NONLINEAR SECOND ORDER ELLIPTIC EQUATIONS WITH VENTTSSEL BOUNDARY CONDITIONS

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The work described in this thesis is my own except otherwise explicitly indicated.
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Abstract

This thesis is concerned with second order elliptic equations with certain second order boundary conditions which are called Venttsel boundary conditions. This sort of boundary condition originally came from probability theory. An example of such a boundary value problem of PDE also arises from a model in three dimensional water wave theory. Venttsel boundary conditions contain Dirichlet, Neumann, oblique and mixed boundary conditions as special cases, and from the probability point of view they are the most general admissible boundary conditions for second order elliptic operators.

The results in this thesis are mainly on classical existence and uniqueness of the solutions of the quasilinear problems in the form

\begin{align*}
(1) & \quad a^{ij}(x, u, Du) D_{ij}u + b(x, u, Du) = 0, \quad \text{in } \Omega, \\
(2) & \quad \alpha^{ij}(x, u, \delta u) \delta_i \delta_j u + \beta(x, u, Du) = 0, \quad \text{on } \partial \Omega,
\end{align*}

where $\Omega$ is a smooth domain in $\mathbb{R}^n$ and $\delta_i, \ i = 1, \ldots, n$, are tangential differential operators on $\partial \Omega$. The boundary condition (2) is called Venttsel when the following conditions are satisfied:

\begin{enumerate}
  \item $\alpha^{ij}(x, z, p) \eta_i \eta_j \geq \kappa |\eta|^2 \forall (x, z, p) \in \partial \Omega \times \mathbb{R} \times \mathbb{R}^n$, and $\forall \eta \in \mathbb{R}^n$ s.t. $\eta \perp \nu$ at $x$, for some function $\kappa \geq 0$, $\nu$ is the inner normal vector field of $\partial \Omega$.
  \item $\beta(x, z, p)$ is differentiable with respect to $p$ and $D_p \beta \cdot \nu \geq \chi$ for some $\chi \geq 0$.
\end{enumerate}

A fundamental result is the unique existence of the solutions of linear problems, that is, in (1) and (2) when $\alpha^{ij}, \alpha^{ij}$ are $z$ and $p$ independent, $b(x, z, p) = b^i(x) p_i + c(x)$.
\[ f(x), \beta(x, z, p) = \beta^i p_i + \gamma(x) z - g(x), \text{ and } \partial \Omega \in C^{2,\alpha}, a^{ij}, b^i, c, \alpha^{ij}, \beta^i, \gamma; f, g \in C^\alpha(\Omega), c, \gamma < 0, \text{ there is a unique solution } u \in C^{2,\alpha}, \text{ provided } \kappa > 0 \text{ in (i)}. \]

In the further studying of linear operators, a boundary version of Aleksandrov maximum principle is established. From it we obtain a boundary weak Harnack inequality which provides a Hölder estimate for the solutions of (1) and (2). All of these are done in some small neighbourhood of those points where the boundary condition is elliptic, i.e. \( \kappa > 0 \) in (i) or partially degenerate elliptic, i.e. \( \kappa = 0 \) but the matrix \{\alpha^{ij}\} has rank \( h \) for some \( 0 < h < n-1 \).

For quasilinear problems we assume the natural structure conditions of Ladyzhenskaya and Uraltseva. When the boundary condition is elliptic we prove an existence result by applying Leray-Schauder theory. The required \( C^{1,\alpha} \) estimate is obtained in following way. We first establish the tangential gradient estimate under an additional assumption that the boundary operator is also oblique, i.e. \( \chi > 0 \) in (ii), and then the Hölder estimate for the tangential gradient. After this we obtain the \( C^{2,\alpha} \) bound by virtue of interpolation. The global \( C^{1,\alpha} \) estimate follows from the classical results on Dirichlet problems with \( C^{2,\alpha} \) boundary data.

When the boundary condition is degenerate elliptic but strictly oblique we prove the existence by the procedure of elliptic regularization. In this case we still have the tangential gradient bound around the partially degenerate points because we have Hölder estimates there. In the neighbourhoods of completely degenerate points, i.e. all the eigenvalues of \{\alpha^{ij}\} are zeros, we obtain an oscillation estimate which plays the same role as the Hölder estimate does, so we get the tangential gradient estimate there, hence everywhere. Continuing this processes we finally obtain a \( C^3 \) estimate on the boundary and then the global \( C^{2,\alpha} \) estimate follows also from the classical results on Dirichlet problems. An existence result for viscosity solutions is proved also under some much weaker assumptions on the coefficients of (1) and (2).
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Chapter 1
Introduction

In this thesis we are concerned with second order elliptic equations with second order boundary conditions of Venttsel type. A Venttsel boundary condition contains derivatives of up to second order and such a boundary condition contains Dirichlet, Neumann, oblique derivative and mixed boundary conditions as special cases.

Let $\Omega$ be a bounded domain in Euclidean $n$-space $\mathbb{R}^n$ with $\partial \Omega \in C^k$ for $k \geq 2$. Let $v = (v^1, ..., v^n)$ be the inner normal vector field on the boundary manifold $\partial \Omega$, so $v \in C^{k-1}$. We define the tangential differential operators

$$
\delta_i = c^{ik} D_k, \quad \delta_{ij} = \delta_{ji} = c^{ik} c^{jl} D_{kl}, \quad i, j = 1, ..., n,
$$

where $c^{ik} = \delta^{ik} - v^i v^k$, $\delta^{ik}$ is the Kronecker symbol. We may extend $v$ to $\Omega$ in a suitable way that $\delta_i, \delta_{ij}$ are defined on $\Omega$.

In this thesis we consider both linear and nonlinear boundary value problems. The linear boundary value problem we will consider is

\begin{align}
\text{(1.1)} & \quad Lu = a^{ij} D_{ij}u + b^i D_iu + cu = f, \quad \text{in } \Omega, \\
\text{(1.2)} & \quad Bu = \alpha^{ij} \delta_{ij}u + \beta^i D_iu + \gamma u = g, \quad \text{on } \partial \Omega.
\end{align}

The interior operator $L$ is an elliptic operator in the normal sense. The boundary operator $B$ is called Venttsel if the following conditions are satisfied.

V1) $\{\alpha^{ij}\}$ is a nonnegative definite symmetric matrix valued function, i.e

\begin{align}
\text{(1.3)} & \quad \alpha^{ij}(x) \xi_i \xi_j \geq 0, \quad \forall x \in \partial \Omega, \quad \xi \in \mathbb{R}^n \quad \text{s.t. } \xi \perp v \text{ at } x.
\end{align}

$\alpha^{ij}(x) \xi_i \xi_j \geq 0, \forall x \in \partial \Omega \text{ and } \forall \xi$. 
V2) The vector field $\beta = (\beta^1, \ldots, \beta^n)$ has the property

(1.4) $\beta \cdot \nu \geq 0$.

If at some points the inequality is strict we will call the boundary operator $B$ elliptic there; otherwise we will call it degenerate elliptic. If (1.4) is strict at some points we will call $B$ oblique there; otherwise we will call it degenerate oblique.

The nonlinear problem we are concerned with is the quasilinear problem

(1.5) $a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du) = 0$, in $\Omega$,

(1.6) $\alpha^{ij}(x, u, \delta u) \delta_{ij} u + \beta(x, u, Du) = 0$, on $\partial \Omega$.

The boundary operator (1.6) is called Venttsel if the functions $\alpha^{ij}(\cdot, z, p)$ satisfy V1) for each $(z, p) \in \mathbb{R} \times \mathbb{R}^n$ and

V2) $\beta(x, z, p)$ is differentiable with respect to $p$ variables and

(1.7) $D_p \beta \cdot \nu \geq 0$, $\forall (x, z, p) \in \partial \Omega \times \mathbb{R} \times \mathbb{R}^n$.

We will still talk about the ellipticity, degenerate ellipticity, obliqueness and degenerate obliqueness of the boundary operator $B$ respectively as in the linear case.

This sort of boundary conditions originally come from probability theory. Given a second order elliptic differential operator in a closed bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, people have been looking for the most general supplementary conditions which will restrict the given operator to the infinitesimal operator of a semigroup which one-to-one corresponds to a Markov process in the domain. In 1959, A.D.Venttsel proved in [27] that when the domain has some special shape the conditions are boundary conditions containing (1.2) as a particular case. Thus from the probability point of view, this kind of boundary condition is the most general
admissible boundary condition for second order elliptic differential operators in the domain. From then on some authors have called it the Venttsel boundary condition. For more information about Venttsel boundary conditions we refer also [7], [28].

As an example of a boundary value problem of partial differential equations with Venttsel type boundary conditions, we have a model in three-dimensional water wave theory, namely

\[
\begin{align*}
D_2 u - D_{11} u &= g(x, u), \quad x_2 = 1, \\
\Delta u &= f(x, u, Du), \quad 0 < x_2 < 1, \\
u &= 0, \quad \text{in } x_2 = 0,
\end{align*}
\]

in an unbounded domain \( \Omega = \{ x \in \mathbb{R}^3 \mid 0 < x_2 < 1 \} \), see [20], [10]. This also provides an example of a nonlinear problem with degenerate elliptic boundary conditions.

Second order elliptic equations with Dirichlet, Neumann, oblique derivative and mixed boundary conditions have been well studied, for example, Gilbarg and Trudinger [6], Ladyzhenskaya and Ural'tseva [15], Friedman [5], Lieberman and Trudinger [13] and Ural'tseva [26] etc. However Venttsel boundary value problems have not been studied generally except that a particular case, the problem (1.8), has recently been considered by P. Korman [10], [11], [12], [13]. In his work the classical existence of periodic solutions of such problems was obtained by using an embedding theorem. As mentioned in [10], an elliptic equation of second order with such a boundary condition is a noncoercive problem because the Lopatinski condition fails here. We would like to point out also that in Venttsel boundary conditions we must have \( \{ \alpha^{ij} \} \geq 0 \) and \( \beta \cdot v \geq 0 \). An example constructed in [11] shows that if \( \{ \alpha^{ij} \} \geq 0, \beta \cdot v < 0 \), or \( \{ \alpha^{ij} \} \leq 0, \beta \cdot v > 0 \) the problem we are considering is ill-posed in some sense. Therefore we are not going to deal with such a problem here.

Our project is to study the solvability and uniqueness of general Venttsel boundary problems, both in a classical and a weak sense. The existence, by using the frame work
of functional analysis, is reduced to a series of a priori estimates, and the uniqueness follows from the comparison principle which is derived from the maximum principle. The maximum principle is the only good property that Venttsel boundary problems share with usual boundary value problems and has been studied by K. Amano [3] in some other situation.

The organization of this thesis is as follows. Chapter 2 deals with the linear problem (1.1), (1.2). In the elliptic boundary case, by regarding the boundary condition as an elliptic equation on the manifold \( \partial \Omega \) we may obtain a \( C^{2,\alpha} \) estimate of the solution \( u \) on \( \partial \Omega \) linearly in terms of \( |D_u|_{\alpha;\Omega} \), where \( D_u \) is the normal derivative operator. Applying the well known interpolation inequality, see Chapter 6.8 of [6], we obtain global \( C^{2,\alpha} \) estimates. The existence then is only a consequence of the classical method of continuity. When the boundary condition is degenerate elliptic, we can not expect Schauder estimates on the boundary. In this case, we use a maximum principle argument to first get a tangential gradient bound by differentiating the equation and the boundary condition. Instead of then estimating the normal derivative we must now go further to estimate the second order tangential derivatives. The normal derivative can be then estimated from the boundary condition under the hypothesis of obliqueness, after obtaining the bound for the second tangential derivatives. In this way we can achieve \( C^k \) estimates of \( u \) for any \( k \) if we assume the coefficients are sufficiently smooth. The existence also follows from the method of continuity.

Chapter 3 deals with some further properties of linear operators. A boundary version of Aleksandrov-Bakelman maximum principle is established here for both elliptic and partially degenerate elliptic boundary conditions. The idea to prove the maximum principle is based on [1], that is, by considering normal mappings on the upper contact set of the solutions. An important feature of this maximum principle is that a boundary type of weak Harnack inequality can be derived from it by an argument closely similar to [21]. As is well known, the weak Harnack inequality plays an important role in Hölder estimates for solutions of nonlinear problems; our boundary Hölder estimate of solutions
of nonlinear problems in the following chapters is obtained from our boundary weak
Harnack inequality in exactly the same way as in the interior case. An Aleksandrov
maximum principle for oblique boundary condition is also proved here, but it cannot
yield a Harnack inequality. A similar result for the oblique boundary condition is proved
by P.-L. Lions, N.S. Trudinger and J.I.E. Urbas in [29], see also [23].

Chapter 4 is concerned with quasilinear equation (1.5) with elliptic quasilinear
boundary condition (1.6). Our scheme for existence is to use the Leray-Schauder
theorem. For this purpose, we need only establish global $C^{1,\alpha}$ estimates for the
solutions. Using the results in Chapter 3 we immediately have a Hölder estimate of the
solution itself. Under the assumption that the boundary condition is also oblique, we
may get a tangential gradient estimate, and then the Hölder norm of the tangential
gradient. By working from the interior of the domain we can obtain an estimate for the
normal derivative and its Hölder norm linearly in terms of $|\delta\phi|^2_{0,\partial\Omega}$ and $|\delta^2\phi|_{0,\partial\Omega}$,
where $\phi$ is the boundary trace. This is done by a sharp barrier argument. Since we
have already got the estimate for the tangential gradient on the boundary, that is $|\delta\phi|_{0,\partial\Omega}$
is bounded, we can obtain the $C^{1,\alpha}$ estimates on the boundary by using the interpolation
inequality mentioned above. The global $C^{1,\alpha}$ estimates then follow from the classical
results on Dirichlet problems with $C^{2,\alpha}$ boundary data, see for example [6], [22].

Chapter 5 is a continuation of Chapter 4. The same equations with degenerate
elliptic boundary conditions are considered here. The situation in this case is more
complicated. First we don't have Hölder estimates around completely degenerate points.
Second, the interpolation inequality doesn't work here so we can not get $C^{2,\alpha}$ estimates
on the boundary directly as above. To overcome the first difficulty we establish some
kind of oscillation estimate which plays the same role in the tangential gradient estimate.
Higher order derivative bounds are obtained step by step as in the linear case until we
have a $C^3$ estimate on the boundary. In doing so we have to make some strong
assumptions on the coefficients of the interior and boundary operators. These restrictions
seem artificial but they are satisfied by a class of operators. The existence is obtained by
an elliptic regularization process. As a special case of our general theory the water wave problem (1.8) is also solved here directly in a class of smooth functions under some weaker assumptions than that in [10]. The last section of this chapter deals with weak solutions of problem (1.5), (1.6) in the viscosity sense of P. -L. Lions and M. Crandall. An existence result is proved here without all the restrictions made for the existence of classical solutions in the beginning of Chapter 5.

Most of our notation is standard, as for example in Gilbarg & Trudinger [6]. Some special notation is given below:

\[ B^+_R \] — the upper half ball \( B_R \cap \mathbb{R}^n_+ \).

\[ B^0_R \] — the n-1 dimensional ball \( B_R \cap \partial \mathbb{R}^n_+ \).

\[ B^h_R \] — the h dimensional ball, for \( 0 < h < n-1 \).

We always write \( x = (x', x_n) \), for \( x \in \mathbb{R}^n \), where \( x' = (x_1, \ldots, x_{n-1}) \), and \( x' = (\xi_1, \xi_2) \), where \( \xi_1 = (x_1, \ldots, x_h) \), \( \xi_2 = (x_{h+1}, \ldots, x_{n-1}) \), for \( 0 < h < n-1 \). We use \( D' \) to denote the partial derivatives with respect to \( x' \). The indices \( s, t, r, \sigma, \tau \), always go from 1 to \( n-1 \), whereas \( i, j, k, l, m, h \), go from 1 to \( n \).
This chapter is primarily concerned with the classical existence and uniqueness of solutions of the linear problem (1.1), (1.2).

According to the definitions in Chapter 1, we will consider elliptic and degenerate elliptic boundary conditions separately. In Section 1 we will establish Schauder estimates in the elliptic case and we will prove the $C^{2,\alpha}$ existence of solutions of (1.1), (1.2) by the method of continuity. In Section 2, instead of Schauder estimates, we will derive higher order derivative estimates for solutions of (1.1), (1.2) when the boundary conditions are degenerate elliptic but strictly oblique, enabling the unique existence to be obtained by elliptic regularization.

1. Elliptic Boundary Conditions

The main result of this section is a Schauder estimate. Owing to the special structure of the boundary conditions, we can consider them as elliptic equations on $\partial \Omega$ so that our assertion is nothing but the direct application of classical Schauder estimates.

Within this section we assume that $\Omega$ is a $C^{2,\alpha}$ domain for some $0 < \alpha < 1$ and besides (1.4) the following conditions hold,

\begin{align}
(1.3)' & \quad \alpha^{ij}(x) \xi_i \xi_j \geq \kappa(x), \quad \forall \xi \in \mathbb{R}^n \text{ s.t. } \xi \perp \Gamma \text{ at } x, \text{ for some } \kappa(x) > 0. \\
(2.1) & \quad |a^{ij}, b^i, c|_{0,\alpha;\Omega} \leq \mu, \quad |\alpha^{ij}, \beta^i, \gamma|_{0,\alpha;\Omega} \leq \mu.
\end{align}

for some constant $\mu > 0$. Since one can extend any $C^{\alpha}$ ($\partial \Omega$) function to a $C^{\alpha}(\overline{\Omega})$ function such that the corresponding norms change only by a positive constant factor, we will use $|\cdot|_{0,\alpha;\Omega}$ and $|\cdot|_{0,\alpha;\overline{\Omega}}$ interchangeably without declaration.
For our convenience we sometimes write the boundary condition (1.2) as

\[
\tilde{\alpha}^{ij} D_{ij} u + \beta^i D_i u + \gamma u = 0,
\]

with \( \tilde{\alpha}^{ij} = \alpha^{kl} c^{ki} c^{lj} \), and we observe that

\[
\tilde{\alpha}^{ij} v^i = 0, \quad \forall \, x \in \partial \Omega \text{ and } \forall \, i.
\]

1.1 Maximum Principles

We will first prove some maximum principles which will play an essential role in our later estimates and guarantee the uniqueness of solutions.

Let \( u \in C^2(\Omega) \). For any point \( x_0 \in \partial \Omega \) we can always find a neighbourhood \( \mathcal{N} \) of \( x_0 \) and a diffeomorphism \( \Psi: \mathcal{N} \to \mathbb{R}^n \) such that \( \Psi(\mathcal{N} \cap \Omega) = B^+ \) and \( \Psi(\mathcal{N} \cap \partial \Omega) = B^0 \). For each continuous function \( v \) in \( \mathcal{N} \cap \Omega \) we denote \( \nabla = v \circ \Psi^{-1} \) which then belongs to \( C(B^+ \cup B^0) \). By the diffeomorphism \( \Psi \) the boundary condition (1.2) is changed into the form

\[
\bar{B} \bar{u} \equiv \tilde{\alpha}^{ij} D_{ij} \bar{u} + \tilde{\beta}^i D_i \bar{u} + \gamma \bar{u}, \quad \text{on } B^0,
\]

where

\[
\tilde{\alpha}^{ij} = \alpha^{kl} c^{ki} c^{lj} \frac{\partial \psi_i}{\partial x_h} \frac{\partial \psi_j}{\partial x_m}, \quad \text{and}
\]

\[
\tilde{\beta}^i = \alpha^{kl} c^{ki} c^{lj} \frac{\partial^2 \psi_i}{\partial x_h \partial x_m} + \beta^k \frac{\partial \psi_i}{\partial x_k}.
\]

By the property (2.3) we see that the matrix \( \{ \tilde{\alpha}^{ij} \} \) has the n-th row and the n-th column both zero and
From the above observation we conclude immediately

Lemma 2.1. Suppose that $B$ is degenerate elliptic and degenerate oblique. If $x_0 \in \partial \Omega$ is a maximum point of some function $u \in C^2(\Omega)$. Then at $x_0$ we have $\{\delta_{ij} u\} \leq 0$, $D_v u \leq 0$, $\delta u = 0$ so that

\[
(2.7) \quad \alpha_{ij} \delta_{ij} u + \beta^i D_i u \leq 0.
\]

Proof. We only prove that $\{\delta_{ij} u\} \leq 0$. The rest of assertion is obvious. By flattening the boundary $\partial \Omega$ locally around $x_0$ we obtain a boundary operator in the form (2.4) on $B^0$. $u$ achieves its maximum at $x_0$, so does $\bar{u}$ at $0$. Hence we have at $0$ $\{D_i \bar{u}\} \leq 0$ and $D_s \bar{u} = 0, 1 \leq s, t \leq n-1$. Since

\[
\delta_{ij} u = c^{ik} c^{jl} D_{kl} u = c^{ik} c^{jl} \left( D_{\sigma \tau} \bar{u} \frac{\partial \psi_\sigma}{\partial x_k} \frac{\partial \psi_\tau}{\partial x_l} + D_{\sigma \tau} \bar{u} \frac{\partial^2 \psi_\sigma}{\partial x_k \partial x_l} \right), \quad c^{ik} \frac{\partial \psi_\sigma}{\partial x_k} = 0, \forall i,
\]

we can see that

\[
\{\delta_{ij} u\} |_{x=x_0} = \left\{ \sum_{\sigma, \tau=1}^{n-1} c^{ik} c^{jl} D_{\sigma \tau} \bar{u} \frac{\partial \psi_\sigma}{\partial x_k} \frac{\partial \psi_\tau}{\partial x_l} \right\}_{y=0} \leq 0.
\]

As a straightforward consequence of Lemma 2.1 we have the following maximum principles.

Lemma 2.2. Suppose that $L$ is degenerate elliptic and $B$ is degenerate elliptic and degenerate oblique. Let $u \in C^2(\Omega)$ satisfy

\[
(2.8) \quad Lu \geq f, \quad \text{in } \Omega; \quad Bu \geq g, \quad \text{on } \partial \Omega.
\]
Then we have the estimate

\begin{equation}
\sup_{\Omega} u \leq \sup_{\partial\Omega} \left| \frac{f}{\gamma} \right| + \sup_{\Omega} \left| \frac{f}{\lambda} \right|,
\end{equation}

provided \( \gamma < 0 \) on \( \partial\Omega \) and \( c < 0 \) in \( \Omega \).

The proof of this lemma is trivial.

**Lemma 2.3.** Suppose that \( L \) is strictly elliptic with ellipticity number \( \lambda \) and \( c \leq 0 \), \( B \) is as in Lemma 2.2. Suppose \( u \in C^2(\Omega) \) satisfies (2.8). Then we have

\begin{equation}
\sup_{\Omega} u \leq \sup_{\partial\Omega} \left| \frac{g}{\gamma} \right| + \sup_{\Omega} \left| \frac{f}{\lambda} \right|,
\end{equation}

where \( C \) is a constant depending on \( n, \text{diam} \Omega, \sup_{\Omega} \frac{|b|}{\lambda} \) and \( \sup_{\partial\Omega} \frac{|\alpha| + |\beta|}{\gamma} \).

**Proof.** We can assume that the origin is in \( \Omega \). Let \( d = \text{diam} \Omega \). Choosing \( \sigma > 0 \) sufficiently large and

\[ v = (e^{\sigma d} - e^{\sigma}) \sup_{\Omega} \frac{|f|}{\lambda} \]

we have \( L(u - v) \geq 0 \) in \( \Omega \). Then by the weak maximum principle we have

\[ \sup_{\Omega} (u - v) \leq \sup_{\partial\Omega} (u - v). \]

At the boundary maximum point, we apply Lemma 2.1 to obtain the desired estimate (2.10).

**Lemma 2.4.** Suppose that \( L \) is as in Lemma 2.2 and that \( B \) is degenerate elliptic but strictly oblique, i.e., \((\beta \cdot v) \geq \chi > 0\), and \( \gamma \leq 0 \). Suppose also \( u \in C^2(\Omega) \) satisfies (2.8). Then we have

\begin{equation}
\sup_{\Omega} u \leq \sup_{\partial\Omega} \left| \frac{f}{\gamma} \right| + C \sup_{\partial\Omega} \frac{|g|}{\chi}.
\end{equation}
where $C$ is a constant depending on $\Omega$, $\sup_{\Omega} \frac{|a_{ij}|}{\infty}$, $\sup_{\Omega} \frac{|b_i|}{\infty}$.

**Proof.** Let $\psi \in C^2(\Omega)$ be a function such that $\psi > 0$ in $\Omega$, $\psi = 0$ on $\partial \Omega$. It is obvious that $\delta_{ij}\psi = 0$ on $\partial \Omega$. Setting $v = u + A\psi$ with $A$ a constant we have on $\partial \Omega$

$$\alpha^{ij} \delta_{ij}v + \beta^i D_i v + \gamma v \geq g + A\beta^i D_i \psi > 0,$$

if $A > \sup_{\partial \Omega} \frac{|g|}{\chi|\nabla \psi|}$. Hence $v$ must take an interior positive maximum in $\Omega$, at which

$$-cu \leq f + A(\alpha^{ij} D_{ij} \psi + b^i D_i \psi).$$

Thus the assertion follows.

**Remark.** Unlike the situation of an interior operator $L$, we can not simultaneously permit $(\beta \cdot v) = 0$ and $\gamma = 0$, by assuming $B$ is strictly elliptic. For example, we take $\beta = 0$, $\gamma = 0$, $g = 0$, and we notice that any constant boundary data $C$ will satisfy the condition $\alpha^{ij} \delta_{ij}u = 0$. If we solve the problem $Lu = f$ in $\Omega$, $u = C$ on $\partial \Omega$, we will see that $\sup u$ can be as large as possible. However we will see in the next chapter that the ellipticity of $B$ with some other restriction on $u$, will also yield a maximum principle.

### 1.2 Schauder Estimates and Existence

Let $u$ be a $C^{2, \alpha}$ solution of (1.1), (1.2). First of all we assume that the boundary is flat, i.e.

$$\alpha^{st} D_{st}u + \beta^i D_i u + \gamma u = g, \quad \text{on } B^0.$$
Setting \( G = g - \beta^u D_n u \) we can rewrite (2.12) as

\[
\alpha^s D_n^u + \beta^s D_n u + \gamma u = G, \quad \text{on } B^0.
\]

Considering (2.13) as an elliptic equation in the \((n-1)\)-dimensional domain \( B^0 \) and applying the Schauder interior estimates, Theorem 6.2 of [6], to this equation we obtain for \( g \in C^\alpha(B^0) \) that

\[
\|u\|_{2,\alpha; B^0} \leq C \left( \|u\|_{0, B^0} + \|G\|_{0,\alpha; B^0} \right),
\]

where \( C = C(n, \alpha, \kappa, \mu) \). The inequality (2.14) is equivalent to

\[
\|u\|_{2,\alpha; \partial \Omega \cap \mathcal{N}} \leq C \left( \|u\|_{1,\alpha; \Omega} + \|g\|_{0,\alpha; \Omega} \right),
\]

where \( \mathcal{N} \) is the diffeomorphism neighbourhood of \( B(0) \). By a suitable choice of finite covering \( \{\mathcal{N}_i\} \) of \( \partial \Omega \) and the corresponding diffeomorphism \( \{\psi_i\} \), we conclude

\[
\|u\|_{2,\alpha; \partial \Omega} \leq C \left( \|u\|_{1,\alpha; \Omega} + \|g\|_{0,\alpha; \Omega} \right).
\]

Recalling the global Schauder estimates for Dirichlet problems, Theorem 6.6 of [6], we have

\[
\|u\|_{2,\alpha; \Omega} \leq C \left( \|u\|_{0,\Omega} + \|u\|_{2,\alpha; \partial \Omega} + \|f\|_{0,\alpha; \Omega} \right).
\]

Now we plug (2.16) into (2.17) to obtain the inequality

\[
\|u\|_{2,\alpha; \Omega} \leq C \left( \|u\|_{1,\alpha; \Omega} + \|g\|_{0,\alpha; \Omega} + \|f\|_{0,\alpha; \Omega} \right).
\]

Finally by the well-known interpolation inequality Lemma 6.35 of [6] we have
Theorem 2.5. Let $\Omega$ be a $C^{2,\alpha}$ domain and $u \in C^{2,\alpha}(\Omega)$ be a solution of (1.1), (1.2), $f, g \in C^\alpha(\Omega)$. Suppose the coefficients of $L$ and $B$ satisfy (1.3), (1.4) with $\kappa > 0, \chi \geq 0$, and (2.3). Then we have

\[
|u|_{2,\alpha;\Omega} \leq C (|u|_{0,\Omega} + |g|_{0,\alpha;\Omega} + |f|_{0,\alpha;\Omega}),
\]

where $C = C(n, \alpha, \lambda, \kappa, \mu, \partial\Omega)$.

Now we turn to the solvability of the problem (1.1), (1.2). Our procedure is to reduce it to a simple case by the method of continuity. To this end let us consider first the boundary value problem of equation (1.1) with boundary condition

\[
\Delta_{\partial\Omega} u - u = g, \quad \text{on } \partial\Omega,
\]

where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on the $C^{2,\alpha}$ manifold $\partial\Omega$ defined by $\Delta_{\partial\Omega} = \delta_1 \delta_i$. The $C^{2,\alpha}$ solvability of this problem can be obtained by various methods, for example, by variational method and the classical regularity theory. For any $g \in C^\alpha(\partial\Omega)$, let $v \in C^{2,\alpha}(\partial\Omega)$ be the unique solution of equation (2.20). Taking this $v$ as boundary data, we can then obtain a function $u \in C^{2,\alpha}(\Omega)$ by solving (1.1) with $u = v$ on $\partial\Omega$. Obviously this $u$ is our desired solution for the problem (1.1), (2.20).

Theorem 2.6. Let $\Omega$ be a $C^{2,\alpha}$ domain and $L$ be an elliptic operator, $B$ be an elliptic Venttsel boundary operator. Suppose that (2.3) is satisfied and $c, \gamma < 0$ in $\bar{\Omega}$. Then the boundary value problem (1.1), (1.2) has a unique $C^{2,\alpha}$ solution for every $f, g \in C^\alpha(\bar{\Omega})$.

Proof. Let $B' = \Delta_{\partial\Omega} u - u$. Consider the family of problems

\[
\begin{align*}
L_t u &= Lu = f, \quad \text{in } \Omega, \\
B_t u &= t Bu + (1 - t) B' u = g, \quad \text{on } \partial\Omega,
\end{align*}
\]

for $t \in [0,1]$. It is obvious that the coefficients of $B_t$ satisfy
\[ \alpha_{ij} \eta_i \eta_j \geq \min \{ 1, \kappa \} |\eta|^2, \quad \forall x \in \partial \Omega \text{ and } \forall \eta \in \mathbb{R}^n \text{ s.t. } \eta \perp v \text{ at } x, \]

and

\[ |\alpha_{ij}, \beta_i, \gamma_t|_{0, \alpha} \leq \max \{ \mu, 1 \}, \quad \gamma_t \leq \gamma_0 < 0 \]

for some constant \( \gamma_0 \). By Lemma 2.2 and theorem 2.5 we can have a \( t \)-independent estimate

\[ |u_t|_{2, \alpha; \Omega} \leq C ( |g|_{0, \alpha; \Omega} + |f|_{0, \alpha; \Omega} ), \]

for the solutions \( u_t \) of (2.21). Therefore by virtue of the method of continuity, the solvability of (1.1), (1.2) is equivalent to that of (1.1), (2.20) which is known.

For Schauder estimates and the method of continuity we refer to Chapter 6 of [6].

### 2. Degenerate Elliptic Boundary Conditions

In this section we derive \( C^k \) estimates for solutions by differentiating the equation and the boundary condition \( k \) times. For this, higher order smoothness of the coefficients of \( L \) and \( B \), and obliqueness of \( B \) are needed.

Throughout this section we assume \( L \) is strictly elliptic and \( B \) satisfies (1.3) and \((\beta \cdot v) \geq \chi > 0 \). By flattening the boundary locally we will only consider the following problem.

\[ a_{ij} D_{ij} u + b^i D_i u + cu = f, \quad \text{in } B^+, \]
\[ \alpha s^s D_s u + \beta^i D_i u + \gamma u = g, \quad \text{on } B^0. \]
It follows from the assumptions above that

\[ a_{ij} \xi_i \xi_j \geq \lambda \| \xi \|^2, \quad \forall x \in B^+, \forall \xi \in \mathbb{R}^n, \]
\[ \alpha^{st} \eta_i \eta_i \geq 0, \quad \forall x \in B^0, \forall \eta \in \mathbb{R}^{n-1}, \]
\[ \beta^n \geq \chi, \quad \forall x \in B^0, \]

for some constants \( \lambda > 0, \chi > 0. \)

2.1 Gradient Estimates

We will adopt Lemma 1.7.1 of [19] without proof, that is,

**Lemma 2.7.** Let \( \{ \alpha^{st} \} \) be nonnegative definite and \( \alpha^{st} \in C^2(\mathbb{R}^{n-1}). \) Then for any \( v \in C^2(\mathbb{R}^{n-1}) \) we have

\[
(D_i \alpha^{st} D_{st} v)^2 \leq C \alpha^{st} D_{st} v D_{st} v,
\]

where the constant \( C \) depends only on the second derivatives of \( \alpha^{st}. \)

**Lemma 2.8.** Let \( u \in C^3(B^+ \cup B^0) \) be a solution of (2.23), (2.24). Suppose that \( \alpha^{st} \in C^2(\overline{B}^+), a_{ij}, b_i, c, \beta^i, \gamma, f, g \in C^1(\overline{B}^+) \) and there is a constant \( \mu > 0 \) such that

\[
|\alpha^{st}|_2, \quad |a_{ij}, b_i, c, \beta^i, \gamma, f, g|_1 \leq \mu.
\]

Then there exists a constant \( C \) depending only on \( \lambda, \mu, \chi, n, M_0 = \sup |u| \) such that

\[
(2.26) \quad \sup_{B^+_1/2} |D'u| \leq C.
\]
Proof. Let

\[ w' = |D'u|^2 = \sum_{s=1}^{n-1} |D_s u|^2, \quad v = |Du|^2, \quad \eta = 1 - |x|^2, \]

and

\[ w = \eta^4 w' + \alpha_1 \eta^2 v x_n + \alpha_2 u^2 \]

with \( \alpha_1, \alpha_2 > 1 \) the constants to be determined later. As long as \( \alpha_1, \alpha_2 \) have been determined from the known quantities and if \( M_0^2 = \sup_{B} \eta^4 w' \leq \alpha_2 M_0^2 \), one can see that with this assumption \( w \) can not achieve its maximum on \( \partial B^+ \setminus B^0 \). Let \( x_0 \) be a maximum point of \( w \).

If \( x_0 \in B^0 \), by applying the operator \( \delta' = D_{s'} u D_{s} \) to (2.24) we have on \( B^0 \)

\[
(2.27) \quad \alpha_{s't} D_{s't} w + \beta |D_{s'} w |^{2} \\
= 2 \eta^4 \alpha_{s't} D_{s't} u D_{s} u - 2 \eta^4 D_{s'} u D_{s'} \alpha_{s't} D_{s't} u + 16 \eta^3 D_{s'} u \alpha_{s't} D_{s} \eta D_{u} u \\
+ 2 (2 \eta \alpha_{s't} D_{s't} \eta + 6 \alpha_{s't} D_{s} \eta D_{r} \eta - \eta^2 \gamma ) \eta^2 w' - 2 \eta^4 (\delta \beta |D_{s'} u + \delta \gamma u - \delta g ) \\
+ \alpha_1 \eta^2 \beta^{n} v + 2 \alpha_2 \alpha_{s't} D_{s'} u D_{s} u + 2 \alpha_2 u ( g - \gamma u ).
\]

By Lemma 2.7 we know that

\[ \eta^4 D_{s'} u D_{s} \alpha_{s't} D_{s't} u \leq \eta^4 \alpha_{s't} D_{s't} u D_{u} u + C(\mu) \eta^4 w', \]

and by the Schwartz inequality

\[ \eta^3 D_{u} u \alpha_{s't} D_{s} \eta D_{u} u \leq \eta^3 D_{s't} u (\alpha_{s't} D_{s} \eta D_{s} \eta)^{1/2} (\alpha_{s't} D_{s't} u D_{u} u)^{1/2} \]
\[ \leq \eta^4 \alpha_{s't} D_{s't} u D_{u} u + C(\mu) \eta^2 w'. \]

Inserting these into (2.27) and applying the Cauchy inequality to the other terms we have
\[ \alpha^i D_{st} w + \beta^i D_i w \geq \eta^2 v \left( \frac{1}{2} \alpha_1 \chi - C_1(\mu, M_0) \right) + \left( \frac{1}{2} \chi M_1^2 \alpha_1 - \alpha_2 C_2(\mu, M_0) \right). \]

which will be positive if \( \alpha_1 > 2C_1/\chi \) and \( M_1^2 > 2 \alpha_2 C_2/\alpha_1 \chi \). Therefore by Lemma 2.1 we cannot have a maximum of \( w \) on \( B^0 \) if \( M_1^2 > 2 \alpha_2 C_2/\alpha_1 \chi \).

If \( x_0 \in B^+ \), we apply \( \delta' \) to equation (2.23) to obtain

\[ (2.28) \quad a_{ij} D_{ij} w + b_i D_i w \]

\[ = 2 \eta^4 a_{ij} D_{ij} u D_{ij} u - 2 \eta^4 D_i u D_{ij} u + 16 \eta^3 D_i u a_{ij} D_i \eta D_{ij} u \]

\[ + 2 ( 2 \eta a_{ij} D_{ij} \eta + 6 a_{ij} D_i \eta D_j \eta - \eta^2 C ) \eta^2 w' + 2 \eta^4 ( \delta' b_i D_i u + \delta' c u - \delta' f ) \]

\[ + 2 \alpha_1 \eta^2 x_n a_{ij} D_{ij} u D_{jk} u + 8 \alpha_1 \eta x_n D_k u a_{ij} D_i \eta D_{jk} u + 4 \alpha \eta^2 D_k u a_{ij} D_{jk} u \]

\[ + 4 \alpha_1 \eta v a_{ij} D_{ij} \eta + 2 \alpha_1 \eta x_n v a_{ij} D_i \eta D_j \eta \]

\[ + 2 \alpha_1 x_n \eta^2 D_k u a_{ij} D_{ij} u + 2 \alpha_1 \eta^2 x_n D_k u b_i D_{ik} u \]

\[ + \alpha_1 \eta^2 v b_n + 2 \alpha_2 a_{ij} D_{ij} u D_{ij} u + 2 \alpha_2 u ( f - cu ). \]

where \( f_k = a_{ij} D_{ijk} u - D_k ( a_{ij} D_{ij} u + b_i D_i u + cu - f ) \). By the ellipticity assumption of \( L \) we have

\[ a_{ij} D_{ij} u D_{ij} u \geq \lambda \| DD' u \|^2 = \lambda \sum_{i=1}^{n} \sum_{r=1}^{m} ( D_{ij} u )^2, \]

\[ a_{ij} D_{ij} u D_{ij} u \geq \lambda \| D^2 u \|^2. \]

From equation (2.23) we may also obtain

\[ \| D_{nn} u \| \leq \frac{1}{\lambda} C ( \mu, M_0 ) ( \| DD' u \| + \| Du \| + 1 ). \]

Using these facts and the similar argument to that in last paragraph we have at the point
which is greater than zero provided $\alpha_2 > C_3 \left( \alpha_1, \mu, M_0 \right) / \lambda$ and $M_1^2 > C_4 / \lambda$. So $w$ cannot take its maximum in $B^+$ if $M_1^2 > C_4 / \lambda$. In summary we have

$$M_1^2 \leq \max \left( \alpha_2 M_0^2, \frac{2\alpha_2 C_2}{\chi \alpha_1}, \frac{C_4}{\lambda} \right).$$

Then (2.26) follows from the definition of $w$ and $M_1$.

**Remark** The assumption that $\alpha^{st} \in C^2$ can be relaxed in the following way. We denote the square root matrix of $\{\alpha^{st}\}$ by $\{\sigma^{st}\}$ so that $\alpha^{st} = \sigma^{st} \sigma^{rt}$. If we suppose $\sigma^{st} \in C^1(\overline{B}^+)$ we will still get (2.25) with the constant depending on $C^1$ bound of $\sigma^{st}$.

Since $\delta$, the tangential gradient operator, is defined over $\overline{\Omega}$, it follows from Lemma 2.8 that

$$\sup_{\overline{\Omega}} |\delta u| \leq C,$$

for some constant $C = C \left( \lambda, \mu, \chi, \partial \Omega, M_0 \right)$. This can be proved easily by piecing together the local boundary estimates and combining the well known interior estimates.

The estimate for the normal derivative will be obtained after the following second order tangential derivatives are bounded.

**Lemma 2.9.** Let $u \in C^4(\overline{B}^+)$ be a solution of (2.23), (2.24). Suppose that there is a constant $\mu$ such that

$$|a^{ij}, b^i, c, \alpha^{st}, \beta^i, \gamma, f, g|_{2,B^+} \leq \mu.$$ 

Suppose also that there are constants $M_1$ and $K$ such that
Then there is a constant $C = C (\lambda, \mu, \chi, M_1, K)$ such that

\begin{equation}
\sup_{B^+} |D'u| + \sup_{B^+} |u| \leq M_1, \quad (1 + |Du|^2)^{1/2} x_n \leq K, \quad \text{in } B^+.
\end{equation}

\begin{equation}
\sup_{B^2} |D^2u| \leq C,
\end{equation}

for some constant $\sigma = \sigma (\lambda, \mu, \chi, M_1, K) < \lambda / 2\mu$.

**Proof.** Let $w_1 = |D^2u|^2, v_1 = |DD'u|^2, w_2 = |D'u|^2, v_2 = (1 + |Du|^2)^{1/2}, \eta = (1 - |x|^2), M_2^2 = \sup_{B^2} \eta^4 w_1,$ and

\[ w = \eta^4 w_1 + \alpha_1 \eta^2 v_1 x_n + \alpha_2 w_2 + \alpha_3 M_2 v_2 x_n + \alpha_4 u^2 + \frac{1}{2\sigma} M_2^2 x_n^2. \]

It is obvious that $w$ cannot achieve its maximum on $(\partial B^+ \setminus B^0) \cap G, G = \{ x \in B^+ | x_n < \sigma \}$ for some $\sigma$ to be determined later, if $M_2^2 > 8 (\alpha_2 + \alpha_4) M_1^2 + 32 (\alpha_3 K)^2$.

Once $\sigma$ has been determined from the known quantities and the maximum of $w$ is obtained in $B^+ \cap G$, then by interior estimates we can have a bound for $M_2$ in terms of $\sigma$, so we are done. Let $x_0 \in B^+$ be a maximum point of $w$.

If $x_0 \in B^0$, we apply the operator $\bar{\delta} + \bar{\delta}' = D_{qr} u D_{qr} + D_r u D_r$ to (2.24) to obtain on $B^0$ the equation

\begin{equation}
(2.31) \quad \alpha^{st} D_{st} w + \beta^i D_i w
\end{equation}

\[ = 2 \eta^4 \alpha^{st} D_{sq} u D_{sq} u + 2 \alpha_2 \alpha^{st} D_{sq} u D_{sq} u - 2 \eta^4 D_{sq} u D_r \alpha^{st} D_{sq} u
\]
\[ + 16 \eta^3 D_{qr} u \alpha^{st} D_t \eta D_{sq} u + (B^+ \eta^4 - 2 \eta^4 \gamma) w_1 - 4 \eta^4 D_{qr} u D_q \beta^i D_{sq} u
\]
\[ - 4 \eta^4 D_{qr} u D_q \gamma D_{rt} u + 2 (\eta^4 \bar{\delta} + \alpha_2 \bar{\delta}') g - 2 (\eta^4 \bar{\delta} + \alpha_2 \bar{\delta}') \gamma u
\]
\[ - 2 (\eta^4 \bar{\delta} + \alpha_2 \bar{\delta}') \beta^i D_i u - 2 (\eta^4 \bar{\delta} + \alpha_2 \bar{\delta}') \alpha^{st} D_{st} u + \alpha_1 \beta^i \eta^2 v_1
\]
\[ + \alpha_3 \beta^i M_2 v_2 - 2 \alpha_2 \gamma w_2 + 2 \alpha_4 \alpha^{st} D_{st} u D_i u + 2 \alpha_4 u (g - \gamma u). \]
where \( B' = \alpha^{st} D_{st} + \beta^i D_i \). 

By the assumption on \( M_2 \) we have \( \eta^4 v_1 > M_2^2 / 4\alpha_1 \) at \( x_0 \). Using this fact and the same argument as in the proof of Lemma 2.8 we have at \( x_0 \)

\[
B'w \geq \eta^2 v_1 \left( \frac{1}{2} \alpha_1 \chi - C_1(\mu, M_1) \right) + M_2 v_2 \left( \alpha_3 \chi - \alpha_2 C_2(\mu, M_1) \right) + \left( \frac{1}{8} \chi M_2^2 - \alpha_4 C_4(\mu, M_1) \right)
\]

Then we choose \( \alpha_1 > 2C_1 / \chi, \alpha_3 > \alpha_2 C_2 / \chi \) and suppose \( M_2^2 > 8 \alpha_4 C_3 / \chi \) so that \( B'w > 0 \), a contradiction.

If \( x_0 \in B^+ \setminus G \), by a calculation we have

\[
L'w = a^{ij} D_{ij}w + b^i D_i w
\]

\[
= 2 \eta^4 a^{ij} D_{qr} u D_{ijr} u + 2 \alpha_1 \eta^2 x_n a^{ij} D_{ikr} u D_{jkr} u + 2 \alpha_2 a^{ij} D_{i} u D_{j} u
\]

\[
- 4 \eta^4 D_{qr} u D_{i} a^{ij} D_{ijq} u + 16 \eta^3 D_{qr} u a^{ij} D_{i} \eta D_{jqr} u
\]

\[
- 4 \eta^4 D_{qr} u \left( D_{q} b^{i} D_{i} u + D_{q} c D_{i} u \right) + \left( L'/\eta^4 - 2 c \eta^4 \right) w_1
\]

\[
- 2 \left( \eta^4 \bar{\delta} + \alpha_2 \delta' \right) \left( a^{ij} D_{ij} u + b^{i} D_{i} u + c u - f \right) - 2 \alpha_2 c w_2
\]

\[
+ 8 \alpha_1 \eta x_n D_{kr} u a^{ij} D_{i} \eta D_{jkr} u + 4 \alpha_1 \eta^2 D_{kr} u a^{nj} D_{jk} u + 2 \alpha_1 \eta^2 x_n D_{kr} u f_{kr}
\]

\[
+ 2 \alpha_1 \eta^2 x_n D_{kr} u b^{i} D_{i} u + 2 \alpha_1 x_n L'(\eta^2) v_1 + 2 \alpha_1 \eta v_1 a^{nj} D_{j} \eta
\]

\[
+ \alpha_3 \frac{x_n}{v^2} a^{ij} g^{kl} D_{ik} u D_{jl} u + 2 \alpha_3 M^2_2 \frac{D_{k} u}{v^2} a^{in} D_{ik} u + \alpha_3 M^2_2 \frac{D_{k} u}{v^2} b^{i} D_{ik} u
\]

\[
+ \alpha_3 M^2_2 b^{n} + \alpha_3 M^2_2 \frac{D_{k} u}{v^2} f_{k} x_n + \alpha_4 a^{ij} D_{i} u D_{j} u
\]

\[
+ 2 \alpha_4 \left( f - c u \right) + \frac{1}{\alpha} M^2_2 a^{nn} + \frac{1}{\alpha} M^2_2 b^{n} x_n,
\]

where \( g^{kl} = \delta^{kl} - \frac{D_{k} u D_{l} u}{(1 + ||D u||^2)} \) is a positive definite matrix and
\[ f_{kr} = a_{ij} D_{ijk} u - D_{kr} (a_{ij} D_{ij} u + b_i D_i u + cu - f), \]
\[ f_k = a_{ij} D_{ijk} u - D_k (a_{ij} D_{ij} u + b_i D_i u + cu - f). \]

Notice that by differentiating (2.23) with respect to \( x_r, r = 1, \ldots, n-1 \), and \( x_k, k = 1, \ldots, n \), we may obtain

(2.33) \( |D'D^2 u|, |D^3 u| \leq \frac{1}{\lambda} C(\mu) (|DD'^2 u| + |DD'u| + |Du| + M_1) \).

Inserting (2.33) into (2.32) and using the same argument as before we can obtain at \( x_0 \) the estimate

\[ Lw \geq v_1^2 (\alpha_2 \lambda - C_4(\mu, \lambda, M_1)) + v_2^2 (\alpha_4 \lambda - C_5(\mu, \lambda, M_1, \alpha_1, \alpha_2, \alpha_3)) \]
\[ + \left( \frac{1}{2} \lambda M_2^2 - C_6(\alpha_3) M_2^2 - \alpha_4 C_7(\mu, M_1) \right) > 0, \]

provided \( \alpha_2 > C_4 / \lambda, \alpha_4 > C_5 / \lambda, \sigma < 1 / 4 C_6 \) and \( M_2^2 > \alpha_4 C_7 / C_6 \). This gives us another contradiction.

**Corollary 2.10.** Under the assumptions of Lemma 2.9 we have

(2.34) \( \sup_{B_{\sigma}} |Du| \leq C, \)

for some constant depending on the same quantities as in Lemma 2.9.

**Proof.** From the boundary condition (2.24) we can obtain

\[ \sup_{B_{\sigma}} |D_n u| \leq \frac{1}{\lambda} \sup_{B_{\sigma}} [ |g| + C (|D'u| + |D^2 u|) ] \leq C. \]

The last inequality follows from Lemma 2.9 and Lemma 2.8. Applying a maximum principle argument to the test function \( w = \eta^2 |Du|^2 + A u^2 \), with \( A \) a constant to be chosen, we then conclude (2.34), because \( w \) is now bounded on \( B_{\sigma} \).
We summarize the results in this subsection into the following theorem.

**Theorem 2.11.** Let \( \partial \Omega \in C^6 \) and \( u \in C^4(\bar{\Omega}) \) be a solution of (1.1), (1.2). Suppose there is a constant \( \mu > 0 \) such that

\[
|a^{ij}, b^i, c, \omega^{ij}, \beta^i, \gamma, f, g|_{L^2; \Omega} \leq \mu.
\]

Suppose also \( \sup |u| \leq M_0. \) Then there exists a constant \( C = C(\lambda, \mu, \chi, M_0, \partial \Omega) \) such that

\[
\sup_{\Omega} |Du|, \sup_{\Omega} |\nabla^2 u| \leq C.
\]

### 2.2 \( C^k \) Estimates

We first introduce some notation. Let \( \mathcal{M} = \{1, 2, \ldots, m\}, \mathcal{N} = \{1, 2, \ldots, n\}, \) and \( P_m = \{p: \mathcal{M} \rightarrow \mathcal{N}\}, \) that is, if \( p \in P_m, \) then \( p = (p_1, \ldots, p_m), p_i \in \mathcal{N} \) for \( i = 1, \ldots, m. \) We define the operator \( D_p \) for \( p \in P_m \) by

\[
D_p u \equiv D_{p_1} D_{p_2} \cdots D_{p_m} u
\]

as an \( m \)-th order differential operator, and \( \delta_p \) by

\[
\delta_p u \equiv c^{p_1q_1} c^{p_2q_2} \cdots c^{p_mq_m} D_{q_1} D_{q_2} \cdots D_{q_m} u, \quad p, q \in P_m,
\]

as an \( m \)-th order tangential differential operator. It is not difficult to see that \( |\delta_m u|^2 = c^{p_1q_1} \cdots c^{p_mq_m} D_p u D_q u \) is equivalent to \( |D^m u|^2 \) when we flatten the boundary locally, where \( D^m u = |D_{s_1} \cdots D_{s_n} u|^2, \ s_i \in \mathcal{N}' = \{1, 2, \ldots, n-1\}, i = 1, 2, \ldots, m. \) The equivalence is in the sense that

\[
|\delta_m u|^2 \equiv C |D^m u|^2, \quad \mod |u|_{m-1; \Omega}.
\]
for some constant $C > 0$.

**Theorem 2.12.** Let $\partial \Omega \in C^{k+1}$ ($k = 2, 3, \ldots$) and $u \in C^{k+2}(\Omega)$ be a solution of (1.1), (1.2). Suppose the following inequality holds

$$|a_{ij}, b^i, c, \alpha^i, \beta^j, \gamma; f, g |_{k; \Omega} \leq \mu,$$

for some constant $\mu$. Suppose also that there is a constant $M_{k-1}$ such that

$$\sup_{\Omega} |\delta_p u| + |u|_{k-1; \Omega} \leq M_{k-1}, \text{ for all } p \in P_{k-1}.$$

Then there exists a constant $M_k$ depending on $\lambda, \mu, \chi, \partial \Omega, M_{k-1}$, such that

$$\sup_{\Omega} |\delta_q u| + |u|_{k-1; \Omega} \leq M_k, \text{ for all } q \in P_k.$$

**Proof.** We argue by induction. For $k = 2$ the theorem is a combination of Lemma 2.9 and 2.10.

Suppose it is true for $k = m-1$. We will prove that it is also true for $k = m$. First of all we make some preparation. Picking a $p \in P_{m-2}$, we differentiate the equation (1.1) with respect to $p$ so that we obtain an equation in the form (1.1) that $v = D_p u$ satisfies. Then we repeat the procedure in the proof of Theorem 6.2 of [6] and use the induction hypothesis to obtain that $|D v| \leq K$, for some constant $K$, where $d = d(x) = \text{dist} (x, \partial \Omega)$. Doing this all over $P_{m-2}$ we conclude that $(1 + |D^{m-1} u|^2)^{1/2} \leq K$ for the same constant $K$ as above.

Now we consider (2.23), (2.24) in $B^+$ with flat boundary $B^0$. The above discussion means that $(1 + |D^{m-1} u|^2)^{1/2} \leq K$. By applying the same argument as in the proof of Lemma 2.10 to the test function

$$w = \eta^4 w_1 + \alpha_1 \eta^2 v_1 x_n + \alpha_2 w_2 + \alpha_3 \overline{M}_m v_2 x_n + \alpha_4 w_3 + \frac{1}{2\sigma} \overline{M}_m x_n^2,$$
where \( w_1 = |D^m u|^2, \ w_2 = |D^{m-1} u|^2, \ w_3 = |D^{m-2} u|^2, \ v_1 = |D^m u|^2, \ v_2 = (|DD^{m-2} u|^2 + 1)^{1/2}, \ M_m^2 = \sup_B |\eta^4 w_1|, \) we can prove that
\[
\sup_{B^r} |D^m u| \leq C.
\]

Finally from the differentiated boundary condition and the differentiated equation we have
\[
\sup_{B^r} |D_n D^{m-2} u| \leq C, \quad \text{and} \quad \sup_{B^r} |D^{m-1} u| \leq C.
\]

Piecing these local estimates together and combining them with interior estimates we are done.

2.3 Existence

Now we can establish our main result of this section.

**Theorem 2.13.** Let \( \partial \Omega \in C^{5, \alpha} \) and \( a^{ij}, b^i, c, \alpha^i, \beta^i, \gamma; f, g \in C^{3, \alpha}(\overline{\Omega}), \ c \leq 0, \gamma \leq 0. \) Then the problem (1.1), (1.2) has a unique solution \( u \in C^2(\overline{\Omega}). \)

**Proof.** We consider the family of problems

\[
(2.37) \quad Lu = f, \quad \text{in } \Omega,
\]
\[
\varepsilon \Delta_{\partial \Omega} u + Bu = g, \quad \text{on } \partial \Omega, \quad \text{for } \varepsilon \in [0,1].
\]

By Theorem 2.6 and Theorem 6.17 of [6] there is a unique solution \( u_\varepsilon \in C^{5, \alpha}(\overline{\Omega}) \) of (2.37) for each \( \varepsilon > 0. \) It follows from Theorem 2.12 that \( |u_\varepsilon|_{3; \partial \Omega} \) is bounded. By the Schauder global estimates for Dirichlet problems with \( C^{2, \alpha} \) boundary data, we can then obtain a \( C^{2, \alpha} \) a priori estimate of \( u_\varepsilon \) for some \( \alpha > 0, \) which is independent of \( \varepsilon. \)

Therefore \( \{ u_\varepsilon \} \) possesses a convergent subsequence \( \{ u_{\varepsilon_k} \} \) which converges to a function \( u \in C^2(\overline{\Omega}) \) as \( \varepsilon_k \) goes to 0. By taking the limit on both sides of (2.37) we...
can see that this \( u \) is a solution of (1.1), (1.2). The uniqueness follows from Lemma 2.4.

We finish this chapter with the following corollary.

**Corollary 2.14.** If \( \Omega \) is a \( C^\infty \) domain and the coefficients of \( L \) and \( B \) together with \( f, g \) are in \( C^\infty(\Omega) \). Then the problem (1.1), (1.2) has a unique \( C^\infty \) solution, provided \( c \leq 0, \gamma \leq 0 \).
Chapter 3
An Aleksandrov - Bakelman Type Maximum Principle And Applications

In this chapter, we deal with some further properties of linear operators.

Existence result for nonlinear problems usually depends strongly upon a priori estimates, and often an essential step is the estimation of moduli of continuity of solutions and their derivatives. The modulus that is generally used is Hölder continuity. One approach to Hölder estimates is to make use of a weak Harnack inequality which may be obtained from the Aleksandrov maximum principle, see, for example, [21]. In Venttsel boundary value problems we need only establish such estimates locally at the boundary because the interior operators are normal elliptic operators and correspondly the interior results are known. Accordingly we have a boundary version of the Aleksandrov - Bakelman maximum principle for Venttsel boundary conditions which will yield a boundary version of weak Harnack inequality and subsequently the desired boundary Hölder estimates.

To handle the conjunction of interior and boundary operators, we introduce a new notion of boundary contact set.

1. The Maximum principle

Let $u$ be a continuous function in $\overline{\mathbb{R}}^n_+$, and suppose $u \leq 0$ in $\mathbb{R}^n_+ \setminus B^+_R$. In the upper half ball $B^+_R$ we define the upper contact set $\Gamma_+(u)$ of $u$ and the normal mapping $\chi$ from $\Gamma_+(u)$ to $\mathcal{A}(\mathbb{R}^n)$, the family of all subset of $\mathbb{R}^n$, in the usual way. We denote the restriction of $u$ to $B^+_R$ by $u_0$ and define also the upper contact set $\Gamma(u_0)$ of $u_0$ and the normal mapping $\chi_0$ from $\Gamma_+(u_0)$ to $\mathcal{A}(\mathbb{R}^{n-1})$ in the same way. For
definitions of contact set and normal mapping we refer to Chapter 9 of [6].

We make an even extension of $u$ to the whole ball $B_R$, that is, we set

$$v = \begin{cases} 
  u(x', x_n), & \text{for } x_n \geq 0; \\
  \bar{u}(x', x_n) = u(x', -x_n), & \text{for } x_n < 0.
\end{cases}$$

Let $\hat{v}$ be the upper convex hull of $v$. It follows that

$$\sup_{B_R^+} \hat{v} = \sup_{B_R^-} v = \sup_{B_R^+} u^+,$$

and that there always exists at least one $x_0$ on $B_R^+$ such that

$$\hat{v}(x_0) = \sup_{B_R^+} \hat{v}.$$

We define the boundary contact set of $u$ to be the set

$$\Gamma(u_0) = \{ x \in B_R^+ | u_0(x) = \hat{v}(x) \},$$

which is obviously contained in $\Gamma_+(u_0)$.

We consider a pair of linear operators

\begin{align*}
Lu & = a^{ij} D_{ij} u + b^i D_i u + c u, & \text{in } \mathbb{R}_+^n, \\
Bu & = \alpha^{st} D_{st} u + \beta^i D_i u + \gamma u, & \text{on } \partial \mathbb{R}_+^n.
\end{align*}

Suppose $L$ is uniformly elliptic and $B$ is Venttsel, $c \leq 0, \gamma \leq 0$. By saying that $u$ is a subsolution of the problem

$$Lu = f, \quad \text{in } \mathbb{R}_+^n.$$
we mean that $u$ satisfies the following differential inequalities

\begin{align}
(3.7) & \quad Lu \geq f, \quad \text{in } \mathbb{R}^n, \\
(3.8) & \quad Bu \geq g, \quad \text{on } \partial \mathbb{R}^n.
\end{align}

For uniformly elliptic boundary operator $B$ we have the following maximum principle.

**Theorem 3.1.** Let $u \in W^{2,n}_{\text{Loc}}(\mathbb{R}^n) \cap W^{2,n-1}_{\text{Loc}}(\partial \mathbb{R}^n) \cap C(\overline{\mathbb{R}^n})$ be a subsolution of (3.5), (3.6), and $u \leq 0$ in $\mathbb{R}^n \setminus B_R$. Suppose $L$ and $B$ are uniformly elliptic and $\beta^s \equiv 0$, $s = 1, \ldots, n-1$. Then we have

\begin{equation}
(3.9) \quad \sup_{B_R} u^+ \leq C \left( \| \frac{g}{\Delta^s} \|_{n-1, \Gamma(a)} + \| \frac{f}{\Delta^s} \|_{n, \Gamma(u)} \right) R,
\end{equation}

where $C$ is a constant depending on $n$, $\| b/\Delta^s \|_n$, and $\Delta^s = (\det \{ \alpha^{st} \})^{1/n-1}$, $\Delta^s = (\det \{ \alpha^{ij} \})^{1/n}$. $\Delta^s$.

The idea used to prove Theorem 3.1 is based on that of the original version of the Aleksandrov maximum principle which is due to Aleksandrov [1]. We first prove a special case of Theorem 3.1.

**Lemma 3.2.** In addition to the assumptions of Theorem 3.1 assume also $b_i \equiv 0$, $i = 1, 2, \ldots, n$. Then (3.9) holds with $C = C(n)$. 

**Proof.** By a standard approximation we may assume $u \in C^2(\overline{\mathbb{R}^n})$. Let $x_0 \in B_R$ be a maximum point of $\hat{u}$ and $M = \hat{u}(x_0)$. We define two functions $W_1$ and $W_2$ whose graphs are the cones with the same vertex $(x_0, M)$ and bases $B_R$ and $B_{2R}(x_0)$ respectively. Let $I_1$ and $I_2$ be the normal images of $W_1$ and $W_2$, and denote
If \( \Gamma(u_0) = \emptyset \), we of course have the following relation

\[
I_{2,1} \subset I_{1,1} \subset \chi_u(B^+_R)
\]

where \( \chi_u(B^+_R) \) is the normal image of \( u \) over \( B^+_R \). Since

\[
|I_{2,1}| = \frac{1}{2} |I_2| = \frac{1}{2} \left( \frac{M}{2R} \right)^n \omega_n,
\]

we conclude from the relation (3.10) that

\[
\frac{1}{2} \left( \frac{M}{2R} \right)^n \omega_n \leq \int_{\Gamma(u_0)} \left( \frac{f^+}{nD_n^*} \right)^n d\Gamma.
\]

Suppose \( \Gamma(u_0) \neq \emptyset \). It is obvious that \( \beta^a D_n u \leq 0 \) on \( \Gamma(u_0) \) by the assumption of degenerate obliqueness of \( B \). So we have on \( \Gamma(u_0) \) that \( \alpha^a D_n u \geq g \). To proceed further we consider the set

\[
\Sigma = I_{2,1} \setminus \chi_u(B^+_R).
\]

An observation shows that for each \( p = (p', p_n) \in \Sigma \) we have \( p' \in \chi_{u_0}(\Gamma(u_0)) \) and \( p_n \geq -M/2R \). Hence we get

\[
\Sigma \subset \chi_{u_0}(\Gamma(u_0)) \times [\frac{-M}{2R}, 0],
\]

so that

\[
I_{l,1} = \{ p = (p', p_n) \in I_{l,1} | p_n \leq 0 \}, \quad i = 1, 2.
\]
From (3.12) we can see

\[
\frac{1}{2} \left( \frac{M}{2R} \right)^n \omega_n \leq \left| \chi_{\omega} (B^+_R) \right| + |\Sigma|.
\]

Combining (3.11) and (3.14) we obtain

\[
\frac{M}{2R} \leq \frac{1}{n} \omega_n^{1/n} \left\| \frac{g}{\Delta^*_{n-1}} \right\|_{L^1(B^+_R)} + \frac{1}{(n-1)^{n/2}} \omega_n^{1/n} \left( \frac{M}{2R} \right)^{1/n} \left\| \frac{g}{\Delta^*_{n-1}} \right\|_{L^1(B^+_R)}.
\]

Using Young's inequality in the second term of the right hand side of above inequality we get the estimate (3.9).

**Remark 1.** This lemma cannot be obtained simply by twice applying the interior Aleksandrov maximum principle to \( L \) and \( B \) respectively, because on the upper contact set \( \Gamma(u_0) \) we cannot expect \( \alpha^* \Delta^* u \geq g \) generally, except on \( \Gamma(u_0) \).

**Proof of Theorem 3.1.** This proof is a modification of the proof of Lemma 9.4 of [6]. We introduce first the weight function

\[
h(p) = \left( |p|^{n/(n-1)} + \mu^{n/(n-1)} \right)^{1-n},
\]

for some constant \( \mu \geq 0 \) to be fixed later. We know from the relation (3.12) that

\[
\int_{\Gamma^*_c} h(p) \, dp + \int_{\chi_{\omega} (B^+_R)} h(p) \, dp \leq \int_{\chi_{\omega} (B^+_R)} h(p) \, dp.
\]

We are going to estimate the integrals on the right hand side of (3.15). For the first one...
it is known, see Lemma 9.4 in [6], that

\[ (3.16) \quad \int h(p) \, dp \leq \frac{1}{n^n} \int (|b|^n + \mu^n \omega) / \mathcal{D} \, dx. \]

To estimate the second integral we split \( h(p) \) into \( h(p) = h_1(p) h_2(p) \), where

\[ h_1(p) = (|p| n / (n-1) + \mu^n / (n-1)) \cdot (n-1)/n, \]
\[ h_2(p) = (|p| n / (n-1) + \mu^n / (n-1)) \cdot (n-1)/n. \]

It is easy to see that

\[ h_1(p) \leq (|p| n / (n-1) + \mu(n-1)/(n-2)) \cdot 2, \quad h_2(p) \leq (|p| + \mu)^{-1}. \]

Therefore by the discussion on \( \Sigma \) in the proof of Lemma 3.2 we have

\[ (3.17) \quad \int h(p) \, dp \leq \int (|p| n / (n-2) + \mu(n-1)/(n-2)) \cdot 2. \]

On the other hand we have

\[ (3.18) \quad \int h(p) \, dp \geq \frac{\omega_n}{2^{n-1}} \log \left( 1 + \left( \frac{M}{2\mu R} \right)^n \right). \]

By taking \( \mu = k (\|f/\mathcal{D}*\|_n, \Gamma(u) + \|g/\Delta*\|_{n-1}, \Gamma(u_0)) \) with \( k \) so large that \( 2^{n-1}/kn-1\omega_n \)
< 1 and combining (3.15), (3.16), (3.17), (3.18) we arrive at

\[ \log \left( 1 + \left( \frac{M}{2\mu R} \right)^n \right) \leq C_1 + \log \left( 1 + \frac{M}{2\mu R} \right). \]

This means

\[ \left( \frac{M}{2\mu R} \right)^n \leq e^{C_1} - 1 + \frac{C_1 M}{2\mu R}. \]

Using Young's inequality we finally obtain

\[ M \leq C \left( \| f/\mathcal{D}^* \|_{n_1, \Gamma(u)} + \| g/\Delta^* \|_{n-1, \Gamma_{(u_0)}} \right), \]

When \( B \) is degenerate elliptic, we will consider the following two types of degeneracy:

I. \( \{ \alpha^{st} \} = \{ \alpha^{st} \}_{h=1} \oplus \{ 0 \}_{[n-(h+1)]x[n-(h+1)]}, 0 < h < n-1, \) and \( \beta^n \geq 0. \)

II. \( \{ \alpha^{st} \} \geq 0, \beta^n \geq \sigma > 0, \) for some constant \( \sigma. \)

In order to state our theorem simply we will assume from now on that \( u \in C^2. \) Of course such a smoothness can be relaxed by suitable choice of Sobolev spaces and approximation. In the first case we have,

**Theorem 3.3.** \( u \in C^2(\overline{\mathbb{R}^n_+}) \) be a subsolution of (3.5), (3.6) and satisfy \( u \leq 0 \) on \( \mathbb{R}^n_+ \setminus B_1^h. \) Suppose \( L \) is uniformly elliptic and \( B \) is degenerate oblique, \( \{ \alpha^{st} \} \) is of type I with positive definite submatrix \( \{ \alpha^{st} \}_{h=1} \), \( b^i \equiv 0, i = 1, \ldots, n, \beta^s \equiv 0, s = 1, \ldots, n-1. \) Then we have

\[ \sup_{B_1^h} u^+ \leq C \left( \| f/\mathcal{D}^* \|_{n_1, \Gamma(u)} + \| G/\Delta^* \|_{h, B_1^h} \right) R, \]

where \( C = C(n), \mathcal{D}^* \) is the same quantity as in Theorem 3.1, \( G \) is a function on \( B_1^h \) defined by
(3.20) \[ G(\mathcal{X}_1) = \sup_{(\mathcal{X}_2 \mid (\mathcal{X}_1, \mathcal{X}_2) \in \mathcal{F}(\mu))} g^-(\mathcal{X}_1, \mathcal{X}_2) / \Delta_h^*(\mathcal{X}_1, \mathcal{X}_2), \]

where \( \Delta_h^* = \sqrt{\det (\alpha_{st})_{hxh}} \).

To prove this theorem we need the following lemma from linear algebra.

**Lemma 3.4.** Let \( A = \{a_{ij}\}_{hxh} \) be a positive definite matrix and \( B = \{b_{ij}\}_{nxn} \) be a nonnegative definite matrix, \( 0 < h < n \). Suppose the inequality \( a_{ij}b_{ij} \geq g \) holds for some number \( g \geq 0 \). Then we have the estimate

\[
(3.21) \quad \det B \leq \left( \frac{n+1}{n} \right)^n \left( \frac{g}{\Delta^*} \right)^h \det B',
\]

where \( B' = \{b_{ij}\}_{i,j \geq n+1} \) and \( \Delta^* = (\det A)^{1/h} \).

**Proof.** Since \( B \) is nonnegative definite, so is \( B' \). For an arbitrary \( \varepsilon > 0 \), we write

\[
B'_\varepsilon = B' + \varepsilon I_{(n-h) \times (n-h)};
\]
\[
B_\varepsilon = B + H_\varepsilon, \text{ with } H_\varepsilon = O_{hxh} \oplus \varepsilon I_{(n-h) \times (n-h)};
\]
\[
A_\varepsilon = A \oplus g_\varepsilon (B'_\varepsilon)^{-1}, \text{ with } g_\varepsilon = g + \varepsilon.
\]

Obviously \( g_\varepsilon \to g \), \( \det B_\varepsilon \to \det B \) and \( \det B'_\varepsilon \to \det B' \) when \( \varepsilon \to 0 \). Because \( B'_\varepsilon \) is positive definite, so is \( (B'_\varepsilon)^{-1} \). Therefore by a calculation we have

\[
\frac{g_\varepsilon^{n-h} \det A}{\det B'_\varepsilon} \det B_\varepsilon = \det (A_\varepsilon B_\varepsilon)
\]
\[
\leq \left[ \frac{\text{tr}(A_\varepsilon B_\varepsilon)}{n} \right]^n = \frac{1}{n^n} \left( a_{ij} b_{ij} + (n-h) g_\varepsilon \right)^n \leq \left( \frac{n - h + 1}{n} \right)^n g_\varepsilon^n
\]

hence
Thus the inequality (3.21) follows by letting $\varepsilon \rightarrow 0$.

**Proof of Theorem 3.3.** The proof differs from that of Lemma 3.2 only in the estimation of $|\Sigma|$.

Let us introduce some new notation. We write for $p \in \mathbb{R}^n$ that $p = (p', p_n)$, and $p' = (\mathcal{P}_1, \mathcal{P}_2)$ with $\mathcal{P}_1 = (p_1, \ldots, p_h)$, $\mathcal{P}_2 = (p_{h+1}, \ldots, p_{n-1})$. We observe that if $p \in \Sigma$, then $0 \geq p_n > -M/2R$, and $p' \in \chi_{u_0}(\Gamma(u_0))$. Let $\pi$ be the orthogonal projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. If we denote $\Sigma' = \pi(\Sigma)$ we will see that

$$\Sigma \subset \Sigma' \times \left[ \frac{-M}{2R}, 0 \right].$$

By the observation above we have that $\Sigma' \subset \chi_{u_0}(\Gamma(u_0))$. Since $\Sigma \subset I_2$, we know that if $p' \in \Sigma'$ then $\mathcal{P}_2 \in B_{M/2R}^{n-h-1}$. Thus for each $\mathcal{X}_1 = (x_1, \ldots, x_h) \in B_R^h$ we have

$$\int \det D^2 u \, d\mathcal{X}_2 \leq |B_{M/2R}^{n-h-1}| = \omega_{n-h-1} \left( \frac{M}{2R} \right)^{n-h-1},$$

where $Q(\mathcal{X}_1) = \{ y = (Y_1, Y_2) \in \mathbb{R}^{n-1} | Y_1 = \mathcal{X}_1 \}$ is the $h$-dimensional hyperplane, and $D'$ stands for the derivatives taken w.r.t. $x_j$, $j = h+1, \ldots, n-1$. Applying Lemma 3.4 we get

$$|\Sigma'| \leq \int_{\chi_{u_0}^1(\Sigma)} \int_{B_R^h} \int_{\chi_{u_0}^2(\Sigma) \cap Q(\mathcal{X}_1)} \left( \frac{g}{\Delta^*} \right)^h \det D^2 u \, d\mathcal{X}_2$$

$$\leq \left( \frac{n+1}{n} \right)^n \omega_{n-h+1} \left( \frac{M}{2R} \right)^{n-h-1} \int_{B_R^h} [G(\mathcal{X}_1)]^h \, d\mathcal{X}_1.$$
So we have the estimate

$$|\Sigma| \leq |\Sigma' \times [\frac{M}{2R}, 0]| \leq C(n) \left(\frac{M}{2R}\right)^{-n} \int_{B_R^\beta} \|G(x_1)\|^n \, dx_1.$$ 

Combining this with the estimate of $\chi_u(B_R^\beta)$ we see

$$\frac{1}{2} \omega_n \left(\frac{M}{2R}\right)^n \leq C \left(\frac{M}{2R}\right)^{-n} \int_{B_R^\beta} \|G(x_1)\|^n \, dx_1 + \frac{1}{n} \int_{\Gamma_0(B_R^\beta)} \left(\frac{f}{\mathcal{D}^*}\right)^n \, dx.$$

Using Young's inequality again in the first term of right hand side of above inequality we conclude (3.19).

Now we look at the case where the boundary condition is of type II.

**Theorem 3.5.** Let $u \in C^2(\mathbb{R}^n_\beta)$ be a subsolution of (3.5), (3.6) and $u \leq 0$ on $\mathbb{R}^n_\beta \setminus \overline{B}_R^\beta$. Suppose $L$ is uniformly elliptic, $B$ is degenerate elliptic but II holds and $|\beta| / \beta^n \leq \mu$ for some constant $\mu$, where $\beta = (\beta^1, ..., \beta^{n-1})$. Then we have

$$(3.22) \quad \sup_{B_R^\beta} u^+ \leq C \left(\sup_{\Gamma_0(u_\beta)} \frac{|g|}{\sigma} + \left\| \frac{f^-}{\mathcal{D}^*} \right\|_{n, \Gamma_\sigma(u)} \right) R,$$

where $C = C(n, \mu)$.

**Proof.** First of all we perform a coordinate transformation $\Phi: x \mapsto y$ such that $y' = x'$, $y_n = \delta x_n$ with constant $\delta$ to be fixed later. Under such a transformation the function $\bar{u} = u(y', y_n / \delta)$ satisfies the inequalities

$$(3.23) \quad \bar{a} \bar{u} D_{ij} \bar{u} \geq \bar{f}, \quad \text{in } D,$$

$$(3.24) \quad \bar{b}^i D_i \bar{u} \geq g, \quad \text{on } B_R^0.$$
where \( D = \Phi(B^+_R), \) \( \{ \hat{a}^{ij} \} = \begin{pmatrix} 1 \ldots 0 \\ \vdots \\ 0 \ldots 0 \end{pmatrix} \) \( \{ a^{ij} \} \begin{pmatrix} 1 \ldots 0 \\ \vdots \\ 0 \ldots 0 \end{pmatrix} \) and \( \hat{\beta}^s \) for \( s = i, \ldots, n-1, \) \( \hat{\beta}^n(y) = \delta \beta^n(y', \frac{1}{\delta} y_n), \) \( \tilde{f}(y) = f(y', \frac{1}{\delta} y_n). \) One can also see that \( \Gamma_+(\tilde{u}_0) = \Gamma_+(u_0) \) and \( \sup_{B_R^+} \tilde{u} = \sup_{B_R^+} u, \) \( \tilde{u} \leq 0 \) outside of \( D. \)

Let \( x_0 \in B^+_R \) be the maximum point of \( \tilde{u}. \) We still use \( I_1, I_2 \) to denote the normal images of functions \( W_1, W_2 \) whose graphs now are the cones with the same vertex \( (x_0, M) \) and basis \( D, B_{2\delta R}(x_0) \) respectively. All the other notation we are going to use in the following is the same as in the proof of Lemma 3.2.

We consider the function

\[
v = \tilde{u} + A x_n
\]

with \( A = \sup \frac{|g|}{\delta \sigma} + \frac{\mu M}{2\delta^2 \sigma R}. \) Because the negative part of \( v \) will not affect our argument, we consider \( v \) in \( B^+_{8R} \) which contains \( D. \) By such a choice of \( A \) we can see that \( D_n v(y) > 0 \) at those points where \( |D'v(y)| \leq M / 2\delta R. \) So we conclude that

\[
(3.25) \quad \chi_0(\tilde{f}(v_0)) \cap B_{M/2\delta R}^{n-1} = \emptyset.
\]

This implies \( \Sigma = \emptyset \) so that \( I_{2*} \subset \chi_v(D). \) By applying Lemma 3.2 to \( v - K, K = \sup_{\partial D B^+_R} v \)

\[
= R \sup_{\tilde{f}(u_0)} \frac{|g|}{\sigma} + \frac{\mu M}{2\delta \sigma}, \quad \text{we get}
\]

\[
M - K \leq \frac{C}{\delta} \| f^* \|_{n, \Gamma_+(u)} R
\]

Now we choose \( \delta \) so large that \( \mu / \delta \sigma \leq 1 \) and hence

\[
M \leq C \left( \sup_{\tilde{f}(u_0)} \frac{|g|}{\sigma} + \| f^* \|_{n, \Gamma_+(u)} R \right).
\]
To generalize Theorem 3.1 and Theorem 3.3 to the case where $\beta^s \neq 0$, $s = 1, \ldots, n-1$, we assume the strict obliqueness condition II. This generalization is of importance in practice, see, for example, the Hölder norm estimates for quasilinear problems in Chapter 5. However we need only to do this generalization for Theorem 3.3, because when the boundary condition is uniformly elliptic, the ellipticity will control the lower order terms, and when the boundary condition is degenerate elliptic, we always assume the strict obliqueness. Otherwise, as is known, the situation in oblique boundary value problems with degenerate obliqueness is extremely complicated.

**Theorem 3.6.** Let $u \in C^2(\mathbb{R}^n_+) \cup \mathbb{R}^n_+ \setminus B^+_{\alpha}$. Suppose $L$ is uniformly elliptic with $b^i \equiv 0$, $i = 1, \ldots, n$, and both conditions I and II hold. Then we have

$$(3.26) \quad \sup_{B^+_{\alpha}} u^+ \leq C \left( \| f \|_{L^\infty}, \Gamma(u) + \| G \|_{L^\infty}, B^+_{\alpha} \right) R,$$

where $C = C(n, \sigma)$, $D^*$ is the same as before, and

$$\mathcal{G}(x) = \sup_{\{x_2 \in (x_1, x_2) \in \Gamma(u) \}} \frac{|g(x_1, x_2)|}{\sqrt{|\det(\alpha^{x_2})|_{x_2}}}.$$

**Proof.** By the same coordinate transformation as in the proof of Theorem 3.5, we consider the domain $D$ with boundary $B^+_{\alpha}$. Because on $\Gamma(u_0)$ both $\alpha^{x_1} D_{x_1} \hat{u} + \beta^s D_{x_2} \hat{u} \geq g$ and $\beta^s D_{x_1} \hat{u} + \beta^s D_{x_2} \hat{u} \geq g$ hold, we have for $p = (p', p_n) \in \Sigma$ that $p' \in \Sigma'$ and

$$0 \geq p_n \geq \inf_{y \in \Gamma(u_0)} D_{x_1} \hat{u}(y) \geq - \left( \sup_{y \in \Gamma(u_0)} \frac{|g(y)|}{\delta \sigma} + \frac{\mu M}{\delta 2\delta R} \right).$$

We denote the last quantity in the above inequality by $h(p')$. Hence
\[
|\Sigma| \leq \int_{\Sigma'} \int dp \, dp' = \int_{\Sigma'} h(p') \, dp' \leq \int_{\Sigma'} H(x') \, |\det D^2 u| \, dx',
\]
where \( H(x') = \frac{|g(x')|}{\delta} + \frac{\mu}{\delta} \frac{M}{2\delta R} \). Applying Lemma 3.4 we have

\[
|\Sigma| \leq \frac{1}{\delta} \left( \frac{n+1}{n} \right)^n \omega_{n-h-1} \left( \frac{M}{2\delta R} \right)^{n-h-1} \int_{\bar{B}_R^h} |G(x')|^{h+1} \, d\omega + \left( \frac{\mu}{\delta} \right)^{h+1} \left( \frac{n+1}{n} \right)^n \omega_{n-h-1} \left( \frac{M}{2\delta R} \right)^n.
\]

Now we let \( \delta \) be such that

\[
\left( \frac{\mu}{\delta} \right)^{h+1} \left( \frac{n+1}{n} \right)^n \omega_{n-h-1} < \frac{1}{4},
\]
and we see that the desired estimate follows.

**Remark 2.** By Lemma 3.4 and the argument in the proof of Theorem 3.3 we may prove also the following result on degenerate elliptic equations which is first obtained by Aleksandrov in [2].

Suppose \( Lu = a_{ij} D_{ij} u \) and \( \{a_{ij}\} = \{a_{ij}\}_{h \times h} \Theta \{0\}_{(n-h) \times (n-h)} \) for some \( 0 < h < n \), with \( \{a_{ij}\}_{h \times h} \) uniformly positive definite. Assume \( u \in C^2(B_R) \) satisfies

\[
Lu \leq g, \quad \text{in } B_R, \quad u \leq 0, \quad \text{on } \partial B_R.
\]

Then we have the estimate

\[
\sup_{B_R} u^+ \leq C R \|G\|_{h, B_R^h},
\]
where \( C \) depends only on \( n, G \) is defined in a similar way as in Theorem 3.3 with \( \Gamma_p(u) \) in the place of \( \Gamma_p(u_0) \).
Remark 3. The restriction in Theorem 3.3 and Theorem 3.6 that \( \{\alpha^{st}\} \) is of Type I is strong, but it can be relaxed in the following way.

Suppose in general that \( \{\alpha^{st}\} \) has rank \( h \) at 0, for some \( 0 < h < n-1 \). By a coordinate transformation we can assume that

\[
A = \{\alpha^{st}\} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where \( A_{11} \) is the \( h \times h \) positive definite minor with the smallest eigenvalue \( \kappa > 0 \) near 0. For \( \delta \in [0, 1) \), we decompose \( A \) into

\[
(3.27) \quad A = (1-\delta) A_{11} 0 + \delta A_{11} A_{12} = A_1 + A_2.
\]

Let \( B = \begin{pmatrix} I - \frac{1}{\delta} A_{11}^{-1} A_{12} \\ 0 \end{pmatrix} \). By the similarity transformation \( A'_2 = B^T A_2 B \) one can see that \( A_2 \geq 0 \) if and only if \( A_{22} - \frac{1}{\delta} A_{21} A_{11}^{-1} A_{12} \geq 0 \), of which the sufficient condition is

\[
(3.28) \quad A_{22} - \frac{1}{\delta \kappa} A_{21} A_{12} \geq 0.
\]

If there exists such \( \delta \) that (3.28) holds near the origin we can write the boundary condition in the form

\[
(3.29) \quad \alpha^{st}_1 D_{st} u + \alpha^{st}_2 D_{st} u + \beta^i D_i u + \gamma u \geq g, \quad \text{on } B_\delta^0,
\]

for some small \( R > 0 \), \( \alpha^{st}_1, \alpha^{st}_2 \) are the entries of \( A_1, A_2 \). Notice that on \( \Gamma (u_0) \) we have \( \alpha^{st}_2 D_{st} u \leq 0 \), so we get

\[
(3.30) \quad \alpha^{st}_1 D_{st} u + \beta^i D_i u + \gamma u \geq g, \quad \text{on } \Gamma (u_0),
\]
where $A_1$ is of Type 1. Therefore Theorem 3.3 and Theorem 3.6 remain true if (3.28) holds.

**Example.** Suppose $\{\alpha^s\}$ is a $2 \times 2$ nonnegative definite matrix, i.e.

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

and $a \geq 1$, $c(0) = 0$, so $b(0) = 0$ also. Assume $c(x) \geq C_1 |x|$ and $|b(x)| \leq C_2 |x|$, for some constants $C_1$, $C_2$. Then (3.28) is satisfied for each $\delta \in (0, 1)$.

This example shows that the assumption (3.28) is a restriction on the behavior of the degeneracy

## 2. Boundary Weak Harnack Inequalities

By means of the maximum principles developed in the previous section we can prove boundary versions of the weak Harnack inequalities.

**Lemma 3.7.** Let $u \in C^2(B_R^+)$ be a nonnegative solution of

$$Lu \leq f, \text{ in } B_R^+, \quad Bu \leq g, \text{ on } B_R^0.$$

Assume that $L$ is uniformly elliptic, $\lambda I \leq \{a^{ij}\} \leq \Lambda I$, and $B$ satisfies $\kappa I \leq \{\alpha^s\} \leq \kappa I$, $\beta^n \geq 0$, $\beta^s \equiv 0$, $s = 1, 2, \ldots, n-1$, $\Lambda / \lambda$, $K / \kappa$, $|\beta^n| / \kappa \leq \mu$ for some constant $\mu$. Assume also $f/\lambda \in L^n(B_R^+)$, $g/\kappa \in L^{n-1}(B_R^0)$. Let \( \overline{u} = u + \| f/\lambda \|_{n, B_R^+} + \| g/\kappa \|_{n-1, B_R^0} \), $\Sigma = \{ x \in B_R^0 \mid \overline{u}(x) > 1 \}$. Then there are constants $\delta \in (0, 1)$, $\nu > 0$ and $\alpha = 1/6(n-1)^{1/2}$ such that if $|\Sigma \cap K_\alpha| \geq \delta |K_\alpha|$ we will have

$$\inf_{K_{3\alpha}} \overline{u} \geq C^{-1},$$

(3.31)
where $K_\alpha = \{ x \in B^n \mid |x_i| < \alpha R, i = 1, \ldots, n-1 \}$. The constants $C, \delta$ depend only on $n, \mu$.

**Proof.** The basic idea of this proof follows from Trudinger [20]. Our contribution here is the construction of cut-off function so that our boundary Aleksandrov maximum principle can be applied. Without loss of generality we may assume $R = 1$. For any $\varepsilon > 0$, we define $w = \ln \frac{1}{u + \varepsilon}$ and $\tilde{f} = \frac{f}{u + \varepsilon}, \tilde{g} = \frac{g}{u + \varepsilon}$. A direct calculation shows

\[
D_i w = - \frac{1}{u + \varepsilon} D_i u,
\]

\[
D_{ij} w = - \frac{1}{u + \varepsilon} D_{ij} u + D_i w D_j w.
\]

Hence

\[
L w \geq - \tilde{f} + a^{ij} D_i w D_j w, \quad \text{in } B^+.
\]

\[
B w \geq - \tilde{g} + \alpha^{st} D_s w D_t w, \quad \text{on } B^0.
\]

Now we introduce a cut-off function

\[
(3.32) \quad \eta(x) = \begin{cases} (2 - 1 \cdot \varepsilon^2 - (x_n + 1)^2)^\beta, & \text{on } \{ x_n \geq 0, 1 \cdot \varepsilon^2 + (x_n + 1)^2 < 2 \}, \\ 0, & \text{otherwise,} \end{cases}
\]

with $\beta \geq 2$ a constant to be chosen later. The function $v = \eta w$ then satisfies

\[
L v \geq - \eta \tilde{f} - \frac{a^{ij} D_i \eta D_j \eta}{\eta} + \frac{L \eta}{\eta}, \quad \text{in } B^+.
\]

\[
B v \geq - \eta \tilde{g} - \frac{\alpha^{st} D_s \eta D_t \eta}{\eta} + \frac{B \eta}{\eta}, \quad \text{on } B^0.
\]
Choosing $\beta$ such that $\beta - 1 > 2\eta\mu / \alpha^2$ we can see that for any $\alpha \in (0, 1)$ $\eta$ has the properties

\[
\begin{align*}
L\eta &\geq 0, \\
B\eta &\geq 0,
\end{align*}
\]

in $B^+$, on $B^0$.

Thus on $\Gamma_+^*$ we have

\[
Lv \geq -|f| - 4\beta^2 \Lambda,
\]

and on $\Gamma (v_0)$ with $v_0 = v(x', 0)$

\[
Bv \geq -|f| - 4\beta^2 K - \sup_{B^0} \frac{B\eta}{\eta} \chi (B^0) v,
\]

$\chi (B^0)$ represents the characteristic function of $B^0$. By Theorem 3.1 we have

\[
\sup_{B^*} v \leq C \left( 1 + \|v^+\|_{H^{-1}, B^0} \right) \leq C \left( 1 + |K^+|^{n-1} \sup_{B^*} v^+ \right),
\]

where $K^+ = \{ x' \in K^+ \mid v > 0 \} = \{ x' \in K^+ \mid \bar{u} + \varepsilon < 1 \}$. If

\[
\frac{|K^+_\alpha|}{|K^-\alpha|} \leq \theta = \frac{1}{2C|K^+\alpha|} = \frac{1}{2C(2\alpha)^{n-1}},
\]

we will have

\[
\sup_{B^*} v \leq 2C'.
\]
When \( \alpha = 1 / \sqrt[n]{n-1} \) it is obvious that \( K_{3\alpha} \subset B_{1/2}^0 \), so we have

\[
\inf_{K_{3\alpha}} \bar{u} + \varepsilon \geq e^{-C}.
\]

Let \( \delta = 1-\theta \). It follows that if \( |\Sigma \cap K_{\alpha} | \geq \delta |K_{\alpha} | \) then \( |K_{\alpha}^*| \leq |K_{\alpha}| \). Finally the estimate (3.31) follows by letting \( \varepsilon \to 0 \) and the constant \( C \) is \( e^C \).

We state here a result on measure theory from [21] without proof.

**Lemma 3.8.** Suppose \( K \) is a cube in \( \mathbb{R}^n \), \( \Gamma \) is a measurable subset of \( K \).

For any \( 0 < \delta < 1 \) we denote

\[
\Gamma_\delta = \bigcup \{ K_{3R}^*(y) \cap K \mid y \in K, R > 0, |\Gamma \cap K_R(y)| \geq \delta |K_R(y)| \}.
\]

Then we have either \( \Gamma_\delta = K \) or \( |\Gamma| \leq \delta |\Gamma_\delta| \).

Utilizing Lemma 3.7, 3.8 we can prove

**Lemma 3.9.** Under the assumptions of Lemma 3.7 except that

\[
|\Gamma \cap K_{\alpha}| \geq \delta^m |K_{\alpha}|,
\]

we have the estimate

\[
(3.33) \quad \inf_{K_{3\alpha}} \bar{u} \geq C m,
\]

with the same constants \( \alpha, \delta, C \) as in Lemma 3.7.

Now our main result, the boundary version of weak Harnack inequality, can be derived from Lemma 3.9.

**Theorem 3.10.** Let \( u \in C^2 (B_\delta^+ \cup B_\delta^-) \) be a nonnegative solution of

\[
Lu \leq f, \quad \text{in } B_\delta^+.
\]
Suppose that \( f / \lambda \in L^n(B^+_R), g / \kappa \in L^{n-1}(B^+_R) \). Then there exist constants \( p > 0, C > 0 \) such that

\[
(3.34) \left( \frac{1}{|B_{R/2}|} \int_{B^+_R} |u|^p \, dx \right)^{1/p} \leq C ( \inf_{B^+_R} u + R \| \frac{g}{\kappa} \|_{n-1, B^+_R} + R \| \frac{f}{\lambda} \|_{n, B^+_R} ),
\]

and \( p, C \) depend only on \( n, \mu \).

For the detailed proof of Lemma 3.8, 3.9 and the deduction of Theorem 3.10 from Lemma 3.7, 3.8, 3.9 we refer to [21].

For degenerate elliptic boundary conditions, we assume that the degeneracy satisfies the condition 1 described in Section 1. In this case, instead of Lemma 3.7 we have the following lemma which also leads us to a weak Harnack inequality.

**Lemma 3.11.** Let \( u \in C^2(B^+_R) \) be a nonnegative solution of

\[
Lu \leq f, \quad \text{in } B^+_R, \\
Bu \leq g, \quad \text{on } B^0_R.
\]

Suppose that \( L, B \) satisfy \( \Lambda I \leq \{a^{ij}\} \leq \Lambda I, \{\alpha^{st}\} = \{\alpha^{st}\}_{0 < h < n-1}, 0 < h < n-1, 0 < k I \leq K I, \beta^s \equiv 0, s = 1, \ldots, n-1, b_i \equiv 0, i = 1, \ldots, n, \Lambda / \lambda, K / k, \beta / \kappa, \lambda / \mu \), for some constant \( \mu \). Suppose also \( f / \lambda \in L^n(B^+_R), G \in L^h(B^h_R) \), where \( G \) is defined in Theorem 3.3. Let \( \bar{u} = u + \| f / \lambda \|_{n, B^+_R} + \| G \|_{h, B^h_R} \) and define \( \bar{U} : B^+_R \rightarrow \mathbb{R} \) by

\[
\bar{U}(x_0) = \sup_{\{x_1, x_2 \in B^+_R \}} \bar{u}(x_1, x_2).
\]
Let $\Sigma = \{ x \in B^h \mid U > 1 \}$. Then there exist $\delta \in (0, 1)$, $C > 0$, $\alpha = 1 / 6(n-1)^{1/2}$ such that if $|\Sigma \cap K_\delta| \geq \delta |K_\delta|$ we have

$$(3.35) \quad \inf_{K_\alpha} \bar{u} \geq C^{-1},$$

where $K_\alpha = \pi^1(K_\alpha)$, $K_\alpha$ is as before and $\pi^1: \mathbb{R}^{n-1} \to \mathbb{R}^h$ is the orthogonal projection, $\delta, C$ depend only on $\mu, n$.

**Proof.** We define $w, \tilde{r}, \tilde{g}$ in the same way as in the proof of Theorem 3.7, and introduce the cut-off function $\eta = \eta_1 \eta_2$,

$$\eta_1(x) = \begin{cases} (2 - |\xi_1|^2 - (x_n + 1)^2)^{\beta}, & \text{if } |\xi_1|^2 + (x_n + 1)^2 < 2, \\ 0, & \text{otherwise}; \end{cases}$$

$$\eta_2(x) = \begin{cases} (1 - |\xi_2|^2 - x_n^2)^{\beta}, & \text{if } |\xi_2|^2 + x_n^2 < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Now we let $v = \eta w$. Then $v$ satisfies

$$L^v \geq -\eta^2 \frac{a_{ij} D_i \eta D_j \eta}{\eta} + \nu \frac{L \eta}{\eta}, \quad \text{in } B^+, \tag{6.7}$$

$$B^v \geq -\eta \tilde{g} \frac{\alpha^{st} D_s \eta D_t \eta}{\eta} + \nu \frac{B \eta}{\eta_1}, \quad \text{on } B^0. \tag{6.9}$$

Writing $X = (x_1, \ldots, x_n, 0, \ldots, 0, x_{n+1})$, $Y = (0, \ldots, 0, x_{n+1}, \ldots, x_n)$ and $\bar{X} = (1 - |\xi_2|^2 - x_n^2) X$, $\bar{Y} = (2 - |\xi_1|^2 - (x_n + 1)^2) Y$, we can see that

$$L \eta = a_{ij} D_i \eta = \eta_2 a_{ij} D_i \eta_1 + a_{ij} D_i \eta_1 D_j \eta_2 + a_{ij} D_i \eta_1 D_j \eta_2 + \eta_1 a_{ij} D_i \eta_2$$

$$= 2 \beta \eta_1^{\beta - 2} \eta_2^{\beta - 2} \left[ 2 (\beta - 1) (a_{ij} \bar{X}_i \bar{X}_j + a_{ij} \bar{Y}_i \bar{Y}_j) + 4 a_{ij} \bar{X}_i \bar{Y}_j \right]$$
If we choose $\beta \geq (8n + 2)\mu + 1$, we will have, by the definition of $X$ and $Y$, that

$$
- (\sum_{i=1}^{n} a_{ii} + a_{nn}) (2 - |x_{n+1}^2 - (x_n + 1)^2|) (1 - |x_{n+2}^2 - x_n|^2)
$$

$$
- \left( \sum_{i=h+1}^{n} \beta_{ii} \right) (1 - |x_{h+2}^2 - x_n^2|) (2 - |x_{n+1}^2 - (x_n + 1)^2|^2)
$$

$$
\geq 2 \beta \eta_{1}^{\beta} \eta_{2}^{\beta} [ (2 (\beta - 1) \lambda - 4 \Lambda) (|X|^2 + |Y|^2)]
$$

$$
- (h + 1) \Lambda (1 - |x_{h+2}^2 - x_n^2|^2) - (n - h) \Lambda (2 - |x_{n+1}^2 - (x_n + 1)^2|).
$$

In summary we have

$$(3.37) \quad L\eta \geq 0, \quad \text{in } B^{+}.$$
\[
\beta - 2 \\
\geq 2 \beta \eta_1 (2 (\beta - 1) \kappa |s_1|^2 - h K - \mu K) \geq 0,
\]

provided \( \beta > (h + 1) \mu / \alpha^2 \), and \( |s_1| > \alpha \). If we take \( \beta = \max\{(8n+2)\mu + 1, \) \( (\eta+1)\mu / \alpha^2\} \), (3.37) and (3.38) will hold simultaneously. Therefore we get from (3.36) that

\begin{align*}
(3.39) & \quad L_v \geq -|\beta| - 24 \beta^2 \Lambda, \quad \text{on } \Gamma_v(v), \\
(3.40) & \quad B_v \geq -|\beta| - 4 \beta^2 K - \sup_{B_\beta} \left| \frac{B_\beta}{\eta_1} \right| (B^h_\alpha) v,
\end{align*}

where \( \chi(B^h_\alpha) \) is the characteristic function of \( B^h_\alpha \). By Theorem 3.3 we have

\[
\sup_{B^h_\alpha} v \leq C \left( 1 + \|V^v\| \right) \leq C \left( 1 + \|K^v_\alpha\|^{1/h} \sup_{B^h_\alpha} v, \right)
\]

where \( V(x) = \sup_{(x_1, x_2) \in B^h_\alpha} v(x_1, x_2) \), and \( K^v_\alpha = \{x_1 \in K^v_\alpha \mid v > 0\} \). The rest of the proof is basically the same as that of Lemma 3.6.

From Lemma 3.11 we may obtain by a similar argument the following estimate

\[
(3.41) \quad \left( \frac{1}{|B_\beta^{R/2}|} \int |U|^p \, d\mathcal{H}^1 \right)^{1/p} \leq C \left( \inf_{B_{R/2}} u + R \|G\|_{h, B^h_\alpha} + R \frac{f}{\lambda_{n, B^h_\alpha}} \right)
\]

for some constants \( p > 0, C > 0 \). \( U(x) = \sup_{(x_1, x_2) \in B^h_\alpha} u(x_1, x_2) \). Notice that

\[
(3.42) \quad \frac{1}{|B^{R/2}|} \int u^p \, dx' \leq \frac{1}{|B^{R/2}|} \int |U|^p \, d\mathcal{H}^1 \leq C(n) \int |U|^p \, d\mathcal{H}^1.
\]

Hence we conclude from (3.41) and (3.42)
Theorem 3.12. Let \( u \in C^2 (B^+_h \cup B^0_h) \) be a nonnegative solution of

\[
Lu \leq f, \quad \text{in } B^+_h, \quad Bu \leq g, \quad \text{on } B^0_h,
\]

and assume that \( L, B, f, g \) satisfy the hypothesis of Lemma 3.11. Then there are constants \( p > 0, C > 0 \) depending only on \( n, \mu \), such that

\[
(3.43) \quad \left( \frac{1}{|B^0_{R/2}|} \int_{\partial B^0_{R/2}} |u|^p \, dx' \right)^{1/p} \leq C \left( \inf_{B^0_{R/2}} \left( u + R \| G \|_{L^\infty(B^0_{R/2})} \right) \right).
\]

For the purpose of application we need to extend Theorem 3.12 to the case where \( B \) satisfies (3.28) and \( \beta^s \neq 0, s = 1, ..., n-1 \), but \( \beta^n \geq \sigma > 0 \). To do so we have to generalize Lemma 3.11 first, and then we can get (3.43) in the same way. Recalling the proof of Lemma 3.11, we find that the only modification we need to do is a suitable choice of cut-off function \( \eta \) such that \( L\eta \geq 0 \) in \( B^+ \) and \( B\eta \geq 0 \) when \( |x_1| > \alpha \). This can be done by setting \( \eta = \eta_1 \eta_2 \), where

\[
\eta_1(x) = \begin{cases} 
\left( \frac{3}{2} - 2 |x_2|^2 - (x_n - 1)^2 \right)^{\beta}, & \text{if } (x_n - 1)^2 + 2 |x_2|^2 \leq \frac{3}{2}, \\
0, & \text{otherwise};
\end{cases}
\]

\[
\eta_2(x) = \begin{cases} 
\left( \frac{1}{4} - |x_1|^2 - x_n^2 \right)^{\beta}, & \text{if } |x_1|^2 + x_n^2 \leq \frac{1}{4}, \\
0, & \text{otherwise}.
\end{cases}
\]

To check that \( \eta \) has the desired properties we assume without loss of generality that

\[
(3.44) \quad \sigma > 3(n-1) \mu, \quad \mu \text{ is the constant such that } |\beta^i|, K \leq \mu.
\]
In fact by a dilation in the $x_n$ direction as done in the proof of Theorem 3.5 we can increase the obliqueness constant by a factor $\delta$ and we can choose $\delta$ as large as we wish. This procedure does not interfere with value on the boundary $B^0$. The fact $L\eta \geq 0$ in $B^+$ follows by the same argument as in the proof of Lemma 3.11. To see $B\eta \geq 0$ when $|\xi_1| > \alpha$ we observe

$$B\eta = \alpha^{st}_1 D_{st}\eta + \beta^i D_i\eta = \alpha^{st}_1 D_{st}\eta + \alpha^{st}_2 D_{st}\eta + \beta^i D_i\eta,$$

where $\{ \alpha^{st}_1 \} + \{ \alpha^{st}_2 \}$ is the decomposition of $\{ \alpha^{st} \}$ in the way of (3.29). So we have

$$B\eta = \eta_1 \left( \alpha^{st}_1 D_{st}\eta_2 + \sum_{i=1}^{h} \beta^i D_i\eta_2 \right) + \eta_1 \alpha^{st}_2 D_{st}\eta_2$$

$$+ \alpha^{st}_2 D_s\eta_2 D_t\eta_1 + \alpha^{st}_2 D_s\eta_1 D_t\eta_2 + \eta_2 \alpha^{st}_2 D_{st}\eta_1 + \eta_2 \sum_{i=h+1}^{n} \beta^i D_i\eta_1$$

$$\geq \eta_1 \left( \alpha^{st}_2 D_{st}\eta_2 + \sum_{i=1}^{h} \beta^i D_i\eta_2 - \beta h \mu \eta_1^2 \beta \eta_1^2 \beta \right) + \beta^i \eta_2 \eta_1 \beta \left( 2\sigma - 6(n-h-1)\mu \right).$$

The second term above is positive by (3.44) and the first one will be positive also for $|\xi_1| > \alpha$ if we choose $\beta$ properly. Therefore we conclude

**Theorem 3.13.** Under the same assumptions of Theorem 3.12 except that $B$ satisfies (3.28) and $\beta^i \neq 0$, $s = 1, \ldots, n-1$, $\beta^n \geq \sigma > 0$. Then we have (3.43) with $G$ defined in Theorem 3.6.

We will see an application of the results here in Chapter 5.
Chapter 4
Quasilinear Problems (I)
Elliptic Boundary Conditions

In this chapter we study the nonlinear Venttsel boundary value problem (1.5), (1.6) under the assumption that the boundary condition (1.6) is uniformly elliptic. We also assume throughout this chapter and the next one that $\alpha^{ij} v_i \equiv 0$, $\forall i$, so that when we locally flatten the boundary, we reduce the problem to the form

\begin{align*}
(4.1) & \quad a^{ij}(x, u, Du) \, D_{ij} u + b(x, u, Du) = 0, \quad \text{in } B^+, \\
(4.2) & \quad \alpha^{st}(x, u, D'u)D_{st} u + \beta(x, u, Du) = 0, \quad \text{on } B^0.
\end{align*}

The general assumptions on (4.1) and (4.2) are the so called natural structure conditions:

(A) \quad $\lambda(x, z, p) \leq a^{ij}(x, z, p) \xi_i \xi_j \leq \Lambda(x, z, p) \, \xi^2$, \\
\hspace{1cm} $\forall \xi \in \mathbb{R}^n$ and $\forall (x, z, p) \in B^+ \times \mathbb{R} \times \mathbb{R}^n$, and $\Lambda \leq \lambda \mu(|z|)$;

(B) \quad $\kappa(x, z, p) \leq \alpha^{st} \eta_s \eta_t \leq K(x, z, p) \, \eta^2$, \\
\hspace{1cm} $\forall \eta \in \mathbb{R}^{n-1}$ and $\forall (x, z, p) \in B^0 \times \mathbb{R} \times \mathbb{R}^{n-1}$, \\
\hspace{1cm} $K \leq \kappa \mu(|z|)$, and $\beta^a = D_p \beta \geq \chi \geq 0$, for some constant $\chi$;

(A1) \quad $|b(x, z, p)| \leq \lambda \mu_1(|z|) \, (1 + |p|^2)$, for $|p| \geq M$;

(B1) \quad $|\beta(x, z, p)| \leq \kappa \mu_2(|z|) \, (1 + |p|)$, for $|p| \geq M$;

(A2) \quad $|D_x a^{ij}(x, z, p)|, |D_x a^{ij}(x, z, p)|, |D_p a^{ij}(x, z, p)| (1 + |p|) \leq \lambda \mu_1(|z|)$,
\[ |D_x b(x, z, p)|, |D_z b(x, z, p)|, |D_p b(x, z, p)| (1 + |p|) \]
\[ \leq \lambda \mu_1(|z|) (1 + |p|^2), \quad \text{for } |p| \geq M; \]

\[ (B2) |D_x \alpha^st(x, z, p)|, |D_z \alpha^st(x, z, p)|, |D_p \alpha^st(x, z, p)| (1 + |p|) \leq \kappa \mu_2(|z|), \]
\[ |D_x \beta(x, z, p)|, |D_z \beta(x, z, p)| \leq \kappa \mu_2(|z|) (1 + |p|), \]
\[ |D_p \beta(x, z, p)| \leq \kappa \mu_2(|z|), \quad \text{for } |p| \geq M. \]

where \( \mu, \mu_1, \mu_2 \) are nondecreasing functions and \( M \) is some positive constant. The structure conditions for the general problem (1.5), (1.6) can be described in exactly the same form as above. We only consider problem (4.1), (4.2) in the following, because global estimates will be obtained by piecing together the local estimates.

1. Hölder Estimates near the Boundary

**Lemma 4.1.** Let \( u \in C^2(B^+ \cup B^0) \) satisfy the differential inequalities

\[ |Lu| = |a_{ij} D_{ij} u| \leq \lambda (|\mu_0|D^2 u + \Phi), \quad \text{in } B^+, \]
\[ |Bu| = |a^{st} D^{st} u + \beta^n D^n u| \leq \kappa (|\mu_0|D^2 u + \Phi), \quad \text{on } B^0, \]

for some nonegative constants \( \mu_0, \Phi \). Suppose there is a constant \( \Lambda \) such that \( \Lambda \leq \lambda \mu, K \leq \kappa \mu, \) and \( \beta^n \geq 0. \) Suppose also \( u \leq M_0. \) Then there exist constants \( \alpha, C \) depending on \( \mu, \mu_0, \Phi, M_0, n \) such that

\[ \sup_{B_{1/4}} u \leq C R^\alpha, \]

provided \( R \leq 1/4. \)

**Proof.** Considering \( \tilde{u} = (1 - e^{-\mu_0 u}) / \mu_0 \) we may assume \( \mu_0 = 0. \) Let \( m_R \)
= \inf_{B_R} u, M_R = \sup_{B_R} u, \omega(R) = M_R - m_R. Applying Theorem 3.10 to the functions \( u - m_R \) and \( M_R - u \) we obtain the inequalities

\[
\left( \frac{1}{|B_{R/2}|} \int_{B_{R/2}}^{0} (u - m_R)^p \, dx' \right)^{1/p} \leq C (m_{R/2} - m_R + \Phi R^2),
\]

\[
\left( \frac{1}{|B_{R/2}|} \int_{B_{R/2}}^{0} (M_R - u)^p \, dx' \right)^{1/p} \leq C (M_R - M_{R/2} + \Phi R^2).
\]

Applying these inequalities we get

\[
\omega(R/2) \leq \gamma \omega(R) + \Phi R^2,
\]

with \( \gamma < 1 \). Then (4.5) follows by a standard argument.

**Theorem 4.2.** Let \( u \) be a solution of (4.1) and (4.2). Suppose (A), (A1), (B), (B1) are satisfied. Suppose also \( |u| \leq M_0 \) for some constant \( M_0 \). Then there are constants \( \alpha \in (0, 1) \) and \( C \) such that

\[
(4.6) \quad [u]_{\alpha, B_{R/2}} \leq C,
\]

where \( \alpha, C \) depend on \( n, \mu, \mu_1, \mu_2, M_0 \).

This theorem follows from Lemma 4.1 immediately if we set \( L = a^{ij} D_{ij}, B = \alpha^{st} D_{st} + \beta^n D_n \), where

\[
\beta^n = \int_0^1 D_{pa} \beta(x, u, t D_n u) \, dt.
\]
2. Tangential Gradient Estimates

We first introduce a lemma about interior estimates for the gradient of solutions of (4.1) which is essentially due to Ladyzhenskaya and Ural'tseva [15], (see also [21]).

**Lemma 4.3.** Let \( u \in C^3 \) be a solution of (4.1). Suppose (A), (A1), (A2) are satisfied. Then there are constants \( C \) and \( \theta \) depending only on \( n, \mu, \mu_1 \) such that if \( B = B_{x_n}(x) \) is any ball in \( B^+ \) and \( \alpha = \text{osc} u \leq \theta \), then

\[
|Du(x)| \leq C \alpha / x_n + M. 
\]

The following theorem is the main result in this section. The proof is based on that of Lieberman and Trudinger [17]. Since at this stage we do not have a normal derivative estimate, we also assume the obliqueness of our boundary conditions.

**Theorem 4.4.** Let \( u \in C^3(B^+ \cup B^0) \) be a solution of (4.1), (4.2). Suppose the operators in (4.1), (4.2) satisfy the structure conditions (A), (A1), (A2), (B), (B1), (B2) and \( \partial_{x_1} \beta \geq \chi > 0 \) for each \( (x, z, \mathbf{p}) \in B^0 \times \mathbb{R} \times \mathbb{R}^n \). Let \( M_0 = \sup_{B_R^+} u \). Then there exist \( R = R(n, \chi, \mu, \mu_1, \mu_2, M) \) and \( C = C(n, \chi, \mu, \mu_1, \mu_2, M) \) such that

\[
\sup_{B_R^+} |D' u| \leq C. 
\]

**Proof.** Let \( R \in (0, 1) \) be a constant and set

\[
w' = |D'u|^2, \quad M_1 = \sup_{B_R^+} \eta^2 w', \quad \eta = (1 - \frac{|x|^2}{R^2})^2.
\]

With \( \alpha_1 > 4, \alpha_2 > 0 \) constants to be chosen we set \( u^* = \exp \alpha_1 (u - M_0), v = (1 + |Du|^2)^{1/2} \) and
\[ w = \eta^2 w' + M_1 u^* / \alpha_1 + \alpha_2 M_1 v x_n. \]

By the estimates (4.6) and (4.7) we may determine \( R \) from \( \alpha_1, \alpha_2, M_1 \), such that

\[ \alpha_1 (M_0 - u) \leq 1, \quad \alpha_2 v x_n \leq 1/4, \quad \text{in } B_\delta^+. \]

If \( M_1 \leq R^{-4} \), then the desired estimate (4.8) is clear once we have determined \( \alpha_1 \) and \( \alpha_2 \). Now we determine suitable \( \alpha_1 \) and \( \alpha_2 \) so that \( M_1 \leq R^{-4} \). Suppose \( M_1 > R^{-4} \).

Let \( x_0 \) be a point in \( \overline{B}_\delta^+ \) where \( w \) attains its maximum. One can easily see from the definition of \( w \) that \( x_0 \notin \partial B_\delta^+ \setminus B_\delta^\sharp \), so that \( \eta(x_0) \neq 0 \) and

\[ \eta^2 w'(x_0) \geq M_1 / 2. \]

If \( x_0 \in B_\delta^\sharp \), by applying the operator \( B \equiv \alpha^{st} D_{st} + (D_{p_1} \beta + D_{p_1} \alpha^{st} D_{st} u) D_i \) to \( w \) and using the differentiated boundary condition we obtain

\[ Bw = 2 \eta^2 \alpha^{st} D_u u D_u u + 4 D_{i} u \alpha^{st} D_i \eta^2 D_u u + w' \alpha^{st} D_u \eta^2 \]
\[ - 2 \eta^2 D_{i} u D_i \alpha^{st} D_u u - 2 \eta^2 D_{i} \alpha^{st} D_i \eta^2 u w' + w' D_{p_1} \alpha^{st} D_{p_1} u D_i \eta^2 \]
\[ + w' \beta_i D_i \eta^2 - 2 w' \eta^2 D_{i} \beta - 2 \eta D_{i} u D_i \beta + M_1 u^* \alpha^{st} D_{si} u \]
\[ + \alpha_1 M_1 u^* \alpha^{st} D_i u D_i u + M_1 u^* \beta_i D_i u + M_1 D_{p_1} \alpha^{st} D_{si} u D_i u \]
\[ + \alpha_2 M_1 v \beta^2. \]

By the assumption (B2) we have at \( x_0 \)

\[ M_1 u^* D_{p_1} \alpha^{st} D_{si} u D_i u \leq (\varepsilon M_1 \alpha^{st} D_{si} u D_u u) / w' + C(\varepsilon) M_1 w'. \]

From (4.9) we can see \( M_1 / w(x_0) \leq 2 \eta^2 \), so that
(4.12) \[(\varepsilon M_1 \alpha^\varepsilon D_{sr}u \ D_{tr}u) / w' \leq 2 \varepsilon \eta \alpha^\varepsilon D_{sr}u \ D_{tr}u.\]

The rest of the terms in (4.10) can also be estimated by means of Cauchy inequality and using the hypothesis \(M_1 > R^4\), (B2). Therefore at \(x_0\) we have

(4.13) \[Bw \geq M_1 w' (\alpha_1 / \varepsilon - C_1) + \chi M_1 |Du| (\alpha_2 - nC_2),\]

for some constants \(C_1\) and \(C_2\). Now we choose \(\alpha_1 > C_1 \varepsilon\) and \(\alpha_2 > C_2\) so that \(Bw(x_0) > 0\). Then we have a contradiction.

If \(x_0 \in B^*_h\), by applying the operator \(\mathcal{Z} \equiv a_{ij} D_{ij} + (D_{pi} b + D_{pi} a_{kl} D_{kl}u) D_i\) to the function \(w\) and differentiating the equation (4.1) with respect to \(x_r\) we get

(4.14) \[
\mathcal{Z}w = 2 \eta^2 a_{ij} D_{tr}u D_{tr}u - 2 \eta^2 D_{sr}u D_{sr}u a_{ij} D_{ij}u + 4 D_{sr}u a_{ij} D_{ij}u D_{tr}u
+ D_{pk} a_{ij} D_{ij}u D_k \eta^2 w' - 2 w' \eta^2 D_{sr}a_{ij} D_{ij}u + w' a_{ij} D_{ij}u \eta^2
+ w' b_i D_i \eta^2 - 2 w' \eta^2 D_{sr}u D_{sr}b + \alpha_1 M_1 u^* a_{ij} D_{ij}u D_{tr}u
+ M_1 u^* a_{ij} D_{ij}u + M_1 u^* b_i D_i u + M_1 u^* (D_{pi} a_{kl} D_{kl}u) D_i u
+ \alpha_2 M_1 \frac{x_n}{\nu} a_{ij} g^{kl} D_{ik} u D_{ij}u + 2 \alpha_2 M_1 \frac{D_{kl} u}{\nu} a^{in} D_{ik} u
+ \alpha_2 M_1 \frac{x_n}{\nu} (-\delta a_{ij} D_{ij}u - \delta b) |D_{sr}u|^2 + \alpha_2 M_1 b_i^{n} v + \alpha_2 M_1 v (D_{pi} a_{ij} D_{ij}u),
\]

where \(g^{kl} = \delta^{kl} - \frac{D_{kl} u D_{li} u}{\nu} \geq 0\), and \(\delta = D_{sr} + \frac{Pr}{|p|} D_{pr}, b_i = D_{pi} b\). Solving \(D_{mn}u\) from the equation (4.1) we obtain

(4.15) \[|D_{mn}u| \leq 1 / \lambda (|b| + \lambda |\mu| |D'Du|).\]

Substituting (4.15) into (4.14) and by a similar consideration as we did in deriving (4.13) we get at \(x_0\).
where $C_3$ depends on $\alpha_2$ and the quantities described in the theorem. If we choose $\alpha_1 > \epsilon \max \{ C_1, C_3 \}$ we will have another contradiction. This completes the proof.

3. Hölder Estimates for the Tangential Gradient

We derive the Hölder estimate for tangential gradient by means of the boundary weak Harnack inequality Theorem 3.10. In our argument, to overcome the difficulties caused by the presence of the normal derivative, we employ more complicated test functions, and then use the technique from [17].

**Theorem 4.5.** Let $u \in C^3(B^+ \cup B^0)$ be a solution of (4.1), (4.2) with $|u|_0 + |D'u|_0 \leq M_1$. Suppose the operators in (4.1), (4.2) satisfy the structure conditions (A), (A1), (A2), (B), (B1), (B2). Then we have

$$
(4.17) \quad [D'u]_{\alpha; B_{1/2}} \leq C,
$$

for some constants $\alpha \in (0, 1)$ and $C$ depending only on $n, \mu, \mu_1, \mu_2, M_1$.

**Proof.** As for the proof of Theorem 4.2 we need only to establish the oscillation estimate for the tangential gradient around the origin similar to (4.5).

In order to make use of Theorem 3.10 we first define the linear operators

$$
L \equiv a^{ij} D_{ij} \quad \text{and} \quad B \equiv \alpha^{st} D_{st} + \beta D_i,
$$

which satisfy the hypothesis of Theorem 3.10. Secondly we introduce some functions: $\nu' = |D'u|^2$, $\nu = (1 + |D'u|^2)^{1/2}$,
\[ w'_x = w'_y = \pm D_r u + \varepsilon v' + A_1 x_n v + A_2 u^2, \]

and \( w = \frac{1}{\delta} e^{-\delta w} \) with \( \delta, \varepsilon, A_1, A_2 \) positive constants to be determined later.

By differentiating the equation (4.1) we obtain

\[(4.18) \quad Lw' = e^{-\delta w} (\delta a^i_j D_i w'_x D_j w'_x - a^i_j D_i w'_y)
- e^{-\delta w} (\delta a^i_j D_i w'_x D_j w'_x - 2 \varepsilon a^i_j D_i u D_j u - 2 A_2 a^i_j D_i u D_j u
- A_1 \frac{x_n}{v} g^{kl} D_{ik} u D_{jl} u - 2 A_1 a^{in} \frac{D_k u}{v} D_{ik} u + 2 A_2 u b + F_r
+ 2 \varepsilon D_i u F_s + A_1 \frac{x_n}{v} D_k u F_k),\]

where \( F_i = D_a^k D_k j u + D_2 a^k D_k u D_i u + D_1 a^k D_k u D_i j u + D_1 j b + a^k D_i u + D_1 j b \) \( D_i j u, \) and \( g^{kl} \) is defined as before. Differentiating the boundary condition (4.2) gives us

\[(4.19) \quad Bw' = e^{-\delta w} (\delta \alpha^{st} D_s w'_x D_t w'_x - \alpha^{st} D_s w'_x - D_p t \beta D_i w'_x)
- e^{-\delta w} [\delta \alpha^{st} D_s w'_x D_t w'_x - 2 \varepsilon \alpha^{st} D_{sg} u D_{tg} u - 2 \alpha^{st} D_s u D_i u
- A_1 D_{pr} \beta v + G_r + 2 \varepsilon D_{sg} u G_s + 2 A_2 u (D_{pr} \beta D_i u - \beta)],\]

where \( G_r = D_r \alpha^{st} D_s u + D_2 \alpha^{st} D_s u D_i u + D_1 \alpha^{st} D_s u D_i j u + D_1 j \beta D_i u + D_{r} \beta \) Without loss of generality we assume that \( u(0) = 0. \) Now we choose \( \delta = \delta(\varepsilon, M_1, \mu_1, \mu_2) \) so small that the following inequalities hold.

\[ \delta a^i_j D_i w'_x D_j w'_y \leq \varepsilon a^i_j D_i u D_j u + \delta (4 A_2^2 u^2 + A_1^2) v + C_1, \]
\[ \delta \alpha^{st} D_s w'_x D_t w'_x \leq \varepsilon \alpha^{st} D_{sg} u D_{tg} u + C_2. \]

Once we have determined \( A_1 \) and \( A_2 \) we can choose \( R \) such that \( A_2 |u| \leq 1. \) Then by the same discussion as in the proof of Theorem 4.4 we get from (4.18) and (4.19) that
\[ a^{ij} D_{ij} w_\pm \leq e^{-\delta w_\pm} (-2 A_2 a^{ij} D_i u D_j u + \lambda C_3 v + C_4), \]
\[ B w_\pm \leq e^{-\delta w_\pm} (\chi v (C_5 - A_1) + C_6). \]

If we let \( A_1 > C_5 \) and \( A_2 > C_3 \) we will have
\[ L w_\pm \leq C_4 \text{ in } B^+_R, \quad B w_\pm \leq C_8 \text{ on } B^0_R, \]

where we can apply Theorem 3.10. Therefore we obtain the Hölder norm estimates of \( w_\pm \), and hence also of \( w'_\pm \). Notice that on \( B^0_R \)
\[ w'_\pm = \pm D_r u + \varepsilon v + A_2 u^2. \]

Since \( u \) is Hölder continuous with bounded Hölder norm, we can see that \( w'_\pm \) behave like \( \pm D_r u \) if \( \varepsilon \) is small enough. We obtain the estimates of Hölder norms of \( D_r u \) for \( r = 1, \ldots, n-1 \) by the standard argument, see [6].

4. \( C^2, \alpha \) Estimates on the Boundary

We start with a sharp form of a normal derivative estimate on the boundary. Considering the linear operator
\[ L u = a^{ij} D_{ij} u \]
with \( \lambda I \leq (a^{ij}) \leq \Lambda I \), and \( \Lambda / \lambda \leq \mu \) for some constant \( \mu \), we have the following estimate.

\textbf{Lemma 4.6.} Let \( u \in C^2(B^+ R) \), \( R \leq 1 \), be a bounded solution of the differential inequality
\[ |Lu| \leq \lambda (\mu_0 |Du|^2 + \Phi), \quad \text{in } B^+_R, \]
\[ u = 0, \quad \text{on } B^+_R, \]

and \( M_0 = \sup_{B^+_R} |u| \). Then there exist constants \( \delta, C_1, C_2 \) depending only on \( n, \mu, \mu_0, M_0 \) such that either

\[ \frac{|u(0, x_n)|}{x_n} \leq C_1 \frac{1}{R}, \quad \text{provided } x_n < \delta R; \]

or

\[ \frac{|u(0, x_n)|}{x_n} \leq C_2 \sqrt{\Phi}. \]

The important feature of this lemma is the dependence on \( \Phi \) of the quantity on the right hand side of the last inequality above. We will see the utility of this feature soon. The idea of the proof is the standard barrier argument as is described in [6]. We write out the detailed proof to show the precise dependence.

**Proof.** We are going to prove the upper bound only, the lower bound follows by the same reasoning.

Let \( x^0 = (0, \ldots, 0, -R) \), \( d = |x - x^0| - R \). Considering the operator \( \mathcal{L} \) defined by

\[ \mathcal{L}w \equiv Lw + \lambda \mu_0 |Dw|^2, \]

we get from (4.20) that

\[ \mathcal{L}u \geq -\lambda \Phi. \]

Now we look at the barrier function
\[ v(x) = \psi(d) = \frac{1}{v} \ln \left( 1 + \frac{\tau}{R} d \right), \]

where \( v, \tau \) are constants to be fixed later. It is easy to see that \( \psi' = -v \psi^2 \) and

\[
\mathcal{L} w = a^i \psi'' \frac{(x_i - x^0_i)(x_j - x^0_j)}{|x - x^0|^2} + a^j \psi' \frac{\delta^{ij} |x - x^0| - (x_i - x^0_i)(x_j - x^0_j)}{|x - x^0|^2} + \lambda \mu_0 \psi^2
\]

\[
\leq \lambda \psi'' + \psi' \frac{(n-1)}{R} \Lambda + \lambda \mu_0 \psi^2 \leq \lambda (\psi'' + 2 \mu_0 \psi^2)
\]

if \( \psi' \geq \frac{(n-1)}{R} \Lambda \). Further more if \( \psi' \geq \sqrt{\Phi} \) and we choose \( v \geq 2 (\mu_0 + 1) \) we will have

\[ \mathcal{L} v \leq -2 \lambda \Phi < -\lambda \Phi \leq \mathcal{L} u. \]

It is obvious that \( v \geq 0 \) on \( B^*_r \) and if we set \( \delta \) such that \( \psi(\delta R) \geq M_0 \), by maximum principle we will have in the domain \( \mathcal{N}_\delta = \{ 0 < d \leq \delta R \} \) that \( u \leq v \), in particular

\[ (4.23) \quad u(0, x_n) \leq \frac{1}{v} \ln \left( 1 + \frac{\tau}{R} x_n \right) \leq \frac{\tau x_n}{R}. \]

The rest of proof is to estimate the quantity \( \tau / v \).

To fulfill the requirement that \( v \geq 2 (\mu_0 + 1) \) and \( \psi(\delta R) \geq M_0 \) we need only take \( v = 2 (\mu_0 + 1), \tau \delta = \mu \exp (\nu M_0) - 1 \). Since

\[ \psi'(d) = \frac{\tau}{\nu (R + \tau d)} \geq \frac{\tau}{\nu R (1 + \tau \delta)} \geq \frac{1}{2 v R \delta}, \text{ when } d \leq \delta R, \]

we can see that in order to get \( \psi'(d) \geq \max \{ \frac{\Lambda (n-1)}{\lambda \mu_0 R}, \sqrt{\Phi} \} \) we may choose \( \delta = \frac{\lambda \mu_0}{2 v \Lambda (n-1)}, \text{ when } \frac{\Lambda (n-1)}{\lambda \mu_0 R} \geq \sqrt{\Phi}; \) and \( \delta = \frac{1}{2 v R \sqrt{\Phi}}, \text{ when } \sqrt{\Phi} > \frac{\Lambda (n-1)}{\lambda \mu_0 R}. \)

In the first case
\[
\frac{\tau}{\nu} = \frac{1}{\nu} (e^{\nu M_0} - 1) = \frac{2(e^{\nu M_0} - 1) \Lambda (n - 1)}{\lambda \mu_0}
\]

which is the constant \( C_1 \) in (4.21). In the second case, if \( x_n \leq \delta \), that is \( x_n \leq \frac{1}{2 \nu \sqrt{\Phi}} \), we have

\[
\frac{\tau}{\nu} = 2(e^{\nu M_0} - 1) \Phi \sqrt{\Phi};
\]

and if \( x_n > \frac{1}{2 \nu \sqrt{\Phi}} \), we have

\[
\frac{|u(0, x_n)|}{x_n} < 2 M_0 \nu \sqrt{\Phi}.
\]

So we can take \( C_2 \) in (4.22) to be \( 2 [\exp(\nu M_0) - 1] \).

The next two lemmas concern the Hölder norm of the normal derivative on the boundary. The first one is due to Krylov [14], see also Lieberman [16] Lemma 5.1. For our convenience we first introduce some notation. For positive constants \( \rho \) and \( R \) we write

\[
G(\rho, R) = \{ x \in \mathbb{R}^n \mid 0 < x_n < \rho R, |x'| < R \},
\]

\[
G'(\rho, R) = \{ x \in \mathbb{R}^n \mid \rho R < x_n < 3 \rho R/2, |x'| < R \}.
\]

**Lemma 4.7.** Let \( u \in C^2(B^+) \) be a bounded solution of

\[
|Lu| \leq \lambda (\mu_0 |Du| + \Phi), \quad \text{in} \; B^+; \quad u = 0, \quad \text{on} \; \partial B.
\]

and \( M_0 = \sup |u| \). Then there exist constants \( \alpha \in (0, 1), C \) depending on \( \mu, \mu_0, M_0 \) such that for \( R < 1/4, \rho = \{ \lambda / 6(n-1) \Lambda \}, \delta \) where \( \delta \) is the constant defined in Lemma 4.6, we have
Proof. We define the functions

\[ v_1 = M_{4R} x_n - u, \quad v_2 = u - m_{4R} x_n. \]

where \( M_R = \sup_{G(p, R)} \frac{u}{x_n} \) and \( m_R = \inf_{G(p, R)} \frac{u}{x_n} \). It is clear that \( v_i \geq 0 \) for \( i = 1, 2 \). By considering

\[ \tilde{v}_i = \frac{1}{2\mu_0} (1 - e^{-2\mu_0 v_i}), \quad i = 1, 2, \]

we can see that

\[
(4.26) \quad L \tilde{v}_1 = e^{-2\mu_0 \tilde{v}_1} (-a^{ij} D_{ij} u - 2\mu_0 a^{ij} D_{ij} u D_{ij} u - 2\mu_0 M_{4R} a^{nn} + 4\mu_0 a^{in} M_{4R} D_{ij} u) \\
\leq \lambda (\Phi + 2\mu_0 \mu M_{4R}^2),
\]

and

\[
(4.27) \quad L \tilde{v}_2 = e^{-2\mu_0 \tilde{v}_2} (a^{ij} D_{ij} u - 2\mu_0 a^{ij} D_{ij} u D_{ij} u - 2\mu_0 a^{nn} m_{4R}^2 + 4\mu_0 a^{in} m_{4R} D_{ij} u) \\
\leq \lambda (\Phi + 2\mu_0 \mu m_{4R}^2).\]

By setting \( R = 1 \) in Lemma 4.6 we obtain

\[
(4.28) \quad m_{4R}, M_{4R} \leq C (\Phi^{1/2} + 1).
\]

Inserting (4.28) into (4.26) and (4.27) and applying the interior weak Harnack inequality, Theorem 9 of [21], we get
(4.29) \[
\left( \frac{1}{|G'(\rho, 2R)|} \int_{G'(\rho, 2R)} v_i^p \right)^{1/p} \leq C \left( \inf_{G'(\rho, 2R)} \frac{v_i}{x_n} R + (\Phi + 1) R^2 \right), \quad i = 1, 2.
\]

Back to \( v_i \), (4.29) implies

(4.30) \[
\left( \frac{1}{|G'(\rho, 2R)|} \int_{G'(\rho, 2R)} v_i^p \right)^{1/p} \leq C e^{2\mu_0 M_0} \left( \inf_{G(\rho, R)} \frac{v_i}{x_n} + (\Phi + 1) R \right) R, \quad i = 1, 2.
\]

In the last inequality we have used Lemma 4.7. Summing up (4.30) over \( i = 1, 2 \) and writing \( \omega (R) = M_R - m_R \) we obtain

\[
\omega (4 R) \leq C (\omega (4 R) - \omega (R) + (\Phi + 1) R),
\]

that is

\[
\omega (R) \leq \gamma \omega (4 R) + (\Phi + 1) R, \quad \text{for some } \gamma \in (0, 1).
\]

From this and by virtue of the standard method it follows that

\[
\omega (R) \leq C R^\alpha (\omega (1/4) + \Phi + 1).
\]

As a consequence of Lemma 4.8 we get the Hölder estimate of normal derivative on the boundary immediately in the homogeneous boundary condition case. Generally, if \( u \in C^2 (\overline{B}^+) \) satisfies

\[
|Lu| \leq \lambda (|Du|^2 + \Phi), \quad \text{in } B^+, \quad u = \phi, \quad \text{on } B^0,
\]

for some \( \phi \in C^2 (\overline{B}^0) \), by substitution \( v = u - \phi \) we will have that \( v \) satisfies the
homogeneous boundary problem

$$|Lv| \leq \lambda (\mu_0 |Dv|^2 + \psi), \text{ in } B^+, \ v = 0, \text{ on } B^0,$$

with a new constant \(\psi = \mu_0 |D\phi|_0^2 + \mu |D^2\phi|_0 + \Phi\). If we apply Lemma 4.6 and Lemma 4.8 to \(v\) we will see that the normal derivative and its Hölder norm can be bounded on \(B^0\) linearly in terms of \(|D^2\phi|_0\).

Now let us return to our quasilinear problem (4.1), (4.2). Suppose \(u\) is a bounded solution of (4.1), (4.2). By the results in Section 2.3 we know that for some positive constant \(R < 1\) we have

\[(4.31) \quad |u|_{1,\alpha; B^0_R} \leq C.\]

Under the structure conditions (A2), (B2) we have

\[(4.32) \quad |\alpha^{st}(\cdot, u(\cdot), D'u(\cdot))|_{\alpha; B^0_R} \leq C \left(1 + |u|_{1,\alpha; B^0_R}\right) \leq C,
\]

\[(4.33) \quad |\beta(\cdot, u(\cdot), Du(\cdot))|_{\alpha; B^0_R} \leq C \left(1 + |Du|_{2,\alpha; B^0_R}\right) \leq C \left(1 + |D^2u|_{0, B^0}\right).\]

Returning to the original domain \(\Omega\), if \(u\) is a bounded solution of the problem (1.5), (1.6), then (4.32) and (4.33) will imply

\[|\alpha^{ij}(\cdot, u(\cdot), \delta u(\cdot))|_{\alpha; \partial\Omega} \leq C,
\]

\[|\beta(\cdot, u(\cdot), Du(\cdot))|_{\alpha; \partial\Omega} \leq C \left(1 + |u|_{2,\alpha; \partial\Omega}\right).
\]

We can now apply the classical Schauder estimate, Theorem 6.6 of [6], to (1.6) on the manifold \(\partial\Omega\) to obtain

\[(4.34) \quad |u|_{2,\alpha; \partial\Omega} \leq C \left(1 + |u|_{2; \partial\Omega}\right).\]
Finally the interpolation inequality, Lemma 6.35 of [6], yields

\[(4.35) \quad |u|_{2, \alpha; \partial \Omega} \leq C.\]

**Theorem 4.9.** Let \( u \in C^3(\bar{\Omega}) \) be a solution of (1.5), (1.6) and \( \partial \Omega \in C^3 \). Suppose that the structure conditions (A), (A1), (A2), (B), (B1), (B2) are satisfied and the boundary condition is also oblique with obliqueness constant \( \chi > 0 \). Suppose also \( \sup \| u \| \leq M_0 \) for some constant \( M_0 \). Then there exist constants \( \alpha \in (0, 1) \) and \( C \) depending on \( n, \mu, \mu_1, \chi, M_0 \) and \( \partial \Omega \) such that

\[(4.36) \quad |u|_{1, \alpha; \Omega} \leq C.\]

**Proof.** By the well known results on \( C^{1, \alpha} \) global estimates for the solutions of Dirichlet problems of equation (1.5) with \( C^{2, \alpha} \) boundary data, this theorem is a straightforward consequence of the estimate (4.35).

5. Existence and Uniqueness

All the estimates in previous sections are based on having a \( L^\infty \) estimate for the solution \( u \), which can be obtained in various ways, for example:

**Lemma 4.10.** Suppose \( a^{ij}, \alpha^{ij}, \beta \) satisfy (A), (B) and \( D_2 b, D_2 \beta \leq -c_0 \) for some positive constant \( c_0 \). Let \( u \) be a solution of (1.5), (1.6). Then

\[(4.37) \quad \sup_{\Omega^2} |u| \leq \frac{1}{c_0} \sup_{\Omega^2} [ |b(x, 0, 0)| + |\beta(x, 0, 0)| ].\]

**Proof.** Let \( x_0 \) be such a point that
If \( x_0 \in \Omega \), then by (A) and \( D \beta \leq -c_0 \) we have

\[
0 = a_{ij}(x_0, M_0, 0) D_{ij} u(x_0) + b(x_0, M_0, 0) \leq b(x_0, M_0, 0)
\]

\[
\leq M_0 \int_0^1 D_2 b(x_0, t M_0, 0) \, dt + b(x_0, 0, 0) \leq -c_0 M_0 + b(x_0, 0, 0).
\]

If \( x_0 \in \partial \Omega \), by (B) and \( D \beta \leq -c_0 \) we have

\[
0 = \alpha_{ij}(x_0, M_0, 0) \delta_{ij} u(x_0) + \beta(x_0, M_0, D_v u(x_0) v)
\]

\[
\leq \int_0^1 D_p \beta(x_0, t M_0, t D_v u) \, dt (D_v u \cdot v) + \int_0^1 D_2 \beta(x_0, t M_0, 0) \, dt M_0 + \beta(x_0, 0, 0)
\]

\[
= -c_0 M_0 + \beta(x_0, 0, 0).
\]

Then (4.37) follows from (4.38) and (4.39).

**Theorem 4.11.** Assume that \( a_{ij}, b; \alpha_{ij}, \beta \) satisfy (A) \( \sim (A2) \), (B) \( \sim (B2) \) and \( \chi > 0 \). Assume also that \( D_2 b, D_2 \beta \leq -c_0 \) for some constant \( c_0 > 0 \). Then there exists a solution \( u \in C^{2,\alpha}(\Omega) \) of the problem (1.5), (1.6), provided \( \partial \Omega \in C^3 \).

**Proof.** We define a map \( T : C^{1,\alpha}(\Omega) \times [0, 1] \to C^{1,\alpha}(\Omega) \) by setting, for each \( v \in C^{1,\alpha}(\Omega) \) and \( \tau \in [0, 1] \),

\[
u = T(v, \tau) \]

\[
(4.40) \quad \tau \left[ a_{ij}(x, v, Dv) D_{ij} u + b(x, v, Dv) + v - u \right] + (1 - \tau) \left( \Delta u - u \right) = 0, \quad \text{in} \ \Omega,
\]

\[
\tau[\alpha_{ij}(x, v, Dv) \delta_{ij} u + \beta(x, v, Dv) + v - u] + (1 - \tau) \left( \Delta_{\partial \Omega} u + D_v u - u \right) = 0, \quad \text{on} \ \partial \Omega.
\]

\]

The existence and uniqueness of solutions of (4.40) is guaranteed by Theorem 2.6, so \( T \) is well defined. It is easy to check that \( T \) is continuous and compact, and \( T(v, 0) = 0 \), for all \( v \in C^2(\Omega) \). Therefore the solvability of (1.5), (1.6) is equivalent to \( T(u, 1) = u \).
and by Leray - Schauder fixed point theory we know that $T(\cdot, 1)$ has a fixed point if and only if $T(\cdot, 0)$ does, but the later is obvious since $T(0, 0) = 0$.

The uniqueness result follows from following comparison principle.

**Lemma 4.12.** Let $u, v \in C^2(\Omega)$ satisfy $Lu \geq Lv$ in $\Omega$, $Bu \geq Bv$ on $\partial\Omega$, where $L$ and $B$ are the operators defined by (1.5) and (1.6) respectively, and

i) $(\alpha^i)$ is elliptic, $(\alpha^i)$ is degenerate elliptic and both are independent of $z$,

ii) $b$ is non-increasing in $z$, and $\beta$ is strictly decreasing in $z$ and degenerate oblique,

iii) $\alpha^i, \beta; \alpha^i, b$ are continuously differentiable with respect to $p$ variables.

It then follows that $u \leq v$ in $\Omega$.

**Proof.** By writting $w = u - v$ we may obtain

$$a^i_{ij} D_{ij} w + b^i D_i w \geq 0, \quad \text{in } \Omega^+,$$

where $\Omega^+ = \{ \mathbf{x} \in \Omega \mid w > 0 \}$, $a^i_{ij}(\mathbf{x}) = a^i_{ij}(\mathbf{x}, D_u(\mathbf{x}))$, and

$$b^i(\mathbf{x}) = \int_0^1 D_p a^{kl}(\mathbf{x}, D_v(x) + t \cdot D w(x)) D_{kl} v(x) \, dt + \int_0^1 D_p b(x, u(x), D v(x) + t \cdot D w(x)) \, dt.$$

Hence the classical weak maximum principle implies that $\sup_{\Omega^+} w = \sup_{\partial \Omega^+} w$. If $\overline{\Omega^+} \cap \partial \Omega \neq \emptyset$, then $w \leq 0$ because $w = 0$ on $\partial \Omega^+$. Now we suppose there is a $x_0 \in \partial \Omega$ such that $w(x_0) = \sup_{\Omega^+} w > 0$. Then by the assumption ii) on $\beta$ we have

$$\alpha^{ij} \delta_{ij} w + \beta^i D_i w > 0, \quad \text{at } x_0,$$

where $\alpha^{ij}(\mathbf{x}) = \alpha^{ij}(\mathbf{x}, \delta u(\mathbf{x}))$, $\beta^i(\mathbf{x}) = \int_0^1 D_p \beta(x, u(x), D v(x) + t \cdot D w(x)) \, dt$. Thus we have
a contradiction which implies that the assertion of this lemma is true.

**Corollary 4.13.** If all the assumption in Theorem 4.11 and Lemma 4.12 hold, the problem (1.5), (1.6) has a unique solution \( u \in C^{2,\alpha}(\Omega) \).
Chapter 5
Quasilinear Problems (II)
Degenerate Boundary Conditions

This is a continuation of the last chapter. Here we deal with the same problem (1.5), (1.6) under the assumption that the boundary condition is degenerate elliptic but strictly oblique. The loss of ellipticity of (1.6) prevents us getting $C^{2,\alpha}$ estimates on the boundary by using the interpolation inequality, so that we have to establish the desired estimates step by step. For this purpose we drop the $z$ and $p$ dependence of functions $\alpha^{ij}$, and assume the natural structure conditions (A), (A1), (A2), (B), (B1), (B2) as in Chapter 4, except that in (B) we assume $\kappa = 0, \chi > 0$ and that in (B1), (B2) the quantity $\kappa$ is replaced by $\chi$, and

\[ (A3) \quad |D_{xix}a^{ij}, D_{xx}a^{ij}, D_{zz}a^{ij}|, (1 + |p|) |D_{xp}a^{ij}, D_{zp}a^{ij}|, (1 + |p|^2) |D_{pp}a^{ij}| \leq \lambda \mu_1(z I), \]
\[ |D_{xx}b, D_{xx}b, D_{xx}b, (1 + |p|) |D_{xp}b, D_{zp}b, (1 + |p|^2) |D_{pp}b| \leq \lambda \mu_1(z I)(1 + |p|^2), \]
for $|p| \geq M, M$ and $\mu_1$ are the same as in (A2);

\[ (B3) \quad |D_{xx}\alpha^{ij}| \leq \chi \mu_2(z I), |D_{xx}B, D_{xx}B, D_{xx}B| \leq \chi \mu_2(z I)(1 + |p|), \]
\[ |D_{zp}B, D_{zp}B|, |D_{pp}B| \leq \chi \mu_2(z I), \]
for $|p| \geq M, M$ and $\mu_2$ are the same as in (B2), and $\chi$ is the obliqueness constant;

\[ (A4) \quad \frac{\partial^2 a^{ij}}{\partial p_k \partial p_l} \nu^i \nu^k \nu^l = 0, \forall i, \text{ and } \frac{\partial a^{ij}}{\partial p_k} \nu^i \nu^j \nu^k = 0, \]
for $x$ in a small neighbourhood of those points where the boundary condition is degenerate elliptic;
Locally our problem is

\begin{align}
\text{(5.1)} & \quad a_{ij}(x, u, Du) D_{ij}u + b(x, u, Du) = 0, \quad \text{in } B^+, \\
\text{(5.2)} & \quad \alpha_{st}(x) D_{st}u + \beta(x, u, Du) = 0, \quad \text{on } B^0,
\end{align}

and (A4), (B4) become

\begin{align}
\text{(A4')} & \quad \frac{\partial^2}{\partial p_n \partial p_n} a^{in} = 0, \quad \forall \, i, \quad \text{and } \frac{\partial}{\partial p_n} a^{nm} = 0, \\
\text{(B4')} & \quad \frac{\partial^2}{\partial p_n \partial p_n} \beta = 0.
\end{align}

The behaviour of the degeneracy could be very complicated. We consider two types. We call \( x \in \partial \Omega \) a type I degenerate point if \( \{ \alpha_{st} \} \) satisfies (3.28) there; and a type II degenerate point if \( \{ \alpha_{st} \} \) satisfies \( 0 \leq \{ \alpha_{st} \} \leq K(x) \) in a small neighbourhood of \( x \) and \( K(x) \) is a function satisfying \( K(x) \leq \mu_3 |x - y| \), for each \( y \) and a constant \( \mu_3 \).

\section{1. Oscillation Estimates}

We have seen in Section 4.2 that Hölder estimate is crucial in estimating the gradient under the natural structure conditions. However we cannot obtain the Hölder estimate generally, for example, near the degenerate points of type II. A useful estimate which can take place of a Hölder estimate is an oscillation estimate. To achieve such an estimate near the degenerate points of type II we need following lemma which we also call a weak Harnack inequality which is a modification of Lemma 2.1 of [17].
Lemma 5.1. Let \( u \in C^2(B^+ \cup B^0) \) satisfy \( u \geq 0 \) and

\[
Lu \equiv a^{ij} D_{ij} u \leq \lambda \left( \mu_0 |Du|^2 + \Phi \right), \quad \text{in } B^+,
\]

\[
Bu \equiv \alpha^{st} D_{st} u + \beta^i D_i u \leq \chi \Psi, \quad \text{on } B^0,
\]

for some nonnegative constants \( \mu_0, \Phi, \Psi \), where \( \lambda > 0 \) is the minimum eigenvalue of \( \{a^{ij}\} \) and \( \chi > 0 \) is the obliqueness constant. Suppose that \( 0 \) is a type II degenerate point of \( B \) and that there are constants \( M_0 > 0, \mu_3 \geq 0 \) such that

\[
u \leq M_0, \quad \Lambda \leq \lambda \mu_3, \quad |\beta|/\chi \leq \mu_3.
\]

Then there exist constants \( \sigma \) and \( \rho \) depending only on \( n, \mu_3 \) and \( C \) depending also on \( \mu_0, M_0 \) such that if \( R \in (0, 1/4) \) then

\[
(5.3) \quad \left( \frac{1}{|G'(\rho, 2R)|} \int_{G'(\rho, 2R)} u^\sigma \right)^{1/\sigma} \leq C \left( \inf_{G(\rho, R)} u + \Phi R^2 + \Psi R \right)
\]

where \( G(\rho, R) \) and \( G'(\rho, R) \) are as defined in Section 4.4.

Proof. Considering \( \tilde{u} = (1 - \exp(-\mu_0 u)) / \mu_0 \) we may assume that \( \mu_0 = 0 \).

With this assumption, we show that for \( \rho = 1/8 \) \( n - 1 \) \( \mu_3 \) we have

\[
(5.4) \quad A = \inf_{G'(\rho, 2R)} u \leq 4 \left( \inf_{G(\rho, R)} u + 4 \Phi (\rho R)^2 + 2 \Psi \rho R \right)
\]

The desired inequality follows by combining this inequality with the interior weak Harnack inequality

\[
\left( \frac{1}{|G'(\rho, 2R)|} \int_{G'(\rho, 2R)} u^\sigma \right)^{1/\sigma} \leq C \left( \inf_{G'(\rho, 2R)} u + \Phi R^2 \right)
\]
which follows from [21]. To prove (5.4) we introduce the functions

\[ w_1 = 4 (\rho R)^2 - x_n^2, \quad w_2 = 2 \rho R - x_n, \quad w_3 = 2 - \frac{x_n^2}{4(\rho R)^2} - \frac{x_n}{2\rho R} + \frac{|x'|^2}{R^2}. \]

Clearly \( w_i \geq 0 \) in \( G(\rho, 2R) \) for \( i = 1, 2, 3 \), and a simple calculation shows that

\[
\begin{align*}
  Lw_1 &\leq -2 \lambda, \quad Lw_2 \leq 0, \quad \text{in} \ G(\rho, 2R), \\
  Bw_1 &\leq 0, \quad Bw_2 \leq -\chi, \quad \text{on} \ B^0,
\end{align*}
\]

and

\[
\begin{align*}
  Lw_3 &\leq \lambda \left[ -\frac{1}{\rho^2} + 8 \mu_3(n-1) \right] \frac{1}{4R^2} \leq \lambda \left[ -\frac{1}{\rho} + 8 \mu_3(n-1) \right] \frac{1}{4R^2} \leq 0, \\
  Bw_3 &\leq \chi \left[ -\frac{1}{\rho} + 2 \mu_3 + 4 \mu_3(n-1) \right] \frac{1}{2R} \leq \chi \left[ -\frac{1}{\rho} + 6 \mu_3(n-1) \right] \frac{1}{2R} \leq 0,
\end{align*}
\]

in \( G(\rho, 2R) \) and on \( B^0 \) respectively, by virtue of our choice of \( \rho \). Furthermore we see that

\[ w_3 \geq 4, \quad \text{on} \ \partial G(\rho, 2R) \cap \{|x'| = 2R\}, \quad w_3 \leq 3, \quad \text{in} \ G(\rho, R). \]

Thus the function \( w = u + \Phi w_1 + \Psi w_2 + \frac{A}{4} w_3 \) satisfies

\[ Lw \leq 0, \quad \text{in} \ G(\rho, 2R); \quad Bw \leq 0, \quad \text{on} \ B^0, \]

and \( w \geq A \) on \( \partial G(\rho, 2R) \setminus B^0 \). Therefore the maximum principle Lemma 2.1 implies that \( w \geq A \) in \( G(\rho, 2R) \). Hence \( w \geq A \) in \( G(\rho, R) \) which gives us
\[ \inf_{G(\rho, R)} u + 4 \Phi (\rho R)^2 + 2 \psi R + \frac{3}{4} A \geq A. \]

This proves our (5.3).

**Remark.** If we check the proof above we will find that the assumption of degenerate point of type II is not necessary. Actually \( K(x) \leq R \) is enough. Of course now the assumption on \( \{ \alpha^\varepsilon \} \) depends on \( R \). We will see later on that in practice, when we apply the elliptic regularization, \( R \) can be fixed before hand, then if we add \( \varepsilon \) times the Laplacian operator to the boundary operator near the type II degenerate point and set \( \varepsilon \) small enough, the result (5.3) remains valid.

**Lemma 5.2.** Let \( u \) be a solution of (5.1), (5.2). Suppose \( a, b, \alpha^\varepsilon, \beta \) satisfy (A), (A1), (B), (B1) and \( 0 \) is a type II degenerate point of (5.2). Suppose also \( \sup |u| \leq M_0 \). Then there are positive constants \( \alpha < 1 \), and \( C \) depending on \( \mu, \mu_1, \mu_2, \mu_3, M_0, n \) such that

\[ \text{osc}_{B_R} u \leq C R^\alpha, \quad \text{for any} \quad R \leq 1/4. \]

The proof of this lemma is quite similar to that of Lemma 4.1, so we omit it here.

For the degenerate points of type I we have the following result on Hölder estimates.

**Theorem 5.3.** Let \( u \) be a solution of (5.1), (5.2). Suppose all the assumptions in Lemma 5.2 are satisfied except that \( 0 \) is a degenerate point of type I. Then there are positive constant \( \alpha < 1 \) and \( C \) depending on \( \mu, \mu_1, \mu_2, M_0, n \) such that

\[ \left[ \frac{u}{\alpha; B_{12}} \right] \leq C. \]

The proof of Theorem 5.3 is exactly the same as that of Theorem 4.2 if we make
use of Theorem 3.13 in the place of Theorem 3.10.

Theorem 4.4 remains true near the degenerate points of either type I or type II, provided we have obtained the oscillation estimates Lemma 5.2 and Theorem 5.3. By the same argument as in the proof of Lemma 5.2 and Theorem 5.3, we can also prove the oscillation estimates for tangential gradient around the type I and type II degenerate points. From now on we assume that the tangential gradient is bounded and its oscillation estimate in the form (5.5) is known.

2. Estimates for Second Order Tangential Derivatives

In this section we shall derive bounds and oscillation estimates for pure tangential second order derivatives independently of bounds for the normal derivative. In order to get around the associated difficulties we make the strong assumptions (A4) and (B4). Fortunately the assumption (A4) is satisfied by mean curvature operator and (B4) is satisfied if $\beta(x, z, p)$ is linear in the $p_n$ variable.

Before we establish our main result we need to prove an auxiliary result on the behavior of the mixed derivatives, i.e. $D_{ni}u$ for $i \neq n$.

**Lemma 5.4.** Let $u \in C^4(B^+)$ be a solution of (5.1). Suppose that the conditions (A), (A1), (A2), (A3), (A4)' are satisfied. For any point $\bar{x} = (\bar{x}, \bar{x}_n) \in B^+$ and $B = B_R(\bar{x})$ with $R = \bar{x}_n$ we denote

$$\theta = \max \{ \text{osc}_B |Du|, \text{osc}_B u \}. $$

Then there are constants $\theta_0, C$ depending only on $\mu, \mu_1, n, M_0 = |u|_0, M_1 = |Du|_0$, such that if $\theta \leq \theta_0$, then

$$|D_{ii}u(\bar{x})| \leq C \theta / \bar{x}_n, \text{ for } i = 1, \ldots, n, \text{ and } r \neq n. $$
Proof. Without loss of generality we assume that $\mathbf{x}' = 0$, and $u(0) = |D'u(0)| = 0$. Let

$$w' = \sum_{i+j<n} |D_{ij}u|^2, \quad v = |D'u|^2, \quad \eta = \left(1 - \frac{|x - \mathbf{x}'|^2}{R}\right),$$

and

$$w = \eta w' + M_2 v / 4 \theta^2 + M_2 u^2 / 4 \theta^2,$$

where $M_2 = \sup_{B^+} \eta w'$. By the differentiation of equation (5.1) we obtain

$$a^{ij} D_{ij} w + b^i D_i w = 2 \eta a^{ij} D_{ikr} u D_{jkr} u + \frac{M_2}{2 \theta} a^{ij} D_{ir} u D_{jr} u + \frac{M_2}{2 \theta} a^{ij} D_{i} u D_{j} u + F w'$$

$$+ 2 \eta \left(\frac{\partial^2 a^{kl}}{\partial x_i \partial x_j} D_{kl} u + \frac{\partial^2 b}{\partial x_i \partial x_j} \right) Y_{ij} - 4 \eta D_{pm} a^{ij} D_{ijk} u D_{im} u D_{kl} u$$

$$- 2 \eta D_2 a^{ij} D_{ijk} u D_{kl} u - 4 \eta D_1 a^{ij} D_{ijk} u D_{kl} u$$

$$- \frac{M_2}{2 \theta} (D_r D_r a^{ij} D_{ij} u + D_r u D_r b + (D_z a^{ij} D_{ij} u + D_z b) v + \frac{2}{\eta} a^{ij} D_j \eta D_r u D_r u$$

$$- u B^i D_i u + u b),$$

where

$$B^i = D_{pi} a^{kl} D_{kl} u + D_{pi} b - \frac{2}{\eta} a^{ij} D_j \eta,$$

$$F = D_{pi} a^{kl} D_{kl} \eta + D_{pi} b D_1 \eta - 2 \eta (D_z a^{ij} D_{ij} u + D_z b)$$

$$- \frac{2}{\eta} a^{ij} D_j \eta D_1 \eta + a^{ij} D_1 \eta,$$

and $X = (X_1, \ldots, X_{2n+1}) = (x_1, \ldots, x_n, z, p_1, \ldots, p_n), Y_{ij} = D_{kr} u \nabla_i D_{kr} u \nabla_j D_r u, D$ is
the gradient operator and \( \nabla = (\sigma, 1, D) \) with \( \sigma = (\sigma_1, \ldots, \sigma_n) \) defined by \( \sigma f_j = \delta^{ij} \) for any vector field \( f = (f_1, \ldots, f_n) \). It is obvious that \( w \) cannot achieve its maximum on \( \partial B \) and if \( x_0 \) is a maximum point then at \( x_0 \) we have \( \eta w' \geq M_2 / 2 \). For our convenience we use the notation \( \mathcal{E} = a^{ij} D_{i} u \ D_{j} u \), \( \mathcal{C} = a^{ij} D_{ir} u \ D_{jr} u \). Then by the assumptions of the theorem we have the following estimates

\[
\begin{align*}
(5.8) \quad F w' &\leq C [(M_2 + M_2^{1/2} / R + 1 / R^2) \mathcal{C} + M_2 \mathcal{C}],  \\
&\leq C \left( M_2 \mathcal{C} + \lambda M_2 + M_2 \mathcal{C} + \frac{1}{R} M_2^{1/2} \mathcal{C} \right),  \\
&\eta (D_{p} a^{ij} D_{ijk} u \ D_{lm} u + D_{q} a^{ij} D_{ijk} u \ D_{lm} u + D_{m} a^{ij} D_{ijk} u) \ D_{kl} u  \\
&\leq \lambda |D^2 D'u|^2 + C (M_2 \mathcal{C} + M_2 \mathcal{C},
\end{align*}
\]

and

\[
(5.9) \quad \frac{M_2}{\theta^2} \left( D_{i} u \ D_{a} a^{ij} D_{ij} u + D_{i} u \ D_{j} b + (D_{a} a^{ij} D_{ij} u + D_{b} b)v  \\
+ \frac{2}{\eta} a^{ij} D_{i} \eta D_{ir} u \ D_{r} u - u B^i D_{i} u + u b)  \\
\leq C \left( \frac{1}{\theta^2} + \frac{1}{\theta R M_2^{1/2}} \right) M_2 \mathcal{C} + C \left( \frac{1}{\theta^2} + \frac{1}{\theta R M_2^{1/2}} \right) M_2 \mathcal{C}.
\]

The constant \( C \) depends only on \( n, \mu, \mu_1, \mu_2, M_0, M_1 \). It then follows from (5.8), (5.9) that

\[
(5.10) \quad a^{ij} D_{ij} w + B^i D_{i} w  \\
\geq \frac{1}{2} M_2 \mathcal{C} \left[ \frac{1}{\theta^2} - C_1 \left( 1 + \frac{1}{RM_2^{1/2}} + \frac{1}{R M_2} + \frac{1}{\theta} + \frac{1}{\theta M_2^{1/2}} \right) \right]  \\
+ \frac{1}{2} M_2 \mathcal{C} \left[ \frac{1}{\theta^2} - C_1 \left( 1 + \frac{1}{R M_2^{1/2}} + \frac{1}{\theta} + \frac{1}{\theta M_2^{1/2}} \right) \right]
\]
Now we can see that if $\theta_0$ is sufficiently small and if $M_2^{1/2} > \theta C_1 / 2 R$ we will have

$$a^{ij} D_{ij} w(x_0) + B^i D_i w(x_0) > 0.$$ 

This contradiction shows that the lemma is true.

**Theorem 5.5.** Let $u \in C^4 (B^+ \cup B^0)$ be a solution of the problem (5.1), (5.2). Suppose that the conditions $(A) \sim (A4)'$, $(B) \sim (B4)'$ are satisfied. Suppose also that $M_0 = \sup_{B_R} u$, $M_1 = \sup_{B_R} D_i u$ are bounded uniformly in $R < 1$, and the estimates

$$\text{osc}_{B_R} u, \text{osc}_{B_R} D^i u \leq K R^\alpha$$

hold for some constants $K$ and $\alpha \in (0, 1)$. Then there exist constants $R, C$ depending only on $\tau, \mu, \mu_1, \mu_2, \chi$ and $K, \alpha$, such that

$$\sup_{B_{R/2}} \sum_{i,j=1}^{n-1} |D_{ij} u| \leq C.$$ 

**Proof.** The argument here is quite similar to that in estimating the tangential gradient. Let $R \in (0, 1)$ be a constant in our disposal and set

$$\tilde{w} = |D^2 u|^2 = \sum_{i,j=1}^{n+1} |D_{ij} u|^2, \quad v_1 = \sqrt{1 + \sum_{i+j<n} |D_{ij} u|^2}, \quad v_2 = \sqrt{1 + |D u|^2},$$

$$\eta(x) = (1 - \frac{|x|^2}{R^2}), \quad M_2 = \sup_{B_R} \eta^4 \tilde{w}.$$ 

With $\alpha_1 > 4n, \alpha_2 > 0$ constants to be chosen, we set

$$u^* = e^{\alpha_1 (u - M_0)} + \sum_{r=1}^{n-1} e^{\alpha_1 (D_r u - M_{1,r})}$$

With the help of Theorem 5.5, we have

$$\text{osc}_{B_R} u^* \leq K R^\alpha$$

and

$$\sup_{B_{R/2}} \sum_{i,j=1}^{n-1} |D_{ij} u^*| \leq C.$$
\[ w = \eta^i \tilde{w} + \frac{M_i}{\alpha_1} u^* + \alpha_2 M_2 x_n (v_1 + v_2). \]

By the assumption (5.11) we can determine \( R \) from \( \alpha_1, \alpha_2 \) such that \( \alpha_1 (M_0 - u) \leq 1, \alpha_2 (M_1, r - D_r u) \leq 1, \) for \( r = 1, \ldots, n-1, \) \( \alpha_2 x_n v_1, \alpha_2 x_n v_2 \leq 1 / 8. \) If \( M_2 \leq R - 4, \) then we are done once we have determined \( \alpha_1 \) and \( \alpha_2. \) Now we want to determine suitable \( \alpha_1 \) and \( \alpha_2 \) such that \( M_2 \leq R - 4. \) Suppose that \( M_2 > R - 4 \) and let \( x_0 \) be a point in \( \bar{B}_r^+ \) where \( w \) attains its maximum. One can see easily from the definition of \( w \) that \( x_0 \not\in \partial B_r^+ \setminus B_r^0. \) Hence at \( x_0 \) we have \( \eta (x_0) \neq 0 \) and

\[(5.16) \quad \eta^i \tilde{w} (x_0) \geq M_2 / 2.\]

If \( x_0 \in B_r^0, \) by tangentially differentiating the equation (5.2) twice we get

\[(5.17) \quad \alpha^{st} D_{st} w + \beta^i D_i w \]
\[= 2 \eta^i \alpha^{st} D_{st} u D_{st} u + \beta^i \alpha_2 M_2 (v_1 + v_2) \]
\[+ M_2 \alpha_1 e^{\alpha_1 (u - M_0)} \alpha^{st} D_{st} u D_{st} u + M_2 \alpha_1 e^{\alpha_1 (D_u u - M_1, r)} \alpha^{st} D_{st} u D_{st} u \]
\[+ (\alpha^{st} D_{st} \eta^4 + D_{st} \beta^i D_i \eta^4 - 2 \eta^4 D_i \beta^i - \frac{2}{\eta^4} \alpha^{st} D_s \eta^4 D_s \eta^4) \tilde{w} \]
\[- 2 \eta^4 \left( \frac{\partial^2 \alpha^{st}}{\partial x_i \partial x_j} D_{st} u + \frac{\partial^2 \beta^i}{\partial x_i \partial x_j} \right) \tilde{w} \]
\[+ M_2 e^{\alpha_1 (u - M_0)} (\beta^i D_i u - \beta) - M_2 e^{\alpha_1 (D_u u - M_1, r)} (D_i \alpha^{st} D_{st} u + D_i \beta + D_i \beta D_i u), \]

where \( \tilde{w} = \frac{\partial^2 \alpha^{st}}{\partial x_i \partial x_j} D_{st} u + \frac{\partial^2 \beta^i}{\partial x_i \partial x_j} \tilde{w} \leq C M_2 (v_1 + v_2), \)

By the hypothesis of the theorem we can estimate the terms in (5.17) as follows.

\[(\alpha^{st} D_{st} \eta^4 + \beta^i D_i \eta^4 - 2 \eta^4 D_i \beta^i - \frac{2}{\eta^4} \alpha^{st} D_s \eta^4 D_s \eta^4) \tilde{w} \]
\[- 2 \eta^4 \left( \frac{\partial^2 \alpha^{st}}{\partial x_i \partial x_j} D_{st} u + \frac{\partial^2 \beta^i}{\partial x_i \partial x_j} \right) \tilde{w} \leq C M_2 (v_1 + v_2). \]
\[ \eta^4 D_1 \alpha^{st} D_{s(t)u} D_{(t)u} \leq \eta^4 \alpha^{st} D_{s(t)u} D_{(t)u} + \eta^4 \tilde{w}, \]

\[ M_2 \epsilon^{(u-M_0)}_i (\beta D_i u - \beta), \quad M_2 \epsilon^{(D_+M_0)}_i (D_i \alpha^{st} D_{s(t)u} + D_i \beta + D_{i2} \beta D_i u) \leq C M_2 (v_1 + v_2) + M_2 \alpha^{st} D_{s(t)u} D_{(t)u}. \]

The second inequality above follows from Lemma 2.7, and the third one follows from (5.16). We insert these inequalities into (5.17) and we see

\[ \alpha^{st} D_{s(t)w} - \beta^i D_i w \geq \chi M_2 (v_1 + v_2) (\alpha_2 - C / \chi). \]

Now we choose \( \alpha_2 > C / \chi \) so that we have a contradiction which means that \( x_0 \) cannot be in \( B \).

If \( x_0 \in B^+ \), by differentiating the equation (5.1) twice and with the notation

\[ \bar{w} = \sum_{i+j \leq n} |D_{ij} w|^2, \quad B^i = (D_{pi} a^{kl} D_{klu} + D_{pi} b - \frac{8}{\eta} a^{ij} D_{j l} \eta), \]

\( \mathcal{E}, \mathcal{C}, g^{kl}, Y_{ij}, \bar{Y}_{ij} \) as before and \( \delta = a^{ij} D_{i(t)u} D_{j(t)u}, \)

\[ G^{kl, hm} = \left\{ \frac{\delta^{kl, hm} - D_{kl} u D_{hm} u}{v_2} \right\} \geq 0, \]

\[ T^{ij} = \frac{\partial^2 a^{kl}_{ij}}{\partial X_i \partial X_j} D_{kl} u + \frac{\partial^2 b}{\partial X_i \partial X_j}, \]

\[ F_k = D_x a^{ij} D_{ij} u D_{klu} + D_k a^{ij} D_{ij} u + D_x b D_{klu} + D_k b + \frac{8}{\eta} a^{ij} D_{i} \eta D_{j} \eta, \]

we get

\[ (5.18) \quad a^{ij} D_{ij} w + B^i D_i w \]
\[ = 2\eta^4 \delta + \alpha_2 M_2 \frac{x_n}{v_1} a^{ij} G^{kl, hm} D_{ikl} u D_{jhm} u + \alpha_2 M_2 \frac{x_n}{v_2} a^{ij} g^{kl} D_{ik} u D_{j} u \]
Before we start to estimate the terms on the right hand side of (5.18) we need to do some preparation in order to handle the higher order normal derivatives. By differentiating the equation (5.1) with respect to $x_s$, $s \neq n$ and $x_n$ respectively we may obtain

\begin{align*}
\text{(S.19)} & \quad \frac{1}{\eta} \mathcal{D} u - \mathcal{D} \mathcal{u} \mathcal{D} u + \mathcal{D} \mathcal{u} \mathcal{D} u - \mathcal{D} \mathcal{u} \mathcal{D} u - \mathcal{D} \mathcal{u} \mathcal{D} u
\end{align*}

By the assumptions of the theorem and the facts above we can have the estimates of the quantities in (5.18) as follows.

\begin{align*}
\text{(S.21)} & \quad (\alpha_1 (u - M_0) + \alpha_2 M_2 e^{-k} D_{i j} u D_{j i} u + \alpha_1 (u - M_0) + \alpha_2 M_2 e^{-k} D_{i j} u D_{j i} u)
\end{align*}

and

\begin{align*}
\text{(S.20)} & \quad (D_{i j} u D_{j i} u + C \mathcal{u} \mathcal{D} \mathcal{u}^2) \leq C \mathcal{w} + C \mathcal{D}^2 u + C \mathcal{w} + C \mathcal{D} u.
\end{align*}

By the assumptions of the theorem and the facts above we can have the estimates of the quantities in (5.18) as follows.

\begin{align*}
\text{(5.19)} & \quad |Du| \leq n^2 \mu |D^2 u| + C (\mathcal{w} + |D u|^2)
\end{align*}

and

\begin{align*}
\text{(5.20)} & \quad |D_{i j} u| \leq C |D^2 u| + C (\mathcal{w} + |D u|^2).
\end{align*}
and
\[ \eta^4 D_{\sigma u} (D_{p_{\ell}} a_{ij} D_{i\sigma} u D_{j\sigma} u + D_{z_{\alpha}} a_{ij} D_{i\sigma} u D_{j\sigma} u + D_{z_{\alpha}} a_{ij} D_{i\sigma} u) \leq \eta^4 \delta + C M_2 (\tau + \varepsilon). \]

By the same consideration above and more precise estimation we get

\[ (5.22) \quad \alpha_2 M_2 \frac{x_n}{v_1} [T^{ij} Y_{ij} - (D_{z} a_{ij} D_{i\sigma} u + D_{z} b) \bar{w}] + \alpha_2 M_2 B^a v_1 \leq \alpha_2 M_2 C (\tau + \varepsilon). \]

Using Lemma 5.4 and (5.19), (5.20) we can have

\[
\alpha_2 M_2 \frac{x_n}{v_1} (2 D_{p_{\ell}} a_{ij} D_{i\sigma} u D_{j\sigma} u + D_{z_{\alpha}} a_{ij} D_{i\sigma} u D_{j\sigma} u + D_{z_{\alpha}} a_{ij} D_{i\sigma} u D_{j\sigma} u )
+ 2 D_{z} a_{ij} D_{i\sigma} u D_{j\sigma} u - \frac{8}{\eta} a_{ij} D_{j} D_{i} D_{ikl} D_{kl} u)
\leq \eta^2 \delta + \alpha_2^2 C M_2 (\tau + \varepsilon).
\]

It is easy to check that the rest terms in (5.18) is less than \( \alpha_2 M_2 C (\tau + \varepsilon + 1) \).

Substituting all the estimates above into (5.18) we conclude

\[ a_{ij} D_{ij} w + B^i D_{i} w \geq e^{-1} M_2 [\tau (\alpha_1 - \alpha_2^2 C_1) + \varepsilon (\alpha_1 - \alpha_2^2 C_2)] > 0, \]

provided \( \alpha_1 \geq \max \{ \alpha_2^2 C_1, \alpha_2^2 C_2 \} \), which is another contradiction. Hence the theorem is proved.

**Corollary 5.6.** *Under the assumptions of above theorem we have*

\[ (5.23) \quad \sup_{B_R} |D_{n} u| \leq C. \]

**Proof.** We write the equation (5.2) into the form
\[
\alpha^{st} D_{st} u + \frac{1}{\alpha} \int_D \beta(x, u, D' u, t \cdot D_n u) \, dt \cdot D_n u = -\beta(x, u, D' u, 0).
\]

By the obliqueness of the boundary condition we obtain

\[
|D_n u| \leq \frac{1}{\lambda} (|\mu| D^2 u| + |\beta(x, u, D' u)|).
\]

Thus (5.23) follows.

A combination of Corollary 5.6 and the interior estimate Lemma 3.1 of [17] gives us immediately the global gradient estimate.

**Lemma 5.7.** Let \( u \in C^4(B^+ \cup B^0) \) be a solution of (5.1), (5.2), Suppose the structure conditions \( (A) \sim (A4)', (B) \sim (B4)' \) are satisfied and \( |u|_0 \leq M_0 \) for some constant \( M_0 \). Then there exist constants \( \alpha \in (0, 1) \) and \( C \) such that

\[
(5.24) \quad \text{osc}_{B_R} D u, \quad \text{osc}_{B_R} D^2 u \leq C,
\]

provided \( 0 \) is either a type I degenerate point or type II.

The idea of the proof is the same as before but it needs long calculation. We only give the test function here and omit the detailed proof.

The operators we are going to use are

\[
L \equiv a_{ij} D_{ij}, \quad \text{in } B^+, \quad \text{and } B \equiv \alpha^{st} D_{st} + D_{p1} \beta D_{1}, \quad \text{on } B^0.
\]

The test function is

\[
w_\pm = \frac{1}{\delta} e^{-\delta w_\pm}
\]

where
\[ w^\pm = w^\pm = \pm D^u + \epsilon v^t + A_1 x_n v_1 + A_2 |D'u|^2, \]
\[ v^t = \sum_{s,t=1}^{n-1} |D^u|^2, \quad v_1 = \sqrt{1 + \overline{w}}, \quad \overline{w} = \sum_{i+j \leq 2n} |D^u|^2, \]

and \( A_1, A_2, \delta \) are constants to be fixed. For the detailed proof we refer to that of Lemma 4.5.

3. \( C^{2,\alpha} \) Estimates and Existence.

We have seen in previous sections that we cannot prove the Hölder estimate around the degenerate points of type II. Our plan to achieve the global \( C^{2,\alpha} \) estimate is to obtain the \( C^{2,\alpha} \) bound on the boundary, then apply the classical result on Dirichlet problems. In order to obtain the boundary \( C^{2,\alpha} \) estimate we have to establish the third order tangential derivative estimates. To this end we assume further

\[ (A5) \quad |D^3 \alpha|, \quad |D^3 \beta| \leq \lambda \mu_1(1|z|), \]
\[ (B5) \quad |D^3 \alpha|, \quad |D^3 \beta| \leq \chi \mu_2(1|z|), \]

where \( D \) means the partial derivatives taken with respect to all variables.

Our procedure is to prove first an auxiliary result on the behavior of third order derivatives near the boundary and then to prove our main theorem. For our convenience we will use the notation \( F(x, z, p, r) = \hat{a}^{ij}(x, z, p) r_{ij} + b(x, z, p), \) and \( \partial_i = D_i + D_i u D_z + D_{im} u D_{pm}. \)

**Lemma 5.8.** Let \( u \in C^5 \) be a solution of (5.1), (5.2). Suppose the conditions (A) \~ (A5) are satisfied. For such a point \( \overline{x} = (\bar{x}', \bar{x}_n) \in B^+ \) that \( B = B_R(\overline{x}) \subset B^+ \) with \( R = \bar{x}_n \) we denote
\[ \theta = \max \{ \text{osc}_B D^2 u, \text{osc}_B D'u, \text{osc}_B u \}. \]

Then there are constants \( \theta_0, C \) depending on \( \mu, \mu_1, n, M_0 = |u|_0, M_1 = |D'u|_0, M_2 \)

\( = |D^2 u|_0 \) such that if \( \theta \leq \theta_0 \), then we have

\[ |D_{s,t} u(\bar{x})| \leq C \frac{\theta}{R} \quad \text{for each } i, s, t \neq n. \]

**Proof.** Without loss of generality we assume that \( \bar{x}' = 0 \), and \( u(0) = |D'u(0)| = |D^2 u(0)| = 0. \) Let

\[ \bar{w} = \sum_{s,t=1}^n \sum_{k=1}^n |D_{k,s,t} u|^2, \quad v_1 = |D^2 u|, \quad v_2 = |D'u|, \quad \eta = \left( 1 - \frac{|x - \bar{x}|^2}{R^2} \right), \]

and

\[ w = \eta^2 \bar{w} + \frac{M_3}{8 \theta^2} (v_1 + v_2) + \frac{1}{2 \theta^2} M_3^{1/2} M_2^{1/2} v_2, \]

where \( M_3 = \sup_{B_R(\bar{x})} \eta^2 \bar{w}, \quad M_2 = \sup_{B_R(\bar{x})} \eta^2 |D'Du| \). By Lemma 5.4 we see that \( M_2^{1/2} \leq \frac{C\theta}{R}. \)

By differentiating (5.1) and denoting

\[ B^i = (D_{p1} F - 4 a_{ij} D_i \eta / \eta) \]

we have

\[ a_{ij} D_{ij} w + B^i D_i w \]

\[ = 2\eta^2 a_{ij} D_{ikst} u D_{kst} u + \frac{M_3}{4 \theta^2} a_{ij} D_{ist} u D_{jst} u + \frac{M_3^{1/2}}{2 \theta^2} (M_3^{1/2} + M_2^{1/2}) a_{ij} D_s u D_t u \]

\[ + (a_{ij} D_i \eta^2 - 2 a_{ij} \frac{D_j \eta^2 D_i \eta^2}{\eta^2} + D_p F D_i \eta^2) \bar{w} \]

\[ + 2 \eta^2 D_{kst} u \left[ 2 \partial_i a_{ij} D_{ijks} u + \partial_k a_{ij} D_{ijst} u + \partial_j a_{ij} D_{ijk} u + \partial_i \partial_k a_{ij} D_{ij} u \right. \]

\[ + \partial_i \left( \frac{\partial^2 F}{\partial X_i \partial X_j} \right) Y_{ij}^{ks} + \frac{\partial^2 F}{\partial X_i \partial X_j} D_i Y_{ij}^{ks} + \partial_i D_p F D_{iks} u + \partial_i D_z F D_{ks} u + D_z F D_{kst} u \right] \]
where $Y_{ij}^{kl} = \nabla_i D_k u \nabla_j D_l u$. If $x_0 \in B$ is such a point that $w$ attains its maximum there, then by a similar consideration as before we may obtain such an estimate

\begin{equation}
A^i D_i w + B^i D_i w \\
\geq \frac{M_3 \delta}{4\theta} \left( \frac{1}{\theta} - C_1 \right) + \frac{M_2 \tilde{C}}{2\theta} \left( \frac{1}{\theta} - C_3 \right) + \frac{M_3^{1/2} M_2^{1/2}}{2\theta} \tilde{C} \left( \frac{1}{\theta} - C_4 \right),
\end{equation}

which will be positive if $\theta_0 \leq \min \left\{ \frac{1}{C_3}, \frac{1}{C_4}, \frac{1}{2C_1} \right\}$ and $M_3^{1/2} > \frac{2C_2\theta}{R}$. This contradiction means that (5.25) is true.

**Theorem 5.9.** Let $u \in C^5$ be a solution of (5.1), (5.2). Under the hypothesis (A) \sim (A5), (B) \sim (B5) we have

\begin{equation}
\sup_{B_R} |D^3 u| \leq C,
\end{equation}

if $R \leq R_0$, $R_0$, $C$ are constants depending on $\mu, \mu_1, \mu_2, n, M_0 = |u|_0$.

The proof of this theorem is done in exactly the same way as that of theorem 5.5, but the test function we employ here is

\[ w = \eta^4 \tilde{w} + \frac{M_3}{\alpha_1^{-2}} u^* + \alpha_2 M_3 x_n (v_1 + v_2) \]

where $\eta$ is as in theorem 5.5.
\[ \hat{w} = |D^3 u|^2, \quad M_3 = \sup_{B_R} \eta^4 \hat{w}, \quad u^* = \sum_{r=1}^{n-1} \alpha_r (D_r u - M_1) + \sum_{s,t=1}^{n-1} \alpha_{1,s} (D_{u_{1,s}} - M_{2}) \]

with \( M_{1,r} = \sup_{B_R} D_r u, \quad M_{2,s,t} = \sup_{B_R} D_{s,t} u \), and

\[ v_1 = \sqrt{1 + \sum_{i+j<n} |D_{ij} u|^2}, \quad v_2 = \sqrt{1 + \sum_{s,t \neq n} |D_{s,t} u|^2}, \]

\( \alpha_1, \alpha_2 \) are positive constants to be chosen. Since the calculation is too long and all the techniques to handle the higher order terms have been seen in the previous proof, we are not going to write down the detailed proof here.

Before we state our existence result we need a regularity theorem.

**Theorem 5.10.** Let \( \Omega \) be a \( C^{k+2,\alpha} \) domain and \( u \in C^2 \) be a solution of the linear problem (1.1), (1.2). Suppose \( L \) and \( B \) defined in (1.1), (1.2) are uniformly elliptic and \( B \) is also degenerate oblique, furthermore the coefficients of \( L \) and \( B \) and \( f, g \) belong to \( C^{k,\alpha} \). Then \( u \in C^{k+2,\alpha} \) for any integer \( k \).

This theorem is a straightforward consequence of the classical regularity theorem, see Theorem 6.19 of [6].

**Theorem 5.11.** Assume that \( \partial \Omega \in C^5 \), \( a^{ij}, b, \alpha^{ij}, \beta \) satisfy the structure conditions (A) \( - (A5) \), (B) \( - (B5) \), and the conditions i) \( - iii \) in Theorem 4.12, and \( D_{x} b, D_{x} \beta \leq c_0 \) for some constant \( c_0 \). Then the problem (1.5), (1.6) has a unique solution \( u \in C^{2,\alpha} (\overline{\Omega}) \) provided the degenerate points of (1.6) are of type I or type II.

**Proof.** This theorem can be proved easily by elliptic regularization. We consider the family of problems

\[ a^{ij} (x, u, Du) D_{ij} u + b (x, u, Du) = 0, \quad \text{in} \ \Omega, \]

\[ \varepsilon \Delta_{\partial \Omega} u + \alpha^{ij} (x) \delta_{ij} u + \beta (x, u, Du) = 0, \quad \text{on} \ \partial \Omega, \quad \varepsilon \in [0, 1]. \]

Theorem 4.11 tells us that for each \( \varepsilon > 0 \) there is a unique \( C^{2,\alpha} \) solution \( u_\varepsilon \) of (5.29),
and the regularity result Theorem 5.10 says that \( u_\varepsilon \) is actually in \( C^5 \). Therefore we can apply our results in previous sections to obtain an \( \varepsilon \)-independent global \( C^{2,\alpha} \) estimates for \( u_\varepsilon \). Passing through a subsequence when \( \varepsilon \to 0 \), we finally get a limit function \( u \in C^2(\Omega) \) which satisfies the problem (1.5) and (1.6).

The uniqueness is also due to Theorem 4.12.

As an example of the application of our general theory, let us look at the boundary value problem

\[(5.30)\]

\[
\begin{align*}
D_2 u - D_{11} u &= g(x_1, x_3, u), \quad \text{when } x_2 = 1, \\
\Delta u &= f(x_1, x_2, x_3, u, Du), \quad \text{when } 0 < x_2 < 1, \\
u &= 0, \quad \text{when } x_2 = 0,
\end{align*}
\]

which arises in three-dimensional water wave theory and has been studied in [10]. We obtain the same existence result under weaker assumptions by a different method.

**Theorem 5.12.** Suppose \( f(x, z, p), g(x, z) \) are in \( C^3(V) \), \( V = \{ x \in \mathbb{R}^3 \mid 0 \leq x_2 \leq 1 \} \), and \( f, g \) are \( 2\pi \) periodic in \( x_1, x_3 \) variables. Suppose also

(i) \( |f|, |f_x|, |f_{x_3}|, (1 + |p|) |f_p| \leq \mu_1(|z|)(1 + |p|^2) \),

(ii) \( |f_{xx}|, |f_{x_3x}|, |f_{x_3z}|, (1 + |p|) (|f_{xp}| + |f_{zp}|), (1 + |p|^2) |f_{pp}| \leq \mu_2(|z|)(1 + |p|^2) \),

(iii) there exist constants \( \alpha < 0 < \beta \) such that \( g(x_1, x_3, \alpha) \geq 0 \), \( f(x_1, x_2, x_3, \alpha, 0) < 0 \); \( g(x_1, x_3, \beta) \leq 0 \), \( f(x_1, x_2, x_3, \beta, 0) > 0 \).

where (i), (ii) hold for \( |p| > M, \mu_1, \mu_2 \) are nondecreasing functions. Then the problem (5.30) has a \( C^2 \) solution \( u \) which is \( 2\pi \) periodic in \( x_1, x_2 \) and \( \alpha \leq u \leq \beta \).

**Proof.** The only difference between our proof and that of [10] is the a priori estimate. The basic idea is the same. We describe the procedure as follows.

(1) By elementary Fourier analysis we have a unique \( 2\pi \) periodic solution \( u \) of
\[(5.31)\]  
\[D_2u - (D_{11}u + D_{33}u) = g, \quad \text{when } x_2 = 1,\]
\[\Delta u = f, \quad \text{when } 0 < x_2 < 1,\]
\[u = 0, \quad \text{when } x_2 = 0.\]

if \(g, f\) are \(2\pi\) periodic in \(x_1, x_2\).

(2) By the method of continuity and the Schauder estimates in Chapter 2 we obtain the solvability of \(2\pi\) periodic solution of

\[(5.32)\]  
\[D_2u - (\alpha^{11} D_{11}u + 2 \alpha^{13} D_{13}u + \alpha^{33} D_{33}u) = g, \quad \text{when } x_2 = 1,\]
\[\Delta u = f, \quad \text{when } 0 < x_2 < 1,\]
\[u = 0, \quad \text{when } x_2 = 0.\]

if \(g, f, \alpha^{st}\) are \(2\pi\) periodic and \(\{\alpha^{st}\}\) is uniformly elliptic. Furthermore we have also the \(C^{2,\alpha}\) regularity of \(u\) from Theorem 2.6 if \(g, f, \alpha^{st} \in C^\alpha\).

(3) Leray - Schauder theory implies that the problem

\[(5.33)\]  
\[D_2u - D_{11}u - \varepsilon (D_{11}u + D_{33}u) = g(x_1, x_3, u), \quad \text{when } x_2 = 1,\]
\[\Delta u = f(x_1, x_2, x_3, u, Du), \quad \text{when } 0 < x_2 < 1,\]
\[u = 0, \quad \text{when } x_2 = 0,\]

is \(C^2, 2\pi\) periodic solvable, and also by Theorem 5.10 the solution is in fact \(C^5\) under the assumptions of the theorem.

(4) Finally applying our estimates developed in this chapter we can get an \(\varepsilon\)-independent \(C^{2,\alpha}\) bound for the solutions \(u_\varepsilon\) of (5.33) for \(\varepsilon \in [0, 1]\). Our estimates are applicable because the solutions of (5.33) are \(2\pi\) periodic in \(x_1, x_2\) so that our local estimates are sufficient though the domain is unbounded in these two directions. Passing through a subsequence when \(\varepsilon \to 0\) we obtain a solution of (5.30).

The mini-maximum bound \(\alpha \leq u \leq \beta\) is guaranteed by condition (ii).
Remark. The restrictions $|f| \leq \mu (1 + |Du|^{2+\delta})$, $0 < \delta \leq 2$, and $g \in C^4$ in [10] have been relaxed by using our estimates.

4. Existence of viscosity solutions

The conditions (A4), (B4) are used to derive higher order derivative estimates. Although (A4) is satisfied by a class of operators, for example, the mean curvature operator, it is still too strong to be applicable to the existence results for more general equations. One way to eliminate these restrictions is to consider the solutions in some weak sense, because the existence of weak solutions usually depends on much weaker estimates. The weak solution in the $W^{k,p}$ sense is widely used in various problems, but it needs the divergence theorem of integration to change the problems into appropriate weak forms. For Venttsel boundary value problems there are some difficulties in integration by parts, so that Sobolev spaces may not be suitable classes of weak solutions of such problems. Recently a notion of solutions in viscosity sense was introduced by Crandall and Lions [4] in the context of first order equations. After that many authors treated the existence and uniqueness of viscosity solutions of fully nonlinear second order elliptic equations, for example Lions [18], Jensen [20], Ishii [8] and Trudinger [24]. Also the regularity of viscosity solutions was studied by Trudinger in [25].

In this section we study the existence of viscosity solutions of our problem (1.5), (1.6). There are many ways to prove the existence of viscosity solutions, as mentioned in [25]. Our approach is the elliptic regularization. An existence result is proved without not only the conditions (A3), (A4), (B3), (B4) but also the restriction of $z$ and $p$ independence of the functions $\alpha^{ij}$, so the result is much more general than that in Section 3. The conditions (A2) and (B2) are used only for the existence of solutions of the approximating equations in the elliptic regularization, so they may be relaxed also if we use some different methods, such as Perron's method.
Definition 1. A function $u \in C(\Omega)$ is said to be a viscosity sub solution (supersolution) of (1.5), (1.6) if $\phi \in C^2(\Omega)$ is an arbitrary function and $u - \phi$ attains its maximum (minimum) at $x_0 \in \Omega$, then

$$a^{ij}(x_0, u(x_0), D\phi(x_0)) D_{ij}\phi(x_0) + b(x_0, u(x_0), D\phi(x_0)) \geq (\leq) 0,$$

when $x_0 \in \Omega$, and

$$\alpha^{ij}(x_0) \delta_{ij}\phi(x_0) + \beta(x_0, u(x_0), D\phi(x_0)) \geq (\leq) 0,$$

when $x_0 \in \partial\Omega$.

Definition 2. A function $u \in C(\Omega)$ is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Theorem 5.13. Let $\Omega$ be a $C^3$ domain. Suppose $a^{ij}, b, \alpha^{ij}, \beta$ satisfy the structure conditions (A), (A1), (A2), (B), (B1), (B2). Suppose all the degenerate points are of either type I or type II. Suppose also that there is a positive constant $c_0$ such that $D_2^b, D_2^\beta \leq -c_0$. Then there exists a viscosity solution of (1.5), (1.6).

Before we prove this theorem we introduce another version of the definition of viscosity sub and supersolutions equivalent to Definition 1 when the interior operator is uniformly elliptic.

Definition 1'. A function $u \in C(\Omega)$ is said to be a viscosity sub solution (supersolution) of (1.5), (1.6) if $\phi \in C^2(\Omega)$ is an arbitrary function and $u - \phi$ attains its maximum (minimum) at $x_0 \in \Omega$, then

$$a^{ij}(x_0, u(x_0), D\phi(x_0)) D_{ij}\phi(x_0) + b(x_0, u(x_0), D\phi(x_0)) \geq (\leq) 0,$$

when $x_0 \in \Omega$, and
To see the equivalence of these two definitions, we consider only the subsolution case, the supersolution case follows by the same reasoning. It is trivial that Definition 1 implies Definition 1'. On the other hand, if $u$ is a subsolution in the sense of Definition 1', we want to show that (5.34) holds. Suppose the contrary, that is, that there is a $\phi \in C^2(\bar{\Omega})$ such that $u - \phi$ attains its maximum and

$$\alpha^{ij}(x_0) \delta_{ij} \phi(x_0) + \beta(x_0, u(x_0), D\phi(x_0)) < 0.$$  \hspace{1cm} (5.37)$$

We define $\psi = x_n - \frac{1}{2\delta} x_n^2$ which is positive when $x_n \leq 2\delta$. It is obvious that $0 \leq D_n \psi = 1 - \frac{x_n}{\delta} \leq 1$ when $x_n \leq \delta$, and $D_{nn} \psi = -\frac{1}{\delta}$. So the point $x_0$ is also a local maximal of $u - \phi_1$, where $\phi_1 = \phi + \psi$. By the uniform ellipticity of $a^{ij}$ we can see that

$$a^{ij}(x_0, u(x_0), D\phi_1(x_0)) D_{ij} \phi_1(x_0) + b(x_0, u(x_0), D\phi_1(x_0))$$

$$\leq -\frac{\lambda}{\delta} + a^{ij}(x_0, u(x_0), D\phi(x_0)) + 1 - \frac{x_n}{\delta} D_{ij} \phi(x_0)$$

$$+ b(x_0, u(x_0), D\phi(x_0) + 1 - \frac{x_n}{\delta})$$

$$\leq \lambda (-\frac{1}{\delta} + \frac{\lambda}{\lambda} |D^2 \phi| + \mu_1 (|u|) (1 + |D\phi|^2)).$$  \hspace{1cm} (5.38)$$

The last inequality above follows from (A1). The right hand side of (5.38) will be negative if we choose $\delta$ sufficiently small, which contradicts to Definition 1'.

**Proof of Theorem 5.13.** We consider the approximating problems

$$a^{ij} D_{ij} u + b = 0, \quad \text{in } \Omega,$$

$$\varepsilon \Delta_{\partial\Omega} u + \alpha^{ij} \delta_{ij} u + \beta = 0, \quad \text{on } \partial\Omega,$$

$$\varepsilon \Delta_{\partial\Omega} \phi + \alpha^{ij} \delta_{ij} \phi + \beta = 0, \quad \text{on } \partial\Omega,$$  \hspace{1cm} (5.40)$$

 astronauts and engineers who were conducting the experiments.
for $\varepsilon \in (0, 1]$. By Theorem 4.11 there is a solution $u_\varepsilon$ for each $\varepsilon$.

Next we want to show that $\{u_\varepsilon\}$ is uniformly bounded and equicontinuous. The uniform bound follows directly from Theorem 4.10. To see the equicontinuity we need to prove that for each $\varepsilon > 0$, there is a $\delta > 0$ s.t. whenever $x, y \in \overline{\Omega}$ and $|x - y| < \delta$ we have

$$|u(x) - u(y)| < \varepsilon.$$  

We consider first the points in $N_R(x_0) = B_R(x_0) \cap \overline{\Omega}$, where $x_0 \in \partial \Omega$ is a type II degenerate point. If $x, y \in N_R(x_0)$ by Lemma 5.2 we have

$$|u(x) - u(y)| < CR^\alpha,$$  

for some $\alpha \in (0, 1)$ and a constant $C$. To fulfill the requirement (5.41) we need only choose $R < (\varepsilon/C)^{1/\alpha}$, and $\delta = R$. Secondly we look at the region $\Omega' = \overline{\Omega} \setminus (\cup N_{R/2}(x_0))$, where $N_{R/2}(x_0)$ is defined as above with fixed $R$ and the union is taken over all the degenerate points of type II. If $x, y \in \Omega'$, by Theorem 4.2 and Theorem 5.3 we have

$$|u(x) - u(y)| \leq C |x - y|^\alpha,$$  

for some $\alpha \in (0, 1)$ and a constant $C$. Then (5.41) follows easily by taking $\delta = (\varepsilon/C)^{1/\alpha}$. Finally we suppose $x \in \Omega'$, $y \in N_R(x_0)$ for some point $x_0$, and $|x - y| < \delta$. If $\delta < R/2$ then we have either $x \in N_{R/2}(x_0)$ or $y \in \Omega'$ which are the cases we have already discussed before. Hence we have proved the equicontinuity. By Ascoli's theorem we can pass through a subsequence of $\{u_\varepsilon\}$ to get a limit function $u \in C(\overline{\Omega})$.

To proceed further we prove that this $u$ satisfies Definition 1'. Let $\phi \in C^2$ be an arbitrary function and suppose $u - \phi$ attains its maximum at $x_0 \in \Omega$. It is easy to check that (5.35) holds when $x_0 \in \Omega$. Suppose $x_0 \in \partial \Omega$. Without loss of generality we may
assume that \( x_0 \) is a strict maximal point of \( u - \phi \). Since \( \{u_{e_n}\} \), the convergent subsequence, approaches to \( u \) as \( n \) tends to infinity, there is a sequence \( \{x_n\} \) converging to \( x_0 \) such that \( x_n \) is the maximal point of \( u_{e_n} - \phi \) for each \( n \). Let \( \{x_k\} \) be such a subsequence of \( \{x_n\} \) that \( \{x_k\} \) contains all and only the \( x_n \)'s which are on \( \partial \Omega \). If \( \{x_k\} \) is finite we have

\[
(5.44) \quad a^{ij}(x_0, u(x_0), D\phi(x_0)) \, D_{ij}\phi(x_0) + b(x_0, u(x_0), D\phi(x_0)) \geq 0.
\]

Otherwise we have

\[
(5.45) \quad \alpha^{ij}(x_0) \, \delta_{ij}\phi(x_0) + \beta(x_0, u(x_0), D\phi(x_0)) \geq 0.
\]

This completes our proof.

We leave further investigation of existence and uniqueness of viscosity solutions to a future work.
References


