BANACH SPACES OF PSEUDOMEASURES ON COMPACT GROUPS
WITH EMPHASIS ON HOMOGENEOUS SPACES

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Professor A.F. Robertson and Dr W.J. Robertson, acting as my supervisors whilst I was in Western Australia, have also been very helpful to me. I have had several interesting and helpful discussions concerning my work, in particular about the theory, and also to Professors

STATEMENT

Except where specifically stated otherwise, the material presented in this thesis is my own work.

J. A. Ward.

J. Ward

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ABSTRACT

Segal algebras and homogeneous Banach algebras, which are generalizations of the group algebra $L^1(G)$ of a compact, or locally compact abelian group G , have been studied in Reiter [1], [2], [3], Burnham [1], [2], [3]; and Wang [1].

In chapter 3 of this thesis a generalized notion of homogeneous Banach space over a compact group is considered. No multiplication is assumed. If B denotes such a space then an operator $*$ may be defined from $M(G) \times B$ to B so that it corresponds to the convolution product in familiar cases. Then

$$\lim_{n \rightarrow \infty} \|k_n * b - b\|_B = 0$$

for any $b \in B$ and any approximate identity $(k_n)_{n \in \mathbb{N}}$ of $L^1(G)$. In fact this property is characteristic of homogeneous Banach spaces amongst those which are translation invariant. It is the basic tool used to study homogeneous Banach spaces.

Chapters 4 and 5 consider, in detail, the family $\mathcal{B}_h$ of homogeneous Banach spaces of pseudomeasures defined on a compact group. In chapter 4 the elements B of $\mathcal{B}_h$ are characterized by means of norms defined on a family of trigonometric polynomials, and also by certain subspaces of $\underline{E}(\Sigma(G))$ which are identified with B^* . If B is also a subspace of $M(G)$ then it is a homogeneous convolution algebra; these algebras are the subject of chapter 5.

A complete description of the homogeneous subalgebras of $L^1(G)$ is easily given. It is proved that the closed two-sided and closed left ideal theory of a homogeneous convolution Banach algebra B is precisely the same as that of one of the homogeneous subalgebras of $L^1(G)$. Moreover, an explicit representation of the ideals is given. One is also

given for the $\ast$ -representations of any symmetric B - again the picture is similar to that of one of the symmetric homogeneous subalgebras of $L^1(G)$.

The seemingly disjoint second chapter considers a problem concerning weighted subspaces of some pointwise algebras of integrable functions defined on a compact abelian group G . This problem leads to the study of the tensor algebras $C(G)_{F_1} \hat{\otimes} C(G)_{F_2}$, where G is now a compact group and F_1, F_2 are subsets of $\Sigma(G)$. These algebras are homogeneous convolution Banach algebras in B_h .

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

(1.1) Introduction

The purpose of this chapter is to introduce, for the sake of completeness, those concepts of the basic theory of the harmonic analysis of compact groups that will be used in the sequel. I do not claim that my treatment is nearly exhaustive, and I urge the reader to consider my principal references which are Hewitt and Ross [1] and Edwards [1]. As I shall refer to these books many times in the text I will denote them by [HR] and [E] respectively; all other references will be cited in full as Edwards [2], Varopoulos [1], etc.

The last two sections of this chapter are more specialized. The first of these, §(1.5), introduces the projective tensor product of some spaces of continuous functions defined on a compact group. The study of this form of tensor product became an important part of the theory of Harmonic Analysis in 1965. In Varopoulos [2] and [3] the author used the tensor product to obtain a simplified proof of Malliavin's result concerning sets of spectral synthesis (see Malliavin [1]). The formal theory of tensor products is covered in great detail in Grothendieck [1] and Schwartz [1]. For some of its applications to Harmonic Analysis see Varopoulos [1]. The final section, §(1.6), gives some background definitions and preliminary results concerning the operational calculus of Banach algebras. The operational calculus is the study of the map which takes some analytic complex functions into the algebra of locally analytic functions with domain and range in the Banach algebra being studied. See Hille and Phillips [1]. Some applications of this theory to certain homogeneous Banach algebras will be given in §(5.5).

(1.2) Preliminaries and Notation

(1.2.1) Assume G is a group. We denote the group operations multiplicatively. A topology on G is *compatible* with the group structure if the map $(x, y) \rightarrow xy^{-1}$ of $G \times G$ to G is continuous. A group with a compatible topology is a *topological group*. It is often unnecessary to mention the topology explicitly, and so we will speak of a topological group G .

The topological group G is called *locally compact* (compact, Hausdorff) iff G is locally compact (compact, Hausdorff) as a topological space.

Let $\{G_1, \dots, G_n\}$ be a finite set of topological groups. Let $G_1 \times \dots \times G_n$ denote the Cartesian product of these groups. If it is endowed with the natural multiplication and the Cartesian product topology then $G_1 \times \dots \times G_n$ is a topological group (see (6.2) of [HR]). If the G_i 's are all locally compact (compact, Hausdorff) then so too is $G_1 \times \dots \times G_n$ (see (6.4) of [HR]) and if the G_i 's are all abelian so too is their product (see (6.5) of [HR]).

(1.2.2) Let G be a compact group. Then $\mathcal{C}(G)$ denotes the set of continuous complex-valued functions defined on G ; it is a linear space which may be normed by putting

$$\|f\|_{\infty} = \sup_{x \in G} |f(x)| \quad \text{for each } f \in \mathcal{C}(G).$$

The endomorphism $*$ of $\mathcal{C}(G)$ is defined by

$$f^*(x) = \overline{f(x^{-1})} \quad \text{for each } x \in G, \text{ and each } f \in \mathcal{C}(G).$$

Then under the operations of pointwise multiplication and $*$, $\mathcal{C}(G)$ is a Banach $*$ -algebra. We denote by $M(G)$ the space of continuous linear

functionals defined on $C(G)$. The Riesz Representation theorem ((14.10) of [HR]) asserts that $M(G)$ may be identified with the space of regular Borel measures on G ; the value of the linear functional $\mu \in M(G)$ at the point $f \in C(G)$ being given by $\mu(f) = \int f d\mu$. With the norm

$$\|\mu\|_1 = \sup_{\|f\|_\infty \leq 1} |\mu(f)| \quad \text{for each } \mu \in M(G),$$

$M(G)$ becomes a Banach space. The map $*$ defined on $M(G)$ by

$$\mu^*(f) = \overline{\mu(f^*)} \quad \text{for all } f \in C(G) \text{ and } \mu \in M(G)$$

is a continuous endomorphism of $M(G)$.

(1.2.3) If $f \in C(G)$ and $x \in G$, the *left x -translate* ${}_x f$ of f is defined by

$${}_x f(y) = f(x^{-1}y) \quad \text{for all } y \in G.$$

The *right x -translate* f_x is similarly defined and shares the following properties given for left translation. Let $f \in C(G)$ and $x \in G$, then ${}_x f$ is an element of $C(G)$ and $\|{}_x f\|_\infty = \|f\|_\infty$. For each $f \in C(G)$ the map defined by

$$L(x) = {}_x f \quad \text{for each } x \in G$$

is uniformly continuous from G to $C(G)$ (see (20.4) of [HR]).

If $\mu \in M(G)$ and $x \in G$, the left x -translate ${}_x \mu$ of μ is defined by

$${}_x \mu(f) = \mu({}_x^{-1} f) \quad \text{for all } f \in C(G).$$

(We similarly define the right x -translate μ_x of μ). Then ${}_x \mu$ is an element of $M(G)$ and $\|{}_x \mu\|_1 = \|\mu\|_1$. However, we note that there exists $\mu \in M(G)$ (for example $\mu = \delta_1$) such that the map defined by

$$L(v) = \int_G v(x) \mu \quad \text{for all } v \in C(G)$$

is not continuous from G to $M(G)$. As we shall see in §(3.3) this means that the Banach space $M(G)$ is not homogeneous while the Banach space $C(G)$ is.

(1.2.4) There exists a unique regular, normalized, left-invariant positive measure λ_G on G which is called the *Haar measure on G* (see (15.8) of [HR]). If G is compact then λ_G is also right invariant and satisfies

$$\lambda_G(f^*) = \overline{\lambda_G(f)} \quad \text{for all } f \in C(G).$$

Since λ_G is unique we will often omit the subscript G in the text unless there is possible confusion. So the Haar measure will be denoted by λ .

Let $p \in [1, \infty]$. Then $L^p(G)$ will denote the usual L^p spaces with respect to the measure λ . The elements of $L^p(G)$ are equivalence classes of functions (functions which are equal almost everywhere) which are p^{th} power integrable or essentially bounded in the case $p = \infty$. The norm on $L^p(G)$ is defined by

$$\|f\|_p = \left(\int |f|^p d\lambda \right)^{1/p} \quad \text{if } p \in [1, \infty)$$

or

$$\|f\|_\infty = \lambda\text{-ess sup}_{x \in G} |f(x)|.$$

As $\lambda(G) = 1$, we have $L^p(G) \subseteq L^q(G)$ for $p \geq q$. Also $C(G) \subseteq L^\infty(G)$ and $L^1(G) \subseteq M(G)$. Let $g \in L^1(G)$. Then g determines a measure μ by

$$\mu(f) = \int fg d\lambda \quad \text{for each } f \in C(G).$$

(1.2.5) Let v and μ be elements of $M(G)$. Their *convolution*

product $\nu * \mu$ is defined by

$$\nu * \mu(f) = \nu(x \rightarrow \mu_x(f)) \quad \text{for each } f \in \mathcal{C}(G).$$

With convolution as multiplication and involution given by $*$ (see (1.2.3)) $M(G)$ is a Banach $*$ -algebra (see (19.6) of [HR]).

For f and g elements of $L^1(G)$, their convolution product may be expressed by

$$f * g(y) = \int f(x) {}_xg(y) \, d\lambda(x) \quad \text{for each } y \in G.$$

Each of the spaces $L^p(G)$, $1 \leq p < \infty$ and $\mathcal{C}(G)$ is an $M(G)$ convolution module; moreover each is also a convolution Banach $*$ -algebra (with involution defined by $*$ in (1.2.3)).

For $p \in [1, \infty]$ let p' satisfy $\frac{1}{p} + \frac{1}{p'} = 1$ ($p' = 1$ if $p = \infty$).
if $p \neq \infty$ and $p' = \infty$ if $p = 1$.
 Then $L^{p'}(G)$ is the dual space of $L^p(G)$. The duality is given by

$$\langle f, g \rangle = \int f \bar{g} d\lambda \quad \text{for all } f \in L^p(G), g \in L^{p'}(G).$$

(1.2.6) Unless G is finite, the convolution algebras $L^p(G)$, $1 \leq p \leq \infty$, and $\mathcal{C}(G)$ do not possess an identity. (However, in $M(G)$, the measure $\mu = \delta_1$ ($\delta_1(f) = f(1)$ for all $f \in \mathcal{C}(G)$) acts as the identity.) It is often possible to use instead an approximate identity. This, in $L^1(G)$, is a sequence $(k_n)_{n \in \mathbf{N}}$ of integrable functions satisfying

$$(a) \quad \sup_{n \in \mathbf{N}} \|k_n\|_1 < \infty$$

$$(b) \quad \lim_{n \rightarrow \infty} \int k_n(x) \, d\lambda(x) = 1$$

$$(c) \quad \lim_{n \rightarrow \infty} \int |k_n(x)| \mathbb{1}_{G \setminus N}(x) \, d\lambda(x) = 0$$

for every neighbourhood N of 1 in G . (For any subset U of G ,

$\mathbb{1}_U$ is the function defined by $\mathbb{1}_U(x) = 1$ if $x \in U$ and $\mathbb{1}_U(x) = 0$ elsewhere. It is called the *indicator* function of U .)

More will be said about approximate identities in (1.4.4).

All of the results contained in §(1.2) are well known. For a more complete treatment the reader is referred to [HR] or [E].

(1.3) Representation Theory

(1.3.1) Assume H is a complex Hilbert space. Then $B(H)$ denotes the linear space of all bounded operators on H ; it may be normed by putting

$$\|T\| = \sup_{\|h\|_H \leq 1} \|Th\|_H \quad \text{for each } T \in B(H).$$

A representation U of a compact group G on H is a continuous homomorphism of G into the multiplicative group of invertible operators in $B(H)$, normed qua subspace of $B(H)$. If $U(x)$ is a unitary operator for each $x \in G$ then the representation U is called *unitary*. The *dimension* of the representation U is defined to be the dimension of H which may be infinite.

(1.3.2) Two representations of G , U_1 acting on H_1 and U_2 on H_2 , are called *equivalent* iff there exists a linear homeomorphism T of H_1 into H_2 such that

$$T U_1(x) = U_2(x) T \quad \text{for each } x \in G.$$

If U_1 and U_2 are unitary representations then they are called *unitarily equivalent* iff the linear homeomorphism T may be chosen to be an isometry.

The representation U of G is called *irreducible* iff the only

closed linear subspaces of H which are invariant under the action of $U(x)$ for each $x \in G$ are $\{0\}$ and H . Otherwise, U is said to be *reducible*. In this case it is easily seen that there exists a proper subspace H_0 of H for which the orthogonal projection P_{H_0} of H onto H_0 commutes with all $U(x)$ for $x \in G$.

The study of the representation theory of compact groups is simplified by the following results. First, every finite dimensional representation of a compact group is equivalent to a finite dimensional unitary representation (see page 431 of Naimark [1]). Next, every irreducible unitary representation of a compact group is finite dimensional (see (22.13) of [HR]). Finally, any two equivalent finite dimensional irreducible representations of a compact group are unitarily equivalent (see (22.13) of [HR]).

(1.3.3) For a compact group G we denote by $\Sigma(G)$ the set of equivalence classes of irreducible unitary representations of G (see (27.2) of [HR]). Thus each $\sigma \in \Sigma(G)$ is a set of representations of G , and consists of all representations that are equivalent to some fixed representation. We shall frequently write U_σ for a representation in the set σ . For a fixed $\sigma \in \Sigma(G)$, all U_σ in σ operate on Hilbert spaces of the same finite dimension. Let d_σ denote this dimension.

The Gelfand-Raikov theorem (see (22.12) of [HR]) asserts that there exist sufficiently many irreducible representations of a compact group G to separate the points of G .

(1.3.4) Let G be a compact group, let σ be any element of $\Sigma(G)$ and let $U_\sigma \in \sigma$. Assume U_σ operates on the Hilbert space H_σ . Then, for each $\xi, \zeta \in H_\sigma$, the map given by

$$p(x) = \langle \xi, U_\sigma(x) \zeta \rangle \quad \text{for all } x \in G \quad (1)$$

is a continuous complex valued function on G . (Note that this notation differs slightly from that of [HR], but the differences are minor because $\langle \xi, U_\sigma(x) \zeta \rangle = \overline{\langle U_\sigma(x) \zeta, \xi \rangle}$ for all $x \in G$.) It is easily seen that the family of functions having the form (1) depends only on the equivalence class to which U_σ belongs (see (27.6) of [HR]). Therefore for σ in $\Sigma(G)$ we define $T(G)_{\{\sigma\}}$ as follows. For $U_\sigma \in \sigma$ with U_σ operating on H_σ , $T(G)_{\{\sigma\}}$ is the set of all finite complex linear combinations of functions of the form (1). Then $T(G)$ is the smallest linear space containing $\cup\{T(G)_{\{\sigma\}} : \sigma \in \Sigma(G)\}$. Functions in $T(G)$ are called *trigonometric polynomials*. The linear space $T(G)$ is a dense subalgebra of the convolution Banach algebras $C(G)$ and $L^p(G)$, $1 \leq p < \infty$ (see 27.39) and (31.4) of [HR]). We shall see in (3.3.10) that the density of a family of trigonometric polynomials in a Banach space B is closely related to the continuity of the map $x \rightarrow x \cdot b$ for each $b \in B$.

For each finite dimensional representation U of a compact group G , the map χ_U defined by

$$\chi_U(x) = \overline{\text{tr}(U(x))} \quad \text{for each } x \in G$$

is a trigonometric polynomial, called the *character* of U . Two representations of G are equivalent iff they have the same character (see (27.32) of [HR]). So, for each $\sigma \in \Sigma(G)$, χ_σ will denote the character of all the representations U_σ in σ , and χ_σ will be called an *irreducible character*.

Let σ and η be elements of $\Sigma(G)$. Then we have the following *orthogonality relations*

$$\int \langle \xi_1, U_\sigma(x) \zeta_1 \rangle \overline{\langle \xi_2, U_\eta(x) \zeta_2 \rangle} d\lambda(x) = d_\sigma^{-1} \delta_{\sigma\eta} \langle \xi_1, \xi_2 \rangle \langle \zeta_1, \zeta_2 \rangle$$

for all $\xi_1, \zeta_1 \in H_\sigma$ and for all $\xi_2, \zeta_2 \in H_\eta$ (see (27.18) of [HR]).

(here $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise).

In particular,

$$\int \chi_{\sigma}(x) \overline{\chi_{\eta}(x)} d\lambda(x) = \delta_{\sigma\eta}.$$

(1.3.5) Let $\{U_i : i \in I\}$ be a set of representations of the compact group G . For each $i \in I$, U_i acts on the Hilbert space H_i . Then the direct sum $\bigoplus_{i \in I} U_i$ of the representations in this set is defined by

$$\left[\bigoplus_{i \in I} U_i \right](x) = \bigoplus_{i \in I} U_i(x) \quad \text{for each } x \in G.$$

It is a representation of G which operates on the Hilbert space $\bigoplus_{i \in I} H_i$. Every representation of a compact group G may be uniquely decomposed into a direct sum of irreducible representations of G (see (27.44) of [HR]).

Let $\{U_1, \dots, U_n\}$ be a finite set of representations of the compact group G . For each $i \in \{1, \dots, n\}$, U_i acts on H_i . Then

$$\chi_{\bigoplus_{i=1, \dots, n} U_i} = \sum_{i=1}^n \chi_{U_i}.$$

Let $H_1 \dot{\otimes} H_2$ denote the Hilbert space tensor product of H_1 and H_2 (see D15 of [HR]). For each $x \in G$, $U_1(x) \otimes U_2(x)$ is a continuous endomorphism of $H_1 \otimes H_2$; hence there exists a unique continuous linear extension $U_1(x) \dot{\otimes} U_2(x)$ which is an endomorphism of $H_1 \dot{\otimes} H_2$. The tensor product $U_1 \dot{\otimes} U_2$ of the representations U_1 and U_2 is defined by

$$U_1 \dot{\otimes} U_2(x) = U_1(x) \dot{\otimes} U_2(x) \quad \text{for each } x \in G.$$

It is a representation of G which operates on the Hilbert space $H_1 \dot{\otimes} H_2$ (see (27.33) of [HR]). If U_1 and U_2 are finite dimensional then $H_1 \dot{\otimes} H_2 = H_1 \otimes H_2$ and so we will write $U_1 \otimes U_2$ for the tensor

product of U_1 and U_2 . In this case,

$$\chi_{U_1 \otimes U_2} = \chi_{U_1} \cdot \chi_{U_2}.$$

The definition of the tensor product can obviously be extended to any finite family of representations of G .

(1.3.6) Let $\{G_1, \dots, G_n\}$ be a set of compact groups. For each $i \in \{1, \dots, n\}$ let $\sigma_i \in \Sigma(G_i)$. Define $\bigotimes_{1 \leq i \leq n} U_{\sigma_i}$ to be the map

$$\left[\bigotimes_{1 \leq i \leq n} U_{\sigma_i} \right](x) = \bigotimes_{1 \leq i \leq n} U_{\sigma_i}(x_i) \quad \text{for } x = (x_1, \dots, x_n)$$

of $G_1 \times \dots \times G_n$ into $B \left(\bigotimes_{1 \leq i \leq n} H_{\sigma_i} \right)$. Then $\bigotimes_{1 \leq i \leq n} U_{\sigma_i}$ is an irreducible representation of $G_1 \times \dots \times G_n$. Moreover, every irreducible representation of $G_1 \times \dots \times G_n$ is equivalent to a representation of this form.

Hence

$$\Sigma(G_1 \times \dots \times G_n) = \bigotimes_{1 \leq i \leq n} \Sigma(G_i)$$

(see (27.43) of [HR]). Also

$$\chi_{\bigotimes_{1 \leq i \leq n} \sigma_i} = \prod_{i=1}^n \chi_{\sigma_i}.$$

(1.3.7) It is a consequence of Schur's lemma (see (27.9) of [HR]) that every irreducible representation of a compact abelian group G is one dimensional. If $\sigma \in \Sigma(G)$ and $U \in \sigma$ operates on the one dimensional Hilbert space H_σ then

$$U(x) = \overline{\chi(x)} \text{Id}_\sigma \quad \text{for all } x \in G$$

where Id_σ denotes the identity endomorphism of H_σ and χ is a multiplicative character of G . The character of U is precisely χ .

Hence $\Sigma(G)$ may be identified with the multiplicative group of characters of G (denoted by X and called the *dual group*).

If G is not abelian then $\Sigma(G)$ cannot be given the structure of a group (see (27.34) and (27.35) of [HR]).

The Pontryagin duality theorem asserts that if G is an abelian group then it is the dual group of its dual group X .

(1.3.8) Let G be a compact group and σ any element of its dual object $\Sigma(G)$. Then there is a generalization of a classical theorem of Burnside which asserts that the linear span of $\{U_\sigma(x) : x \in G\}$ is all of $B(H_\sigma)$ for each element U_σ of σ which operates on H_σ , and in particular, there are d_σ^2 linearly independent operators among the $U_\sigma(x)$'s (see (38.32) of [HR]).

(1.3.9) We complete this section on representation theory by introducing the notion of a $*$ -representation of a Banach $*$ -algebra B . This theory is well set out in §21 of [HR]; see also the references contained in that section.

A $*$ -representation R of the Banach $*$ -algebra B in the complex Hilbert space H is a continuous homomorphism from B to $B(H)$ that preserves involution. The notions of *equivalence*, *irreducibility* and of *unitary representations* are as in (1.3.1) and (1.3.2).

It is a consequence of (22.3) in [HR] that each representation U of the compact group G which operates on the Hilbert space H induces a $*$ -representation R of a Banach $*$ -algebra B of measures on G by means of the relationship

$$\langle \xi, R(\mu)\zeta \rangle = \int \langle \xi, U_\sigma(x)\zeta \rangle d\mu(x) \quad \text{for all } \xi, \zeta \in H$$

for each $\mu \in B$. In general, however, not all $*$ -representations of B

may be obtained by means of this type of relationship. In (22.7) of [HR] there is an example of a family of subalgebras of $M(G)$ for which every $*$ -representation is determined by a representation of G .

We will see in theorem (5.6.9) another example of a family of algebras which also has this property.

(1.4) Fourier Transforms

(1.4.1) Let G be a compact group. Before introducing the idea of the Fourier transform of a measure $\mu \in M(G)$ we introduce certain spaces which will be needed.

Let F be a subset of $\Sigma(G)$, and let $\sigma \in F$. We fix a representation $U_\sigma \in \sigma$, and assume it operates on the Hilbert space H_σ . Then define $\underline{\underline{E}}(F)$ by

$$\underline{\underline{E}}(F) = \prod_{\sigma \in F} B(H_\sigma).$$

For $1 \leq p < \infty$, $\underline{\underline{E}}^p(F)$ denotes the subspace of $\underline{\underline{E}}(F)$ consisting of those $A = (A_\sigma)_{\sigma \in F}$ for which

$$\|A\|_p = \left(\sum_{\sigma \in F} d_\sigma \|A_\sigma\|_{\phi_p}^p \right)^{1/p} < \infty,$$

(see appendix D of [HR] for a definition of the ϕ_p -norms on $B(H)$).

For $p = \infty$, $\underline{\underline{E}}^\infty(F)$ denotes the subspace of $\underline{\underline{E}}(F)$ consisting of those $A = (A_\sigma)_{\sigma \in F}$ for which

$$\|A\|_\infty = \sup\{\|A_\sigma\| : \sigma \in F\} < \infty,$$

(see (1.3.1) for the definition of $\|T\|$ in $B(H)$).

We will also need to use the subspaces $\underline{\underline{E}}^{oo}(F)$ and $\underline{\underline{E}}^o(F)$ given by

$$\underline{\underline{E}}^{oo}(F) = \{A = (A_\sigma)_{\sigma \in F} \text{ in } \underline{\underline{E}}(F) : \{\sigma \in F : A_\sigma \neq 0\} \text{ is finite}\}$$

$$\underline{E}^0(F) = \{A = (A_\sigma)_{\sigma \in F} \text{ in } \underline{E}(F) : \text{for all } \varepsilon > 0, \{\sigma \in F : \|A_\sigma\| \geq \varepsilon\} \text{ is finite}\}.$$

Then, for $p \leq q$, $\underline{E}^{00}(F) \subseteq \underline{E}^p(F) \subseteq \underline{E}^q(F) \subseteq \underline{E}^0(F) \subseteq \underline{E}^\infty(F)$. For $1 \leq p \leq \infty$, $(\underline{E}^p(F), \|\cdot\|_p)$ is a Banach $*$ -algebra when multiplication by scalar, addition, multiplication and the adjoint of an element are defined coordinatewise (see (28.25) and (28.26) of [HR]).

For $1 \leq p < \infty$ the linear space $\underline{E}^p(F)^*$ of continuous linear functionals on $\underline{E}^p(F)$ is $\underline{E}^{p'}(F)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. The duality is given by

$$\langle A_1, A_2 \rangle = \sum_{\sigma \in F} d_\sigma \operatorname{Tr} (A_{1\sigma}^* A_{2\sigma}) \text{ for all } A_1 = (A_{1\sigma})_{\sigma \in F} \text{ in } \underline{E}^{p'}(F)$$

and all $A_2 = (A_{2\sigma})_{\sigma \in F}$ in $\underline{E}^p(F)$,

(see (28.28) of [HR]).

If G is an abelian group the $\underline{E}^p$ spaces are precisely the familiar ℓ^p spaces.

(1.4.2) We preserve the notation of (1.4.1). Let $\mu \in M(G)$. The Fourier transform $\hat{\mu}$ of μ is defined by

$$\langle \xi, \hat{\mu}(\sigma)\zeta \rangle = \int \langle \xi, U_\sigma(x)\zeta \rangle d\mu(x) \text{ for all } \xi, \zeta \in H_\sigma,$$

for each $\sigma \in \Sigma(G)$.

(This definition differs slightly from the corresponding definition in [HR], but the difference is purely formal and so is of no real consequence.) Clearly $\hat{\mu}$ is an element of $\underline{E}(\Sigma(G))$. We will sometimes write

$$\hat{\mu}(\sigma) = \int U_\sigma(x) d\mu(x) \text{ for each } \sigma \in \Sigma(G)$$

where the integral is a Bochner integral (see Edwards [1], §8.14).

We emphasize the definition of the Fourier transform depends on the initial choice of U_σ 's. However, as it is seen in (28.37) of [HR], many of the useful properties of Fourier transforms remain unchanged by different choices of U_σ 's. For that reason, *we will assume the U_σ 's to be fixed from this point on.*

If B is any subset of $M(G)$ and F is a subset of $\Sigma(G)$ then B_F denotes the set

$$B_F = \{\mu \in B : \text{supp } \hat{\mu} \subseteq F\}.$$

The map $\mu \rightarrow \hat{\mu}$ is a norm-decreasing (in the wide sense) $*$ -isomorphism of $M(G)$ into $\underline{\underline{E}}^\infty(\Sigma(G))$. It induces a norm-decreasing $*$ -isomorphism of $L^1(G)$ onto a dense subalgebra of $\underline{\underline{E}}^0(\Sigma(G))$, (see (28.40) of [HR]), and an inner-product preserving linear map of $L^2(G)$ onto $\underline{\underline{E}}^2(\Sigma(G))$, (see (28.43) of [HR]). For each $f \in L^1(G)$, $\{\sigma \in \Sigma(G) : \hat{f}(\sigma) \neq 0\}$ is countable (see (28.40) of [HR]). Finally, we note that for each $\sigma \in \Sigma(G)$ the map $\mu \rightarrow \hat{\mu}(\sigma)$ is a $*$ -representation of $M(G)$ in $B(H_\sigma)$ (see (28.36) of [HR]) and that it induces an isomorphism between $T(G)_{\{\sigma\}}$ and $\mathfrak{B}(H_\sigma)$, (see (28.39) of [HR]).

(1.4.3) Let G be a compact group. Then $A(G)$ denotes the set of $f \in L^1(G)$ such that $\hat{f} \in \underline{\underline{E}}^1(\Sigma(G))$. In fact $A(G)$ is a subset of $C(G)$ and is isomorphic to $\underline{\underline{E}}^1(\Sigma(G))$. It becomes a Banach $*$ -algebra with the structure induced by $\underline{\underline{E}}^1(\Sigma(G))$. The dual space of $\underline{\underline{E}}^1(\Sigma(G))$ is $\underline{\underline{E}}^\infty(\Sigma(G))$ (see (1.4.1)); we will let $PM(G)$ denote the dual space of $A(G)$. Then every element s of $PM(G)$ may be identified with an element $\hat{s} = (\hat{s}(\sigma))_{\sigma \in \Sigma(G)}$ of $\underline{\underline{E}}^\infty(\Sigma(G))$ which is defined by

$$\langle \xi, \hat{s}(\sigma)\zeta \rangle = s\left[x \rightarrow \langle \xi, U_\sigma(x)\zeta \rangle\right]$$

for all $\xi, \zeta \in H_\sigma$ and for all $\sigma \in \Sigma(G)$, (cf. the definition of the

Fourier transform of a measure μ , (1.4.2)). We will call $\hat{S}$ the *Fourier transform of S* and we note that the duality between $PM(G)$ and $A(G)$ is given by

$$\langle S, f \rangle = \langle \hat{S}, \hat{f} \rangle \quad \text{for all } S \in PM(G) \text{ and } f \in A(G)$$

(see (1.4.1) for the definition of $\langle \hat{S}, \hat{f} \rangle$).

Every element of $E^\infty(\Sigma(G))$ determines a distinct continuous linear functional of $A(G)$.

The elements of $PM(G)$ are called *pseudomeasures*; they have been extensively studied when $G = \mathbb{T}$ in Kahane [1].

As in (1.4.2), if B is any subset of $PM(G)$ and F is a subset of $\Sigma(G)$ then B_F denotes the set

$$B_F = \{S \in B : \text{supp } \hat{S} \subseteq F\}. \quad (1)$$

If $S \in PM(G)$ and $x \in G$ then the *left x-translate* ${}_xS$ of S is defined by

$${}_xS(f) = S({}_{x^{-1}}f) \quad \text{for all } f \in A(G).$$

(We could similarly define the right x -translate S_x of S .) Then ${}_xS$ is an element of $PM(G)$ and $\|{}_xS\|_{PM} = \|S\|_{PM}$.

Referring back to (1), a subset F of $\Sigma(G)$ is called a *Sidon set* if $C(G)_F$ is a subset of $A(G)$, or equivalently if there exists a constant K such that

$$\|\hat{f}\|_1 \leq K \|f\|_\infty \quad \text{for all } f \in C(G)_F$$

(see (37.2) of [HR]).

(1.4.4) Let $ZM(G)$ denote $\{\mu \in M(G) : \mu * \nu = \nu * \mu \text{ for all } \nu \in M(G)\}$.

The elements of $ZM(G)$ are called *central measures*, and $ZM(G)$ is

called the *centre of* $M(G)$. It is shown in (28.48) of [HR] that an element μ of $M(G)$ is central iff

$$\hat{\mu}(\sigma) = \alpha(\mu, \sigma) \text{Id}_\sigma \quad \text{for each } \sigma \in \Sigma(G),$$

where Id_σ denotes the identity map on H_σ and $\alpha(\mu, \sigma)$ is a complex number depending on μ and σ .

In the text we will make particular use of the fact that there exists an approximate identity $(k_n)_{n \in \mathbf{N}}$ of $L^1(G)$, (see (1.2.6) for the definition) which possesses the additional property that for each $n \in \mathbf{N}$ k_n is a central non-negative trigonometric polynomial of unit norm in $L^1(G)$, (see (28.53) of [HR]).

(1.5) Tensor Algebras

(1.5.1) In Chapter 2 we will study the space formed by the projective tensor product of certain closed subspaces of $C(G)$. The concept of tensor products of vector spaces has been fully described in, for example, Schwartz [1] or Grothendieck [1].

If E_1 and E_2 are vector spaces, we shall denote their *tensor product* by $E_1 \otimes E_2$ and the *tensor product of two elements* $e_1 \in E_1$ and $e_2 \in E_2$ by $e_1 \otimes e_2$. Then $E_1 \otimes E_2$ is generated by these elements $e_1 \otimes e_2$. If H_1 and H_2 are also vector spaces and T_1 is a linear map from E_1 to H_1 , T_2 from E_2 to H_2 , then the tensor product $T = T_1 \otimes T_2$ is the linear map from $E_1 \otimes E_2$ to $H_1 \otimes H_2$ that takes each $e_1 \otimes e_2$ to $T_1(e_1) \otimes T_2(e_2)$.

When E_1 and E_2 are normed spaces, we can give $E_1 \otimes E_2$ the *projective norm*; its completion for this norm is denoted by $E_1 \hat{\otimes} E_2$. If H_1 and H_2 are also normed and T_1 and T_2 are continuous then the linear map $T_1 \otimes T_2$ extends by continuity to a linear map $T_1 \hat{\otimes} T_2$ from $E_1 \hat{\otimes} E_2$ to $H_1 \hat{\otimes} H_2$ so that $\|T_1 \hat{\otimes} T_2\| \leq \|T_1\| \|T_2\|$.

If E_1 and E_2 are Banach spaces then every element $e \in E = E_1 \hat{\otimes} E_2$ can be represented as

$$e = \sum_{n \in \mathbf{N}} x_n \otimes y_n$$

with $x_n \in E_1$, $y_n \in E_2$ and $\sum_{n \in \mathbf{N}} \|x_n\|_1 \|y_n\|_2$ finite.

Moreover,

$$\|e\|_E = \inf \sum_{n \in \mathbf{N}} \|x_n\|_1 \|y_n\|_2$$

taken over all such representations.

It will be useful for us to note that if E_1 and E_2 are Banach spaces and T is a continuous bilinear functional on $E_1 \times E_2$ then there exists a unique bounded linear functional $\tilde{T}$ of $E_1 \hat{\otimes} E_2$ such that

$$(a) \quad \tilde{T}(e_1 \otimes e_2) = T(e_1, e_2) \quad \text{for all } e_1 \in E_1 \text{ and } e_2 \in E_2$$

$$\text{and (b) } \|\tilde{T}\| \leq \sup\{|T(e_1, e_2)| : e_1 \in B_{E_1}(0,1) \text{ and } e_2 \in B_{E_2}(0,1)\}.$$

Conversely, every bounded linear functional of $E_1 \hat{\otimes} E_2$ determines a continuous bilinear functional of $E_1 \times E_2$.

(1.5.2) Now let G be a compact group and let E_1, E_2 be Banach spaces of continuous complex-valued functions on G . Then each element $f \otimes g \in E_1 \otimes E_2$ may be identified with the complex-valued function $f \otimes g : (x, y) \rightarrow f(x) g(y)$ on $G \times G$, so $E_1 \otimes E_2$ is the space of functions of the form

$$(x, y) \rightarrow \sum_{i=1}^n f_i(x) g_i(y),$$

a vector subspace of $C(G \times G)$. The natural mapping of $E_1 \otimes E_2$ (with its projective norm) into $C(G \times G)$ is continuous, linear, and of

norm at most 1. Hence it extends to a continuous, linear, norm-decreasing map J of $E_1 \hat{\otimes} E_2$ into $C(G \times G)$. Then the element

$$\sum_{n \in \mathbf{N}} f_n \otimes g_n \text{ of } E_1 \hat{\otimes} E_2$$

is mapped to the continuous function

$$(x, y) \rightarrow \sum_{n \in \mathbf{N}} f_n(x) g_n(y) .$$

By théorème 3 of Schwartz [1], Exposé 14, J is an injection iff E_2 has the Banach approximation property (the identity mapping on E_2 is the limit, uniformly on compact sets, of operators of finite rank).

If J is an injection then $E_1 \hat{\otimes} E_2$ is the space of continuous functions h on $G \times G$ which have the representation

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n$$

with $f_n \otimes g_n \in E_1 \otimes E_2$, and $\sum_{n \in \mathbf{N}} \|f_n\|_1 \|g_n\|_2 < \infty$.

Moreover,

$$\|h\|_{E_1 \hat{\otimes} E_2} = \inf \sum_{n \in \mathbf{N}} \|f_n\|_1 \|g_n\|_2$$

where the inf is taken over all such representations.

Because $E_1 \otimes E_2$ is a dense subspace of $E_1 \hat{\otimes} E_2$, the space of all $h \in C(G \times G)$ which may be represented as a finite sum of elements $f \otimes g$ is a dense subspace of $E_1 \hat{\otimes} E_2$.

(1.5.3) We shall be interested in the case E_2 is a Banach space of continuous complex-valued functions on G which also satisfies (P):

There exists an approximate identity $(k_n)_{n \in \mathbf{N}}$ in $L^1(G)$, (see (1.2.6)) for which each $k_n \in T(G)$, and which satisfies

$$\lim_{n \rightarrow \infty} \|k_n * f - f\|_2 = 0 \text{ for all } f \in E_2.$$

LEMMA : Let E be a Banach space of continuous complex-valued functions on G such that $\|f\|_E \leq K\|f\|_\infty$ for all $f \in E$. Then E has the Banach approximation property if it satisfies (P).

Proof : For each $n \in \mathbb{N}$ define the map T_n from E to E by $T_n(f) = k_n * f$ for all $f \in E$. Then the image of T_n , $\text{im } T_n$, is finite dimensional and by (P)

$$\lim_{n \rightarrow \infty} \|T_n f - f\|_E = 0. \quad (1)$$

But

$$\lim_{n \rightarrow \infty} \|k_n - \delta_1\|_1 = 0$$

so necessarily by (1)

$$\lim_{n \rightarrow \infty} \|T_n - \text{Id}\| = 0$$

and E possesses the Banach approximation property. $\square$

(1.5.4) Let G be a compact abelian group, and F_1, F_2 be subsets of X (see (1.3.7)). Then we obtain the following extension of an assertion made on page 58 of Varcopoulos [1]. Recall the definition of $A(G)$ from (1.4.3), and of $A(G)_{F_1}$ taken from (1.4.2).

$$\underline{\text{LEMMA}} : A(G)_{F_1} \hat{\otimes} A(G)_{F_2} = A(G \times G)_{F_1 \times F_2}.$$

Proof : Assume that $h \in A(G)_{F_1} \hat{\otimes} A(G)_{F_2}$ and has the representation

$$h = \sum_{n \in \mathbb{N}} f_n \otimes g_n$$

where $f_n \in A(G)_{F_1}$ and $g_n \in A(G)_{F_2}$, and

$$\sum_{n \in \mathbf{N}} \|f_n\|_A \|g_n\|_A < \infty. \quad (1)$$

For each $n \in \mathbf{N}$, assume that

$$f_n = \sum_{\chi \in F_1} \alpha_n(\chi) \chi \quad \text{and} \quad g_n = \sum_{\eta \in F_2} \beta_n(\eta) \eta$$

where $(\alpha_n(\chi))_{\chi \in F_1} \in \ell^1(F_1)$ and $(\beta_n(\eta))_{\eta \in F_2} \in \ell^1(F_2)$.

From (1)

$$\begin{aligned} h &= \sum_{n \in \mathbf{N}} \left(\sum_{\chi \in F_1} \alpha_n(\chi) \chi \right) \otimes \left(\sum_{\eta \in F_2} \beta_n(\eta) \eta \right) \\ &= \sum_{(\chi, \eta) \in F_1 \times F_2} \left(\sum_{n \in \mathbf{N}} \alpha_n(\chi) \beta_n(\eta) \right) \chi \otimes \eta. \end{aligned}$$

Hence $Th \in A(G \times G)_{F_1 \times F_2}$.

On the other hand assume that $h \in A(G \times G)_{F_1 \times F_2}$. Then h is given by

$$h = \sum_{(\chi, \eta) \in F_1 \times F_2} \alpha(\chi, \eta) \chi \otimes \eta$$

where $(\alpha(\chi, \eta))_{(\chi, \eta) \in F_1 \times F_2} \in \ell^1(F_1 \times F_2)$. For each $\eta \in F_2$ write

$$f_\eta = \sum_{\chi \in F_1} \alpha(\chi, \eta) \chi,$$

then $f_\eta \in A(G)_{F_1}$. Hence

$$h = \sum_{\eta \in F_2} f_\eta \otimes \eta,$$

where $\sum_{\eta \in F_2} \|f_\eta\|_A \|\eta\|_A \leq \|h\|_A$. But $\hat{f}_\eta$ is non-zero for at most a countable subset of F_2 (see (1.4.2)). Hence $h \in A(G)_{F_1} \hat{\otimes} A(G)_{F_2}$. $\square$

(1.6) Banach Algebras and the Operational Calculus

The material presented in this section is well known. See, for example, Chapter 24 of Hille and Phillips [1], and the references given there. The results summarized here have been presented only because each will be used, particularly in §(5.5).

(1.6.1) We assume B is a Banach algebra which is not taken to be commutative or to possess an identity. For elements b, c of B , the *cross product* is defined by $b \times c = b + c - bc$.

If $b \times c = 0$ then b is called *right reversible* with *right reverse* c , and c is *left reversible* with *left reverse* b . If $b \times c = 0 = c \times b$ then $b(c)$ is *reversible* with *reverse* $c(b)$. Reverses are necessarily unique. If B does contain an identity e then the element b of B is invertible iff $(e - b)$ is reversible and the reverse of $e - b$ is $e - b^{-1}$. If we adjoin an identity to B in the usual way then reverses in B correspond to inverses in the extended algebra with identity.

The element b of B is left reversible if and only if for each maximal ideal M of B there exists an element c of B such that $b + c - c * b$ is in M , (see theorem (1.1.8) of Larsen [1]). A similar characterization is possible for right reversible elements of B .

(1.6.2) The set $\mathbb{C}$ of complex numbers may be partitioned into two-disjoint pieces $\delta(b)$ and $\alpha(b)$ corresponding to any $b \in B$; $\delta(b)$ is called the *dissolvent set* of b and $\alpha(b)$ the *spectrum* of b . If $b \neq 0$ then $\alpha(b)$ is a compact, non-empty subset of $\mathbb{C}$. The non-zero complex number ξ is in $\delta(b)$ iff $\xi^{-1}b$ is reversible; $\xi = 0$ is an element of $\delta(b)$ iff there exists $c, f \in B$ such that

$$f^2 = f, \quad cf = fc = c, \quad bf = fb = b$$

and

$$bc = f = cb.$$

If $\xi \in \delta(b) \setminus \{0\}$ then the reverse of $\xi^{-1}b$ is denoted by $D(\xi, b)$ and is called the *dissolvent* of b ; if $0 \in \delta(b)$ then $D(0, b) = f$.

If B contains an identity e then for any element b of B the spectra $\alpha(b)$ and $\sigma(b)$ ($\sigma(b) = \{\xi \in \mathbb{C} : \xi e - b \text{ is singular}\}$) are identical except perhaps for $\xi = 0$ which may be an element of $\sigma(b) \setminus \alpha(b)$. This occurs iff $D(0, b) \neq e$. For any $\xi \in \alpha(b) \setminus \{0\}$

$$(\xi e - b)^{-1} = \xi^{-1}e - \xi^{-1}D(\xi, b).$$

Hence

$$(1.6.3) \quad b + \xi D(\xi, b) = bD(\xi, b) = D(\xi, b)b \text{ for all } \xi \in \delta(b).$$

(1.6.4) Assume ϕ is a function with domain and image in the Banach space B . In (3.17) of Hille-Phillips [1] the authors define ϕ to be *analytic* in the open subset Δ of the domain of ϕ if it is single valued, locally bounded (in the usual sense) and Gateaux differentiable in Δ . The authors observe that if Δ is not connected then ϕ will, in general, define distinct analytic functions (in a sense compatible with the definition of Weierstrass) on distinct components. This distinction is made by calling ϕ *locally analytic* in Δ .

The main concern in the theory of the operational calculus is the study of the mapping of locally-analytic complex-valued functions into the set of locally analytic functions with domain and image in B .

(1.6.5) Let Δ be a neighbourhood of zero in $\mathbb{C}$, and let ϕ be a complex-valued function which is locally analytic on Δ . If ρ is the radius of the largest ball centred on zero in Δ then

$$\phi(\xi) = \sum_{n \in \mathbb{N}} \frac{\phi^{(n)}(0)}{n!} \xi^n \text{ for } |\xi| < \rho.$$

The *principal extension* of ϕ to B , which will also be denoted ϕ , is defined by

$$\phi(b) = \sum_{n \in \mathbf{N}} \frac{\phi^{(n)}(0)}{n!} b^n \quad \text{for } \|b\|_B < \rho.$$

In order to extend ϕ further (that is, to define ϕ on a larger subset of B than $\{b \in B : \|b\|_B < \rho\}$) it is necessary to introduce the notion of an *oriented envelope* Γ of a compact subset Φ of Δ with respect to ϕ , (see (5.2) of Hille and Phillips [1]). Basically Δ is the oriented boundary of some open set Ω satisfying $\Phi \subseteq \Omega \subseteq \Delta$. For each $\lambda \in \Omega$ the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi$$

exists and is independent of Γ if $\lambda \in \Phi$. It is this integral which will enable further extension of ϕ , (see (1.6.6)).

(1.6.6) Recall the definition of $\alpha(b)$ for any $b \in B$ from (1.6.2).

Define $\mathcal{C}_B(\Delta)$ to be

$$\mathcal{C}_B(\Delta) = \{b \in B : \alpha(b) \subseteq \Delta\}$$

and $\mathcal{D}_B(\Delta)$ to be the principal component of $\mathcal{C}_B(\Delta)$.

It is a consequence of the theory of locally analytic functions that the principal extension of ϕ defined in (1.6.5) may be uniquely extended to $\mathcal{D}_B(\Delta)$ in B , (see (3.18) of Hille and Phillips [1]). We will also call this extension principal.

If $\phi(0) = 0$, then for each element b of $\mathcal{C}_B(\Delta)$ the integral

$$- \frac{1}{2\pi i} \int_{\Gamma_b} \phi(\lambda) D(\lambda, b) \frac{d\lambda}{\lambda} \quad (1)$$

(Γ_b is an oriented envelope of $\alpha(b)$ with respect to ϕ) exists and

coincides with the principal extension of ϕ on $\mathcal{D}_B(\Delta)$. In fact, it is shown in (24.5.2) of Hille and Phillips [1] that the map

$$b \rightarrow -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\lambda) D(\lambda, b) \frac{d\lambda}{\lambda}$$

from $\mathcal{C}_B(\Delta)$ into B is locally analytic.

If B contains an identity element e , we put $\mathcal{E}_B(\Delta)$ equal to the set

$$\mathcal{E}_B(\Delta) = \{b \in B : \sigma(b) \subseteq \Delta\}.$$

For any element $b \in \mathcal{E}_B(\Delta)$ the integral

$$\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) (\xi e - b)^{-1} d\xi \quad (2)$$

(Γ_b is an oriented envelope of $\sigma(b)$ with respect to ϕ) exists and is equal to the principal extension of ϕ on the principal component of $\mathcal{E}_B(\Delta)$.

Finally we note that it is shown in (24.5.3) of Hille and Phillips [1] that if B contains an identity element e and if $b \in \mathcal{C}_B(\Delta) \cap \mathcal{E}_B(\Delta)$ then the integrals (1) and (2) coincide.

CHAPTER 2

BEURLING ALGEBRAS AND TENSOR ALGEBRAS

(2.1) Introduction

The work of this thesis grew out of a specific problem described below. My first attempts to solve this problem led me to study certain tensor algebras; this approach did not, in fact, lead to a neat solution but many interesting results appeared on the way. Later I realized that these tensor algebras are examples of the homogeneous algebras (and of homogeneous spaces if the multiplication is ignored), which are defined and extensively studied in Chapters 3-5; examples which motivated the work in these chapters. In this chapter I first introduce a generalized form of the specific problem; then I describe its solution; and finally I present the work on tensor algebras.

The problem referred to above was concerned with Beurling algebras.

Let ω be a measurable positive function on $\mathbb{R}$, bounded away from zero, with the following property

$$\omega(\chi + \eta) \leq \omega(\chi)\omega(\eta) \text{ for all } \chi, \eta \in \mathbb{R}. \quad (1)$$

Denote by $L^1(\mathbb{R})_\omega$ the space of integrable functions f on $\mathbb{R}$ such that

$$\|f\|_\omega = \int |\hat{f}(\chi)| \omega(\chi) d\lambda(\chi)$$

is finite. Then $L^1(\mathbb{R})_\omega$, equipped with $\|\cdot\|_\omega$, is a Banach algebra under pointwise multiplication; it was studied initially by A. Beurling, mainly in unpublished work (see also Beurling [1] and Wermer [1]).

There is a natural extension to functions on any locally compact group G with dual X which is discussed in Domar [1]. If ω is a

non-negative function defined on X , must it necessarily satisfy the conditions implied above, that is

$$\inf_{\chi \in X} \omega(\chi) > 0 \quad \text{and} \quad \omega(\chi + \eta) \leq \omega(\chi)\omega(\eta) \quad \text{for } \chi, \eta \in X,$$

if $L^1(G)_\omega$ is to be an algebra under pointwise multiplication?

Another possible direction of generalization is to restrict the functions to some Banach algebra B (under pointwise multiplication) of integrable functions on G ; then the corresponding space B_ω is the space of functions $f \in B$ for which $\int |\hat{f}(\chi)| \omega(\chi) d\lambda(\chi)$ is finite.

The specific problem can now be described in terms of this framework. Take $G = \mathbb{T}$, the circle group, so that $X = \mathbb{Z}$. Let $\omega = (\omega_n)_{n \in \mathbb{N}}$ be any sequence of 1's and 0's, so that ω corresponds to the indicator function of some subset F of X . Finally take for B the space $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$. Then the problem is: what conditions on F are equivalent to the condition that $B_\omega \stackrel{\text{df}}{=} C^F(\mathbb{T})$ is an algebra under pointwise multiplication? In §(2.3) I consider this problem and show how it leads to the introduction of the tensor algebras

$$C(G)_{X \setminus F} \hat{\otimes} C(G)_{X \setminus F} \quad \text{and} \quad C(G)_{X \setminus F} \hat{\otimes} A(G)_F.$$

First however, in §(2.2) I let G denote a compact abelian group and consider under what conditions a non-negative function ω defined on X has the property that B_ω is a subalgebra of B whenever B is a Banach algebra similar to $L^1(G)$ (for example, if B is either $A(G)$ or $C(G)$). The properties which B must have are given in (2.2.1). The specific problem described above and considered in §(2.3) is completely solved by (2.2.10). The principal results in this section are (2.2.5) and (2.2.10) in which it is proved that B_ω is a subalgebra of B if and only if

$$\omega(\chi + \eta) \leq \text{const } \omega(\chi)\omega(\eta) \quad \text{for all } \chi, \eta \in X$$

(if $\inf_{\chi \in X} \omega(\chi) > 0$,

or if and only if

$$B_{\omega} = B \text{ or } A(G)$$

(if $\liminf_{\chi \in X} \omega(\chi) = 0$ and $\omega \in \ell^{\infty}(X)$).

Tensor algebras became an important part of Harmonic Analysis in 1965 when N. Th. Varopoulos used them to produce an entirely new proof of Malliavin's theorem (see Malliavin [1]) concerning the existence of sets of non-synthesis in any non-discrete locally compact ^{abelian} group. Since that time they have been the subject of many papers, both on their own account and as related to Fourier algebras (see Varopoulos [1], [2], [4], Stegeman [1], Graham [1], [2], [3]). In each of these papers the author considers the space $C(E) \hat{\otimes} C(E)$ where E is a subset of a locally compact abelian group G , and $C(E)$ is the space of continuous functions defined on E . Then $C(E) \hat{\otimes} C(E)$ is a Banach algebra under pointwise multiplication. However, I am interested instead in the Banach spaces $V_{F_1, F_2}(G) \stackrel{\text{df}}{=} C(G)_{F_1} \hat{\otimes} C(G)_{F_2}$ where G is taken to be compact and F_1, F_2 are subsets of its dual space. If G is also abelian then it is easily seen that $V_{F_1, F_2}(G)$ is an algebra under pointwise multiplication iff F_1, F_2 are semigroups (see (2.4.1)). But for any compact group G , $V_{F_1, F_2}(G)$ is a convolution Banach algebra (see (2.4.2)). In §(2.4) I prove that each $V_{F_1, F_2}(G)$ possesses the properties which will allow me to use it as an example of a Banach algebra in $\mathcal{B}_h$ (see (3.4.5) for the definition of $\mathcal{B}_h$).

In §(2.5) and §(2.6) G is assumed to be abelian. The first of these sections is devoted to the question of which non-negative functions ω defined on $X \times X$ have the property that if $V(G)$ denotes $V_{X, X}(G)$ then $V(G)_{\omega} = V(G)$. In the second I consider the problem of estimating

the norms of elements of $V_{F_1, F_2}(G)$. In particular, I show in (2.6.4) that $V_{F, F}(G)$ is equal to $C(G \times G)_{F \times F}$ iff F is a finite subset of X ; and in (2.6.6) and (2.6.7) that if F_2 is a Sidon set then every element h of $V_{F_1, F_2}(G)$ has a representation

$$h = \sum_{n \in \mathbf{N}} H_n \otimes \chi_n,$$

where $H_n \in C(G)_{F_1}$ and $\chi_n \in F_2$ for $n \in \mathbf{N}$ and $\sum_{n \in \mathbf{N}} \|H_n\|_\infty$ is finite, such that

$$K \sum_{n \in \mathbf{N}} \|H_n\|_\infty \leq \|h\|_{F_1, F_2} \leq \sum_{n \in \mathbf{N}} \|H_n\|_\infty$$

(K^{-1} is the Sidon constant of F_2 and $\|\cdot\|_{F_1, F_2}$ is the norm in $V_{F_1, F_2}(G)$).

The problems treated in §(2.5) and §(2.6) may easily be posed if G is an arbitrary compact group; however I have not yet considered this possibility.

(2.2) Initial Problem

(2.2.1) Notation: Throughout this section G denotes a compact abelian group; X its dual group and B denotes a Banach algebra of Haar-integrable functions defined on G . The multiplication in B is point-wise multiplication and B is assumed to satisfy

(a) $A(G)$ is a dense subset of B ,

(b) there exists $\kappa > 0$ such that $\|b\|_{PM} \leq \kappa \|b\|_B$ for all $b \in B$,

and (c) $\sup_{\chi \in X} \|\chi\|_B$ is finite.

By (1.3.7) each $\chi \in X$ is the character of an irreducible representation of G and so is a trigonometric polynomial. By (a) χ is an element

of B and then (c) asserts that the set of characters is bounded in B . For any G , $B = A(G)$ or $C(G)$ satisfies each of these conditions.

Let ω be a complex-valued function defined on X .

Define the subset B_ω of B by

$$B_\omega = \{b \in B : \hat{b}\omega \in \ell^1(X)\}.$$

Then the set B_ω is a linear space. As I have already said we will study, in this section, conditions under which B_ω is a subalgebra of B . But first we have some preliminaries.

(2.2.2) LEMMA : The following hold

$$(i) \quad B_\omega = B_{|\omega|},$$

$$(ii) \quad B_{\lambda\omega} = B_\omega \quad \text{for all } \lambda \neq 0,$$

$$(iii) \quad B_\omega = B_{\omega+\omega'}, \quad \text{for any complex-valued function } \omega' \text{ defined on } X \text{ which has finite support, and}$$

$$(iv) \quad T(G) \subseteq B_\omega.$$

Proof : The proof follows immediately from (2.2.1). □

A consequence of (2.2.2) is that it is sufficient to consider only real-valued non-negative functions ω on X . We consider first the case $\liminf_{\chi \in X} \omega(\chi) > 0$ (see (2.2.5)) and then $\liminf_{\chi \in X} \omega(\chi) = 0$. In particular, (2.2.10) gives the complete solution if $\liminf_{\chi \in X} \omega(\chi) = 0$ and $\omega \in \ell^\infty(X)$; and finally, in (2.2.11), we make some comments about the remaining case $\liminf_{\chi \in X} \omega(\chi) = 0$ and $\omega \notin \ell^\infty(X)$.

(2.2.3) LEMMA : Let E_1 and E_2 be subspaces of E which are Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Assume that there exist constants K_1 and $K_2 > 0$ such that

$$\|b\|_B \leq \kappa_1 \|b\|_1 \text{ for } b \in E_1 \text{ and } \|b\|_B \leq \kappa_2 \|b\|_2 \text{ for } b \in E_2 .$$

Then $E_1 \cdot E_2$ is a subset of B_ω if and only if there exists a positive constant κ such that for all $b_1 \in E_1$ and $b_2 \in E_2$

$$\|\widehat{b_1 b_2} \omega\|_1 \leq \kappa \|b_1\|_1 \|b_2\|_2 .$$

Proof : Assume that $E_1 \cdot E_2 \subseteq B_\omega$. For each $b_2 \in E_2$ define the complex-valued function T_{b_2} on E_1 by

$$T_{b_2}(b_1) = \|\widehat{b_1 b_2} \omega\|_1 \text{ for all } b_1 \in E_1 .$$

Then T_{b_2} is a seminorm on E_1 . It is lower-semicontinuous because it is the upper envelope of the seminorms

$$b_1 \rightarrow \|\widehat{b_1 b_2} \omega \Pi_I\|_1 ,$$

where I is a finite subset of X , which are continuous by (2.2.1)(b).

By the Uniform Boundedness Principle

$$\sup\{\|\widehat{b_1 b_2} \omega\|_1 : \|b_1\|_1 \leq 1\} < \infty , \quad (1)$$

and so for each b_1 in E_1

$$\|\widehat{b_1 b_2} \omega\|_1 \leq \|b_1\|_1 \sup\{\|\widehat{b_1 b_2} \omega\|_1 : \|b_1\|_1 \leq 1\} . \quad (2)$$

Now define the complex-valued function T on E_2 by

$$T(b_2) = \sup\{\|\widehat{b_1 b_2} \omega\|_1 : \|b_1\|_1 \leq 1\} \text{ for all } b_2 \in E_2 .$$

Then T is a seminorm on E_2 . It is the upper envelope of the seminorms

$$b_2 \rightarrow \|\widehat{b_1 b_2} \omega\|_1$$

for $\|b_1\|_1 \leq 1$. But each of these is lower-semicontinuous, being itself the upper envelope of the continuous seminorms

$$b_2 \longrightarrow \|\widehat{b_1 b_2} \omega \cdot \mathbb{1}_I\|_1$$

as I ranges over finite subsets of X . Hence T is lower semi-continuous. A second application of the Uniform Boundedness Principle guarantees the existence of a positive constant K such that we may write in (2)

$$\|\widehat{b_1 b_2} \omega\|_1 \leq K \|b_1\|_1 \|b_2\|_2$$

for all $b_1 \in E_1$ and $b_2 \in E_2$.

The converse is trivial. □

(2.2.4) Assume that $\liminf_{\chi \in X} \omega(\chi) > 0$. Then $\omega(\chi) = 0$ for at most finitely many χ . By (2.2.2)(iii) we may assume that $\inf_{\chi \in X} \omega(\chi) = \delta$ is positive. For each element b of B_ω put $\|b\|_\omega = \|\widehat{b\omega}\|_1$. Then $\|\cdot\|_\omega$ is a norm on B_ω and

$$\|b\|_\omega \geq \delta \|\widehat{b}\|_1 = \delta \|b\|_A \text{ for all } b \in B_\omega.$$

LEMMA : The normed space B_ω is complete with $\|\cdot\|_\omega$. Moreover

$$\|b\|_B \leq \text{const} \|b\|_\omega \text{ for all } b \in B_\omega.$$

Proof : By (2.2.1)(a) $A(G)$ is a subset of B . We begin by proving that

$$\|b\|_B \leq \text{const} \|b\|_A \text{ for all } b \in A(G), \quad (1)$$

and so we will have the inequality

$$\|b\|_B \leq \text{const} \|b\|_\omega \text{ for all } b \in B_\omega. \quad (2)$$

Let $(b_n)_{n \in \mathbb{N}}$ be a sequence extracted from $A(G)$ which satisfies

$$\lim_{n \rightarrow \infty} \|b_n\|_A = 0. \quad (3)$$

Assume that there exists an element y of B such that

$$\lim_{n \rightarrow \infty} \|b_n - y\|_B = 0. \quad (4)$$

It follows from (3) that the sequence converges to 0 in $PM(G)$ and from (2.2.1)(b) and (4) that it converges to y in $PM(G)$. Hence $y = 0$. Then by the Closed Graph theorem (1) holds and so also (2).

But (1) is immediate from (2.2.1)(a) and (c).

To prove that B_ω is complete, let $(b_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in B_ω and let $\{n_i\}_{i \in \mathbb{N}}$ be a subsequence of $\mathbb{N}$ such that for each $i \in \mathbb{N}$

$$\|b_{n_{i+1}} - b_{n_i}\|_\omega < 2^{-i}. \quad (5)$$

The sequence $(b_{n_i})_{i \in \mathbb{N}}$ is Cauchy in B by (2) and so there exists an element b of B such that $\lim_{i \rightarrow \infty} \|b - b_{n_i}\|_B = 0$. Then by (2.2.1)(b)

$$\lim_{i \rightarrow \infty} \|b - b_{n_i}\|_{PM} = 0. \quad (6)$$

For each $j \in \mathbb{N}$, $b_{n_j} = b_{n_1} + \sum_{i=1}^{j-1} (b_{n_{i+1}} - b_{n_i})$ and so

$$\|b_{n_j}\|_\omega \leq \|b_{n_1}\|_\omega + \sum_{i=1}^{j-1} \|b_{n_{i+1}} - b_{n_i}\|_\omega. \quad (7)$$

Then (5), (6) and (7) are sufficient to prove that $b \in B_\omega$. Hence B_ω is complete. $\square$

(2.2.5) We now obtain an analogue of an assertion in Domar [1].

PROPOSITION : Notation is taken from (2.2.1). If $\inf_{\chi \in X} \omega(\chi) > 0$ then B_ω is a subalgebra of B if and only if there exists a positive constant K such that

$$\omega(\chi + \eta) \leq K\omega(\chi) + \omega(\eta) \text{ for all } \chi, \eta \text{ in } X.$$

Proof : By (2.2.4) B_ω is a Banach space with $\|\cdot\|_\omega$. If B_ω is a subalgebra of B then appealing to (2.2.3) we see that there exists a positive constant K such that for all $b_1, b_2 \in B_\omega$

$$\|b_1 \cdot b_2\|_\omega \leq K \|b_1\|_\omega \|b_2\|_\omega. \quad (1)$$

Let $\chi, \eta \in X$. We saw in (1.3.7) that each is the character of an irreducible representation of G and so is an element of $T(G)$. But since $\hat{\chi}(\eta) = \delta_{\chi\eta}$ for any $\chi, \eta \in X$ it follows from (1) that

$$\omega(\chi + \eta) \leq K \omega(\chi) \omega(\eta). \quad (2)$$

Conversely if ω satisfies (2) for all $\chi, \eta \in X$ then for each element b_1 and b_2 of B_ω

$$\begin{aligned} \|\widehat{b_1 b_2}\|_1 &= \sum_{\chi \in X} \omega(\chi) \left| \sum_{\eta \in X} \hat{b}_1(\chi - \eta) \hat{b}_2(\eta) \right| \\ &\leq \sum_{\chi \in X} \omega(\chi) \sum_{\eta \in X} |\hat{b}_1(\chi - \eta)| |\hat{b}_2(\eta)| \\ &\leq K \sum_{\chi, \eta \in X} \omega(\chi - \eta) \omega(\eta) |\hat{b}_1(\chi - \eta)| |\hat{b}_2(\eta)| \\ &= K \|b_1\|_\omega \|b_2\|_\omega. \end{aligned}$$

Hence by (2.2.3) B_ω is a subalgebra of B . □

(2.2.6) If we allow ω to take the value 0 on an infinite subset of X then we will not be able to replace it by an ω which is bounded away from zero and so $\|\cdot\|_\omega$ (defined in (2.2.4)) will not be a proper norm on B_ω . (This would be the case if for example $\omega = \mathbb{1}_F$, the indicator function of a co-infinite subset F of X .) Instead, for each $b \in B_\omega$ put

$$\|b\|_\omega = \|b\|_B + \|\hat{b}\omega\|_1. \quad (1)$$

Then $\|\cdot\|_\omega$ is clearly a norm on B_ω .

LEMMA : The normed space B_ω is complete with $\|\cdot\|_\omega$.

Proof : By (1) for all $b \in B_\omega$

$$\|b\|_B \leq \|b\|_\omega. \quad (2)$$

The proof is easily completed as in (2.2.4) by using (2) to show that every Cauchy sequence in B_ω converges in B_ω . $\square$

(2.2.7) Let E be a finite neighbourhood of zero in X and let $\varepsilon > 0$ be given. Then define the subset $X(E, \varepsilon)$ of X by

$$X(E, \varepsilon) = \{\eta \in X : \max_{\chi \in E} \omega(\eta - \chi) \leq \varepsilon\}.$$

PROPOSITION : Let $\liminf_{\chi \in X} \omega(\chi) = 0$. Assume that the set $X(E, \varepsilon)$ is not empty for each $\varepsilon > 0$ and each finite neighbourhood E of zero in X . Then B_ω is a subalgebra of B if and only if $B_\omega = B$.

Proof : In view of (2.2.6), lemma (2.2.3) proves that if B_ω is normed as in (2.2.6) then it is a subalgebra of B iff there exists a constant $K > 0$ such that for all $b_1, b_2 \in B_\omega$

$$\|\widehat{b_1 b_2} \omega\|_1 \leq K \|b_1\|_\omega \|b_2\|_\omega. \quad (1)$$

Let b be a non-zero element of $T(G)$ and put E equal to $\text{supp } \hat{b}$. Then E is a finite subset of X and there exists $\rho \in X$ such that $T_\rho E = \{\chi - \rho : \chi \in E\}$ is a finite neighbourhood of zero. By (2.2.2)(iv) b is an element of B_ω . Let η be an element of $X(T_\rho E, \varepsilon|E|^{-1})$. Then $b = b\bar{\eta} \cdot \eta$ where $b\bar{\eta}$ and η are also in B_ω by (2.2.2)(iv).

If B_ω is a subalgebra of B then it follows from (1) that for all $b \in B_\omega$

$$\|\hat{b}\omega\|_1 \leq K(\|b\bar{\eta}\|_B + \|\hat{b}\bar{\eta}\omega\|_1)(\|\eta\|_B + \|\hat{\eta}\omega\|_1). \quad (2)$$

In view of (2.2.1), (2) leads to

$$\|\hat{b}\omega\|_1 \leq K(K' \|b\|_B \sup_{\chi \in X} \|\chi\|_B + \|\hat{b}\bar{\eta}\omega\|_1) \left(\sup_{\chi \in X} \|\chi\|_B + \varepsilon \right). \quad (3)$$

However,

$$\|\hat{b}\bar{\eta}\omega\|_1 = \sum_{\chi \in X} \omega(\chi - \eta) |\hat{b}(\chi)| = \sum_{\chi \in X} \omega(\eta - (\chi - \rho)) |\hat{b}(\rho - \chi)| \quad (4)$$

and so by (4)

$$\|\hat{b}\bar{\eta}\omega\|_1 \leq \|b\|_B \frac{\varepsilon}{|E|} |E| = \varepsilon \|b\|_B. \quad (5)$$

Then appealing to (3) and (5) gives

$$\|\hat{b}\omega\|_1 \leq \text{const } \|b\|_B. \quad (6)$$

But we know from (2.2.1)(a) that $A(G)$ is a dense subset of B . By the Closed Graph theorem as in (2.2.4) $\|b\|_B \leq \text{const } \|b\|_A$ for all $b \in A(G)$. Hence the set of trigonometric polynomials, which is a dense subset of $A(G)$, is a dense subset of B . Therefore (6) will hold for all b in B and so $B_\omega = B$.

The converse is trivial. □

(2.2.8) I have not been able to solve completely the case left unanswered by (2.2.5) and (2.2.7); namely, that in which $\liminf_{\chi \in X} \omega(\chi) = 0$ and $X(E, \varepsilon)$ is an empty set for some $\varepsilon > 0$ and some finite neighbourhood E of zero in X . However, I do have two results in this direction. In the first proposition I assume that ω is any element of $\ell^\infty(X)$; then B_ω is a subalgebra of B iff $B_\omega = A(G)$, (see (2.2.9)). In the second, which deals with $G = \mathbb{T}$, I show that if ω is sufficiently well-behaved then B_ω will be a subalgebra of B but will not be equal to either B or $A(G)$ (see (2.2.11)).

(2.2.9) PROPOSITION : Let $\omega \in \ell^\infty(X)$ and assume $X(E, \epsilon)$ is empty for some $\epsilon > 0$ and some finite neighbourhood E of 0 . Then B_ω is a subalgebra of B if and only if $B_\omega = A(G)$.

Proof : Recall the definition of $X(E, \epsilon)$ from (2.2.7). Let X_ϵ denote the set $X_\epsilon = \{\chi \in X : \omega(\chi) > \epsilon\}$. We will prove that

$$X = \bigcup \{T_{-\eta} X_\epsilon : \eta \in E\}. \quad (1)$$

Clearly X contains the finite union. On the other hand assume ρ is an element of X which does not belong to $T_{-\eta} X_\epsilon$ for any $\eta \in E$. Then $\rho - \eta \notin X_\epsilon$ for each $\eta \in E$. Hence $\rho \in X(E, \epsilon)$ which contradicts the assumption that $X(E, \epsilon)$ is empty and so (1) holds.

Assume B_ω is a subalgebra of B . Let $b \in B_\omega$. Then $\|\hat{b}\omega\|_1$ is finite. For each $\eta \in E$, $b\eta$ is an element of B_ω and since $\hat{b\eta}(\chi) = \hat{b}(\chi - \eta)$ for each $\chi \in X$ it satisfies

$$\|\hat{b\eta}\omega\|_1 \geq \|\hat{b\eta}\omega \mathbb{1}_{X_\epsilon}\|_1 \geq \epsilon \|\hat{b\eta} \mathbb{1}_{X_\epsilon}\|_1 = \epsilon \|\hat{b} \mathbb{1}_{T_{-\eta} X_\epsilon}\|_1. \quad (2)$$

Thus by (1) and (2)

$$\sum_{\eta \in E} \|\hat{b\eta}\omega\|_1 \geq \epsilon \sum_{\eta \in E} \|\hat{b} \mathbb{1}_{T_{-\eta} X_\epsilon}\|_1 \geq \epsilon \|b\|_A.$$

Hence $B_\omega \subseteq A(G)$. If $b \in A(G)$ then $\|\hat{b}\omega\|_1 \leq \|\omega\|_\infty \|b\|_A$ so b is an element of B_ω . Therefore $B_\omega = A(G)$.

The converse is trivial because $A(G)$ is an algebra under point-wise multiplication. □

(2.2.10) COROLLARY to (2.2.7) and (2.2.9): If $\omega \in \ell^\infty(X)$ then B_ω is a subalgebra of B if and only if $B_\omega = B$ or $B_\omega = A(G)$.

Proof : If $\omega \in \ell^\infty(X)$ then either (2.2.7) or (2.2.9) apply so the proof is immediate. □

Remark : It follows from (2.2.5) that if $\omega \in \ell^\infty(X)$ and $\liminf_{\chi \in X} \omega(\chi) > 0$ then B_ω is a subalgebra of B iff there exists a constant $K > 0$ such that

$$\omega(\chi + \eta) \leq K\omega(\chi)\omega(\eta) \quad \text{for all } \chi, \eta \in X.$$

By (2.2.2)(iii) we may take $\inf_{\chi \in X} \omega(\chi) = \delta > 0$. It is easily checked that if $K = \delta^{-2} \|\omega\|_\infty$ then ω satisfies this inequality. So if $\omega \in \ell^\infty(X)$ and $\liminf_{\chi \in X} \omega(\chi) > 0$ then B_ω is always a subalgebra of B , and by (2.2.10) must equal B or $A(G)$.

(2.2.11) PROPOSITION : Let $G = \mathbf{T}$. For any real-valued non-negative sequence $(\omega_n)_{n \in \mathbf{Z}}$ which satisfies

$$(i) \quad \omega_n = 0 \text{ for } n < 0, \text{ and } \inf_{n \geq 0} \omega_n = \delta > 0,$$

$$(ii) \quad \text{the sequence increases unboundedly on } \mathbf{N} \cup \{0\},$$

$$(iii) \quad \omega_n \geq \sum_{m=1}^{n-1} \omega_m \text{ for all } n \geq 0, \text{ and}$$

$$(iv) \quad \omega_{m+n} \leq \text{const } \omega_m \omega_n \text{ for all } m, n \geq 0,$$

the Banach space B_ω is a subalgebra of B and is a proper subspace of $A(G)$.

Proof : It follows from (ii) that there exists a non-negative integer N such that $\omega_n \geq 1$ for all $n \geq N$. Let $b \in B_\omega$. We now show that it may be written as the unique sum of an element in $B_{-\mathbf{N}}$ and of one in $(B_\omega)_{\mathbf{N}^\circ}$ (where $\mathbf{N}^\circ = \mathbf{N} \cup \{0\}$). Then $B_\omega = B_{-\mathbf{N}} \oplus (B_\omega)_{\mathbf{N}^\circ}$. (It is an easy consequence of (i) that B_ω contains the direct sum.) We have that

$$\sum_{n \in \mathbf{N}^\circ} \omega_n |\hat{b}(n)| \geq \sum_{0 \leq n < N} \omega_n |\hat{b}(n)| + \sum_{n \geq N} |\hat{b}(n)| \quad (1)$$

and so by (1),

$$\|\hat{b} \mathbf{1}_{\mathbf{N}^\circ}\|_1 < \infty. \quad (2)$$

Define b^+ by $b^+ = \sum_{n \in \mathbb{N}^0} \hat{b}(n) \chi_n$. Then b^+ is an element of $A(\mathbb{T})_{\mathbb{N}^0}$ and so also of $B_{\mathbb{N}^0}$. Also $b^- = b - b^+$ is an element of $B_{-\mathbb{N}}$. We have $\|\hat{b}^+ \omega\|_1 = \|\hat{b} \omega\|_1$ and so b^+ is an element of $(B_\omega)_{\mathbb{N}^0}$. Hence

$$B_\omega = B_{-\mathbb{N}} \oplus (B_\omega)_{\mathbb{N}^0}. \quad (3)$$

By (3) B_ω is a subalgebra of B iff both $B_{-\mathbb{N}}$ and $(B_\omega)_{\mathbb{N}^0}$ are, and also

$$B_{-\mathbb{N}} \cdot (B_\omega)_{\mathbb{N}^0} \subseteq B_\omega. \quad (4)$$

We first note that $B_{-\mathbb{N}}$ is a subalgebra because $-\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$, and also that $(B_\omega)_{\mathbb{N}^0} \cdot (B_\omega)_{\mathbb{N}^0} \subseteq B_{\mathbb{N}^0}$. But, for $b_1, b_2 \in (B_\omega)_{\mathbb{N}^0}$ it follows from (iv) that

$$\begin{aligned} \|\widehat{b_1 b_2} \omega\|_1 &\leq \text{const} \sum_{m, n \in \mathbb{Z}} \omega_{m-n} \omega_n |\hat{b}_1(m-n)| |\hat{b}_2(n)| \\ &= \text{const} \|\hat{b}_1 \omega\|_1 \|\hat{b}_2 \omega\|_1 < \infty. \end{aligned}$$

Hence $(B_\omega)_{\mathbb{N}^0}$ is also a subalgebra of B .

We now only need to prove (4). The subset $B_{-\mathbb{N}}$ is closed in B and so is complete with $\|\cdot\|_B$. For each $b \in (B_\omega)_{\mathbb{N}^0}$ let $\|b\|_\omega = \|\hat{b} \omega\|_1$. Then $\|\cdot\|_\omega$ is a norm on $(B_\omega)_{\mathbb{N}^0}$. Another look at the proof of (2.2.4) shows that it also proves that $(B_\omega)_{\mathbb{N}^0}$ is complete with $\|\cdot\|_\omega$ and that $\|b\|_B \leq \text{const} \|b\|_\omega$ for all $b \in (B_\omega)_{\mathbb{N}^0}$. Hence by (2.2.3), (4) holds iff there exists a constant $K > 0$ such that for all $b_1 \in B_{-\mathbb{N}}$ and $b_2 \in (B_\omega)_{\mathbb{N}^0}$

$$\|\widehat{b_1 b_2} \omega\|_1 \leq K \|b_1\|_B \|b_2\|_\omega. \quad (5)$$

Since $A(\mathbb{T})$ is a dense subset of B so also is $T(\mathbb{T})$, (see the proof of (2.2.4)). Hence (5) holds iff it holds for all $b_1 \in T(\mathbb{T})_{-\mathbb{N}}$ and all $b_2 \in (B_\omega)_{\mathbb{N}^0}$.

Let $b_1 \in T(\mathbb{T})_{-\mathbb{N}}$ and $b_2 \in (B_\omega)_{\mathbb{N}^0}$. Then

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \omega_n |\widehat{b_2 b_1}(n)| &= \sum_{n \in \mathbb{N}^\bullet} \omega_n \left| \sum_{m \in \mathbb{Z}} \widehat{b_1}(n-m) \widehat{b_2}(m) \right| \\ &\leq \sum_{m \in \mathbb{Z}} |\widehat{b_2}(m)| \sum_{n \in \mathbb{N}^\bullet} \omega_n |\widehat{b_1}(n-m)|. \end{aligned}$$

But for $n \geq m$ $\widehat{b_1}(n-m) = 0$, and so

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \omega_n |\widehat{b_1 b_2}(n)| &\leq \sum_{m \in \mathbb{Z}} |\widehat{b_2}(m)| \sum_{0 \leq n < m} \omega_n |\widehat{b_1}(n-m)| \\ &\leq \kappa \|b_1\|_B \sum_{m \in \mathbb{Z}} |\widehat{b_2}(m)| \omega_m \quad \text{by (iii)} \\ &= \kappa \|b_1\|_B \|b_2\|_\omega < \infty. \end{aligned}$$

Hence (5) holds and so B_ω is a subalgebra of B

Since $(\omega_n)_{n \in \mathbb{N}^\bullet}$ increases unboundedly there exists an element b of $A(\pi)_{\mathbb{N}^\bullet}$ such that $\omega \widehat{b}$ is not an element of $\ell^1(\mathbb{N}^\bullet)$. Hence, by (3) $B_\omega \neq B$ or $A(\pi)$. □

Remark : It is easy to verify that the sequence $(\omega_n)_{n \in \mathbb{Z}}$ defined by

$$\omega_n = \begin{cases} 0 & \text{if } n < 0 \\ 2^n & \text{if } n \geq 0 \end{cases}$$

satisfies conditions (i)-(iv) of (2.2.11). Hence B_ω is a subalgebra of B for any Banach space as in (2.2.1). (This example was provided by R.E. Edwards).

I am grateful to C. Graham of Northwestern University, Evanston, for several helpful discussions concerning the results in this section.

(2.3) Alternative Approach to the Simplified Problem

The simplified problem which was mentioned in §(2.1) may be completely solved by applying (2.2.10). I now present the alternative

approach which led me to the study of tensor algebras which fills the remaining sections of this chapter. For completeness I begin with a statement of the problem, replacing $\mathbb{T}$ by an arbitrary compact abelian group.

(2.3.1) Problem : Let G denote a compact abelian group with dual X . Let F be any subset of X , and $C(G)$ denote the set of continuous complex-valued functions defined on G (see (1.2.2.)). Define the subset $C^F(G)$ of $C(G)$ by

$$C^F(G) = \{f \in C(G) : \hat{f} \mathbb{1}_F \in \ell^1(X)\}$$

(that is, $C^F(G) = C(G)_{\mathbb{1}_F}$). Clearly $A(G)$ is a subset of $C^F(G)$. The problem we shall consider is when is $C^F(G)$ a subalgebra of the point-wise algebra $C(G)$.

Remark : By (2.2.10) $C^F(G)$ is a subalgebra iff $C^F(G) = C(G)$ or $C^F(G) = A(G)$. It is easily seen that the first possibility coincides with F finite and the second with F coSidon in X .

We begin with some preliminaries.

(2.3.2) LEMMA : The linear space $C^F(G)$ may be written as the direct sum

$$C^F(G) = C(G)_{X \setminus F} \oplus A(G)_F.$$

Proof : Clearly $C(G)_{X \setminus F} \oplus A(G)_F \subseteq C^F(G)$.

Conversely let $b \in C^F(G)$. Then $\hat{b} \mathbb{1}_F \in \ell^1(X)$ and b^+ defined by $b^+ = \sum_{\chi \in F} \hat{b}(\chi) \chi$ is an element of $A(G)_F$ and so of $C(G)_F$. Hence $b^- = b - b^+$ is an element of $C(G)$. But $\hat{b}^- = \hat{b} - \hat{b}^+$ and so b^- is an element of $C(G)_{X \setminus F}$. Hence $b \in C(G)_{X \setminus F} \oplus A(G)_F$. $\square$

(2.3.3) COROLLARY : The linear space $C^F(G)$ is a subalgebra of $C(G)$ if and only if

$$(i) \quad C(G)_{X \setminus F} C(G)_{X \setminus F} \subseteq C^F(G),$$

and $(ii) \quad C(G)_{X \setminus F} A(G)_F \subseteq C^F(G).$

Proof : This is an immediate consequence of (2.3.2) because

$$A(G)_F A(G)_F \subseteq A(G) \subseteq C^F(G). \quad \square$$

(2.3.4) COROLLARY : The linear space $C^F(G)$ is a subalgebra of $C(G)$ if and only if there exist positive constants K_1 and K_2 such that

$$(i) \quad \|\widehat{f_1 f_2} \mathbb{1}_F\|_1 \leq K_1 \|f_1\|_\infty \|f_2\|_\infty \text{ for all } f_1, f_2 \in C(G)_{X \setminus F},$$

and $(ii) \quad \|\widehat{f_1 f_2} \mathbb{1}_F\|_1 \leq K_2 \|f_1\|_\infty \|f_2\|_A \text{ for all } f_1 \in C(G)_{X \setminus F}$

and $f_2 \in A(G)_F.$

Proof : The linear space $C(G)_{X \setminus F}$ is a closed subspace of $C(G)$ and so is complete with $\|\cdot\|_\infty$; similarly $A(G)_F$ is complete with $\|\cdot\|_A$.

The result now follows immediately from (2.3.3) by (2.2.3). $\square$

(2.3.5) Let ϕ be an element of $\ell^\infty(X)$ which vanishes on $X \setminus F$, and let f_1, f_2 be elements of $C(G)_{X \setminus F}$. Then

$$\sum_{\chi \in X} \phi(\chi) \widehat{f_1 f_2}(\chi) = \sum_{\chi + \eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta).$$

From this we obtain the next result.

PROPOSITION : Let K be a positive constant. Then for all

$$f_1, f_2 \in C(G)_{X \setminus F}$$

$$\|\widehat{f_1 f_2} \mathbb{1}_F\|_1 \leq K \|f_1\|_\infty \|f_2\|_\infty \quad (1)$$

if and only if

$$\left| \sum_{\chi+\eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta) \right| \leq K \|\phi\|_{\infty} \|f_1\|_{\infty} \|f_2\|_{\infty} \quad (2)$$

for every $\phi \in \ell^{\infty}(X)$ which vanishes on $X \setminus F$.

Proof : Assume (1) holds for f_1, f_2 in $T(G)_{X \setminus F}$ and let ϕ be an element of $\ell^{\infty}(X)$ which vanishes on $X \setminus F$. Then

$$\begin{aligned} \left| \sum_{\chi+\eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta) \right| &\leq \|\phi\|_{\infty} \sum_{\chi \in F} |\widehat{f_1 f_2}(\chi)| \\ &\leq K \|\phi\|_{\infty} \|f_1\|_{\infty} \|f_2\|_{\infty}. \end{aligned}$$

But $T(G)_{X \setminus F}$ is a dense subset of $C(G)_{X \setminus F}$ and so (2) holds for all $f_1, f_2 \in C(G)_{X \setminus F}$.

Conversely suppose that (2) holds and again let $f_1, f_2 \in T(G)_{X \setminus F}$. Define ϕ on X by

$$\phi(\chi) = \begin{cases} \text{sign } \widehat{f_1 f_2}(\chi) & \text{if } \chi \in F \\ 0 & \text{if } \chi \notin F. \end{cases}$$

Then ϕ is an element of $\ell^{\infty}(X)$ which vanishes on $X \setminus F$ and has norm at most 1. So in (2)

$$\left| \sum_{\chi \in F} \text{sign } \widehat{f_1 f_2}(\chi) \cdot \widehat{f_1 f_2}(\chi) \right| \leq K \|f_1\|_{\infty} \|f_2\|_{\infty}. \quad (3)$$

But LHS of (3) is equal to $\|\widehat{f_1 f_2} \mathbb{1}_F\|_1$ and so (1) holds for all $f_1, f_2 \in T(G)_{X \setminus F}$. Again since $T(G)_{X \setminus F}$ is a dense subset of $C(G)_{X \setminus F}$ it will hold for all $f_1, f_2 \in C(G)_{X \setminus F}$. $\square$

(2.3.6) Similarly we obtain the proposition,

PROPOSITION : Let K be a positive constant. Then for all

$$f_1 \in C(G)_{X \setminus F}, f_2 \in A(G)_F$$

$$\|\widehat{f_1 f_2} \mathbb{1}_F\|_1 \leq K \|f_1\|_{\infty} \|f_2\|_A \quad (1)$$

if and only if

$$\left| \sum_{\chi+\eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta) \right| \leq K \|\phi\|_{\infty} \|f_1\|_{\infty} \|f_2\|_A \quad (2)$$

for every $\phi \in \ell^{\infty}(X)$ which vanishes on $X \setminus F$.

Proof : This proof is similar to that of (2.3.6) and so is not set out in detail. $\square$

Assume that $C^F(G)$ is a subalgebra of $C(G)$.

(2.3.7) Let ϕ be an element of $\ell^{\infty}(X)$ which vanishes on $X \setminus F$.

Define the complex-valued maps S_{ϕ} on $C(G)_{X \setminus F} \times C(G)_{X \setminus F}$ and T_{ϕ} on $C(G)_{X \setminus F} \times A(G)_F$ by

$$S_{\phi}(f_1, f_2) = \sum_{\chi+\eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta) \quad \text{for } f_1, f_2 \in C(G)_{X \setminus F}$$

and

$$T_{\phi}(f_1, f_2) = \sum_{\chi+\eta \in F} \phi(\chi + \eta) \hat{f}_1(\chi) \hat{f}_2(\eta) \quad \text{for } f_1 \in C(G)_{X \setminus F}$$

$$\text{and } f_2 \in A(G)_F.$$

(2.3.3), (2.3.4),

Then S_{ϕ} and T_{ϕ} are bilinear and, by (2.3.5) and (2.3.6), are continuous. As we have already noted in (2.3.4) both $C(G)_{X \setminus F}$ and $A(G)_F$ are Banach spaces. Hence there exist unique bounded linear functionals $\tilde{S}_{\phi}$ on $C(G)_{X \setminus F} \hat{\otimes} C(G)_{X \setminus F}$ and $\tilde{T}_{\phi}$ on $C(G)_{X \setminus F} \hat{\otimes} A(G)_F$ such that

$$\tilde{S}_{\phi}(f_1 \otimes f_2) = S_{\phi}(f_1, f_2) \quad \text{for all } f_1, f_2 \in C(G)_{X \setminus F},$$

$$\|\tilde{S}_{\phi}\| \leq \sup\{|S_{\phi}(f_1, f_2)| : f_1, f_2 \in C(G)_{X \setminus F} \text{ and } \|f_1\|_{\infty}, \|f_2\|_{\infty} \leq 1\}$$

$$\text{and } \tilde{T}_{\phi}(f_1 \otimes f_2) = T_{\phi}(f_1, f_2) \quad \text{for all } f_1 \in C(G)_{X \setminus F} \text{ and } f_2 \in A(G)_F,$$

$$\|\tilde{T}_{\phi}\| \leq \sup\{|T_{\phi}(f_1, f_2)| : f_1 \in C(G)_{X \setminus F}, f_2 \in A(G)_F \text{ and}$$

$$\|f_1\|_{\infty}, \|f_2\|_A \leq 1\}$$

(see (1.5.1)).

The Banach space $A(G)_{X \setminus F}$ is a subset of $C(G)_{X \setminus F}$. Hence if h is an element of $A(G)_{X \setminus F} \hat{\otimes} A(G)_{X \setminus F}$ or of $A(G)_{X \setminus F} \hat{\otimes} A(G)_F$ then h is

an element of $C(G)_{X \setminus F} \hat{\otimes} C(G)_{X \setminus F}$ or of $C(G)_{X \setminus F} \hat{\otimes} A(G)_F$ respectively; moreover the natural injections are norm decreasing. By (1.5.4)

$$A(G)_{X \setminus F} \hat{\otimes} A(G)_{X \setminus F} = A(G \times G)_{(X \setminus F) \times (X \setminus F)}$$

and
$$A(G)_{X \setminus F} \hat{\otimes} A(G)_F = A(G \times G)_{(X \setminus F) \times F},$$

so by the Hahn-Banach theorem $\tilde{S}_\phi$ and $\tilde{T}_\phi$ are pseudomeasures on $G \times G$.

Hence we may determine their Fourier transforms as in (1.4.3). If

$(\chi, \eta) \in (X \setminus F) \times (X \setminus F)$ then

$$\langle \widehat{\chi \otimes \eta}, \hat{\tilde{S}}_\phi \rangle = \tilde{S}_\phi(\chi \otimes \eta) = S_\phi(\chi, \eta) = \phi(\chi + \eta).$$

Hence

$$\hat{\tilde{S}}_\phi(\chi, \eta) = \overline{\phi(\chi + \eta)} \quad \text{for all } (\chi, \eta) \in (X \setminus F) \times (X \setminus F).$$

Similarly,

$$\hat{\tilde{T}}_\phi(\chi, \eta) = \overline{\phi(\chi + \eta)} \quad \text{for all } (\chi, \eta) \in (X \setminus F) \times F.$$

This leads to our final result.

PROPOSITION : The linear space $C^F(G)$ is a subalgebra of $C(G)$ if and only if for each $\phi \in \ell^\infty(X)$ which vanishes on $(X \setminus F)$, the map

$$(\chi, \eta) \longrightarrow \overline{\phi(\chi + \eta)}$$

agrees on $(X \setminus F) \times (X \setminus F)$ with the Fourier transform of a linear functional in $(C(G)_{X \setminus F} \hat{\otimes} C(G)_{X \setminus F})^*$ and on $(X \setminus F) \times F$ with the Fourier transform of a linear functional in $(C(G)_{X \setminus F} \hat{\otimes} A(G)_F)^*$.

Proof : The proof follows immediately from (2.3.6) and the argument given in the preceding paragraph. □

Remark : It is clear that Proposition (2.3.7) does not really help to solve the original problem. However, it does suggest that it might be worthwhile to look at the Banach spaces $C(G)_{F_1} \hat{\otimes} C(G)_{F_2}$ (and perhaps

$C(G)_{F_1} \hat{\otimes} A(G)_{F_2}$ for subsets F_1, F_2 of X . This I do in the remaining sections of the chapter. Where I have found it possible I allow G to be an arbitrary compact group and let F_1 and F_2 be subsets of $\Sigma(G)$.

Finally I should note that the results contained in this section are valid if $C(G)$ is replaced by, say, $A(G)$, but there seems to be little point in presenting any generalized results as the purpose of this section is purely one of motivation.

(2.4) General Properties of $V_{F_1, F_2}(G)$

Let G be an arbitrary compact group and let F_1 and F_2 be subsets of $\Sigma(G)$. Then $C(G)_{F_1}$ and $C(G)_{F_2}$ are closed subspaces of $C(G)$ and so are complete. Define $V_{F_1, F_2}(G)$ by

$$V_{F_1, F_2}(G) = C(G)_{F_1} \hat{\otimes} C(G)_{F_2}.$$

(This notation is a little misleading because it follows from (1.5.2) that the elements of $V_{F_1, F_2}(G)$ may be thought of as continuous functions on $G \times G$, and not on G . However, I will use it as it is consistent with that in Varopoulos [1].) By (1.5.2) $V_{F_1, F_2}(G)$ is identifiable with the set of all functions h which are continuous on $G \times G$ and which may be represented by

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n,$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and $\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty$ is finite. Then,

$$\|h\|_{V_{F_1, F_2}} = \inf \sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty$$

taken over all possible representations. We denote this norm by

$\|\cdot\|_{F_1, F_2}$ for simplicity.

(2.4.1) PROPOSITION : Let G be a compact abelian group and F_1, F_2 be subsets of X . Then $V_{F_1, F_2}(G)$ is a subalgebra of $C(G \times G)$ under pointwise multiplication if and only if F_1 and F_2 are semigroups in X .

Proof : Assume that F_1 and F_2 are semigroups in X , and let h_1 and h_2 be elements of $V_{F_1, F_2}(G)$ with representations

$$h_i = \sum_{n \in \mathbf{N}} f_{in} \otimes g_{in},$$

where $f_{in} \in C(G)_{F_1}$ and $g_{in} \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and

$\sum_{n \in \mathbf{N}} \|f_{in}\|_{\infty} \|g_{in}\|_{\infty}$ is finite, for $i = 1, 2$. For each $m, n \in \mathbf{N}$,

$f_{1m} f_{2n}$ is an element of $C(G)_{F_1}$ and $g_{1m} g_{2n}$ is an element of $C(G)_{F_2}$, and

$$\sum_{m, n \in \mathbf{N}} \|f_{1m} f_{2n}\|_{\infty} \|g_{1m} g_{2n}\|_{\infty} \leq \sum_{m \in \mathbf{N}} \|f_{1m}\|_{\infty} \|g_{1m}\|_{\infty} \sum_{n \in \mathbf{N}} \|f_{2n}\|_{\infty} \|g_{2n}\|_{\infty}$$

which is finite. Hence

$$h_1 h_2 = \sum_{m, n \in \mathbf{N}} f_{1m} f_{2n} \otimes g_{1m} g_{2n},$$

and $h_1 h_2$ is an element of $V_{F_1, F_2}(G)$.

On the other hand suppose, as we may without loss of generality, that F_1 is not a semigroup. Then there exists χ_1, χ_2 in F_1 such that $\chi_1 + \chi_2 \notin F_1$. Hence, for any $f \in C(G)_{F_2}$, $\chi_1 \otimes f$ and $\chi_2 \otimes f$ are elements of $V_{F_1, F_2}(G)$ but their product is not. $\square$

(2.4.2) PROPOSITION : Let G be a compact group and F_1, F_2 be subsets of $\Sigma(G)$. Then $V_{F_1, F_2}(G)$ is a convolution algebra.

Proof : Let h_1 and h_2 be elements of $V_{F_1, F_2}(G)$ and let $\epsilon > 0$.

Then there exists representations

$$h_i = \sum_{n \in \mathbf{N}} f_{in} \otimes g_{in} ,$$

where $f_{in} \in C(G)_{F_1}$ and $g_{in} \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and

$$\sum_{n \in \mathbf{N}} \|f_{in}\|_{\infty} \|g_{in}\|_{\infty} < \|h_i\|_{F_1, F_2} + \varepsilon , \quad (1)$$

for $i = 1, 2$. For $m, n \in \mathbf{N}$ and $x, y \in G$,

$$\begin{aligned} (f_{1m} \otimes g_{1m}) * (f_{2n} \otimes g_{2n})(x, y) &= \iint f_{1m}(a^{-1}x) f_{2n}(a) g_{1m}(a'^{-1}y) g_{2n}(a') d\lambda(a) d\lambda(a') \\ &= (f_{1m} * f_{2n}) \otimes (g_{1m} * g_{2n})(x, y). \end{aligned} \quad (2)$$

The double series

$$\sum_{m, n \in \mathbf{N}} (f_{1m} * f_{2n}) \otimes (g_{1m} * g_{2n})$$

converges uniformly on $G \times G$ because for any $x, y \in G$

$$\begin{aligned} &\sum_{m, n \in \mathbf{N}} |f_{1m} * f_{2n}(x) \cdot g_{1m} * g_{2n}(y)| \\ &\leq \sum_{m, n \in \mathbf{N}} \|f_{1m} * f_{2n}\|_{\infty} \|g_{1m} * g_{2n}\|_{\infty} \\ &\leq \sum_{m \in \mathbf{N}} \|f_{1m}\|_{\infty} \|g_{1m}\|_{\infty} \sum_{n \in \mathbf{N}} \|f_{2n}\|_{\infty} \|g_{2n}\|_{\infty} \\ &\leq (\|h_1\|_{F_1, F_2} + \varepsilon) (\|h_2\|_{F_1, F_2} + \varepsilon) \end{aligned}$$

by (1). Hence we can rearrange the double series to obtain

$$h_1 * h_2 = \sum_{m, n \in \mathbf{N}} (f_{1m} * f_{2n}) \otimes (g_{1m} * g_{2n})$$

and

$$\|h_1 * h_2\|_{F_1, F_2} \leq (\|h_1\|_{F_1, F_2} + \varepsilon) (\|h_2\|_{F_1, F_2} + \varepsilon) .$$

Therefore $h_1 * h_2$ is an element of $V_{F_1, F_2}(G)$ and

$$\|h_1 * h_2\|_{F_1, F_2} \leq \|h_1\|_{F_1, F_2} \|h_2\|_{F_1, F_2} . \quad \square$$

(2.4.3) Referring to (1.2.3) we have that for each $(x, y) \in G \times G$ and each $h \in C(G \times G)$ the left (x, y) -translate $(x, y)^h$ of h is defined by

$$(x, y)^h(a, a') = h(x^{-1}a, y^{-1}a') \quad \text{for all } (a, a') \in G \times G.$$

Similarly the right (x, y) -translate of h is defined by

$$h_{(x, y)}(a, a') = h(ax^{-1}, a'y^{-1}) \quad \text{for all } (a, a') \in G \times G.$$

The next proposition proves that $V_{F_1, F_2}(G)$ is a left-translation invariant subset of $C(G \times G)$, that each of the left translation operators is a continuous endomorphism of $V_{F_1, F_2}(G)$ and that the left regular representation of $G \times G$ on $V_{F_1, F_2}(G)$ is continuous. A similar result holds for right translations, but this will not be presented.

PROPOSITION : Let G be a compact group and F_1, F_2 be subsets of $\Sigma(G)$. For any $h \in V_{F_1, F_2}(G)$ and any $x, y \in G$, $(x, y)^h$ is an element of $V_{F_1, F_2}(G)$ and

$$\|(x, y)^h\|_{F_1, F_2} \leq \|h\|_{F_1, F_2}.$$

The map $(x, y) \longrightarrow (x, y)^h$ is continuous from $G \times G$ to $V_{F_1, F_2}(G)$ for each $h \in V_{F_1, F_2}(G)$.

Proof : Let h have the representation

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n,$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and $\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty$ is finite.

For any subset F of $\Sigma(G)$ the Banach space $C(G)_F$ is left translation invariant because $\hat{x} \hat{f}(\sigma) = \hat{U}_\sigma(x) \hat{f}(\sigma)$ for each $\sigma \in \Sigma(G)$.

Moreover $\|_x f\|_\infty \leq \|f\|_\infty$ for all $x \in G$ and all $f \in C(G)_F$. Hence

$$\sum_{n \in \mathbf{N}} \|_x f_n\|_\infty \|_y g_n\|_\infty \leq \sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty < \infty, \quad (1)$$

and so

$$(x, y)^h = \sum_{n \in \mathbf{N}} x^{f_n} \otimes y^{g_n}$$

where $x^{f_n} \in C(G)_{F_1}$ and $y^{g_n} \in C(G)_{F_2}$ for $n \in \mathbf{N}$. Therefore $(x, y)^h$ is an element of $V_{F_1, F_2}(G)$ and

$$\|(x, y)^h\|_{F_1, F_2} \leq \|h\|_{F_1, F_2}, \quad (2)$$

because for any $\varepsilon > 0$ we may have chosen the representation of h so that

$$\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty < \|h\|_{F_1, F_2} + \varepsilon.$$

Let $\varepsilon > 0$ be given. There exists $h_0 \in C(G)_{F_1} \otimes C(G)_{F_2}$ such that

$$\|h - h_0\|_{F_1, F_2} < \frac{1}{3} \varepsilon, \quad (3)$$

(see (1.5.2)). By (2) and (3) for all $x, y \in G$

$$\|(x, y)^h - (x, y)^{h_0}\|_{F_1, F_2} < \frac{1}{3} \varepsilon. \quad (4)$$

It is sufficient to prove that the map $(x, y) \longrightarrow (x, y)^h$ is continuous at $(1, 1)$. Now for each $x, y \in G$,

$$\begin{aligned} \|h - (x, y)^h\|_{F_1, F_2} &\leq \|h - h_0\|_{F_1, F_2} + \|h_0 - (x, y)^{h_0}\|_{F_1, F_2} + \\ &\quad + \|(x, y)^{h_0} - (x, y)^h\|_{F_1, F_2} \\ &\leq \frac{2}{3} \varepsilon + \|h_0 - (x, y)^{h_0}\|_{F_1, F_2} \quad \text{by (3) and (4).} \end{aligned}$$

Hence, it is sufficient to prove that the map $(x, y) \longrightarrow (x, y)^{h_0}$ is continuous at $(1, 1)$ for all $h_0 \in C(G)_{F_1} \otimes C(G)_{F_2}$.

Any such h_0 may be represented as a finite series

$$h_0 = \sum_{n=1}^N f_n \otimes g_n ,$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $1 \leq n \leq N$. Hence, for any $x, y \in G$

$$h_0 - (x, y) h_0 = \sum_{n=1}^N ((f_n \otimes g_n) - (x f_n \otimes y g_n)),$$

and

$$\|h_0 - (x, y) h_0\|_{F_1, F_2} \leq \sum_{n=1}^N (\|f_n\|_\infty \|g_n - y g_n\|_\infty + \|y g_n\|_\infty \|f_n - x f_n\|_\infty).$$

..... (5)

The left regular representation of G on $C(G)_F$ (F any subset of $\Sigma(G)$) is continuous so there exists a neighbourhood U of 1 such that if $x, y \in U$

$$\max_{1 \leq n \leq N} \{\|g_n - y g_n\|_\infty, \|f_n - x f_n\|_\infty\} < \frac{1}{3} \frac{\epsilon}{\sum_{n=1}^N (\|f_n\|_\infty + \|g_n\|_\infty)} . \quad (6)$$

Hence by (5) and (6) the map is continuous. $\square$

(2.4.4) The Fourier series of an element h of $V_{F_1, F_2}(G)$ may be determined from its representation as a series. By (1.3.6)

$\Sigma(G \times G) \simeq \Sigma(G) \otimes \Sigma(G)$. To simplify notation we will write $\hat{f}(\sigma \otimes \eta)$ as $\hat{f}(\sigma, \eta)$ for all $\sigma, \eta \in \Sigma(G)$.

PROPOSITION : Let G be a compact group and F_1, F_2 subsets of $\Sigma(G)$.

If h has the representation

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n ,$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and $\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty$ is finite, then for any $\sigma, \eta \in \Sigma(G)$

$$\hat{h}(\sigma, \eta) = \sum_{n \in \mathbf{N}} \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta)$$

in $B(H_\sigma \otimes H_\eta)$. In particular, $\text{supp } \hat{h} \subseteq F_1 \otimes F_2$ (which we write as $F_1 \times F_2$) and $\|h\|_{PM} \leq \|h\|_{F_1, F_2}$.

Proof : First let h be an element of $C(G)_{F_1} \otimes C(G)_{F_2}$. Then h has a representation

$$h = \sum_{n=1}^N f_n \otimes g_n$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $1 \leq n \leq N$. Let $\sigma, \eta \in \Sigma(G)$.

Then

$$\begin{aligned} \hat{h}(\sigma, \eta) &= \iint h(x, y) U_\sigma(x) \otimes U_\eta(y) d\lambda(x) d\lambda(y) \\ &= \sum_{n=1}^N \iint f_n(x) g_n(y) U_\sigma(x) \otimes U_\eta(y) d\lambda(x) d\lambda(y), \text{ and so} \\ \hat{h}(\sigma, \eta) &= \sum_{n=1}^N \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta). \end{aligned} \quad (1)$$

Now let $h \in V_{F_1, F_2}(G)$ and $\epsilon > 0$ be given. Then if

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n,$$

where $f_n \in C(G)_{F_1}$ and $g_n \in C(G)_{F_2}$ for $n \in \mathbf{N}$ and $\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty$ is finite, then there exists $N > 0$ such that for $m > N$

$$\|h - h_m\|_{F_1, F_2} < \epsilon, \quad (2)$$

where h_m denotes the m^{th} partial sum of the representation given for h . By (1)

$$\hat{h}_m(\sigma, \eta) = \sum_{n=1}^m \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta).$$

Hence for each $\sigma, \eta \in \Sigma(G)$ the sequence $(\hat{h}_m(\sigma, \eta))_{m \in \mathbf{N}}$ is bounded in $B(H_\sigma \otimes H_\eta)$ by

$$\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty. \quad (3)$$

It is also easily seen that $\sum_{n \in \mathbf{N}} \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta)$ is an element of

$B(H_\sigma \otimes H_\eta)$ with norm less than the sum in (3). Hence

$$\lim_{m \rightarrow \infty} \left\| \sum_{n \in \mathbb{N}} \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta) - \hat{h}_m(\sigma, \eta) \right\| = 0. \quad (4)$$

But from (2) $\lim_{m \rightarrow \infty} \|h - h_m\|_\infty = 0$, and so

$$\lim_{m \rightarrow \infty} \|h - h_m\|_{PM} = 0. \quad (5)$$

By (4) and (5)

$$\hat{h}(\sigma, \eta) = \sum_{n \in \mathbb{N}} \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta) \quad \text{for all } \sigma, \eta \in \Sigma(G).$$

It is clear from this that $\text{supp } \hat{h} \subseteq F_1 \times F_2$.

Finally we see that

$$\begin{aligned} & \sup_{\sigma, \eta \in \Sigma(G)} \left\| \sum_{n \in \mathbb{N}} \hat{f}_n(\sigma) \otimes \hat{g}_n(\eta) \right\| \\ & \leq \sup_{\sigma, \eta \in \Sigma(G)} \sum_{n \in \mathbb{N}} \|\hat{f}_n\|_\infty \|\hat{g}_n\|_\infty \\ & \leq \|h\|_{F_1, F_2} + \epsilon. \end{aligned}$$

Hence $\|h\|_{PM} \leq \|h\|_{F_1, F_2}$. □

Remark : We shall see in chapter 3 that propositions (2.4.3) and (2.4.4) prove that $V_{F_1, F_2}(G)$ is a homogeneous space in $\mathcal{B}_h$ (see (3.4.5) for the definition of $\mathcal{B}_h$); in fact with (2.4.2) they prove it is a homogeneous convolution Banach algebra and, as such, it is studied further in chapter 5.

It is easy to extend the results in this section to the case where $C(G)_{F_1}$ and $C(G)_{F_2}$ are replaced by arbitrary homogeneous Banach spaces of continuous functions in $\mathcal{B}_h$.

(2.5) Properties of the Transform of Elements of $V(G)$

Throughout this section G denotes a compact abelian group unless otherwise stated, and $V(G)$ denotes $V_{X, X}(G)$ (or $V_{\Sigma(G), \Sigma(G)}(G)$)

if G is an arbitrary compact group). Let ω be a non-negative function defined on $X \times X$. Recalling the notation in (2.2.1)

$$C(G \times G)_\omega = \{h \in C(G \times G) : \hat{h}\omega \in \ell^1(X \times X)\}.$$

It is shown in (36.15) of [HR] that $C(G \times G)_\omega = C(G \times G)$ if and only if ω is an element of $\ell^2(X \times X)$. In this section I consider the problem, when is $V(G)_\omega$ equal to $V(G)$? (This is really a question of $(V(G) - \ell^1(X))$ multipliers (see chapter 35 of [HR])). The results in (2.5.3) and (2.5.5) show that if G is infinite then there exist many functions ω , not in $\ell^2(X \times X)$, with $V(G)_\omega$ equal to $V(G)$. One consequence of this is that $V(G)$ is a proper subspace of $C(G \times G)$ if G is infinite. (If G is finite then $V(G) = C(G \times G)$.) Each of the propositions in this section is valid if X is replaced by a Sidon set F and $V(G)$ is replaced by $V_{F,F}(G) = C(G)_F \hat{\otimes} C(G)_F$.

We begin with some preliminaries.

(2.5.1) LEMMA : Let ω be a non-negative function on $X \times X$.

Then $V(G)_\omega = V(G)$ if and only if $(\hat{f} \otimes \hat{g})\omega \in \ell^1(X \times X)$ for all $f, g \in C(G)$.

Proof : Assume that for all $f, g \in C(G)$

$$(\hat{f} \otimes \hat{g})\omega \in \ell^1(X \times X). \quad (1)$$

It follows from (2.3.3) (taking $B = C(G \times G)$, $E_1 = C(G) \otimes \{1\}$ and $E_2 = \{1\} \otimes C(G)$) that (1) holds iff there exists a constant $K > 0$ such that for all $f, g \in C(G)$

$$\|(\hat{f} \otimes \hat{g})\omega\|_1 \leq K \|f\|_\infty \|g\|_\infty. \quad (2)$$

By (2) the map T_ω defined on $C(G) \times C(G)$ by

$$T_\omega(f, g) = \sum_{\chi, \eta \in X} \omega(\chi, \eta) \hat{f}(\chi) \hat{g}(\eta) \quad \text{for all } f, g \in C(G) \quad (3)$$

is continuous. It is easily seen to be bilinear, and so it corresponds to a bounded linear map $\tilde{T}_\omega$ on $V(G)$ which satisfies

$$\tilde{T}_\omega(f \otimes g) = T_\omega(f, g) \quad \text{for all } f \otimes g \in V(G). \quad (4)$$

Let $h \in V(G)$. We will show h is an element of $V(G)_\omega$ (clearly $V(G)_\omega$ is a subset of $V(G)$). Let h have the representation

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n$$

where $f_n, g_n \in C(G)$ for $n \in \mathbf{N}$ and

$$\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty < \infty. \quad (5)$$

By (2), (3), (4) and (5) the series

$$\sum_{n \in \mathbf{N}} \tilde{T}_\omega(f_n \otimes g_n) = \sum_{n \in \mathbf{N}} T_\omega(f_n, g_n)$$

is absolutely convergent, and so by (2.4.4),

$$\tilde{T}_\omega(h) = \sum_{\chi, \eta \in X} \omega(\chi, \eta) \hat{h}(\chi, \eta)$$

and

$$\|\omega \hat{h}\|_1 \leq \|\tilde{T}_\omega\| \|h\|_{X, X} < \infty.$$

Hence $V(G)_\omega = V(G)$.

The converse is trivial. □

(2.5.2) LEMMA : For any non-negative function ω on $X \times X$ the following are equivalent:

(i) $V(G)_\omega = V(G),$

(ii) $(\hat{f} \otimes \hat{g})\omega \in \ell^1(X \times X)$ for all $f, g \in L^2(G)$

and (iii) there exists a positive constant K such that for all

$$f, g \in L^2(G)$$

$$\|(\hat{f} \otimes \hat{g})\omega\|_1 \leq K \|f\|_2 \|g\|_2.$$

Proof : The implication (iii) $\Rightarrow$ (ii) is immediate; the implication (ii) $\Rightarrow$ (i) is a consequence of (2.5.1).

Now assume (i) holds. We will first show that (ii) must hold, and then also (iii). By (2.5.1) $(\hat{f} \otimes \hat{g})\omega$ is an element of $\ell^1(X \times X)$ for all f, g in $C(G)$. Therefore for each $f, g \in C(G)$ the series

$$\sum_{\chi \in X} |\hat{f}(\chi)| \left| \sum_{\eta \in X} \hat{g}(\eta) \omega(\chi, \eta) \right| \quad (1)$$

is finite, being bounded by $\|(\hat{f} \otimes \hat{g})\omega\|_1$. But then it follows that the sequence

$$\left(\sum_{\eta \in X} \hat{g}(\eta) \omega(\chi, \eta) \right)_{\chi \in X}$$

is an element of $\ell^2(X)$, (see introductory paragraph to §(2.5)), and if this is the case then the series in (1) is finite for all $f \in L^2(G)$ (see (35.4) of [HR]). Similarly, for each $f \in L^2(G)$, $g \in C(G)$ the series

$$\sum_{\eta \in X} |\hat{g}(\eta)| \left| \sum_{\chi \in X} \hat{f}(\chi) \omega(\chi, \eta) \right| \quad (2)$$

is bounded and so it will follow that the series in (2) is finite for all $g \in L^2(G)$. Therefore (ii) holds.

Finally to prove that (iii) holds let $f \in L^2(G)$ and define the map T_f from $L^2(G)$ to $\mathbb{C}$ by

$$T_f(g) = \sum_{\chi, \eta \in X} \omega(\chi, \eta) |\hat{f}(\chi)| |\hat{g}(\eta)| \quad \text{for all } g \in L^2(G).$$

Then T_f is a seminorm on $L^2(G)$. It is lower semicontinuous because it is the upper envelope of the continuous seminorms

$$g \longrightarrow \sum_{(\chi, \eta) \in I} \omega(\chi, \eta) |\hat{f}(\chi)| |\hat{g}(\eta)|$$

as I ranges over the finite subsets of $X \times X$. Therefore, by the

Uniform Boundedness principle T_f is continuous. Define the map T from $L^2(G)$ to $\mathbb{C}$ by

$$T(f) = \sup \left\{ \sum_{\chi, \eta \in X} \omega(\chi, \eta) |\hat{f}(\chi)| |\hat{g}(\eta)| : \|g\|_2 \leq 1 \right\} \text{ for } f \in L^2(G).$$

Then T is a seminorm on $L^2(G)$ and it is lower semicontinuous, being the upper envelope of the continuous seminorms

$$f \longrightarrow \sum_{(\chi, \eta) \in I} \omega(\chi, \eta) |\hat{f}(\chi)| |\hat{g}(\eta)|,$$

as I ranges over finite subsets of $X \times X$ and g over the unit ball in $L^2(G)$. Hence, a second application of the Uniform Boundedness principle proves that T is continuous and so (iii) holds. $\square$

(2.5.3) We now use (2.5.2), with the help of theorem 292 of Hardy, Littlewood and Polya [1], to characterize those non-negative functions ω on $X \times X$ with $V(G)_\omega = V(G)$.

PROPOSITION : Let G be a compact abelian group and ω a non-negative function on $X \times X$. Then $V(G)_\omega = V(G)$ if and only if

$$(i) \quad \sum_{\chi \in X} \omega(\chi, \eta)^2 \text{ is finite for each } \eta \in X, \text{ and}$$

$$(ii) \quad \sup \left\{ \sum_{\eta \in X} \left| \sum_{\chi \in X} \omega(\chi, \eta) \phi(\chi) \right|^2 : \|\phi\|_2 \leq 1 \right\} \text{ is finite.}$$

Proof : By (2.5.2) $V(G)_\omega = V(G)$ iff the bilinear operator T_ω defined on $L^2(G) \times L^2(G)$ by

$$T_\omega(f, g) = \sum_{\chi, \eta \in X} \omega(\chi, \eta) \hat{f}(\chi) \hat{g}(\eta) \text{ for } f, g \in L^2(G)$$

is continuous. Appealing to theorem 292 of Hardy, Littlewood and Polya [1] this holds iff conditions (i) and (ii) are valid. $\square$

Remark : The proposition (2.5.3) does not really give us an easy method of checking whether a given weight function on $X \times X$ has the property that $V(G)_\omega = V(G)$. In this direction, in (2.5.4) we give a necessary condition for ω to have this property, and in (2.5.5) we characterize those subsets F of $X \times X$ whose indicator function $\mathbb{1}_F$ has the property.

(2.5.4) PROPOSITION : Let G be a compact abelian group and ω a non-negative function on $X \times X$. Assume that $V(G)_\omega = V(G)$ and that

$$K^2 = \sup \left\{ \sum_{\eta \in X} \left| \sum_{\chi \in X} \omega(\chi, \eta) \phi(\chi) \right|^2 : \|\phi\|_2 \leq 1 \right\}.$$

Then

$$(i) \quad \sum_{\chi \in X} \omega(\chi, \eta)^2 \leq K \text{ for all } \eta \in X, \text{ and}$$

$$(ii) \quad \sum_{\eta \in X} \omega(\chi, \eta)^2 \leq K \text{ for all } \chi \in X.$$

Proof : It follows from (2.5.2) and Plancherel's theorem that the complex-valued map S_ω , defined on $\ell^2(X) \times \ell^2(X)$ by

$$S_\omega(\phi, \psi) = \sum_{\chi, \eta \in X} \omega(\chi, \eta) \phi(\chi) \psi(\eta),$$

is bounded; that is, for all $\phi, \psi \in \ell^2(X)$

$$\left| \sum_{\chi, \eta \in X} \omega(\chi, \eta) \phi(\chi) \psi(\eta) \right| \leq K \|\phi\|_2 \|\psi\|_2. \quad (1)$$

Let $\chi \in X$ and let I be a finite subset of X . Then $\phi = \mathbb{1}_{\{\chi\}}$ and $\psi = \mathbb{1}_I$ (indicator functions of subsets of X) are elements of $\ell^2(X)$ and by (1)

$$\left| \sum_{\eta \in I} \omega(\chi, \eta) \right| \leq K \|\mathbb{1}_I\|_2. \quad (2)$$

By theorem 15 of Hardy, Littlewood and Polya [1], (2) gives

$$\sum_{\eta \in X} |\omega(\chi, \eta)|^2 \leq K \text{ for each } \chi \in X.$$

Similarly,

$$\sum_{\chi \in X} |\omega(\chi, \eta)|^2 \leq K \text{ for each } \eta \in X. \quad \square$$

(2.5.5) Let F be a subset of $X \times X$. For each $\chi \in X$ we define the sets F_χ and F_χ^* by

$$F_\chi = \{\eta \in X : (\eta, \chi) \in F\}$$

$$F_\chi^* = \{\eta \in X : (\chi, \eta) \in F\}.$$

Let ω denote the indicator function of the subset F in $X \times X$.

PROPOSITION : Let G be a compact abelian group. Then $V(G)_\omega = V(G)$ if and only if there exists a positive constant K such that

$$(i) \quad |F_\chi| \leq K \text{ for each } \chi \in X, \text{ and}$$

$$(ii) \quad |F_\chi^*| \leq K \text{ for each } \chi \in X.$$

Proof : Assume F satisfies (i) and (ii). It follows from (2.5.2) that it is sufficient to prove that for any $\phi, \psi \in \ell^2(X)$ with finite supports

$$\sum_{(\chi, \eta) \in F} \phi(\chi) \psi(\eta) \leq K \|\phi\|_2 \|\psi\|_2. \quad (1)$$

But

$$\begin{aligned} \sum_{(\chi, \eta) \in F} \phi(\chi) \psi(\eta) &= \sum_{\chi \in X} \phi(\chi) \left(\sum_{\eta \in F_\chi^*} \psi(\eta) \right) \\ &\leq \sum_{\chi \in X} \phi(\chi) |F_\chi^*|^{\frac{1}{2}} \left(\sum_{\eta \in F_\chi^*} \psi(\eta)^2 \right)^{\frac{1}{2}} \\ &\leq K^{\frac{1}{2}} \sum_{\chi \in X} \phi(\chi) \left(\sum_{\eta \in F_\chi^*} \psi(\eta)^2 \right)^{\frac{1}{2}} \quad \text{by (ii)} \\ &\leq K^{\frac{1}{2}} \left(\sum_{\chi \in X} \phi(\chi)^2 \right)^{\frac{1}{2}} \left(\sum_{\chi \in X} \sum_{\eta \in F_\chi^*} \psi(\eta)^2 \right)^{\frac{1}{2}} \quad \text{and so} \\ \sum_{(\chi, \eta) \in F} \phi(\chi) \psi(\eta) &\leq K^{\frac{1}{2}} \|\phi\|_2 \left(\sum_{\chi \in X} \sum_{\eta \in F_\chi^*} \psi(\eta)^2 \right)^{\frac{1}{2}}; \end{aligned} \quad (2)$$

and,

$$\begin{aligned}
 \sum_{\chi \in X} \sum_{\eta \in F_{\chi}^*} \psi(\eta)^2 &= \sum_{\chi \in X} \sum_{\eta \in X} \mathbb{1}_{F_{\chi}^*}(\eta) \psi(\eta)^2 \\
 &= \sum_{\chi, \eta \in X} \mathbb{1}_F(\chi, \eta) \psi(\eta)^2 \\
 &= \sum_{\eta \in X} \psi(\eta)^2 \sum_{\chi \in X} \mathbb{1}_F(\chi, \eta) \\
 &= \sum_{\eta \in X} |F_{\eta}| \psi(\eta)^2, \text{ and so}
 \end{aligned}$$

$$\sum_{\chi \in X} \sum_{\eta \in F_{\chi}^*} \psi(\eta)^2 \leq K \|\psi\|_2^2. \quad (3)$$

Then by (2) and (3), inequality (1) holds.

The converse follows immediately from (2.5.4). $\square$

Remark : Let G be a compact abelian group, and let F be the subset of $X \times X$ given by

$$F = \{(\chi, \chi) : \chi \in X\}.$$

(If G is infinite then so too is X and therefore F .) Let ω denote the indicator function $\mathbb{1}_F$ of F in $X \times X$. By (2.5.5), $V(G)_{\omega} = V(G)$. However if G is infinite then

$$C(G \times G)_{\omega} \subsetneq C(G \times G).$$

Hence

$$V(G) \neq C(G \times G). \quad (1)$$

Now let ω denote any positive function on $X \times X$ satisfying

$$V(G) \subseteq C(G \times G)_{\omega}.$$

By (1.5.4) $A(G \times G) = A(G) \hat{\otimes} A(G)$ and so $A(G \times G) \subsetneq V(G) \subseteq C(G \times G)_{\omega}$.

According to (35.4) of [HR], ω must be an element of $\ell^{\infty}(X \times X)$. Hence

if G is infinite we may apply (2.2.10) to conclude that

$V(G) \neq C(G \times G)_\omega$ because by (1) $V(G) \neq C(G \times G)$.

(2.5.6) As we already noted in the introductory paragraph

$C(G \times G)_\omega = C(G \times G)$ if and only if ω is an element of $\ell^2(X \times X)$.

The following ^{proposition} shows that there does not exist a similar result for $V(G)$.

PROPOSITION : Let G be a compact abelian group and let p be greater than 2. Then there exists a function ω in $\ell^p(X \times X)$ such that $V(G)_\omega \neq V(G)$.

Proof : Let γ be a non-negative function in $\ell^p(X)$ but not in $\ell^2(X)$. Then there exists a function f in $C(G)$ such that $f \notin C(G)_\gamma$. Define the non-negative function ω on $X \times X$ by

$$\omega(\chi, \eta) = \begin{cases} \gamma(\chi) & \text{if } \eta = 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Then ω is an element of $\ell^p(X \times X)$. Define the element h of $V(G)$ by $h = f \otimes 1$. By (2.4.4)

$$\hat{h}(\chi, \eta) = \begin{cases} \hat{f}(\chi) & \text{if } \eta = 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Hence h is not in $V(G)_\omega$ because f is not in $C(G)_\gamma$. □

(2.6) Estimation of Projective Norms

Throughout this section, with the exception of (2.6.9), G denotes a compact abelian group with dual X and F_1, F_2 are subsets of X . It follows from (1.5.2) and (2.4.4) that

$$V_{F_1, F_2}(G) \subseteq V(G)_{F_1 \times F_2} \subseteq C(G \times G)_{F_1 \times F_2}.$$

In this section I present some partial results concerning the possibility of equality existing in each of these inclusions (see (2.6.4) and (2.6.9)). The remainder of the section is devoted to estimating the norms of some elements of $V_{F_1, F_2}(G)$. In particular I show in (2.6.6) and (2.6.7) that if either F_1 or F_2 is a Sidon subset of X then the norm of every element of $V_{F_1, F_2}(G)$ can be simply bounded both above and below.

I begin by assuming that $F_1 = F = F_2$ for some subset F of X and considering certain restrictions of the maps M and P introduced in Hertz [1]. They will be used in (2.6.4) to show that $V_{F, F}(G)$ is equal to $C(G \times G)_{F \times F}$ if and only if F is a finite subset of X .

(2.6.1) DEFINITION : Let f be an element of $C(G)_F$ and h an element of $C(G \times G)_{F \times F}$. Define the complex-valued function Mf on $G \times G$ by

$$Mf(x, y) = f(x + y) \quad \text{for all } x, y \in G,$$

and the complex-valued function Ph on G by

$$Ph(x) = \int h(x - y, y) d\lambda(y) \quad \text{for all } y \in G.$$

Then Mf is an element of $C(G \times G)$ because the group operation is continuous, and Ph is an element of $C(G)$ because h is uniformly continuous on $G \times G$.

(2.6.2) LEMMA : Mf is an element of $C(G \times G)_{F \times F}$ for all $f \in C(G)_F$ and Ph is an element of $C(G)_F$ for all $h \in C(G \times G)_{F \times F}$. The map $P \circ M$ is the identity on $C(G)_F$.

Proof : Let $\chi, \eta \in X$. Then

$$\begin{aligned}
 \widehat{Mf}(\chi, \eta) &= \iint Mf(x, y) \overline{\chi(x)} \overline{\eta(y)} d\lambda(x) d\lambda(y) \\
 &= \iint f(x+y) \overline{\chi(x)} \overline{\eta(y)} d\lambda(x) d\lambda(y) \\
 &= \delta_{\chi\eta} \iint f(x+y) \overline{\chi(x+y)} d\lambda(x) d\lambda(y) \\
 &= \delta_{\chi\eta} \widehat{f}(\chi),
 \end{aligned}$$

and so $\widehat{Mf}$ has support in $F \times F$. Also

$$\begin{aligned}
 \widehat{Ph}(\chi) &= \int Ph(x) \overline{\chi(x)} d\lambda(x) \\
 &= \iint h(x-y, y) \overline{\chi(x)} d\lambda(y) d\lambda(x) \\
 &= \iint h(x-y, y) \overline{\chi(x-y)} \overline{\chi(y)} d\lambda(y) d\lambda(x) \\
 &= \widehat{h}(\chi, \chi),
 \end{aligned}$$

and so $\widehat{Ph}$ has support in F . It is an immediate consequence of the definition (2.6.1) that PoM is the identity map. $\square$

(2.6.3) PROPOSITION : Let G be a compact abelian group and F a subset of X . Then Mf lies in $V_{F,F}(G)$ with

$$\|Mf\|_{F,F} \leq \|f\|_A \quad \text{for all } f \in A(G)_F;$$

and Ph lies in $A(G)_F$ with

$$\|Ph\|_A \leq \|h\|_{F,F} \quad \text{for all } h \in V_{F,F}(G).$$

Also P maps $V_{F,F}(G)$ onto $A(G)_F$.

Proof : Let $f \in A(G)_F$. Write

$$f = \sum_{\chi \in F} \alpha_\chi \chi$$

where the sequence $(\alpha_\chi)_{\chi \in F}$ is in $\ell^1(F)$. Then for all $x, y \in G$

$$Mf(x, y) = f(x+y) = \sum_{\chi \in F} \alpha_\chi \chi(x+y) = \sum_{\chi \in F} \alpha_\chi \chi \otimes \chi(x, y).$$

Hence Mf is in $V_{F,F}(G)$ and

$$\|Mf\|_{F,F} \leq \sum_{\chi \in F} |\alpha_\chi| = \|f\|_A.$$

Now let $h \in V_{F,F}(G)$ and $\varepsilon > 0$ be given. Then h may be represented by the series

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n,$$

where $f_n, g_n \in C(G)_F$ for $n \in \mathbf{N}$ and

$$\sum_{n \in \mathbf{N}} \|f_n\|_\infty \|g_n\|_\infty < \|h\|_{F,F} + \varepsilon. \quad (1)$$

Then for all $x \in G$

$$\begin{aligned} Ph(x) &= \int h(x-y, y) d\lambda(y) \\ &= \sum_{n \in \mathbf{N}} \int f_n(x-y) g_n(y) d\lambda(y) \end{aligned}$$

because, by (1), the series representing h is uniformly convergent on $G \times G$. Hence

$$Ph(x) = \sum_{n \in \mathbf{N}} f_n * g_n(x). \quad (2)$$

But for each $f, g \in C(G)_F$

$$f * g \in A(G)_F \text{ and } \|f * g\| \leq \|f\|_\infty \|g\|_\infty. \quad (3)$$

By (1), (2), (3) and (2.6.2), Ph is an element of $A(G)_F$ and

$$\|Ph\|_A \leq \|h\|_{F,F}.$$

The map P is a surjection because for each f in $A(G)_F$, Mf is an element of $V_{F,F}(G)$ and $P(Mf) = f$. $\square$

Finally we obtain

(2.6.4) PROPOSITION : Let G be a compact abelian group and F a subset of X . Then $V_{F,F}(G) = C(G \times G)_{F \times F}$ if and only if F is finite.

Proof : Assume first that F is not a Sidon subset of X . Then $A(G)_F$ is a proper subspace of $C(G)_F$. Hence $V_{F,F}(G) \subsetneq C(G \times G)_{F \times F}$. This is because otherwise $P(V_{F,F}(G)) = P(C(G \times G)_{F \times F})$ and so, by (2.6.3) and (2.6.2), $A(G)_F = C(G)_F$, a contradiction.

If F is a Sidon subset of X then

$$V_{F,F}(G) = C(G)_F \hat{\otimes} C(G)_F = A(G)_F \hat{\otimes} A(G)_F = A(G \times G)_{F \times F}.$$

But $A(G \times G)_{F \times F}$ is equal to $C(G \times G)_{F \times F}$ iff $F \times F$ is a Sidon subset of $X \times X$, and it is proved in (3.5) of Edwards, Hewitt and Ross [1] that this occurs iff F is a finite subset of X .

The converse is immediate. □

(2.6.5) We now let F_1 and F_2 be subsets of X . In the next proposition we present an estimation of the norm of an element h in $V_{F_1, F_2}(G)$ which permits the representation

$$\left. \begin{aligned} h &= \sum_{n \in \mathbf{N}} H_n \otimes \chi_n \text{ with } H_n \in C(G)_{F_1}, \chi_n \in F_2 \text{ for } n \in \mathbf{N} \\ \text{and } \sum_{n \in \mathbf{N}} \|H_n\|_{\infty} &\text{ is finite.} \end{aligned} \right\} \quad (1)$$

PROPOSITION : Let G be a compact abelian group, and F_1, F_2 be subsets of X . Let h be an element of $V_{F_1, F_2}(G)$ which permits a representation as in (1). Then

$$\left(\sum_{n \in \mathbf{N}} \|H_n\|_{\infty}^2 \right)^{\frac{1}{2}} \leq \|h\|_{F_1, F_2} \leq \sum_{n \in \mathbf{N}} \|H_n\|_{\infty}.$$

Proof : The second inequality is an immediate consequence of the definition of the norm in $V_{F_1, F_2}(G)$.

Now let $N \in \mathbf{N}$ and let $x_1, \dots, x_N$ be elements of G and $\alpha_1, \dots, \alpha_N$ be complex numbers. Then

$$\left| \sum_{n=1}^N \alpha_n H_n(x_n) \right| = \left| \sum_{n=1}^N \alpha_n \int h(x_n, x) \overline{\chi_n(x)} d\lambda(x) \right|. \quad (1)$$

Let $\epsilon > 0$. Assume that h has the representation

$$h = \sum_{i \in \mathbf{N}} f_i \otimes g_i, \quad (2)$$

where $f_i \in C(G)_{F_1}$ and $g_i \in C(G)_{F_2}$ for $i \in \mathbf{N}$, and

$$\sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \|g_i\|_{\infty} < \|h\|_{F_1, F_2} + \epsilon. \quad (3)$$

Then by (3) the series representing h in (2) is uniformly convergent on $G \times G$. Hence by (1)

$$\begin{aligned} \left| \sum_{n=1}^N \alpha_n H_n(x_n) \right| &= \left| \sum_{n=1}^N \alpha_n \sum_{i \in \mathbf{N}} f_i(x_n) \hat{g}_i(\chi_n) \right| \\ &\leq \sum_{i \in \mathbf{N}} \left| \sum_{n=1}^N \alpha_n f_i(x_n) \hat{g}_i(\chi_n) \right| \\ &\leq \sum_{i \in \mathbf{N}} \left(\sum_{n=1}^N |\alpha_n f_i(x_n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^N |\hat{g}_i(\chi_n)|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \|g_i\|_{\infty} \left(\sum_{n=1}^N |\alpha_n|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and so by (3)

$$\left(\sum_{n=1}^N |H_n(x_n)|^2 \right)^{\frac{1}{2}} \leq \sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \|g_i\|_{\infty} < \|h\|_{F_1, F_2} + \epsilon.$$

Therefore,

$$\left(\sum_{n=1}^N \|H_n\|_{\infty}^2 \right)^{\frac{1}{2}} < \|h\|_{F_1, F_2} + \varepsilon ,$$

and so,

$$\left(\sum_{n=1}^N \|H_n\|_{\infty}^2 \right)^{\frac{1}{2}} \leq \|h\|_{F_1, F_2} . \quad (4)$$

But the R.H.S. of (4) is independent of N so we can conclude that

$$\left(\sum_{n \in \mathbf{N}} \|H_n\|_{\infty}^2 \right)^{\frac{1}{2}} \leq \|h\|_{F_1, F_2} . \quad \square$$

(2.6.6) COROLLARY : Let G be a compact abelian group and F_1, F_2 be subsets of X . Assume that F_2 is Sidon and let h be an element of $V_{F_1, F_2}(G)$ which permits a representation as in (1). Then

$$K \sum_{n \in \mathbf{N}} \|H_n\|_{\infty} \leq \|h\|_{F_1, F_2} \leq \sum_{n \in \mathbf{N}} \|H_n\|_{\infty} ,$$

where K^{-1} is the Sidon constant of F_2 .

Proof : In view of the proposition (2.6.5) it is sufficient to prove the first inequality. With the notation of the proof of (2.6.5)

$$\begin{aligned} \left| \sum_{n=1}^N \alpha_n H_n(x_n) \right| &= \left| \sum_{n=1}^N \alpha_n \sum_{i \in \mathbf{N}} f_i(x_n) \hat{g}_i(\chi_n) \right| \\ &\leq \sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \sup_{1 \leq n \leq N} |\alpha_n| \left(\sum_{n=1}^N |\hat{g}_i(\chi_n)| \right) \\ &\leq K^{-1} \sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \|g_i\|_{\infty} \sup_{1 \leq n \leq N} |\alpha_n| \end{aligned}$$

where K^{-1} is the Sidon constant of F_2 , and so

$$\sum_{n=1}^N \|H_n\|_{\infty} \leq K^{-1} \sum_{i \in \mathbf{N}} \|f_i\|_{\infty} \|g_i\|_{\infty} < K^{-1} (\|h\|_{F_1, F_2} + \varepsilon)$$

for all $\varepsilon > 0$. Hence

$$\sum_{n=1}^N \|H_n\|_{\infty} \leq K^{-1} \|h\|_{F_1, F_2}. \quad (1)$$

But the R.H.S. of (1) is independent of N and so

$$K \sum_{n \in \mathbf{N}} \|H_n\|_{\infty} \leq \|h\|_{F_1, F_2}. \quad \square$$

(2.6.7) We might ask when does every element h of $V_{F_1, F_2}(G)$ have a representation of the form given in (1) of (2.6.5). We prove that this is the case if F_2 is a Sidon subset of X . But first we give a definition.

DEFINITION : Let h be an element of $C(G \times G)$ and χ be an element of X . Define the complex-valued function H_{χ} on G by

$$H_{\chi}(x) = \int h(x, y) \bar{\chi}(y) d\lambda(y) \quad \text{for each } x \in G.$$

It is easily seen that H_{χ} is an element of $C(G)$ because h is uniformly continuous on $G \times G$, and for all $x \in G$

$$\begin{aligned} |H_{\chi}(x) - H_{\chi}(1)| &= \left| \int (h(x, y) - h(1, y)) \bar{\chi}(y) d\lambda(y) \right| \\ &\leq \sup_{y \in G} |h(x, y) - h(1, y)|. \end{aligned}$$

If h permits a representation of the form (1) in (2.6.5) then $H_{\chi} = H_n$ if $\chi = \chi_n$ and $H_{\chi} = 0$ elsewhere.

PROPOSITION : Let G be a compact abelian group and F_1, F_2 be subsets of X , with F_2 a Sidon set. Then if h is an element of $V_{F_1, F_2}(G)$

$$h = \sum_{\chi \in F_2} H_{\chi} \otimes \chi \quad (1)$$

and $\sum_{\chi \in F_2} \|H_\chi\|_\infty$ is finite. At most a countable number of the terms in the sum in (1) are non-zero.

Proof : Let h be an element of $V_{F_1, F_2}(G)$. We know that $T(G)_{F_1}$ is a dense subspace of $C(G)_{F_1}$ and $T(G)_{F_2}$ of $C(G)_{F_2}$ and so $T(G)_{F_1} \otimes T(G)_{F_2} = T(G \times G)_{F_1 \times F_2}$ is a dense subspace of $V_{F_1, F_2}(G)$. Let $(p_n)_{n \in \mathbb{N}}$ be a sequence extracted from $T(G \times G)_{F_1 \times F_2}$ such that

$$\lim_{n \rightarrow \infty} \|h - p_n\|_{F_1, F_2} = 0. \quad (2)$$

For each $n \in \mathbb{N}$, the support of $\hat{p}_n$ is finite, and so if

$$H_{n\chi}(x) = \int p_n(x, y) \overline{\chi(y)} d\lambda(y) \quad \text{for all } x \in G,$$

then $H_{n\chi}$ is non-zero for at most a finite number of χ . So

$$p_n = \sum_{\chi \in F_2} H_{n\chi} \otimes \chi, \quad (3)$$

and, by (2.6.6), for each $n \in \mathbb{N}$

$$K \sum_{\chi \in F_2} \|H_{n\chi}\|_\infty \leq \|p_n\|_{F_1, F_2}. \quad (4)$$

By (2), given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|h - p_n\|_\infty < \varepsilon$ for $n > N$, and so for $n > N$,

$$\|H_\chi - H_{n\chi}\|_\infty = \sup_{x \in G} \left| \int (h(x, y) - p_n(x, y)) \overline{\chi(y)} d\lambda(y) \right| < \varepsilon. \quad (5)$$

It follows from (4) and (5) that for any finite subset I of F_2

$$K \sum_{\chi \in I} \|H_\chi\|_\infty \leq \lim_{n \rightarrow \infty} \|p_n\|_{F_1, F_2}, \quad (6)$$

and so, because the RHS of (6) is independent of I and is finite,

$$K \sum_{\chi \in F_2} \|H_\chi\|_\infty \leq \lim_{n \rightarrow \infty} \|p_n\|_{F_1, F_2}.$$

By (2), (3) and (5) h has a representation as in (1). The series has at most a countable number of non-zero terms because the support of $\hat{h}$ must be countable (see (1.4.2)). $\square$

(2.6.8) The following result was suggested to me, in the case

$F_1 = F_2 = X$, by R.E. Edwards. In it I prove that every element of $A(G \times G)_{F_1 \times F_2}$ (which is a subspace of $V_{F_1, F_2}(G)$) has a representation of the form in (1) of (2.6.5).

PROPOSITION : Let G be a compact abelian group and F_1, F_2 be subsets of X . Let h be an element of $A(G \times G)_{F_1 \times F_2}$. Then

$$h = \sum_{\chi \in F_2} H_\chi \otimes \chi$$

where $\sum_{\chi \in F_2} \|H_\chi\|_\infty$ is finite.

Proof : The function h may be written in the form

$$h = \sum_{(\eta, \chi) \in F_1 \times F_2} \hat{h}(\eta, \chi) \eta \otimes \chi, \quad (1)$$

with

$$\sum_{(\eta, \chi) \in F_1 \times F_2} |\hat{h}(\eta, \chi)| < \infty. \quad (2)$$

By (2), the function H^χ defined on G by

$$H^\chi(x) = \sum_{\eta \in F_1} \hat{h}(\eta, \chi) \eta(x) \quad \text{for each } x \in G, \quad (3)$$

is an element of $A(G)_{F_1}$, and so of $C(G)_{F_1}$, for all $\chi \in F_2$.

We have

$$\|H^\chi\|_\infty \leq \sum_{\eta \in F_1} |\hat{h}(\eta, \chi)| \quad (4)$$

which is finite by (2). Hence by (1) and (2)

$$h = \sum_{\chi \in F_2} H^\chi \otimes \chi . \quad (5)$$

Now for each $\rho \in F_2$ and $x \in G$

$$\begin{aligned} H_\rho(x) &= \int h(z, y) \overline{\rho(y)} d\lambda(y) \\ &= \sum_{\chi \in F_2} \int H^\chi(x) \chi(y) \overline{\rho(y)} d\lambda(y) \quad \text{by (2), (4) and (5)} \\ &= \sum_{\chi \in F_2} H^\chi(x) \delta_{\chi\rho} \\ &= H^\rho(x) . \end{aligned}$$

Hence $H_\rho = H^\rho$ and so (2), (4) and (5) complete the proof. $\square$

Remark : The results in (2.6.5), (2.6.6), (2.6.7) and (2.6.8) have obvious analogues when the roles of F_1 and F_2 are interchanged.

(2.6.9) We complete this chapter with a proposition which is valid for an arbitrary compact group G .

For each subset F of $\Sigma(G)$ define the element $\mathbb{I}_F$ of $\underline{E}^\infty(\Sigma(G))$ by

$$\mathbb{I}_F(\sigma) = \begin{cases} \text{Id}_\sigma & \text{if } \sigma \in F \\ 0 & \text{if } \sigma \notin F . \end{cases}$$

PROPOSITION : Let G be a compact group and F_1, F_2 be subsets of $\Sigma(G)$. Assume that there exist $\mu_1, \mu_2 \in M(G)$ such that

$$\hat{\mu}_1 = \mathbb{I}_{F_1} \quad \text{and} \quad \hat{\mu}_2 = \mathbb{I}_{F_2} .$$

Then $V_{F_1, F_2}(G) = V(G)_{F_1 \times F_2}$.

Proof : By (2.4.4) $V_{F_1, F_2}(G)$ is a subspace of $V(G)_{F_1 \times F_2}$.

Now let h be an element of $V(G) (= V_{\Sigma(G), \Sigma(G)}(G))$ which has the representation

$$h = \sum_{n \in \mathbf{N}} f_n \otimes g_n, \quad (1)$$

with $f_n, g_n \in C(G)$ for $n \in \mathbf{N}$ and

$$\sum_{n \in \mathbf{N}} \|f_n\|_{\infty} \|g_n\|_{\infty} < \infty. \quad (2)$$

Let $\mu = \mu_1 \otimes \mu_2$. Then μ is an element of $M(G \times G)$. It is easy to see that $\text{supp } \hat{\mu} = F_1 \times F_2$. Hence $\mu * h$ is an element of $C(G \times G)_{F_1 \times F_2}$. Also $\mu_1 * f_n$ is an element of $C(G)_{F_1}$ and $\mu_2 * g_n$ is an element of $C(G)_{F_2}$ for all $n \in \mathbf{N}$; by (2)

$$\sum_{n \in \mathbf{N}} \|\mu_1 * f_n\|_{\infty} \|\mu_2 * g_n\|_{\infty} < \infty, \quad (3)$$

and

$$\mu * h = \sum_{n \in \mathbf{N}} (\mu_1 * f_n) \otimes (\mu_2 * g_n). \quad (4)$$

By (3) and (4) $\mu * h$ is an element of $V_{F_1, F_2}(G)$.

Define the map T from $V(G)$ to $V_{F_1, F_2}(G)$ by

$$T(h) = \mu * h \text{ for all } h \in V(G). \quad (5)$$

We will show that T has a closed graph, and so by the Closed Graph Theorem T will be continuous. Hence there exists a positive constant such that for all $h \in V(G)$

$$\|\mu * h\|_{F_1, F_2} \leq K \|h\|_{\Sigma(G), \Sigma(G)}. \quad (6)$$

However, if $h \in V(G)_{F_1 \times F_2}$ then $\mu * h = h$, and so by (6)

$$\|h\|_{F_1, F_2} \leq K \|h\|_{\Sigma(G), \Sigma(G)} \text{ for all } h \in V(G)_{F_1 \times F_2}. \text{ Hence}$$

$$V(G)_{F_1 \times F_2} \subseteq V_{F_1, F_2}(G).$$

To complete the proof it only remains to prove that the map T defined in (5) has a closed graph. Let $(h_n)_{n \in \mathbf{N}}$ be a sequence in $V(G)$ which converges to 0 in $V(G)$, and such that there exists H in $V_{F_1, F_2}(G)$ satisfying

$$\lim_{n \rightarrow \infty} \|Th_n - H\|_{F_1, F_2} = \lim_{n \rightarrow \infty} \|\mu * h_n - H\|_{F_1, F_2} = 0. \quad (7)$$

By (2.4.4) and (7)

$$\lim_{n \rightarrow \infty} \|\mu * h_n - H\|_{PM} = 0. \quad (8)$$

But $(\mu * h_n)^\wedge = \hat{\mu} \hat{h}_n$ and so $(\mu * h_n)_{n \in \mathbf{N}}$ must tend to 0 in $PM(G)$ because $(h_n)_{n \in \mathbf{N}}$ does. Hence by (8) $H = 0$ and T has a closed graph. $\square$

CHAPTER 3

EXAMPLES AND BASIC PROPERTIES
OF HOMOGENEOUS BANACH SPACES(3.1) Introduction

Let G denote a compact group, and let F_1, F_2 denote subsets of $\Sigma(G)$. I proved in (2.4.3) that the convolution Banach algebra

$V_{F_1, F_2}(G)$ has the following properties:

(a) it is translation invariant (that is, $(x, y)^h \in V_{F_1, F_2}(G)$ for each x, y in G and h in $V_{F_1, F_2}(G)$), and

(b) the map $(x, y) \rightarrow (x, y)^h$ is continuous from $G \times G$ to $V_{F_1, F_2}(G)$ for each h in $V_{F_1, F_2}(G)$.

These properties were probably first isolated as crucial to the structure of many algebras in Segal [1]. They were not exploited to full advantage, however, until 1965 when H. Reiter in Reiter [3], and subsequently in Reiter [1] and [2], introduced and studied Segal algebras. These algebras are dense subspaces of $L^1(G)$ which possess the obvious analogues of properties (a) and (b).

One of the most notable of Reiter's results, proved initially for a locally compact abelian group and extended to the compact non-abelian case by Burnham in Burnham [1], [2], is his ideal theorem. This asserts that the left ideal theory of a Segal algebra is the same as that of $L^1(G)$. I will consider Reiter's theorem and its possible generalizations in Chapter 5.

In the process of proving the more general version of this theorem Burnham introduced A -Segal algebras, one generalization of Segal algebras. Another possible generalization is that of a normed ideal which was introduced in Cigler [1] and further studied in Riemersma [1], but which I do not consider. Yet another extension of this class of algebras is

the class of homogeneous Banach algebras which actually dates back to Silov [1]. In fact the algebraic structure may be neglected and a study be made of homogeneous Banach spaces. In [E] and Katznelson [1] the authors present much of the classical Fourier series theory for the algebras $C(G)$ and $L^p(G)$, $1 \leq p < \infty$ (for G compact and $G = \mathbb{T}$ respectively) by studying a generic homogeneous algebra. Homogeneous Banach spaces and Banach algebras are the subject of the monograph Wang [1] and with one extension provide the setting for the main part of the remainder of this thesis.

Segal algebras and homogeneous Banach spaces have a common property which is an important tool in their study. It is that each includes a set of trigonometric polynomials which is a dense subset. This property has been exploited numerous times in the papers mentioned above and I also take advantage of it in the sequel. In fact in (4.2.1) I show exactly what form this set of trigonometric polynomials must have.

Throughout this chapter G denotes an arbitrary compact group. The extension to which I refer above is that I omit the assumption that all spaces are contained in $L^1(G)$, and rather will allow them to consist of pseudomeasures defined on G . It is interesting to note that many of the known results concerning homogeneous Banach spaces extend easily to this more general setting.

To begin with I let B denote a Banach space which has the property that there exists a representation L of G in the space of continuous endomorphisms of B . Within this framework I introduce the notion of continuously L -translation invariance and L -homogeneity, and consider some elementary properties of these spaces. This is not an original approach as it is mentioned, in passing, in Wang [1]. In particular I prove that there exists a subspace $P(B)$ of B which

plays the role of the trigonometric polynomials; that is, it is a dense subset of B if B is L -homogeneous. This is an important corollary of (3.3.10) which is the principal result of this chapter, and is used repeatedly throughout the remainder of this thesis.

Theorem (3.3.10) asserts that B is L -homogeneous if and only if for each b in B

$$\lim_{n \rightarrow \infty} \|k_n * b - b\|_B = 0,$$

where $(k_n)_{n \in \mathbb{N}}$ is an approximate identity in $L^1(G)$ and $k_n * b$ is defined in (3.3.6).

A large, but natural, family of examples arises if I assume that B consists of pseudomeasures defined on G and that L is the left regular representation of G defined in the space of continuous endomorphisms of B . Chapters 4 and 5 are devoted to the study of Banach spaces in this family. But first in sections (3.4) and (3.5) there are many examples of such spaces together with several examples of generalized L -homogeneous spaces. I show that if B consists of pseudomeasures then $P(B) = B \cap T(G)$ and so $P(B)$ really does play the role of the trigonometric polynomials (see (3.5.2)). The results of these sections are summarized in (3.5.9).

(3.2) Continuously L -translation Invariant Banach Spaces

Throughout this chapter G denotes a compact group and $\Sigma(G)$ its dual object (see (1.3.3)). Also B denotes a Banach space with norm $\|\cdot\|_B$.

(3.2.1) DEFINITION : B is called a *continuously L -translation invariant Banach space* (or is said to be *continuously L -translation invariant* (c.l.t.i)) iff there exists a homomorphism (representation)

L of G into the space of continuous endomorphisms of B .

For each $x \in G$, L_x will denote the endomorphism $L(x)$ of B .

Remark : It is clear that a Banach space B may be continuously translation invariant with respect to several representations L of G . For instance, if B is one of $A(G)$, $C(G)$, $L^p(G)$ $1 \leq p \leq \infty$ or $V(G)$, and if L_1 denotes the left regular representation of G ($G \times G$ if $B = V(G)$) on B and L_2 denotes the right regular representation of G ($G \times G$ if $B = V(G)$) on B then B is both $c.L_1.t.i$ and $c.L_2.t.i$, and L_1 and L_2 are distinct if G is not abelian.

(3.2.2) DEFINITION : Let B^* denote the space of continuous linear functionals on B . Assume B is $c.L.t.i$. For each $x \in G$ we define $\mathcal{L}_x$ to be the *adjoint* of $L_{x^{-1}}$; that is

$$\mathcal{L}_x M(b) = M(L_{x^{-1}} b) \text{ for all } b \in B \text{ and } M \in B^*.$$

(3.2.3) It is immediately seen that

$$\|\mathcal{L}_x M\|_{B^*} = \sup_{b \in B_B(0,1)} |M(L_{x^{-1}} b)| \leq \|M\|_{B^*} \sup_{b \in B_B(0,1)} \|L_{x^{-1}} b\|_B,$$

and so :

LEMMA : B^* is continuously $\mathcal{L}$ -translation invariant. □

For typographical reasons we will write L in place of $\mathcal{L}$.

Hence for each $M \in B^*$ and $b \in B$

$$L_x M(b) = M(L_{x^{-1}} b)$$

and B^* is continuously L -translation invariant.

(3.2.4) DEFINITION : Let $M \in B^*$ and $b \in B$. Define the function

$\Gamma_M(b)$ from G to $\mathbb{C}$ by

$$\Gamma_M(b)(x) = L_x M(b) \text{ for each } x \in G.$$

Denote by Γ_M the map $b \rightarrow \Gamma_M(b)$ and now define the subspace B'' of B^* to be

$$B'' = \{M \in B^* : \Gamma_M(b) \in \mathbb{C}(G) \text{ for each } b \in B \text{ and } \Gamma_M \in \mathcal{B}(B, \mathbb{C}(G))\}.$$

For each M in B'' let $\|M\|_{B''} = \|\Gamma_M\|_{\mathcal{B}(B, \mathbb{C}(G))}$ and so

$$\|M\|_{B''} = \sup_{b \in B_B(0,1)} \|\Gamma_M(b)\|_{\infty}.$$

Clearly the definition of B'' depends on L ; we don't however emphasize this dependence because we assume that L is fixed in any argument.

(3.2.5) We see that $\|\cdot\|_{B''}$ is a norm on B'' . Furthermore, for each M in B''

$$\|M\|_{B''} \geq \sup_{b \in B_B(0,1)} |\Gamma_M(b)(1)| = \|M\|_{B^*}.$$

(3.2.6) DEFINITION : We define $B^\#$ to be the set of M in B'' for which Γ_M is a compact operator from B to $\mathbb{C}(G)$.

(3.2.7) Remark : Each of B'' and $B^\#$ may be isometrically identified with a subspace of $\mathcal{B}(B, \mathbb{C}(G))$ by means of the map $M \rightarrow \Gamma_M$. We occasionally use this fact without specifically saying so.

We hope to be able to use results concerning $\mathbb{C}(G)$ to obtain information about the spaces B'' and $B^\#$, and so to obtain more information about B itself.

(3.2.8) We first prove that both B'' and $B^\#$ share the translation invariance properties of the underlying space B if we consider the adjoint representation.

PROPOSITION : (i) $(B'', \| \cdot \|_{B''})$ is a c.l.t.i. Banach space, and
(ii) $E^\#$ is a closed c.l.t.i. subspace of B'' , and so
 $(B^\#, \| \cdot \|_{B''})$ is also a c.l.t.i. Banach space.

Proof : It is easy to see that B'' and $B^\#$ are normed subspaces of B^* . By (3.2.7) B'' will be complete if $\{\Gamma_M : M \in B''\}$ is closed in $B(B, C(G))$. Therefore let $(M_n)_{n \in \mathbb{N}}$ be a sequence taken from B'' such that there exists an operator F in $B(B, C(G))$ satisfying

$$\lim_{n \rightarrow \infty} \sup_{b \in B_B(0,1)} \| F(b) - \Gamma_{M_n}(b) \|_\infty = 0. \quad (1)$$

By (3.2.4) and (3.2.5) the sequence $(M_n)_{n \in \mathbb{N}}$ is Cauchy in B^* and so converges to an element M of B^* . Hence by (3.2.3), for each $x \in G$

$$\lim_{n \rightarrow \infty} \sup_{b \in B_B(0,1)} | M(L_{x^{-1}}b) - M_n(L_{x^{-1}}b) | = 0, \quad (2)$$

and so by (1) and (2)

$$F(b)(x) = M(L_{x^{-1}}b) = \Gamma_M(b)(x).$$

Therefore $\Gamma_M = F$ and $\{\Gamma_M : M \in B''\}$ is closed in $B(B, C(G))$.

Now we prove that B'' is c.l.t.i. Let $M \in B''$ and $x \in G$. Then for each $b \in B$ and $y \in G$

$$\Gamma_{L_x M}(b)(y) = L_y L_x M(b) = \Gamma_M(b)(yx),$$

and so $\Gamma_{L_x M} \in C(G)$ and

$$\| \Gamma_{L_x M}(b) \|_\infty = \sup_{y \in G} | \Gamma_M(b)(yx) | = \| \Gamma_M(b) \|_\infty. \quad (3)$$

It follows that B'' is c.l.t.i.

Appealing to (9.2.6) of Edwards [2] and to (3.2.7) we have that $B^\#$ is a closed subspace of B'' . Therefore it will be c.l.t.i.

if $\Gamma_{L_x M}$ is a compact operator from B to $C(G)$ for each M in $B^\#$

and x in G . It is easily seen from (3) that

$$\{\Gamma_{L_M}^{(b)} : b \in B_B(0,1)\} \quad (4)$$

is a bounded subset of $C(G)$. Let $y \in G$. Then

$$|\Gamma_{L_M}^{(b)}(y) - \Gamma_{L_M}^{(b)}(1)| = |\Gamma_M^{(b)}(yx) - \Gamma_M^{(b)}(x)|,$$

and so the set in (4) is equicontinuous at 1 and hence for all $y \in G$ because Γ_M is compact. Applying Ascoli's theorem we see it must be relatively compact and so $\Gamma_{L_M}^{(b)}$ is compact. Thus $(B^\#, \|\cdot\|_B)$ is a c.l.t.i. Banach space. $\square$

(3.3) L-Homogeneous Banach Spaces

We begin by introducing the concept of L -homogeneity which is an extension of the property (b) of §(3.1).

(3.3.1) DEFINITION : A c.l.t.i. Banach space B is L -homogeneous (L.h) iff for each $b \in B$ the map $x \rightarrow L_x b$ is continuous from G to B , (hence the representation L is continuous).

If B is (L.h) we have that for each $b \in B$

$$\sup_{x \in G} \|L_x b\|_B < \infty \quad (1)$$

because G is compact. Define the function p from B to $\mathbb{C}$ so that $p(b)$ is the supremum in (1). Clearly p is a seminorm on B . Also for each $x \in G$ the map $b \rightarrow \|L_x b\|_B$ is continuous from B to $\mathbb{C}$, so p must be lower-semicontinuous. Appealing to the Uniform Boundedness Principle, there exists a positive constant K such that

$$(3.3.2) \quad \sup_{x \in G} \|L_x b\|_B \leq K \|b\|_B \quad \text{for all } b \in B.$$

(3.3.3) LEMMA : If B is L -homogeneous then $B'' = B^*$ and $\| \cdot \|_{B^*}$ and $\| \cdot \|_{B''}$ are equivalent.

Proof : Clearly $B'' \subseteq B^*$. On the other hand let $M \in B^*$ and let $b \in B$ and $x, y \in G$. Then

$$|\Gamma_M(b)(x) - \Gamma_M(b)(y)| = |M(L_{x^{-1}}b) - M(L_{y^{-1}}b)| \leq \|M\|_{B^*} \|L_{x^{-1}}b - L_{y^{-1}}b\|_B,$$

and so $\Gamma_M(b) \in C(G)$. Furthermore,

$$\|\Gamma_M(b)\|_\infty = \sup_{x \in G} |M(L_{x^{-1}}b)| \leq \|M\|_{B^*} \sup_{x \in G} \|L_{x^{-1}}b\|_B \leq K \|M\|_{B^*} \|b\|_B$$

by (3.3.2).

Therefore $B^* \subseteq B''$ and, in view of (3.2.5), $\| \cdot \|_{B^*}$ and $\| \cdot \|_{B''}$ are equivalent. $\square$

Rather than pause here to give some examples of $c.l.t.i.$ and $L.h$ Banach spaces I have opted to devote sections (3.4) and (3.5) to this exercise. I will then be in a position to give a more complete set of examples.

(3.3.4) PROPOSITION : If B is $c.l.t.i.$ then $B^\#$ is L -homogeneous.

Proof : In view of (3.2.8) it suffices to prove that for each M in $B^\#$

$$\lim_{x \rightarrow 1} \|L_x^M - M\|_{B''} = 0. \quad (1)$$

Let $x \in G$. Then

$$\begin{aligned}
\|L_x^M - M\|_{B''} &= \sup_{b \in B_B(0,1)} \|\Gamma_{L_x^M}(b) - \Gamma_M(b)\|_\infty \\
&= \sup_{b \in B_B(0,1)} \sup_{y \in G} |L_{yx}^M(b) - L_y^M(b)| \\
&= \sup_{b \in B_B(0,1)} \|\Gamma_M(b) - x^{-1}(\Gamma_M(b))\|_\infty.
\end{aligned}$$

But $\{\Gamma_M(b) : b \in B_B(0,1)\}$ is relatively compact in $C(G)$ and so is equicontinuous. Therefore (1) holds. $\square$

One of the useful properties possessed by the homogeneous Banach spaces B studied in Wang [1] is that $T(G) \cap B$ is a dense subspace of B (see (2.17) of Wang [1]). In fact it is shown in (3.4) of Wang [1] that B must be a convolution module over $M(G)$; and in (3.7) that if $(k_n)_{n \in \mathbb{N}}$ is an approximate identity in $L^1(G)$ then

$$\lim_{n \rightarrow \infty} \|k_n * b - b\|_B = 0 \text{ for all } b \in B. \quad (1)$$

Appealing to (1.4.4) we see that it is an immediate consequence of this last property that $T(G) \cap B$ is a dense subspace of B .

In (3.3.7) we prove that any L -homogeneous Banach space B is a convolution module over $M(G)$ when convolution is defined as in (3.3.6); and in (3.3.10) that (1) holds for any such Banach space B and any approximate identity $(k_n)_{n \in \mathbb{N}}$ in $L^1(G)$. In fact we prove that this property is characteristic of L -homogeneous spaces.

First, however, we introduce the space $P(B)$ for any c.l.t.i. Banach space B . It will play the role of $B \cap T(G)$. In (3.5.1) we prove that if B is a continuously left invariant Banach space of pseudomeasures then $P(B) = B \cap T(G)$. It follows from (3.3.10) that $P(B)$ is a dense subspace of B if B is L -homogeneous.

(3.3.5) DEFINITION : Let B be c.l.t.i. and $b \in B$. Define V_b to be the set $\text{span}\{L_x b : x \in G\}$, and $P(B)$ to be

$$P(B) = \{b \in B : V_b \text{ is finite dimensional}\}.$$

Clearly $P(B)$ is a linear subspace of B .

(3.3.6) DEFINITION : Let B be L.h and $b \in B$. For each $\mu \in M(G)$ denote by $\mu * b$ the Bochner integral $\int L_x b d\mu(x)$.

Our use of convolution notation is not accidental because for those homogeneous spaces studied in Wang [1], (3.3.6) gives the usual definition of the convolution product of a measure μ and an integrable function b .

(3.3.7) PROPOSITION : *If B is L-homogeneous then for each $b \in B$ and $\mu \in M(G)$ $\mu * b$ is in B and*

$$\|\mu * b\|_B \leq K \|\mu\|_1 \|b\|_B.$$

Proof : By (8.14.14) and (8.15.2) of Edwards [2] $\mu * b$ is in B if the map g from G to B , $x \rightarrow L_x b$, is measurable and

$$\int \|g(x)\|_B d\mu(x) < \infty. \quad (1)$$

Now by (3.3.1) g is continuous and so is measurable. Furthermore

$$\begin{aligned} \int \|g(x)\|_B d\mu(x) &= \int \|L_x b\|_B d\mu(x) \leq K \|b\|_B \int d\mu(x) \text{ by (3.3.2)} \\ &\leq K \|b\|_B \|\mu\|_1. \end{aligned} \quad (2)$$

Therefore (1) holds and $\mu * b$ is in B . Finally (8.14.6) of Edwards [2] and (2) ensure that

$$\|\mu * b\|_B \leq K \|\mu\|_1 \|b\|_B. \quad \square$$

If B is L.h and if $b \in B$, $\mu \in M(G)$ and $x \in G$ we now determine $L_x \mu * b$. Let $M \in B^*$. Then in the weak topology on B

$$\langle L_x \mu * b, M \rangle = \langle \mu * b, L_x^* M \rangle.$$

Therefore, by the definition of the Bochner integral in (8.14.1) of

Edwards [2],

$$\begin{aligned}
 \langle L_x \mu * b, M \rangle &= \int \langle L_y b, L_x^* M \rangle d\mu(y) \\
 &= \int \langle L_{xy} b, M \rangle d\mu(y) \\
 &= \left\langle \int L_y b d_{x\mu}(y), M \right\rangle \\
 &= \langle x^\mu * b, M \rangle,
 \end{aligned}$$

and so for each $x \in G$

$$(3.3.8) \quad L_x \mu * b = x^\mu * b.$$

(3.3.9) PROPOSITION : Assume B is L -homogeneous. Let $b \in B$ and $k \in T(G)$. Then $k * b$ is in $P(B)$.

Proof : Recall the definition of $P(B)$ from (3.3.5). The assertion is an immediate consequence of (3.3.8) because

$V_k = \{x^\mu : x \in G\}$ is finite dimensional. □

(3.3.10) Finally we obtain the approximation theorem.

THEOREM : Assume B is c.l.t.i. The following properties are equivalent:

- (i) B is L -homogeneous, and
- (ii) if $(k_n)_{n \in \mathbf{N}}$ is an approximate identity in $L^1(G)$ then

$$\lim_{n \rightarrow \infty} \|k_n * b - b\|_B = 0 \text{ for all } b \in B.$$

Proof : (i) $\rightarrow$ (ii). Assume that B is L .h and that $(k_n)_{n \in \mathbf{N}}$ is an approximate identity in $L^1(G)$. By (3.3.7) for each n , $k_n * b$ is in B . Furthermore

$$k_n * b - b = \int k_n(x) (L_x b - b) d\lambda(x). \quad (1)$$

Now for each $\varepsilon > 0$ there exists a neighbourhood U of 1 in G such that for all $x \in U$

$$\|L_x b - b\|_B < \varepsilon. \quad (2)$$

It follows from (1.2.6)(c) that there exists a positive integer N such that for $n > N$

$$\|k_n\|_{G \setminus U} < \varepsilon \left[\sup_{x \in G} \|L_x b - b\|_B \right]^{-1}. \quad (3)$$

Therefore, by (1), (2), (3) and (8.14.6) of Edwards [2], for $n > N$

$$\|k_n * b - b\|_B < \varepsilon (\|k_n\|_1 + 1),$$

and so by (1.2.6)(a) (ii) holds.

(ii) $\rightarrow$ (i). Let $(k_n)_{n \in \mathbb{N}}$ be an approximate identity of $L^1(G)$ taken from $T(G)$ (see (1.4.4)). For each n and each $b \in B$ the map $x \rightarrow L_x k_n * b$ is continuous from G to B . This is so because, by (3.3.7) and (3.3.8),

$$\|L_x k_n * b - k_n * b\|_B \leq K \|x k_n - k_n\|_1 \|b\|_B. \quad (4)$$

Define the map T_b from G to B by $T_b(x) = L_x b$. Then B is L.h. if T_b is continuous for each $b \in B$. By (4) T_b is continuous for all b in a dense subspace of B .

By the Baire Category theorem for each b in B the set of discontinuities of T_b is of the first category, and so is not the whole of G . As G is compact this proves that B is L.h. To see this, let x be a point of continuity of T_b . Then for each $\varepsilon > 0$ there exists a neighbourhood U of x such that for $y \in U$

$$\|L_y b - L_x b\|_B < \varepsilon. \quad (5)$$

Furthermore, there exists elements $x_1, \dots, x_n$ of G such that

$$G = \bigcup_{i=1}^n T_{x_i} U,$$

where $T_{x_i} u$ denotes the left x_i -translate of u . For each $z \in G$ there is an open set V and an $i \in \{1, \dots, n\}$ such that $z \in V \subseteq T_{x_i} u$.

Then for each $y \in V$

$$\begin{aligned} \|L_y b - L_z b\|_B &\leq \|L_y b - L_{x_i^{-1} x} b\|_B + \|L_{x_i^{-1} x} b - L_z b\|_B \\ &\leq \sup_{b \in B_B(0,1)} \|L_{x_i^{-1} b}\|_B (\|L_{x_i y} b - L_x b\|_B + \|L_x b - L_{x_i z} b\|_B). \end{aligned} \quad (6)$$

But $x_i y$ and $x_i z$ are both elements of u and so it follows from (5) and (6) that

$$\|L_y b - L_z b\|_B \leq 2\epsilon \max_{i=1, \dots, n} \sup_{b \in B_B(0,1)} \|L_{x_i^{-1} b}\|_B.$$

Therefore T_b is continuous and B is L.h. □

Remark : Appealing to (3.3.9) we see that if B is L.h and if each k_n is a trigonometric polynomial then $k_n * b$ is in $P(B)$ for each b in B . Hence by (3.3.10) $P(B)$ is a dense subspace of B .

(3.4) Examples of c.l.t.i. and L.h Banach spaces

I now present some examples which will show that I am considering more than just the usual spaces $A(G)$, $C(G)$, $L^p(G)$ $1 \leq p \leq \infty$ and $M(G)$. We begin with a compact group G . It is easy to see that each Segal algebra on G (see Reiter [1], [2], [3]), and each homogeneous Banach algebra on G (see Wang [1]) is L -homogeneous if L is taken to be the appropriate left regular representation of G . Each semi-homogeneous Banach algebra on G (see Wang [1]) is continuously L -translation invariant. Many examples are given in each of these references.

(3.4.1) Recall from (1.4.1) the definitions of $\underline{E}(\Sigma(G))$, $\underline{E}^0(\Sigma(G))$ and $\underline{E}^p(\Sigma(G))$, $1 \leq p \leq \infty$.

DEFINITION : For each $x \in G$ and $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))$ define $L_x A = ((L_x A)_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))$ by

$$(L_x A)_\sigma = U_\sigma(x) A_\sigma \text{ for each } \sigma \in \Sigma(G).$$

(3.4.2) Let B denote one of the Banach spaces $\underline{E}^p(\Sigma(G))$, $1 \leq p \leq \infty$ or $p = 0$. Define the map L from G to B by

$$L(x)A = L_x A \text{ for each } A \in B.$$

PROPOSITION : Each of the Banach spaces $\underline{E}^p(\Sigma(G))$, $1 \leq p \leq \infty$ or $p = 0$, is continuously L -translation invariant, and if $1 \leq p < \infty$ or $p = 0$ then it is L -homogeneous.

Proof : Let $p \in \{0\} \cup [1, \infty]$, $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ be an element of $\underline{E}^p(\Sigma(G))$ and $x \in G$. Define U_x in $\underline{E}^\infty(\Sigma(G))$ to be $(U_\sigma(x))_{\sigma \in \Sigma(G)}$. Then $\|U_x\|_\infty = 1$. Appealing to (28.27) and (28.32)(iii) of [HR] we see that $L_x A = U_x \cdot A$, and so $L_x A$ is in $\underline{E}^p(\Sigma(G))$ and $\|L_x A\|_p \leq \|U_x\|_\infty \|A\|_p \leq \|A\|_p$. Therefore $\underline{E}^p(\Sigma(G))$ is c.l.t.i.

Assume $p \neq \infty$. Then $\underline{E}^p(\Sigma(G))$ will be $L.h$ if the map $x \rightarrow L_x A$ is continuous from G to $\underline{E}^p(\Sigma(G))$ for each $A \in \underline{E}^p(\Sigma(G))$. Let $p \in [1, \infty)$ then

$$\|L_x A - A\|_p^p = \sum_{\sigma \in \Sigma(G)} d_\sigma \|U_\sigma(x) A_\sigma - A_\sigma\|_{\phi_p}^p. \quad (1)$$

For any $\epsilon > 0$ there exists a finite subset F of $\Sigma(G)$ such that

$$\sum_{\sigma \in \Sigma(G) \setminus F} d_\sigma \|A_\sigma\|_{\phi_p}^p < \epsilon^p. \quad (2)$$

But each representation U_σ is continuous, so (1) and (2) combine to prove that the map $x \rightarrow L_x A$ is continuous.

If $p = 0$ then

$$\|L_x A - A\|_\infty = \sup_{\sigma \in \Sigma(G)} \| (U_\sigma(x) - \text{Id}_\sigma) A_\sigma \| . \quad (3)$$

For any $\epsilon > 0$ there exists a finite set F of $\Sigma(G)$ such that

$$\sup_{\sigma \in \Sigma(G) \setminus F} \|A_\sigma\| < \epsilon. \quad (4)$$

Then again using the continuity of the representations U_σ , (4) and (5) prove that the map $x \rightarrow L_x A$ is continuous. $\square$

We may construct many more examples of c.l.t.i. and L.h Banach spaces from those given in (3.4.2) by using lemma (3.4.4) to follow. First we give a definition.

(3.4.3) DEFINITION : For each $\sigma \in \Sigma(G)$ let E_σ be a subspace of H_σ and let $\mathcal{G} = \{E_\sigma : \sigma \in \Sigma(G)\}$. Let B be one of the Banach spaces given in (3.4.2). Then $B_{\mathcal{G}}$ denotes

$$B_{\mathcal{G}} = \{A = (A_\sigma)_{\sigma \in \Sigma(G)} \in B : A_\sigma(E_\sigma) = \{0\} \text{ for all } \sigma \in \Sigma(G)\}.$$

In particular, if $E_\sigma = \{0\}$ for all $\sigma \in F$ and $E_\sigma = H_\sigma$ for $\sigma \in \Sigma(G) \setminus F$ then $B_{\mathcal{G}}$ is the set of all elements of B supported by F .

(3.4.4) LEMMA : If $B_{\mathcal{G}}$ is normed as a subspace of B then it is a continuously L-translation invariant Banach space. If B is L-homogeneous then so too is $B_{\mathcal{G}}$.

Proof : Clearly $B_{\mathcal{G}}$ is a normed subspace of B . For each $x \in G$ and $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $B_{\mathcal{G}}$ we have that, for $\sigma \in \Sigma(G)$, $(L_x A)_\sigma = U_\sigma(x)A_\sigma$, and so it is easy to see that $L_x A$ is in $B_{\mathcal{G}}$. Furthermore $B_{\mathcal{G}}$ is c.l.t.i. because

$$\|L_x A\|_{B_{\mathcal{G}}} = \|L_x A\|_B \leq \|A\|_B = \|A\|_{B_{\mathcal{G}}}.$$

It only remains to prove that $B_{\mathcal{G}}$ is complete. This will be so if it is closed in B . Therefore let $(A_n)_{n \in \mathbb{N}}$ be a sequence taken from $B_{\mathcal{G}}$, and assume that it converges to A in B . Then it must converge to A in $\underline{E}^\infty(\Sigma(G))$, and in particular the coordinate sequences

$(A_{n_\sigma})_{n \in \mathbf{N}}$ converge to A_σ in $B(H_\sigma)$ for each σ . But for each n , $A_{n_\sigma}(E_\sigma) = \{0\}$ and so we must have $A_\sigma(E_\sigma) = \{0\}$. Thus B_g is a c.l.t.i. Banach space.

If B is L.h then it follows immediately that B_g is also. $\square$

Before proceeding with the task of obtaining more examples I introduce the family $\mathcal{B}$ of c.l.t.i. Banach spaces which will be the principal object of study in chapters 4 and 5.

(3.4.5) DEFINITION : Let B denote a Banach space of pseudomeasures defined on G (see (1.4.3) for the definition of a pseudomeasure). We say that B belongs to the family $\mathcal{B}$ iff it satisfies the conditions:

- (a) the natural injection $J : B \rightarrow PM(G)$, $b \mapsto b$, is a continuous operator, and
- (b) for each b in B and x in G , ${}_x b$ is an element of B and $\|{}_x b\|_B \leq \text{const } \|b\|_B$.

(Then $\mathcal{B}$ actually depends on the underlying group G and so should be more correctly written $\mathcal{B}(G)$. However, I prefer to denote it simply by $\mathcal{B}$ as the underlying group G will be fixed during any argument.)

If we define L from G to the space of continuous endomorphisms of B by $L(x)(b) = {}_x b$ for all b in B and x in G , then (b) ensures that B is c.l.t.i. No confusion should arise from my use of L to denote this representation and also that defined in (3.4.1) because for each $\sigma \in \Sigma(G)$, $x \in G$ and $b \in B$

$$L(x)(b)^\wedge(\sigma) = ({}_x b)^\wedge(\sigma) = U_\sigma(x)\hat{b}(\sigma) = (L_x \hat{b})_\sigma,$$

and so

$$(3.4.6) \quad (L(x)(b))^\wedge = L_x \hat{b}.$$

DEFINITION : Denote by $\mathcal{B}_h$ the subset of $\mathcal{B}$ consisting of all L -homogeneous Banach spaces.

In the sequel we ~~do~~ not refer to L when considering spaces in $\mathcal{B}$ or $\mathcal{B}_h$; rather we call them continuously translation invariant and homogeneous, respectively.

Remark : We might easily have used the right regular representation of G in (3.4.1) and (3.4.5) in which case all of the results to follow possess obvious analogues. However, L always denotes the left regular representation.

(3.4.7) PROPOSITION : Each of the Banach spaces $PM(G)$; $M(G)$; and $L^\infty(G)$ is in $\mathcal{B}$; and each of the Banach spaces $PF(G)$; $L^p(G)$; $1 \leq p < \infty$; $U^p(G) = \{f \in L^1(G) : \hat{f} \in \underline{E}^p(\Sigma(G))\}$ (with $\|f\|_{U^p} = \|f\|_1 + \|\hat{f}\|_p$), $1 \leq p < \infty$, $C(G)$ and $A(G)$ is an element of $\mathcal{B}_h$. The Banach space $V_{F_1, F_2}(G)$, with F_1 and F_2 subsets of $\Sigma(G)$, is in $\mathcal{B}_h(G \times G)$.

Proof : The case of $V_{F_1, F_2}(G)$ is an immediate consequence of (2.4.3) and (2.4.4).

It is easily seen that each of the remaining Banach spaces is in

$\mathcal{B}$

It follows from (20.4) of [HR] that $L^p(G)$, $1 \leq p < \infty$, is an element of $\mathcal{B}_h$; from (34.26) of [HR] that $A(G)$ is an element of $\mathcal{B}_h$; and from (2.1.4) of [E] that $C(G)$ is also in $\mathcal{B}_h$. Appealing to (3.4.2) and (3.4.6) deals with $PF(G)$, which only leaves the spaces $U^p(G)$, $1 \leq p < \infty$, to consider. Let $x \in G$ and $f \in U^p(G)$. Then

$$\|x^f - f\|_{U^p} = \|x^f - f\|_1 + \|L_x \hat{f} - \hat{f}\|_p. \quad (1)$$

But $L^1(G)$ is in $\mathcal{B}_h$ and, by (3.4.2), $\underline{E}^p(\Sigma(G))$ is L -homogeneous so

it follows from (1) that the map $x \rightarrow_x f$ is continuous from G to $U^p(G)$ for each f in $U^p(G)$. Therefore $U^p(G)$ is also in $\mathcal{B}_h$. $\square$

As in (3.4.4) the examples in (3.4.7) may be used to generate many more Banach spaces belonging to either $\mathcal{B}$ or $\mathcal{B}_h$. In the case of $V_{F_1, F_2}(G)$, where F_1 and F_2 are subsets of $\Sigma(G)$, it is important to remember that the underlying group is $G \times G$ not G . Even in this case the following type of construction is valid.

(3.4.8) DEFINITION : For each σ in $\Sigma(G)$ let E_σ denote a subspace of H_σ and let $\mathcal{J}$ denote the set $\mathcal{J} = \{E_\sigma : \sigma \in \Sigma(G)\}$. For any set B of pseudomeasures defined on G , $B_{\mathcal{J}}$ will denote the set

$$B_{\mathcal{J}} = \{b \in B : \hat{b}(\sigma)(E_\sigma) = \{0\} \text{ for all } \sigma \in \Sigma(G)\}.$$

In particular, if $E_\sigma = H_\sigma$ for $\sigma \in \Sigma(G) \setminus F$ (F some subset of $\Sigma(G)$) and $E_\sigma = \{0\}$ for $\sigma \in F$ then $B_{\mathcal{J}}$ is precisely the set of pseudomeasures in B whose Fourier transform is supported by F . We then write $B_{\mathcal{J}} = B_F$.

(3.4.9) LEMMA : Let $B \in \mathcal{B}$ and norm $B_{\mathcal{J}}$ as a subspace of B . Then $B_{\mathcal{J}}$ is in $\mathcal{B}$. If B is in $\mathcal{B}_h$ then so too is $B_{\mathcal{J}}$.

Proof : By (3.4.5)(a) and (3.4.6) this follows immediately by the argument used in (3.4.4). $\square$

It is, in fact, a consequence of (4.2.1) that (3.4.8) and (3.4.9) give a complete description of the closed subspaces of a homogeneous Banach space B , which are themselves homogeneous when they inherit the topology and translation operators of B .

The final family of examples we consider is motivated by the algebras of type C studied in Silov [1]. Let B denote a regular, commutative algebra without radical, and $M(B)$ its maximal ideal space. Then B is of type C if its norm $\|\cdot\|_B$ is equivalent to $\|\cdot\|$

defined by

$$\|b\| = \sup_{m \in M(B)} \inf_{b' \in B} \{\|b'\|_B : \hat{b}' = \hat{b} \text{ on } N(m)\}$$

where $\hat{}$ denotes the Gelfand transformation, and $N(m)$ is some neighbourhood of m in $M(B)$. Clearly $\| \cdot \| \leq \| \cdot \|_B$.

(3.4.10) First let G denote a compact abelian group and X the set of continuous characters of G . If B is either a Segal algebra or a (Wang) homogeneous algebra then $M(B) = X \cap B$ (see Reiter [1] and Wang [1]). In either case B is of type C iff its norm is equivalent to $\| \cdot \|$ defined by

$$\|b\| = \sup_{\chi \in \text{sp } B} \inf_{b' \in B} \{\|b'\|_B : \hat{b}'(\chi) = \hat{b}(\chi)\}. \quad (1)$$

DEFINITION : Let $B \in \mathcal{B}$. It is of type C iff its norm is equivalent to $\| \cdot \|$ defined in (1).

(3.4.11) PROPOSITION : Assume G is compact and abelian, and that B is in $\mathcal{B}_h$. Then B is of type C if and only if there exists a positive function δ defined on $\text{sp } B$ such that δ is bounded away from 0, and $\| \cdot \|_B$ is equivalent to $\| \cdot \|_\delta$ defined by

$$\|b\|_\delta = \sup_{\chi \in \text{sp } B} \delta(\chi) |\hat{b}(\chi)|.$$

Proof : Assume B is of type C and let $\chi \in \text{sp } B$, and $b \in B$ such that $\hat{b}(\chi) \neq 0$. Then

$$\begin{aligned} \inf_{b' \in B} \{\|b'\|_B : \hat{b}'(\chi) = \hat{b}(\chi)\} &= \inf_{b' \in B} \{\|\hat{b}(\chi)\chi + b'\|_B : \hat{b}'(\chi) = 0\} \\ &= |\hat{b}(\chi)| \inf_{b' \in B} \{\|\chi + b'\|_B : \hat{b}'(\chi) = 0\}. \end{aligned} \quad (1)$$

Therefore define $\delta(\chi)$ to be the infimum given in (1). It follows from (3.4.5) that $\delta : \chi \rightarrow \delta(\chi)$ is bounded away from 0 on $\text{sp } B$ because for each b in B with $\hat{b}(\chi) = 0$,

$$\|x + b\|_B \geq K \|x + b\|_{PM} \geq K.$$

Clearly now, by (1)

$$\sup_{x \in \text{sp} B} \inf_{b' \in B} \{\|b'\|_B : \hat{b}'(x) = \hat{b}(x)\} = \sup_{x \in \text{sp} B} \delta(x) |\hat{b}(x)|, \quad (2)$$

and so $\|\cdot\|_B$ and $\|\cdot\|_\delta$ are equivalent.

The converse follows immediately from (2) and (3.4.10). $\square$

(3.4.12) Suppose now that G denotes an arbitrary compact group. Then (3.4.11) suggests that the following construction may produce homogeneous Banach spaces. Let F be a subset of $\Sigma(G)$. It is easy to see that $\underline{\underline{E}}(F)$ and $\underline{\underline{E}}(\Sigma(G))_F$ are isomorphic.

DEFINITION : Let $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ be an element of $\underline{\underline{E}}(F)$. Assume that each A_σ is invertible and that $A^{-1} = (A_\sigma^{-1})_{\sigma \in \Sigma(G)}$ is in $\underline{\underline{E}}^\infty(F)$. For any $p \in [1, \infty] \cup \{0\}$ define $PM(G)^{A,p}$ by

$$PM(G)^{A,p} = \{S \in PM(G)_F : \hat{S}A \in \underline{\underline{E}}^p(F)\},$$

and for each S in $PM(G)^{A,p}$ put $\|S\|_{A,p} = \|\hat{S}A\|_p$.

These spaces are similar, in definition, to the Segal algebra $S(\alpha)$ studied in Riemersma [1] for abelian groups, and to the Segal algebra $A_p(G)$ which is studied in Larsen, Liu and Wang [1], also for abelian groups.

(3.4.13) It is clear that $(PM(G)^{A,p}, \|\cdot\|_{A,p})$ is a normed space.

PROPOSITION : If $p \neq \infty$ then $PM(G)^{A,p}$ is in $\mathcal{B}_h$.

Proof : We first prove that the injection J of $PM(G)^{A,p}$ into $PM(G)$, $S \mapsto S$, is continuous. Let $S \in PM(G)^{A,p}$. Then

$$\|S\|_{A,p} = \|\hat{S}A\|_p \geq \|\hat{S}A\|_\infty = \sup_{\sigma \in F} \|\hat{S}(\sigma) A_\sigma\|.$$

But for each σ in F $\hat{S}(\sigma) = \hat{S}(\sigma) A_\sigma \cdot A_\sigma^{-1}$ and so

$$\|\hat{S}(\sigma)A_\sigma\| \geq \|\hat{S}(\sigma)\| \|A_\sigma^{-1}\|^{-1} \geq \|\hat{S}(\sigma)\| \left(\sup_{\sigma \in F} \|A_\sigma^{-1}\|\right)^{-1}.$$

Hence J is continuous.

We use an argument similar to that in (2.2.4) to prove that $PM(G)^{A,p}$ is a Banach space. If $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence taken from $PM(G)^{A,p}$ then it is Cauchy in $PM(G)$, and so converges to some element S of $PM(G)$ in $PM(G)$. Let $(S_{n_j})_{j \in \mathbb{N}}$ be a subsequence such that for each j

$$\|S_{n_{j+1}} - S_{n_j}\|_{A,p} < 2^{-j}.$$

Then for each $N \in \mathbb{N}$

$$S_{n_N} = S_{n_1} + \sum_{j=1}^{N-1} (S_{n_{j+1}} - S_{n_j}),$$

and so

$$\begin{aligned} \|S_{n_N}\|_{A,p} &\leq \|S_{n_1}\|_{A,p} + \sum_{j=1}^{N-1} \|S_{n_{j+1}} - S_{n_j}\|_{A,p} \\ &\leq \|S_{n_1}\|_{A,p} + \sum_{j=1}^{N-1} 2^{-j}. \end{aligned}$$

Therefore S must be in $PM(G)^{A,p}$ and so it is a Banach space.

It follows from (3.4.7), (3.4.6) and (3.4.2) that S_x is in $PM(G)^{A,p}$ for each $x \in G$ and $S \in PM(G)^{A,p}$. Also by (3.4.6)

$$\|S_x\|_{A,p} = \|(\hat{S}^A)_x\|_p = \|L_x \hat{S} A\|_p,$$

and so by (3.4.2)

$$\|S_x\|_{A,p} \leq \|\hat{S} A\|_p = \|S\|_{A,p}.$$

Hence $PM(G)^{A,p} \in \mathcal{B}$.

By (3.4.2) the map $x \mapsto L_x \hat{S} A$ is continuous from G to $\underline{E}^p(F)$. In view of (3.4.6) $PM(G)^{A,p}$ is easily seen to be in $\mathcal{B}_h$.

(3.4.14) If $p \in [1, \infty] \cup \{0\}$ and $q \in [1, \infty] \cup \{0\}$ define r by

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} & \text{if } p \text{ and } q \text{ are non-zero} \\ \frac{1}{p} & \text{if } q = 0 \text{ (or } \frac{1}{q} \text{ if } p = 0). \end{cases}$$

Let $A^{-1} = (A^{-1}_{\sigma})_{\sigma \in F}$ be an element of $\underline{E}^q(F)$. Then

$$\{\hat{S} : S \in PM(G)^{A,p}\} \subseteq \underline{E}^r(F).$$

To see this, let $S \in PM(G)^{A,p}$. Then S is in $PM(G)_F$ and $\hat{S} A$ is in $\underline{E}^p(F)$. It follows from (28.27), (28.32) and (28.33) of [HR] that $\underline{E}^p(F) \cdot \underline{E}^q(F)$ is contained in $\underline{E}^r(F)$, and so $\hat{S} = \hat{S} A \cdot A^{-1}$ is in $\underline{E}^r(F)$.

From (3.4.12), A^{-1} is always an element of $\underline{E}^{\infty}(F)$ so $\{\hat{S} : S \in PM(G)^{A,p}\} \subseteq \underline{E}^p(F)$. If $A \in \underline{E}^{\infty}(F)$ then these two sets are actually equal because $\underline{E}^p(F) \cdot \underline{E}^{\infty}(F) \subseteq \underline{E}^p(F)$.

The spaces $PM(G)^{A,p}$ are sparse in the sense of the final lemma.

LEMMA : If $PM(G)^{A,p} \subseteq L^{\infty}(G)_F$ then $PM(G)^{A,p} \subseteq A(G)_F$.

Proof : Let $S_1 \in PM(G)_F$ and $S_2 \in PM(G)^{A,p}$. Then $S_1 * S_2$ is in $PM(G)_F$ and

$$\|\hat{S}_1 \hat{S}_2 A\|_p \leq \|\hat{S}_1\|_{\infty} \|\hat{S}_2 A\|_p = \|\hat{S}_1\|_{\infty} \|S_2\|_{A,p}.$$

Therefore $S_1 * S_2$ is in $PM(G)^{A,p}$. By (D39) of [HR] for each σ in F there exists $S_{\sigma} \in B(H_{\sigma})$ such that $\|S_{\sigma}\| = 1$ and

$$\|\hat{S}_2(\sigma)\| = \text{Tr}(S_{\sigma} \hat{S}_2(\sigma)). \quad (1)$$

Define the element S in $PM(G)_F$ by $\hat{S}(\sigma) = S_{\sigma}$ for each σ in F .

Then $S * S_2 \in PM(G)^{A,p} \subseteq L^{\infty}(G)_F$.

But by (1) for each σ

$$\text{Tr}(S_{\sigma} \hat{S}_2(\sigma)) = \text{Tr}(|\hat{S}_2(\sigma)|) \geq 0, \quad (2)$$

and so it follows from (1), (2) and (34.9) of [HR] that

$$\sum_{\sigma \in F} d_{\sigma} \operatorname{Tr} ((S * S_2)^{\wedge}(\sigma)) \leq \|S_2\|_{\infty}.$$

Hence S_2 is in $A(G)_F$. □

(3.5) Determination of $B^{\#}$ for some c.l.t.i. Banach spaces.

The preceding section includes many examples of c.l.t.i. and l.h Banach spaces. We now determine the space $B^{\#}$ (see (3.2.6)) for some of the examples. These results, summarized in (3.5.9), motivate §4.2.

We first note that the examples constructed by using (3.4.2) and (3.4.4) are, in view of (3.4.6), essentially included in those constructed by using (3.4.13) and (3.4.9). It is easy to see that for each $p \in [1, \infty] \cup \{0\}$ and any family of subspaces $\mathcal{G}, \underline{E}^p(\Sigma(G))_{\mathcal{G}}$ can be written as

$$\{\hat{S} : S \in \operatorname{PM}(G)^{I,p} \text{ and } \hat{S}(\sigma)(E_{\sigma}) = \{0\} \text{ for all } \sigma \in \Sigma(G)\}$$

where $I = (\operatorname{Id}_{\sigma})_{\sigma \in \Sigma(G)}$. Hence the spaces $\underline{E}^p(\Sigma(G))_{\mathcal{G}}$ and $(\operatorname{PM}(G)^{I,p})_{\mathcal{G}}$ are isomorphic. Clearly the isomorphism is an isometry. Therefore we only consider examples which are in $\mathcal{B}$.

We begin by proving that if B is in $\mathcal{B}$ then $P(B) = B \cap T(G)$. But first we have a lemma.

(3.5.1) Let $S \in \operatorname{PM}(G)$. Then $V_S = \operatorname{span} \{x_S : x \in G\}$.

LEMMA : $\dim V_S < \infty$ if and only if $S \in T(G)$.

Proof : Clearly V_S is finite dimensional if S is in $T(G)$.

Conversely, let $S \in \operatorname{PM}(G)$ and assume that V_S is finite dimensional and $\operatorname{supp} \hat{S}$ is countable. Let $k \in ZA(G)$, the centre of $A(G)$. Then $k * S = \int k(x)_x S d\lambda(x)$ is the usual convolution product and so $k * S$ is an element of $C(G)$. By (1.4.4), $\hat{k}(\sigma)$ is a

multiple of Id_σ for each $\sigma \in \Sigma(G)$. Hence for each $y \in G$

$${}_y(k * S) = {}_y(S * k) = {}_y S * k, \quad (1)$$

and so V_{k*S} is finite dimensional. Therefore $k*S$ is an element of $T(G)$. Define the linear map Φ_S from $ZA(G)$ to $T(G)$ by

$$\Phi_S(k) = k * S \text{ for all } k \in ZA(G).$$

Then Φ_S is continuous. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of $\Sigma(G)$ such that $\text{supp } \hat{S} = \bigcup \{F_n : n \in \mathbb{N}\}$ and for each n put

$$A_n = \{k \in ZA(G) : \text{supp } (k * S)^\wedge \subseteq F_n\}.$$

Each A_n is a closed subspace of $ZA(G)$, and $ZA(G) = \bigcup_{n \in \mathbb{N}} A_n$. Hence by the Baire Category theorem there exists a positive integer N such that A_N has interior points in $ZA(G)$, and so $A_N = ZA(G)$ and $k * S$ is an element of $T(G)_{F_N}$ for all $k \in ZA(G)$. Thus S is in $T(G)$.

To complete the proof we must consider the case where S is in $PM(G)$ with V_S having finite dimension but $\text{supp } \hat{S}$ being possibly uncountable. For any such S and any $k \in ZA(G)$, $k * S$ is in $PM(G)$ by (3.3.7) and its transform has countable support.

As in (1) V_{k*S} is finite dimensional and so $k * S$ is in $T(G)$. Therefore S must also be in $T(G)$. This is because if Q is a countably infinite subset of $\text{supp } \hat{S}$ then there exists k in $ZA(G)$ such that $\hat{k}$ does not vanish on Q ; but $k * S$ is not in $T(G)$ which is a contradiction. $\square$

(3.5.2) COROLLARY : If B is a continuously translation invariant Banach space of pseudomeasures then $P(B) = B \cap T(G)$.

Proof : This result is an immediate consequence of (3.5.1). $\square$

(3.5.3) By (3.3.3) B'' is isomorphic to B^* if B is an element of $\mathcal{B}_h$. In that case it follows from (3.3.10) and (3.3.4) that $B^\#$ is

equal to the closure in B^* of $P(B^\#)$. We now consider a lemma which will be used to determine $P(B^\#)$ for some of the examples given in the previous section.

LEMMA : Let $B \in \mathcal{B}_h$ and $\mathcal{G} = \{E_\sigma = \bigcap_{b \in B} \ker \hat{b}(\sigma) : \sigma \in \Sigma(G)\}$. Assume that $P(B) = T(G)_\mathcal{G}$. For each M in B^* there exists an element $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ of $\underline{E}(\Sigma(G))_\mathcal{G}$ such that, for each σ and each b in B , $\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) A_\sigma^*$. Then for each b in $T(G)_\mathcal{G}$

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)).$$

Proof : Let $M \in B^*$ and $\sigma \in \Sigma(G)$. Then $B_{\{\sigma\}}$ is isomorphic to $J(E_\sigma)$ which is, by definition, $\{T \in B(H_\sigma) : T(E_\sigma) = \{0\}\}$. Define the map $\bar{M}$ from $J(E_\sigma)$ to $\mathbb{C}$ by putting

$$\bar{M}(\hat{b}(\sigma)) = M(b) \text{ for all } b \text{ in } B_{\{\sigma\}}.$$

Clearly $\bar{M}$ is a linear map on $J(E_\sigma)$. But it is easy to see that $B(H_\sigma) = J(E_\sigma) \oplus J(E_\sigma^\perp)$ where $J(E_\sigma^\perp)$ is the set $\{T \in B(H_\sigma) : T(E_\sigma^\perp) = \{0\}\}$. Hence the set of linear functionals on $J(E_\sigma)$ is identifiable with $B(H_\sigma)/J(E_\sigma^\perp) \cong J(E_\sigma)$, and so there exists $A_\sigma \in J(E_\sigma)$ such that for all $b \in B_{\{\sigma\}}$

$$\bar{M}(\hat{b}(\sigma)) = d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)). \quad (1)$$

It follows from (1) that for all b in $T(G)_\mathcal{G}$

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)).$$

Let $b \in B$ and $(b_n)_{n \in \mathbb{N}}$ be a sequence extracted from $T(G)_\mathcal{G}$ such that

$$\lim_{n \rightarrow \infty} \|b - b_n\|_B = 0. \quad (2)$$

Then

$$\lim_{n \rightarrow \infty} |M(b - b_n)| = 0, \quad (3)$$

and so for all $n \in \mathbb{N}$ and $x \in G$

$$|\Gamma_M(b)(x) - \Gamma_M(b_n)(x)| = |L_x^M(b - b_n)| \leq |M(x^{-1}(b - b_n))|. \quad (4)$$

By (2), (4) and (3.3.2)

$$\lim_{n \rightarrow \infty} \|\Gamma_M(b) - \Gamma_M(b_n)\|_\infty = 0. \quad (5)$$

It follows from (1) that for each $n \in \mathbb{N}$

$$\Gamma_M(b_n)(x) = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr}(\hat{b}_n(\sigma) A_\sigma^* U_\sigma(x)^*),$$

and so for each $\sigma \in \Sigma(G)$

$$\Gamma_M(b_n)^\wedge(\sigma) = \hat{b}_n(\sigma) A_\sigma^*. \quad (6)$$

In view of (3.4.5) and (2)

$$\lim_{n \rightarrow \infty} \|b - b_n\|_{PM} = 0. \quad (7)$$

By (6) and (7) for each $\sigma \in \Sigma(G)$

$$\lim_{n \rightarrow \infty} \|\hat{b}(\sigma) A_\sigma^* - \Gamma_M(b_n)^\wedge(\sigma)\| = 0.$$

Hence appealing to (5) we see that

$$\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) A_\sigma^* \text{ for all } \sigma \in \Sigma(G). \quad \square$$

Note that by (4.2.1) we could omit the hypothesis $P(B) = T(G)_g$.

Remark : Referring to (35.1) of [HR] for notation, we have just proved that each M in B^* determines a right $(B, C(G))$ -multiplier by putting $\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) A_\sigma^*$ for all $\sigma \in \Sigma(G)$ and $b \in B$.

We now obtain a partial converse to this result.

LEMMA : If $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ is an element of $\underline{E}(\Sigma(G))_g$ such that A^* is a right $(B, C(G))$ -multiplier, then there exists a linear functional M in B^* such that $\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) A_\sigma^*$ for each $\sigma \in \Sigma(G)$ and $b \in B$.

Proof : We first notice that for each b in B there exists a function h_b in $C(G)$ such that $\hat{h}_b = \hat{b}A^*$. Define the map M from B to C by

$$M(b) = h_b(1) \quad \text{for all } b \text{ in } B. \quad (1)$$

It is clear that M is linear. Appealing to the Closed Graph theorem, in view of (3.4.5) (a), there must exist a positive constant K such that for all $b \in B$

$$\|h_b\|_\infty \leq K \|b\|_B. \quad (2)$$

Then by (1) and (2), M is in B^* , and so by (3.3.3) $\Gamma_M(b)$ is in $C(G)$ for each b in B . In fact, we will prove that $\Gamma_M(b) = h_b$, and so $\Gamma_M(b)^\wedge(\sigma) = \hat{h}_b(\sigma) = \hat{b}(\sigma)A_\sigma^*$ for each $\sigma \in \Sigma(G)$. But

$$\begin{aligned} \Gamma_M(b)(x) &= M(x^{-1}b) = h_{x^{-1}b}(1) \quad \text{for each } x \in G \text{ and} \\ \hat{h}_{x^{-1}b} &= x^{-1}\hat{b}A^* = U^*(x)\hat{b}A = (x^{-1}\hat{h}_b); \quad \text{therefore} \end{aligned}$$

$$\Gamma_M(b)(x) = x^{-1}h_b(1) = h_b(x). \quad \square$$

(3.5.4) Let $B \in \mathcal{B}_h$ and assume $P(B) = T(G)_g$. Also assume that $B^* \subseteq PM(G)$. Then for each $M \in B^*$ and $b \in T(G)_g$

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr}(\hat{M}(\sigma)^* \hat{b}(\sigma)).$$

It is clear that for each $\sigma \in \Sigma(G)$, if A_σ is determined as in

(3.5.3) then $A_\sigma = \hat{M}(\sigma) P_{E_\sigma}^\perp$. Hence

$$\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) (\hat{M}(\sigma) P_{E_\sigma}^\perp)^*. \quad (1)$$

If M is in $B^* \cap T(G)$ and if F denotes the support of $\hat{M}$ then F is a finite subset of $\Sigma(G)$. By (1), $\Gamma_M(b)$ is in $T(G)_F$ for each b in B . Hence Γ_M is compact and M is in $P(B^\#)$.

Clearly $B^\#$ is contained in $PM(G)$. Therefore by (2.2.8) and (3.5.2) $P(B^\#) = B^\# \cap T(G)$. But the RHS must equal $T(G)_g$ because $\hat{b}(\sigma)(E_\sigma) = \{0\}$ for all σ , and so by (3.3.10) and (3.3.3) $B^\#$ is

the closure in B^* of $T(G)_g$.

The results obtained by applying this to the homogeneous spaces listed in (3.4.7) are summarized in (3.5.9).

(3.5.5) The results in (3.5.4) do not apply to the Banach spaces $L^\infty(G)$, $M(G)$ or $PM(G)$, all of which are continuously left translation invariant. These can, however, be dealt with by the following proposition.

First, recall that any Banach space may be embedded in B^{**} by putting $\theta_b(M) = M(b)$ for all M in B^* .

PROPOSITION : Let $B \in \mathcal{B}_h$. Assume that $P(B) = T(G)$ and that $A(G)$ is contained in B^* . Also assume that $\|f\|_{B^*} \leq K \|f\|_A$ for all f in $A(G)$. Then $(B^*)^\# = B$.

Proof : We first prove that B is a subspace of $(B^*)^\#$. Let $b \in B$. We denote the image of b , under θ , also by b . Then for $x \in G$ and $M \in B^*$

$$\Gamma_b(M)(x) = {}_x b(M) = M({}_x b). \quad (1)$$

Therefore

$$|\Gamma_b(M)(x) - \Gamma_b(M)(1)| = |M({}_x b - b)| \leq \|M\|_{B^*} \|{}_x b - b\|_B. \quad (2)$$

By (2), $\Gamma_b(M)$ is in $C(G)$ and b is in $(B^*)''$. By (1) the set $\{\Gamma_b(M) : M \in B_{B^*}(0,1)\}$ is bounded, and by (2) it is equicontinuous. Hence it is relatively compact in $C(G)$, and so B is a subspace of $(B^*)^\#$.

Our hypotheses ensure that B^{**} is in $PM(G)$ and that $\|\Pi\|_{PM} \leq \text{const} \|\Pi\|_{B^{**}}$ for all Π in B^{**} . It is easily seen, by appealing to (3.3.4), that $(B^*)^\#$ is in $\mathcal{B}_h$. By (3.5.2)

$$T(G) = B \cap T(G) \subseteq (B^*)^\# \cap T(G) = P((B^*)^\#),$$

and so it follows from (3.3.10) that $(B^*)^\#$ is the closure in $(B^*)''$ of $T(G)$.

To complete the proof we show that $\| \cdot \|_B$ and $\| \cdot \|_{(B^*)''}$ are equivalent on B . From this $(B^*)^\#$ must be precisely B .

Let $b \in B$. Then

$$\begin{aligned} \| b \|_{(B^*)''} &= \sup_{M \in B_{B^*}(0,1)} \sup_{x \in G} |M(xb)| \\ &\leq \sup_{x \in G} \| x^b \|_B \leq K \| b \|_B \text{ by (3.3.2).} \end{aligned} \quad (3)$$

Returning to (3),

$$\| b \|_{(B^*)''} \geq \sup_{M \in B_{B^*}(0,1)} |M(b)|. \quad (4)$$

By the Hahn-Banach theorem there exists M in B^* with $\| M \|_{B^*} = 1$ and $|M(b)| = \| b \|_B$. Hence (4) proves that $\| b \|_{(B^*)''} \geq \| b \|_B$. $\square$

We again summarize in (3.5.9) the results obtained by applying (3.5.5) to the Banach spaces mentioned above.

(3.5.6) Our final examples are the spaces $PM(G)^{A,P}$ defined in (3.4.12).

The map $S \rightarrow \hat{S}A$ is a surjection from $PM(G)_F$ to $\underline{E}^P(F)$. To see this, let $D \in \underline{E}^P(F)$. Then $DA^{-1} \in \underline{E}^P(F) \cdot \underline{E}^\infty(F) \subseteq \underline{E}^\infty(F)$, and so there exists S in $PM(G)_F$ such that $\hat{S} = DA^{-1}$. But $\hat{S}A = DA^{-1}A = D$.

It follows that $(PM(G)^{A,P})^*$ is isomorphic to the space $\underline{E}(F)^{(A^{-1})^*, p'}$ which is defined by

$$\underline{E}(F)^{(A^{-1})^*, p'} = \{ D \in \underline{E}(F) : D(A^{-1})^* \in \underline{E}^{p'}(F) \}$$

and normed by $\| D \|_{(A^{-1})^*, p'} = \| D(A^{-1})^* \|_{p'}$. The duality is given by

$$\langle\langle D, S \rangle\rangle = \langle D(A^{-1})^*, \hat{S}A \rangle.$$

By (3.3.3) and (3.4.13) $(PM(G)^{A,P})''$ is isomorphic to $\underline{E}(F)^{(A^{-1})^*, p'}$, and so $\underline{E}(F)^{(A^{-1})^*, p'}$ is c.l.t.i.. We now prove that

$$\begin{aligned} (PM(G)^{A,P})^\# &= \underline{E}(F)^{(A^{-1})^*, p'} \text{ if } p \neq 1, \text{ and that} \\ (PM(G)^{A,1})^\# &= \underline{E}(F)^{(A^{-1})^*, 0}. \end{aligned}$$

(3.5.7) LEMMA : $\underline{E}^{\circ\circ}(F)$ is a subspace of $P(\underline{E}(F)^{(A^{-1})^*, p'})$ for any $p \in [1, \infty] \cup \{0\}$. If $p \neq 1$ then it is a dense subspace of $\underline{E}(F)^{(A^{-1})^*, p'}$ and if $p = 1$ it is dense in $\underline{E}(F)^{(A^{-1})^*, 0}$.

Proof : Clearly $\underline{E}^{\circ\circ}(F)$ is a subspace of $\underline{E}(F)^{(A^{-1})^*, p'}$. Now let $D \in \underline{E}^{\circ\circ}(F)$. Then for each $\sigma \in F$ and $x \in G$

$$(L_x D)_\sigma = U_\sigma(x) D_\sigma. \quad (1)$$

By (1.3.8) the linear span of the $U_\sigma(x)$'s is finite dimensional for each σ in F . Therefore it follows from (1) that the set $\{L_x D : x \in G\}$ spans a finite dimensional subspace of $\underline{E}^{\circ\circ}(F)$ because only a finite number of D_σ 's are non-zero. Hence $\underline{E}^{\circ\circ}(F)$ is a subset of $P(\underline{E}(F)^{(A^{-1})^*, p'})$.

The final assertions are immediate consequences of (28.27) of [HR] because $\underline{E}^{\circ\circ}(F) \cdot (A^{-1})^* = \underline{E}^{\circ\circ}(F)$. $\square$

(3.5.8) PROPOSITION : If $p \in (1, \infty) \cup \{0\}$ then $(PM(G)^{A, p})^\# = \underline{E}(F)^{(A^{-1})^*, p'}$. Also $(PM(G)^{A, 1})^\# = \underline{E}(F)^{(A^{-1})^*, 0}$.

Proof : By (3.3.10) $P((PM(G)^{A, p})^\#)$ is a dense subspace of $(PM(G)^{A, p})^\#$. Therefore, in view of (3.5.7), we need only prove that $\underline{E}^{\circ\circ}(F)$ is in $(PM(G)^{A, p})^\#$.

Let $D \in \underline{E}^{\circ\circ}(F)$. For each $x \in G$ and $S \in PM(G)^{A, p}$

$$\Gamma_D(S)(x) = \langle\langle L_x D, S \rangle\rangle = \langle L_x D (A^{-1})^*, \hat{S} A \rangle. \quad (1)$$

Therefore $\|\Gamma_D(S)\|_\infty = \|D\|_{(A^{-1})^*, p'}$. If $x \in G$ then

$$\begin{aligned} |\Gamma_D(S)(x) - \Gamma_D(S)(1)| &= \langle L_x D (A^{-1})^* - D (A^{-1})^*, \hat{S} A \rangle \\ &= \sum_{\sigma \in F} d_\sigma \text{Tr} (A_\sigma^{-1} D_\sigma^* (U_\sigma(x)^* - \text{Id}_\sigma) \hat{S}(\sigma) A_\sigma) \\ &= \sum_{\sigma \in F} d_\sigma \text{Tr} (\hat{S}(\sigma) D_\sigma^* (U_\sigma(x^{-1}) - \text{Id}_\sigma)). \end{aligned} \quad (2)$$

Since only a finite number of D_σ 's are non-zero, (2) ensures that $\{\Gamma_D(s) : s \in PM(G)^{A,p}, \|s\|_{A,p} \leq 1\}$ is equicontinuous. It is clearly bounded so it must be relatively compact. Hence D is in $(PM(G)^{A,p})^*$. □

(3.5.9) We now summarize the results of this section in the form of a table of examples where h indicates homogeneous.

	B	$B^\#$	h	Reference
1.	$A(G)$	$PF(G)$	✓	(3.5.4)
2.	$C(G)$	$L^1(G)$	✓	(3.5.4)
3.	$L^\infty(G)$	$L^1(G)$		(3.5.5)
4.	$L^p(G) \ 1 < p < \infty$	$L^{p'}(G)$	✓	(3.5.4)
5.	$U^p(G) \ 1 < p < \infty$	$C(G) \oplus \underline{E}^{p'}(\Sigma(G))$	✓	(3.5.4) and Gregory [1]
6.	$L^1(G)$	$C(G)$	✓	(3.5.4)
7.	$M(G)$	$C(G)$		(3.5.5)
8.	$PF(G)$	$A(G)$	✓	(3.5.4)
9.	$PM(G)$	$A(G)$		(3.5.5)
10.	$PM(G)^{A,p} \ 1 < p < \infty$	$\underline{E}(F) (A^{-1})^*, p'$	✓	(3.5.8)
11.	$PM(G)^{A,1}$	$\underline{E}(F) (A^{-1})^*, 0$	✓	(3.5.8)
12.	$PM(G)^{A,0}$	$\underline{E}(F) (A^{-1})^*, 1$	✓	(3.5.8)
13.	$V_{F_1, F_2}(G)$	closure in $V_{F_1, F_2}(G)$ of $T(G \times G)_{F_1 \times F_2}$	✓	(3.5.10) and (3.5.2)

(3.5.10) We can use (3.5.9) to obtain information about some of the spaces constructed by means of (3.4.8).

Recall that if B is in $\mathcal{B}_h$ and $\mathcal{G}$ is a family of subspaces $\{E_\sigma \subseteq H_\sigma : \sigma \in \Sigma(G)\}$ then by (3.4.9) $B_{\mathcal{G}}$ is also in $\mathcal{B}_h$. Since it is

closed in B we have

$$(B_{\mathcal{G}})^* \cong B^* / B_{\mathcal{G}}^{\perp} \quad (1)$$

where $B_{\mathcal{G}}^{\perp} = \{M \in B^* : M(b) = 0 \text{ for all } b \in B_{\mathcal{G}}\}$.

There is one case in which it is possible to explicitly represent $(B_{\mathcal{G}})^*$ and $(B_{\mathcal{G}})^{\#}$. We assume that $P(B) = T(G)$ and that $B = B_{\mathcal{G}} \oplus B_{\mathcal{G}^{\perp}}$, where $\mathcal{G}^{\perp} = \{E_{\sigma}^{\perp} : E_{\sigma} \in \mathcal{G}\}$. If there exists a continuous projection of B onto $B_{\mathcal{G}}$ then this second condition holds.

By (3.5.3), if M is in $(B_{\mathcal{G}})^*$ then there exists $A = (A_{\sigma})_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))_{\mathcal{G}}$ such that for each b in $B_{\mathcal{G}}$

$$\Gamma_M(b)^{\wedge}(\sigma) = \hat{b}(\sigma) A_{\sigma}^* \text{ for } \sigma \in \Sigma(G), \quad (2)$$

and for each b in $T(G)_{\mathcal{G}}$

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_{\sigma} \text{Tr}(A_{\sigma}^* \hat{b}(\sigma)). \quad (3)$$

If M is in B^* then there exists $A = (A_{\sigma})_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))$ which satisfies (2) for all b in B , and (3) for all b in $T(G)$.

Let $(B^*)_{\mathcal{G}}$ denote the set of all M in B^* such that the corresponding A is in $\underline{E}(\Sigma(G))_{\mathcal{G}}$. Denote by $(B^{\#})_{\mathcal{G}}$ the same subset of $B^{\#}$.

PROPOSITION : $(B_{\mathcal{G}})^* = (B^*)_{\mathcal{G}}$ and $(B_{\mathcal{G}})^{\#} = (B^{\#})_{\mathcal{G}}$.

Proof : Let $M \in (B^*)_{\mathcal{G}}$. By (1) M is in $(B_{\mathcal{G}})^*$ if $M(b) = 0$ for all b in $B_{\mathcal{G}}$. Now $T(G)_{\mathcal{G}}$ is a dense subspace of $B_{\mathcal{G}}$, and so it is sufficient to prove that $T(G)_{\mathcal{G}} \subseteq \ker M$. This, however, follows immediately from (3) because $A \in \underline{E}(\Sigma(G))_{\mathcal{G}}$ iff the image of A_{σ}^* is contained in $E_{\sigma}^{\perp}$ for each σ .

On the other hand, let M be in $(B_{\mathcal{G}})^*$, and let A be the element of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ corresponding to M . Let $b \in B$. Then b has a unique decomposition, $b = b_1 + b_2$, where b_1 is in $B_{\mathcal{G}}$ and b_2 is

$B_{\mathcal{G}}$. Define $\bar{M}$ on B by $\bar{M}(b) = M(b_1)$ for all b in B . Clearly $\bar{M}$ is linear and continuous. For b in B

$$\Gamma_{\bar{M}}(b)^{\wedge}(\sigma) = A_{\sigma}^* \hat{b}_1(\sigma) \quad \text{for } \sigma \text{ in } \Sigma(G). \quad (4)$$

But if b is in $B_{\mathcal{G}}$ then $A_{\sigma}^* \hat{b}(\sigma) = 0$; so we can write

$$\Gamma_{\bar{M}}(b)^{\wedge}(\sigma) = A_{\sigma}^* \hat{b}(\sigma) \quad \text{for all } \sigma \text{ in } \Sigma(G).$$

Hence $\bar{M}$ is in $(B^*)_{\mathcal{G}}$ as required.

To prove the second equality, first let M be in $(B_{\mathcal{G}})^{\#}$. Then it is in $(B_{\mathcal{G}})^*$ and so it can be extended to a functional $\bar{M}$ in $(B^*)_{\mathcal{G}}$. It follows from (4) that $\Gamma_{\bar{M}}$ is a compact operator on B because Γ_M is compact on $B_{\mathcal{G}}$. Therefore $\bar{M}$ is in $(B^{\#})_{\mathcal{G}}$.

On the other hand it is easy to see that the restriction of any operator which is compact, is again compact. Hence $(B^{\#})_{\mathcal{G}} \subseteq (B_{\mathcal{G}})^{\#}$. $\square$

If $B = A(G)$, $L^2(G)$ or $PF(G)$ then (3.5.10) may be applied for any family $\mathcal{G}$.

CHAPTER 4

HOMOGENEOUS BANACH SPACES OF PSEUDOMEASURES

(4.1) Introduction

The table of examples given in (3.5.9) illustrates an interesting phenomenon. If B is a homogeneous Banach space then $(B^\#)^\#$ (which I write as $B^{\#\#}$) is precisely B ; and if B is not homogeneous then $B^{\#\#}$ is a proper subspace of B . I begin this chapter by proving that this characterizes homogeneous Banach spaces of pseudomeasures - that is, B is homogeneous if and only if B and $B^{\#\#}$ are equivalent Banach spaces (see (4.2.11)). The proof depends intrinsically on the fact that the elements of B are pseudomeasures. I have not yet determined whether the characterization holds for arbitrary L -homogeneous Banach spaces.

In de Leeuw [1] the author proves that there is a natural correspondence between the family of all homogeneous algebras B of continuous functions defined on a compact abelian group G , and a family of G -invariant norms on $B \cap T(G)$. de Leeuw makes explicit use of the algebraic structure of B . In §(4.3) I prove that even if it is neither assumed that B is an algebra nor that G is abelian, then it is possible to associate with B a norm N defined on $B \cap T(G)$ which has the properties listed in (4.3.5). N need not be G -invariant. On the other hand, every such norm determines a Banach space in $\mathcal{B}_h$ (see (4.3.8)).

If B is a homogeneous Banach space in $\mathcal{B}_h$, and if $\mathcal{J}$ denotes the set $\mathcal{J} = \{E_\sigma = \bigcap_{b \in B} \ker \hat{b}(\sigma) : \sigma \in \Sigma(G)\}$, then I proved in (3.5.3) that B^* may be identified with a subset Λ of $\underline{E}(\Sigma(G))_{\mathcal{J}}$. In fact, employing the notation of §35 of [HR], the elements of Λ are

precisely the adjoints of the right $(B, C(G))$ -multipliers. Hence the set of right $(B, C(G))$ -multipliers is isomorphic to B^* . In §(4.4) I study the vector space Λ and show that it must have the properties listed in (4.4.11). On the other hand, every such subspace of $\underline{E}(\Sigma(G))$ determines a homogeneous Banach space in $\mathcal{B}_h$.

The final section of this chapter includes a characterization of p -Sidon sets, $1 \leq p < 2$, in terms of homogeneous spaces. This could be considered one possible answer to question 4 posed in Burnham [3]. I introduce homogeneous convolution Banach algebras. These will be studied in chapter 5. I prove that every B in $\mathcal{B}_h$ which contains only measures is, in fact, a homogeneous convolution Banach algebra.

(4.2) Equality of B and $B^{##}$ if $B \in \mathcal{B}_h$.

The notation for this chapter is taken from (3.4.5). G again denotes a compact group.

We begin with a lemma which, in view of (3.3.10), proves that if B is in $\mathcal{B}_h$ then the subspaces $B_{\mathcal{G}}$, defined by means of (3.4.8), are the only closed subspaces of B which are also in $\mathcal{B}_h$ when they inherit the topology and translation operators of B .

(4.2.1) LEMMA : Let $B \in \mathcal{B}_h$. For each $\sigma \in \Sigma(G)$ let $E_{\sigma} = \bigcap_{b \in B} \ker \hat{b}(\sigma)$, and $\mathcal{G} = \{E_{\sigma} : \sigma \in \Sigma(G)\}$. Then $P(B) = T(G)_{\mathcal{G}}$.

Proof : By (3.5.2) $P(B) = B \cap T(G)$. Therefore it is clear that $P(B)$ is a subspace of $T(G)_{\mathcal{G}}$.

On the other hand we first prove that for each $\sigma \in \Sigma(G)$ the set $I_{\sigma} = \{\hat{b}(\sigma) : b \in B\}$ is a left ideal in B . This allows us to write $I_{\sigma} = \{T \in B(H_{\sigma}) : T(E_{\sigma}) = \{0\}\}$. Now, by (1.3.8) there exist elements $x_1, \dots, x_{d_{\sigma}^2}$ of G such that $\{U_{\sigma}(x_i), i=1, \dots, d_{\sigma}^2\}$ is a basis for $B(H_{\sigma})$. Let $T \in B(H_{\sigma})$. Then there exists constants $\alpha_i(T)$ in

$C, i=1, \dots, d_\sigma^2$, such that

$$T = \sum_{i=1}^{d_\sigma^2} \alpha_i(T) U_\sigma(x_i),$$

and so for any $b \in B$

$$\begin{aligned} T \hat{b}(\sigma) &= \left(\sum_{i=1}^{d_\sigma^2} \alpha_i(T) U_\sigma(x_i) \right) \hat{b}(\sigma) \\ &= \sum_{i=1}^{d_\sigma^2} \alpha_i(T) x_i \hat{b}(\sigma). \end{aligned}$$

But $\sum_{i=1}^{d_\sigma^2} \alpha_i(T) x_i b$ is an element of B so I_σ is a left ideal in

$B(H_\sigma)$. Therefore, by (38.11) of [HR], there exists a subspace Y_σ of H_σ such that $I_\sigma = \{T \in B(H_\sigma) : T(Y_\sigma) = \{0\}\}$. Clearly Y_σ must equal E_σ . We complete the proof by proving that if $k \in T(G)_{\{\sigma\}}$ such that $\hat{k}(\sigma)(E_\sigma) = \{0\}$ then k is in $P(B)$. The operator $\hat{k}(\sigma) \in I_\sigma$, and so there exists an element b of B such that $\hat{b}(\sigma) = \hat{k}(\sigma)$. Let χ_σ be the character of σ . By (3.3.9) $\chi_\sigma * b$ is in $P(B)$. But $\chi_\sigma * b = k$ and so k is in $P(B)$. Hence $T(G)_\sigma \subseteq P(B)$. $\square$

(4.2.2) Notation : It follows from (4.2.1) that $P(B)$ is the linear span of the set of trigonometric polynomials of the form

$$x \rightarrow \langle \xi, U_\sigma(x) \zeta \rangle$$

where $\xi \in H_\sigma$, $\zeta \in E_\sigma^\perp$ and $\sigma \in \Sigma(G)$.

Let $\{e_i^\sigma, i=1, \dots, d_\sigma\}$ denote an orthonormal basis for H_σ and assume that $\{e_i^\sigma, i=1, \dots, k_\sigma\}$ spans E_σ . Define Ω_σ to be

$$\Omega_\sigma = \{1, \dots, d_\sigma\} \times \{k_\sigma+1, \dots, d_\sigma\}$$

(where $\Omega_\sigma = \{\}$ if $k_\sigma = d_\sigma$), and for each $(i,j) \in \Omega_\sigma$ define the trigonometric polynomial u_{ij}^σ by

$$u_{ij}^\sigma(x) = d_\sigma^{-\frac{1}{2}} \langle e_i^\sigma, U_\sigma(x) e_j^\sigma \rangle \quad \text{for all } x \in G.$$

Then $\{u_{ij}^\sigma : (i,j) \in \Omega_\sigma, \sigma \in \Sigma(G)\}$ is a basis for $P(B)$. Furthermore,

for each b in $P(B)$

$$b = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} \left[\int b(x) \overline{u_{ij}^\sigma(x)} d\lambda(x) \right] u_{ij}^\sigma,$$

where the sum has only a finite number of non-zero terms. However,

$$\begin{aligned} \int b(x) \overline{u_{ij}^\sigma(x)} d\lambda(x) &= d_\sigma^{\frac{1}{2}} \int b(x) \left\langle \overline{e_i^\sigma}, u_\sigma(x) e_j^\sigma \right\rangle d\lambda(x) \\ &= d_\sigma^{\frac{1}{2}} \int b(x) \left\langle u_\sigma(x) e_j^\sigma, e_i^\sigma \right\rangle d\lambda(x) \\ &= d_\sigma^{\frac{1}{2}} \left\langle \hat{b}(\sigma) e_j^\sigma, e_i^\sigma \right\rangle, \end{aligned}$$

and so for $b \in P(B)$

$$b = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} d_\sigma^{\frac{1}{2}} \left\langle \hat{b}(\sigma) e_j^\sigma, e_i^\sigma \right\rangle u_{ij}^\sigma.$$

It is our intention to prove in this section that if B is in $\mathcal{B}_h$ then the Banach spaces B and $B^{\#\#}$ are isomorphic and their norms are equivalent. In view of (3.3.4) this provides a characterization of homogeneous Banach spaces. To obtain this result we first determine a basis for $P(B^{\#})$ from that given for $P(B)$ in (4.2.2) (see (4.2.5)). This basis is then used to construct a basis for $P(B^{\#\#})$ in (4.2.8), and finally in (4.2.11) we prove that the spaces $P(B)$ and $P(B^{\#\#})$ are isomorphic and that $\|\cdot\|_B$ and $\|\cdot\|_{B^{\#}}$ are essentially equivalent. By (3.3.10) this completes our argument.

(4.2.3) DEFINITION : For each $\sigma \in \Sigma(G)$ and $(i,j) \in \Omega_\sigma$ define the functional M_{ij}^σ from B to $\mathbb{C}$ by

$$M_{ij}^\sigma(b) = d_\sigma^{\frac{1}{2}} \left\langle \hat{b}(\sigma) e_j^\sigma, e_i^\sigma \right\rangle \text{ for each } b \in B.$$

Clearly each functional is linear on B . Furthermore, for any $\eta \in \Sigma(G)$ and $(k,l) \in \Omega_\eta$

$$M_{ij}^\sigma(u_{kl}^\eta) = \delta_{\sigma\eta} \delta_{ik} \delta_{jl}$$

because it is easily seen that

$$\begin{aligned}
M_{ij}^\sigma(u_{kl}^\eta) &= d_\sigma^{\frac{1}{2}} \langle \hat{u}_{kl}^\eta(\sigma) e_j^\sigma, e_i^\sigma \rangle \\
&= d_\sigma^{\frac{1}{2}} d_\eta^{\frac{1}{2}} \int \langle e_k^\eta, u_\eta(x) e_l^\eta \rangle \overline{\langle e_i^\sigma, u_\sigma(x) e_j^\sigma \rangle} d\lambda(x),
\end{aligned}$$

and so the assertion follows immediately from (1.3.4).

(4.2.4) LEMMA : Let $B \in \mathcal{B}_h$. Then each linear functional M_{ij}^σ is an element of $P(B^*) \cap B^\#$.

Proof : In view of (3.4.5) (a) M_{ij}^σ is continuous because

$$|M_{ij}^\sigma(b)| \leq d_\sigma^{\frac{1}{2}} \|\hat{b}(\sigma)\| \quad \text{for each } b \in B.$$

Let $x \in G$. Then

$$L_x M_{ij}^\sigma(b) = M_{ij}^\sigma(x^{-1}b) = d_\sigma^{\frac{1}{2}} \langle u_\sigma(x^{-1}) \hat{b}(\sigma) e_j^\sigma, e_i^\sigma \rangle, \quad (1)$$

and so, by (1.3.8) and (1), the linear span of the set of translates of M_{ij}^σ is finite dimensional. Thus $M_{ij}^\sigma \in P(B^*)$.

By (3.3.3) $B'' = B^*$; therefore M_{ij}^σ is in $B^\#$ iff the corresponding operator is compact from B to $C(G)$. We recall that for each $b \in B$ and $x \in G$

$$\Gamma_{M_{ij}^\sigma}(b)(x) = L_x M_{ij}^\sigma(b) = M_{ij}^\sigma(x^{-1}b).$$

Appealing to (3.3.2) we see that the set

$$\{\Gamma_{M_{ij}^\sigma}(b) : b \in B_B(0,1)\}$$

is a bounded subset of $C(G)$. In fact it follows easily from (1) that it is equicontinuous because for any $x, y \in G$ and $b \in B$

$$|\Gamma_{M_{ij}^\sigma}(b)(x) - \Gamma_{M_{ij}^\sigma}(b)(y)| \leq d_\sigma^{\frac{1}{2}} \|u_\sigma(x^{-1}) - u_\sigma(y^{-1})\| \|\hat{b}(\sigma)\|.$$

Therefore it is an immediate consequence of Ascoli's theorem that the set is relatively compact in $C(G)$, and so $M_{ij}^\sigma \in B^\#$. $\square$

(4.2.5) PROPOSITION : Let $B \in \mathcal{B}_h$. Then $\{M_{ij}^\sigma : (i,j) \in \Omega_\sigma, \sigma \in \Sigma(G)\}$ is a basis for $P(B^\#)$. Furthermore, if M is in $P(B^\#)$ then

$$M = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} M(u_{ij}^\sigma) M_{ij}^\sigma,$$

where the sum contains only finitely many non-zero terms.

Proof : Let $b \in P(B)$. From (4.2.2) and (4.2.3)

$$b = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} M_{ij}^\sigma(b) u_{ij}^\sigma, \quad (1)$$

and so if M is in $B^\#$ then

$$M(b) = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} M_{ij}^\sigma(b) M(u_{ij}^\sigma). \quad (2)$$

Now it follows from (4.2.3) and (4.2.4) that the set of M_{ij}^σ 's is a linearly independent subset of $P(B^\#)$. In view of (2) and (3.3.10) it is a basis for $P(B^\#)$ if the linear subspace generated by $\{b \in P(B) : M(b) \neq 0\}$ is finite dimensional for each M in $P(B^\#)$. It will also follow from (2) that M permits a representation as a sum of the type required.

Therefore we consider $\{b \in P(B) : M(b) \neq 0\}$. It certainly generates a finite dimensional subspace of $P(B)$ if

$$\text{card } \{\sigma \in \Sigma(G) : M(b) \neq 0 \text{ for some } b \in B_{\{\sigma\}}\} < \infty. \quad (3)$$

Let W denote the linear span of the set $\{L_x M : x \in G\}$. Then

$$\dim W < \infty \quad (4)$$

because $M \in P(B^\#)$, and so W is closed. It is clearly continuously L -translation invariant. Hence $(W, \|\cdot\|_{B^\#})$ is L -homogeneous. By (3.3.7) for each $\mu \in M(G)$.

$$\mu * M = \int L_x M d\mu(x) \in W. \quad (5)$$

In particular, if b_1 and b_2 are elements of $B \cap T(G)$ then

$$\begin{aligned}
 b_1 * M(b_2) &= \int b_1(x) L_x M(b_2) d\lambda(x) \\
 &= M \left(\int b_1(x) x b_2 d\lambda(x) \right).
 \end{aligned}
 \tag{6}$$

If b_1 is in $B_{\{\sigma\}}$ and $\hat{b}_2(\sigma) = 0$ then

$$b_1 * M(b_2) = 0 \tag{7}$$

because $\int b_1(x) x b_2 d\lambda(x) = 0$.

Let $E_\sigma = \cap \{\ker \hat{b}(\sigma) : b \in B\}$, and define the trigonometric polynomial ω_σ by

$$\omega_\sigma(x) = d_\sigma \operatorname{Tr} (P_{E_\sigma^\perp} U_\sigma^*(x)) \text{ for each } x \in G.$$

Then it follows easily from (6) that

$$b_1 * M(\omega_\sigma) = M(b_1). \tag{8}$$

Now assume (3) is false. Then there is an infinite subset $\{\sigma_n, n \in \mathbf{N}\}$ of $\Sigma(G)$ such that for each n there is an element b_n in $B_{\{\sigma_n\}}$ satisfying $M(b_n) \neq 0$. Define the linear functional M_n to be $b_n * M$. By (5) $M_n \in W$, and it follows from (8) that M_n is non-zero because $M_n(\omega_{\sigma_n}) \neq 0$. Hence $\{M_n : n \in \mathbf{N}\}$ is linearly independent in W . To see this let

$$\sum_{n=1}^N \alpha_n M_n = 0.$$

Then in particular

$$\sum_{n=1}^N \alpha_n M_n(\omega_{\sigma_1}) = \alpha_1 M_1(\omega_{\sigma_1}) = 0.$$

Thus $\alpha_1 = 0$. Similarly $\alpha_2, \dots, \alpha_N$ must all be zero. But this contradicts (4), so (3) must be true. $\square$

(4.2.6) We now construct a basis for $P(B^{##})$. It follows from (3.3.4) and (3.3.10) that $P(B^{\#})$ is a dense subspace of $B^{\#}$. Therefore if $\Pi \in (B^{\#})^*$ and $M \in P(B^{\#})$ then by (4.2.5)

$$\Pi(M) = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_{\sigma}} M(u_{ij}^{\sigma}) \Pi(M_{ij}^{\sigma}),$$

and Π is completely determined by this sum in which at most a finite number of terms are non-zero. This suggests that we should give the following definition.

DEFINITION : For each $\sigma \in \Sigma(G)$ and $(i,j) \in \Omega_{\sigma}$ define the functional Π_{ij}^{σ} from $B^{\#}$ to $\mathbb{C}$ by

$$\Pi_{ij}^{\sigma}(M) = M(u_{ij}^{\sigma}) \text{ for all } M \in B^{\#}.$$

(4.2.7) LEMMA : Let $B \in \mathcal{B}_h$. Then each functional Π_{ij}^{σ} is an element of $P(B^{##})$.

Proof : It is clear that Π_{ij}^{σ} is a continuous linear functional on $B^{\#}$. Let $x \in G$ and $M \in B^{\#}$. Then

$$L_x \Pi_{ij}^{\sigma}(M) = \Pi_{ij}^{\sigma}(L_{x^{-1}} M) = M(x u_{ij}^{\sigma}),$$

and so by (1.3.8) Π_{ij}^{σ} is in $P((B^{\#})^*)$. As in the proof of (4.2.4) it is easy to see that Π_{ij}^{σ} is an element of $B^{##}$ because the set

$$\{\Gamma_{\Pi_{ij}^{\sigma}}(M) : M \in B_{B^{\#}}(0,1)\}$$

is a bounded equicontinuous subset of $C(G)$. Therefore Π_{ij}^{σ} is in $P((B^{\#})^*) \cap B^{##}$ which is a subspace of $P(B^{##})$. $\square$

(4.2.8) In much the same way as we proved (4.2.5) we can now prove that the set $\{\Pi_{ij}^{\sigma} : (i,j) \in \Omega_{\sigma}, \sigma \in \Sigma(G)\}$ is a basis for $P(B^{##})$.

PROPOSITION : Let $B \in \mathcal{B}_h$. Then $\{\Pi_{ij}^{\sigma} : (i,j) \in \Omega_{\sigma}, \sigma \in \Sigma(G)\}$ is a basis for $P(B^{##})$. Furthermore, if Π is in $P(B^{##})$ then

$$\Pi = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} \Pi(M_{ij}^\sigma) \Pi_{ij}^\sigma,$$

where the sum contains only finitely many non-zero terms.

Proof : Let $M \in P(B^\#)$ and $\Pi \in P^{\#\#}$. By (4.2.5)

$$\Pi(M) = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} \Pi(M_{ij}^\sigma) M(u_{ij}^\sigma). \quad (1)$$

Appealing to (4.2.3) and (4.2.6) we see that for any $\eta \in \Sigma(G)$,

$$(k,1) \in \Omega_\eta$$

$$\Pi_{ij}^\sigma (M_{kl}^\eta) = \delta_{\sigma\eta} \delta_{ik} \delta_{jl}.$$

Hence it follows from (4.2.7) that the set of Π_{ij}^σ 's is a linearly independent subset of $P(B^{\#\#})$. Therefore, in view of (1), (3.3.10) and (3.3.4), it is a basis for $P(B^{\#\#})$ if for each $\Pi \in P(B^{\#\#})$

$$\dim \text{span} \{M \in P(B^\#) : \Pi(M) \neq 0\} < \infty. \quad (2)$$

As far as is possible we emulate the proof of (4.2.5). Let W denote the linear span of the set $\{L_x \Pi : x \in G\}$. Then

$$\dim W < \infty \quad (3)$$

because $\Pi \in P(B^{\#\#})$, and so W is closed. It is clearly continuously L -translation invariant. Hence $(W, \| \cdot \|_{(B^*)})$ is L -homogeneous.

By (3.3.7) for each $\mu \in M(G)$

$$\mu * \Pi = \int L_x \Pi d\mu(x) \in W. \quad (4)$$

In particular, if $b \in T(G)$ and $M \in P(B^\#)$ then by (3.3.4) and (3.3.9)

$$\begin{aligned} b * \Pi(M) &= \int b(x) L_x \Pi(M) d\lambda(x) \\ &= \Pi \left(\int b(x) L_x M d\lambda(x) \right). \end{aligned} \quad (5)$$

If $b \in T(G)_{\{\sigma\}}$ and $M \in P(B^\#)$ such that $B_{\{\sigma\}} \subseteq \ker M$ then, by (5)

$$b * \Pi(M) = 0. \quad (6)$$

It also follows from (5) that

$$\chi_\sigma * \Pi(M_{ij}^\sigma) = \Pi(M_{ij}^\sigma). \quad (7)$$

Now assume that (2) is false. Since the set of M_{ij}^σ 's is a basis for $P(B^\#)$, and since $\text{card } \Omega_\sigma$ is finite for each $\sigma \in \Sigma(G)$, there must exist an infinite subset $\{\sigma_n, n \in \mathbb{N}\}$ of $\Sigma(G)$ and pairs (i_n, j_n) in Ω_{σ_n} such that $\Pi(M_{i_n j_n}^{\sigma_n}) \neq 0$. Define the linear functional Π_n to be $\chi_{\sigma_n} * \Pi$. By (4) $\Pi_n \in W$, and it follows from (7) that it is non-zero. By (6) the set of Π_n 's must be linearly independent in W . To see this let

$$\sum_{n=1}^N \alpha_n \Pi_n = 0.$$

Then, in particular

$$\sum_{n=1}^N \alpha_n \Pi_n(M_{i_1 j_1}^{\sigma_1}) = \alpha_1 \Pi_1(M_{i_1 j_1}^{\sigma_1}) = 0.$$

Thus $\alpha_1 = 0$. Similarly $\alpha_2, \dots, \alpha_N$ must all be zero. But this contradicts (3) and so (2) must be true. $\square$

(4.2.9) COROLLARY : Let $\Pi \in P(B^{\#\#})$. Then

$$b_\Pi = \sum_{\sigma \in \Sigma(G)} \sum_{(i,j) \in \Omega_\sigma} \Pi(M_{ij}^\sigma) u_{ij}^\sigma$$

is an element of $P(B)$, and Π corresponds to the evaluation map at b_Π .

Proof : By (4.2.8) $\{\sigma : \Pi(M_{ij}^\sigma) \neq 0 \text{ for some } (i,j) \in \Omega_\sigma\}$ is finite. Therefore by (4.2.2) b_Π is in $P(B) = B \cap T(G)$. It is easily seen from (4.2.8) that Π is the b_Π evaluation map. $\square$

(4.2.10) Define the operator Θ from B to B^{**} by $b \mapsto \Theta_b$, where $\Theta_b(M) = M(b)$ for all $M \in B^*$. Then by (3.2.5) each functional Θ_b is also in $(B^\#)^*$. In fact we have :

LEMMA : Let $B \in \mathcal{B}_h$. Then $\theta_b \in B^{\#\#}$ and $\|\theta_b\|_{(B^\#)^{**}} \leq \|b\|_B$ for all $b \in B$.

Proof : Let $M \in B^\#$. Then for each $x \in G$

$$\Gamma_{\theta_b}(M)(x) = L_x \theta_b(M) = L_{x^{-1}} M(b) = \Gamma_M(b)(x^{-1}). \quad (1)$$

Hence $\Gamma_{\theta_b}(M)$ is in $C(G)$ and its infinity norm is bounded by

$\|M\|_{B^\#} \|b\|_B$. It then follows easily from (1) that the set

$$\{\Gamma_{\theta_b}(M) : M \in B_{B^\#}(0,1)\}$$

is bounded and equicontinuous in $C(G)$, and so Γ_{θ_b} is compact.

Therefore θ_b is in $B^{\#\#}$ and

$$\|\theta_b\|_{(B^\#)^{**}} \leq \|b\|_B. \quad \square$$

Remark : Clearly if $b \in P(B)$ then θ_b is in $P(B^{\#\#})$. By (4.2.9) θ maps $P(B)$ onto $P(B^{\#\#})$.

(4.2.11) We now complete the characterization of spaces in $\mathcal{B}_h$.

THEOREM : Let $B \in \mathcal{B}_h$. The map θ is an isomorphism of B onto $B^{\#\#}$. Furthermore, there exists a positive constant K such that

$$\|b\|_B \leq K^2 \|\theta_b\|_{(B^\#)^{**}} \quad b \in B. \quad (1)$$

Proof : We first prove that (1) holds for all $b \in P(B)$. Let $b \in P(B)$. Then

$$\begin{aligned} \|\theta_b\|_{(B^\#)^{**}} &= \sup_{M \in B_{B^\#}(0,1)} \|\Gamma_{\theta_b}(M)\|_\infty \\ &= \sup_{M \in B_{B^\#}(0,1)} \sup_{x \in G} |L_x \theta_b(M)| \\ &\geq \sup_{M \in B_{B^\#}(0,1)} |\theta_b(M)|, \end{aligned}$$

and so

$$\| \theta_b \|_{(B^\#)''} \geq \sup_{M \in B_{B^\#}(0,1)} |M(b)|. \quad (2)$$

By the Hahn-Banach theorem there exists a functional M_0 in B^* , M_0 depending on b , such that $\|M_0\|_{B^*} = 1$ and

$$|M_0(b)| = \|b\|_B. \quad (3)$$

Appealing to (3.2.5) and (2) we see that (1) holds for each b in $P(B)$ if, for each M in B^* with $\|M\|_{B^*} = 1$, there exists a sequence $(M_n)_{n \in \mathbb{N}}$ extracted from $B^\#$ such that

$$\left. \begin{aligned} \sup_{n \in \mathbb{N}} \|M_n\|_{B''} &\leq K^2, & \text{and} \\ \lim_{n \rightarrow \infty} |M_n(b) - M(b)| &= 0 \text{ for all } b \in B. \end{aligned} \right\} \quad (4)$$

To see that such a sequence exists let $(k_n)_{n \in \mathbb{N}}$ be an approximate identity of $L^1(G)$ taken from $T(G)$. We will prove first that the linear functionals M_n defined by

$$M_n(b) = M(\overset{\vee}{k}_n * b) \text{ for each } b \in B \quad (5)$$

are in $B^\#$. By $\overset{(3.3.7)}{(3.3.8)}$ and $\overset{\vee}{k}_n * b$ is in $P(B)$ and

$$\|\overset{\vee}{k}_n * b\|_B \leq K \|k_n\|_1 \|b\|_B. \quad (6)$$

Then (5) and (6) prove that M_n is in B^* , and so in B'' by (3.3.3).

Let F_n denote the support of $\overset{\Delta}{k}_n$. Then by (3.5.5), for each $b \in B$, $\overset{\vee}{k}_n * b$ is in $T(G)_{F_n}$. Therefore it follows from (5) that M_n is in $B^\#$. We now prove that the sequence $(M_n)_{n \in \mathbb{N}}$ satisfies (4).

For each $n \in \mathbb{N}$

$$\begin{aligned} \|M_n\|_{B''} &= \sup_{b \in B_B(0,1)} \|\Gamma_{M_n}(b)\|_\infty \\ &= \sup_{b \in B_B(0,1)} \sup_{x \in G} |M(\overset{\vee}{k}_n * x^{-1}b)| \text{ by (5),} \\ &\leq \|M\|_{B^*} \sup_{b \in B_B(0,1)} \sup_{x \in G} K \|k_n\|_1 \|x^{-1}b\|_B \text{ by (6),} \\ &\leq K^2 \text{ by (3.3.2) and (3.3.3).} \end{aligned}$$

Now let $b \in B$. Then

$$\begin{aligned} |M_n(b) - M(b)| &= |M(\bigvee k_n * b - b)| \\ &\leq \|M\|_{B^*} \|\bigvee k_n * b - b\|_B, \end{aligned}$$

and so it follows from (3.3.10) that $\lim_{n \rightarrow \infty} |M_n(b) - M(b)| = 0$. Hence

$(M_n)_{n \in \mathbb{N}}$ satisfies (4), and (1) holds for all $b \in B$.

We now prove that θ maps B onto $B^{##}$. Let $\Pi \in B^{##}$. By (3.3.10) there exists a sequence $(\Pi_n)_{n \in \mathbb{N}}$ in $P(B^{##})$ such that

$$\lim_{n \rightarrow \infty} \|\Pi - \Pi_n\|_{(B^{\#})''} = 0. \quad (7)$$

Furthermore, by (4.2.9), for each n there exists b_n in $P(B)$ such that

$$\theta_{b_n} = \Pi_n. \quad (8)$$

By (7) and (1) the sequence $(b_n)_{n \in \mathbb{N}}$ is Cauchy in B , and so it converges to an element b of B . But θ is continuous and so

$$\lim_{n \rightarrow \infty} \|\theta_b - \theta_{b_n}\|_{(B^{\#})''} = 0. \quad (9)$$

Hence by (7) and (9) $\Pi = \theta_b$ and θ is a surjection.

Finally it follows from (3.3.4) and (3.3.10) that (1) holds for all $b \in B$. □

(4.3) Characterization of $\mathcal{B}_h$ by Norms on Sets of Trigonometric Polynomials

Let $B \in \mathcal{B}_h$. For each $\sigma \in \Sigma(G)$ let $E_\sigma = \bigcap \{\ker \hat{b}(\sigma) : b \in B\}$ and let $\mathcal{J} = \{E_\sigma : \sigma \in \Sigma(G)\}$. By (4.2.1) $P(B) = T(G)_{\mathcal{J}}$, and so by (3.3.10) $T(G)_{\mathcal{J}}$ is a dense subspace of B . Let N denote the norm of B restricted to $T(G)_{\mathcal{J}}$. In this section we characterize Banach

spaces in $\mathcal{B}_h$ by means of the norm N defined on $T(G)_g$.

(4.3.1) It is an immediate consequence of (3.4.5) that

- (a) $\|b\|_{PM} \leq \text{const } N(b)$ for each $b \in T(G)_g$, and
- (b) each translation operator is a continuous endomorphism of $T(G)_g$, and the operator $x \rightarrow_x b$ is continuous from G to $T(G)_g$ for each $b \in T(G)_g$.

(4.3.2) The third property which N must possess is explained in the following proposition.

PROPOSITION : Let $B \in \mathcal{B}_h$ and $(b_n)_{n \in \mathbf{N}}$ be a sequence taken from $T(G)_g$ which satisfies :

- (i) $\lim_{n \rightarrow \infty} \|b_n\|_{PM} = 0$, and
- (ii) it is a Cauchy sequence with respect to N .

Then $\lim_{n \rightarrow \infty} N(b_n) = 0$.

Proof : By (ii) the sequence is Cauchy in B and so converges in B to an element b of B . Therefore by (3.4.5) it converges to b in $PM(G)$. Hence, by (i) $b = 0$. It then follows immediately from the definition of N that $\lim_{n \rightarrow \infty} N(b_n) = 0$. $\square$

Notation : It will help to simplify many of the statements in the remainder of this section if we pause to introduce a little notation.

Let $S \in PM(G)$ and let $(b_n)_{n \in \mathbf{N}}$ be a sequence of elements of B .

Then $(b_n)_{n \in \mathbf{N}}$ satisfies $\mathcal{P}(S)$ iff each b_n is in $T(G)_g$ and

$$\lim_{n \rightarrow \infty} \|S - b_n\|_{PM} = 0.$$

We will show in (4.3.4) that B is precisely the set of pseudo-measures S for which there exists a sequence $(b_n)_{n \in \mathbf{N}}$ in B satisfying $\mathcal{P}(S)$. First nowever we have a lemma.

(4.3.3) LEMMA : Let $S \in PM(G)$ and let $(b_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ be two sequences taken from B which satisfy $\mathcal{P}(S)$. Then

$$\lim_{n \rightarrow \infty} N(b_n) = \lim_{n \rightarrow \infty} N(c_n).$$

Proof : For each $n \in \mathbb{N}$ let $d_n = b_n - c_n$. It is clear that $d_n \in T(G)_g$ and $\lim_{n \rightarrow \infty} \|d_n\|_{PM} = 0$. Because N is subadditive the sequence $(d_n)_{n \in \mathbb{N}}$ is Cauchy in $T(G)_g$. Therefore by (4.3.2) $\lim_{n \rightarrow \infty} N(d_n) = 0$, and so

$$\lim_{n \rightarrow \infty} N(b_n) = \lim_{n \rightarrow \infty} N(c_n). \quad \square$$

(4.3.4) PROPOSITION : Let $B \in \mathcal{B}_h$. Then B is precisely the set $\{S \in PM(G) : \text{there is a sequence } (b_n)_{n \in \mathbb{N}} \text{ in } B \text{ satisfying } \mathcal{P}(S)\}$. (1)

For each $b \in B$ and every sequence satisfying $\mathcal{P}(b)$

$$\|b\|_B = \lim_{n \rightarrow \infty} N(b_n).$$

Proof : Let $b \in B$ and let $(k_n)_{n \in \mathbb{N}}$ be the approximate identity of $L^1(G)$ given in (1.4.4). Then b belongs to the set given in (1) because the sequence $(k_n * b)_{n \in \mathbb{N}}$ satisfies $\mathcal{P}(b)$. To see this we first note that by (3.3.9) and (4.2.1) each $k_n * b \in T(G)_g$. Furthermore by (3.3.10)

$$\lim_{n \rightarrow \infty} \|b - k_n * b\|_B = 0,$$

and so the sequence $(k_n * b)_{n \in \mathbb{N}}$ is Cauchy in $T(G)_g$. In view of (3.4.5) it satisfies $\mathcal{P}(b)$.

On the other hand let S belong to the set given in (1), and let $(b_n)_{n \in \mathbb{N}}$ be a sequence taken from B and satisfying $\mathcal{P}(S)$. Since it is Cauchy in B there exists an element b of B such that the sequence tends to b in B . Therefore, by (3.4.5) $(b_n)_{n \in \mathbb{N}}$ tends to

b in $PM(G)$, and so $b = S$ and S is an element of B .

Finally let $b \in B$ and $(b_n)_{n \in \mathbb{N}}$ be a sequence satisfying $\varphi(b)$. In view of (4.3.3) our proof will be complete if we prove that

$$\|b\|_B = \lim_{n \rightarrow \infty} N(b_n). \quad (2)$$

It is easy to see that $\lim_{n \rightarrow \infty} \|b - b_n\|_B = 0$, and that for $n \in \mathbb{N}$

$$\left| \|b\|_B - \|b_n\|_B \right| = \left| \|b\|_B - N(b_n) \right| \leq \|b - b_n\|_B,$$

and so (2) holds. $\square$

(4.3.5) For each $\sigma \in \Sigma(G)$ let E_σ be a subspace of H_σ , and let $\mathcal{G} = \{E_\sigma : \sigma \in \Sigma(G)\}$. Let N be a norm defined on $T(G)_\mathcal{G}$ which satisfies

- (a) $\|b\|_{PM} \leq \text{const } N(b)$ for all $b \in T(G)_\mathcal{G}$,
- (b) each translation operator is a continuous endomorphism of $T(G)_\mathcal{G}$, and the operator $x \rightarrow_x b$ is continuous from G to $T(G)_\mathcal{G}$ for each $b \in T(G)_\mathcal{G}$, and
- (c) if $(b_n)_{n \in \mathbb{N}}$ is a Cauchy sequence taken from $T(G)_\mathcal{G}$, which converges to 0 in $PM(G)$, then it converges to 0 in $T(G)_\mathcal{G}$.

The remainder of this section is devoted to the proof that there exists a Banach space B in $\mathcal{B}_h$ such that $P(B) = T(G)_\mathcal{G}$ and $\|\cdot\|_B$ and N coincide on $T(G)_\mathcal{G}$. This will complete the promised characterization of Banach spaces in $\mathcal{B}_h$.

(4.3.6) For the rest of this section we regard $\mathcal{G}$ and N as fixed.

Let $S \in PM(G)$ and let $(b_n)_{n \in \mathbb{N}}$ be a Cauchy sequence taken from $T(G)_\mathcal{G}$ which converges to S in $PM(G)$. Then we say the sequence satisfies $\varphi(S)$.

DEFINITION : The subset B of $PM(G)$, which depends on $\mathcal{J}$ and N , is defined to be

$$B = \{S \in PM(G) : \text{there is a sequence } (b_n)_{n \in \mathbb{N}} \text{ satisfying } \mathcal{P}(S)\}.$$

For each S in B let $\| \cdot \|_B = \lim_{n \rightarrow \infty} N(b_n)$ (an argument similar to that used in (4.3.3) proves that $\| \cdot \|_B$ is single-valued).

(4.3.7) PROPOSITION : $(B, \| \cdot \|_B)$ is a Banach space.

Proof : Clearly B is a linear subspace of $PM(G)$ and $\| \cdot \|_B$ is a seminorm on B . Therefore B is a normed space if the only pseudo-measure S in B satisfying $\| S \|_B = 0$ is 0 . To see that this is true let $(b_n)_{n \in \mathbb{N}}$ be a sequence satisfying $\mathcal{P}(S)$. Since $\| S \|_B = 0$ we must have $\lim_{n \rightarrow \infty} N(b_n) = 0$ and so $\lim_{n \rightarrow \infty} \| b_n \|_{PM} = 0$. Hence the sequence converges to 0 in $PM(G)$. But it converges to S in $PM(G)$ by definition and so $S = 0$.

To see that B is complete let $(S_n)_{n \in \mathbb{N}}$ be a Cauchy sequence taken from B . For each n there is a $b_n \in T(G)_{\mathcal{J}}$ such that

$$\| S_n - b_n \|_B < n^{-1}. \quad (1)$$

For each $\varepsilon > 0$ there exists an integer N_0 in $\mathbb{N}$ such that for $m, n > N_0$

$$\| S_n - S_m \|_B < \varepsilon. \quad (2)$$

But for any $m, n \in \mathbb{N}$

$$N(b_n - b_m) \leq \| b_n - S_n \|_B + \| S_n - S_m \|_B + \| S_m - b_m \|_B,$$

and so, by (1) and (2), the sequence $(b_n)_{n \in \mathbb{N}}$ is Cauchy in $T(G)_{\mathcal{J}}$. Now by (4.3.5) (a) the sequence $(S_n)_{n \in \mathbb{N}}$ is Cauchy in $PM(G)$ and therefore converges to an element S of $PM(G)$. By (1) and (4.3.5) (a) again, the sequence $(b_n)_{n \in \mathbb{N}}$ converges to S in $PM(G)$, and so S is in B .

It only remains to prove that

$$\lim_{n \rightarrow \infty} \|S - S_n\|_B = 0. \quad (3)$$

For a fixed m in $\mathbf{N}$ the sequence $(b_n - b_m)_{n \in \mathbf{N}}$ is Cauchy in $T(G)_g$ and converges in $PM(G)$ to $S - b_m$. Hence each $S - b_m$ is in B , and by (4.3.6)

$$\|S - b_m\|_B = \lim_{n \rightarrow \infty} N(b_n - b_m) = \epsilon_m \text{ say.}$$

Since the sequence $(b_n)_{n \in \mathbf{N}}$ is Cauchy in $T(G)_g$ it follows that

$$\lim_{m \rightarrow \infty} \epsilon_m = 0. \quad (4)$$

By (1) and (4) equation (3) holds. $\square$

(4.3.8) PROPOSITION : If our notation is taken from (4.3.6) then $B \in \mathcal{B}_h$.

Proof : We first prove that B is an element of $\mathcal{B}$. Let $b \in B$, $x \in G$ and let $(b_n)_{n \in \mathbf{N}}$ be a sequence satisfying $\mathcal{P}(b)$. The space $T(G)_g$ is left translation invariant, so for each n xb_n is in $T(G)_g$. Each left translation operator is an isometry of $PM(G)$, and so it follows from (4.3.5)(b) that the sequence $(xb_n)_{n \in \mathbf{N}}$ satisfies $\mathcal{P}(xS)$. Since each left translation operator is a continuous endomorphism of $T(G)_g$, it is easy to see that each is also a continuous endomorphism of B . Therefore B is in $\mathcal{B}$.

To prove that B is in $\mathcal{B}_h$ we need to prove that the operator $x \rightarrow xS$ is continuous from G to B for each S in B . It is a consequence of definition (4.3.6) that $T(G)_g$ is a dense subspace of B ; furthermore by (4.3.5)(b) the operator $x \rightarrow xS$ is continuous from G to B for each S in $T(G)_g$. As in the proof of (3.3.10) the Baire Category theorem may be used to conclude that B is in $\mathcal{B}_h$. $\square$

(4.3.9) Finally we prove that B is the homogeneous Banach space conjectured in (4.3.5).

PROPOSITION : With the notation of (4.3.6) we have that $P(B) = T(G)_g$, and $\| \cdot \|_B$ and N coincide on $T(G)_g$.

Proof : If b is in $T(G)_g$ then the trivial sequence, in which each $b_n = b$, satisfies $\mathcal{P}(b)$ and so $N(b) = \|b\|_B$.

By (3.5.5) $P(B) = B \cap T(G)$ and hence $T(G)_g \subseteq P(B)$.

On the other hand assume b is in $P(B) \setminus T(G)_g$. Then there exists $\sigma \in \Sigma(G)$ such that $\hat{b}(\sigma)(E_\sigma) \neq \{0\}$. Choose ξ in E_σ such that $\hat{b}(\sigma)\xi \neq 0$. If b is in B then there exists a sequence $(b_n)_{n \in \mathbb{N}}$ in $T(G)_g$ which satisfies $\mathcal{P}(b)$, and so

$$\|b - b_n\|_{PM} \geq \|\hat{b}(\sigma) - \hat{b}_n(\sigma)\| \geq \sup_{\zeta \in E_\sigma} \|\hat{b}(\sigma)\zeta\|_{H_\sigma} > 0.$$

This contradicts the choice of $(b_n)_{n \in \mathbb{N}}$, and so $P(B) \setminus T(G)_g$ is empty. $\square$

(4.4) Alternative Characterization

Let B be in $\mathcal{B}_h$. For each $\sigma \in \Sigma(G)$ let $E_\sigma = \cap \{\ker \hat{b}(\sigma) : b \in B\}$. By (4.2.1) $P(B) = T(G)_g$. By (3.5.3) for each M in B^* there exists an element $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))_g$ such that

$$\Gamma_M(b)^\wedge(\sigma) = \hat{b}(\sigma) A_\sigma^* \text{ for each } \sigma \in \Sigma(G).$$

Therefore B^* may be identified with a subset Λ of $\underline{E}(\Sigma(G))_g$. In this section we study the properties possessed by Λ . Then, in (4.4.11), we prove that if Λ is any subset of $\underline{E}(\Sigma(G))_g$ which possesses the properties stated therein, there exists a Banach space B in $\mathcal{B}_h$ such that B^* may be identified with Λ . In doing so we have

given another characterization of the elements of $\mathcal{B}_h$.

(4.4.1) Lemma (3.5.3) suggests that Λ should be specified as follows:

DEFINITION : Λ is the set of all $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))_{\mathcal{G}}$ such that

$$\sup \left\{ \left| \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)) \right| : b \in B_B(0,1) \cap T(G) \right\} \quad (1)$$

is finite. For each A in Λ define $\|A\|_\Lambda$ to be the supremum in (1).

(4.4.2) It is clear that Λ is a linear subspace of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ which depends on B . Furthermore $\|\cdot\|_\Lambda$ is a seminorm on Λ .

LEMMA : $(\Lambda, \|\cdot\|_\Lambda)$ is a normed space.

Proof : It is sufficient to prove that if A is in Λ and $\|A\|_\Lambda = 0$ then A must be 0. So assume $\|A\|_\Lambda = 0$. For each $\sigma \in \Sigma(G)$ define b_σ from G to $\mathbb{C}$ by

$$b_\sigma(x) = d_\sigma \operatorname{Tr} (A_\sigma U_\sigma(x)^*) \text{ for each } x \in G.$$

Then $b_\sigma \in T(G)_{\mathcal{G}}$ and $b_\sigma \|b_\sigma\|_B^{-1}$ is in $B_B(0,1) \cap T(G)$. By (4.4.1) for each $\sigma \in \Sigma(G)$

$$\begin{aligned} 0 &= \left| \sum_{\eta \in \Sigma(G)} d_\eta \operatorname{Tr} (A_\eta^* \hat{b}_\sigma(\eta) \|b_\sigma\|_B^{-1}) \right| \\ &= d_\sigma \|b_\sigma\|_B^{-1} \operatorname{Tr} (|A_\sigma|^2). \end{aligned}$$

Therefore $|A_\sigma|^2 = 0$ for each $\sigma \in \Sigma(G)$; hence $A_\sigma = 0$ for each σ , and so $A = 0$. $\square$

(4.4.3) DEFINITION : Let $M \in B^*$. By (3.5.3) there exists

$A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))_{\mathcal{G}}$ such that

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \text{ for each } b \in T(G)_{\mathcal{G}}.$$

Denote by ψ the map taking M to A and write $\psi M = A$.

(4.4.4) The map ψ operates from B^* to Λ .

PROPOSITION : ψ is an isometric isomorphism of B^* onto Λ .

Proof : Let $M \in B^*$. It follows from (3.5.3) and (4.4.1) that ψM is an element of Λ and that $\|\psi M\|_{\Lambda} = \|M\|_{B^*}$.

The proof will be complete if we show that ψ is a surjection. Therefore let $A = (A_{\sigma})_{\sigma \in \Sigma(G)}$ be an element of Λ . Define the operator M from $T(G)_{\mathcal{G}}$ to $\mathbb{C}$ by

$$M(b) = \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \text{ for each } b \in T(G)_{\mathcal{G}}. \quad (1)$$

Clearly M is linear. Furthermore

$$\begin{aligned} \sup_{b \in B_B(0,1) \cap T(G)} |M(b)| &= \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \right| \\ &= \|A\|_{\Lambda}, \end{aligned}$$

and so for each b in $T(G)_{\mathcal{G}}$

$$|M(b)| \leq \|A\|_{\Lambda} \|b\|_B. \quad (2)$$

By (3.3.10) and (2) M may be extended to a continuous linear functional on B , and by (1) $\psi M = A$, so the proof is complete. $\square$

(4.4.5) COROLLARY : $(\Lambda, \|\cdot\|_{\Lambda})$ is a Banach space. $\square$

We now consider the properties possessed by Λ which will be used when characterizing those subsets of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ which may be associated with a Banach space in $\mathcal{B}_h$ (see (4.4.11)). In (4.4.6) to (4.4.10) we regard B as a fixed element of $\mathcal{B}_h$ such that $P(B) = T(G)_{\mathcal{G}}$, and we let Λ denote the subset of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ determined in (4.4.1).

(4.4.6) PROPOSITION : $\underline{E}^1(\Sigma(G))_{\mathcal{G}}$ is a subspace of Λ and for each A in $\underline{E}^1(\Sigma(G))_{\mathcal{G}}$

$$\|A\|_{\Lambda} \leq \text{const } \|A\|_1.$$

Proof : Let $A = (A_{\sigma})_{\sigma \in \Sigma(G)}$ be an element of $\underline{E}^1(\Sigma(G))_{\mathcal{G}}$. Then

$$\begin{aligned} & \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \text{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \right| \\ & \leq \sup_{b \in B_B(0,1) \cap T(G)} \sum_{\sigma \in \Sigma(G)} d_{\sigma} |\text{Tr} (A_{\sigma}^* \hat{b}(\sigma))| \\ & \leq \sup_{b \in B_B(0,1) \cap T(G)} \sum_{\sigma \in \Sigma(G)} d_{\sigma} \|A_{\sigma}\|_{\phi_{\sigma}} \|\hat{b}(\sigma)\| \\ & \leq \text{const} \sup_{b \in B_B(0,1) \cap T(G)} \|A\|_1 \|b\|_B \\ & \leq \text{const } \|A\|_1. \end{aligned}$$

Therefore A is in Λ and $\|A\|_{\Lambda} \leq \text{const } \|A\|_1$. $\square$

(4.4.7) PROPOSITION : Let $x \in G$ and $A = (A_{\sigma})_{\sigma \in \Sigma(G)}$ be in Λ . Then $L_x A$ is in Λ , and each operator L_x is a continuous endomorphism of Λ .

Proof : Recall from (3.4.1) that $L_x A = (U_{\sigma}(x) A_{\sigma})_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))$.

So

$$\begin{aligned} & \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \text{Tr} ((U_{\sigma}(x) A_{\sigma})^* \hat{b}(\sigma)) \right| \\ & = \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \text{Tr} (A_{\sigma}^* x^{-1} \hat{b}(\sigma)) \right|. \end{aligned} \quad (1)$$

The proposition now follows immediately from (4.4.1), (3.3.2) and (1). $\square$

The next property is purely topological. But first we have a definition.

(4.4.8) DEFINITION : Define the bilinear form $\langle , \rangle$ from $\underline{\underline{E}}(\Sigma(G))_{\mathcal{G}} \times T(G)_{\mathcal{G}}$ to $\mathbb{C}$ by

$$\langle A, b \rangle = \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma))$$

for all $A \in \underline{\underline{E}}(\Sigma(G))_{\mathcal{G}}$ and $b \in T(G)_{\mathcal{G}}$.

(4.4.9) It is easy to see that $(\underline{\underline{E}}(\Sigma(G))_{\mathcal{G}}, T(G)_{\mathcal{G}}, \langle , \rangle)$ is a dual system.

PROPOSITION : The unit ball $B_{\wedge}(0,1) = \{A \in \wedge : \|A\|_{\wedge} \leq 1\}$ is $\sigma(\underline{\underline{E}}(\Sigma(G))_{\mathcal{G}}, T(G)_{\mathcal{G}})$ - closed.

Proof : If $(A_n)_{n \in \mathbb{N}}$ is a sequence taken from $B_{\wedge}(0,1)$, which converges to an element A of $\underline{\underline{E}}(\Sigma(G))_{\mathcal{G}}$ with respect to $\sigma(\underline{\underline{E}}(\Sigma(G))_{\mathcal{G}}, T(G)_{\mathcal{G}})$, then

$$\begin{aligned} \limsup_{n \in \mathbb{N}} \{ |\langle A_n, b \rangle| : b \in B_B(0,1) \cap T(G) \} \\ = \sup \{ |\langle A, b \rangle| : b \in B_B(0,1) \cap T(G) \}. \end{aligned} \quad (1)$$

Therefore $A \in \wedge$. Since for each n $\|A_n\|_{\wedge} \leq 1$, (1) proves that A is in $B_{\wedge}(0,1)$. $\square$

(4.4.10) The final property we consider is motivated by (4.3.2).

PROPOSITION : Let $(A_n)_{n \in \mathbb{N}}$ be a sequence taken from $\underline{\underline{E}}^{\circ\circ}(\Sigma(G))_{\mathcal{G}}$ which converges to zero in $\underline{\underline{E}}^{\circ\circ}(\Sigma(G))_{\mathcal{G}}$ and which satisfies

$$\lim_{m, n \rightarrow \infty} \sup_{A \in B_{\wedge}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* (A_{n_{\sigma}} - A_{m_{\sigma}})) \right| = 0.$$

Then

$$\lim_{n \rightarrow \infty} \sup_{A \in B_{\wedge}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* A_{n_{\sigma}}) \right| = 0. \quad (1)$$

Proof : Each element of $\underline{\underline{E}}^{\circ\circ}(\Sigma(G))_{\mathcal{G}}$ is the Fourier transform of an element of $T(G)_{\mathcal{G}}$. For each $n \in \mathbb{N}$ let b_n denote the trigonometric

polynomial with Fourier transform A_n . By (4.4.1) for each n

$$N(b_n) = \|b_n\|_B = \sup_{A \in B_{\Lambda}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \right|.$$

Therefore the sequence $(A_n)_{n \in \mathbb{N}}$ satisfies the hypotheses of (4.4.10) iff the sequence $(b_n)_{n \in \mathbb{N}}$ converges to 0 in $PM(G)$ and is Cauchy in $T(G)_{\mathcal{G}}$. Hence by (4.3.2) $(b_n)_{n \in \mathbb{N}}$ tends to 0 in B , and so (1) holds. $\square$

(4.4.11) For each $\sigma \in \Sigma(G)$ let E_{σ} be a subspace of H_{σ} , and let $\mathcal{G} = \{E_{\sigma} : \sigma \in \Sigma(G)\}$. Let Λ be a subspace of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ which possesses the properties obtained in (4.4.5), (4.4.6), (4.4.7), (4.4.9) and (4.4.10) for some norm $\|\cdot\|_{\Lambda}$. We complete this section by proving that there exists a Banach space B in $\mathcal{B}_h$ with the property that B^* is isometrically isomorphic to Λ (see (4.4.4)). We do this by using Λ to define a norm N on $T(G)_{\mathcal{G}}$, and then by applying (4.3.6) and (4.3.8).

Theorem : Let Λ be a linear subspace of $\underline{E}(\Sigma(G))_{\mathcal{G}}$ which becomes a Banach space when equipped with a norm $\|\cdot\|_{\Lambda}$. Assume that Λ possesses the following properties :

- (i) $\underline{E}^1(\Sigma(G))_{\mathcal{G}}$ is a subspace of Λ , and for each $A \in \underline{E}^1(\Sigma(G))_{\mathcal{G}}$ $\|A\|_{\Lambda} \leq \text{const } \|A\|_1$,
- (ii) for each $A \in \Lambda$ and $x \in G$ $L_x A$ is in Λ , and the operator L_x is a continuous endomorphism of Λ ,
- (iii) $B_{\Lambda}(0,1)$ is $\sigma(\underline{E}(\Sigma(G))_{\mathcal{G}}, T(G)_{\mathcal{G}})$ -closed, and
- (iv) if $(A_n)_{n \in \mathbb{N}}$ is a sequence taken from $\underline{E}^{\circ\circ}(\Sigma(G))_{\mathcal{G}}$ which converges to 0 in $\underline{E}^{\infty}(\Sigma(G))_{\mathcal{G}}$ and which satisfies

$$\lim_{m,n \rightarrow \infty} \sup_{A \in B_{\Lambda}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* (A_{n_{\sigma}} - A_{m_{\sigma}})) \right| = 0$$

then

$$\lim_{n \rightarrow \infty} \sup_{A \in B_{\wedge}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* A_{n_{\sigma}}) \right| = 0.$$

Then there exists a Banach space B in $\mathcal{B}_h$ such that B^* is isometrically isomorphic to $\wedge$.

Proof : For each b in $T(G)_{\mathcal{G}}$ define $N(b)$ by

$$N(b) = \sup_{A \in B_{\wedge}(0,1)} \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) \right|. \quad (1)$$

It is easy to see that N is a seminorm on $T(G)_{\mathcal{G}}$. That it is a norm follows from the type of argument used in (4.4.2) with b_{σ} replaced by $A^{\sigma} = (\delta_{\sigma\eta} \hat{b}(\eta))_{\eta \in \Sigma(G)}$.

Before applying (4.3.6) we must verify that N satisfies (a), (b) and (c) of (4.3.5).

By (D39) of [HR] for each $\sigma \in \Sigma(G)$ and $b \in T(G)_{\mathcal{G}}$ there exists an operator T , depending on σ and b , in $B(H_{\sigma})$ such that $\|T\| = 1$ and $T^* \hat{b}(\sigma) = |\hat{b}(\sigma)|$. Let $T' = TP$ where P denotes the projection of H_{σ} onto $E_{\sigma}^{\perp}$. Then T' is in $B(H_{\sigma})$; it satisfies $T'(E_{\sigma}) = \{0\}$ and $\|T'\| \leq \|T\| = 1$. Furthermore

$$\begin{aligned} \operatorname{Tr} (T'^* \hat{b}(\sigma)) &= \operatorname{Tr} (PT^* \hat{b}(\sigma)) \\ &= \operatorname{Tr} (T^* \hat{b}(\sigma) P) \\ &= \operatorname{Tr} (T^* \hat{b}(\sigma)) = \|\hat{b}(\sigma)\|. \end{aligned}$$

Now by (1) there exists an element A of $\wedge$ such $A_{\sigma} = \delta_{\sigma\eta} T$, and so it follows that

$$\|b\|_{PM} \leq \operatorname{const} N(b).$$

To check property (b) let $x \in G$ and $b \in T(G)_{\mathcal{G}}$. For each A in $B_{\wedge}(0,1)$ and each σ in $\Sigma(G)$

$$\operatorname{Tr} (A_{\sigma}^* \hat{b}(\sigma)) = \operatorname{Tr} ((U_{\sigma}(x) A_{\sigma})^* \hat{b}(\sigma)),$$

and so it follows from (1) and (ii) that (b) holds.

Finally let $(b_n)_{n \in \mathbb{N}}$ be a Cauchy sequence, taken from $T(G)_g$, which converges to 0 in $PM(G)$. It is easily seen that the sequence $(\hat{b}_n)_{n \in \mathbb{N}}$ satisfies the hypothesis in (iv). Therefore by (iv) and (1) $(b_n)_{n \in \mathbb{N}}$ converges to 0 in $T(G)_g$.

Hence, appealing to (4.3.6), there must exist a Banach space B in $\mathcal{B}_h$ such that $P(B) = T(G)_g$, and N and $\|\cdot\|_B$ coincide on $T(G)_g$. Now define $\Lambda(B)$ as in (4.4.1). That is, $\Lambda(B)$ is the set of all $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}(\Sigma(G))_g$ such that

$$\sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)) \right| \quad (2)$$

is finite, and $\|A\|_{\Lambda(B)}$ is the supremum in (2). By (4.4.4) B^* is isometrically isomorphic to $\Lambda(B)$. Therefore to complete the proof it is sufficient to show that Λ and $\Lambda(B)$ are isometrically isomorphic.

By (iii) $B_{\Lambda}(0,1)$ is weakly closed. It is convex and balanced and so $B_{\Lambda}(0,1) = B_{\Lambda}(0,1)^{\circ\circ}$, the bipolar of $B_{\Lambda}(0,1)$. But

$$\begin{aligned} B_{\Lambda}(0,1)^{\circ} &= \{b \in T(G)_g : |\langle A, b \rangle| \leq 1 \text{ for all } A \in B_{\Lambda}(0,1)\} \\ &= B_B(0,1) \cap T(G), \end{aligned}$$

and so

$$\begin{aligned} B_{\Lambda}(0,1)^{\circ\circ} &= \{A \in \underline{E}(\Sigma(G))_g : |\langle A, b \rangle| \leq 1 \text{ for all } b \in B_B(0,1) \cap T(G)\} \\ &= B_{\Lambda(B)}(0,1). \end{aligned}$$

This completes the proof. □

(4.5) Applications

We complete this chapter by applying the results of §(4.4) to characterize p -Sidon subsets of $\Sigma(G)$ (see Edwards and Ross [1]). We also introduce homogeneous Banach algebras which we study in greater detail in chapter 5.

(4.5.1) DEFINITION : Let F be a subset of $\Sigma(G)$ and let $p \in [1, 2)$. Then F is p -Sidon iff for every element $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ of $\underline{E}^{p'}(\Sigma(G))$ there exists μ in $M(G)$ such that $\hat{\mu}(\sigma) = A_\sigma$ for all $\sigma \in F$.

(4.5.2) Recall from (4.4.1) the definition of the subspace $\Lambda(B)$ of $\underline{E}(\Sigma(G))$ (B an element of $\mathcal{B}_h$) which, by (4.4.4), is isometrically isomorphic to B^* .

PROPOSITION : The following are equivalent :

- (i) F is p -Sidon,
- (ii) $\underline{E}^{p'}(\Sigma(G))_F \cdot \Lambda(C(G)_F) \subseteq \Lambda(C(G)_F)$, and
- (iii) $\underline{E}^{p'}(\Sigma(G))_F \cdot \Lambda(B) \subseteq \Lambda(B)$ for every Banach space B in $\mathcal{B}_h$ with $P(B) = T(G)_F$.

Proof : We first prove that (i) $\rightarrow$ (iii). Assume (i) holds and let B be in $\mathcal{B}_h$ with $P(B) = T(G)_F$. Let $D = (D_\sigma)_{\sigma \in \Sigma(G)}$ and $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ be elements of $\underline{E}^{p'}(\Sigma(G))_F$ and $\Lambda(B)$ respectively. By (4.5.1) there exists $\mu \in M(G)$ such that $\hat{\mu}(\sigma) = D_\sigma$ for each σ in F . Also

$$\begin{aligned} & \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} ((D_\sigma A_\sigma)^* \hat{b}(\sigma)) \right| \\ &= \sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* D_\sigma^* \hat{b}(\sigma)) \right|. \end{aligned} \quad (1)$$

Furthermore μ^* is a measure satisfying $\hat{\mu}^*(\sigma) = D_\sigma^*$ for each σ in F , and by (3.3.7) $\mu^* * b \in B$ and

$$\| \mu^* * b \|_B \leq K \| \mu^* \|_1 \| b \|_B. \quad (2)$$

But $T(G)$ is a convolution module over $M(G)$, and so $\mu^* * b$ is in $T(G)$ if b is in $T(G)$. Now (1) and (2) show that

$$\sup_{b \in B_B(0,1) \cap T(G)} \left| \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} ((D_\sigma A_\sigma)^* \hat{b}(\sigma)) \right| \leq K \|\mu^*\|_1 \|b\|_B.$$

Hence (iii) holds.

Clearly (iii) $\rightarrow$ (ii) so it only remains to prove that (ii) $\rightarrow$ (i). If (ii) holds then, applying the Uniform Boundedness Principle twice as in (2.2.3), we see that there exists a constant K' such that for each $D = (D_\sigma)_{\sigma \in \Sigma(G)}$ and $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\underline{E}^{p'}(\Sigma(G))_F$ and $\wedge(C(G)_F)$ respectively

$$\|DA\|_{\wedge(C(G)_F)} \leq K' \|D\|_{p'} \|A\|_{\wedge(C(G)_F)}. \quad (3)$$

Define the endomorphism T of $T(G)_F$ by

$$T(b)(x) = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (D_\sigma^* \hat{b}(\sigma) U_\sigma(x)^*). \quad (4)$$

By (3.3.10) $T(G)_F$ is a dense subspace of $C(G)_F$. Define the bilinear form $\langle \cdot, \cdot \rangle$ on $T(G)_F \times \wedge(C(G)_F)$ by

$$\langle b, A \rangle = \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{Tr} (A_\sigma^* \hat{b}(\sigma)). \quad (5)$$

Then $(T(G)_F, \wedge(C(G)_F), \langle \cdot, \cdot \rangle)$ is a dual system. Furthermore for each $A = (A_\sigma)_{\sigma \in \Sigma(G)}$ in $\wedge(C(G)_F)$ and each $b \in T(G)_F$, by (4)

$$|\langle T(b), A \rangle| = |\langle b, DA \rangle| \leq \|b\|_\infty \|DA\|_{\wedge(C(G)_F)}.$$

Therefore by (3)

$$|\langle T(b), A \rangle| \leq K' \|b\|_\infty \|D\|_{p'} \|A\|_{\wedge(C(G)_F)}.$$

By the Hahn-Banach theorem

$$\|T(b)\|_\infty = \sup_{A \in B_{\wedge(C(G)_F)}(0,1)} |\langle T(b), A \rangle|$$

and so, in particular,

$$|T(b)(0)| \leq K' \|b\|_{\infty} \|D\|_{p'}.$$

Referring back to (3) it is seen that K' is independent of D , and so by (4), for all $D \in \underline{\underline{E}}^{p'}(\Sigma(G))_F$,

$$\left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr}(D_{\sigma}^* \hat{b}(\sigma)) \right| \leq K' \|b\|_{\infty} \|d\|_{p'}.$$

The spaces $\underline{\underline{E}}^p(\Sigma(G))_F$ and $\underline{\underline{E}}^{p'}(\Sigma(G))_F$ form a dual system with the action defined in (5) (with $\hat{b}$ replaced by an element A of $\underline{\underline{E}}^p(\Sigma(G))_F$). For each $b \in T(G)_F$ $\hat{b}$ is an element of $\underline{\underline{E}}^p(\Sigma(G))_F$, so by the Hahn-Banach theorem

$$\|\hat{b}\|_p = \sup \left\{ \left| \sum_{\sigma \in \Sigma(G)} d_{\sigma} \operatorname{Tr}(D_{\sigma}^* \hat{b}(\sigma)) \right| : D \in \underline{\underline{E}}^{p'}(\Sigma(G))_F, \|D\|_{p'} \leq 1 \right\}.$$

Hence $\|\hat{b}\|_p \leq K' \|b\|_{\infty}$. Now it is easy to see that F is p -Sidon. □

(4.5.3) We end this section with a few comments on homogeneous Banach algebras. These algebras are the subject of chapter 5.

DEFINITION : Let $B \in \mathcal{B}$. Then B is a *translation invariant convolution Banach algebra* iff it is a Banach algebra under convolution. If it also belongs to $\mathcal{B}_h$ then it is called a *homogeneous Banach algebra*.

Examples 1-9 and 13 of (3.5.9) are all Banach algebras under convolution. Therefore they inherit the translation properties of the underlying Banach space.

We note that definition (4.5.3) could easily be stated with the generality of (3.2.1) and (3.3.1), but we do not do this as we only consider those algebras which are in $\mathcal{B}$ or $\mathcal{B}_h$.

(4.5.4) It is interesting to note that every Banach space B in $\mathcal{B}_h$ which contains only measures is a convolution Banach algebra.

PROPOSITION : If $B \in \mathcal{B}_h$ and $B \subseteq M(G)$ then B is a convolution Banach algebra.

Proof : In view of (3.3.7) (recalling that L is the left regular representation) we have that $\mu * b \in B$ for each $b \in B$ and $\mu \in M(G)$. Furthermore, $\|\mu * b\|_B \leq K \|\mu\|_1 \|b\|_B$. Let T denote the natural injection of B into $M(G)$. By the Closed Graph theorem T is continuous, and so there exists a constant K' such that $\|\mu\|_1 \leq K' \|\mu\|_B$ for all $\mu \in B$. It is now easy to see that B must be a convolution Banach algebra. $\square$

Remark : Recall that in §(4.3) we characterized Banach spaces B in $\mathcal{B}_h$ by norms on $B \cap T(G)$, and in §(4.4) by subspaces of $\underline{E}(\Sigma(G))$ which correspond to B^* . It is easily seen that B is a homogeneous convolution Banach algebra iff the norm N defined in §(4.3) satisfies

$$N(b_1 * b_2) \leq K N(b_1)N(b_2) \quad \text{for all } b_1, b_2 \in B \cap T(G).$$

Also, it is a homogeneous convolution Banach algebra iff the subspace Λ defined in §(4.4) is a Banach algebra under pointwise multiplication.

CHAPTER 5

HOMOGENEOUS CONVOLUTION BANACH ALGEBRAS

(5.1) Introduction

As I observed in §(3.1), Reiter's Ideal theorem, and Burnham's extension, prove that the closed ideal theory of any Segal algebra and of the convolution algebra $L^1(G)$ are the same. If G is non-compact, but locally compact, there is still very little known about the ideal theory of $L^1(G)$ (see sections 39-42 of [HR]). However, if G is compact then the outlook is much better; there exists a very elegant and complete theory of the closed ideal structure of $L^1(G)$. This theory extends easily to some of the subalgebras of $L^1(G)$, all of which contain $T(G)$ as a dense subspace, (see (38.7) and (38.13) of [HR]).

With this in mind I now consider the closed ideal structure of any homogeneous convolution Banach algebra B in $\mathcal{B}_h$. In (5.4.2) I give a complete description of the closed two-sided ideals of B , and in (5.4.5) of the closed left ideals of B . If B is symmetric then (5.4.5) also permits a description of the closed right ideals of B .

In this way I obtain a generalization of Reiter's theorem when G is compact. Let B denote a homogeneous convolution algebra in $\mathcal{B}_h$ and let $\mathcal{J} = \{ \bigcap_{b \in B} \ker \hat{b}(\sigma) : \sigma \in \Sigma(G) \}$. If I is a closed two-sided (left) ideal of B then there exists a unique closed two-sided (left) ideal J of $L^1(G)_{\mathcal{J}}$ such that J is the closure in $L^1(G)$ of $I \cap L^1(G)$. Conversely, if J is a closed two-sided (left) ideal of $L^1(G)_{\mathcal{J}}$ then there exists a unique ideal I of B such that J is the closure in $L^1(G)$ of $I \cap L^1(G)$. This result is given in (5.4.7).

Section (5.4) may be used to prove that for any homogeneous Banach algebra B in $\mathcal{B}_h$, if b is in B and α is a non-zero complex number for which $b(\sigma) - \alpha \text{Id}_{\sigma}$ is invertible for each σ , then there exists c in B such that

$$\hat{b}(\sigma) [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} = \hat{c}(\sigma) = [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} \hat{b}(\sigma)$$

for all σ in $\text{sp } B$. (See (5.5.4).)

If G is a compact abelian group then this result may be restated so as to be a result concerning the functions which operate on the algebra $\hat{B} = \{\hat{b} : b \in B\}$. If G is not abelian then it is not clear what is meant by a function operating on $\hat{B}$ because the elements of $\hat{B}$ are then in $\underline{E}^\infty(\Sigma(G))$. One approach to the problem was used in Segal [1] in 1947, when the author attempted to use the homogeneous properties of $L^1(G)$ (G compact) to determine a family of complex-valued complex functions which acted on $L^1(G)$. The complex functions are extended to $L^1(G)$ -valued functions defined on a subset of $L^1(G)$ - see §24 of Hille and Phillips [1], and §(1.6). In §(5.5) I extend Segal's methods to a symmetric homogeneous Banach algebra B in $\mathcal{B}_h$.

I complete this chapter, and the thesis, by considering the $*$ -representation theory of symmetric homogeneous convolution algebras in $\mathcal{B}_h$. In (38.21) of [HR] it is proved that if G is compact then the $*$ -representations of $L^p(G)$, $1 \leq p < \infty$, are in 1-1 correspondence with the unitary representations of G . See also Loomis [1]. In (22.3) of [HR] the authors extend this result to a larger family of symmetric subalgebras B of $M(G)$ where G may be locally compact. They prove that if R is a $*$ -representation of B which acts on H then there exists a unitary representation U of G which also acts on H , and which satisfies

$$\langle \zeta, R(b)\xi \rangle = \int \langle \zeta, U(x)\xi \rangle b(x) d\lambda(x) \quad (1)$$

for all $\zeta, \xi \in H$ and $b \in B$. In (5.6.7) I show that if B is a symmetric homogeneous convolution algebra in $\mathcal{B}_h$ and if R is a $*$ -representation of B which acts on H then there exists a unitary representation U of G , which also acts on H , and which satisfies

$$R(b)\xi = \lim_{n \rightarrow \infty} \int (k_n * b)(x) U(x)\xi d\lambda(x) \quad (2)$$

for all $\xi \in H$ and $b \in B$. Here $(k_n)_{n \in \mathbb{N}}$ denotes an approximate identity of $L^1(G)$ consisting of trigonometric polynomials. From (2) I can obtain an equality similar to (1).

The introductory sections (5.2) and (5.3) concern the ideal theory of some algebras of pseudomeasures after the fashion of the early results of §38 of [HR].

Throughout this chapter G denotes an arbitrary compact group.

(5.2) Two-sided Ideals in Elements of $\mathcal{R}$

(5.2.1) Definition : Let $\mathcal{R}$ denote the family of convolution algebras B of pseudomeasures defined by $B \in \mathcal{R}$ iff

- (i) for each $\sigma \in \Sigma(G)$ there exists a subspace E_σ of H_σ such that $B_{\{\sigma\}}$ is the linear span of the set of trigonometric polynomials $\{x \rightarrow \langle \xi, U_\sigma(x)\zeta \rangle : \xi \in H_\sigma \text{ and } \zeta \in E_\sigma^\perp\}$, and
- (ii) $\hat{B}(\sigma) = \{\hat{b}(\sigma) : b \in B\}$ is precisely $\{T \in B(H_\sigma) : T(E_\sigma) = \{0\}\}$, (we shall denote this subspace of $B(H_\sigma)$ by $J(E_\sigma)$).

If $E_\sigma = H_\sigma$ then $B_{\{\sigma\}} = 0$; if $E_\sigma = \{0\}$ then $B_{\{\sigma\}} = T(G)_{\{\sigma\}}$ and $J(E_\sigma) = B(H_\sigma)$.

By (3.3.10) and (4.2.1) any convolution Banach algebra in $\mathcal{B}_h$ is an element of $\mathcal{R}$. Obviously the converse is not true because $B = M(G)$ is in $\mathcal{R}$ but not in $\mathcal{B}_h$.

In this section we study the two-sided ideals of each algebra B

in $\mathcal{R}$. In particular, in (5.2.12) we give a complete description of the minimal two-sided ideals in B .

We begin by presenting two algebraic lemmas which we use to determine a family of two-sided ideals in B . The notation for these results is introduced in (5.2.2). I would like to thank T. Dickson and C. Praeger, of the University of Western Australia, for discussions leading to simplified proofs of (5.2.3) and (5.2.5). (The proofs given are original although I feel sure the results are not new.)

(5.2.2) Notation : Let H denote an n -dimensional Hilbert space, and E an m -dimensional subspace of H , $0 \leq m \leq n$. Let $\{e_1, \dots, e_n\}$ denote an orthonormal basis for H such that $\{e_1, \dots, e_m\}$ is a basis for E , and put $J(E) = \{T \in B(H) : T(E) = \{0\}\}$. Then $J(E)$ is a subalgebra of $B(H)$ because $T_1 T_2(E) = \{0\}$ for all T_1, T_2 in $J(E)$. Let I be a non-zero two-sided ideal in $J(E)$.

(5.2.3) LEMMA : If there exists an operator T in I such that

$$\langle T e_y, e_x \rangle = 0 \quad \text{for some } x, y > m, \text{ then } I = J(E).$$

Proof : For each $i \in \{1, \dots, n\}$ the operator E_{ix} , defined by

$$E_{ix}(\xi) = \langle \xi, e_x \rangle e_i, \text{ is in } J(E). \text{ Hence}$$

$$E_{ix}T = \sum_{l > m} \langle T e_l, e_x \rangle E_{il}$$

is non-zero and is in I . For each $j \in \{m+1, \dots, n\}$, the operator E_{yj} is also in $J(E)$, and so

$$E_{ix}T E_{yj} = \langle T e_y, e_x \rangle E_{ij}$$

is in I . Hence E_{ij} is in I for all $i \in \{1, \dots, n\}$ and $j \in \{m+1, \dots, n\}$. Therefore $I = J(E)$ because $J(E)$ is precisely the set of linear combinations of the family of E_{ij} 's with $i \in \{1, \dots, n\}$ and $j \in \{m+1, \dots, n\}$. $\square$

(5.2.4) COROLLARY : If $E = \{0\}$ then $I = B(H)$. If E is a proper subspace of H and if $I \neq J(E)$ then $\text{im } T \subseteq E$ for each $T \in I$.

Proof : This is an immediate consequence of (5.2.3) and the definition of $J(E)$ in (5.2.1). $\square$

(5.2.5) LEMMA : Assume $E \neq \{0\}$. If I is a proper two-sided ideal in $J(E)$ then there exists a non-zero subspace E_0 of E such that

$$I = \{T \in J(E) : \text{im } T \subseteq E_0\} . \quad (1)$$

Every set of the form (1) is a proper two-sided ideal of $J(E)$.

Distinct subspaces of E determine distinct ideals.

Proof : By (5.2.4), $\text{im } T \subseteq E$ for each T in I . To prove that I has the form in (1) we must first determine a likely subspace E_0 of E . Let E_0 denote the intersection of all subspaces of E which contain $\cup\{\text{im } T : T \in I\}$. Then E is one such subspace and so E_0 is a subspace of E . It is non-zero because it contains the non-zero set $\cup\{\text{im } T : T \in I\}$. Clearly then $I \subseteq \{T \in J(E) : \text{im } T \subseteq E_0\}$.

To prove the opposite inclusion let $T \in J(E)$ such that $\text{im } T \subseteq E_0$. There exists a non-zero element T_1 of I such that $\dim(\text{im } T \cap \text{im } T_1) \geq 1$ (otherwise $E_0 - \text{im } T$ is a smaller subspace of E containing $\cup\{\text{im } T : T \in I\}$). If $\text{im } T$ is a subset of $\text{im } T_1$ then T is in I because there exists an element $D \in J(E)$ such that $T = T_1 D$. If, however, $\text{im } T$ is not a subset of $\text{im } T_1$, then there still exists an operator D_1 in $J(E)$ such that

$$T = T_1 D_1 \text{ on } \text{im } T \cap \text{im } T_1 .$$

Then $T - T_1 D_1$ is in $J(E)$ and satisfies $\text{im}(T - T_1 D_1) \subseteq E_0$ and

$$\dim(\text{im}(T - T_1 D_1)) < \dim(\text{im } T) \leq n .$$

By repeating the argument first on $T - T_1 D_1$, then again if necessary, there exists $p \in \mathbf{N}$ and operators $T_i \in I$ and $D_i \in J(E)$ for $i = 2, \dots, p$ such that

$$\dim (\operatorname{im}(T - \sum_{i=1}^p T_i D_i)) = 0.$$

Therefore $T = \sum_{i=1}^p T_i D_i$ and T is in I . Hence I is given by (1).

Now let E_0 be a non-zero subspace of E . Denote by I_{E_0} the set

$$I_{E_0} = \{T \in J(E) : \operatorname{im} T \subseteq E_0\}.$$

Then I_{E_0} is non-zero because for any $\zeta \in E^\perp$ and $\xi \in E_0$ there exists an operator T in $J(E)$ such that $T\zeta = \xi$; it is a linear subspace of $J(E)$, and so will be a non-zero two-sided ideal in $J(E)$ if $I_{E_0} \cdot J(E) \subseteq I_{E_0}$ and $J(E) \cdot I_{E_0} \subseteq I_{E_0}$. (It will be proper because $E_0 \neq H$). Let $T_1 \in J(E)$ and $T_2 \in I_{E_0}$. Then $T_1 T_2$ and $T_2 T_1$ are in $J(E)$ (see (5.2.2)). Moreover,

$$\operatorname{im}(T_2 T_1) \subseteq \operatorname{im} T_2 \subseteq E_0,$$

and so $T_1 T_2$ is in I_{E_0} ; also $\operatorname{im}(T_1 T_2) = \{0\}$ because $\operatorname{im} T_2 \subseteq E$ and $T_1(E) = \{0\}$. Hence $T_2 T_1$ is in I_{E_0} and therefore I_{E_0} is a proper two-sided ideal of $J(E)$.

Let E_1, E_2 be distinct subspaces of E , and let $\xi \in E_1 - E_2$ be non-zero. For any $\zeta \in E^\perp$ there exists an operator T in $J(E)$ such that $T\zeta = \xi$ and $T(H) \subseteq \operatorname{span} \{\xi\}$. Then T is in $I_{E_1} \setminus I_{E_2}$ and so I_{E_1} and I_{E_2} are distinct. $\square$

(5.2.6) COROLLARY : There are no proper two-sided ideals in $B(H)$.

If $E \neq \{0\}$ then I is a proper minimal two-sided ideal in $J(E)$ if and only if

$$I = \{T \in J(E) : \operatorname{im} T \subseteq E_0\} \quad (1)$$

for a one-dimensional subspace E_0 of E , and I is a proper maximal two-sided ideal of $J(E)$ if and only if I is given by (1) for $E_0 = E$.

Proof : If $E = \{0\}$ then by (5.2.5) the only two-sided ideals in $J(E) = B(H)$ are $B(H)$ and $\{0\}$. The remaining assertions follow from (5.2.4) and (5.2.5). $\square$

Now let B be in $\mathcal{R}$ (recall the definition of $\mathcal{R}$ from (5.2.1)). There are some subspaces of B which will correspond to certain types of ideals in B , and which will appear often in this section, and also in §(5.3) and §(5.4). We pause here to introduce the notation which we use.

(5.2.7) DEFINITION : Let B denote a subset of $PM(G)$ and let $\sigma \in \Sigma(G)$. If Y_σ and Z_σ are subspaces of H_σ then we define $B[Y_\sigma, Z_\sigma]$ by

$$B[Y_\sigma, Z_\sigma] = \{b \in B : \hat{b}(\sigma)Y_\sigma \subseteq Z_\sigma\}.$$

If $Z_\sigma = \{0\}$ then we put

$$B[Y_\sigma] = B[Y_\sigma, \{0\}] = \{b \in B : \hat{b}(\sigma)Y_\sigma = \{0\}\}.$$

If we now let $B \in \mathcal{B}_h$ and $\mathcal{J} = \{Y_\sigma : \sigma \in \Sigma(G)\}$ then $B_{\mathcal{J}}$, defined in (3.4.8), is precisely

$$B_{\mathcal{J}} = \bigcap_{\sigma \in \Sigma(G)} B[Y_\sigma].$$

The other particular case we are interested in is when $Y_\sigma = H_\sigma$.

Then $B[H_\sigma, Z_\sigma] = \{b \in B : \text{im } \hat{b}(\sigma) \subseteq Z_\sigma\}$. We will see in (5.2.9) that if $B \in \mathcal{R}$ and E_σ is found as in (5.2.1), then $B[H_\sigma, Z_\sigma]$ is a two-sided ideal in B if $Z_\sigma = H_\sigma$ (in which case $B[H_\sigma, H_\sigma] = B$), or if Z_σ is a subspace of E_σ . Hence if $F \subseteq \Sigma(G)$, and for each $\sigma \in F$ Z_σ is a subspace of E_σ , then $\bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma]$ is also a two-sided ideal

in B . In general not every two-sided ideal of B is of this form.

However, in (5.4.2) we see that if $B \in \mathcal{B}_h \cap \mathcal{R}$ then each of these ideals is closed, and every closed two-sided ideal is of this form.

(5.2.8) We now assume that B is an algebra in $\mathcal{R}$, and that for each $\sigma \in \Sigma(G)$, $\hat{B}(\sigma) = J(E_\sigma)$ (see 5.2.1)). Then define $\text{sp } B$ to be that subset of $\Sigma(G)$ given by

$$\text{sp } B = \{\sigma \in \Sigma(G) : \hat{B}(\sigma) \neq \{0\}\},$$

and let

$$\overline{\text{sp } B} = \{\sigma \in \text{sp } B : E_\sigma = \{0\}\},$$

and

$$\underline{\text{sp } B} = \{\sigma \in \text{sp } B : E_\sigma \text{ is non-zero}\} = \text{sp } B \setminus \overline{\text{sp } B}.$$

(5.2.9) Let F be a subset of $\Sigma(G)$. Recalling that for any subset A of $\text{PM}(G)$

$$A_F = \{S \in A : \hat{S}(\sigma) = 0 \text{ if } \sigma \notin F\},$$

it follows immediately from (5.2.7) that, if $\sigma \in \Sigma(G)$ and Y_σ and Z_σ are subspaces of H_σ , then

$$(B[Y_\sigma, Z_\sigma])_F = B_F[Y_\sigma, Z_\sigma].$$

If $\sigma \notin F$ then $B_F[Y_\sigma, Z_\sigma] = B_F$. Hence assume $\sigma \in F$. Then

PROPOSITION : $B_F[H_\sigma, Z_\sigma]$ is a two-sided ideal in B if and only if $Z_\sigma = H_\sigma$ or Z_σ is a subspace of E_σ .

Proof : Clearly $B_F * B$ and $B * B_F$ are subsets of B_F . Hence $B_F[H_\sigma, Z_\sigma]$ is a two-sided ideal in B if for each $b_1 \in B$ and $b_2 \in B_F[H_\sigma, Z_\sigma]$ we have

$$\text{im}(b_1 * b_2)^\wedge(\sigma) \subseteq Z_\sigma \text{ and } \text{im}(b_2 * b_1)^\wedge(\sigma) \subseteq Z_\sigma. \quad (1)$$

It is only necessary to consider Z_σ a subspace of E_σ , because if $Z_\sigma = H_\sigma$ then $B_F[H_\sigma, H_\sigma] = B_F$. But (1) holds because $(b_1 * b_2)^\wedge(\sigma) = \hat{b}_1(\sigma) \hat{b}_2(\sigma)$ and $(b_2 * b_1)^\wedge(\sigma) = \hat{b}_2(\sigma) \hat{b}_1(\sigma)$, and so

$$\text{im}(b_1 * b_2)^\wedge(\sigma) \subseteq \hat{b}_1(\sigma) Z_\sigma = \{0\} \quad \text{and} \quad \text{im}(b_1 * b_2)^\wedge(\sigma) \subseteq \text{im } \hat{b}_2(\sigma) \subseteq Z_\sigma.$$

Conversely, assume that Z_σ is a subspace of H_σ and that $B_F[H_\sigma, Z_\sigma]$ is a two-sided ideal in B . Then it is easy to see that $\{\hat{b}(\sigma) : b \in B_F[H_\sigma, Z_\sigma]\}$ is a two-sided ideal in $J(E_\sigma)$. Hence, by (5.2.6), either it equals the whole of $J(E_\sigma)$, in which case $Z_\sigma = H_\sigma$, or there exists a subspace Z'_σ of E_σ such that it is

$$\{T \in J(E_\sigma) : \text{im } T \subseteq Z'_\sigma\}.$$

It then follows that Z'_σ must contain Z_σ , and so Z_σ is a subspace of E_σ . □

We now consider the non-zero minimal two-sided ideals in B . In (5.2.12) we give a complete description of these ideals for any B in $\mathcal{R}$. First we have some preliminary results.

(5.2.10) PROPOSITION : Let $B \in \mathcal{R}$ and $\sigma \in \overline{\text{sp } B}$. Then

$B_{\{\sigma\}} = B_{\{\sigma\}} [H_\sigma, H_\sigma]$ is a non-zero minimal two-sided ideal in B . If I is a non-zero two-sided ideal in B , and if $\sigma \in \overline{\text{sp } B}$, then $B_{\{\sigma\}}$ is a subset of I if and only if $\hat{b}(\sigma) \neq 0$ for some $b \in I$.

Proof : The proof is a repeat of the argument used in (38.3) of [HR], if we note that

$$(i) \quad \text{PM}(G) * T(G)_{\{\sigma\}} = T(G)_{\{\sigma\}} = T(G)_{\{\sigma\}} * \text{PM}(G), \text{ and}$$

$$(ii) \quad B_{\{\sigma\}} = T(G)_{\{\sigma\}} \quad \text{by (5.2.1) and (5.2.8).} \quad \square$$

(5.2.11) PROPOSITION : Let $B \in \mathcal{R}$ and $\sigma \in \underline{\text{sp}} B$. Let Z_σ be a one-dimensional subspace of E_σ . Then $B_{\{\sigma\}}[H_\sigma, Z_\sigma]$ is a non-zero minimal two-sided ideal in B . If I is a non-zero two-sided ideal in B , and if $\sigma \in \underline{\text{sp}} B$ and Z_σ is a subspace of E_σ , then $B_{\{\sigma\}}[H_\sigma, Z_\sigma]$ is a subset of I if $Z_\sigma \subseteq \text{im } \hat{b}(\sigma)$ for some $b \in I$.

Proof : By (5.2.9) $B_{\{\sigma\}}[H_\sigma, Z_\sigma]$ is a two-sided ideal in B , and by (5.2.1) it is proper. It is easily seen to be a two-sided ideal in $B_{\{\sigma\}}$ and so, by (5.2.1) and (5.2.6), it is minimal if $\dim Z_\sigma = 1$.

Now let I be a non-zero two-sided ideal in B , let $\sigma \in \underline{\text{sp}} B$ and let $Z_\sigma \subseteq \text{im } \hat{b}(\sigma)$ for some $b \in I$. Then there exists a subspace Y_σ of $E_\sigma^\perp$ such that

$$\dim Y_\sigma = \dim Z_\sigma \quad \text{and} \quad \hat{b}(\sigma) Y_\sigma = Z_\sigma. \quad (1)$$

Define the trigonometric polynomial b_1 on G by

$$b_1(x) = d_\sigma \text{Tr}(P_{Y_\sigma} U_\sigma(x)^*) \quad \text{for each } x \in G.$$

Then b_1 is in $T(G)_{\{\sigma\}} \cap B = B_{\{\sigma\}}$, and

$$(b * b_1)^\wedge(\sigma) = \hat{b}(\sigma) P_{Y_\sigma}. \quad (2)$$

By (1) $\text{im}(b * b_1)^\wedge(\sigma) \subseteq Z_\sigma$ and so

$$b * b_1 \in I \cap B_{\{\sigma\}}[H_\sigma, Z_\sigma]. \quad (3)$$

It follows from (2) and (3) that $I \cap B_{\{\sigma\}}[H_\sigma, Z_\sigma]$ is a non-zero two-sided ideal in B and also in $B_{\{\sigma\}}$. It is proper because Z_σ is a proper subspace of H_σ . Hence, by (5.2.5) there exists a subspace Z'_σ of E_σ such that

$$I \cap B_{\{\sigma\}}[H_\sigma, Z_\sigma] = B_{\{\sigma\}}[H_\sigma, Z'_\sigma]. \quad (4)$$

By (3) Z_σ is a subspace of Z'_σ . If $Z_\sigma \neq Z'_\sigma$ then the trigonometric polynomial b_2 , defined by

$$b_2(x) = d_\sigma \operatorname{Tr}(P_{Z'_\sigma - Z_\sigma} U_\sigma(x)^*) \quad \text{for each } x \in G,$$

is in $B_{\{\sigma\}}[H_\sigma, Z'_\sigma]$ but not in $B_{\{\sigma\}}[H_\sigma, Z_\sigma]$. This contradicts (4), and so $Z_\sigma = Z'_\sigma$ and

$$I \cap B_{\{\sigma\}}[H_\sigma, Z_\sigma] = B_{\{\sigma\}}[H_\sigma, Z_\sigma]. \quad \square$$

Remark : The converse of the last assertion in (5.2.11) is not true.

To see this, let G be a compact group which has a representation U_σ of dimension 3 (for example, $G = \mathrm{SU}(2)$). Let H_σ be the representation space of U_σ , and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H_σ . Let $E_\sigma = \operatorname{span}\{e_1, e_2\}$, and let

$$B = \{p \in T(G)_{\{\sigma\}} : \hat{p}(\sigma)E_\sigma = \{0\}\}.$$

Then, if B is normed as a closed subspace of the convolution algebra $C(G)$, it is in $\mathcal{R}$. Let I be given by

$$I = \{b \in B : \operatorname{im} \hat{b}(\sigma) \subseteq Z_\sigma\}$$

where $Z_\sigma = \operatorname{span}\{e_2, e_3\}$. Then $I = B[H_\sigma, Z_\sigma]$ and so is a two-sided ideal in B . However there does not exist an element b of I such that $Z_\sigma \subseteq \operatorname{im} b(\sigma)$, because $\dim(\operatorname{im} \hat{b}(\sigma)) \leq 1$ for all $b \in B$.

(5.2.12) The propositions (5.2.10) and (5.2.11) enable us to give a complete description of the non-zero minimal two-sided ideals of each algebra B in $\mathcal{R}$.

PROPOSITION : Let $B \in \mathcal{R}$ and let I be a non-zero two-sided ideal in B . Then one of the following conditions hold:

$$(1) \quad B_{\{\sigma\}} \subseteq I \quad \text{for some } \sigma \in \overline{\operatorname{sp} B}, \text{ or}$$

$$(2) \quad B_{\{\sigma\}}[H_\sigma, Z_\sigma] \subseteq I \quad \text{for some } \sigma \in \operatorname{sp} B \text{ and a non-zero subspace } Z_\sigma \text{ of } E_\sigma.$$

If (2) holds then either $B_{\{\sigma\}} \subseteq I$, or there is a maximal subspace Z_σ of E_σ such that $B_{\{\sigma\}} [H_\sigma, Z_\sigma] \subseteq I$. Hence, every non-zero minimal two-sided ideal of B is of the form $B_{\{\sigma\}}$ for some $\sigma \in \overline{\text{sp } B}$, or $B_{\{\sigma\}} [H_\sigma, Z_\sigma]$ for some $\sigma \in \underline{\text{sp } B}$ and a one-dimensional subspace Z_σ of E_σ .

Proof : The ideal I is non-zero, and so there exists b in I such that $\hat{b}(\sigma) \neq 0$ for some σ in $\Sigma(G)$. If $\sigma \in \overline{\text{sp } B}$ then by (5.2.10) $B_{\{\sigma\}}$ is a subset of I and so (1) holds. If $\sigma \in \underline{\text{sp } B}$ then either

- (i) $\text{im } \hat{b}(\sigma) \subseteq E_\sigma^\perp$, or
- (ii) $\text{im } \hat{b}(\sigma) \cap E_\sigma$ is non-zero.

Assume (i) holds. It is easily seen that $\hat{I}(\sigma) = \{\hat{b}(\sigma) : b \in I\}$ is a non-zero two-sided ideal in $J(E_\sigma)$. By (i) and (5.2.3) it must equal $J(E_\sigma)$, and so $B_{\{\sigma\}} = I * (T(G)_{\{\sigma\}} \cap B)$ is contained in I . Hence $B_{\{\sigma\}} [H_\sigma, E_\sigma]$ is also in I . If (ii) holds and Z_σ is a non-zero subspace of $\text{im } \hat{b}(\sigma) \cap E_\sigma$ then, by (5.2.11), $B_{\{\sigma\}} [H_\sigma, Z_\sigma]$ is a subset of I .

Assume now that (2) holds and that $B_{\{\sigma\}}$ is not a subset of I . Let Y_σ denote the subspace of H_σ generated by all subspaces Z_σ of H_σ such that $B_{\{\sigma\}} [H_\sigma, Z_\sigma] \subseteq I$. By (5.2.9) each Z_σ is a subspace of E_σ and therefore so is Y_σ . If $B_{\{\sigma\}} [H_\sigma, Y_\sigma] \subseteq I$ then Y_σ is the maximal subspace of H_σ with this property. To check this, let $b \in B_{\{\sigma\}} [H_\sigma, Y_\sigma]$; then if there exists a subspace Z_σ of E_σ such that $B_{\{\sigma\}} [H_\sigma, Z_\sigma] \subseteq I$ and $b \in B_{\{\sigma\}} [H_\sigma, Z_\sigma]$, clearly b is in I . So we assume b is not in any such ideal. The finite dimensionality of H_σ ensures the existence of subspaces $Z_\sigma^1, \dots, Z_\sigma^n$ of E_σ , such that $B_{\{\sigma\}} [H_\sigma, Z_\sigma^i] \subseteq I$ for $i = 1, \dots, n$, and $\text{im } \hat{b}(\sigma)$ is a subset of the subspace generated by the Z_σ^i 's. Denote by $(Z_\sigma^j)'$ the subspace generated by $\{Z_\sigma^i, i = 1, \dots, j-1\}$ for $j = 1, \dots, n$ (take $(Z_\sigma^1)' = \{0\}$), and define the trigonometric polynomial b_j on G by

$$b_j(x) = d_\sigma \operatorname{Tr}(P_{Z_\sigma^j} - (Z_\sigma^j)^* U_\sigma(x)^*) \quad \text{for each } x \in G.$$

Then b_j is in $B_{\{\sigma\}}[H_\sigma, Z_\sigma^j]$ for $j = 1, \dots, n$, and

$$b = \sum_{j=1}^n b_j * b.$$

Hence, b is in I and so $B_{\{\sigma\}}[H_\sigma, Y_\sigma]$ is a subset of I .

It follows immediately that the non-zero minimal two-sided ideals in B are precisely $B_{\{\sigma\}}$, for $\sigma \in \overline{\operatorname{sp} B}$, and any $B_{\{\sigma\}}[H_\sigma, Z_\sigma]$ where Z_σ is a one-dimensional subspace of E_σ and $\sigma \in \operatorname{sp} B$. $\square$

(5.2.13) We end this section by taking a brief look at the family of maximal two-sided ideals of any algebra B in $\mathcal{R}$ which has the property that there exists distinct $\sigma, \eta \in \Sigma(G)$ such that $B \cap T(G)_{\{\sigma\}}$ and $B \cap T(G)_{\{\eta\}}$ are non-zero. If this is not the case then B is a subalgebra of $T(G)_{\{\sigma\}}$ for some $\sigma \in \Sigma(G)$, and (5.2.6) may be applied.

PROPOSITION : Let $B \in \mathcal{R}$ and suppose that B possesses the additional property just mentioned. Let $\sigma \in \operatorname{sp} B$. With the notation of (5.2.8), $B[H_\sigma, E_\sigma]$ is a proper two-sided ideal in B . The map ϕ of B , given by

$$\phi(b) = \hat{b}(\sigma) \quad \text{for each } b \in B$$

is a homomorphism onto $J(E_\sigma)$. The ideal $B[H_\sigma, E_\sigma]$ is both regular and maximal.

Proof : By (5.2.9) $B[H_\sigma, E_\sigma]$ is a two-sided ideal in B . It is non-zero because there exists $\eta \neq \sigma$ in $\Sigma(G)$ such that $B \cap T(G)_{\{\eta\}}$ is a non-zero subset of $B[H_\sigma, E_\sigma]$, and it is proper because it does not contain the trigonometric polynomial defined by

$$b(x) = d_\sigma \operatorname{Tr}(P_{E_\sigma^\perp} U_\sigma(x)^*) \quad \text{for each } x \in G.$$

It follows from (5.2.1) that ϕ is a surjection. It is clearly a homomorphism.

By (5.2.6) the set $\{T \in J(E_\sigma) : \text{im } T \subseteq E_\sigma\}$ is a proper maximal two-sided ideal in $J(E_\sigma)$ if $E_\sigma \neq \{0\}$ (that is, if $\sigma \in \underline{\text{sp } B}$). If $E_\sigma = \{0\}$ then it is maximal in the sense that the only ideal containing it is $J(E_\sigma)$. The set $B[H_\sigma, E_\sigma]$ is the inverse image of this ideal under the map ϕ , and so it is maximal in B .

To prove that it is regular we prove that ω_σ , defined by

$$\omega_\sigma(x) = d_\sigma \text{Tr} \left(P_{E_\sigma^\perp} U_\sigma(x)^* \right) \text{ for each } x \in G,$$

is an identity of B modulo $B[H_\sigma, E_\sigma]$. It follows from (5.2.1) that ω_σ is in $B_{\{\sigma\}} = T(G)_{\{\sigma\}} \cap B$. Let $b \in B$. Then $\omega_\sigma * b$ and $b * \omega_\sigma$ are in $B_{\{\sigma\}}$ because $B_{\{\sigma\}}$ is a two-sided ideal in B (see (5.2.9)). For any $\eta \in \Sigma(G)$,

$$(b * \omega_\sigma)^\wedge(\eta) = \delta_{\sigma\eta} \hat{b}(\eta) \text{ and } (\omega_\sigma * b)^\wedge(\eta) = \delta_{\sigma\eta} P_{E_\sigma^\perp} \hat{b}(\sigma),$$

and so $b - b * \omega_\sigma$ and $b * \omega_\sigma - b$ are in $B[H_\sigma, E_\sigma]$. □

(5.2.14) COROLLARY : Let $B \in \mathcal{R}$ and assume B is a Banach algebra under some norm. For each $\sigma \in \text{sp } B$, $B[H_\sigma, E_\sigma]$ is closed in the norm topology. Hence if F is a subset of $\Sigma(G)$ then

$$\bigcap_{\sigma \in F} B[H_\sigma, E_\sigma] = \{b \in B : \text{im } \hat{b}(\sigma) \subseteq E_\sigma \text{ for } \sigma \in F\}$$

is a closed two-sided ideal in B .

Proof : In view of (Cl6) of [HR] the first assertion is a direct consequence of (5.2.13), and the second assertion follows from the first and (5.2.13) again. □

(5.3) The Left Ideals in Elements of $\mathcal{R}$

We again let $\mathcal{R}$ denote the family of convolution algebras introduced in (5.2.1). In this section we consider the left ideals of each of the algebras B in $\mathcal{R}$. In general we can not determine all the left ideals in B , and in fact, we are not even able to determine all the minimal left ideals. As we would expect it is more difficult to write down (closed) left ideals than it was to write down (closed) two-sided ideals, and for this reason this section appears rather technical. Before considering which subspaces of B are candidates for left ideals, I present three algebraic results, analogues of (5.2.3) to (5.2.6), which are the basic tools of this section. Of these, (5.3.1) and (5.3.2) were suggested to me by L. Kovacs of the Australian National University.

(5.3.1) Take the notation from (5.2.2). Each operator in $J(E)$ may be regarded as a map defined on the quotient space H/E . Let Ω be a subset of $J(E)$. Let $\mathcal{V}$ be a subspace of H containing E such that

$$\begin{aligned} \mathcal{V}/E &= \text{Annihilator of } \Omega \text{ in } H/E \\ &\stackrel{\text{def}}{=} \{ \xi + E : T\xi \in E \text{ for all } T \in \Omega \}. \end{aligned}$$

Clearly Ω is a subset of the annihilator of $\mathcal{V}/E$ in $J(E)$. Also,

LEMMA : If Ω is a left ideal in $J(E)$ then the annihilator of $\mathcal{V}$ in $J(E)$ is a subset of Ω .

Proof : Let $T \in$ annihilator of $\mathcal{V}$ in $J(E)$. Then T is in Ω if there exists $T_1 \in J(E)$ and $T_2 \in \Omega$ such that $T = T_1 T_2$. The operator T vanishes on $\mathcal{V}$. Let $R_1 \in \Omega$. Because Ω is a subset of the annihilator of $\mathcal{V}/E$ in $J(E)$, there exists a subspace Z_1 of $\mathcal{V}^\perp$ which is also a subspace of $E^\perp$, and satisfies

$$R_1 \xi \neq 0 \text{ for all non-zero } \xi \text{ in } Z_1.$$

Hence, R_1 is invertible on $R_1(Z_1)$. If $Z_1 = Y^\perp$ define T_1 and T_2

$$T_1 = T R_1^{-1} P_{R_1(Y^\perp)} \quad \text{and} \quad T_2 = P_{R_1(Y^\perp)} R_1,$$

where R_1^{-1} denotes the inverse of R_1 on $R_1(Y^\perp)$. Then T_1 is in $J(E)$, T_2 is in Ω and $T = T_1 T_2$.

If Z_1 is a proper subspace of $Y^\perp$ then we repeat the argument on $Y^\perp - Z_1$. Because $\dim Y^\perp$ is finite there exists $n \geq 1$, maps R_i in Ω and subspaces Z_i of $Y^\perp - \bigoplus_{j=1}^{i-1} Z_j$ such that $R_i \xi \neq 0$ for all non-zero ξ in Z_i , $i = 1, \dots, n-1$, and $Z_n = Y^\perp - \bigoplus_{j=1}^{n-1} Z_j$. Then define T_1 and T_2 by

$$T_1 = T \left(\sum_{i=1}^n R_i^{-1} P_{R_i(Z_i)} \right) \quad \text{and} \quad T_2 = \sum_{i=1}^n P_{R_i(Z_i)} R_i.$$

It follows that $T_1 \in J(E)$, $T_2 \in \Omega$ and $T = T_1 T_2$. □

As a converse to (5.3.1) we obtain the following lemma.

(5.3.2) LEMMA : Let V be a subspace of H which contains E . Then every subspace Ω of $J(E)$ satisfying:

annihilator of V in $J(E) \subseteq \Omega \subseteq$ annihilator of V/E in $J(E)$
is a left ideal in $J(E)$.

Proof : The subspace Ω is a left ideal in $J(E)$ if for each $T_1 \in J(E)$ and $T_2 \in \Omega$, $T_1 T_2$ is an operator in the annihilator of V in $J(E)$. But $T_1 T_2(V) \subseteq T_1(E)$ because Ω is a subspace of the annihilator of V/E in $J(E)$, and so $T_1 T_2(V) = \{0\}$. Hence $T_1 T_2$ is in the annihilator of V in $J(E)$, and so Ω is a left ideal of $J(E)$. □

(5.3.3) COROLLARY : Let Ω be a left ideal in $B(H)$. Then there exists a subspace Y of H such that

$$\Omega = \{T \in B(H) : T(Y) = \{0\}\} .$$

Every set of this form is a left ideal in $B(H)$.

proof : If $E = \{0\}$ then $J(E) = B(H)$, and $H/E = H$. Hence the assertions follow from (5.3.1) and (5.3.2). $\square$

Corollary (5.3.3) is also proved directly in (38.11) of [HR]

We can use (5.3.1) and (5.3.2) to write down a complete list of the left ideals in $J(E)$. However, unless $E = \{0\}$, in which case (5.3.3) applies, the process is a little technical. Before doing this we introduce some notation.

(5.3.4) Notation : We again begin with the notation of (5.2.2). Let Y denote a subspace of H that contains E . Then $\{e_1, \dots, e_m\}$, which is an orthonormal basis for E , may be extended to the set $\{e_1, \dots, e_m, f_1, \dots, f_\ell\}$, where $m + \ell = \dim Y$, which is an orthonormal basis for Y . We will denote by $\beta(Y)$ the integer ℓ .m. To each operator T in $J(E)$ there corresponds a basis dependent vector T_Y in $\mathbb{C}^{\beta(Y)}$. It is determined as follows. If $j \in \{1, \dots, \beta(Y)\}$ and

$$j = a + bm \text{ where } 0 \leq a < m ,$$

Then the j^{th} co-ordinate of T_Y is $\langle T f_{b+1}, e_{a+1} \rangle$.

(5.3.5) Let Y be a subspace of H which contains E . We now give an explicit representation of the subspaces Ω of $J(E)$ which satisfy (1) of (5.3.2), and which are therefore left ideals of $J(E)$.

LEMMA : Assume that Ω is a subset of $J(E)$ which satisfies (1) of (5.3.2). Then Ω is a subspace if and only if there exists a continuous endomorphism W of $\mathbb{C}^{\beta(Y)}$ such that

$$\Omega = \{T \in J(E) : T(Y) \subseteq E \text{ and } \mathcal{T}_Y \in \ker W\}.$$

Proof : It easily follows from (5.3.4) that for any operators S, T in Ω , $(S \pm T)_Y = \mathcal{S}_Y \pm \mathcal{T}_Y$. Therefore, if Ω is a linear space, the set $\{\mathcal{T}_Y : T \in \Omega\}$ is a linear subspace of $\mathcal{C}^{\beta(Y)}$. Hence there exists a continuous endomorphism W of $\mathcal{C}^{\beta(Y)}$ such that

$$\ker W = \{\mathcal{T}_Y : T \in \Omega\} \quad (1)$$

Then $\Omega \subseteq \{T \in J(E) : T(Y) \subseteq E \text{ and } \mathcal{T}_Y \in \ker W\}$.

To prove the reverse inclusion assume that T is an operation in $J(E)$ satisfying $T(Y) \subseteq E$ and $\mathcal{T}_Y \in \ker W$. The operator $TP_Y^\perp$ is in annihilator of Y in $J(E)$, and so is in Ω . Further, by (1), there exists T_1 in Ω such that $\mathcal{T}_{1Y} = \mathcal{T}_Y$. Then $TP_Y = T_1P_Y$, and so $TP_Y (= T_1 - T_1P_Y^\perp)$ is in Ω . It follows that T is also in Ω .

Conversely, if Ω is a set of this form then it is easily seen to be a subspace of $J(E)$. $\square$

(5.3.6) COROLLARY : Let Ω be a subset of $J(E)$. Then it is a left ideal in $J(E)$ if and only if there exists a subspace Y of H , with $E \subseteq Y$, and a continuous endomorphism W on $\mathcal{C}^{\beta(Y)}$ such that

$$\Omega = \{T \in J(E) : T(Y) \subseteq E \text{ and } \mathcal{T}_Y \in \ker W\}. \quad (1)$$

Proof : If Ω is of the form in (1), then by (5.3.5) it is a subspace of $J(E)$. Again by the argument used in (5.3.5) this will mean that Ω is a left ideal in $J(E)$ if the annihilator of Y in $J(E)$ is in Ω .

Let T belong to this annihilator. Then we need only check that $T_Y \in \ker W$. But $T_Y = 0$ so it is in $\ker W$.

Conversely, if Ω is a left ideal, let Y be the subspace

$$Y = \{\xi \in H : T\xi \in E \text{ for all } T \in \Omega\}.$$

Then by (5.3.1) the annihilator of Y in $J(E)$ is a subset of Ω , and so by (5.3.5) there exists a continuous endomorphism W of $\mathbb{C}^{\beta(Y)}$ such that Ω is given as in (1). $\square$

(5.3.7) We will refer to (5.2.7) and (5.2.8) for notation. Now let $B \in \mathcal{B}_h \cap \mathcal{R}$ and let $\sigma \in \text{sp } B$. By (4.2.1) there exists a subspace E_σ of H_σ such that $B_{\{\sigma\}} \cong J(E_\sigma)$. In (4.2.2) we specified an orthonormal basis $\{e_1^\sigma, \dots, e_{k_\sigma}^\sigma\}$ for E_σ . Let Y_σ be a subspace of H_σ which contains E_σ . Then we can extend this basis to an orthonormal basis

$$\{e_1^\sigma, \dots, e_{k_\sigma}^\sigma, f_1^\sigma, \dots, f_{\dim Y_\sigma \cap E_\sigma^\perp}^\sigma\}$$

of Y_σ . Denote by $\beta(Y_\sigma)$ the integer

$$\beta(Y_\sigma) = \dim(Y_\sigma \cap E_\sigma^\perp) \cdot \dim E_\sigma.$$

Finally let W_σ be a continuous endomorphism of $\mathbb{C}^{\beta(Y_\sigma)}$. Then we have

PROPOSITION : The set I given by

$$I = \{b \in B[Y_\sigma, E_\sigma] : \hat{b}(\sigma)_{Y_\sigma} \in \ker W_\sigma\} \quad (1)$$

is a left ideal in B .

Proof : Clearly I is a left ideal in B if $\{\hat{b}(\sigma) : b \in I\}$ is a left ideal in $J(E_\sigma)$. But this is so by (5.3.6), because $\{\hat{b}(\sigma) : b \in I\}$ is $\{T \in J(E_\sigma) : T(Y_\sigma) \subseteq E \text{ and } T_{Y_\sigma} \in \ker W_\sigma\}$. $\square$

(5.3.8) PROPOSITION : Let $B \in \mathcal{R}$ and I be a left ideal in B .

Let $\sigma \in \Sigma(G)$. Then there exists a subspace Y_σ of H_σ , with $E_\sigma \subseteq Y_\sigma$, and a continuous endomorphism W_σ on $\mathbb{C}^{\beta(Y_\sigma)}$ such that

$$I \cap B_{\{\sigma\}} = \{b \in B_{\{\sigma\}}[Y_\sigma, E_\sigma] : \hat{b}(\sigma)_{Y_\sigma} \in \ker W_\sigma\}.$$

If I is non-zero, then either $I \cap B_{\{\sigma\}}$ is non-zero for some $\sigma \in \text{sp } B$, or $\text{sp } B$ is non-empty and

$$\text{im } \hat{b}(\sigma) \subseteq E_\sigma \text{ for all } \sigma \in \text{sp } B \text{ and all } b \in I.$$

Proof : It follows from (5.2.9) that $B_{\{\sigma\}}$ is a left ideal in B , and so $I \cap B_{\{\sigma\}}$ is a left ideal in B . It is easily seen to be a left ideal in $B_{\{\sigma\}}$. Hence $\{\hat{b}(\sigma) : b \in I \cap B_{\{\sigma\}}\}$ is a left ideal in $J(E_\sigma)$. By (5.3.6) there exists a subspace Y_σ of H_σ , with $E_\sigma \subseteq Y_\sigma$, and a continuous endomorphism W_σ on $\mathbb{C}^{\beta(Y_\sigma)}$ such that

$$\{\hat{b}(\sigma) : b \in I \cap B_{\{\sigma\}}\} = \{T \in J(E_\sigma) : T(Y_\sigma) \subseteq E_\sigma \text{ and } T_{Y_\sigma} \in \ker W_\sigma\}.$$

Hence $I \cap B_{\{\sigma\}} = \{b \in B_{\{\sigma\}}[Y_\sigma, E_\sigma] : \hat{b}(\sigma)_{Y_\sigma} \in \ker W_\sigma\}.$

Now assume that I is a non-zero left ideal of B , and let b be a non-zero element of I . Then one of the following conditions must hold:

- (i) $\text{sp } B \neq \{\}$ and $\text{im } \hat{b}(\sigma) \subseteq E_\sigma$ for all $b \in I$ and all $\sigma \in \Sigma(G)$, or
- (ii) there exists a non-zero $b \in I$ and $\sigma \in \Sigma(G)$ such that $\text{im } \hat{b}(\sigma) \cap E_\sigma^\perp \neq \{0\}.$

If (ii) holds then by (5.2.1) the trigonometric polynomial b_1 defined by

$$b_1(x) = d_\sigma \text{Tr} (P_{\text{im } \hat{b}(\sigma) \cap E_\sigma^\perp} U_\sigma(x)^*) \text{ for each } x \in G$$

is a non-zero element of $B_{\{\sigma\}}$. Hence $b_1 * b$ is in $I \cap B_{\{\sigma\}}$. But $b_1 * b$ is non-zero because

$$(b_1 * b)^\wedge(\sigma) = \hat{b}_1(\sigma) \hat{b}(\sigma) = P_{\text{im} \hat{b}(\sigma) \cap E_\sigma^\perp} \hat{b}(\sigma) . \quad \square$$

(5.3.9) COROLLARY : Let $B \in \mathcal{R}$ and I be a left ideal in B . Let $\sigma \in \overline{\text{sp } B}$. Then there exists a subspace Y_σ of H_σ such that

$$I \cap B_{\{\sigma\}} = B_{\{\sigma\}} [Y_\sigma] .$$

Proof : Recall from (5.2.7) that $B_{\{\sigma\}} [Y_\sigma] = \{b \in B_{\{\sigma\}} : \hat{b}(\sigma) Y_\sigma = \{0\}\}$.

If $\sigma \in \overline{\text{sp } B}$ then $E_\sigma = \{0\}$, and so for any non-zero $b \in I$ $\text{im } \hat{b}(\sigma) \cap E_\sigma^\perp$ is non-zero. So, if $I \cap B_{\{\sigma\}}$ is zero let $Y_\sigma = H_\sigma$, and if it is non-zero apply (5.3.8), and observe that W_σ must be trivial. $\square$

We complete this section by finding a partial analogue of (5.2.12).

(5.3.10) PROPOSITION : Let $B \in \mathcal{R}$ and I be a non-zero minimal left ideal in B . Then one of the following conditions holds:

- (i) there exists $\sigma \in \overline{\text{sp } B}$ and a $(d_\sigma - 1)$ -dimensional subspace Y_σ of H_σ such that $I = B_{\{\sigma\}} [Y_\sigma]$, or
- (ii) $\hat{b}(\sigma) = 0$ for all $\sigma \in \overline{\text{sp } B}$ and all $b \in I$.

If (ii) holds, then for each $\sigma \in \underline{\text{sp } B}$, for which there exists $b \in I$ with $\hat{b}(\sigma) \neq 0$, there exists a continuous endomorphism W_σ of $\mathbb{C}^{\beta(H_\sigma)}$, with rank $(\beta(H_\sigma) - 1)$, such that

$$\{\hat{b}(\sigma) : b \in I\} = \{T \in J(E_\sigma) : \text{im } T \subseteq E_\sigma \text{ and } T_{H_\sigma} \in \ker W_\sigma\} .$$

Conversely if $\sigma \in \overline{\text{sp } B}$ then $B_{\{\sigma\}} [Y_\sigma]$ is a non-zero minimal left ideal for each $(d_\sigma - 1)$ -dimensional subspace Y_σ of H_σ , and if $\sigma \in \underline{\text{sp } B}$ and W_σ is a continuous endomorphism of $\mathbb{C}^{\beta(Y_\sigma)}$ of rank $(\beta(H_\sigma) - 1)$, then

$$\{b \in B_{\{\sigma\}} [H_\sigma, E_\sigma] : \hat{b}(\sigma)_{H_\sigma} \in \ker W_\sigma\}$$

is a non-zero minimal left ideal in B .

Proof : By (5.3.8), either $I \cap B_{\{\sigma\}}$ is non-zero for some $\sigma \in \text{sp } B$, or $\text{sp } B$ is non-empty and $\text{im } \hat{b}(\sigma) \subseteq E_\sigma$ for all $\sigma \in \text{sp } B$ and all $b \in I$. In the first case we must have $I = I \cap B_{\{\sigma\}}$ since I is minimal. If $\sigma \in \overline{\text{sp } B}$ then by (5.3.8) there exists a $(d_\sigma - 1)$ -dimensional subspace Y_σ of H_σ such that $I = B_{\{\sigma\}} [Y_\sigma]$, and so either (i) or (ii) must hold.

Assume now that (ii) holds, and let $\sigma \in \text{sp } B$ such that $\hat{b}(\sigma)$ is non-zero for some $b \in I$. Then the set $\{\hat{b}(\sigma) : b \in I\}$ is a non-zero left ideal in $J(E_\sigma)$. Hence, by (5.3.6), there exists a subspace Y_σ of H_σ , with $E_\sigma \subseteq Y_\sigma$, and a continuous endomorphism w_σ of $\mathbb{C}^{\beta(Y_\sigma)}$ such that

$$\{\hat{b}(\sigma) : b \in I\} = \{T \in J(E_\sigma) : T(Y_\sigma) \subseteq E_\sigma \text{ and } T_{Y_\sigma} \in \ker w_\sigma\}.$$

To prove the converse first let $\sigma \in \overline{\text{sp } B}$ and Y_σ be a $(d_\sigma - 1)$ -dimensional subspace of H_σ . Then it follows from (5.3.9) that $B_{\{\sigma\}} [Y_\sigma]$ is a non-zero left ideal of B . Again by (5.3.9) it must be minimal because the only subspace properly containing Y_σ is H_σ , and $B_{\{\sigma\}} [H_\sigma] = \{0\}$. Now let $\sigma \in \text{sp } B$ and let w_σ be a continuous endomorphism of $\mathbb{C}^{\beta(H_\sigma)}$ of rank $(\beta(H_\sigma) - 1)$. Then

$$\begin{aligned} \{b \in B_{\{\sigma\}} [H_\sigma, E_\sigma] : \hat{b}(\sigma)_{\sim H_\sigma} \in \ker w_\sigma\} \approx \\ \{T \in J(E_\sigma) : \text{im } T \subseteq E_\sigma \text{ and } T_{H_\sigma} \in \ker w_\sigma\}. \end{aligned} \quad (1)$$

So by (5.3.6) it is a left ideal in B , and clearly it must be non-zero. Again by (5.3.6) it must be minimal because the only set of form (1) which is contained in $\{\hat{b}(\sigma) : b \in B_{\{\sigma\}} [H_\sigma, E_\sigma] \text{ and } \hat{b}(\sigma)_{\sim H_\sigma} \in \ker w_\sigma\}$ is $\{0\}$, where w_σ has rank d_σ . $\square$

(5.4) The Ideal Theory of Convolution Banach Algebras in $\mathcal{B}_h$

We noted in (5.2.1) that if B is a convolution Banach algebra in $\mathcal{B}_h$ then it is in $\mathcal{R}$. Hence the results of §(5.2) and §(5.3) may be applied to B . However, the homogeneity property of B may be used, as we shall see in (5.4.2) and (5.4.5), to identify all the closed two-sided and closed left ideals of B . These results generalize some of the early results in §38 of [HR].

We will first deal with the closed two-sided ideals in B . Recall the notation of (5.2.1) and (5.2.7).

(5.4.1) LEMMA : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and let $\sigma \in \Sigma(G)$. Let Z_σ be a subspace of H_σ . Then $B[H_\sigma, Z_\sigma]$ is closed in the norm topology.

Proof : Recall that $B[H_\sigma, Z_\sigma] = \{b \in B : \text{im } \hat{b}(\sigma) \subseteq Z_\sigma\}$. Let $(b_n)_{n \in \mathbb{N}}$ be a sequence extracted from $B[H_\sigma, Z_\sigma]$ which converges to b in B . Then by (3.4.5) for each $\sigma \in \text{sp } B$

$$\lim_{n \rightarrow \infty} \|\hat{b}_n(\sigma) - \hat{b}(\sigma)\| = 0. \quad (1)$$

Let $\xi \in H_\sigma$. Then $\hat{b}_n(\sigma)\xi \in Z_\sigma$ for all $n \in \mathbb{N}$. But Z_σ is finite dimensional, and so by (1) $\hat{b}(\sigma)\xi \in Z_\sigma$. Hence $b \in B[H_\sigma, Z_\sigma]$ and $B[H_\sigma, Z_\sigma]$ is closed. $\square$

(5.4.2) Using (5.4.1) we can give a complete description of the closed two-sided ideals in B .

THEOREM : Let $B \in \mathcal{B}_h \cap \mathcal{R}$. Let F be a subset of $\text{sp } B$. Then if Z_σ is a subspace of E_σ for each $\sigma \in F$, the set

$$\bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma] = \{b \in B : \text{im } \hat{b}(\sigma) \subseteq Z_\sigma \text{ for all } \sigma \in F\} \quad (1)$$

is a closed two-sided ideal in B . Conversely, every set of this form

is a closed two-sided ideal of B . Moreover, distinct subsets of $\text{sp } B$ and distinct subspaces determine distinct ideals.

Proof : By (5.2.14) the set given in (1) is a closed two-sided ideal in B .

Now let I be a closed two-sided ideal in B and let F denote the set

$$F = \{\sigma \in \text{sp } B : \{\hat{b}(\sigma) : b \in I\} \neq J(E_\sigma)\}.$$

For each $\sigma \in F$ let Z_σ denote the subspace of H_σ generated by $U\{\text{im } \hat{b}(\sigma) : b \in I\}$. By (4.2.3) $\text{im } \hat{b}(\sigma) \subseteq E_\sigma$ for each $b \in I$ and each $\sigma \in F$, and so Z_σ is a subspace of E_σ for each $\sigma \in F$. Then it is clear that

$$I \subseteq \bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma]. \quad (1)$$

To prove the reverse inclusion in (1) we use (3.3.10).

Let $b \in \bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma]$, and let $(k_n)_{n \in \mathbb{N}}$ be the approximate identity of $L^1(G)$ consisting of central trigonometric polynomials. By (3.3.10) $\lim_{n \rightarrow \infty} \|k_n * b - b\|_B = 0$. Hence, if $k_n * b$ is in I for each $n \in \mathbb{N}$ then we must have b in I , and so we will have $I = \bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma]$. Now, for each $n \in \mathbb{N}$ and $\sigma \in \Sigma(G)$, $\hat{k}_n(\sigma)$ is a non-negative multiple of Id_σ , and so $(k_n * b)^\wedge(\sigma)$ is a non-negative multiple of $\hat{b}(\sigma)$. By (3.3.9) $k_n * b$ is in $P(B)$ for each $n \in \mathbb{N}$.

By (5.2.10) $B_{\{\sigma\}}$ is a subset of I for each $\sigma \in \text{sp } B \setminus F$.

It follows from (5.2.12) that

$$B_{\{\sigma\}}[H_\sigma, Z_\sigma] \subseteq I \quad (2)$$

because $B_{\{\sigma\}}[H_\sigma, \text{im } \hat{b}(\sigma)] \subseteq I$ for each $b \in I$, and so the maximal subspace determined in (5.2.12) contains Z_σ . Therefore by (2) $k_n * b$

may be written as a finite linear combination of functions in I , and so is in I for each $n \in \mathbf{N}$.

Finally we note that

$$T(G) \cap \bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma] = \bigcap_{\sigma \in F} (B[H_\sigma, Z_\sigma] \cap T_F(G)) ,$$

and so distinct subsets of $\text{sp } B$ and distinct subspaces determine distinct ideals. $\square$

(5.4.3) In (5.2.12) we obtained a complete description of the minimal two-sided ideals of any algebra B in $\mathcal{R}$. By (5.4.2), if $B \in \mathcal{B}_h \cap \mathcal{R}$ then each of these is closed. We now give a complete picture, for this case, of the proper regular maximal two-sided ideals of B . (Each of these ideals must be closed).

PROPOSITION : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and let $\sigma \in \text{sp } B$. Then $B[H_\sigma, E_\sigma]$ is a regular maximal two-sided ideal in B . Every proper regular maximal two-sided ideal in B has this form. The quotient algebra $B/B[H_\sigma, E_\sigma]$ is isomorphic to $J(E_\sigma)$ for each $\sigma \in \text{sp } B$.

Proof : By (5.2.13) $B[H_\sigma, E_\sigma]$ is a regular maximal two-sided ideal in B .

Now let I be a proper-regular maximal two-sided ideal in B . Then it must be closed, and so by (5.4.2) there exists a subset F of $\text{sp } B$ and a subspace Z_σ of E_σ for each $\sigma \in F$ such that

$$I = \bigcap_{\sigma \in F} B[H_\sigma, Z_\sigma] .$$

Clearly F must be a singleton subset of $\text{sp } B$. This is because I is a subset of $B[H_\sigma, Z_\sigma]$ and of $B[H_\eta, Z_\eta]$ if $F \supseteq \{\sigma, \eta\}$ which contradicts the maximality of I . Hence

$$I = B[H_\sigma, Z_\sigma] \quad \text{for some } \sigma \in \text{sp } B.$$

Clearly $B[H_\sigma, Z_\sigma] \subsetneq B[H_\sigma, E_\sigma]$ if Z_σ is a proper subspace of E_σ so we have $I = B[H_\sigma, E_\sigma]$.

The final statement is a special case of (5.2.13). $\square$

Before finding explicitly the family of closed left ideals in B we require a lemma. Recall the notation of (5.3.4) and (5.3.7).

(5.4.4) LEMMA : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and let $\sigma \in \Sigma(G)$. Let Y_σ be a subspace of H_σ , with $E_\sigma \subseteq Y_\sigma$, and w_σ is a continuous endomorphism of $\mathbb{C}^{\beta(Y_\sigma)}$. Then

$$\{b \in B[Y_\sigma, E_\sigma] : \hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker w_\sigma\} \quad (1)$$

is closed in the norm topology.

Proof : Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of elements in

$\{b \in B[Y_\sigma, E_\sigma] : \hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker w_\sigma\}$ which converges to b in B . By (3.4.5) for all $\eta \in \Sigma(G)$

$$\lim_{n \rightarrow \infty} \|\hat{b}(\eta) - \hat{b}_n(\eta)\| = 0. \quad (2)$$

For each $n \in \mathbb{N}$ $\hat{b}_n(\sigma)_{Y_\sigma} \subseteq E_\sigma$. Since E_σ is finite dimensional, by (2), $\hat{b}(\sigma)_{Y_\sigma} \subseteq E_\sigma$. Recall from (5.2.7) that if $j \in \{1, \dots, \beta(Y_\sigma)\}$ and $j = a + b(\dim E_\sigma)$ then

$$\hat{b}_{\sim}(\sigma)_{Y_\sigma} = \langle \hat{b}_n(\sigma) f_{b+1}^\sigma, e_{a+1}^\sigma \rangle,$$

and so

$$\begin{aligned} \left| \hat{b}_{\sim}(\sigma)_{Y_\sigma} - \hat{b}_{n\sim}(\sigma)_{Y_\sigma} \right| &= \left| \langle (\hat{b}(\sigma) - \hat{b}_n(\sigma)) f_{b+1}^\sigma, e_{a+1}^\sigma \rangle \right| \\ &\leq \|\hat{b}(\sigma) - \hat{b}_n(\sigma)\|. \end{aligned}$$

Hence, by (2) $\hat{b}_{n\sim}(\sigma)_{Y_\sigma}$ tends to $\hat{b}_{\sim}(\sigma)_{Y_\sigma}$ in $\mathbb{C}^{\beta(Y_\sigma)}$. But $\hat{b}_{n\sim}(\sigma)_{Y_\sigma} \in \ker W_\sigma$ for each $n \in \mathbb{N}$, and so $\hat{b}_{\sim}(\sigma)_{Y_\sigma}$ must also be in $\ker W_\sigma$ because $\ker W_\sigma$ is finite dimensional. Therefore the set given in (1) is closed. $\square$

(5.4.5) THEOREM : Let $B \in \mathcal{B}_h \cap \mathcal{R}$. For each $\sigma \in \Sigma(G)$ let Y_σ be a subspace of H_σ , containing E_σ , and let W_σ be a continuous endomorphism of $\mathbb{C}^{\beta(Y_\sigma)}$. Then the set

$$\bigcap_{\sigma \in \Sigma(G)} \{b \in B[Y_\sigma, E_\sigma] : \hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker W_\sigma\} \quad (1)$$

is a closed left ideal of B . Every closed left ideal in B is of the form given in (1).

Proof : It follows from (5.3.7) and (5.4.4) that the set given in (1) is a closed left ideal of B .

Conversely let I be a closed left ideal in B . Then for each $\sigma \in \Sigma(G)$ $I \cap B_{\{\sigma\}}$ is a left ideal in B , and by (5.3.8) there exists a subspace Y_σ of H_σ , with $E_\sigma \subseteq Y_\sigma$, and a continuous endomorphism W_σ of $\mathbb{C}^{\beta(Y_\sigma)}$ such that

$$I \cap B_{\{\sigma\}} = \{b \in B_{\{\sigma\}}[Y_\sigma, E_\sigma] : \hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker W_\sigma\}. \quad (2)$$

By (3.5.2) $P(B) = B \cap T(G)$, and by (3.3.10) $B \cap T(G)$ is a dense subspace of B . Hence $I \cap T(G)$ is a dense subspace of I , and so by (2) we must have $b \in B[Y_\sigma, E_\sigma]$ and $\hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker W_\sigma$ for each $b \in I$, because both H_σ and $\ker W_\sigma$ are finite dimensional vector spaces.

Therefore

$$I \subseteq \bigcap_{\sigma \in \Sigma(G)} \{b \in B[Y_\sigma, E_\sigma] : \hat{b}_{\sim}(\sigma)_{Y_\sigma} \in \ker W_\sigma\}. \quad (3)$$

To prove the reverse inclusion in (3) let b_1 be an element of

$\bigcap_{\sigma \in \Sigma(G)} \{b \in B[Y_\sigma, E_\sigma] : \hat{b}(\sigma) \underset{\sim}{y}_\sigma \in \ker W_\sigma\}$, and let $(k_n)_{n \in \mathbb{N}}$ denote the approximate identity of $L^1(G)$ consisting of central trigonometric polynomials. By (3.3.10)

$$\lim_{n \rightarrow \infty} \|b_1 - k_n * b_1\|_B = 0,$$

so b_1 will be in I if $k_n * b_1$ is in I for each $n \in \mathbb{N}$. Again, as in (5.4.2), $(k_n * b_1)^\wedge(\sigma) \underset{\sim}{y}_\sigma$ is a non-negative multiple of $\hat{b}_1(\sigma) \underset{\sim}{y}_\sigma$ for each n and each σ . By (3.3.9) $k_n * b_1$ is in $B \cap T(G)$ and so

$$k_n * b_1 \in B[Y_\sigma, E_\sigma] \text{ for each } \sigma \in \Sigma(G). \quad (4)$$

It is clear that for each $n \in \mathbb{N}$, $(k_n * b_1)^\wedge(\sigma) \underset{\sim}{y}_\sigma$ is a non-negative multiple of $\hat{b}_1(\sigma) \underset{\sim}{y}_\sigma$, and so

$$(k_n * b_1)^\wedge(\sigma) \underset{\sim}{y}_\sigma \in \ker W_\sigma \text{ for each } \sigma \in \Sigma(G). \quad (5)$$

Hence, by (4) and (5), each $k_n * b_1$ may be expressed as a finite linear combination of elements in the sets

$$\{b \in B_{\{\sigma\}}[Y_\sigma, E_\sigma] : \hat{b}(\sigma) \underset{\sim}{y}_\sigma \in \ker W_\sigma\},$$

and so by (2) $k_n * b_1$ is in I . Therefore b_1 is also in I and equality holds in (3).

The converse follows from (5.4.4). □

Theorem (5.4.5) reduces to (38.13) of [HR] if we assume that B is a subalgebra of $L^1(G)$ in which $T(G)$ is a dense subspace. Under these conditions B will be homogeneous, and for each $\sigma \in \Sigma(G)$ $E_\sigma = \{0\}$, and so W_σ must be the trivial map defined on the zero dimensional space $\mathbb{C}^\circ$.

(5.4.6) COROLLARY : Let $B \in \mathcal{B}_h \cap \mathcal{R}$. Then every closed left ideal in B is the intersection of the maximal left ideals containing it. Every

maximal proper closed left ideal in B is of the form $B[Y_\sigma, E_\sigma]$ where Y_σ is a subspace of H_σ , which contains E_σ , and has dimension equal to $\dim E_\sigma + 1$.

Proof : This is an immediate consequence of (5.4.5). $\square$

(5.4.7) Remarks : It follows from (5.4.2) and (5.4.5) that the closed two-sided and closed left ideal structure of any convolution Banach algebra B in $\mathcal{B}_h$ is precisely the same as that of $L^1(G)$, or of one of its subalgebras $L^1(G)_\mathcal{G}$ (see (3.4.8)). If we let

$$\mathcal{G} = \left\{ \bigcap_{b \in B} \ker \hat{b}(\sigma) : \sigma \in \Sigma(G) \right\},$$

and if I denotes a closed two-sided (left) ideal of B then the closure of $I \cap L^1(G)$ in $L^1(G)$ is a closed two-sided (left) ideal of $L^1(G)_\mathcal{G}$. Conversely, if J is a closed two-sided (left) ideal of $L^1(G)_\mathcal{G}$ then there exists exactly one closed two-sided (left) ideal I of B such that J is the closure of $I \cap L^1(G)$ in $L^1(G)$. So (5.4.2) and (5.4.5) provide generalizations of Reiter's closed ideal theorem for Segal algebras (see Reiter [2]).

If B is a symmetric algebra then I is a left ideal of B if and only if I^* is a right ideal of B because

$$I^* * B = I^* * B^* = (B * I)^*.$$

Therefore, in this case, (5.4.5) will also give a complete description of the closed right ideal structure of B .

We could, of course, also consider the right ideal structure of the algebras in $\mathcal{R}$, and of those in $\mathcal{B}_h \cap \mathcal{R}$. The procedure would be the same as that in §(5.3) and §(5.4), using instead of (5.3.6) an analogous result about the right ideals in $J(E)$.

(5.4.8) To complete this section we will consider the ideal theory of the symmetric homogeneous Banach algebra $V_{F_1, F_2}(G)$, where F_1 and F_2 are subsets of $\Sigma(G)$. This algebra was introduced and studied in chapter 2. Because of its symmetry we need only consider its closed two-sided and closed left ideals. Recall that $V_{F_1, F_2}(G)$ is a subset of $C(G \times G)_{F_1 \times F_2}$. For each $(\sigma, \eta) \in F_1 \times F_2$

$$V_{F_1, F_2}(G) \cap T(G \times G)_{\{(\sigma, \eta)\}} = T(G \times G)_{\{(\sigma, \eta)\}},$$

and so $E_{(\sigma, \eta)} = \{0\}$; if $(\sigma, \eta) \notin F_1 \times F_2$ then this intersection is empty and so $E_{(\sigma, \eta)} = H_\sigma \otimes H_\eta$.

Hence by (5.4.2) the closed two-sided ideals of $V_{F_1, F_2}(G)$ are precisely

$$\bigcap_{(\sigma, \eta) \in F} V_{F_1, F_2}(G) [H_\sigma \otimes H_\eta] = \{h \in V_{F_1, F_2}(G) : \hat{h}(\sigma, \eta) = 0 \text{ for } (\sigma, \eta) \in F\}$$

where F is a subset of $F_1 \times F_2$.

By (5.4.5) the closed left ideals of $V_{F_1, F_2}(G)$ are precisely the sets

$$\bigcap_{(\sigma, \eta) \in F} V_{F_1, F_2}(G) [Y_\sigma \otimes Z_\eta],$$

where Y_σ and Z_η are subspaces of H_σ and H_η respectively and F is a subset of $F_1 \times F_2$.

(5.5) Symbolic Calculus

Let B denote a convolution Banach algebra in $\mathcal{B}_h$ (see (3.4.5)).

Recall that by (4.2.1), if $E_\sigma = \bigcap_{b \in B} \ker \hat{b}(\sigma)$ for each $\sigma \in \Sigma(G)$ then

$P(B)$ is equal to the set $\{k \in T(G) : \hat{k}(\sigma) E_\sigma = \{0\} \text{ for all } \sigma \in \Sigma(G)\}$,

and so by (3.3.10) $\hat{b}(\sigma) E_\sigma = \{0\}$ for all $\sigma \in \Sigma(G)$ and all $b \in B$.

We assume in this section, and in §(5.6), that E_σ either equals $\{0\}$ or H_σ for each $\sigma \in \Sigma(G)$. Hence, in the notation of (5.2.8) $\overline{\text{sp } B} = \text{sp } B$. With this restriction we obtain from (5.4.6):

(5.5.1) LEMMA : Every proper maximal left ideal in B is of the form

$$B[\text{Span } \{\xi\}] = \{b \in B : \hat{b}(\sigma)\xi = 0\}$$

for some $\sigma \in \text{sp } B$ and some non-zero $\xi \in H_\sigma$. □

This lemma enables me to generalize to an arbitrary compact group G , and for any such algebra B , (11.4.13) of Edwards [3]. This generalization, stated in (5.5.4), gives a sufficient condition for the reversibility of $\alpha^{-1}b$ in B , where α is a non-zero complex number and b is in B . We recall that reversibility, which was introduced in §(1.6), is an interesting property because many of the algebras we have been studying do not possess an identity.

We give two proofs of (5.5.4); one is in (5.5.4) and the second follows from (5.5.10).

We begin with some preliminaries.

(5.5.2) PROPOSITION : Let $B \in \mathcal{B}_h \cap \mathcal{R}$. Assume $\overline{\text{sp } B} = \text{sp } B$. Let $b \in B$. Then b is left reversible if and only if 1 is not an eigenvalue of $\hat{b}(\sigma)$ for each $\sigma \in \text{sp } B$.

Proof : The element b is left reversible iff for each maximal regular left ideal M of B there exists an element c of B such that $b + c - c * b$ is in M (see (1.6.1)). It follows from (5.5.1) that this holds iff for each $\sigma \in \text{sp } B$ and $\xi \in H_\sigma$ there exists an

element c of B such that

$$(\hat{b}(\sigma) + \hat{c}(\sigma) - \hat{c}(\sigma) \hat{b}(\sigma))\xi = 0 . \quad (1)$$

But (1) may be rewritten as

$$(\hat{c}(\sigma) - \text{Id}_\sigma)(\hat{b}(\sigma) - \text{Id}_\sigma)\xi = \xi . \quad (2)$$

If b is left reversible then by (2) the operator $\hat{b}(\sigma) - \text{Id}_\sigma$ must be invertible (that is, 1 is not an eigenvalue of $\hat{b}(\sigma)$), because if not, and if $\xi \in \ker(\hat{b}(\sigma) - \text{Id}_\sigma)$, then (2) has no solution. Conversely, if $(\hat{b}(\sigma) - \text{Id}_\sigma)$ is invertible then clearly (2) will always be solvable, and so b is left reversible. $\square$

(5.5.3) PROPOSITION : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and assume that $\overline{\text{sp } B} = \text{sp } B$.

Then an element b of B is left reversible if and only if it is right reversible. Therefore, if b has a left reverse, it has a reverse.

Proof : Assume b is left reversible. Then there exists an element c of B such that $c + b - c * b = 0$. Hence for each $\sigma \in \text{sp } B$

$$(\hat{c}(\sigma) - \text{Id}_\sigma)(\hat{b}(\sigma) - \text{Id}_\sigma) = \text{Id}_\sigma . \quad (1)$$

But H_σ is a finite dimensional Hilbert space, and so by (1)

$$(\hat{b}(\sigma) - \text{Id}_\sigma)(\hat{c}(\sigma) - \text{Id}_\sigma) = \text{Id}_\sigma . \quad (2)$$

By the uniqueness of Fourier transforms (2) ensures that

$b + c - b * c = 0$, and so c is a right reverse of b . The converse is easy to see from this argument, and so c must be a (two-sided) reverse of b (see (1.6.1)). $\square$

(5.5.4) THEOREM : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and assume that $\overline{\text{sp } B} = \text{sp } B$.

Let $b \in B$ and α be a non-zero complex number such that $\hat{b}(\sigma) - \alpha \text{Id}_\sigma$

is an invertible operator for each $\sigma \in \text{sp } B$. Then there exists an element c of B such that for all $\sigma \in \text{sp } B$

$$\hat{b}(\sigma) [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} = \hat{c}(\sigma) = [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} \hat{b}(\sigma) .$$

Proof : It follows from (5.5.2) that $\alpha^{-1}b$ is left reversible (and so reversible by (5.5.3)) because $(\alpha^{-1} \hat{b}(\sigma) - \text{Id}_\sigma) = \alpha^{-1}(\hat{b}(\sigma) - \alpha \text{Id}_\sigma)$ is an invertible operator for each $\sigma \in \text{sp } B$. Hence there exists an element c of B such that $b + \alpha c - c * b = 0 = b + \alpha c - b * c$. Then for each $\sigma \in \text{sp } B$

$$\hat{b}(\sigma) + \alpha \hat{c}(\sigma) - c(\sigma) \hat{b}(\sigma) = 0 = \hat{b}(\sigma) + \alpha \hat{c}(\sigma) - \hat{b}(\sigma) \hat{c}(\sigma) ,$$

and so for all $\sigma \in \text{sp } B$

$$\hat{b}(\sigma) [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} = \hat{c}(\sigma) = [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1} \hat{b}(\sigma) . \quad \square$$

(5.5.5) It is possible to restate (5.5.4) so as to suggest another problem which I have considered.

Let α be a non-zero complex number. Define the complex-valued function ϕ on $\mathbb{C} \setminus \{\alpha\}$ by

$$\phi(\lambda) = \lambda(\lambda - \alpha)^{-1} . \quad (1)$$

Then ϕ is an analytic function on $\mathbb{C} \setminus \{\alpha\}$.

In (1.6.5) and (1.6.6) we discussed various ways in which ϕ could be extended to an A -valued function on a Banach algebra A . Then if f is an element of A which, in this case, has spectrum contained in $\mathbb{C} \setminus \{\alpha\}$, we can define $\phi(f)$ in A so that the map $f \longrightarrow \phi(f)$ is locally analytic (see (1.6.4)).

In particular now, we could rewrite (5.5.4) as follows.

THEOREM : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and assume $\overline{\text{sp } B} = \text{sp } B$. Let ϕ be given by (1). Then for each $b \in B$ such that the spectrum of $\hat{b}(\sigma)$ is a subset of $\mathbb{C} \setminus \{\alpha\}$ for each $\sigma \in \text{sp } B$, there exists an element c of B such that for each $\sigma \in \text{sp } B$

$$\hat{c}(\sigma) = \phi(\hat{b}(\sigma)) = \hat{b}(\sigma) [\hat{b}(\sigma) - \alpha \text{Id}_\sigma]^{-1}.$$

As we shall see in (5.5.10) we could actually write $\phi(b)$ in place of c .

The question that we now consider is whether there exist other analytic complex-valued functions ϕ which have this property. The notation is taken from §(1.6). We begin with some preliminary results.

(5.5.6) Let B be an algebra in $\mathcal{B}_h \cap \mathcal{R}$ with $\overline{\text{sp } B} = \text{sp } B$. Let b be an element of B . Recall from (1.6.2) that a non-zero complex number ξ is in the spectrum $\alpha(b)$ of b iff $\xi^{-1}b$ is not reversible. For any $\sigma \in \text{sp } B$, $\text{sp}(\hat{b}(\sigma))$ will denote the set of eigenvalues of the operator $\hat{b}(\sigma)$ in $B(H_\sigma)$.

LEMMA : If ξ is a non-zero complex number then $\xi \in \alpha(b)$ if and only if $\xi \in \bigcup_{\sigma \in \text{sp } B} \text{sp}(\hat{b}(\sigma))$.

Proof : If ξ is a non-zero element of $\text{sp}(\hat{b}(\sigma))$ for some $\sigma \in \Sigma(G)$ then it follows from (5.5.2) and (5.5.3) that $\xi^{-1}b$ is not reversible. Hence $\xi \in \alpha(b)$.

Conversely let ξ be a non-zero element of $\alpha(b)$. Then $\xi^{-1}b$ is not reversible, and so it follows again from (5.5.2) and (5.5.3) that there exists $\sigma \in \text{sp } B$ such that 1 is an eigenvalue of $\xi^{-1}\hat{b}(\sigma)$. So ξ is an eigenvalue of $\hat{b}(\sigma)$ and $\xi \in \text{sp}(\hat{b}(\sigma))$. $\square$

(5.5.7) Let Δ be an open subset of $\mathbb{C}$ which contains zero. For any Banach algebra A the subsets $\mathcal{C}_A(\Delta)$ and $\mathcal{D}_A(\Delta)$ were defined in (1.6.6) to be

$$\mathcal{C}_A(\Delta) = \{f \in A : \alpha(f) \subseteq \Delta\} \quad (1)$$

and $\mathcal{D}_A(\Delta) = \text{principal component of } \mathcal{C}_A(\Delta)$.

The principal extension of any complex-valued function ϕ , which is analytic on Δ and for which $\phi(0) = 0$, is uniquely defined on $\mathcal{D}_A(\Delta)$.

LEMMA : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ with $\overline{\text{sp } B} = \text{sp } B$. Then b is in $\mathcal{C}_B(\Delta)$ if and only if $\hat{b}(\sigma) \in \mathcal{C}_{B(H_\sigma)}(\Delta)$ for each $\sigma \in \text{sp } B$, and b is in $\mathcal{D}_B(\Delta)$ if and only if $\hat{b}(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta)$ for each $\sigma \in \text{sp } B$.

Proof : The first equivalence is an immediate consequence of (1) and (5.5.6).

Now let $b_1 \in \mathcal{D}_B(\Delta)$ and assume that there exists σ in $\text{sp } B$ such that $\hat{b}_1(\sigma)$ is not an element of $\mathcal{D}_{B(H_\sigma)}(\Delta)$. Then there exists disjoint open sets A_1 and A_2 in $\mathcal{C}_{B(H_\sigma)}(\Delta)$ such that $0 \in A_1$ and $\hat{b}_1(\sigma) \in A_2$. But the map $b \rightarrow \hat{b}(\sigma)$ is continuous from B to $B(H_\sigma)$ by (3.4.5), and so the sets A'_1 and A'_2 , defined by

$$A'_1 = \{b \in B : \hat{b}(\sigma) \in A_1\} \text{ and } A'_2 = \{b \in B : \hat{b}(\sigma) \in A_2\},$$

are open in B . Therefore the sets $A'_1 \cap \mathcal{C}_B(\Delta)$ and $A'_2 \cap \mathcal{C}_B(\Delta)$ are open in $\mathcal{C}_B(\Delta)$; they are clearly disjoint and we have that 0 is an element of $A'_1 \cap \mathcal{C}_B(\Delta)$ and b_1 is in $A'_2 \cap \mathcal{C}_B(\Delta)$. This contradicts our assertion that b_1 is in $\mathcal{D}_B(\Delta)$. Hence $\hat{b}_1(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta)$ for all $\sigma \in \text{sp } B$.

We will now prove that if an element b_1 of B has the property that $\hat{b}_1(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta)$ for each $\sigma \in \text{sp } B$ then b is in $\mathcal{D}_B(\Delta)$.

We first note that because $\mathcal{D}_{B(H_\sigma)}(\Delta) \subseteq \mathcal{C}_{B(H_\sigma)}(\Delta)$ for each σ in $\text{sp } B$, we must have that b belongs to $\mathcal{C}_B(\Delta)$. It is also clear that

$$\xi \hat{b}_1(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta) \text{ for } |\xi| \leq 1. \quad (2)$$

Let $(k_n)_{n \in \mathbf{N}}$ denote the approximate identity of $L^1(G)$, consisting of central trigonometric polynomials, discussed in (1.4.4). Then by (2)

$(k_n * b_1)^\wedge(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta)$ for each $n \in \mathbf{N}$ and $\sigma \in \text{sp } B$ because $\hat{k}_n(\sigma) = \lambda \text{Id}_\sigma$ for some $\lambda \in [0,1)$. If we can show that $k_n * b_1$ is in $\mathcal{D}_B(\Delta)$ for each $n \in \mathbf{N}$, then we must have that b_1 is in $\mathcal{D}_B(\Delta)$ because it is a closed subset of $\mathcal{C}_B(\Delta)$ (see (3.3.10)).

Hence it is sufficient to prove that if b_1 is in $B \cap T(G)$ and if $\hat{b}_1(\sigma) \in \mathcal{D}_{B(H_\sigma)}(\Delta)$ for all $\sigma \in \text{sp } B$, then $b_1 \in \mathcal{D}_B(\Delta)$.

The algebra $B(H_\sigma)$ is finite dimensional and so $\mathcal{D}_{B(H_\sigma)}(\Delta)$ must be arcwise connected for each $\sigma \in \text{sp } B$. Therefore for each $\sigma \in \text{sp } B$ there exists a continuous map γ_σ from $[0,1]$ to $B(H_\sigma)$ such that

$$\gamma_\sigma(0) = 0, \quad \gamma_\sigma(1) = \hat{b}_1(\sigma) \text{ and } \text{im } \gamma_\sigma \subseteq \mathcal{C}_{B(H_\sigma)}(\Delta). \quad (3)$$

We certainly have that b_1 is an element of $\mathcal{D}_B(\Delta)$ if there exists a continuous map γ from $[0,1]$ to B such that

$$\gamma(0) = 0, \quad \gamma(1) = b_1 \text{ and } \text{im } \gamma \subseteq \mathcal{C}_B(\Delta). \quad (4)$$

Consider the map γ defined on $[0,1]$ by

$$\gamma(t)(x) = \sum_{\sigma \in \text{supp } \hat{b}_1} d_\sigma \text{Tr}(\gamma_\sigma(t) U_\sigma(x)^*)$$

for all $x \in G$ and $t \in [0,1]$. For each $t \in [0,1]$ $\gamma(t)$ is in $T(G)_{\text{sp } B}$, and so is in B . It follows from (3) that

$$\bigcup_{\sigma \in \text{supp } \hat{b}_1} \text{sp}(\gamma_\sigma(t)) \subseteq \Delta,$$

and so

$$\bigcup_{\sigma \in \text{supp } \hat{b}_1} \text{sp}(\gamma(t)^{\wedge}(\sigma)) \subseteq \Delta .$$

By (5.5.6) $\alpha(\gamma(t)) \subseteq \Delta$ and so $\text{im } \gamma \subseteq \mathcal{C}_B(\Delta)$. It is easily seen that γ must be continuous and that $\gamma(0) = 0$ and $\gamma(1) = b_1$. Hence (4) holds and b_1 is in $\mathcal{D}_B(\Delta)$. $\square$

(5.5.8) It is perhaps worthwhile at this stage to summarize what we have achieved.

We let Δ be an open subset of $\mathbb{C}$ which contains 0, and ϕ a complex-valued function which is analytic on Δ and for which $\phi(0) = 0$. We let B denote an algebra in $\mathcal{B}_h \cap \mathcal{R}$. The general theory of extensions of ϕ given in §(1.6) ensures that if b is in $\mathcal{D}_B(\Delta)$ then $\phi(b)$ may be uniquely defined in B by putting

$$\phi(b) = -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) D(\xi, b) \xi^{-1} d\xi, \quad (1)$$

where Γ_b denotes an oriented envelope of $\alpha(b)$ with respect to ϕ . Similarly if $\hat{b}(\sigma)$ is in $\mathcal{D}_{B(H_\sigma)}(\Delta)$ for each $\sigma \in \text{sp } B$ then $\phi(\hat{b}(\sigma))$ may be uniquely defined in $B(H_\sigma)$ by putting

$$\phi(\hat{b}(\sigma)) = -\frac{1}{2\pi i} \int_{\Gamma_{\hat{b}(\sigma)}} \phi(\xi) D(\xi, \hat{b}(\sigma)) \xi^{-1} d\xi, \quad (2)$$

where $\Gamma_{\hat{b}(\sigma)}$ denotes an oriented envelope of $\text{sp}(\hat{b}(\sigma))$ with respect to ϕ .

It is easily checked from the definition of an oriented envelope and from (5.5.6), that Γ is an oriented envelope of $\alpha(b)$ if and only if it is an oriented envelope of $\text{sp } \hat{b}(\sigma)$ with respect to ϕ for each $\sigma \in \text{sp } B$.

Hence by (5.5.7) the integral in (1) exists if and only if the integral in (2) exists for each $\sigma \in \text{sp } B$. However, at this stage we are not able to connect them. What we would like is

$$\phi(b)^{\wedge}(\sigma) = \phi(\hat{b}(\sigma)) \quad \text{for each } \sigma \in \text{sp } B.$$

It is the next lemma which will enable us to write this.

(5.5.9) Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and assume that $\overline{\text{sp } B} = \text{sp } B$. Let b be an element of B . We recall from (1.6.2) that $\delta(b)$ is the complement of $\alpha(b)$ in $\mathbb{C}$, and that $D(\xi, b)$ denotes the reverse of $\xi^{-1}b$ if $\xi \neq 0$. See (1.6.2) for the definition of $D(0, b)$ if $0 \in \delta(b)$.

LEMMA : For each $\sigma \in \text{sp } B$, $D(\xi, b)^{\wedge}(\sigma) = D(\xi, \hat{b}(\sigma))$.

Proof : Let $\xi \in \delta(b)$. By (1.6.3)

$$b + \xi D(\xi, b) - b * D(\xi, b) = 0 = b + \xi D(\xi, b) - D(\xi, b) * b.$$

Hence if $\sigma \in \text{sp } B$

$$\hat{b}(\sigma) + \xi D(\xi, b)^{\wedge}(\sigma) - \hat{b}(\sigma) D(\xi, b)^{\wedge}(\sigma) = 0, \quad \text{and}$$

$$\hat{b}(\sigma) + \xi D(\xi, b)^{\wedge}(\sigma) - D(\xi, b)^{\wedge}(\sigma) \hat{b}(\sigma) = 0.$$

But this ensures that $D(\xi, b)^{\wedge}(\sigma) = D(\xi, \hat{b}(\sigma))$ for all $\sigma \in \text{sp } B$. $\square$

(5.5.10) THEOREM : Let $B \in \mathcal{B}_h \cap \mathcal{R}$ and assume that $\overline{\text{sp } B} = \text{sp } B$. Let Δ be an open subset of $\mathbb{C}$ which contains 0, and let ϕ be an analytic complex valued function defined on Δ with $\phi(0) = 0$. Let $b \in \mathcal{D}_B(\Delta)$. Then

$$\phi(b) = -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) D(\xi, b) \xi^{-1} d\xi,$$

where Γ_b is an oriented envelope of $\alpha(b)$ with respect to ϕ , is an element of B . For each $\sigma \in \text{sp } B$

$$\phi(b)^{\wedge}(\sigma) = \phi(\hat{b}(\sigma)) = -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) D(\xi, \hat{b}(\sigma)) \xi^{-1} d\xi.$$

Proof : In view of the arguments given in (5.5.8) it is only necessary to prove the last assertion. By (3.4.5) the map T_σ , defined from B to $B(H_\sigma)$ by $T_\sigma(b) = \hat{b}(\sigma)$, is continuous for each $\sigma \in \text{sp } B$. Hence by (8.14.5) of Edwards [2]

$$T_\sigma(\phi(b)) = -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) T_\sigma(D(\xi, b)) \xi^{-1} d\xi,$$

and so by (5.5.9)

$$\begin{aligned} \phi(b)^\wedge(\sigma) &= -\frac{1}{2\pi i} \int_{\Gamma_b} \phi(\xi) D(\xi, \hat{b}(\sigma)) \xi^{-1} d\xi \\ &= \phi(\hat{b}(\sigma)). \end{aligned}$$

□

Remark : As I have already mentioned, (5.5.4) is in fact a corollary of (5.5.10).

To illustrate the preceding result, suppose the series $\sum_{n \in \mathbf{N}} c_n \xi^n$ is convergent for all $\xi \in \mathbf{C}$ with $|\xi| < \rho$. Denote its sum at the point ξ by $\phi(\xi)$. Let B denote any of the examples 1, 2, 4, 5, 6, 8, 10, 11, 12 or 13 of (3.5.9), and let $b \in \mathcal{A}_B(\Delta)$ where $\Delta = \{\xi \in \mathbf{C} : |\xi| < \rho\}$. Then

$$\phi(b) = \sum_{n \in \mathbf{N}} c_n b^n \quad (\text{convolution product})$$

is in B , and for each $\sigma \in \text{sp } B$

$$\phi(b)^\wedge(\sigma) = \sum_{n \in \mathbf{N}} c_n (\hat{b}(\sigma))^n \quad (\text{composition product}).$$

(5.6) The Representation Theory of Some Homogeneous Algebras

In §(5.4) I proved that the ideal structure of a homogeneous convolution algebra B in $\mathcal{B}_h$ is the same as that of $L^1(G)_g$, where

$$g = \left\{ \bigcap_{b \in B} \ker \hat{b}(\sigma), \sigma \in \Sigma(G) \right\},$$

if G is an arbitrary compact group. In this section I show that if B is symmetric then it shares more the structure of $L^1(G)$. I say that B is symmetric if b^* is in B for each $b \in B$. It is easy to see that if B is symmetric then either

$$\bigcap_{b \in B} \ker \hat{b}(\sigma) = \{0\} \quad \text{or} \quad \bigcap_{b \in B} \ker \hat{b}(\sigma) = H_\sigma$$

for each $\sigma \in \Sigma(G)$, and so $\overline{\text{sp } B} = \text{sp } B$. From (1.3.9) a $*$ -representation R of B in the complex Hilbert space H is a continuous homomorphism of B to $B(H)$ such that $R(b^*) = R(b)^*$ for all $b \in B$. In (38.21) of [HR] the authors show that if $B = L^p(G)$, $1 \leq p < \infty$, and R has the property that for every non-zero $\xi \in H$ there exists $b \in B : R(b)\xi \neq 0$, then there exists a continuous unitary representation U of G such that for all $b \in B$ and $\xi, \zeta \in H$

$$\langle R(b)\xi, \zeta \rangle = \int \langle U(x)\xi, \zeta \rangle b(x) d\lambda(x) .$$

I show that this holds for any symmetric homogeneous convolution algebra (see (5.6.6)). First we require some lemmas.

(5.6.1). Let $\sigma \in \text{sp } B$. Then by (5.4.2) $B_{\{\sigma\}}$ is a closed two-sided ideal in B and so is a closed subalgebra of B . Since $E_\sigma = \{0\}$ it is isomorphic to $B(H_\sigma)$. The restriction of R to $B_{\{\sigma\}}$ is a $*$ -representation of $B_{\{\sigma\}}$ in $B(H)$. Define Q_σ from $B(H_\sigma)$ to $B(H)$ by

$$Q_\sigma(T) = R(x \longrightarrow d_\sigma \text{Tr}(TU_\sigma(x)^*)) \quad \text{for each } T \in B(H_\sigma) ,$$

that is, $Q_\sigma(\hat{b}(\sigma)) = R(b)$.

LEMMA : For each $\sigma \in \text{sp } B$ Q_σ is a $*$ -representation of $B(H_\sigma)$ in $B(H)$.

Proof : This follows immediately from the definition of $\mathcal{Q}_\sigma$ because

$$(b_1 * b_2)^\wedge(\sigma) = \hat{b}_1(\sigma) \hat{b}_2(\sigma) \quad \text{and} \quad \hat{b}_1^*(\sigma) = (\hat{b}_1(\sigma))^* \quad \text{for all } b_1, b_2 \in B. \quad \square$$

(5.6.2) There is a lot known about the $*$ -representations of $B(K)$ in $B(H)$ where K is a finite dimensional Hilbert space (see Naimark [1]).

Of greatest significance to us is theorem 1 of §(17.3) of Naimark [1] from which it follows that

(a) $\mathcal{Q}_\sigma$ is continuous for each $\sigma \in \text{sp } B$,

and §(22.2) of Naimark [1] from which we see that for each $\sigma \in \text{sp } B$

(b) there exists a family of mutually orthogonal $\mathcal{Q}_\sigma$ -invariant closed subspaces K_0^σ, K_i^σ $i \in \mathbf{N}$ of H such that

$$(i) \quad H = \bigoplus_{i \in \mathbf{N}} K_i^\sigma \oplus K_0^\sigma,$$

$$(ii) \quad \dim K_i^\sigma = d_\sigma \quad \text{for each } i \in \mathbf{N}, \quad \text{and}$$

$$(iii) \quad K_0^\sigma = \bigcap_{T \in B(H_\sigma)} \ker \mathcal{Q}_\sigma(T),$$

and there exists a family of unitary operators V_i^σ from H_σ to K_i^σ for each $i \in \mathbf{N}$, such that if P_i^σ denotes the projection of H onto K_i^σ then

$$\mathcal{Q}_\sigma(T) = \sum_{i \in \mathbf{N}} V_i^\sigma T (V_i^\sigma)^{-1} P_i^\sigma \quad \text{in } B(H) \quad \text{for each } T \in B(H_\sigma). \quad (1)$$

(5.6.3) LEMMA : $P_i^\eta P_j^\sigma = 0$ for all $i, j \in \mathbf{N}$ if $\eta \neq \sigma$ and $\eta, \sigma \in \text{sp } B$.

Proof : If $\eta \neq \sigma$ then $B_{\{\eta\}} * B_{\{\sigma\}} = \{0\}$, and so if $b_1 \in B_{\{\eta\}}$ and $b_2 \in B_{\{\sigma\}}$ then $R(b_1 * b_2) = 0$. Hence

$$\mathcal{Q}_\sigma(T_1) \mathcal{Q}_\sigma(T_2) = 0 \quad \text{for all } T_1 \in B(H_\eta) \quad \text{and} \quad T_2 \in B(H_\sigma). \quad (1)$$

Now for each $\sigma \in \text{sp } B$

$$Q_\sigma(\text{Id}_\sigma) = \sum_{i \in \mathbf{N}} v_i^\sigma \text{Id}_\sigma (v_i^\sigma)^{-1} P_i^\sigma = \sum_{i \in \mathbf{N}} P_i^\sigma, \quad (2)$$

and so if $\xi \in K_i^\sigma$, by (1) and (2), $Q_\eta(T_1)\xi = 0$ and so $\xi \in K_0^\eta$.

Therefore $K_i^\sigma \subseteq K_0^\eta$ for each $i \in \mathbf{N}$ and hence $P_i^\eta P_j^\sigma = 0$. $\square$

(5.6.4) We now see how (1) in (5.6.2) can be used to determine a similar expansion for $R(b)$ for each $b \in B$.

PROPOSITION : Let B denote a symmetric homogeneous convolution algebra, then

$$R(b)\xi = \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} P_i^\sigma \xi \text{ for each } \xi \in H \text{ and } b \in B,$$

and

$$\|R(b)\| \leq \|b\|_{p_M} \leq \text{const } \|b\|_B \text{ for all } b \in B.$$

Proof : Assume that b is an element of $P(B)$ ($= B \cap T(G)$). Then b has a representation

$$b(x) = \sum_{\sigma \in \text{sp } B} d_\sigma \text{Tr} (\hat{b}(\sigma) U_\sigma(x)^*) \text{ for all } x \in G, \quad (1)$$

where only a finite number of the terms in the series are non-zero.

Hence,

$$R(b) = \sum_{\sigma \in \text{sp } B} R(x \longrightarrow d_\sigma \text{Tr} (\hat{b}(\sigma) U_\sigma(x)^*)),$$

and so

$$R(b) = \sum_{\sigma \in \text{sp } B} Q_\sigma(\hat{b}(\sigma)). \quad (2)$$

By (1) in (5.6.2), (2) gives

$$R(b) = \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} P_i^\sigma \text{ in } B(H). \quad (3)$$

Now let $\xi \in H$. Then by (3)

$$\begin{aligned} \|R(b)\xi\|_H^2 &\leq \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} \|v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} p_i^\sigma \xi\|_H^2 \\ &\leq \max_{\sigma \in \text{sp } B} \|\hat{b}(\sigma)\|^2 \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} \|p_i^\sigma \xi\|_H^2, \end{aligned}$$

and so by (5.6.3)

$$\|R(b)\xi\|_H \leq \|b\|_{pM} \|\xi\|_H.$$

Hence by (3.4.5)

$$\|R(b)\| \leq \|b\|_{pM} \leq \text{const } \|b\|_B. \quad (4)$$

We now prove that (4) holds for any element b of B . Let $(k_n)_{n \in \mathbf{N}}$ denote an approximate identity of $L^1(G)$ consisting of trigonometric polynomials. By (3.3.10)

$$\lim_{n \rightarrow \infty} \|b - k_n * b\|_B = 0. \quad (5)$$

By (21.22) of [HR] R is strongly continuous, and

$$\|R(b)\| \leq \|b\|_B \text{ for all } b \in B. \quad (6)$$

Then by (5) and (6)

$$\lim_{n \rightarrow \infty} \|R(b) - R(k_n * b)\| = 0. \quad (7)$$

It follows from (3.3.9) that $k_n * b$ is in $B \cap T(G)$ for each $n \in \mathbf{N}$ and so by (3)

$$R(k_n * b) = \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (k_n * b)^\wedge(\sigma) (v_i^\sigma)^{-1} p_i^\sigma. \quad (8)$$

For each $b \in B$, $\sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} p_i^\sigma$ is an operator in $B(H)$. To see this, let $\xi \in H$. Then

$$\begin{aligned} \left\| \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} p_i^\sigma \xi \right\|_H^2 &\leq \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} \|v_i^\sigma \hat{b}(\sigma) (v_i^\sigma)^{-1} p_i^\sigma \xi\|_H^2 \\ &\leq \|b\|_{p_M}^2 \|\xi\|_H^2 . \end{aligned}$$

In view of (7) and (8) $R(b)$ is given by (3) if

$$\lim_{n \rightarrow \infty} \left\| \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (\hat{b}(\sigma) - (k_n * b)^\wedge(\sigma)) (v_i^\sigma)^{-1} p_i^\sigma \right\| = 0 . \quad (9)$$

But this follows immediately from (3.3.10) and (3.4.5) because

$$\left\| \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (\hat{b}(\sigma) - (k_n * b)^\wedge(\sigma)) (v_i^\sigma)^{-1} p_i^\sigma \xi \right\| \leq \|b - k_n * b\|_{p_M} \|\xi\|_H .$$

It is easily seen that (4) must hold for all $b \in B$. □

(5.6.5) As mentioned in the introduction, our intention is to associate with R a unitary representation U of G such that

$$\langle \zeta, R(b)\xi \rangle = b \left[x \longrightarrow \langle \zeta, U(x)\xi \rangle \right] \text{ for all } \zeta, \xi \in H, b \in B .$$

(If $b \in M(G)$ then this condition may be reduced to one similar to that in (38.21) of [HR].)

Take the notation (5.6.2). Let $x \in G$. Then

$\sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (U_\sigma(x)) (v_i^\sigma)^{-1} p_i^\sigma$ is a bounded operator on H because

$$\left\| \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (U_\sigma(x)) (v_i^\sigma)^{-1} p_i^\sigma \xi \right\|_H^2 \leq \sup_{\sigma \in \text{sp } B} \|U_\sigma(x)\|^2 \|\xi\|_H^2 \leq \|\xi\|_H^2$$

for each $\xi \in H$. Define the operator $U(x)$ in $B(H)$ by

$$U(x) = \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} v_i^\sigma (U_\sigma(x)) (v_i^\sigma)^{-1} p_i^\sigma .$$

LEMMA : The map U from G to $B(H)$ defined by $U : x \longrightarrow U(x)$ is a unitary representation of G in $B(H)$.

Proof : Clearly for each $x \in G$, $U(x)$ is a unitary operator. It is also easy to see that $U(xy) = U(x) U(y)$ for all $x, y \in G$. To prove that U is a unitary representation it only remains to show that U is continuous from G to $B(H)$.

For each $i \in \mathbf{N}$ and $\sigma \in \text{sp } B$ the map $x \longrightarrow V_i^\sigma U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma$ is continuous from G to $B(H)$ since

$$\|V_i^\sigma (U_\sigma(x) - U_\sigma(y)) (V_i^\sigma)^{-1} P_i^\sigma\| \leq \|U_\sigma(x) - U_\sigma(y)\| \text{ for all } x, y \in G. \quad (1)$$

Because U is unitary it is sufficient to prove that it is weakly continuous from G to $B(H)$. By (1) this is so if the series

$$\sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} V_i^\sigma U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma \xi$$

is uniformly convergent on G for each $\xi \in H$. This is easily seen to hold because

$$\|V_i^\sigma (U_\sigma(x)) (V_i^\sigma)^{-1} P_i^\sigma \xi\|_H \leq \|P_i^\sigma \xi\|_H$$

for all $i \in \mathbf{N}$ and $\sigma \in \text{sp } B$, and

$$\sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} \|P_i^\sigma \xi\|_H^2 \leq \|\xi\|_H^2. \quad \square$$

(5.6.6) PROPOSITION : Let B be a symmetric algebra in $\mathcal{B}_H \cap \mathcal{R}$ and let $b \in T(G) \cap B$. Then $\int b(x) U(x) \xi \, d\lambda(x)$ is an element of H for each ξ in H , and

$$R(b)\xi = \int b(x) U(x) \xi \, d\lambda(x).$$

Hence $\langle \zeta, R(b)\xi \rangle = \int \langle \zeta, U(x)\xi \rangle b(x) \, d\lambda(x)$ for all $\xi, \zeta \in H$.

Proof : By (8.15.2) and (8.14.14) of Edwards [2], $\int b(x) U(x) \xi \, d\lambda(x)$ is an element of H if the map g from G to H , defined by

$$g(x) = b(x) U(x)\xi \quad \text{for all } x \in G,$$

is measurable and $\int \|g(x)\|_H d\lambda(x)$ is finite. It follows from (5.6.5) that g is continuous. The second condition holds because

$$\begin{aligned} \int \|g(x)\|_H d\lambda(x) &\leq \int |b(x)| \|U(x)\xi\|_H d\lambda(x) \\ &\leq \sup_{x \in G} \|U(x)\xi\|_H \|b\|_1, \end{aligned}$$

which is finite since G is compact and U is continuous.

By (5.6.4)

$$\begin{aligned} R(b) &= \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} V_i^\sigma \hat{b}(\sigma) (V_i^\sigma)^{-1} P_i^\sigma \xi \\ &= \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} V_i^\sigma \left(\int b(x) U_\sigma(x) d\lambda(x) \right) (V_i^\sigma)^{-1} P_i^\sigma \xi. \end{aligned} \quad (1)$$

Consider the series

$$\sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} V_i^\sigma b(x) U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma \xi \quad (2)$$

in H . Each of its terms is a continuous function from G to H .

Hence we can interchange the order of summation and integration in (1)

if the series in (2) is uniformly convergent on G . Then it will follow that

$$\begin{aligned} R(b)\xi &= \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} \left(\int V_i^\sigma b(x) U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma \xi d\lambda(x) \right) \\ &= \int \sum_{\sigma \in \text{sp } B} \sum_{i \in \mathbf{N}} V_i^\sigma b(x) U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma d\lambda(x) \\ &= \int b(x) U(x)\xi d\lambda(x). \end{aligned}$$

The series in (2) is uniformly convergent because

$$\|V_i^\sigma b(x) U_\sigma(x) (V_i^\sigma)^{-1} P_i^\sigma \xi\|_H \leq |b(x)| \|P_i^\sigma \xi\|_H \leq \|b\|_\infty \|P_i^\sigma \xi\|_H$$

for each $\sigma \in \text{sp } B$ and $i \in \mathbf{N}$. The last assertion is actually a statement of the definition of the integral in H .

(5.6.7) THEOREM : Let B be a symmetric homogeneous algebra in $\mathcal{B}_h$ and let R be a $*$ -representation of B in the complex Hilbert space H . Let $(k_n)_{n \in \mathbf{N}}$ be an approximate identity of $L^1(G)$ consisting of trigonometric polynomials, and let U be defined as in (5.6.5). Then

$$R(b)\xi = \lim_{n \rightarrow \infty} \int k_n * b(x) U(x)\xi \, d\lambda(x)$$

for all $\xi \in H$ and $b \in B$, and

$$\langle \zeta, R(b)\xi \rangle = b \left[x \rightarrow \langle \zeta, U(x)\xi \rangle \right]$$

for all $\zeta, \xi \in H$ and b in B .

Proof : By (3.3.10)

$$\lim_{n \rightarrow \infty} \|b - k_n * b\|_B = 0. \quad (1)$$

Hence, as R is a continuous operator from B to $B(H)$,

$$\lim_{n \rightarrow \infty} \|R(b) - R(k_n * b)\| = 0. \quad (2)$$

By (3.3.9) $k_n * b$ is in $P(B) = B \cap T(G)$ for each $n \in \mathbf{N}$, and so by

(5.6.6)

$$R(k_n * b)(\xi) = \int k_n * b(x) U(x)\xi \, d\lambda(x)$$

for all $\xi \in H$ and $n \in \mathbf{N}$. Therefore, by (2) for each $\xi \in H$ and $b \in B$,

$$R(b)\xi = \lim_{n \rightarrow \infty} \int k_n * b(x) U(x)\xi \, d\lambda(x).$$

The second assertion follows from (5.6.6) and (1), because we may write

$$\int \langle \zeta, U(x)\xi \rangle b(x) \, d\lambda(x) = b \left[x \rightarrow \langle \zeta, U(x)\xi \rangle \right]$$

if b is then regarded as a continuous linear functional on $A(G)$. $\square$

As I have already noted, if $B = L^p(G)$, $1 \leq p < \infty$, then (5.6.7) reduces to (38.21) of [HR]. However (5.6.7) may be applied to, for example, any of the algebras $A(G)_F$; $C(G)_F$; $L^p(G)_F$ $1 \leq p < \infty$; $U^p(G)_F$ $1 \leq p < \infty$; $PF(G)_F$; $V_{F_1, F_2}(G)$, where F, F_1 and F_2 are subsets of $\Sigma(G)$. See (22.3) of [HR] for another result of this type.

(5.6.8) We recall from (1.3.2) that a representation U of G is irreducible iff the only subspaces of the representation space H which are invariant under the action of each of the operators $U(x)$ are $\{0\}$ and H . A similar definition applies for any $*$ -representation of a symmetric homogeneous algebra B . The construction in (5.6.7) preserves irreducibility.

PROPOSITION : Let B be a symmetric homogeneous convolution algebra in $\mathcal{B}_h$ and let R be a $*$ -representation of B in the Hilbert space H . Let U denote the representation of G constructed in (5.6.5). Then R is irreducible if and only if U is irreducible.

Proof : Assume R is reducible. For each finite subset F of $\text{sp } B$ define ω_F from G to $\mathbb{C}$ by

$$\omega_F(x) = \sum_{\sigma \in F} d_\sigma \text{Tr}(U_\sigma(x)^*) \quad \text{for each } x \in G.$$

Then ω_F is in $T(G)_{\text{sp } B}$, and so is in B . It follows from (5.6.6) that the limit over all such finite subsets F of

$$\|U(x)\xi - R(\omega_F)\xi\|_H = 0. \quad \text{There exists a closed subspace } K \text{ of } H$$

such that $R(b)K \subseteq K$ for all $b \in B$. In particular,

$$R(\omega_F)K \subseteq K \quad \text{for all } x \in G \text{ and all finite subsets } F \text{ of } \Sigma(G),$$

and so $U(x)K \subseteq K$ for all $x \in G$. Hence U is reducible.

Conversely, assume U is reducible. Then there exists a closed subspace K of H such that $U(x)K \subseteq K$ for all $x \in G$. Hence if P_K denotes the projection from H to K , we have $P_K U(x) = U(x) P_K$ for all $x \in G$. By (5.6.7) for all $\xi \in H$ and $b \in B$,

$$R(b)\xi = \lim_{n \rightarrow \infty} \int (k_n * b)(x) U(x)\xi \, d\lambda(x) \quad \text{in } H.$$

$((k_n)_{n \in \mathbb{N}}$ is any approximate identity of $L^1(G)$ consisting of trigonometric polynomials.) Hence

$$\begin{aligned} P_K R(b)\xi &= \lim_{n \rightarrow \infty} P_K \int (k_n * b)(x) U(x)\xi \, d\lambda(x) \quad \text{in } H \\ &= \lim_{n \rightarrow \infty} \int (k_n * b)(x) P_K U(x)\xi \, d\lambda(x) \quad \text{in } H \\ &= \lim_{n \rightarrow \infty} \int (k_n * b)(x) U(x) P_K \xi \, d\lambda(x) \quad \text{in } H \\ &= R(b) P_K \xi, \end{aligned}$$

and so R is reducible. □

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subjected to more than one interpretation, and which are used throughout the thesis. If terms or symbols are undefined then they are to be given their usual meaning, otherwise the definition is found on the page indicated.

$A(n)$	14	$\mathcal{G}_p(A)$	24
$a(b), a(b')$	21, 22	$\mathcal{G}(T) (A^{-1})^* P^0$	101
B	88	X, P_X	50
B_n	89	$I_n, I_n^{(1)}$	77
B^*	77	Γ	12
B^T	77	$\mathcal{K}_0$	7
B_F	14, 15	$\mathcal{K}_1$	47
B_g	87, 90, 104	$\mathcal{K}_0$	13
B_0	29	λ (identity of a compact group)	
$B(n)$	6	$\mathcal{B}$ (identity of an abelian group)	
$B(Y_0, Z_0), B(Y_0)$	142	$I(t)$	139
$B_F(Y_0, Z_0), B_F(Y_0)$	143	$\mathcal{K} : \mathcal{K} \rightarrow \mathcal{K}(t)$	
$C(a)$	7	$L^p(a), 1 \leq p < \infty$	4
$C^p(a)$	40	L/L_X	75, 76, 80
c.l.t.t.	75	$L, \mathcal{L}$	75
$C_p(A)$	23	$\mathcal{L}, \mathcal{L}_X$	76
$\delta(b)$	21	λ_0, λ	6
d_0	7	Δ	129
$D(\xi, b)$	20	$\mathcal{E}_0$	8
$\mathcal{D}_g(A)$	23	$\mathcal{K}(a)$	3
$\mathcal{D}(T), \mathcal{D}^p(T), 1 \leq p < \infty$		$\mathcal{H}_1$	102
$\mathcal{D}^p(X), \mathcal{D}^{\infty}(X)$		$\mathcal{H}(\text{form})$	119

INDEX OF SYMBOLS

I give a list of symbols which may not be in common use, or which may be subjected to more than one interpretation, and which are used throughout the thesis. If terms or symbols are undefined then they are to be given their usual meaning, otherwise the definition is found on the page indicated.

$A(G)$	14	$\mathcal{E}_B(\Delta)$	24
$\alpha(b), \sigma(b)$	21, 22	$\underline{E}(F) (A^{-1})^*, p'$	101
$\mathcal{B}$	88	F_X, F_X^*	58
$\mathcal{B}_h$	89	$\Gamma_M, \Gamma_M(b)$	77
B''	77	Γ	23
$B^\#$	77	H_σ	7
B_F	14, 15	H_X	67
B_g	87, 90, 104	Id_σ	10
B_ω	29	1 (identity of a compact group)	
$B(H)$	6	0 (identity of an abelian group)	
$B[Y_\sigma, Z_\sigma], B[Y_\sigma]$	142	$J(E)$	139
$B_F[Y_\sigma, Z_\sigma], B_F[Y_\sigma]$	143	$\bigvee k : x \longrightarrow k(x^{-1})$	
$C(G)$	2	$L^p(G), 1 \leq p \leq \infty$	4
$C^F(G)$	40	L, L_x	75, 76, 86
$c.l.t.i.$	75	$L.h$	79
$\mathcal{C}_B(\Delta)$	23	$\mathcal{L}, \mathcal{L}_x$	76
$\delta(b)$	21	λ_G, λ	4
d_σ	7	$\wedge$	125
$D(\xi, b)$	22	Π_U	5
$\mathcal{D}_B(\Delta)$	23	$M(G)$	2
$\underline{E}(F); \underline{E}^p(F) \ 1 \leq p \leq \infty$	12	M_{ij}^σ	109
$\underline{E}^0(F); \underline{E}^{\infty}(F)$		N (norm)	118

Ω_σ	108	χ_u, χ_σ	8
$PM(G), PM(G)^{A,p}$	14, 92	$*$ (involution)	2, 3
P (property)	61	$*$ (convolution)	5
$P(B)$	81	$*$ (product)	82
$\mathcal{P}(b), \mathcal{P}(s)$	119, 121	$\ \ $	6, 91
Π_{ij}^σ	113	$\ \ _{B''}$	77
ψ	126	$\ \ _{A,p}$	91
Q_σ	176	$\ \ _{(A^{-1})^*, p'}$	101
$\otimes$ (tensor product)		$\ \ _{F_1, F_2}$	45
$\hat{\otimes}$ (projective tensor product)		$\ \ _\omega$	31, 33
$\langle \cdot \rangle$	5, 13, 15, 128	$\ \ _\wedge$	125
$\langle\langle \cdot \rangle\rangle$	101	$\ \ _{\phi_p}$ $1 \leq p < \infty$	12
$\Sigma(G)$	7	$\ \ _1$	3, 4, 12
$\mathcal{G}$	87	$\ \ _p$ $1 < p < \infty$	4, 12
$sp B, \overline{sp B}, \underline{sp B}$	143	$\ \ _\infty$	2, 4, 12
$T(G)_{\{\sigma\}}, T(G)$	8		
Tr (trace)			
type C	90		
Θ	100		
$\mathbb{T}_Y$	152		
u_σ, u	6, 7, 180		
$U^p(G)$ $1 \leq p < \infty$	78		
u_{ij}^σ	108		
$V(G)$	52		
$V_{F_1, F_2}^{(G)}$	45		
V_b	81		
ω (weight function)			
$x^f, x^\mu, x^S, f_x, \mu_x, S_x$	3, 15		
$X(E, \varepsilon)$	34		
$Z(M(G))$	15		