (2EG)^{-1}(E^2 + G^2), \quad \operatorname{Im}z_1 < 0,$$

$$S_2 : -1 < \operatorname{Re}z_2 = b < (2FG)^{-1}(F^2 + G^2), \quad \operatorname{Im}z_2 < 0,$$

$$S_3 : -1 < \operatorname{Re}z_3 = c < (2EH)^{-1}(E^2 + H^2), \quad \operatorname{Im}z_3 < 0,$$

$$S_4 : -1 < \operatorname{Re}z_4 = d < (2FH)^{-1}(F^2 + H^2), \quad \operatorname{Im}z_4 < 0.$$

As was shown in sects. 4-3 and 4-8, $I(x, y, a, b, c, d)$ is defined when (5-3) holds and $x < -1$, $y < -1$, and its value can be obtained from $I(\sigma, \tau, z_1, z_2, z_3, z_4)$ by letting $\operatorname{Im}z_i \uparrow 0$

($i = 1, \dots, 4$), $\operatorname{Im}\sigma \downarrow 0$, $\operatorname{Im}\tau \downarrow 0$ with a, b, c, d, x and y held fixed. In fact, when a, b, c and d are such that eqs. (5-1)

and (5-2) hold, the continuation can be carried out with real

$z_1 (= a)$, $z_2 (= b)$, $z_3 (= c)$ and $z_4 (= d)$. It is only when

$a + b \leq 0$ (resp. $c + d \leq 0$) that we need to take z_1 and z_2

(resp. z_3 and z_4) into the lower half complex planes and then back

to the real axes. As mentioned earlier, in this section and in sects.

5-3 and 5-4, we take σ and τ to be real and further restrict

$x = \operatorname{Re}\sigma$, $y = \operatorname{Re}\tau$ to the intervals $x < -1$, $y < -1$.

The integration over η in eq. (4-13) (or eq. (4-94)) can be carried out and from eqs. (4-91), (4-92) and (4-93) we see that

$$I(x, y) = \int_1^{\infty} \frac{d\xi}{\xi-x} X(\xi, y) \quad (5-5)$$

where, if $y \leq f_-(\xi)$, that is, $F(\xi, y) \geq 0$ (see the discussion after eq. (4-92)),

$$X(\xi, y) = \frac{\frac{1}{2}}{\sqrt{F(\xi, y)}} \ln E(\xi, y) \quad (5-6)$$

$$E(\xi, y) = \frac{G(\xi, y) + H(\xi, y)}{G(\xi, y) - H(\xi, y)}, \quad (5-7)$$

$$G(\xi, y) = -(\xi^2 - 1)y + (\xi - 1)(ad + bc) + (a + b)(c + d), \quad (5-8)$$

$$H(\xi, y) = \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)} \quad (5-9)$$

and $F(\xi, y)$ is given in eq. (4-12). The function $G(\xi, y)$ is also given in eq. (4-A5). If, however, $F(\xi, y) < 0$,

$$X(\xi, y) = \frac{1}{\sqrt{-F(\xi, y)}} \left[\frac{\pi}{2} - \arctan \frac{G(\xi, y)}{K(\xi, y)} \right] \quad (5-10)$$

where

$$K(\xi, y) = \sqrt{\xi^2 - 1} \sqrt{-F(\xi, y)}. \quad (5-11)$$

At this stage, we examine some of the properties of the functions $G(\xi, y)$ and $F(\xi, y)$ and then establish the continuity of $X(\xi, y)$. We see from eqs. (4-12) and (5-8) that $F(\xi, y)$ and $G(\xi, y)$ are quadratic functions of ξ which are positive for large ξ when $y < -1$. The solutions of $F(\xi, y) = 0$ are $\xi = g_{\pm}(y)$ where $g_{\pm}(y)$ are given in eq. (4-36), and we denote the solutions of $G(\xi, y) = 0$ by $\gamma_{\pm}(y)$ where $\gamma_{\pm}(y)$ are either complex conjugates or are real and satisfy $\gamma_+(y) \geq \gamma_-(y)$. From eqs. (5-9) and (4-A4) it follows that

$$\begin{aligned} [G(\xi, y)]^2 - [H(\xi, y)]^2 &= [G(\xi, y)]^2 - (\xi^2 - 1)F(\xi, y) \\ &= (\xi - \xi_{1+})(\xi - \xi_{1-})(\xi - \xi_{2+})(\xi - \xi_{2-}) \end{aligned} \quad (5-12)$$

where the quantities $\xi_{1\pm}$ are given in eqs. (3-55) and (4-32) and

$\xi_{2\pm}$ in eq. (4-42). As discussed in sects. 3-5, 3-7 and 4-8, when $(1-a)(1-b) < 0$, $\xi_{1\pm}$ are complex conjugates and when $(1-a)(1-b) \geq 0$, $\xi_{1\pm}$ satisfy the inequality in (3-57). A similar statement holds for $\xi_{2\pm}$ (with $a \rightarrow c$, $b \rightarrow d$). Thus, $\xi_{1+} = 1$ if and only if $a + b = 0$, $\xi_{2+} = 1$ if and only if $c + d = 0$ and the right hand side of eq. (5-12) is non-negative for all $\xi \geq 1$.

We shall need to know the sign of $G(g_{\pm}(y), y)$ when $g_{\pm}(y) \geq 1$ to establish whether $X(\xi, y)$ is continuous at $\xi = g_{\pm}(y)$ for cases II and III as well as for case I. We now list the various cases. From eq. (5-8) we see that

$$G(1, y) = (a+b)(c+d)$$

so that when $(a+b)(c+d) > 0$, then $G(\xi, y) > 0$ for all $\xi \geq 1$ if $\gamma_{\pm}(y)$ are complex conjugates or if $\gamma_{-}(y) \leq \gamma_{+}(y) < 1$. If $1 < \gamma_{-}(y) \leq \gamma_{+}(y)$, then it follows from eq. (5-12) that

$$g_{-}(y) < \gamma_{-}(y) \leq \gamma_{+}(y) < g_{+}(y)$$

and $G(g_{\pm}(y), y) > 0$. If $(a+b)(c+d) < 0$, then $\gamma_{-}(y) < 1 < \gamma_{+}(y)$ and using eq. (5-12),

$$g_{-}(y) < \gamma_{+}(y) < g_{+}(y)$$

and $G(g_{+}(y), y) > 0$. Also, from eq. (4-A10) it follows that

$g_{-}(y) < 1$. If $(a+b) = 0$ or $(c+d) = 0$ then from eqs. (5-12) and

(4-A10) we see that $G(g_{+}(y), y) > 0$ and $g_{-}(y) < 1 < g_{+}(y)$, provided

that $(a+b)$ and $(c+d)$ are not simultaneously zero. Except for the degenerate cases, where some of the masses are the same, the continuation in the external masses can be carried out in such a way that $(a+b)$

and $(c+d)$ are not simultaneously zero. (We do not consider the degenerate cases here.) We have now shown that $G(g_+(y), y) > 0$ (resp. $G(g_-(y), y) > 0$) when $g_+(y) \geq 1$ (resp. $g_-(y) \geq 1$). The continuity of $X(\xi, y)$ at $\xi = g_{\pm}(y)$ when $g_{\pm}(y) \geq 1$, can now be established by using an argument similar to the one that led to eqs. (3-81) and (3-82). We find that

$$X(g_{\pm}(y), y) = \frac{2\sqrt{g_{\pm}(y)-1}\sqrt{g_{\pm}(y)+1}}{G(g_{\pm}(y), y)} \quad (5-13)$$

when $g_{\pm}(y) \geq 1$.

Interpreting the logarithm in eq. (5-6) and the square roots in eqs. (5-6) and (5-9) in a more general sense we find that

$$\begin{aligned} \frac{\frac{1}{2}}{[F(\xi, y)]^{\frac{1}{2}}} \log \left[\frac{G(\xi, y) + H(\xi, y)}{G(\xi, y) - H(\xi, y)} \right] &= \frac{\arg[G(\xi, y) + iK(\xi, y)]}{\sqrt{-F(\xi, y)}} \\ &= \frac{1}{\sqrt{-F(\xi, y)}} \left[\frac{\pi}{2} - \arctan \frac{G(\xi, y)}{K(\xi, y)} \right] \end{aligned} \quad (5-14)$$

when $F(\xi, y) < 0$. With this interpretation of the logarithm and square roots, we see that the results in eqs. (5-10) and (5-6) can be summarized by eq. (5-6) when $x < -1$, $y < -1$. In particular, $I(x, y)$ is given by eqs. (5-5) and (5-6) for all $F(\xi, y)$ when $(A+B) \leq (E+F)$, $(C+D) \leq (E+F)$, $x < -1$, $y < -1$.

To do the continuation in the external masses, we replace a, b, c and d by z_1, z_2, z_3 and z_4 respectively in eqs. (5-5) to (5-12) and in eq. (4-12), the logarithm in eq. (5-6) and the square roots in eqs. (5-6) and (5-9) being now interpreted in a more general sense. From eqs. (5-7) and (5-12) we see that $X(\xi, y)$ is singular when the right hand side of eq. (5-12) vanishes, that is, when $\xi = \xi_{1\pm}$,

$\xi = \xi_{2\pm}$ where

$$\xi_{1\pm} = -z_1 z_2 \pm \left[\left(1 - z_1^2 \right) \left(1 - z_2^2 \right) \right]^{\frac{1}{2}} \quad (5-15)$$

$$\xi_{2\pm} = -z_3 z_4 \pm \left[\left(1 - z_3^2 \right) \left(1 - z_4^2 \right) \right]^{\frac{1}{2}} \quad (5-16)$$

Using the argument following eq. (5-12) we see that provided eq. (5-1) continues to hold, we can carry out the continuation in real $z_1 (= a)$, real $z_2 (= b)$ from the case $(A+B) \leq (E+F)$ and the singularities $\xi_{1\pm}$ will not appear on the path of integration in eq. (5-5). Similarly, provided eq. (5-2) continues to hold, we can change c and d from the case when $(C+D) \leq (E+F)$ and the singularities $\xi_{2\pm}$ will not appear on the path of integration $[1, +\infty)$. Further, the arguments leading to eq. (5-13) show that $X(\xi, y)$ is not singular at $\xi = g_+(y)$ (resp. $\xi = g_-(y)$) when $g_+(y) \geq 1$ (resp. $g_-(y) \geq 1$). Thus the right hand side of eq. (5-5) continues to define $I(x, y)$, with the logarithm in eq. (5-6) and the square roots in eqs. (5-6) and (5-9) interpreted more generally as remarked earlier.

We now check that this prescription reproduces the expressions for $I(x, y)$ given in sects. 4-6 and 4-8 for case I where $(a+b) > 0$, $(c+d) > 0$. (We do not consider the degenerate cases for which at least one of the quantities $(a+b)$, $(c+d)$ is zero; these can be obtained by a modification of the analysis which will be used to obtain cases II and III (ii) and III (iii). See, for example, the method used to obtain a spectral representation for the transition case in sect. 3-8.) When $F(\xi, y) \geq 0$ and $\xi > 1$, we see from eq. (5-12) that $|G(\xi, y)| > H(\xi, y)$, so that $E(\xi, y) > 0$ and the logarithm is the ordinary logarithm. Thus $X(\xi, y)$ is again given by eqs. (5-6) and (5-7) and $I(x, y)$ is given by eq. (5-5) and hence by

eqs. (4-94) and (4-115). If $F(\xi, y) < 0$ and $\xi > 1$ we use eq. (5-14) to write $X(\xi, y)$ as in eq. (5-10), and again $I(x, y)$ is given by eqs. (4-94) and (4-115).

5-3. Continuation in external masses: Case II

To carry out the continuation of $I(x, y)$ from case I to case II (i) where $(a+b) > 0$, $(c+d) < 0$, we use a method similar to the one used in sect. 3-8. (We do not consider the degenerate case for which $a + b = 0$, $c + d < 0$.) Then, as discussed in sect. 3-8, we need to carry out the continuation using complex z_3 and z_4 which replace c and d . We fix $z_1 (= a)$ and $z_2 (= b)$ such that $a + b > 0$ and $\xi_{1\pm}$ are given in eq. (4-32).

We choose to start the continuation from a subcase of case I for which $|c| < 1$, $|d| < 1$, $\xi_{1\pm}$ are real and

$$\xi_{1+} < \xi_{2+},$$

and for which

$$g_-(y) \leq \xi_{2-} < \xi_{2+} < 1 < g_+(y). \quad (5-17)$$

The inequalities in eq. (5-17) can be satisfied for case I by taking $(c+d)$ sufficiently small to start with. This also ensures that

$\xi_{1+} < \xi_{2+}$. For case II (i), eq. (5-17) always holds, as can be seen

from eqs. (4-A9) and (4-A10). Initially we shall continue to a

subcase of case II (i) for which $\xi_{1+} < \xi_{2+}$. The functions defined

in eqs. (5-5) to (5-12) and in eqs. (4-12) and (4-36) are now considered

to depend on z_3 and z_4 which replace c and d . Then, for

sufficiently small $\text{Im}z_3, \text{Im}z_4$,

$$\text{Re}g_-(y) < \text{Re}\xi_{2+} \quad (5-18)$$

throughout the continuation procedure. Eq. (5-18) ensures that $g_-(y)$ never lies on the (possibly deformed) path of integration. Further, we deform the path of integration in such a way that $g_+(y)$ only lies on the path of integration when z_3 and z_4 are real (see the discussion given in sect. 3-8).

To study the behaviour of $\xi_{2\pm}$ as we take z_3, z_4 into the lower half planes and then back to the real axes, we introduce the quantities

$$\gamma = \arccos c, \quad \delta = \arccos d$$

which are also given in eq. (4-113) and define the quantities

$$\theta_3 = \arccos z_3, \quad \theta_4 = \arccos z_4. \quad (5-19)$$

Then

$$\xi_{2\pm} = -\cos(\theta_3 \pm \theta_4). \quad (5-20)$$

The steps (a), (b), (c) and (d) in the continuation are then as given in sect. 3-8 with the replacements $z_1 \rightarrow z_3$, $z_2 \rightarrow z_4$, $\alpha \rightarrow \gamma$, $\beta \rightarrow \delta$, $a \rightarrow c$, $b \rightarrow d$ where these quantities appear explicitly in the steps. The expressions for $\operatorname{Re}\xi_{2+}$, $\operatorname{Im}\xi_{2+}$ can then be obtained from eq. (3-101) by making the above replacements and the path taken by ξ_{2+} as steps (a)-(d) are taken is as shown in fig. 3-3.

An argument similar to the one which led to eq. (3-102) (with the above replacements and $g_{\pm} \rightarrow g_{\pm}(y)$) shows that the path of integration used to define $I(x, y)$ in eq. (5-5) must be deformed; the modified path C is shown in fig. 3-4. The only new point to note is that the restriction $\xi_{1+} < \xi_{2+}$, when z_3 and z_4 are real, ensures that the path of integration is not pinched between these two singularities.

The branch cuts from the singularities ξ_{1+} , ξ_{2-} are taken to run from the respective points parallel to the real axis to $-\infty$, and the branch cut associated with ξ_{2+} is as described in sect. 3-8 for ξ_{1+} (with $\xi_{1+} \rightarrow \xi_{2+}$). We note from eq. (4-A3) that

$$F(\xi, y) = (y^2 - 1) (\xi - g_+(y)) (\xi - g_-(y))$$

and we take the branch cut of $[(\xi - g_+(y)) (\xi - g_-(y))]^{\frac{1}{2}}$ between $g_-(y)$ and $g_+(y)$ in such a way that it does not pass through C (see fig. 3-4). Similarly the branch cut of $[\xi^2 - 1]^{\frac{1}{2}}$ in eq. (5-9) is taken between -1 and 1 in such a way that it does not pass through C (except at the point $(1, 0)$).

At step (d), ξ_{2+} approaches the real axis from below, and again we may deform C to the segment of the real axis from ξ_{2+} to $+\infty$, with the interval $[\xi_{2+}, 1]$ being traversed *in both directions*. Thus

$$I(x, y) = \int_1^{\infty} \frac{d\xi X(\xi, y)}{\xi - x} + \int_{\xi_{2+}}^1 \frac{d\xi \text{disc.}[X(\xi, y)]}{\xi - x} \quad (5-21)$$

where

$$\begin{aligned} \text{disc.}[X(\xi, y)] = & \left[(\text{value of } X(\xi, y) \text{ below cut from } \xi_{2+} \text{ to } 1) \right. \\ & \left. - (\text{value of } X(\xi, y) \text{ above cut from } \xi_{2+} \text{ to } 1) \right], \end{aligned}$$

and it remains to evaluate $X(\xi, y)$ and $\text{disc.}[X(\xi, y)]$. When $\xi \geq 1$, the arguments of sect. 5-2 apply to the evaluation of $X(\xi, y)$, and hence the contribution to $I(x, y)$ from $\xi \geq 1$ is given by the right hand side of eq. (4-94).

Next we evaluate $\text{disc.}[X(\xi, y)]$ for $\xi_{2+} < \xi < 1$. An adaptation of the argument that led to eq. (3-104) shows that with

$$g_-(y) < \xi_{2+} < \xi < 1 < g_+(y) ,$$

$$[F(\xi, y)]^{\frac{1}{2}} = -i \sqrt{-F(\xi, y)} \quad (5-22)$$

and

$$H(\xi, y) = -M(\xi, y)$$

where

$$M(\xi, y) = \sqrt{y^2-1} \sqrt{g_+(y)-\xi} \sqrt{\xi-g_-(y)} \sqrt{1-\xi} \sqrt{\xi+1} > 0 . \quad (5-23)$$

From eq. (5-12) it follows that

$$[G(\xi_{2+}, y)]^2 - [M(\xi_{2+}, y)]^2 = 0$$

and using the fact that for this particular subcase of case II (i),

$\xi_{1+} < \xi_{2+}$, we have

$$[G(\xi, y)]^2 \geq [M(\xi, y)]^2 > 0 \quad \text{for} \quad \xi_{2+} \leq \xi < 1 ,$$

so that $G(\xi, y)$ cannot vanish in this interval. Since from eq. (5-8), $G(1, y) = (a+b)(c+d) < 0$ it follows that

$$G(\xi_{2+}, y) = (c\sqrt{1-d^2} + d\sqrt{1-c^2})h_2(y) < 0 , \quad (5-24)$$

and so $G(\xi_{2+}, y) + M(\xi_{2+}, y) = 0$. Here $h_2(y)$ is given in eq.

(4-47). Thus $E(\xi, y)$ (eq. (5-7)) may be put into the form

$$E(\xi, y) = \frac{[G(\xi, y) - M(\xi, y)]^2}{(\xi - \xi_{1+})(\xi - \xi_{1-})(\xi - \xi_{2+})(\xi - \xi_{2-})} \quad (5-25)$$

and an adaptation of the argument that led to eq. (3-107) shows that

$$\text{disc.}[X(\xi, y)] = \frac{\pi}{\sqrt{-F(\xi, y)}} = Y(\xi, y) . \quad (5-26)$$

This result does not depend on which square root of (y^2-1) is chosen.

Finally,

$$\begin{aligned}
 I(x, y) &= \int_1^{\infty} \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_{2+}}^1 \frac{d\xi}{\xi-x} \frac{\pi}{\sqrt{-F(\xi, y)}} \\
 &= \int_1^{\infty} \frac{d\xi}{\xi-x} X(\xi, y) + \int_{\xi_{2+}}^1 \frac{d\xi}{\xi-x} Y(\xi, y). \quad (5-27)
 \end{aligned}$$

Eq. (5-27) has been shown to be valid when $\xi_{1+} < \xi_{2+}$. To obtain the rest of case II (i), we note that $Y(\xi, y)$ is singular at $\xi = g_{\pm}(y)$ but not at $\xi = \xi_{1\pm}$, $\xi = \xi_{2\pm}$. Further, eq. (5-17) holds whenever $(a+b) > 0$, $(c+d) < 0$ and $y < -1$ and so we can continue in real $z_1 (= a)$, $z_2 (= b)$, and we find that eq. (5-27) is valid whenever $(a+b) > 0$, $(c+d) < 0$, $y < -1$, $x < -1$. We see that eq. (5-27) is just eq. (4-99) so that for case II (i), $I(x, y)$ is again given by eqs. (4-116) and (4-117). For case II (ii), the results of case II (i) again apply with $a \rightarrow c$, $b \rightarrow d$, $\xi_{1+} \leftrightarrow \xi_{2+}$.

5-4. Continuation in external masses: Case III (ii) and (iii)

We begin this section by summarizing the properties of case III (ii) defined by $(a+b) < 0$, $(c+d) < 0$, $h_1(y) \leq 0$, $h_2(y) > 0$ when $y < -1$. The quantities $h_1(y)$ and $h_2(y)$ are given in eqs. (4-38) and (4-47). We showed (eq. (4-107)) that for case III (ii), $h_2(-1) > 0$ and thus

$$f_{\pm}(\xi_{1+}) \leq y < -1 \leq f_{\pm}(\xi_{2+}) \quad (5-28)$$

where $f_{\pm}(\xi_{1+})$ is given in eq. (4-109) and $f_{\pm}(\xi_{2+})$ in eq. (4-110).

We also showed at the end of sect. 4-4 that for case III (ii)

$$\xi_{2+} < \xi_{1+}. \quad (5-29)$$

Eqs. (5-28) and (5-29) can also be established directly; in sect. 5-7

we establish these equations when we examine the five branches of the set of points $\{(\xi, y) \mid F(\xi, y) = 0\}$ in the case $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$.

The transition case between case II (i) where $(a+b) > 0$, $(c+d) < 0$ and case III where $(a+b) < 0$, $(c+d) < 0$ is defined by $(a+b) = 0$, $(c+d) < 0$. For this case we see from eq. (4-38) that

$$h_1(y) = (c+d)\sqrt{1-a^2} < 0.$$

Thus, *whatever* value of y we choose to the left of -1 , as a, b are changed from $(a+b) = 0+$ to $(a+b) = 0$ to $(a+b) = 0-$, we in fact change from case II (i) through the transition case to case III (ii). (It is then possible to obtain case III (i) by continuing in a, b, c and d , but in fact, we shall obtain case III (i) by continuing in y .)

The continuation from case II (i) to case III (ii) is carried out using complex z_1 and z_2 while z_3 and z_4 are real and fixed such that $z_3 + z_4 = c + d < 0$. We choose to start the continuation from a subcase of case II (i) for which eq. (5-29) holds. Initially we continue to a subcase of case III (ii) for which eq. (5-17) holds. Then for sufficiently small $\text{Im}z_1, \text{Im}z_2$,

$$\text{Reg}_-(y) < \xi_{2+} < 1 < \text{Reg}_+(y) \quad (5-30)$$

throughout the continuation procedure. These restrictions ensure that the singularities $g_{\pm}(y)$ do not move onto the segment $[\xi_{2+}, 1]$ of the real axis and hence $Y(\xi, y)$ is non singular for $\xi \in [\xi_{2+}, 1]$.

The steps (a), (b), (c) and (d) in the continuation are then as given in sect. 3-8 and the path of integration from 1 to $+\infty$ along the real ξ axis must be deformed into the path C shown in fig. 3-4.

Notice that it is the path of integration used to define the first term on the right hand side of eq. (5-21) or (5-27) which must be deformed. The path of integration in the second integral on the right hand side of eq. (5-27) remains undeformed since ξ_{1+} is not a singularity of $Y(\xi, y)$.

Again, at step (d), C may be deformed into a segment of the real axis from ξ_{1+} to $+\infty$, with the interval $[\xi_{1+}, 1]$ being traversed in both directions, and again the contribution to $I(x, y)$ from $\xi \geq 1$ is given by the right hand side of eq. (4-94). Taking the branch cuts as described in sect. 5-3 with $\xi_{1\pm} \leftrightarrow \xi_{2\pm}$ we find that, with

$$g_-(y) < \xi_{1+} < \xi < 1 < g_+(y) ,$$

eq. (5-22) holds and $H(\xi, y) = -M(\xi, y)$ where $M(\xi, y)$ is given in eq. (5-23). In this case

$$[G(\xi, y)]^2 \geq [M(\xi, y)]^2 > 0 \quad \text{for} \quad \xi_{1+} \leq \xi < 1 ,$$

$$G(1, y) = (a+b)(c+d) > 0$$

and

$$G(\xi_{1+}, y) = (a\sqrt{1-b^2} + b\sqrt{1-d^2})h_1(y) > 0 ,$$

and so $G(\xi_{1+}, y) - M(\xi_{1+}, y) = 0$. (We do not here consider the case $h_1(y) = 0$.) Thus $E(\xi, y)$ (eq. (5-7)) may be put in the form

$$E(\xi, y) = \frac{(\xi - \xi_{1+})(\xi - \xi_{1-})(\xi - \xi_{2+})(\xi - \xi_{2-})}{[G(\xi, y) + M(\xi, y)]^2}$$

and hence

$$\text{disc.}[\log E(\xi, y)] = 2\pi i \quad (5-31)$$

where the discontinuity is again computed as

$$[(\text{value below cut}) - (\text{value above cut})].$$

Combining eqs. (5-31) and (5-22) we find that the contribution to $I(x, y)$, when the interval $[\xi_{1+}, 1]$ of C is traversed in both directions, is

$$- \int_{\xi_{1+}}^1 \frac{d\xi}{\xi-x} Y(\xi, y)$$

where $Y(\xi, y)$ is given in eq. (5-26). Thus, using eq. (5-27) we see that

$$I(x, y) = \int_1^{\infty} \frac{d\xi}{\xi-x} X(\xi, y) + \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-x} Y(\xi, y) \quad (5-32)$$

where $X(\xi, y)$ is given in eq. (5-6) or (5-10). Eq. (5-32) has been established for the subcase of case III (ii) for which eq. (5-17) holds. To obtain the rest of case III (ii), we note that $Y(\xi, y)$ is singular at $\xi = g_{\pm}(y)$ but not at $\xi = \xi_{1\pm}$, $\xi = \xi_{2\pm}$. Further, we see from eqs. (4-A7), (4-A9) and (5-29) that

$$g_-(y) \leq \xi_{2-} < \xi_{2+} < \xi_{1+} < g_+(y)$$

when $h_1(y) < 0$, $h_2(y) > 0$. Thus, provided these inequalities hold, we can change a, b, c and d , and eq. (5-32) continues to be valid for $a + b < 0$, $c + d < 0$. Finally, $I(x, y)$ is again given by eq. (4-106) and hence by eq. (4-120) for case III (ii). For case III (iii) the results of case III (ii) apply with $\xi_{1+} \leftrightarrow \xi_{2+}$.

5-5. Continuation in x and y : Case I

Since, as discussed in sect. 5-2, $I(\sigma, \tau)$ is an analytic function of (σ, τ) in the Cartesian product $S_{\sigma} \times S_{\tau}$ when the stability conditions in eq. (5-3) hold, we see that for case I where $(a+b) > 0$, $(c+d) > 0$, $I(\sigma, \tau)$ is given by eq. (4-115) with the

replacements $x \rightarrow \sigma$, $y \rightarrow \tau$ for all σ, τ such that $\text{Im}\sigma > 0$, $\text{Im}\tau > 0$. Next, we want to obtain $I(\sigma, \tau)$, for case I, when at least one of σ, τ is real. For subcase (1a) (defined in sect. 4-8), $(a+c)(b+d) > 0$ in addition to $(a+b)(c+d) > 0$, and the region of integration in eq. (4-115) is the region R_{1a} shown in fig. 4-4. Then, as discussed in sect. 4-8, $I(x, y)$ is defined by eq. (4-115) for $x < 1$, $y < 1$. Similarly for subcase (1b) where $(a+c)(b+d) < 0$ in addition to $(a+b)(c+d) > 0$, the region of integration is R_{1b} shown in fig. 4-4, and $I(x, y)$ is defined by eq. (4-115) for $x < 1$, $y < \eta_\alpha$ where η_α is defined in a similar way to ξ_α (eq. (4-108) with $\xi \rightarrow \eta$).

For subcase (1a), it follows from the properties of the region R_{1a} in fig. 4-4 that if $\sigma \in S_\sigma$ or if σ is real and < 1 but $y > 1$, then

$$I(\sigma, y+i\cdot 0) = \int_1^{g_+(y)} \frac{d\xi}{\xi-\sigma} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{g_+(y)}^{\infty} \frac{d\xi}{\xi-\sigma} \left[P \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \frac{i\pi}{\sqrt{F(\xi, y)}} \right], \quad (5-33)$$

where P denotes the Cauchy principal value. Thus

$$P \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \equiv \lim_{\epsilon \downarrow 0} \left[\int_{f_+(\xi)}^{y-\epsilon} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{y+\epsilon}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \right] = \frac{\frac{1}{2}}{\sqrt{F(\xi, y)}} \ln \left[\frac{G(\xi, y) + \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}}{G(\xi, y) - \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}} \right]. \quad (5-34)$$

In deriving eq. (5-34) we use the fact that

$$G(\xi, y) = (\xi^2 - 1) \left[-y + \frac{1}{2} f_+(\xi) + \frac{1}{2} f_-(\xi) \right]$$

where $G(\xi, y)$ is given in eq. (5-8) and $f_{\pm}(\xi)$ in eq. (4-11). The method of evaluating an integral like the inner integral in eq. (5-33), when the pole singularity approaches a point on the path of integration from above, is often written symbolically in the form

$$\frac{1}{\eta - (y + i \cdot 0)} = P \frac{1}{\eta - y} + i\pi\delta(\eta - y). \quad (5-35)$$

The interpretation of this symbolic equation is illustrated in eqs.

(5-33) and (5-34). If we define $X(\xi, y + i \cdot 0)$ by

$$X(\xi, y + i \cdot 0) = \begin{cases} \frac{\frac{1}{2}}{\sqrt{F(\xi, y)}} \ln \left[\frac{G(\xi, y) + \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}}{G(\xi, y) - \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}} \right] & \text{when } y \leq f_-(\xi), \\ \frac{1}{\sqrt{-F(\xi, y)}} \left[\frac{\pi}{2} - \arctan \frac{G(\xi, y)}{\sqrt{\xi^2 - 1} \sqrt{-F(\xi, y)}} \right] & \text{when } f_-(\xi) < y < f_+(\xi), \\ \frac{\frac{1}{2}}{\sqrt{F(\xi, y)}} \left[2\pi i + \ln \left[\frac{G(\xi, y) + \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}}{G(\xi, y) - \sqrt{\xi^2 - 1} \sqrt{F(\xi, y)}} \right] \right] & \text{when } y > f_+(\xi), \end{cases} \quad (5-36)$$

then

$$I(\sigma, y + i \cdot 0) = \int_1^{\infty} \frac{d\xi}{\xi - \sigma} X(\xi, y + i \cdot 0). \quad (5-37)$$

Notice that the definition of $X(\xi, y + i \cdot 0)$ in eq. (5-36) is an extension of that given previously in eqs. (5-6) and (5-10). Further, eq. (5-37) is in fact valid for $\sigma \notin [1, +\infty)$ and for all $-\infty < y < +\infty$ except for $y = 1$; $I(\sigma, y + i \cdot 0)$ has a cut in the σ plane from 1 to $+\infty$ along the real axis. In a similar way we can obtain $I(x + i \cdot 0, \tau)$ for $\tau \notin [1, +\infty)$ and for all $-\infty < x < +\infty$, except for $x = 1$.

For subcase (1b) we see from the properties of the region R_{1b}

in fig. 4-4, that if $\sigma \in S_\sigma$ or if σ is real and < 1 but

$\eta_\alpha < y < 1$ then

$$\begin{aligned}
 I(\sigma, y+i\cdot 0) = & \int_1^{g_+(y)} \frac{d\xi}{\xi-\sigma} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \\
 & + \int_{g_+(y)}^{g_-(y)} \frac{d\xi}{\xi-\sigma} \left[P \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \frac{i\pi}{\sqrt{F(\xi, y)}} \right] \\
 & + \int_{g_-(y)}^{\infty} \frac{d\xi}{\xi-\sigma} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \quad (5-38)
 \end{aligned}$$

and again $I(\sigma, y+i\cdot 0)$ is given by eqs. (5-37) and (5-36). If

$\sigma \in S_\sigma$ or if σ is real and < 1 but $y > 1$, then $I(\sigma, y+i\cdot 0)$ is

again given by eq. (5-33) and hence by eqs. (5-37) and (5-36). Eq.

(5-37) is, in fact, valid for $\sigma \notin [1, +\infty)$ and for all $-\infty < y < +\infty$

except at most for $y \in \{1, \eta_\alpha\}$. The method of obtaining

$I(x+i\cdot 0, \tau)$ for $\tau \notin [\eta_\alpha, +\infty)$ and for all $-\infty < x < +\infty$ except for

$x = 1$, is similar to that given in the next section for case II D (2a).

5-6. Continuation in x and y : Case II D

In this section we study the continuation in x and y of the spectral representation for case II D, a particular subcase of case II. We define case II D by $(a+b)(c+d) < 0$ and in addition either $\max\{\alpha, b, c, d\} > 1$ or $-1 < a \leq 1$, $-1 < b \leq 1$, $-1 < c \leq 1$, $-1 < d \leq 1$ and $(\alpha+\beta+\gamma+\delta) \leq 2\pi$, where α, β, γ and δ are given in eq. (4-113). Then we see from eq. (4-116), and a similar equation for case II (ii) that, for case II D, $I(x, y)$ is given by eq. (4-116) for the range of values of x and y given there. Thus, it can also be written in the form

$$I(x, y) = \int_1^{\infty} \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^{\infty} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \int_{\xi_\alpha}^1 \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} , \quad (5-39)$$

where ξ_α is given in eq. (4-108). Again $I(\sigma, \tau)$ is given by eqs.

(4-116) and (5-39) for all σ, τ such that $\text{Im}\sigma > 0$, $\text{Im}\tau > 0$.

Also, as discussed in sect. 4-8, $I(x, y)$ is defined by eqs. (4-116)

and (5-39) for $x < \xi_\alpha$, $y < 1$ in the subcase (2a) for which

$(a+c)(b+d) > 0$ in addition to $(a+b)(c+d) < 0$. For this subcase the

region of integration is R_{2a} shown in fig. 4-4. Similarly, $I(x, y)$

is defined by eq. (4-116) and (5-39) for $x < \xi_\alpha$, $y < \eta_\alpha$ in subcase

(2b) for which $(a+c)(b+d) < 0$ and then the region of integration is

R_{2b} shown in fig. 4-4.

For subcase (2a), an argument similar to the one that led to eq.

(5-33) shows that if $\sigma \in S_\sigma$ or if σ is real and $< \xi_\alpha$ but

$1 < y \leq f_+(1)$ then

$$I(\sigma, y+i\cdot 0) = \int_1^{\infty} \frac{d\xi}{\xi-\sigma} X(\xi, y+i\cdot 0) + \int_{\xi_\alpha}^1 \frac{d\xi}{\xi-\sigma} Y(\xi, y+i\cdot 0) . \quad (5-40)$$

Here, $X(\xi, y+i\cdot 0)$ is given in eq. (5-36) and, for reasons which will

become clear later, we define $Y(\xi, y+i\cdot 0)$ by

$$Y(\xi, y+i\cdot 0) = \begin{cases} \frac{\pi}{\sqrt{-F(\xi, y)}} & \text{when } y < f_+(\xi) , \\ \frac{\pi i}{\sqrt{F(\xi, y)}} & \text{when } f_+(\xi) < y < f_-(\xi) , \\ \frac{-\pi}{\sqrt{-F(\xi, y)}} & \text{when } y > f_-(\xi) . \end{cases} \quad (5-41)$$

Notice that the definition of $Y(\xi, y+i\cdot 0)$ in eq. (5-41) is an

extension of that given previously in eq. (5-26). When the above

restrictions on σ and y hold, then $Y(\xi, y+i\cdot 0)$ is given exclusively by eq. (5-26). When $\sigma \in S_\sigma$ or σ is real and $< \xi_\alpha$ but $f_+(1) < y < +\infty$ then

$$\begin{aligned}
 I(\sigma, y+i\cdot 0) = & \int_1^\infty \frac{d\xi}{\xi-\sigma} \left[P \int_{f_+(\xi)}^\infty \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \frac{i\pi}{\sqrt{F(\xi, y)}} \right] \\
 & + \int_{\xi_\alpha}^{g_+(y)} \frac{d\xi}{\xi-\sigma} \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} \\
 & + \int_{g_+(y)}^1 \frac{d\xi}{\xi-\sigma} \left[P \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} + \frac{i\pi}{\sqrt{F(\xi, y)}} \right]. \quad (5-42)
 \end{aligned}$$

Now

$$P \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} = 0$$

and

$$\int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}} = Y(\xi, y+i\cdot 0)$$

where $Y(\xi, y+i\cdot 0)$ is given in eq. (5-41). Thus $I(\sigma, y+i\cdot 0)$ is again given by eqs. (5-40), (5-36) and (5-41). Eq. (5-40) is, in fact, valid for $\sigma \notin [\xi_\alpha, +\infty)$ and for all $-\infty < y < +\infty$ except for $y = 1$; $I(\sigma, y+i\cdot 0)$ has a cut in the σ plane from ξ_α to $+\infty$ along the real axis. The method of obtaining $I(x+i\cdot 0, \tau)$ for $\tau \notin [1, +\infty)$ and for all $-\infty < x < +\infty$ except at most for $x \in \{1, \xi_\alpha\}$, is similar to that given in sect. 5-5 for case I (1b).

For subcase (2b), we noted in sect. 4-8 that the ordering of the points 1 , $g_+(1)$ and $g_\pm(\eta_\alpha)$, and also of 1 , $f_+(1)$ and

$f_{\pm}(\xi_{\alpha})$, varies depending on the values of a, b, c and d . We start by obtaining $I(\sigma, y+i\cdot 0)$, for $\sigma \in S_{\sigma}$ or σ real and $< \xi_{\alpha}$, when the region R_{2b} is as shown in fig. 4-4. If $\eta_{\alpha} < y < 1$, then the contribution to $I(\sigma, y+i\cdot 0)$ from $\xi \geq 1$ is given by the right hand side of eq. (5-38) and the contribution from $\xi_{\alpha} \leq \xi < 1$ is given by the second term on the right hand side of eq. (5-39). Thus $I(\sigma, y+i\cdot 0)$ is given by eqs. (5-40), (5-36) and (5-41). If $1 < y \leq f_{+}(1)$ then the contribution to $I(\sigma, y+i\cdot 0)$ from $\xi \geq 1$ is given by the right hand side of eq. (5-33) and the contribution from $\xi_{\alpha} \leq \xi < 1$ is given by the second term on the right hand side of eq. (5-39), and $I(\sigma, y+i\cdot 0)$ is again given by eqs. (5-40), (5-36) and (5-41). Finally, if $f_{+}(1) < y < +\infty$, then $I(\sigma, y+i\cdot 0)$ is given by eq. (5-42) and hence by eq. (5-40). In fact, $I(\sigma, y+i\cdot 0)$ is given by eqs. (5-40), (5-36) and (5-41), for $\sigma \notin [\xi_{\alpha}, +\infty)$ and for all $-\infty < y < +\infty$ except at most for $y \in \{1, \eta_{\alpha}\}$, irrespective of the ordering of the points $1, g_{+}(1)$ and $g_{\pm}(\eta_{\alpha})$, and of $1, f_{+}(1)$ and $f_{\pm}(\xi_{\alpha})$. In a similar way, we can obtain $I(x+i\cdot 0, \tau)$ for $\tau \notin [\eta_{\alpha}, +\infty)$ and for all $-\infty < x < +\infty$ except at most for $x \in \{1, \xi_{\alpha}\}$.

5-7. Properties of the curve Γ

We established in sect. 4-8 that for case III and for the subcase of case II for which $-1 < a \leq 1$, $-1 < b \leq 1$, $-1 < c \leq 1$, $-1 < d \leq 1$ and $\alpha + \beta + \gamma + \delta > 2\pi$, $I(x, y)$ cannot be written as a double spectral representation. The above subcase of case II we call case II S. Notice that case II D, defined at the beginning of sect. 5-6, and case II S together make up the whole of case II. Further, we

found in sect. 4-8 that, for cases II S and III, $I(x, y)$ could be written as the sum of a double spectral representation and a single spectral representation, at least for $x < -1$, $y < -1$. The double spectral representation can be continued in x and y , exactly as done in sects. 5-5 and 5-6. However, to continue the single spectral representation in y , we need to know the behaviour of $g_{\pm}(y)$ (eq. (4-36)) for all $-\infty < y < \infty$. A useful way of exhibiting the properties of $g_{\pm}(y)$ is to draw the graph of the branches of the set of points $\{(\xi, y) \mid F(\xi, y) = 0\}$, which we call the curve Γ . (The curve Γ was first studied in detail by Tarski (1960).)

One factorization of $F(\xi, y)$ is given in eq. (4-12), the functions $f_{\pm}(\xi)$ being given by eq. (4-11) or alternatively by

$$f_{\pm}(\xi) = (\xi^2 - 1)^{-1} \left\{ (\xi - 1)(ad + bc) + (a + b)(c + d) \right. \\ \left. \pm [(\xi - \xi_{1+})(\xi - \xi_{1-})(\xi - \xi_{2+})(\xi - \xi_{2-})]^{\frac{1}{2}} \right\}, \quad (5-43)$$

where $\xi_{1\pm}$ are given in eq. (4-32) and $\xi_{2\pm}$ in eq. (4-42). From eq. (4-A3) we have an alternative factorization of $F(\xi, y)$, the functions $g_{\pm}(y)$ being given by eq. (4-36) or alternatively by

$$g_{\pm}(y) = (y^2 - 1)^{-1} \left\{ (y - 1)(ad + bc) + (a + c)(b + d) \right. \\ \left. \pm [(y - \eta_{1+})(y - \eta_{1-})(y - \eta_{2+})(y - \eta_{2-})]^{\frac{1}{2}} \right\}, \quad (5-44)$$

where $\eta_{1\pm}$ and $\eta_{2\pm}$ are given in eq. (4-A2). From the definition of case II S above, and since $(a + b) < 0$, $(c + d) < 0$ for case III, we see that for these cases $-1 < a \leq 1$, $-1 < b \leq 1$, $-1 < c \leq 1$, $-1 < d \leq 1$ and consequently

$$\begin{aligned} -1 \leq \xi_{1-} < \xi_{1+} \leq 1, \\ -1 \leq \eta_{1-} < \eta_{1+} \leq 1, \end{aligned} \quad (5-45)$$

The same inequalities hold for $\xi_{2\pm}$ and $\eta_{2\pm}$ with $1 \rightarrow 2$. At this stage we introduce the quantities $\xi_\alpha, \xi_\beta, \xi_\gamma$ and ξ_δ given by

$$\{\xi_\alpha, \xi_\beta, \xi_\gamma, \xi_\delta\} = \{\xi_{1\pm}, \xi_{2\pm}\},$$

$$\xi_\delta < \xi_\gamma < \xi_\beta < \xi_\alpha. \quad (5-46)$$

Similarly $\eta_\alpha, \eta_\beta, \eta_\gamma$ and η_δ given by the above relations with

$\xi \rightarrow \eta$. We do not consider the degenerate cases, for which some of

the quantities in (5-46) are equal. Now, when $y \geq \eta_\alpha$, the term inside

the square root in eq. (5-44) is positive and the square root is just

the ordinary square root. In general, if $\eta_0 \in \{\eta_{1\pm}, \eta_{2\pm}\}$, then we

choose the square root by continuity as follows:

$$[y+i\cdot 0-\eta_0]^{\frac{1}{2}} = \begin{cases} \sqrt{y-\eta_0} & \text{when } y \geq \eta_0 \\ i\sqrt{\eta_0-y} & \text{when } y < \eta_0. \end{cases} \quad (5-47)$$

The method of determining the behaviour of the real branches of Γ is similar to that given in sect. 4-8 for the branch Γ_1 of Γ .

Firstly, we note from eqs. (5-44), (5-45), (5-46) (with $\xi \rightarrow \eta$) and (5-47) that $g_\pm(y)$ have the following properties:

- (i) $g_\pm(y)$ are real for $y \geq \eta_\alpha$,
- (ii) $\text{Im}g_+(y+i\cdot 0) = -\text{Im}g_-(y+i\cdot 0) < 0$ for $\eta_\beta < y < \eta_\alpha$,
- (iii) $g_\pm(y)$ are real for $\eta_\gamma \leq y \leq \eta_\beta$,
- (iv) $\text{Im}g_+(y+i\cdot 0) = -\text{Im}g_-(y+i\cdot 0) > 0$ for $\eta_\delta < y < \eta_\gamma$,
- (v) $g_\pm(y)$ are real for $y \leq \eta_\delta$. (5-48)

The functions $f_\pm(\xi)$ have similar properties. Next we recall from

sect. 4-8 that when $(a+b)(c+d) > 0$, $f_+(\xi) \rightarrow \pm\infty$ as $\xi \rightarrow 1^\pm$,

$g_+(y) \rightarrow 1^\pm$ as $y \rightarrow \pm\infty$ and

$$f_{-}(1) = 1 + [2(a+b)(c+d)]^{-1}[(ad-bc)^2 - (a+b+c+d)^2] . \quad (5-49)$$

When $(a+b)(c+d) < 0$, $f_{-}(\xi) \rightarrow \mp\infty$ as $\xi \rightarrow 1^{\pm}$, $g_{+}(y) \rightarrow 1^{\mp}$ as $y \rightarrow \pm\infty$ and $f_{+}(1)$ is given by the right hand side of eq. (5-49).

Similarly, when $(a+c)(b+d) > 0$, $g_{+}(y) \rightarrow \pm\infty$ as $y \rightarrow 1^{\pm}$, $f_{+}(\xi) \rightarrow 1^{\pm}$ as $\xi \rightarrow \pm\infty$ and $g_{-}(1)$ is given by the right hand side of eq. (5-49) with $b \leftrightarrow c$. When $(a+c)(b+d) < 0$, $g_{-}(y) \rightarrow \mp\infty$ as $y \rightarrow 1^{\pm}$, $f_{+}(\xi) \rightarrow 1^{\mp}$ as $\xi \rightarrow \pm\infty$ and $g_{+}(1)$ is given by the right hand side of eq. (5-49) with $b \leftrightarrow c$.

We also see from eqs. (4-11) and (4-36) that when $(a-b)(c-d) > 0$, $f_{+}(\xi) \rightarrow \mp\infty$ as $\xi \rightarrow -1^{\pm}$, $g_{-}(y) \rightarrow -1^{\mp}$ as $y \rightarrow \pm\infty$ and

$$f_{-}(-1) = 1 + [2(a-b)(c-d)]^{-1}[(ad-bc)^2 - (a-b+c-d)^2] . \quad (5-50)$$

When $(a-b)(c-d) < 0$, $f_{-}(\xi) \rightarrow \pm\infty$ as $\xi \rightarrow -1^{\pm}$, $g_{-}(y) \rightarrow -1^{\pm}$ as $y \rightarrow \pm\infty$ and $f_{+}(-1)$ is given by the right hand side of eq. (5-50).

Similarly, when $(a-c)(b-d) > 0$, $g_{+}(y) \rightarrow \mp\infty$ as $y \rightarrow -1^{\pm}$, $f_{-}(\xi) \rightarrow -1^{\mp}$ as $\xi \rightarrow \pm\infty$ and $g_{-}(-1)$ is given by the right hand side of eq. (5-50) with $b \leftrightarrow c$. When $(a-c)(b-d) < 0$, $g_{-}(y) \rightarrow \pm\infty$ as $y \rightarrow -1^{\pm}$, $f_{-}(\xi) \rightarrow -1^{\pm}$ as $\xi \rightarrow \pm\infty$ and $g_{+}(-1)$ is given by the right hand side of eq. (5-50) with $b \leftrightarrow c$.

The expressions for $f_{\pm}(\xi_{1+})$ and $f_{\pm}(\xi_{2+})$ are again given by eqs. (4-109) and (4-110) respectively, and the expressions for $g_{\pm}(\eta_{1+})$ and $g_{\pm}(\eta_{2+})$ given in sect. 4-8 again hold. Similar expressions hold for $f_{\pm}(\xi_{1-})$, $f_{\pm}(\xi_{2-})$, $g_{\pm}(\eta_{1-})$ and $g_{\pm}(\eta_{2-})$. The expression for $f_{\pm}(\xi_{1-})$ can be obtained from the right hand side of eq. (4-109)

by making the replacements $b \rightarrow -b$, $d \rightarrow -d$.

Arguments similar to the ones used to establish the behaviour of Γ_1 in sect. 4-8 can now be employed to determine the behaviour of the real branches of Γ for the various mass configurations discussed in the last two paragraphs. We do not consider the degenerate cases for which at least one of the quantities $(a+b)$, $(c+d)$, $(a+c)$, $(b+d)$, $(a-b)$, $(c-d)$, $(a-c)$, $(b-d)$ is zero. Further, we do not consider the cases for which some of the quantities in (5-46) are equal. There are then sixteen cases of which four are shown in figs. 5-1, 5-2, 5-3 and 5-4. As shown, the curve Γ has five real branches Γ_1 , Γ_2 , Γ_3 , Γ_4 and Γ_5 . Fig. 5-1 corresponds to subcase (1a) $\{(a+b)(c+d) > 0$, $(a+c)(b+d) > 0\}$ and in addition $(a-b)(c-d) < 0$, $(a-c)(b-d) > 0$. Fig. 5-2 corresponds to subcase (1b) $\{(a+b)(c+d) > 0$, $(a+c)(b+d) < 0\}$ and in addition $(a-b)(c-d) < 0$, $(a-c)(b-d) < 0$. Fig. 5-3 corresponds to subcase (2a) $\{(a+b)(c+d) < 0$, $(a+c)(b+d) > 0\}$ and in addition $(a-b)(c-d) > 0$, $(a-c)(b-d) > 0$. Fig. 5-4 corresponds to subcase (2b) $\{(a+b)(c+d) < 0$, $(a+c)(b+d) < 0\}$ and in addition $(a-b)(c-d) > 0$, $(a-c)(b-d) < 0$. By suitably combining the five branches shown in these figures we can generate the curve Γ for the sixteen cases, four for each of subcases (1a), (1b), (2a) and (2b).

Next, we want to find the possible orderings of the quantities $\xi_{1\pm}$, $\xi_{2\pm}$ and of $\eta_{1\pm}$, $\eta_{2\pm}$. In the following two sections, we shall be mainly concerned with the subcases of case II S and case III for which $\xi_{2+} < \xi_{1+}$. We call these subcases case II S (i) and case III A

respectively. Notice that for case II S (i), $(a+b) > 0$, $(c+d) < 0$ and the spectral representation for this case is given in eq. (4-117)

Fig. 5-1.

The five branches of the curve Γ for subcase (1a) $\{(a+b)(c+d) > 0$, $(a+c)(b+d) > 0\}$ when, in addition, $(a-b)(c-d) < 0$, $(a-c)(b-d) > 0$.

Fig. 5-2.

The five branches of the curve Γ for subcase (1b) $\{(a+b)(c+d) > 0$, $(a+c)(b+d) < 0\}$ when, in addition, $(a-b)(c-d) < 0$, $(a-c)(b-d) < 0$.

Fig. 5-3.

The five branches of the curve Γ for subcase (2a) $\{(a+b)(c+d) < 0$, $(a+c)(b+d) > 0\}$ when, in addition, $(a-b)(c-d) > 0$, $(a-c)(b-d) > 0$.

Fig. 5-4.

The five branches of the curve Γ for subcase (2b) $\{(a+b)(c+d) < 0, (a+c)(b+d) < 0\}$ when, in addition, $(a-b)(c-d) > 0, (a-c)(b-d) < 0$.

for $x < -1$, $y < -1$. That is, case II S (i) is the subcase of case II (i) for which $I(x, y)$ no longer has a double spectral representation. Also, case III A, for which $(a+b) < 0$, $(c+d) < 0$, includes the whole of case III (ii) and also the part of case III (i) for which $\xi_{2+} < \xi_{1+}$.

Now, suppose that $\xi_{2+} < \xi_{1+}$ and that $\eta_{2+} < \eta_{1+}$. If, in addition $\xi_{2+} < \xi_{1-}$, then

$$\sqrt{(1-c^2)(1-d^2)} + \sqrt{(1-a^2)(1-b^2)} < cd - ab$$

and hence

$$\sqrt{(1-c^2)(1-a^2)} + \sqrt{(1-b^2)(1-d^2)} < |ac-bd|.$$

That is, either $\eta_{1+} < \eta_{2-}$ or $\eta_{2+} < \eta_{1-}$. The inequality $\eta_{2+} < \eta_{1+}$ then implies that $\eta_{2+} < \eta_{1-}$. Similarly $\eta_{2+} < \eta_{1-}$ implies that $\xi_{2+} < \xi_{1-}$.

We now have the following possible orderings:

$$\begin{aligned} (\alpha) : \quad & -1 < \xi_{\delta} < \xi_{\gamma} < \xi_{2+} < \xi_{1+} < 1, \\ & -1 < \eta_{\delta} < \eta_{\gamma} < \eta_{2+} < \eta_{1+} < 1, \end{aligned} \tag{5-51}$$

$$\begin{aligned} (\beta) : \quad & -1 < \xi_{2-} < \xi_{2+} < \xi_{1-} < \xi_{1+} < 1, \\ & -1 < \eta_{2-} < \eta_{2+} < \eta_{1-} < \eta_{1+} < 1. \end{aligned} \tag{5-52}$$

Finally, before leaving this section, we want to establish eqs. (5-28) and (5-29) from the properties of Γ . From figs. 5-1 and 5-2 and from eqs. (5-45) and (5-46), we see that for subcases (1a) and (1b), the inequality $\xi_{2+} < \xi_{1+}$ implies that $f_{\pm}(\xi_{1+}) < f_{\pm}(\xi_{2+})$ and $-1 < f_{\pm}(\xi_{2+})$. The same statement holds with $\xi_{1+} \leftrightarrow \xi_{2+}$, and hence for case III (ii) eqs. (5-28) and (5-29) hold.

5-8. Continuation in x and y : Case II S

We established in sect. 4-8 and confirmed in sects. 5-3 and 5-4 that, for case II S (i), $I(x, y)$ is given by eq. (4-117) for $x < -1$, $y < -1$ and, for case III (ii), $I(x, y)$ is given by eq. (4-120) for $x < -1$, $y < -1$. Thus for these two cases, $I(x, y)$ can be written as

$$I(x, y) = I_D(x, y) + I_S(x, y) \quad (5-53)$$

where

$$I_D(x, y) = \iint_R \frac{d\xi}{\xi-x} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}}, \quad (5-54)$$

$$I_S(x, y) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-x} \frac{\pi}{\sqrt{-F(\xi, y)}} \quad (5-55)$$

when $x < -1$, $y < -1$. Here $R = R_1$ for case III (ii) and $R = R_2$ for case II S (i) where R_1 and R_2 are given in eqs. (4-111) and (4-112) respectively. We recall that case III (ii) is defined by $(a+b) < 0$, $(c+d) < 0$, $h_1(y) \leq 0$, $h_2(y) > 0$ and that for this case $\xi_{2+} < \xi_{1+}$ and $\alpha + \beta + \gamma + \delta > 2\pi$. Here $h_1(y)$ and $h_2(y)$ are given in eqs. (4-38) and (4-47) respectively. Further, case III (ii) is a subcase of case III A where case III A is defined by $(a+b) < 0$, $(c+d) < 0$ and $\xi_{2+} < \xi_{1+}$. Also, case II S (i) is defined by $(a+b) > 0$, $(c+d) < 0$, $-1 < a \leq 1$, $-1 < b \leq 1$ and $\alpha + \beta + \gamma + \delta > 2\pi$ and for this case $\xi_{2+} < \xi_{1+}$.

As discussed in sect. 5-2, $I(\sigma, \tau)$ is an analytic function of (σ, τ) in the Cartesian product $S_\sigma \times S_\tau$ when the stability conditions in eq. (5-3) hold. Thus $I(\sigma, \tau)$ is given by eqs. (5-53), (5-54) and (5-55) with $x \rightarrow \sigma$, $y \rightarrow \tau$ and $R = R_1$ for case III A

and by the same equations with $x \rightarrow \sigma$, $y \rightarrow \tau$ and $R = R_2$ for case II S (i). The square root in eq. (5-55) is now interpreted more generally. Further, $I_D(\sigma, \tau)$ is an analytic function of (σ, τ) in $S_\sigma \times S_\tau$ when the stability conditions hold, and hence, $I_S(\sigma, \tau)$ is also an analytic function of (σ, τ) in $S_\sigma \times S_\tau$ when the stability conditions hold; it is given by

$$I_S(\sigma, \tau) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{[-F(\xi, \tau)]^{\frac{1}{2}}}. \quad (5-56)$$

To obtain $I_D(\sigma, y+i\cdot 0)$ for case II S, we proceed exactly as in sect. 5-6 for case II D. That is, the expressions for $I(\sigma, y+i\cdot 0)$ given there apply equally well to $I_D(\sigma, y+i\cdot 0)$. Similarly, the expressions given in sect. 5-5 for $I(\sigma, y+i\cdot 0)$ in case I apply to $I_D(\sigma, y+i\cdot 0)$ in case III. Our aim in this and the next section is to obtain $I_S(\sigma, y+i\cdot 0)$ for all $\sigma \in S_\sigma$ and for almost all $-\infty < y < +\infty$. In this section we consider case II S and in the next section we consider case III.

We begin by considering the subcase of case II S (i) for which $(a+c)(b+d) > 0$ in addition to $(a+b)(c+d) < 0$ and for which eq. (5-51) holds. This subcase we call subcase (2a)(α). The branch Γ_1 of Γ is then as shown in fig. 5-3. The branches Γ_2, Γ_3 and Γ_4 can have a number of different configurations, Γ_5 remaining essentially the same as in fig. 5-3. Γ_2, Γ_3 and Γ_4 can be as in fig. 5-3, or Γ_2 can be as in fig. 5-3 with Γ_3 and Γ_4 as in fig. 5-1, or Γ_2 and Γ_3 can be as in fig. 5-4 with Γ_4 as in fig. 5-3, or Γ_2 can

be as in fig. 5-4 with Γ_3 as in fig. 5-2 and Γ_4 as in fig. 5-1.

We note that when $\xi = \xi_0 \in \{\xi_{1\pm}, \xi_{2\pm}\}$ is a vertical tangent to Γ_i ($i = 1, \dots, 4$) and $y = \eta_0 \in \{\eta_{1\pm}, \eta_{2\pm}\}$ is a horizontal tangent to Γ_i , then the precise shape of Γ_i can vary considerably, depending on the values of a, b, c and d . This was noted in sect. 4-8 for the branch Γ_1 bounding the region R_{2b} .

To simplify the continuation procedure, we take $\sigma \in S_\sigma$ and $\text{Re} \sigma = x \ll -1$. Then for $y < -1$, $I_S(\sigma, y+i\cdot 0)$ is given by the right hand side of eq. (5-55) with $x \rightarrow \sigma$, and the singularities of the weight function are $g_\pm(y)$ given in eq. (5-44). We now see from fig. 5-3 (and the three similar figures which can be constructed as described in the previous paragraph) and from the properties of $g_\pm(y)$ given in eq. (5-48), that for subcase (2a)(α) the singularities $g_\pm(y+i\cdot 0)$ do not appear on the path of integration from ξ_{2+} to ξ_{1+} as long as

$$y < f_\pm(\xi_{2+}) . \quad (5-57)$$

Thus, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-55) with $x \rightarrow \sigma$ whenever eq. (5-57) holds or, equivalently, since $c + d < 0$, when

$$h_2(y) > 0$$

where $h_2(y)$ is given in eq. (4-47).

To obtain $I_S(\sigma, y+i\cdot 0)$ when $y > f_\pm(\xi_{2+})$ ($h_2(y) < 0$), we need to examine the path taken by $g_+(\tau)$ in the complex plane when $\tau = y + i\varepsilon$ and ε is small and > 0 . From eqs. (4-36), (5-47) and (5-44), we see that when $\eta_Y \leq y \leq \eta_{2+}$,

$$g_{\pm}(y+i\cdot 0) = (y^2-1)^{-1} \left\{ (y-1)(ad+bc) + (a+c)(b+d) \right. \\ \left. \mp [- (y^2-1+2(y-1)ac + (a+c)^2)]^{\frac{1}{2}} [- (y^2-1+2(y-1)bd + (b+d)^2)]^{\frac{1}{2}} \right\}$$

and, using eq. (4-42),

$$g_{\pm}(y+i\cdot 0) - \xi_{2+} = \sqrt{1-c^2} \sqrt{1-d^2} \left[-1 + \frac{(yc+a)(yd+b)}{(y^2-1)\sqrt{1-c^2}\sqrt{1-d^2}} \right. \\ \left. \pm \sqrt{1 - \frac{(yc+a)^2}{(1-y^2)(1-c^2)}} \sqrt{1 - \frac{(yd+b)^2}{(1-y^2)(1-d^2)}} \right].$$

(We suppose in the following, that none of the quantities a, b, c and d is equal to 1, that is, $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$.) Next, we introduce the quantities $\phi_1(y), \phi_2(y), \phi_3(y)$

and $\phi_4(y)$ defined by

$$\phi_1(y) = \arccos \left[\frac{ya+c}{\sqrt{1-y^2}\sqrt{1-a^2}} \right], \quad \phi_2(y) = \arccos \left[\frac{yb+d}{\sqrt{1-y^2}\sqrt{1-b^2}} \right] \\ \phi_3(y) = \arccos \left[\frac{yc+a}{\sqrt{1-y^2}\sqrt{1-c^2}} \right], \quad \phi_4(y) = \arccos \left[\frac{yd+b}{\sqrt{1-y^2}\sqrt{1-d^2}} \right], \quad (5-58)$$

where

$$0 < \phi_i(y) < \pi; \quad i = 1, \dots, 4.$$

Now

$$g_{\pm}(y+i\cdot 0) - \xi_{2+} = \sqrt{1-c^2} \sqrt{1-d^2} [-1 - \cos(\phi_3(y) \pm \phi_4(y))] . \quad (5-59)$$

We also define $\phi_3(\tau)$ and $\phi_4(\tau)$ by eq. (5-58) with

$y \rightarrow \tau = y + i\varepsilon$, where ε is small and ≥ 0 , and the square root is now interpreted more generally. Then $g_{\pm}(\tau) - \xi_{2+}$ are given by eq.

(5-59) with $y \rightarrow \tau$. With $\tau = y + i\varepsilon$, we see that for small $\varepsilon \geq 0$,

$$\operatorname{Re}(\phi_3(\tau) + \phi_4(\tau)) = \phi_3(y) + \phi_4(y) + \varepsilon^2 A(\varepsilon, y, a, b, c, d), \quad (5-60)$$

$$\operatorname{Im}\{\phi_3(\tau)+\phi_4(\tau)\} = \frac{-\varepsilon}{1-y^2} \{\cot\phi_1(y)+\cot\phi_2(y)\} + \varepsilon^3 B(\varepsilon, y, a, b, c, d) \quad (5-61)$$

where $|A(\varepsilon, y, a, b, c, d)| < \infty$, $|B(\varepsilon, y, a, b, c, d)| < \infty$ for $\eta_Y < y < \eta_{2+}$ ($< \eta_{1+}$).

Now, from eq. (3-100) we find that

$$g_+(\tau) - \xi_{2+} = \sqrt{1-c^2} \sqrt{1-d^2} [-1 - \cos\{\operatorname{Re}\{\phi_3(\tau)+\phi_4(\tau)\}\}] \cdot \cosh\{\operatorname{Im}\{\phi_3(\tau)+\phi_4(\tau)\}\} + i \sin\{\operatorname{Re}\{\phi_3(\tau)+\phi_4(\tau)\}\} \sinh\{\operatorname{Im}\{\phi_3(\tau)+\phi_4(\tau)\}\} \quad (5-62)$$

The steps in the continuation are then as follows.

(A) Start with y such that $y < f_{\pm}(\xi_{2+})$ ($h_2(y) > 0$), that is,

$$0 < \phi_3(y) + \phi_4(y) < \pi,$$

(B) make $\varepsilon > 0$,

(C) fix ε and change y until $y > f_{\pm}(\xi_{2+})$ ($h_2(y) < 0$), that is,

$$\pi < \phi_3(y) + \phi_4(y) < 2\pi,$$

(D) let $\varepsilon \downarrow 0$.

From fig. 5-3 (and the three related figures) we see that

$\eta_{2+} < 1 < f_{\pm}(\xi_{1+})$, and since $(a+b) > 0$, we have $h_1(y) < 0$ when $y < \eta_{2+}$, where $h_1(y)$ is given in eq. (4-38). Thus,

$$\cos\phi_1(y) + \cos\phi_2(y) < 0$$

and

$$\cot\phi_1(y) + \cot\phi_2(y) < 0 \quad (5-63)$$

when $\eta_Y \leq y \leq \eta_{2+}$. From eqs. (5-61) and (5-63), we see that we can

choose ε sufficiently small, so that

$$\operatorname{Im}\{\phi_3(\tau)+\phi_4(\tau)\} > 0$$

at steps (B) and (C) of the continuation procedure. Further, we can choose ϵ sufficiently small so that at step (B)

$$\text{sign}\{\text{Re}(\phi_3(\tau) + \phi_4(\tau) - \pi)\} = \text{sign}\{\phi_3(y) + \phi_4(y) - \pi\}.$$

Similarly, when step (C) has been taken, the above equation again holds for sufficiently small ϵ .

We now see that as steps (A)-(D) are taken, the path taken by $g_+(\tau)$ in the complex plane is as shown in fig. 5-5. (If only terms up to second order in ϵ are retained, then the path taken by $g_+(\tau)$ is still essentially the same. The expressions in eq. (5-62) now have to be expanded and only terms up to second order in ϵ retained. The argument is very similar to that leading to eq. (3-101). The main point to note in eq. (3-101a) is that the sign of the term multiplying $\sin(\alpha+\beta)$ is unimportant since this term vanishes when $\alpha + \beta = \pi$. A similar observation should be made for $\text{Re}(g_+(\tau) - \xi_{2+})$ when it is expanded.) Once $g_+(\tau)$ crosses the real axis between ξ_{2+} and ξ_{1+} , and moves into the lower half-plane, the path of integration between ξ_{2+} and ξ_{1+} must be deformed; the modified path is shown in fig. 5-6. Notice that for sufficiently small ϵ , $\text{Reg}_-(\tau)$ is always less than ξ_{2+} and also $\text{Reg}_-(\tau) < \text{Reg}_+(\tau)$ for $\eta_Y < y < \eta_{2+}$. Using a similar argument to that used in sects. 3-8, 5-3 and 5-4, we see from Cauchy's theorem that the required continuation of $I_S(\sigma, \tau)$ for step (C) is given by

$$I_S(\sigma, \tau) = \int_P \frac{d\xi}{\xi - \sigma} \frac{\pi}{[-F(\xi, \tau)]^{\frac{1}{2}}}, \quad (5-64)$$

where P is the path shown in fig. 5-6. The branch cut of

Fig. 5-5.

Fig. 5-6.

$[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of $[-F(\xi, \tau)]^{\frac{1}{2}}$, is chosen in the ξ plane to connect $g_+(\tau)$ and $g_-(\tau)$ in such a way that it does not pass through P (except at the point $(\xi_{2+}, 0)$), and at step (C), it is as shown in fig. 5-6; that is, it consists of the broken line from $g_-(\tau)$ to ξ_{2+} plus the broken line from ξ_{2+} to $g_+(\tau)$. At step (D), $g_+(\tau)$ approaches the real axis from below and the path P is forced into the real axis at $\xi = g_+(y+i\cdot 0)$. It may then be deformed to the segment of the real axis from $g_+(y+i\cdot 0)$ to ξ_{1+} with the interval $[g_+(y+i\cdot 0), \xi_{2+}]$ being traversed in both directions.

The contribution to $I_S(\sigma, y+i\cdot 0)$ when $\xi_{2+} \leq \xi \leq \xi_{1+}$ is again given by the right hand side of eq. (5-55) with $x \rightarrow \sigma$. When $g_+(y+i\cdot 0) < \xi < \xi_{2+}$,

$$\text{disc.}([-F(\xi, y)]^{-\frac{1}{2}}) = \frac{1}{\sqrt{-F(\xi, y)}} - \frac{-1}{\sqrt{-F(\xi, y)}} = \frac{2}{\sqrt{-F(\xi, y)}} \quad (5-65)$$

where the discontinuity is again computed as [(value below cut) - (value above cut)]. Thus,

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + 2 \int_{g_+(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \quad (5-66)$$

when $f_{\pm}(\xi_{2+}) < y < \eta_{2+}$.

Next, we want to find $I_S(\sigma, y+i\cdot 0)$ when $\eta_{2+} < y < \eta_{1+}$. We start with the spectral representation for $I_S(\sigma, \tau)$ given in eq. (5-64) at step (C) and do the continuation in $\tau = y + i\varepsilon$. When

$$|\eta_{2+} - y| \ll \min\{(\eta_{1+} - y), (y - \eta_{1-}), (y - \eta_{2+})\}$$

and

$$\epsilon \ll \min\{(\eta_{1+}-y), (y-\eta_{1-}), (y-\eta_{2+})\},$$

we let

$$\eta_{2+} - y - i\epsilon = Re^{-i\theta},$$

and from eqs. (5-44) and (5-47), we see that

$$g_-(\tau) - g_+(\tau) \approx - \frac{2R^{\frac{1}{2}} \exp\left(-\frac{i}{2}\theta\right)}{(1-\eta_{2+}^2)} \sqrt{\eta_{1+}-\eta_{2+}} \sqrt{\eta_{2+}-\eta_{1-}} \sqrt{\eta_{2+}-\eta_{2-}}. \quad (5-67)$$

Thus, as $\eta_{2+} - y$ changes from $\eta_{2+} - y \gg \epsilon$ through zero to $y - \eta_{2+} \gg \epsilon$, $g_-(\tau)$ moves in an arc about $g_+(\tau)$, from being to the left of $g_+(\tau)$ to being above $g_+(\tau)$. The motion of $g_-(\tau)$ relative to $g_+(\tau)$ is shown in fig. 5-6. Notice that throughout the continuation $R > 0$ and hence P is not pinched between the two singularities $g_{\pm}(\tau)$. Once $y > \eta_{2+}$, we can take the limit $\epsilon \downarrow 0$, and, as shown in eq. (5-48), $g_-(y+i\cdot 0)$ then lies in the upper half ξ plane and $g_+(y+i\cdot 0)$ has moved into the lower half plane, deforming the path of integration P as shown in fig. 5-7. Thus $I_S(\sigma, y+i\cdot 0)$ is given by

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + 2 \int_{g_+(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{[-F(\xi, y)]^{\frac{1}{2}}} \quad (5-68)$$

for $\eta_{2+} < y < \eta_{1+}$. The path of integration from $g_+(y+i\cdot 0)$ to ξ_{2+} and the square root of the complex number $\{-F(\xi, y)\}$ when ξ is in the lower half plane are to be determined from fig. 5-7. The branch cut of $[-F(\xi, y)]^{\frac{1}{2}}$ is taken to be the straight line segment from $g_+(y+i\cdot 0)$ to ξ_{2+} plus the straight line segment from ξ_{2+} to

Fig. 5-7.

$g_-(y+i\cdot 0)$ and $[-F(\xi, y)]^{\frac{1}{2}} = \sqrt{-F(\xi, y)}$ for ξ on the real axis to the right of the point $(\xi_{2+}, 0)$. The square root $[-F(\xi, y)]^{\frac{1}{2}}$ in eq. (5-68), when ξ lies in the lower half plane, is then determined by continuity.

We now consider the continuation into the region $\eta_{1+} < y < 1$.

As usual, the continuation is carried out in $\tau = y + i\varepsilon$ and then the limit $\varepsilon \downarrow 0$ is taken. An argument similar to the one leading to eq. (5-67) shows that as $\eta_{1+} - y$ changes from $\eta_{1+} - y \gg \varepsilon$ through zero to $y - \eta_{1+} \gg \varepsilon$, $g_-(\tau)$ describes the path about $g_+(\tau)$ shown in fig. 5-8. Again P is not pinched between $g_+(\tau)$ and $g_-(\tau)$ for $\varepsilon > 0$. Once $y > \eta_{1+}$, we can take the limit $\varepsilon \downarrow 0$ and with the branch cuts of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of $[-F(\xi, \tau)]^{\frac{1}{2}}$, as shown in broken line in fig. 5-8, it follows that

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + 2 \int_{g_-(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \\ + 2 \int_{g_+(y+i\cdot 0)}^{g_-(y+i\cdot 0)} \frac{d\xi}{\xi - \sigma} \frac{\pi i}{\sqrt{F(\xi, y)}} \quad (5-69)$$

for $\eta_{1+} < y < 1$.

The continuation into the region $1 < y < f_{\pm}(\xi_{1+})$ is again carried out with $\tau = y + i\varepsilon$ and then taking the limit $\varepsilon \downarrow 0$. When $y \approx 1$, we see from eq. (4-36) and the fact that for this subcase $(a+c)(b+d) > 0$, that when $\tau = y + i\varepsilon$ with ε small and > 0 ,

$$g_+(\tau) \approx \frac{y-1-i\varepsilon}{(y-1)^2 + \varepsilon^2} (a+c)(b+d). \quad (5-70)$$

Fig. 5-8.

Thus, as $1 - y$ changes from $1 - y \gg \epsilon$ through zero to $y - 1 \gg \epsilon$, $g_+(\tau)$ moves in a large anticlockwise arc in the lower half plane from the situation shown in fig. 5-8 to that shown in fig. 5-9. It is necessary to deform the path of integration P in such a way that at no stage does $g_+(\tau)$ pass through it. Notice that for $\epsilon > 0$, $g_+(\tau)$ is always finite. Once $y > 1$, we can take the limit $\epsilon \downarrow 0$. The branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ is as shown in broken line in fig. 5-9, and the path P defining $I(\sigma, \tau)$ in eq. (5-64) is also shown in fig. 5-9. Thus, using the fact that

$$[(1 - (y + i \cdot 0))(1 + y + i \cdot 0)]^{\frac{1}{2}} = -i \sqrt{y-1} \sqrt{1+y} \quad (5-71)$$

when $y > 1$, we find that $I_S(\sigma, y + i \cdot 0)$ is again given by eq. (5-66) for $1 < y < f_{\pm}(\xi_{1+})$.

To obtain $I_S(\sigma, y + i \cdot 0)$ when $y > f_{\pm}(\xi_{1+})$ or, since $a + b > 0$ for case II S (i), when $h_1(y) > 0$, we need to examine the path taken by $g_+(\tau)$ in the complex plane, when $\tau = y + i\epsilon$ and ϵ is small and $\epsilon > 0$. Now, $g_+(y) - \xi_{1+}$ is given by eq. (4-A7) with $\chi_1(y)$ and $\chi_2(y)$ defined by eq. (4-A6). We also define the quantities $\chi_3(y)$ and $\chi_4(y)$ by

$$\chi_3(y) = \text{arc sinh} \left[\frac{yc+a}{\sqrt{y^2-1}\sqrt{1-c^2}} \right], \quad \chi_4(y) = \text{arc sinh} \left[\frac{yd+b}{\sqrt{y^2-1}\sqrt{1-d^2}} \right]. \quad (5-72)$$

The quantities $\chi_1(\tau)$, $\chi_2(\tau)$ are given by eq. (4-A6), with $y \rightarrow \tau$ and the square roots being interpreted more generally. Thus, $g_+(\tau) - \xi_{1+}$ is given by eq. (4-A7) with $y \rightarrow \tau$. For small $\epsilon \geq 0$, we find that

Fig. 5-9.

$$\operatorname{Re}\{\chi_1(\tau)+\chi_2(\tau)\} = \chi_1(y) + \chi_2(y) + \varepsilon^2 C(\varepsilon, y, a, b, c, d), \quad (5-73)$$

$$\begin{aligned} \operatorname{Im}\{\chi_1(\tau)+\chi_2(\tau)\} &= \frac{-\varepsilon}{y^2-1} \{\tanh\chi_3(y)+\tanh\chi_4(y)\} \\ &\quad + \varepsilon^3 D(\varepsilon, y, a, b, c, d) \end{aligned} \quad (5-74)$$

where $|C(\varepsilon, y, a, b, c, d)| < \infty$, $|D(\varepsilon, y, a, b, c, d)| < \infty$ for $y > 1$.

Now since

$$\cosh\theta = \cosh(\operatorname{Re}\theta)\cos(\operatorname{Im}\theta) + i\sinh(\operatorname{Re}\theta)\sin(\operatorname{Im}\theta), \quad (5-75)$$

we find that

$$\begin{aligned} g_+(\tau) - \xi_{1+} &= \sqrt{1-a^2} \sqrt{1-b^2} [-1 + \cosh\{\operatorname{Re}\{\chi_1(\tau)+\chi_2(\tau)\}\} \cdot \\ &\quad \cdot \cos\{\operatorname{Im}\{\chi_1(\tau)+\chi_2(\tau)\}\} \\ &\quad + i\sinh\{\operatorname{Re}\{\chi_1(\tau)+\chi_2(\tau)\}\}\sin\{\operatorname{Im}\{\chi_1(\tau)+\chi_2(\tau)\}\}] \end{aligned} \quad (5-76)$$

Notice that, since $(a+b) > 0$, $y < f_{\pm}(\xi_{1+})$ if and only if $h_1(y) < 0$.

The steps in the continuation are then as follows.

(1) Start with y such that $h_1(y) < 0$, that is,

$$\chi_1(y) + \chi_2(y) < 0,$$

(2) make $\varepsilon > 0$,

(3) fix ε and change y until $h_1(y) > 0$, that is,

$$\chi_1(y) + \chi_2(y) > 0,$$

(4) let $\varepsilon \downarrow 0$.

As mentioned earlier, and as can be seen from the facts that $f_{\pm}(\xi_{2+}) < 1$

and $c+d < 0$, $h_2(y) < 0$ throughout the continuation procedure.

Thus,

$$\sinh\chi_3(y) + \sinh\chi_4(y) < 0$$

and

$$\tanh \chi_3(y) + \tanh \chi_4(y) < 0. \quad (5-77)$$

From eqs. (5-73), (5-74) and (5-77), we see that for sufficiently small ϵ , $\{\text{Im}(\chi_1(\tau) + \chi_2(\tau))\}$ is small and positive at steps (2) and (3), and that at step (2),

$$\text{sign}\{\text{Re}(\chi_1(\tau) + \chi_2(\tau))\} = \text{sign}\{\chi_1(y) + \chi_2(y)\}.$$

Similarly, when step (3) has been taken, the above equation holds for sufficiently small ϵ .

It now follows that as steps (1)-(4) are taken, the path taken by $g_+(\tau)$ in the complex plane is as shown in fig. 5-10. Once $g_+(\tau)$ crosses the real axis between ξ_{2+} and ξ_{1+} and moves into the upper half plane, the path of integration P shown in fig. 5-9, can be deformed into the segment of the real axis between ξ_{2+} and ξ_{1+} as shown in fig. 5-11. Notice, from the expression for $g_-(y) - \xi_{1-}$ given in appendix 4A, and from eqs. (4-A8) and (4-A9), that for sufficiently small ϵ , $\text{Reg}_-(\tau) < \xi_\gamma < \text{Reg}_+(\tau)$. Taking the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ as shown in fig. 5-11, and using eq. (5-71), we find that

$$\begin{aligned} I_S(\sigma, y+i\cdot 0) &= - \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \\ &= \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + 2 \int_{\xi_{1+}}^{g_+(y+i\cdot 0)} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \\ &\quad + 2 \int_{g_+(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}}. \end{aligned} \quad (5-78)$$

The expressions obtained for $I_S(\sigma, y+i\cdot 0)$ for subcase (2a)(α),

Fig. 5-10.

Fig. 5-11.

where $(a+b)(c+d) < 0$, $(a+c)(b+d) > 0$ and eq. (5-51) holds, can now be summarized. We find that

$$\begin{aligned}
 I_S(\sigma, y+i\cdot 0) = & \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} \\
 & + 2\theta(-h_2(y)) \int_{g_+(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} \\
 & + 2\theta(h_1(y)) \int_{\xi_{1+}}^{g_+(y+i\cdot 0)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} \quad (5-79)
 \end{aligned}$$

for all $-\infty < y < \infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$. In eq.

(5-79),

$$\theta(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x \leq 0 \end{cases} \quad (5-80)$$

and

$$\{-F(\xi, y)\}^{\frac{1}{2}} = \begin{cases} \sqrt{-F(\xi, y)} & \text{when } F(\xi, y) \leq 0, \\ -i\sqrt{F(\xi, y)} & \text{when } F(\xi, y) > 0, \\ [-F(\xi, y)]^{\frac{1}{2}} & \text{when } F(\xi, y) \text{ is complex.} \end{cases} \quad (5-81)$$

When $g_+(y+i\cdot 0)$ lies in the lower half plane and $F(\xi, y)$ is complex, then the path of integration and the square root of $\{-F(\xi, y)\}$ are to be determined as described after eq. (5-68). The quantities $h_1(y)$ and $h_2(y)$ are given in eqs. (4-38) and (4-47), and $g_+(y+i\cdot 0)$ is given in eq. (5-44) with the square root interpreted as in eq. (5-47).

Next, we consider the subcase (2b)(a) of case II S (i) for which $(a+c)(b+d) < 0$ in addition to $(a+b)(c+d) < 0$ and for which eq. (5-51) holds. The branch Γ_1 is then as shown in fig. 5-4. We recall from

sect. 4-8 that the exact ordering of the points 1 , $g_+(1)$ and $g_\pm(\eta_\alpha)$ and of 1 , $f_+(1)$ and $f_\pm(\xi_\alpha)$ depends on the values of a , b , c and d . As for subcase (2a)(α), the branches Γ_2 , Γ_3 and Γ_4 can have a number of different configurations. One of the four different configurations of Γ is shown in fig. 5-4.

When $y < \eta_{1+}$, the arguments and expressions for $I_S(\sigma, y+i\cdot 0)$ given for subcase (2a)(α) apply. We now obtain $I_S(\sigma, y+i\cdot 0)$ when $\eta_{1+} < y < \min[1, f_\pm(\xi_{1+})]$. The continuation is carried out in $\tau = y + i\varepsilon$ with ε small and > 0 and then the limit $\varepsilon \downarrow 0$ is taken. An argument similar to the one leading to eq. (5-67) shows that as $\eta_{1+} - y$ changes from $\eta_{1+} - y \gg \varepsilon$ through zero to $y - \eta_{1+} \gg \varepsilon$, the motion of $g_-(\tau)$ relative to $g_+(\tau)$ is as shown in fig. 5-12, and the path P is not pinched between the singularities since $\varepsilon > 0$. Once $y > \eta_{1+}$, we take the limit $\varepsilon \downarrow 0$. With the path P as shown in fig. 5-12 and the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of $[-F(\xi, \tau)]^{\frac{1}{2}}$ as shown in the broken line in fig. 5-12, it follows that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) for $\eta_{1+} < y < \min[1, f_\pm(\xi_{1+})]$.

For the case where $f_\pm(\xi_{1+}) > 1$ we again carry out the continuation into the region $1 < y < f_\pm(\xi_{1+})$ in $\tau = y + i\varepsilon$ and then let $\varepsilon \downarrow 0$. When $y \approx 1$, we see from eq. (4-36) and the fact that for this subcase $(a+c)(b+d) < 0$, that $g_-(\tau)$ is approximated by the right hand side of eq. (5-70) when $\varepsilon > 0$. Consequently, as $1 - y$ changes from $1 - y \gg \varepsilon$ through zero to $y - 1 \gg \varepsilon$, $g_-(\tau)$ moves in a

Fig. 5-12.

large anticlockwise arc in the upper half plane. Once $y > 1$, we take the limit $\epsilon \downarrow 0$ and, with the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and the path P as shown in fig. 5-9, we find on using eq. (5-71), that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) for $1 < y < f_{\pm}(\xi_{1+})$. Using the argument that led to eq. (5-78) for subcase (2a)(α), we find that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-78) for $y > f_{\pm}(\xi_{1+})$.

For the case where $f_{\pm}(\xi_{1+}) < 1$, the continuation into the region $f_{\pm}(\xi_{1+}) < y < 1$ is again carried out in $\tau = y + i\epsilon$ and then letting $\epsilon \downarrow 0$. We find that $g_{\pm}(\tau) - \xi_{1+}$ are given in eq. (4-A12) with $y \rightarrow \tau$ and $\kappa_1(\tau)$, $\kappa_2(\tau)$ are given by eq. (4-A11) with $y \rightarrow \tau$ in the case $(a+c) > 0$, $(b+d) < 0$. When $(a+c) < 0$, $(b+d) > 0$, the right hand sides of the equations defining $\kappa_1(\tau)$ and $\kappa_2(\tau)$ are both multiplied by a minus sign. On using eq. (5-75) we find that as the steps

- (1) start with y such that $h_1(y) < 0$,
- (2) make $\epsilon > 0$,
- (3) fix ϵ and change y until $h_1(y) > 0$,
- (4) let $\epsilon \downarrow 0$,

are taken, $g_+(\tau)$ moves in the complex plane as shown in fig. 5-10, for sufficiently small ϵ . We illustrate the argument leading to this behaviour of $g_+(\tau)$, when we consider the behaviour of $g_-(\tau)$ in subcase (2a)(β). Throughout the continuation procedure, $\text{Reg}_-(\tau) > \max[\xi_{1+}, \text{Reg}_+(\tau)]$ for sufficiently small ϵ . At step (4), the path of integration can be deformed to the segment of the real axis from ξ_{2+} to ξ_{1+} as shown in fig. 5-11. The branch cut of

$[(\xi - g_+(y))(\xi - g_-(y))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of $[-F(\xi, y)]^{\frac{1}{2}}$, is chosen to lie between $g_+(y)$ and $g_-(y)$ along the real axis where $\xi_{1+} < g_+(y) < g_-(y)$. (Note that in this case the branch cut is not as shown in fig. 5-11.) Thus $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-78) for $f_{\pm}(\xi_{1+}) < y < 1$. When $y \approx 1$, $g_-(\tau)$ is approximated by the right hand side of eq. (5-70), when $\tau = y + i\epsilon$ and ϵ is small and > 0 . Thus, as $1 - y$ changes from $1 - y \gg \epsilon$ through zero to $y - 1 \gg \epsilon$, $g_-(\tau)$ moves in a large anticlockwise arc in the upper half plane. For $y > 1$, the path P and the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ are as shown in fig. 5-11, and using eq. (5-71) we find that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-78) for $y > 1$.

Finally, we see that the expressions obtained for $I(\sigma, y+i\cdot 0)$ for subcase (2b)(α) can also be summarized as in eqs. (5-79), (5-80) and (5-81). (Notice that for this subcase, $F(\xi, y) \leq 0$ or $F(\xi, y)$ is complex.) Thus, for case II S (i), $I_S(\sigma, y+i\cdot 0)$ is given by eqs. (5-79), (5-80) and (5-81) whenever eq. (5-51) holds.

We now consider subcase (2a)(β), for which $(a+c)(b+d) > 0$ in addition to $(a+b)(c+d) < 0$ and for which eq. (5-52) holds. From fig. 5-3 (and the three similar figures which can be constructed as described for subcase (2a)(α)), we see that provided $y < \eta_{2+}$, the singularities $g_{\pm}(y+i\cdot 0)$ do not appear on the path of integration in eq. (5-55), and $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-55) with $x \rightarrow \sigma$. From eqs. (5-48) and (5-52) we see that when $\eta_{2-} < y < \eta_{2+}$, $g_{\pm}(y+i\cdot 0)$ are complex conjugates with $g_+(y+i\cdot 0)$ lying in the upper

half plane, and $g_-(y+i\cdot 0)$ lying in the lower half plane. As

$y \uparrow \eta_{2+}$, $g_+(y+i\cdot 0)$ approaches the point $g_+(\eta_{2+})$ from the upper

half plane and $g_-(y+i\cdot 0)$ approaches $g_-(\eta_{2+})$ from the lower half

plane, where from fig. 5-3, we see that $\xi_{2+} < g_+(\eta_{2+}) < \xi_{1-}$. Thus,

in order to continue into the region $y > \eta_{2+}$, we must do the

continuation in $\tau = y + i\varepsilon$ with ε small and > 0 . An argument

similar to the one leading to eq. (5-67) shows that, as $\eta_{2+} - y$

changes from $\eta_{2+} - y \gg \varepsilon$ through zero to $y - \eta_{2+} \gg \varepsilon$, the motion

of $g_+(\tau)$ relative to $g_-(\tau)$ is as shown in fig. 5-13. Thus, the

path P is not pinched between the two singularities $g_{\pm}(\tau)$ when

$\varepsilon > 0$. Once $y > \eta_{2+}$, we can take the limit $\varepsilon \downarrow 0$ and with the

branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of

$[-F(\xi, \tau)]^{\frac{1}{2}}$, as shown in broken line in fig. 5-13, we find that

$$I_S(\sigma, y+i\cdot 0) = \int_{g_+(y+i\cdot 0)}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + \int_{g_-(y+i\cdot 0)}^{g_+(y+i\cdot 0)} \frac{d\xi}{\xi-\sigma} \frac{\pi i}{\sqrt{F(\xi, y)}} \\ + \int_{\xi_{2+}}^{g_-(y+i\cdot 0)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \quad (5-82)$$

for $\eta_{2+} < y < f_{\pm}(\xi_{2+})$.

To obtain $I_S(\sigma, y+i\cdot 0)$ when $f_{\pm}(\xi_{2+}) < y < \eta_{1-}$, we need to

examine the path taken by $g_-(\tau)$ in the complex plane when $\tau = y + i\varepsilon$

and ε is small and > 0 . From eqs. (5-44), (5-47) and (4-42), we

see that when $\eta_{2+} < y < \eta_{1-}$,

Fig. 5-13.

$$g_-(y+i\cdot 0) - \xi_{2+} = \sqrt{1-c^2} \sqrt{1-d^2} \left[-1 + \frac{(yc+a)(yd+b)}{(y^2-1)\sqrt{1-c^2}\sqrt{1-d^2}} \right. \\ \left. - \sqrt{\frac{(yc+a)^2}{(1-y^2)(1-c^2)}} - 1 \sqrt{\frac{(yd+b)^2}{(1-y^2)(1-d^2)}} - 1 \right] . \quad (5-83)$$

We note from eq. (4-47) that when $h_2(y) = 0$, then $(yc+a)(yd+b) < 0$

and since

$$(yc+a)^2 - (1-y^2)(1-c^2) > 0, \quad (yd+b)^2 - (1-y^2)(1-d^2) > 0$$

for $\eta_{2+} < y < \eta_{1-}$, it follows that $(yc+a)(yd+b) < 0$ for

$\eta_{2+} < y < \eta_{1-}$. Suppose that $yc+a > 0$, $yd+b < 0$; then we

define $\kappa_3(y)$ and $\kappa_4(y)$ by

$$\cosh \kappa_3(y) = \frac{yc+a}{\sqrt{1-y^2}\sqrt{1-c^2}}, \quad \cosh \kappa_4(y) = -\frac{yd+b}{\sqrt{1-y^2}\sqrt{1-d^2}}. \quad (5-84)$$

Now

$$g_-(\tau) - \xi_{2+} = \sqrt{1-c^2}\sqrt{1-d^2} [-1 + \cosh(\kappa_3(\tau) - \kappa_4(\tau))] \quad (5-85)$$

where $\kappa_3(\tau)$, $\kappa_4(\tau)$ are given in eq. (5-84), with $y \rightarrow \tau$ and the

square root interpreted more generally. With $\tau = y + i\epsilon$, we find

that for small $\epsilon \geq 0$,

$$\operatorname{Re}(\kappa_3(\tau) - \kappa_4(\tau)) = \kappa_3(y) - \kappa_4(y) + \epsilon^2 E(\epsilon, y, a, b, c, d), \quad (5-86)$$

$$\operatorname{Im}(\kappa_3(\tau) - \kappa_4(\tau))$$

$$= \frac{\epsilon}{1-y^2} \left[\frac{ya+c}{\sqrt{(ya+c)^2 - (1-y^2)(1-a^2)}} + \frac{yb+d}{\sqrt{(yb+d)^2 - (1-y^2)(1-b^2)}} \right] \\ + \epsilon^3 F(\epsilon, y, a, b, c, d) \quad (5-87)$$

where $|E(\epsilon, y, a, b, c, d)| < \infty$, $|F(\epsilon, y, a, b, c, d)| < \infty$ for

$\eta_{2+} < y < \eta_{1-}$.

Now, using eq. (5-75), we find that

$$g_-(\tau) - \xi_{2+} = \sqrt{1-c^2} \sqrt{1-d^2} [-1 + \cosh\{\operatorname{Re}\{\kappa_3(\tau) - \kappa_4(\tau)\}\} \cdot \cos\{\operatorname{Im}\{\kappa_3(\tau) - \kappa_4(\tau)\}\} + i \sinh\{\operatorname{Re}\{\kappa_3(\tau) - \kappa_4(\tau)\}\} \sin\{\operatorname{Im}\{\kappa_3(\tau) - \kappa_4(\tau)\}\}] \quad (5-88)$$

The steps in the continuation are then as follows.

- (i) Start with y such that $y < f_{\pm}(\xi_{2+})$ ($h_2(y) > 0$),
that is, $\kappa_3(y) - \kappa_4(y) > 0$,
- (ii) make $\epsilon > 0$,
- (iii) fix ϵ and change y until $y > f_{\pm}(\xi_{2+})$ ($h_2(y) < 0$),
that is, $\kappa_3(y) - \kappa_4(y) < 0$,
- (iv) let $\epsilon \downarrow 0$.

Since $y < f_{\pm}(\xi_{1+})$, and hence $h_1(y) < 0$, throughout the continuation procedure it follows that

$$\frac{ya+c}{\sqrt{(ya+c)^2 - (1-y^2)(1-a^2)}} + \frac{yb+d}{\sqrt{(yb+d)^2 - (1-y^2)(1-b^2)}} < 0 \quad (5-89)$$

From eqs. (5-86), (5-87) and (5-89), we see that for sufficiently small ϵ , $\{\operatorname{Im}\{\kappa_3(\tau) - \kappa_4(\tau)\}\}$ is small and negative at steps (ii) and (iii), and that at step (ii)

$$\operatorname{sign}\{\operatorname{Re}\{\kappa_3(\tau) - \kappa_4(\tau)\}\} = \operatorname{sign}\{\kappa_3(y) - \kappa_4(y)\}.$$

Similarly, when step (iii) has been taken, the above equation holds for sufficiently small ϵ .

It now follows from eq. (5-88) that, as steps (i)-(iv) are taken, the path taken by $g_-(\tau)$ in the complex plane is as shown in fig.

5-14. Also, from eqs. (5-44), (5-47) and (4-42), (see also fig. 5-3) we see that for sufficiently small ϵ , $\operatorname{Reg}_+(\tau) > \max\{\xi_{2+}, \operatorname{Reg}_-(\tau)\}$

Fig. 5-14.

for $\eta_{2+} < y < \eta_{1-}$. At step (iii), the path P and the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ and, since $1 - y^2 > 0$, of $[-F(\xi, \tau)]^{\frac{1}{2}}$, are as shown in fig. 5-15. Consequently,

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{g_-(y+i\cdot 0)} \frac{d\xi}{\xi-\sigma} \frac{(-\pi)}{\sqrt{-F(\xi, y)}} + \int_{g_-(y+i\cdot 0)}^{g_+(y+i\cdot 0)} \frac{d\xi}{\xi-\sigma} \frac{\pi i}{\sqrt{F(\xi, y)}} + \int_{g_+(y)}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \quad (5-90)$$

for $f_{\pm}(\xi_{2+}) < y < \eta_{1-}$. Notice that $I_S(\sigma, y+i\cdot 0)$ can also be written in the form

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} + 2 \int_{g_-(y+i\cdot 0)}^{\xi_{2+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} \quad (5-91)$$

where $\{-F(\xi, y)\}^{\frac{1}{2}}$ is defined in eq. (5-81). (In this case $F(\xi, y)$ is, of course, real.)

On continuing into the region $\eta_{1-} < y < \eta_{1+}$, $g_+(y+i\cdot 0)$ moves into the lower half plane, deforming the path P as shown in fig. 5-7 and $g_-(y+i\cdot 0)$ moves into the upper half plane. We find that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-68) for $\eta_{1-} < y < \eta_{1+}$. To obtain $I_S(\sigma, y+i\cdot 0)$ for $y > \eta_{1+}$, we proceed as for subcase (2a)(α) and the results given for that subcase again apply.

We see from eq. (5-91) that our choice of square roots in eq. (5-47) and the consequent labelling of $g_+(y+i\cdot 0)$ and $g_-(y+i\cdot 0)$ makes eq. (5-79) unsuitable for summarizing the results for subcase (2a)(β) but suits subcases (2a)(α) and (2b)(α). However, by choosing the square roots differently, we can summarize our results in a way similar

Fig. 5-15.

to that given in eq. (5-79). We define the quantities $g_{\pm}(y)$ by

$$g_{\pm}(y) = \{y^2 - 1\}^{-1} [(y-1)(ad+bc) + (a+c)(b+d) \pm \{y^2 - 1 + 2(y-1)ac + (a+c)^2\}^{\frac{1}{2}} \{y^2 - 1 + 2(y-1)bd + (b+d)^2\}^{\frac{1}{2}}] \quad (5-92)$$

where

$$\{y^2 - 1 + 2(y-1)ac + (a+c)^2\}^{\frac{1}{2}} = \begin{cases} \sqrt{y^2 - 1 + 2(y-1)ac + (a+c)^2} & \text{if } y^2 - 1 + 2(y-1)ac + (a+c)^2 \geq 0, \\ i\sqrt{-(y^2 - 1 + 2(y-1)ac + (a+c)^2)} & \text{if } y^2 - 1 + 2(y-1)ac + (a+c)^2 < 0 \end{cases} \quad (5-93)$$

and the other square root in eq. (5-92) is defined in the same way

with $a \rightarrow b$, $c \rightarrow d$. The quantities $g_{\pm}(y)$ are just the quantities

$g_{\pm}(y)$ but with a different choice of square roots. The expressions obtained for $I_S(\sigma, y+i\cdot 0)$ for subcase (2a)(β), as well as for subcases (2a)(α) and (2b)(α), can now be summarized. We find that

$$I_S(\sigma, y+i\cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} + 2\theta(-h_2(y)) \int_{g_1(y)}^{\xi_{2+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} + 2\theta(h_1(y)) \int_{\xi_{1+}}^{g_1(y)} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\{-F(\xi, y)\}^{\frac{1}{2}}} \quad (5-94)$$

for all $-\infty < y < \infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$. Here

$\{-F(\xi, y)\}^{\frac{1}{2}}$ is defined in eq. (5-81), $\theta(x)$ is defined in eq.

(5-80) and $h_1(y)$ and $h_2(y)$ are given in eqs. (4-38) and (4-47).

Next, we consider subcase (2b)(β), for which $(a+c)(b+d) < 0$ in addition to $(a+b)(c+d) < 0$ and for which eq. (5-52) holds. To obtain

$I_S(\sigma, y+i\cdot 0)$ for $y < \eta_{1+}$, we proceed as for subcase (2a)(β) and the results given for that subcase again apply. Similarly to obtain $I_S(\sigma, y+i\cdot 0)$ for $y > \eta_{1+}$, we proceed as for subcase (2b)(α) and the results for that subcase apply. Thus, $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-94).

The restriction $\operatorname{Re} \sigma \ll -1$ and the restrictions on $\eta_{1\pm}, \eta_{2\pm}$ in eqs. (5-51) and (5-52) can now be removed, and we find that for case II S (i), defined by $(a+b) > 0$, $(c+d) < 0$, $-1 < a \leq 1$, $-1 < b \leq 1$ and $\alpha + \beta + \gamma + \delta > 2\pi$, and for which $\xi_{2+} < \xi_{1+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) for $\sigma \in S_\sigma$, $-\infty < y < +\infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$. For case II S (ii) defined by $(a+b) < 0$, $(c+d) > 0$, $-1 < c \leq 1$, $-1 < d \leq 1$ and $\alpha + \beta + \gamma + \delta > 2\pi$, and for which $\xi_{1+} < \xi_{2+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) with $\xi_{1+} \leftrightarrow \xi_{2+}$, $h_1(y) \leftrightarrow h_2(y)$, for $\sigma \in S_\sigma$, $-\infty < y < +\infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$.

5-9. Continuation in x and y : Case III

In this section, we obtain $I_S(\sigma, y+i\cdot 0)$ for case III, defined by $(a+b) < 0$, $(c+d) < 0$, for all $\sigma \in S_\sigma$ and for almost all $-\infty < y < +\infty$. As discussed in sect. 5-8, $I_S(x, y)$ is given by eq. (5-55) for case III (ii) when $x < -1$, $y < -1$. We recall that case III (ii) is defined by $(a+b) < 0$, $(c+d) < 0$, $h_1(y) \leq 0$ and $h_2(y) > 0$, and that for this case $\xi_{2+} < \xi_{1+}$ (eq. (5-29)). Also, case III (ii) is a subcase of case III A for which $a + b < 0$, $c + d < 0$ and $\xi_{2+} < \xi_{1+}$.

We begin by considering the subcase of case III A for which $(a+c)(b+d) > 0$ in addition to $(a+b)(c+d) > 0$ and for which eq. (5-51) holds. This subcase we call subcase (1a)(α). The branch Γ_1 of Γ is then as shown in fig. 5-1. The branches Γ_2 , Γ_3 and Γ_4 can again have a number of different configurations, and one of the four possible configurations of Γ is shown in fig. 5-1.

To simplify the continuation procedure, we again take $\sigma \in S_\sigma$ and $\text{Re}\sigma = x \ll -1$. Firstly we suppose that $(a-c)(b-d) > 0$ so that Γ_2 is as shown in fig. 5-1. From fig. 5-1 we see that $f_\pm(\xi_{1+}) < -1 < f_\pm(\xi_{2+})$ and thus $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-55) with $x \rightarrow \sigma$ for $f_\pm(\xi_{1+}) < y < -1$.

To obtain $I_S(\sigma, y+i\cdot 0)$ when $y < f_\pm(\xi_{1+})$ or since $a+b < 0$, when $h_1(y) > 0$, we need to examine the path taken by $g_+(\tau)$ in the complex plane when $\tau = y + i\varepsilon$ and ε is small and > 0 . We find that $g_+(\tau) - \xi_{1+}$ is again given by eqs. (5-76), (5-73) and (5-74). However, in this case, $y < f_\pm(\xi_{2+})$ ($h_2(y) > 0$) and hence, instead of eq. (5-77), we have

$$\tanh\chi_3(y) + \tanh\chi_4(y) > 0.$$

Thus, as steps (1), (2), (3) and (4) after eq. (5-76) are taken, $g_+(\tau)$ moves from point (4) to point (3) to point (2) to point (1) in fig. 5-10. Also, for sufficiently small ε , we see from eqs. (4-A8) and (4-A9) that $\text{Reg}_-(\tau) < \xi_Y < \text{Reg}_+(\tau)$. The path of integration between ξ_{2+} and ξ_{1+} along the real axis is deformed before $g_+(\tau)$ crosses into the lower half plane. At step (3) of the continuation

procedure when $g_+(\tau)$ is at point (2) in fig. 5-10, the path P is as shown in fig. 5-9. Taking the branch cut of $[(\xi - g_+(\tau))(\xi - g_-(\tau))]^{\frac{1}{2}}$ as shown in fig. 5-9, and using the fact that

$$[(1 - (y + i \cdot 0))(1 + y + i \cdot 0)]^{\frac{1}{2}} = i\sqrt{-(1+y)}\sqrt{1-y},$$

we find that

$$I_S(\sigma, y + i \cdot 0) = \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + 2 \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{\xi - \sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} \quad (5-95)$$

for $-\infty < y < f_{\pm}(\xi_{1+})$. Thus, $I_S(\sigma, y + i \cdot 0)$ is given by eq. (5-95)

when $h_1(y) > 0$, $h_2(y) > 0$ and $\xi_{2+} < \xi_{1+}$. This is just the subcase of case III (i) for which $\xi_{2+} < \xi_{1+}$. In fact on using eq. (5-95) (with $\sigma \rightarrow x < \xi_{2+}$) and eq. (5-54) in eq. (5-53), we see that the resultant expression for $I(x, y)$ is just that given in eq. (4-118) when $\xi_{2+} < \xi_{1+}$.

Next, we continue from the region $f_{\pm}(\xi_{1+}) < y < -1$ into the region $-1 < y < +\infty$. The method used and the results obtained for $I_S(\sigma, y + i \cdot 0)$ are just those given for case II S (i) (2a)(\alpha), except that now $f_{\pm}(\xi_{1+}) \neq 1$, so that $I_S(\sigma, y + i \cdot 0)$ is given by eq. (5-66) for $1 < y < +\infty$, rather than just for $1 < y < f_{\pm}(\xi_{1+})$. Thus, $I_S(\sigma, y + i \cdot 0)$ is given by eq. (5-55) with $x \rightarrow \sigma$ for $f_{\pm}(\xi_{1+}) < y < f_{\pm}(\xi_{2+})$, by eq. (5-66) for $f_{\pm}(\xi_{2+}) < y < \eta_{2+}$, by eq. (5-68) for $\eta_{2+} < y < \eta_{1+}$, by eq. (5-69) for $\eta_{1+} < y < 1$ and by eq. (5-66) for $1 < y < +\infty$.

Next we suppose that $(a-c)(b-d) < 0$, so that the branch Γ_2 is as shown in fig. 5-2. We note that in this case it is possible to

have $f_{\pm}(\xi_{1+}) < -1$ or $-1 \leq f_{\pm}(\xi_{1+}) < \eta_{\delta}$. If $f_{\pm}(\xi_{1+}) < -1$, then the continuation procedure is just that given for subcase (1a)(α) above, where in addition $(a-c)(b-d) > 0$. However, if $-1 \leq f_{\pm}(\xi_{1+}) < \eta_{\delta}$, then we need to start with $I_S(\sigma, y+i\cdot 0)$ for a case when $f_{\pm}(\xi_{1+}) < -1$ and continue in the external masses squared. We note that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-55), with $x \rightarrow \sigma$, for $f_{\pm}(\xi_{1+}) < y < f_{\pm}(\xi_{2+})$, and in particular for $f_{\pm}(\xi_{1+}) < y < \eta_Y$, when $f_{\pm}(\xi_{1+}) < -1$. On continuing in the external masses squared, we find that $I_S(\sigma, y+i\cdot 0)$ is still given by eq. (5-55), with $x \rightarrow \sigma$, for $f_{\pm}(\xi_{1+}) < y < \eta_Y$ when $-1 \leq f_{\pm}(\xi_{1+}) < \eta_{\delta}$.

The method of continuation into the region $-1 < y < f_{\pm}(\xi_{1+})$ is then similar to that described for the continuation into the region $f_{\pm}(\xi_{1+}) < y < 1$ for case II S (i) (2b)(α). As steps (1), (2), (3) and (4) are taken, $g_+(\tau)$ moves from point (4) to point (3) to point (2) to point (1) in fig. 5-10, and $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-95). Continuing into the region $-\infty < y \leq -1$, we find that $I_S(\sigma, y+i\cdot 0)$ is still given by eq. (5-95). The continuation into the region $\eta_Y < y < +\infty$ is then the same as described for subcase (1a)(α) when in addition $(a-c)(b-d) > 0$. Thus, we find that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-79) and by eq. (5-94) for subcase (1a)(α) of case III A.

We now consider subcase (1b)(α) for which $(a+c)(b+d) < 0$ in addition to $(a+b)(c+d) > 0$ and for which eq. (5-51) holds. The branch Γ_1 is then as shown in fig. 5-2. The arguments of subcase

(1a)(α) show that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-55) with $x \rightarrow \sigma$ for $f_{\pm}(\xi_{1+}) < y < f_{\pm}(\xi_{2+})$ and by eq. (5-95) for $-\infty < y < f_{\pm}(\xi_{1+})$. The method of continuation into the region $f_{\pm}(\xi_{2+}) < y < +\infty$ is the same as that used for case II S (i) (2b)(α), except that now $f_{\pm}(\xi_{1+}) \nmid \eta_{1+}$, so that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) for $\eta_{1+} < y < +\infty$. For $f_{\pm}(\xi_{2+}) < y < \eta_{1+}$, the expressions for $I_S(\sigma, y+i\cdot 0)$ are just those given for case II S (i) (2b)(α). Thus $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-79) and by eq. (5-94) for subcase (1b)(α).

For subcase (1a)(β), where $(a+c)(b+d) > 0$, $(a+b)(c+d) > 0$ and for which eq. (5-52) holds, Γ_1 is as shown in fig. 5-1. The arguments of subcase (1a)(α) show that $I_S(\sigma, y+i\cdot 0)$ is again given by eq. (5-55) with $x \rightarrow \sigma$ for $f_{\pm}(\xi_{1+}) < y < \eta_{2+}$ and by eq. (5-95) for $-\infty < y < f_{\pm}(\xi_{1+})$. The method of continuation into the region $\eta_{2+} < y < +\infty$ and the results obtained for $I_S(\sigma, y+i\cdot 0)$ are just those given for case II S (i) (2a)(β), except that now $f_{\pm}(\xi_{1+}) \nmid 1$, so that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) for $1 < y < +\infty$ instead of just $1 < y < f_{\pm}(\xi_{1+})$. Again, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) for subcase (1a)(β).

For subcase (1b)(β), where $(a+c)(b+d) < 0$, $(a+b)(c+d) > 0$ and for which eq. (5-52) holds, Γ_1 is as shown in fig. 5-2. The arguments of subcase (1a)(α) again show that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-55) with $x \rightarrow \sigma$ for $f_{\pm}(\xi_{1+}) < y < \eta_{2+}$ and by eq. (5-95)

for $-\infty < y < f_{\pm}(\xi_{1+})$. The method of continuation into the region $\eta_{2+} < y < +\infty$ and the results obtained for $I_S(\sigma, y+i\cdot 0)$ are just those given for case II S (i) (2b)(β), except that now $f_{\pm}(\xi_{1+}) \neq \eta_{1+}$, so that $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) for $\eta_{1+} < y < +\infty$ instead of just $\eta_{1+} < y < f_{\pm}(\xi_{1+})$. Again, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) for subcase (1b)(β).

The restriction $\text{Re}\sigma \ll -1$ and the restrictions on $\eta_{1\pm}, \eta_{2\pm}$ in eqs. (5-51) and (5-52) can now be removed, and we find that for case III A defined by $(a+b) < 0$, $(c+d) < 0$ and $\xi_{2+} < \xi_{1+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) for $\sigma \in S_{\sigma}$, $-\infty < y < +\infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$. For case III B defined by $(a+b) < 0$, $(c+d) < 0$ and $\xi_{1+} < \xi_{2+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-94) with $\xi_{1+} \leftrightarrow \xi_{2+}$, $h_1(y) \leftrightarrow h_2(y)$ for $\sigma \in S_{\sigma}$, $-\infty < y < +\infty$ except at most for $y \in \{\eta_{1+}, \eta_{2+}, 1\}$.

5-10. Conclusion

In this section, we compare the results obtained in this chapter with those given in GDD. As mentioned in sect. 5-1, some of our results disagree with those of GDD. Further, many of the details of the method, in particular details of the movement of singularities at the critical points and the precautions that need to be taken to ensure that the path of integration is not pinched between the singularities, are not given by GDD. They started their continuation from a normal threshold spectral representation for the box diagram amplitude, equivalent to that given in eqs. (5-5) and (5-6). Mandelstam (1959 a) had obtained this spectral representation by using a prescription for calculating

the imaginary part of the box diagram amplitude (in eq. (2-8)). GDD carried out the continuation in the external masses squared to obtain results for the box diagram equivalent to those given for case II in eq. (5-27) (eq. (29) of GDD) and for case III (ii) in eq. (5-32) (eq. (57) of GDD).

We recall that, in sect. 5-8, we wrote $I(x, y)$ as the sum of a double integral part, $I_D(x, y)$, and a single integral part, $I_S(x, y)$, and carried out the continuation, in x and y , of these two parts separately. GDD made essentially the same separation as in eqs. (5-53), (5-54) and (5-55) for case III A but made a different separation for case II S (i). For case II S (i), $I(x, y)$ can also be written as

$$I(x, y) = I_N(x, y) + I_G(x, y) \quad (5-96)$$

where

$$I_N(x, y) = \int_1^\infty \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^\infty \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}}, \quad (5-97)$$

$$I_G(x, y) = I_S(x, y) + \int_{\xi_{1+}}^1 \frac{d\xi}{\xi-x} \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-y} \frac{1}{\sqrt{F(\xi, \eta)}}, \quad (5-98)$$

and $I_S(x, y)$ is given in eq. (5-55). Notice that $I_N(x, y)$ is just the first term on the right hand side of eq. (5-39) and can be continued in x and y as discussed in sect. 5-6. Also, the second term on the right hand side of eq. (5-98) is just the second term on the right hand side of eq. (5-39) (with $\xi_\alpha = \xi_{1+}$) and, thus, can be continued in x and y as discussed in sect. 5-6. In particular, the inner integral in the second term in eq. (5-98) gives, on continuing in y (with y approaching the real axis from above), $Y(\xi, y+i\cdot 0)$ in eq. (5-41).

GDD considered the continuation in x and y of a function $A_{an}(-x, -y)$ related to $I_G(x, y)$, for $y < -1$, $x < -1$, by

$$I_G(x, y) = -\frac{\pi}{2} A_{an}(-x, -y). \quad (5-99)$$

We start by comparing our results with those of GDD for case II S (i) (2a)(α). For $y \leq f_{\pm}(\xi_{2+})$, we see from eqs. (5-55) and (5-41) together with fig. 5-3 (which shows that $y < f_{+}(\xi)$ for $\xi_{1+} \leq \xi \leq 1$) that our result in eq. (5-98) (with $x \rightarrow \sigma$) agrees with that given in eq. (42) of GDD. For $f_{\pm}(\xi_{2+}) < y < \eta_{2+}$, eqs. (5-66) and (5-41) (together with fig. 5-3) show that eq. (5-98) (with $x \rightarrow \sigma$) again agrees with eq. (42) of GDD. For $\eta_{2+} < y < \eta_{1+}$, $I_G(\sigma, y+i\cdot 0)$ can be obtained from eqs. (5-68), (5-41) and (5-98) (together with fig. 5-3). This result for $I_G(\sigma, y+i\cdot 0)$ agrees with that indicated in sect. 3 of GDD. For $\eta_{1+} < y < 1$, eqs. (5-69) and (5-41) (together with fig. 5-3) show that eq. (5-98) (with $x \rightarrow \sigma$) agrees with eqs. (49) and (50) of GDD. For $1 < y \leq f_{+}(1)$, we see from eqs. (5-66) and (5-41) (together with fig. 5-3) that $I_G(\sigma, y+i\cdot 0)$ can be written as

$$I_G(\sigma, y+i\cdot 0) = - \int_{\xi_{2+}}^1 \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} - 2 \int_1^{g_{+}(y)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}}, \quad (5-100)$$

which agrees with eq. (55) of GDD. It is further claimed in GDD that eq. (55) of GDD holds for all $y > 1$. However, from fig. 5-3, we see that, for $f_{+}(1) < y < f_{\pm}(\xi_{1+})$,

$$\begin{aligned}
& \int_{\xi_{1+}}^1 \frac{d\xi}{\xi-\sigma} \int_{f_+(\xi)}^{f_-(\xi)} \frac{d\eta}{\eta-(y+i\cdot 0)} \frac{1}{\sqrt{F(\xi, \eta)}} \\
&= \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + \int_{g_+(y)}^1 \frac{d\xi}{\xi-\sigma} \frac{\pi i}{\sqrt{F(\xi, y)}} \quad (5-101)
\end{aligned}$$

so that, using eq. (5-66), we find that

$$I_G(\sigma, y+i\cdot 0) = - \int_{\xi_{2+}}^{g_+(y)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} + \int_{g_+(y)}^1 \frac{d\xi}{\xi-\sigma} \frac{\pi i}{\sqrt{F(\xi, y)}}. \quad (5-102)$$

This equation disagrees with eq. (55) of GDD. For $f_{\pm}(\xi_{1+}) \leq y < +\infty$, we find that eq. (5-101) holds with the first term on the right hand side multiplied by a minus sign (see eq. (5-41)). Thus, from eq. (5-78) we see that $I_G(\sigma, y+i\cdot 0)$ is again given by eq. (5-102), for $f_{\pm}(\xi_{1+}) \leq y < +\infty$, which disagrees with eq. (55) of GDD.

Next, we compare our results with those of GDD for case II S (i) (2b)(α). When $y < \eta_{1+}$, the arguments of case II S (i) (2a)(α) apply and our expressions for $I_G(\sigma, y+i\cdot 0)$ agree with those of GDD. When $\eta_{1+} < y < \min[1, f_{\pm}(\xi_{1+})]$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66). However, the expression for the second term in eq. (5-98) (with $x \rightarrow \sigma$) depends on the particular shape of the curve Γ_1 . If Γ_1 is as shown in fig. 5-4, so that $1 < f_+(1) < f_{\pm}(\xi_{1+})$, then we find that $I_G(\sigma, y+i\cdot 0)$ is given in eq. (5-100), which agrees with eq. (54) of GDD. However, if Γ_1 is such that $f_+(1) < 1 < f_{\pm}(\xi_{1+})$, then $I_G(\sigma, y+i\cdot 0)$ is given by eq. (5-100) for $\eta_{1+} < y \leq f_+(1)$, in agreement with eq. (54) of GDD, and by eq. (5-102) for $f_+(1) < y < 1$, which disagrees with eq. (54) of GDD. In general, we find that for

$\eta_{1+} < y < +\infty$, our results for $I_G(\sigma, y+i\cdot 0)$ agree with those in eqs. (54) and (55) of GDD when $g_+(y) \geq 1$, and disagree when $g_+(y) < 1$.

Next, we compare our results with those of GDD for case II S (i) (2a)(β). When $y < \eta_{2+}$, the arguments of case II S (i) (2a)(α) apply, and our expressions for $I_G(\sigma, y+i\cdot 0)$ agree with those of GDD. For $\eta_{2+} < y \leq f_{\pm}(\xi_{2+})$, eqs. (5-82) and (5-41) (together with fig. 5-3) show that eq. (5-98) (with $x \rightarrow \sigma$) agrees with eq. (43) of GDD. Also, when $f_{\pm}(\xi_{2+}) < y < \eta_{1-}$, eqs. (5-90) and (5-41) (together with fig. 5-3) show that eq. (5-98) (with $x \rightarrow \sigma$) again agrees with eq. (43) of GDD. (Note that there is a misprint in eq. (48) of GDD.) When $\eta_{1-} < y < \eta_{1+}$, $I_G(\sigma, y+i\cdot 0)$ can be obtained from eqs. (5-68), (5-41) and (5-98) (together with fig. 5-3). This result agrees with that indicated in sect. 3 of GDD. For $\eta_{1+} < y < +\infty$, the arguments of case II S (i) (2a)(α) apply.

For case II S (i) (2b)(β), the arguments of case II S (i) (2a)(β) apply for $y < \eta_{1+}$ and the arguments of case II S (i) (2b)(α) apply for $y > \eta_{1+}$.

We now compare our results for $I(\sigma, y+i\cdot 0)$ with those of GDD for case III A. GDD considered the continuation, in x and y , of a function $A_{an}(-x, -y)$ related to $I_S(x, y)$ (eq. (5-55)), for $x < -1$, $f_{\pm}(\xi_{1+}) < y < -1$, by

$$I_S(x, y) = -\frac{\pi}{2} A_{an}(-x, -y). \quad (5-103)$$

We start by comparing our results with those of GDD for case III A (1a)(α). For $y \leq f_{\pm}(\xi_{1+})$, eq. (5-95) agrees with eq. (58) of GDD.

When $f_{\pm}(\xi_{1+}) < y \leq f_{\pm}(\xi_{2+})$, eq. (5-55) (with $x \rightarrow \sigma$) agrees with eqs. (57) and (59) of GDD (except that GDD do not consider the case when $-1 \leq f_{\pm}(\xi_{1+}) < \eta_{\delta}$). For $f_{\pm}(\xi_{2+}) < y < \eta_{2+}$, eq. (5-66) agrees with eq. (59) of GDD and for $\eta_{2+} < y < \eta_{1+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-68), which agrees with the result indicated in sect. 4 of GDD. For $\eta_{1+} < y < 1$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-69), which agrees with eq. (61) of GDD. For $y > 1$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) which can also be written in the form

$$I_S(\sigma, y+i\cdot 0) = - \int_{\xi_{2+}}^{\xi_{1+}} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}} - 2 \int_{\xi_{1+}}^{g_+(y)} \frac{d\xi}{\xi-\sigma} \frac{\pi}{\sqrt{-F(\xi, y)}}. \quad (5-104)$$

Thus, for $y > 1$, eq. (5-66) or (5-104) disagrees with eq. (63) of GDD.

Next we consider case III A (1b)(α). For $y < \eta_{1+}$, the argument of case III A (1a)(α) applies. For $y > \eta_{1+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-66) or (5-104). This result agrees with eq. (62) of GDD when $\eta_{1+} < y < 1$, except that the θ function used there is not needed. However, for $1 < y < +\infty$, eq. (5-66) or (5-104) disagrees with eq. (63).

For case III A (1a)(β), the arguments of case III A (1a)(α) apply for $y < \eta_{2+}$. For $\eta_{2+} < y \leq f_{\pm}(\xi_{2+})$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-82), which disagrees with eq. (60) of GDD. When $f_{\pm}(\xi_{2+}) < y < \eta_{1-}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-90), which again disagrees with eq. (60) of GDD. When $\eta_{1-} < y < \eta_{1+}$, $I_S(\sigma, y+i\cdot 0)$ is given by eq. (5-68), which agrees with the result

indicated in sect. 4 of GDD. For $y > \eta_{1+}$, the argument of case III

A (1a)(α) applies.

For case III A (1b)(β), the argument of case III A (1a)(β) applies for $y < \eta_{1+}$ and the argument of case III A (1b)(α) applies for $y > \eta_{1+}$.

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