PERISTALTIC MOTION

by

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The material reported in this thesis is the author's work except where references to other sources are explicitly stated.

J Parkes

1. Introduction and Summary

The study of the flow of an incompressible viscous liquid is greatly simplified if discussion is limited to Stokes flow in which the Reynolds number is small enough for the non-linear inertia forces to be neglected in comparison with the viscous forces so that the equations of motion become linear. The Stokes flow approximation is a suitable model to take for peristaltic motion since the velocities met in practice are small and conditions at infinity are not considered.

In this thesis, two dimensional flow through a symmetrical channel and axially symmetric flow through a pipe of circular cross-section are considered. In each case the boundary varies sinusoidally.

Two causes of motion are studied. Firstly, it will be assumed that there is a prescribed pressure gradient along the pipe or channel and secondly that a progressive wave passes along the walls. If the prescribed pressure gradient is constant, and if the progressive wave velocity is small enough, this peristaltic motion is governed by the usual equations for steady Stokes flow. Thus the two extreme cases, of motion caused solely by the variation in cross-section and of motion under a constant pressure gradient when the walls are fixed can be treated together. Moreover, the two cases of pipe flow and channel flow can be treated together by taking advantage of the notation of generalized axi-symmetric potential theory to develop the theory in a form which is applicable to each case, leaving only the detailed calculations to be carried out separately. This simpler case is treated first.

If the prescribed pressure gradient varies sinusoidally with time then the motion is governed by the unsteady Stokes flow equations. In this case there is nothing to be gained by imposing the condition that the progressive wave velocity should be small and so the general case of peristalsis with sinusoidal pressure gradient has been solved.

It will be convenient, where it is not necessary to distinguish between channel flow and pipe flow, to use "tube" to denote either the symmetric channel or the axi-symmetric pipe and "radius of the tube" to denote half the breadth of the channel or the radius of the pipe.

The problem is solved by expanding the stream function, which determines the flow, as a Fourier series involving two infinite sets of unknown coefficients, real in the simple case but complex in general. The boundary conditions on the wall of the tube give a set of linear equations which can be solved for these coefficients. Following closely the method used by Taylor (1951) in a similar problem, a perturbation solution is found in which these coefficients are derived as power series in η/h, the ratio of the amplitude of the variation of the tube radius to the mean tube radius. The solution is used to derive an expression for the average flux through the tube in the general case. This average flux is calculated numerically in some particular cases, as are the streamlines and the velocity distributions along and normal to the axis of the tube.

An alternative method of direct calculation of the unknown coefficients has been devised and tested in particular cases. This method works well for small $\ell\eta$, where it would be expected to do so, and agrees with the perturbation method. It should work better than the perturbation method for large $\ell\eta$ and it is proposed to complete the necessary calculations on a digital computer.

2. Historical Survey

Peristaltic motion of a viscous fluid through pipes and channels does not appear to have been discussed previously mathematically although the particular case of flow under a prescribed pressure gradient through a fixed tube whose walls vary sinusoidally has been treated by several authors. Langlois (1964) discussed flow along channels of varying breadth and obtained approximate solutions in several different cases. Gheorgita (1959) found solutions to the first order in η for symmetrical channels in which the breadth varies along the length according to a cosine law and also gave first order solutions in cases in which the distance of each wall from the centre plane varies periodically along the length with the same frequency but the channel is not symmetrical. Belinfante(1962) considered flow of a viscous fluid along pipes and channels in which the radius or breadth varies along the length according to a cosine law. He also obtained solutions for the Stokes flow approximation correct to the first order in η/h. He used these solutions as a basis for solutions of the Navier-Stokes equations in powers of the Reynolds number of the flow. He remarks that he has also obtained solutions of the Stokes flow problem to the second order in η/h for both pipe and channel flow. Burns (1965) used the methods of this thesis to obtain results for both pipes and channels with fixed walls, which is a special case of the general problem of peristaltic flow with sinusoidal pressure gradient considered here. It is also, of course, a special case of peristaltic flow with constant pressure gradient which was considered by Burns and Parkes (1967), in a paper which presented the earlier results of the present work.

The analysis of peristaltic motion may have several important practical applications. Peristalsis occurs naturally in several parts of the body. According to Wright (1961) "Bayliss and Starling defined true peristalsis as a coordinated reaction in which a wave of contraction preceded by a wave of relaxation passes down a hollow viscus; the contents of the viscus as they are propelled along would thus always enter a segment which had actively relaxed and enlarged to receive them. This type of movement was thought to be responsible for transferring the contents of the alimentary canal from the oesophagus through the stomach, small and large intestine and finally to the anus." Peristaltic waves also occur in the ureter.

The flow of corrosive fluids along pipes is often brought about by the use of peristaltic pumps which are designed so that the fluid does not come into contact with the pump itself. Latham and Shapiro (1966) have done some work on peristaltic pumping but this paper is not yet available.

The flow of blood in arteries, while not being caused by peristaltic motion of the artery walls, has similarities, particularly if it is assumed that the artery walls are sufficiently flexible so that they oscillate in sympathy with the pressure gradient. For this reason it seemed logical in this thesis to consider the general case of peristalsis with sinusoidal pressure gradient. However it must be realised that the Reynolds number associated with blood flow in arteries is not small, and ranges from about 300 to 10,000 in the human body.

The particular case of flow through a fixed pipe of constant cross section under a prescribed sinusoidal pressure gradient has also been treated by several authors but the more general case of a pipe whose walls vary sinusoidally does not appear to have been considered. Womersley was very prolific on this subject both theoretically and experimentally. Amongst other things, he (1955) considered the oscillatory motion of a viscous liquid in a rigid tube under a simple harmonic pressure gradient and also similar motion in a thin walled elastic tube with particular reference to the flow of blood in arteries. In the latter case he showed that the longitudinal oscillation of the walls of the tube, caused by viscous drag on its inner surface, is important in determining the rate of flow, which may be 10% greater than that in a rigid tube under the same pressure gradient. Olsen and Shapiro (1967) considered large amplitude, unsteady flow in liquid filled elastic tubes but only where the wave-length is long compared with the diameter ($\ell h < 0{\cdot}06$) so that a one dimensional model can be used. Lance (1956) considered the flow through a pipe or channel of constant cross section due to a pressure gradient and a series of pulses acting in the opposite direction. He showed that it is possible for the flow to be arrested when the pipe is subjected to one or more pulses of sufficient strength. (This problem was suggested by reports that the engines of a certain jet aircraft fail when the guns are fired. The failure could be due to fuel starvation.) Sanyal (1956) studied the flow in a circular tube under pressure gradients which rise or fall exponentially with time. She found that for small diameter tubes the velocity distribution in a cross section of the tube is parabolic in both cases but for large diameter tubes the two motions are quite different. In the first case the flow has the boundary layer character while this characteristic is completely absent in the second case and the velocity depends on the wall distance.

3. Statement of the Problem

It is convenient to use "tube" to denote either the symmetrical channel or the axi-symmetric pipe and "radius of tube" to denote half the breadth of the channel or the radius of the pipe.

The wall of the tube is defined by the equation

$$y = h + \eta \cos \ell(x - \sigma t) \qquad (1)$$

so that a progressive wave of amplitude η, velocity σ and wave-length $\lambda = 2\pi/\ell$ passes along the tube in the positive x-direction. It will frequently be convenient to write $z = x - \sigma t$. If $\sigma = 0$, the wall of the tube is a fixed cosine wave. The x-y plane is a meridian plane of the tube, the axis OX being along the axis of symmetry and the axis OY normal to OX. Let the velocity components in the x,y directions be u,v respectively (see Figure 1).

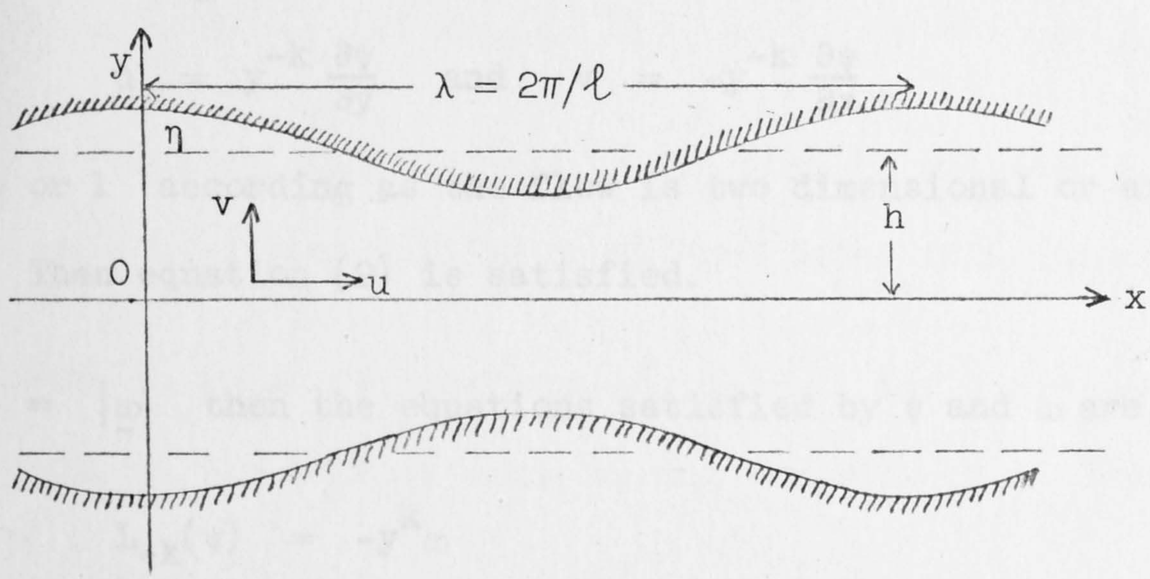

Figure 1.

The equations of motion governing the flow of an incompressible fluid through the tube are the continuity equation and the Navier-Stokes equation. These, in the absence of body forces, are

$$\nabla \cdot \underset{\sim}{v} = 0 , \tag{2}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \nabla(\tfrac{1}{2}\underset{\sim}{v}^2) - \underset{\sim}{v} \times \underset{\sim}{\omega} = -\frac{1}{\rho} \nabla p - \nu\nabla \times \underset{\sim}{\omega} , \tag{3}$$

where $\underset{\sim}{v}$ is the velocity, ν is the kinematic viscosity, p is the pressure, ρ is the density and $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v}$ is the vorticity (Rosenhead 1963).

If squares of velocities can be neglected then equation (3) reduces to

$$\frac{\partial \underset{\sim}{v}}{\partial t} = - \frac{1}{\rho} \nabla p - \nu\nabla \times \underset{\sim}{\omega}. \tag{4}$$

It is possible to treat the two dimensional flow and the axi-symmetric flow together by using the notation of generalized axially symmetric potential theory (Weinstein 1953).

Let ψ be a stream function such that

$$u = y^{-k} \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -y^{-k} \frac{\partial\psi}{\partial x} \tag{5}$$

where k = 0 or 1 according as the flow is two dimensional or axi-symmetric. Then equation (2) is satisfied.

If $\omega = |\underset{\sim}{\omega}|$ then the equations satisfied by ψ and ω are

$$L_{-k}(\psi) = -y^k\omega \tag{6}$$

and

$$L_{-k}(y^k\omega) = \frac{1}{\nu} \frac{\partial}{\partial t} (y^k\omega) , \tag{7}$$

where

$$L_{-k} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{k}{y}\frac{\partial}{\partial y} .$$

Also equation (4) produces the following pressure relations:

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t} - \mu y^{-k} \frac{\partial}{\partial y}(y^k \omega) , \tag{8}$$

$$\frac{\partial p}{\partial y} = -\rho \frac{\partial v}{\partial t} + \mu y^{-k} \frac{\partial}{\partial x}(y^k \omega) , \tag{9}$$

where $\mu = \rho\nu$ is the viscosity.

The following conditions must be satisfied. On the axis of symmetry $\psi = 0$ and $\omega \to 0$ as $y \to 0$. On the outer boundary of the flow, the fluid must have the same velocity as the wall of the tube. It will be assumed in the first instance that the particles of the tube wall move in straight lines perpendicular to the axis of the tube so that the boundary condition is

$$y^{-k}\frac{\partial \psi}{\partial y} = u = 0 ,$$

$$-y^{-k}\frac{\partial \psi}{\partial z} = v = \frac{\partial y}{\partial t} = \ell\sigma\eta \sin \ell z \quad \text{on} \quad y = h + \eta \cos \ell z. \tag{10}$$

This condition requires that the wall of the tube be extensible. A modified boundary condition will be considered later in Section 7.

The problem is to solve equations (6) and (7) for ψ and ω subject to these conditions on the axis of symmetry and the wall of the tube.

It is necessary to prescribe the pressure gradient which produces the flow and, in the general case, it will be assumed that the pressure drop over a wave-length has the value P cos ft. In particular, P = 0 corresponds to peristalsis with zero pressure gradient and f = 0 corresponds to peristalsis with constant pressure gradient.

4. Solution of the problem

It is convenient to deal with a particular case first and then consider the general case. It is assumed that the flow has existed long enough for all transient terms to have decayed so that only the steady state solution is considered.

4.1 Peristalsis with constant pressure gradient and σ small i.e. $f = 0$ and $\sigma/\ell\nu = o(1)$

This is the case of peristalsis with constant pressure gradient where the progressive wave velocity is small enough for the term $\frac{\partial \underset{\sim}{v}}{\partial t}$ in equation (4) to be neglected (Rosenhead 1963). An account of this has already been published (Burns and Parkes 1967).

The equations to be solved are now

$$L_{-k}(\psi) = -y^{k}\omega \tag{6}$$

and

$$L_{-k}(y^{k}\omega) = 0 \tag{7A}$$

with the conditions $\psi = 0$ on $y = 0$ and $\omega \to 0$ on $y = 0$.

The boundary conditions (10) are unchanged. The pressure relations (8) and (9) become

$$\frac{\partial p}{\partial x} = -\mu y^{-k}\frac{\partial}{\partial y}(y^{k}\omega) \tag{8A}$$

and

$$\frac{\partial p}{\partial y} = \mu y^{-k}\frac{\partial}{\partial x}(y^{k}\omega) \tag{9A}$$

Since the boundary varies periodically with z and is symmetrical about $z = 0$, it follows that both ψ and ω are even periodic functions of z with wave-length λ, and so can be expressed in the form

$$\psi(z,y) = \sum_{n=0}^{\infty} \psi_n(y) \cos n\ell z , \tag{11}$$

$$\omega(z,y) = \sum_{n=0}^{\infty} \omega_n(y) \cos n\ell z . \tag{12}$$

If (11) and (12) are substituted in equations (6) and (7A) and the coefficients of terms in cos $n\ell z$ compared, then for $n \geq 0$, $\psi_n(y)$ and $\omega_n(y)$ satisfy the equations

$$\frac{d^2\psi_n}{dy^2} - \frac{k}{y}\frac{d\psi_n}{dy} - n^2\ell^2\psi_n = -y^k\omega_n , \tag{13}$$

$$\frac{d^2(y^k\omega_n)}{dy^2} - \frac{k}{y}\frac{d(y^k\omega_n)}{dy} - n^2\ell^2(y^k\omega_n) = 0. \tag{14}$$

These equations have to be solved under the conditions that $\psi_n \to 0$ and $\omega_n \to 0$ as $y \to 0$.

The solutions of (14) satisfying these conditions are

$$y^k\omega_0 = -A_0 y^{k+1}$$

and for $n \geq 1$

$$y^k\omega_n = -A_n y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(n\ell y) \tag{15}$$

where A_0, A_n are arbitrary constants and $I_\nu(x)$ is a modified Bessel function of the first kind of order ν.

When ω_n in equation (13) is replaced by the expressions given in (15) then the resulting equations are of a type for which particular integrals can be found (Burns (1966)) and the complementary functions are of course the general solution of equation (14). The functions $\psi_n(y)$ which satisfy these equations and the condition $\psi_n \to 0$ as $y \to 0$ are found without difficulty and when these are substituted in (11) the resulting expression for $\psi(z,y)$ is

$$\psi(z,y) = \frac{A_o}{2(k+3)} y^{k+3} + B_o y^{k+1}$$
$$+ \sum_{n=1}^{\infty} y^{\frac{k+1}{2}} \left[\frac{A_n}{2n\ell} y I_{\frac{k-1}{2}}(n\ell y) + B_n I_{\frac{k+1}{2}}(n\ell y) \right] \cos n\ell z. \qquad (16)$$

The arbitrary constants A_n, B_n for $n \geq 0$ must be determined from the condition (10), that the fluid on the boundary has the same velocity as the wall of the tube, together with the condition that the pressure drop per wave-length is P. It will be seen later, that, with this assumption, the mean flux consists of terms proportional to P and terms independent of P but proportional to σ.

It can be seen from equation (9A) that $\frac{\partial p}{\partial y}$ is an odd periodic function of z. Hence $\frac{\partial p}{\partial y} = 0$, i.e. p is constant, on the sections $z = 0, \lambda, 2\lambda \ldots$ of the tube. It can also be seen from equation (8A) that $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z}$ will be an even periodic function of z and so the pressure difference between successive points of maximum cross section is always the same. In this case it is assumed to be P.

Since the pressure is constant over the sections $z = 0, \lambda$ it follows that the pressure drop between these sections can be obtained by integrating $\frac{\partial p}{\partial z}$ along the line $y = 0$.

Equation (8A) gives

$$\frac{\partial p}{\partial z} = -\mu y^{-k} \frac{\partial}{\partial y} (y^k \omega)$$

and this, with (12) and (15), leads to

$$\left(\frac{\partial p}{\partial z}\right)_{y=0} = \mu(k + 1) A_o + \sum_{n=1}^{\infty} f_n \cos n\ell z , \tag{17}$$

where the coefficients f_n are constants. Integration of (17) from $z = 0$ to $z = \lambda$ gives

$$A_o = - \frac{P\ell}{2\mu\pi(k + 1)} . \tag{18}$$

Thus the constant A_o is known in terms of the prescribed pressure gradient. The remaining boundary condition leads to equations sufficient to determine the constants B_o, A_n, B_n $(n \geqq 1)$ in terms of A_o and σ.

At this point it becomes convenient to give separate (although closely similar) discussions of the two cases of channel flow and pipe flow.

For channel flow, $k = 0$ and the stream function becomes

$$\psi(z,y) = \frac{A_o}{6} y^3 + B_o y + \left(\frac{2}{\pi\ell}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{2}} \left\{\frac{A_n}{2n\ell} y \cosh n\ell y + B_n \sinh n\ell y\right\} \cos n\ell z . \tag{19}$$

The replacement of the Bessel functions $I_{\frac{1}{2}}(n\ell y)$ and $I_{-\frac{1}{2}}(n\ell y)$ by expressions involving $\cosh n\ell y$ and $\sinh n\ell y$ leads to a considerable simplification in the detailed analysis which follows.

For pipe flow, k = 1 and the stream function becomes

$$\psi(z,y) = \frac{A_o}{8} y^4 + B_o y^2$$

$$+ \sum_{n=1}^{\infty} y \left\{ \frac{A_n}{2n\ell} y I_o(n\ell y) + B_n I_1(n\ell y) \right\} \cos n\ell z. \tag{20}$$

It is convenient to introduce new coefficients as follows:

For channel flow, let

$$a_o = \tfrac{1}{2} A_o \ , \quad b_o = B_o,$$

and, for n ≧ 1, (21)

$$a_n = \left(\frac{2}{n\pi\ell}\right)^{\frac{1}{2}} \frac{A_n}{2n\ell} \ , \quad b_n = \left(\frac{2}{n\pi\ell}\right)^{\frac{1}{2}} B_n \ ;$$

and for pipe flow, let

$$a_o = \tfrac{1}{2} A_o \ , \quad b_o = 2B_o,$$

and, for n ≧ 1, (22)

$$a_n = \frac{A_n}{2n\ell} \ , \quad b_n = B_n.$$

The stream function for channel flow then becomes

$$\psi(z,y) = \tfrac{1}{3} a_o y^3 + b_o y$$

$$+ \sum_{n=1}^{\infty} \left\{ a_n y \cosh n\ell y + b_n \sinh n\ell y \right\} \cos n\ell z \tag{23}$$

where b_o, a_n, b_n (n ≧ 1) are to be found in terms of a_o and σ from the conditions

$$\frac{\partial\psi}{\partial y} = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial z} = - \ell\sigma\eta \sin \ell z \quad \text{on} \quad y = h + \eta \cos \ell z \qquad (24)$$

which are obtained from (10) by putting k = 0.

The stream function for pipe flow becomes

$$\psi(z,y) = \tfrac{1}{4} a_o y^4 + \tfrac{1}{2} b_o y^2 + \sum_{n=1}^{\infty} y \left\{ a_n y \, I_o(n\ell y) + b_n I_1(n\ell y) \right\} \cos n\ell z \qquad (25)$$

where b_o, a_n, b_n ($n \geqq 1$) are to be found in terms of a_o and σ from the conditions

$$\frac{1}{y}\frac{\partial\psi}{\partial y} = 0 \quad \text{and} \quad \frac{1}{y}\frac{\partial\psi}{\partial z} = - \ell\sigma\eta \sin \ell z \quad \text{on} \quad y = h + \eta \cos \ell z \qquad (26)$$

which are obtained from (10) by putting k = 1.

4.1.1 <u>Evaluation of the coefficients a_n, b_n.</u>

The boundary conditions (24) and (26) at the tube wall, which in each case has equation $y = h + \eta \cos \ell z = y_1$ say, lead to the following equations:

For channel flow:

$$a_o y_1^2 + b_o + \sum_{n=1}^{\infty} [(a_n + n\ell b_n) \cosh n\ell y_1 + n\ell a_n y_1 \sinh n\ell y_1] \cos n\ell z = 0 \qquad (27)$$

and

$$\sum_{n=1}^{\infty} [a_n y_1 \cosh n\ell y_1 + b_n \sinh n\ell y_1]\, n \sin n\ell z = \eta\sigma \sin \ell z; \qquad (28)$$

for pipe flow:

$$a_o y_1^2 + b_o + \sum_{n=1}^{\infty} [(2a_n + n\ell b_n)\, I_o(n\ell y_1) + n\ell a_n y_1\, I_1(n\ell y_1)] \cos n\ell z = 0 \qquad (29)$$

and

$$\sum_{n=1}^{\infty} [a_n y_1 I_o(n\ell y_1) + b_n I_1(n\ell y_1)]\, n \sin n\ell z = \eta\sigma \sin \ell z. \qquad (30)$$

For channel flow, $\cosh n\ell y_1$ and $\sinh n\ell y_1$ and for pipe flow, $I_o(n\ell y_1)$ and $I_1(n\ell y_1)$ are expanded in powers of $\cos \ell z$. Substitution in (27), (28) and (29), (30) leads, in each case, to terms of the form $\cos^p \ell z \cos n\ell z$ and $\cos^p \ell z \sin n\ell z$ which are expanded in Fourier cosine and sine series respectively. Finally, the coefficients of terms in $\cos r\ell z$ and $\sin r\ell z$ in the resulting equations are equated and linear equations for b_o, a_n, b_n $(n \geq 1)$ are obtained. In each case, these are of the form

$$\sum_{n=1}^{\infty} (p_{n1} a_n + q_{n1} b_n) = \eta\sigma , \quad r = 1,$$

$$\sum_{n=1}^{\infty} (p_{nr} a_n + q_{nr} b_n) = 0 , \quad r = 2, 3, 4 \ldots ,$$

$$b_o + \sum_{n=1}^{\infty} (h_{no} a_n + k_{no} b_n) = c_o ,$$

$$\sum_{n=1}^{\infty} (h_{nr} a_n + k_{nr} b_n) = c_r , \quad r = 1, 2, 3 \ldots , \tag{31}$$

where all the coefficients p_{nr}, q_{nr}, h_{nr}, k_{nr}, c_r are known. These can be calculated to any required accuracy, so that direct numerical solution of the equations for as many coefficients a_n, b_n as are necessary to give a desired accuracy is possible. This method is discussed more fully in Section 6.

For the purposes of the perturbation solution used here, the coefficients p_{nr}, q_{nr}, h_{nr}, k_{nr}, c_r may be expanded in powers of $\ell\eta$ and the leading term of the series for each of the first four involves $\ell\eta$ raised to the power $|n - r|$ while c_r is of the order $(\ell\eta)^r$. It follows that a_n and b_n are of order $(\ell\eta)^n$ and it turns out that they are obtained as series in the form

$$\begin{aligned} a_n &= \sum_{t=0}^{\infty} \alpha_{n,n+2t} (\ell\eta)^{n+2t} , \quad n \geqq 1 ; \\ b_n &= \sum_{t=0}^{\infty} \beta_{n,n+2t} (\ell\eta)^{n+2t} , \quad n \geqq 0. \end{aligned} \tag{32}$$

At this stage it is assumed that the non-dimensional quantity, $\ell\eta$, is small and $\psi(z,y)$ is to be calculated to order $(\ell\eta)^n$. Imposing this restriction, and comparing coefficients of powers $\ell\eta$ in equations (31) gives a set of equations which can be solved for α_{nr} and β_{nr}.

To find $\psi(z,y)$ to order $(\ell\eta)^n$, $\frac{1}{2}(n + 1)(n + 2)$ equations are needed but these can be solved in pairs. Thus to find ψ to order $(\ell\eta)^4$ needs

15 equations. This is the order of many of the calculations in this thesis.

An alternative approach, which leads to the same results for a_n and b_n, is given in Appendix A. There it is assumed at the start that $\ell\eta$ is small so that only a few terms in the expansions of $\cosh n\ell y_1$, $\sinh n\ell y_1$, $I_0(n\ell y_1)$ and $I_1(n\ell y_1)$ need be considered.

4.1.2 Calculation of the flux through the tube

To find the flux through the tube it is necessary to evaluate the stream function $\psi(z,y)$ at a point on the boundary $y = h + \eta \cos \ell z$. For any value of x, this flux varies periodically with the time. Since a_n, b_n are determined as power series in $\ell\eta$ it follows that the flux ψ is also a power series in $\ell\eta$. Alternatively we can write ψ as a power series in η/h, i.e.,

$$\psi = \psi_0 + \psi_1(\eta/h) + \psi_2(\eta/h)^2 + \quad \ldots$$

where ψ_n is a periodic function of z.

What is wanted is the average flux per cycle and this can be found by integrating ψ at a point on the boundary over one period. Doing this removes all the odd powers of η/h for it is easily seen (Appendix B) that

$$\int_0^\lambda \psi_1 \, dz = \int_0^\lambda \psi_3 \, dz = \ldots = 0.$$

The mean flux is then

$$\bar{\psi} = \bar{\psi}_0 + \bar{\psi}_2(\eta/h)^2 + \bar{\psi}_4(\eta/h)^4 + \quad \ldots \tag{33}$$

and the expressions obtained for $\bar{\psi}_0$, $\bar{\psi}_2$, $\bar{\psi}_4$ are as follows:

(a) For channel flow:

$$\bar{\psi}_0 = -\tfrac{2}{3} a_0 h^3, \quad \bar{\psi}_2 = -h^3[a_0 + \ell^2\alpha_{11} \sinh \ell h + \tfrac{1}{2}\ell^2\sigma] ,$$

$$\bar{\psi}_4 = -\ell^3 h^4[\tfrac{1}{8}\alpha_{11}(\ell h \sinh \ell h + 2\cosh \ell h) + \alpha_{13}\, \ell h \sinh \ell h$$

$$+ \tfrac{1}{2}\alpha_{22}(2\ell h \cosh 2\ell h + \sinh 2\ell h)] ; \qquad (34)$$

(b) For pipe flow:

$$\bar{\psi}_0 = -\tfrac{1}{4} a_0 h^4, \quad \bar{\psi}_2 = -\tfrac{1}{4} h^4[3a_0 + 2\ell^2\alpha_{11} I_1(\ell h) + \ell^2\sigma] ,$$

$$\bar{\psi}_4 = -h^4[\tfrac{3}{32} a_0 + \tfrac{1}{16}\ell^3 h\alpha_{11}(3I_0(\ell h) + \ell h I_1(\ell h)) + \tfrac{1}{2}\ell^4 h^2\alpha_{13} I_1(\ell h)$$

$$+ \tfrac{1}{4}\ell^3 h\alpha_{22}(2\ell h\, I_0(2\ell h) + I_1(2\ell h)) + \tfrac{1}{16}\ell^2\sigma]. \qquad (35)$$

In each case α_{11}, α_{13}, α_{22} are obtained as indicated in section 4.1.1. The expressions for some of these are given in Appendix A, where it can be seen that they are all linear functions of a_0 and σ.

In both cases therefore, the flux consists of two distinct parts, one due to the pressure gradient only, the other due to the movement of the walls only. There are no interaction terms which makes computing much easier. The numerical results are discussed in Section 5.

4.2 The general case of peristalsis with sinusoidal pressure gradient and with σ large

The equations to be solved are

$$L_{-k}(\psi) = -y^k\omega , \qquad (6)$$

$$L_{-k}(y^k\omega) = \frac{1}{\nu}\frac{\partial}{\partial t}(y^k\omega), \tag{7}$$

with the conditions $\psi = 0$ on $y = 0$ and $\omega \to 0$ on $y = 0$. The boundary conditions (10) hold, i.e.,

$$u = y^{-k}\frac{\partial\psi}{\partial y} = 0$$

and

$$v = -y^{-k}\frac{\partial\psi}{\partial z} = \ell\sigma\eta \sin \ell z$$

on $y = h + \eta \cos \ell z$;

and so do the pressure relations

$$\frac{\partial p}{\partial x} = -\rho\frac{\partial u}{\partial t} - \mu y^{-k}\frac{\partial}{\partial y}(y^k\omega) \tag{8}$$

and

$$\frac{\partial p}{\partial y} = -\rho\frac{\partial v}{\partial t} + \mu y^{-k}\frac{\partial}{\partial y}(y^k\omega). \tag{9}$$

It can easily be seen that, because of the term on the right hand side of equation (7), ψ and ω can no longer be even, periodic functions of z and a more general form must be taken. If the pressure gradient is to vary sinusoidally with time with frequency f then clearly ψ and ω must also be periodic functions of t with frequency f. The most general form satisfying these conditions will be taken for ψ and ω, namely,

$$\psi(z,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_{nm}(y)\cos mft + B_{nm}(y)\sin mft)\cos n\ell z + (C_{nm}(y)\cos mft + D_{nm}(y)\sin mft)\sin n\ell z$$

with a similar expression for $\omega(z,y,t)$.

This form of solution, however, is not convenient for solving equations (6) and (7) because it leads to a set of simultaneous differential equations and so it will be replaced by the equivalent form

$$\psi(z,y,t) = \mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\psi_{nm}(y)e^{i(n\ell z+mft)} + \Psi_{nm}(y)e^{i(n\ell z-mft)}\right) \quad (36)$$

and

$$\omega(z,y,t) = \mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\omega_{nm}(y)e^{i(n\ell z+mft)} + \Omega_{nm}(y)e^{i(n\ell z-mft)}\right), \quad (37)$$

where $\psi_{nm}(y)$, $\Psi_{nm}(y)$, $\omega_{nm}(y)$ and $\Omega_{nm}(y)$ are complex functions of y only. Note that $z = x - \sigma t$ and is real.

If (37) is substituted in equation (7) and the coefficients of $e^{i(n\ell z+mft)}$ and $e^{i(n\ell z-mft)}$ are compared, then $y^k\omega_{nm}$ and $y^k\Omega_{nm}$ satisfy the equations

$$\frac{d^2}{dy^2}(y^k\omega_{nm}) - \frac{k}{y}\frac{d}{dy}(y^k\omega_{nm}) - K^2_{nm}y^k\omega_{nm} = 0 \quad (38)$$

and

$$\frac{d^2}{dy^2}(y^k\Omega_{nm}) - \frac{k}{y}\frac{d}{dy}(y^k\Omega_{nm}) - L^2_{nm}y^k\Omega_{nm} = 0, \quad (39)$$

where

$$K^2_{nm} = n^2\ell^2 - \frac{i(n\ell\sigma - mf)}{\nu}$$

and (40)

$$L^2_{nm} = n^2\ell^2 - \frac{i(n\ell\sigma + mf)}{\nu},$$

for $n = 0, 1, 2 \ldots$ and $m = 0, 1, 2 \ldots$.

Equations (38) and (39) are similar to equation (14) but now ω_{nm} and Ω_{nm} are complex and $n^2\ell^2$ in (14) is replaced by the complex quantity K^2_{nm} in (38) and by L^2_{nm} in (39). The solution therefore of

(38) is

$$y^k \omega_{oo} = -A_{oo} y^{k+1}$$

and

$$y^k \omega_{nm} = -A_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{nm}y) \quad \text{for } n = 0, 1, 2 \ldots \quad (41)$$

$$m = 0, 1, 2 \ldots$$

but not $n = m = 0$.

Similarly the solution of (39) is

$$y^k \Omega_{oo} = -C_{oo} y^{k+1}$$

and

$$y^k \Omega_{nm} = -C_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{nm}y) \quad \text{for } n = 0, 1, 2 \ldots \quad (42)$$

$$m = 0, 1, 2 \ldots$$

but not $n = m = 0$.

If these are substituted in (37) then

$$y^k \omega = \text{Re}\left\{-(A_{oo} + C_{oo})y^{k+1} - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{nm}y) e^{i(n\ell z+mft)} + C_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{nm}y) e^{i(n\ell z-mft)}\right)\right\}$$

where the summation does not include the case $n = m = 0$.

There is no loss of generality by taking $C_{oo} = 0$ or by taking A_{oo} to be real, so that

$$y^k \omega = -A_{oo} y^{k+1} - \text{Re}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{nm}y) e^{i(n\ell z+mft)} + C_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{nm}y) e^{i(n\ell z-mft)}\right)\right\} \quad (43)$$

If (43) is now substituted in equation (6), and if the coefficients of $e^{i(n\ell z+mft)}$ and $e^{i(n\ell z-mft)}$ are compared, then ψ_{nm} and Ψ_{nm} can be shown to satisfy the following equations:

$$\frac{d^2}{dy^2}\psi_{nm} - \frac{k}{y}\frac{d}{dy}\psi_{nm} - n^2\ell^2\psi_{nm} = A_{nm}y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{nm}y) \qquad (44)$$

for n = 0, 1, 2 ... , m = 0, 1, 2 ... , except n = m = 0,

$$\frac{d^2}{dy^2}\Psi_{nm} - \frac{k}{y}\frac{d}{dy}\Psi_{nm} - n^2\ell^2\Psi_{nm} = C_{nm}y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{nm}y) \qquad (45)$$

for n = 0, 1, 2 ... , m = 0, 1, 2 ... , except n = m = 0,

and

$$\frac{d^2}{dy^2}(\psi_{oo} + \Psi_{oo}) - \frac{k}{y}\frac{d}{dy}(\psi_{oo} + \Psi_{oo}) = A_{oo}y^{k+1} . \qquad (46)$$

The solution of equation (44) which satisfies the condition $\psi_{nm} = 0$ when $y = 0$ is

$$\psi_{nm} = y^{\frac{k+1}{2}}\left(\frac{A_{nm}}{K_{nm}^2 - n^2\ell^2} I_{\frac{k+1}{2}}(K_{nm}y) + B_{nm} I_{\frac{k+1}{2}}(n\ell y)\right)$$

for n = 1, 2 ... and m = 0, 1, 2 ...

and

$$\psi_{om} = \frac{A_{om}}{K_{om}^2} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{om}y) + B_{om}y^{k+1}$$

for m = 1, 2 ... .

Similarly the solution of equation (45) satisfying the condition $\Psi_{nm} = 0$ when $y = 0$ is

$$\Psi_{nm} = y^{\frac{k+1}{2}}\left(\frac{C_{nm}}{L_{nm}^2 - n^2\ell^2} I_{\frac{k+1}{2}}(L_{nm}y) + D_{nm} I_{\frac{k+1}{2}}(n\ell y)\right)$$

for n = 1, 2 ... and m = 0, 1, 2 ...

and

$$\Psi_{om} = \frac{C_{om}}{L_{om}^2} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{om}y) + D_{om}y^{k+1}$$

for m = 1, 2 ... .

The appropriate solution of equation (46) is

$$\psi_{oo} + \Psi_{oo} = \frac{A_{oo}y^{k+3}}{2(k+3)} + B_{oo}y^{k+1}$$

where B_{oo} is real.

If these solutions are substituted in (36) then

$$\psi = \frac{A_{oo}}{2(k+3)} y^{k+3} + B_{oo}y^{k+1} + \text{Re}\left\{\sum_{m=1}^{\infty}\left(\frac{A_{om}}{K_{om}^2} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{om}y) + B_{om}y^{k+1}\right)e^{imft}\right.$$

$$+ \sum_{m=1}^{\infty}\left(\frac{C_{om}}{L_{om}^2} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(L_{om}y) + D_{om}y^{k+1}\right)e^{-imft}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} y^{\frac{k+1}{2}}\left(\frac{A_{nm}}{K_{nm}^2 - n^2\ell^2} I_{\frac{k+1}{2}}(K_{nm}y) + B_{nm} I_{\frac{k+1}{2}}(n\ell y)\right)e^{i(n\ell z+mft)}$$

$$\left. + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} y^{\frac{k+1}{2}}\left(\frac{C_{nm}}{L_{nm}^2 - n^2\ell^2} I_{\frac{k+1}{2}}(L_{nm}y) + D_{nm} I_{\frac{k+1}{2}}(n\ell y)\right)e^{i(n\ell z-mft)}\right\} \qquad (47)$$

It is easy to see, from (40), that $K^2_{no} = L^2_{no}$ for $n = 1, 2 \ldots$. It follows, therefore, that the terms involving C_{no} and D_{no} can be absorbed into those involving A_{no} and B_{no} respectively so there is no loss of generality in taking $C_{no} = D_{no} = 0$. Similarly $K^2_{om} = -L^2_{om}$ for $m = 1, 2 \ldots$ and it follows that there is no loss of generality in taking $C_{om} = D_{om} = 0$.

A simpler expression for ψ which is still perfectly general is therefore

$$\psi = \frac{A_{oo}}{2(k+3)} y^{k+3} + B_{oo} y^{k+1} + \text{Re}\left\{ \sum_{m=1}^{\infty} \left(\frac{A_{om}}{K^2_{om}} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(K_{om} y) + B_{om} y^{k+1} \right) e^{imft} \right.$$

$$+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} y^{\frac{k+1}{2}} \left(\frac{A_{nm}}{K^2_{nm} - n^2\ell^2} I_{\frac{k+1}{2}}(K_{nm} y) + B_{nm} I_{\frac{k+1}{2}}(n\ell y) \right) e^{i(n\ell z + mft)}$$

$$\left. + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y^{\frac{k+1}{2}} \left(\frac{C_{nm}}{L^2_{nm} - n^2\ell^2} I_{\frac{k+1}{2}}(L_{nm} y) + D_{nm} I_{\frac{k+1}{2}}(n\ell y) \right) e^{i(n\ell z - mft)} \right\} . \quad (48)$$

It is assumed in this solution that $K^2_{nm} - n^2\ell^2 \neq 0$ and that $L^2_{nm} - n^2\ell^2 \neq 0$. From (40) it can be seen that

$$K^2_{nm} - n^2\ell^2 = -\frac{i(n\ell\sigma - mf)}{\nu} \quad \text{and} \quad L^2_{nm} - n^2\ell^2 = -\frac{i(n\ell\sigma + mf)}{\nu}$$

so that, if $n\ell\sigma - mf = 0$ then $K^2_{nm} - n^2\ell^2 = 0$, and if $n\ell\sigma + mf = 0$ then $L^2_{nm} - n^2\ell^2 = 0$. The first of these conditions is possible but the second is not since all the quantities concerned are positive. This condition will be dealt with later in section 4.3.6.

The arbitrary complex coefficients A_{nm}, B_{nm}, C_{nm} and D_{nm} in equation (48) can be determined from the condition (10) namely, that the fluid on the boundary has the same velocity as the boundary, together with the condition that the pressure drop per wavelength is P cos ft.

If the expressions for $y^k\omega$ and ψ given in (43) and (48) are substituted in equation (8) then the following result is obtained:

$$\frac{\partial p}{\partial x} = \mu A_{oo}(k+1) - \rho \mathrm{Re}\left\{\sum_{m=1}^{\infty} imfB_{om}(k+1)e^{imft} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_m(y)e^{i(n\ell z+mft)} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} g_m(y)e^{i(n\ell z-mft)}\right\}$$

$$+ \mu \mathrm{Re}\left\{\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(F_m(y)e^{i(n\ell z+mft)} + G_m(y)e^{i(n\ell z-mft)}\right)\right\},$$

where $f_m(y)$, $g_m(y)$, $F_m(y)$ and $G_m(y)$ are functions of y only.

It follows that

$$\int_0^\lambda \frac{\partial p}{\partial x}\,dz = \lambda\left(\mu A_{oo}(k+1) - \rho \mathrm{Re}\left\{imfB_{om}(k+1)e^{imft}\right\}\right)$$

if y is kept constant during the integration. Hence for all values of y, $\int_0^\lambda \frac{\partial p}{\partial x}\,dz$ is the same, i.e. the change in pressure per wavelength is the same for all y. If this change in pressure is -P cos ft then

$$\lambda\left(\mu A_{oo}(k+1) - \rho \mathrm{Re}\left\{\sum_{m=1}^{\infty} imfB_{om}(k+1)e^{imft}\right\}\right) = -P\cos ft. \tag{49}$$

This result is similar to that obtained in the simpler case treated in 4.1 which leads to equation (18).

If the coefficients of cos mft and sin mft in (49) are compared then the following results are obtained:

$$A_{oo} = 0, \quad B_{om} = 0 \quad \text{for } m = 2, 3 \ldots ,$$

and (50)

$$B_{o1} = - \frac{iP}{\lambda \rho f(k+1)} \quad .$$

Thus the constant B_{o1} is known in terms of the prescribed pressure gradient, i.e. in terms of P and f. The remaining boundary condition (10) at the wall leads to equations sufficient to determine the remaining arbitrary complex coefficients in terms of B_{o1} and σ.

It becomes convenient at this point to give separate discussions of the two cases of channel flow and pipe flow. It is also convenient to define new coefficients.

For channel flow, k = 0 and the stream function in (48) becomes

$$\psi = b_{oo}y + \frac{Py}{\lambda\rho f}\sin ft + \text{Re}\left\{\sum_{m=1}^{\infty} a_{om}\sinh K_{om}y e^{imft}\right.$$

$$+ \sum_{n=1}^{\infty}\left(a_{no}\sinh K_{no}y + b_{no}\sinh n\ell y\right)e^{in\ell z}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(a_{nm}\sinh K_{nm}y + b_{nm}\sinh n\ell y\right)e^{i(n\ell z+mft)}$$

$$\left. + \left(c_{nm}\sinh L_{nm}y + d_{nm}\sinh n\ell y\right)e^{i(n\ell z-mft)}\right\} , \qquad (51)$$

where $a_{om} = \frac{A_{om}}{K^2_{om}} \sqrt{\frac{2}{K_{om}\pi}}$ for m = 1, 2 ... , $b_{oo} = B_{oo}$,

$$a_{nm} = \frac{A_{nm}}{K^2_{nm} - n^2\ell^2} \sqrt{\frac{2}{K_{nm}\pi}}\,, \quad b_{nm} = B_{nm} \sqrt{\frac{2}{n\ell\pi}} \quad \text{for } n = 1, 2 \ldots \quad m = 0, 1 \ldots, \tag{52}$$

$$c_{nm} = \frac{C_{nm}}{L^2_{nm} - n^2\ell^2} \sqrt{\frac{2}{L_{nm}\pi}}\,, \quad d_{nm} = D_{nm} \sqrt{\frac{2}{n\ell\pi}} \quad \text{for } n = 1, 2 \ldots \quad m = 1, 2 \ldots$$

For pipe flow, k = 1 and the stream function in (48) becomes

$$\psi = \frac{b_{oo}}{2} y^2 + \frac{Py^2}{2\lambda\rho f} \sin ft + \mathrm{Re}\left\{ \sum_{m=1}^{\infty} a_{om} y\, I_1(K_{om} y) e^{imft} \right.$$

$$+ \sum_{n=1}^{\infty} y\left(a_{no}\, I_1(K_{no} y) + b_{no}\, I_1(n\ell y)\right) e^{in\ell z}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y\left(a_{nm}\, I_1(K_{nm} y) + b_{nm}\, I_1(n\ell y)\right) e^{i(n\ell z + mft)}$$

$$\left. + y\left(c_{nm}\, I_1(L_{nm} y) + d_{nm}\, I_1(n\ell y)\right) e^{i(n\ell z - mft)} \right\}, \tag{53}$$

where $a_{om} = \frac{A_{om}}{K^2_{om}}$ for m = 1, 2 ... , $b_{oo} = 2B_{oo}$,

$$a_{nm} = \frac{A_{nm}}{K^2_{nm} - n^2\ell^2}\,, \quad b_{nm} = B_{nm}\,, \quad \text{for } n = 1, 2 \ldots \quad m = 0, 1 \ldots, \tag{54}$$

$$c_{nm} = \frac{C_{nm}}{L^2_{nm} - n^2\ell^2}\,, \quad d_{nm} = D_{nm}\,, \quad \text{for } n = 1, 2 \ldots \quad m = 0, 1 \ldots .$$

4.2.1 Evaluation of the complex coefficients a_{nm}, b_{nm}, c_{nm}, d_{nm}.

The remaining complex coefficients a_{nm}, b_{nm}, c_{nm} and d_{nm} can be determined in terms of P, f and σ by using the boundary conditions (10) on the wall of the tube in a similar way to that developed in 4.1.1 and Appendix A. However two factors make this analysis more complicated.

Firstly, if σ is not assumed small, then the expression for ψ contains cos $n\ell z$ and sin $n\ell z$ terms and this doubles the number of coefficients. This means that instead of having two infinite sets of coefficients a_n and b_n there would be four infinite sets a_n, b_n, c_n and d_n. The presence of both cos $n\ell z$ and sin $n\ell z$ terms also doubles the number of equations obtained when comparing coefficients of cos $n\ell z$ and sin $n\ell z$ so that a solution is still possible.

Secondly, if the pressure gradient varies sinusoidally with time with frequency f and ψ is assumed to contain cos mft and sin mft terms, for all m, then the four infinite sets of coefficients a_n, b_n, c_n and d_n become the four doubly infinite sets a_{nm}, b_{nm}, c_{nm} and d_{nm}.

When the boundary conditions (10) are applied then two equations are obtained similar to equations (27) and (28) or (29) and (30) of 4.1.1. If the coefficients of cos mft and sin mft are compared in these two equations then each one gives two more equations so that for each value of m there are four equations which are sufficient to determine the coefficients needed in the solution. For the perturbation solution an analysis similar to that of Appendix A shows that

$$a_{nm} = \sum_{t=0}^{\infty} \alpha_{nmn+2t}(\ell\eta)^{n+2t}, \quad b_{nm} = \sum_{t=0}^{\infty} \beta_{nmn+2t}(\ell\eta)^{n+2t},$$

$$c_{nm} = \sum_{t=0}^{\infty} \gamma_{nmn+2t}(\ell\eta)^{n+2t}, \quad d_{nm} = \sum_{t=0}^{\infty} \delta_{nmn+2t}(\ell\eta)^{n+2t}. \tag{55}$$

The perturbation solution for the stream function to order $(\ell\eta)^2$ has been found for both channel flow and pipe flow. ~~and~~ The ~~relevant~~ coefficients ~~are as follows:~~ required in calculating the average flux per wavelength, which is done in 4.2.2, are as follows:

For channel flow:

$$\beta_{000} = 0, \tag{56}$$

$$\alpha_{010} = \frac{iP}{\lambda\rho f K_{01} \cosh K_{01} h}, \quad \alpha_{0m0} = 0 \quad \text{for } m \geq 2, \tag{57}$$

$$\alpha_{101} = \frac{-\sigma \cosh \ell h}{K_{10} \cosh K_{10} h \sinh \ell h - \ell \sinh K_{10} h \cosh \ell h}, \tag{58}$$

$$\beta_{101} = \frac{\sigma K_{10} \cosh K_{10} h}{\ell(K_{10} \cosh K_{10} h \sinh \ell h - \ell \sinh K_{10} h \cosh \ell h)}, \tag{59}$$

$$\alpha_{111} = \frac{-iPK_{01} \tanh K_{01} h \sinh \ell h}{2\lambda\rho f \ell(K_{11} \cosh K_{11} h \sinh \ell h - \ell \sinh K_{11} h \cosh \ell h)},$$

$$\alpha_{1m1} = 0 \quad \text{for } m \geq 2, \tag{60}$$

$$\beta_{111} = \frac{iPK_{01} \tanh K_{01} h \sinh K_{11} h}{2\lambda\rho f \ell(K_{11} \cosh K_{11} h \sinh \ell h - \ell \sinh K_{11} h \cosh \ell h)},$$

$$\beta_{1m1} = 0 \quad \text{for } m \geq 2, \tag{61}$$

$$\gamma_{111} = \frac{iPL_{01} \tanh L_{01} h \sinh \ell h}{2\lambda\rho f \ell(L_{11} \cosh L_{11} h \sinh \ell h - \ell \sinh L_{11} h \cosh \ell h)},$$

$$\gamma_{1m1} = 0 \quad \text{for } m \geq 2, \tag{62}$$

$$\delta_{111} = \frac{-iPL_{01} \tanh L_{01}h \sinh L_{11}h}{2\lambda\rho f\ell(L_{11} \cosh L_{11}h \sinh \ell h - \ell \sinh L_{11}h \cosh \ell h)},$$

$$\delta_{111} = 0 \quad \text{for } m \geqq 2, \tag{63}$$

$$2\beta_{002}\ell + \text{Re}\left\{\alpha_{101}K_{10}^2 \sinh K_{10}h + \beta_{101}\ell^2 \sinh \ell h\right\} = 0, \tag{64}$$

$$\alpha_{012}\ell^2 K_{01} \cosh K_{01}h = \frac{P}{4\lambda\mu} - \frac{\ell}{2}(K_{11}^2 - \ell^2)\alpha_{111} \sinh K_{11}h$$

$$- \frac{\ell}{2}(L_{11}^2 - \ell^2)\,\bar{\delta}_{111} \sinh \ell h, \tag{65}$$

where $\bar{\delta}_{111}$ is the complex conjugate of δ_{111},

and

$$\alpha_{om2} = 0 \quad \text{for } m \geqq 2. \tag{66}$$

For pipe flow:

$$\beta_{000} = 0, \tag{67}$$

$$\alpha_{010} = \frac{iP}{\lambda\rho f K_{01} I_0(K_{01}h)}, \quad \alpha_{omo} = 0 \quad \text{for } m \geqq 2, \tag{68}$$

$$\alpha_{101} = \frac{-\sigma I_0(\ell h)}{K_{10} I_0(K_{10}h) I_1(\ell h) - \ell I_1(K_{10}h) I_0(\ell h)}, \tag{69}$$

$$\beta_{101} = \frac{\sigma K_{10} I_0(K_{10}h)}{\ell\{K_{10} I_0(K_{10}h) I_1(\ell h) - \ell I_1(K_{10}h) I_0(\ell h)\}}, \tag{70}$$

$$\alpha_{111} = \frac{-iPK_{01} I_1(K_{01}h) I_1(\ell h)}{2\lambda\rho f\ell I_0(K_{01}h)\{K_{11} I_0(K_{11}h) I_1(\ell h) - \ell I_1(K_{11}h) I_0(\ell h)\}},$$

$$\alpha_{1m1} = 0 \quad \text{for } m \geqq 2, \tag{71}$$

$$\beta_{111} = \frac{iPK_{01} I_1(K_{01}h) I_1(K_{11}h)}{2\lambda\rho f\ell I_0(K_{01}h)\{K_{11} I_0(K_{11}h) I_1(\ell h) - \ell I_1(K_{11}h) I_0(\ell h)\}},$$

$$\beta_{1m1} = 0 \quad \text{for } m \geqq 2, \tag{72}$$

$$\gamma_{111} = \frac{iPL_{01}\, I_1(L_{01}h)\, I_1(\ell h)}{2\lambda\rho f\ell\, I_0(L_{01}h)\{L_{11}\, I_0(L_{11}h)\, I_1(\ell h) - \ell\, I_1(L_{11}h)\, I_0(\ell h)\}},$$

$$\gamma_{1m1} = 0 \quad \text{for } m \geqq 2, \tag{73}$$

$$\delta_{111} = \frac{-iPL_{01}\, I_1(L_{01}h)\, I_1(L_{11}h)}{2\lambda\rho f\ell\, I_0(L_{01}h)\{L_{11}\, I_0(L_{11}h)\, I_1(\ell h) - \ell\, I_1(L_{11}h)\, I_0(\ell h)\}},$$

$$\delta_{1m1} = 0 \quad \text{for } m \geqq 2, \tag{74}$$

$$2\beta_{002}\ell + \text{Re}\left\{\alpha_{101} K_{10}^2\, I_1(K_{10}h) + \beta_{101}\ell^2\, I_1(\ell h)\right\} = 0, \tag{75}$$

$$\alpha_{012}\ell^2 K_{01}\, I_0(K_{01}h) = \frac{P}{4\lambda\mu}\left\{1 - \frac{I_1(K_{01}h)}{K_{01}h\, I_0(K_{01}h)}\right\}$$
$$- \frac{\ell}{2}(K_{11}^2 - \ell^2)\alpha_{111}\, I_1(K_{11}h)$$
$$- \frac{\ell}{2}(L_{11}^2 - \ell^2)\bar{\delta}_{111}\, I_1(\ell h), \tag{76}$$

and

$$\alpha_{0m2} = 0 \quad \text{for } m \geqq 2. \tag{77}$$

Since α_{0m0}, α_{1m1} etc. are all zero for $m \geqq 2$, it follows that ψ to order $(\eta/h)^2$ does not contain any terms in cos mft and sin mft for $m \geqq 2$. It can be shown that this is also true in the perturbation solution for ψ to any power of η/h so that some simplification could have been achieved by assuming that m takes only the values 0 and 1 in the expression (36) for ψ and (37) for ω.

$K_{10}^2 = \ell^2 - i\ell\sigma/_\nu$ so that the denominator of the expressions for α_{101} and β_{101} can be zero only if $\ell = 0$, which is of no interest, or if $\sigma = 0$, the fixed wall case, which will be considered later in 4.3.3.

The denominator of the expressions for α_{111} and β_{111} is zero only if $K_{11}^2 = \ell^2$, i.e. if $\ell\sigma - f = 0$ and in this case the alternative solution discussed later in 4.3.6 must be used.

The denominator of the expressions for γ_{111} and δ_{111} is never zero, if ℓ is non zero.

4.2.2 Calculation of the flux through the tube

The average flux per wavelength, $\bar{\psi}$, can be calculated by evaluating ψ on the boundary and integrating it with respect to z over one wave-length. In general, this average flux will consist of two parts, one independent of t which will be called the net flux, the other a linear function of cos ft and sin ft. The net flux is due to the motion of the walls and the remainder is due to the prescribed pressure gradient.

The expression for $\bar{\psi}$ thus obtained is similar to that of 4.1.2 namely,

$$\bar{\psi} = \bar{\psi}_0 + \bar{\psi}_2(\eta/h)^2 + \bar{\psi}_4(\eta/h)^4 + \ldots$$

The expressions for $\bar{\psi}_0$ and $\bar{\psi}_2$ have been calculated for the general case and are as follows:

For channel flow:

$$\bar{\psi}_0 = \frac{Ph}{\lambda\rho f}\sin ft + \operatorname{Re}\left\{\frac{iP\tanh K_{01}h}{\lambda\rho f K_{01}}e^{ift}\right\}, \tag{78}$$

$$\bar{\psi}_2 = -h^2\operatorname{Re}\left\{\frac{\ell h}{2}\left(\ell^2\sinh\ell h\,\beta_{101} + K_{10}^2\sinh K_{10}h\,\alpha_{101}\right)\right.$$
$$+\frac{K_{01}\tanh K_{01}h}{2f}e^{ift}\left(\frac{iP}{\lambda\rho}\right.$$
$$\left.\left.+\ell\sinh\ell h\left[(\ell\sigma - f)\beta_{111} - (\ell\sigma + f)\bar{\delta}_{111}\right]\right)\right\}; \tag{79}$$

For pipe flow:

$$\bar{\psi}_0 = \frac{Ph^2}{2\lambda\rho f} \sin ft + \text{Re}\left\{\frac{iPh\ I_1(K_{01}h)}{\lambda\rho f K_{01}\ I_0(K_{01}h)} e^{ift}\right\}, \tag{80}$$

$$\bar{\psi}_2 = -h^2\ \text{Re}\left\{\frac{\ell h^2}{4}\left(\ell^2\ I_1(\ell h)\beta_{101} + K_{10}^2\ I_1(K_{10}h)\alpha_{101}\right)\right.$$

$$-\frac{iP\ I_1^2(K_{01}h)}{4\lambda\rho f\ I_0^2(K_{01}h)} e^{ift} + \frac{K_{01}h\ I_1(K_{01}h)}{2f\ I_0(K_{01}h)} e^{ift}\left(\frac{iP}{\lambda\rho}\right.$$

$$\left.\left.+\ \ell\ I_1(\ell h)\left[(\ell\sigma - f)\beta_{111} - (\ell\sigma + f)\bar{\delta}_{111}\right]\right)\right\}, \tag{81}$$

where the expressions for α_{101}, β_{101}, β_{111} and δ_{111} are given in 4.2.1.

It can be seen, from (40), that $K_{01}^2 = {}^{if}/_{\nu}$ so that, in both channel flow and pipe flow, $\bar{\psi}_0$ is dependent on P and f but independent of σ which is consistent with the simpler case of peristalsis with constant pressure gradient.

It can also be seen that $K_{10}^2 = \ell^2 - {}^{i\ell\sigma}/_{\nu}$ so that the first term in $\bar{\psi}_2$ for channel flow and pipe flow depends on σ but not on P and f. However, $K_{11}^2 = \ell^2 - i(\ell\sigma - f)/_{\nu}$ and $L_{11}^2 = \ell^2 - i(\ell\sigma + f)/_{\nu}$ so that β_{111} and $\bar{\delta}_{111}$ depend on both σ and p. This means that there are interaction terms in $\bar{\psi}_2$ as well as terms which depend only on σ or only on P and f.

The net flux in this general case for channel flow is:

$$-\frac{\ell h^3}{2}\ \text{Re}\left\{\ell^2\ \sinh \ell h\ \beta_{101} + K_{10}^2\ \sinh K_{10}h\ \alpha_{101}\right\}(\eta/h)^2 \tag{82}$$

where α_{101} and β_{101} are given by (58) and (59).

For pipe flow the net flux is

$$-\frac{\ell h^4}{4}\,\mathrm{Re}\left\{\ell^2\, I_1(\ell h)\beta_{101} + K_{10}^2\, I_1(K_{10}h)\alpha_{101}\right\}, \tag{83}$$

where α_{101} and β_{101} are given by (69) and (70).

4.3 Average flux in particular cases

It is now useful to consider the average flux through the tube in the particular cases which can be derived from the general case already discussed. These cases are listed in Table 1 below, together with the sections in which they are considered.

Table 1

	Peristalsis			Fixed Wall	
	Osc pg	Const pg	Zero pg	Osc pg	Const pg
σ large	4.2.2	4.3.1	4.3.2	4.3.3	4.1.2
σ small	4.3.4	4.1.2	4.1.2	4.3.3	4.1.2

One other particular case, namely the flux through a uniform pipe, is discussed in 4.3.5.

It was shown in 4.2 that the solution for ψ was not valid when $n\ell\sigma - mf = 0$; the solution which is valid in this case is given in 4.3.6.

4.3.1 Peristalsis, constant pressure gradient, σ large

This particular case can be derived from the general case by letting $f \to 0$. If this is done, then, for channel flow $\bar{\psi}_o$ from (78) tends to

$\frac{Ph^3}{3\lambda\mu}$ and for pipe flow, $\bar{\psi}_o$ from (80) tends to $\frac{Ph^4}{16\lambda\mu}$ and these agree with the values obtained in 4.1.2 for the σ small case.

The first term in ψ_2 is independent of P and f so is not affected by letting f tend to zero. The limit as f tends to zero of the second part of $\bar{\psi}_2$ can be found easily and it turns out that,

for channel flow:

$$\bar{\psi} = \frac{Ph^3}{3\lambda\mu} + \bar{\psi}_2(\eta/h)^2, \tag{84}$$

$$\text{where } \bar{\psi}_2 = -h^3 \operatorname{Re}\left\{\frac{\ell}{2}\left(\ell^2 \sinh \ell h\, \beta_{101} + K_{10}^2 \sinh K_{10}h\, \alpha_{101}\right)\right.$$

$$\left. - \frac{P}{2\lambda\mu}\left(1 + \frac{i\sigma\, \ell h \sinh K_{10}h \sinh \ell h}{\nu(K_{10}\cosh K_{10}h \sinh \ell h - \ell \sinh K_{10}h \cosh \ell h)}\right)\right\}; \tag{85}$$

for pipe flow:

$$\bar{\psi} = \frac{Ph^4}{16\lambda\mu} + \bar{\psi}_2(\eta/h)^2, \tag{86}$$

$$\text{where } \bar{\psi}_2 = -h^4 \operatorname{Re}\left\{\frac{\ell h^2}{4}\left(\ell^2 I_1(\ell h)\beta_{101} + K_{10}^2 I_1(K_{10}h)\alpha_{101}\right)\right.$$

$$\left. - \frac{P}{16\lambda\mu}\left(3 + \frac{2i\sigma\, \ell h\, I_1(K_{10}h)\, I_1(\ell h)}{\nu[K_{10} I_o(K_{10}h) I_1(\ell h) - \ell I_1(K_{10}h) I_o(\ell h)]}\right)\right\}. \tag{87}$$

This average flux is independent of time and so it is also the net flux. If the approximation $\sigma/\ell\nu = o(1)$ is now made then the resulting average flux agrees with that given by Burns and Parkes (1967) and in section 4.1.2.

4.3.2 Peristalsis, zero pressure gradient, σ large

This particular case can be derived either from the general case or from the case discussed in 4.3.1 by putting $P = 0$. If this is done then for channel flow

$$\bar{\psi} = -\frac{\ell h^3}{2} \operatorname{Re}\left\{\ell^2 \sinh \ell h \beta_{101} + K_{10}^2 \sinh K_{10} h\, \alpha_{101}\right\} (\eta/h)^2, \tag{88}$$

and for pipe flow,

$$\bar{\psi} = -\frac{\ell h^4}{4} \operatorname{Re}\left\{\ell^2 I_1(\ell h)\beta_{101} + K_{10}^2 I_1(K_{10}h)\, \alpha_{101}\right\} (\eta/h)^2. \tag{89}$$

4.3.3 Fixed wall, sinusoidal pressure gradient

This particular case can be derived from the general one by letting σ tend to zero. If this is done then $\bar{\psi}_0$ given by (78) or (80) is not affected and the first term in $\bar{\psi}_2$ given by (79) or (81) tends to zero. The limit as σ tends to zero of the second part of $\bar{\psi}_2$ can be found easily giving the following results:

For channel flow,

$$\bar{\psi}_2 = -h^2 \operatorname{Re}\left\{\frac{iPK_{01} \tanh K_{01}}{2\lambda\rho f} e^{ift}\left(1 - \frac{K_{01} \tanh K_{01} h \sinh K_{11} h \sinh \ell h}{K_{11} \cosh K_{11} h \sinh \ell h - \ell \sinh K_{11} h \cosh \ell h}\right)\right\}, \tag{90}$$

For pipe flow,

$$\bar{\psi}_2 = -h^2 \operatorname{Re}\left\{\frac{iPK_{01} h I_1(K_{01}h)}{2\lambda\rho f\, I_0(K_{01}h)} e^{ift}\left(1 - \frac{K_{01} I_1(K_{01}h) I_1(K_{11}h) I_1(\ell h)}{I_0(K_{01}h)[K_{11} I_0(K_{11}h) I_1(\ell h) - \ell I_1(K_{11}h) I_0(\ell h)]} - \frac{I_1(K_{01}h)}{2K_{01}h\, I_0(K_{01}h)}\right)\right\} \tag{91}$$

Thus the average flux to order $(\eta/h)^2$ through a tube with fixed sinusoidal walls and sinusoidal pressure gradient is $\bar{\psi} = \bar{\psi}_o + \bar{\psi}_2(\eta/h)^2$ where, for channel flow, $\bar{\psi}_o$ is given by (78) and $\bar{\psi}_2$ by (90) while for pipe flow, $\bar{\psi}_o$ is given by (80) and $\bar{\psi}_2$ by (91). In this case the average flux is oscillatory and there is no net flux.

4.3.4 Peristalsis, oscillatory pressure gradient, σ small

This particular case is derived from the general case by taking $\sigma/\ell\nu = o(1)$ and $f/\ell^2\nu = O(1)$. In both channel flow and pipe flow $\bar{\psi}_o$ is independent of σ and so is not affected by this approximation.

From (40),
$$K_{11}^2 = \ell^2 - i\ell^2\left(\frac{\sigma}{\ell\nu} - \frac{f}{\ell^2\nu}\right) \rightarrow \ell^2 + \frac{if}{\nu}$$
and
$$L_{11}^2 = \ell^2 - i\ell^2\left(\frac{\sigma}{\ell\nu} + \frac{f}{\ell^2\nu}\right) \rightarrow \ell^2 - \frac{if}{\nu},$$

so that K_{11} and L_{11} are the same as in the $\sigma = 0$ case (4.3.3).

$$K_{10}^2 = \ell^2 - i\ell\sigma/\nu \quad \text{so that } K_{10} \approx \ell - i\sigma/2\nu,$$

$$\sinh K_{10}h \approx \sinh \ell h - \frac{i\sigma h}{2\nu}\cosh \ell h,$$

$$\cosh K_{10}h \approx \cosh \ell h - \frac{i\sigma h}{2\nu}\sinh \ell h,$$

$$I_o(K_{10}h) \approx I_o(\ell h) - \frac{i\sigma h}{2\nu} I_o'(\ell h),$$

and
$$I_1(K_{10}h) \approx I_1(\ell h) - \frac{i\sigma h}{2\nu} I_1'(\ell h).$$

If these expressions are substituted in $\bar{\psi}_2$ given by (79) or (81), and σ^2 is neglected, the following results are obtained:

For channel flow,

$$\bar{\psi} = \bar{\psi}_o + \bar{\psi}_2(\eta/h)^2$$

where $\bar{\psi}_o$ is given by (78) and

$$\bar{\psi}_2 = -h^2 \mathrm{Re}\left\{\frac{\sigma\ell^2 h}{2}\left(\frac{2\ell h + \sinh 2\ell h}{2\ell h - \sinh 2\ell h}\right) + \frac{iPK_{o1}\tanh K_{o1}h}{2\lambda\rho f}\, e^{ift}\left(1 - \frac{K_{o1}\tanh K_{o1}h \sinh K_{11}h \sinh \ell h}{K_{11}\cosh K_{11}h \sinh \ell h - \ell \sinh K_{11}h \cosh \ell h}\right)\right\}; \quad (92)$$

For pipe flow,

$\bar{\psi}_o$ is given by (80) and

$$\bar{\psi}_2 = -h^2 \mathrm{Re}\left\{\frac{\sigma\ell^2 h^2}{4}\left(\frac{I_1^2(\ell h) - I_o^2(\ell h)}{I_1^2(\ell h) - I_o(\ell h)\, I_2(\ell h)}\right) + \frac{iPK_{o1}h\, I_1(K_{o1}h)}{2\lambda\rho f\, I_o(K_{o1}h)}\, e^{ift}\left(1 - \frac{K_{o1}\, I_1(K_{o1}h)\, I_1(K_{11}h)\, I_1(\ell h)}{I_o(K_{o1}h)\,[K_{11}\, I_o(K_{11}h)\, I_1(\ell h) - \ell\, I_1(K_{11}h)\, I_o(\ell h)]} - \frac{I_1(K_{o1}h)}{2K_{o1}h\, I_o(K_{o1}h)}\right)\right\}. \quad (93)$$

In this case, with σ small, the interaction term disappears and so the average flux consists of two parts, the net flux due to the moving walls and an oscillatory flux due to the oscillatory pressure gradient. If now $f \to 0$ then the average flux given in this section tends to the simpler case of peristalsis with constant pressure gradient discussed in 4.1.2.

4.3.5 Flux through a uniform pipe

The flux due to a sinusoidal pressure gradient through a uniform pipe of radius h can be found from (80), i.e., $\bar{\psi}_o$, by replacing P by $P_o\lambda$ where P_o is the pressure drop per unit length. If this is done, then the flux through the pipe reduces to

$$\frac{\pi h^2 P_o}{\rho f}\left[\sin ft + \mathrm{Re}\left\{\frac{2i\, I_1(K_{o1}h)}{K_{o1}h\, I_o(K_{o1}h)}\, e^{ift}\right\}\right]. \tag{94}$$

This expression agrees with that derived by Womersley and given in McDonald and Taylor (1959) if the appropriate changes in notation are used.

4.3.6 Solution when $n\ell\sigma - mf = 0$

It was shown in 4.2 that equation (48) is only valid if $K^2_{nm} - n^2\ell^2 \neq 0$ and $L^2_{nm} - n^2\ell^2 \neq 0$ and it was also shown that $K^2_{nm} - n^2\ell^2 = 0$ if $n\ell\sigma - mf = 0$ but that $L^2_{nm} - n^2\ell^2$ is never zero.

If $n\ell\sigma - mf = 0$ then (40) gives $K_{nm} = n\ell$ and (44) is changed to

$$\frac{d^2\psi_{nm}}{dy^2} - \frac{k}{y}\frac{d\psi_{nm}}{dy} - n^2\ell^2\psi_{nm} = A_{nm} y^{\frac{k+1}{2}} I_{\frac{k+1}{2}}(n\ell y) \tag{95}$$

but only for the critical values of n and m satisfying $n\ell\sigma - mf = 0$.
The solution of (95) is

$$\psi_{nm} = y^{\frac{k+1}{2}}\left\{\frac{A_{nm}}{2n\ell} y\, I_{\frac{k-1}{2}}(n\ell y) + B_{nm}\, I_{\frac{k+1}{2}}(n\ell y)\right\}$$

which, by appropriate change of coefficients gives, for channel flow,

$$\psi_{nm} = a_{nm} y \cosh n\ell y + b_{nm} \sinh n\ell y \tag{96}$$

and, for pipe flow,

$$\psi_{nm} = y\Big(a_{nm} y\, I_o(n\ell y) + b_{nm}\, I_1(n\ell y)\Big). \tag{97}$$

The stream function, therefore, for channel flow is the expression given in (51) for all m and n except those critical values satisfying the condition $n\ell\sigma - mf = 0$, and then, for these values only, expression (96) must be used.

Similarly for pipe flow, the stream function is the expression given in (53) for all m and n except the critical values for which (97) must be used.

If $m = 0$ is one critical value then $n = 0$ must be the other and this is not applicable. If $m = 1$ is one critical value then $n = f/\ell\sigma$ and so the critical value for n could be any positive whole number. However, in calculating average flux to order η^2, the only critical value that needs to be considered is $n = 1$. Thus the special case $f = \ell\sigma$ is the critical one and this is the case when the frequency of the pressure fluctuation is equal to the frequency of oscillation of the wall. If both of these frequencies are zero then the problem reduces to that of flow through a tube with fixed walls and constant pressure gradient discussed by Burns (1965) using solutions of this kind.

The only coefficients that need to be changed are α_{111} and β_{111} and these become, for channel flow,

$$\alpha_{111} = -\frac{\alpha_{010}\,K_{01}^2 \sinh K_{01}h \sinh \ell h}{\ell(\sinh 2\ell h - 2\ell h)}$$

and (98)

$$\beta_{111} = \frac{\alpha_{010}\,K_{01}^2 h \sinh K_{01}h \cosh \ell h}{\ell(\sinh 2\ell h - 2\ell h)}$$

and for pipe flow,

$$\alpha_{111} = - \frac{\alpha_{010} K_{01}^2 \; I_1(K_{01} h) \; I_1(\ell h)}{2\ell^2 h [I_1^2(\ell h) - I_o(\ell h) \; I_2(\ell h)]}$$

and (99)

$$\beta_{111} = \frac{\alpha_{010} K_{01}^2 \; h \; I_1(K_{01} h) \; I_o(\ell h)}{2\ell^2 h [I_1^2(\ell h) - I_o(\ell h) \; I_2(\ell h)]} .$$

The average flux $\bar{\psi}$ is still given by

$$\bar{\psi} = \bar{\psi}_o + \bar{\psi}_2 (\eta/h)^2 + \dots ,$$

with $\bar{\psi}_o$ and $\bar{\psi}_2$ given by (78) and (79) for channel flow and (80) and (81) for pipe flow but with β_{111} as defined in (98) and (99).

5. Numerical Calculations of Flux, Streamlines and Velocity Distribution

The computations done so far have been of an exploratory nature. As was to be expected, they have shown that more detailed calculations are needed to obtain the full range of results available from the theory. It is hoped that these calculations will be carried out later.

In the first part of the numerical calculations dealing with the simple case 4.1 the two causes of motion have been treated separately, i.e. peristaltic flow with no pressure gradient, given by putting $P = 0$ and flow through a fixed tube with a prescribed pressure gradient given by putting $\sigma = 0$.

5.1 Peristalsis, Zero pressure gradient, σ small

5.1.1 Flux through tube

In the case of channel flow the non-dimensional flux $\ell\bar{\psi}/\sigma$ has been calculated both to order $(\eta/h)^2$ and to order $(\eta/h)^4$ for a range of values of the two non-dimensional parameters $\ell\eta$ and ℓh. These results are displayed in Figure 2 by showing the graphs of $\ell\bar{\psi}/\sigma$ against the ratio η/h for values of ℓh ranging from 0·25 to 2·0.

For pipe flow, the non-dimensional flux $\ell^2\bar{\psi}/\sigma$ has been similarly calculated and the results are shown in the same way in Figure 3.

Throughout the development of the theory there has been an implicit assumption that conditions ensuring the convergence of the various processes are satisfied. It is clear that, for a given value of ℓh, a perturbation solution in powers of η/h can be expected to converge only for values of η/h not exceeding some value depending on ℓh. Physically of course $\eta/h < 1$.

It can be seen from Figures 2 and 3 that, for each value of ℓh, as η/h increases, the curves for the non-dimensional flux to order $(\eta/h)^2$ and to order $(\eta/h)^4$ begin to diverge appreciably which suggests that the limit to the convergence of the process has been reached. The calculations have also been carried out to order $(\eta/h)^6$ and the curve for $\ell\bar{\psi}/\sigma$ to this order lies between the curves for $\ell\bar{\psi}/\sigma$ to order $(\eta/h)^2$ and to order $(\eta/h)^4$. This suggests that the curve for order $(\eta/h)^2$ is the upper bound and the curve for order $(\eta/h)^4$ is the lower bound.

A point has been indicated for each value of ℓh, where the $(\eta/h)^4$ term first becomes one tenth of the $(\eta/h)^2$ term. Hence if the application of the perturbation theory is limited to values of η/h below those indicated then the flux through the tube will be known to an accuracy of better than 10%. This is an arbitrary criterion of course and Taylor (1951) uses 25% instead of 10%.

It will be seen from Figures 2 and 3 that the indicated value of η/h decreases as ℓh increases.

Figure 2 shows that if η/h is constant, the non-dimensional flux per unit length normal to the plane of motion through a channel of mean breadth h, for a given wave velocity σ and given wave-length λ, is roughly proportional to h (i.e. to the area of the cross-section). If η is constant then the flux per unit length is roughly inversely proportional to h.

Figure 3 shows that if η/h is constant, the flux through a pipe of mean radius h is similarly roughly proportional to h^2 (i.e. to the area of cross-section) and that if η is constant then the flux is roughly independent of h.

These results can be expressed in a different way be defining a mean velocity $\bar{u}$ related to the average flux $\bar{\psi}$ by $\bar{\psi} = A\bar{u}$ where A is πh^2, the mean cross-sectional area for pipe flow, and is h, the mean cross-sectional area per unit depth normal to the plane of motion for channel flow. A plot of $\bar{u}/\sigma$ as a function of η/h for channel flow and pipe flow is displayed in Figure 4.

5.1.2 Streamlines

Figure 5 shows the streamlines in two dimensional flow for the case $\ell h = 0{\cdot}25$ and $\eta/h = 0{\cdot}1$ taking ψ to order $(\eta/h)^2$. The streamlines have only been drawn for positive y but of course they are symmetrical about the z-axis which has been taken as $\psi = 0$. The streamline $\psi = 0$ in addition to lying along the z-axis, also runs approximately perpendicular to the z-axis at $\ell z = 0{\cdot}56\pi$ and $\ell z = 1{\cdot}44\pi$ where $z = 0$ corresponds to a peak on the boundary.

At the boundary, $y = h + \eta \cos \ell z$, the streamlines are parallel to the y-axis.

5.1.3 Velocity Distributions

Figures 6 and 7 show the distribution of the velocity parallel to and perpendicular to the axis of the channel in two dimensional flow for the case $\ell h = 0{\cdot}25$ and $\eta/h = 0{\cdot}1$. In both directions the flow is symmetrical about $\ell z = \pi$ (i.e. a trough in the boundary). The maximum velocity for the case considered is $0{\cdot}16\sigma$ along the axis of the channel and $0{\cdot}25\sigma$ at the boundary, perpendicular to the axis.

5.2 Fixed boundary with prescribed constant pressure gradient (i.e. $\sigma = 0$)

5.2.1 Flux through tube

The ratio $\bar{\psi}/\bar{\psi}_o$ has been computed for both channel flow and pipe flow for several values of ℓh and plotted against η/h in Figures 8 and 9. In each case the flux is given to order $(\eta/h)^2$ and to order $(\eta/h)^4$.

As in the case of peristaltic flow the curves for $\bar{\psi}/\bar{\psi}_o$ to order $(\eta/h)^2$ and to order $(\eta/h)^4$ diverge appreciably. Points have been indicated on the curves where the $(\eta/h)^4$ term first becomes one-tenth of the $(\eta/h)^2$ term so that, for values of η/h below those indicated, the flux through the tube will be known to better than 10%.

It should be noted that increasing the amplitude η for a given h and ℓ has opposite effects in the two cases considered. In peristaltic motion, the flux through the tube increases with η but with a prescribed pressure gradient and fixed boundary, the flux decreases as η is increased.

The flux of a viscous fluid through a uniform two-dimensional channel or through a uniform pipe of circular cross-section is given by $\bar{\psi}_o$ which is the value of $\bar{\psi}$ when $\eta = 0$.

For two dimensional flow the flux per unit length normal to the plane of motion through a uniform channel of breadth 2h is $2\bar{\psi}_o$, while for axi-symmetrical flow the flux through a uniform pipe of radius h is $2\pi\bar{\psi}_o$. $\bar{\psi}_o$ is given in the two cases by (34) and (35) and the values given there can be expressed in terms of the original quantities by using (21), (22) and (18) noting however that P, the pressure drop over a wave length should now be replaced by $P_o\, 2\pi/\ell$ where P_o is the pressure drop per unit length. The resulting values of the flux are $2P_o h^3/3\mu$ for the channel and $\pi P_o h^4/8\mu$ for the pipe.

These are the well-known values obtained by solving the full Navier-Stokes equations of motion. For this simple type of flow, all the non-linear terms vanish and the Stokes equations give the flow exactly.

5.2.2 Streamlines

The streamlines in this case of steady flow follow the expected pattern, i.e. approximate cosine curves whose amplitude increases from zero on the axis of symmetry to η on the boundary. Consequently they are not included in graphical form. Neither are the velocity distributions.

5.3 Peristalsis, Zero pressure gradient, σ large

5.3.1 Average flux through tube

The average flux through the tube in this case for channel flow was derived in 4.3.2 and is given by

$$\bar{\psi} = -\frac{\ell h^3}{2}\,\mathrm{Re}\left\{\ell^2 \sinh \ell h\, \beta_{1 0 1} + K_{1 0}^2 \sinh K_{1 0} h\, \alpha_{1 0 1}\right\}, \tag{87}$$

where α_{101} and β_{101} are defined in (58) and (59).

Here $K^2_{10} = \ell^2 - i\ell\sigma/\nu$ and so $\bar{\psi}$ is not simply proportional to σ as it was in 5.1 with σ small. (87) can be written in a convenient non-dimensional form as

$$\frac{\bar{\psi}}{\nu} = - \operatorname{Re}\left\{\frac{\sigma h}{\nu}\frac{(\ell h)^2}{2}\left(1 + \frac{i\sigma h}{\nu}\frac{1}{K_{10}h \coth K_{10}h \tanh \ell h - \ell h}\right)\right\}(\eta/h)^2.$$

This non-dimensional flux can then be computed for various values of the non-dimensional parameters ℓh and $\sigma h/\nu$. These computations have been done for two values of ℓh, namely 0·25 and 1, and for several values of $\sigma h/\nu$ from 0·1 to 10.

With $\ell h = 0{\cdot}25$ the non-dimensional flux to order $(\eta/h)^2$ for σ large does not differ significantly from the σ small case (see Figure 2) for $\sigma h/\nu$ up to 10. With $\ell h = 1$ the non-dimensional flux to order $(\eta/h)^2$ for σ large is bigger than in the σ small case (see Figure 2). This difference is negligible for $\sigma h/\nu = 1$ but increases to 17% for $\sigma h/\nu = 10$.

Similar calculations will be done for pipe flow, but previous work suggests that the results will be similar to channel flow.

The average flux to order $(\eta/h)^4$ has not been derived for the general case simply because of the tedious algebra involved. However, in view of what has been done so far, it seems unlikely that there will be much difference in $\bar{\psi}_4$ for the two cases, σ large and σ small.

Several references to peristaltic motion in the body have been found but none of them gives the amplitude of the progressive wave. Houssay (1955) gives the wave velocity for the ureter as 2 to 3 cm/sec. with frequency 3 cycles/min. which corresponds to a wave-length of about 8 cm or $\ell = 0{\cdot}8\ \text{cm}^{-1}$. The calibre of the ureter apparently varies along its length, but an average value of the radius (i.e. h) would be 1 cm. The viscosity of the fluid also varies but an average value would be about 0·2 in cgs units. Using these rather rough values gives $\ell h = 0{\cdot}8$ and $^{\sigma h}/_{\nu} = 12{\cdot}5$ so that for this case the "σ small" approximation given in 4.1 and 5.1 would be good enough.

It should be remembered that the expressions derived may not be valid if σ is very large because this may produce velocities which are so large that "squares of velocities" in the Navier-Stokes equation (3) cannot be neglected.

5.4 Fixed boundary, sinusoidal pressure gradient

5.4.1 Average flux through tube

The average flux to order $(\eta/h)^2$ for channel flow was derived in 4.3.3 and is $\bar{\psi} = \bar{\psi}_o + \bar{\psi}_2(\eta/h)^2$ where $\bar{\psi}_o$ is given by (78) and $\bar{\psi}_2$ by (90). It can be written in non-dimensional form as follows:

$$\frac{\mu}{Ph^2}\bar{\psi} = \frac{\ell h}{2\pi}\frac{\nu}{fh^2}\left[\sin ft + \text{Re}\left\{\frac{i\tanh K_{o1}h}{K_{o1}h}e^{ift}\right\}\right.$$
$$- \text{Re}\left\{\frac{iK_{o1}h\tanh K_{o1}h}{2}e^{ift}\left(1\right.\right.$$
$$\left.\left.\left.- \frac{K_{o1}h\tanh K_{o1}h}{K_{11}h\coth K_{11}h - \ell h\coth \ell h}\right)\right\}(\eta/h)^2\right],$$

where $K_{01}^2 = {}^{if}/\nu$ and $K_{11}^2 = \ell^2 + {}^{if}/\nu$.

This non-dimensional flux can be computed for various values of the non-dimensional parameters ℓh and ${}^{fh^2}/\nu$. The computations have been done for ${}^{fh^2}/\nu = 0{\cdot}125$, $\ell h = 0{\cdot}25$ and 1, to order $(\eta/h)^2$ and $(\eta/h)^4$. They have also been done for ${}^{fh^2}/\nu = 0{\cdot}5$, $\ell h = 0{\cdot}25$ and 1, to order $(\eta/h)^2$ only.

This non-dimensional flux is of the form A cos ft + B sin ft or $\sqrt{A^2 + B^2}\cos(ft - \epsilon)$ where A and B are power series in $(\eta/h)^2$ and $\epsilon = \tan^{-1}{}^{B}/_{A}$. Hence a pressure difference of -P cos ft per wave-length produces an average non dimensional flux of $\sqrt{A^2 + B^2}\cos(ft - \epsilon)$ and there is no net flux.

The amplitude of the non-dimensional flux is displayed in Figure 10 as a function of η/h to order $(\eta/h)^2$ and $(\eta/h)^4$ where appropriate. The curves begin to diverge appreciably just as they did in the previous cases showing that the limit to the convergence of the various processes has been reached. This range of convergence decreases as ${}^{fh^2}/\nu$ increases. Over the range of values of η/h and ℓh considered, the phase angle ϵ is almost constant but depends on ${}^{fh^2}/\nu$. For ${}^{fh^2}/\nu = 0{\cdot}125$, $\epsilon = 2{\cdot}5°$ while for ${}^{fh^2}/\nu = 0{\cdot}5$, $\epsilon = 11°$ so that the phase angle is approximately proportional to ${}^{fh^2}/\nu$ over this range.

For pipe flow the non-dimensional flux $\frac{\mu\bar{\psi}}{Ph^3}$ can be computed for various values of the non-dimensional parameters ℓh and ${}^{fh^2}/\nu$. These computations have been done for ${}^{fh^2}/\nu = 0{\cdot}125$, $0{\cdot}5$ and $\ell h = 0{\cdot}25$, 1 to order $(\eta/h)^2$ only and are shown in Figure 11. The phase angle ϵ, defined as for channel flow, is 0 for $fh^2/\nu = 0{\cdot}125$ and $4°$ for $fh^2/\nu = 0{\cdot}5$.

It can be seen in Figures 10 and 11 that the effect of increasing η/h, i.e. of making the walls more deeply corrugated, is to reduce the to and fro surge of the fluid in response to the fluctuating pressure gradient.

It can also be seen from Figures 10 and 11, that the effect of increasing fh^2/ν is to decrease the surge in the case of channel flow but to increase the surge in pipe flow. Further, more detailed, calculations may prove this effect to be spurious.

5.5 Peristalsis with sinusoidal pressure gradient

In order to calculate the average non-dimensional flux $\frac{\mu\bar{\psi}}{Ph^2}$ through a channel derived in 4.2.2 for σ large and in 4.3.4 for σ small, it is necessary to specify the four quantities ℓh, $\sigma h/\nu$, fh^2/ν and $\mu\nu/Ph^2$. These calculations will be done later.

6. Direct Calculation of the Coefficients a_n, b_n

The perturbation solution is only valid for η/h up to about $0{\cdot}4$. It would be interesting to have a solution which is valid for larger values of η/h so that more detailed information could be gained about the nature of the flow for large η.

The problem is to calculate the coefficients a_n and b_n in the stream function (23) for channel flow and (25) for pipe flow. A method of doing this has been tried for the two cases of (a) peristaltic flow with zero pressure gradient along a channel assuming σ small, and (b) fixed walls with constant pressure gradient.

The method was to compute the coefficients c_{nr}, h_{nr}, k_{nr}, p_{nr} and q_{nr} in the set of equations (31) and then to solve the equations for a_n and b_n. This set of equations is a doubly infinite set of linear algebraic equations, but as a_n, b_n are Fourier coefficients and tend to zero as n increases, it seems reasonable to seek approximate solutions by retaining only a small number of the coefficients and enough equations to solve for them. It should be possible to judge the accuracy of these solutions by comparing the results as the number of coefficients retained is increased.

6.1 Evaluation of the coefficients c_{nr}, etc.

For channel flow the boundary conditions on the wall lead to the following equations:

$$a_o y_1^2 + b_o + \sum_{n=1}^{\infty} \Big\{ a_n (\cosh n\ell y_1 + n\ell y_1 \sinh n\ell y_1)$$

$$+ n\ell b_n \cosh n\ell y_1 \Big\} \cos n\ell z = 0, \qquad (27)$$

$$\sum_{n=1}^{\infty} \Big\{ a_n n\ell y_1 \cosh n\ell y_1 + n\ell b_n \sinh n\ell y_1 \Big\} \sin n\ell z = \ell\eta\sigma \sin \ell z, \qquad (28)$$

where $y_1 = h + \eta \cos \ell z$.

These equations can be written as

$$b_o + \sum_{n=1}^{\infty} (a_n H_n + \ell b_n K_n) \cos n\ell z = - a_o y_1^2, \qquad (100)$$

$$\sum_{n=1}^{\infty} (a_n P_n + \ell b_n Q_n) \sin n\ell z = \ell\eta\sigma \sin \ell z, \qquad (101)$$

where $H_n = \cosh n\ell y_1 + n\ell y_1 \sinh n\ell y_1$,

$$K_n = n \cosh n\ell y_1,$$

$$P_n = n\ell y_1 \cosh n\ell y_1,$$

$$Q_n = n \sinh n\ell y_1.$$

Expanding $\cosh n\ell y_1$ and $\sinh n\ell y_1$ gives

$$H_n = (\cosh n\ell h + n\ell y_1 \sinh n\ell h) \cosh(n\ell\eta \cos \ell z)$$

$$+ (\sinh n\ell h + n\ell y_1 \cosh n\ell h) \sinh(n\ell\eta \cos \ell z),$$

$$K_n = n \cosh n\ell h \cosh(n\ell\eta \cos \ell z) + n \sinh n\ell h \sinh(n\ell\eta \cos \ell z),$$

$$P_n = n\ell y_1 \cosh n\ell h \cosh(n\ell\eta \cos \ell z) + n\ell y_1 \sinh n\ell h \sinh(n\ell\eta \cos \ell z),$$

$$Q_n = n \sinh n\ell h \cosh(n\ell\eta \cos \ell z) + n \cosh n\ell h \sinh(n\ell\eta \cos \ell z). \quad (102)$$

The coefficients of a_n and ℓb_n in (100) and (101) can be expanded as Fourier cosine and sine series as follows:

$$H_n \cos n\ell z = \sum_{r=0}^{\infty} h_{nr} \cos r\ell z,$$

$$K_n \cos n\ell z = \sum_{r=0}^{\infty} k_{nr} \cos r\ell z,$$

$$P_n \sin n\ell z = \sum_{r=1}^{\infty} p_{nr} \sin r\ell z,$$

$$Q_n \sin n\ell z = \sum_{r=1}^{\infty} q_{nr} \sin r\ell z. \quad (103)$$

If these series are substituted in (100) and (101) and rearranged so that terms in sin $r\ell z$ and cos $r\ell z$ are collected together, then the following equations are obtained:

$$b_o + \sum_{n=1}^{\infty} (a_n h_{no} + \ell b_n k_{no}) + \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (a_n h_{nr} + \ell b_n k_{nr}) \cos r\ell z$$

$$= -a_o \left\{ (h^2 + \tfrac{1}{2}\eta^2) + 2h\eta \cos \ell z + \tfrac{1}{2}\eta^2 \cos 2\ell z \right\}, \quad (104)$$

$$\sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (a_n p_{nr} + \ell b_n q_{nr}) \sin r\ell z = \ell\eta\sigma \sin \ell z. \quad (105)$$

Finally equating coefficients of cos $r\ell z$ and sin $r\ell z$ in (104) and (105) gives the following equations:

$$b_o + \sum_{n=1}^{\infty} (a_n h_{no} + \ell b_n k_{no}) = -a_o(h^2 + \tfrac{1}{2}\eta^2),$$

$$\sum_{n=1}^{\infty} (a_n h_{n1} + \ell b_n k_{n1}) = -2a_o h\eta,$$

$$\sum_{n=1}^{\infty} (a_n h_{n2} + \ell b_n k_{n2}) = -\tfrac{1}{2}a_o\eta^2,$$

$$\sum_{n=1}^{\infty} (a_n h_{nr} + \ell b_n k_{nr}) = 0 \quad \text{for } r = 3, 4, 5 \ldots,$$

$$\sum_{n=1}^{\infty} (a_n p_{n1} + \ell b_n q_{n1}) = \ell\eta\sigma,$$

$$\sum_{n=1}^{\infty} (a_n p_{nr} + \ell b_n q_{nr} = 0 \quad \text{for } r = 2, 3, 4 \ldots \quad . \qquad (106)$$

These equations which appeared in slightly less detail as (31), are the set of equations which needs to be solved for a_n and b_n.

Burns (1965) pointed out that

$$\cosh(p\cos x)\begin{cases}\cos 2tx\\ \sin 2tx\end{cases} = \begin{matrix} I_{2t}(p)\\ 0\end{cases} + \sum_{m=1}^{\infty}\left[I_{2|t-m|}(p) \pm I_{2(t+m)}(p)\right]\begin{cases}\cos 2mx\\ \sin 2mx\end{cases},$$

$$\sinh(p\cos x)\begin{cases}\cos 2tx\\ \sin 2tx\end{cases} = \sum_{m=0}^{\infty}\left[I_{|2t-2m-1|}(p) \pm I_{2t+2m+1}(p)\right]\begin{cases}\cos(2m+1)x\\ \sin(2m+1)x\end{cases},$$

$$\cosh(p\cos x)\begin{cases}\cos(2t+1)x\\ \sin(2t+1)x\end{cases} = \sum_{m=0}^{\infty}\left[I_{2|t-m|}(p) \pm I_{2(t+m+1)}(p)\right]\begin{cases}\cos(2m+1)x\\ \sin(2m+1)x\end{cases},$$

$$\sinh(p\cos x)\begin{cases}\cos(2t+1)x\\ \sin(2t+1)x\end{cases} = \begin{cases}I_{2t+1}(p)\\ 0\end{cases}$$

$$+\sum_{m=1}^{\infty}\left[I_{|2t-2m+1|}(p) \pm I_{2t+2m+1}(p)\right]\begin{cases}\cos 2mx\\ \sin 2mx\end{cases}. \qquad (107)$$

Substitution of (107) in (102) and comparison then with (103) gives the coefficients h_{nr}, k_{nr}, p_{nr} and q_{nr} for any r or n. They can be calculated to any desired accuracy and so the infinite set of equations (106) can be solved to any desired accuracy simply by taking n large enough in the approximating finite set. However it is not clear how large n should be in any particular case.

6.2 Application of the method

The method was investigated by considering firstly the case of peristaltic flow with zero pressure gradient in a channel with $\ell h = 0{\cdot}5$, $\ell\eta = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}4$, and secondly the case of fixed walls with constant pressure gradient for the same values of ℓh and $\ell\eta$.

Initially it was assumed that only values of $n \leq 1$ and $r \leq 1$ need be considered and the three equations to which (106) reduces under this assumption (with $a_o = 0$), namely,

$$a_1 p_{11} + \ell b_1 q_{11} = \ell\eta\sigma,$$

$$a_1 h_{11} + \ell b_1 k_{11} = 0,$$

and $$b_o + a_1 h_{10} + \ell b_1 k_{10} = 0,$$

were solved for a_1/σ, $\ell b_1/\sigma$ and b_o/σ. Having solved these equations, the average non-dimensional flux $\ell\bar{\psi}/\sigma$ was then calculated.

The next step was to consider values of $n \leqq 2$ and $r \leqq 2$ and then (106) reduces to five equations, namely,

$$\begin{aligned}
a_1 p_{11} + \ell b_1 q_{11} + a_2 p_{21} + \ell b_2 q_{21} &= \ell\eta\sigma, \\
a_1 p_{12} + \ell b_1 q_{12} + a_2 p_{22} + \ell b_2 q_{22} &= 0, \\
a_1 h_{11} + \ell b_1 k_{11} + a_2 h_{21} + \ell b_2 k_{21} &= 0, \\
a_1 h_{12} + \ell b_1 k_{12} + a_2 h_{22} + \ell b_2 k_{22} &= 0, \\
b_o + a_1 h_{10} + \ell b_1 k_{10} + a_2 h_{20} + \ell b_2 k_{20} &= 0.
\end{aligned}$$

These equations were solved for a_1/σ, a_2/σ, $\ell b_1/\sigma$, $\ell b_2/\sigma$ and b_o/σ, and then $\ell\bar{\psi}/\sigma$ was calculated.

The method was repeated for $n = 3$ and $n = 4$. The coefficients and flux in each case are given in Table 2. It can be seen from this table that the method works, and that the values of the coefficients and flux converge at a rate which depends on η/h. For $\eta/h = 0{\cdot}2$ the convergence is rapid as is apparent from the first two lines of the table; for $\eta/h = 0{\cdot}4$ the convergence is not so rapid and all four lines of the table are needed; for $\eta/h = 0{\cdot}8$ stability has not been reached.

The flux is also given in Table 3 together with the flux derived from the perturbation method. For $\eta/h = 0{\cdot}2$ and $0{\cdot}4$ the two estimates agree very well and differ by less than 2%. When comparing the flux for $\eta/h = 0{\cdot}8$ it should be remembered that the perturbation method is not valid in this case. However, it was suggested in 5.1.1 that the true flux should lie between the values to order $(\eta/h)^2$ and to order $(\eta/h)^4$ and the values given by the direct method do this.

The velocity along the tube has also been calculated using the coefficients in Table 2 and converges more slowly than the flux. This is illustrated in Table 2 by giving the velocity on the axis at $z = 0$.

The method has also been used for the case of fixed walls with constant pressure gradient and the results are given in Tables 4 and 5. Conclusions similar to those for peristalsis apply also to these results.

7. Other Boundary Conditions

The boundary conditions (10) used throughout the thesis, require that the wall of the tube be extensible. In the case of peristalsis with constant pressure gradient and σ small, the equations have also been solved for boundary conditions similar to those used by Taylor (1951). In the case of two dimensional flow these correspond to an inextensible wall.

These boundary conditions are, to order $(\ell\eta)^4$,

$$\frac{u}{\sigma} = -\frac{(\ell\eta)^4}{32} - \left(\frac{(\ell\eta)^2}{4} - \frac{(\ell\eta)^4}{8}\right)\cos 2\ell z - \frac{3(\ell\eta)^4}{64}\cos 4\ell z$$

and (108)

$$\frac{v}{\sigma} = \left(\ell\eta - \frac{(\ell\eta)^3}{8}\right)\sin \ell z + \frac{(\ell\eta)^3}{8}\sin 3\ell z$$

on $y = h + \eta \cos \ell z$.

The difference between the two solutions is very small and can be ignored.

8. Conclusion

The general case of peristaltic motion of a viscous fluid with a sinusoidal pressure gradient has been investigated assuming Stokes flow. A perturbation solution has been derived for the stream function in the two cases of channel flow and axi-symmetric pipe flow. Various particular solutions have then been derived from the general one. Some numerical results have been obtained which give an indication of the range of convergence of the solutions.

The flux through a tube has been derived as a series in powers of the square of the relative variation in the radius, i.e. $(\eta/h)^2$. It has been shown that, for a moderate range of η/h, a reasonable estimate of the flux can be obtained by using the first two terms of the series. The streamlines and velocity distributions have also been calculated for a particular case of peristaltic flow through a channel with zero pressure gradient.

For applications to peristaltic motion this moderate range of convergence may not be an irksome restriction although no data on the relative amplitudes in naturally occurring peristalsis has yet been found. For the flow of blood along arteries a ratio of $\eta/h = 0{\cdot}04$ is given by McDonald and Taylor (1959). Nevertheless, particularly for flow through fixed tubes with constant pressure gradient and for peristaltic flow with zero pressure gradient, it would be desirable to have solutions for values of η/h outside the range provided by the perturbation solution.

A more direct numerical approach has been tried for determining the coefficients in the Fourier series for the stream function. The method appears to be promising. For small η/h not many coefficients are needed and this method and the perturbation method agree very closely. For larger η/h, where the perturbation solution is invalid, more coefficients are needed and it becomes impractical to do all the calculations without the aid of a digital computer.

Appendix A

Evaluation of the coefficients a_n and b_n

A.1. Channel Flow

The boundary conditions on the tube wall lead to the following equations for channel flow:

$$a_o y_1^2 + b_o + \sum_{n=1}^{\infty} \left[(a_n + n\ell b_n) \cosh n\ell y_1 + n\ell a_n y_1 \sinh n\ell y_1 \right] \cos n\ell z = 0 \qquad (27)$$

and

$$\sum_{n=1}^{\infty} \left[a_n y_1 \cosh n\ell y_1 + b_n \sinh n\ell y_1 \right] n \sin n\ell z = \eta\sigma \sin \ell z , \qquad (28)$$

where $y_1 = h + \eta \cos \ell z$.

Cosh $n\ell y_1$ and sinh $n\ell y_1$ can be expanded as powers of cos ℓz as follows:

$$\cosh n\ell y_1 = \cosh n\ell h \sum_{r=0}^{\infty} \frac{(n\ell\eta \cos \ell z)^{2r}}{(2r)!} + \sinh n\ell h \sum_{r=0}^{\infty} \frac{(n\ell\eta \cos \ell z)^{2r+1}}{(2r+1)!} ,$$

$$\sinh n\ell y_1 = \sinh n\ell h \sum_{r=0}^{\infty} \frac{(n\ell\eta \cos \ell z)^{2r}}{(2r)!} + \cosh n\ell h \sum_{r=0}^{\infty} \frac{(n\ell\eta \cos \ell z)^{2r+1}}{(2r+1)!} .$$

If these expressions are substituted in equations (27) and (28) then the left hand side of (27) and (28) will consist of terms of the form $\cos^p \ell z \cos n\ell z$ and $\cos^p \ell z \sin n\ell z$ respectively. These in turn can be expressed as cosine and sine series and coefficients of corresponding terms compared giving equations which can be solved for a_n and b_n as power series in $\ell\eta$. To illustrate the method a_n and b_n will be determined in detail as far as $\ell\eta$.

Assume $(\ell\eta)^2$ and higher powers can be neglected. Then equation (27) leads to

$$a_o(h^2 + 2h\eta \cos \ell z) + b_o + \Big[(a_1 + \ell b_1)(\cosh \ell h + \ell\eta \sinh \ell h \cos \ell z)$$

$$+ \ell a_1(h + \eta \cos \ell z)(\sinh \ell h + \ell\eta \cosh \ell h \cos \ell z)\Big] \cos \ell z$$

$$+ \Big[(a_2 + 2\ell b_2)(\cosh 2\ell h + 2\ell\eta \sinh 2\ell h \cos \ell z)$$

$$+ 2\ell a_2(h + \eta \cos \ell z)(\sinh 2\ell h + 2\ell\eta \cosh 2\ell h \cos \ell z)\Big] \cos 2\ell z$$

$$+ \ldots = 0.$$

It can easily be seen that the following terms appear:

constant,

$\cos \ell z$,

$\cos^2 \ell z = \frac{1}{2}(1 + \cos 2\ell z)$,

$\cos 2\ell z$,

$\cos 2\ell z \cos \ell z = \frac{1}{2}(\cos \ell z + \cos 3\ell z)$,

$\cos 3\ell z$,

$\cos 3\ell z \cos \ell z = \frac{1}{2}(\cos 2\ell z + \cos 4\ell z)$,

etc.

Equating the coefficient of the constant term to zero therefore gives

$$a_o h^2 + b_o + \tfrac{1}{2}(a_1 + \ell b_1)\ell\eta \sinh \ell h + \tfrac{1}{2}\ell a_1(\ell h \cosh \ell h + \sinh \ell h)\eta = 0. \qquad \text{(A1)}$$

Similarly, equating to zero the coefficients of $\cos \ell z$ and $\cos 2\ell z$ gives

$$2h\eta a_o + (a_1 + \ell b_1) \cosh \ell h + a_1 \ell h \sinh \ell h + (a_2 + 2\ell b_2)\ell\eta \sinh 2\ell h$$

$$+ a_2 \ell\eta(2\ell h \cosh 2\ell h + \sinh 2\ell h) = 0, \qquad \text{(A2)}$$

$$\tfrac{1}{2}(a_1 + \ell b_1)\ell\eta \sinh \ell h + \tfrac{1}{2}\ell a_1 \eta(\ell h \cosh \ell h + \sinh \ell h)$$

$$+ (a_2 + 2\ell b_2) \cosh 2\ell h + 2\ell h a_2 \sinh 2\ell h + \tfrac{1}{2}(a_3 + 3\ell b_3)3\ell\eta \sinh 3\ell h$$

$$+ \tfrac{1}{2}\, 3\ell a_3 \eta(3\ell h \cosh 3\ell h + \sinh 3\ell h) = 0. \qquad \text{(A3)}$$

Equation (28) leads to

$$\sin \ell z \Big[a_1(h + \eta \cos \ell z)(\cosh \ell h + \ell\eta \sinh \ell h \cos \ell z)$$

$$+ b_1(\sinh \ell h + \ell\eta \cosh \ell h \cos \ell z) \Big]$$

$$+ 2 \sin 2\ell z \Big[a_2(h + \eta \cos \ell z)(\cosh 2\ell h + 2\ell\eta \sinh 2\ell h \cos \ell z)$$

$$+ b_2(\sinh 2\ell h + 2\ell\eta \cosh 2\ell h \cos \ell z) \Big]$$

$$+ 3 \sin 3\ell z \Big[a_3(h + \eta \cos \ell z)(\cosh 3\ell h + 3\ell\eta \sinh 3\ell h \cos \ell z)$$

$$+ b_3(\sinh 3\ell h + 3\ell\eta \cosh 3\ell h \cos \ell z) \Big]$$

$$+ \ldots = \eta\sigma \sin \ell z.$$

It can easily be seen that the following terms appear:

$\sin \ell z$,

$\sin \ell z \cos \ell z = \tfrac{1}{2} \sin 2\ell z$,

$\sin 2\ell z$,

$\sin 2\ell z \cos \ell z = \tfrac{1}{2}(\sin \ell z + \sin 3\ell z)$,

$\sin 3\ell z$,

$\sin 3\ell z \cos \ell z = \tfrac{1}{2}(\sin 2\ell z + \sin 4\ell z)$,

etc.

The equations obtained by equating coefficients of $\sin \ell z$ and $\sin 2\ell z$ are

$$a_1 h \cosh \ell h + b_1 \sinh \ell h + a_2 \eta(2\ell h \sinh 2\ell h + \cosh 2\ell h) + 2b_2 \ell\eta \cosh 2\ell h = \sigma\eta, \tag{A4}$$

and

$$\tfrac{1}{2} a_1 \eta(\ell h \sinh \ell h + \cosh \ell h) + \tfrac{1}{2} b_1 \ell\eta \cosh \ell h + 2a_2 h \cosh 2\ell h + 2b_2 \sinh 2\ell h + \tfrac{3}{2} a_3 \eta(3\ell h \sinh 3\ell h + \cosh 3\ell h) + \tfrac{3}{2} b_3 3\ell\eta \cosh 3\ell h = 0. \tag{A5}$$

Now let $a_n = \alpha_{no} + \alpha_{n1}\ell\eta + \dots$, for $n = 1, 2, 3 \dots$,

and $b_n = \beta_{no} + \beta_{n1}\ell\eta + \dots$, for $n = 0, 1, 2 \dots$.

Then if the coefficients of powers of $\ell\eta$ are compared, equations (A1), (A2), (A3), (A4) and (A5) each lead to a pair of equations, i.e. to the following ten equations:

$$a_o h^2 + \beta_{oo} = 0, \tag{A6}$$

$$\beta_{o1} + \tfrac{1}{2}(\alpha_{1o} + \ell\beta_{1o}) \sinh \ell h + \tfrac{1}{2} \alpha_{1o}(\ell h \cosh \ell h + \sinh \ell h) = 0, \tag{A7}$$

$$(\alpha_{1o} + \ell\beta_{1o}) \cosh \ell h + \alpha_{1o}\ell h \sinh \ell h = 0, \tag{A8}$$

$$\frac{2ha_o}{\ell} + (\alpha_{11} + \ell\beta_{11}) \cosh \ell h + \alpha_{11} \ell h \sinh \ell h + (\alpha_{2o} + 2\ell\beta_{2o}) \sinh 2\ell h + \alpha_{2o}(2\ell h \cosh 2\ell h + \sinh 2\ell h) = 0, \tag{A9}$$

$$(\alpha_{2o} + 2\ell\beta_{2o}) \cosh 2\ell h + \alpha_{2o} 2\ell h \sinh 2\ell h = 0, \tag{A10}$$

$$\tfrac{1}{2}(\alpha_{1o} + \ell\beta_{1o}) \sinh \ell h + \tfrac{1}{2} \alpha_{1o}(\ell h \cosh \ell h + \sinh \ell h) + (\alpha_{21} + 2\ell\beta_{21}) \cosh 2\ell h + 2\ell h\alpha_{21} \sinh 2\ell h + \tfrac{1}{2}(\alpha_{3o} + 3\ell\beta_{3o}) 3 \sinh 3\ell h + \tfrac{3}{2} \alpha_{3o}(3\ell h \cosh 3\ell h + \sinh 3\ell h) = 0, \tag{A11}$$

$$\alpha_{1o} h \cosh \ell h + \beta_{1o} \sinh \ell h = 0, \tag{A12}$$

$$\alpha_{11} h \cosh \ell h + \beta_{11} \sinh \ell h + \frac{\alpha_{2o}}{\ell}(2\ell h \sinh 2\ell h + \cosh 2\ell h)$$
$$+ 2\beta_{2o} \cosh 2\ell h = \frac{\sigma}{\ell}, \tag{A13}$$

$$2\alpha_{2o} h \cosh 2\ell h + 2\beta_{2o} \sinh 2\ell h = 0, \tag{A14}$$

$$\frac{1}{2}\frac{\alpha_{1o}}{\ell}(\ell h \sinh \ell h + \cosh \ell h) + \frac{1}{2}\beta_{1o} \cosh \ell h + 2\alpha_{21} h \cosh 2\ell h$$
$$+ 2\beta_{21} \sinh 2\ell h + \frac{3}{2}\frac{\alpha_{3o}}{\ell}(3\ell h \sinh 3\ell h + \cosh 3\ell h)$$
$$+ \frac{3}{2}\beta_{3o}\, 3 \cosh 3\ell h = 0. \tag{A15}$$

Clearly (A8) and (A12) show that α_{1o} and β_{1o} are zero; (A7) then shows that β_{o1} is zero; (A10) and (A14) show that α_{2o} and β_{2o} are zero. β_{oo} is given in terms of a_o from equation (A6); α_{11} and β_{11} are given in terms of a_o and σ by equations (A9) and (A13).

This method can be extended to give a_n and b_n as power series in $\ell\eta$ for all n and to any desired power of $\ell\eta$ and if this is done it turns out that

$$a_n = \sum_{t=0}^{\infty} \alpha_{n,n+2t} (\ell\eta)^{n+2t}, \quad \text{for } n = 1, 2, 3 \ldots,$$

and

$$b_n = \sum_{t=0}^{\infty} \beta_{n,n+2t} (\ell\eta)^{n+2t}, \quad \text{for } n = 0, 1, 2 \ldots .$$

If the coefficients a_n and b_n are needed to order $(\ell\eta)^n$ then it follows that $\frac{(n+1)(n+2)}{2}$ equations are needed but it turns out that these can be solved in pairs. Detailed calculations will be given for determining the coefficients to order $(\ell\eta)^2$.

The six equations needed are

$$a_o h^2 + \beta_{oo} = 0, \tag{A6}$$

$$\ell\alpha_{11}(\cosh \ell h + \ell h \sinh \ell h) + \ell^2\beta_{11} \cosh \ell h = -2ha_o, \tag{A16}$$

$$\alpha_{11}\ell h \cosh \ell h + \ell\beta_{11} \sinh \ell h = \sigma, \tag{A17}$$

$$\ell^2\alpha_{11}(2 \sinh \ell h + \ell h \cosh \ell h) + \ell^3\beta_{11} \sinh \ell h + a_o + 2\beta_{o2} = 0, \tag{A18}$$

$$\ell^2\alpha_{11}(2 \sinh \ell h + \ell h \cosh \ell h) + \ell^3\beta_{11} \sinh \ell h + a_o + 2\ell^2\alpha_{22}(\cosh 2\ell h + 2\ell h \sinh 2\ell h) + 4\ell^3\beta_{22} \cosh 2\ell h = 0, \tag{A19}$$

$$\ell\alpha_{11}(\cosh \ell h + \ell h \sinh \ell h) + \ell^2\beta_{11} \cosh \ell h + 4\ell^2\alpha_{22} h \cosh 2\ell h + 4\ell^2\beta_{22} \sinh 2\ell h = 0. \tag{A20}$$

Clearly (A6) gives $\beta_{oo} = -a_o h^2$,

(A16) and (A17) give
$$\alpha_{11} = -\frac{2(2ha_o \sinh \ell h + \sigma\ell \cosh \ell h)}{\ell(\sinh 2\ell h - 2\ell h)} \tag{A21}$$

and
$$\beta_{11} = \frac{2[2h^2 a_o \cosh \ell h + \sigma(\cosh \ell h + \ell h \sinh \ell h)]}{\ell(\sinh 2\ell h - 2\ell h)} \tag{A22}$$

(A18) then gives β_{o2} in terms of a_o and σ. (A19) and (A20) can then be solved for α_{22} and β_{22} in terms of a_o and σ. These expressions are long and cumbersome and so will not be included here.

A.2 Pipe Flow

The boundary conditions on the tube wall lead to the following equations for pipe flow.

$$a_o y_1^2 + b_o + \sum_{n=1}^{\infty} \left[(2a_n + n\ell b_n) I_o(n\ell y_1) + n\ell a_n y_1 \, I_1(n\ell y_1) \right] \cos n\ell z = 0, \tag{29}$$

$$\sum_{n=1}^{\infty} \left[a_n y_1 \, I_o(n\ell y_1) + b_n \, I_1(n\ell y_1) \right] n \sin n\ell z = \eta\sigma \sin \ell z. \tag{30}$$

$I_o(n\ell y_1)$ and $I_1(n\ell y_1)$ can be expanded as powers of $\cos \ell z$ as follows:

$$I_o(n\ell y_1) = \sum_{r=0}^{\infty} I_o^{(r)}(n\ell h) \frac{(n\ell\eta \cos \ell z)^r}{r!} \quad , \quad I_1(n\ell y_1) = \sum_{r=0}^{\infty} I_1^{(r)}(n\ell h) \frac{(n\ell\eta \cos \ell z)^r}{r!}$$

where $I_o^{(r)}(n\ell h)$ means $\dfrac{d^r}{d(n\ell h)^r} I_o(n\ell h)$ etc.

If these expressions are substituted in equations (29) and (30) then a similar analysis to that done for channel flow leads to the same conclusions, i.e.

$$a_n = \sum_{t=0}^{\infty} \alpha_{n,n+2t} (\ell\eta)^{n+2t} \quad , \quad n = 1, 2, 3 \ldots ,$$

and

$$b_n = \sum_{t=0}^{\infty} \beta_{n,n+2t} (\ell\eta)^{n+2t} \quad , \quad n = 0, 1, 2 \ldots .$$

It also leads to similar expressions for β_{oo}, α_{11} and β_{11}, i.e.

$$\beta_{oo} = -a_o h^2, \tag{A23}$$

$$\alpha_{11} = -\frac{[2a_o h\, I_1(\ell h) + \ell I_o(\ell h)\sigma]}{\ell^2 h[I_1^2(\ell h) - I_o(\ell h) I_2(\ell h)]} \tag{A24}$$

$$\beta_{11} = \frac{2a_o h^2\, I_o(\ell h) + [2I_o(\ell h) + \ell h\, I_1(\ell h)]\sigma}{\ell^2 h[I_1^2(\ell h) - I_o(\ell h) I_2(\ell h)]} . \tag{A25}$$

The expressions for the higher coefficients are more cumbersome.

Appendix B

Calculation of the flux through the tube

B.1 Channel Flow

The flux through the channel is found by evaluating the stream function $\psi(z,y)$ at a point on the boundary $y_1 = h + \eta \cos \ell z$, and subtracting from it the flux on the axis $\psi(z,0)$ which was made zero. By symmetry, then, the total flux through the tube is $2\psi(z,y_1)$.

Equation (23) gives

$$\psi(z,y_1) = \tfrac{1}{3} a_o y_1^3 + b_o y_1 + \sum_{n=1}^{\infty} \left\{ a_n y_1 \cosh n\ell y_1 + b_n \sinh n\ell y_1 \right\} \cos n\ell z$$

which gives, assuming $(\ell\eta)^3$ can be neglected,

$$\psi(z,y_1) = \tfrac{1}{3} a_o h^3 + \tfrac{1}{2} a_o h\eta^2 + \beta_{oo} h + \beta_{o2} h\ell^2\eta^2$$

$$+ \tfrac{1}{2} \alpha_{11} \ell\eta^2 (\ell h \sinh \ell h + \cosh \ell h) + \tfrac{1}{2} \beta_{11} \ell^2 \eta^2 \cosh \ell h$$

$$+ f(\eta) \cos \ell z + g(\eta) \cos 2\ell z,$$

where $f(\eta)$ and $g(\eta)$ are functions of η and independent of z.

This gives the flux as a function of z. A more useful quantity is the average flux per cycle and this can be found by integrating $\psi(z,y_1)$ over one period. There are two ways of looking at this which give the same result: for a given t integrate $\psi(z,y_1)$ with respect to x from $x = 0$ to $x = \lambda$ or for a given x integrate $\psi(z,y_1)$ with respect to t from $t = 0$ to $^{2\pi}/\ell\sigma$.

If this is done then $\int_0^\lambda f(\eta)\cos \ell z\,dz = \int_0^\lambda g(\eta)\cos 2\ell z\,dz = 0$

and so

$$\bar{\psi} = \frac{1}{\lambda}\int_0^\lambda \psi(z,y_1)\,dz$$

$$= -\frac{2}{3}a_o h^3 - h\eta^2\left(a_o + \ell^2\alpha_{11}\sinh \ell h + \frac{\ell^2\sigma}{2}\right).$$

If (A21) is substituted for α_{11} then

$$\bar{\psi} = -\frac{2}{3}a_o h^3 - h^3\left[a_o\left(\frac{\sinh 2\ell h - 2\ell h\cosh 2\ell h}{\sinh 2\ell h - 2\ell h}\right)\right.$$

$$\left. - \frac{\ell^2\sigma}{2}\left(\frac{\sinh 2\ell h + 2\ell h}{\sinh 2\ell h - 2\ell h}\right)\right]\left(\frac{\eta}{h}\right)^2, \qquad \text{(B1)}$$

where $a_o = -\frac{P\ell}{4\pi\mu}$.

B.2 Pipe Flow

The flux through the pipe is found in a similar way. If $(\ell\eta)^3$ is neglected then equation (25) reduces to

$$\psi(z,y_1) = \frac{1}{4}a_o h^4 + \frac{3}{4}a_o h^2\eta^2 + \frac{1}{2}\beta_{oo}h^2 + \frac{1}{4}\beta_{oo}\eta^2 + \frac{1}{2}\beta_{o2}\ell^2\eta^2h^2$$

$$+ \frac{1}{2}\ell h\eta^2\alpha_{11}\left[2I_o(\ell h) + \ell h\, I_1(\ell h)\right] + \frac{1}{2}\ell\eta^2\beta_{11}\ell h\, I_o(\ell h)$$

$$+ f(\eta)\cos \ell z + g(\eta)\cos 2\ell z.$$

As in (B1),

$$\bar{\psi}(z,y_1) = \frac{1}{\lambda}\int_0^\lambda \psi(z,y_1)\,dz$$

$$= -\frac{1}{4}a_o h^4 + \frac{h^4}{4}\left[a_o\left(\frac{I_1^2(\ell h) + 3I_o(\ell h)I_2(\ell h)}{I_1^2(\ell h) - I_o(\ell h)I_2(\ell h)}\right)\right.$$

$$\left. + \ell^2\sigma\left(\frac{I_o^2(\ell h) - I_1^2(\ell h)}{I_1^2(\ell h) - I_o(\ell h)I_2(\ell h)}\right)\right]\left(\frac{\eta}{h}\right)^2 \qquad \text{(B2)}$$

where $a_o = -\frac{P\ell}{8\pi}$.

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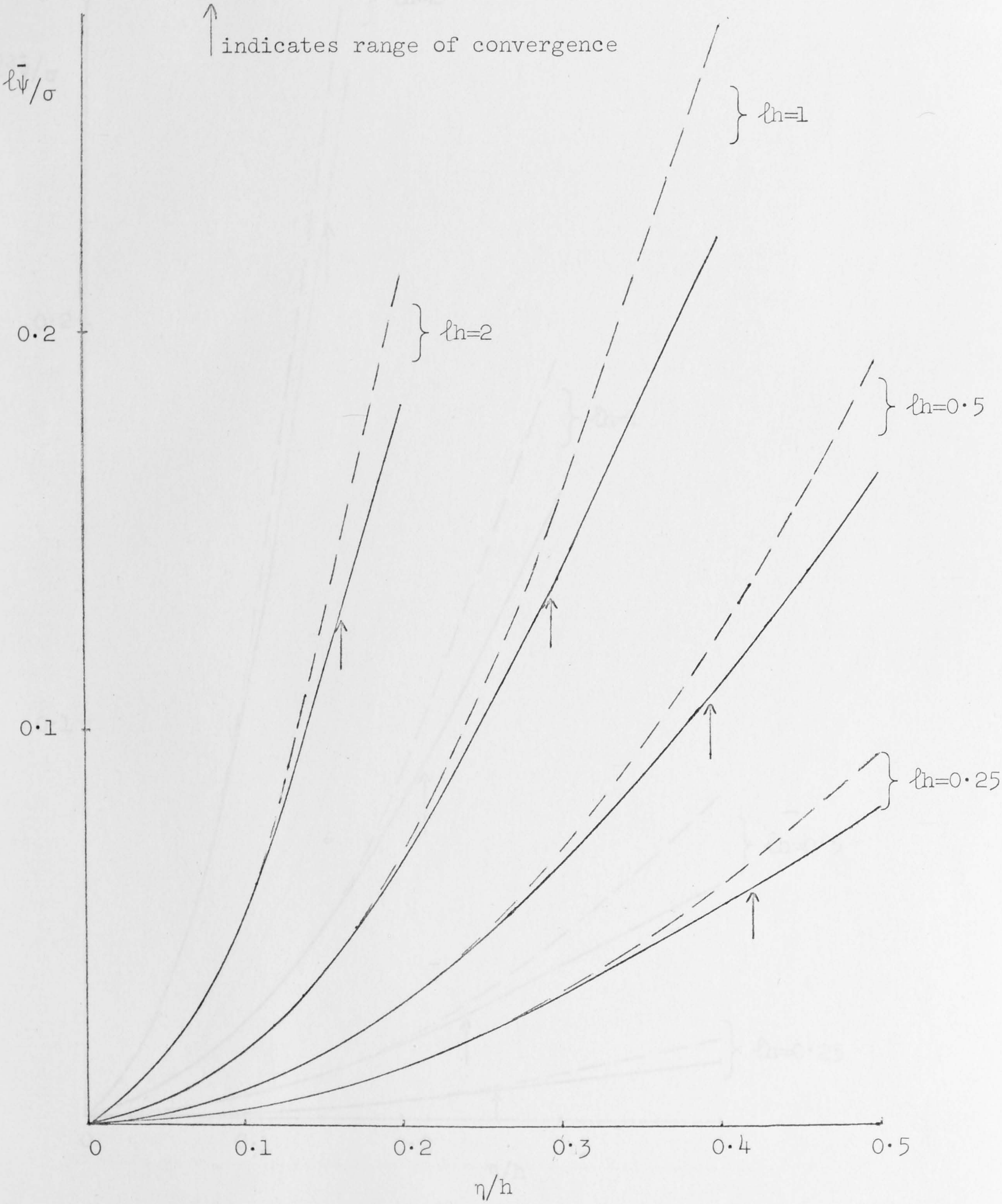

Figure 2. Peristalsis. Flux through channel

- - - - $O(\eta/h)^2$ ——— $O(\eta/h)^4$

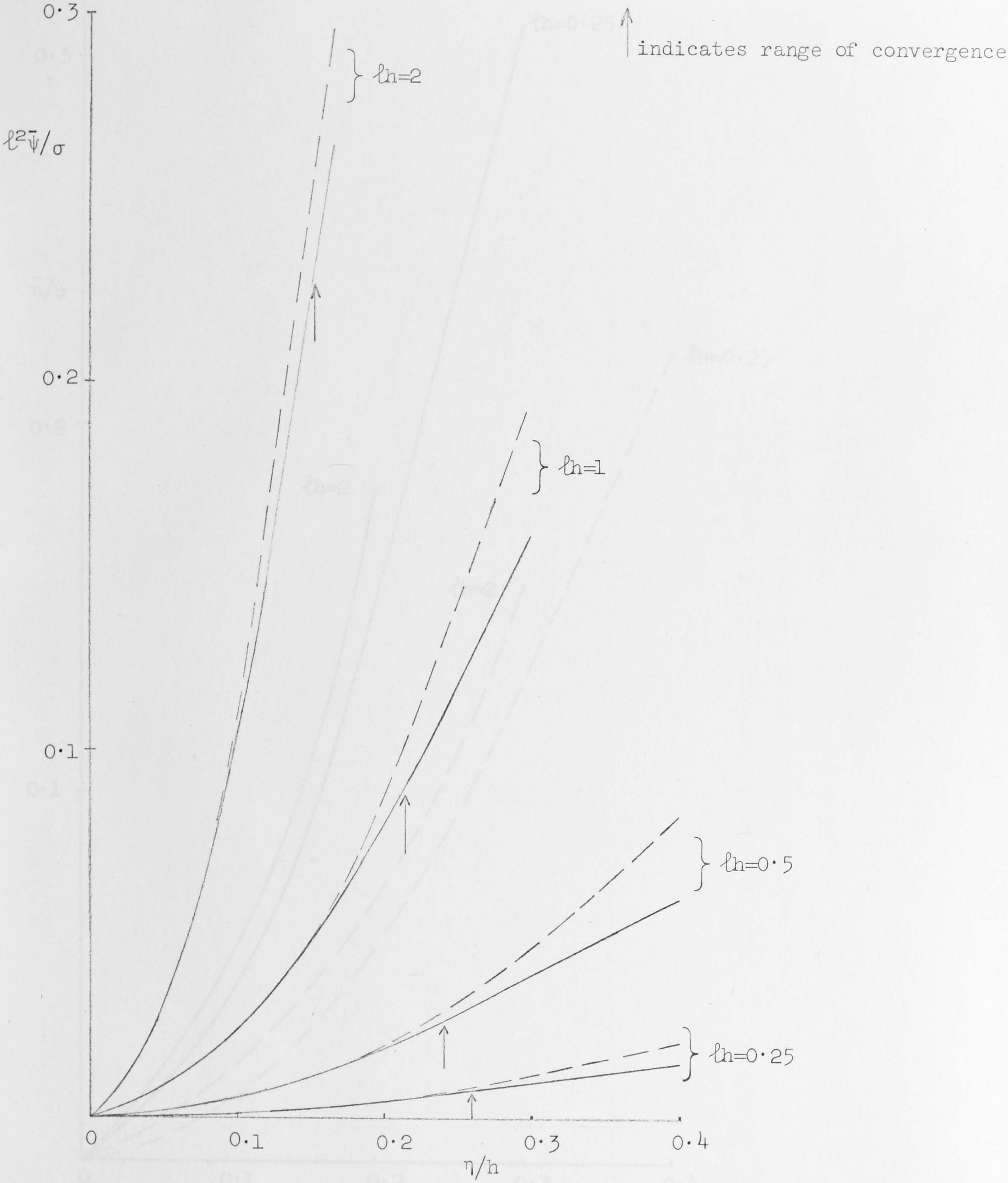

Figure 3. Peristalsis. Flux through pipe.
- - - - $O(\eta/h)^2$ ——— $O(\eta/h)^4$

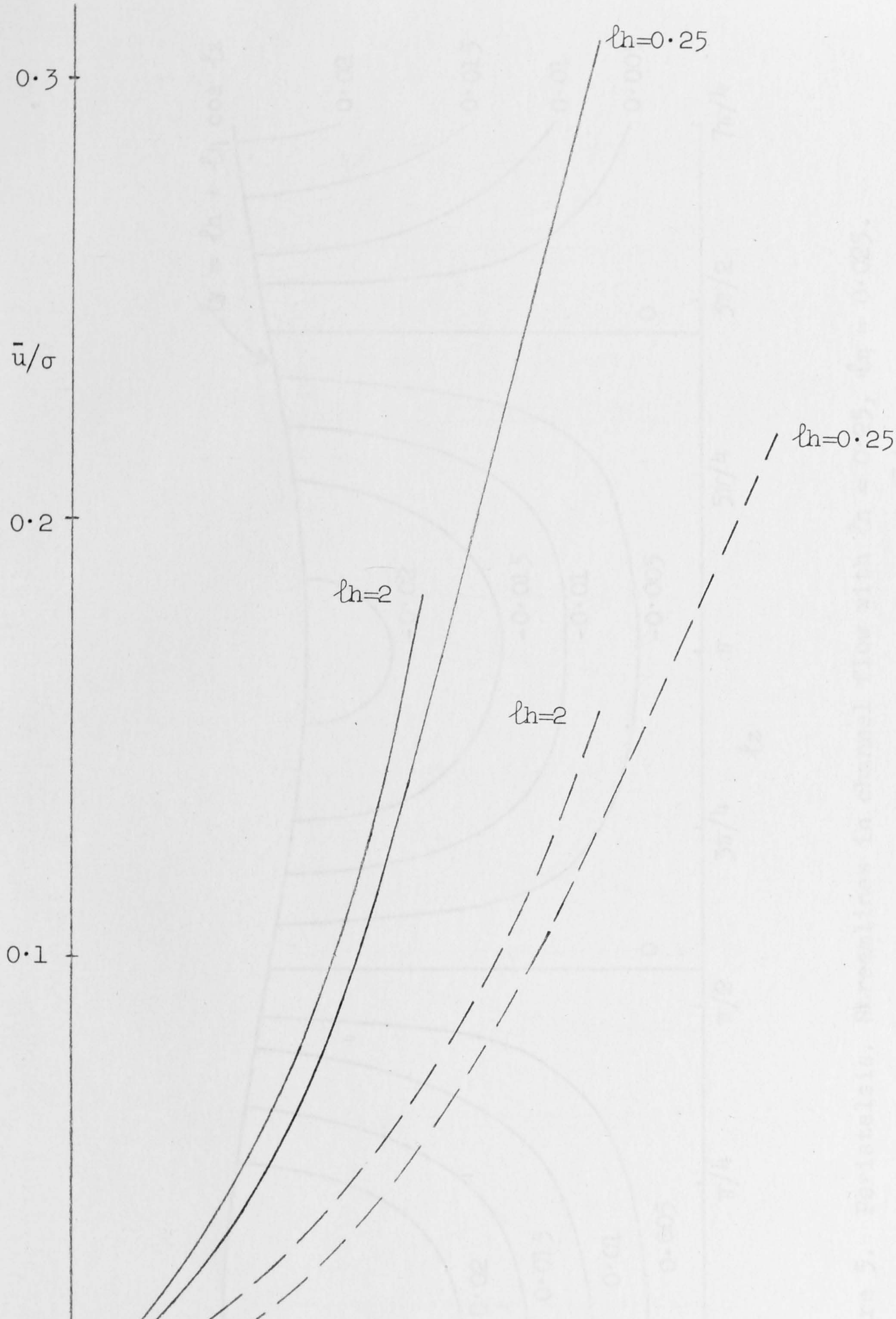

Figure 4. Peristalsis. Mean velocity.

- - - Channel Flow ——— Pipe Flow

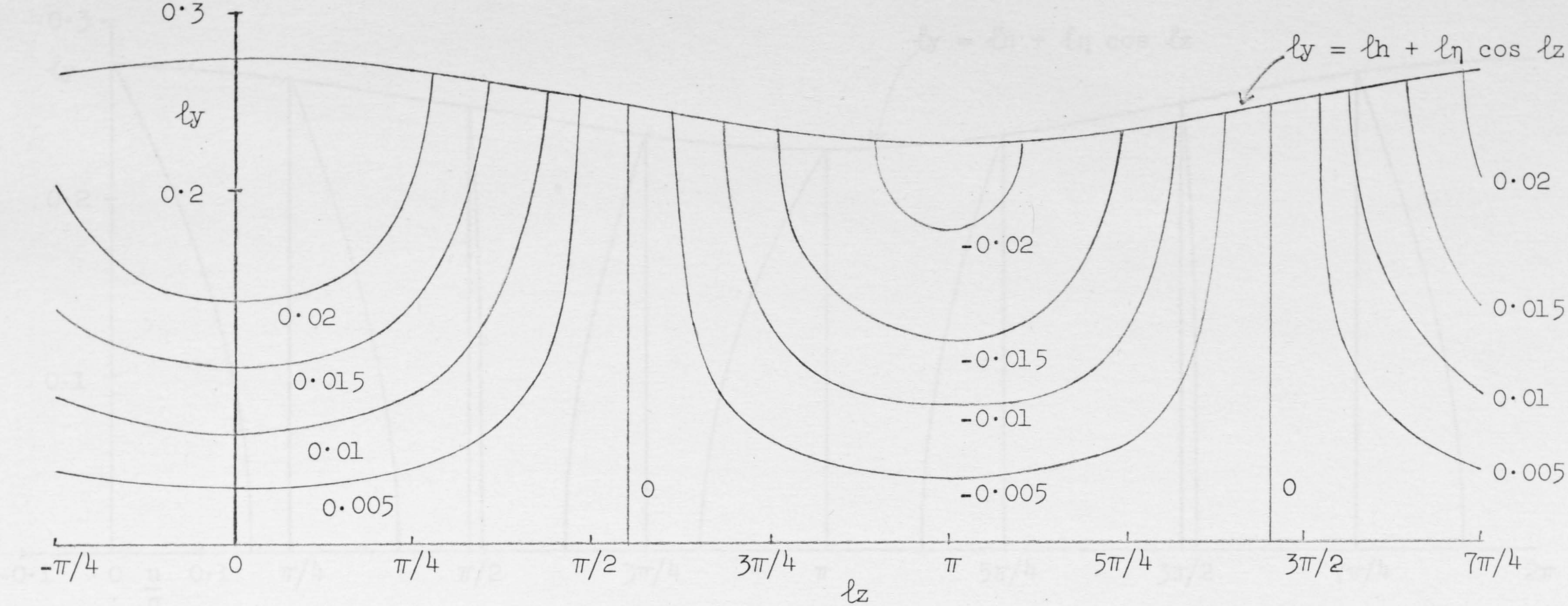

Figure 5. Peristalsis. Streamlines in channel flow with $\ell h = 0{\cdot}25$, $\ell\eta = 0{\cdot}025$. Streamlines correspond to indicated values of $\ell\bar{\psi}/\sigma$.

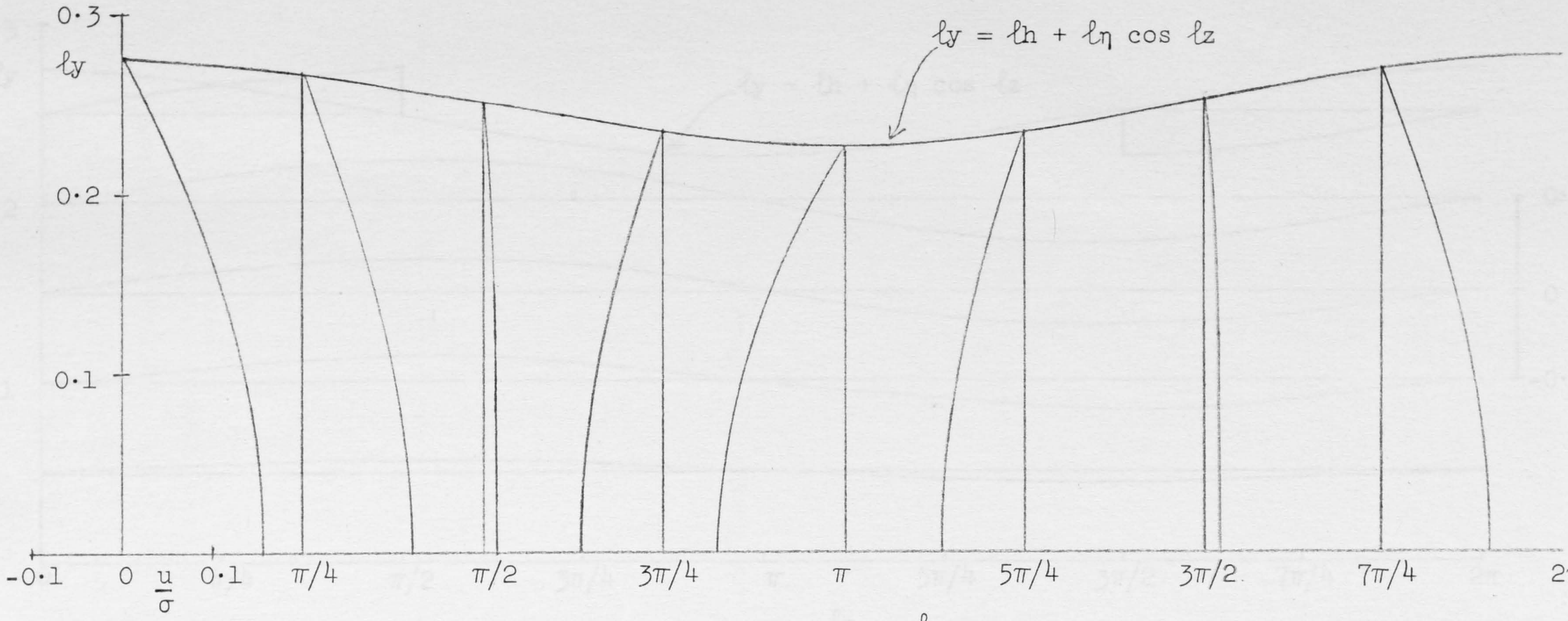

Figure 6. Peristalsis. Velocity distribution along axis in channel flow.

$\ell h = 0{\cdot}25 \quad \ell\eta = 0{\cdot}025$

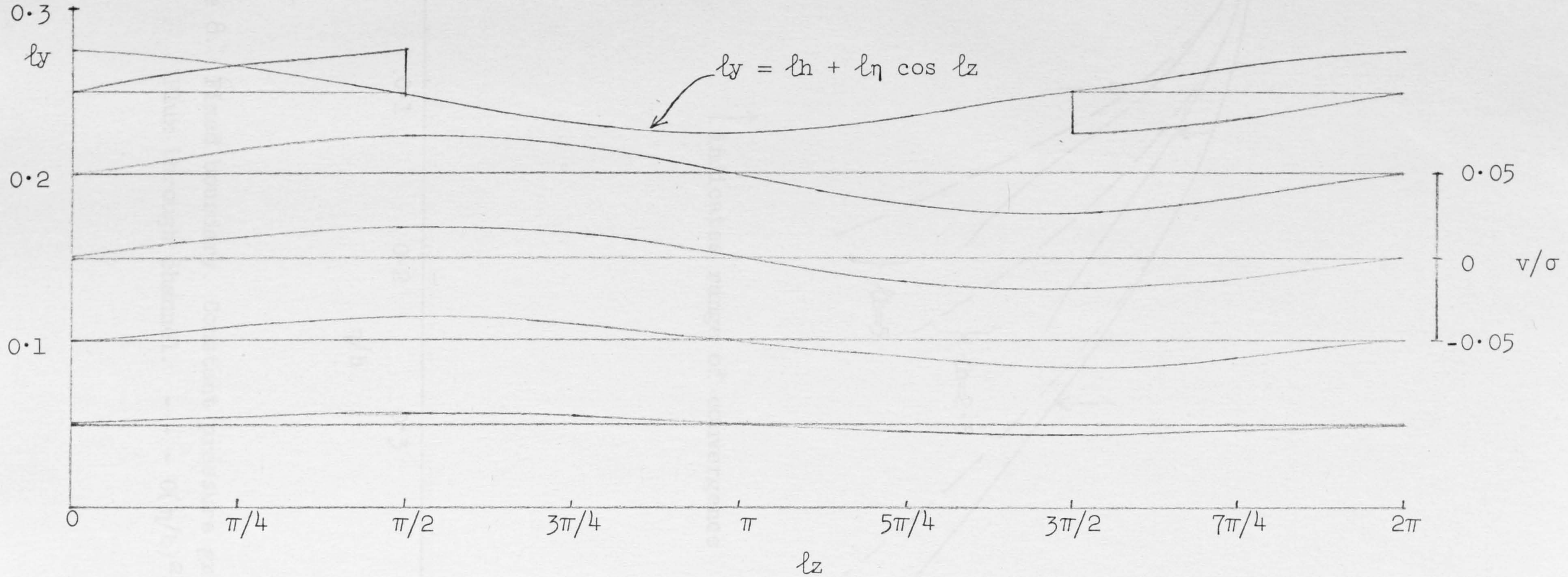

Figure 7. Peristalsis. Velocity distribution normal to axis in channel flow with $\ell h = 0{\cdot}25$, $\ell\eta = 0{\cdot}025$

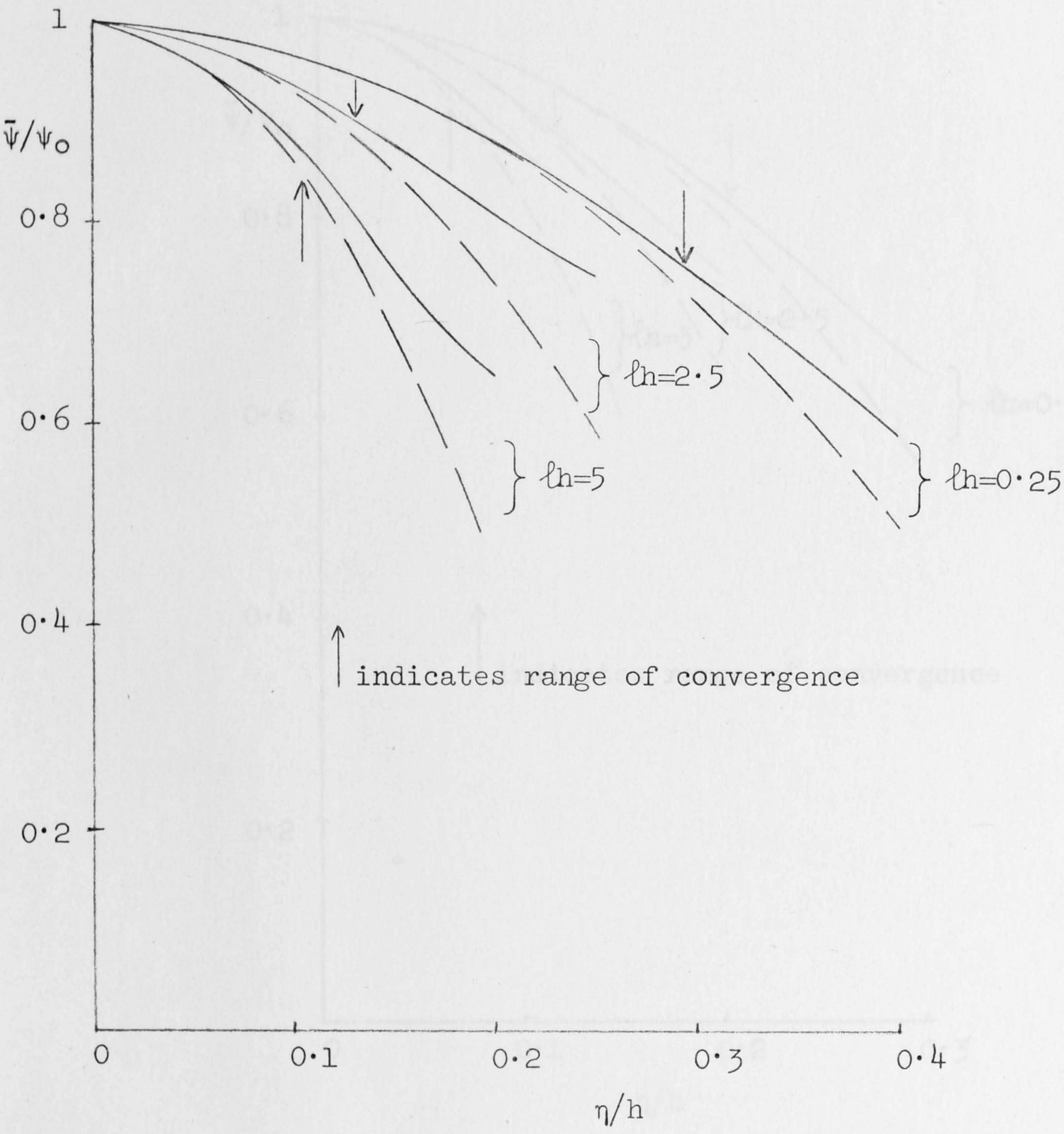

Figure 8. Fixed boundary. Constant pressure gradient. Flux through channel. - - - $O(\eta/h)^2$ _____ $O(\eta/h)^4$

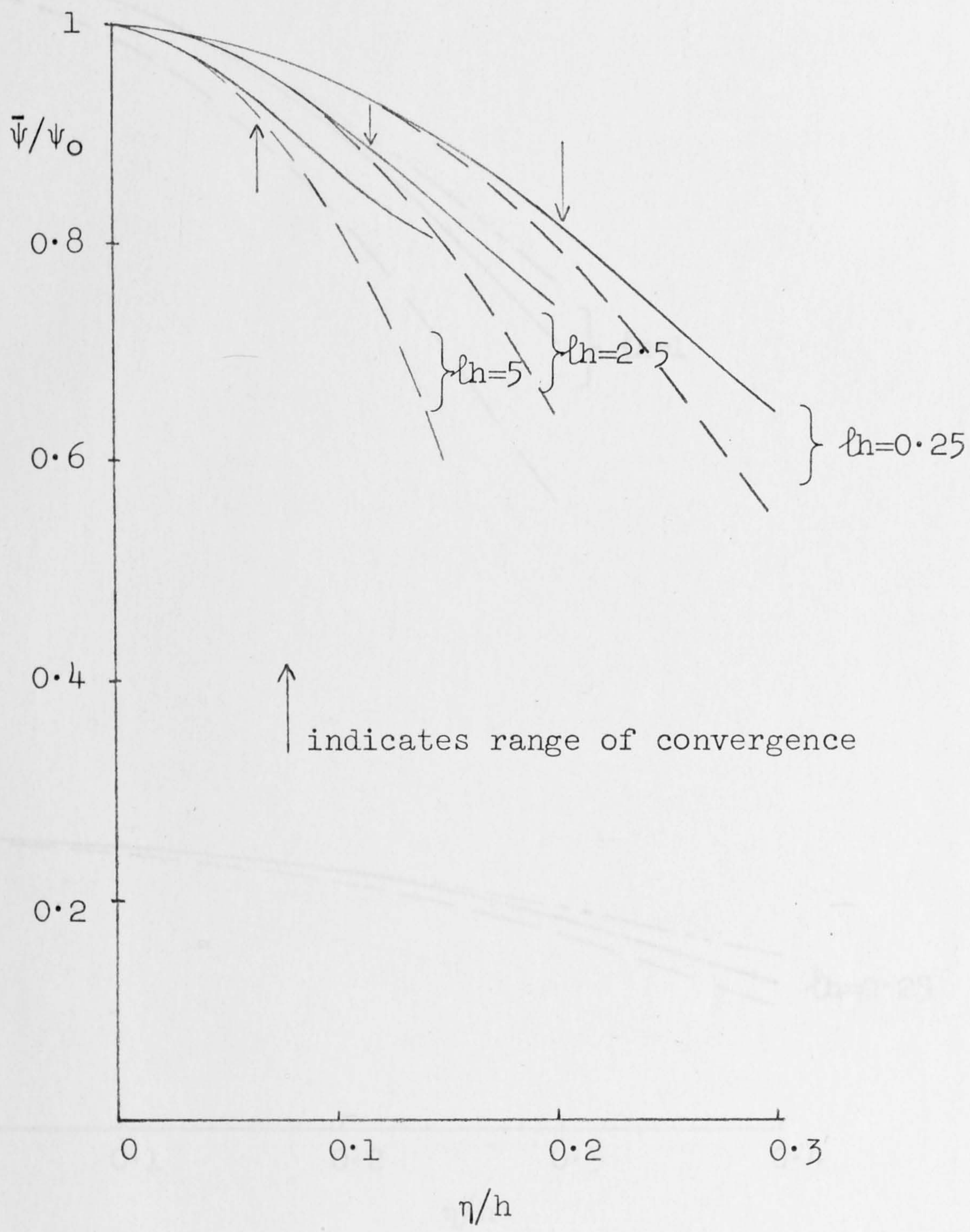

Figure 9. Fixed boundary. Constant pressure gradient.
Flux through pipe. - - - $O(\eta/h)^2$ ——— $O(\eta/h)^4$

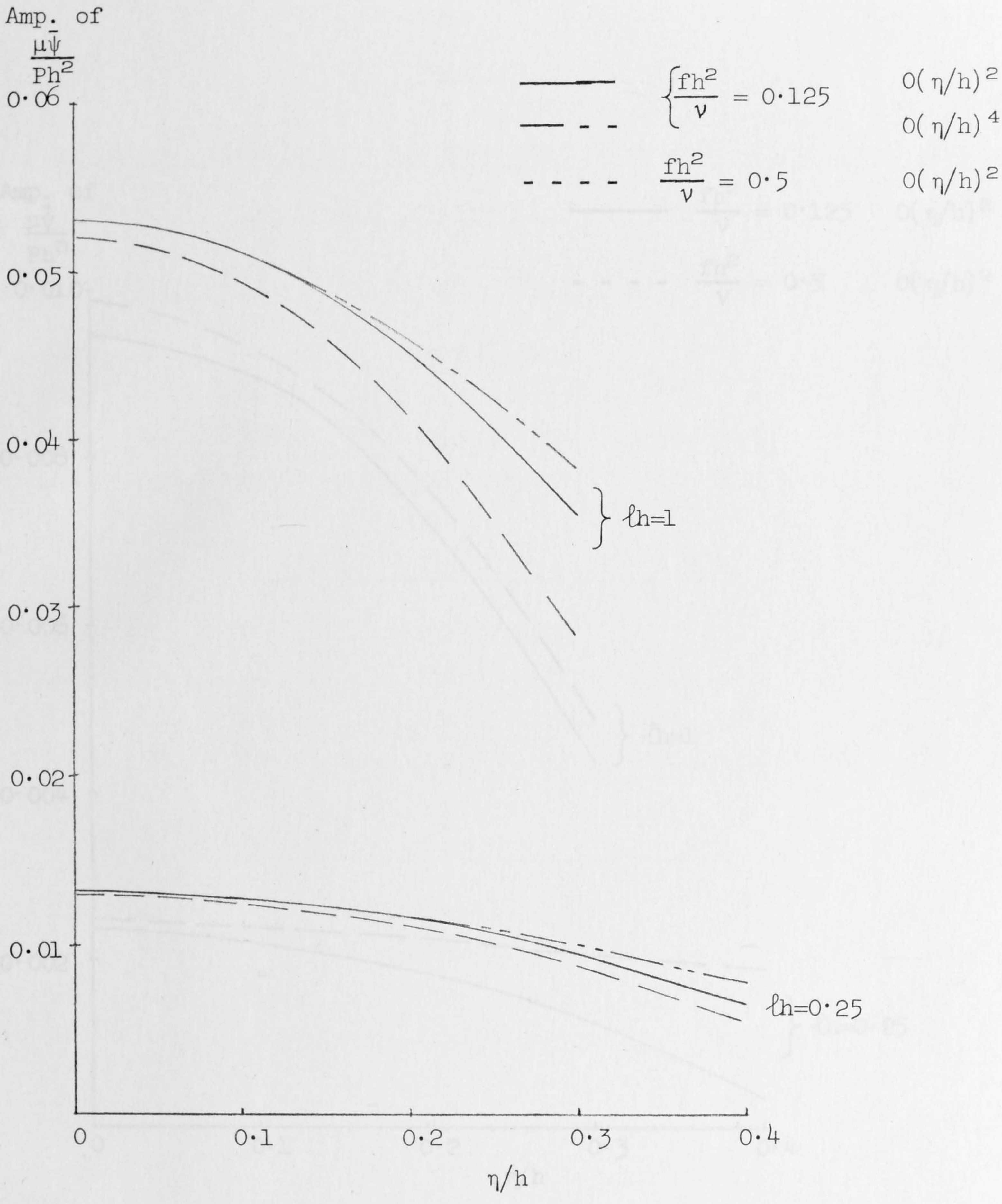

Figure 10. Fixed boundary. Sinusoidal pressure gradient. Amplitude of flux through channel.

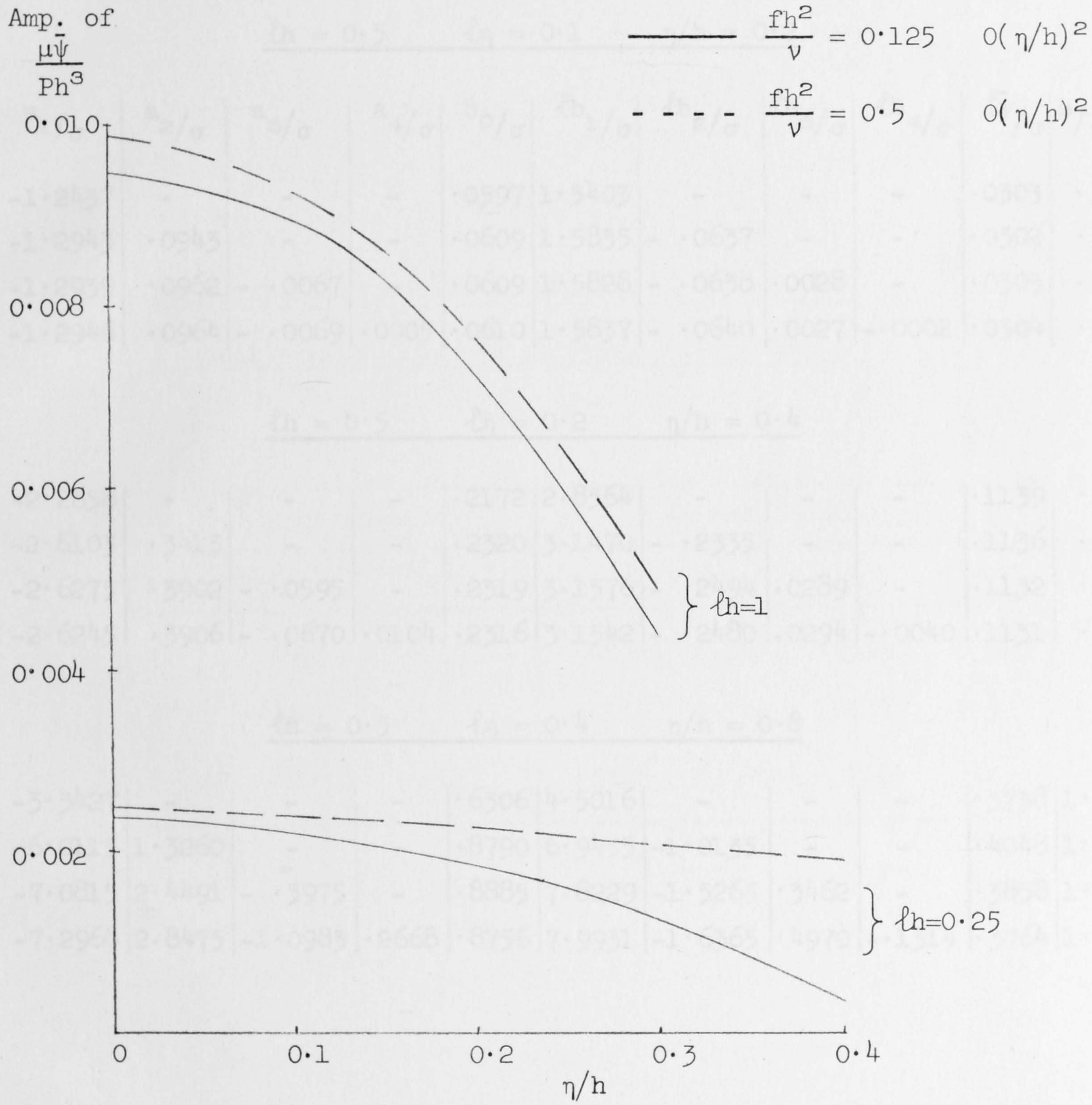

Figure 11. Fixed boundary. Sinusoidal pressure gradient. Amplitude of flux through pipe.

Table 2

Peristalsis, coefficients and flux through channel, direct method

a_1/σ	a_2/σ	a_3/σ	a_4/σ	b_0/σ	$\ell b_1/\sigma$	$\ell b_2/\sigma$	$\ell b_3/\sigma$	$\ell b_4/\sigma$	$\ell\bar{\psi}/\sigma$	$\mu/\sigma(0,0)$
$\ell h = 0\cdot5$	$\ell\eta = 0\cdot1$	$\eta/h = 0\cdot2$								
-1·2437	-	-	-	·0597	1·5403	-	-	-	·0303	·3563
-1·2943	·0943	-	-	·0609	1·5835	- ·0637	-	-	·0302	·3170
-1·2939	·0962	- ·0067	-	·0609	1·5828	- ·0638	·0028	-	·0303	·3200
-1·2946	·0964	- ·0069	·0005	·0610	1·5837	- ·0640	·0027	-·0002	·0304	·3196
$\ell h = 0\cdot5$	$\ell\eta = 0\cdot2$	$\eta/h = 0\cdot4$								
-2·2658	-	-	-	·2172	2·8564	-	-	-	·1139	·8078
-2·6103	·3413	-	-	·2320	3·1470	- ·2335	-	-	·1136	·6430
-2·6275	·3902	- ·0595	-	·2319	3·1576	- ·2494	·0289	-	·1132	·6806
-2·6245	·3906	- ·0670	·0104	·2316	3·1542	- ·2480	·0294	-·0040	·1131	·6717
$\ell h = 0\cdot5$	$\ell\eta = 0\cdot4$	$\eta/h = 0\cdot8$								
-3·3427	-	-	-	·6306	4·5016	-	-	-	·3738	1·7895
-6·0115	1·3260	-	-	·8790	6·9455	-1·0135	-	-	·4048	1·1120
-7·0815	2·4491	- ·5975	-	·8885	7·8229	-1·5265	·3462	-	·3858	1·4671
-7·2968	2·8475	-1·0983	·2668	·8756	7·9931	-1·6365	·4970	-·1314	·3764	1·2803

Table 3

Peristaltic flow through channel with $\ell h = 0{\cdot}5$

Comparison of flux ($\ell\bar{\psi}/\sigma$) using both methods

η/h	Perturbation		Direct Method			
	$O(\eta/h)^2$	$O(\eta/h)^4$	n = 1	n = 2	n = 3	n = 4
0·2	·0310	·0302	·0303	·0302	·0303	·0304
0·4	·1241	·1114	·1139	·1136	·1132	·1131
0·8	·4964	·2933	·3738	·4048	·3858	·3764

Table 4

Fixed walls. Constant pressure gradient

Flux through channel and coefficients, direct method

$\ell h = 0\cdot 5 \qquad \ell\eta = 0\cdot 1 \qquad \eta/h = 0\cdot 2$

a_1/a_0	a_2/a_0	a_3/a_0	a_4/a_0	b_0/a_0	b_1/a_0	b_2/a_0	b_3/a_0	b_4/a_0	$\bar{\psi}/a_0$	$u/a_0(0,0)$
-·5737	-	-	-	-·2251	·6222	-	-	-	-·0732	-·1766
-·5844	·0293	-	-	-·2250	·6290	-·0178	-	-	-·0733	-·1868
-·5842	·0296	-·0019	-	-·2250	·6287	-·0176	·0008	-	-·0733	-·1866
-·5845	·0296	-·0018	·0004	-·2250	·6290	-·0177	·0006	-·0002	-·0733	-·1866
$\ell h = 0\cdot 5$				$\ell\eta = 0\cdot 4$		$\eta/h = 0\cdot 8$				
-1·4857	-	-	-	-·0142	1·6662	-	-	-	·0238	·1664
-1·8159	·2500	-	-	-·0090	1·8727	-·1665	-	-	-·0815	-·0352
-1·8654	·3615	-·0734	-	-·0153	1·8918	-·2022	·0386	-	-·0014	·0106
-1·8479	·3779	-·1138	·0250	-·0200	1·8706	-·1985	·0469	-·0114	-·0033	-·0102

N.B. a_0 is negative

Table 5

Fixed walls. Constant pressure gradient $\ell h = 0\cdot5$

Comparions of flux ($\bar{\psi}/a_0$) through channel using both methods

η/h	Perturbation		Direct Method			
	$O(\eta/h)^2$	$O(\eta/h)^4$	n = 1	n = 2	n = 3	n = 4
·2	−·0728	−·0734	−·0732	−·0733	−·0733	−·0733
·8	+·0846	−·0546	+·0238	−·0815	−·0014	−·0033

N.B. a_0 is negative.