Proceedings of the

CENTRE FOR
MATHEMATICAL ANALYSIS

AUSTRALIAN NATIONAL UNIVERSITY

Volume 3, 1983

Lectures on Geometric Measure Theory

Leon Simon

PROCEEDINGS

OF

THE CENTRE FOR MATHEMATICAL ANALYSIS

AUSTRALIAN NATIONAL UNIVERSITY

VOLUME 3, 1983

LECTURES ON GEOMETRIC MEASURE THEORY

LEON SIMON

First published in Australia 1984

National Library of Australia
Cataloguing-in-Publication entry

Simon, L.M. (Leon Melvin), 1945- .
Lectures on geometric measure theory.

ISBN 0 86784 429 9.

1. Geometric measure theory.
I. Australian National University.
Centre for Mathematical Analysis.
II Title.

516'.1

CONTENTS

INTRODUCTION

These notes grew out of lectures given by the author at the Institut für Angewandte Mathematik, Heidelberg University, and at the Centre for Mathematical Analysis, Australian National University.

A central aim was to give the basic ideas of Geometric Measure Theory in a style readily accessible to analysts. I have tried to keep the notes as brief as possible, subject to the constraint of covering the really important and central ideas. There have of course been omissions; in an expanded version of these notes (which I hope to write in the near future), topics which would obviously have a high priority for inclusion are the theory of flat chains, further applications of G.M.T. to geometric variational problems, P.D.E. aspects of the theory, and boundary regularity theory.

I am indebted to many mathematicians for helpful conversations concerning these notes. In particular C. Gerhardt for his invitation to lecture on this material at Heidelberg, K. Ecker (who read thoroughly an earlier draft of the first few chapters), R. Hardt for many helpful conversations over a number of years. Most especially I want to thank J. Hutchinson for numerous constructive and enlightening conversations.

As far as *content* of these notes is concerned, I have drawn heavily from the standard references Federer [FH1] and Allard [AW1], although the reader will see that the presentation and point of view often differs from these references.

An outline of the notes is as follows. Chapter 1 consists of basic measure theory (from the Caratheodory viewpoint of outer measure). Most of the results are by now quite classical. For a more extensive treatment of some of the topics covered, and for some bibliographical remarks, the reader is referred to Chapter 2 of Federer's book [FH1], which was in any case the basic source used for most of the material of Chapter 1.

Chapter 2 develops further basic preliminaries from analysis. In preparing the discussion of the area and co-area formulae we found Hardt's Melbourne notes [HR1] particularly useful. There is only a short section on BV functions, but it comfortably suffices for all the later applications. We found Giusti's Canberra notes [G] useful in preparing this material (especially in relation to the later material on sets of locally finite perimeter).

Chapter 3 is the first specialized chapter, and gives a concise treatment of the most important aspects of countably n-rectifiable sets. There are much more general results in Federer's book [FH1], but hopefully the reader will find the discussion here suitable for most applications, and a good starting point for any extensions which might occasionally be needed.

In Chapters 4, 5 we develop the basic theory of rectifiable varifolds and prove Allard's regularity theorem. ([AW1].) Our treatment here is formally much more concrete than Allard's; in fact the entire argument is given in the concrete setting of rectifiable varifolds, considered as countably n-rectifiable sets equipped with a locally $\mathcal{H}^n$-integrable multiplicity function. Hopefully this will make it easier for the reader to see the important ideas involved in the regularity theorem (and in the preliminary theory involving monotonicity formulae etc.).

Chapter 6 contains the basic theory of currents, including integer multiplicity rectifiable currents, but not including a discussion of flat chains. The basic references for this chapter are the original paper of Federer and Fleming [FF] and Federer's book [FH1], although in a number of respects our treatment is a little different from these references.

In Chapter 7 there is a discussion of the basic theory of minimizing currents. The theorem 36.4, the proof of which is more or less standard, does not seem to appear elsewhere in the literature. In the last section we develop the regularity theory for codimension 1 minimizing currents. A feature of this section is that we treat the case when the currents in question are actually codimension 1 in some smooth submanifold. (This was of course generally known, but does not explicitly appear elsewhere in the literature.)

Finally in Chapter 8 we describe Allard's theory of general varifolds, which originally appeared in [AW1]. (Important aspects of the theory of varifolds had earlier been developed by Almgren [A3].)

In conclusion I want to express my sincere gratitude to Dorothy Nash, who did such a superb job in typing these lectures from a manuscript that was often messy and which frequently had to be corrected.

NOTATION

The following notation is frequently used without explanation in the text.

$\bar{A}$ = closure of a subset A (usually in a Euclidean space)

$B \sim A$ = $\{x \in B : x \notin A\}$

χ_A = characteristic function of A

$\underline{\underline{1}}_A$ = identity map $A \to A$

$\mathcal{L}^n$ = Lebesgue measure in $\mathbb{R}^n$

$B_\rho(x)$ = open(*) ball with centre x radius ρ

$\bar{B}_\rho(x)$ = closed ball ;

(If we wish to emphasize that these balls are in the balls in $\mathbb{R}^P$, we write $B^P_\rho(x)$, $\bar{B}^P_\rho(x)$.)

ω_n = $\mathcal{L}^n(B_1(0))$

$\eta_{x,\lambda} : \mathbb{R}^P \to \mathbb{R}^P$

(for $\lambda > 0$, $x \in \mathbb{R}^P$) is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$;
thus $\eta_{x,1}$ is translation $y \mapsto y-x$, and $\eta_{0,\lambda}$ is homothety $y \mapsto \lambda^{-1}y$)

$W \subset\subset U$ (U an open subset of $\mathbb{R}^P$)

shall always mean that W is *open* and $\bar{W}$ is a compact subset of U .

$C^k(U,V)$ (U,V open subsets of finite dimensional vector spaces) denotes the space of C^k maps from U into V .

$C^k_c(U,V) = \{\phi \in C^k(U,V) : \phi$ has compact support$\}$.

(*) In Chapter 1 $B_\rho(x)$ denotes the *closed* ball.

ERRATA

Please send further corrections/comments to:
Leon Simon
Centre for Mathematical Analysis
Australian National University, GPO Box 4,
Canberra ACT 2601 AUSTRALIA.

p17 line 13 H is a *finite dimensional* Hilbert space

p21 line 9 [RH] should be [Roy]

p33 line 11 η_k converges uniformly to zero on bounded subsets of A.

p51 line -1 "if 9.3 holds" should be "if $\int_M \operatorname{div}_M X = 0$"

p65 line -1 $\delta/2$ should be $\delta/4$.

p70 line -9 "ordered by inclusion" should be "ordered by the relation $R < S \Leftrightarrow R \subset S$ and $\mathcal{H}^n(S \sim R) > 0$".

p87 Note that the Remark 17.9(1) refers to the case $\underline{\underline{H}} \in L^p_{\text{loc}}(\mu)$, $p > n$.

p96 line -5 Chapter 10 should be Chapter 8

p127 line 8 $\delta^{3/4}$ in place of $\delta^{1/2}$

line 10 $\delta^{1/2}$ in place of $\delta^{1/4}$

line -5 $\delta^{1/4}$ in place of $\delta^{1/8}$.

p130 line -7 25.1 should be $dx^j(f) = e_j \cdot f$, $f \in C^\infty(U;\mathbb{R}^P)$.

p140 line -8 σ^{-n} should be σ^{-P}.

p143 In Remark 26.28 we must justify that θ_{σ_k} is bounded in $L^1(B)$ for each ball $B \subset\subset U$. Indeed by 6.4 and $\underline{\underline{M}}_B(\partial T) < \infty$, there are constants c_k such that $\theta_{\sigma_k} - c_k$ is bounded in $L^1(B)$, and hence $T_{\sigma_k} - c_k[\![B]\!]$ has bounded mass in B. But $T_{\sigma_k} \rightharpoonup T$ and hence $\{c_k\}$ is bounded.

p149 line 9 $P = n+1$ should be $P = n$.

p171 line -6 $(\partial T)_\rho = (\partial T) \llcorner L_{k-1}(a;\rho)$.

p176 In (31), Q should be (∂Q) in both terms on right side.

p215 line 13 (*) should be $T = \partial[\![E]\!]$

p169 line 1 $\neq$ line 2 unless $k = 2$. But with $L = L_{k-2}(a_F)$, $\operatorname{dist}(y,L)/\operatorname{dist}(x,p_F^{-1}(L)) = |y-a|/|x-a|$ by similarity, and $p_F^{-1}(L) \subset L_{k-1}(a)$, so $|\bar{D}\tilde{\psi}(y)| \le c|x-a|/\operatorname{dist}(x,L_{k-1}(a))$ as required.

p191 In line -2, replace T by $T_{j'}$, where $\{j'\} \subset \{j\}$ and $\rho>0$ are chosen so that (i) $\eta_{x,\lambda_j\#}T_{j'} \rightharpoonup \theta(x)[\![T_xM]\!]$ (O.K. by (10) and the fact that $T_j \rightharpoonup T$), and so that (ii) lines -4, -5 remain valid with $T_{j'}$ in place of T (O.K. by 28.5(1) and a selection argument as in 10.7(2)).

CHAPTER 1

PRELIMINARY MEASURE THEORY

In this chapter we briefly review the basic theory of outer measure (with Caratheodory's definition of measurability). Hausdorff measure is discussed, including the main results concerning n-dimensional densities and the way in which they relate more general measures to Hausdorff measures. The final section of the chapter gives the basic theory of Radon measures (including the Riesz representation theorem and the differentiation theory).

Throughout the chapter X will denote a metric space with metric d. In the last section X satisfies the additional requirements of being locally compact and separable.

§1. BASIC NOTIONS

Recall that an *outer measure* (henceforth simply called a *measure*) on X is a *monotone subadditive* function $\mu : 2^X \to [0,\infty]$ with $\mu(\phi) = 0$. Thus $\mu(\phi) = 0$ and

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j) \quad \text{whenever} \quad A \subset \bigcup_{j=1}^{\infty} A_j$$

with $A, A_1, A_2, \ldots$ any countable collection of subsets of X. Of course this in particular implies $\mu(A) \le \mu(B)$ whenever $A \subset B$.

We adopt Caratheodory's notion of *measurability*:

A subset $A \subset X$ is said to be μ-measurable if

$$\mu(S) = \mu(S\sim A) + \mu(S\cap A)$$

for each subset $S \subset X$. Of course by subadditivity of μ we only actually have to check that

1.1 $$\mu(S) \geq \mu(S\sim A) + \mu(S\cap A)$$

for each subset $S \subset X$ with $\mu(S) < \infty$. One readily checks (see for example [M] or [FH1]) that the collection $\mathcal{S}$ of all measurable subsets forms a *σ-algebra*; that is

(1) ϕ, $X \in \mathcal{S}$

(2) If $A_1, A_2, \ldots \in \mathcal{S}$ then $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j \in \mathcal{S}$

(3) If $A \in \mathcal{S}$ then $X \sim A \in \mathcal{S}$.

Furthermore all sets of μ-measure zero are trivially μ-measurable (because 1.1 holds trivially in case $\mu(A) = 0$). If $A_1, A_2, \ldots$ are pairwise disjoint μ-measurable subsets of X, then $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$. If $A_1 \subset A_2 \subset \ldots.$ are μ-measurable then $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$ and if $A_1 \supset A_2 \supset \ldots.$ are μ-measurable then $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i=1}^{\infty} A_i)$ provided $\mu(A_1) < \infty$.

A measure μ on X is said to be *regular* if for each subset $A \subset X$ there is a μ-measurable subset $B \supset A$ with $\mu(B) = \mu(A)$. One readily checks that for a regular measure μ the relation $\lim_{i\to\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$ is valid provided $A_i \subset A_{i+1}$ $\forall i$, *even if the* A_i *are not assumed to be* *μ-measurable.*

A measure X is said to be *Borel-regular* if all Borel sets are μ-measurable and if for each subset $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. (Notice that this does *not* imply $\mu(B\sim A) = 0$ unless A

is μ-measurable and $\mu(A) < \infty$.)

Given any subset $A \subset X$ and any measure μ on X, we can define a new measure $\mu \llcorner A$ on X by

$$(\mu \llcorner A)(Z) = \mu(A \cap Z) \ , \ Z \subset X \ .$$

One readily checks that all μ-measurable subsets are also $(\mu \llcorner A)$-measurable (even if A is *not* μ-measurable). It is also easy to check that $\mu \llcorner A$ is Borel regular whenever μ is, provided A is μ-measurable.

The following theorem, due to Caratheodory, is particularly useful. In the statement we use the notation

$$d(A,B) = \operatorname{dist}(A,B) = \inf\{d(a,b) : a \in A \ , \ b \in B\} \ .$$

1.2 THEOREM (Caratheodory's Criterion) *If* μ *is a measure on* X *such that*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever A, B *are subsets of* X *with* $d(A,B) > 0$, *then all Borel sets are* μ*-measurable.*

Proof Since the measurable sets form a σ-algebra, it is enough to prove that all closed sets are μ-measurable, so that by 1.1 we have only to check that

$$(1) \qquad \mu(S) \geq \mu(S \sim C) + \mu(S \cap C)$$

whenever $\mu(S) < \infty$ and C is closed.

Let $C_j = \{x \in X : \operatorname{dist}(x,C) \leq 1/j\}$. Then $d(S \sim C_j, S \cap C) > 0$, hence

$$\mu(S) \geq \mu((S \sim C_j) \cup (S \cap C)) = \mu(S \sim C_j) + \mu(S \cap C) \ ,$$

and we will have (1) if we can show $\lim_{j\to\infty} \mu(S\sim C_j) = \mu(S\sim C)$. To check this, note that since C is closed we can write

$$S \sim C = (S\sim C_j) \cup \left(\bigcup_{k=j}^{\infty} R_k\right)$$

where $R_k = \{x \in S : \frac{1}{k+1} < \text{dist}(x,C) \le \frac{1}{k}\}$. But then by subadditivity of μ we have

$$\mu(S\sim C_j) \le \mu(S\sim C) \le \mu(S\sim C_j) + \sum_{k=j}^{\infty} \mu(R_k) ,$$

and hence we will have $\lim_{j\to\infty} \mu(S\sim C_j) = \mu(S\sim C)$ as required, provided only that $\sum_{k=1}^{\infty} \mu(R_k) < \infty$.

To check this we note that $d(R_i,R_j) > 0$ if $j \ge i+2$, and hence by the hypothesis of the theorem and induction on N we have for each integer $N \ge 1$

$$\sum_{k=1}^{N} \mu(R_{2k}) = \mu\left(\bigcup_{k=1}^{N} R_{2k}\right) \le \mu(S) < \infty$$

and

$$\sum_{k=1}^{N} \mu(R_{2k-1}) = \mu\left(\bigcup_{k=1}^{N} R_{2k-1}\right) \le \mu(S) < \infty .$$

The following regularity properties of Borel-regular measures are of basic importance.

1.3 THEOREM *Suppose* μ *is a Borel-regular measure on* X *and* $X = \bigcup_{j=1}^{\infty} V_j$, *where* $\mu(V_j) < \infty$ *and* V_j *is open for each* $j = 1,2,\ldots$. *Then*

(1) $$\mu(A) = \inf_{U \text{ open}, U\supset A} \mu(U)$$

for each subset $A \subset X$, *and*

(2) $$\mu(A) = \sup_{C \text{ closed}, C\subset A} \mu(C)$$

for each μ*-measurable subset* $A \subset X$.

1.4 REMARK In case the metric space X is *locally compact* and *separable*, the condition $X = \bigcup_{j=1}^{\infty} V_j$ with V_j open and $\mu(V_j) < \infty$ is *automatically satisfied provided* $\mu(K) < \infty$ *for each compact* K . Furthermore in this case we have from 1.3(2) that

$$\mu(A) = \sup_{K \text{ compact}, K\subset A} \mu(K)$$

for each μ-measurable subset $A \subset X$ with $\mu(A) < \infty$, because under the above conditions on X any closed set C can be written $C = \bigcup_{i=1}^{\infty} K_i$, compact.

Proof of Theorem 1.3 First note that 1.3(2) follows quite easily from 1.3(1). To prove 1.3(1), we assume first that $\mu(X) < \infty$. By Borel regularity of the measure μ , it is enough to prove (1) in case A is a Borel set. Then let

$$\mathcal{A} = \{\text{Borel sets } A : 1.3(1) \text{ holds}\} .$$

Trivially $\mathcal{A}$ contains all open sets and one readily checks that $\mathcal{A}$ is closed under both countable unions and intersections; in particular $\mathcal{A}$ must also contain the *closed sets*, because any closed set in X can be written as a countable intersection of open sets. Thus if we let $\tilde{\mathcal{A}} = \{A \in \mathcal{A} : X\sim A \in \mathcal{A}\}$ then $\tilde{\mathcal{A}}$ is a σ-algebra containing all the closed sets, and hence $\tilde{\mathcal{A}}$ contains all the Borel sets. Thus $\mathcal{A}$ contains all the Borel sets and 1.3(1) is proved in case $\mu(X) < \infty$.

In the general case $(\mu(X) \le \infty)$ it still suffices to prove 1.3(1) when A is a Borel set. For each $j = 1,2,\ldots$ apply the previous case to the measure $\mu \llcorner V_j$, $j = 1,2,\ldots$. Then for each $\varepsilon > 0$ we can select an open

$U_j \supset A$ such that

$$\mu(U_j \cap V_j \sim A \cap V_j) < \varepsilon/2^j ,$$

so that

$$\mu(U_j \cap V_j \sim A) < \varepsilon/2^j ,$$

and hence (summing over j)

$$\mu\Big(\bigcup_{j=1}^{\infty} (U_j \cap V_j) \sim A\Big) < \varepsilon .$$

Since $\bigcup_{j=1}^{\infty} (U_j \cap V_j)$ is open and contains A, this completes the proof.

§2. HAUSDORFF MEASURE

If m is a non-negative real number, we define m-dimensional Hausdorff measure by

2.1 $$H^m(A) = \lim_{\delta \downarrow 0} H^m_\delta(A) , \quad A \subset X ,$$

where for each $\delta > 0$, $H^m_\delta(A)$ is defined by

2.2 $$H^m_\delta(A) = \inf \sum_{j=1}^{\infty} \omega_m \left(\frac{\operatorname{diam} C_j}{2}\right)^m$$

(ω_m = volume of unit ball in $\mathbb{R}^m$ in case m is a positive integer; ω_m any convenient constant > 0 otherwise), where the inf is taken over all countable collections $C_1, C_2, \ldots$ of subsets of X such that $\operatorname{diam} C_j < \delta$ and $A \subset \bigcup_{j=1}^{\infty} C_j$.

Notice that the limit in 2.1 always exists (although it may be $+\infty$) because $H^m_\delta(A)$ is a decreasing function of δ; thus $H^m(A) = \sup_{0<\delta} H^m_\delta(A)$.

2.3 REMARKS

(1) Since $\operatorname{diam} C_j = \operatorname{diam} \bar{C}_j$ we can add the additional requirement in definition 2.2 that the C_j be *closed* without changing the value of $H^m(A)$; indeed since for any $\varepsilon > 0$ we can find an open set $U_j \supset C_j$ with $\operatorname{diam} U_j < \operatorname{diam} C_j + \varepsilon/2^j$, we could also take the C_j to be *open* , *except in case* $m = 0$.

(2) Evidently $H^m_\delta(A) < \infty$ $\forall m \geq 0$, $\delta > 0$ in case A is a totally bounded subset of X .

One easily checks from the definition of H^m_δ that

$$H^m_\delta(A \cup B) = H^m_\delta(A) + H^m_\delta(B) \quad \text{if} \quad d(A,B) > 2\delta ,$$

hence

$$H^m(A \cup B) = H^m(A) + H^m(B) \quad \text{whenever} \quad d(A,B) > 0 ,$$

and therefore all Borel sets are H^m-measurable by the Caratheodory criterion 1.2. It follows from this and Remark 2.3(1) that *each of the measures* H^m *is Borel-regular.*

Note: It is *not* true in general that the Borel sets are H^m_δ-measurable for $\delta > 0$; for instance if $n \geq 2$ then one easily checks that the half-space $\{x = (x^1,\ldots,x^n) \in \mathbb{R}^n : x_n > 0\}$ is not H^1_δ-measurable.

We will later show that for each integer $n \geq 1$ H^n agrees with the usual definition of n-dimensional volume measure on an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$, $k \geq 0$. As a first step we want to prove that H^n and L^n (n-dimensional Lebesgue measure) agree on $\mathbb{R}^n$. First recall one of the standard definitions of L^n :

If K denotes the collection of all "n-dimensional cubes" I of the form $I = (a_1, a_1+\ell) \times (a_2, a_2+\ell) \times \ldots \times (a_n, a_n+\ell)$, where $a_i \in \mathbb{R}$ and $\ell > 0$, and if $|I|$ = volume of $I = \ell^n$, then

2.4 $$L^n(A) = \inf \sum_j |I_j|$$

where the inf is taken over all countable (or finite) collections $\{I_1, I_2, \ldots\} \subset K$ with $A \subset \bigcup_j I_j$. One easily checks that L^n is *uniquely characterized* among measures on $\mathbb{R}^n$ by the properties

$$L^n(I) = |I| \ \forall I \in K, \quad L^n(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} L^n(U) \ \forall A \subset \mathbb{R}^n .$$

We can now show

(*) $$H^n_\delta(A) \le L^n(A) \quad \forall \delta > 0$$

as follows. Let $\varepsilon > 0$ and choose $I_1, I_2, \ldots \in K$ so that $A \subset \bigcup_k I_k$ and

$$\sum_k |I_k| \le L^n(A) + \varepsilon .$$

Now for each bounded open set $U \subset \mathbb{R}^n$ and each $\delta > 0$ we can select a pair-wise disjoint family of closed balls $B_1, B_2, \ldots$ with $\bigcup_{j=1}^\infty B_j \subset U$, diam $B_j < \delta$, and $L^n(U \sim \bigcup_{j=1}^\infty B_j) = 0$. (To see this first decompose U as a union $\bigcup_{j=1}^\infty C_j$ of closed cubes C_j of diameter $< \delta$ and with pairwise disjoint interiors, and for each $j \ge 1$ select a ball $B_j \subset$ interior C_j with diam $B_j > \frac{1}{2}$ edge-length of C_j. Then $L^n(B_j) > \theta_n L^n(C_j)$, $\theta_n = \omega_n/4^n$, and it follows $L^n(U \sim \bigcup_{j=1}^\infty B_j) < (1-\theta_n) L^n(U)$. Thus $L^n(U \sim \bigcup_{j=1}^N B_j) \le (1-\theta_n) L^n(U)$ for suitable $\theta_n \in (0,1)$. Since $U \sim \bigcup_{j=1}^N B_j$ is open, we can repeat the argument inductively to obtain the required collection of balls.) Then take $U = I_k$ and such a collection of balls $\{B_j\}$. Since $L^n(Z) = 0 \Rightarrow H^n_\delta(Z) = 0$ for each subset $Z \subset X$ (by definitions 2.2, 2.4) we then have (writing ρ_j = radius B_j)

$$H^n_\delta(I_k) = H^n_\delta(\bigcup_{j=1}^{\infty} B_j) \le \sum_{j=1}^{\infty} \omega_n \rho_j^n$$

$$= \sum_{j=1}^{\infty} L^n(B_j) = L^n(\bigcup_{j=1}^{\infty} B_j) = L^n(I_k) = |I_k| ,$$

and hence

$$H^n_\delta(A) \le H^n_\delta(\bigcup_k I_k) \le \sum_k H^n_\delta(I_k) \le L^n(A) + \varepsilon .$$

Thus 2.5 is established.

To prove the reverse inequality

(**) $$L^n(A) \le H^n_\delta(A) \quad \forall \delta > 0 , \quad A \subset \mathbb{R}^n ,$$

we are going to need the inequality

2.5 $$L^n(A) \le \omega_n \left(\frac{\text{diam } A}{2}\right)^n \quad \forall A \subset \mathbb{R}^n .$$

This is called the *isodiametric inequality* ; it asserts that among all sets $A \subset \mathbb{R}^n$ with a given diameter ρ , the ball with diameter ρ has the largest L^n measure. It is proved by *Steiner symmetrization* (see [HR] or [FH1] for the details).

Now suppose $\delta > 0$, $A \subset \mathbb{R}^n$, and let $C_1, C_2, \ldots$ be any countable collection with $A \subset \bigcup_j C_j$ and diam $C_j < \delta$. Then

$$L^n(A) \le L^n(\bigcup_j C_j) \le \sum_j L^n(C_j)$$

$$\le \sum_j \omega_n \left(\frac{\text{diam } C_j}{2}\right)^n \quad \text{(by 2.5)} .$$

Taking the inf over all such collections $\{C_j\}$ we have (**).

Thus we have proved:

2.6 THEOREM

$$\mathcal{L}^n(A) = \mathcal{H}^n(A) = \mathcal{H}^n_\delta(A) \quad \textit{for every} \quad A \subset \mathbb{R}^n \ \textit{and} \ \delta > 0 .$$

§3. DENSITIES

Next we want to introduce the notion of n-dimensional density of a measure μ on X. For any measure μ on X, any subset $A \subset X$, and any point $x \in X$, we define the n-dimensional upper and lower densities $\Theta^{*n}(\mu,A,x)$, $\Theta^n_*(\mu,A,x)$ by

$$\Theta^{*n}(\mu,A,x) = \limsup_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}$$

$$\Theta^n_*(\mu,A,x) = \liminf_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}$$

(where $B_\rho(x)$ denotes the closed ball). In case $A = X$ we simply write $\Theta^{*n}(\mu,x)$ and $\Theta^n_*(\mu,x)$ to denote these quantities, so that $\Theta^{*n}(\mu,A,x) = \Theta^{*n}(\mu \llcorner A,x)$, $\Theta^n_*(\mu,A,x) = \Theta^n_*(\mu \llcorner A,x)$.

3.1 REMARK One readily checks that if all Borel sets are μ-measurable then $\mu(A \cap B_\rho(x)) \geq \limsup_{y \to x} \mu(A \cap B_\rho(y))$ for each fixed $\rho > 0$, so that $\mu(A \cap B_\rho(x))$ is a Borel-measurable function of x for each fixed $\rho > 0$. Hence $\Theta^{*n}(\mu,A,x)$ and $\Theta^n_*(\mu,A,x)$ are both Borel measurable (and hence μ-measurable) functions of $x \in X$. Notice that it is *not* necessary that A be μ-measurable.

If $\Theta^{*n}(\mu,A,x) = \Theta^n_*(\mu,A,x)$ then the common value will be denoted $\Theta^n(\mu,A,x)$.

Appropriate information about the upper density gives connections between μ and $\mathcal{H}^n$. Specifically we have

3.2 THEOREM *Let* μ *be a Borel-regular measure on* X *and* $t \geq 0$.

(1) *If* $A_1 \subset A_2 \subset X$ *and* $\Theta^{*n}(\mu, A_2, x) \geq t$ *for all* $x \in A_1$, *then* $t\,\mathcal{H}^n(A_1) \leq \mu(A_2)$.

(2) *If* $A \subset X$ *and* $\Theta^{*n}(\mu, A, x) \leq t$ *for all* $x \in A$, *then* $\mu(A) \leq 2^n t\,\mathcal{H}^n(A)$.

An important case of (1) is when $A_1 = A_2$. Notice that we do *not* assume A , A_1 , A_2 are μ-measurable.

Of the two propositions above, (2) is the more elementary and we could prove it immediately. (1) requires a covering lemma, so we defer both proofs until we have discussed this.

In the following covering theorem and its proof, we use the notation that if B is a ball $B_\rho(x) \subset X$, then $\hat{B} = B_{5\rho}(x)$.

3.3 THEOREM *If* $\mathcal{B}$ *is any family of closed balls in* X *with* $R \equiv \sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty$, *then there is a pairwise disjoint subcollection* $\mathcal{B}' \subset \mathcal{B}$ *such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in\mathcal{B}'} \hat{B} \; ;$$

in fact we can arrange the stronger property

$$(*) \qquad B \in \mathcal{B} \Rightarrow \exists\, S \in \mathcal{B}' \textit{ with } S \cap B \neq \phi \textit{ and } \hat{S} \supset B .$$

Proof For $j = 1,2,\ldots$ let $\mathcal{B}_j = \{B \in \mathcal{B} : R/2^j < \operatorname{diam} B \leq R/2^{j-1}\}$, so that $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$. Proceed to define $\mathcal{B}_j' \subset \mathcal{B}_j$ as follows:

(i) Let $\mathcal{B}_1'$ be any maximal pairwise disjoint subcollection of $\mathcal{B}_1$.

(ii) Assuming $j \geq 2$ and that $\mathcal{B}_1', \ldots, \mathcal{B}_{j-1}'$ are defined, let $\mathcal{B}_j'$ be a maximal pairwise disjoint subcollection of $\{B \in \mathcal{B}_j : B \cap B' = \phi$ whenever $B' \in \bigcup_{i=1}^{j-1} \mathcal{B}_i'\}$.

Then evidently if $j \geq 1$ and $B \in \mathcal{B}_j$ we must have

$$B \cap B' \neq \phi \text{ for some } B' \in \bigcup_{i=1}^{j} \mathcal{B}_i'$$

(otherwise we contradict maximality of $\mathcal{B}_j'$), and for such a pair B, B' we have $\operatorname{diam} B \leq R/2^{j-1} = 2R/2^j \leq 2 \operatorname{diam} B'$, so that $B \subset \hat{B}'$.

Thus we may take $\mathcal{B}' = \bigcup_{i=1}^{\infty} \mathcal{B}_i'$.

In the following corollary we use the terminology that a subset $A \subset X$ is covered *finely* by a collection $\mathcal{B}$ of balls, meaning that for each $x \in A$ and each $\varepsilon > 0$, there is a ball $B \in \mathcal{B}$ with $x \in B$ and $\operatorname{diam} B < \varepsilon$.

3.4 COROLLARY *If* $\mathcal{B}$ *is as in* 3.3, *if* A *is a subset of* X *covered finely by* $\mathcal{B}$, *and if* $\mathcal{B}' \subset \mathcal{B}$ *is as in* 3.3, *then*

$$A \sim \bigcup_{j=1}^{N} B_j \subset \bigcup_{B \in \mathcal{B}' \sim \{B_1, \ldots, B_N\}} \hat{B}$$

for each finite subcollection $\{B_1, \ldots, B_N\} \subset \mathcal{B}'$.

Proof If $x \in A \sim \bigcup_{j=1}^{N} B_j$, since $\mathcal{B}$ covers A finely and since $X \sim \bigcup_{j=1}^{N} B_j$ is open, we can then find $B \in \mathcal{B}$ with $B \cap (\bigcup_{j=1}^{N} B_j) = \phi$ and $x \in B$, and (by $(*)$) find $S \in \mathcal{B}'$ with $S \cap B \neq \phi$ and $\hat{S} \supset B$. Evidently then $S \neq B_j \; \forall j = 1, \ldots, N$, and hence $x \in \bigcup_{S \in \mathcal{B}' \sim \{B_1, \ldots, B_N\}} \hat{S}$.

Proof of (1) of Theorem 3.2 We can assume $\mu(A_2) < \infty$ and $t > 0$, otherwise the result is trivial. We can also assume the strict inequality

$$\Theta^{*n}(\mu, A_2, x) > t \quad \text{for} \quad x \in A_1$$

(because to obtain the conclusion of (1) for t equal to a given $t_1 > 0$, it clearly suffices to prove it for each $t < t_1$).

For $\delta > 0$, let $\mathcal{B}$ (depending on δ) be the collection $\{\text{closed balls } B_\rho(x) : x \in A_1 , 0<\rho<\delta/2 , \mu(A_2 \cap B_\rho(x)) \ge t\,\omega_n\rho^n\}$. Evidently $\mathcal{B}$ covers A_1 finely and hence there is a disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ so that 3.3(*) holds. Since $\mu(A_2 \cap B) > 0$ for each $B \in \mathcal{B}$ and since $\mu(A_2) < \infty$ it follows that $\mathcal{B}'$ is a countable collection $\{B_1, B_2, \ldots\}$ and hence 3.4 implies

$$A_1 \sim \bigcup_{j=1}^{N} B_j \subset \bigcup_{j=N+1}^{\infty} \hat{B}_j \qquad \forall N \ge 1 .$$

Thus $A_1 \subset (\bigcup_{j=1}^{N} B_j) \cup (\bigcup_{j=N+1}^{\infty} \hat{B}_j)$ and hence by the definition 2.2 of $\mathcal{H}^n_\delta$ we have

$$\mathcal{H}^n_{5\delta}(A_1) \le \sum_{j=1}^{N} \omega_n\rho_j^n + 5^n \sum_{j=N+1}^{\infty} \omega_n\rho_j^n \quad \left(\rho_j = \frac{\operatorname{diam} B_j}{2}\right) .$$

Since $B_j \in \mathcal{B}$, we have

$$\sum_{j=1}^{\infty} \omega_n\rho_j^n \le t^{-1} \sum_{j=1}^{\infty} \mu(A_2 \cap B_j) = t^{-1} \sum_{j=1}^{\infty} (\mu \llcorner A_2)(B_j)$$

$$= t^{-1}(\mu \llcorner A_2)(\bigcup_{j=1}^{\infty} B_j) \le t^{-1}\mu(A_2) < \infty ,$$

and hence letting $N \to \infty$ we deduce

$$\mathcal{H}^n_{5\delta}(A_1) \le t^{-1}\mu(A_2) .$$

Letting $\delta \downarrow 0$, we then have the required result.

Proof of (2) of Theorem 3.2 We may assume that

$$\Theta^{*n}(\mu,A,x) < t \quad \text{for all} \quad x \in A$$

because to prove the conclusion of (2) for a given $t = t_1 > 0$, it is clearly enough to prove it for each $t > t_1$. Thus if

$$A_k = \{x \in A : \mu(A \cap B_\rho(x)) \le t\omega_n\rho^n \quad \forall 0 < \rho < 1/k\}$$

then $A = \bigcup_{n=1}^{\infty} A_k$ and $A_{k+1} \supset A_k$, $k = 1,2,\ldots$. The A_k are not necessarily μ-measurable, but we still have $\lim_{k\to\infty} \mu(A_k) = \mu(\bigcup_{k=1}^{\infty} A_k)$ by virtue of the regularity of μ. Thus we will be finished if we can prove

$$\mu(A_k) \le 2^n t \mathcal{H}^n(A_k) \quad \forall k \ge 1 .$$

Let $\delta \in (0,1/2k)$ and let $C_1, C_2, \ldots$ be any countable cover for A_k with diam $C_j < \delta$ and $C_j \cap A_k \ne \phi$ $\forall j$. For each j we can find an $x_j \in A_k$ so that $B_{2\rho_j}(x_j) \supset C_j$, $\rho_j = \frac{\text{diam } C_j}{2}$. Then since $2\rho_j < 1/k$ we have by definition of A_k that

$$\mu(C_j) \le \mu(B_{2\rho_j}(x_j)) \le 2^n t\omega_n\rho_j^n .$$

Hence

$$\mu(A_k) \le 2^n t \sum_{j=1} \omega_n \left(\frac{\text{diam } C_j}{2}\right)^n .$$

Taking inf over all such covers $\{C_j\}$ we then have (by definition of $\mathcal{H}^n_\delta$) that $\mu(A_k) \le 2^n t\, \mathcal{H}^n_\delta(A_k)$. Thus we have the required inequality by letting $\delta \downarrow 0$.

As a corollary to Theorem 3.2 (1) we can easily prove the following.

3.5 THEOREM *If* μ *is Borel regular, if* A *is a* μ*-measurable subset of* X *and if* $\mu(A) < \infty$, *then*

$$\Theta^{*n}(\mu,A,x) = 0 \quad \textit{for} \quad H^n\text{ - a.e.} \quad x \in X \sim A \ .$$

REMARK Of course $\mu = H^n$ is an important case.

Proof For $t > 0$ let $C_t = \{x \in X \sim A : \Theta^{*n}(\mu,A,x) \geq t\}$ and if $H^n(C_t) > 0$ we can (by Theorem 1.3(2)) find a closed set $E \subset A$ such that

$$\text{(1)} \qquad \mu(A \sim E) < t\, H^n(C_t) \ .$$

Since $X \sim E$ is open and $C_t \subset X \sim A \subset X \sim E$ we have $\Theta^{*n}(\mu, A \cap (X \sim E), x) = \Theta^{*n}(\mu,A,x) \geq t$ for $x \in C_t$. Thus we can apply Theorem 3.2(1) with $\mu \llcorner A$, C_t, $X \sim E$ in place of μ, A_1, A_2 to give $t\, H^n(C_t) \leq \mu(A \sim E)$, thus contradicting (1). Thus we conclude that $H^n(C_t) = 0$ $\forall t > 0$. In particular $H^n(\bigcup_{k=1}^{\infty} C_{1/k}) = 0$. Thus $\Theta^{*n}(\mu,A,x) = 0$ for H^n - a.e. $x \in X \sim A$.

We conclude this section with two important bounds for densities with respect to Hausdorff measure.

3.6 THEOREM *Suppose* A *is any subset of* X.

(1) *If* $H^n(A) < \infty$, *then* $\Theta^{*n}(H^n,A,x) \leq 1$ *for* H^n - a.e. $x \in A$.

(2) *If* $H^n_\delta(A) < \infty$ *for each* $\delta > 0$ *(note this is automatic if* A *is a totally bounded subset of* X*), then* $\Theta^{*n}(H^n_\infty,A,x) \geq 2^{-n}$ *for* H^n - a.e. $x \in A$.

3.7 REMARK Since $H^n \geq H^n_\delta \geq H^n_\infty$ (by definitions 2.1, 2.2) this theorem implies

$$2^{-n} \leq \Theta^{*n}(H^n,A,x) \leq 1 \quad \text{for} \quad H^n\text{ - a.e.} \quad x \in A$$

if $H^n(A) < \infty$.

Proof of 3.6 To prove (1), let $\varepsilon, t > 0$, let $A_t = \{x \in A : \Theta^{*n}(\mathcal{H}^n, A, x) \geq t\}$ and (using Theorem 1.3(1) with $\mu = \mathcal{H}^n \llcorner A$) choose an open set $U \supset A_t$ such that

$$\mathcal{H}^n(U \cap A) < \mathcal{H}^n(A_t) + \varepsilon .$$

Since U is open and since $A_t \subset U$ we have $\Theta^*(\mathcal{H}^n, A \cap U, x) \geq t$ for each $x \in A_t$. Hence Theorem 3.2(1) (with $\mathcal{H}^n \llcorner A, A_t, A \cap U$ in place of μ, A_1, A_2) implies that

$$t\, \mathcal{H}^n(A_t) \leq \mathcal{H}^n(A \cap U) \leq \mathcal{H}^n(A_t) + \varepsilon .$$

We thus have $\mathcal{H}^n(A_t) = 0$ for each $t > 1$. Since $\{x : \Theta^{*n}(\mathcal{H}^n, A, x) > 1\} = \bigcup_{j=1}^{\infty} A_{t_j}$ for any strictly decreasing sequence $\{t_j\}$ with $\lim t_j = 1$, we thus have $\mathcal{H}^n\{x : \Theta^{*n}(\mathcal{H}^n, A, x) > 1\} = 0$, as required.

To prove (2), suppose for contradiction that $\Theta^{*n}(\mathcal{H}^n_\infty \llcorner A, x) < 2^{-n}$ for each x in a set $B_0 \subset A$ with $\mathcal{H}^n(B_0) > 0$. Then for each $x \in B_0$ (by definition) we can select $\delta_x \in (0,1)$ such that

$$\mathcal{H}^n_\infty(A \cap B_\rho(x)) \leq \frac{1-\delta_x}{2^n}\, \omega_n \rho^n , \quad 0 < \rho < \delta_x .$$

Therefore, since $B_0 = \bigcup_{j=1}^{\infty} \{x \in B_0 : \delta_x > 1/j\}$ and since $\mathcal{H}^n_\delta(A \cap B_\rho(x)) \equiv \mathcal{H}^n_\infty(A \cap B_\rho(x))$ for any $\rho < \delta/2$ (by definition 2.2), we can select $\delta > 0$ and $B \subset B_0$ with $\mathcal{H}^n(B) > 0$ and

(1) $$\mathcal{H}^n_\delta(A \cap B_\rho(x)) \leq \frac{1-\delta}{2^n}\, \omega_n \rho^n , \quad 0 < \rho < \delta/2 , \quad x \in B .$$

Now using 2.2 again, we can choose sets $C_1, C_2, \ldots$ with $B \subset \bigcup_{j=1}^{\infty} C_j$, $C_j \cap B \neq \phi \ \forall j$, and

(2) $$\sum_j \omega_n (\rho_j/2)^n < \frac{1}{1-\delta}\, \mathcal{H}^n_\delta(B) , \quad \rho_j = \operatorname{diam} C_j .$$

Now take $x_j \in C_j \cap B$, so that $B \subset A \cap (\bigcup_{j=1}^{\infty} B_{\rho_j}(x_j))$, and we conclude from (1), (2) that $\mathcal{H}^n_\delta(B) = 0$, hence $\mathcal{H}^n(B) = 0$, contradicting our choice of B .

§4. RADON MEASURES

In this section X is assumed to be locally compact and separable. On such a space we say that μ is a *Radon measure* if μ is Borel regular and if μ is finite on compact subsets of X . Notice that (by 1.3, 1.4) such a measure μ automatically has the properties

$$\mu(A) = \inf_{\substack{U \supset A \\ U \text{ open}}} \mu(U) , \quad A \subset X \text{ arbitrary}$$

and

$$\mu(A) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \mu(K) , \quad A \subset X \ \mu\text{-measurable.}$$

The finiteness of Radon measures μ on compact subsets enables us to integrate continuous functions with compact support. Indeed if H is a Hilbert space with inner product $(,)$ and if $K(X,H)$ denotes the space of continuous functions $X \to H$ with compact support, then associated with each Radon measure μ and each μ-measurable H-valued function $\nu : X \to H$ satisfying $|\nu| = 1$, μ - a.e. , we have the linear functional $L : K(X,H) \to \mathbb{R}$ defined by

$$L(f) = \int_X (f,\nu)\, d\mu .$$

The following *Riesz representation theorem* shows that *every* linear functional $L : K(X,H) \to \mathbb{R}$ is obtained as above, provided

(*) $$\sup\{L(f) : f \in K(X,H) , |f| \le 1 , \text{spt } f \subset K\} < \infty$$

for each compact $K \subset X$.

4.1 THEOREM *Let* L *be any linear functional on* $K(X,H)$ *satisfying* (*) *above. Then there is a Radon measure* μ *on* X *and a* μ*-measurable function* $\nu : X \to H$ *such that* $|\nu(x)| = 1$ *for* μ-a.e. $x \in X$ *and*

$$L(f) = \int_X (f,\nu)\, d\mu \qquad \forall f \in K(X,H) .$$

4.2 REMARK Notice that (as one readily checks by using Lusin's theorem to exhaust μ-almost all of X by an increasing sequence of compact sets on which ν is continuous), we have

$$\sup\{L(f) : f \in K(X,H) , |f| \le 1 , \text{spt } f \subset V\} = \mu(V)$$

for every open $V \subset X$, assuming μ , ν are as in the theorem. For this reason the measure μ is called the *total variation measure* associated with the functional L .

Proof of 4.1 First *define* $\mu(V)$ on open sets V according to the identity of 4.2 above, and then for an arbitrary subset $A \subset X$ let

(1) $$\mu(A) = \inf_{\substack{A\subset V\\ V \text{ open}}} \mu(V) .$$

(Of course these definitions are not contradictory when A itself is open.) To check that μ is a Radon measure we proceed as follows. First, if $V, V_1, V_2, \ldots$ are open sets in X with $V \subset \bigcup_{j=1}^{\infty} V_j$, and if ω is any element of $K(X,H)$ with $\sup_X|\omega| \le 1$ and support $\omega \subset V$, then, by using the definition of μ and a partition of unity of support ω subordinate to to the sets $\{V_j\}_{j=1,2,\ldots}$, we have

$$|L(\omega)| \le \sum_{j=1}^{\infty} \mu(V_j) .$$

Taking sup over all such ω we thus get $\mu(V) \le \sum_{j=1}^{\infty} \mu(V_j)$. Then by (1) we see that

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A, A_1, A_2, \ldots$ are subsets of X with $A \subset \bigcup_{j=1}^{\infty} A_j$. Thus μ is a measure on X . It is also clear from the definition of μ that

$$\mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2)$$

whenever V_1, V_2 are open subsets of X with $d(V_1,V_2) > 0$. Then by (1) we see that μ satisfies the Caratheodory criterion, and hence all Borel sets are measurable by Theorem 1.2. Thus we can conclude that μ is a Borel regular measure and since it is evidently finite on compact sets (by (*)) it is then a Radon measure.

Next let $K(X) = K(X,\mathbb{R})$ and $K_+(X) = \{f \in K(X) : f \ge 0\}$.

Define

$$\lambda(f) = \sup_{\substack{|\omega| \le f \\ \omega \in K(X,H)}} |L(\omega)| , \quad f \in K_+(X) .$$

Then by definition of μ we have

$$\sup_{\substack{f \in K_+(X) \\ \text{support } f \subset U}} \lambda(f) = \mu(U) \quad \forall \text{ open } U \subset X .$$

We in fact claim

$$\lambda(f) = \int f d\mu , \quad f \in K_+(X) . \tag{2}$$

To see this we first note that $\lambda(cf) = c\,\lambda(f)$, c constant ≥ 0 , $f \in K_+(X)$. Further we claim that $\lambda(f+g) = \lambda(f) + \lambda(g)$ $\forall f,g \in K_+(X)$. Indeed the inequality

$\lambda(f+g) \geq \lambda(f) + \lambda(g)$ is obvious, and we prove the reverse inequality as follows. Let $\omega \in K(X,H)$ with $|\omega| \leq f+g$, and define ω_1, ω_2 by

$$\omega_1 = \begin{cases} \frac{f}{f+g}\,\omega & \text{if } f+g > 0 \\ 0 & \text{if } f+g = 0 \end{cases} \qquad \omega_2 = \begin{cases} \frac{g}{f+g}\,\omega & \text{if } f+g > 0 \\ 0 & \text{if } f+g = 0 \end{cases}.$$

One easily checks that then $\omega_1, \omega_2 \in K(X,H)$. Then since $\omega = \omega_1 + \omega_2$ and $|\omega_1| \leq f$, $|\omega_2| \leq g$, we have $|L(\omega)| \leq \lambda(f) + \lambda(g)$. Taking sup over all such ω we then have $\lambda(f+g) = \lambda(f) + \lambda(g)$. To complete the proof that (2) holds we let $\varepsilon > 0$ and choose $t_0 = 0 < t_1 < \ldots < t_N$, $t_N > \sup f$, such that $t_i - t_{i-1} < \varepsilon$ and $\mu(f^{-1}(t_j)) = 0 \quad \forall j = 1,\ldots,N$. (This is of course possible, because $\{t \in \mathbb{R} : \mu(f^{-1}(t)) > 0\}$ is clearly countable.) Write $U_j = \{x \in X : t_{j-1} < f < t_j\}$, $j = 1,\ldots,N$.

Now, by definition of μ, for each $\varepsilon > 0$ we can choose $h_j \in K_+(X)$ with support $h_j \subset U_j$, $h_j \leq 1$,

(3) $$\lambda(h_j) \geq \mu(U_j) - \varepsilon/N$$

and

(4) $$\mu(U_j \sim \{x : h_j(x) = 1\}) < \varepsilon/N .$$

Evidently (4) together with the definitions of λ, μ implies

$$\lambda(f - f\sum_{j=1}^{N} h_j) \leq \sup|f|\,\mu\{x : \sum_{i=1}^{N} U_i \sim \{x : h_j(x) = 1\}\}$$

$$\leq \sup|f|\ \varepsilon ,$$

and it readily follows that

$$\sum_{j=1}^{N} t_{j-1}\mu(U_j) - 2\varepsilon \sup|f| \leq \lambda(f \sum_{j=1}^{N} h_j) \leq \lambda(f) \leq \lambda(f \sum_{j=1}^{N} h_j) + \varepsilon \sup|f|$$

$$\leq \sum_{j=1}^{N} t_j\mu(U_j) + \varepsilon \sup|f| .$$

Since

$$\sum_{j=1}^{N} t_{j-1}\mu(U_j) \le \int f d\mu \le \sum_{j=1}^{N} t_j\mu(U_j)$$

we then have $\left|\lambda(f) - \int f d\mu\right| < 2\varepsilon \sup|f|$, and hence (2).

To complete the proof of the theorem, let $e \in H$ with $|e| = 1$, and consider the linear functional λ_e on $K(X)$ defined by $\lambda_e(f) = T(fe)$. Evidently by (2) ,

$$|\lambda_e(f)| \le \int |f| d\mu \qquad \forall f \in K(X)$$

and hence λ_e extends uniquely to a linear functional on $L^1(\mu)$. By the Riesz Representation Theorem for $L^1(\mu)$ functions (see e.g. [RH] for details - the proof is based on the Radon-Nikodym theorem) we have a bounded μ-measurable (in fact Borel-measurable) function ν_e on X such that

$$L(fe) = \int f \, \nu_e \, d\mu \qquad \forall f \in K(X) .$$

Taking $e_1, \ldots, e_n$ to be an orthonormal basis for H , and defining $\nu = \sum_{j=1}^{n} \nu^j e_j$, $\nu^i \equiv \nu_{e_i}$, one then easily checks that $L(g) = \int (g,\nu) d\mu$ for each $g \in K(X,H)$, as required. Furthermore (Cf. Remark 4.2) for each open $U \subset X$ we have

$$\text{(5)} \qquad \sup\{L(g) : g \in K(X,H) , |g| \le 1 , \operatorname{spt} g \subset U\} = \int_U |\nu| d\mu .$$

On the other hand the left side of (5) is $\mu(U)$ by definition of μ . Hence (from the arbitraryness of U) we conclude $|\nu| = 1$ μ - a.e. This completes the proof of Theorem 4.1.

4.3 REMARK Note that in case $H = \mathbb{R}$, Theorem 4.2 asserts that the linear functional L can be represented

$$L(f) = \int_X f\nu\, d\mu \qquad \forall F \in K(X,\mathbb{R}) ,$$

where $\nu(x) = \pm 1$ for μ-a.e. $x \in X$. In the special case when L is *non-negative*, i.e. $L(f) \geq 0$ if $f \geq 0$, then one easily checks that $\nu \equiv +1$, so that the theorem gives

$$L(f) = \int_X f\, d\mu$$

in this case. Thus *we can identify the Radon measures on* X *with the non-negative linear functionals on* $K(X,\mathbb{R})$. (Note (*) is automatic if L is non-negative.)

Now for $U \subset X$ with U open and $\bar{U}$ compact, let L_U^+ denote the set of bounded (real-valued) linear functionals on $K_U(X) = \{$continuous functions $f : X \to \mathbb{R}$ with $\operatorname{spt} f \subset U\}$ which are non-negative on $K_U^+(X) = \{f \in K_U(X) : f \geq 0\}$. The Banach-Alaoglu theorem (see e.g. [Roy]) tells us that $\{\lambda \in L_U^+ : \|\lambda\| \leq 1\}$ is weak* compact. That is, given a sequence $\{\lambda_k\} \subset L_U^+$ with $\sup_{k\geq 1}\|\lambda_k\| < \infty$, we can find a subsequence $\{\lambda_{k'}\}$ and $\lambda \in L_U^+$ such that $\lim \lambda_{k'}(f) = \lambda(f)$ for each fixed $f \in K_U^+(X)$. Using the above Riesz Representation Theorem (and in particular Remark 4.3) together with an exhaustion of X by an increasing sequence $\{U_i\}$ of open sets with $\bar{U}_i$ compact $\forall i$, this evidently implies the following assertion concerning sequences of Radon measures on X.

4.4 THEOREM *Suppose* $\{\mu_k\}$ *is a sequence of Radon measures on* X *with* $\sup_{k\geq 1} \mu_k(U) < \infty$ *for each open* $U \subset X$ *with* $\bar{U}$ *compact. Then there is a subsequence* $\{\mu_{k'}\}$ *which converges to a Radon measure* μ *on* X *in the sense that*

$$\lim \mu_{k'}(f) = \mu(f) \quad \textit{for each} \quad f \in K(X) ,$$

where $K(X) = \{f : f$ *is a real-valued continuous function with compact support on* $X\}$. *Here we used the notation*

$$\mu(f) = \int_X f\, d\mu .$$

Now let μ be any Radon measure on X. We say that X has the *symmetric Vitali property* relative to μ if for every collection $\mathcal{B}$ of balls which covers its *set of centres* $A \equiv \{x : B_\rho(x) \in \mathcal{B}$ for some $\rho > 0\}$ *finely* (i.e. for each $x \in A$ we have $\inf\{\rho : B_\rho(x) \in \mathcal{B}\} = 0$), there is a countable pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ covering μ-almost all of A, provided $\mu(A) < \infty$.

4.5 REMARKS

(1) It is easy to see (from Corollary 3.4) that the locally compact separable metric space X has this property with respect to μ, provided $\mu(B_{5\rho}(x)) \le c\,\mu(B_\rho(x))$ whenever $B_\rho(x) \subset X$, where c is a fixed constant independent of x and ρ.

(2) Most importantly, in the special case when $X = \mathbb{R}^n$, we have the *symmetric Vitali property* with respect to μ for any Radon measure μ.

To justify this last remark we need first to recall the following *Besicovitch covering lemma* (see [FH1] or [HR] for a proof).

4.6 LEMMA *Suppose* $\mathcal{B}$ *is a collection of closed balls in* $\mathbb{R}^n$, *let* A *be the set of centres, and suppose the set of all radii of balls in* $\mathcal{B}$ *is a bounded set. Then there are sub-collections* $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ $(N=N(n))$ *such that each* $\mathcal{B}_j$ *is a pairwise disjoint (or empty) collection, and* $\bigcup_{j=1}^N \mathcal{B}_j$ *still covers* A : $A \subset \bigcup_{j=1}^N \left(\bigcup_{B\in\mathcal{B}_j} B\right)$.

We emphasize that N is a certain fixed constant depending only on n.

Using this lemma we can easily justify Remark 4.5(2): For a given Radon measure μ in $\mathbb{R}^n$ and for a given collection of balls $\mathcal{B}$ covering its own set of centres A finely, we first choose (from the set $\{B \in \mathcal{B} : \text{radius } B \le 1\}$)

pairwise disjoint collections $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ such that $\bigcup_{j=1}^{N} \mathcal{B}_j$ covers A. Then for at least one $j \in \{1,\ldots,N\}$ we get

$$\mu(A \sim \bigcup_{B\in\mathcal{B}_j} B) \leq (1 - 1/N)\mu(A)$$

and hence taking a suitable finite subcollection $\{B_1, \ldots, B_Q\} \subset \mathcal{B}_j$,

$$\mu(A \sim \bigcup_{k=1}^{Q} B_k) \leq (1 - 1/2N)\mu(A) .$$

Since $\mathcal{B}$ covers A finely, and since $\bigcup_{k=1}^{Q} B_k$ is closed, the collection $\tilde{\mathcal{B}} = \{B \in \mathcal{B} : B \cap (\bigcup_{k=1}^{Q} B_k) = \phi\}$ covers $A \sim \bigcup_{k=1}^{Q} B_k$ finely. Thus we can repeat the argument with $\tilde{\mathcal{B}}$ in place of $\mathcal{B}$ and $A \sim \bigcup_{k=1}^{Q} B_k$ in place of A in order to select new balls $B_{Q+1}, \ldots, B_P \in \tilde{\mathcal{B}}$ such that

$$\mu(A \sim \bigcup_{k=1}^{P} B_k) \leq (1 - \frac{1}{2N})\, \mu(A \sim \bigcup_{k=1}^{Q} B_k)$$

$$\leq (1 - \frac{1}{2N})^2\, \mu(A) .$$

Continuing (inductively) in this way, we conclude that if $\mu(A) < \infty$ there is a pairwise disjoint sequence $B_1, B_2, \ldots$ of balls in $\mathcal{B}$ such that

$$\mu(A \sim \bigcup_{k=1}^{\infty} B_k) = 0 .$$

Thus Remark 4.5(2) is established.

4.7 THEOREM *Suppose* μ_1, μ_2 *are Radon measures on* X, *where* X *has the symmetric Vitali property with respect to* μ_1. *Then*

$$D_{\mu_1}\mu_2(x) \equiv \lim_{\rho\downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

exists μ_1*-almost everywhere and is* μ_1*-measurable. Furthermore for any Borel set* $A \subset X$

(1) $$\mu_2(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1 + \mu_2^*(A) ,$$

where

$$\mu_2^* = \mu_2 \llcorner Z ,$$

where Z *is a Borel set of* μ_1*-measure zero* (Z *independent of* A).

In case X *also has the symmetric Vitali property with respect to* μ_2 *then* $D_{\mu_1}\mu_2$ *also exists* μ_2*-almost everywhere and*

(2) $$\mu_2^* = \mu_2 \llcorner \{x : D_{\mu_1}\mu_2(x) = +\infty\} .$$

(i.e. we may take $Z = \{x : D_{\mu_1}\mu_2(x) = +\infty\}$ in this case.)

4.8 REMARKS

(1) Of course by Remark 4.5(2), we always have 4.7(2) if $X = \mathbb{R}^n$.

(2) μ_2^* is called the singular part of μ_2 with respect to μ_1 . One readily checks that $\mu_2^* = 0$ if and only if all sets of μ_1-measure zero also have μ_2-measure zero. In this case we say that μ_2 is *absolutely continuous* with respect to μ_1 . 4.7(1) then simply says

(*) $$\mu_2(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1 , \quad A \subset X , \quad A \text{ a Borel set.}$$

Proof We can of course assume $\mu_1(X) < \infty$, $\mu_2(X) < \infty$ since μ_1 , μ_2 are Radon measures and X is locally compact and separable.

First consider the case when all sets of μ_1-measure zero also have μ_2-measure zero. In this case we want to prove (*) , and we have that X also has the symmetric Vitali property relative to μ_2 .

Let $\tilde{X} = X \sim \{x : \mu_1(B_\sigma(x)) = 0 \text{ for some } \sigma > 0\}$. Evidently $\tilde{X}$ is closed and (by separability) $\mu_1(X\sim\tilde{X}) = 0$, $\mu_1 = \mu_1 \llcorner \tilde{X}$. Let $\underline{D}_{\mu_1}\mu_2$ and

$\bar{D}_{\mu_1}\mu_2$ be defined on $\tilde{X}$ by

$$\underline{D}_{\mu_1}\mu_2(x) = \liminf_{\rho \downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

$$\bar{D}_{\mu_1}\mu_2(x) = \limsup_{\rho \downarrow 0} \frac{\mu_2(B_\rho(x))}{\mu_1(B_\rho(x))}$$

and define $\bar{D}_{\mu_1}\mu_2$, $\underline{D}_{\mu_1}\mu_2 \equiv \infty$ on $X \sim \tilde{X}$. Notice that $\bar{D}_{\mu_1}\mu_2$ and $\underline{D}_{\mu_1}\mu_2$ are Borel measurable.

We first prove that if $\alpha \in (0,\infty)$ then for any Borel set $A \subset X$,

(1) $$A \subset \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < \alpha\} \Rightarrow \mu_2(A) \le \alpha\, \mu_1(A)$$

(2) $$A \subset \{x \in \tilde{X} : \bar{D}_{\mu_1}\mu_2(x) > \alpha\} \Rightarrow \mu_2(A) \ge \alpha\, \mu_1(A) .$$

To prove (1) we simply note that if $A \subset \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) > \alpha\}$, then for any open $V \supset A$ the collection $\mathcal{B} = \{B_\rho(x) : x \in A, B_\rho(x) \subset V ,$ $\mu_2(B_\rho(x)) \le \alpha\, \mu_1(B_\rho(x))\}$ covers A finely, so there is a countable disjoint subcollection $\{B_1, B_2, \ldots\} \subset \mathcal{B}$ which covers μ_1-almost all of A (and hence μ_2-almost all of A).

Then

$$\mu_2(A) \le \mu_2(\bigcup_{j=1}^{\infty} B_j) \le \sum_{j=1}^{\infty} \mu_2\ (Bj) \le \alpha \sum_{j=1}^{\infty} \mu_1(B_j)$$

$$= \alpha\, \mu_1\ (\bigcup_{j=1}^{\infty} B_j) \le \alpha\, \mu_1(V) .$$

Taking inf over all such V , by (1.31) we have (1) as required.

The proof of (2) is almost identical and is left to the reader.

Notice particularly that if we let $\alpha \to \infty$ in (1) and use

$\mu_1(X\sim\tilde{X}) = 0$, then we deduce

(3) $$\mu_1\{x \in X : \bar{D}_{\mu_1}\mu_2(x) = +\infty\} = 0 .$$

Now let $a < b$ and $A = \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < a < b < \bar{D}_{\mu_1}\mu_2(x)\}$. Then by (1), (2) above we have

$$\mu_2(A) \le a\,\mu_1(A) \quad \text{and also} \quad b\,\mu_1(A) \le \mu_2(A) ,$$

which implies that $\mu_1(A) = \mu_2(A) = 0$. Thus, by (3) together with the fact that $\{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < \bar{D}_{\mu_1}\mu_2(x)\} = \bigcup_{a,b \text{ rational}, a<b} \{x \in \tilde{X} : \underline{D}_{\mu_1}\mu_2(x) < a < b < \bar{D}_{\mu_1}\mu_2(x)\}$, we have that $\underline{D}_{\mu_1}\mu_2(x) = \bar{D}_{\mu_1}\mu_2(x)$ $(= D_{\mu_1}\mu_2(x)) < \infty$ for μ_1 almost all $x \in X$.

Next, to establish (*) we proceed as follows. For any Borel set $A \subset X$ let

$$\nu(A) = \int_A (D_{\mu_1}\mu_2)\,d\mu_1$$

and for any subset $A \subset X$ let $\nu(A) = \inf_{\substack{B \supset A \\ B \text{ Borel}}} \nu(B)$.

Then ν is evidently a Radon measure and

$$t_1\,\mu_1(A_{t_1,t_2}) \le \mu(A_{t_1,t_2}) \le t_2\,\mu_1(A_{t_1,t_2})$$

for any $0 < t_1 \le t_2$, $A_{t_1,t_2} = \{x \in A : t_1 < D_{\mu_1}\mu_2(x) < t_2\}$, A any Borel set. By then by (1), (2) we have

$$\frac{t_1}{t_2}\,\mu_2(A_{t_1,t_2}) \le \nu(A_{t_1,t_2}) \le \frac{t_2}{t_1}\,\mu_2(A_{t_1,t_2})$$

and it readily follows that $\nu = \mu_2$. Thus (*) is established.

In the general case (when it may be that $\mu_2(A) > 0$ when $\mu_1(A) = 0$)

select a Borel set B from the collection $\mathcal{A} = \{A \subset X : A$ is Borel, $\mu_1(X \sim A) = 0\}$ such that $\mu_2(B) = \inf_{A \in \mathcal{A}} \mu_2(A)$. (Take $B = \bigcap_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{A}$, $\lim \mu_2(A_i) = \inf_{A \in \mathcal{A}} \mu_2(A)$.) Now if $A \subset B$ with $\mu_1(A) = 0$ then we must have $\mu_2(A) = 0$ also, otherwise we contradict the minimality of $\mu_2(B)$. Then we can apply the previous argument to the measure $\tilde{\mu}_2 = \mu_2 \llcorner B$, thus giving

$$\mu_2(A \cap B) = \int_A (D_{\mu_1} \mu_2) \, d\mu_1 \quad \forall \text{ Borel set } A \subset X .$$

Thus 4.7(2) holds with $\mu_2^* = \mu_2 \llcorner (X \sim B)$.

Finally, in case X also has the symmetric Vitali property with respect to μ_2 , the first part of the argument above establishes that $D_{\mu_1} \mu_2$ exists μ_2-almost everywhere (as well as μ_1-almost everywhere) in $\tilde{X}$ and (1) shows that if $A \subset \{x \in \tilde{X} : D_{\mu_1} \mu_2(x) < \infty\}$ $(= \bigcup_{n=1}^{\infty} \{x \in \tilde{X} : D_{\mu_1} \mu_2(x) < n\})$ and if $\mu_1(A) = 0$, then also $\mu_2(A) = 0$. We can therefore apply the above argument to $\tilde{\mu}_2 = \mu_2 \llcorner \{x \in \tilde{X} : D_{\mu_1} \mu_2(x) < \infty\}$. Since we trivially have $D_{\mu_1} \mu_2(x) = \infty$ for μ_2-a.e. $x \in X \sim \tilde{X}$, we then deduce 4.7(1) with μ_2^* as in 4.7(2).

CHAPTER 2

SOME FURTHER PRELIMINARIES FROM ANALYSIS

Here we develop the necessary further analytical background material needed for later developments. In particular we prove some basic results about Lipschitz and BV functions, and we also present the basic facts concerning C^k submanifolds of Euclidean space. There is also a brief treatment of the area and co-area formula and a discussion of first and second variation formulae for C^2 submanifolds of Euclidean space. These latter topics will be discussed in a much more general context later.

§5. LIPSCHITZ FUNCTIONS

Recall that a function $f : X \to \mathbb{R}$ is said to be Lipschitz if there is $L < \infty$ such that (if d is the metric on X)

$$|f(x) - f(y)| \le L\, d(x,y) \qquad \forall\, x,y \in X .$$

$\mathrm{Lip}\, f$ denotes the least such constant L .

First we have the following trivial extension theorem.

5.1 THEOREM *If* $A \subset X$ *and* $f : A \to \mathbb{R}$ *is Lipschitz, then* $\exists\, \bar{f} : X \to \mathbb{R}$ *with* $\mathrm{Lip}\, \bar{f} = \mathrm{Lip}\, f$, *and* $f = \bar{f}|A$.

Proof Simply define

$$\bar{f}(x) = \inf_{y\in A}(f(y) + L\, d(x,y)) \ , \quad L = \mathrm{Lip}\, f \ .$$

Since $f(y) + L\, d(x,y) \ge f(z) - L\, d(x,z) \ \ \forall\, x \in X$, $y,z \in A$, we see that $\bar{f}$

is real-valued and $\bar{f}(x) = f(x)$ for $x \in A$. Furthermore

$$\bar{f}(x_1) - \bar{f}(x_2) = \sup_{y_2 \in A} \inf_{y_1 \in A} (f(y_1) + L\,d(x_1,y_1) - f(y_2) - L\,d(x_2,y_2))$$

$$\leq \sup_{y_2 \in A} (L\, d(x_1,y_2) - L\, d(x_2,y_2))$$

$$\leq L\, d(x_1,x_2) \qquad \forall\, x_1,x_2 \in X .$$

Next we need the theorem of Rademacher concerning differentiability of Lipschitz functions on $\mathbb{R}^n$. (The proof given here is due to C.B. Morrey.)

5.2 THEOREM *If* f *is Lipschitz on* $\mathbb{R}^n$, *then* f *is differentiable* L^n*-almost everywhere; that is,* $\operatorname{grad} f(x) = (D_1 f(x), \ldots, D_n f(x))$ *exists and*

$$\lim_{y \to x} \frac{f(y) - f(x) - \operatorname{grad} f(x) \cdot (y-x)}{|y-x|} = 0$$

for L^n - a.e. $x \in \mathbb{R}^n$.

Proof Let $v \in S^{n-1}$, and whenever it exists let $D_v f(x)$ denote the directional derivative $\left.\frac{d}{dt} f(x+tv)\right|_{t=0}$. Since $D_v f(x)$ exists precisely when the Borel-measurable functions $\limsup_{t \to 0} \frac{f(x+tv)-f(x)}{t}$ and $\liminf_{t \to 0} \frac{f(x+tv)-f(x)}{t}$ coincide, the set A_v on which $D_v f$ fails to exist is L^n-measurable. However $\phi(t) = f(x+tv)$ is an absolutely continuous function of $t \in \mathbb{R}$ for any fixed x and v, and hence is differentiable for almost all t. Thus A_v intersects every line L which is parallel to v in a set of H^1 measure zero. Thus for each $v \in S^{n-1}$

(1) $$D_v f(x) \ \textit{exists}\ L^n\text{ - a.e. } x \in \mathbb{R}^n .$$

Now take any $C_0^\infty(\mathbb{R}^n)$ function ζ and note that for any $h > 0$

$$\int_{\mathbb{R}^n} \frac{f(x+hv)-f(x)}{h}\ \zeta(x)\,d\mathcal{L}^n(x) = -\int_{\mathbb{R}^n} \frac{\zeta(x)-\zeta(x-hv)}{h}\ f(x)\ d\mathcal{L}^n(x)$$

(by the change of variable $z = x+hv$ in the first part of the integral on the left). Using the dominated convergence theorem and (1) we then get

$$\begin{aligned}\int D_v f\,\zeta &= -\int f\,D_v\,\zeta = -\int f\ v\cdot\operatorname{grad}\zeta\\ &= -\sum_{j=1}^{n} v^j \int f\,D_j\,\zeta = +\sum_{j=1}^{n} v^j \int \zeta\,D_j\,f\\ &= \int \zeta\ v\cdot\operatorname{grad} f\ ,\end{aligned}$$

where all integrals are with respect to Lebesgue measure on $\mathbb{R}^n$, and where we have used Fubini's theorem and the absolute continuity of f on lines to justify the integration by parts. Since ζ is arbitrary we thus have

(2) $$D_v f(x) = v\cdot\operatorname{grad} f(x)\ ,\ \mathcal{L}^n\text{- a.e. } x \in \mathbb{R}^n .$$

Now let $v_1, v_2, \ldots$ be a countable dense subset of S^{n-1}, and let $A_k = \{x : \operatorname{grad} f(x), D_{v_k} f(x)$ exist and $D_{v_k} f(x) = v_k\cdot\operatorname{grad} f(x)\}$. Then with $A = \bigcap_{k=1}^{\infty} A_k$ we have by (2) that

(3) $$\mathcal{L}^n(\mathbb{R}^n \sim A) = 0\ ,\ D_{v_k} f(x) = v_k\cdot\operatorname{grad} f(x) \quad \forall\, x \in A\ ,\ k = 1,2,\ldots\ .$$

Using this, we are now going to prove that f is differentiable at each point x of A. To see this, for any $x \in A$, $v \in S^{n-1}$ and $h > 0$ define

$$Q(x,v,h) = \frac{f(x+hv)-f(x)}{h} - v\cdot\operatorname{grad} f(x)\ .$$

Evidently for any $x \in A$, $v,v' \in S^{n-1}$, $h > 0$,

(4) $$|Q(x,v,h) - Q(x,v',h)| \le (n+1)L\,|v-v'|\ ,\quad L = \operatorname{Lip} f\ .$$

Now let $\varepsilon > 0$ be given and select P large enough so that

(5) $\quad v \in S^{n-1} \Rightarrow |v-v_k| < \frac{\varepsilon}{2(n+1)L}$ *for some* $k \in \{1,\ldots,P\}$.

Since $\lim_{h \downarrow 0} Q(x,v_\ell,h) = 0$, $\forall\, \ell = 1,2,\ldots, x \in A$, (by (2)), we see that for a given $x_0 \in A$ we can choose $\delta > 0$ so that

(6) $\quad |Q(x_0,v_k,h)| < \varepsilon/2$ whenever $0<h<\delta$ and $k \in \{1,\ldots,P\}$.

Since $|Q(x_0,v,h)| \le |Q(x_0,v_k,h)| + |Q(x_0,v,h) - Q(x_0,v_k,h)|$ for each $k \in \{1,\ldots,P\}$, we then have (by (4), (5), (6)) that

$$|Q(x_0,v,h)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $v \in S^{n-1}$ and $0 < h < \delta$. Thus the theorem is proved.

Finally we shall need the following consequence of the Whitney Extension Theorem.

5.3 THEOREM *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is Lipschitz. Then for each* $\varepsilon > 0$ *there is a* $C^1(\mathbb{R})$ *function* g *with*

$$\mathcal{L}^n(\{x : f(x) \neq g(x)\} \cup \{x : \text{grad}\, f(x) \neq \text{grad}\, g(x)\}) < \varepsilon .$$

Proof First recall Whitney's extension theorem for C^1 functions:

If $A \subset \mathbb{R}^n$ is closed and if $h : A \to \mathbb{R}$ and $\nu : A \to \mathbb{R}^n$ are continuous, and if

(*) $$\lim_{\substack{x\to x_0,\, y\to x_0 \\ x,y\in A,\, x\neq y}} R(x,y) = 0 \quad \forall\, x_0 \in A ,$$

where

(**) $$R(x,y) = \frac{h(y)-h(x)-\nu(x)\cdot(y-x)}{|x-y|} ,$$

then there is a C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ such that $g = h$ and $\text{grad}\, g = \nu$ on A . (For the proof see for example [SE] or [FH1]; for the case $n=1$, see Remark 5.4(3) below.)

Now by Rademacher's Theorem f is differentiable on a set $B \subset \mathbb{R}^n$ with $L^n(\mathbb{R}^n \sim B) = 0$. By Lusin's theorem (which applies to sets of infinite measure for L^n) there is a closed set $C \subset B$ such that $\operatorname{grad} f|C$ is continuous and $L^n(\mathbb{R}^n \sim C) < \varepsilon/2$. On C we define $h(x) = f(x)$, $\nu(x) = \operatorname{grad} f(x)$ and $R(x,y)$ for $x,y \in C$ is as defined in (**). Evidently (since $C \subset B$) we have $\lim_{\substack{y\to x\\ y\in C}} R(x,y) = 0 \quad \forall\, x \in C$, but *not* necessarily (*). We therefore proceed as follows. For each $k = 1,2,\ldots$ let

$$\eta_k(x) = \sup\{|R(x,y)| : y \in C \cap (B_{\frac{1}{k}}(x) \sim \{x\})\} .$$

Then $\eta_k \downarrow 0$ pointwise in C, and hence by *Egoroff's Theorem* there is a closed set $A \subset C$ such that $L^n(C \sim A) < \varepsilon/2$ and η_k converges uniformly to zero on A. One now readily checks that (*) holds. Hence we can apply the Whitney Theorem.

5.4 REMARKS

(1) The reader will see that without any significant change the above proof establishes the following: *If* $U \subset \mathbb{R}^n$ *is open and if* $f : U \to \mathbb{R}$ *is differentiable* L^n-a.e. *in* U, *then for each* $\varepsilon > 0$ *there is a closed set* $A \subset U$ *and a* C^1 *function* $g : \mathbb{R}^n \to \mathbb{R}$ *such that* $L^n(U \sim A) < \varepsilon$ *and* $f(x) = g(x)$, $\operatorname{grad} f(x) = \operatorname{grad} g(x)$ *for each* $x \in A$.

(2) The hypothesis (*) above cannot be weakened to the requirement that $\lim_{\substack{y\to x\\ y\in A}} R(x,y) = 0 \quad \forall\, x \in A$. For instance we have the example (for $n=1$) when $A = \{0\} \cup \{\frac{1}{k} : k = 1,2,\ldots\}$ and $h(0) = 0$, $h(\frac{1}{k}) = (-1)^k/k^{3/2}$, $\nu \equiv 0$. Evidently in this case we have $\lim_{\substack{y\to x\\ y\in A}} R(x,y) = 0 \quad \forall\, x \in A$, but there is no C^1 extension because $\dfrac{|h(\frac{1}{k}) - h(\frac{1}{k+1})|}{(\frac{1}{k} - \frac{1}{k+1})} \to \infty$ as $k \to \infty$.

(3) In the case $n = 1$, the Whitney Extension Theorem used above has a simple proof. Namely in this case define

$$R(x,y) = \frac{h(y) - h(x)}{y - x} - \nu(x)$$

and note that the hypothesis (*) guarantees that for each compact subset C of A we have a function ε_C with $\varepsilon_C(t) \downarrow 0$ as $t \downarrow 0$, and

(i) $$|R(x,y)| \le \varepsilon_C(|x-y|) \quad \forall\, x,y \in C .$$

Notice in particular this implies

(ii) $$|\nu(x) - \nu(y)| \le 2\, \varepsilon_C(|x-y|) \quad \forall\, x,y \in C .$$

Also $\mathbb{R} \sim A$ is a countable disjoint union of open intervals $I_1, I_2, \ldots$. If $I_j = (a,b)$, we then select $g_j \in C^1([a,b])$ as follows:

(iii) $g_j(a) = h(a)$, $g_j(b) = h(b)$, $g_j'(a) = \nu(a)$, $g_j'(b) = \nu(b)$

and

(iv) $$\sup_{x \in I_j} |g_j'(x) - \nu(a)| \le 3\, \varepsilon_C(b-a) , \quad C = [a-1,b+1] \cap A .$$

This is possible by (i), (ii), with $(x,y) = (a,b)$. One now defines $g(x) = g_j(x)$ $\forall\, x \in I_j$, $j = 1,2,\ldots$, and $g(x) = h(x)$ $\forall\, x \in A$. It is then easy to check $g \in C^1(\mathbb{R})$ and $g' = \nu$ on A by virtue of (i) - (iv).

§6. BV FUNCTIONS

In this section we gather together the basic facts about locally BV functions which will be needed later.

First recall that if U is open in $\mathbb{R}^n$ and if $u \in L^1_{loc}(U)$, then u

is said to be in $BV_{loc}(U)$ if for each $W \subset\subset U$ there is a constant $c(W) < \infty$ such that

$$\int_W u \operatorname{div} g \, d\mathcal{L}^n \leq c(W) \sup|g|$$

for all vector functions $g = (g^1,\ldots,g^n)$, $g^j \in C_c^\infty(W)$. Notice that this means that the functional $\int_U u \operatorname{div} g$ extends uniquely to give a (real-valued) linear functional on $K(U,\mathbb{R}^n) \equiv \{$continuous $g = (g^1,\ldots,g^n) : U \to \mathbb{R}^n$, support $|g|$ compact$\}$ which is bounded on $K_W(U,\mathbb{R}^n) \equiv \{g \in K(U,\mathbb{R}^n) : \operatorname{spt}|g| \subset W\}$ for every $W \subset\subset U$. Then, by the Riesz representation theorem 4.1, there is a Radon measure μ on U and a μ-measurable function $\nu = (\nu^1,\ldots,\nu^n)$, $|\nu| = 1$ a.e. , such that

6.1 $$\int_U u \operatorname{div} g \, d\mathcal{L}^n = \int_U g \cdot \nu \, d\mu \, .$$

Thus, in the language of distribution theory, the generalized derivatives $D_j u$ of u are represented by the *signed measures* $\nu_j \, d\mu$, $j = 1,\ldots,n$. For this reason we often denote the total variation measure μ (see 4.2) by $|Du|$. (In fact if $u \in W^{1,1}_{loc}(U)$ we evidently do have $d\mu = |Du| \, d\,\mathcal{L}^n$ and

$$\nu_j = \begin{cases} \dfrac{D_j u}{|Du|} & \text{if } |Du| \neq 0 \\ 0 & \text{if } |Du| = 0 \, . \end{cases}$$)

Thus for $u \in BV_{loc}(U)$, $|Du|$ will denote the Radon measure on U which is uniquely characterized by

$$|Du|(W) = \sup_{\substack{|g| \leq 1, \operatorname{spt}|g| \subset W \\ g \text{ smooth}}} \int u \operatorname{div} g \, d\mathcal{L}^n \, , \quad W \text{ open} \subset U \, .$$

The left side here is more usually denoted $\int_W |Du|$. Indeed if f is any non-negative Borel measurable function function on U , then $\int f \, d|Du|$ is more usually denoted simply by $\int f|Du|$ $(\equiv \int f|Du| \, d\mathcal{L}^n$ in case $u \in W^{1,1}_{loc}(U))$.

We shall henceforth adopt this notation.

There are a number of important results about BV functions which can be obtained by mollification. We let $\phi_\sigma(x) = \sigma^{-n}\phi(\frac{x}{\sigma})$, where ϕ is a symmetric mollifier (so that $\phi \in C_c^\infty(\mathbb{R}^n)$, $\phi \geq 0$, $\operatorname{spt} \phi \subset B_1(0)$, $\int_{\mathbb{R}^n} \phi = 1$, and $\phi(x) = \phi(-x)$) , and for $u \in L^1_{loc}(U)$ let $u^{(\sigma)} = \phi_\sigma * u$ be the mollified functions, where we set $\tilde{u} = u$ on U_σ , $\tilde{u} = 0$ outside U_σ , $U_\sigma = \{x \in U : \operatorname{dist}(x, \partial U) > \sigma\}$. A key result concerning mollification is then as follows:

6.2 LEMMA *If* $u \in BV_{loc}(U)$, *then* $u^{(\sigma)} \to u$ *in* $L^1_{loc}(U)$ *and* $|Du^{(\sigma)}| \to |Du|$ *in the sense of Radon measures in* U (see 4.4) *as* $\sigma \downarrow 0$.

Proof The convergence of $u^{(\sigma)}$ to u in $L^1_{loc}(U)$ is standard. Thus it remains to prove

$$\text{(1)} \qquad \lim_{\sigma\downarrow 0} \int f|Du^{(\sigma)}| = \int f|Du|$$

for each $f \in C_c^0(U)$, $f \geq 0$. In fact by definition of $|Du|$ it is rather easy to prove that

$$\int f|Du| \leq \liminf_{\sigma\downarrow 0} \int f|Du^{(\sigma)}| ,$$

so we only have to check

$$\text{(2)} \qquad \limsup_{\sigma\downarrow 0} \int f|Du^{(\sigma)}| \leq \int f|Du|$$

for each $f \in C_c^0(U)$, $f \geq 0$.

This is achieved as follows: First note that

$$\text{(3)} \qquad \int f|Du^{(\sigma)}| = \sup_{|g|\leq f, g \text{ smooth}} \int g\cdot \operatorname{grad} u^{(\sigma)} \, dL^n .$$

On the other hand for fixed g with g smooth and $|g| \leq f$, and for

$\sigma < \operatorname{dist}(\operatorname{spt} f, \partial U)$, we have

$$\begin{aligned}\int g \cdot \operatorname{grad} u^{(\sigma)} \, dL^n &= -\int u^{(\sigma)} \operatorname{div} g \, dL^n \\ &= -\int \phi_\sigma * u \operatorname{div} g \, dL^n \\ &= -\int u(\phi_\sigma * \operatorname{div} g)\, dL^n \\ &= -\int u \operatorname{div}(\phi_\sigma * g)\, dL^n .\end{aligned}$$

On the other hand by definition of $|Du|$, the right side here is

$$\le \int_{W_\sigma} (f+\varepsilon(\sigma))\,|Du|$$

where $\varepsilon(\sigma) \downarrow 0$, where $W = \operatorname{spt} f$, $W_\sigma = \{x \in U : \operatorname{dist}(x,W) < \sigma\}$, because

$$\begin{aligned}|\phi_\sigma * g| &\equiv |(\phi_\sigma * g^1, \ldots, \phi_\sigma * g^n)| \\ &\le \phi_\sigma * |g| \le \phi_\sigma * f\end{aligned}$$

and because $\phi_\sigma * f \to f$ uniformly in W_{σ_0} as $\sigma \downarrow 0$, where $\sigma_0 < \operatorname{dist}(W, \partial U)$. Thus (2) follows from (3).

6.3 THEOREM (Compactness Theorem for BV function)

If $\{u_k\}$ *is a sequence of* $BV_{loc}(U)$ *functions satisfying*

$$\sup_{k\ge 1}\left(\|u_k\|_{L^1(W)} + \int_W |Du_k|\right) < \infty$$

for each $W \subset\subset U$ *, then there is a subsequence* $\{u_{k'}\} \subset \{u_k\}$ *and a* $BV_{loc}(U)$ *function* u *such that* $u_{k'} \to u$ *in* $L^1_{loc}(U)$ *and*

$$\int_W |Du| \le \liminf \int_W |Du_{k'}| \qquad \forall\, W \subset\subset U .$$

Proof By virtue of the previous lemma, in order to prove $u_{k'} \to u$ in

$L^1_{loc}(U)$ for some subsequence $\{u_{k'}\}$, it is enough to prove that the sets

$$\left\{u \in C^\infty(U) : \int_W (|u|+|Du|)\,dL^n \leq c(W)\right\}, \quad W \subset\subset U ,$$

(for given constants $c(W) < \infty$) are precompact in $L^1_{loc}(U)$. For the simple proof of this (involving mollification and Arzela's theorem) see for example [GT, Theorem 7.22].

Finally the fact that $\int_W |Du| \leq \liminf \int_W |Du_{k'}|$ is a direct consequence of the definition of $|Du|$, $|Du_{k'}|$.

Next we have the Poincaré inequality for BV functions.

6.4 LEMMA *Suppose* U *is bounded, open and convex,* $u \in BV_{loc}(\mathbb{R}^n)$ *with* $\operatorname{spt} u \subset \bar{U}$. *Then for any* $\theta \in (0,1)$ *and any* $\beta \in \mathbb{R}$ *with*

$$(*) \qquad \min\{L^n\{x \in U : u(x) \geq \beta\}, L^n\{x \in U : u(x) \leq \beta\}\} \geq \theta\, L^n(U) .$$

we have

$$\int_U |u-\beta| \quad dL^n \leq c \int_U |Du| ,$$

where $c = c(\theta, U)$.

Proof Let β, θ be as in $(*)$ and choose convex $W \subset U$ such that

$$(1) \qquad |Du|(\partial W) = 0 , \quad \int_W |u-\beta| \; dL^n \geq \frac{1}{2}\int_U |u-\beta| dL^n$$

and such that $(*)$ holds with W in place of U and $\theta/2$ in place of θ. (For example we may take $W = \{x \in U : \operatorname{dist}(x,\partial U) > \eta\}$ with η small.)

Letting $u^{(\sigma)}$ denote the mollified functions corresponding to u, note that for sufficiently small σ we must then have $(*)$ with $u^{(\sigma)}$ in place of u, $\theta/4$ in place of θ, $\beta^{(\sigma)} \to \beta$ in place of β, and W in place of U. Hence by the usual Poincaré inequality for smooth functions (see e.g. [GT]) we have

$$\int_W |u^{(\sigma)} - \beta^{(\sigma)}| dL^n \le c \int_W |Du^{(\sigma)}| \; dL^n ,$$

$c = c(n,\theta,W)$, for all sufficiently small σ . The required inequality now follows by letting $\sigma \downarrow 0$ and using (1) above together with 6.2.

6.5 LEMMA *Suppose* U *is bounded, open and convex,* $u \in BV_{loc}(\mathbb{R}^n)$ *with* $\operatorname{spt} u \subset \bar{U}$. *Then*

$$\int_{\mathbb{R}^n} |Du| \left(\equiv \int_{\bar{U}} |Du| \right) \le c\left(\int_U |Du| + \int_U |u| \; dL^n \right) ,$$

where $c = c(U)$.

6.6 REMARK Note that by combining this with the Poincaré inequality 6.4, we conclude

$$\int_{\mathbb{R}^n} |D(u-\beta\chi_U)| \le c \int_U |Du| ,$$

$c = c(\theta,U)$, whenever β is as in (*) of 6.4.

Proof of 6.5 Let $U_\delta = \{x \in U : \operatorname{dist}(x,\partial U) > \delta\}$ and (for δ small) let ϕ_δ be a $C_c^\infty(\mathbb{R}^n)$ function satisfying

(1) $$\phi_\delta = \begin{cases} 1 \text{ in } U_\delta \\ 0 \text{ in } \mathbb{R}^n \sim U_{\delta/2} \end{cases}$$

(2) $$0 \le \phi_\delta \le 1 \quad \text{in} \quad \mathbb{R}^n ,$$

and (for a given point $a \in U$)

(3) $$|D\phi_\delta(x)| \le - c(x-a) \cdot D\phi_\delta(x) \; , \quad x \in U ,$$

where $c = c(U,a)$ is independent of δ . (One easily checks that such ϕ_δ

exist, for sufficiently small δ, because of the convexity of U.)

Now by definition of $|Dw|$ for $BV_{loc}(\mathbb{R}^n)$ functions w, we have

(4) $$\int_{\mathbb{R}^n} |D(\phi_\delta u)| \leq \int_{\mathbb{R}^n} |D\phi_\delta|\ |u|\ dL^n + \int_{\mathbb{R}^n} \phi_\delta |Du|$$

and by (3)

(5) $$\begin{aligned} c^{-1}\int_{\mathbb{R}^n} |D\phi_\delta|\ |u|\ dL^n &\leq -\int (x-a)\cdot D\phi_\delta\ |u|\ dL^n \\ &\equiv -\int (|u|\ \mathrm{div}((x-a)\phi_\delta) + n|u|\phi_\delta)\ dL^n \\ &\leq c\left(\int_U |D|u|| + \int_{\mathbb{R}^n} |u|\ dL^n\right) \end{aligned}$$

(by definition of $|D|u||$)

$$\leq c\left(\int_U |Du| + \int_{\mathbb{R}^n} |u|\ dL^n\right)$$

(because $|D|u|| \leq |Du|$ by virtue of 6.2 and the fact that $|D|u|| \leq \liminf_{\sigma\downarrow 0} |D|u^{(\sigma)}||$).

Finally, to complete the proof of 6.5, we note that (using the definition of $|Dw|$ for the $BV_{loc}(\mathbb{R}^n)$ functions $w = u, \phi_\delta u$, together with the fact that $\phi_\delta u \to u$ in $L^1(\mathbb{R}^n)$)

$$\int_{\mathbb{R}^n} |Du| \leq \liminf_{\delta\downarrow 0} \int_{\mathbb{R}^n} |D(\phi_\delta u)| .$$

Then 6.5 follows from (4), (5).

7. SUBMANIFOLDS OF $\mathbb{R}^{n+k}$

Let M denote an n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$, $0 \leq k$, $r \geq 1$. By this we mean that for each $y \in M$ there are open sets $U, V \subset \mathbb{R}^{n+k}$ with

$y \in U$, $0 \in V$ and a C^r diffeomorphism $\phi : U \to V$ such that $\phi(y) = 0$ and

$$\phi(M \cap U) = W = V \cap \mathbb{R}^n .$$

(Here and subsequently we identify $\mathbb{R}^n$ with the subspace of $\mathbb{R}^{n+k}$ consisting of all points $x = (x^1,\dots,x^{n+k})$ such that $x^j = 0$, $j = n+1,\dots,n+k$.)

In particular we have *local representations*

$$\psi : W \to \mathbb{R}^{n+k}, \quad \psi(W) = M \cap V, \quad \psi(0) = y$$

such that $\frac{\partial\psi}{\partial x^1}(0), \dots, \frac{\partial\psi}{\partial x^n}(0)$ are linearly independent vectors in $\mathbb{R}^{n+k}$. (For example we can take $\psi = \phi^{-1}|W$.) The tangent space T_yM of M at y is the subspace of $\mathbb{R}^{n+k}$ consisting of those $\tau \in \mathbb{R}^{n+k}$ such that

$$\tau = \dot{\gamma}(0) \text{ for some } C^1 \text{ curve } \gamma : (-1,1) \to \mathbb{R}^{n+k}, \gamma(-1,1) \subset M, \gamma(0) = y .$$

One readily checks that T_yM has a basis $\frac{\partial\psi}{\partial x^1}(0), \dots, \frac{\partial\psi}{\partial x^n}(0)$ for a local representation ψ as above.

A function $f : M \to \mathbb{R}^N$ $(N \ge 1)$ is said to be C^ℓ $(\ell \le r)$ on M if f is the restriction to M of a C^ℓ function $\bar{f} : U \to \mathbb{R}^N$, where U is an open set in $\mathbb{R}^{n+k}$ such that $M \subset U$.

Given $\tau \in T_yM$ the directional derivative $D_\tau f \in \mathbb{R}^N$ is defined by $\frac{d}{dt} f(\gamma(t))\Big|_{t=0}$ for any C^1 curve $\gamma : (-1,1) \to M$ with $\gamma(0) = y$, $\dot{\gamma}(0) = \tau$. Of course it is easy to see that this definition is independent of the particular curve γ we use to represent τ. The induced linear map $df_y : T_yM \to \mathbb{R}^N$ is defined by $df_y(\tau) = D_\tau f$, $\tau \in T_yM$. (Evidently df_y is exactly characterized by being the "best linear approximation" to f at y in the obvious sense.)

In case f is real-valued (i.e. $N=1$) then we define the gradient $\nabla^M f$ of f by

$$\nabla^M f(y) = \sum_{j=1}^{n} (D_{\tau_j} f)\tau_j \ , \quad y \in T_y M \ ,$$

$\tau_1,\ldots,\tau_n$ any orthonormal basis for $T_y M$. If we let $\nabla_j^M f \equiv e_j \cdot \nabla^M f$ (e_j = j-th standard basis vector in $\mathbb{R}^{n+k}$, $j=1,\ldots,n+k$) then

$$\nabla^M f(y) = \sum_{j=1}^{n+k} \nabla_j^M f(y) e_j \ .$$

If f is the restriction to M of a $C^1(U)$ function $\bar{f}$, where U is an open subset of $\mathbb{R}^{n+k}$ containing M, then

$$\nabla^M f(y) = (\text{grad}_{\mathbb{R}^{n+k}} \bar{f}(y))^T \ , \quad y \in M \ ,$$

where $\text{grad}_{\mathbb{R}^{n+k}} \bar{f}$ is the usual $\mathbb{R}^{n+k}$ gradient $(D_1\bar{f},\ldots,D_{n+k}\bar{f})$ on U, and where $(\ \ )^T$ means orthogonal projection of $\mathbb{R}^{n+k}$ onto $T_y M$.

Now given a vector function ("vector field") $X = (X^1,\ldots,X^{n+k}) : M \to \mathbb{R}^{n+k}$ with $X^j \in C^1(M)$, $j = 1,\ldots,n+k$, we define

$$\text{div}_M X = \sum_{j=1}^{n+k} \nabla_j^M X^j$$

on M. (Notice that we do *not* require $X_y \in T_y M$.) Then, at $y \in M$, we have

$$\begin{aligned}\text{div}_M X &= \sum_{j=1}^{n+k} e_j \cdot (\nabla^M X^j) \\ &= \sum_{j=1}^{n+k} e_j \cdot \left(\sum_{i=1}^{n} (D_{\tau_j} X^j)\tau_i\right) ,\end{aligned}$$

so that (since $X = \sum_{j=1}^{n+k} X^j e_j$)

$$\text{div}_M X = \sum_{i=1}^{n} (D_{\tau_i} X) \cdot \tau_i \ ,$$

where $\tau_1,\ldots,\tau_n$ is any orthonormal basis for T_yM .

The *divergence theorem* states that if the closure $\bar{M}$ of M is a smooth compact manifold with boundary $\partial M = \bar{M} \sim M$, and if $X_y \in T_yM \ \forall\, y \in M$, then

7.1 $$\int_M \operatorname{div}_M X \, dH^n = -\int_{\partial M} X\cdot\eta \, dH^{n-1}$$

where η is the inward pointing unit co-normal of ∂M ; that is, $|\eta| = 1$, η is normal to ∂M , tangent to M , and points into M at each point of ∂M .

7.2 REMARKS

(1) M need *not* be orientable here.

(2) In general the closure $\bar{M}$ of M will not be a nice manifold with boundary; indeed it can certainly happen that $H^n(\bar{M}\sim M) > 0$. (For example consider the case when $M = \bigcup_{k=1}^{\infty} \{(x,y) \in \mathbb{R}^2 : y = x^2/k\} \sim \{0\}$. M is a C^r 1-dimensional submanifold of $\mathbb{R}^2$ $\forall\, r$ in the sense of the above definitions, but $\bar{M} \sim M$ is the whole x-coordinate axis.) Nevertheless in the general case we still have (in place of 7.1)

$$\int_M \operatorname{div}_M X = 0$$

provided support $X \cap M$ *is a compact subset of* M and $X_y \in T_yM \quad \forall\, y \in M$.

In case M is at least C^2 we define *the second fundamental form* of M at y to be the bilinear form

$$B_y : T_yM \times T_yM \to (T_yM)^{\perp}$$

such that

7.3 $$B_y(\tau,\eta) = -\sum_{\alpha=1}^{k} (\eta\cdot D_\tau \nu^\alpha)\nu^\alpha\Big|_y \ , \quad \tau,\eta \in T_yM \ ,$$

where $\nu^1,\ldots,\nu^k$ are (locally defined, near y) vector fields with $\nu^\alpha(z)\cdot\nu^\beta(z) = \delta_{\alpha\beta}$ and $\nu^\alpha(z) \in (T_zM)^\perp$ for every z in some neighbourhood of y. The geometric significance of B is as follows: If $\tau \in T_yM$ with $|\tau| = 1$ and $\gamma : (-1,1) \to \mathbb{R}^{n+k}$ is a C^2 curve with $\gamma(0) = y$, $\gamma(-1,1) \subset M$, and $\dot\gamma(0) = \tau$, then

$$B_y(\tau,\tau) = (\ddot\gamma(0))^\perp ,$$

which is just the normal component (relative to M) of the *curvature* of γ at 0, γ being considered as an ordinary space-curve in $\mathbb{R}^{n+k}$. (Thus $B_y(\tau,\tau)$ measures the "normal curvature" of M in the direction τ.) To check this, simply note that $\nu^\alpha(\gamma(t)) \cdot \dot\gamma(t) \equiv 0$, $|t| < 1$, because $\dot\gamma(t) \in T_{\gamma(t)}M$ and $\nu^\alpha(\gamma(t)) \in (T_{\gamma(t)}M)^\perp$. Differentiating this relation with respect to t, we get

(after setting $t = 0$)

$$\nu^\alpha(y) \cdot \ddot\gamma(0) = -(D_\tau\nu^\alpha)\cdot\tau$$

and hence (multiplying by $\nu^\alpha(y)$ and summing over α) we have

$$(\ddot\gamma(0))^\perp = -\sum_{\alpha=1}^{k} (\tau\cdot D_\tau\nu^\alpha)\nu^\alpha(y)$$

$$= B_y(\tau,\tau)$$

as required. (Note that the parameter t here need *not* be arc-length for γ; it suffices that $\dot\gamma(0) = \tau$, $|\tau| = 1$.) More generally, by a similar argument, if $\tau, \eta \in T_yM$ and if $\phi : U \to \mathbb{R}^{n+k}$ is a C^2 mapping of a neighbourhood U of 0 in $\mathbb{R}^2$ such that $\phi(U) \subset M$, $\phi(0) = y$, $\frac{\partial\phi}{\partial x^1}(0,0) = \tau$, $\frac{\partial\phi}{\partial x^2}(0,0) = \eta$, then

$$B_y(\tau,\eta) = \left(\frac{\partial^2\phi}{\partial x^1\partial x^2}(0,0)\right)^\perp .$$

In particular $B_y(\tau,\eta) = B_y(\eta,\tau)$; that is B_y is a *symmetric* bilinear form with values in $(T_yM)^\perp$.

We define the mean curvature vector $\underline{\underline{H}}$ of M at y to be trace B_y ; thus

7.4 $$\underline{\underline{H}}(y) = \sum_{i=1}^{n} B_y(\tau_i,\tau_i) \in (T_yM)^\perp ,$$

where $\tau_1, \ldots, \tau_n$ is any orthonormal basis for T_yM . Notice that then (if $\nu^1,\ldots,\nu^k$ are as above)

$$\underline{\underline{H}}(y) = -\sum_{\alpha=1}^{k}\sum_{i=1}^{n} (\tau_i\cdot D_{\tau_i}\nu^\alpha)\nu^\alpha(y)$$

so that

7.5 $$\underline{\underline{H}} = -\sum_{\alpha=1}^{k} (\mathrm{div}_M \nu^\alpha)\nu^\alpha$$

near y .

Returning for a moment to 7.1 (in case $\bar{M}$ is a compact C^2 manifold with smooth $(n-1)$-dimensional boundary $\partial M = \bar{M} \sim M$) it is interesting to compute $\int_M \mathrm{div}_M X$ in case the condition $X_y \in T_yM$ is *dropped*. To compute this, we decompose X into its tangent and normal parts:

$$X = X^\top + X^\perp ,$$

where (at least locally, in the notation introduced above)

$$X^\perp = \sum_{\alpha=1}^{k} (\nu^\alpha\cdot X)\nu^\alpha .$$

Then we have (near y)

$$\mathrm{div}_M X^\perp = \sum_{\alpha=1}^{k} (\nu^\alpha\cdot X)\,\mathrm{div}\,\nu^\alpha ,$$

so that by 7.5

7.5' $$\operatorname{div}_M X^{\perp} = - X \cdot \underline{\underline{H}}$$

at each point of M. On the other hand $\int_M \operatorname{div}_M X^{\top} = - \int_{\partial M} X \cdot \eta$ by 7.1. Hence, since $\operatorname{div}_M X = \operatorname{div}_M X^{\top} + \operatorname{div}_M X^{\perp}$, we obtain

7.6 $$\int_M \operatorname{div}_M X \, d\mathcal{H}^n = - \int_M X \cdot \underline{\underline{H}} \, d\mathcal{H}^n - \int_{\partial M} X \cdot \eta \, d\mathcal{H}^{n-1} .$$

§8. THE AREA FORMULA

Recall that if λ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, then $\mathcal{L}^n(\lambda(A)) = |\det \lambda| \, \mathcal{L}^n(A)$. More generally if $\lambda : \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n$, then $\lambda(\mathbb{R}^n) \subset F$ where F is a n-dimensional subspace of $\mathbb{R}^N$, and hence choosing an orthogonal transformation q of $\mathbb{R}^N$ such that $q(F) = \mathbb{R}^n$, we see that $q \circ \lambda : \mathbb{R}^n \to \mathbb{R}^n$ and hence $\mathcal{L}^n(q\lambda(A)) = |\det(q\lambda)| \mathcal{L}^n(A)$ for $A \subset \mathbb{R}^n$. One readily checks, since q is orthogonal, that $|\det(q\lambda)| = \sqrt{\det \lambda^* \circ \lambda}$, where $\lambda^* : \mathbb{R}^N \to \mathbb{R}^n$ is the adjoint of λ. Since $\mathcal{H}^n(q(B)) = \mathcal{H}^n(B)$ (by definition of $\mathcal{H}^n$) we have by Theorem 2.8 that $\mathcal{L}^n(q\lambda(A)) = \mathcal{H}^n(q\lambda(A)) = \mathcal{H}^n(\lambda(A))$, and hence we obtain the area formula

8.1 $$\mathcal{H}^n(\lambda(A)) = \sqrt{\det \lambda^* \circ \lambda} \; \mathcal{H}^n(A) , \quad A \subset \mathbb{R}^n ,$$

whenever λ is a linear map $\mathbb{R}^n \to \mathbb{R}^N$, $n \leq N$.

More generally given a 1:1 C^1 map $f : M \to \mathbb{R}^N$ (M an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$) we have, by an approximation argument based on the linear case 8.1 (see [HR] or [FH1] for details) that

8.2 $$\mathcal{H}^n(f(A)) = \int_A Jf \, d\mathcal{H}^n \quad \forall \; \mathcal{H}^n\text{-measurable set } A \subset M ,$$

where Jf is the Jacobian of f (or area magnification factor of f) defined by

8.3 $$Jf(y) = \sqrt{\det (df_y)^* \circ (df_y)} \ .$$

Here $df_y : T_yM \to \mathbb{R}^N$ is the induced linear map described in §7, and $(df_y)^* : \mathbb{R}^N \to T_yM$ denotes the adjoint transformation.

If f is not 1:1 we have the general area formula (which actually follows quite easily from 8.2)

8.4 $$\int_{\mathbb{R}^N} H^0(f^{-1}(y) \cap A)\, dH^n(y) = \int_A Jf \, dH^n \ , \quad \forall \ H^n\text{-measurable } A \subset M \ ,$$

where H^0 is 0-dimensional Hausdorff measure i.e. "counting measure".

(Thus $H^0(B) = 0$ if $B = \emptyset$, $H^0(B) =$ the number of elements of the set B if B is a finite non-empty set, and $H^0(B) = \infty$ if B is not finite). More generally still, if g is a non-negative H^n-measurable function on M, then

8.5 $$\int_{\mathbb{R}^N} \int_{f^{-1}(y)} g \, dH^0 \, dH^n(y) = \int_M (Jf) \, g \, dH^n \ .$$

This follows directly from 8.4 if we approximate g by simple functions.

8.6 EXAMPLES

(1) *Space curves.* Using the above area formula we first check that H^1-measure agrees with the usual arc-length measure for C^1 curves in $\mathbb{R}^n$. In fact if $\gamma : [a,b] \to \mathbb{R}^n$ is a 1:1 C^1 map then the Jacobian is just $\sqrt{|\dot{\gamma}|^2} = |\dot{\gamma}|$, so that 8.2 gives

$$H^1(\gamma(A)) = \int_A |\dot{\gamma}| \, dL^1$$

as required.

(2) *Submanifolds of* $\mathbb{R}^{n+k}$. If M is any n-dimensional C^1 manifold of $\mathbb{R}^{n+k}$, we want to check that $\mathcal{H}^n$ agrees with the usual n-dimensional volume measure on M. It is enough to check this in a region where a local coodinate representation as in §7 applies. If

$$\psi : W \to \mathbb{R}^{n+k}, \quad \psi(W) = M \cap U$$

is a local representation for M as in §7 then the usual definition of the n-dimensional volume of a Borel set $A \subset M \cap U$ is

$$\mu(A) = \int_{\tilde{A}} \sqrt{g}\, d\mathcal{L}^n ,$$

where $\tilde{A} = \psi^{-1}(A)$ and $g = \det(g_{ij})$, $g_{ij} = \frac{\partial\psi}{\partial x^i} \cdot \frac{\partial\psi}{\partial x^j}$, $i,j = 1,\ldots,n$. However one easily checks that then $\sqrt{g}$ is precisely $J\psi$, the Jacobian of $\psi : W \to \mathbb{R}^{n+k}$, defined as above. Hence we have by the area formula 8.2 that $\int_{\tilde{A}} \sqrt{g}\, d\mathcal{L}^n = \mathcal{H}^n(\psi(\tilde{A})) = \mathcal{H}^n(A)$, so that $\mu(A) = \mathcal{H}^n(A)$.

(3) *n-dimensional graphs in* $\mathbb{R}^{n+1}$. If Ω is a domain in $\mathbb{R}^n$ and if $M = \text{graph } u$, where $u \in C^1(\Omega)$, then M is globally represented by the map $\psi : x \mapsto (x,u(x))$; in this case $J\psi(x) \equiv \sqrt{\det(\frac{\partial\psi}{\partial x^i} \cdot \frac{\partial\psi}{\partial x^j})}$

$$\equiv \sqrt{\det(\delta_{ij} + D_i u D_j u)} = \sqrt{1 + |Du|^2} ,$$

so $\mathcal{H}^n(M) = \int_\Omega \sqrt{1 + |Du|^2}\, dx$ (by (2) above).

§9. FIRST AND SECOND VARIATION FORMULAE

Suppose that M is an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$ and let U be an open subset of $\mathbb{R}^{n+k}$ such that $U \cap M \neq \emptyset$ and $\mathcal{H}^n(K \cap M) < \infty$ for each compact $K \subset U$. Also, let $\{\phi_t\}_{0 \le t \le 1}$ be a 1-parameter family of diffeomorphisms $U \to U$ such that

$$9.1 \quad \begin{cases} (1) & \phi(t,x)\,(\equiv \phi_t(x)) \text{ is a } C^2 \text{ map}: (-1,1)\times U \to U \\ (2) & \phi_0(x) \equiv x\ ,\quad x \in U \\ (3) & \phi_t(x) \equiv x \quad \forall\, t \in (-1,1)\ ,\ x \in U\sim K\ , \text{ where } K \subset U \end{cases}$$

is a compact subset of U .

Also, let X , Z denote the initial velocity and acceleration vectors for ϕ_t : thus $X_x = \left.\frac{\partial\phi(t,x)}{\partial t}\right|_{t=0}$, $Z_x = \left.\frac{\partial^2\phi(t,x)}{\partial t^2}\right|_{t=0}$.

Then

$$9.2 \qquad \phi_t(x) = x + tX_x + \frac{t^2}{2} Z_x + O(t^3)$$

and X , Z have supports which are compact subsets of U . Let $M_t = \phi_t(M \cap K)$ (K as in 9.1 (3)); thus M_t is a 1-parameter family of manifolds such that $M_0 = M \cap K$ and M_t agrees with M outside some compact subset of U . We want to compute $\left.\frac{d}{dt} \mathcal{H}^n(M_t)\right|_{t=0}$ and $\left.\frac{d^2}{dt^2} \mathcal{H}^n(M_t)\right|_{t=0}$ (i.e. the "first and second variation" of M). The area formula is particularly useful here because it gives (with K as in 9.1 (3))

$$\mathcal{H}^n(\phi_t(M \cap K)) = \int_{M\cap K} J\psi_t \, d\mathcal{H}^n\ ,\quad \psi_t = \phi_t | M \cap U\ ,$$

and hence to compute the first and second variation we can differentiate under the integral. Thus the computation reduces to calculation of $\left.\frac{\partial}{\partial t} J\psi_t\right|_{t=0}$ and $\left.\frac{\partial^2}{\partial t^2} J\psi_t\right|_{t=0}$.

To calculate we first want to get a manageable expression for $J\psi_t$. First note that (for fixed t)

$$\begin{aligned} d\psi_t|_x(\tau) &= D_\tau \psi_t \qquad (\tau \in T_x M) \\ &= \tau + t\, D_\tau X + \frac{t^2}{2} D_\tau Z + O(t^3) \qquad \text{(by 9.2)} . \end{aligned}$$

Hence, relative to the bases $\tau_1,\ldots,\tau_n$ for T_xM and $e_1,\ldots,e_{n+k}$ for $\mathbb{R}^{n+k}$, the map $d\psi_t|_x : T_xM \to \mathbb{R}^{n+k}$, has matrix

$$a_{\ell i} = \tau_i^\ell + t\, D_{\tau_i} X^\ell + \frac{t^2}{2} D_{\tau_i} Z^\ell + O(t^3)$$

for $i = 1,\ldots,n$, $\ell = 1,\ldots,n+k$. Then $(d\psi_t|_x)^* \circ (d\psi_t|_x)$ has matrix $\left(\sum_{\ell=1}^{n+k} a_{\ell i} a_{\ell j}\right)_{i,j=1,\ldots,n} \equiv (b_{ij})$, where

$$\begin{aligned} b_{ij} = \delta_{ij} &+ t(\tau_i \cdot D_{\tau_j} X + \tau_j \cdot D_{\tau_i} X) \\ &+ t^2(\tfrac{1}{2}(\tau_i \cdot D_{\tau_j} Z + \tau_j \cdot D_{\tau_i} Z) + (D_{\tau_i} X) \cdot (D_{\tau_j} X)) \\ &+ O(t^3), \end{aligned}$$

so that (by the general formula $\det(I+tA+t^2B) = 1 + t \operatorname{trace} A + t^2(\operatorname{trace} B + \frac{1}{2}(\operatorname{trace} A)^2 - \frac{1}{2}\operatorname{trace}(A^2)) + O(t^3)$) we have

$$\begin{aligned} (J\psi_t)^2 &= 1 + 2t\, \operatorname{div}_M X + t^2(\operatorname{div}_M Z + \sum_{i=1}^n |D_{\tau_i} X|^2 \\ &\qquad + 2(\operatorname{div}_M X)^2 - \frac{1}{2}\sum_{i,j=1}^n (\tau_i \cdot D_{\tau_j} X + \tau_j \cdot D_{\tau_i} X)^2 + O(t^3) \\ &= 1 + 2t\, \operatorname{div}_M X + t^2(\operatorname{div}_M Z + \sum_{i=1}^n |(D_{\tau_i} X)^\perp|^2 \\ &\qquad + 2(\operatorname{div}_M X)^2 - \sum_{i,j=1}^n (\tau_i \cdot D_{\tau_j} X)(\tau_j \cdot D_{\tau_i} X)) + O(t^3), \end{aligned}$$

where $(D_{\tau_i} X)^\perp$ ($\equiv$ normal part of $D_{\tau_i} X$) $= D_{\tau_i} X - \sum_{j=1}^n (\tau_j \cdot D_{\tau_i} X)\tau_j$.

Using $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$, we thus get

$$\begin{aligned} J\psi_t = 1 + t\, \operatorname{div}_M X + \frac{t^2}{2}(\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^n |(D_{\tau_i} X)^\perp|^2 \\ - \sum_{i,j=1}^n (\tau_i \cdot D_{\tau_j} X)(\tau_j \cdot D_{\tau_i} X)) + O(t^3). \end{aligned}$$

Thus the area formula immediately yields the first and second variation formulae:

9.3
$$\frac{d}{dt} \mathcal{H}^n(M_t)\Big|_{t=0} = \int_M \operatorname{div}_M X \, d\mathcal{H}^n$$

and

9.4
$$\frac{d^2}{dt^2} \mathcal{H}^n(M_t)\Big|_{t=0} = \int_M (\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^n |(D_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^n (\tau_i \cdot D_{\tau_j} X)(\tau_j \cdot D_{\tau_i} X)) .$$

We shall use the terminology that M *is stationary in* U if $\mathcal{H}^n(M \cap K) < \infty$ for each compact $K \subset U$ and if $\frac{d}{dt} \mathcal{H}^n(M_t)\Big|_{t=0} = 0$ whenever $M_t = \phi_t(M \cap K)$, K, ϕ_t as in 9.1. Thus in view of 9.3 we see that M is *stationary in* U *if and only if* $\int_M \operatorname{div}_M X \, d\mathcal{H}^n = 0$ whenever X is C^1 on U with support X a compact subset of U.

In view of 7.6 we also have the following

9.5 LEMMA

(1) *If* M *is a* C^2 *submanifold of* $\mathbb{R}^{n+k}$ *and* $\bar{M}$ *is a* C^2 *submanifold with smooth* $(n-1)$*-dimensional boundary* $\partial M = \bar{M} \sim M$, *then* M *is stationary in* U *if and only if* $\underline{\underline{H}} \equiv 0$ *on* $M \cap U$ *and* $\partial M \cap U = \emptyset$.

(2) *Generally, if* M *is an arbitrary* C^2 *submanifold of* $\mathbb{R}^{n+k}$ *and* $\bar{U} \cap M$ *is a compact subset of* M , *then* M *is stationary in* U *if and only if* $\underline{\underline{H}} \equiv 0$ *on* $M \cap U$.

(In both parts (1), (2) above $\underline{\underline{H}}$ denotes the mean curvature vector of M .)

For later reference we also want to mention an important modification of the idea that M be stationary in U, U open in $\mathbb{R}^{n+k}$. Namely, suppose N is a C^2 $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$, $0 \le k_1 \le k$, and *suppose* U *is an open subset of* N *and* $M \subset N$. Then *we say that* M *is stationary in* U *if* 9.3 *holds whenever* $X_y \in T_y N$ $\forall\, y \in M$. This is evidently equivalent to the requirement that $\frac{d}{dt} \mathcal{H}^n(\phi_t(M \cap K))\Big|_{t=0} = 0$ whenever ϕ_t satisfies the conditions 9.1 (bearing in mind that U is required now to be an open subset of N rather than an open subset of $\mathbb{R}^{n+k}$ as before). If we let $\nu^1, \ldots, \nu^k$ be an orthonormal family (defined locally near a point $y \in M$) of vector fields normal to M, such that $\nu^1, \ldots, \nu^{k_1}$ are tangent to N and $\nu^{k_1+1}, \ldots, \nu^k$ are normal to N, then for any vector field X on M we can write $X = X^{(1)} + X^{(2)}$, where $X_z^{(1)} \in T_z N$ and $X^{(2)} = \sum_{j=k_1+1}^{n} (\nu^j \cdot X)\nu^j$ (= part of X normal to N). Then if $\tau_1, \ldots, \tau_n$ is any orthonormal basis for $T_y M$, we have

$$\begin{aligned} \operatorname{div}_M X &= \operatorname{div}_M X^{(1)} + \sum_{j=k_1+1}^{n} (\nu^j \cdot X) \operatorname{div}_M \nu^j \\ &\equiv \operatorname{div}_M X^{(1)} + \sum_{i=1}^{n} X \cdot \bar{B}_y(\tau_i, \tau_i) , \end{aligned}$$

where $\bar{B}_y$ is the second fundamental form of N at y,

Thus we conclude

9.6 LEMMA *If* N *is an* $(n+k_1)$*-dimensional submanifold of* $\mathbb{R}^{n+k}$, *if* $M \subset N$ *and if* U *is an open subset of* N *such that* $\mathcal{H}^n(M \cap K) < \infty$ *whenever* K *is a compact subset of* U, *then* M *is stationary in* U *if and only if*

$$\int_M \operatorname{div}_M X = - \int_M \bar{\underline{\underline{H}}}_M \cdot X$$

for each C^1 *vector field* X *with compact support contained in* U; *here* $\bar{\underline{\underline{H}}}_M\big|_y = \sum_{i=1}^{n} \bar{B}_y(\tau_i, \tau_i)$, $y \in M$, *where* $\bar{B}_y$ *denotes the second fundamental form of* N *at* y *and* $\tau_1, \ldots, \tau_n$ *is any orthonormal basis of* $T_y M$.

Finally, we shall need later the following important fact about second variation formula 9.4.

9.7 LEMMA *If* M *is* C^2 *, stationary in* U *,* U *open in* $\mathbb{R}^{n+k}$ *with* $(\bar{M} \sim M) \cap U = \emptyset$ *, and if* X *as in* 9.4 *has compact support in* U *with* $X_y \in (T_y M)^{\perp}$ $\forall\, y \in M$ *, then* 9.4 *says*

$$\frac{d^2}{dt^2} \mathcal{H}^n(M_t)\Big|_{t=0} = \int_M \left(\sum_{i=1}^n |(D_{\tau_i} X)^{\perp}|^2 - \sum_{i,j=1}^n (X \cdot B(\tau_i, \tau_j))^2 \right) d\mathcal{H}^n .$$

9.8 REMARK In case $k = 1$ and M is orientable, with continuous unit normal ν , then $X = \zeta\nu$ for some scalar function ζ with compact support on M , and the above identity has the simple form

$$\frac{d^2}{dt^2} \mathcal{H}^n(M_t)\Big|_{t=0} = \int_M (|\nabla^M \zeta|^2 - \zeta^2 |B|^2)\, d\mathcal{H}^n ,$$

where $|B|^2 = \sum_{i,j=1}^n |B(\tau_i, \tau_j)|^2 \equiv \sum_{i,j=1}^n |\nu \cdot B(\tau_i, \tau_j)|^2$. This is clear, because $(D_{\tau_i}(\nu\zeta))^{\perp} = \nu\, D_{\tau_i} \zeta$ by virtue of the fact that $D_{\tau_i} \nu \big|_y \in T_y M$ $\forall\, y \in M$.

Proof of Lemma 9.7 First we note that $\int_M \operatorname{div}_M Z\, d\mathcal{H}^n = 0$ by virtue of the fact that M is stationary in U , and second we note that $\operatorname{div}_M X = -X \cdot \underline{\underline{H}} = 0$ by virtue of 7.5' and 9.5(2) and the fact that X is normal to M . The proof is completed by noting that $\tau_i \cdot D_{\tau_j} X \equiv X \cdot B(\tau_i, \tau_j)$ by virtue of 7.3 and the fact that X is normal to M .

§10. CO-AREA FORMULA

As in our discussion of the area formula, we begin by looking at linear maps $\lambda : \mathbb{R}^n \to \mathbb{R}^N$, but here we assume $N < n$. Let us first look at the special case when λ is the orthogonal projection p of $\mathbb{R}^n$ onto $\mathbb{R}^N$. (As before, we identify $\mathbb{R}^N$ with the subspace of $\mathbb{R}^n$ consisting of all points

$(x^1,\ldots,x^n)$ with $x^j = 0$, $j = n-N+1,\ldots,n$.) The orthogonal projection p has the property that, for each $y \in \mathbb{R}^N$, $p^{-1}(y)$ is an (N-n)-dimensional affine space; each of these spaces is a translate of the (N-n)-dimensional subspace $p^{-1}(0)$. Thus the inverse images $p^{-1}(y)$ decompose all of $\mathbb{R}^n$ into parallel "(n-N)-dimensional slices" and by Fubini's Theorem

$$10.1 \qquad \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(p^{-1}(y) \cap A)\,dy = \mathcal{H}^n(A)$$

whenever A is an $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$.

This formula (which, we emphasize again, is just Fubini's Theorem) is a special case of a more general formula known as the co-area formula. We first derive this in case of an arbitrary linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^N$ with rank $\lambda = N$.

Let $F = \lambda^{-1}(0)$. (Then for each $y \in \mathbb{R}^N$, $\lambda^{-1}(y)$ is an (n-N)-dimensional affine space which is a translate of F ; the sets $\lambda^{-1}(y)$ thus decompose all of $\mathbb{R}^n$ into parallel (n-N)-dimensional slices.)

Take an orthogonal transformation $q = \mathbb{R}^n \to \mathbb{R}^n$ such that $q\,F^{\perp} = \mathbb{R}^N$, $q\,F = \mathbb{R}^{n-N}$. Then λ can be represented in the form $\lambda = \sigma \circ p \circ q$, where p is the orthogonal projection $\mathbb{R}^n$ onto $\mathbb{R}^N$ and σ is a non-singular transformation of $\mathbb{R}^N$. (This is easily checked by considering the action of λ on suitable basis vectors.) By 10.1 above, for any $\mathcal{H}^n$-measurable $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) = \mathcal{L}^n(q(A)) = \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(q(A) \cap p^{-1}(y))\,d\mathcal{L}^N(y)$$

$$= \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(A \cap q^{-1}(p^{-1}(y)))\,d\mathcal{L}^N(y) \ .$$

making the change of variable $z = \sigma(y)$ $(dy = |\det\sigma|^{-1}dz)$, we thus get

$$|\det\sigma| L^n(A) = \int_{\mathbb{R}^N} H^{n-N}(A \cap q^{-1}(p^{-1}(\sigma^{-1}(z)))) dL^N(z)$$

$$\equiv \int_{\mathbb{R}^N} H^{n-N}(A \cap \lambda^{-1}(z)) dL^N(z) .$$

Also, since $q^*q = 1_{\mathbb{R}^n}$ and $pp^* = 1_{\mathbb{R}^N}$, we have $\lambda \circ \lambda^* = \sigma \circ \sigma^* : \mathbb{R}^N \to \mathbb{R}^N$, so that $|\det\sigma| = \sqrt{\det \lambda\circ\lambda^*}$.

Thus finally

10.2 $$\sqrt{\det \lambda\circ\lambda^*}\ L^n(A) = \int_{\mathbb{R}^N} H^{n-N}(A \cap \lambda^{-1}(z)) dL^N(z) .$$

This is the co-area formula for linear maps. (Note that it is trivially valid, with both sides zero, in case rank $\lambda < N$.)

Generally, given a C^1 map $f : M \to \mathbb{R}^N$, where M is an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$, we can define

$$J^*f(x) = \sqrt{\det(df_x) \circ (df_x)^*} ,$$

where, as usual, $df_x : T_xM \to \mathbb{R}^N$ denotes the induced linear map. Then for any Borel set $A \subset M$

10.3 $$\int_A J^*f\ dH^n = \int_{\mathbb{R}^N} H^{n-N}(A \cap f^{-1}(y))\ dL^N(y) .$$

This is the general co-area formula. Its proof uses an approximation argument based on the linear case 10.2. (See [HR1] or [FH1] for the details.)

An important consequence of 10.3 is that if $C = \{x \in M : J^*f(x) = 0\}$, then (by using 10.3 with $A = C$) $H^{n-N}(C \cap f^{-1}(y)) = 0$ for L^N-a.e. $y \in \mathbb{R}^N$. Since $J^*f(x) \neq 0$ precisely when df_x has rank N, it is clear from the implicit function theorem that $x \in f^{-1}(y) \sim C \Rightarrow \exists$ a neighbourhood V of x

such that $V \cap f^{-1}(y)$ is an $(n-N)$-dimensional C^1 submanifold. In summary we thus have the following important result.

10.4 THEOREM (C^1 Sard-type theorem.)

Suppose $f : M \to \mathbb{R}^N$, $N < n$, *is* C^1. *Then for* $\mathcal{L}^N$ - a.e. $y \in f(M)$, $f^{-1}(y)$ *decomposes into an* $(n-N)$*-dimensional* C^1 *submanifold and a closed set of* $\mathcal{H}^{n-N}$*-measure zero. Specifically,*

$$f^{-1}(y) = (f^{-1}(y) \sim C) \cup (f^{-1}(y) \cap C) ,$$

$C = \{x \in M : J^*f(x) = 0\}$ $(\equiv \{x \in M : \operatorname{rank}(df_x) < N\})$, $\mathcal{H}^{n-N}(f^{-1}(y) \cap C) = 0$, $\mathcal{L}^N$ - a.e. y, *and* $f^{-1}(y) \sim C$ *is an* $(n-N)$*-dimensional* C^1 *submanifold.*

10.5 REMARK If f and M are of class C^{n-N+1}, then *Sard's Theorem* asserts the stronger result that in fact $f^{-1}(y) \cap C = \emptyset$ for $\mathcal{L}^N$ - a.e. $y \in \mathbb{R}^N$, so that $f^{-1}(y)$ is an $(n-N)$-dimensional C^{n-N+1} submanifold for $\mathcal{L}^N$ - a.e. $y \in \mathbb{R}^N$.

A useful generalization of 10.3 is as follows: If g is a non-negative $\mathcal{H}^n$-measurable function on M, then

$$\text{10.6} \qquad \int_M (J^*f) g \, d\mathcal{H}^n = \int_{\mathbb{R}^N} \int_{f^{-1}(y)} g \, d\mathcal{H}^{n-N} \, d\mathcal{L}^N(y) .$$

10.7 REMARKS

(1) Notice that the above formulae enable us to bound the $\mathcal{H}^{n-N}$ measure of the "slices" $f^{-1}(y)$ for a good set of y. Specifically if $|f| \le R$ and g is as in 10.6 ($g \equiv 1$ is an important case), then there must be a set $S \subset B_R(0)$ $(\subset \mathbb{R}^N)$, $S = S(g,f,M)$, with $\mathcal{L}^N(S) \ge \frac{1}{2} \mathcal{L}^N(B_R(0))$ and with

$$\int_{f^{-1}(y)} g \, d\mathcal{H}^{n-N} \le \frac{2}{\mathcal{L}^N(B_R(0))} \int_M g \, J^*f \, d\mathcal{H}^n$$

for each $y \in S$. For otherwise there would be a set $T \subset B_R(0)$ with $\mathcal{L}^N(T) > \frac{1}{2}\mathcal{L}^N(B_R(0))$ and

$$\int_{f^{-1}(y)} g \, d\mathcal{H}^{n-N} \geq \frac{2}{\mathcal{L}^N(B_R(0))} \int_M g \, J^*f \, d\mathcal{H}^n \ , \quad y \in T \ ,$$

so that, integrating over T we obtain a contradiction to 10.6 if $\int_M g \, J^*f \, d\mathcal{H}^n > 0$. On the other hand, if $\int_M g \, J^*f \, d\mathcal{H}^n = 0$ then the required result is a trivial consequence of 10.6.

(2) The above has an important extension to the case when we have $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ and *sequences* $\{M_j\}$, $\{g_j\}$ satisfying the conditions of M , g above. In this case there is a set $S \subset B_R(0)$ with $\mathcal{L}^N(S) \geq \frac{1}{2}\mathcal{L}^N(B_R(0))$ such that for each $y \in S$ there is a *subsequence* $\{j'\}$ (depending on y) with

$$\int_{M_{j'} \cap f^{-1}(y)} g_{j'} \, d\mathcal{H}^{n-N} \leq \frac{2}{\mathcal{L}^N(B_R(0))} \int_{M_{j'}} g_{j'} \, J^*f \, d\mathcal{H}^n \ .$$

Indeed otherwise there is a set T with $\mathcal{L}^N(T) > \frac{1}{2}\mathcal{L}^N(B_R(0))$ so that for each $y \in T$ there is $\ell(y)$ such that

$$(*) \qquad \int_{M_j \cap f^{-1}(y)} g_j \, d\mathcal{H}^{n-N} > \frac{2}{\mathcal{L}^N(B_R(0))} \int_{M_j} g_j \, J^*f \, d\mathcal{H}^n$$

for each $j > \ell(y)$. But $T = \bigcup_{j=1}^{\infty} T_j$, $T_j = \{y \in T : \ell(y) \leq j\}$, and hence there must exist j so that $\mathcal{L}^N(T_j) > \frac{1}{2}\mathcal{L}^N(B_R(0))$. Then, integrating (*) over $y \in T_j$, we obtain a contradiction to 10.6 as before.

CHAPTER 3

COUNTABLY n-RECTIFIABLE SETS

The countably n-rectifiable sets, the theory of which we develop in this chapter, provide the appropriate notion of "generalized surface"; they are the sets on which rectifiable currents and varifolds live (see later).

In the first section of this chapter we give some basic definitions, and prove the important result that countably n-rectifiable sets are essentially characterized by the property of having a suitable "approximate tangent space" almost everywhere.

In later sections we show that the area and co-area formula (see §§8,10 of Chapter 2) extend naturally to the case when M is merely countably n-rectifiable rather than a C^1 submanifold, we make a brief discussion of Federer's structure theorem (for the proof we refer to [FH1] or [RM]), and finally we discuss sets of finite perimeter, which play an important role in later developments.

§11. BASIC NOTIONS, TANGENT PROPERTIES

Firstly, a set $M \subset \mathbb{R}^{n+k}$ is said to be countably n-rectifiable if $M \subset M_0 \cup (\bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n))$, where $\mathcal{H}^n(M_0) = 0$ and $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ are Lipschitz functions for $j = 1,2,\ldots$ *. Notice that by the extension theorem 5.1 this is equivalent to saying

$$M = M_0 \cup (\bigcup_{j=1}^{\infty} F_j(A_j))$$

---

* Notice that this differs slightly from the terminology of [FH1] in that we allow the set M_0 of $\mathcal{H}^n$-measure zero.

where $\mathcal{H}^n(M_0) = 0$, $F_j : A_j \to \mathbb{R}^{n+k}$ Lipschitz, $A_j \subset \mathbb{R}^n$. More importantly, we have the following lemma.

11.1 LEMMA *M is countably n-rectifiable if and only if $M \subset \bigcup_{j=0}^{\infty} N_j$, where $\mathcal{H}^n(N_0) = 0$ and where each N_j , $j \geq 1$, is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+k}$.*

Proof The "if" part is essentially trivial and is left to the reader. The "only if" part is an easy consequence of Theorem 5.3 as follows. By Theorem 5.3 we can choose C^1 functions $g_1^{(j)}, g_2^{(j)}, \ldots$ such that, if F_j are Lipschitz functions as in the above definition of countably n-rectifiable, then

$$F_j(\mathbb{R}^n) \subset E_j \cup \Big(\bigcup_{i=1}^{\infty} g_i^{(j)}(\mathbb{R}^n) \Big) , \quad j = 1,2\ldots$$

where $\mathcal{H}^n(E_j) = 0$. Then we let

$$N_0 = \Big(\bigcup_{j=1}^{\infty} E_j \Big) \cup \Big(\bigcup_{i,j=1}^{\infty} g_i^{(j)}(C_{ij}) \Big) ,$$

where $C_{ij} = \{x \in \mathbb{R}^n : J\, g_i^{(j)}(x) = 0\}$ and $J\, g_i^{(j)}$ denotes the Jacobian of $g_i^{(j)}$ as in §8. By the area formula (see §8) we have $\mathcal{H}^n\big(\bigcup_{i,j=1}^{\infty} g_i^{(j)}(C_{ij})\big) = 0$ and hence $\mathcal{H}^n(N_0) = 0$.

Now for each $x \in \mathbb{R}^n \sim C_{ij}$ we let $U_{ij}(x)$ be an open subset of $\mathbb{R}^n \sim C_{ij}$ containing x and such that $g_i^{(j)} | U_{ij}(x)$ is 1:1 . Such $U_{ij}(x)$ exists by the inverse function theorem (since $J\, g_i^{(j)}(x) > 0 \Rightarrow d\, g_i^{(j)}(x)$ has rank n) , and the inverse function theorem also guarantees that $g_i^{(j)}(U_{ij}(x)) \equiv N_{ij}(x)$, say, is an n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$ in the sense of §7. We can evidently choose a countable collection $x_1, x_2, \ldots$ of points of $\mathbb{R}^n \sim C_{ij}$ such that $\bigcup_{k=1}^{\infty} U_{ij}(x_k) = \mathbb{R}^n \sim C_{ij}$, hence $\bigcup_{k=1}^{\infty} N_{ij}(x_k) \supset g_i^{(j)}(\mathbb{R}^n \sim C_{ij})$, so we have $F_j(\mathbb{R}^n) \sim N_0 \subset \bigcup_{i,k=1}^{\infty} N_{ij}(x_k)$ for each j . The required result now evidently follows.

We now want to give an important characterization of countably n-rectifiable sets in terms of approximate tangent spaces, which we first define.

11.2 DEFINITION If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ with $\mathcal{H}^n(M \cap K) < \infty$ $\forall$ compact K, then we say that an n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M at x (x a given point in $\mathbb{R}^{n+k}$) if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y)\, d\mathcal{H}^n(y) = \int_P f(y)\, d\mathcal{H}^n(y) \qquad \forall\, f \in C_c^0(\mathbb{R}^{n+k})$$

(Recall $\eta_{x,\lambda} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$, $x,y \in \mathbb{R}^{n+k}$, $\lambda > 0$.)

11.3 REMARK Of course P is unique if it exists; we shall denote it by $T_x M$.

It is often convenient to be able to relax the condition $\mathcal{H}^n(M \cap K) < \infty$ $\forall$ compact K in 11.2; we can in fact define $T_x M$ in case we merely assume the existence of a positive locally $\mathcal{H}^n$-integrable function θ on M (the existence of such a θ is evidently equivalent to the requirement that M can be expressed as the countable union of $\mathcal{H}^n$-measurable sets with locally finite $\mathcal{H}^n$-measure).

11.4 DEFINITION If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ and θ is a positive locally $\mathcal{H}^n$-integrable function on M, then we say that a given n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M at x with respect to θ if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda} M} f(y)\, \theta(x+\lambda y)\, d\mathcal{H}^n(y) = \theta(x) \int_P f(y)\, d\mathcal{H}^n(y) \qquad \forall\, f \in C_c^0(\mathbb{R}^{n+k}) .$$

(By change of variable $z = \lambda y + x$, this is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{-n} \int_M f(\lambda^{-1}(z-x))\theta(z)\,d\mathcal{H}^n(z) = \theta(x) \int_P f(y)\,d\mathcal{H}^n(y) \quad \forall\, f \in C_c^0(\mathbb{R}^{n+k}) .$$

11.5 REMARK Notice that if $\mu = \mathcal{H}^n \llcorner \theta$ and if $M_\eta = \{x \in M : \theta(x) > \eta\}$, then $\mathcal{H}^n(M_\eta \cap K) < \infty$ for each compact K and $\Theta^{*n}(\mu, M\sim M_\eta, x) = 0$ for $\mathcal{H}^n$-a.e. $x \in M_\eta$ (by 3.5). Hence for $\mathcal{H}^n$-a.e. $x \in M_\eta$ the approximate tangent space for M with respect to θ coincides with $T_x M_\eta$ (as defined in 11.2) if the latter exists. It follows that the approximate tangent spaces of M with respect to two different positive $\mathcal{H}^n$-integrable functions $\theta, \tilde{\theta}$ coincide $\mathcal{H}^n$-a.e. in M. For this reason we often still denote the approximate tangent space defined in 11.4 by $T_x M$ (without indicating the dependence on θ).

The following theorem gives the important characterization of countably n-rectifiable sets in terms of existence of approximate tangent spaces.

11.6 THEOREM *Suppose* M *is* $\mathcal{H}^n$*-measurable. Then* M *is countably n-rectifiable if and only if there is a positive locally* $\mathcal{H}^n$*-integrable function* θ *on* M *with respect to which the approximate tangent space* $T_x M$ *exists for* $\mathcal{H}^n$*-a.e.* $x \in M$.

11.7 REMARK If M is $\mathcal{H}^n$-measurable, countable n-rectifiable, then we can write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $\mathcal{H}^n(M_0) = 0$, M_j is $\mathcal{H}^n$-measurable, and $M_j \subset N_j$, $j \geq 1$, with N_j an embedded n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$. (To achieve this, just define the M_j inductively by $M_j = M \cap N_j \sim \bigcup_{i=0}^{j-1} M_i$, $j \geq 1$, where N_j are C^1 submanifolds with $M_0 \equiv M \sim \bigcup_{j=1}^{\infty} N_j$ having $\mathcal{H}^n$-measure zero; such N_j exist by 11.1.) We shall show below (in the proof of the "only if" part of Theorem 11.6) that then

$$(*) \qquad T_x M = T_x N_j , \ \mathcal{H}^n\text{-a.e. } x \in M_j .$$

This is a very useful fact.

Proof of "only if" part of Theorem 11.6 As described in 11.7 above, we may write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, $j \geq 1$, N_j embedded C^1 submanifolds of dimension n, and M_j $\mathcal{H}^n$-measurable. Let $\mu = \mathcal{H}^n \llcorner \theta$, where θ is any positive locally $\mathcal{H}^n$-integrable function on M (e.g. put $\theta = 1/2^j$ on M_j, assuming, without loss of generality, that $\mathcal{H}^n(N_j) < \infty$ $\forall$ j).

Now, by 3.5,

$$\Theta^{*n}(\mu, M \sim M_j, x) = 0, \quad \mathcal{H}^n\text{-a.e. } x \in M_j. \tag{1}$$

Also, since N_j is C^1, we have (by the differentiation theorem 4.7)

$$\Theta^n(\mu, M_j, x) = \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x) \cap M_j)}{\mathcal{H}^n(B_\rho \cap N_j)} = \theta(x), \quad \mathcal{H}^n\text{-a.e. } x \in M_j \tag{2}$$

From (1), (2) and the fact that N_j is C^1, it now easily follows that the approximate tangent space for M with respect to θ exists for $\mathcal{H}^n$-a.e. $x \in M_j$, and agrees with $T_x N_j$.

Rather than just proving the "if" part of Theorem 11.6, we prove the following slightly more general result. (The "if" part of Theorem 11.6 corresponds to the case $\mu = \mathcal{H}^n \llcorner \theta$ in this more general result - see Remark 11.9 below.)

11.8 THEOREM *Suppose* μ *is a Radon measure on* $\mathbb{R}^{n+k}$, *and for* $x \in \mathbb{R}^{n+k}$, $\lambda > 0$ *let* $\mu_{x,\lambda}$ *be the measure given by* $\mu_{x,\lambda} A = \lambda^{-n}\mu(x+\lambda A)$. *Suppose that for* μ-a.e. x *there is* $\theta(x) \in (0,\infty)$ *and an* n-*dimensional subspace* $P \subset \mathbb{R}^{n+k}$ *with*

$$\lim_{\lambda \downarrow 0} \int f(y)\, d\mu_{x,\lambda}(y) = \theta(x) \int_P f(y)\, d\mathcal{H}^n(y). \tag{*}$$

(P is called the approximate tangent space for μ at x, and $\theta(x)$ is called the multiplicity.) *Let* $M = \{x : (*)$ *holds for some* P *and some*

$\theta(x) \in (0,\infty)\}$, *and set* $\theta(x) = 0$, $x \in \mathbb{R}^{n+k} \sim M$.

Then M *is countably* n-*rectifiable,* θ *is* $\mathcal{H}^n$*-measurable on* $\mathbb{R}^{n+k}$ *, and* $\mu = \mathcal{H}^n \llcorner \theta$.

11.9 REMARK Notice that in case $\mu = \mathcal{H}^n \llcorner \theta$, where θ is a non-negative locally $\mathcal{H}^n$-integrable function on $\mathbb{R}^{n+k}$, then

$$\int f \, d\mu_{x,\lambda} = \int_{\eta_{x,\lambda}(M)} f(z)\,\theta(x+\lambda y)\, d\mathcal{H}^n(y) ,$$

where $M = \{x : \theta(x) > 0\}$, so the approximate tangent space for μ at x is just the approximate tangent space T_xM with respect to θ (in the sense of 11.4). Thus we get the "if" part of Theorem 11.6 in this special case.

Proof of Theorem 11.8 Replacing μ by $\mu \llcorner B_R(0)$ (R chosen so that $\mu(\partial B_R(0)) = 0$) , we may assume that $\mu(\mathbb{R}^{n+k}) < \infty$. First note that (by (*)) we have

$$\text{(1)} \qquad \theta(x) = \lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \quad (\equiv \Theta^n(\mu,x)) \quad \mu\text{-a.e. } x \in \mathbb{R}^{n+k} ,$$

and hence, by Remark 3.1,

$$\text{(2)} \qquad \theta \text{ is } \mathcal{H}^n\text{-measurable.}$$

Given any k-dimensional subspace $\pi \subset \mathbb{R}^{n+k}$ and any $\alpha \in (0,1)$ let p_π denote the orthogonal projection of $\mathbb{R}^{n+k}$ onto π and $X_\alpha(\pi,x)$ denote the cone

$$X_\alpha(\pi,x) = \{y \in \mathbb{R}^{n+k} : |p_\pi(y-x)| \geq \alpha|y-x|\} .$$

For k-dimensional subspaces π , π' we define the distance between π , π' , denoted $\text{dist}(\pi,\pi')$, by

$$\text{dist}(\pi,\pi') = \sup_{|x|=1} |p_\pi(x) - p_{\pi'}(x)| .$$

Choose $\theta_0 > 0$ and a Borel-measurable subset $F \subset \mathbb{R}^{n+k}$ such that

(3) $$\mu(\mathbb{R}^{n+k} \sim F) \le \frac{1}{4}\mu(\mathbb{R}^{n+k})$$

and such that for each $x \in F$, μ has an approximate tangent space P_x at x with multiplicity $\theta(x) \ge \theta_0$. Thus in particular for $x \in F$ (by (1) and (*))

(4) $$\lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \ge \theta_0$$

and

(5) $$\lim_{\rho \downarrow 0} \frac{\mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_\rho(x))}{\omega_n \rho^n} = 0 ,$$

where $\pi_x = (P_x)^\perp$.

For $k = 1,2,\ldots$ and $x \in F$, define

$$f_k(x) = \inf_{0<\rho<\frac{1}{k}} \frac{\mu(B_\rho(x))}{\omega_n \rho^n}$$

and

$$q_k(x) = \sup_{0<\rho<\frac{1}{k}} \frac{\mu(X_{\frac{1}{2}}(\pi_x, x) \cap B_\rho(x))}{\omega_n \rho^n} .$$

Then

(6) $$\lim f_k(x) \ge \theta_0 \quad \text{and} \quad \lim q_k(x) = 0 \qquad \forall\, x \in F ,$$

and hence by Egoroff's Theorem we can choose a μ-measurable $E \subset F$ with

(7) $$\mu(F \sim E) \le \frac{1}{4}\mu(\mathbb{R}^{n+k})$$

and with (6) holding *uniformly* for $x \in E$. Thus for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(8)\qquad \frac{\mu(B_\rho(x))}{\omega_n\rho^n} \ge \theta_0 - \varepsilon\ ,\quad \frac{\mu(X_{\frac{1}{2}}(\pi_x,x)\cap B_\rho(x))}{\omega_n\rho^n} \le \varepsilon$$

$x \in E\ ,\quad 0 < \rho < \delta\ .$

Now choose k-dimensional subspaces $\pi_1,\ldots,\pi_N$ of $\mathbb{R}^{n+k}$ $(N=N(n,k))$ such that for *each* k-dimensional subspace π of $\mathbb{R}^{n+k}$, there is a $j\in\{1,\ldots,N\}$ such that $d(\pi,\pi_j) < \frac{1}{16}$, and let $E_1,\ldots,E_N$ be the subsets of E defined by

$$E_j = \{x\in E : d(\pi_j,\pi_x) < \frac{1}{16}\}\ .$$

Then $E = \bigcup_{j=1}^N E_j$ and we claim that if we take $\varepsilon = \theta_0/16^n$ and let $\delta > 0$ be such that (8) holds, then

$$(9)\qquad X_{\frac{3}{4}}(\pi_j,x)\cap E_j \cap B_{\delta/2}(x) = \{x\}\ ,\quad \forall\, x\in E_j\ ,\ j = 1,\ldots,N\ .$$

Indeed otherwise we could find a point $x \in E_j$ and a $y \in X_{\frac{3}{4}}(\pi_j,x)\cap E_j\cap \partial B_\rho(x)$ for some $0 < \rho \le \delta/2$. But since $x \in E$ and $2\rho \le \delta$, we have (by (8))

$$(10)\qquad \mu(X_{\frac{1}{2}}(\pi_x,x)\cap B_{2\rho}(x)) < \varepsilon\,\omega_n(2\rho)^n$$

and (since $B_{\rho/8}(y) \subset X_{\frac{1}{2}}(\pi_j,x)\cap B_{2\rho}(x)$) we have also (again by (8))

$$\begin{aligned}\mu(X_{\frac{1}{2}}(\pi_x,x)\cap B_{2\rho}(x)) &\ge \mu(B_{\rho/8}(y))\\ &\ge \theta_0\,\omega_n(\rho/8)^n\ ,\end{aligned}$$

which contradicts (8), since $\varepsilon = \theta_0/16^n$. We have therefore proved (9). Now for any fixed $x_0 \in E_j$ it is easy to check that (9), taken together with the extension theorem 2.1, implies

$$E_j \cap B_{\delta/2}(x_0) \subset q(\text{graph } f)$$

where q is an orthogonal transformation of $\mathbb{R}^{n+k}$ with $q(\pi_j) = \mathbb{R}^k$, and where $f = (f^1,\dots,f^k)$ is Lipschitz.

Since $j \in \{1,\dots,N\}$ and $x_0 \in E_j$ are arbitrary, we can then evidently select Lipschitz functions $f_1,\dots,f_Q : \mathbb{R}^n \to \mathbb{R}^k$ and orthogonal transformations $q_1,\dots,q_Q$ of $\mathbb{R}^{n+k}$ such that

$$E \subset \bigcup_{j=1}^{Q} q_j(\text{graph } f_j) .$$

Thus by (3), (7) we have

$$\mu(\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{Q} q_j(\text{graph } f_j)) \le \frac{1}{2}\mu(\mathbb{R}^{n+k}) .$$

Since we can now repeat the same argument, starting with $\mu \llcorner (\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{Q} q_j(\text{graph } f_j))$ in place of μ, we thus deduce that there are countably many Lipschitz graphs $F_j = \text{graph } f_j$, $f_j : \mathbb{R}^n \to \mathbb{R}^k$, and that $\mu(\mathbb{R}^{n+k} \sim \bigcup_{j=1}^{\infty} F_j) = 0$. By (1) and 3.2(1) we then deduce $\mathcal{H}^n(M \sim \bigcup_{j=1}^{\infty} F_j) = 0$, so that (by definition) M is countably n-rectifiable. Thus by 11.1 (see in particular Remark 11.7) we can write M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$, where $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, N_j being n-dimensional C^1 submanifolds of $\mathbb{R}^{n+k}$. Then (1) evidently implies that $\lim_{\rho\downarrow 0} \dfrac{\mu(B_\rho(x))}{\mathcal{H}^n(B_\rho(x)\cap N_j)} = \theta(x)$, $\mathcal{H}^n$ a.e. $x \in M_j$; then by the differentiation theorem 4.7 we have $\mu = \mathcal{H}^n \llcorner \theta$ as required.

§12. GRADIENTS, JACOBIANS, AREA, CO-AREA

Throughout this section M is supposed to be $\mathcal{H}^n$-measurable and countably n-rectifiable, so that we can express M as the disjoint union $\bigcup_{j=0}^{\infty} M_j$ (as in

11.7), where $\mathcal{H}^n(M_0) = 0$, M_j is $\mathcal{H}^n$-measurable, $M_j \subset N_j$, $j \geq 1$, where N_j are embedded n-dimensional C^1 submanifolds of $\mathbb{R}^{n+k}$.

Let f be a locally Lipschitz function on U , where U is an open set in $\mathbb{R}^{n+k}$ containing M . Then we can define the gradient of f , $\nabla^M f$, $\mathcal{H}^n$-a.e. on M according to :

12.1 DEFINITION

$$\nabla^M f(x) = \nabla^{N_j} f(x)$$

whenever $x \in M_j$ and $f|N_j$ is differentiable (which is true $\mathcal{H}^n$-a.e. $x \in M_j$ by virtue of Rademacher's Theorem 5.2 together with the fact that N_j is C^1).

12.2 REMARK Note that (by 11.7) $\nabla^M f(x) \in T_x M$ for $\mathcal{H}^n$-a.e. $x \in M$, and is, up to a set of $\mathcal{H}^n$-measure zero in M , independent of the particular decomposition $\bigcup_{j=0}^{\infty} M_j$ used in the definition. (i.e. $\nabla^M f$ is well-defined as an L^1 function on subsets of M with finite $\mathcal{H}^n$-measure). Indeed we can easily check that, at all points x where $f|N_j$ is differentiable, we have $f|L$ is differentiable on the affine space $L = x + T_x N_j$ at the point x , and gradient $f|L\,(x) = \nabla^{N_j} f(x)$. Since $T_x N_j = T_x M$ for $\mathcal{H}^n$-a.e. $x \in M_j$ (see 11.7), and since $T_x M$ is independent of the particular decomposition $\bigcup_{j=0}^{\infty} M_j$, we thus deduce that $\nabla^M f$ is also independent of the decomposition up to a set of measure zero, as required.

Having defined $\nabla^M f$, we can now define the linear map $d^M f_x : T_x M \to \mathbb{R}$ induced by f by setting

$$d^M f_x(\tau) = \langle \tau, \nabla^M f(x) \rangle , \quad \tau \in T_x M$$

at all points where $T_x M$ and $\nabla^M f(x)$ exist. If $f = (f^1, \ldots, f^N)$ takes values in $\mathbb{R}^N$ (f^j still locally Lipschitz on U , $j = 1, \ldots, N$) , we define $d^M f_x : T_x M \to \mathbb{R}^N$ by

12.3 $$d^M f_x(\tau) = \sum_{j=1}^{N} < \tau, \nabla^M f^j(x) > e_j \ .$$

With such an f, in case $N \geq n$, we define the Jacobian $J_M f(x)$ for $\mathcal{H}^n$ - a.e. $x \in M$ by

$$J_M f(x) = \sqrt{\det(d^M f_x)^* \circ (d^M f_x)}$$

(Cf. the smooth case 8.3), where $(d^M f_x)^* : \mathbb{R}^N \to T_x M$ denotes the adjoint of $d^M f_x$. Then we have the general area formula

12.4 $$\int_A J_M f \, d\mathcal{H}^n = \int_{\mathbb{R}^N} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^N(y)$$

for any $\mathcal{H}^n$-measurable set $A \subset M$. The proof of this is as follows: We may suppose (decomposing $\mathcal{H}^n$-almost all M_j as a countable union if necessary and using the C^1 approximation theorem 5.3) that $f|M_j = g_j|M_j$, where g_j is C^1 on $\mathbb{R}^{n+k}$, $j \geq 1$.

By virtue of the definition 12.1, 12.3, we then have

$$J_M f(x) = J_{N_j} g_j(x) \ , \quad \mathcal{H}^n \text{- a.e. } \quad x \in M_j \ .$$

Thus $J_M f$ is $\mathcal{H}^n$ - measurable, and by the smooth case 8.4 of the area formula (with N_j in place of M, $A \cap M_j$ in place of A and g_j in place of f), we have

$$\int_{A \cap M_j} J_M f \, d\mathcal{H}^n = \int_{\mathbb{R}^N} \mathcal{H}^0(A \cap M_j \cap f^{-1}(y)) \, d\mathcal{H}^N \ .$$

We now conclude 12.4 by summing over $j \geq 1$ and using the (easily checked) fact that if $\psi : U \to \mathbb{R}^N$ is locally Lipschitz and B has $\mathcal{H}^n$-measure zero, then $\mathcal{H}^n(\psi(B)) = 0$.

We note also that if g is any non-negative $\mathcal{H}^n$-measurable function on M, then, by approximation of g by simple functions, 12.4 implies the more general formula

$$\int_M g\, J_M f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} \left(\int_{f^{-1}(y)} g\, d\mathcal{H}^0 \right) d\mathcal{H}^N(y) .$$

In case f is 1:1 on M, this becomes

12.5 $$\int_M g\, J_M f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} g \circ f^{-1}\, d\mathcal{H}^N .$$

There is also a version of the co-area formula in case M is merely $\mathcal{H}^n$-measurable, countably n-rectifiable and $f : U \to \mathbb{R}^N$ is locally Lipschitz with $N < n$. (U open, $M \subset U$ as before).

In fact we can define (Cf. the smooth case described in §10)

$$J_M^* f(x) = \sqrt{\det(d^M f_x) \circ (d^M f_x)^*}$$

with $d^M f_x$ as in 12.3 and $(d^M f_x)^* =$ adjoint of $d^M f_x$. Then, for any $\mathcal{H}^n$-measurable set $A \subset M$,

12.6 $$\int_A J_M^* f\, d\mathcal{H}^n = \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(A \cap f^{-1}(y))\, d\mathcal{L}^N(y) .$$

This follows from the C^1 case (see §10) by using the decomposition $M = \bigcup_{j=0}^{\infty} M_j$ and the approximation theorem 5.3 in a similar manner to the procedure used for the discussion of the area formula above.

As for the smooth case, approximating a given non-negative $\mathcal{H}^n$-measurable function g by simple functions, we deduce from 12.6 the more general formula

12.7 $$\int_A g\, J_M^* f \, dH^n = \int_{\mathbb{R}^n} \int_{f^{-1}(y)\cap M} g \, dH^{n-N} \, dL^N(y) \, .$$

12.8 REMARKS

(1) Note that the remarks 10.7 carry over without change to this setting.

(2) The "slices" $M \cap f^{-1}(y)$ are countably (n-N)-rectifiable subsets of $\mathbb{R}^{n+k}$ for L^N-a.e. $y \in \mathbb{R}^N$. This follows directly from the decomposition $M = \bigcup_{j=0}^{\infty} M_j$, together with the C^1 Sard-type theorem 10.4 and the approximation theorem 5.3.

§13 THE STRUCTURE THEOREM

Notice that an arbitrary subset A of $\mathbb{R}^{n+k}$ which can be written as the countable union $\bigcup_{j=1}^{\infty} A_j$ of sets of finite measure, is always decomposible into a disjoint union

13.1 $$A = R \cup P \, ,$$

where R is countably n-rectifiable and P is *purely* n-*unrectifiable;* that is P contains no countably n-rectifiable subsets of positive H^n-measure.

To prove 13.1, we simply let R be a *maximal* element of the collection of all countably n-rectifiable subsets of A (ordered by inclusion); such R exists by the Hausdorff maximal principal.

A very non-trivial theorem (the Structure Theorem) due to Besicovitch [B] in case $n = k = 1$ and Federer [FH2] in general, says that the purely unrectifiable sets Q of $\mathbb{R}^{n+k}$ which (like the subset P in 13.1) can be written as the countable union of sets of finite H^n-measure, are characterized by the fact that they have H^n-null projection via almost all orthogonal projections onto n-dimensional subspaces of $\mathbb{R}^{n+k}$. More precisely:

13.2 THEOREM *Suppose* Q *is a purely* n*-unrectifiable subset of* $\mathbb{R}^{n+k}$ *with* $Q = \bigcup_{j=1}^{\infty} Q_j$, $\mathcal{H}^n(Q_j) < \infty\ \forall j$. *Then* $\mathcal{H}^n(p(Q)) = 0$ *for* σ*-almost all* $p \in O(n+k,n)$. *Here* σ *is Haar measure for* $O(n+k,n)$, *the orthogonal projections of* $\mathbb{R}^{n+k}$ *onto* n*-dimensional subspaces of* $\mathbb{R}^{n+k}$.

For the proof of this theorem see [FH1] or [RM].

13.3 REMARK Of course only the purely n-unrectifiable subsets could possibly have the null projection property described in 13.2. Indeed (by 11.1) if Q were not purely n-unrectifiable then there would be an n-dimensional C^1 submanifold M of $\mathbb{R}^{n+k}$ such that $\mathcal{H}^n(M \cap Q) > 0$. It is then an easy matter to check that if we select any $x \in M$ with $\Theta^{*n}(\mathcal{H}^n, M \cap Q, x) > 0$, then $\mathcal{H}^n(p(M \cap Q)) > 0$ for all orthogonal projections p of $\mathbb{R}^{n+k}$ onto an n-dimensional subspace S which is not orthogonal to $T_x M$.

Notice that, by combining 13.1 and 13.2, we get the following useful *rectifiability theorem:*

13.4 THEOREM *If* A *is an arbitrary subset of* $\mathbb{R}^{n+k}$ *which can be written as a countable union* $\bigcup_{j=1}^{\infty} A_j$ *with* $\mathcal{H}^n(A_j) < \infty\ \forall j$, *and if every subset* $B \subset A$ *with positive* $\mathcal{H}^n$*-measure has the property that* $\mathcal{H}^n(p(B)) > 0$ *for a set of* $p \in O(n+k,n)$ *with* σ*-measure* > 0, *then* A *is countably* n*-rectifiable.*

§14 SETS OF LOCALLY FINITE PERIMETER

An important class of countably n-rectifiable sets in $\mathbb{R}^{n+1}$ comes from the sets of locally finite perimeter. (Or Caccipoli sets - see De Giorgi [DG], Giusti [G].) First we need some definitions.

If $U \subset \mathbb{R}^{n+1}$ is open and if E is an $\mathcal{L}^{n+1}$-measurable subset of $\mathbb{R}^{n+1}$, we say that E has *locally finite perimeter* in U if the characteristic function χ_E of E is in $BV_{loc}(U)$. (See §6.)

Thus E has locally finite perimeter in U if there is a Radon measure μ_E ($= |D\chi_E|$ in the notation of §6) on U and a μ_E-measurable function $\nu = (\nu^1,\ldots,\nu^{n+1})$ with $|\nu| = 1$ μ_E-a.e. in U, such that

14.1 $$\int_{E\cap U} \operatorname{div} g \, dL^{n+1} = -\int_U g\cdot\nu \, d\mu_E$$

for each $g = (g^1,\ldots,g^{n+1})$ with $g^j \in C^1_c(U)$, $j = 1,\ldots,n+1$. Notice that if E is open and $\partial E \cap U$ is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+1}$, then the divergence theorem tells us that 14.1 holds with $\mu_E = H^n \llcorner (\partial E \cap U)$ and with ν = the inward pointing unit normal to ∂E. Thus in general we interpret μ_E as a "generalized boundary measure" and ν as a "generalized inward unit normal". It turns out (see Theorem 14.3 below) that in fact this interpretation is quite generally correct in a rather precise (and concrete) sense.

We now want to define the *reduced boundary* ∂^*E of a set E of finite perimeter by

14.2 $$\partial^*E = \left\{x \in U : \lim_{\rho\downarrow 0} \frac{\int_{B_\rho(x)} \nu d\mu_E}{\mu_E(B_\rho(x))} \text{ exists and has length } 1\right\}.$$

Since $|\nu| = 1$ μ_E-a.e. in U, by virtue of the differentiation theorem 4.7 we have $\mu_E(U \sim \partial^*E) = 0$, so that $\mu_E = \mu_E \llcorner \partial^*E$. We in fact claim much more :

14.3 THEOREM (De Giorgi) *Suppose* E *has locally finite perimeter in* U. *Then* ∂^*E *is countably* n-*rectifiable and* $\mu_E = H^n \llcorner \partial^*E$. *In fact at each point* $x \in \partial^*E$ *the approximate tangent space* T_x *of* μ_E *exists, has multiplicity* 1, *and is given by*

(1) $$T_x = \{y \in \mathbb{R}^{n+1} : y \cdot \nu_E(x) = 0\} ,$$

where $\nu_E(x) = \lim_{\rho \downarrow 0} \dfrac{\int_{B_\rho(x)} \nu \, d\mu_E}{\mu_E(B_\rho(x))}$ *(so that* $|\nu_E(x)| = 1$ *by* 14.2). *Furthermore at any such point* $x \in \partial^* E$ *we have that* $\nu_E(x)$ *is the "inward pointing unit normal for* E *" in the sense that*

(2) $$E_{x,\lambda} \equiv \{\lambda^{-1}(y-x) : y \in E\} \to \{y \in \mathbb{R}^{n+1} : y \cdot \nu_E(x) > 0\}$$

in the $L^1_{loc}(\mathbb{R}^{n+1})$ *sense.*

Proof By 11.6 and 3.5, the first part of the theorem follows from (1), which we now establish. (2) will also appear as a "by product" of the proof of (1). Assume without loss of generality $\nu \equiv \nu_E$ on $\partial^* E$.

Take any $y \in \partial^* E$. For convenience of notation we suppose that $y = 0$ and $\nu(0) = (0,\ldots,0,1)$. Then we have

(1) $$\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(0)} \nu_{n+1} \, d\mu_E}{\mu_E(B_\rho(0))} = 1$$

and hence (since $|\nu| = 1$ μ_E - a.e.)

(2) $$\lim_{\rho \downarrow 0} \frac{\int_{B_\rho(0)} |\nu_i| \, d\mu_E}{\mu_E(B_\rho(0))} = 0 , \quad i = 1,\ldots,n .$$

Further if $\zeta \in C^1_0(U)$ has support in $B_\rho(0) \subset U$, then by 14.1

(3) $$\int_U \nu_{n+1} \zeta \, d\mu_E = - \int_U \chi_E D_{n+1} \zeta \, dL^{n+1}$$
$$\leq \int_E |D\zeta| \, dL^{n+1} .$$

Now (taking $B_\rho(0)$ to be the closed ball) replace ζ by a decreasing sequence $\{\zeta_k\}$ converging pointwise to the characteristic function of $B_\rho(0)$ and satisfying

(4) $$\lim_{k\to\infty} \int_E |D\zeta_k| = \frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0)) .$$

(Notice that this can be done whenever the right side exists, which is $\mathcal{L}^1$-a.e. ρ .) Then (3) gives

(5) $$\int_{B_\rho(0)} \nu_{n+1}\, d\mu_E \le \frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0))$$

for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_0)$, $\rho_0 = \operatorname{dist}(0,\partial U)$. Then by (1) we have, for suitable $\rho_1 \in (0,\rho_0)$,

(6) $$\begin{aligned}\mu_E(B_\rho(0)) &\le 2\frac{d}{d\rho} \mathcal{L}^{n+1}(E\cap B_\rho(0)) \equiv 2\, \mathcal{H}^n(E\cap\partial B_\rho(0)) \\ &\le 2(n+1)\omega_{n+1}\, \rho^n\end{aligned}$$

for $\mathcal{L}^1$-a.e. $\rho \in (0,\rho_1)$.

Then by the compactness theorem 6.3, it follows that we can select a sequence $\rho_k \downarrow 0$ so that $\chi_{\rho_k^{-1}E} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^{n+1})$, where F is a set of locally finite perimeter in $\mathbb{R}^{n+1}$. Hence in particular for any non-negative $\zeta \in C^1_0(\mathbb{R}^{n+1})$

(7) $$\lim_{k\to\infty} \int_{\rho_k^{-1}E} D_i\, \zeta\, d\mathcal{L}^{n+1} = \int_F D_i\, \zeta\, d\mathcal{L}^{n+1} .$$

Now write $\zeta_k(x) = \zeta(\rho_k^{-1}x)$ and change variable $x \to \rho_k x$; then

$$\int_{\rho_k^{-1}E} D_i\, \zeta\, d\mathcal{L}^{n+1} = \rho_k^{-n} \int_E D_i\, \zeta_k\, d\mathcal{L}^{n+1} \equiv -\rho_k^{-n} \int_U \zeta_k\, \nu_i\, d\mu_E$$

(by 14.1), so that $\int_{\rho_k^{-1}E} D_i\, \zeta\, d\mathcal{L}^{n+1} \to 0$ by (2) for $i = 1,\dots,n$. Thus (7) gives

$$\int_F D_i\,\zeta\,dL^{n+1} = 0 \quad \forall\ \zeta \in C_0^1(\mathbb{R}^{n+1})\ , \quad i = 1,\dots,n\ ,$$

and it follows that $F = \mathbb{R}^n \times H$ for some L^1-measurable subset H of $\mathbb{R}$.

On the other hand by 14.1 with $g = \zeta_k\, e_{n+1}$ and by (1) we have, for k sufficiently large and $\zeta \geq 0$,

$$0 \leq \rho_k^{-n} \int_U \zeta_k\, \nu_{n+1}\, d\mu_E = \int_{\rho_k^{-1}E} D_{n+1}\,\zeta$$

$$\to \int_F D_{n+1}\,\zeta \equiv \int_{\mathbb{R}^n} \left(\int_H \frac{\partial \zeta}{\partial x_{n+1}} (x', x_{n+1})\, dx_{n+1} \right) dx'$$

as $k \to \infty$, so that χ_H is *non-decreasing* on $\mathbb{R}$, hence

(8) $$F = \{x \in \mathbb{R}^{n+1} : x_{n+1} < \lambda\}$$

for some λ. We have next to show that $\lambda = 0$. To check this we use the Sobolev inequality (see e.g. [GT]) to deduce that, if $\zeta \geq 0$, $\operatorname{spt} \zeta \subset U$ and $\sigma < \operatorname{dist}(\operatorname{spt} \zeta, \partial U)$, then

$$\left(\int_U (\zeta\, \phi_\sigma * \chi_E)^{\frac{n+1}{n}}\, dL^{n+1} \right)^{\frac{n}{n+1}} \leq c \int_U |D(\zeta\, \phi_\sigma * \chi_E)|\, dL^{n+1}$$

$$\leq c \left(\int_U \zeta\, |D(\phi_\sigma * \chi_E)|\, dL^{n+1} + \int_U \phi_\sigma * \chi_E |D\zeta|\, dL^{n+1} \right) .$$

Then by 6.2 it follows that

$$\left(\int_E \zeta^{\frac{n+1}{n}}\, dL^{n+1} \right)^{\frac{n}{n+1}} \leq c \left(\int_U \zeta\, d\mu_E + \int_E |D\zeta|\, dL^{n+1} \right) ,$$

and replacing ζ by a sequence ζ_k as in (4), we get for a.e. $\rho \in (0, \rho_1)$

$$\left(\mathcal{L}^{n+1}(E\cap B_\rho(0))\right)^{\frac{n}{n+1}} \le c\left(\mu_E(B_\rho(0)) + \frac{d}{d\rho}\mathcal{L}^{n+1}(E\cap B_\rho(0))\right) ,$$

which by (6) gives

$$\left(\mathcal{L}^{n+1}(E\cap B_\rho(0))\right)^{\frac{n}{n+1}} \le c\,\frac{d}{d\rho}\mathcal{L}^{n+1}(E\cap B_\rho(0)) \qquad \text{a.e. } \rho \in (0,\rho_1) .$$

Integration (using the fact that $\mathcal{L}^{n+1}(E\cap B_\rho(0))$ is non-decreasing) then implies

(9) $$\mathcal{L}^{n+1}(E\cap B_\rho(0)) \ge c\,\rho^{n+1}$$

for all sufficiently small ρ . Repeating the same argument with $U \sim E$ in place of E , we also deduce

(10) $$\mathcal{L}^{n+1}(B_\rho(0)\sim E) \ge c\,\rho^{n+1}$$

for all sufficiently small ρ . (9) and (10) evidently tell us that $\lambda = 0$ in (8).

Now given any sequence $\rho_k \downarrow 0$, the argument above guarantees we can select a subsequence $\rho_{k'}$ such that $\chi_{\rho_{k'}^{-1}E} \to \chi_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}}$ in $L^1_{loc}(\mathbb{R}^{n+1})$. Hence $\chi_{\rho^{-1}E} \to \chi_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}}$ and (2) of the theorem is established. Then by 14.1, (1) and (2) we have $\mu_{\rho^{-1}E} \to \mu_{\{x\in\mathbb{R}^{n+1}:\, x_{n+1}<0\}} \equiv \mathcal{H}^n \llcorner \{x\in\mathbb{R}^{n+1} : x_{n+1} = 0\}$ as $\rho \downarrow 0$ and the proof is complete.

CHAPTER 4

THEORY OF RECTIFIABLE n-VARIFOLDS

Let M be a countably n-rectifiable, $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$, and let θ be a positive locally $\mathcal{H}^n$-integrable function on M. Corresponding to such a pair (M, θ) we define the rectifiable n-varifold $\underline{\underline{v}}(M,\theta)$ to be simply the equivalence class of all pairs $(\tilde{M},\tilde{\theta})$, where $\tilde{M}$ is countably n-rectifiable with $\mathcal{H}^n((M\sim\tilde{M}) \cup (\tilde{M}\sim M)) = 0$ and where $\tilde{\theta} = \theta$ $\mathcal{H}^n$-a.e. on $M \cap \tilde{M}$.* θ is called the *multiplicity function* of $\underline{\underline{v}}(M,\theta)$. $\underline{\underline{v}}(M,\theta)$ is called an integer multiplicity rectifiable n-varifold (more briefly, an *integer* n-*varifold*) if the multiplicity function is integer-valued $\mathcal{H}^n$-a.e.

In this chapter and in Chapter 5 we develop the theory of general n-rectifiable varifolds, particularly concentrating on *stationary* (see §16) rectifiable n-varifolds, which generalize the notion of classical minimal submanifolds of $\mathbb{R}^{n+k}$. The key section is §17, in which we obtain the monotonicity formulae; much of the subsequent theory is based on these and closely related formulae.

§15. BASIC DEFINITIONS AND PROPERTIES

Associated to a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$ (as described above) there is a Radon measure μ_V (called the *weight measure* of V) defined by

15.1 $$\mu_V = \mathcal{H}^n \mathbin{\llcorner} \theta ,$$

---

* We shall see later that this is essentially equivalent to Allard's ([AW1]) notion of n-dimensional rectifiable varifold. In case $M \subset U$, U open in $\mathbb{R}^{n+k}$ and θ is locally $\mathcal{H}^n$-integrable in U, we say $V = \underline{\underline{v}}(M,\theta)$ (as defined above) is *a rectifiable* n-*varifold in* U.

where we adopt the convention that $\theta \equiv 0$ on $\mathbb{R}^{n+k} \sim M$. Thus for $\mathcal{H}^n$-measurable A,

$$\mu_V(A) = \int_{A \cap M} \theta \, d\mathcal{H}^n ,$$

The *mass* (or weight) of V, $\underline{\underline{M}}(V)$, is defined by

15.2 $$\underline{\underline{M}}(V) = \mu_V(\mathbb{R}^{n+k}) .$$

Notice that, by virtue of Theorem 11.8, an abstract Radon measure μ is actually μ_V for some rectifiable varifold V if and only if μ has an approximate tangent space T_x with multiplicity $\theta(x) \in (0,\infty)$ for μ-a.e. $x \in \mathbb{R}^{n+k}$. (See the statement of Theorem 11.8 for the terminology.) In this case $V = \underline{\underline{v}}(M,\theta)$, where $M = \{x : \Theta^{*n}(\mu,x) > 0\}$.

Given a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$, we define the tangent space T_xV to be the approximate tangent space of μ_V (as defined in the statement of Theorem 11.8) whenever this exists. Thus

15.3 $$T_xV = T_xM \quad \mathcal{H}^n\text{-a.e. } x \in M$$

where T_xM is the approximate tangent space of M with respect to the multiplicity θ. (See 11.4, 11.5.)

We also define, for $V = \underline{\underline{v}}(M,\theta)$,

15.4 $$\operatorname{spt} V = \operatorname{spt} \mu_V ,$$

and for any $\mathcal{H}^n$-measurable subset $A \subset \mathbb{R}^{n+k}$, $V \llcorner A$ is the rectifiable n-varifold defined by

15.5 $$V \llcorner A = \underline{\underline{v}}(M \cap A, \theta|(M \cap A)) .$$

Given $V = \underline{\underline{v}}(M,\theta)$ and a sequence $V_k = \underline{\underline{v}}(M_k,\theta_k)$ of rectifiable n-varifolds, we say that $V_k \to V$ provided $\mu_{V_k} \to \mu_V$ in the usual sense of Radon measures. (Notice that this is *not* varifold convergence in the sense of Chapter 8.)

Next we want to discuss the notion of mapping a rectifiable n-varifold relative to a Lipschitz map. Suppose $V = \underline{\underline{v}}(M,\theta)$, $M \subset U$, U open in $\mathbb{R}^{n+k}$, W open in $\mathbb{R}^{n+k_1}$ and suppose $f : \operatorname{spt} V \cap U \to W$ is proper*, Lipschitz and 1:1. Then we define the *image* varifold $f_\# V$ by

15.6 $$f_\# V = \underline{\underline{v}}(f(M), \theta \circ f^{-1}) .$$

We leave it to the reader to check using 12.5 that $f(M)$ is countably n-rectifiable and that $\theta \circ f^{-1}$ is locally H^n-integrable in W, and therefore that 15.6 does define a rectifiable n-varifold in W. More generally if f satisfies the conditions above, except that f is not necessarily 1:1, then we define $f_\# V$ by

$$f_\# V = \underline{\underline{v}}(f(M), \tilde{\theta}) ,$$

where $\tilde{\theta}$ is defined on $f(M)$ by $\sum_{x \in f^{-1}(y) \cap M} \theta(x) \left(\equiv \int_{f^{-1}(y) \cap M} \theta \, dH^0\right)$. Notice that $\tilde{\theta}$ is locally H^n-integrable in W by virtue of the area formula (see §12), and in fact

15.7 $$\underline{\underline{M}}(f_\# V) = \int_{f(M)} \tilde{\theta} \, dH^n$$

$$\equiv \int_M J_M f \, \theta \, dH^n ,$$

where $J_M f$ is the Jacobian of f relative to M as defined in §12; that is

$$J_M f = \sqrt{\det (d^M f_x)^* \circ d^M f_x}$$

---

* i.e. $f^{-1}(K) \cap \operatorname{spt} V$ is compact whenever K is a compact subset of W.

where $d^M f_x : T_x M \to \mathbb{R}^{n+k_1}$ is the linear map induced by f as described in §12.

§16. FIRST VARIATION

Suppose $\{\phi_t\}_{-\varepsilon<t<\varepsilon}$ $(\varepsilon>0)$ is a 1-parameter family of diffeomorphisms of an open set U of $\mathbb{R}^{n+k}$ satisfying

16.1
(i) $\phi_0 = \underline{\underline{1}}_U$, $\exists$ compact $K \subset U$ such that $\phi_t | U \sim K = \underline{\underline{1}}_{U\sim K}$ $\forall$ $t \in (-\varepsilon,\varepsilon)$

(ii) $(x,t) \to \phi_t(x)$ is a smooth map $U \times (-\varepsilon,\varepsilon) \to U$.

Then if $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold and if $K \subset U$ is compact as in (i) above, we have, according to 15.7 above,

$$\underline{\underline{M}}(\phi_{t\#}(V \llcorner K)) = \int_{M\cap K} J_M\, \phi_t\, \theta\, d\mathcal{H}^n ,$$

and we can compute the *first variation* $\frac{d}{dt}\underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0}$ exactly as in §9. We thus deduce

16.2
$$\frac{d}{dt}\underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0} = \int_M \operatorname{div}_M X \, d\mu_V ,$$

where $X_{|x} = \frac{\partial}{\partial t}\phi(t,x)\Big|_{t=0}$ is the initial velocity vector for the family $\{\phi_t\}$ and where $\operatorname{div}_M X$ is as in §7:

$$\operatorname{div}_M X = \nabla_j^M x^j (\equiv e_j \cdot (\nabla^M x^j)) .$$

($\nabla^M x^j$ as in §12) we can therefore make the following definition.

16.3 DEFINITION $V = \underline{\underline{v}}(M,\theta)$ is *stationary in* U if $\int \operatorname{div}_M X \, d\mu_V = 0$ for any C^1 vector field X on U having compact support in U .

More generally if N is an $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$ $(k_1 \le k)$, if U is an open subset of N, if $M \subset N$, and if the family $\{\phi_t\}$ is as in 16.1, then by Lemma 9.6 it is reasonable to make the following definition (in which $\bar{B}$ is the second fundamental form of N).

16.4 DEFINITION If $U \subset N$ is open and $M \subset N$ is as above, then we say $V = \underline{\underline{v}}(M,\theta)$ is *stationary in* U if

$$\int_U \operatorname{div}_M X \, d\mu_V = -\int_U X \cdot \underline{\underline{\bar{H}}}_M \, d\mu_V$$

whenever X is a C^1 vector field in U with compact support in U; here $\underline{\underline{\bar{H}}}_M = \sum_{i=1}^n \bar{B}_x(\tau_i,\tau_i)$, $\tau_1,\ldots,\tau_n$ any orthonormal basis for the approximate tangent space T_xM of M at x. (Notice that by 16.2, which remains valid when $U \subset N$, this is equivalent to $\frac{d}{dt}\underline{\underline{M}}(\phi_{t\#}(V \llcorner K))\Big|_{t=0} = 0$ whenever $\{\phi_t\}$ are as in 16.1 with $U \subset N$.)

It will be convenient to generalize these notions of stationarity in the following way:

16.5 DEFINITION Suppose $\underline{\underline{H}}$ is a locally μ_V-integrable function on $M \cap U$ with values in $\mathbb{R}^{n+k}$. We say that $V (= \underline{\underline{v}}(M,\theta))$ has *generalized mean curvature* $\underline{\underline{H}}$ in U (U open in $\mathbb{R}^{n+k}$) if

$$\int_U \operatorname{div}_M X \, d\mu_V = -\int_U X \cdot \underline{\underline{H}} \, d\mu_V$$

whenever X is a C^1 vector field on U with compact support in U.

16.6 REMARKS

(1) Notice that in case M is smooth with $(\bar{M} \sim M) \cap U = \emptyset$, and when $\theta \equiv 1$, the generalized mean curvature of V is exactly the ordinary mean

curvature of M as described in §7 (see in particular 7.6).

(2) V is stationary in U (U open in $\mathbb{R}^{n+k}$) in the sense of 16.3 precisely when it has zero generalized mean curvature in U, and V is stationary in U (U open in N) in the sense of 16.4 precisely when it has generalized mean curvature $\bar{\underline{\underline{H}}}_M$.

§17. MONOTONICITY FORMULAE AND BASIC CONSEQUENCES

In this section we assume that U is open in $\mathbb{R}^{n+k}$, $V = \underline{\underline{v}}(M,\theta)$ has generalized mean curvature $\underline{\underline{H}}$ in U (see 16.5), and we write μ for μ_V ($= \mathcal{H}^n \llcorner \theta$ as in 15.1).

Our aim is to obtain information about V by making appropriate choices of X in the formula (see 16.5)

17.1 $$\int \operatorname{div}_M X \, d\mu = - \int X \cdot \underline{\underline{H}} \, d\mu \,, \quad X \in C_c^1(U;\mathbb{R}^{n+k}) \,.$$

First we choose $X_x = \gamma(r)(x-\xi)$, where $\xi \in U$ is fixed, $r = |x-\xi|$, and γ is a $C^1(\mathbb{R})$ function with

$$\gamma'(t) \le 0 \quad \forall\, t \,, \quad \gamma(t) \equiv 1 \quad \text{for} \quad t \le \rho/2 \,, \quad \gamma(t) \equiv 0 \quad \text{for} \quad t > \rho \,,$$

where $\rho > 0$ is such that $\bar{B}_\rho(\xi) \subset U$. (Here and subsequently $B_\rho(\xi)$ denotes the *open* ball in $\mathbb{R}^{n+k}$ with centre ξ and radius ρ.)

For any $f \in C^1(U)$ and any $x \in M$ such that T_xM exists (see 11.4 - 11.6) we have (by 12.1) $\nabla^M f(x) = \sum_{j,\ell=1}^{n+k} e^{j\ell} D_\ell f(x) e_j$, where $D_\ell f$ denotes the partial derivative $\frac{\partial f}{\partial x^\ell}$ of f taken in U and where $(e^{j\ell})$ is the matrix of the orthogonal projection of $\mathbb{R}^{n+k}$ onto T_xM . Thus, writing $\nabla_j^M = e_j \cdot \nabla^M$ (as in §16), with the above choice of X we deduce

$$\operatorname{div}_M X \equiv \sum_{j=1}^{n+k} \nabla_j^M X^j = \gamma(r) \sum_{j=1}^{n+k} e^{jj} + r\gamma'(r) \sum_{j,\ =1}^{n+k} \frac{(x^j-\xi^j)}{r} \frac{(x^\ell-\xi^\ell)}{r} e^{j\ell} .$$

Since $(e^{j\ell})$ represents orthogonal projection onto T_xM we have $\sum_{j=1}^{n+k} e^{jj} = n$ and $\sum_{j,\ell=1}^{n+k} \frac{(x^j-\xi^j)}{r} \frac{(x^\ell-\xi^\ell)}{r} e^{j\ell} = 1 - |D^\perp r|^2$, where $D^\perp r$ denotes the orthogonal projection of Dr (which is a vector of length = 1) onto $(T_xM)^\perp$. The formula 17.1 thus yields

$$(*) \qquad n \int \gamma(r)\,d\mu + \int r\gamma'(r)\,d\mu = - \int \underline{\underline{H}} \cdot (x-\xi)\ \gamma(r)\,d\mu + \int r\gamma'(r)\,|(Dr)^\perp|^2 d\mu$$

provided $\bar{B}_\rho(\xi) \subset U$. Now take ϕ such that $\phi(t) \equiv 1$ for $t \le 1/2$, $\phi(t) = 0$ for $t \ge 1$ and $\phi'(t) \le 0$ for all t . Then we can use (*) with $\gamma(r) = \phi(r/\rho)$. Since $r\gamma'(r) = r\rho^{-1}\phi'(r/\rho) = -\rho \frac{\partial}{\partial\rho} [\phi(r/\rho)]$ this gives

$$n\, I(\rho) - \rho\, I'(\rho) = J'(\rho) - L(\rho)$$

where

$$I(\rho) = \int \phi(r/\rho)\,d\mu , \quad L(\rho) = \int \phi(r/\rho)(x-\xi)\cdot \underline{\underline{H}}\ d\mu$$

$$J(\rho) = \int \phi(r/\rho)\,|(Dr)^\perp|^2 d\mu .$$

Thus, multiply by ρ^{-n-1} and rearranging we have

$$\text{17.2} \qquad \frac{d}{d\rho} [\rho^{-n} I(\rho)] = \rho^{-n} \quad J'(\rho) + \rho^{-n-1} L(\rho) .$$

Thus letting ϕ increase to the characteristic function of the interval $(-\infty,1)$, we obtain, in the distribution sense,

$$17.3 \qquad \frac{d}{d\rho}\left(\rho^{-n}\mu(B_\rho(\xi))\right) = \frac{d}{d\rho}\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu + \rho^{-n-1}\int_{B_\rho(\xi)} (x-\xi)\cdot\underline{\underline{H}}\,d\mu \ .$$

This is the fundamental *monotonicity identity*; since $\mu(B_\rho(\xi))$ and $\int_{B_\rho(\xi)} \frac{|D^\perp r|^2}{r^n}$ are *increasing* in ρ it also holds in the *classical sense* for a.e. $\rho > 0$ such that $\bar{B}_\rho(\xi) \subset U$. Notice that if $\underline{\underline{H}} \equiv 0$ then 17.3 tells us that the ratio $\rho^{-n}\mu(B_\rho(\xi))$ is non-decreasing in ρ. Generally, by integrating with respect to ρ in 17.3 we get the identity

$$17.4 \qquad \sigma^{-n}\,\mu(B_\sigma(\xi)) = \rho^{-n}\,\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu$$

$$- \frac{1}{n}\int_{B_\rho(\xi)} (x-\xi)\cdot\underline{\underline{H}}\left(\frac{1}{r_\sigma^n} - \frac{1}{\rho^n}\right) d\mu \ ,$$

for all $0 < \sigma \le \rho$ with $\bar{B}_\rho(\xi) \subset U$, where $r_\sigma = \max\{r,\sigma\}$, so that if $\underline{\underline{H}} \equiv 0$ we have the particularly interesting identity

$$17.5 \qquad \sigma^{-n}\,\mu(B_\sigma(\xi)) = \rho^{-n}\,\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu \ .$$

We now want to examine the important question of what 17.3 tells us in case we assume boundedness and L^p conditions on $\underline{\underline{H}}$.

17.6 THEOREM *If* $\xi \in U$, $0 < \alpha \le 1$, $\Lambda \ge 0$, *and if*

$$(*) \qquad \alpha^{-1}\int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu \le \Lambda(\rho/R)^{\alpha-1}\,\mu(B_\rho(\xi)) \quad \text{for all} \quad \rho \in (0,R) \ ,$$

where $\bar{B}_R(\xi) \subset U$, *then* $e^{\Lambda R^{1-\alpha}\rho^\alpha}\,\rho^{-n}\mu(B_\rho(\xi))$ *is a non-decreasing function of* $\rho \in (0,R)$, *and in fact*

$$(1) \qquad e^{\Lambda R^{1-\alpha}\sigma^\alpha}\,\sigma^{-n}\mu(B_\sigma(\xi)) \le e^{\Lambda R^{1-\alpha}\rho^\alpha}\,\rho^{-n}\mu(B_\rho(\xi)) - \int_{B_\rho(\xi)\sim B_\sigma(\xi)} \frac{|D^\perp r|^2}{r^n}\,d\mu$$

whenever $0 < \sigma < \rho \le R$. *Also,*

$$(2) \qquad e^{-\Lambda R^{1-\alpha}\sigma^{\alpha}} \sigma^{-n}\mu(B_{\sigma}(\xi)) \ge e^{-\Lambda R^{1-\alpha}\rho^{\alpha}} \rho^{-n}\mu(B_{\rho}(\xi)) - \int_{B_{\rho}(\xi)\sim B_{\sigma}(\xi)} \frac{|D^{\perp}r|^2}{r^n}\, d\mu$$

whenever $0 < \sigma < \rho \le R$.

Proof To get (1) we simply multiply the identity 17.3 by the integrating factor $e^{\Lambda R^{1-\alpha}\rho^{\alpha}}$, whereupon, after using (*), we obtain

$$\frac{d}{d\rho}\left(e^{\Lambda R^{1-\alpha}\rho^{\alpha}} \rho^{-n}\mu(B_{\rho}(\xi))\right) \ge \frac{d}{d\rho}\int_{B_{\rho}(\xi)} \frac{|D^{\perp}r|^2}{r^n}\, d\mu \text{ , in the sense of distributions.}$$

(2) is proved similarly except that this time we multiply through in 17.3 by the integrating factor $e^{-\Lambda R^{1-\alpha}\rho^{\alpha}}$.

17.7 THEOREM *If* $\xi \in U$, *and* $\left(\int_{B_R(\xi)} |\underline{\underline{H}}|^p\, d\mu\right)^{1/p} \le \Gamma$, *where* $\overline{B_R}(\xi) \subset U$ *and* $p > n$, *then*

$$(\sigma^{-n}\mu(B_{\sigma}(\xi)))^{1/p} \le (\rho^{-n}\mu(B_{\rho}(\xi)))^{1/p} + \frac{\Gamma}{p-n}\,(\rho^{1-n/p}-\sigma^{1-n/p})$$

whenever $0 < \sigma < \rho \le R$.

Proof Using the Hölder inequality, we obtain from 17.2 that

$$\frac{d}{d\rho}\,(\rho^{-n}I(\rho)) \ge -\,\rho^{-n}\Gamma(I(\rho))^{1-1/p}$$

for L^1-a.e. $\rho \in (0,R)$. Hence

$$\frac{d}{d\rho}\,(\rho^{-n}I(\rho))^{1/p} \ge -\,p\,\rho^{-n/p}\,\Gamma \text{ .}$$

Thus, integrating over (σ,ρ) and letting ϕ increase to the characteristic function of $(-\infty,1)$ as before, we deduce the required inequality.

17.8 COROLLARY *If* $\underline{\underline{H}} \in L^p_{loc}(\mu)$ *in* U *for some* $p > n$, *then the density*

$\Theta^n(\mu,x) = \lim_{\rho\downarrow 0} \dfrac{\mu(\bar{B}_\rho(x))}{\omega_n\rho^n}$ *exists at every point* $x \in U$, *and* $\Theta^n(\mu,\cdot)$ *is an upper-semi-continuous function in* U :

$$\Theta^n(\mu,x) \geq \limsup_{y\to x} \Theta^n(\mu,y) \qquad \forall\, x \in U .$$

Proof The inequality 17.7 tells us that $(\rho^{-n}\mu(B_\rho(\xi))^{1/p} + \frac{1}{p-n}\Gamma\,\rho^{1-n/p}$ is a non-decreasing function of ρ; hence $\lim_{\rho\downarrow 0} \rho^{-n}\mu(B_\rho(\xi))$ exists (and is the same as $\lim_{\rho\downarrow 0} \rho^{-n}\mu(\bar{B}_\rho(\xi)))$. We also deduce that

$$(\sigma^{-n}\mu(B_\sigma(y)))^{1/p} \leq (\rho^{-n}\mu(B_\rho(y)))^{1/p} + c\,\rho^{1-n/p}$$

$$\leq (\rho^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p} + c\,\rho^{1-n/p}$$

whenever $\sigma < \rho$, $\varepsilon > 0$, $B_{\rho+\varepsilon}(x) \subset U$ and $|y-x| < \varepsilon$. Letting $\sigma \downarrow 0$ we thus have

$$(\Theta^n(\mu,y))^{1/p} \leq (\omega_n^{-1}(\rho+\varepsilon)^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p}(1+\varepsilon/\rho)^{n/p} + c\,\rho^{1-n/p} .$$

Now let $\delta > 0$ be given and choose $\varepsilon << \rho < \delta$ so that

$$(\omega_n^{-1}(\rho+\varepsilon)^{-n}\mu(B_{\rho+\varepsilon}(x)))^{1/p}(1+\varepsilon/\rho)^{n/p} < (\Theta^n(\mu,x))^{1/p} + \delta .$$

Then the above inequality gives

$$(\Theta^n(\mu,y))^{1/p} \leq (\Theta^n(\mu,x))^{1/p} + c\,\rho^{1-n/p}$$

(c depends on x but is independent of δ, ε) provided $|y-x| < \varepsilon$. Thus the required upper-semi-continuity is proved.

17.9 REMARKS

(1) If $\theta \geq 1$ μ-a.e. in U, then $\Theta^n(\mu,x) \geq 1$ at *each* point of $\operatorname{spt}\mu \cap U$, and hence we can write $V \llcorner U = \underline{\underline{v}}(M_*,\theta_*)$ where $M_* = \operatorname{spt}\mu \cap U$, $\theta_*(x) = \Theta^n(\mu,x)$, $x \in U$. Thus $V \llcorner U$ is represented in terms of a relatively *closed* countably n-rectifiable set with *upper-semi-continuous* multiplicity function.

(2) If $\xi \in U$, $\Theta^n(\mu,\xi) \geq 1$, and $\left(\omega_n^{-1}\int_{B_R(\xi)} |\underline{\underline{H}}|^p d\mu\right)^{1/p} \leq \Gamma(1-n/p)$, where $\bar{B}_R(\xi) \subset U$ and $p > n$, then both inequalities 17.6(1), (2) hold with $\Lambda = 2\Gamma R^{-n/p}$ and $\alpha = 1-n/p$, provided $\Gamma\rho^{1-n/p} \leq 1/2$. To see this, just use Hölder's inequality to give

$$(*) \qquad \int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu \leq \Gamma(\mu(B_\rho(\xi)))^{1-1/p} = \Gamma\,\mu(B_\rho(\xi))(\mu(B_\rho(\xi)))^{-1/p} .$$

On the other hand, letting $\sigma \downarrow 0$ in 17.7 we have

$$\mu(B_\rho(\xi)) \geq \omega_n\rho^n(1-\Gamma\rho^{1-n/p})^p ,$$

so that $\mu(B_\rho(\xi)) \geq \frac{1}{2}p\,\omega_n\rho^n$ for $\Gamma\rho^{1-n/p} \leq \frac{1}{2}$, and (*) gives $\int_{B_\rho(\xi)} |\underline{\underline{H}}|\,d\mu \leq 2\,\Gamma\,\mu(B_\rho(\xi))\rho^{-n/p}$. Thus the hypotheses of 17.6 hold with $\Lambda = 2\,\Gamma\,R^{-n/p}$.

(3) Notice that either 17.6(1) or 17.7 give bounds of the form $\mu(B_\sigma(\xi)) \leq \beta\sigma^n$, $0 < \sigma < R$ for suitable constant β. Such an inequality implies

$$\int_{B_\rho(\xi)} |x-\xi|^{\alpha-n}d\mu \leq n\beta\alpha^{-1}\rho^\alpha$$

for any $\rho \in (0,R)$ and $0 < \alpha < n$. This is proved by using the following general fact with $f(t) = t^{-1}$, $t_0 = \rho^{-1}$, and with $n-\alpha$ in place of α.

17.10 LEMMA *If* X *is an abstract space,* μ *is a measure on* X , $\alpha > 0$, $f \in L^1(\mu)$, $f \geq 0$, *and if* $A_t = \{x \in X : f(x) > t\}$, *then*

$$\int_0^\infty t^{\alpha-1}\mu(A_t)\,dt = \alpha^{-1}\int_{A_0} f^\alpha\,d\mu\ .$$

More generally

$$\int_{t_0}^\infty t^{\alpha-1}\mu(A_t)\,dt = \alpha^{-1}\int_{A_{t_0}} (f^\alpha - t_0^\alpha)\,d\mu$$

for each $t_0 \geq 0$.

This is proved simply by applying Fubini's theorem on the product space $A_{t_0} \times [t_0,\infty)$ for $t_0 > 0$.

The observation of the following lemma is important.

17.11 LEMMA *Suppose* $\theta \geq 1$ μ-a.e. *in* U , $\underline{\underline{H}} \in L^p_{loc}(\mu)$ *in* U *for some* $p > n$. *If the approximate tangent space* T_xV (see §15) *exists at a given point* $x \in U$, *then* T_xV *is a "classical" tangent plane for* $\operatorname{spt}\mu$ *in the sense that*

$$\lim_{\rho\downarrow 0}(\sup\{\rho^{-1}\operatorname{dist}(y,T_xV) : y \in \operatorname{spt}\mu \cap B_\rho(x)\}) = 0\ .$$

Proof For sufficiently small R (with $B_{2R}(x) \subset U$), 17.7, 17.8 (with $\sigma \downarrow 0$) evidently imply

(1) $$\omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)) \geq 1/2\ ,\quad 0 < \rho < R\ ,\quad \xi \in \operatorname{spt}\mu \cap B_R(x)\ .$$

Using this we are going to prove that if $\alpha \in (0,1/2)$ and $\rho \in (0,R)$ then

(2) $$\mu(B_\rho(x) \sim \{y : \operatorname{dist}(y,T_xV) < \varepsilon\rho\}) < \frac{\omega_n}{2}\alpha^n\rho^n \Rightarrow$$

$$\operatorname{spt}\mu \cap B_{\rho/2}(x) \subset \{y : \operatorname{dist}(y,T_xV) < (\varepsilon+\alpha)\rho\}\ .$$

Indeed if $\xi \in \{y : \operatorname{dist}(y, T_x V) \geq (\varepsilon+\alpha)\rho\} \cap B_{\rho/2}(x)$, then $B_{\alpha\rho}(\xi) \subset B_\rho(x) \sim \{y : \operatorname{dist}(y, T_x V) < \varepsilon\rho\}$ and hence the hypothesis of (2) implies $\mu(B_{\alpha\rho}(\xi)) < \frac{1}{2}\omega_n\alpha^n\rho^n$. On the other hand (1) implies $\mu(B_{\alpha\rho}(\xi)) \geq \frac{1}{2}\omega_n\alpha^n\rho^n$, so we have a contradiction. Thus (2) is proved, and (2) evidently leads immediately to the required result.

§18. POINCARÉ AND SOBOLEV INEQUALITIES (*)

In this section we continue to assume that $V = \underline{\underline{v}}(M,\theta)$ has generalized mean curvature $\underline{\underline{H}}$ in U , and we again write μ for μ_V . We shall also assume $\theta \geq 1$ μ - a.e. $x \in U$ (so that (by 17.9) $\Theta^n(\mu,x) \geq 1$ everywhere in $\operatorname{spt}\mu \cap U$ if $\underline{\underline{H}} \in L^p_{loc}(\mu)$ for some $p > n$).

We begin by considering the possibility of repeating the argument of the previous section, but with $X_x = h(x)\gamma(r)(x-\xi)$ (rather than $X_x = \gamma(r)(x-\xi)$ as before), where h is a non-negative function in $C^1(U)$. In computing $\operatorname{div}_M X$ we will get the additional term $\gamma(r)(x-\xi)\cdot\nabla^M h$, and other terms will be as before with an additional factor $h(x)$ everywhere. Thus in place of 17.2 we get

18.1
$$\frac{\partial}{\partial\rho}(\rho^{-n}\tilde{I}(\rho)) = \rho^{-n}\frac{\partial}{\partial\rho}\int |(Dr)^\perp|^2 h\,\phi(r/\rho)\,d\mu + \rho^{-n-1}\int (x-\xi)\cdot[\nabla^M h + \underline{\underline{H}}h]\phi(r/\rho)\,d\mu$$

where now $\tilde{I}(\rho) = \int \phi(r/\rho)h\,d\mu$.

Thus

$$\frac{\partial}{\partial\rho}[\rho^{-n}\tilde{I}(\rho)] \geq \rho^{-n-1}\int (x-\xi)\cdot(\nabla^M h + \underline{\underline{H}}h)\phi(r/\rho)\,d\mu$$
$$\equiv R \text{ say .}$$

(*) The results of this section are not needed in the sequel.

We can estimate the right-side R here in two ways: if $|\underline{\underline{H}}| \le \Lambda$ we have

$$(*)\qquad R \ge -\rho^{-n-1}\int r|\nabla^M h|\,\phi(r/\rho)\,d\mu - (\Lambda\rho)\rho^{-n}\tilde{I}(\rho)\ .$$

Alternatively, without any assumption on $\underline{\underline{H}}$ we can clearly estimate

$$(**)\qquad R \ge -\rho^{-n-1}\int r(|\nabla^M h| + h|\underline{\underline{H}}|)\phi(r/\rho)\,d\mu\ .$$

If we use $(*)$ in 18.1 and integrate (making use of 17.10) we obtain (after letting ϕ increase to the characteristic function of $(-\infty,1)$ as before)

$$18.2\qquad \frac{\int_{B_\sigma(\xi)} h\,d\mu}{\omega_n\sigma^n} \le e^{\Lambda\rho}\left[\frac{\int_{B_\rho(\xi)} h\,d\mu}{\omega_n\rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)}\frac{|\nabla^M h|}{|x-\xi|^{n-1}}\,d\mu\right],$$

provided $B_\rho(\xi) \subset U$ and $0 < \sigma < \rho$.

If instead we use $(**)$ then we similarly get

$$\frac{\int_{B_\sigma(\xi)} h\,d\mu}{\omega_n\sigma^n} \le \frac{\int_{B_\rho(\xi)} h\,d\mu}{\omega_n\rho^n} + \omega_n^{-1}\int_\sigma^\rho \tau^{-n-1}\int_{B_\tau(\xi)} r(|\nabla^M h| + h|\underline{\underline{H}}|)\,d\mu\,d\tau\ .$$

and hence (by 17.10 again)

$$18.3\qquad \frac{\int_{B_\sigma(\xi)} h\,d\mu}{\omega_n\sigma^n} \le \frac{\int_{B_\rho(\xi)} h\,d\mu}{\omega_n\rho^n} + (n\omega_n)^{-1}\int_{B_\rho(\xi)}\frac{(|\nabla^M h| + h|\underline{\underline{H}}|)}{|x-\xi|^{n-1}}\,d\mu$$

provided $B_\rho(\xi) \subset U$ and $0 < \sigma < \rho$.

If we let $\sigma \downarrow 0$ in 18.2 then we get (since $\Theta(\mu,\xi) \ge 1$ for $\xi \in \operatorname{spt}\mu$)

$$h(\xi) \le e^{\Lambda\rho}\left(\frac{\int_{B_\rho(\xi)} h\,d\mu}{\omega_n\rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)}\frac{|\nabla^M h|}{|x-\xi|^{n-1}}\right),\ \xi \in \operatorname{spt}\mu\ ,\ B_\rho(\xi) \subset U\ .$$

We now state our Poincaré-type inequality.

18.4 THEOREM *Suppose* $h \in C^1(U)$, $h \ge 0$, $B_{2\rho}(\xi) \subset U$, $|\underline{\underline{H}}| \le \Lambda$, $\theta \ge 1$ μ-a.e. *in* U *and*

$$\text{(i)} \qquad \mu\{x \in B_\rho(\xi) : h(x) > 0\} \le (1-\alpha)\omega_n \rho^n , \quad e^{\Lambda\rho} \le 1+\alpha$$

for some $\alpha \in (0,1)$. *Suppose also that*

$$\text{(ii)} \qquad \mu(B_{2\rho}(\xi)) \le \Gamma \rho^n , \quad \Gamma > 0 .$$

Then there are constants $\beta = \beta(n,\alpha,\Gamma) \in (0,1/2)$ *and* $c = c(n,\alpha,\Gamma) > 0$ *such that*

$$\int_{B_{\beta\rho}(\xi)} h \, d\mu \le c\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu .$$

Proof To begin we take β to be an arbitrary parameter in $(0,1/2)$ and apply 18.2 with $\eta \in B_{\beta\rho}(\xi) \cap \operatorname{spt}\mu$ in place of ξ . This gives

$$\text{(1)} \qquad h(\eta) \le e^{\Lambda(1-\beta)\rho}\left(\frac{\int_{B_{(1-\beta)\rho}(\eta)} h \, d\mu}{\omega_n((1-\beta)\rho)^n} + \frac{1}{n\omega_n}\int_{B_{(1-\beta)\rho}(\xi)} \frac{|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \right)$$

$$\le e^{\Lambda\rho}\left((1-\beta)^{-n} \frac{\int_{B_\rho(\xi)} h \, d\mu}{\omega_n \rho^n} + \frac{1}{n\omega_n}\int_{B_\rho(\xi)} \frac{|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \right) .$$

Now let γ be a fixed C^1 non-decreasing function on $\mathbb{R}$ with $\gamma(t) = 0$ for $t \le 0$ and $\gamma(t) \le 1$ everywhere, and apply (1) with $\gamma(h-t)$ in place of h , where $t \ge 0$ is fixed. Then by (1)

$$\gamma(h(\eta)-t) \le \frac{1+\alpha}{n\omega_n}\int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu + (1-\alpha^2)(1-\beta)^{-n} .$$

Selecting β small enough so that $(1-\beta)^{-n}(1-\alpha^2) \le 1-\alpha^2/2$, we thus get

$$(2) \qquad \frac{\alpha^2}{2} \le \frac{1+\alpha}{n\omega_n} \int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu$$

for any $\eta \in B_{\beta\rho}(\xi) \cap \operatorname{spt}\mu$ such that $\gamma(h(\eta)-t) \ge 1$. Now let $\varepsilon > 0$ and choose γ such that $\gamma(t) \equiv 1$ for $t \ge 1+\varepsilon$. Then (2) implies

$$(3) \qquad 1 \le c \int_{B_\rho(\xi)} \frac{\gamma'(h-t)|\nabla^M h|}{|x-\eta|^{n-1}} \, d\mu \, , \quad \eta \in B_{\beta\rho}(\xi) \cap A_{t+\varepsilon} \, ,$$

where $A_\tau = \{y \in \operatorname{spt}\mu : h(y) > \tau\}$. Integrating over $A_{t+\varepsilon} \cap B_{\beta\rho}(\xi)$ we thus get (after interchanging the order of integration on the right)

$$(A_{t+\varepsilon} \cap B_{\beta\rho}(\xi)) \le c \int_{B_\rho(\xi)} \gamma'(h(x)-t)|\nabla^M h(x)| \left(\int_{B_{\beta\rho}(\xi)} \frac{1}{|x-\eta|^{n-1}} \, d\mu(\eta) \right) d\mu(x)$$

$$\le c\Gamma\rho \int_{B_\rho(\xi)} \gamma'(h-t)|\nabla^M h| \, d\mu$$

by hypothesis (ii) and Remark 17.9(3). Since $\gamma'(h(x)-t) = -\frac{\partial}{\partial t}\gamma(h(x)-t)$ we can now integrate over $t \in (0,\infty)$ to obtain (from 17.10) that

$$\int_{A_\varepsilon \cap B_{\beta\rho}(\xi)} (h-\varepsilon) \le c\Gamma\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu \, .$$

Letting $\varepsilon \downarrow 0$, we have the required inequality.

18.5 REMARK If we drop the assumption that $\theta \ge 1$, then the above argument still yields

$$\int_{\{x:\theta(x)\ge 1\} \cap B_{\beta\rho}(\xi)} h \, d\mu \le c\rho \int_{B_\rho(\xi)} |\nabla^M h| \, d\mu \, .$$

We can also prove a Sobolev inequality as follows.

18.6 THEOREM *Suppose* $h \in C_0^1(U)$, $h \ge 0$, *and* $\theta \ge 1$ μ-a.e. *in* U . *Then*

$$\left(\int h^{\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} \le c\int (|\nabla^M h| + h|\underline{\underline{H}}|)\,d\mu\ , \quad c = c(n)\ .$$

Note: c does not depend on k .

Proof In the proof we shall need the following simple calculus lemma.

18.7 LEMMA *Suppose* f , g *are bounded and non-decreasing on* $(0,\infty)$ *and*

(1) $$1 \le \sigma^{-n}f(\sigma) \le \rho^{-n}f(\rho) + \int_0^\rho \tau^{-n}g(\tau)\,d\tau\ , \quad 0 < \sigma < \rho < \infty\ .$$

then $\exists\ \rho$ *with* $0 < \rho < \rho_0 \equiv 2(f(\infty))^{1/n}$ $(f(\infty) = \lim_{\rho\uparrow\infty} f(\rho))$ *such that*

(2) $$f(5\rho) \le \frac{1}{2}\, 5^n \rho_0\ \ g(\rho)\ .$$

Proof of Lemma Suppose (2) is false for each $\rho \in (0,\rho_0)$. Then (1) $\Rightarrow$

$$\begin{aligned}
1 \le \sup_{0<\sigma<\rho_0} \sigma^{-n}f(\sigma) &\le \rho_0^{-n}f(\rho_0) + \frac{2.5^{-n}}{\rho_0}\int_0^{\rho_0} \rho^{-n}f(5\rho)\,d\rho \\
&\equiv \rho_0^{-n}f(\rho_0) + \frac{2}{5\rho_0}\int_0^{5\rho_0} \rho^{-n}f(\rho)\,d\rho \\
&\equiv \rho_0^{-n}f(\rho_0) + \frac{2}{5\rho_0}\left(\int_0^{\rho_0} \rho^{-n}f(\rho)\,d\rho + \int_{\rho_0}^{5\rho_0} \rho^{-n}f(\rho)\,d\rho\right) \\
&\le \rho_0^{-n}f(\infty) + \frac{2}{5}\sup_{0<\rho<\rho_0} \rho^{-n}f(\rho) + \frac{2}{5(n-1)}\,\rho_0^{-n}f(\infty)\ .
\end{aligned}$$

Thus

$$\frac{1}{2} \le \frac{1}{2}\sup_{0<\sigma<\rho_0} \sigma^{-n}f(\sigma) < 2\rho_0^{-n}f(\infty) = 2^{-n}\ ,$$ which is a

contradiction.

Continuation of the proof of Theorem 18.6

First note that because h has compact support in U, the formula 18.3 is actually valid here for all $0 < \sigma < \rho < \infty$. Hence we can apply the above lemma with the choices

$$f(\rho) = \omega_n^{-1} \int_{B_\rho(\xi)} h \, d\mu ,$$

$$g(\rho) = \omega_n^{-1} \int_{B_\rho(\xi)} (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu ,$$

provided that $\xi \in \operatorname{spt}\mu$ and $h(\xi) \geq 1$.

Thus for each $\xi \in \{x \in \operatorname{spt}\mu : h(x) \geq 1\}$ we have $\rho < 2(\omega_n^{-1} \int_M h \, d\mu)^{1/n}$ such that

$$(1) \qquad \int_{B_{5\rho}(\xi)} h \, d\mu \leq 5^n (\omega_n^{-1} \int_M h \, d\mu)^{1/n} \int_{B_\rho(\xi)} (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu .$$

Using the covering Lemma (Theorem 3.3) we can select disjoint balls $B_{\rho_1}(\xi_1)$, $B_{\rho_2}(\xi_2)$, ..., $\xi_i \in \{\xi \in \operatorname{spt}\mu : h(\xi) \geq 1\}$ such that $\{\xi \in M : h(\xi) \geq 1\} \subset \bigcup_{j=1}^{\infty} B_{5\rho_j}(\xi_j)$. Then applying (1) and summing over j we have

$$\int_{\{x \in \operatorname{spt}\mu : h(x) \geq 1\}} h \, d\mu \leq 5^n \left(\omega_n^{-1} \int_M h \, d\mu\right)^{1/n} \int_M (|\nabla^M h| + h|\underline{\underline{H}}|) \, d\mu .$$

Next let γ be a non-decreasing $C^1(\mathbb{R})$ function such that $\gamma(t) \equiv 1$ for $t > \varepsilon$ and $\gamma(t) \equiv 0$ for $t < 0$, and use this with $\gamma(h-t)$, $t \geq 0$, in place of h. This gives

$$\mu(M_{t+\varepsilon}) \leq 5^n (\omega_n \quad (\mu(M_t))^{1/n} \int_M (\gamma'(h-t)|\nabla^M h| + \gamma(h-t)|\underline{\underline{H}}|) \, d\mu ,$$

where

$$M_\alpha = \{x \in M : h(x) > \alpha\} , \quad \alpha \geq 0 .$$

Multiplying this inequality by $(t+\varepsilon)^{\frac{1}{n-1}}$ and using the trivial inequality $(t+\varepsilon)^{\frac{1}{n-1}}\mu(M_t) \le \int_{M_t}(h+\varepsilon)^{\frac{1}{n-1}}\,d\mu$ on the right, we then get

$$(t+\varepsilon)^{\frac{1}{n-1}}\mu(M_{t+\varepsilon}) \le 5^n\omega_n^{-1/n}\left(\int_M (h+\varepsilon)^{\frac{n}{n-1}}\,d\mu\right)^{1/n}\left(-\frac{d}{dt}\int_M \gamma(\xi-t)|\nabla^M h| + \int_{M_t}|\underline{\underline{H}}|\,d\mu\right).$$

Now integrate of $t \in (0,\infty)$ and use 17.10. This then gives

$$\int_{M_\varepsilon}\left(h^{\frac{n}{n-1}} - \varepsilon^{\frac{n}{n-1}}\right)d\mu \le 5^{n+1}\omega_n^{-1/n}\left(\int_M (h+\varepsilon)^{\frac{n}{n-1}}\right)^{1/n}\int_M (|\nabla^M h| + h|\underline{\underline{H}}|)\,d\mu .$$

The theorem (with $c = 5^{n+1}\omega_n^{-1/n}$) now follows by letting $\varepsilon \downarrow 0$.

18.8 REMARK Note that the inequality of Theorem 18.6 is valid without any boundedness hypothesis on $\underline{\underline{H}}$: it suffices that $\underline{\underline{H}}$ is merely in $L^1_{loc}(\mu)$.

§19. MISCELLANEOUS ADDITIONAL CONSEQUENCES OF THE MONOTONICITY FORMULAE

Here $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold in $\mathbb{R}^{n+k}$ and we continue to assume V has an $L^1_{loc}(\mu_V)$ mean curvature $\underline{\underline{H}}$ in U , U open in $\mathbb{R}^{n+k}$.

We first want to derive convex hull properties for V in case $\underline{\underline{H}}$ is bounded.

19.1 LEMMA *Suppose* $U = \mathbb{R}^{n+k} \sim \bar{B}_R(\xi)$ *and* $n^{-1}|\underline{\underline{H}}(x)\cdot(x-\xi)| < 1$ μ_V - a.e. $x \in U$, *and suppose* spt V *is compact. Then*

$$\operatorname{spt} V \subset \bar{B}_R(\xi) .$$

(i.e. $V \llcorner U = 0$.)

Proof Since spt V is compact it is easily checked that the formulae (see §17)

$$n\int \gamma(r)\,d\mu_V + \int r\gamma'(r)(1-|D^{\perp}r|^2)\,d\mu_V = -\int \underline{\underline{H}}(x)\cdot(x-\xi)\gamma(r)\,d\mu_V(x)$$

(where $r = |x-\xi|$) actually holds for any non-negative non-decreasing $C^1(\mathbb{R})$ function γ with $\gamma(t) = 0$ for $t \le R+\varepsilon$. ($\varepsilon > 0$ arbitrary.) We see this as in §17, by substituting $X(x) = \psi(x)\ \gamma(r)(x-\xi)$, where $\psi \equiv 1$ in a neighbourhood of spt V. Since $1 - |D^{\perp}r|^2 \ge 0$ and $|\underline{\underline{H}}\cdot(x-\xi)| < n$ μ_V-a.e., we thus deduce $\int \gamma(r)\,d\mu_V = 0$ for any such γ. Since we may select γ so that $\gamma(t) > 0$ for $t > R+\varepsilon$, we thus conclude spt V ($\equiv$ spt μ_V) $\subset \bar{B}_{R+\varepsilon}(\xi)$. Because $\varepsilon > 0$ was arbitrary, this proves the lemma.

19.2 THEOREM (Convex hull property for stationary varifolds)

Suppose spt V *is compact and* V *is stationary in* $\mathbb{R}^{n+k} \sim K$, K *compact. Then*

$$\text{spt } V \subset \text{convex hull of } K .$$

Proof The convex hull of K can be written as the intersection of all balls $B_R(\xi)$ with $K \subset B_R(\xi)$. Hence the result follows immediately from 19.1.

Next we want to derive a rather important fact concerning existence of "tangent cones" for V in U. We will actually derive much more general theorems of this type later (in Chapter 10); the present simple result suffices for our applications to minimizing currents in Chapter 7.

The main idea here is to consider the possibility of getting a cone (or a plane) as the limit when we take a sequence of enlargements near a given point $\xi \in U$. Specifically, we use the transformation $\eta_{\xi,\lambda} : x \mapsto \lambda^{-1}(x-\xi)$, and we consider the sequence $V_j = \eta_{\xi,\lambda_j \#} V$ (see 15.6 for notation) of "enlargements" of V centred at ξ for a sequence $\lambda_j \downarrow 0$.

19.3 THEOREM *Suppose* $\xi \in U$, $\Theta^n(\mu_V,\xi) = \lim_{\rho\downarrow 0} \dfrac{\mu_V(\bar{B}_\rho(\xi))}{\omega_n\rho^n}$ *exists, and, with* $V_j = \eta_{\xi,\lambda_j \#} V$ *as above, suppose* $\mu_{V_j} \to \mu_W$ *in the sense of Radon measures in* $\mathbb{R}^{n+k}$, *where* W *is a rectifiable* n*-varifold which is stationary in all of* $\mathbb{R}^{n+k}$. *Then* W *is a* *<u>cone</u>, in the sense that* $W = \underline{\underline{v}}(C,\psi)$, *where* C *is a countably* n*-rectifiable set invariant under all homotheties* $x \to \lambda^{-1}x$, $\lambda > 0$, *and* ψ *is a positive locally* $\mathcal{H}^n$*-integrable function on* C *with* $\psi(x) \equiv \psi(\lambda^{-1}x)$ *for* $x \in C$, $\lambda > 0$.

19.4 REMARK We do not need to assume V has a generalized mean curvature here. However note that (by 17.8) generalized mean curvature in $L^p_{loc}(\mu_V)$, $p > n$, guarantees the hypothesis that $\Theta^n(\mu,x)$ exists. Furthermore, in later applications the fact that the limit varifold W is stationary will often be a *consequence* of the fact that V has a generalized mean curvature which satisfies suitable restrictions near ξ.

Proof of 19.3 Whenever $\mu_W(\partial B_\sigma(0)) = 0$ (which is true except possibly for countably many σ) we have

$$\text{(1)} \qquad \sigma^{-n}\mu_W(B_\sigma(0)) = \lim_{j\to\infty} \sigma^{-n}\mu_{V_j}(\bar{B}_\sigma(0))$$

$$= \lim_{j\to\infty} (\lambda_j\sigma)^{-n}\,\mu_V(\bar{B}_{\lambda_j\sigma}(\xi)) \text{ (by definition of } V_j)$$

$$= \omega_n\Theta^n(\mu_V,\xi),$$

independent of σ.

On the other hand since W is stationary in $\mathbb{R}^{n+k}$ we know by 17.5 that (with $r = |x|$)

$$\sigma^{-n}\mu_W(B_\sigma(0)) = \rho^{-n}\mu_W(B_\rho(0)) - \int_{B_\rho(0)\sim B_\sigma(0)} \frac{|D^\perp r|^2}{r^n}\, d\mu_W ,$$

so that from (1) we deduce

(2) $$|D^\perp r|^2 = 0 \quad \mu_W\text{-a.e.}$$

But recall that (letting grad denote gradient taken in $\mathbb{R}^{n+k}$)

$$D^\perp r(x) = q_x(\text{grad } r(x)) \quad (\equiv r^{-1}q_x(x)) , \quad \mu_W\text{-a.e. } x ,$$

where q_x denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto $(T_xW)^\perp$, T_xW the tangent space of W at x (see §15). Therefore (2) implies

$$q_x(x) = 0 , \quad \mu_W\text{-a.e. } x ;$$

in other words

(3) $$x \in T_xW \quad \mu_W\text{-a.e. } x .$$

Next note that if h is a $C^1(\mathbb{R}^{n+k}\sim\{0\})$ homogeneous function of degree zero, so that $h(x) \equiv h(\frac{x}{|x|})$, then $x\cdot\text{grad } h(x) = 0$, $x \neq 0$, and so, for such a function h, (3) implies

(4) $$x \cdot \nabla^W h = 0$$

$(\nabla^W h(x) = p_{T_xW}(\text{grad } h(x)))$.

Thus for any homogeneous degree zero function h we see from (2), (4) and 18.1 that

(5) $$\rho^{-n}\int_{B_\rho(0)} h\, d\mu_W = \text{const. (independent of } \rho\text{)}.$$

(Notice the fact that it is valid to substitute h in 18.1, even though h is not C^1 at 0, is a consequence of a simple approximation argument, using the fact that $\sigma^{-n}\,\mu(B_\sigma(0))$ is constant.)

It is easy to check that (5) (for arbitrary non-negative $C^1\,(\mathbb{R}^{n+k}\sim\{0\})$ homogeneous degree zero functions) implies that μ_W is invariant under homotheties in the sense that $\lambda^{-n}\,\mu_W(\lambda A) = \mu_W(A)$ for any subset $A \subset \mathbb{R}^{n+k}$.

Thus the theorem is proved by taking

$$C = \{x : \Theta^n(\mu_W, x) > 0\}\ ,$$

$$\psi(x) \equiv \Theta^n(\mu_W, x)\ .$$

Finally we wish to prove a technical lemma concerning densities which we shall need in the next chapter.

19.5 LEMMA *Suppose* $0 < \ell, \beta < 1$, $R > 0$, $\bar{B}_R(0) \subset U$, $p > n$,

$$(*) \qquad \left(\omega_n^{-1}\int_{B_R(0)} |\underline{\underline{H}}|^p d\mu_V\right)^{1/p} \le (1-n/p)\Gamma\ , \quad \Gamma R^{1-n/p} \le 1/2$$

and suppose $y, z \in B_{\beta R}(0)$ *with* $|y-z| \ge \beta R/4$, $\Theta^n(\mu_V, y)$, $\Theta^n(\mu_V, z) \ge 1$, *and* $|q(y-z)| \ge \ell|y-z|$, *where* q *is the orthogonal projection of* $\mathbb{R}^{n+k}$ *onto* $\mathbb{R}^k$. *Then*

$$\Theta^n(\mu_V, y) + \Theta^n(\mu_V, z) \le (1+c(\ell\beta)^{-n}\Gamma R^{1-n/p})(1-\beta)^{-n}\omega_n^{-1}R^{-n}\mu_V(B_R(0))$$

$$+\ c(\ell\beta)^{-n-1}R^{-n}\int_{B_R(0)} \|p-p_x\|\,d\mu_V\ ,$$

where $c = c(n,k,p)$, $p = p_{\mathbb{R}^n}$, $p_x = p_{T_xV}(\equiv p_{T_xM}\ \mu_V$ a.e. $x)$.

19.6 REMARK By (*) and Remark 17.9(2) we can use the monotonicity formulae 17.6 with $\Lambda = 2\Gamma R^{-n/p}$, $\alpha = 1-n/p$, and $\xi = y$ or z. Notice that in fact the quantity $\Lambda R^{1-\alpha}\rho^\alpha$ is then just $2\Gamma\rho^{1-n/p}$ and, since $e^t \le 1+2t$ for

$t \le 1$, we have by 17.6(1) that

$$(**) \qquad \omega_n^{-1}\tau^{-n}\mu(B_\tau(\xi)) \le (1+4\Gamma\sigma^{1-n/p})\omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\xi))$$

whenever $B_\sigma(\xi) \subset B_R(0)$, $0 < \tau < \sigma$, and $\Theta^n(\mu,\xi) \ge 1$, where we write μ for μ_V .

Proof of 19.5 First note that by 18.3 we have

$$\sigma^{-n}\int_{B_\sigma(\xi)} h\, d\mu \le \rho^{-n}\int_{B_\rho(\xi)} h\, d\mu + \int_\sigma^\rho \tau^{-n}\int_{B_\tau(\xi)} (|\nabla^M h| + |\underline{\underline{H}}|h)\, d\mu\, d\tau$$

for any non-negative $C^1(\mathbb{R}^{n+k})$ function h , provided $0<\sigma<\rho<(1-\beta)R$ and $\xi = y$ or z . We make a special choice of h such that $h = f(|q(x-\xi)|)$, where f is $C^1(\mathbb{R})$ with:

$$f(t) \equiv 1 \text{ for } |t| < \ell\beta R/16, f(t) \equiv 0 \text{ for } |t| > \ell\beta R/8 , |f'(t)| \le 3(\ell\beta R)^{-1} \text{ and}$$
$$0 \le f(t) \le 1 \quad \forall\ t .$$

Then, since $|\nabla_j^M(q(x-\xi))| \le |p_x\circ q| \equiv |(p_x-p)\circ q| \le |p_x-p| \le \sqrt{n+k}\,\|p_x-p\|$ for $j = 1,\ldots,n+k$ (where $\nabla_j^M = e_j\cdot\nabla^M$ as in §12), we deduce, with $\sigma \le \ell\beta R/2$, $\rho \le (1-\beta)R$

$$(1) \quad \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\xi)) \le \omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)\cap\{x : |q(x-\xi)| \le \ell\beta R/8\})$$
$$+ c\,\sigma^{-n}(\ell\beta R)^{-1}\rho\int_{B_\rho(\xi)} \|p_x-p\|\, d\mu$$
$$+ c\,\sigma^{-n}\rho\int_{B_\rho(\xi)} |\underline{\underline{H}}|\, d\mu .$$

Now (see 17.9(2)) from (*) we have

$$(2) \qquad \int_{B_\rho(\xi)} |\underline{\underline{H}}|\, d\mu \le 2\Gamma\rho^{-n/p}\mu(B_\rho(\xi)) .$$

Taking alternately $\xi = y$, $\xi = z$ and adding the resultant inequalities in (1), (2) and 19.6 (**), we deduce the required result (upon letting $\tau \downarrow 0$ in 19.6 (**) and taking $\sigma = \ell\beta R/8$ and $\rho = (1-\beta)R$ in all inequalities).

CHAPTER 5

THE ALLARD REGULARITY THEOREM

Here we discuss Allard's ([AW1]) regularity theorem, which says roughly that if the generalized mean curvature of a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$ is in $L^p_{loc}(\mu_V)$ in U, $p > n$, if $\theta \ge 1$ μ_V a.e. in U, if $\xi \in \operatorname{spt} V \cap U$, and if $\omega_n^{-1} \rho^{-n} \mu_V(B_\rho(\xi))$ is sufficiently close to 1 for *some* sufficiently small* ρ, then V is *regular* near V in the sense that $\operatorname{spt} V$ is a $C^{1,1-n/p}$ n-dimensional submanifold near ξ.

A key idea of the proof is to show that V is well-approximated by the graph of a harmonic function near ξ. The background results needed for this are given in §20 (where it is shown that it is possible to approximate $\operatorname{spt} V$ by the graph of a Lipschitz function) and in §21 (which gives the relevant results about approximation by harmonic functions). The actual harmonic approximation is made as a key step in proving the central "tilt-excess decay" theorem in §22.

The idea of approximating by harmonic functions (in roughly the sense used here) goes back to De Giorgi [DG] who proved a special case of the above theorem (when $k=1$ and when V corresponds to the reduced boundary of a set of least perimeter - see the previous discussion in §14 and the discussion in §37 below. Almgren used analogous approximations in his work [A1] for arbitrary $k \ge 1$. Reifenberg [R1, R2] used approximation by harmonic functions in a rather different way in his work or regularity of minimal surfaces.

---

* Depending on $\|\underline{\underline{H}}\|_{L^p(\mu_V)}$

§20 LIPSCHITZ APPROXIMATION

In this section $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold with generalized mean curvature $\underline{\underline{H}}$ in U (see 16.5), and we assume $p > n$, and

$$\text{20.1} \qquad \begin{cases} 0 \in \operatorname{spt}\mu_V \ , \quad \bar{B}_R(0) \subset U \\ \left(\omega_n^{-1}\int_{B_R(0)} |\underline{\underline{H}}|^p d\mu_V\right)^{1/p} \le (1-n/p)\Gamma \ , \quad \Gamma R^{1-n/p} \le 1/2 \\ \theta \ge 1 \ , \quad \omega_n^{-1} R^{-n} \mu_V(B_R(0)) \le 2(1-\alpha) \ , \end{cases}$$

where $\alpha \in (0,1)$. We also subsequently write μ for μ_V, and

$$E = R^{-n}\int_{B_R(0)} \|p_x - p\|^2 d\mu + \left(\Gamma R^{1-n/p}\right)^2 ,$$

where $p = p_{\mathbb{R}^n}$, $p_x = p_{T_xV}$ ($= p_{T_xM}$ μ-a.e. x). Notice that then the first term in the definition of E measures the "mean-square deviation" of T_xV away from $\mathbb{R}^n$ over $B_R(0)$. (This is called the "tilt-excess" of V over $B_R(0)$ - see §22).

20.2 THEOREM *Assuming* 20.1, *there is a constant* $\gamma = \gamma(n,\alpha,k,p) \in (0,1/2)$ *such that if* $\ell \in (0,1]$ *then there is a Lipschitz function* $f = (f^1,\ldots,f^k) : B^n_{\gamma R}(0) \to \mathbb{R}^k$ *with*

$$\operatorname{Lip} f \le \ell \ , \quad \sup|f| \le c\, E^{\frac{1}{2n+2}} R$$

and

$$\mathcal{H}^n(((\operatorname{graph} f \sim \operatorname{spt} V) \cup (\operatorname{spt} V \sim \operatorname{graph} f)) \cap B_{\gamma R}(0)) \le c\ell^{-2n-2} E ,$$

where $c = c(n,\alpha,k,p)$.

20.3 REMARK Notice that this is trivial (by setting $f \equiv 0$ and taking suitable c) unless $\ell^{-2n-2}E$ is small. In particular we may assume $E \le \delta\ell^{2n+2}$, which δ is as small as we please, so long as our eventual choice of δ depends only on n, k, α, p.

Proof of 20.2 By virtue of the above remark we can assume

$$E \le \delta_0^2 , \tag{1}$$

δ_0 to be chosen depending only on n, k, α, p. Set

$$\ell_0 = (\delta_0^{-2}E)^{\frac{1}{2n+2}} < 1 ,$$

and take any two points $x,y \in B_{\beta R}(0) \cap \operatorname{spt} V$ with $|q(x-y)| \ge \ell_0|x-y|$, $|x-y| \ge \beta R/4$, where $\beta \in (0,1/2)$ is for the moment arbitrary. By Lemma 19.5 we have

$$\Theta^n(\mu,x) + \Theta^n(\mu,y) \le (1+c(\ell_0\beta)^{-n}\Gamma R^{1-n/p})(1-\beta)^{-n}\omega_n^{-1}R^{-n}\mu(B_R(0))$$

$$+ c(\ell_0\beta)^{-n-1}R^{-n}\int_{B_R(0)} \|p_x-p\|\,d\mu .$$

Using Cauchy inequality $ab \le \frac{\alpha}{4}a^2 + \frac{1}{\alpha}b^2$ in the last term, together with the assumption (in 20.1) that $\omega_n^{-1}R^{-n}\mu(B_R(0)) \le 2(1-\alpha)$, this gives

$$\Theta^n(\mu,x) + \Theta^n(\mu,y) \le 2(1+c(\ell_0\beta)^{-n}\sqrt{E})(1-\beta)^{-n}(1-\alpha)$$

$$+ \frac{\alpha}{2} + \frac{c}{\alpha}(\ell_0\beta)^{-2n-2}E .$$

Since $\ell_0^{2n+2} = \delta_0^{-2}E$ and $\Theta^n(\mu,\xi) \ge 1 \quad \forall\, \xi \in \operatorname{spt} V \cap U$ (by 17.8 and the assumption that $\theta \ge 1$ μ-a.e.) this gives

$$2 \le 2(1+c\delta_0)(1-\beta)^{-n}(1-\alpha)$$

$$+ \frac{\alpha}{2} + c\alpha^{-1}\delta_0\beta^{-2n-2}$$

which is clearly impossible if we take $\beta = \beta(n,k,p,\alpha)$ and $\delta_0 = \delta_0(n,k,\beta,p,\alpha)$ small enough. Thus for such a choice of β, δ_0 we have

$$(2) \qquad |q(x-y)| \le cE^{\frac{1}{2n+2}} R\ ,\ x,y \in \operatorname{spt}\mu \cap B_{\beta R}(0)\ ,\ |x-y| \ge \beta R/4\ ,$$

where $c = c(n,k,p,\alpha)$, $\beta = \beta(n,k,p,\alpha)$. (Formally we derived this subject to assumption (1), but if (1) fails then (2) is trivial with $c = \delta_0^{-1}$.) Noting the arbitrariness of x,y in (2) and noting also that $0 \in \operatorname{spt}\mu$ and that $\operatorname{spt}\mu \cap \partial B_{\beta R/2}(0) \ne \emptyset$ (which follows for example by selecting suitable ϕ in 17.2), we conclude (after replacing β by $\beta/4$)

$$(3) \qquad |q(x)| \le c\, E^{\frac{1}{2n+2}} R\ ,\quad x \in B_{\beta R}(0) \cap \operatorname{spt} V\ ,\ \beta = \beta(n,k,p,\alpha) \in (0,1)\ .$$

Next let $\delta, \ell \in (0,1]$ be arbitrary and assume

$$(4) \qquad \left(\Gamma R^{1-n/p}\right)^2 \le \ell^{2n+2}\delta$$

(which we can do by Remark 20.3, provided we eventually choose $\delta = \delta(n,k,\alpha,p)$). Set $E_0(\sigma,\xi) = \sigma^{-n}\int_{B_\sigma(\xi)} \|p_x - p\|^2 d\mu(x)$ for any $\xi \in \operatorname{spt} V$, $B_\sigma(\xi) \subset B_R(0)$, and define

$$G = \{\xi \in \operatorname{spt} V \cap B_{\beta R/2}(0) : E_0(\sigma,\xi) \le \delta\ell^{2n+2} \quad \forall\, \sigma \in (0,R/2)\}\ .$$

Notice that if $\xi \in \operatorname{spt} V \cap B_{\beta R}(0)$ then by (4) and the monotonicity formula 17.6(1) (see Remark 17.9(2) to justify the application of 17.6(1))

$$\begin{aligned}(5) \qquad \tfrac{1}{2} \le \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\xi)) &\le (1+c\delta)\omega_n^{-1}((1-\beta)R)^{-n}\mu(B_{(1-\beta)R}(\xi)) \\ &\le (1+c\delta)(1-\beta)^{-n}\omega_n^{-1}R^{-n}\mu(B_R(0)) \\ &\le 2(1+c\delta)(1-\beta)^{-n}(1-\alpha) \\ &\le 2(1-\alpha/2)\ ,\end{aligned}$$

for δ, β small enough (depending on n,k,p,α).

Now let $x, y \in G$. In view of (4), (5) we may now apply the previous argument with $\alpha/2$, $\beta^{-1}|x-y|/2$, x in place of α , R , 0 in order to deduce from (3) that

$$(6) \qquad |q(x-y)| \le c\ell\delta^{1/(2n+2)}|x-y| \ , \ x,y \in G \ , \ c = c(n,k,p,\alpha)$$

(because $E_0(\sigma,x) + \left(\Gamma\sigma^{1-n/p}\right)^2 \le 2\delta\ell^{2n+2}$, $\sigma = \beta^{-1}|x-y|/2$, by virtue of (4) and the fact that $x \in G$) .

Choosing δ so that $2\delta^{1/(2n+2)}(1+c)(n+k) < 1$ (c as in (6)), we thus deduce

$$|q(x-y)| \le \frac{\ell}{2(n+k)} |x-y| \ , \ x,y \in G \ , \quad c = c(n,k,p,\alpha) \ .$$

Since $|x-y| \le |q(x-y)| + |p(x-y)|$, this implies

$$(7) \qquad |q(x) - q(y)| \le \frac{\ell}{(n+k)} |p(x) - p(y)|$$

and so (by the extension theorem 5.1)

$$G \subset \text{graph } f \ ,$$

where f is a Lipschitz function $B_{\beta R/2}(0) \to \mathbb{R}^k$ with $\text{Lip } f \le \ell$. By virtue of (3) we can assume (by truncating f if necessary) that $\sup|f| \le cE^{\frac{1}{2n+2}}R$.

Next we note that (by definition of G) for each $\xi \in (B_{\beta R/2}(0) \sim G) \cap \text{spt } V$ we have $\sigma(\xi) \in (0,R/10)$ such that

$$\ell^{2n+2}\delta\sigma(\xi)^n \le \int_{B_{\sigma(\xi)}(\xi)} \|p_x - p\|^2 d\mu(x)$$

and by (5) we therefore have

$$\mu(\bar{B}_{5\sigma(\xi)}(\xi)) \le c\ell^{-2n-2}\delta^{-1} \int_{B_{\sigma(\xi)}(\xi)} \|p_x - p\|^2 d\mu(x) \ .$$

By definition the collection of balls $\{B_{\sigma(\xi)}(\xi)\}_{\xi \in B_{\beta R/2}(0) \sim G}$ is a cover for $B_{\beta R/2}(0) \sim G$, and hence by the covering Theorem 3.3 we can select points $\xi_1, \xi_2, \ldots, \in B_{\beta R/2}(0) \sim G$ such that $\{B_{\sigma_j}(\xi_j)\}$ is a disjoint collection $(\sigma_j = \sigma(\xi_j))$ and $\{\bar{B}_{5\sigma_j}(\xi_j)\}$ still covers $B_{\beta R/2}(0) \sim G$. Then setting $\xi = \xi_j$ and summing over j, we conclude

$$\text{(8)} \qquad \mu(B_{\beta R/2}(0) \sim G) \le c\, \ell^{-2n-2}\delta^{-1} \int_{B_R(0)} \|p_x - p\|^2 d\mu(x) \ .$$

Since $\Theta^n(\mu,\xi) \ge 1$ for $\xi \in \operatorname{spt} V \cap U$ we have

$$\mathcal{H}^n((\operatorname{spt}\mu \sim \operatorname{graph} f) \cap B_{\beta R/2}(0)) \le \mu(B_{\beta R/2}(0) \sim \operatorname{graph} f)$$

(by Theorem 3.2(1)) and it thus remains only to prove

$$\text{(9)} \qquad \mathcal{H}^n((\operatorname{graph} f \sim \operatorname{spt}\mu) \cap B_{\beta R/2}(0)) \le c\, \ell^{-2n-2} E_R{}^n \ .$$

(Then the theorem will be established with $\gamma = \beta/2$.)

To check this, take any $\eta \in (\operatorname{graph} f \sim \operatorname{spt}\mu) \cap B_{\beta R/4}(0)$ and let $\sigma \in (0, \beta R/2)$ be such that $B_{\sigma/2}(\eta) \cap \operatorname{spt}\mu = \emptyset$ and $B_{3\sigma/4}(\eta) \cap \operatorname{spt}\mu \ne \emptyset$. (Such σ exists because $0 \in \operatorname{spt}\mu$.) Then the monotonicity formula 17.6(2) (See Remark 17.9(2)) implies

$$\mu(B_\sigma(\eta)) \le c\, \sigma^n \int_{B_\sigma(\eta) \sim B_{\sigma/2}(\eta)} |x-\eta|^{-n} \left| p_{(T_x M)^\perp}\left(\frac{x-\eta}{|x-\eta|}\right)\right|^2 d\mu$$

$$\le c \int_{B_\sigma(\eta)} \left| p_{(T_x M)^\perp}\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu$$

$$\le c \left(\int_{B_\sigma(\eta)} \left| p_{(\mathbb{R}^n)^\perp}\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu + \int_{B_\sigma(\eta)} \|p_{T_x M} - p_{\mathbb{R}^n}\|^2 d\mu \right)$$

$$\leq c\left(\int_{B_\sigma(\eta)\cap F}\left|p_{(\mathbb{R}^n)^\perp}\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu + \mu(B_\sigma(\eta)\sim F)\right.$$

$$\left. + \int_{B_\sigma(\eta)} \|p_{T_xM} - p_{\mathbb{R}^n}\|^2 d\mu\right) ,$$

where $F = \text{graph } f$, and where we used $p_{T^\perp}(x) = x - p_T(x)$ for any subspace $T \subset \mathbb{R}^{n+k}$. Since $\left|p_{(\mathbb{R}^n)^\perp}\left(\frac{x-y}{\sigma}\right)\right| \leq c\,\ell$ for $x,y \in F \cap B_\sigma(\eta)$ (because $\text{Lip } f \leq \ell$) , this implies

$$\mu(B_\sigma(\eta)) \leq c\left(\ell\mu(B_\sigma(\eta)) + \mu(B_\sigma(\eta)\sim F) + \int_{B_\sigma(\eta)} \|p_{T_xM} - p_{\mathbb{R}^n}\|^2 d\mu\right) .$$

Since we can take $c\,\ell \leq 1/2$ (notice again the validity of the theorem in this case automatically implies its validity for larger values of $\ell \in (0,1]$) , we thus get

$$\mu(B_\sigma(\eta)) \leq c\left(\mu(B_\sigma(\eta)\sim F) + \int_{B_\sigma(\eta)} \|p_{T_xM} - p_{\mathbb{R}^n}\|^2 \, d\mu\right) , \tag{10}$$

where $F = \text{graph } f$. Now since $\text{spt } \mu \cap B_{3\sigma/4}(\eta) \neq \emptyset$, the monotonicity (5) implies $\mu(B_\sigma(\eta)) \geq \frac{1}{2}\sigma^n$, and hence (10) gives

$$\sigma^n \leq c\,T , \tag{11}$$

where T is the expression on the right of (10). Thus, writing $\eta' = p_{\mathbb{R}^n}(\eta)$, we get

$$\begin{aligned} L^n(B^n_{5\sigma}(\eta')) &\leq c\,T \\ &\leq c\left(\mu((B^n_\sigma(\eta') \times \mathbb{R}^k) \cap B_{\beta R/2}(0) \sim F)\right. \\ &\qquad \left. + \int_{(B^n_\sigma(\eta')\times\mathbb{R}^k)\cap B_{\beta R/2}(0)} \|p_{T_xM} - p_{\mathbb{R}^n}\|^2 \, d\mu\right) . \end{aligned}$$

Since we have this for each $\eta \in (\text{graph } f \sim \text{spt } \mu) \cap B_{\beta R/4}(0)$, it follows from the Covering Theorem 3.3 in the usual way that

$$L^n(p_{\mathbb{R}^n}((\text{graph } f \sim \text{spt } \mu) \cap B_{\beta R/4}(0)) \le c\, \mu(B_{\beta R/2}(0) \sim F)$$

$$+ c \int_{B_{\beta R/2}(0)} \|p_{T_x M} - p_{\mathbb{R}^n}\|^2 \, d\mu$$

$$\le c\, \ell^{-2n-2} E_R{}^n \quad \text{by} \quad (8) .$$

Since $\text{Lip } f \le 1$, this gives (9) with $\beta/4$ in place of β . Thus the theorem is established for suitable γ depending only on n, k, α, p .

§21. APPROXIMATION BY HARMONIC FUNCTIONS

The main result we shall need is given in the following lemma, which is an almost trivial consequence of Rellich's theorem:

21.1 LEMMA *Given any* $\varepsilon > 0$ *there is a constant* $\delta = \delta(n,\varepsilon) > 0$ *such that if* $f \in W^{1,2}(B)$, $B \equiv B_1(0) =$ *open unit ball in* $\mathbb{R}^n$, *satisfies*

$$\int_B |\text{grad } f|^2 \le 1$$

$$\left| \int_B \text{grad } f \cdot \text{grad } \zeta \; dL^n \right| \le \delta \sup|\text{grad } \zeta|$$

for any $\zeta \in C_c^\infty(B)$, *then there is a harmonic function* u *on* B *such that*

$$\int_B |\text{grad } u|^2 \le 1$$

and

$$\int_B (u-f)^2 \le \varepsilon .$$

Proof Suppose the lemma is false. Then we can find $\varepsilon > 0$ and a sequence $\{f_k\} \in W^{1,2}(B)$ such that

$$(1) \qquad \left| \int_B \operatorname{grad} f_k \cdot \operatorname{grad} \zeta \, dL^n \right| \le k^{-1} \sup |\operatorname{grad} \zeta|$$

for each $\zeta \in C_c^\infty(B)$, and

$$(2) \qquad \int_B |\operatorname{grad} f_k|^2 \le 1 ,$$

but so that

$$(3) \qquad \int_B |f_k - u|^2 > \varepsilon$$

whenever u is a harmonic function on B with $\int_B |\operatorname{grad} u|^2 \le 1$.

Let $\lambda_k = \omega_n^{-1} \int_B f_k \, dL^n$. Then by the Poincaré inequality (see e.g. [GT]) we have

$$(4) \qquad \int_B |f_k - \lambda_k|^2 \le c \int_B |\operatorname{grad} f_k|^2 \le c ,$$

and hence, by Rellich's theorem (see [GT]), we have a subsequence $\{k'\} \subset \{k\}$ such that $f_{k'} - \lambda_{k'} \to w$ in $L^2(B)$, where $w \in W^{1,2}(B)$ with $\int_B |\operatorname{grad} w|^2 \le 1$. Also by (1) we evidently have

$$\int \operatorname{grad} w \cdot \operatorname{grad} \zeta \, dL^n = 0$$

for each $\zeta \in C_c^\infty(B)$. Thus w is harmonic in B and $\int_B |f_{k'} - w - \lambda_{k'}|^2 \to 0$. Since $w + \lambda_{k'}$ is harmonic, this contradicts (3).

We also recall the following standard estimates for harmonic functions (which follow directly from the mean-value property - see e.g. [GT]): If u is harmonic on $B \equiv B_1(0)$, then

$$\sup_{B_{\frac{1}{2}}(0)} |D^q(u)| \le c \|u\|_{L^1(B)}$$

for each integer $q \ge 1$, where $c = c(q,n)$. Indeed applying this with Du in place of u we get

21.2 $$\sup_{B_{\frac{1}{2}}(0)} |D^q u| \le c\|Du\|_{L^1(B)} \quad (\le c'\|Du\|_{L^2(B)})$$

for $q \ge 2$. Using an order 2 Taylor series expansion for u , we see that this implies

21.3 $$\sup_{B_\eta(0)} |u-\ell| \le c\eta^2\|Du\|_{L^2(B)}$$

for each $\eta \in (0,1/2]$, where $c = c(n)$ is independent of η and where ℓ is the affine function given by $\ell(x) = u(0) + x \cdot \operatorname{grad} u(0)$.

§22. THE TILT-EXCESS DECAY LEMMA

We define tilt-excess $E(\xi,\rho,T)$ (relative to the rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta))$ by

$$E(\xi,\rho,T) = \frac{1}{2}\rho^{-n}\int_{B_\rho(\xi)} |p_{T_xM}-p_T|^2 d\mu_V \ , \ *$$

whenever $\rho > 0$, $\xi \in \mathbb{R}^{n+k}$ and T is an n-dimensional subspace of $\mathbb{R}^{n+k}$. Thus E measures the mean-square deviation of the approximate tangent space T_xM away from the given subspace T . Notice that if we have $T = \mathbb{R}^n$ then $|p_{T_xM}-p_{\mathbb{R}^n}|^2$ is just $2\sum_{j=1}^k |\nabla^M x^{n+j}|^2$, so that in this case

22.1 $$E(\xi,\rho,T) = \rho^{-n}\int_{B_\rho(\xi)} \sum_{j=1}^k |\nabla^M x^{n+j}|^2 \, d\mu_V$$

(∇^M = gradient operator on M as defined in §12.)

---

* $|p_{T_xM}-p|^2$ denotes the inner product norm trace $(p_{T_x}-p)^2$; this differs from $\|p_{T_xM}-p\|^2$ by at most a constant factor depending on $n+k$.

In this section we continue to assume V has generalized mean curvature $\underline{\underline{H}} \in L^1_{loc}(\mu_V)$ in U, and we write μ for μ_V.

We shall need the following simple lemma relating tilt-excess and height; note that we do not need $\theta \geq 1$ for this.

22.2 LEMMA *Suppose* $B_\rho(\xi) \subset U$. *Then for any* n*-dimensional subspace* $T \subset \mathbb{R}^{n+k}$ *we have*

$$E(\xi,\rho/2,T) \leq c\left[\rho^{-n}\int_{B_\rho(\xi)}\left(\frac{\operatorname{dist}(x-\xi,T)}{\rho}\right)^2 d\mu + \rho^{2-n}\int_{B_\rho(\xi)}|\underline{\underline{H}}|^2\, d\mu\right].$$

22.3 REMARK Note that in case $\rho^{-n}\mu(B_\rho(\xi)) \leq c$, we can use the Hölder inequality to estimate the term $\int_{B_\rho(\xi)}|\underline{\underline{H}}|^2\, d\mu$, giving $\rho^{2-n}\int_{B_\rho(\xi)}|\underline{\underline{H}}|^2 d\mu \leq c\left[\left(\int_{B_\rho(\xi)}|\underline{\underline{H}}|^p d\mu\right)^{1/p}\rho^{1-n/p}\right]^2$, $p > 2$. Thus 22.2 gives

$$E(\xi,\rho/2,T) \leq c\left[\rho^{-n}\int_{B_\rho(\xi)}\left(\frac{\operatorname{dist}(x-\xi,T)}{\rho}\right)^2 d\mu + \left(\left(\int_{B_\rho(\xi)}|\underline{\underline{H}}|^p d\mu\right)^{1/p}\rho^{1-n/p}\right)^2\right],$$

$p \geq 2$.

Proof of 22.2 It evidently suffices to prove the result with $\xi = 0$ and $T = \mathbb{R}^n$. The proof simply involves making a suitable choice of X in the formula of 16.5. In fact we take

$$X_x = \zeta^2(x)x', \quad x' = (0,x^{n+1},\ldots,x^{n+k})$$

for $x = (x^1,\ldots,x^{n+k}) \in U$, where $\zeta \in C_0^\infty(U)$ will be chosen.

By the definition of div_M (see §12) we have

$$\operatorname{div}_M x' = \sum_{i=n+1}^{n+k} e^{ii}, \quad \mu\text{-a.e. } x \in M,$$

where (e^{ij}) is the matrix of the projection p_{T_xM} (relative to the usual orthonormal basis for $\mathbb{R}^{n+k}$). Thus by the definition 16.5 of $\underline{\underline{H}}$ we have

$$(1) \qquad \int \sigma\zeta^2\, d\mu = \int \left(-2\zeta \sum_{i=n+1}^{n+k} \sum_{j=1}^{n+k} x^i e^{ij} D_j\zeta + \zeta^2\, x'\cdot\underline{\underline{H}}\right) d\mu ,$$

with

$$(2) \qquad \sigma \equiv \sum_{i=n+1}^{n+k} e^{ii} = \frac{1}{2}\sum_{i,j=1}^{n+k} (e^{ij}-\varepsilon^{ij})^2 = \frac{1}{2}\,|p_{T_xM}-p_{\mathbb{R}^n}|^2 ,$$

where (ε^{ij}) = matrix of $p_{\mathbb{R}^n}$ and where we used $(e^{ij})^2 = (e^{ij})$ and $\operatorname{trace}(e^{ij}) = n$. We thus have for $\zeta \ge 0$

$$\int \sigma\zeta^2\, d\mu \le \int (2\sqrt{\sigma}|x'|\,|\operatorname{grad}\zeta|\zeta + |x'|\,|\underline{\underline{H}}|\zeta^2)\, d\mu ,$$

and hence (using $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$)

$$\int \sigma\zeta^2\, d\mu \le 4\int (|x'|^2|\operatorname{grad}\zeta|^2 + |x'|\,|\underline{\underline{H}}|\zeta^2)\, d\mu ,$$

The lemma now follows by choosing $\zeta \equiv 1$ in $B_{\rho/2}(0)$, $\zeta \equiv 0$ outside $B_\rho(0)$, and $|\operatorname{grad}\zeta| \le 3/\rho$, and then noting that $|x'|\,|\underline{\underline{H}}| = (\rho^{-1}|x'|)\,(|\underline{\underline{H}}|\rho) \le \frac{1}{2}\rho^{-2}|x'|^2 + \frac{1}{2}\,(|\underline{\underline{H}}|\rho)^2$.

We are now ready to discuss the following *tilt-excess decay theorem*, which is the main result concerning tilt-excess needed for the regularity theorem of the next section. (The Lipschitz approximation result of the previous section will play an important rôle in the proof.)

In order to state this result in a convenient manner, we let $\varepsilon,\alpha \in (0,1)$, $\rho > 0$, $p > n$, and T, an n-dimensional subspace of $\mathbb{R}^{n+k}$, be fixed, and we shall consider the hypotheses

22.4
$$\begin{cases} 1 \le \theta \le 1+\varepsilon & \mu\text{-a.e. in } U \\[2ex] \xi \in \operatorname{spt}\mu\ ,\ B_\rho(\xi) \subset U\ , & \dfrac{\mu(B_\rho(\xi))}{\omega_n\rho^n} \le 2(1-\alpha)\ , \\[2ex] & E_*(\xi,\rho,T) \le \varepsilon\ , \end{cases}$$

where $E_*(\xi,\rho,T) = \max\left\{E(\xi,\rho,T)\ ,\ \varepsilon^{-1}\left(\int_{B_\rho(\xi)} |\underline{\underline{H}}|^p d\mu\right)^{2/p} \rho^{2(1-n/p)}\right\}$.

22.5 THEOREM *For any* $\alpha \in (0,1)$, $p > n$ *there are constants* $\eta,\varepsilon \in (0,1/2)$, *depending only on* n , k , α , p , *such that if hypotheses* 22.4 *hold, then*

$$E_*(\xi,\eta\rho,S) \le \eta^{2(1-n/p)} E_*(\xi,\rho,T)$$

for some n*-dimensional subspace* $S \subset \mathbb{R}^{n+k}$

22.6 REMARK Notice that any such S automatically satisfies

(*) $$|p_S - p_T|^2 \le c(\eta) E_*(\xi,\rho,T)\ .$$

Indeed we trivially have

$$(\eta\rho)^{-n} \int_{B_{\eta\rho}(\xi)} |p_{T_xM} - p_T|^2 d\mu \le \eta^{-n} E(\xi,\rho,T)\ ,$$

while by 22.5 we have

$$(\eta\rho)^{-n} \int_{B_{\eta\rho}(\xi)} |p_{T_xM} - p_S|^2 d\mu \le E_*(\xi,\rho,T)\ ,$$

and hence by adding these inequalities and using the fact that $\mu(B_{\eta\rho}(\xi)) \ge c\rho^n$ (see 19.6) we get (*) as required.

Proof of Theorem 22.5 Throughout the proof, $c = c(n,k,\alpha,p)$. We can suppose $\xi = 0$, $T = \mathbb{R}^n$. By the Lipschitz approximation theorem 20.2 there is a $\beta = \beta(n,k,\alpha,p) > 0$ and a Lipschitz function $f : B^n_{\beta\rho}(0) \to \mathbb{R}^k$ with

(1) $$\operatorname{Lip} f \le 1 \ , \ \sup|f| \le c\, E_*^{\frac{1}{2n+2}} \rho \le c\, \varepsilon^{\frac{1}{2n+2}} \rho$$

and

(2) $$\mathcal{H}^n(((\operatorname{spt}\mu \sim \operatorname{graph} f) \cup (\operatorname{graph} f \sim \operatorname{spt}\mu)) \cap B_{\beta\rho}(0)) \le c\, E_* \rho^n \ ,$$

where $E_* = E_*(0,\rho,\mathbb{R}^n) \left(\equiv \max\left\{ \rho^{-n} \int_{B_\rho(0)} |p_{T_xM} - p_{\mathbb{R}^n}|^2 d\mu \ , \right.\right.$ $\left.\left. \varepsilon^{-1}\left(\int_{B_\rho(0)} |\underline{\underline{H}}|^p d\mu \right)^{2/p} \rho^{2(1-n/p)} \right\}\right)$. Furthermore by the height estimate (3) in the proof of 20.2 we have

(3) $$\sup_{B_{\beta\rho}(0)\cap \operatorname{spt}\mu} |x^j| \le c\, E_*^{\frac{1}{2n+2}} \rho \le c\, \varepsilon^{\frac{1}{2n+2}} \rho \ ,$$

$j = n+1,\ldots,n+k$. Let us agree that

(4) $$c\, \varepsilon^{\frac{1}{2n+2}} \le \beta/4 \qquad (c \text{ as in } (3)) \ .$$

Then (3) implies

(5) $$\sup_{B_{\beta\rho}(0)\cap \operatorname{spt}\mu} |x^j| \le \beta\rho/4 \ ,$$

so that

(6) $$\mathbb{R}^k \times B^n_{\beta\rho/2}(0) \cap \operatorname{spt}\mu \cap \partial B_{\beta\rho}(0) = \emptyset \ .$$

Our aim now is to prove that each component of the approximating function f is well-approximated by a harmonic function. Preparatory to this, note that the defining identity for $\underline{\underline{H}}$ (see 16.5), with $X = \zeta\, e_{n+j}$, implies

$$\int_M \nabla^M_{n+j} \zeta \, d\mu = - \int e_{n+j} \cdot \underline{\underline{H}}\, \zeta \, d\mu \ , \quad \zeta \in C^1_0(U) \ ,$$

$j = 1,\ldots,k$, where $\nabla^M_{n+j} = e_{n+j} \cdot \nabla^M = p_{T_xM}(e_{n+j}) \cdot \nabla^M = (\nabla^M x^{n+j}) \cdot \nabla^M$ (∇^M = gradient operator for M as in §12) . Thus we can write

(7) $$\int_M (\nabla^M x^{n+j}) \cdot \nabla^M \zeta \, d\mu = - \int_M e_{n+j} \cdot \underline{\underline{H}}\, \zeta \, d\mu \ .$$

Since $x^{n+j} \equiv \tilde{f}^j(x)$ on $M_1 = M \cap \text{graph } f$ (where $\tilde{f}^j$ is defined on $\mathbb{R}^{n+k}$ by $\tilde{f}^j(x^1,\ldots,x^{n+k}) = f^j(x^1,\ldots,x^n)$ for $x = (x^1,\ldots,x^{n+k}) \in \mathbb{R}^{n+k}$), we have by the definition of ∇^M (see §12) that

$$(8) \qquad \nabla^M x^{n+j} = \nabla^M \tilde{f}^j(x) \quad \mu \text{ a.e. } x \in M_1 = M \cap \text{graph } f .$$

Hence (7) can be written

$$\int_{M_1} (\nabla^M \tilde{f}^j) \cdot \nabla^M \zeta \, d\mu = - \int_{M \sim M_1} (\nabla^M x^{n+j}) \cdot \nabla^M \zeta \, d\mu - \int_M e_{n+j} \cdot \underset{=}{H} \zeta \, d\mu ,$$

and hence by (2), together with the fact that (by 22.4)

$$(9) \qquad \int_{B_\rho(\xi)} |\underset{=}{H}| \, d\mu \le \left(\int_{B_\rho(\xi)} |\underset{=}{H}|^p d\mu\right)^{1/p} (\mu(B_\rho(\xi)))^{1-1/p} \le c\varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} \rho^{n-1} ,$$

we obtain

$$(10) \qquad \rho^{-n} \int_{M_1} (\nabla^M \tilde{f}^j) \cdot \nabla^M \zeta \, d\mu \le c(\rho^{-1} \sup|\zeta| + \sup|\text{grad}\zeta|) \varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}}$$
$$\le c \sup|\text{grad}\zeta| \varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} ,$$

for any smooth ζ with $\text{spt } \zeta \subset B_{\beta\rho}(0)$.

Furthermore by (8), 22.1, we evidently have

$$(11) \qquad \rho^{-n} \int_{M_1 \cap B_{\beta\rho}(0)} |\nabla^M \tilde{f}^j|^2 d\mu \le E_* .$$

Now suppose that ζ_1 is an arbitrary $C_c^1(B^n_{\beta\rho/2}(0))$ function, and note that (by (6)) there is a function $\zeta \in C_c^1(B_{\beta\rho}(0))$ such that $\zeta \equiv \tilde{\zeta}_1$ in some neighbourhood of $B^n_{\beta\rho/2}(0) \times \mathbb{R}^k \cap \text{spt } \mu \cap B_{\beta\rho}(0)$ where $\tilde{\zeta}_1(x^1,\ldots,x^{n+k}) \equiv \zeta_1(x^1,\ldots x^n)$. Hence (10) holds with $\tilde{\zeta}_1$ in place of ζ . Also, since $p_{\mathbb{R}^n} \text{grad } \tilde{\zeta}_1 = \text{grad } \tilde{\zeta}_1$ and $p_{\mathbb{R}^n} \text{grad } \tilde{f}^j = \text{grad } \tilde{f}^j$, we have

$$(12)\qquad |\nabla^M \tilde{f}^j \cdot \nabla^M \tilde{\zeta}_1 - \operatorname{grad} f^j \cdot \operatorname{grad} \zeta_1|$$

$$\equiv |p_{(T_xM)^\perp}(\operatorname{grad} \tilde{f}^j) \cdot p_{(T_xM)^\perp}(\operatorname{grad} \tilde{\zeta}_1)|$$

$$\equiv \left|\left(p_{(T_xM)^\perp} \circ p_{\mathbb{R}^n}(\operatorname{grad} \tilde{f}^j)\right) \cdot \left(p_{(T_xM)^\perp} \circ p_{\mathbb{R}^n}(\operatorname{grad} \tilde{\zeta}_1)\right)\right|$$

$$\leq |p_{(T_xM)^\perp} \circ p_{\mathbb{R}^n}|^2 |\operatorname{grad} \tilde{f}^j|\ |\operatorname{grad} \tilde{\zeta}_1|$$

$$\leq |p_{T_xM} - p_{\mathbb{R}^n}|^2 |\operatorname{grad} \tilde{f}^j|\ |\operatorname{grad} \tilde{\zeta}_1|$$

μ-a.e. on $\operatorname{spt} \mu \cap B_{\beta\rho}(0)$, and hence (10) implies

$$(13)\qquad \left|\rho^{-n} \int_{M_1} \operatorname{grad} \tilde{f}^j \cdot \operatorname{grad} \tilde{\zeta}_1 \, d\mu \right| \leq c\, \varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} \sup|\operatorname{grad} \zeta_1| .$$

Also since (12) is valid with $\zeta_1 = \tilde{f}^j$, we conclude from (11) that

$$(14)\qquad \rho^{-n} \int_{M_1 \cap B_{\beta\rho}} |\operatorname{grad} \tilde{f}^j|^2 \, d\mu \leq c\, E_* .$$

From (13), (14) and the area formula 8.5 we then have (using also (1),(2))

$$(15)\qquad \left|\rho^{-n} \int_{B^n_{\beta\rho}(0)} \operatorname{grad} f^j \cdot \operatorname{grad} \zeta_1 \quad \theta\circ F \quad J(F)\ dL^n\right|$$

$$\leq c\, \varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} \sup|\operatorname{grad} \zeta_1|$$

and

$$(16)\qquad \rho^{-n} \int_{B^n_{\beta\rho}(0)} |\operatorname{grad} f^j|^2 \ \theta\circ F \ \ J(F)\ dL^n \leq c\, E_* ,$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n+k}$ is defined by $F(x) = (x, f(x))$, $x \in B^n_{\beta\rho}(0)$, and where $J(F)$ is the Jacobian $(\det((dF_x)^* \circ dF_x))^{1/2}$ as in §8. Since $1 \leq J(F) \leq 1 + c|\operatorname{grad} f|^2$ on $B^n_{\beta\rho}(0)$, as one checks by directly computing the matrix of dF_x (relative to the usual orthonormal bases for $\mathbb{R}^n$, $\mathbb{R}^{n+k}$) in terms of the partial derivatives of f, and since $1 \leq \theta \leq 1+\varepsilon$, we conclude

$$(17)\qquad \left|\rho^{-n}\int_{B^n_{\beta\rho}(0)} \operatorname{grad} f^j \cdot \operatorname{grad} \zeta_1 \, dL^n\right|$$

$$\le c\left(\varepsilon^{\frac{1}{2}}E_*^{\frac{1}{2}} + \varepsilon\rho^{-n}\int_{B^n_{\beta\rho/2}(0)} |\operatorname{grad} f^j|\, dL^n\right)\sup|\operatorname{grad}\zeta_1|$$

$$\le c\,\varepsilon^{\frac{1}{2}}E_*^{\frac{1}{2}}\,\sup|\operatorname{grad}\zeta_1|$$

by (16), because by (16) (and the fact that $\theta \ge 1$, $J(F) \ge 1$) we have

$$(18)\qquad \rho^{-n}\int_{B^n_{\beta\rho}(0)} |\operatorname{grad} f^j|^2\, dL^n \le c\, E_* .$$

Now (17), (18) and the harmonic approximation lemma 21.1 (with $E_*^{-\frac{1}{2}}f^j$ in place of f) we know that for any given $\delta > 0$ there is $\varepsilon_0 = \varepsilon_0(n,k,\delta)$ such that, if the hypotheses 22.1 hold with $\varepsilon \le \varepsilon_0$, there are harmonic functions $u^1,\ldots,u^k$ on $B_{\beta\rho/2}(0)$ such that

$$(19)\qquad \sigma^{-n}\int_{B^n_\sigma(0)} |\operatorname{grad} u^j|^2\, dL^n \le c\, E_*$$

and

$$(20)\qquad \sigma^{-n-2}\int_{B^n_\sigma(0)} |f^j-u^j|^2\, dL^n \le \delta\, E_* ,$$

where $\sigma = \beta\rho/2$.

Using 21.3 we then conclude that

$$(21)\qquad (\eta\sigma)^{-n-2}\int_{B^n_{\eta\sigma}(0)} |f^j-\ell^j|^2\, dL^n \le 2\eta^{-n-2}\,\delta\, E_* +$$

$$+ c\eta^2\sigma^{-n}\int_{B^n_\sigma(0)} |\operatorname{grad} u^j|^2\, dL^n$$

$$\le 2\eta^{-n-2}\,\delta\, E_* + c\eta^2\, E_* \quad \text{(by (19))} ,$$

where $\ell^j(x) = u^j(0) + x\cdot\text{grad}\, u^j(0)$. Notice that, since $\sup|f| \le c\, E_*^{\frac{1}{2n+2}} \rho$, (19), (20) in particular imply (using 21.3 again)

$$(22)\qquad \sum_{j=1}^{k} |\ell^j(0)| \le c\, E_*^{\frac{1}{2n+2}} \rho \le c\, \varepsilon^{\frac{1}{2n+2}} \rho .$$

Now let $\ell = (\ell^1,\ldots,\ell^k) : \mathbb{R}^n \to \mathbb{R}^k$ and let S be the n-dimensional subspace graph $(\ell-\ell(0))$. In view of (1), (2), (3) and (22) it is clear that (21) implies

$$(23)\qquad (\eta\sigma)^{-n-2} \int_{B_{\eta\sigma/2}(\tau)} \text{dist}(x-\tau,S)^2 d\mu \le c\eta^{-n-2}\delta E_* + c\eta^2 E_* ,$$

where $\tau = (0,\ell(0))$, provided $c\varepsilon^{\frac{1}{2n+2}} < \eta/2$. Then by 22.3 we get

$$E(\tau,\eta\sigma/2,S) \le c\eta^{-n-2}\delta E_* + c\eta^2 E_* .$$

If we in fact require

$$(24)\qquad (1+c)\varepsilon^{\frac{1}{2n+2}} < \eta \qquad (c \text{ as in } (22)) .$$

then $B_{\eta\sigma/4}(0) \subset B_{\eta\sigma/2}(\tau)$ (by (22)) and this gives

$$(25)\qquad E(0,\eta\sigma/4,S) \le c\eta^{-n-2}\delta E_* + c\eta^2 E_* .$$

The proof of the theorem is now completed as follows:

First select η so that $c\eta^2 \le \frac{1}{2}(\eta\beta/8)^{2(1-n/p)}$ (with c as in (25)), then select δ so that $c\eta^{-n-2}\delta < \frac{1}{2}(\eta\beta/8)^{2(1-n/p)}$ (c again as in (25)). Then, provided the hypotheses 22.4 hold with ε satisfying the conditions required during the above argument (in particular (4), (24) must hold, and $\varepsilon \le \varepsilon_0(n,k,\delta)$, $\varepsilon_0(n,k,\delta)$ as in the discussion leading to (19)) we get

$$E(0,\tilde{\eta}\rho,S) \le \tilde{\eta}^{2(1-n/p)} E_* ,$$

where $\tilde\eta = \eta\beta/8$. Since we trivially have

$$\left(\int_{B_{\tilde\eta\rho}(0)} |\underline{\underline{H}}|^p d\mu\right)^{1/p} (\tilde\eta\rho)^{1-n/p} \le \tilde\eta^{1-n/p}\left(\int_{B_\rho(0)} |\underline{\underline{H}}|^p d\mu\right)^{1/p} \rho^{1-n/p}$$

we thus conclude that

$$E_*(0,\tilde\eta\rho,S) \le \tilde\eta^{2(1-n/p)} E_*(0,\rho,T)$$

as required.

This completes the proof of 22.5 (with $\tilde\eta$ in place of η).

§23. MAIN REGULARITY THEOREM: FIRST VERSION

We here show that one useful form of Allard's theorem follows very directly from the tilt-excess decay theorem 22.5 of the previous section.

23.1 THEOREM *Suppose* $\alpha \in (0,1)$ *and* $p > n$ *are given. There are constants* $\varepsilon = \varepsilon(n,k,\alpha,p)$, $\gamma = \gamma(n,k,\alpha,p) \in (0,1)$ *such that if hypotheses* 22.4 *hold with* $T = \mathbb{R}^n$ *and* $\xi = 0$, *then there is a* $C^{1,1-n/p}$ *function* $u = (u^1,\ldots,u^k) : \bar B^n_{\gamma\rho}(0) \to \mathbb{R}^k$ *such that* $u(0) = 0$,

(1) $$\operatorname{spt} V \cap B_{\gamma\rho}(0) = \operatorname{graph} u \cap B_{\gamma\rho}(0) ,$$

and

(2) $$\rho^{-1}\sup|u| + \sup|Du| + \rho^{1-n/p}\sup_{\substack{x,y\in B^n_{\gamma\rho}(0)\\ x\neq y}} |x-y|^{-(1-n/p)}|Du(x)-Du(y)|$$

$$\le c\left\{E^{\frac12}(0,\rho,\mathbb{R}^n) + \left(\int_{B_\rho(0)} |\underline{\underline{H}}|^p d\mu\right)^{1/p}\rho^{1-n/p}\right\} .$$

Before giving the proof we make a couple of important remarks concerning removability of the hypothesis $\theta \le 1+\varepsilon$ in 22.4: *

23.2 REMARKS

(1) The monotonicity formula in 17.6(1), together with Remark 17.9(1), evidently implies that if $\left(\omega_n^{-1}\int_{B_\rho(\xi)}|\underline{\underline{H}}|^p d\mu\right)^{1/p}\rho^{1-n/p} \le \varepsilon < \frac{1}{2}$, then, for $0<\sigma<\tau<(1-\beta)\rho$

$$(*)\qquad \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) \le (1+c\varepsilon)\omega_n^{-1}\tau^{-n}\mu(B_\tau(\zeta))$$

$$\le (1+c\varepsilon)^2\,\omega_n^{-1}((1-\beta)\rho)^{-n}\,\mu(B_{(1-\beta)\rho}(\zeta))$$

$$\le (1+c\varepsilon)^2\,(1-\beta)^{-n}\,\omega_n^{-1}\,\rho^{-n}\,\mu(B_\rho(\xi))$$

provided $\zeta \in \operatorname{spt} V \cap B_{\beta\rho}(\xi)$. Then the hypothesis $\omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)) \le 2(1-\alpha)$ (in 22.4) gives

$$(**)\qquad \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) \le 2(1-\alpha/2)\ ,\quad 0<\sigma<\rho/2\ ,\ \zeta\in \operatorname{spt} V\cap B_{\beta\rho}(\xi)\ ,$$

provided $\beta = \beta(n,k,\alpha,p)$ is sufficiently small. Thus letting $\sigma \downarrow 0$ we have

$$\theta(\zeta) \le 2(1-\alpha/2) \qquad \mu\text{-a.e. } \zeta \in B_{\beta\rho}(\xi)\ .$$

If θ is integer-valued (i.e. if V is an integer multiplicity n-varifold) then this evidently implies $\theta = 1$ μ-a.e. in $B_{\beta\rho}(\xi)$. Thus, with $\beta\rho$ in place of ρ, the hypothesis $\theta \le 1+\varepsilon$ in 22.4 is automatically satisfied, hence the conclusion of Theorem 23.1 holds with $\beta\rho$ in place of ρ, even without the hypothesis $\theta \le 1+\varepsilon$.

---

* J. Duggan in his Ph.D. thesis [DJ] has shown that in any case the hypothesis $\theta \le 1+\varepsilon$ can be dropped entirely.

(2) Quite generally, even if θ is not necessarily integer-valued, we note that if we make the stronger hypothesis $\omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)) < 1+\varepsilon$ (instead of $\omega_n^{-1}\rho^{-n}\mu(B_\rho(\xi)) \le 2(1-\alpha))$, then (*) above gives (taking $\beta=\varepsilon$)

$$\omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) \le 1+c\varepsilon \ , \ 0<\sigma<\rho/2 \ , \ \zeta\in B_{\varepsilon\rho}(\xi)\cap \operatorname{spt} V \ .$$

Thus again we can drop the restriction $\theta\le 1+\varepsilon$ in 22.4, provided we make the assumption $\omega_n^{-1}\sigma^{-n}\mu(B_\rho(\xi)) < 1+\varepsilon$; then Theorem 23.1 holds with $\varepsilon\rho$ in place of ρ.

Proof of 23.1 Throughout the proof $c = c(n,k,\alpha,p) > 0$. We are assuming

(1) $$1\le\theta\le 1+\varepsilon \quad \mu-\text{a.e. in } B_\rho(0)\cap \operatorname{spt} V$$

(ε to be chosen) and by Remark 23.2(1) (**) we can select $\varepsilon=\varepsilon(n,k,\alpha,p)$ and $\beta=\beta(n,k,\alpha,p)$ such that

(2) $$\omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) \le 2(1-\alpha/2) \ , \ 0<\sigma\le\rho/2 \ , \ \zeta\in B_{\beta\rho}(0)\cap \operatorname{spt} V \ .$$

By (1), (2) and the tilt excess decay theorem 22.5 (with σ in place of ρ, $\alpha/2$ in place of α, ζ in place of ξ) we then know that there are $\varepsilon=\varepsilon(n,k,\alpha,p)$, $\eta=\eta(n,k,\alpha,p)$ so that, for $\sigma<\rho/2$, $\zeta\in \operatorname{spt} B_{\beta\rho}(0)\cap \operatorname{spt} V$,

(3) $$E_*(\zeta,\sigma,S_0) < \varepsilon \ \Rightarrow\ E_*(\zeta,\eta\sigma,S_1) < \eta^{2(1-n/p)}\,E_*(\zeta,\sigma,S_0)$$

for suitable S_1. Notice that this can be repeated; by induction we prove that if $\zeta\in \operatorname{spt} V\cap B_{\beta\rho}(0)$, $\sigma<\rho/2$, and $E_*(\zeta,\sigma,S_0)<\varepsilon$, then there is a sequence $S_1,S_2\ldots$ of n-dimensional subspaces such that

(4) $$E_*(\zeta,\eta^j\sigma,S_j) \le \eta^{2(1-n/p)}E_*(\zeta,\eta^{j-1}\sigma,S_{j-1}) \le \eta^{2(1-n/p)j}E_*(\zeta,\sigma,S_0)$$

for each $j\ge 1$, and (by Remark 22.6)

$$(5)\qquad |p_{S_j}-p_{S_{j-1}}|^2 \le cE_*(\zeta,\eta^{j-1}\sigma,S_{j-1}) \le c\eta^{2(1-n/p)j}E_*(\zeta,\sigma,S_0)\ .$$

Next we note that $E_*(\zeta,\rho/2,\mathbb{R}^n) \le 2^n\, E_*(0,\rho,\mathbb{R}^n)$ for $\zeta \in B_{\rho/2}(0)$, and hence the above discussion shows that if 22.4 holds with $\xi = 0$, $T = \mathbb{R}^n$ and $2^{-n}\varepsilon$ in place of ε (ε as above) then (4), (5) hold with $S_0 = \mathbb{R}^n$ and $\sigma = \rho/2$. Thus

$$(6)\qquad E(\zeta,\eta^j\rho/2,S_j) \le \eta^{2(1-n/p)j}\, E_*(\zeta,\rho/2,\mathbb{R}^n) \le c\eta^{2(1-n/p)j}\, E_*(0,\rho,\mathbb{R}^n)$$

and

$$(7)\qquad |p_{S_j}-p_{S_{j-1}}|^2 \le c\eta^{2(1-n/p)j}\, E_*(0,\rho,\mathbb{R}^n)$$

for each $j \ge 1$ (with $S_0 = \mathbb{R}^n$) . Notice that (7) gives

$$(8)\qquad |p_{S_\ell}-p_{S_j}|^2 \le c\eta^{2(1-n/p)j}\, E_*(0,\rho,\mathbb{R}^n)$$

for $\ell \ge j \ge 0$. It evidently follows from (8) that there is $S(\zeta)$ such that

$$(9)\qquad |p_{S(\zeta)}-p_{S_j}|^2 \le c\eta^{2(1-n/p)j}\, E_*(0,\rho,\mathbb{R}^n)\ .$$

In particular (setting $j=0$)

$$(10)\qquad |p_{S(\zeta)}-p_{\mathbb{R}^n}|^2 \le c\, E_*(0,\rho,\mathbb{R}^n)\ .$$

Now if $r \in (0,\rho)$ is arbitrary we can choose $j \ge 0$ such that $\eta^{j+1}\rho < r \le \eta^j\rho$. Then (6) and (9) evidently imply

$$(11)\qquad E_*(\zeta,r,S(\zeta)) \le c(r/\rho)^{2(1-n/p)}\, E_*(0,\rho,\mathbb{R}^n)$$

for each $\zeta \in B_{\beta\rho}(0) \cap \operatorname{spt} V$ and each $0 < r \le \rho$. Notice also that (10), (11) and (2), with $\sigma = r$, imply

$$(12)\qquad E_*(\zeta,r,\mathbb{R}^n) \le c\, E_*(0,\rho,\mathbb{R}^n) (\le c\varepsilon)\ .$$

Hence for sufficiently small ε we have from (12) that if G is as in the proof of Theorem 20.2 (with $\ell = \varepsilon^{\frac{1}{2n+3}}$) then $\mu(B_{\beta\rho} \sim G) = 0$ $(\beta = \beta(n,k,\alpha,p)$, $\varepsilon = \varepsilon(n,k,\alpha,p)$ sufficiently small). That is

$$\text{(13)} \qquad \operatorname{spt} V \cap B_{\beta\rho}(0) \subset \operatorname{graph} f$$

for $\varepsilon = \varepsilon(n,k,\alpha,p)$ and $\beta = \beta(n,k,\alpha,p)$ sufficiently small, where f is a Lipschitz function $B^n_{\beta\rho}(0) \to \mathbb{R}^k$ with

$$\text{(14)} \qquad \operatorname{Lip} f \le \varepsilon^{\frac{1}{2n+3}}, \quad \sup|f| \le c\, \varepsilon^{\frac{1}{2n+2}} \rho .$$

Now we claim that in fact

$$\text{(15)} \qquad \operatorname{spt} V \cap B_{\beta\rho}(0) = \operatorname{graph} f \cap B_{\beta\rho}(0) .$$

Indeed otherwise by (13) we could choose $\zeta \in B^n_{\beta\rho/2}(0)$ and $0 < \sigma < \beta\rho/2$ such that

$$\text{(16)} \qquad \begin{cases} (B^n_\sigma(\zeta) \times \mathbb{R}^k) \cap B_{\beta\rho}(0) \cap \operatorname{spt} V = \emptyset \\ (\bar{B}^n_\sigma(\zeta) \times \mathbb{R}^k) \cap B_{\beta\rho}(0) \cap \operatorname{spt} V \neq \emptyset . \end{cases}$$

Then taking $\zeta_* \in (\bar{B}^n_\sigma(\zeta) \times \mathbb{R}^k) \cap B_{\beta\rho}(0) \cap \operatorname{spt} V$ and using (1), (13), (14), (16) we would evidently have $\Theta^n(\mu,\zeta_*) < 1$ (if ε is sufficiently small). This contradicts the fact that $\Theta^n(\mu,\zeta) \ge 1 \quad \forall\, \zeta \in \operatorname{spt} V \cap B_\rho(0)$.

Having established (15) we can now easily check (using the area formulae) that for any linear subspace $S = \operatorname{graph} \ell$, where $\ell = (\ell^1,\dots,\ell^k) : \mathbb{R}^n \to \mathbb{R}^k$ is linear and $|\operatorname{grad} \ell^j| \le 1$ for each j, we have

$$\text{(17)} \qquad \sigma^{-n} \int_{B^n_{\sigma/2}(p_{\mathbb{R}^n}(\zeta))} \sum_{j=1}^k |\operatorname{grad} f^j(x) - \operatorname{grad} \ell^j|^2 \, dL^n(x) \le c\, E(\zeta,\sigma,S)$$

for $\sigma \in (0,\beta\rho/2)$ (again provided ε in (14) is sufficiently small). Using

(17) and (11) we conclude that for each $\zeta \in B_{\beta\rho/2}(0) \cap \operatorname{spt} V$ there is a linear function $\ell_\zeta = (\ell_\zeta^1, \ldots, \ell_\zeta^k) : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$(18) \qquad r^{-n} \int_{B_r^n(p_{\mathbb{R}^n}(\zeta))} \sum_{j=1}^k |\operatorname{grad} f^j(x) - \operatorname{grad} \ell_\zeta^j|^2 dL^n(x)$$

$$\le c(r/\rho)^{2(1-n/p)} E_*(0,\rho,\mathbb{R}^n)$$

for $0 < r < \beta\rho/4$. It evidently follows from this, by letting $r \downarrow 0$ in (18), that $\operatorname{grad} f_j(p_{\mathbb{R}^n}(\zeta)) = \operatorname{grad} \ell_\zeta^j$ for μ-a.e. $\zeta \in \operatorname{spt} V \cap B_{\beta\rho/4}(0)$. Hence using (18) again, we easily conclude $|\operatorname{grad} f^j(x_1) - \operatorname{grad} f^j(x_2)| \le c(r/\rho)^{1-n/p} E_*(0,\rho,\mathbb{R}^n)^{\frac{1}{2}}$ for $x_1, x_2 \in B_r^n(0)$, and so

$$(19) \qquad |\operatorname{grad} f^j(x_1) - \operatorname{grad} f^j(x_2)| \le c\left(\frac{|x_1 - x_2|}{\rho}\right)^{1-n/p} E_*(0,\rho,\mathbb{R}^n)^{\frac{1}{2}}$$

for L^n-a.e. $x_1, x_2 \in B_{\beta\rho/8}^n(0)$. Since f is Lipschitz it follows from this that $f \in C^{1,1-n/p}$ with (19) holding for *every* $x_1, x_2 \in B_{\beta\rho/8}^n(0)$. The theorem now follows with $f = u$ and $\gamma = \beta/8$.

§24. MAIN REGULARITY THEOREM: SECOND VERSION

In this section we continue to assume $V = \underline{\underline{v}}(M,\theta)$ is a rectifiable n-varifold with generalized mean curvature $\underline{\underline{H}}$ in U. With $\delta \in (0,1/2)$ a constant to be specified below, we consider the hypotheses:

$$24.1 \qquad \begin{cases} 1 \le \theta \quad \mu\text{-a.e.}, \quad 0 \in \operatorname{spt} V, \quad B_\rho(0) \subset U \\ \omega_n^{-1}\rho^{-n}\mu(B_\rho(0)) \le 1+\delta, \quad \left(\int_{B_\rho(0)} |\underline{\underline{H}}|^p d\mu\right)^{1/p} \rho^{1-n/p} \le \delta . \end{cases}$$

24.2 THEOREM *If* $p > n$ *is arbitrary, then there are* $\delta = \delta(n,k,p)$, $\gamma = \gamma(n,k,p) \in (0,1)$ *such that the hypotheses* 24.1 *imply the existence of a linear isometry* q *of* $\mathbb{R}^{n+k}$ *and a* $u = (u^1,\ldots,u^k) \in C^{1,1-n/p}(\bar{B}^n_{\gamma\rho}(0);\mathbb{R}^k)$ *with* $u(0) = 0$, $\operatorname{spt} V \cap B_{\gamma\rho}(0) = q(\operatorname{graph} u) \cap B_{\gamma\rho}(0)$, *and*

$$\rho^{-1}\sup|u| + \sup|Du| + \rho^{1-n/p}\sup_{\substack{x,y\in B^n_{\gamma\rho}(0)\\ x\neq y}}|x-y|^{-(1-n/p)}|Du(x)-Du(y)| \le c\delta^{1/4n},$$

$c = c(n,k,p)$.

Before giving the proof of 24.2, we shall need the following lemma.

24.3 LEMMA *Suppose* $\delta \in (0,1/2)$ *and that* 24.1 *holds. Then there is* $\beta = \beta(n,k,p,\delta) \in (0,1/2)$ *such that*

$$(1+c\delta)^{-1} < \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) < 1+c\delta \ , \quad 0<\sigma\le\beta\rho \ , \quad \zeta\in \operatorname{spt} V\cap B_{\beta\rho}(0)$$

and such that, for any $\zeta \in \operatorname{spt} V \cap B_{\beta\rho}(0)$, $\sigma \in (0,\beta\rho)$ *there is an* n-*dimensional subspace* $T = T(\zeta,\sigma)$ *with*

$$\sigma^{-1}\sup\{\operatorname{dist}(x,T) \ : \ x\in \operatorname{spt} V\cap B_\sigma(\zeta)\} \le c\delta^{1/4n} .$$

Proof First note that by the monotonicity formulae of §17 (see in particular 23.2(1)(*)) we have, subject to 24.1, that

$$(1)\qquad (1+c\delta)^{-1} \le \omega_n^{-1}\sigma^{-n}\mu(B_\sigma(\zeta)) \le 1+c\delta \ , \quad 0<\sigma<\rho/2 \ ,$$

$\zeta \in \operatorname{spt} V \cap B_{\beta\rho}(0)$, $\beta = \beta(n,k,p,\delta) \in (0,1/4)$, so the first part of the lemma is proved.

Now take any fixed $\sigma \in (0,\beta\rho)$ and suppose for convenience of notation (by changing scale and translating the origin) that $\sigma = 1/2$ and $\zeta = 0$. Then by (1) and 17.6(1) (see in particular Remark 17.9(1)) we have

$$(2) \qquad \int_{B_{1/2}(0)} |p_{T_x^{\perp}}(x-\zeta)|^2 \le \int_{B_1(\zeta)} |p_{T_x^{\perp}}(x-\zeta)|^2 \, |x-\zeta|^{-n-2} d\mu \le c\delta$$

for $\zeta \in \operatorname{spt} V \cap B_{1/2}(0)$, where $T_x^{\perp} = (T_x M)^{\perp}$. Next note that we can select N points $\zeta_1,\ldots,\zeta_N \in \operatorname{spt} V \cap B_{1/2}(0) \sim B_{\delta^{1/4n}}(0)$, $N \le c\delta^{-1/4}$, such that

$$(3) \qquad \operatorname{spt} V \cap B_{1/2}(0) \sim B_{\delta^{1/4n}}(0) \subset \bigcup_{j=1}^{N} B_{\delta^{1/4n}}(\zeta_j) .$$

(To achieve this, just take a *maximal* disjoint collection of balls of radius $\delta^{1/4n}/4$ centred in $\operatorname{spt} V \cap B_{1/2}(0) \sim B_{\delta^{1/4n}}(0)$.) Then by using (2) with $\zeta = \zeta_j$ we have

$$\int_{B_{1/2}(0)} \sum_{j=1}^{N} |p_{T_x^{\perp}}(x-\zeta_j)|^2 d\mu \le c\,\delta\, N \le c'\delta^{1/2} ,$$

so that for any given $R \ge 1$ we have

$$(4) \qquad \sum_{j=1}^{N} |p_{T_x^{\perp}}(x-\zeta_j)|^2 \le R\, \delta^{1/4}$$

except possibly for a set of $x \in B_{1/2}(0) \cap \operatorname{spt} V$ of μ-measure $\le c\, R^{-1}\delta^{1/4}$. Taking $R = R(n,k)$ sufficiently large and noting $\mu(B_{\delta^{1/4n}}(0)) \ge c\delta^{1/4}$ (by (1)), we can therefore find $x_0 \in \operatorname{spt} V \cap B_{\delta^{1/4n}}(0)$ such that

$$|p_{T_{x_0}^{\perp}}(x_0-\zeta_j)| \le c\delta^{1/8} , \quad j = 1,\ldots,N .$$

Since $|x_0| < \delta^{1/4n}$, we then have

$$(5) \qquad |p_{T_{x_0}^{\perp}}\, \zeta_j| \le c\delta^{1/4n} , \quad j = 1,\ldots,N .$$

That is, all points $\zeta_1,\ldots,\zeta_N$ are in the $c\delta^{1/4n}$ neighbourhood of the subspace T_{x_0} , and the required result now follows from (3).

Proof of Theorem 24.2 Theorem 24.2 in fact now follows directly from Theorem 23.1, because by combining Lemma 22.2 and the above lemma we see that for any $\varepsilon > 0$ there is $\delta = c\,\varepsilon^{2n}$ $(c = c(n,k,p))$ such that the hypotheses 24.1 imply 22.4 with $\xi = 0$, ρ replaced by $\beta\rho$ and with suitable T.

CHAPTER 6

CURRENTS

This chapter provides an introduction to the basic theory of currents, with particular emphasis on integer multiplicity rectifiable n-currents (briefly called integer multiplicity currents), which are essentially just integer n-varifolds equipped with an orientation.* The concept of such currents was introduced in the historic paper [FF] of Federer and Fleming; their advantage is that they are at once able to be represented as "generalized surfaces" (in terms of a countably n-rectifiable set with an integer multiplicity) and at the same time have nice compactness properties (see 27.3 below).

§25. PRELIMINARIES: VECTORS, CO-VECTORS, AND FORMS

$e_1,\ldots,e_P$ denote the standard orthonormal basis for $\mathbb{R}^P$ and $\omega^1,\ldots,\omega^P$ the dual basis for the dual space $\Lambda^1(\mathbb{R}^P)$ of $\mathbb{R}^P$. $\Lambda_n(\mathbb{R}^P)$, $\Lambda^n(\mathbb{R}^P)$ denote the spaces of n-vectors and n-covectors respectively. Thus $v \in \Lambda_n(\mathbb{R}^P)$ can be represented

$$v = \sum_{1\le i_1<\ldots<i_n\le P} a_{i_1\ldots i_n}\, e_{i_1}\wedge\ldots\wedge e_{i_n}$$

$$= \sum_{\alpha\in I_{n,P}} a_\alpha\, e_\alpha ,$$

using "multi-index" notation in which $\alpha = (i_1,\ldots,i_n) \in \mathbb{Z}^n_+ \equiv \{(j_1,\ldots,j_n) :$ each j_ℓ is a positive integer$\}$ and $I_{n,P} = \{\alpha = (i_1,\ldots,i_n) \in \mathbb{Z}^n_+ :$ $1 \le i_1<\ldots<i_n \le P\}$. Similarly any $w \in \Lambda^n(\mathbb{R}^P)$ can be represented as

---

* These are precisely the currents called *locally rectifiable* in the literature (see [FF], [FH1]); we have adopted the present terminology both because it seems more natural and also because it is consistent with the varifold terminology of Allard (see Chapter 4, Chapter 8).

$$w = \sum_{\alpha \in I_{n,P}} a_\alpha \, \omega^\alpha ,$$

where $\omega^\alpha = \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}$ if $\alpha = (i_1, \ldots, i_n) \in I_{n,P}$. Such a v (respectively w) is called *simple* if it can be expressed $v_1 \wedge \ldots \wedge v_n$ with $v_j \in \mathbb{R}^P$ (respectively $w_1 \wedge \ldots \wedge w_n$ with $w_j \in \Lambda^1(\mathbb{R}^P)$) . We assume $\Lambda_n(\mathbb{R}^P)$, $\Lambda^n(\mathbb{R}^P)$ are equipped with the inner products $< , >$ naturally induced from $\mathbb{R}^P$ (making $\{e_\alpha\}_{\alpha \in I_{n,P}}$, $\{\omega^\alpha\}_{\alpha \in I_{n,P}}$ orthonormal bases) . Thus

$$\left\langle \sum_{\alpha \in I_{n,P}} a_\alpha e_\alpha , \sum_{\alpha \in I_{n,P}} b_\alpha e_\alpha \right\rangle = \sum_{\alpha \in I_{n,P}} a_\alpha b_\alpha$$

and

$$\left\langle \sum_{\alpha \in I_{n,P}} a_\alpha \omega^\alpha , \sum_{\alpha \in I_{n,P}} b_\alpha \omega^\alpha \right\rangle = \sum_{\alpha \in I_{n,P}} a_\alpha b_\alpha .$$

For open $U \subset \mathbb{R}^P$, $E^n(U)$ denotes the set of smooth n-forms $\omega = \sum_{\alpha \in I_{n,P}} a_\alpha dx^\alpha$ where $a_\alpha \in C^\infty(U)$ and $dx^\alpha = dx^{i_1} \wedge \ldots \wedge dx^{i_n}$ if $\alpha = (i_1, \ldots, i_n) \in I_{n,P}$. dx^j as usual denotes the 1-form given by

25.1 $$dx^j(f) = \frac{\partial f}{\partial x^j} , \; f \in C^\infty(U) .$$

If we make the usual identifications of $T_x \mathbb{R}^P$ and $\Lambda^1(T_x\mathbb{R}^P)$ with $\mathbb{R}^P$ and $\Lambda^1(\mathbb{R}^P)$, we are able to interpret $\omega \in E^n(U)$ as an element of $C^\infty(U;\Lambda^n\mathbb{R}^P)$; we shall do this frequently in the sequel.

The exterior derivative $E^n(U) \to E^{n+1}(U)$ is defined as usual by

25.2 $$d\omega = \sum_{j=1}^{P} \sum_{\alpha \in I_{n,P}} \frac{\partial a_\alpha}{\partial x^j} dx^j \wedge dx^\alpha$$

if $\omega = \sum_{\alpha \in I_{n,P}} a_\alpha dx^\alpha$. By direct computation (using $\dfrac{\partial^2 a_\alpha}{\partial x^i \partial x^j} = \dfrac{\partial^2 a_\alpha}{\partial x^j \partial x^i}$

and $dx^i \wedge dx^j = - dx^j \wedge dx^i$) one checks that

25.3 $$d^2\omega = 0 \quad \forall\, \omega \in \mathcal{E}^n(U) .$$

Given $\omega = \sum_{\alpha \in I_{n,Q}} a_\alpha(y)\, dy^\alpha \in \mathcal{E}^n(V)$, $V \subset \mathbb{R}^Q$ open, and a smooth map $f : U \to V$, we define the "pulled back" form $f^\# \omega \in \mathcal{E}^n(U)$ by

25.4 $$f^\# \omega = \sum_{\alpha = (i_1, \ldots, i_n) \in I_{n,Q}} a_\alpha \circ f \; df^{i_1} \wedge \ldots \wedge df^{i_n} ,$$

where df^j is $\sum_{i=1}^{P} \frac{\partial f^j}{\partial x^i} dx^i$, $j = 1, \ldots, Q$.

Notice that the exterior derivative commutes with pulling back:

25.5 $$df^\# = f^\# d .$$

We let $\mathcal{D}^n(U)$ denote the set of $\omega = \sum_{\alpha \in I_{n,P}} a_\alpha dx^\alpha \in \mathcal{E}^n(U)$ such that each a_α has compact support. We topologize $\mathcal{D}^n(U)$ with the usual locally convex topology, characterized by the assertion that $\omega^k = \sum_{\alpha \in I_{n,P}} a_\alpha^{(k)} dx^\alpha \to \omega = \sum_{\alpha \in I_{n,P}} a_\alpha\, dx^\alpha$ if there is a fixed compact $K \subset U$ such that $\operatorname{spt} a_\alpha^{(k)} \subset K$ $\forall\, \alpha \in I_{n,P}$, $k \geq 1$, and if $\lim D^\beta a_\alpha^{(k)} = D^\beta a_\alpha$ $\forall\, \alpha \in I_{n,P}$ and every multi-index β . For any $\omega \in \mathcal{D}^n(U)$, we define

25.6 $$|\omega| = \sup_{x \in U} \langle \omega(x), \omega(x) \rangle^{\frac{1}{2}} .$$

Notice that if $f : U \to V$ is smooth (U, V open in $\mathbb{R}^P$, $\mathbb{R}^Q$) and if f is *proper* (i.e. $f^{-1}(K)$ is a compact subset of U whenever K is a compact subset of V) then $f^\# \omega \in \mathcal{D}^n(U)$ whenever $\omega \in \mathcal{D}^n(V)$.

§26. GENERAL CURRENTS

Throughout this section U is an open subset of $\mathbb{R}^P$.

26.1 DEFINITION An n-dimensional current (briefly called an n-current) in U is a continuous linear functional on $\mathcal{D}^n(U)$. The set of such n-currents will be denoted $\mathcal{D}_n(U)$.

Note that in case $n=0$ the n-currents in U are just the Schwartz distributions on U. More importantly though, the n-currents, $n \geq 1$, can be interpreted as a generalization of the n-dimensional oriented submanifolds M having locally finite $\mathcal{H}^n$-measure in U. Indeed given such an $M \subset U$ with orientation ξ (thus $\xi(x)$ is continuous on M with $\xi(x) = \pm\, \tau_1 \wedge \ldots \wedge \tau_n$ $\forall\, x \in M$, where $\tau_1, \ldots, \tau_n$ is an orthonormal basis for $T_xM)^*$, then there is a corresponding n-current $[\![M]\!] \in \mathcal{D}_n(U)$ defined by

$$[\![M]\!](\omega) = \int_M \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^n(x) \;, \quad \omega \in \mathcal{D}^n(U) \;, \tag{26.2}$$

where $\langle\, ,\, \rangle$ denotes the dual pairing for $\Lambda^n(\mathbb{R}^P)$, $\Lambda_n(\mathbb{R}^P)$. (That is, the n-current $[\![M]\!]$ is obtained by integration of n-forms over M in the usual sense of differential geometry: $[\![M]\!](\omega) = \int_M \omega$ in the usual notation of differential geometry.)

Motivated by the classical Stokes' theorem ($\int_M d\omega = \int_{\partial M} \omega$ if M is a compact smooth manifold with smooth boundary) we are led (by 26.2) to quite generally define the boundary ∂T of an n-current $T \in \mathcal{D}_n(U)$ by

$$\partial T(\omega) = T(d\omega) \;, \quad \omega \in \mathcal{D}^n(U) \tag{26.3}$$

---

* Thus $\xi(x) \in \Lambda_n(T_xM)$; notice this differs from the usual convention of differential geometry where we would take $\xi(x) \in \Lambda^n(T_xM)$.

(and $\partial T = 0$ if $n = 0$) ; thus $\partial T \in \mathcal{D}_{n-1}(U)$ if $T \in \mathcal{D}_n(U)$. Here and subsequently we define $\mathcal{D}_{n-1}(U) = 0$ in case $n = 0$.

Notice that $\partial^2 T = 0$ by 25.3

Again motivated by the special example $T = [\![M]\!]$ as in 26.2 we define the *mass* of T , $\underline{\underline{M}}(T)$, for $T \in \mathcal{D}_n(U)$ by

26.4
$$\underline{\underline{M}}(T) = \sup_{|\omega|\leq 1, \omega\in\mathcal{D}^n(U)} T(\omega)$$

(so that $\underline{\underline{M}}(T) = \mathcal{H}^n(M)$ in case $T = [\![M]\!]$ as in 26.2). More generally for any open $W \subset U$ we define

26.5
$$\underline{\underline{M}}_W(T) = \sup_{\substack{|\omega|\leq 1, \omega\in\mathcal{D}^n(U) \\ \operatorname{spt}\omega\subset W}} T(\omega)$$

26.6 REMARK Notice that there is some flexibility in the definition of $\underline{\underline{M}}$; we would still get the "correct" value $\mathcal{H}^n(M)$ for the case $T = [\![M]\!]$ if we were to make the definition $\underline{\underline{M}}(T) = \sup_{\substack{\|\omega(x)\|\leq 1 \\ \omega\in\mathcal{D}^n(U)}} T(\omega)$,

provided only that $\| \ \|$ is a norm for $\Lambda^n(\mathbb{R}^P)$ with the properties:

(1) $\langle\omega,\xi\rangle \leq \|\omega\| \ |\xi|$ whenever $\xi \in \Lambda_n(\mathbb{R}^P)$ is *simple*

and

(2) for each fixed simple $\xi \in \Lambda_n(\mathbb{R}^P)$, equality holds in (1) for some $\omega \neq 0$.

(Evidently $\| \ \| = | \ |$ is one such norm.) Notice that the *smallest* possible norm for $\Lambda^n(\mathbb{R}^P)$ having these properties is defined by

$$\|\omega\| = \sup_{\substack{\xi\in\Lambda_n(\mathbb{R}^P), |\xi|=1 \\ \xi \text{ simple}}} \langle\omega,\xi\rangle$$

($\| \ \|$ is called the co-mass norm for $\Lambda^n(\mathbb{R}^P)$.) There is a good argument to say that one should adopt this norm in the definition of $\underline{\underline{M}}(T)$ (and indeed this is usually done - see e.g. [FF], [FH1]) since, by virtue of the consequent *maximality* of $\underline{\underline{M}}(T)$ it is more likely to yield *equality* in the general inequality $\underline{\underline{M}}(T) \le \liminf \underline{\underline{M}}([\![M_j]\!])$, if $\{M_j\}$ is a sequence of C^1 submanifolds with weak limit T (see 26.12 below). Nevertheless we will here stick to the definition 26.4, because it has certain advantages (e.g. the application of the Riesz representation theorem - see below - is cleaner, and 26.4 does yield the "correct" value in the most important case when T is an integer multiplicity current as in §27.)

Notice that by the Riesz Representation Theorem 4.1 we have that if $T \in \mathcal{D}_n(U)$ satisfies $\underline{\underline{M}}_W(T) < \infty \quad \forall\, W \subset\subset U$, then there is a Radon measure μ_T on U and μ_T-measurable function $\vec{T}$ with values in $\Lambda_n(\mathbb{R}^P)$, $|\vec{T}| = 1$ μ_T-a.e., such that

26.7 $$T(\omega) = \int <\omega(x), \vec{T}(x)> d\mu_T(x) .$$

μ_T (the total variation measure associated with T) is characterized by

26.8 $$\mu_T(W) = \sup_{\substack{\omega\in\mathcal{D}^n(U),\,|\omega|\le 1\\ \operatorname{spt}\omega\subset W}} T(\omega) \quad (\equiv \underline{\underline{M}}_W(T))$$

for any open $W \subset U$. In particular

$$\mu_T(U) = \underline{\underline{M}}(T) .$$

Notice that for such a T we can define, for any μ_T-measurable subset A of U (and in particular for any Borel set $A \subset U$), a new current $T \llcorner A \in \mathcal{D}_n(U)$ by

26.9 $$(T \llcorner A)(\omega) = \int_A <\omega, \vec{T}> d\mu_T .$$

More generally, if ϕ is any locally μ_T-integrable function on U then we can define $T \llcorner \phi \in \mathcal{D}_n(U)$ by

26.10 $$(T \llcorner \phi)(\omega) = \int \phi \langle \omega, \xi \rangle \, d\mu_T \, .$$

Given $T \in \mathcal{D}_n(U)$ we define the *support* spt T of T to be the relatively closed subset of U defined by

26.11 $$\operatorname{spt} T = U \sim \cup W$$

where the union is over all open sets W such that $T(\omega) = 0$ whenever $\omega \in \mathcal{D}^n(U)$ with $\operatorname{spt} \omega \subset W$. Notice that if $\underline{\underline{M}}_W(T) < \infty$ for each $W \subset\subset U$ and if μ_T is the corresponding total variation measure (as in 26.7, 26.8) then

$$\operatorname{spt} T = \operatorname{spt} \mu_T$$

where $\operatorname{spt} \mu_T$ is the support of μ_T in the usual sense of Radon measures in U.

Given a sequence $\{T_q\} \subset \mathcal{D}_n(U)$, we write $T_q \rightharpoonup T$ in U ($T \in \mathcal{D}_n(U)$) if $\{T_q\}$ converges weakly to T in the usual sense of distributions:

26.12 $$T_q \rightharpoonup T \iff \lim T_q(\omega) = T(\omega) \quad \forall\, \omega \in \mathcal{D}^n(U) \, .$$

Notice that mass is trivially lower semi-continuous with respect to weak convergence: if $T_q \rightharpoonup T$ in U then

26.13 $$\underline{\underline{M}}_W(T) \le \liminf_{q \to \infty} \underline{\underline{M}}_W(T_q) \qquad \forall \text{ open } W \subset U \, .$$

Notice also that by applying the standard Banach-Alaoglu theorem [Roy] (in the Banach spaces $\mathcal{M}_n(W) = \{T \in \mathcal{D}_n(W) : \underline{\underline{M}}_W(T) < \infty\}$, $W \subset\subset U$) we deduce

26.14 LEMMA *If* $\{T_q\} \subset \mathcal{D}_n(U)$ *and* $\sup_{q \ge 1} \underline{\underline{M}}_W(T_q) < \infty$ *for each* $W \subset\subset U$, *then there is a subsequence* $\{T_{q'}\}$ *and a* $T \in \mathcal{D}_n(U)$ *such that* $T_{q'} \rightharpoonup T$ *in* U.

The following terminology will be used frequently:

26.15 TERMINOLOGY Given $T_1 \in \mathcal{D}_n(U_1)$, $T_2 \in \mathcal{D}_n(U_2)$ and an open $W \subset U_1 \cap U_2$, we say $T_1 = T_2$ in W if $T_1(\omega) = T_2(\omega)$ whenever ω is a smooth n-form in $\mathbb{R}^{n+k}$ with $\operatorname{spt}\omega \subset W$.

Next we want to describe the cartesian product of currents $T_1 \in \mathcal{D}_r(U_1)$, $T_2 \in \mathcal{D}_s(U_2)$, $U_1 \subset \mathbb{R}^{P_1}$, $U_2 \subset \mathbb{R}^{P_2}$ open. We are motivated by the case when $T_1 = [\![M_1]\!]$ and $T_2 = [\![M_2]\!]$ (Cf. 26.2) where M_1, M_2 are oriented submanifolds of dimension r, s respectively. We want to define $T_1 \times T_2 \in \mathcal{D}_{r+s}(U_1 \times U_2)$ in such a way that for this special case (when $T_j = [\![M_j]\!]$) we get $[\![M_1]\!] \times [\![M_2]\!] = [\![M_1 \times M_2]\!]$. We are thus inevitably led to the following

26.16 DEFINITION If $\omega \in \mathcal{D}^{r+s}(U_1 \times U_2)$ is written in the form

$$\omega = \sum_{\substack{(\alpha,\beta)\in I_{r',P_1}\times I_{s',P_2} \\ r'+s'=r+s}} a_{\alpha\beta}(x,y)\,dx^\alpha \wedge dy^\beta \quad \text{(using multi-index notation as in §26)}$$

then we define

$$T_1 \times T_2(\omega) = T_1\left(\sum_{\alpha\in I_{r,P_1}} T_2\left(\sum_{\beta\in I_{s,P_2}} a_{\alpha\beta}(x,y)\,dy^\beta\right) dx^\alpha\right).$$

(Notice in particular this gives $T_1 \times T_2(\omega_1 \wedge \omega_2) = 0$ if $\omega_1 \in \mathcal{D}^{r'}(U_1)$, $\omega_2 \in \mathcal{D}^{s'}(U_2)$ with $r' + s' = r + s$ but $(r',s') \neq (r,s)$.)

One readily checks, using this definition and the definition of ∂ (in 26.3) that

26.17 $$\partial(T_1 \times T_2) = (\partial T_1) \times T_2 + (-1)^r T_1 \times \partial T_2 .$$

(Notice this is valid also in case r or $s = 0$ if we interpret the appropriate terms as zero; e.g. if $r = 0$ then $\partial(T_1 \times T_2) = T_1 \times \partial T_2$.)

An important special case of 26.17 occurs when we take $T \in \mathcal{D}_n(U)$, $U \subset \mathbb{R}^P$, and we let $[\![(0,1)]\!]$ be the 1-current defined as in 26.3 with $M = (0,1) \subset \mathbb{R}$ ($(0,1)$ having its usual orientation). Then 26.17 gives

26.18 $$\partial([\![(0,1)]\!]\times T) = (\{1\}-\{0\})\times T - [\![(0,1)]\!]\times\partial T$$

$$\equiv \{1\}\times T - \{0\}\times T - [\![(0,1)]\!]\times\partial T\ .$$

Here and subsequently $\{p\}$, for a point $p \in U$, means the 0-current $\in \mathcal{D}_0(U)$ defined by

26.19 $$\{p\}(\omega) = \omega(p)\ ,\quad \omega \in \mathcal{D}^0(U)\quad (\equiv C_c^\infty(U))\ .$$

Next we want to discuss the notion of "pushing forward" a current T via a smooth map $f : U \to V$, $U \subset \mathbb{R}^P$, $V \subset \mathbb{R}^Q$ open. The main restriction needed is that $f|\operatorname{spt} T$ is *proper;* that is $f^{-1}(K) \cap \operatorname{spt} T$ is a compact subset of U whenever K is a compact subset of V . Assuming this, we can define

26.20 $$f_\# T(\omega) = T(\zeta f^\# \omega)\qquad \forall\ \omega \in \mathcal{D}^n(V)\ ,$$

where ζ is any function $\in C_c^\infty(U)$ such that $\zeta \equiv 1$ in a neighbourhood of $\operatorname{spt} T \cap \operatorname{spt} f^\#\omega$. One easily checks that the definition of $f_\# T$ in 26.20 is independent of ζ . (Of course such ζ exist and $\zeta f^\#\omega \in \mathcal{D}^n(U)$ because $f|\operatorname{spt} T$ is proper and $\operatorname{spt}\omega$ is a compact subset of V .)

26.21 REMARKS

(1) Notice that $\partial f_\# T = f_\# \partial T$ whenever f , T are as in 26.20.

(2) If $\underline{\underline{M}}_W(T) < \infty$ for each $W \subset\subset U$, so that T has a representation as in 26.7, then it is straightforward to check that $f_\# T$ is given explicitly by

$$f_{\#}T(\omega) = \int < f^{\#}\omega, \vec{T} > d\mu_T$$

$$= \int < \omega(f(x)), df_{x\#}\vec{T}(x) > d\mu_T(x) \ . \quad *$$

Notice that we can thus make sense of $f_{\#}T$ in case f is merely C^1 (with $f|\operatorname{spt} T$ proper).

(3) If $T = [\![M]\!]$ as in 26.2, then the above remark (2) tells us that if $f|(\bar{M} \cap U)$ is proper,

$$(*) \qquad f_{\#}T(\omega) = \int_M < \omega(x), df_{x\#}\xi(x) > d\mathcal{H}^n(x) \ ,$$

where ξ is the orientation for M. Notice that this makes sense if f is only Lipschitz (by virtue of Rademacher's Theorem 5.2). If f is 1:1 and if Jf is the Jacobian of f as in 8.3, then the area formula evidently tells us that (since $df_{x\#}\xi(x) = Jf(x)\tau(f(x))$, where τ is the orientation for $f(M_+)$, $M_+ = \{x \in M : Jf(x) > 0\}$, induced by f)

$$f_{\#}T(\omega) = \int_{f(M_+)} < \omega(y) , \tau(y) > d\mathcal{H}^n(y) \ .$$

(Which confirms that our definition of $f_{\#}T$ is "correct".)

By using the above notions we can derive the important homotopy formula for currents as follows:

If $f, g : U \to V$ are smooth ($V \subset \mathbb{R}^Q$) and $h : [0,1] \times U \to V$ is smooth

---

* For a linear map $\ell : \mathbb{R}^P \to \mathbb{R}^Q$ and for $v = \sum_{\alpha \in I_{n,P}} a_\alpha e_\alpha \in \Lambda_n(\mathbb{R}^P)$ we define $\ell_{\#}v \in \Lambda_n(\mathbb{R}^Q)$ by $\ell_{\#}v = \sum_{\alpha \in I_{n,P}} a_\alpha \ell_{\#}e_\alpha = \sum_{\alpha=(i_1,\ldots,i_n)\in I_{n,P}} a_\alpha \ell(e_{i_1}) \wedge \ldots \wedge \ell(e_{i_n})$.

Then $<w, \ell_{\#}v> = <\ell^{\#}w, v>$, $w \in \Lambda^n(\mathbb{R}^Q)$.

with $h(0,x) \equiv f(x)$, $h(1,x) \equiv g(x)$, if $T \in \mathcal{D}_n(U)$, and if $h \mid [0,1] \times \operatorname{spt} T$ is proper, then (by the above discussion) $h_\#([\![(0,1)]\!] \times T)$ is well defined ($\in \mathcal{D}_{n+1}(V)$) and

$$\begin{aligned}\partial h_\#([\![(0,1)]\!] \times T) &= h_\# \partial([\![(0,1)]\!] \times T) \\ &= h_\#(\{1\} \times T - \{0\} \times T - [\![(0,1)]\!] \times \partial T) \\ &\equiv g_\# T - f_\# T - h_\#([\![(0,1)]\!] \times \partial T) .\end{aligned}$$

Thus we obtain the *homotopy formula*

26.22 $$g_\# T - f_\# T = \partial h_\#([\![(0,1)]\!] \times T) + h_\#([\![(0,1)]\!] \times \partial T) .$$

Notice that an important case of the above is given by

(*) $$h(t,x) = tg(x) + (1-t)f(x) = f(x) + t(g(x) - f(x))$$

(i.e. h is an "affine homotopy" from f to g). In this case we note that by using the integral representation 26.7 and Remark 26.21(2) above that

26.23 $$\underline{\underline{M}}(h_\# [\![(0,1)]\!] \times T)) \le \sup_{\operatorname{spt} T} |f-g| \cdot \sup_{x \in \operatorname{spt} T}(|df_x| + |dg_x|)\underline{\underline{M}}(T) .$$

(Indeed $\overrightarrow{[\![(0,1)]\!] \times T} = e_1 \wedge \vec{T}$ and $\mu_{[\![(0,1)]\!] \times T} = \mathcal{L}^1 \times \mu_T$, so by Remark 26.21(2) we have

$$\begin{aligned}h_\#([\![(0,1)]\!] \times T)(\omega) &= \int \langle \omega(h(t,x)), df_{(t,x)\#}\, e_1 \wedge \vec{T}(x) \rangle \, d\mu_T(x)\, dt \\ &= \int \langle \omega(h(t,x)), (g(x) - f(x)) \wedge (t df_x + (1-t) df_x)_\# \vec{T}(x) \rangle \\ &\qquad d\mu_T(x)\, dt\end{aligned}$$

and 26.23 follows immediately.)

We now give a couple of important applications of the above homotopy formula.

26.24 LEMMA *If* $T \in \mathcal{D}_n(U)$, $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$ *and if* $f, g : U \to V$ *are* C^1 *with* $f \mid \operatorname{spt} T = g \mid \operatorname{spt} T$ *proper , then* $f_\# T = g_\# T$. (Note that $f_\# T$, $g_\# T$ are well-defined by 26.21(2).)

Proof By the homotopy formula 26.22 we have, with $h(t,x) = tg(x) + (1-t)f(x)$,

$$g_\# T(\omega) - f_\# T(\omega) = \partial h_\# ([\![(0,1)]\!] \times T)(\omega) + h_\# ([\![(0,1)]\!] \times \partial T)(\omega)$$

$$= h_\# ([\![(0,1)]\!] \times T)(d\omega) + h_\# ([\![(0,1)]\!] \times \partial T)(\omega) ,$$

so that, by 26.23,

$$|f_\# T(\omega) - g_\# T(\omega)| \le c(\underline{\underline{M}}(T)\,|d\omega| + \underline{\underline{M}}(\partial T)\,|\omega|) \sup_{x \in \operatorname{spt} T} |f-g|$$

$$= 0 , \quad \text{since } f = g \text{ on } \operatorname{spt} T .$$

The homotopy formula also enables us to define $f_\# T$ in case f is merely Lipschitz, provided $f \mid \operatorname{spt} T$ is proper and $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$. In the following lemma we let $f^{(\sigma)} = f * \phi_\sigma$, $\phi_\sigma(x) = \sigma^{-n}\phi(\sigma^{-1}x)$, with ϕ a mollifier as in §6.

26.25 LEMMA *If* $T \in \mathcal{D}_n(U)$, $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$, *and if* $f : U \to V$ *is Lipschitz with* $f \mid \operatorname{spt} T$ *proper, then* $\lim_{\sigma \downarrow 0} f^{(\sigma)}_\# T(\omega)$ *exists for each* $\omega \in \mathcal{D}^n(V)$; $f_\# T(\omega)$ *is defined to be this limit; then* $\operatorname{spt} f_\# T \subset f(\operatorname{spt} T)$ *and* $\underline{\underline{M}}_W(f_\# T) \le (\operatorname{ess\,sup}_{f^{-1}(W)} |Df|)^n \underline{\underline{M}}_{f^{-1}(W)}(T)$ $\forall\, W \subset\subset V$.

Proof If σ, τ are sufficiently small (depending on ω) then the homotopy formula gives

$$f_{\sigma\#}T(\omega) - f_{\tau\#}T(\omega) = h_{\#}([\![(0,1)]\!] \times T)(d\omega) + h_{\#}([\![(0,1)]\!] \times \partial T)(\omega)$$

where $h : [0,1] \times U \to V$ is defined by $h(t,x) = t\, f_\sigma(x) + (1-t) f_\tau(x)$. Then by 26.23, for sufficiently small σ, τ , we have

$$|f_{\sigma\#}T(\omega) - f_{\tau\#}T(\omega)| \le c \sup_{f^{-1}(K) \cap \operatorname{spt} T} |f_\sigma - f_\tau| \cdot \operatorname{Lip} f \, ,$$

where K is a compact subset of V with $\operatorname{spt} \omega \subset \operatorname{interior}(K)$. Since $f_\sigma \to f$ uniformly on compact subsets of U , the result now clearly follows.

Next we want to define the notion of the *cone* over a given current $T \in \mathcal{D}_n(U)$. We want to define this in such a way that if $T = [\![M]\!]$ where M is a submanifold of $S^{P-1} \subset \mathbb{R}^P$ then the cone over T is just $[\![C_M]\!]$, $C_M = \{\lambda x : x \in M, 0 < \lambda \le 1\}$. We are thus led generally to make the definition that the cone over T , denoted $0 \,⨳\, T$, is defined by

26.26 $$0 \,⨳\, T = h_{\#}([\![(0,1)]\!] \times T)$$

whenever $T \in \mathcal{D}_n(U)$ with U star-shaped relative to 0 and $\operatorname{spt} T$ compact, where $h : \mathbb{R} \times \mathbb{R}^P \to \mathbb{R}^P$ is defined by $h(t,x) = tx$. Thus $0 \,⨳\, T \in \mathcal{D}_{n+1}(U)$ and (by the homotopy formula)

$$\partial\, 0 \,⨳\, T = T - 0 \,⨳\, \partial T \, .$$

The following *Constancy Theorem* is very useful:

26.27 THEOREM *If* U *is open in* $\mathbb{R}^n$ (i.e. $P = n$), *if* U *is connected, if* $T \in \mathcal{D}_n(U)$ *and* $\partial T = 0$, *then there is a constant* c *such that* $T = c[\![U]\!]$ *(using the notation of* 26.2 *in the special case* $n = P$, $M = U$; U *is of course equipped with the standard orientation* $e_1 \wedge \ldots \wedge e_n$) .

Proof We are given

(1) $$T(d\omega) = 0 \quad \text{whenever} \quad \omega \in \mathcal{D}^{n-1}(U) \, .$$

Let $\phi_\sigma(x) = \sigma^{-n}\phi(\sigma^{-1}x)$, with ϕ a mollifier as in §6, and define T_σ by

$$T_\sigma(\omega) = T(\phi_\sigma * \omega)$$

if $\operatorname{dist}(\operatorname{spt}\omega, \partial U) > \sigma$. ($\phi_\sigma * \omega$ means $(\phi_\sigma * a)\, dx^1 \wedge \ldots \wedge dx^n$ if $\omega = a\, dx^1 \wedge \ldots \wedge dx^n$, $a \in C_c^\infty(U)$; since $P = n$, any $\omega \in \mathcal{D}^n(U)$ has this form.)

Now if $W \subset\subset U$ and $\sigma < \operatorname{dist}(W, \partial U)$, we claim there is a constant $c = c(T, W, \sigma)$ such that

(2) $$|T_\sigma(\omega)| \le c \int_U |\omega|\, d\mathcal{L}^n .$$

Indeed this follows directly from the fact that for fixed σ, W the set $S = \{\phi_\sigma * \omega : \omega \in \mathcal{D}^n(U) , \operatorname{spt}\omega \subset W , \int_U |\omega|\, d\mathcal{L}^n \le 1\}$ is compact in $\mathcal{D}^n(U)$, relative to the norm $|\ |$. By the Riesz Representation Theorem 4.1, we see that (1) implies

(3) $$T_\sigma(\omega) = \int a\, \theta_\sigma\, d\mathcal{L}^n , \quad \omega = a\, dx^1 \wedge \ldots \wedge dx^n ,$$

$a \in C_c^\infty(W)$.

On the other hand if $\operatorname{spt}\omega \subset W$, $\omega \in \mathcal{D}^{n-1}(U)$, then

$$T_\sigma(d\omega) = T(\phi_\sigma * d\omega) = T(d\phi_\sigma * \omega) = \partial T(\phi_\sigma * \omega) = 0$$

by (1) . In particular, taking $\omega = a\, dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n$, so that $d\omega = \pm \partial a/\partial x^j dx^1 \wedge \ldots \wedge dx^n$, and using (3) we have

$$\int D_j a\, \theta_\sigma\, d\mathcal{L}^n = 0 , \quad j = 1, \ldots, n ,$$

for $a \in C_c^\infty(U)$ with $\operatorname{spt} a \subset W$. This evidently implies that θ_σ = constant (depending on σ) on each component of W . The required result now follows from (3) by letting $\sigma \downarrow 0$ and $W \uparrow U$.

26.28 REMARK Notice that if we merely have $\underline{\underline{M}}(\partial T) < \infty$ then the obvious modifications of the above argument (note that (3) still holds) give first that

$$\left| \int D_j a \, \theta_\sigma \, d\mathcal{L}^n \right| \le c \sup|a| \, \underline{\underline{M}}(\partial T)$$

with c independent of σ, for $a \in C_c^\infty(U)$ such that dist(spt a, ∂U) $> \sigma$. Thus (see §6 and in particular Theorem 6.3) we deduce that $\theta_{\sigma_k} \to \theta$ in $L^1_{loc}(U)$ (for some sequence $\sigma_k \downarrow 0$), with $\theta \in BV_{loc}(U)$, and (from (3))

$$(*) \qquad T\omega = \int a \, \theta \, d\mathcal{L}^n , \quad \omega = a \, dx^1 \wedge \ldots \wedge dx^n \in \mathcal{D}^n(U) .$$

Using the definition of $\underline{\underline{M}}(\partial T)$, we easily then check that $\underline{\underline{M}}_W(\partial T) = |D\theta|(W)$ for each open $W \subset U$ (and $\underline{\underline{M}}_W(T) = \int_W |\theta| \, d\mathcal{L}^n$). Indeed in the present case $n = P$, any $\omega \in \mathcal{D}^{n-1}(U)$ can be written $\omega = \sum_{j=1}^n (-1)^j a_j dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n$ for suitable $a_j \in C_c^\infty(U)$, and $d\omega = \operatorname{div} \underline{a} \, dx^1 \wedge \ldots \wedge dx^n$ for such ω ($\underline{a} = (a_1, \ldots, a_n)$). Therefore by (*) above we have

$$\partial T(\omega) = T(d\omega) = \int \operatorname{div} \underline{a} \, \theta \, d\mathcal{L}^n$$

and the assertion $\underline{\underline{M}}_W(\partial T) = |D\theta|(W)$ then follows directly from the definition of $\underline{\underline{M}}_W(\partial T)$ and $|D\theta|$ (in §6).

In the following lemma, for $\alpha = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ with $1 \le i_1 < i_2 < \ldots < i_n \le P$, we let p_α denote the orthogonal projection of $\mathbb{R}^P$ onto $\mathbb{R}^n$ given by

$$(x^1, \ldots, x^P) \mapsto (x^{i_1}, \ldots, x^{i_n}) .$$

26.29 LEMMA *Suppose* E *is a closed subset of* U, U *open in* $\mathbb{R}^P$, *with* $\mathcal{L}^n(p_\alpha(E)) = 0$ *for each multi-index* $\alpha = (i_1, \ldots, i_n)$, $1 \le i_1 < i_2 < \ldots < i_n \le P$. *Then* $T \llcorner E = 0$ *whenever* $T \in \mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ *for every* $W \subset\subset U$.

26.30 REMARK The hypothesis $L^n(p_\alpha(E)) = 0$ is trivially satisfied if $H^n(E) = 0$, so in particular we deduce $T \llcorner E = 0$ if $T \in \mathcal{D}_n(U)$ with $\underline{\underline{M}}_W(T)$, $\underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$ and $H^n(E) = 0$.

Proof of 26.29 Let $\omega \in \mathcal{D}^n(U)$. Then we can write $\omega = \sum_{\alpha \in I_{n,P}} \omega_\alpha \, dx^\alpha$, $\omega_\alpha \in C_c^\infty(U)$, so that

$$\begin{aligned} T(\omega) &= \sum_\alpha T(\omega_\alpha dx^\alpha) = \sum_\alpha (T \llcorner \omega_\alpha)(dx^\alpha) \\ &= \sum_\alpha (T \llcorner \omega_\alpha) p_\alpha^\# \, dy \end{aligned}$$

($dy = dy^1 \wedge \ldots \wedge dy^n$, $y^1, \ldots, y^n$ the standard coordinate functions in $\mathbb{R}^n$.)

Thus

$$T(\omega) = \sum_\alpha p_{\alpha\#}(T \llcorner \omega_\alpha)(dy) \tag{1}$$

(which makes sense because $\operatorname{spt} T \llcorner \omega_\alpha \subset \operatorname{spt} \omega_\alpha$ = compact subset of U).

On the other hand

$$\begin{aligned} \underline{\underline{M}}(\partial p_{\alpha\#}(T \llcorner \omega_\alpha)) &= \underline{\underline{M}}(p_{\alpha\#}\partial(T \llcorner \omega_\alpha)) \\ &\le \underline{\underline{M}}(\partial(T \llcorner \omega_\alpha)) < \infty, \end{aligned}$$

(because for any $\tau \in \mathcal{D}^{n-1}(U)$,

$$\begin{aligned} \partial(T \llcorner \omega_\alpha)(\tau) &= (T \llcorner \omega_\alpha)(d\tau) \\ &= T(\omega_\alpha d\tau) \\ &= T(d(\omega_\alpha \tau)) - T(d\omega_\alpha \wedge \tau) \\ &= \partial T(\omega_\alpha \tau) - T(d\omega_\alpha \wedge \tau); \end{aligned}$$

thus in fact

$$\underline{\underline{M}}_W(\partial(T \llcorner \omega_\alpha)) \le \underline{\underline{M}}_W(\partial T)\,|\omega_\alpha|$$

$$+ \underline{\underline{M}}_W(T)\,|d\omega_\alpha| \ .)$$

Therefore by Remark 26.28 we have $\theta_\alpha \in BV(p_\alpha(U))$ such that $p_{\alpha\#}(T \llcorner \omega_\alpha)(\tau) = \int_{p_\alpha(U)} < \tau , e_1 \wedge \ldots \wedge e_n > \theta_\alpha dL^n$, and hence $p_{\alpha\#}(T \llcorner \omega_\alpha) \llcorner p_\alpha(E) = 0$ because $L^n(p_\alpha(E)) = 0$. Then, assuming without loss of generality that E is closed,

$$(2) \qquad \underline{\underline{M}}(p_{\alpha\#}(T \llcorner \omega_\alpha)) \le \underline{\underline{M}}(p_{\alpha\#}(T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^n \sim p_\alpha(E)))$$

$$= \underline{\underline{M}}(p_{\alpha\#}((T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha(E))))$$

$$\le \underline{\underline{M}}((T \llcorner \omega_\alpha) \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha E))$$

$$\le \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim p_\alpha^{-1} p_\alpha E)) \cdot |\omega_\alpha|$$

$$\le \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E)) \cdot |\omega_\alpha|$$

for any W such that $\operatorname{spt} \omega \subset W \subset U$.

Combining (1) and (2) we then have

$$\underline{\underline{M}}_W(T) \le c\, \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E))$$

so that in particular

$$(3) \qquad \underline{\underline{M}}_W(T \llcorner E) \le c\, \underline{\underline{M}}_W(T \llcorner (\mathbb{R}^P \sim E)) \ .$$

Letting K be an arbitrary compact subset of E , we can choose $\{W_q\}$ so that $W_q \subset\subset U$, $W_{q+1} \subset W_q$, $\bigcap_{q=1}^{\infty} W_q = K$; using (3) with $W = W_q$ then gives $\underline{\underline{M}}(T \llcorner K) = 0$. Thus $\underline{\underline{M}}(T \llcorner E) = 0$ as required.

§27. INTEGER MULTIPLICITY RECTIFIABLE CURRENTS

In this section we want to develop the theory of integer multiplicity currents $T \in \mathcal{D}_n(U)$, which, roughly speaking are those currents obtained by assigning (in a $\mathcal{H}^n$-measurable fashion) an orientation to the tangent spaces $T_x V$ of an integer multiplicity varifold V . (See Chapter 4 for terminology.)

These currents are precisely those called locally rectifiable by Federer and Fleming [FF], [FH1].

Throughout this section $n \geq 1$, $k \geq 1$ are integers and U is an open subset of $\mathbb{R}^{n+k}$.

27.1 DEFINITION If $T \in \mathcal{D}_n(U)$ we say that T is an integer multiplicity rectifiable n-current (briefly an integer multiplicity current) if it can be expressed

$$(*) \qquad T(\omega) = \int_M < \omega(x), \xi(x) > \theta(x) d\mathcal{H}^n(x) \ , \quad \omega \in \mathcal{D}^n(U) \ ,$$

where M is an $\mathcal{H}^n$-measurable countably n-rectifiable subset of U , θ is a locally $\mathcal{H}^n$-integrable positive integer-valued function, and $\xi : M \to \Lambda_n(\mathbb{R}^{n+k})$ is a $\mathcal{H}^n$-measurable function such that for $\mathcal{H}^n$-a.e. point $x \in M$, $\xi(x)$ can be expressed in the form $\tau_1 \wedge \ldots \wedge \tau_n$, where $\tau_1, \ldots, \tau_n$ form an orthonormal basis for the approximate tangent space $T_x M$. (See Chapter 3,4.) Thus $\xi (= \vec{T})$ orients the approximate tangent spaces of M in an $\mathcal{H}^n$-measurable way. The function θ in (*) is called the *multiplicity* and ξ is called the orientation for T . If T is as in (*) we shall often write $T = \underline{\underline{\tau}}(M,\theta,\xi)$. Notice that there is associated with any such T the integer multiplicity varifold $V = \underline{\underline{v}}(M,\theta)$ in U .

27.2 REMARKS

(1) If $T_1, T_2 \in \mathcal{D}_n(U)$ are integer multiplicity, then so is $p_1T_1 + p_2T_2$, $p_1, p_2 \in \mathbb{Z}$.

(2) If $T_1 = \underline{\underline{\tau}}(M_1,\theta_1,\xi_1) \in \mathcal{D}_r(U)$, $T_2 = \underline{\underline{\tau}}(M_2,\theta_2,\xi_2) \in \mathcal{D}_s(V)$ ($V \subset \mathbb{R}^Q$ open), then $T_1 \times T_2 \in \mathcal{D}_{r+s}(U \times V)$ is also integer multiplicity, and in fact

$$T_1 \times T_2 = \underline{\underline{\tau}}(M_1 \times M_2, \theta_1\theta_2, \xi_1 \wedge \xi_2) .$$

(3) If $f : U \to V$ is Lipschitz, $T = \underline{\underline{\tau}}(M,\theta,\xi) \in \mathcal{D}_n(U)$ $(M \subset U)$ and $f \mid \operatorname{spt} T$ is proper, then we can define $f_\# T \in \mathcal{D}_n(V)$ by

$$(*) \qquad f_\# T(\omega) = \int_M < \omega(f(x)) , d^M f_{x\#}\xi(x) > \theta(x) \, d\mathcal{H}^n(x) .$$

Since $|d^M f_{x\#} \, \xi(x)| = J^M f(x)$ (as in §12) by the area formula this can be written

$$(**) \qquad f_\# T(\omega) = \int_{f(M)} \Big\langle \omega(y) , \sum_{x \in f^{-1}(y) \cap M_+} \theta(x) \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} \Big\rangle \, d\mathcal{H}^n(y) ,$$

where $M_+ = \{x \in M : J_M f(x) > 0\}$. Furthermore at points y where the approximate tangent space $T_y(f(M))$ exists (which is $\mathcal{H}^n$ - a.e. y by virtue of the fact that $f(M)$ is countably n-rectifiable) and where T_xM, $d^M f_x$ exist $\forall\, x \in f^{-1}(y)$ (which is again for $\mathcal{H}^n$ - a.e. y because T_xM, $d^M f_x$ exist for $\mathcal{H}^n$ - a.e. $x \in M_+$), we have

$$(***) \qquad \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} = \pm\, \tau_1 \wedge \ldots \wedge \tau_n ,$$

where $\tau_1,\ldots,\tau_n$ is an orthonormal basis for $T_y(f(M))$. Hence (**) gives

$$f_\# T(\omega) = \int_{f(M)} < \omega(y) , \eta(y) > N(y) \, d\mathcal{H}^n(y)$$

where $\eta(y)$ is a suitable orientation for the approximate tangent space $T_y(f(M))$ and $N(y)$ is a non-negative integer. N, η in fact satisfy

$$\sum_{x\in f^{-1}(y)\cap M_+} \theta(x) \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} = N(y)\,\eta(y) ,$$

so that for $\mathcal{H}^n$-a.e. $y \in f(M)$ we have

$$N(y) \le \sum_{x\in f^{-1}(y)\cap M_+} \theta(x)$$

and

$$N(y) \equiv \sum_{x\in f^{-1}(y)\cap M_+} \theta(x) \quad (\mathrm{mod}\ 2) .$$

Notice that, in case f is C^1, $f_\# T$ agrees with the previous definition in 26.20 (see also 26.21(2)). Notice also that if $f : U \to W$ is Lipschitz and if $V = \underline{\underline{v}}(M,\theta)$ is the varifold associated with $T = \underline{\underline{\tau}}(M,\theta,\xi)$, then

$$\mu_{f_\# T} \le \mu_{f_\# V}$$

(in the sense of measures) with equality if and only if, for $\mathcal{H}^n$-a.e. $y \in f(M)$, the sign in (***) above remains constant as x varies over $f^{-1}(y) \cap M_+$. In particular we have $\mu_{f_\# T} = \mu_{f_\# V}$ in case f is 1:1.

A fact of central importance concerning integer multiplicity currents is the following compactness theorem, first proved by Federer and Fleming [FF].

27.3 THEOREM *If* $\{T_j\} \subset \mathcal{D}_n(U)$ *is a sequence of integer multiplicity currents with*

$$\sup_{j\geq 1}(\underset{=}{M}_W(T_j) + \underset{=}{M}_W(\partial T_j)) < \infty \quad \forall\, W \subset\subset U ,$$

then there is an integer multiplicity $T \in \mathcal{D}_n(U)$ *and a subsequence* $\{T_{j'}\}$ *such that* $T_{j'} \rightharpoonup T$ *in* U .

We shall give the proof of this in Chapter 8. Notice that the existence of a $T \in \mathcal{D}_n(U)$ and a subsequence $\{T_{j'}\}$ with $T_{j'} \rightharpoonup T$ is a consequence of the elementary lemma 26.14; only the fact that T is an integer multiplicity current is non-trivial.

27.4 REMARK Note that the proof of 27.3 in the codimension 1 case (when $P = n+1$) is a direct consequence of the Remark 26.28 and the compactness theorem 6.3 for BV functions.

In contrast to the difficulty in proving 27.3, it is quite straightforward to prove that if T_j converges to T in the strong sense that $\lim \underset{=}{M}_W(T_j - T) = 0 \quad \forall\, W \subset\subset U$, and if T_j are integer multiplicity $\forall j$, then T is integer multiplicity. Indeed we have the following lemma.

27.5 LEMMA *The set of integer multiplicity currents in* $\mathcal{D}_n(U)$ *is complete with respect to the topology given by the family* $\{\underset{=}{M}_W\}_{W\subset\subset U}$ *of semi-norms.*

Proof Let $\{T_Q\}$ be a sequence of integer multiplicity currents in $\mathcal{D}_n(U)$, and $\{T_Q\}$ is Cauchy with respect to the semi-norms $\underset{=}{M}_W$, $W \subset\subset U$. Suppose $T_Q = \underline{\underline{\tau}}(M_Q, \theta_Q, \xi_Q)$ (θ_Q positive integer-valued on M_Q , M_Q countably n-rectifiable, $\mathcal{H}^n(M_Q \cap W) < \infty$ for each $W \subset\subset U$.) Then

$$(1) \qquad \underset{\approx}{M}_W(T_Q-T_P) \equiv \int_W |\theta_P\xi_P-\theta_Q\xi_Q|\,d\mathcal{H}^n < \varepsilon_W(Q)$$

$\forall\ P \ge Q$, where $\varepsilon_W(Q) \downarrow 0$ as $Q \to \infty$ and where we adopt the convention $\xi_P = 0$, $\theta_P = 0$ on $U \sim M_P$. In particular, since $|\xi_P| = 1$ on M_P, we get

$$(2) \qquad \int_W |\theta_P-\theta_Q|\,d\mathcal{H}^n < \varepsilon_W(Q) \qquad \forall\ P \ge Q\ ,$$

and hence θ_P converges in $L^1(\mathcal{H}^n)$ locally in U to an integer-valued function θ. Of course (2) implies

$$(3) \qquad \mathcal{H}^n(((M_+ \sim M_Q) \cup (M_Q \sim M_+)) \cap W) \le \varepsilon_W(Q)\ ,$$

where $M_+ = \{x \in U : \theta(x) > 0\}$. (1), (2) also imply

$$(4) \qquad \int_W \theta_P|\xi_P-\xi_Q|\,d\mathcal{H}^n \le 2\varepsilon_W(Q) \qquad \forall\ P \ge Q\ ,$$

and hence by (3) ξ_P converges in $L^1(\mathcal{H}^n)$ locally in U to a function ξ with values in $\Lambda_n(\mathbb{R}^{n+k})$ with $|\xi| = 1$ and ξ simple on M_+.

Now $\xi_Q(x) \in \Lambda_n(T_xM_Q)$, $\mathcal{H}^n$-a.e. $x \in M_Q$, and (by (3)) $T_xM_+ = T_xM_Q$ except for a set of measure $\le \varepsilon_W(Q)$ in $M_+ \cap W$. It follows that $\xi(x) \in \Lambda_n(T_xM_+)$ for $\mathcal{H}^n$-a.e. $x \in M_+$ and we have shown that $\underset{\approx}{M}_W(T_P-T) \to 0$, where $T = \underset{\approx}{\tau}(M_+,\theta,\xi)$ is an integer n-current in U.

Finally, we shall need the following useful *decomposition theorem* for codimension 1 integer multiplicity currents.

27.6 THEOREM *Suppose* $P = n+1$ *(i.e.* U *is open in* $\mathbb{R}^{n+1}$ *) and* R *is an integer multiplicity current in* $\mathcal{D}_{n+1}(U)$ *with* $\underline{\underline{M}}_W(\partial R) < \infty$ $\forall\ W \subset\subset U$. *Then* $T = \partial R$ *is integer multiplicity, and in fact we can find a decreasing sequence of* $\mathcal{L}^{n+1}$*-measurable sets* $\{U_j\}_{j=-\infty}^{\infty}$ *of locally finite perimeter in* U *such that (in* U *)*

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=-\infty}^{0} [\![V_j]\!] , \quad V_j = U \sim U_j , \ j \le 0 ,$$

$$T = \sum_{j=-\infty}^{\infty} \partial [\![U_j]\!] ,$$

and

$$\mu_T = \sum_{j=-\infty}^{\infty} \mu_{\partial [\![U_j]\!]} ;$$

in particular

$$\underline{\underline{M}}_W(T) = \sum_{j=-\infty}^{\infty} \underline{\underline{M}}_W(\partial [\![U_j]\!]) \qquad \forall\ W \subset\subset U .$$

27.7 REMARK Let $* : C_c^{\infty}(U;\mathbb{R}^{n+1}) \to \mathcal{D}^n(U)$ be defined by $*g = \sum_{j=1}^{n+1} (-1)^{j-1} g_j dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^{n+1}$, so that $d*g = \operatorname{div} g\, dx^1 \wedge \ldots \wedge dx^{n+1}$. Then for any $\mathcal{L}^{n+1}$-measurable $A \subset U$ we have

$$\begin{aligned} \partial [\![A]\!] (*g) &= [\![A]\!] (d*g) \\ &= \int_U \chi_A \operatorname{div} g \, d^{\,n+1} , \end{aligned}$$

and hence by definition of $|D\chi_A|$ (in §6) and $\underline{\underline{M}}(T)$ (in §26) we see that

(*) A has locally finite perimeter in $U \Leftrightarrow \underline{\underline{M}}_W(\partial [\![A]\!]) < \infty \qquad \forall\ W \subset\subset U$,

and in this case

$$(**) \qquad \begin{cases} \underline{\underline{M}}_W(\partial[\![A]\!]) = \int_W |D\chi_A| & \forall\ W \subset\subset U \\ \overrightarrow{\partial[\![A]\!]} = *\nu_A\ , \ |D\chi_A| \text{ a.e. in } U\ . \end{cases}$$

Here ν_A is the inward unit normal function for A (defined on the reduced boundary ∂^*A as in 14.3).

Proof of 27.6 R must have the form

$$R = \underline{\underline{\tau}}(V,\theta,\xi)\ ,$$

where V is an $\mathcal{L}^{n+1}$-measurable subset of U and $\xi(x) = \pm e_1 \wedge \ldots \wedge e_{n+1}$ for each $x \in V$. Thus letting

$$\tilde{\theta}(x) = \begin{cases} \theta(x) & \text{when } x \in V \text{ and } \xi(x) = +e_1 \wedge \ldots \wedge e_{n+1} \\ -\theta(x) & \text{when } x \in V \text{ and } \xi(x) = -e_1 \wedge \ldots \wedge e_{n+1} \\ 0 & \text{when } x \notin V\ , \end{cases}$$

we have

$$(1) \qquad R(\omega) = \int_V a\tilde{\theta}\, d\mathcal{L}^{n+1}\ ,$$

$\omega = a\, dx^1 \wedge \ldots \wedge dx^{n+1} \in \mathcal{D}^{n+1}(U)$ and (cf. 26.28)

$$(2) \qquad \underline{\underline{M}}_W(R) = \int_W |\tilde{\theta}|\, d\mathcal{L}^{n+1}\ ,\ \underline{\underline{M}}_W(T) = \int_W |D\tilde{\theta}| \qquad \forall\ W \subset\subset U$$

(and $\tilde{\theta} \in BV_{loc}(U)$) .

Define

$$U_j = \{x \in U : \tilde{\theta}(x) \geq j\}\ ,\ j \in \mathbb{Z}$$

$$V_j = \{x \in U : \tilde{\theta}(x) \leq -1-j\}\ ,\ j \geq 0$$

$$(\ \equiv\ U \sim U_{-j}\ )\ .$$

Then one checks directly that

$$\tilde{\theta} = \sum_{j=1}^{\infty} \chi_{U_j} - \sum_{j=0}^{\infty} \chi_{V_j} \tag{3}$$

(χ_A = characteristic function of A, $A \subset U$), and hence by (1)

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=0}^{\infty} [\![V_j]\!] \quad \text{in } U . \tag{4}$$

Since $T(\omega) = \partial R(\omega) = R(d\omega)$, $\omega \in \mathcal{D}^n(U)$, we then have

$$T = \partial R = \sum_{j=1}^{\infty} \partial[\![U_j]\!] - \sum_{j=0}^{\infty} \partial[\![V_j]\!] \tag{5}$$

$$= \sum_{j=-\infty}^{\infty} \partial[\![U_j]\!] ,$$

so we have the required decomposition, and it remains only to prove that each U_j has locally finite perimeter in U and that the corresponding measures add.

To check this, take $\psi_j \in C^1(\mathbb{R})$ with

$$\begin{cases} \psi_j(t) = 0 \quad \text{for} \quad t \le j-1+\varepsilon , \quad \psi_j(t) = 1 , \quad t \ge j-\varepsilon \\ 0 \le \psi_j \le 1 , \qquad \sup|\psi_j'| \le 1+3\varepsilon , \end{cases}$$

where $\varepsilon \in (0,\frac{1}{2})$. Then if $a \in C_c^{\infty}(U)$ and $g = (g^1,\ldots,g^{n+1})$, $g^j \in C_c^{\infty}(U)$, with $|g| \le a$, we have (since $\chi_{U_j} = \psi_j \circ \tilde{\theta}\ \forall j$) that for any $M \le N$

$$\int_U \operatorname{div} g \sum_{j=M}^{N} \chi_{U_j}\, dL^{n+1} = \int_U \operatorname{div} g \sum_{j=M}^{N} \psi_j \circ \tilde{\theta}\, dL^{n+1} \tag{6}$$

$$= \lim_{\sigma\downarrow 0} \int_U \operatorname{div} g \sum_{j=M}^{N} \psi_j \circ \tilde{\theta}^{(\sigma)}\, dL^{n+1}$$

$$= -\lim_{\sigma\downarrow 0} \int_U g\cdot \operatorname{grad}\tilde{\theta}^{(\sigma)} \psi_j'(\tilde{\theta}^{(\sigma)})\, dL^{n+1}$$

$$\le (1+3\varepsilon) \lim_{\sigma\downarrow 0} \int_U a|\operatorname{grad}\tilde{\theta}^{(\sigma)}|\, dL^{n+1}$$

$$= (1+3\varepsilon) \int_U a|D\tilde{\theta}| = (1+3\varepsilon) \int_U a\, d\mu_T$$

by Lemma 6.2 and (2). (Here $\tilde{\theta}^{(\sigma)}$ are the mollified functions corresponding to $\tilde{\theta}$ as in 6.2.)

Then, taking $M = N$, we deduce that each U_j has locally finite perimeter in U. On the other hand taking $M = -N$ and defining $R_N = \sum_{j=1}^{N} [\![U_j]\!] - \sum_{j=0}^{N} [\![V_j]\!]$ we see that (with g as in 27.7) (6) implies

$$|R_N(d{*}g)| \le (1+3\varepsilon) \int_U a\, d\mu_T ,$$

and hence, with $T_N = \partial R_N$,

$$\text{(7)} \qquad \int_U a\, d\mu_{T_N} \le \int_U a\, d\mu_T \qquad \forall\ N \ge 1 ,$$

$a \ge 0$, $a \in C_c^\infty(U)$. On the other hand by 14.1 we have

$$\text{(8)} \qquad R_N(d{*}g) = \sum_{j=-N}^{N} \int_U \operatorname{div} g\ \chi_{U_j}\ dL^{n+1}$$

$$= -\sum_{j=-N}^{N} \int_{\partial^* U_j} \nu_j \cdot g\ dH^n ,$$

where ν_j is the inward unit normal for U_j and $\partial^* U_j$ is the reduced boundary for U_j (see §14 and in particular Lemma 14.3). By virtue of the fact that $U_{j+1} \subset U_j$ we see from 14.3(2) that $\nu_j \equiv \nu_k$ on $\partial^* U_j \cap \partial^* U_k$ $\forall\ j,k$. Hence (8) can be written

$$T_N(*g) = -\int_U \nu \cdot g\ h_N\ dH^n ,$$

where $h_N = \sum_{j=-N}^{N} \chi_{\partial^* U_j}$ and where ν is defined on $\bigcup_{j=-\infty}^{\infty} \partial^* U_j$ by $\nu = \nu_j$ on $\partial^* U_j$. Since $|\nu| \equiv 1$ on $\bigcup_{j=-\infty}^{\infty} \partial^* U_j$ this evidently gives

$$\int a\ d\mu_{T_N} = \int a\ h_N\ dH^n$$

$$= \sum_{j=-N}^{N} \int_{\partial^* U_j} a\ dH^n$$

$$= \sum_{j=-N}^{N} \int a\ d\mu_{\partial [\![U_j]\!]} .$$

Letting $N \to \infty$ we thus have (by (7))

$$\mu_T \geq \sum_{j=-\infty}^{\infty} \mu_{\partial [\![U_j]\!]} .$$

Since the reverse inequality follows directly from (5), the proof is complete.

27.8 COROLLARY *Let* R *be integer multiplicity* $\in \mathcal{D}_{n+1}(U)$, $U \subset \mathbb{R}^P$, $P \geq n+1$, *and suppose there is an* $(n+1)$*-dimensional* C^1 *submanifold* N *of* $\mathbb{R}^P$ *with* $\operatorname{spt} R \subset N \cap U$. *Suppose further that* $T = \partial R$ *and* $\underline{\underline{M}}_W(T) < \infty$ $\forall\, W \subset\subset U$. *Then* T $(\in \mathcal{D}_n(U))$ *is integer multiplicity and for each point* $y \in N \cap U$ *there is* $W_y \subset\subset U$, $y \in W_y$, *and* H^{n+1} *measurable subsets* $\{U_j\}_{j=-\infty}^{\infty}$ *with* $U_{j+1} \subset U_j \subset N \cap U$, $\underline{\underline{M}}_{W_y}(\partial [\![U_j]\!]) < \infty$ $\forall j$, *and with the following identities holding in* W_y :

$$R = \sum_{j=1}^{\infty} [\![U_j]\!] - \sum_{j=0}^{\infty} [\![U \sim U_{-j}]\!]$$

$$T = \sum_{j=-\infty}^{\infty} \partial [\![U_j]\!]$$

$$\mu_T = \sum_{j=-\infty}^{\infty} \mu_{\partial [\![U_j]\!]} .$$

Proof The proof is an easy consequence of 27.6 using local coordinate representations for N .

§28. SLICING

We first want to define the notion of slice for integer multiplicity currents. Preparatory to this we have the following lemma:

28.1 LEMMA *If* M *is* $\mathcal{H}^n$*-measurable, countably* n*-rectifiable,* f *is Lipschitz on* $\mathbb{R}^{n+k}$ *and* $M_+ = \{x \in M : |\nabla^M f(x)| > 0\}$ *, then for* L^1*-almost all* $t \in \mathbb{R}$ *the following statements hold:*

(1) $M_t \equiv f^{-1}(t) \cap M_+$ *is countably* $\mathcal{H}^{n-1}$*-rectifiable*

(2) *For* $\mathcal{H}^{n-1}$-a.e. $x \in M_t$, $T_x M_t$ *and* $T_x M$ *both exist,* $T_x M_t$ *is an* (n-1)*-dimensional subspace of* $T_x M$ *, and in fact*

(*) $$T_x M = \{y + \lambda \nabla^M f(x) \; : \; y \in T_x M_t , \lambda \in \mathbb{R}\}.$$

Furthermore for any non-negative $\mathcal{H}^n$*-measurable function* g *on* M *we have*

$$\int_{-\infty}^{\infty} \left(\int_{M_t} g \, d\mathcal{H}^{n-1} \right) dt = \int_M |\nabla^M f| \, g \, d\mathcal{H}^n .$$

Proof In fact (1) is just a restatement of Remark 12.8(2), and (2) follows from 11.6 together with the facts that for L^1-a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{n-1}$-a.e. $x \in M_t$

(1) $$\nabla^M f(x) \in T_x M \quad \text{(by definition of } \nabla^M f \text{ in §12)}$$

and

(2) $$\langle \nabla^M f(x), \tau \rangle = 0 \qquad \forall \; \tau \in T_x M_t .$$

(This last follows for example from the definition 12.1 of $\nabla^M f(x)$.)

The last part of the lemma is just a restatement of the appropriate version of the co-area formula (discussed in §12).

28.2 REMARK Note that by replacing g (in 28.1 above) by $g \times$ characteristic function of $\{x : f(x) < t\}$ we get the identity

$$\int_{M\cap\{f(x)<t\}} |\nabla^M f| g \, dH^n = \int_{-\infty}^{t} \int_{M_s} g \, dH^{n-1} ds$$

so that the left side as an absolutely continuous function of t and

$$\frac{d}{dt}\int_{M\cap\{f(x)<t\}} |\nabla^M f| g \, dH^n = \int_{M_t} g \, dH^{n-1} \,, \quad \text{a.e. } t \in \mathbb{R} \,.$$

Now let $T = \underline{\underline{\tau}}(M,\theta,\xi)$ be an integer multiplicity current in U (U open in $\mathbb{R}^{n+k}$, $M \subset U$), let f be Lipschitz in U and let θ_+ be defined H^n- a.e. in M by

$$\theta_+(x) = \begin{cases} 0 & \text{if} \quad \nabla^M f(x) = 0 \\ \theta(x) & \text{if} \quad \nabla^M f(x) \neq 0 \,. \end{cases}$$

For the (L^1-almost all) $t \in \mathbb{R}$ such that T_xM, T_xM_t exist for H^{n-1}- a.e. $x \in M_t$ and such that (*) of 28.1 holds, we have

28.3 $\quad \xi(x) \llcorner \nabla^M f(x)/|\nabla^M f(x)|$ *is simple* $\in \Lambda_{n-1}(T_xM_t) \subset \Lambda_{n-1}(T_xM)$

and has unit length (for H^{n-1}- a.e. $x \in M_t$). Here we use the notation that if $v \in \Lambda_n(T_xM)$ and $w \in T_xM$, then $v \llcorner w \in \Lambda_{n-1}(T_xM)$ is defined by

$$\langle v \llcorner w, a\rangle = \langle v, w \wedge a\rangle \,, \; a \in \Lambda_{n-1}(T_xM) \,.$$

Using this notation we can now define the notion of a slice of T by f; we continue to assume $T \in \mathcal{D}_n(U)$ is given by $T = \underline{\underline{\tau}}(M,\theta,\xi)$ as above.

28.4 DEFINITION For the (L^1-almost all) $t \in \mathbb{R}$ since that T_xM, T_xM_t exist and 28.1(*) holds H^{n-1}- a.e. $x \in M_t$, with the notation introduced above (and bearing in mind 28.3) we define the integer multiplicity current $\langle T,f,t\rangle \in \mathcal{D}_{n-1}(U)$ by

$$\langle T,f,t\rangle = \underline{\underline{\tau}}(M_t,\theta_t,\xi_t) \,,$$

where

$$\xi_t(x) = \xi(x) \llcorner \nabla^M f(x)/|\nabla^M f(x)| \ , \quad \theta_t = \theta_+|M_t \ .$$

So defined, $\langle T,f,t\rangle$ is called *the slice of* T *by* f *at* t .

The main facts concerning the slices $\langle T,f,t\rangle$ are given in the following lemma:

28.5 LEMMA

(1) *For each open* $W \subset U$

$$\int_{-\infty}^{\infty} \underline{\underline{M}}_W(\langle T,f,t\rangle)\,dt = \int_{M\cap W} |\nabla^M f|\theta \, d\mathcal{H}^n \le (\text{ess sup}_{M\cap W}|\nabla^M f|)\underline{\underline{M}}_W(T) \ .$$

(2) *If* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, *then for* $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\langle T,f,t\rangle = \partial[T \llcorner \{f<t\}] - (\partial T)\llcorner \{f<t\} \ .$$

(3) *If* ∂T *is integer multiplicity in* $\mathcal{D}_{n-1}(U)$, *then for* $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\langle \partial T,f,t\rangle = -\partial\langle T,f,t\rangle \ .$$

Proof (1) is a direct consequence of the last part of Lemma 28.1 (with $g = \theta_+$).

To prove (2) we first recall that, since M is countably n-rectifiable, we can write (see Remark 11.7)

$$\text{(1)} \qquad M = \bigcup_{j=0}^{\infty} M_j \ ,$$

where $M_i \cap M_j = \emptyset \quad \forall\, i \neq j$, $\mathcal{H}^n(M_0) = 0$, and $M_j \subset N_j \quad j \ge 1$, with N_j an embedded C^1 submanifold of $\mathbb{R}^{n+k}$. By virtue of this decomposition and the definition of ∇^M (in §12) it easily follows that if h is Lipschitz on $\mathbb{R}^{n+k}$ and if $h^{(\sigma)}$ are the mollified functions (as in §6) then, as $\sigma \downarrow 0$,

(2) $$v\cdot\nabla^M h^{(\sigma)} \to v\cdot\nabla^M h \qquad \text{(weak convergence in } L^2(\mu_T))$$

for any fixed bounded H^n-measurable v with values in $\mathbb{R}^{n+k}$. (Indeed to check this, we have merely to check that (2) holds with N_j in place of M_j and with v vanishing on $\mathbb{R}^{n+k} \sim M_j$; since N_j is C^1 this follows fairly easily by approximating v by smooth functions and using the fact that $h^{(\sigma)}$ converges to h uniformly.)

Next let $\varepsilon > 0$ and let γ be the Lipschitz function on $\mathbb{R}$ defined by

$$\gamma(s) = \begin{cases} 1, & s<t-\varepsilon \\ \text{linear}, & t-\varepsilon\le s\le t \\ 0, & s>t \end{cases}$$

and apply the above to $h = \gamma\circ f$. Then letting $\omega \in \mathcal{D}^n(U)$ we have

$$\begin{aligned} \partial T(h^{(\sigma)}\omega) &= T(d(h^{(\sigma)}\omega)) \\ &= T(dh^{(\sigma)}\wedge\omega) + T(h^{(\sigma)}d\omega) . \end{aligned}$$

Then using the integral representations of the form 26.7 for ∂T we see that

(3) $$(\partial T \llcorner h)(\omega) = \lim_{\sigma\downarrow 0} T(dh^{(\sigma)}\wedge\omega) + (T\llcorner h)(d\omega) .$$

Since $\xi(x)$ orients T_xM, we have

(4) $$\begin{aligned} \langle\xi(x), dh^{(\sigma)}\wedge\omega\rangle &= \langle\xi(x), (dh^{(\sigma)}(x))^T\wedge\omega^T\rangle \\ &= \langle\xi(x), (dh^{(\sigma)}(x))^T\wedge\omega\rangle \end{aligned}$$

(where $(\ )^T$ denotes the orthogonal projection of $\Lambda^q(\mathbb{R}^{n+k})$ onto $\Lambda^q(T_xM)$). Thus

$$\begin{aligned} T(dh^{(\sigma)}\wedge\omega) &= \int_M \langle\xi(x), (dh^{(\sigma)}(x))^T\wedge\omega\rangle\theta\, dH^n \\ &= \int_M \langle\xi(x)\llcorner\nabla^M h^{(\sigma)}(x), \omega\rangle\theta\, dH^n \end{aligned}$$

so that by (2)

$$\lim_{\sigma\downarrow 0} T(dh^{(\sigma)} \wedge \omega) = \int_M \langle \xi(x) \llcorner \nabla^M h(x), \omega\rangle\, \theta \, d\mathcal{H}^n . \tag{5}$$

By definition 12.1 of $\nabla^M h$ and by the chain rule for the composition of Lipschitz functions we have

$$\nabla^M h = \gamma'(f)\nabla^M f \qquad \mathcal{H}^n\text{-a.e. on } M \tag{6}$$

(where we set $\gamma'(f) = 0$ when f takes the "bad" values t or $t-\varepsilon$; note that $\nabla^M h(x) = \nabla^M f(x) = 0$ for $\mathcal{H}^n$-a.e. in $\{x \in M : f(x) = c\}$, c any given constant).

Using (5), (6) in (3), we thus deduce

$$(\partial T \llcorner h)(\omega) = -\varepsilon^{-1} \int_{M \cap \{t-\varepsilon<f<t\}} \langle \xi \llcorner \nabla^M f, \omega\rangle\, \theta \, d\mathcal{H}^n$$

$$+ (T \llcorner h)(d\omega) .$$

Finally we let $\varepsilon \downarrow 0$ and we use Remark 28.2 with $g = \theta \langle \xi \llcorner \nabla^M f / |\nabla^M f|, \omega\rangle$ in order to complete the proof of (2); by considering a countable dense set of $\omega \in \mathcal{D}^n(U)$ one can of course show that 28.2 is applicable with $g = \theta \langle \xi \llcorner \nabla^M f/|\nabla^M f|, \omega\rangle$ except for a set F of t having $\mathcal{L}^1$-measure zero, with F independent of ω .

Finally to prove part (3) of the theorem, we first apply part (2) with ∂T in place of T . Since $\partial^2 T = 0$, this gives

$$\langle \partial T, f, t\rangle = \partial[(\partial T) \llcorner \{f<t\}] .$$

On the other hand, applying ∂ to each side of the original identity (for T) of (2), we get

$$\partial[(\partial T) \llcorner \{f<t\}] = -\partial\langle T,f,t\rangle$$

and hence (3) is established.

Motivated by the above discussion we are led to define slices for an arbitrary current $\in \mathcal{D}_n(U)$ which, together with its boundary, has locally finite mass in U. Specifically, suppose $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$. Then we define "slices"

28.6 $$\langle T,f,t_-\rangle = \partial(T \llcorner \{f<t\}) - (\partial T) \llcorner \{f<t\}$$

and

28.7 $$\langle T,f,t_+\rangle = -\partial(T \llcorner \{f>t\}) + (\partial T) \llcorner \{f>t\} .$$

Of course $\langle T,f,t_+\rangle = \langle T,f,t_-\rangle$ (and the common value is denoted $\langle T,f,t\rangle$) for all but the countably many values of t such that $\underline{\underline{M}}(T \llcorner \{f=t\}) + \underline{\underline{M}}((\partial T) \llcorner \{f=t\}) > 0$.

The important properties of the above slices are that if f is Lipschitz on U (and if we continue to assume $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U)$, then

28.8 $$\operatorname{spt}\langle T,f,t_\pm\rangle \subset \operatorname{spt} T \cap \{x : f(x)=t\}$$

and, $\forall$ open $W \subset U$,

28.9 $$\begin{cases} \underline{\underline{M}}_W(\langle T,f,t_+\rangle) \le \operatorname{ess\,sup}_W|Df| \displaystyle\liminf_{h\downarrow 0} h^{-1}\, \underline{\underline{M}}_W(T \llcorner \{t<f<t+h\}) \\[2ex] \underline{\underline{M}}_W(\langle T,f,t_-\rangle) \le \operatorname{ess\,sup}_W|Df| \displaystyle\liminf_{h\downarrow 0} h^{-1}\, \underline{\underline{M}}_W(T \llcorner \{t-h<f<t\}) . \end{cases}$$

Notice that $\underline{\underline{M}}_W(T \llcorner \{f<t\})$ is increasing in t, hence is differentiable for $\mathcal{L}^1$- a.e. $t \in \mathbb{R}$ and $\int_a^b \frac{d}{dt}\underline{\underline{M}}_W(T \llcorner \{f<t\})\,dt \le \underline{\underline{M}}_W(T \llcorner \{a<f<b\})$. Thus 28.9 gives

28.10 $$\int_a^{*b} \underline{\underline{M}}_W(\langle T,f,t_\pm\rangle)\,dt \le \operatorname{ess\,sup}_W|Df|\ \underline{\underline{M}}_W(T \llcorner \{a<f<b\})$$

for every open $W \subset U$.

To prove 28.8 and 28.9 we consider first the case when f is C^1 and take any smooth increasing function $\gamma : \mathbb{R} \to \mathbb{R}_+$ and note that

$$(*)\qquad \begin{aligned} &\partial(T \llcorner \gamma\circ f)(\omega) - ((\partial T) \llcorner \gamma\circ f)(\omega) \\ &= (T \llcorner \gamma\circ f)(d\omega) - ((\partial T) \llcorner \gamma\circ f)(\omega) \\ &= T(\gamma\circ f d\omega) - T(d(\gamma\circ f\omega)) \\ &= -T(\gamma'(f)\,df \wedge \omega)\ . \end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary and choose γ such that

$$\gamma(t) = 0 \quad \text{for} \quad t<a\ ,\ \gamma(t)=1 \quad \text{for} \quad t>b\ ,\ 0 \le \gamma'(t) \le \frac{1+\varepsilon}{b-a} \quad \text{for} \quad a<t<b\ .$$

Then the left side of (*) converges to $\langle T,f,a_+\rangle$ if we let $b \downarrow a$, and hence 28.8 follows because $\operatorname{spt}\gamma' \subset [a,b]$. Furthermore the right side R of (*) evidently satisfies

$$|R| \le (\sup_W|Df|)\left(\frac{1+\varepsilon}{b-a}\right) \underline{\underline{M}}_W(T \llcorner \{a<f<b\})\,|\omega| \qquad (\operatorname{spt}\omega\subset W)$$

and so we also conclude the first part of 28.9 for $f \in C^1$. We similarly establish the second part for $f \in C^1$. To handle general Lipschitz f we simply use $f^{(\sigma)}$ in place of f in 28.6, 28.7 and in the above proof, then let $\sigma \downarrow 0$ where appropriate.

§29. THE DEFORMATION THEOREM

The deformation theorem, given below in Theorem 29.1 and Corollary 29.3 is a central result in the theory of currents, and was first proved by Federer and Fleming [FF].

The special notation for this section is as follows:

$1 \le n$, $1 \le k$,

$C = [0,1] \times \ldots \times [0,1]$ (Standard unit cube in $\mathbb{R}^{n+k}$)

$\mathbb{Z}^{n+k} = \{z = (z^1, \ldots, z^{n+k}) : z^j \in \mathbb{Z}\}$ $(\subset \mathbb{R}^{n+k})$

L_j = j-skeleton of the decomposition $\bigcup_{z \in \mathbb{Z}^{n+k}} (z+C)$

of $\mathbb{R}^{n+k}$

$\mathcal{L}_j$ = collection of j-faces in L_j

$= \{z + F : z \in \mathbb{Z}^{n+k}$, F is a closed j-face of $C\}$

$\mathcal{L}_j(\rho) = \{\rho F : F \in \mathcal{L}_j\}$, $\rho > 0$

$S_1, \ldots, S_N$ $\left(N = \binom{n+k}{n+1} = \binom{n+k}{k-1}\right)$ denote the

(n+1)-dimensional subspaces of $\mathbb{R}^{n+k}$ which contain an (n+1)-face of the standard cube C .

p_j denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto S_j , $j = 1, \ldots, n$.

29.1 THEOREM (Deformation Theorem, unscaled version)

Suppose T *is an* n-*current in* $\mathbb{R}^{n+k}$ (i.e. $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$) *with* $\underline{\underline{M}}(T) + \underline{\underline{M}}(\partial T) < \infty$. *Then we can write*

$$T - P = \partial R + S ,$$

where P, R, S *satisfy*

$$P = \sum_{F \in \mathcal{L}_n} \beta_F [\![F]\!] \qquad (\beta_F \in \mathbb{R}) ,$$

with

$$\underline{\underline{M}}(P) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(\partial P) \le c\underline{\underline{M}}(\partial T)$$

$$\underline{\underline{M}}(R) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(S) \le c\underline{\underline{M}}(\partial T)$$

$(c = c(n,k))$, *and*

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{n+k}\}$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{n+k}\} \ .$$

In case T *is an integer multiplicity current, then* P , R *can be chosen to be integer multiplicity currents (and the* β_F *appearing in the definition of* P *are integers). If in addition* ∂T *is integer multiplicity**, *then* S *can be chosen to be integer multiplicity.*

29.2 REMARKS

(1) Note that this is slightly sharper than the corresponding theorem in [FF], [FH1], because there is no term involving $\underline{\underline{M}}(\partial T)$ in the bound for $\underline{\underline{M}}(P)$.

(2) It follows automatically from the other conclusions of the theorem that $\underline{\underline{M}}(\partial S) \le c\underline{\underline{M}}(\partial T)$. Also, it follows from the inequalities $\underline{\underline{M}}(\partial P), \underline{\underline{M}}(S) \le c\underline{\underline{M}}(\partial T)$ that $S = 0$ and $\partial P = 0$ when $\partial T = 0$.

The following "scaled version" of 29.1 is obtained from the above by first changing scale $x \to \rho^{-1}x$, then applying 29.1, then changing scale back by $x \to \rho x$.

---

* Actually ∂T *automatically* is integer multiplicity if T is integer multiplicity and $\underline{\underline{M}}(\partial T) < \infty$, see Theorem 30.3.

29.3 COROLLARY (Deformation Theorem, scaled version)

Suppose T, ∂T *are as in* 29.1, *and* $\rho > 0$. *Then*

$$T - P = \partial R + S\ ,$$

where P, R, S *satisfy*

$$P = \sum_{F \in L_j(\rho)} \beta_F [\![F]\!] \qquad\qquad (\beta_F \in \mathbb{R})$$

$$\underline{\underline{M}}(P) \le c\underline{\underline{M}}(T)\ ,\ \underline{\underline{M}}(\partial P) \le c\underline{\underline{M}}(\partial T)$$

$$\underline{\underline{M}}(R) \le c\rho\underline{\underline{M}}(T)\ ,\ \underline{\underline{M}}(S) \le c\rho\underline{\underline{M}}(\partial T)\ ,$$

and

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{n+k}\ \rho\}$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{n+k}\ \rho\}\ .$$

As in 29.1, *in case* T *is integer multiplicity, so are* P, R ; *if* ∂T *is integer multiplicity then so is* S.

The main step in the proof of the deformation theorem will involve "pushing" T onto the n-skeleton L_n via a certain retraction map ψ. We first have to establish the existence of a suitable class of retraction maps. This is done in the following lemma, in which we use the notation:

$$q = \text{centre point of } C = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\ ,$$

$$L_{k-1}(a) = a + L_{k-1} \quad (a \text{ a given point in } B_{\frac{1}{4}}(q))\ ,$$

$$L_{k-1}(a;\rho) = \{x \in \mathbb{R}^{n+k} : \operatorname{dist}(x, L_{k-1}(a)) < \rho\} \quad (\rho \in (0, \tfrac{1}{4}))\ .$$

Note that $\operatorname{dist}(L_{k-1}(a), L_n) \ge \frac{1}{4}$ for any point $a \in B_{\frac{1}{4}}(q)$.

29.4 LEMMA *For every* $a \in B_{\frac{1}{4}}(q)$ *there is a locally Lipschitz map*

$$\psi : \mathbb{R}^{n+k} \sim L_{k-1}(a) \to \mathbb{R}^{n+k} \sim L_{k-1}(a)$$

such that

$$\psi(C \sim L_{k-1}(a)) = C \cap L_n \ , \ \psi | C \cap L_n = \underset{=}{1}_{C \cap L_n} \ ,$$

$$|D\psi(x)| \leq c/\rho \ , \ \mathcal{L}^{n+k} \text{ - a.e. } x \in C \sim L_{k-1}(a;\rho) \ , \ \rho \in (0,\tfrac{1}{4}) \ ,$$

$(c = c(n,k))$, *and such that*

$$\psi(z+x) = z+\psi(x) \ , \ x \in \mathbb{R}^{n+k} \sim L_{k-1}(a) \ , \ z \in \mathbb{Z}^{n+k} \ .$$

Proof We first construct a locally Lipschitz retraction $\psi_0 : C \sim L_{k-1}(a)$ onto the n-faces of C . This is done as follows:

Firstly for each j-face F of C , $j \geq n+1$, let $a_F \in F$ denote the orthogonal projection of a onto F , and let ψ_F denote the retraction of $\bar{F} \sim \{a_F\}$ onto ∂F which takes a point $x \in \bar{F} \sim \{a_F\}$ to the point $y \in \partial F$ such that $x \in \{a_F + \lambda(y-a_F) : \lambda \in (0,1]\}$. (Thus ψ_F is the "radial retraction" of F with a_F as origin.) Of course $\psi_F | \partial F = \underset{=}{1}_{\partial F}$. Notice also that for any j-face F of C , $j \geq n+1$, the line segment $\overline{aa_F}$ is contained in $L_{k-1}(a)$; in fact if J_F denotes the set of ℓ such that S_ℓ (see notation prior to 29.1) is parallel to an (n+1)-face of F , then (because $\overline{aa_F}$ is orthogonal to F , hence orthogonal to each S_ℓ , $\ell \in J_F$) we have

$$\overline{aa_F} \subset \bigcap_{\ell \in J_F} p_\ell^{-1}(p_\ell(a)) \ , \tag{1}$$

and this is contained in $L_{k-1}(a)$, because (by definition)

$$L_{k-1}(a) = \bigcup_{\ell=1}^{N} \ \bigcup_{z \in \mathbb{Z}^{n+k}} (z+p_\ell^{-1}(p_\ell(a))) \ . \tag{2}$$

Next, for each $j \geq n+1$, define

$$\psi^{(j)} : \cup\{\bar{F} \sim \{a_F\} : F \text{ is a } j\text{-face of } C\}$$

$$\to \cup\{\bar{G} : G \text{ is a } (j-1)\text{-face of } C\}$$

by setting

$$\psi^{(j)} | \bar{F} \sim \{a_F\} = \psi_F .$$

(Notice that then $\psi^{(j)}$ is locally Lipschitz on its domain by virtue of the fact that each ψ_F is the identity on ∂F, F any j-face of C.)

Then the composite $\psi^{(n+1)} \circ \psi^{(n+2)} \circ \ldots \circ \psi^{(n+k)}$ makes sense on $C \sim L_{k-1}(a)$ (by (1)), so we can set

$$\psi_0 = \psi^{(n+1)} \circ \psi^{(n+2)} \circ \ldots \circ \psi^{(n+k)} \mid C \sim L_{k-1}(a) .$$

Notice that ψ_0 has the additional property that if

$$z \in \mathbb{Z}^{n+k} \text{ and } x, z+x \in C, \text{ then } \psi_0(z+x) = z+\psi_0(x) .$$

(Indeed $x, z+x \in C$ means that either $x, z+x$ are in L_n (where ψ_0 is the identity) or else lie in the interior of parallel j-faces $F_1, F_2 = z+F_1$ ($j \geq n+1$) of C with z orthogonal to F_1 and $a_{F_2} = z+a_{F_1}$.) It follows that we can then define a retraction ψ of all of $C \sim L_{k-1}(a)$ onto L_n by setting

$$\psi(z+x) = z+\psi_0(x) , \; x \in C \sim L_{k-1}(a) , \; z \in \mathbb{Z}^{n+k} .$$

We now claim that

(3) $$\sup|D\psi| \leq c/\rho \text{ on } \mathbb{R}^{n+k} \sim L_{k-1}(a,\rho) , \; c = c(n,k) .$$

(This will evidently complete the proof of the lemma.)

We can prove (3) by induction on k as follows. First note that (3) is evident from construction in case $k=1$. Hence assume $k \geq 2$ and assume (3) holds in case $k-1$ replaces k in the above construction. Let x be any point of interior $(C) \sim L_{k-1}(a;\rho)$, let $y = \psi^{n+k}(x)$ (ψ^{n+k} is the radial retraction of $C \sim \{a\}$ onto ∂C), and let F be any closed $(n+k-1)$-face of C which contains y.

Suppose now new coordinates are selected so that $F \subset \mathbb{R}^{n+k-1} \times \{0\} \subset \mathbb{R}^{n+k}$, and also let $\tilde{L}_{k-2}(a) = L_{k-1}(a) \cap \mathbb{R}^{n+k-1} \times \{0\})$. By virtue of (1) we have $a_F \in L_{k-1}(a)$, hence

$$(4) \qquad |y-a_F| \geq \operatorname{dist}(y, L_{k-1}(a)) .$$

Let p_F be the orthogonal projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^{n+k-1} \times \{0\}$ ($\supset F$), so that $a_F = p_F(a)$. Evidently $|p_F(x) - a_F| \geq \operatorname{dist}(x, p_F^{-1}(p_F(a)))$ and hence by (2) we deduce

$$(5) \qquad |p_F(x) - a_F| \geq \operatorname{dist}(x, L_{k-1}(a)) .$$

Furthermore by definition of y we know that $y-a = \frac{|y-a|}{|x-a|}(x-a)$ and hence, applying p_F, we have

$$y-a_F = \frac{|y-a|}{|x-a|} p_F(x-a) .$$

Hence since $|y-a| \geq 3/4$, we have

$$(6) \qquad |y-a_F| \geq (3/4)\,|p_F(x-a)|/|x-a| .$$

Now let $\tilde{\psi}$ be the retraction of $F \sim \tilde{L}_{k-2}(a)$ onto the n-faces of F ($\tilde{\psi}$ defined as for ψ but with $(k-1)$ in place of k, a_F in place of a, $\mathbb{R}^{n+k-1}$ in place of $\mathbb{R}^{n+k}$ and $\tilde{L}_{k-2}(a) = L_{k-2}(a_F)$ in place of $L_{k-1}(a)$). By the inductive hypothesis, together with (4), (5), (6) we have

$$(7) \qquad |\bar{D}\tilde{\psi}(y)| \le \frac{c}{\operatorname{dist}(y,\tilde{L}_{k-2}(a))}, \quad \left(|\bar{D}\tilde{\psi}(y)| = \limsup_{z\to y} \frac{|\tilde{\psi}(z)-\tilde{\psi}(y)|}{|z-y|}\right)$$

$$\le \frac{c}{|y-a_F|} (4/3)\, c \frac{|x-a|}{|p_F(x-a)|}$$

$$\le (4/3)c \frac{|x-a|}{\operatorname{dist}(x,L_{k-1}(a))} .$$

Also, by the definition of ψ^{n+k} we have that

$$(8) \qquad |\bar{D}\psi^{n+k}(x)| \le \frac{c}{|x-a|}, \; |\bar{D}\psi^{n+k}(x)| = \limsup_{y\to x} \frac{|\psi^{n+k}(y)-\psi^{n+k}(x)|}{|y-x|} .$$

Since $\psi(x) = \tilde{\psi}\circ\psi^{n+k}(x)$, we have by (7), (8) and the chain rule that

$$|\bar{D}\psi(x)| \le |\bar{D}\tilde{\psi}(y)|\; |\bar{D}\psi^{n+k}(x)| \le \frac{c}{|x-a|} \; \frac{|x-a|}{\operatorname{dist}(x,L_{k-1}(a))}$$

$$= \frac{c}{\operatorname{dist}(x,L_{k-1}(a))} .$$

Proof of Deformation Theorem

We use the subspaces $S_1,\ldots,S_N$ and projections $p_1,\ldots,p_N$ introduced at the beginning of the section. Let $F_j = C \cap S_j$ (so that F_j is a closed $(n+1)$-dimensional face of C), let x_j be the central point of F_j, and for each $j = 1,\ldots,N$ define a "good" subset $G_j \subset F_j \cap B_{\frac{1}{4}}(x_j)$ by $g \in G_j \Leftrightarrow g \in F_j \cap B_{\frac{1}{4}}(x_j)$ and

$$(1) \qquad \underline{\underline{M}}\Big(T \llcorner \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_\rho(g+z))\Big) \le \beta\rho^{n+1}\underline{\underline{M}}(T) \qquad \forall\, \rho \in (0,\tfrac{1}{4})$$

(β to be chosen, $G_j = G_j(\beta)$).

We now claim that the "bad" set $B_j = F_j \cap B_{\frac{1}{4}}(x_j) \sim G_j$ in fact has L^{n+1}-measure (taken in S_j) small; in fact we claim

(2) $$L^{n+1}(B_j) \le 20^{n+1}\beta^{-1}\omega_{n+1}(\tfrac{1}{4})^{n+1} \quad (\omega_{n+1} = L^{n+1}(B_1(0))) \ ,$$

which is indeed small if we choose large β. To see (2), we argue as follows. For each $b \in B_j$ there is (by definition) a $\rho_b \in (0,\frac{1}{4})$ such that

(3) $$\underline{\underline{M}}(T \llcorner \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_{\rho_b}(b+z))) \ge \beta\rho_b^{n+1}\underline{\underline{M}}(T) \ ,$$

and by the covering theorem 3.3 there is a pairwise disjoint subcollection $\{B_{\rho_\ell}(b_\ell)\}_{\ell=1,2\ldots}(\rho_\ell=\rho_{b_\ell})$ of the collection $\{B_{\rho_b}(b)\}_{b\in B_j}$ such that

(4) $$B_j \subset \bigcup_\ell B_{5\rho_\ell}(b_\ell) \ .$$

But then, setting $b = b_\ell$ in (3) and summing, we get

$$\beta(\sum_\ell \rho_\ell^{n+1})\ \underline{\underline{M}}(T) \le \underline{\underline{M}}(T) \quad (\text{i.e. } \sum_\ell \rho_\ell^{n+1} \le \beta^{-1}) \ ,^{(*)}$$

(using the fact that $\{p_j^{-1}B_{\rho_\ell}(b_\ell+z)\}_{\substack{\ell=1,2,\ldots \\ z\in\mathbb{Z}^{n+k}\cap S_j}}$ is a pairwise disjoint collection for fixed j). Thus by (4) we conclude

$$L^{n+1}(B_j) \le \beta^{-1}\ 5^{n+1}\omega_{n+1} \ ,$$

which after trivial re-arrangement gives (2) as required. Thus we have

$$L^{n+1}(G_j) \ge (1-20^{n+1}\beta^{-1})\omega_{n+1}(\tfrac{1}{4})^{n+1} \ ,$$

and it follows that

(5) $$L^{n+k}(p_j^{-1}(G_j) \cap B_{\frac{1}{4}}(q)) \ge \left(1 - \frac{\omega_{n+1}}{\omega_{n+k}}\, 20^{n+1}\beta^{-1}\right)\omega_{n+k}(\tfrac{1}{4})^{n+k} \ ,$$

where q is the centre point $(\frac{1}{2},\ldots,\frac{1}{2})$ of C. (So $p_j(q) = x_j$.)

(*) We of course assume $T \ne 0$.

Then selecting β large enough so that $20^{n+1}\omega_{n+1}N\beta^{-1} < \omega_{n+k}/(n+k)$, we see from (5) that we can choose a point $a \in \bigcap_{j=1}^{N} p_j^{-1}(G_j) \cap B_{\frac{1}{4}}(q)$. Next let $L_{k-1}(a) = a + L_{k-1}$, $L_{k-1}(a;\rho) = \{x \in \mathbb{R}^{n+k}: \operatorname{dist}(x, L_{k-1}(a)) < \rho\}$ (as in the proof of 29.4) and note that in fact

$$L_{k-1}(a;\rho) = \bigcup_{j=1}^{N} \bigcup_{z\in\mathbb{Z}^{n+k}\cap S_j} p_j^{-1}(B_\rho(p_j(a)+z)) .$$

Then since $p_j(a) \in G_j$ we have (by definition of G_j)

$$\text{(6)} \qquad \underline{\underline{M}}(T \llcorner L_{k-1}(a;\rho)) \le N\beta\rho^{n+1}\underline{\underline{M}}(T) \qquad \forall \rho \in (0,\tfrac{1}{4}) .$$

Indeed let us suppose that we take β such that $20^{n+1}\omega_{n+1}N\beta^{-1} < \omega_{n+k}/2(n+k)$. Then more than half the ball $B_{\frac{1}{4}}(q)$ is in the set $\bigcap_{j=1}^{N} p_j^{-1}(G_j)$ and hence, repeating the whole argument above with ∂T in place of T, we can actually select a so that, *in addition to* (6), we also have

$$\text{(7)} \qquad \underline{\underline{M}}(\partial T \llcorner L_{k-1}(a;\rho)) \le N\beta\rho^{n+1}\underline{\underline{M}}(\partial T) \qquad \forall \rho \in (0,\tfrac{1}{4}) .$$

Now let ψ be the retraction of $\mathbb{R}^{n+k} \sim L_{k-1}(a)$ onto L_n given in Lemma 29.4, and let

$$\text{(8)} \qquad T_\rho = T \llcorner L_{k-1}(a;\rho) , \quad (\partial T)_\rho = T \llcorner L_{k-1}(a;\rho) ,$$

so that by (6), (7)

$$\text{(9)} \qquad \underline{\underline{M}}(T_\rho) \le c\rho^{n+1}\underline{\underline{M}}(T) , \quad \underline{\underline{M}}((\partial T)_\rho) \le c\rho^{n+1}\underline{\underline{M}}(\partial T) .$$

Furthermore by 28.10 we know that for each $\rho \in (0,\frac{1}{4})$ we can find $\rho^* \in (\rho/2,\rho)$ such that

$$\text{(10)} \qquad \underline{\underline{M}}(\langle T,d,\rho^*\rangle) \le \frac{c}{\rho}\,\underline{\underline{M}}(T_\rho - T_{\rho/2}) \le c\rho^{n}\underline{\underline{M}}(T) ,$$

where d is the (Lipschitz) distance function to $L_{k-1}(a)$ $(d(x) = \mathrm{dist}(x, L_{k-1}(a))$, $\mathrm{Lip}(d) = 1)$ and $\langle T, d, \rho^*\rangle$ is the slice of T by d at ρ^*. (Notice that we can choose ρ^* such that (10) holds and such that $\langle T, d, \rho^*\rangle$ is integer multiplicity in case T is integer multiplicity - see Lemma 28.5 and the following discussion.)

We now want to apply the homotopy formula 26.22 to the case when $h(x,t) = x+t(\psi(x)-x)$, $x \in \mathbb{R}^{n+k} \sim L_{k-1}(a;\sigma)$, $\sigma > 0$. Notice that h is only Lipschitz on $\mathbb{R}^{n+k} \sim L_{k-1}(a;\sigma)$, so we define $h_\#$, $\psi_\#$ as in Lemma 26.25. (We shall apply $h_\#$, $\psi_\#$ only to currents supported away from $[0,1] \times L_{k-1}(a)$ and $L_{k-1}(a)$ respectively.)

Keeping this in mind we note that by 29.4, (6) and (7) we have

(11) $$\underline{\underline{M}}(\psi_\#(T_\rho - T_{\rho/2})) \le \frac{c}{\rho^n}\rho^{n+1}\underline{\underline{M}}(T) \le c\rho\underline{\underline{M}}(T)$$

and

(12) $$\underline{\underline{M}}(\psi_\#((\partial T)_\rho - (\partial T)_{\rho/2})) \le \frac{c}{\rho^{n-1}}\rho^{n+1}\underline{\underline{M}}(\partial T) \le c\rho\underline{\underline{M}}(\partial T) .$$

Similarly by the homotopy formula 26.22, together with 26.23 and (6), (7) above, we have

(13) $$\underline{\underline{M}}(h_\#([\![(0,1)]\!] \times (T_\rho - T_{\rho/2}))) \le c\rho\underline{\underline{M}}(T)$$

and

(14) $$\underline{\underline{M}}(h_\#([\![(0,1)]\!] \times ((\partial T)_\rho - (\partial T)_{\rho/2})) \le c\rho\underline{\underline{M}}(\partial T) .$$

Notice also that by (6), (10) and 26.23 we have

(15) $$\underline{\underline{M}}(\psi_\#\langle T, d, \rho^*\rangle) \le c\rho\underline{\underline{M}}(T)$$

and

$$(16) \qquad \underline{\underline{M}}(h_{\#}([\![(0,1)]\!] \times \langle T,d,\rho^*\rangle)) \le c\rho\underline{\underline{M}}(T) \ .$$

Next note that by iteration (11), (12) imply

$$(17) \qquad \begin{cases} \underline{\underline{M}}(\psi_{\#}(T_\rho - T_{\rho/2^\nu})) \le 2c\rho\underline{\underline{M}}(T) \\ \underline{\underline{M}}(\psi_{\#}((\partial T)_\rho - (\partial T)_{\rho/2^\nu})) \le 2c\rho\underline{\underline{M}}(\partial T) \end{cases}$$

for each integer $\nu \ge 1$, where c is as in (11), (12) (c independent of ν). Selecting $\rho = \frac{1}{2}$ and using the arbitrariness of ν, it follows that

$$(18) \qquad \begin{cases} \underline{\underline{M}}(\psi_{\#}(T - T_\sigma)) \le c\underline{\underline{M}}(T) \\ \underline{\underline{M}}(\psi_{\#}(\partial T - (\partial T)_\sigma)) \le c\underline{\underline{M}}(\partial T) \end{cases}$$

for each $\sigma \in (0,1)$ (with c independent of σ) .

Now select $\rho = \rho_\nu \equiv 2^{-\nu}$ and $\rho_\nu^* \in [2^{-\nu-1}, 2^{-\nu}]$ such that (10), (15), (16) hold with ρ_ν^* in place of ρ^* ; then by (15), (16), (17), (18) we have that

$$\psi_{\#}(T - T_{\rho_\nu^*}) \ , \ h_{\#}([\![(0,1)]\!] \times (T - T_{\rho_\nu^*})) \ ,$$

$$\psi_{\#}(\partial T - \partial T_{\rho_\nu^*}) \ , \ h_{\#}([\![0,1)]\!] \times \partial(T - T_{\rho_\nu^*}))$$

are Cauchy sequences relative to $\underline{\underline{M}}$, and $\underline{\underline{M}}(\langle T,d,\rho_\nu^*\rangle) + \underline{\underline{M}}(\psi_{\#}\langle T,d,\rho_\nu^* \ ) \to 0$. Hence there are currents Q , $S_1 \in \mathcal{D}_n(\mathbb{R}^{n+k})$ and $R_1 \in \mathcal{D}_{n+1}(\mathbb{R}^{n+k})$ such that

$$(19) \qquad \begin{cases} \lim \underline{\underline{M}}(Q - \psi_{\#}(T - T_{\rho_\nu^*})) = 0 \\ \lim \underline{\underline{M}}(S_1 - h_{\#}([\![(0,1)]\!] \times \partial(T - T_{\rho_\nu^*}))) = 0 \\ \lim \underline{\underline{M}}(R_1 - h_{\#}([\![(0,1)]\!] \times (T - T_{\rho_\nu^*})) = 0 \ . \end{cases}$$

Furthermore by the homotopy formula and 26.23 we have for each ν

$$(20) \qquad T - T_{\rho_\nu^*} - \psi_\#(T - T_{\rho_\nu^*})$$
$$= \partial(h_\#([\![(0,1)]\!] \times (T - T_{\rho_\nu^*}))$$
$$- h_\#([\![(0,1)]\!] \times \partial(T - T_{\rho_\nu^*})) \ .$$

Since $\partial T_{\rho_\nu^*} = (\partial T)_{\rho_\nu^*} - \langle T, d, \rho_\nu^* \rangle$ (by the definition 28.6, 28.7 of slice) we thus get that

$$(21) \qquad T - Q = \partial R_1 + S_1 \ .$$

(Notice that Q, R_1 are integer multiplicity by (19), 28.4, 28.5 and 27.5 in case T is integer multiplicity; similarly S_1 is integer multiplicity if ∂T is.)

Using the fact that ψ retracts $\mathbb{R}^{n+k} \sim L_{k-1}(a)$ onto L_n we know (by 26.23) that $\operatorname{spt} \psi_\#(T - T_{\rho_\nu^*}) \subset L_n$, and hence

$$(22) \qquad \operatorname{spt} Q \subset L_n \ .$$

We also have (since $\psi(z+C) \subset z+C \quad \forall\ z \in \mathbb{Z}^{n+k}$) that

$$(23) \qquad \begin{cases} \operatorname{spt} R_1 \cup \operatorname{spt} Q \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < \sqrt{n+k}\ \} \\ \\ \operatorname{spt} S_1 \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < \sqrt{n+k}\ \} \end{cases}$$

and, by (18), (19), we have

$$(24) \qquad \begin{cases} \underline{\underline{M}}(Q) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(R_1) \le c\underline{\underline{M}}(T) \\ \\ \underline{\underline{M}}(S_1) \le c\underline{\underline{M}}(\partial T) \ . \end{cases}$$

Also by (18) and the semi-continuity of $\underline{\underline{M}}$ under weak convergence, we have

$$(25) \qquad \underline{\underline{M}}(\partial Q) \le \liminf \underline{\underline{M}}(\partial\psi_{\#}(T-T_{\rho_\nu^*}))$$
$$= \liminf \underline{\underline{M}}(\psi_{\#}\partial(T-T_{\rho_\nu^*}))$$
$$\le c\underline{\underline{M}}(\partial T) \ .$$

Now let F be a given face of L_n (i.e. $F \in \mathcal{L}_n$) and let $\mathring{F}$ = interior of F. Assume for the moment that $F \subset \mathbb{R}^n \times \{0\}$ $(\subset \mathbb{R}^{n+k})$, and let p be the orthogonal projection onto $\mathbb{R}^n \times \{0\}$. By construction of ψ we know that $p\circ\psi = \psi$ in a neighbourhood of any point $y \in \mathring{F}$. We therefore have (since Q is given by (18)) that

$$(26) \qquad p_{\#}(Q \llcorner \mathring{F}) = Q \llcorner \mathring{F} \ .$$

It then follows, by the obvious modifications of the arguments in the proof of the constancy theorem (Theorem 26.27) and in Remark 26.28, that

$$(27) \qquad (Q \llcorner \mathring{F})(\omega) = \int_{\mathring{F}} \langle e_1 \wedge \ldots \wedge e_n, \omega(x) \rangle \theta_F(x)\, d\mathcal{L}^n(x)$$

$\forall\ \omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, for some $BV_{loc}(\mathbb{R}^n)$ function θ_F, and

$$(28) \qquad \underline{\underline{M}}(Q \llcorner \mathring{F}) = \int_{\mathring{F}} |\theta_F|\, d\mathcal{L}^n \ , \ \underline{\underline{M}}((\partial Q) \llcorner \mathring{F}) = \int_{\mathring{F}} |D\theta_F| \ .$$

Furthermore, since

$$(Q \llcorner \mathring{F} - \beta[\![F]\!])(\omega) = \int_{\mathring{F}} (\theta_F - \beta) \langle e_1 \wedge \ldots \wedge e_n, \omega(x) \rangle d\mathcal{L}^n(x)$$

(by (27)), we have (again using the reasoning of 26.28)

$$(29) \qquad \begin{cases} \underline{\underline{M}}(Q \llcorner \mathring{F} - \beta[\![F]\!]) = \displaystyle\int_{\mathring{F}} |\theta_F - \beta|\, d\mathcal{L}^n \\ \underline{\underline{M}}(\partial(Q \llcorner \mathring{F} - \beta[\![F]\!])) = \displaystyle\int_{\mathbb{R}^n} |D(\chi_{\mathring{F}}(\theta_F - \beta))| \ , \end{cases}$$

where $\chi_{\mathring{F}}$ = characteristic function of $\mathring{F}$.

Thus taking $\beta = \beta_F$ such that

$$(30) \qquad \min\{L^n\{x \in \mathring{F} : \theta_F \geq \beta\}, L^n\{x \in \mathring{F} : \theta_F(x) \leq \beta\} \geq \frac{1}{2}$$

(which we can do because $L^n(\mathring{F}) = 1$; notice that we can take $\beta_F \in \mathbb{Z}$ if θ_F is integer-valued), we have by 6.4, 6.6, (28) and (29) that

$$(31) \qquad \begin{cases} \underline{\underline{M}}(Q \llcorner \mathring{F} - \beta[\![F]\!]) \leq c\int_{\mathring{F}} |D\theta_F| = c\underline{\underline{M}}(Q \llcorner \mathring{F}) \\ \underline{\underline{M}}(\partial(Q \llcorner \mathring{F} - \beta[\![F]\!])) \leq c\int_{\mathring{F}} |D\theta_F| = c\underline{\underline{M}}(Q \llcorner \mathring{F}) . \end{cases}$$

We also have by 26.30

$$(32) \qquad Q \llcorner \partial F = 0 .$$

Then summing over $F \in L_n$ and using (31), (32) we have, with $P = \sum_{F \in L_n} \beta_F[\![F]\!]$, that

$$(33) \qquad \begin{cases} \underline{\underline{M}}(Q-P) \leq c\underline{\underline{M}}(\partial Q) \\ \underline{\underline{M}}(\partial Q - \partial P) \leq c\underline{\underline{M}}(\partial Q) . \end{cases}$$

Actually by (30) we have

$$(34) \qquad |\beta_F| \leq 2\int_{\mathring{F}} |\theta_F| \, dL^n ,$$

and hence (using again the first part of (28)), since $\underline{\underline{M}}(P) = \sum_F |\beta_F|$,

$$(35) \qquad \underline{\underline{M}}(P) \leq c\underline{\underline{M}}(Q) .$$

Notice that the second part of (33) gives

$$(36) \qquad \underline{\underline{M}}(\partial P) \leq c\underline{\underline{M}}(\partial Q) .$$

Finally we note that (21) can be written

(37) $$T-P = \partial R_1 + (S_1+(Q-P)) \ .$$

Setting $R = R_1$, $S = S_1 + (Q-P)$, the theorem now follows immediately from (23), (24), (25) and (33), (35), (36), (37); the fact that P, R are integer multiplicity if T is should be evident from the remarks during the course of the above proof, as should be the fact that S is integer multiplicity if T, ∂T are.

§30. APPLICATIONS OF THE DEFORMATION THEOREM

We here establish a couple of simple (but very important) applications of the deformation theorem, namely the isoperimetric theorem and the weak polyhedral approximation theorem. This latter theorem, when combined with the compactness theorem 27.3 implies the important "boundary rectifiability theorem" (30.3 below), which asserts that if T is an integer multiplicity current in $\mathcal{D}_n(U)$ and if $\underline{\underline{M}}_W(\partial T) < \infty \ \forall\, W \subset\subset U$, then $\partial T (\in \mathcal{D}_{n-1}(U))$ is integer mutiplicity. (Notice that in the case $k=0$, this has already been established in 27.6.)

30.1 THEOREM (Isoperimetric Theorem)

Suppose $T \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ *is integer multiplicity,* $n \geq 2$, $\operatorname{spt} T$ *is compact and* $\partial T = 0$. *Then there is an integer multiplicity current* $R \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* $\operatorname{spt} R$ *compact,* $\partial R = T$, *and*

$$\underline{\underline{M}}(R)^{\frac{n-1}{n}} \leq c\, \underline{\underline{M}}(T) \ ,$$

where $c = c(n,k)$.

Proof The case $T = 0$ is trivial, so assume $T \neq 0$. Let P, R, S be integer multiplicity currents as in 29.3, where for the moment $\rho > 0$ is arbitrary, and note that $S = 0$ because $\partial T = 0$. Evidently (since $\mathcal{H}^{n-1}(F) = \rho^{n-1} \quad \forall\, F \in \mathcal{F}_{n-1}(\rho)$) we have

$$(*) \qquad \underline{\underline{M}}(P) = N(\rho)\rho^{n-1}$$

for some non-negative *integer* $N(\rho)$. But since $\underline{\underline{M}}(P) \le c\,\underline{\underline{M}}(T)$ (from 29.3) we see that necessarily $N(\rho) = 0$ in (*) if we choose $\rho = (2c\underline{\underline{M}}(T))^{\frac{1}{n-1}}$. Then $P = 0$, and 29.3 gives $T = \partial R$ for some integer multiplicity current R with $\operatorname{spt} R$ compact and $\underline{\underline{M}}(R) \le c\rho\underline{\underline{M}}(T) = c'(\underline{\underline{M}}(T))^{\frac{1}{n-1}}$.

30.2 THEOREM (Weak polyhedral approximation theorem)

Given any integer multiplicity $T \in \mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U$, *there is a sequence* $\{P_k\}$ *of currents of the form*

$$(**) \qquad P_k = \sum_{F \in \mathcal{F}_n(\rho_k)} \beta_F^{(k)} [\![F]\!] , \qquad (\beta_F^{(k)} \in \mathbb{Z}) , \quad \rho_k \downarrow 0 ,$$

such that $P_k \rightharpoonup T$ *(and hence also* $\partial P_k \rightharpoonup \partial T$*) in* U *(in the sense of* 26.12).

Proof First consider the case $U = \mathbb{R}^{n+k}$ and $\underline{\underline{M}}(T), \underline{\underline{M}}(\partial T) < \infty$. In this case we simply use the deformation theorem: for any sequence $\rho_k \downarrow 0$, the scaled version of the deformation theorem (with $\rho = \rho_k$) gives P_k as in (**) such that

$$(1) \qquad T - P_k = \partial R_k + S_k$$

for some R_k, S_k such that

$$(2) \qquad \begin{cases} \underline{\underline{M}}(R_k) \le c\rho_k \underline{\underline{M}}(T) \to 0 \\ \underline{\underline{M}}(S_k) \le c\rho_k \underline{\underline{M}}(\partial T) \to 0 \end{cases}$$

and

$$\underline{\underline{M}}(P_k) \le c\underline{\underline{M}}(T) \ , \ \underline{\underline{M}}(\partial P_k) \le c\underline{\underline{M}}(\partial T) \ .$$

Evidently (1), (2) give $P_k(\omega) \to T_k(\omega) \quad \forall\ \omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, and $\partial P_k = 0$ if $\partial T = 0$, so the theorem is proved in case $U = \mathbb{R}^{n+k}$ and $T, \partial T$ are of finite mass.

In the general case we take any Lipschitz function ϕ on $\mathbb{R}^{n+k}$ such that $\phi > 0$ in U, $\phi \equiv 0$ in $\mathbb{R}^{n+k} \sim U$ and such that $\{x = \phi(x) > \lambda\} \subset\subset U$ $\forall\ \lambda > 0$. For $\mathcal{L}^1$-a.e. $\lambda > 0$, 28.5 implies that $T_\lambda \equiv T \llcorner \{x : \phi(x) > \lambda\}$ is such that $\underline{\underline{M}}(\partial T_\lambda) < \infty$. Since $\operatorname{spt} T_\lambda \subset\subset U$, we can apply the argument above to approximate T_λ for any such λ. Taking a suitable sequence $\lambda_j \downarrow 0$, the required approximation then immediately follows.

30.3 THEOREM (Boundary rectifiability theorem)

Suppose T *is an integer multiplicity current in* $\mathcal{D}_n(U)$ *with* $\underline{\underline{M}}_W(\partial T) < \infty \quad \forall\ W \subset\subset U$. *Then* $\partial T (\in \mathcal{D}_{n-1}(U))$ *is an integer multiplicity current.*

Proof A direct consequence of 30.2 above and the compactness theorem 27.3.

30.4 REMARK Notice that only the case $\partial T_j = 0 \quad \forall j$ of 27.3 is needed in the above proof.

§31. THE FLAT METRIC(*) TOPOLOGY

The main result to be proved here is the equivalence of weak convergence and "flat metric" convergence (see below for terminology) for a sequence of

---

(*) Note that the word "flat" here has *no* physical or geometric significance, but relates rather to Whitney's use of the symbol $\flat$ (the "flat" symbol in musical notation) in his work. We mention this because it is often a source of confusion.

integer multiplicity currents $\{T_j\} \subset \mathcal{D}_n(U)$ such that $\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \quad \forall\, W \subset\subset U$.

We let U denote (as usual) an arbitrary open subset of $\mathbb{R}^{n+k}$,

$$\mathcal{I} = \{T \in \mathcal{D}_n(U) : T \text{ is integer multiplicity and } \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\, W \subset\subset U\},$$

and

$$\mathcal{I}_{M,W} = \{T \in \mathcal{I} : \operatorname{spt} T \subset \overline{W},\ \underline{\underline{M}}(T) + \underline{\underline{M}}(\partial T) \leq M\}$$

for any $M > 0$ and $W \subset\subset U$.

On $\mathcal{I}$ we define a family of pseudometrics $\{d_W\}_{W\subset\subset U}$ by

31.1 $$d_W(T_1, T_2) = \inf\{\underline{\underline{M}}_W(S) + \underline{\underline{M}}_W(R) : T_1 - T_2 = \partial R + S, \text{ where } R \in \mathcal{D}_{n+1}(U),\ S \in \mathcal{D}_n(U) \text{ are integer multiplicity}\}$$

We henceforth assume $\mathcal{I}$ is equipped with the topology given (in the usual way) by the family $\{d_W\}_{W\subset\subset U}$ of pseudometrics. This topology is called the "flat metric topology" for $\mathcal{I}$: there is a countable base of neighbourhoods at each point, and $T_j \to T$ in this topology if and only if $d_W(T_j, T) \to 0 \quad \forall\, W \subset\subset U$.

31.2 THEOREM *Let* T, $\{T_j\} \subset \mathcal{D}_n(U)$ *be integer multiplicity with* $\sup_{j\geq 1}\{\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)\} < \infty \quad \forall\, W \subset\subset U$. *Then* $T_j \to T$ *(in the sense of 26.12) if and only if* $d_W(T_j, T) \to 0$ *for each* $W \subset\subset U$.

31.3 REMARK Notice that no use is made of the compactness theorem 27.3 in this theorem; however if we combine the compactness theorem with it, then we get the statement that for any family of positive (finite) constants

$\{c(W)\}_{W\subset\subset U}$ the set $\{T \in \mathcal{I} : \underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) \le c(W) \quad \forall\ W \subset\subset U\}$ is sequentially compact when equipped with the flat metric topology.

Proof of 31.2 First note that the "if" part of the theorem is trivial (indeed for a given $W \subset\subset U$, the statement $d_W(T_j,T) \to 0$ evidently implies $(T_j - T)(\omega) \to 0$ for any fixed $\omega \in \mathcal{D}^n(U)$ with $\operatorname{spt} \omega \subset W$).

For the "only if" part of the theorem, the main difficulty is to establish the appropriate "total boundedness" property; specifically we show that for any given $\varepsilon > 0$ and $W \subset\subset \tilde{W} \subset\subset U$, we can find $N = N(\varepsilon,W,\tilde{W},M)$ and integer multiplicity currents $P_1,\ldots,P_N \in \mathcal{D}_n(U)$ such that

$$(1) \qquad \mathcal{I}_{M,W} \subset \sum_{j=1}^{N} B_{\varepsilon,\tilde{W}}(P_j) ,$$

where, for any $P \in \mathcal{I}$, $B_{\varepsilon,\tilde{W}}(P) = \{S \in \mathcal{I} : d_{\tilde{W}}(S,P) < \varepsilon\}$. This is an easy consequence of the deformation theorem: in fact for any $\rho > 0$, 29.3 guarantees that for $T \in \mathcal{I}_{M,W}$ we can find integer multiplicity P, R, S such that

$$(2) \qquad T - P = \partial R + S$$

$$(3) \qquad P = \sum_{F\in\mathcal{F}_n(\rho)} \beta_F [\![F]\!] , \quad \beta_F \in \mathbb{Z}$$

$$(4) \qquad \operatorname{spt} P \subset \{x : \operatorname{dist}(x,\operatorname{spt} T) < 2\sqrt{n+k}\,\rho\}$$

$$(5) \qquad \underline{\underline{M}}(P) \left(\equiv \sum_{F\in\mathcal{F}_n(\rho)} |\beta_F|\rho^n\right) \le c\underline{\underline{M}}(T) \le cM$$

$$(6) \qquad \begin{cases} \operatorname{spt} R \cup \operatorname{spt} S \subset \{x : \operatorname{dist}(x,\operatorname{spt} T) < 2\sqrt{n+k}\,\rho\} \\[2ex] \underline{\underline{M}}(R) + \underline{\underline{M}}(S) \le c\rho\underline{\underline{M}}(T) \le c\rho M . \end{cases}$$

Then for ρ small enough to ensure $2\sqrt{n+k}\,\rho < \operatorname{dist}(W,\partial\tilde{W})$, we see from (2), (6) that

$$d_{\tilde{W}}(T,P) \le c\rho M .$$

Hence, since there are only finitely many $P_1,\ldots,P_N$ currents P as in (3), (4), (5) (N depends only on M,W,n,k,ρ), we have (1) as required.

Next note that (by 28.5 (1), (2) and an argument as in 10.7(2)) we can find a subsequence $\{T_{j'}\} \subset \{T_j\}$ and a sequence $\{W_i\}$, $W_i \subset\subset W_{i+1} \subset\subset U$, $\bigcup_{i=1}^{\infty} W_i = U$, such that $\sup_{j'\ge 1} \underline{\underline{M}}(\partial(T_{j'} \llcorner W_i)) < \infty \quad \forall i$. Thus from now on we can assume without loss of generality that $W \subset\subset U$ and

$$\text{(7)} \qquad \operatorname{spt} T_j \subset \bar{W} \quad \forall j .$$

Then take any $\tilde{W}$ such that $W \subset\subset \tilde{W} \subset\subset U$ and apply (1) with $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}$ etc. to extract a subsequence $\left\{T_{j_r}\right\}_{r=1,2,\ldots}$ from $\{T_j\}$ such that

$$d_{\tilde{W}}(T_{j_{r+1}}, T_{j_r}) < 2^{-r}$$

and hence

$$\text{(8)} \qquad T_{j_{r+1}} - T_{j_r} = \partial R_r + S_r$$

where R_r, S_r are integer multiplicity,

$$\operatorname{spt} R_r \cup \operatorname{spt} S_r \subset \tilde{W}$$

$$\underline{\underline{M}}(R_r) + \underline{\underline{M}}(S_r) \le \frac{1}{2^r} .$$

Therefore by 27.5 we can define integer multiplicity $R^{(\ell)}$, $S^{(\ell)}$ by the $\underline{\underline{M}}$- absolutely convergent series

$$R^{(\ell)} = \sum_{r=\ell}^{\infty} R_r , \quad S^{(\ell)} = \sum_{r=\ell}^{\infty} S_r ;$$

then

$$\underline{\underline{M}}(R^{(\ell)}) + \underline{\underline{M}}(S^{(\ell)}) \le 2^{-\ell+1}$$

and (from (8))

$$T - T_{j_\ell} = \partial R^{(\ell)} + S^{(\ell)} .$$

Thus we have a subsequence $\{T_{j_\ell}\}$ of $\{T_j\}$ such that $d_{\tilde{W}}(T, T_{j_\ell}) \to 0$. Since we can thus extract a subsequence converging relative to $d_{\tilde{W}}$ from any given subsequence of $\{T_j\}$, we then have $d_{\tilde{W}}(T, T_j) \to 0$; since this can be repeated with $W = W_i$, $\tilde{W} = W_{i+1}$ $\forall i$ (W_i as above), the required result evidently follows.

§32. RECTIFIABILITY THEOREM, AND PROOF OF THE COMPACTNESS THEOREM.

Here we prove the important rectifiability theorem for currents T which, together with ∂T, have locally finite mass and which have the additional property that $\Theta^{*n}(\mu_T, x) > 0$ for μ_T - a.e. x. The main tool of the proof is the structure theorem 13.2. Having established the rectifiability theorem, we show (in 32.2, 32.3) that it is then straightforward to establish the compactness theorem 27.3. Although this proof of compactness theorem has the advantage of being conceptually straightforward, it is rather lengthy if one takes into account the effort needed to prove the structure theorem. Recently B. Solomon [SB] showed that it is possible to prove the compactness theorem (and to develop the whole theory of integer multiplicity currents) without use of the structure theorm.

32.1 THEOREM (Rectifiability Theorem)

Suppose $T \in \mathcal{D}_n(U)$ *is such that* $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty$ $\forall\, W \subset\subset U$, *and* $\Theta^{*n}(\mu_T, x) > 0$ *for* μ_T - a.e. $x \in U$. *Then* T *is* *__rectifiable__*; *that is*

$$T = \underline{\underline{\tau}}(M,\theta,\xi)\ ,^{(*)}$$

where M *is countably* n-*rectifiable*, $\mathcal{H}^n$-*measurable*, θ *is a positive locally* $\mathcal{H}^n$-*integrable function on* M , *and* $\xi(x)$ *orients the approximate tangent space* T_xM *of* M *for* $\mathcal{H}^n$- a.e. $x \in M$.

Proof First note that (by Theorem 3.2(1))

(1) $$\mathcal{H}^n\{x \in W : \Theta^{*n}(\mu_T, x) > K\} \le K^{-1}\underline{\underline{M}}_W(T)$$

for $W \subset\subset U$, and hence

(2) $$\mathcal{H}^n\{x \in U : \Theta^{*n}(\mu_T, x) = \infty\} = 0\ .$$

Notice that the same argument applies with ∂T in place of T in order to give

(3) $$\mathcal{H}^n\{x \in U : \Theta^{*n}(\mu_{\partial T}, x) = \infty\} = 0\ .$$

(Notice we could also conclude $\mathcal{H}^d\{x \in U : \Theta^{*d}(\mu_{\partial T}, x) = \infty\} = 0$ for any $d > 0$ by 3.2(1).)

Next notice that, because $\underline{\underline{M}}_W(T) + \underline{\underline{M}}_W(\partial T) < \infty \quad \forall\ W \subset\subset U$, we know from 26.29 (see in particular Remark 26.30) that (by (2))

(4) $$\mu_T\{x \in U : \Theta^{*n}(\mu_T, x) = \infty\} = 0\ ,$$

and (by (3))

(5) $$\mu_T\{x \in U : \Theta^{*n}(\mu_{\partial T}, x) = \infty\} = 0\ .$$

(*) The notation here is as for integer multiplicity rectifiable currents (§27):

$$\underline{\underline{\tau}}(M,\theta,\xi)(\omega) = \int_M \langle\xi,\omega\rangle\theta\ d\mathcal{H}^n\ ,$$

although of course θ is not assumed to be integer-valued here.

Now let

$$M = \{x \in U : \Theta^{*n}(\mu_T, x) > 0\}$$

and note by (1) that M is the countable union of sets of finite $\mathcal{H}^n$-measure. Furthermore by 26.29 we know that $\mu_T(P) = 0$ for each purely unrectifiable subset of M, and hence

(6) $\qquad \mathcal{H}^n(P) = 0 \quad \forall$ purely unrectifiable $P \subset M$

by virtue of 3.2(1) and the fact that $\Theta^{*n}(\mu_T, x) > 0$ for every $x \in M$ (by definition of M). Then by the structure theorem 13.2 we deduce that

(7) $\qquad M$ *is countably* n-*rectifiable.*

Furthermore (since $\Theta^{*n}(\mu_T, x) > 0$ for μ_T - a.e. $x \in U$ by assumption), we have

(8) $$T = T \llcorner M .$$

Next we note that μ_T is absolutely continuous with respect to $\mathcal{H}^n$ (by (4) and 3.2(2)), and hence by the differentiation theorem 4.7 we have

$$\mu_T = \mathcal{H}^n \llcorner \theta$$

where θ is a positive locally $\mathcal{H}^n$-integrable function on M and $\theta \equiv 0$ on $U \sim M$. Then by the Riesz representation theorem 4.1 we have

(9) $$T(\omega) = \int_U \langle \xi, \omega \rangle \theta \, d\mathcal{H}^n ,$$

for some $\mathcal{H}^n$-measurable, $\Lambda_n(\mathbb{R}^{n+k})$-valued function ξ, $|\xi| = 1$.

It thus remains only to prove that $\xi(x)$ orients T_xM for $\mathcal{H}^n$ - a.e. $x \in M$. (i.e. $\xi(x) = \pm\tau_1 \wedge \ldots \wedge \tau_n$ for $\mathcal{H}^n$ - a.e. $x \in M$, where $\tau_1, \ldots, \tau_n$ is any orthonormal basis for the approximate tangent space T_xM of M.) To see this, write $M = \bigcup_{j=0}^{\infty} M_j$, M_j pairwise disjoint, $\mathcal{H}^n(M_0) = 0$, $M_j \subset N_j$, N_j a C^1 submanifold of $\mathbb{R}^{n+k}$, $j \geq 1$. Now, by 3.5, if $j \geq 1$ we have, for $\mathcal{H}^n$ - a.e. $x \in M_j$,

$$\Theta^{*n}(\mu , \bigcup_{r \neq j} M_r , x) = 0 . \tag{10}$$

Hence, writing as usual $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$, we have for any $\omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$ that, for all $x \in M_j$ such that (10) holds, and for λ small enough to ensure that $\operatorname{spt} \omega \subset \eta_{x,\lambda}(U)$,

$$\begin{aligned}\eta_{x,\lambda\#}T(\omega) &= T(\eta_{x,\lambda}^{\#}\omega) \\ &= \int_{N_j} <\xi, \eta_{x,\lambda}^{\#}\omega> \theta d\mathcal{H}^n + \varepsilon(\lambda) ,\end{aligned}$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \downarrow 0$. ($\varepsilon(\lambda)$ depending on x and ω .) That is

$$\eta_{x,\lambda\#}T(\omega) = \int_{\eta_{x,\lambda}(N_j)} <\xi(x+\lambda z) , \omega(z)> \theta(x+\lambda z) d\mathcal{H}^n(z) + \varepsilon(\lambda)$$

for all $x \in M_j$ such that (10) holds. Since N_j is C^1 , this gives

$$\lim_{\lambda \downarrow 0} \eta_{x,\lambda\#}T(\omega) = \theta(x) \int_P <\xi(x) , \omega(z)> d\mathcal{H}^n(z) \tag{11}$$

for $\mathcal{H}^n$ - a.e. $x \in M_j$ (independent of ω) , where P is the tangent space T_xN_j of N_j at x . Thus (by definition of T_xM - see §12) we have (11) with $P = T_xM$ for $\mathcal{H}^n$ - a.e. $x \in M_j$. On the other hand by (5) we have

$$\begin{aligned}\partial\eta_{x,\lambda\#}T(\omega) &= \eta_{x,\lambda\#}\partial T(\omega) = \partial T(\eta_{x,\lambda}^{\#}\omega) \\ &= o(\lambda) \quad \text{as} \quad \lambda \downarrow 0\end{aligned}$$

for $\mathcal{H}^n$ - a.e. $x \in M_j$ (independent of ω) . Thus for such x

$$\lim_{\lambda \downarrow 0} (\partial \eta_{x,\lambda \#} T)(\omega) = 0 . \tag{12}$$

On the other hand for μ_T - a.e. $x \in U$, for any $W \subset\subset \mathbb{R}^{n+k}$, we have by (4) that

$$\limsup_{\lambda \downarrow 0} \underline{\underline{M}}_W (\eta_{x,\lambda \#} T) < \infty . \tag{13}$$

Thus (by (11), (12), (13)), for $\mathcal{H}^n$ - a.e. $x \in M$, we can find a sequence $\lambda_\ell \downarrow 0$ such that

$$\eta_{x,\lambda_\ell \#} T \rightharpoonup S_x \;, \; \partial S_x = 0 \;,$$

where $S_x \in \mathcal{D}_n(\mathbb{R}^{n+k})$ is defined by

$$S_x(\omega) = \theta(x) \int_P \langle \xi(x), \omega(z) \rangle \, d\mathcal{H}^n(z) , \tag{14}$$

$\omega \in \mathcal{D}^n(\mathbb{R}^{n+k})$, $P = T_x M$. We now claim that (14), taken together with the fact that $\partial S_x = 0$, implies that $\xi(x)$ orients P. To see this, assume (without loss of generality) that $P = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ and select $\omega \in \mathcal{D}^{n-1}(\mathbb{R}^{n+k})$ so that $\omega(y) = y^j \phi(y) dy^{i_1} \wedge \ldots \wedge dy^{i_{n-1}}$, where $y = (y^1, \ldots, y^{n+k})$, $j \geq n+1$, $\{i_1, \ldots, i_{n-1}\} \subset \{1, \ldots, n+k\}$, and $\phi \in C_c^\infty(\mathbb{R}^{n+k})$. Then since $y_j \equiv 0$ on $\mathbb{R}^n \times \{0\}$ we deduce, from (14) and the fact that $\partial S_x = 0$,

$$0 = \partial S_x(\omega) = S_x(d\omega) = \theta(x) \int_P \phi(y) \langle \xi(x), dy^j \wedge dy^{i_1} \wedge \ldots \wedge dy^{i_{n-1}} \rangle$$

$$= \theta(x) \int_P \phi(y) \xi(x) \cdot (e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n-1}}) \, d\mathcal{H}^n(y) .$$

That is, since $\phi \in C_c^\infty(\mathbb{R}^{n+k})$ is arbitrary, we deduce that $\xi(x) \cdot (e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_{n-1}}) = 0$ whenever $j \geq n+1$ and $\{i_1, \ldots, i_{n-1}\} \subset \{1, \ldots, n+k\}$. Thus we must have (since $|\xi(x)| = 1$), $\xi(x) = \pm e_1 \wedge \ldots \wedge e_n$ as required.

We can now give the proof of the compactness theorem 27.3. For convenience we first re-state the theorem in a slightly weaker form. (See the remark (2) following the statement for the proof that the previous version 27.3 follows.)

32.2 THEOREM *Suppose* $\{T_j\} \subset \mathcal{D}_n(U)$ *, suppose* T_j *,* ∂T_j *are integer multiplicity for each* j *,*

$$(*) \qquad \sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \qquad \forall\ W \subset\subset U\ ,$$

and suppose $T_j \rightharpoonup T \in \mathcal{D}_n(U)$ *. Then* T *is an integer multiplicity current.*

32.3 REMARKS

(1) Note that the general case of the theorem follows from the special case when $U = \mathbb{R}^P$ and $\operatorname{spt} T_j \subset K$ for some fixed compact K ; in fact if T_j are as in the theorem and if $\xi \in U$, then by 28.5 (1), (2) and an argument like that in Remark 10.7(2) we know that, for $\mathcal{L}^1$-a.e. $r > 0$, $\partial(T_j \llcorner B_r(\xi))$ are integer multiplicity and (*) holds with $T_j \llcorner B_r(\xi)$ in place of T_j for some subsequence $\{j'\} \subset \{j\}$ (depending on r) .

(2) The previous (formally slightly stronger) version 27.3 of the above theorem follows by using 30.3. (Note that the proof of 30.3 needed only the weaker version of the compactness theorem given above in 32.2; indeed, as mentioned in Remark 30.4, it used only the case $\partial T_j = 0$ of 27.3.

Proof of 32.2 We shall use induction on n with $U \subset \mathbb{R}^P$ $(U,P$ fixed independent of $n)$. First note that the theorem is trivial in case $n = 0$. Then assume $n \geq 1$ and suppose the theorem is true with $n-1$ in place of n.

By the above remark (1) we shall assume without loss of generality that $\operatorname{spt} T_j \subset K$ for some fixed compact K , and that $U = \mathbb{R}^P$. Furthermore, by remark (1) in combination with the inductive hypothesis, for each $\xi \in \mathbb{R}^P$ we have

(1) $\qquad \partial(T \llcorner B_r(\xi))$ *is an integer multiplicity current*

(in $\mathcal{D}_{n-1}(\mathbb{R}^P)$) for $\mathcal{L}^1$-a.e. $r > 0$.

From the above assumptions $U = \mathbb{R}^P$, spt $T_j \subset K$ we know that $0 ⨳ \partial T - T$ zero boundary and is the weak limit of $0 ⨳ \partial T_j - T_j$; since $0 ⨳ \partial T$ is integer multiplicity (by the inductive hypothesis) we thus see that the general case of the theorem follows from the special case when $\partial T = 0$. We shall therefore henceforth also assume $\partial T = 0$.

Next, define (for $\xi \in \mathbb{R}^P$ fixed)

$$f(r) = \underline{\underline{M}}(T \llcorner B_r(\xi)), \quad r > 0.$$

By virtue of 28.9 we have (since $\partial T = 0$)

(2) $$\underline{\underline{M}}(\partial(T \llcorner B_r(\xi))) \le f'(r), \quad \mathcal{L}^1\text{-a.e. } r > 0.$$

(Notice that $f'(r)$ exists a.e. $r > 0$ because $f(r)$ is increasing.) On the other hand if $\Theta^{*n}(\mu_T, \xi) < \eta$ ($\eta > 0$ a given constant), then $\limsup_{\rho \downarrow 0} \frac{f(\rho)}{\omega_n \rho^n} < \eta$, and hence for each $\delta > 0$ we can arrange

(3) $$\frac{d}{dr}(f^{1/n}(r)) \le 2\omega_n^{1/n}\eta$$

for a set of $r \in (0,\delta)$ of positive $\mathcal{L}^1$-measure. (Because $\delta^{-1}\int_0^\delta \frac{d}{dr}(f^{1/n}(r))dr \le \delta^{-1} f^{1/n}(\delta) \le \omega_n^{1/n}\eta$ for all sufficiently small $\delta > 0$.)

Now by (1) and the isoperimetric theorem, we can find an integer multiplicity $S_r \in \mathcal{D}_n(\mathbb{R}^P)$ such that $\partial S_r = \partial(T \llcorner B_r(\xi))$ and

$$(4)\qquad \underline{\underline{M}}(S_r)^{\frac{n-1}{n}} \le c\underline{\underline{M}}(\partial(T \llcorner B_r(\xi)))$$

$$\le c\eta\underline{\underline{M}}(T \llcorner B_r(\xi))^{\frac{n-1}{n}} \qquad \text{(by (2),(3))}$$

for a set of r of positive L^1-measure in $(0,\delta)$.* Since δ was arbitrary we then have both (1), (4) for a sequence of $r \downarrow 0$. But then (since we can repeat this for any ξ such that $\Theta^{*n}(\mu_T,\xi) < \eta$) if C is any compact subset of $\{x \in \mathbb{R}^P : \Theta^{*n}(\mu_T,x) < \eta\}$, by Remark 4.5(2) we get for each given $\rho > 0$ a pairwise disjoint family $B_j = \bar{B}_{r_j}(\xi_j)$ of closed balls covering μ_T-almost all of C, with

$$(5)\qquad \bigcup_j B_j \subset \{x : \mathrm{dist}(x,C) < \rho\}$$

and with

$$(6)\qquad \underline{\underline{M}}(S_j^{(\rho)}) \le c\eta\underline{\underline{M}}(T \llcorner B_j)$$

for some integer multiplicity $S_j^{(\rho)}$ with

$$(7)\qquad \partial S_j^{(\rho)} = \partial(T \llcorner B_j) .$$

Now because of (7) we have $S_j^{(\rho)} - T \llcorner B_j = \partial(\{\xi_j\} \times\!\!\!\!\times (S_j^{(\rho)} - T \llcorner B_j))$, and hence (by 26.23, 26.26) we have for $\omega \in \mathcal{D}^n(\mathbb{R}^P)$

$$(8)\qquad |(S_j^{(\rho)} - T \llcorner B_j)(\omega)| \le c\rho\underline{\underline{M}}(S_j^{(\rho)} - T \llcorner B_j)|d\omega|$$

$$\le c\rho\underline{\underline{M}}(T \llcorner B_j)|d\omega| \qquad \text{(by (6))}.$$

Therefore we have $\sum_j (S_j^{(\rho)} - T \llcorner B_j) \rightharpoonup 0$ as $\rho \downarrow 0$, and hence

$$(9)\qquad T + \sum_j (S_j^{(\rho)} - T \llcorner B_j) \rightharpoonup T$$

as $\rho \downarrow 0$. However since the series $\sum_j S_j^{(\rho)}$ and $\sum_j T \llcorner B_j$ are $\underline{\underline{M}}$-absolutely convergent (by (6) and the fact that the B_j are disjoint), we deduce that the left side in (9) can be written $T \llcorner (\mathbb{R}^P \sim \bigcup_j B_j) + \sum_j S_j^{(\rho)}$ and hence

---

* In case $n = 1$, (1), (2), (3) (for $\eta < \frac{1}{2}$) imply $\partial(T \llcorner B_r(\xi)) = 0$, hence we get, in place of (4), $\underline{\underline{M}}(S_r) \le \underline{\underline{M}}(T \llcorner B_r(\xi))$ trivially by taking $S_r = 0$.

(using (6) again, together with the lower-semicontinuity of $\underline{\underline{M}}_W$ (W open) under weak convergence)

$$\mu_T(\{x : \mathrm{dist}(x,C) < \rho\}) \le \mu_T(\{x : \mathrm{dist}(x,C) < \rho\} \sim C) +$$

$$c\eta\mu_T(\{x : \mathrm{dist}(x,C) < \rho\}) .$$

Choosing η such that $c\eta \le \frac{1}{2}$, this gives

$$\mu_T(\{x : \mathrm{dist}(x,C) < \rho) \le 2\mu_T(\{x : \mathrm{dist}(x,C) < \rho\} \sim C\} .$$

Letting $\rho \downarrow 0$, we get $\mu_T(C) = 0$.

Thus we have shown that $\Theta^{*n}(\mu_T,x) > 0$ for μ_T - a.e. $x \in \mathbb{R}^P$. We can therefore apply 32.1 in order to conclude that $T = \underline{\underline{\tau}}(M,\theta,\xi)$ as in 32.1. It thus remains only to prove that θ is integer-valued. This is achieved as follows:

First note that for L^n - a.e. $x \in M$ we have (cf. the argument leading to (11) in the proof of 32.1)

$$\eta_{x,\lambda\#}T \rightharpoonup \theta(x)[\![T_xM]\!] \quad \text{as} \quad \lambda \downarrow 0 , \tag{10}$$

where $[\![T_xM]\!]$ is oriented by $\xi(x)$. Assuming without loss of generality that $T_xM = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^P$ and setting $d(y) = \mathrm{dist}(y, \mathbb{R}^n \times \{0\})$, by 28.5(1) we can find a sequence $\lambda_j \downarrow 0$ and a $\rho > 0$ such that $\langle \eta_{x,\lambda_j\#}T, d, \rho\rangle$ is integer multiplicity with

$$\underline{\underline{M}}_\Omega(\langle\eta_{x,\lambda_j\#}T,d,\rho\rangle) \le c \quad (\text{independent of } j\,)$$

where $\Omega = B_1^n(0) \times \mathbb{R}^{P-n} \subset \mathbb{R}^P$. Then by 28.5(2) we have $S_j \equiv (\eta_{x,\lambda_j\#}T) \mathbin{\llcorner} \{y : d(y) < \rho\}$ is such that, writing $\Omega = B_1^n(0) \times \mathbb{R}^{P-n} \subset \mathbb{R}^P$,

$$\sup_{j\ge 1}(\underline{\underline{M}}_\Omega(S_j) + \underline{\underline{M}}_\Omega(\partial S_j)) < \infty . \tag{11}$$

Now let p denote the restriction to Ω of the orthogonal projection of $\mathbb{R}^P$ onto $\mathbb{R}^n$; and let $\tilde{S}_j$ be the current in $\mathcal{D}_n(\Omega)$ obtained by setting $\tilde{S}_j(\omega) = S_j(\tilde{\omega})$, $\omega \in \mathcal{D}^n(\Omega)$, $\tilde{\omega} \in \mathcal{D}^n(\mathbb{R}^P)$ such that $\tilde{\omega} = \omega$ in Ω and $\tilde{\omega} \equiv 0$ on $\mathbb{R}^P \sim \Omega$. Then we have $p_\# \tilde{S}_j \in \mathcal{D}_n(B_1^n(0))$, and hence, by 26.28 and (11) above,

$$p_\# \tilde{S}_j(\omega) = \int_{B_1^n(0)} a\theta_j \, d\mathcal{L}^n \, , \; \omega = a dx^1 \wedge \ldots \wedge dx^n \, , \; a \in C_c^\infty(\mathbb{R}^n) \, ,$$

for some integer-valued $BV_{loc}(B_1^n(0))$ function θ_j with

$$(12) \qquad \begin{cases} \underline{\underline{M}}_{B_1^n(0)}(p_\# \tilde{S}_j) = \displaystyle\int_{B_1^n(0)} |\theta_j| \, d\mathcal{L}^n \\ \underline{\underline{M}}_{B_1^n(0)}(\partial p_\# \tilde{S}_j) = \displaystyle\int_{B_1^n(0)} |D\theta_j| \, . \end{cases}$$

Then by (11), (12) we deduce $\int_{B_1^n(0)} |D\theta_j| + \int_{B_1^n(0)} |\theta_j| \, d\mathcal{L}^n \le c$, c independent of j , and hence by the compactness theorem 6.3 we know θ_j converges strongly in L^1 in $B_1^n(0)$ to an integer-valued BV function θ_* . On the other hand $S_j \rightharpoonup \theta(x)[\![\mathbb{R}^n \times \{0\}]\!]$ by (10), and hence $p_\# \tilde{S}_j \rightharpoonup \theta(x) p_\# [\![\mathbb{R}^n \times \{0\}]\!] = \theta(x)[\![\mathbb{R}^n]\!]$ in $B_1^n(0)$. We thus deduce that $\theta_* \equiv \theta(x)$ in $B_1^n(0)$; thus $\theta(x) \in \mathbb{Z}$ as required.

CHAPTER 7

AREA MINIMIZING CURRENTS

This chapter provides an introduction to the theory of area minimizing currents. In the first section (§33) of the chapter we derive some basic preliminary properties, and in particular we discuss the fact that the integer multiplicity varifold corresponding to a minimizing current is stable (and indeed minimizing in a certain sense). In §34 there are some existence and compactness results, including the important theorem that if $\{T_j\}$ is a sequence of minimizing currents in U with $\sup_{j\geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty \quad \forall\, W \subset\subset U$, and if $T_j \rightharpoonup T \in \mathcal{D}_n(U)$, then T is also minimizing in U and the corresponding varifolds converge in the measure theoretic sense of §15. This enables us to discuss tangent cones and densities in §35, and in particular make some regularity statements for minimizing currents in §36. Finally, in §37 we develop the standard codimension 1 regularity theory, due originally to De Giorgi [DG], Fleming [FW], Almgren [A4], J. Simons [SJ] and Federer [FH2].

§33. BASIC CONCEPTS

Suppose A is any subset of $\mathbb{R}^{n+k}$, $A \subset U$, U open in $\mathbb{R}^{n+k}$, and $T \in \mathcal{D}_n(U)$ an integer multiplicity current.

33.1 DEFINITION We say that T is minimizing in A if

$$\underline{\underline{M}}_W(T) \leq \underline{\underline{M}}_W(S)$$

whenever $W \subset\subset U$, $\partial S = \partial T$ (in U) and $\mathrm{spt}(S-T)$ is a compact subset of $A \cap W$.

There are two especially important cases of this definition:

(1) when $A = U$

(2) when $A = N \cap U$, N an $(n+k_1)$-dimensional embedded submanifold of $\mathbb{R}^{n+k}$ (in the sense of §7).

As a matter of fact, these are the only cases we are interested in here.

Corresponding to the current $T = \underline{\underline{\tau}}(M,\theta,\xi) \in \mathcal{D}_n(U)$ we have the integer multiplicity varifold $V = \underline{\underline{v}}(M,\theta)$. As one would expect, V is stationary in U if T is minimizing in U and $\partial T = 0$; indeed we show more:

33.2 LEMMA *Suppose* T *is minimizing in* $N \cap U$, *where* N *is an* $(n+k_1)$*-dimensional* C^2 *submanifold of* $\mathbb{R}^{n+k}$ $(k_1 \le k)$ *and suppose* $\partial T = 0$ *in* U. *Then* V *is stationary in* $N \cap U$ *in the sense of* 16.4, *so that in particular* V *has locally bounded generalized mean curvature in* U *(in the sense of* 16.5*).*

In fact V *is minimizing in* $N \cap U$ *in the sense that*

$$(*) \qquad \underline{\underline{M}}_W(V) \le \underline{\underline{M}}_W(\phi_\# V),$$

whenever $W \subset\subset U$ *and* ϕ *is a diffeomorphism of* U *such that* $\phi(N\cap U) \subset N\cap U$ *and* $\phi | U \sim K = \underline{\underline{1}}_{U\sim K}$ *for some compact* $K \subset W \cap N$.

<u>Note</u>: Of course $N = U$ (when $k_1 = k$) is an important special case; then V is stationary and in fact stable in U.

33.3 REMARK In view of 33.2 (together with the fact that $\theta \ge 1$) we can apply the theory of chapters 4 and 5 to V; in particular we can represent $T = \underline{\underline{\tau}}(M_*,\theta_*,\xi)$ where M_* is a relatively *closed* countably n-rectifiable subset of U, and θ_* is an upper semi-continuous function on M_* with $\theta_* \ge 1$ everywhere on M_* (and θ_* integer-valued $\mathcal{H}^n$-a.e. on M_*).

Proof of 33.2 Evidently (in view of the discussion of §16) the first claim in 33.2 follows from (*) (by taking $\phi = \phi_t$ in (*), ϕ_t is in 16.1 with $U \cap N$ in place of U).

To prove (*) we first note that, for any W, ϕ as in the statement of the theorem,

$$\text{(1)} \qquad \underline{\underline{M}}_W(\phi_\# V) = \underline{\underline{M}}_W(\phi_\# T)$$

by Remark 27.2(3). Also, since $\partial T = 0$ (in U), we have

$$\text{(2)} \qquad \partial \phi_\# T = \phi_\# \partial T = 0 .$$

Finally,

$$\text{(3)} \qquad \operatorname{spt}(T - \phi_\# T) \subset K \subset W .$$

By virtue of (2), (3) we are able to use the inequality of 33.1 with $S = \phi_\# T$. This gives (*) as required by virtue of (1).

We conclude this section with the following useful *decomposition lemma:*

33.4 LEMMA *Suppose* $T_1, T_2 \in \mathcal{D}_n(U)$ *are integer multiplicity and suppose* $T_1 + T_2$ *is minimizing in* A, $A \subset U$, *and*

$$\underline{\underline{M}}_W(T_1+T_2) = \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2)$$

for each $W \subset\subset U$. *Then* T_1, T_2 *are both minimizing in* A.

Proof Let $X \in \mathcal{D}_n(U)$ be integer multiplicity with $\operatorname{spt} X \subset K$, K a compact subset of $A \cap W$, and with $\partial X = 0$. Because $T_1 + T_2$ is minimizing in A we have (by Definition 33.1)

$$\underline{\underline{M}}_W(T_1+T_2+X) \ge \underline{\underline{M}}_W(T_1+T_2) .$$

However since $\underline{\underline{M}}_W(T_1+T_2) = \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2)$, and $\underline{\underline{M}}_W(T_1+T_2+X) \le \underline{\underline{M}}_W(T_1+X) + \underline{\underline{M}}_W(T_2)$, this gives

$$\underline{\underline{M}}_W(T_1) \le \underline{\underline{M}}_W(T_1+X) .$$

In view of the arbitrariness of X , this establishes that T_1 is minimizing in $A \cap W$ (in accordance with Definition 33.1). Interchanging T_1, T_2 in the above argument, we likewise deduce that T_2 is minimizing in $A \cap W$.

§34. EXISTENCE AND COMPACTNESS RESULTS

We begin with a result which establishes the rich abundance of area minimizing currents in Euclidean space.

34.1 LEMMA *Let* $S \in \mathcal{D}_{n-1}(\mathbb{R}^{n+k})$ *be integer multiplicity with* spt S *compact and* $\partial S = 0$. *Then there is an integer multiplicity current* $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *such that* spt T *is compact and* $\underline{\underline{M}}(T) \le \underline{\underline{M}}(R)$ *for each integer multiplicity* $R \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* spt R *compact and* $\partial R = S$.

34.2 REMARKS

(1) Of course T is minimizing in $\mathbb{R}^{n+k}$ in the sense of Definition 33.1.

(2) By virtue of 33.2 and the convex hull property 19.2 we have automatically that spt $T \subset$ convex hull of spt S .

(3) $$\underline{\underline{M}}(T)^{\frac{n-1}{n}} \le c\underline{\underline{M}}(S)$$

by virtue of the isoperimetric theorem 30.1.

Proof of 34.1 Let

$$\mathcal{I}_S = \{R \in \mathcal{D}_n(\mathbb{R}^{n+k}) : R \text{ is integer multiplicity, spt } R \text{ compact, } \partial R = S\} .$$

Evidently $\mathcal{I}_S \neq \emptyset$. (e.g. $0 \times S \in \mathcal{I}_S$.) Take any sequence $\{R_q\} \subset \mathcal{I}_S$ with

(1) $$\lim_{q\to\infty} \underline{\underline{M}}(R_q) = \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) ,$$

let $B_R(0)$ be any ball in $\mathbb{R}^{n+k}$ such that $\operatorname{spt} S \subset B_R(0)$, and let $f : \mathbb{R}^{n+k} \to \bar{B}_R(0)$ be the nearest point (radial) retract of $\mathbb{R}^{n+k}$ onto $\bar{B}_R(0)$. Then $\operatorname{Lip} f = 1$ and hence

(2) $$\underline{\underline{M}}(f_\# R_q) \le \underline{\underline{M}}(R_q) .$$

On the other hand $\partial f_\# R_q = f_\# \partial R_q = f_\# S = S$, because $f|B_R(0) = \underline{\underline{1}}_{B_R(0)}$ and $\operatorname{spt} S \subset B_R(0)$. Thus $f_\# R_q \subset \mathcal{I}_S$ and by (1), (2) we have

(3) $$\lim_{q\to\infty} \underline{\underline{M}}(f_\# R_q) = \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) .$$

Now by the compactness theorem 27.3 there is a subsequence $\{q'\} \subset \{q\}$ and an integer multiplicity current $T \in \mathcal{D}_n(\mathbb{R}^{n+k})$ such that $f_\# R_{q'} \rightharpoonup T$ and (by (3) and lower semi-continuity of mass with respect to weak convergence)

(4) $$\underline{\underline{M}}(T) \le \inf_{R\in\mathcal{I}_S} \underline{\underline{M}}(R) .$$

However $\operatorname{spt} T \subset \overline{B_R}(0)$ and $\partial T = \lim \partial f_\# R_{q'} = \lim f_\# \partial R_{q'} = S$, so that $T \in \mathcal{I}_S$, and the lemma is established (by (4)).

The proof of the following lemma is similar to that of 34.1 (and again based on 27.3), and its proof is left to the reader.

34.3 LEMMA *Suppose* N *is an* $(n+k_1)$*-dimensional compact* C^1 *submanifold embedded in* $\mathbb{R}^{n+k}$ *and suppose* $R_1 \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is given such that* $\partial R_1 = 0$, $\operatorname{spt} R_1 \subset N$ *and*

$$\mathcal{I}_{R_1} = \{R \in \mathcal{D}_n(\mathbb{R}^{n+k}) : R - R_1 = \partial S$$

for some integer multiplicity $S \in \mathcal{D}_{n+1}(\mathbb{R}^{n+k})$ *with* $\operatorname{spt} S \subset N\} \neq \emptyset$.

Then there is $T \in \mathcal{I}_{R_1}$ *such that*

$$\underline{\underline{M}}(T) = \inf_{R \in \mathcal{I}_{R_1}} \underline{\underline{M}}(R)$$

34.4 REMARKS

(1) $R - R_1 = \partial S$ with S integer multiplicity and $\operatorname{spt} S \subset N$ means that R, R_1 represent homologous cycles in the n-th singular homology class (with integer coefficients) of N . (See [FH1] or [FF] for discussion.)

(2) It is quite easy to see that T is *locally* minimizing in N ; thus for each $\xi \in \operatorname{spt} T$ there is a neighbourhood U of ξ such that T is minimizing in $N \cap U$.

We conclude this section with the following important compactness theorem for minimizing currents:

34.5 THEOREM *Suppose* $\{T_j\}$ *is a sequence of minimizing currents in* U *with* $\sup_{j \geq 1}(\underline{\underline{M}}_W(T_j) + \underline{\underline{M}}_W(\partial T_j)) < \infty$ *for each* $W \subset\subset U$ *, and suppose* $T_j \rightharpoonup T \in \mathcal{D}_n(U)$. *Then* T *is minimizing in* U *and* $\mu_{T_j} \to \mu_T$ *(in the usual sense of Radon measures in* U*)* .

34.6 REMARKS

(1) Note that $\mu_{T_j} \to \mu_T$ means the corresponding sequence of varifolds converge in the measure theoretic sense of §15 to the varifold associated with T . (T is automatically integer multiplicity by 27.3.)

(2) If the hypotheses are as in the theorem, except that $\operatorname{spt} T_j \subset N_j \subset U$ and T_j is minimizing in N_j , $\{N_j\}$ a sequence of C^1 embedded $(n+k_1)$-dimensional submanifolds of $\mathbb{R}^{n+k}$ converging in the C^1 sense to

N , $N \subset U$ an embedded $(n+k_1)$-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$,(*) then T minimizes in N (and we still have $\mu_{T_j} \to \mu_T$ in the sense of Radon measures in U) . We leave this modification of 34.5 to the reader. (It is easily checked by using suitable local representations for the N_j and by obvious modifications of the proof of 34.5 given below.)

Proof of 34.5 Let $K \subset U$ be an arbitrary compact set and choose a smooth $\phi : U \to [0,1]$ such that $\phi \equiv 1$ in some neighbourhood of K , and $\operatorname{spt} \phi \subset \{x \in U : \operatorname{dist}(x,K) < \varepsilon\}$, where $0 < \varepsilon < \operatorname{dist}(K,\partial U)$ is arbitrary. For $0 < \lambda < 1$, let

$$W_\lambda = \{x \in U : \phi(x) > \lambda\} .$$

Then

(1) $$K \subset W_\lambda \subset\subset U$$

for each λ , $0 \le \lambda < 1$.

By virtue of 31.2 we know that $d_W(T_j,T) \to 0$ for each $W \subset\subset U$, hence in particular we have

(2) $$T - T_j = \partial R_j + S_j \ , \ \underline{\underline{M}}_{W_0}(R_j) + \underline{\underline{M}}_{W_0}(S_j) \to 0$$

$(W_0 = \{x \in U : \phi(x) > 0\})$.

By the slicing theory (and in particular by 28.5) we can choose $0 < \alpha < 1$ and a subsequence $\{j'\} \subset \{j\}$ (subsequently denoted simply by $\{j\}$) such that

(3) $$\partial(R_j \llcorner W_\alpha) = (\partial R_j) \llcorner W_\alpha + P_j$$

where $\operatorname{spt} P_j \subset \partial W_\alpha$, P_j is integer multiplicity, and

---

(*) Thus $\exists\ \psi_j : U \to U$, $\psi_j | N_j$ in a diffeomorphism onto N, and $\psi_j \to \underline{\underline{1}}_U$ locally in U with respect to the C^1 metric.

(4) $$\underline{\underline{M}}(P_j) \to 0 .$$

We can also of course choose α to be such that

(5) $$\underline{\underline{M}}(T_j \llcorner \partial W_\alpha) = 0 \qquad \forall j \quad \textit{and} \quad \underline{\underline{M}}(T \llcorner \partial W_\alpha) = 0 .$$

Thus, combining (2), (3), (4) we have

(6) $$T \llcorner W_\alpha = T_j \llcorner W_\alpha + \partial \tilde{R}_j + \tilde{S}_j$$

with $\tilde{R}_j$, $\tilde{S}_j$ integer multiplicity $(\tilde{R}_j = R_j \llcorner W_\alpha , \tilde{S}_j = S_j \llcorner W_\alpha + P_j)$ with

(7) $$\underline{\underline{M}}(\tilde{R}_j) + \underline{\underline{M}}(\tilde{S}_j) \to 0 .$$

Now let $X \in \mathcal{D}_n(U)$ be any integer multiplicity current with $\partial X = 0$ and $\operatorname{spt} X \subset K$. We want to prove

(8) $$\underline{\underline{M}}_{W_\alpha}(T) \le \underline{\underline{M}}_{W_\alpha}(T+X) .$$

(In view of the arbitrariness of K , X this will evidently establish the fact that T is minimizing in U .)

By (6), we have

(9) $$\begin{aligned} \underline{\underline{M}}_{W_\alpha}(T+X) &= \underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j+\tilde{S}_j) \\ &\ge \underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j) - \underline{\underline{M}}(\tilde{S}_j) . \end{aligned}$$

Now since T_j is minimizing and $\partial(X+\partial\tilde{R}_j) = 0$ with $\operatorname{spt}(X+\partial\tilde{R}_j) \subset \bar{W}_\alpha$, we have

(10) $$\underline{\underline{M}}_{W_\lambda}(T_j+X+\partial\tilde{R}_j) \ge \underline{\underline{M}}_{W_\lambda}(T_j)$$

for $\lambda > \alpha$. But by (3) we have $\underline{\underline{M}}(\partial\tilde{R}_j \llcorner \partial W_\alpha) = \underline{\underline{M}}(P_j) \to 0$, and by (5) $\underline{\underline{M}}(T_j \llcorner \partial W_\alpha) = 0$, $\underline{\underline{M}}(T \llcorner \partial W_\alpha) = 0$. Hence letting $\lambda \downarrow \alpha$ in (10) we get

$$\underline{\underline{M}}_{W_\alpha}(T_j+X+\partial\tilde{R}_j) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \underline{\underline{M}}(P_j) ,$$

and therefore from (9) we obtain

(11) $$\underline{\underline{M}}_{W_\alpha}(T+X) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \varepsilon_j , \quad \varepsilon_j \downarrow 0 .$$

In particular, setting $X = 0$, we have

(12) $$\underline{\underline{M}}_{W_\alpha}(T) \geq \underline{\underline{M}}_{W_\alpha}(T_j) - \varepsilon_j , \quad \varepsilon_j \downarrow 0 .$$

Using the lower semi-continuity of mass with respect to weak convergence in (11), we then have (8) as required.

It thus remains only to prove that $\mu_{T_j} \to \mu_T$ in the sense of Radon measures in U . First note that by (12) we have

$$\limsup \underline{\underline{M}}_{W_\alpha}(T_j) \leq \underline{\underline{M}}_{W_\alpha}(T) ,$$

so that (since $K \subset W_\alpha \subset \{x : \mathrm{dist}(x,K)<\varepsilon\}$ by construction)

$$\limsup \mu_{T_j}(K) \leq \underline{\underline{M}}_{\{x:\mathrm{dist}(x,K)<\varepsilon\}}(T) .$$

Hence, letting $\varepsilon \downarrow 0$

(13) $$\limsup \mu_{T_j}(K) \leq \mu_T(K) .$$

(We actually only proved this for some subsequence, but we can repeat the argument for a subsequence of any given subsequence, hence it holds for the original sequence $\{T_j\}$.)

By the lower semi-continuity of mass with respect to weak convergence, we have

(14) $$\mu_T(W) \leq \liminf \mu_{T_j}(W) \qquad \forall \text{ open } W \subset\subset U .$$

Since (13), (14) hold for arbitrary compact K and open $W \subset U$, it now easily follows (by a standard approximation argument) that $\int f d\mu_{T_j} \to \int f d\mu_T$ for each continuous f with compact support in U, as required.

§35. TANGENT CONES AND DENSITIES

In this section we prove the basic results concerning tangent cones and densities of area minimizing currents. All results depend on the fact that (by virtue of 33.2) the varifold associated with a minimizing current is stationary. This enables us to bring into play the important monotonicity results of Chapter 4.

Subsequently we take N to be a smooth (at least C^2) embedded $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$ $(k_1 \le k)$, U open in $\mathbb{R}^{n+k}$ and $(\bar{N} \sim N) \cap U = \emptyset$. Notice that an important case is when $N = U$ (when $k_1 = k$).

35.1 THEOREM *Suppose* $T \in \mathcal{D}_n(U)$ *is minimizing in* $U \cap N$, $\operatorname{spt} T \subset U \cap N$, *and* $\partial T = 0$ *in* U. *Then*

(1) $\Theta^n(\mu_T, x)$ *exists everywhere in* U *and* $\Theta^n(\mu_T, \cdot)$ *is upper semi-continuous in* U;

(2) *For each* $x \in \operatorname{spt} T$ *and each sequence* $\{\lambda_j\} \downarrow 0$, *there is a subsequence* $\{\lambda_{j'}\}$ *such that* $\eta_{x,\lambda_{j'}\#} T \rightharpoonup C$ *in* $\mathbb{R}^{n+k}$, *where* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is integer multiplicity and minimizing in* $\mathbb{R}^{n+k}$, $\eta_{0,\lambda\#} C = C$ $\forall\, \lambda > 0$, *and* $\Theta^n(\mu_C, 0) = \Theta^n(\mu_T, x)$.

35.2 REMARKS

If C is as in (2) above, we say that C is a *tangent cone* for T

at x. If spt C is an n-dimensional subspace P (notice that since C is integer multiplicity and $\partial C = 0$, it then follows from 26.27 that $C = m[\![P]\!]$ for some $m \in \mathbb{Z}$, assuming P has constant orientation) then we call C a *tangent plane* for T at x.

(2) Notice that is *not* clear whether or not there is a *unique* tangent cone for T at x; thus it is an open question whether or not C depends on the particular sequence $\{\lambda_j\}$ or subsequence $\{\lambda_{j'}\}$ used in its definition. Recently it has been shown ([SL3]) that if C is a tangent cone of T at x such that $\Theta^n(\mu_C, x) = 1$ for *all* $x \in \operatorname{spt} C \sim \{0\}$, then C is the unique tangent cone for T at x, and hence $\eta_{x,\lambda\#}T \rightharpoonup C$ as $\lambda \downarrow 0$. Also B. White [WB] has shown in case $n = 2$ that C is always unique (with spt C consisting of a union of 2-planes meeting transversely at 0).

Proof of 35.1 By virtue of Lemma 33.2 we can apply the monotonicity formula of 17.6 (with $\alpha = 1$) and Corollary 17.8 in order to deduce that $\Theta^n(\mu_T, x)$ exists for every $x \in U$ and is an upper semi-continuous function of x in U.

Similarly the existence of C as in part (2) of 35.1 follows directly from Theorem 19.3(*) and the compactness theorem 34.5 (more particularly from Remark 34.6 with $N_j = \eta_{x,\lambda_j\#}N$). Notice that Remark 34.6 establishes first that C is minimizing only in the $(n+k_1)$-dimensional subspace $T_xN \subset \mathbb{R}^{n+k}$. However since orthogonal projection of $\mathbb{R}^{n+k}$ onto T_xN does not increase area, and since spt $C \subset T_xN$, it then follows that C is area minimizing in $\mathbb{R}^{n+k}$.

(*) Actually 19.3 gives $\eta_{0,\lambda\#}V_C = V_C$ for the varifold V_C associated with C, but then $x \wedge \vec{C}(x) = 0$ and hence $\eta_{0,\lambda\#}C = C$ by 26.22 with $h(t,x) = t\lambda x + (1-t)x$.

35.3 THEOREM* *Suppose* $T \in \mathcal{D}_n(U)$ *is minimizing in* $U \cap N$, $\operatorname{spt} T \subset U \cap N$, *and* $\partial T = 0$ (*in* U). *Then*

(1) $\Theta^n(\mu_T, x) \in \mathbb{Z}$ *for all* $x \in U \sim E$, *where* $\mathcal{H}^{n-3+\alpha}(E) = 0 \quad \forall\, \alpha > 0$;

(2) *There is a set* $F \subset E$ (E *as in* (1)) *with* $\mathcal{H}^{n-2+\alpha}(F) = 0$ $\forall\, \alpha > 0$ *and such that for each* $x \in \operatorname{spt} T \sim F$ *there is a tangent plane (see* 35.2(1) *above for terminology) for* T *at* x.

Note: We do not claim E, F are *closed.*

The proof of both parts is based on the abstract dimension reducing argument of Appendix A. In order to apply this in the context of currents we need the observation of the following remark.

35.4 REMARK Given an integer multiplicity current $S \in \mathcal{D}_n(\mathbb{R}^{n+k})$, there is an associated function $\phi_S = (\phi_S^0, \phi_S^1, \ldots, \phi_S^N) : \mathbb{R}^{n+k} \to \mathbb{R}^{N+1}$, where $N = \binom{n+k}{n}$, such that (writing $\theta_S(x) = \Theta^{*n}(\mu_S, x)$)

$$\phi_S^0(x) = \theta_S(x)\ ,\ \phi_S^j(x) = \theta_S(x)\xi_S^j(x)\ ,\ j = 1, \ldots, N\ ,$$

where $\xi_S^j(x)$ is the j^{th} component of the orientation $\vec{S}(x)$ relative to the usual orthonormal basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $1 \le i_1 < i_2 < \ldots < i_n \le n+k$ for $\Lambda_n(\mathbb{R}^{n+k})$ (ordered in any convenient manner). Evidently, for any $x \in \mathbb{R}^{n+k}$,

$$\phi_S(x+\lambda y) = \phi_{\eta_{x,\lambda\#}S}(y)\ ,\ y \in \mathbb{R}^{n+k}\ ,$$

and, given a sequence $\{S_i\} \subset \mathcal{D}_n(I+R^{n+k})$ of such integer multiplicity currents, we trivially have

$$\phi_{S_i}^j\, d\mathcal{H}^n \to \phi_S^j\, d\mathcal{H}^n\ \ \forall\, j \in \{1, \ldots, N\} \Leftrightarrow S_i \rightharpoonup S$$

---

* Cf. Almgren [A2]

and

$$\phi^0_{S_i}\, dH^n \to \phi^0_S\, dH^n \Leftrightarrow \mu_{S_i} \to \mu_S$$

(where $\psi_i dH^n \to \psi dH^n$ means $\int f\psi_i dH^n \to \int f\psi dH^n\ \forall\ f \in C_c(\mathbb{R}^{n+k})$) .

We shall also need the following simple lemma, the proof of which is left to the reader.

35.5 LEMMA *Suppose* S *is minimizing in* $\mathbb{R}^{n+k}$, $\partial S = 0$, *and*

$$\eta_{x,1\#}S = S \qquad \forall\ x \in \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{n+k}$$

for some positive integer $m < n$. (*Recall* $\eta_{x,1} : y \mapsto y-x\ ,\ y \in \mathbb{R}^{n+k}$.) *Then*

$$S = [\![\mathbb{R}^m]\!] \times S_0 \ ,$$

where $\partial S_0 = 0$ *and* S_0 *is minimizing in* $\mathbb{R}^{n+k-m}$.

Furthermore if S *is a cone* (i.e. $\eta_{0,\lambda\#}S = S$ *for each* $\lambda > 0$) , *then so is* S_0 .

Proof of 35.3(1) For each positive integer m and $\beta \in (0,\frac{1}{2})$ let

$$U_{m,\beta} = \{x \in U : \Theta^n(\mu_{T,x}) < m-\beta\} \ .$$

Now T is minimizing in $U \cap N$, so by the monotonicity formula of 17.6 (which can be applied by virtue of 33.2) we have, firstly, that $U_{m,\beta}$ is open, and secondly that for each $x \in U_{m,\beta}$, there is some ball $B_{2\rho}(x) \subset U_{m,\beta}$ such that

$$(1) \qquad \frac{\mu_T(B_\sigma(y))}{\omega_n \sigma^n} \le m-\beta/2 \qquad \forall\ \sigma < \rho\ ,\quad y \in B_\rho(x) \ .$$

We ultimately want to prove

$$\mathcal{H}^{n-3+\alpha}\left(\bigcup_{m=1}^{\infty}\{x\in U_{m,\beta} : m-1+\beta < \Theta^n(\mu_T,x) < m-\beta\}\right) = 0$$

for each sufficiently small $\alpha, \beta > 0$, and, in view of (1), by a rescaling and translation it will evidently suffice to assume

(2) $$B_2(0) = U , \quad \frac{\mu_T(B_\sigma(y))}{\omega_n \sigma^n} \le m-\beta \quad \forall\, \sigma < 1 ,\ y \in B_1(0) ,$$

and then prove

(3) $$\mathcal{H}^{n-3+\alpha}\{x\in B_1(0) : \Theta^n(\mu_T,x) \ge m-1+\beta\} = 0 .$$

We consider the set $\mathcal{T}$ of weak limit points of sequences $S_i = \eta_{x_i,\lambda_i\#}T$ where $|x_i| < 1-\lambda_i$, $0 < \lambda_i < 1$, with $\lim x_i \in \overline{B_1}(0)$ and $\lim \lambda_i = \lambda \ge 0$ both existing. For any such sequence S_i we have (by (2))

$$\limsup \underline{\underline{M}}_W(S_i) < \infty$$

for each $W \subset\subset \eta_{x,\lambda}(U)$ in case $\lambda > 0$, and for each $W \subset\subset \mathbb{R}^{n+k}$ in case $\lambda = 0$. Hence we can apply the compactness theorem 34.5 to conclude that each element S of $\mathcal{T}$ is integer multiplicity and

(4) S minimizes in $\eta_{x,\lambda}U \cap \eta_{x,\lambda}N$ in case $S = \lim \eta_{x_i,\lambda_i\#}T$

with $\lim x_i = x$ and $\lim \lambda_i = \lambda > 0$, and

(5) S minimizes in all of $\mathbb{R}^{n+k}$ in case $S = \lim \eta_{x_i,\lambda_i\#}T$

with $\lim x_i = x$ and $\lim \lambda_i = 0$. (Cf. the discussion in the proof of 35.1(2).)

For convenience we define

$$(6) \qquad U_S = \begin{cases} \eta_{x,\lambda} U & \text{in case } \lim \lambda_i > 0 \text{ (as in (4))} \\ \mathbb{R}^{n+k} & \text{in case } \lim \lambda_i = 0 \text{ (as in (5))} , \end{cases}$$

so that $S \in \mathcal{D}_n(U_S)$ for each $S \in \mathcal{T}$.

Now by definition one readily checks that

$$(7) \qquad \eta_{x,\lambda\#}\mathcal{T} = \mathcal{T} , \quad 0 < \lambda < 1 , \quad |x| < 1-\lambda ,$$

and, by (2),

$$(8) \qquad \Theta^n(\mu_S, y) \le m-\beta \qquad \forall\, y \in U_S , \; S \in \mathcal{T} .$$

Furthermore by using 34.5 together with the monotonicity formula 17.6, one readily checks that if $S_i \rightharpoonup S$ $(S_i, S \in \mathcal{T})$ and if $y, y_i \in B_1(0)$ with $\lim y_i = y$, then

$$(9) \qquad \Theta^n(\mu_S, y) \ge \limsup \Theta^n(\mu_{S_i}, y_i) .$$

It now follows from (7), (8), (9) and 34.5 that all the hypotheses of Theorem A.4 (of Appendix A) are satisfied with (using notation of Remark 35.4)

$$F = \{\phi_S : S \in \mathcal{T}\}$$

and with sing defined by

$$\operatorname{sing} \phi_S = \{x \in U_S : \Theta^n(\mu_S, \cdot) \ge m-1+\beta\}$$

for $S \in \mathcal{T}$. We claim that in this case the additional hypothesis is satisfied with $d = n-3$. Indeed suppose $d \ge n-2$; then there is $S \in \mathcal{T}$ and $\eta_{y,\lambda\#}S = S$ $\forall\, y \in L$, $\lambda > 0$ with L an $(n-2)$-dimensional subspace of $\mathbb{R}^{n+k}$, $L \subset \operatorname{sing} \phi_S$. Since we can make a rotation of $\mathbb{R}^{n+k}$ to bring L into coincidence with $\mathbb{R}^{n-2} \times \{0\}$, we assume that $L = \mathbb{R}^{n-2} \times \{0\}$. Then by Lemma 35.4 we have

$$S = [\![\mathbb{R}^{n-2}]\!] \times S_0 \ ,$$

where $S_0 \in \mathcal{D}_2(\mathbb{R}^N)$, $N = 2+k$, with S_0 a 2-dimensional area minimizing cone in $\mathbb{R}^N$. Then $\operatorname{spt} S_0$ is contained in a finite union $\bigcup_{i=1}^{q} P_i$ of 2-planes, with $P_i \cap P_j = \{0\}$ $\forall i \neq j$. (For a formal proof of this characterization of 2 dimensional area minimizing cones, see for example [WB].) In particular, since $\Theta^n(\mu_S,\cdot)$ is constant on $P_i \sim \{0\}$ (by the constancy theorem 26.27), we have that $\Theta^n(\mu_S,y) \in \mathbb{Z}$ for *every* $y \in \mathbb{R}^{n+k}$, and by (8) it follows that $\Theta^n(\mu_S,y) \le m-1$ $\forall\, y$ $\mathbb{R}^{n+k}$. That is, $\operatorname{sing} \phi_S = \emptyset$, a contradiction, hence we can take $d = n-3$ as claimed. We have thus established (3) as required.

Proof of 35.3(2) The proof goes similarly to 35.3(1). This time we assume (again without loss of generality) that

(1) $$U = B_2(0) \ ,$$

and we prove that T has a tangent plane at all points of $\operatorname{spt} T \cap B_1(0)$ except for a set $F \subset \operatorname{spt} T \cap B_1(0)$ with

(2) $$\mathcal{H}^{n-2+\alpha}(F) = 0 \qquad \forall\, \alpha > 0 \ .$$

$\mathcal{T}$ is as described in the proof of 35.3(1), and for any $S \in \mathcal{T}$ and $\beta > 0$ we let

$$\begin{aligned} R_\beta(S) = \{x \in \operatorname{spt} S :\ & \overline{B_\rho}(x) \subset U_S \ \text{ and} \\ & h(\operatorname{spt} S, L, \rho, x) < \beta\rho \ \text{ for some } \rho > 0 \\ & \text{and some } n\text{-dimensional subspace } L \text{ of } \mathbb{R}^{n+k}\} \ , \end{aligned}$$

where U_S is as in the proof of 35.3(1) (so that $S \in \mathcal{D}_n(U_S)$), and where we define

$$h(\operatorname{spt} S, L, \rho, x) = \sup_{y \in \operatorname{spt} S \cap B_\rho(x)} |q(y-x)| ,$$

with q the orthogonal projection of $\mathbb{R}^{n+k}$ onto $L^\perp$.

Now notice that (Cf. the proof of 35.3(1))

$$(2) \qquad \eta_{x,\lambda\#}\mathcal{T} = \mathcal{T} \qquad \forall\ 0<\lambda<1\ ,\ |x|<1-\lambda\ ,$$

and

$$(3) \qquad \eta_{x,\lambda}R_\beta(S) = R_\beta(\eta_{x,\lambda\#}S)\ ,\ S \in \mathcal{T}\ .$$

Furthermore if $S_j \to S$, $S_j, S \in \mathcal{T}$, then by the monotonicity formula 17.6 it is quite easy to check that if $y \in R_\beta(S)$ and if $y_j \in \operatorname{spt} S_j$ with $y_j \to y$, then $y_j \in R_\beta(S_j)$ for all sufficiently large j. Because of this, and because of (2), (3) above, it is now straightforward to check that the hypotheses of Theorem A.4 hold with (again in notation of Remark 35.4)

$$\mathcal{F} = \{\phi_S : S \in \mathcal{T}\}$$

and

$$\operatorname{sing} \phi_S = \operatorname{spt} \Theta^n(\mu_S, \cdot) \cap U_S \sim R_\beta(S)\ .$$

(Notice that $R_\beta(S)$ is completely determined by $\Theta^n(\mu_S,\cdot)$, and hence this makes sense.) In this case we claim that $d \le n-2$. Indeed if $d > n-2$ (i.e. $d = n-1$) then $\exists\ S \in \mathcal{T}$ such that

$$\eta_{x,\lambda\#}S = S \qquad \forall\ x \in L\ ,\ \lambda > 0\ ,\ \text{and}\ \ L \subset \operatorname{sing} \phi_S$$

where L is an $(n-1)$-dimensional subspace. Then, supposing without loss of generality that $L = \mathbb{R}^{n-1} \times \{0\}$, we have by Lemma 35.5 that

$$(3) \qquad S = [\![\mathbb{R}^{n-1}]\!] \times S_0\ ,$$

where S_0 is a 1-dimensional minimizing cone in $\mathbb{R}^{k+1}$. However it is easy to check that such a 1-dimensional minimizing cone necessarily has the form

$$S_0 = m[\![\ell]\!] \, ,$$

where $m \in \mathbb{Z}$ and ℓ is a 1-dimensional subspace of $\mathbb{R}^{k+1}$. Thus (3) gives that $S = m[\![L]\!]$ where L is an n-dimensional subspace and hence $\operatorname{sing} \phi_S = \emptyset$, a contradiction, so $d \le n-2$ as claimed.

We therefore conclude from Theorem A.4 that for each $S \in \mathcal{T}$

$$\mathcal{H}^{n-2+\alpha}(\operatorname{spt} S \sim R_\beta(S) \cap B_1(0)) = 0 \qquad \forall\, \alpha > 0 \, .$$

If $\beta_j \downarrow 0$ we thus conclude in particular that

$$\text{(4)} \qquad \mathcal{H}^{n-2+\alpha}(\operatorname{spt} T \sim \bigcap_{j=1}^{\infty} R_{\beta_j}(T) \cap B_1(0)) = 0 \qquad \forall\, \alpha > 0 \, .$$

However by (1) we see that

$$x \in \bigcap_{j=1}^{\infty} R_{\beta_j}(T) \Leftrightarrow T \text{ has a tangent plane at } x \, ,$$

and therefore (4) gives (2) as required.

§36. SOME REGULARITY RESULTS (Arbitrary Codimension)

In this section, for $T \in \mathcal{D}_n(U)$ any integer multiplicity current, we define a relatively closed subset $\operatorname{sing} T$ of U by

$$\text{36.1} \qquad \operatorname{sing} T = \operatorname{spt} T \sim \operatorname{reg} T \, ,$$

where $\operatorname{reg} T$ denotes the set of points $\xi \in \operatorname{spt} T$ such that for some $\rho > 0$ there is an $m \in \mathbb{Z}$ and an embedded n-dimensional oriented C^1 submanifold M of $\mathbb{R}^{n+k}$ with $T = m[\![M]\!]$ in $B_\rho(\xi)$.

Recently F.J. Almgren [A2] has proved the very important theorem that $\mathcal{H}^{n-2+\alpha}(\text{sing } T) = 0 \ \ \forall \alpha > 0$ in case $\text{spt } T \subset N$, $\partial T = 0$ and T is minimizing in N, where N is a smooth embedded $(n+k_1)$-dimensional submanifold of $\mathbb{R}^{n+k}$. The proof is very non-trivial and requires development of a whole new range of results for minimizing currents. We here restrict ourselves to more elementary results.

Firstly, the following theorem is an immediate consequence of Theorem 24.4 and Lemma 33.2.

36.2 THEOREM *Suppose* $T \in \mathcal{D}_n(U)$ *is integer multiplicity and minimizing in* $U \cap N$ *for some embedded* C^2 $(n+k_1)$*-dimensional submanifold* N *of* $\mathbb{R}^{n+k}$, $(\bar{N} \sim N) \cap U = \emptyset$, *and suppose* $\text{spt } T \subset U \cap N$, $\partial T = 0$ *(in* U*)*. *Then* $\text{reg } T$ *is dense in* $\text{spt } T$.

(Note that by definition $\text{reg } T$ is relatively open in $\text{spt } T$.)

The following is a useful fact; however its applicability is limited by the hypothesis that $\Theta^n(\mu_T, y) = 1$.

36.3 THEOREM *Suppose* $\{T_i\} \subset \mathcal{D}_n(U)$, $T \in \mathcal{D}_n(U)$ *are integer multiplicity currents with* T_i *minimizing in* $U \cap N_i$, T *minimizing in* $U \cap N$, N, N_i *embedded* $(n+k_1)$*-dimensional* C^2 *submanifolds, and* $\text{spt } T_i \subset N_i$, $\text{spt } T \subset N$, $\partial T_i = \partial T = 0$ *(in* U*)*. *Suppose also that* N_i *converges to* N *in the* C^2 *sense in* U, $T_j \rightharpoonup T$ *in* $\mathcal{D}_n(U)$, *and suppose* $y \in N \cap U$ *with* $\Theta^n(\mu_T, y) = 1$, $y = \lim y_j$, *where* y_j *is a sequence such that* $y_j \in \text{spt } T_j \ \ \forall j$. *Then* $y \in \text{reg } T$ *and* $y_j \in \text{reg } T_j$ *for all sufficiently large* j.

Proof By virtue of the monotonicity formula 17.6(1) (which is applicable by 33.2) it is easily checked that

$$\limsup \Theta^n(\mu_{T_j}, y_j) \le \Theta^n(\mu_T, y) = 1 ,$$

hence (since $\Theta^n(\mu_{T_j}, y_j) \ge 1$ by 17.8) we conclude $\Theta^n(\mu_{T_j}, y_j) \to \Theta^n(\mu_T, y) = 1$. Hence by Allard's theorem 24.2 we have $y \in \text{reg } T$ and $y_j \in \text{reg } T_j$ for all sufficiently large j. (33.2 justifies the use of 24.2.)

Next we have the following consequence of Theorem A.4 of Appendix A.

36.4 THEOREM *Suppose* T *is as in* 36.2, *and in addition suppose* $\xi \in \text{spt } T$ *is such that* $\Theta^n(\mu_T, \xi) < 2$. *Then there is a* $\rho > 0$ *such that*

$$\mathcal{H}^{n-2+\alpha}(\text{sing } T \cap B_\rho(\xi)) = 0 \qquad \forall\, \alpha > 0 .$$

Proof Let $\alpha = \frac{1}{2}(2-\Theta^n(\mu_T, \xi))$ and let $B_\rho(\xi)$ be such that $B_{2\rho}(\xi) \subset U$ and

$$\text{(1)} \qquad \frac{\mu_T(B_\sigma(\zeta))}{\omega_n \sigma^n} < 2(1-\alpha/2)$$

$\forall\, \zeta \in \text{spt } T \cap B_\rho(\xi)$, $0 < \sigma < \rho$. (Notice that such ρ exists by virtue of the monotonicity formula 17.6(1), which can be applied by 33.2.) Assume without loss of generality that $\xi = 0$, $\rho = 1$ and $U = B_2(0)$, and define $\mathcal{T}$ to be the set of weak limits S of sequences $\{S_i\}$ of the form $S_i = \eta_{x_i, \lambda_i \#} T$, $|x_i| < (1-\lambda_i)$, $0 < \lambda_i < 1$, where $\lim x_i$ and $\lim \lambda_i \equiv \lambda$ are assumed to exist. Notice that

$$\limsup \underline{\underline{M}}_W(S_i) < \infty$$

for each $W \subset\subset \eta_{x,\lambda}(U)$ in case $\lambda > 0$ and for each $W \subset\subset \mathbb{R}^{n+k}$ in case $\lambda = 0$. Hence by the compactness theorem 34.5 any such S is integer multiplicity in U_S

$$(U_S = \eta_{x,\lambda} U \text{ in case } \lambda > 0 ,\ U_S = \mathbb{R}^{n+k} \text{ in case } \lambda = 0)$$

and (Cf. the proof of 35.1(2))

(2) S minimizes in $\eta_{x,\lambda}U \cap \eta_{x,\lambda}N$ in case $\lambda > 0$

(3) S minimizes in $\mathbb{R}^{n+k}$ in case $\lambda = 0$.

One readily checks that, by definition of $\mathcal{T}$,

(4) $$\eta_{y,\tau\#}\mathcal{T} = \mathcal{T} \ , \ 0 < \tau < 1 \ , \ |y| < 1-\tau$$

Furthermore we note that (by (1))

(5) $$\Theta^n(\mu_S, x) = 1 \ , \ \mu_S\text{-a.e. } x \in U_S \ ,$$

and by Allard's theorem 24.2 there is $\delta > 0$ such that

(6) $$\text{sing } S = \{x \in U_S : \Theta^n(\mu_S, x) \geq 1+\delta\} \ , \ S \in \mathcal{T} \ .$$

Now in view of (2), (3), (4), (5), (6) and the upper semi-continuity of Θ^n as in (9) of the proof of 35.3(1), all the hypotheses of Theorem A.4 of Appendix A are satisfied with $\mathcal{F} = \{\phi_S : S \in \mathcal{T}\}$ (notation as in Remark 35.4) and with $\text{sing } \phi_S = \{x \in U_S : \Theta^n(\mu_S, x) \geq 1+\delta\}$ ($\equiv$ sing S by (6)). In fact we claim that in this case we may take $d = n-2$, because if $d = n-1$ $\exists$ $S \in \mathcal{T}$ and $\eta_{x,\lambda\#}S = S$ $\forall\, x \in L$, $\lambda > 0$, where $L \subset \text{sing } S$ is an $(n-1)$dimensional subspace of $\mathbb{R}^{n+k}$, then (Cf. the last part of the proof of 35.3(2)) we have $S = m[\![Q]\!]$ for some n-dimensional subspace Q . Hence sing $S = \emptyset$, a contradiction.

The following lemma is often useful:

36.5 THEOREM *Suppose* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *is minimizing in* $\mathbb{R}^{n+k}$, $\partial C = 0$, *and* C *is a cone:* $\eta_{0,\lambda\#}C = C$ $\forall\, \lambda > 0$. *Suppose further that* spt $C \subset \bar{H}$ *where* H *is an open* $\frac{1}{2}$*-space of* $\mathbb{R}^{n+k}$ *with* $0 \in \partial H$. *Then* spt $C \subset \partial H$.

36.6 REMARK The reader will see that the theorem here is actually valid with any stationary rectifiable varifold V in $\mathbb{R}^{n+k}$ satisfying $\eta_{0,\lambda\#}V = V$ in place of C .

Proof of 36.5 Since the varifold V associated with C is stationary (by 33.2) in $\mathbb{R}^{n+k}$ we have by 18.1 (since $(Dr)^{\perp} = 0$ by virtue of the fact that C is a cone),

$$(1) \qquad \frac{d}{d\rho}\left(\rho^{-n}\int_{\mathbb{R}^{n+k}} h\phi(r/\rho)\,d\mu_C\right) = \rho^{-n-1}\int_{\mathbb{R}^{n+k}} x\cdot(\nabla^C h)\,\phi(r/\rho)\,d\mu_C$$

for each $\rho > 0$, where $r = |x|$ and ϕ is a non-negative C^1 function on $\mathbb{R}$ with compact support, and h is an arbitrary $C^1(\mathbb{R}^{n+k})$ function. ($\nabla^C h(x)$ denotes the orthogonal projection of $\operatorname{grad}_{\mathbb{R}^{n+k}} h(x)$ onto the tangent space T_xV of V at x.)

Now suppose without loss of generality that $H = \{x = (x^1,\ldots,x^{n+k}) : x^1 > 0\}$ and select $h(x) \equiv x^1$. Then $x\cdot\nabla^C h = e_1^T\cdot x = e_1\cdot x^T = re_1\cdot\nabla^C r$, where v^T denotes orthogonal projection of v onto T_xV. Thus the term on the right side of (1) can be written $-\int_{\mathbb{R}^{n+k}}(e_1\cdot\nabla^C r)(r\phi(r/\rho))\,d\mu_C$, which in turn can be written $-\int_{\mathbb{R}^{n+k}} e_1\cdot\nabla^C\psi_\rho\,d\mu_C$, where $\psi_\rho(x) = \int_{|x|}^{\infty} r\phi(r/\rho)\,dr$. (Thus ψ_ρ has compact support in $\mathbb{R}^{n+k}$.) But $e_1\cdot\nabla^C\psi_\rho \equiv \operatorname{div}_V(\psi_\rho e_1)$, and hence the term on the right of (1) actually vanishes by virtue of the fact that V is stationary. Thus (1) gives

$$\rho^{-n}\int_{\mathbb{R}^{n+k}} x_1\phi(r/\rho)\,d\mu_C = \text{const.},\ 0 < \rho < \infty .$$

In view of the arbitrariness of ϕ, this implies

$$\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C \equiv \text{const.}$$

However trivially we have $\lim\limits_{\rho\downarrow 0}\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C = 0$, and hence we deduce

$$\rho^{-n}\int_{B_\rho(0)} x_1\,d\mu_C = 0 \qquad \forall\ \rho > 0 .$$

Thus since $x_1 \geq 0$ on spt C $(\subset \bar{H})$, we conclude spt $C \subset \partial H$ $(=\{x : x^1 = 0\})$.

The following corollary of 36.5 follows directly by combining 36.5 and 35.1(2).

36.6 COROLLARY *If* T *is as in* 36.2, *if* $\xi \in \operatorname{spt} T$, *if* Q *is a* C^1 *hypersurface in* $\mathbb{R}^{n+k}$ *such that* $\xi \in Q$ *and if* spt T *is locally on one side of* Q *near* ξ, *then all tangent cones* C *of* T *at* ξ *satisfy* spt $C \subset T_\xi Q \cap T_\xi N$.

§37. CODIMENSION 1 MINIMIZING CURRENTS

We begin by looking at those integer multiplicity currents $T \in \mathcal{D}_n(U)$ with spt $T \subset N \cap U$, N an $(n+1)$-dimensional oriented embedded submanifold of $\mathbb{R}^{n+k}$ with $(\bar{N} \sim N) \cap U = \emptyset$ and such that

$$(*) \qquad \partial T = [\![E]\!]$$

(in U), where E is an $\mathcal{H}^{n+1}$-measurable subset of N. (We know by 27.8, 33.4 that all minimizing currents $T \in \mathcal{D}_n(U)$ with $\partial T = 0$ and spt T in N can be locally decomposed into minimizing currents of this special form.)

37.1 REMARK The fact that T has the form $(*)$ and T is integer multiplicity evidently is equivalent to the requirement that if $V \subset U$ is open, and if ϕ is a C^2 diffeomorphism of V onto an open subset of $\mathbb{R}^{n+k}$ such that $\phi(V \cap N) = G$, G open in $\mathbb{R}^{n+1}$, then $\phi(E)$ has locally finite perimeter in G. This is an easy consequence of Remark 26.28, and in fact we see from this and Theorem 14.3 that any T of the form $(*)$ with $\underline{\underline{M}}_W(T) < \infty$ $\forall\, W \subset\subset U$ is automatically integer multiplicity with

(**) $$\Theta^n(T,x) = 1 \ , \ \mu_T\text{-a.e. } x \in U \ .$$

We shall here develop the theory of minimizing currents of the form (*) ; indeed we show this is naturally done using only the more elementary facts about currents. In particular we shall not in this section have any need for the compactness theorem 27.3 (instead we use only the elementary compactness theorem 6.3 for BV functions), nor shall we need the deformation theorem and the subsequent material of Chapter 6.

The following theorem could be derived from the general compactness theorem 34.5, but here (as we mentioned above) we can give a more elementary treatment. In this theorem, and subsequently, we take $U \subset \mathbb{R}^{n+k}$ to be open, and $\mathcal{O}$ will denote the collection of $(n+1)$-dimensional oriented embedded C^2 submanifolds N of $\mathbb{R}^{n+k}$ with $(\bar{N} \sim N) \cap U = \emptyset$, $N \cap U \neq \emptyset$. A sequence $\{N_j\} \subset \mathcal{O}$ is said to converge to $N \in \mathcal{O}$ in the C^2 sense in U if there are orientation preserving C^2 embeddings $\psi_j : N \cap U \to N_j$ with $\psi_j \to \mathbb{1}_{N \cap U}$ locally relative to the C^2 metric in $N \cap U$. In particular if $x \in N$ then $\eta_{x,\lambda} N$ converges to $T_x N$ in the C^2 sense in W as $\lambda \downarrow 0$, for each $W \subset\subset \mathbb{R}^{n+k}$.

In the following theorem p is a proper C^2 map $U \to N \cap U$ such that, in some neighbourhood $V \subset U$ of $N \cap U$, p coincides with the nearest point projection of V onto N . (Since the nearest point projection is C^2 in some neighbourhood of $N \cap U$ it is clear that such p exists.)

37.2 THEOREM (Compactness theorem for minimizing T as in (*))

Suppose $T_j \in \mathcal{D}_n(U)$, $T_j = \partial[\![E_j]\!]$ *(in* U*)* , E_j $\mathcal{H}^{n+1}$*-measurable subsets of* $N_j \cap U$, $N_j \in \mathcal{O}$, $N_j \to N \in \mathcal{O}$ *in the* C^2 *sense described above, and suppose* T_j *is integer multiplicity and minimizing in* $U \cap N_j$.

Then there is a subsequence $\{T_{j'}\}$ *with* $T_{j'} \rightharpoonup T$ *in* $\mathcal{D}_n(U)$, T *integer multiplicity,* $T = \partial[\![E]\!]$ *(in* U*)* , $\chi_{p(E_{j'})} \to \chi_E$ *in* $L^1_{loc}(\mathcal{H}^{n+1}, U)$, $\mu_{T_{j'}} \to \mu_T$ *(in the usual sense of Radon measures) in* U *, and* T *is minimizing in* $N \cap U$.

37.3 REMARKS

(1) Recall (from Remark 37.1) that the hypothesis that T_j is integer multiplicity is automatic if we assume merely that $\underline{\underline{M}}_W(T_j) < \infty \quad \forall\, W \subset\subset U$.

(2) We make no *a-priori* assumptions on local boundedness of the mass of the T_j (we see in the proof that this is automatic for minimizing currents as in (*)).

(3) Let $h(x,t) = x + t(p(x) - x)$, $x \in U$, $0 \le t \le 1$. Using the homotopy formula 26.22 (and in particular the inequality 26.23) together with the fact that $N_j \to N$ in the C^2 sense in U , it is straightforward to check that

$$T_j - T = \partial R_j \ , \ R_j = h_\#([\![(0,1)]\!] \times T_j) + p_\#[\![E_j]\!] - [\![E]\!]$$

with

$$\underline{\underline{M}}_W(R_{j'}) \to 0 \qquad \forall\, W \subset\subset U \ ,$$

provided that $\chi_{p(E_{j'})} \to \chi_E$ as claimed in the theorem. Thus once we establish $\chi_{p(E_{j'})} \to \chi_E$ for some E , then we can use the argument of 34.5 (with $S_j = 0$) in order to conclude

(1) T is minimizing in U

(2) $\mu_{T_{j'}} \to \mu_T$ in U .

(Notice we have not had to use the deformation theorem here.)

In the following proof we therefore concentrate on proving $\chi_{P(E_{j'})} \to \chi_E$ in $L^1_{loc}(H^{n+1}, N \cap U)$ for some subsequence $\{j'\}$ and some E such that $\partial[\![E]\!]$ has locally finite mass in U. (T is then automatically integer multiplicity by Remark 37.1.)

Proof of 37.2 We first establish a local mass bound for the T_j in U: if $\xi \in N$ and $B_{\rho_0}(\xi) \subset U$, then

$$(1) \qquad \underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \frac{1}{2} H^n(\partial B_\rho(\xi) \cap N) \ , \ L^1 \text{ a.e. } \rho \in (0,\rho_0) \ .$$

This is proved by simple area comparison as follows:

With $r(x) = |x-\xi|$, by the elementary slicing theory of 28.5(1), (2) we have that, for L^1-a.e. $\rho \in (0,\rho_0)$, the slice $\langle [\![E_j]\!], r, \rho \rangle$ (i.e. the slice of $[\![E_j]\!]$ by $\partial B_\rho(\xi)$) is integer multiplicity, and (using $T_j = \partial[\![E_j]\!]$),

$$\partial[\![E_j \cap B_\rho(\xi)]\!] = T_j \llcorner B_\rho(\xi) + \langle [\![E_j]\!], r, \rho \rangle \ .$$

Hence (applying ∂ to this identity)

$$\partial(T_j \llcorner B_\rho(\xi)) = -\partial\langle [\![E_j]\!], r, \rho \rangle \ , \ L^1\text{-a.e. } \rho \in (0,\rho_0) \ .$$

But by definition 33.1 of minimizing we then have

$$\underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \underline{\underline{M}}\langle [\![E_j]\!], r, \rho \rangle \ , \ L^1\text{-a.e. } \rho \in (0,\rho_0) \ .$$

Similarly, since $-T_j$ is also minimizing in $N \cap U$,

$$\underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \underline{\underline{M}}\langle [\![\tilde{E}_j]\!], r, \rho \rangle \ , \ L^1\text{-a.e. } \rho \in (0,\rho_0) \ ,$$

where $\tilde{E}_j = N \cap U \sim E_j$. Thus

$$(2) \qquad \underline{\underline{M}}(T_j \llcorner B_\rho(\xi)) \le \min\{\underline{\underline{M}}\langle [\![E_j]\!], r, \rho \rangle \ , \ \underline{\underline{M}}\langle [\![\tilde{E}_j]\!], r, \rho \rangle\}$$

for L^1-a.e. $\rho \in (0,\rho_0)$. Now of course $[\![\tilde{E}_j]\!] + [\![E_j]\!] = [\![N \cap U]\!]$, so that (for a.e. $\rho \in (0,\rho_0)$)

$$<[\![E_j]\!],r,\rho> + <[\![\tilde{E}_j]\!],r,\rho> = <N,r,\rho>$$

and hence (2) gives (1) as required (because $\underline{\underline{M}}(<N,r,\rho>) \le H^n(N \cap \partial B_\rho(\xi))$ by virtue of the fact that $|Dr| = 1$, hence $|\nabla^N r| \le 1$).

Now by virtue of (1) and Remark 37.1 we deduce from the BV compactness theorem 6.3 that some subsequence $\{\chi_{p(E_{j'})}\}$ of $\{\chi_{p(E_j)}\}$ converges in $L^1_{loc}(H^{n+1}, N \cap U)$ to χ_E, where $E \subset N$ is H^{n+1}-measurable and such that $\partial[\![E]\!]$ is integer multiplicity (in U). The remainder of the theorem now follows as described in Remark 37.3(3).

37.4 THEOREM (Existence of tangent cones)

Suppose $T = \partial[\![E]\!] \in \mathcal{D}_n(U)$ *is integer multiplicity, with* $E \subset N \cap U$, $N \in \mathcal{O}$, *and* T *is minimizing in* $U \cap N$. *Then for each* $x \in \operatorname{spt} T$ *and each sequence* $\{\lambda_j\} \downarrow 0$ *there is a subsequence* $\{\lambda_{j'}\}$ *and an integer multiplicity* $C \in \mathcal{D}_n(\mathbb{R}^{n+k})$ *with* C *minimizing in* $\mathbb{R}^{n+k}$, $0 \in \operatorname{spt} C \subset T_x N$, $\Theta^n(\mu_C, 0) = \Theta^n(\mu_T, x)$, $C = \partial[\![F]\!]$, F *an* H^{n+1}*-measurable subset of* $T_x N$,

(1) $\mu_{\eta_{x,\lambda_{j'}\#}T} \to \mu_C$ *in* $\mathbb{R}^{n+k}$, $\chi_{p(\eta_{x,\lambda_{j'}}(E))} \to \chi_F$ *in* $L^1_{loc}(H^{n+1}, T_x N)$,

where p *is the orthogonal projection of* $\mathbb{R}^{n+k}$ *onto* $T_x N$, *and*

(2) $\qquad \eta_{0,\lambda\#}C = C$, $\eta_{0,\lambda}F = F \qquad \forall\, \lambda > 0$.

37.5 REMARK The proof given here is independent of the general tangent cone existence theorem 35.1.

Proof of Theorem 37.4 As we remarked prior to Theorem 37.2, $\eta_{x,\lambda_j}N$ converges to $T_x N$ in the C^2 sense in W for each $W \subset\subset \mathbb{R}^{n+k}$. By the compactness theorem 37.2 we then have a subsequence λ_{j}, such that all the required conclusions, except possibly for 37.4(2) and the fact that $0 \in \operatorname{spt} C$, hold. To check that $0 \in \operatorname{spt} C$ and that 37.4(2) is valid, we first note by 33.2 that the varifold V associated with T is stationary in $N \cap U$ (and that V therefore has locally bounded generalized mean curvature $\underline{\underline{H}}$ in $N \cap U$). Therefore by the monotonicity formula 17.6(1), and by 17.8, we have

$$(1) \qquad \Theta^{n}(\mu_{V}, x) \text{ exists and is } \geq 1 .$$

Since $\mu_{\eta_{x,\lambda_{j}}\#T} \to \mu_{C}$, we then have $\Theta^{n}(\mu_{C}, 0) = \Theta^{n}(\mu_{T}, x) \geq 1$, so $0 \in \operatorname{spt} C$, and by 19.3 we deduce that the varifold V_{C} associated with C is a cone. Then in particular $x \wedge \vec{C}(x) = 0$ for μ_{C}-a.e. $x \in \mathbb{R}^{n+k}$ and hence, if we let h be the homotopy $h(t,x) = tx + (1-t)\lambda x$, we have $h_{\#}([\![(0,1)]\!] \times C) = 0$, and then by the homotopy formula 26.22 (since $\partial C = 0$) we have $\eta_{0,\lambda\#}C = C$ as required. Finally since $\operatorname{spt} C$ has locally finite $\mathcal{H}^{n}$-measure (indeed by 17.8 $\operatorname{spt} C$ is the closed set $\{y \in \mathbb{R}^{n+k} : \Theta^{n}(\mu_{C}, y) \geq 1\}$), we have

$$[\![F]\!] = [\![\tilde{F}]\!] ,$$

where $\tilde{F}$ is the (open) set $\{y \in T_{x}N \sim \operatorname{spt} C : \Theta^{n+1}(\mathcal{H}^{n+1}, T_{x}N, y) = 1\}$. Evidently $\eta_{0,\lambda}(\tilde{F}) = \tilde{F}$ (because $\eta_{0,\lambda}(\operatorname{spt} C) = \operatorname{spt} C$). Hence the required result is established with $\tilde{F}$ in place of F.

37.6 COROLLARY* *Suppose* T *is as in* 37.4 *and in addition suppose there is an* n*-dimensional submanifold* Σ *embedded in* $\mathbb{R}^{n+k}$ *with* $x \in \Sigma \subset N \cap U$ *for some* $x \in \operatorname{spt} T$, *and suppose* $\operatorname{spt} T \sim \Sigma$ *lies locally, near* x, *on one side of* Σ. *Then* $x \in \operatorname{reg} T$. (reg T is as in 36.1.)

Proof Let $C = \partial [\![F]\!]$ $(F \subset T_{x}N)$ be any tangent cone for T at x. By assumption, $\operatorname{spt}[\![F]\!] \subset \bar{H}$, where H is an open $\frac{1}{2}$-space in $T_{x}N$ with $0 \in \partial H$. Then, by 36.5, $\operatorname{spt} C \subset \partial H$ and hence by the constancy theorem 26.27,

---

* Cf. Miranda [MM1]

since C is integer multiplicity rectifiable, it follows that $C = \pm\partial[\![H]\!]$. However $\operatorname{spt}[\![F]\!] \subset \bar{H}$, hence $C = +\partial[\![H]\!]$. Then $\Theta^n(\mu_C, y) \equiv 1$ for $y \in \partial H$, and in particular $\Theta^n(\mu_C, 0)$ $(= \Theta^n(\mu_T, x)) = 1$, so that $x \in \operatorname{reg} T$ (by Allard's theorem 24.2) as required.

We next want to prove the main regularity theorem for codimension 1 currents. We continue to define $\operatorname{sing} T$, $\operatorname{reg} T$ as in 36.1.

37.7 THEOREM *Suppose* $T = \partial[\![E]\!] \in \mathcal{D}_n(U)$ *is integer multiplicity, with* $E \subset N \cap U$, $N \in \mathcal{O}$, *and* T *minimizing in* $N \cap U$. *Then* $\operatorname{sing} T = \emptyset$ *for* $n \le 6$, $\operatorname{sing} T$ *is locally finite in* U *for* $n = 7$, *and* $\mathcal{H}^{n-7+\alpha}(\operatorname{sing} T) = 0$ $\forall\, \alpha > 0$ *in case* $n > 7$.

Proof We are going to use the abstract dimension reducing argument of Appendix A (Cf. the proof of Theorem 36.4).

To begin we note that it is enough (by re-scaling, translation, and restriction) to assume that

$$U = B_2(0) \tag{1}$$

and to prove that

$$\begin{cases} \operatorname{sing} T \cap B_1(0) = \emptyset \quad \text{if} \quad n \le 6\,, \ \operatorname{sing} T \cap B_1(0) \text{ discrete if } n = 7\,, \\ \mathcal{H}^{n-7+\alpha}(\operatorname{sing} T \cap B_1(0)) = 0 \qquad \forall\, \alpha > 0 \quad \text{if} \quad n > 7\,. \end{cases} \tag{2}$$

Let $\mathcal{T}$ be the set of currents as defined in the proof of 36.4,* and for each $S \in \mathcal{T}$ let ϕ_S be the function : $\mathbb{R}^{n+k} \to \mathbb{R}^{n+1}$ associated with S as in Remark 35.4. Also, let

$$\mathcal{F} = \{\phi_S : S \in \mathcal{T}\}$$

and define

---

* We still have $\Theta^n(\mu_S, x) = 1$ for μ_S-a.e. $x \in U_S$, this time by 37.2 and 37.1 (**).

$$\operatorname{sing} \phi_S = \operatorname{sing} S .$$

(sing S as defined in 36.1.)

By Theorem A.4 we then have either $\operatorname{sing} S = \emptyset$ for all $S \in \mathcal{T}$ (and hence $\operatorname{sing} T = \emptyset$) or

(3) $$\dim B_1(0) \cap \operatorname{sing} S \le d ,$$

where $d \in [0,n-1]$ is the integer such that

$$\dim B_1(0) \cap \operatorname{sing} S \le d \quad \text{for all} \quad S \in \mathcal{T}$$

and such that there is $S \in \mathcal{T}$ and a d-dimensional subspace L of $\mathbb{R}^{n+k}$ such that

$$\eta_{x,\lambda\#} S = S \qquad \forall\, x \in L ,\ \lambda > 0$$

and

(4) $$\operatorname{sing} S = L .$$

Supposing without loss of generality that $L = \mathbb{R}^d \times \{0\}$, we then (by Lemma 35.5) have

(5) $$S = [\![\mathbb{R}^d]\!] \times S_0$$

where $\partial S_0 = 0$, S_0 is minimizing in $\mathbb{R}^{n+k-1}$, and $\operatorname{sing} S_0 = \{0\}$. (With S as in (5), $\operatorname{sing} S_0 = \{0\} \Leftrightarrow$ (4).) Also, by definition of $\mathcal{T}$, spt $S \subset$ some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+k}$, hence without loss of generality we have that S_0 is an $(n-d)$-dimensional minimizing cone in $\mathbb{R}^{n-d+1}$ with $\operatorname{sing} S_0 = \{0\}$. Then by the result of J.Simons (see Appendix B) we have $n-d > 6$; i.e. $d \le n-7$. Notice that this contradicts $d \ge 0$ in case $n < 7$.

Thus for $n < 7$ we must have $\operatorname{sing} T = \emptyset$ as required. If $n = 7$, $\operatorname{sing} T$ is discrete by the last part of Theorem A.4.

37.8 COROLLARY *If* T *is as in* 37.7, *and if* $T_1 \in \mathcal{D}_n(U)$ *is obtained by equipping a component of* $\operatorname{reg} T$ *with multiplicity* 1 *and with the orientation of* T, *then* $\partial T_1 = 0$ (*in* U) *and* T_1 *is minimizing in* $U \cap N$.

37.9 REMARK Notice that this means we can write

$$(*)\qquad T = \sum_{j=1}^{\infty} T_j ,$$

where each T_j is obtained by equipping a component M_j of $\operatorname{reg} T$ with multiplicity 1 and with the orientation of T; then $M_i \cap M_j = \emptyset$ $\forall i \neq j$, $\partial T_j = 0$, and T_j is minimizing in U $\forall j$. Furthermore (since $\mu_{T_j}(B_\rho(x)) \geq c\rho^n$ for $B_\rho(x) \subset U$ and $x \in \operatorname{spt} T_j$ by virtue of 33.2 and the monotonicity formula 17.6(1)) only finitely many T_j can have support intersecting a given compact subset of U.

Proof of 37.8 The main point is to prove

$$(1)\qquad \partial T_1 = 0 \text{ in } U .$$

The fact that T_1 is minimizing in U will then follow from 33.4 and the fact that $\underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T-T_1) = \underline{\underline{M}}_W(T)$ $\forall\, W \subset\subset U$.

To check (1) let $\omega \in \mathcal{D}^{n-1}(U)$ be arbitrary and note that if $\zeta \equiv 0$ in some neighbourhood of $\operatorname{spt} T \sim M_1$

$$(2)\qquad T_1(d(\zeta\omega)) = T(d(\zeta\omega)) = \partial T(\zeta\omega) = 0 .$$

Now corresponding to any $\varepsilon > 0$ we construct ζ as follows: since $\mathcal{H}^{n-1}(\operatorname{sing} T) = 0$ (by 37.7) and since $\operatorname{sing} T \cap \operatorname{spt} \omega$ is compact, we can find a finite collection $\{B_{\rho_j}(\xi_j)\}_{j=1,\ldots,P}$ of balls with $\xi_j \in \operatorname{sing} T \cap \operatorname{spt} \omega$

and $\sum_{j=1}^{P} \rho_j^{n-1} < \varepsilon$. For each $j=1,\ldots,P$ let $\phi_j \in C_c^\infty(\mathbb{R}^{n+k})$ be such that $\phi_j \equiv 1$ on $\bar{B}_{\rho_j}(\xi_j)$, $\phi_j = 0$ on $\mathbb{R}^{n+k} \sim B_{2\rho_j}(\xi_j)$, and $0 \le \phi_j \le 1$ everywhere. Now choose $\zeta = \prod_{j=1}^{P} \phi_j$ in a neighbourhood of spt T_1 and so that $\zeta \equiv 0$ in a neighbourhood of spt $T \sim$ spt T_1 . Then $d\zeta = \sum_{i=1}^{P} \prod_{j\neq i}^{P} \phi_j d\phi_i$ on spt T_1 , and hence

$$|d(\zeta\omega) - \zeta d\omega| \le c|\omega| \sum_{j=1}^{P} \rho_j^{n-1} \le c\varepsilon|\omega| \quad \text{on spt } T_1 .$$

Then letting $\varepsilon \downarrow 0$ in (2), and noting that $\zeta d\omega \to d\omega$ $\mathcal{H}^n$ - a.e. in spt $T_1 \cap N \cap$ spt ω (and using $|\zeta| \le 1$) , we conclude $T_1(d\omega) = 0$. That is $\partial T_1 = 0$ in U as required.

Finally we have the following lemma.

37.10 LEMMA *If* $T_1 = \partial[\![E_1]\!]$, $T_2 = \partial[\![E_2]\!] \in \mathcal{D}_n(U)$, U *bounded,* $E_1, E_2 \subset U \cap N$, N *of class* C^4 , $N \in \mathcal{O}$, T_1 , T_2 *minimizing in* $U \cap N$, reg T_1 , reg T_2 *are connected, and* $E_1 \cap V \subset E_2 \cap V$ *for some neighbourhood* V *of* ∂U , *then* spt $[\![E_1]\!] \subset$ spt$[\![E_2]\!]$ *and either* $[\![E_1]\!] = [\![E_2]\!]$ *or* spt $T_1 \cap$ spt $T_2 \subset$ sing $T_1 \cap$ sing T_2 .

Proof Since $\mathcal{H}^{n+1}(\text{spt } T_j) = 0$ (in fact spt T_j has locally finite $\mathcal{H}^n$-measure in U by virtue of the fact that $\Theta^n(\mu_{T_j}, x) \ge 1$ $\forall\, x \in$ spt T_j) , we may assume that E_1 and E_2 are open with $U \cap \partial E_j = U \cap \partial \bar{E}_j =$ spt T_j , $j = 1,2$.

Let S_1 , $S_2 \in \mathcal{D}_n(U)$ be the currents defined by

$$S_1 = \partial[\![E_1 \cap E_2]\!] , \quad S_2 = \partial[\![E_1 \cup E_2]\!] .$$

Using the hypothesis concerning V we have

$$\text{(1)} \qquad S_j \llcorner (V \cap U) = T_j \llcorner (V \cap U) , \quad j = 1,2 .$$

On the other hand we trivially have

$$[\![E_1 \cap E_2]\!] + [\![E_1 \cup E_2]\!] = [\![E_1]\!] + [\![E_2]\!] ,$$

so (applying ∂) we get

(2) $$S_1 + S_2 = T_1 + T_2 .$$

Furthermore $E_1 \cap E_2 \subset E_1 \cup E_2$, so

(3) $$\begin{aligned} \underline{\underline{M}}_W(S_1) + \underline{\underline{M}}_W(S_2) &= \underline{\underline{M}}_W(S_1+S_2) \\ &= \underline{\underline{M}}_W(T_1+T_2) \quad \text{(by (2))} \\ &\le \underline{\underline{M}}_W(T_1) + \underline{\underline{M}}_W(T_2) \end{aligned}$$

$\forall\, W \subset\subset U$. On the other hand, choosing an open V_0 so that $\partial U \subset V_0 \subset\subset V$, and using (1) together with the fact that T_1 is minimizing, we have

$$\underline{\underline{M}}_W(S_1) \ge \underline{\underline{M}}_W(T_1) , \ W = U \sim \bar{V}_0 ,$$

and hence (combining this with (3))

$$\underline{\underline{M}}_W(S_2) \le \underline{\underline{M}}_W(T_2)$$

for $W = U \sim \bar{V}_0$. Thus (using (1) with $j=2$) S_2 is minimizing in U. Likewise S_1 is minimizing in U.

We next want to prove that either $T_1 = T_2$ or $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 = \emptyset$. Suppose $\operatorname{reg} T_1 \cap \operatorname{reg} T_2 \ne \emptyset$. If the tangent spaces of $\operatorname{reg} T_1$ and $\operatorname{reg} T_2$ coincide at every point of their intersection, then using suitable local coordinates $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ for N near a point $\xi \in \operatorname{reg} T_1 \cap \operatorname{reg} T_2$, we can write

$$\operatorname{reg} T_j = \operatorname{graph} u_j , \ j = 1,2 ,$$

where $Du_1 = Du_2$ at each point where $u_1 = u_2$, and where both u_1, u_2 are (weak) C^1 solutions of the equation

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x,u,Du)\right) - \frac{\partial F}{\partial z}(x,u,Du) = 0 ,$$

where $F = F(x,z,p)$, $(x,z,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, is the area functional for graphs $z = u(x)$ relative to the local coordinates x, z for N. Since N is C^4 we then deduce (from standard quasilinear elliptic theory - see e.g. [GT]) that u_1, u_2 are $C^{3,\alpha}$. Now the difference $u_1 - u_2$ of the solutions evidently satisfies an equation of the general form

$$D_j(a_{ij}D_i u) + b_i D_i u + cu = 0 ,$$

where a_{ij}, b_i, c are $C^{2,\alpha}$. By standard unique continuation results (see e.g. [PM]) we then see that $Du_1 = Du_2$ at each point where $u_1 = u_2$ is impossible if $u_1 - u_2$ changes sign. On the other hand the Harnack inequality ([GT]) tells us that either $u_1 \equiv u_2$ or $|u_1 - u_2| > 0$ in case $u_1 - u_2$ does not change sign. Thus we deduce that either $T_1 = T_2$ or $\text{reg } T_1 \cap \text{reg } T_2 = \emptyset$ or there is a point $\xi \in \text{reg } T_1 \cap \text{reg } T_2$ such that $\text{reg } T_1$ and $\text{reg } T_2$ intersect *transversely* at ξ. But then we would have $\mathcal{H}^{n-1}(\text{sing } \partial[\![E_1 \cap E_2]\!]) > 0$, which by virtue of 37.7 contradicts the fact (established above) that $\partial[\![E_1 \cap E_2]\!]$ is minimizing in U.

Thus either $T_1 = T_2$ or $\text{reg } T_1 \cap \text{reg } T_2 = \emptyset$, and it follows in either case that $E_1 \subset E_2$. On the other hand we then have $\text{sing } T_1 \cap \text{reg } T_2 = \emptyset$ and $\text{sing } T_2 \cap \text{reg } T_1 = \emptyset$ by virtue of Corollary 37.6. Thus we conclude that $E_1 \subset E_2$ and $\text{spt } T_1 \cap \text{spt } T_2 \subset \text{sing } T_1 \cap \text{sing } T_2$ as required.

CHAPTER 8

THEORY OF GENERAL VARIFOLDS

Here we describe the theory of general varifolds, essentially following W.K. Allard [AW1].

General varifolds in U (U open in $\mathbb{R}^{n+k}$) are simply Radon measures on $G_n(U) = \{(x,S) : x \in U$ and S is an n-dimensional subspace of $\mathbb{R}^{n+k}\}$. One basic motivating point for our interest in such objects is described as follows:

Suppose $\{T_j\}$ is a sequence of integer multiplicity currents (see §27) such that the corresponding integer multiplicity varifolds (as in Chapter 4) are stationary in U (U open in $\mathbb{R}^{n+k}$), and suppose $\partial T_j = 0$ and there is a mass bound $\sup_{j\geq 1} \underline{M}_W(T_j) < \infty \;\; \forall\, W \subset\subset U$. By the compactness theorem 27.3 we can assert that $T_{j'} \rightharpoonup T$ for some integer multiplicity T. However it is *not* clear that T is stationary; the chief difficulty is that it is *not* generally true that the corresponding sequence of measures $\mu_{T_{j'}}$ converge to μ_T. Indeed if $\mu_{T_{j'}}$ converges to μ_T (as they would by 34.5 in case the T_j are minimizing in U) then it is not hard to prove that T is stationary in U. This leads one to consider measure theoretic convergence rather than weak convergence of the currents. However if we take a limit (in the sense of Radon measures) of some sub-sequence $\{\mu_{T_{j'}}\}$ of the $\{\mu_{T_j}\}$ then we get merely an abstract Radon measure on U, and first variation of this does not make sense.

To resolve these difficulties, we associate with each T_j a Radon measure V_j on the Grassmaniann $G_n(U)$ ($G_n(U)$ is naturally equipped with a suitable metric - see below); V_j is in fact defined by

$$V_j(A) = \mu_{T_j}(\pi_j(A)) ,$$

where $\pi_j(A)$ denotes $\{x \in U : (x, \langle \vec{T}_j(x) \rangle) \in A\}$ for any subset A of $G_n(U)$. ($\langle \vec{T}_j(x) \rangle$ denotes the n-dimensional subspace determined by $\vec{T}_j(x)$.) One then uses the compactness theorem 4.4 to give $V_{j'} \to V$ for some subsequence $\{j'\}$ and some Radon measure V on $G_n(U)$. It turns out to be possible to define a notion of *stationarity* for such Radon measures (i.e. varifolds) V on $G_n(U)$ and, for example, in the circumstances above V turns out to correspond to a stationary rectifiable varifold (in the sense of Chapter 4). The reader will see that these claims follow easily from the rectifiability and compactness theorems of §42.

§38. BASICS, FIRST RECTIFIABILITY THEOREM

We let $G(n+k,n)$ denote the collection of all n-dimensional subspaces of $\mathbb{R}^{n+k}$, equipped with the metric $\rho(S,T) = |p_S - p_T| = \left(\sum_{i,j=1}^{n+k} (p_S^{ij} - p_T^{ij})^2\right)^{\frac{1}{2}}$, where p_S, p_T denote the orthogonal projections of $\mathbb{R}^{n+k}$ onto S, T respectively, and $p_S^{ij} = e_i \cdot p_S(e_j)$, $p_T^{ij} = e_i \cdot p_T(e_j)$ are the corresponding matrices with respect to the standard orthonormal basis $e_1, \ldots, e_{n+k}$ for $\mathbb{R}^{n+k}$.

For a subset $A \subset \mathbb{R}^{n+k}$ we define

$$G_n(A) = A \times G(n+k,n) ,$$

equipped with the product metric. Of course then $G_n(K)$ is compact for each compact $K \subset \mathbb{R}^{n+k}$. $G_n(\mathbb{R}^{n+k})$ is locally homeomorphic to a Euclidean space of dimension $n+k + nk$.

By an n-*varifold* we mean simply any Radon measure V on $G_n(\mathbb{R}^{n+k})$. By an n-varifold on U (U open in $\mathbb{R}^{n+k}$) we mean any Radon measure V on $G_n(U)$. Given such an n-varifold V on U, there corresponds a Radon measure $\mu = \mu_V$ on U (called the *weight* of V) defined by

$$\mu(A) = V(\pi^{-1}(A)) \ , \ A \subset U \ ,$$

where, here and subsequently, π is the projection $(x,S) \mapsto x$ of $G_n(U)$ onto U. The *mass* $\underline{\underline{M}}(V)$ of V is defined by

$$\underline{\underline{M}}(V) = \mu_V(U) \qquad (= V(G_n(U))) \ .$$

For any Borel subset $A \subset U$ we use the usual terminology $V \llcorner G_n(A)$ to denote the restriction of V to $G_n(A)$; thus

$$(V \llcorner G_n(A))(B) = V(B \cap G_n(A)) \ , \ B \subset G_n(U) \ .$$

Given an n-rectifiable varifold $\underline{\underline{v}}(M,\theta)$ on U (in the sense of Chapter 4) there is a coresponding n-varifold V (also denoted by $\underline{\underline{v}}(M,\theta)$, or simply $\underline{\underline{v}}(M)$ in case $\theta \equiv 1$ on M), defined by

$$V(A) = \mu(\pi(TM \cap A)) \ , \ A \subset G_n(U) \ ,$$

where $\mu = \mathcal{H}^n \llcorner \theta$ and $TM = \{(x, T_xM) : x \in M_*\}$, with M_* the set of $x \in M$ such that M has an approximate tangent space T_xM with respect to θ at x in the sense of 11.4. Evidently V, so defined, has weight measure $\mu_V = \mathcal{H}^n \llcorner \theta = \mu$.

The question of when a general n-varifold actually corresponds to an n-rectifiable varifold in this way is satisfactorily answered in the next theorem. Before stating this we need a definition:

38.1 DEFINITION Given $T \in G(n+k,n)$, $x \in U$, and $\theta \in (0,\infty)$, we say that an n-varifold V on U has tangent space T with multiplicity θ at x if

$$(*) \qquad \lim_{\lambda \downarrow 0} V_{x,\lambda} = \theta \underline{\underline{v}}(T) ,$$

where the limit is in the usual sense of Radon measures on $G_n(\mathbb{R}^{n+k})$. In (*) we use the notation that $V_{x,\lambda}$ is the n-varifold defined by

$$V_{x,\lambda}(A) = \lambda^{-n} V(\{(\lambda y+x,S) : (y,S) \in A\} \cap G_n(U))$$

for $A \subset G_n(\mathbb{R}^{n+k})$.

38.2 REMARK Note that 38.1(*) implies that the weight measure μ_V has approximate tangent space T with multiplicity θ at x in the sense of 11.8.

38.3 THEOREM (First Rectifiability Theorem)

Suppose V *is an* n*-varifold on* U *which has a tangent space* T_x *with multiplicity* $\theta(x) \in (0,\infty)$ *for* μ_V - a.e. $x \in U$. *Then* V *is* n*-rectifiable: in fact* $M \equiv \{x \in U : T_x, \theta(x) \text{ exist}\}$ *is* H^n*-measurable, countably* n*-rectifiable,* θ *is locally* H^n*-integrable on* M, *and* $V = \underline{\underline{v}}(M,\theta)$.

In the proof of 38.3 (and also subsequently) we shall need the following technical lemma:

38.4 LEMMA *Let* V *be any* n*-varifold on* U. *Then for* μ_V - a.e. $x \in U$ *there is a Radon measure* $\eta_V^{(x)}$ *on* $G(n+k,n)$ *such that, for any continuous* β *on* $G(n+k,n)$,

$$\int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S) = \lim_{\rho \downarrow 0} \frac{\displaystyle\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} .$$

Furthermore for any Borel set $A \subset U$,

$$\int_{G_n(A)} \beta(S)\, dV(x,S) = \int_A \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S)\, d\mu_V(x)$$

provided $\beta \geq 0$.

Proof The proof is a simple consequence of the differentiation theory for Radon measures and the separability of $K(X,\mathbb{R})$ (notation as in §4) for compact separable metric spaces X . Specifically, write $K = K(G(n+k,n),\mathbb{R})$, $K^+ = \{\beta \in K : \beta \geq 0\}$, and let $\beta_1,\beta_2,\ldots \in K^+$ be dense in K^+ . By the differentiation theorem 4.7 we know that (since there is a Radon measure γ_j on $\mathbb{R}^{n+k}$ characterized by $\gamma_j(B) = \int_{G_n(B)} \beta_j(S)\, dV(y,S)$ for Borel sets $B \subset \mathbb{R}^{n+k}$)

(1) $$e(x,j) = \lim_{\rho \downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta_j(S)\, dV(y,S)}{\mu_V(B_\rho(x))}$$

exists for each $x \in \mathbb{R}^{n+k} \sim Z_j$, where Z_j is a Borel set with $\mu_V(Z_j) = 0$, and $e(x,j)$ is a μ_V-measurable function of x , with

(2) $$\int_A e(x,j)\, d\mu_V(x) = \int_{G_n(A)} \beta_j(S)\, dV(y,S)$$

for any Borel set $A \subset \mathbb{R}^{n+k}$.

Now let $\varepsilon > 0$, $\beta \in K^+$, $x \in \mathbb{R}^{n+k} \sim \left(\bigcup_{j=1}^{\infty} Z_j\right)$, and choose β_j such that $\sup|\beta-\beta_j| < \varepsilon$. Then for any $\rho > 0$

(3) $$\left| \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} - \frac{\int_{G_n(B_\rho(x))} \beta_j(S)\, dV(y,S)}{\mu_V(B_\rho(x))} \right|$$

$$\leq \varepsilon \frac{V(G_n(B_\rho(x)))}{\mu_V(B_\rho(x))} = \varepsilon ,$$

and hence by (1) we conclude that

$$\tilde{\eta}_V^{(x)}(\beta) \equiv \lim_{\rho\downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))}$$

exists for all $\beta \in K^+$ and all $x \in \mathbb{R}^{n+k} \sim \left(\bigcup_{j=1}^{\infty} Z_j\right)$. Of course, since $|\tilde{\eta}_V^{(x)}(\beta)| \le \sup|\beta| \quad \forall\, \beta \in K^+$, by the Riesz representation theorem 4.1 we have $\tilde{\eta}_V^{(x)}(\beta) = \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S)$, where $\eta_V^{(x)}$ is the total variation measure associated with $\tilde{\eta}_V^{(x)}$.

Finally the last part of the lemma follows directly from (2), (3) if we keep in mind that $e(x,j)$ in (1) is exactly $\tilde{\eta}_V^{(x)}(\beta_j) = \int_{G(n+k,n)} \beta_j(S)\, d\eta_V^{(x)}(S)$.

We are now able to give the proof of Theorem 38.3.

Proof of Theorem 38.3 As mentioned in Remark 38.2, μ_V has approximate tangent space T_x with multiplicity $\theta(x)$ in the sense of 11.8 for μ_V-a.e. $x \in U$. Hence by Theorem 11.8 we have that M is $\mathcal{H}^n$-measurable countably n-rectifiable, θ is locally $\mathcal{H}^n$-integrable on M and in fact $\mu_V = \mathcal{H}^n \llcorner \theta$ in U (if we set $\theta \equiv 0$ in $U \sim M$).

Now if $x \in M$ is one of the μ_V-almost all points such that $\eta_V^{(x)}$ exists, and if β is a non-negative continuous function on $G(n+k,n)$, then we evidently have $\eta_V^{(x)}(\beta) = \theta(x)\beta(T_x)$ and hence by the second part of 38.4 we have

$$\int_{G_n(A)} \beta(S)\, dV(x,S) = \int_{M\cap A} \beta(T_x)\, d\mu_V(x)$$

for any Borel set $A \subset U$. From the arbitrariness of A and β it then easily follows that

$$\int_{G_n(U)} f(x,S)\,dV(x,S) = \int_M f(x,T_x)\,d\mu_V(x)$$

for any non-negative $f \in C_c(G_n(U))$, and hence we have shown $V = \underline{\underline{v}}(M,\theta)$ as required (because $\mu_V = \mathcal{H}^n \llcorner \theta$ as mentioned above).

§39. FIRST VARIATION

We can make sense of first variation for a general varifold V on U . We first need to discuss *mapping* of such a general n-varifold. Suppose $U, \tilde{U}$ open $\subset \mathbb{R}^{n+k}$ and $f : U \to \tilde{U}$ is C^1 with $f|\operatorname{spt}\mu_V \cap U$ *proper*. Then we define the image varifold $f_\# V$ on $\tilde{U}$ by

39.1 $$f_\# V(A) = \int_{F^{-1}(A)} J_S f(x)\,dV(x,S) \ , \ A \text{ Borel, } A \subset G_n(\tilde{U}) \ ,$$

where $F : G_n^+(U) \to G_n(\tilde{U})$ is defined by $F(x,S) = (f(x), df_x(S))$ and where

$$J_S f(x) = (\det((df_x|S)^* \circ (df_x|S)))^{\frac{1}{2}} \ , \ (x,S) \in G_n(U) \ .$$

$$G_n^+(U) = \{(x,S) \in G_n(U) : J_S f(x) \neq 0\} \ .$$

(Notice that this agrees with our previous definition given in §15 in case $V = \underline{\underline{v}}(M,\theta)$.)

Now given any n-varifold V on U we define the *first variation* δV , which is a linear functional on $\mathcal{K}(U,\mathbb{R}^{n+k})$ (notation as in §4) by

$$\delta V(X) = \frac{d}{dt} \underline{\underline{M}}(\phi_{t\#} V \llcorner G_n(K))\Big|_{t=0} \ ,$$

where $\{\phi_t\}_{-1<t<1}$ is any 1-parameter family as in 9.1 (and K is as in 9.1(3)). Of course we can compute $\delta V(X)$ explicitly by differentiation under the integral in 39.1. This gives (by *exactly* the computations of §9)

39.2 $$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x,S) \ ,$$

where, for any $S \in G(n+k,n)$,

$$\operatorname{div}_S X = \sum_{i=1}^{n+k} \nabla_i^S X^i$$

$$= \sum_{i=1}^{n} \langle \tau_i, D_{\tau_i} X \rangle \ ,$$

where $\tau_1,\ldots,\tau_n$ is an orthonormal basis for S and $\nabla_i^S = e_i \cdot \nabla^S$, with $\nabla^S f(x) = p_S(\operatorname{grad}_{\mathbb{R}^{n+k}} f(x))$, $f \in C^1(U)$. (p_S is the orthogonal projection of $\mathbb{R}^{n+k}$ onto S .)

By analogy with 16.3 we then say that V is *stationary in* U if $\delta V(X) = 0 \quad \forall\ X \in K(U,\mathbb{R}^{n+k})$.

More generally V is said to have *locally bounded first variation in* U if for each $W \subset\subset U$ there is a constant $c < \infty$ such that $|\delta V(X)| \le c \sup_U |X| \quad \forall\ X \in K(U,\mathbb{R}^{n+k})$ with $\operatorname{spt}|X| \subset W$. Evidently, by the general Riesz representation theorem 4.1, this is equivalent to the requirement that there is a Radon measure $\|\delta V\|$ (the total variation measure of δV) on U characterized by

39.3 $$\|\delta V\|(W) = \sup_{\substack{X \in K(U,\mathbb{R}^{n+k}) \\ |X| \le 1,\ \operatorname{spt}|X| \subset W}} |\delta V(X)| \qquad (<\infty)$$

for any open $W \subset\subset U$. Notice that then by Theorem 4.1 we can write

$$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x,S) \equiv - \int_U \nu \cdot X d\|\delta V\| \ ,$$

where ν is $\|\delta V\|$-measurable with $|\nu| = 1$ $\|\delta V\|$ - a.e. in U . By the differentiation theory of 4.7 we know furthermore that

$$D_{\mu_V}\|\delta V\|(x) \equiv \lim_{\rho\downarrow 0} \frac{\|\delta V\|(B_\rho(x))}{\mu_V(B_\rho(x))}$$

exists μ_V - a.e. and that (writing $\underline{\underline{H}}(x) = D_{\mu_V}\|\delta V\|(x)\nu(x)$)

$$\int_U \nu\cdot X d\|\delta V\| = \int_U \underline{\underline{H}}\cdot X d\mu_V + \int_U \nu\cdot X d\sigma \ ,$$

with

$$\sigma = \|\delta V\| \llcorner Z \ , \ Z = \{x\in U : D_{\mu_V}\|\delta V\|(x) = +\infty\} \ . \ (\mu_V(Z) = 0\ .)$$

Thus we can write

39.4
$$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x,S)$$
$$= -\int_U \underline{\underline{H}}\cdot X d\mu_V - \int_Z \nu\cdot X d\sigma$$

for $X \in K(U,\mathbb{R}^{n+k})$.

By analogy with the classical identity 7.6 we call $\underline{\underline{H}}$ *the generalized mean curvature of* V *,* Z *the generalized boundary of* V *,* σ *the generalized boundary measure of* V *, and* $\nu|Z$ *the generalized unit co-normal of* V .

§40. MONOTONICITY AND CONSEQUENCES

In this section we assume that V is an n-varifold in U with locally bounded first variation in U (as in 39.3).

We first consider a point $x \in U$ such that there is $0 < \rho_0 < \operatorname{dist}(x,\partial U)$ and $\Lambda \ge 0$ with

40.1 $$\|\delta V\|(B_\rho(x)) \le \Lambda\mu_V(B_\rho(x)) \ , \ 0<\rho<\rho_0 \ .$$

Subject to 40.1 we can choose (in 39.2) $X_y = \gamma(r)(y-x)$, $r=|y-x|$, $y\in U$ as in §17 and note that (by essentially the same computation as in §17)

$$\operatorname{div}_S X = n\gamma(r) + r\gamma'(r)\sum_{i,j=1}^{n+k} e_S^{ij}\,\frac{x^i-y^i}{r}\,\frac{x^j-y^j}{r} \ ,$$

where (e_S^{ij}) is the matrix of the orthogonal projection p_S of $\mathbb{R}^{n+k}$ onto the n-dimensional subspace S . We can then take $\gamma(r) = \phi(r/\rho)$ (again as in §17) and, noting that $\sum_{i,j=1}^{n+k} e_S^{ij}\,\frac{x^i-y^i}{r}\,\frac{x^j-y^j}{r} = 1 - |p_{S^\perp}(\frac{y-x}{r})|^2$, conclude (Cf. 17.6(1) with $\alpha=1$) that $e^{\Lambda\rho}\rho^{-n}\mu_V(B_\rho(x))$ is increasing in ρ , $0<\rho<\rho_0$, and, for $0<\sigma\le\rho<\rho_0$,

40.2 $$\Theta^n(\mu_V,x) \le e^{\Lambda\sigma}\omega_n^{-1}\sigma^{-n}\mu_V(B_\sigma(x)) \le e^{\Lambda\rho}\omega_n^{-1}\rho^{-n}\mu_V(B_\rho(x)) - \omega_n^{-1}\int_{G_n(B_\rho(x)\sim B_\sigma(x))} r^{-n-2}|p_{S^\perp}(y-x)|^2 dV(y,S) \ .$$

In fact if $\Lambda = 0$ (so that V is stationary in $B_{\rho_0}(x)$) we get the precise identity

40.3 $$\Theta^n(\mu_V,x) = \omega_n^{-1}\rho^{-n}\mu_V(B_\rho(x)) - \omega_n^{-1}\int_{G_n(B_\rho(x))} r^{-n-2}|p_{S^\perp}(y-x)|^2 dV(y,S) \ ,$$

for $0<\rho<\rho_0$.

Using $X_y = h(y)\gamma(r)(y-x)$ $(r=|y-x|)$ in 39.2 we also deduce the following analogue of 18.1:

40.4 $$\frac{d}{d\rho}(\rho^{-n}\tilde{I}(\rho)) = \rho^{-n}\frac{d}{d\rho}\int |p_{S^\perp}(y-x)/r|^2\phi(r/\rho)h(y)\,dV(y,S) + \rho^{-n-1}\left(\delta V(X) + \int (y-x)\cdot\nabla^S h(y)\phi(r/\rho)\,dV(y,S)\right) ,$$

where $\tilde{I}(\rho) = \int \phi(r/\rho)h\,d\mu_V$.

40.5 LEMMA *Suppose* V *has locally bounded first variation in* U . *Then, for* μ_V - a.e. $x \in U$, $\Theta^n(\mu_V,x)$ *exists and is real-valued; in fact* $\Theta^n(\mu_V,x)$ *exists whenever there is a constant* $\Lambda(x) < \infty$ *such that*

$$(*) \qquad \|\delta V\|(B_\rho(x)) \le \Lambda(x)\mu_V(B_\rho(x)) \ , \ 0 < \rho < \tfrac{1}{2}\,\mathrm{dist}(x,\partial U) \ .$$

(Such a constant $\Lambda(x)$ exists for μ_V - a.e. $x \in U$ by virtue of the differentiation theorem 4.7.)

Furthermore $\Theta^n(\mu_V,x)$ *is a* μ_V*-measurable function of* x .

Proof The first part of the lemma follows directly from the monotonicity formula 40.2. The μ_V-measurability of $\Theta^n(\mu_V,\cdot)$ follows from the fact that $\mu_V(\bar{B}_\rho(x)) \ge \limsup_{y\to x} \mu_V(\bar{B}_\rho(y))$, which guarantees that $\mu_V(B_\rho(x))/(\omega_n\rho^n)$ is Borel measurable and hence μ_V-measurable for each fixed ρ . Since $\Theta^n(\mu_V,x) = \lim_{\rho\downarrow 0} (\omega_n\rho^n)^{-1}\mu_V(B_\rho(x))$ for μ_V - a.e. $x \in U$, we then have μ_V-measurability of $\Theta^n(\mu_V,\cdot)$ as claimed.

40.6 THEOREM (Semi-continuity of Θ^n under varifold convergence.)

Suppose $V_i \to V$ *(as Radon measures in* $G_n(U)$*) and* $\Theta^n(V_i,y) \ge 1$ *except on a set* $B_i \subset U$ *with* $\mu_{V_i}(B_i \cap W) \to 0$ *for each* $W \subset\subset U$, *and suppose that each* V_i *has locally bounded first variation in* U *with* $\liminf \|\delta V_i\|(W) < \infty$ *for each* $W \subset\subset U$. *Then* $\|\delta V\|(W) \le \liminf\|\delta V_i\|(W)$ $\forall\, W \subset\subset U$ *and* $\Theta^n(\mu_V,y) \ge 1$ μ_V *- a.e. in* U .

40.7 REMARKS

(1) The fact that $\|\delta V\|(W) \le \liminf\|\delta V_i\|(W)$ is a trivial consequence of the definitions of $\|\delta V\|$, $\|\delta V_i\|$ and the fact that $V_i \to V$, so we have only to prove the last conclusion that $\Theta^n(\mu_V,y) \ge 1$ μ_V - a.e.

(2) The proof that $\Theta^n(\mu_V,y) \ge 1$ μ_V - a.e. to be given below is slightly complicated; the reader should note that if $\|\delta V\| \le \Lambda\mu_V$ in U

(i.e. if V has generalized boundary measure $\sigma = 0$ and bounded $\underline{\underline{H}}$ - see 39.4), then the result is a very easy consequence of the monotonicity formula 40.2.

Proof of Theorem 40.6 Set $\mu_i = \mu_{V_i}$, $\mu = \mu_V$, and take any $W \subset\subset U$ and $\rho_0 \in (0, \operatorname{dist}(W, \partial U))$. For $i,j \geq 1$, consider the set $A_{i,j}$ consisting of all points $y \in W \sim B_i$ such that

$$(1) \qquad \|\delta V_i\|(\bar{B}_\rho(y)) \leq j\mu_i(\bar{B}_\rho(y)), \quad 0 < \rho < \rho_0,$$

and let $B_{i,j} = W \sim A_{i,j}$. Then if $x \in B_{i,j}$ we have *either* $x \in B_i \cap W$ *or*

$$(2) \qquad \mu_i(\bar{B}_\sigma(x)) \leq j^{-1}\|\delta V_i\|(\bar{B}_\sigma(x)) \quad \text{for some } \sigma \in (0, \rho_0).$$

Let $\mathcal{B}$ be the collection of balls $\bar{B}_\sigma(x)$ with $x \in B_{i,j}$, $\sigma \in (0, \rho_0)$, and with (2) holding. By the Besicovitch covering lemma 4.6 there are families $\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{B}$ with $N = N(n+k)$, with $B_{i,j} \sim B_i \subset \bigcup_{\ell=1}^{N} \left(\bigcup_{B \in \mathcal{B}_\ell} B\right)$ and with each $\mathcal{B}_\ell$ a pairwise disjoint family. Hence if we sum in (2) over balls $B \in \bigcup_{\ell=1}^{N} \mathcal{B}_\ell$, we get

$$\mu_i(B_{i,j}) \leq Nj^{-1}\|\delta V_i\|(\tilde{W}) + \mu_i(B_i \cap W)$$

($\tilde{W} = \{x \in U : \operatorname{dist}(x, W) < \rho_0\}$), so

$$(3) \qquad \mu_i(B_{i,j}) \leq cj^{-1} + \mu_i(B_i \cap W),$$

with c independent of i,j. In particular for each $i,j \geq 1$

$$(4) \qquad \mu\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \leq \liminf_{q\to\infty} \mu_q\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \leq cj^{-1},$$

since $\mu_q(B_q \cap W) \to 0$ as $q \to \infty$.

Now let $j \in \{1,2,\ldots\}$ and consider the possibility that there is a point $x \in W$ such that $x \in W \sim \text{interior}\left(\bigcap_{q=i}^{\infty} B_{q,j}\right)$ for *each* $i = 1,2,\ldots$. Then we could select, for each $i = 1,2,\ldots$, $y_i \in W \sim \bigcap_{q=i}^{\infty} B_{q,j}$ with $|y_i - x| < 1/i$. Thus there are sequences $y_i \to x$ and $q_i \to \infty$ such that $y_i \notin B_{q_i,j}$ for each $i = 1,2,\ldots$. Then $y_i \in A_{q_i,j}$ and hence (by (1))

$$\|\delta V_{q_i}\|(\bar{B}_\rho(y_i)) \le j\mu_{q_i}(\bar{B}_\rho(y_i)) \, , \quad 0 < \rho < \rho_0 \, ,$$

for all $i = 1,2,\ldots$. Then by the monotonicity formula 40.2 (with $\Lambda = j$) together with the fact that $\Theta^n(\mu_{q_i}, y_i) \ge 1$ we have

$$\mu_{q_i}(\bar{B}_\rho(y_i)) \ge e^{-j\rho}\omega_n\rho^n \, , \quad 0 < \rho < \rho_0 \, , \; i = 1,2,\ldots \, ,$$

and hence

$$\mu(\bar{B}_\rho(x)) \ge e^{-j\rho}\omega_n\rho^n \, , \quad 0 < \rho < \rho_0 \, ,$$

so that $\Theta^n(\mu,x) \ge 1$ for such an x. Thus we have proved $\Theta^n(\mu,x) \ge 1$ for each x with $x \in W \sim \left(\bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right)$ for some $j \in \{1,2,\ldots\}$. That is

$$\text{(5)} \qquad \Theta^n(\mu,x) \ge 1 \qquad \forall\, x \in W \sim \left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) .$$

However

$$\begin{aligned} \mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) &\le \mu\left(\bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \qquad \forall\, j \ge 1 \\ &= \lim_{i\to\infty} \mu\left(\text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) \\ &\le cj^{-1} \quad \text{by (4)} \, , \end{aligned}$$

so $\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \text{interior}\left(\bigcap_{\ell=i}^{\infty} B_{\ell,j}\right)\right) = 0$ and the theorem is established (by (5)).

§41. THE CONSTANCY THEOREM

41.1 THEOREM (Constancy Theorem)

Suppose V *is an* n-*varifold in* U , V *is stationary in* U , *and* $U \cap \operatorname{spt} \mu_V \subset M$, *where* M *is a connected* n-*dimensional* C^2 *submanifold of* $\mathbb{R}^{n+k}$. *Then* $V = \theta_0 \underline{\underline{v}}(M)$ *for some constant* θ_0 .

41.2 REMARKS

(1) Notice in particular this implies $(\bar{M} \sim M) \cap U = \emptyset$ (if $V \neq 0$) ; this is not *a-priori* obvious from the assumptions of the theorem.

(2) J. Duggan in his PhD thesis [DJ] has recently extended 41.1 to the case when M is merely Lipschitz.

(3) The reader will see that, with only minor modifications to the proof to be given below, the theorem continues to hold if N is an embedded $(n+k_1)$-dimensional C^2 submanifold of $\mathbb{R}^{n+k}$ and if V is stationary in $U \cap N$ in the sense that $\delta V(X) = 0$ $\forall\, X \in K(U;\mathbb{R}^{n+k})$ with $X_x \in T_xN$ $\forall\, x \in N$, *provided we are given* $\operatorname{spt} V \subset \{(x,S) : x \in N \text{ and } S \subset T_xN\}$. (This last is equivalent to $\operatorname{spt} \mu_V \subset N$ and $p_{\#}V = V$, where $p : U \to U \cap N$ coincides with the nearest point projection onto $U \cap N$ in some neighbourhood of $U \cap N$.)

Proof of 41.1 We first want to argue that $V = \underline{\underline{v}}(M,\theta)$ for some positive locally H^n-integrable function θ on M .

To do this first take any $f \in C_c^2(U)$ with $M \subset \{x \in U : f(x) = 0\}$ and note that by 39.2

$$(1) \qquad \delta V(f \operatorname{grad} f) = \int |p_S(\operatorname{grad} f)|^2 dV(x,S) ,$$

because (using notation as in 39.2)

$$\operatorname{div}_S(f \operatorname{grad} f) = \nabla^S f \cdot \operatorname{grad} f + f \operatorname{div}_S \operatorname{grad} f$$

$$= |p_S(\operatorname{grad} f)|^2 \quad \text{on} \quad M ,$$

where we used $f \equiv 0$ on M. Since $\delta V = 0$, we conclude from (1) that

(2) $$p_S(\operatorname{grad} f(x)) = 0 \quad \text{for all} \quad (x,S) \in \operatorname{spt} V .$$

Now let $\xi \in M$ be arbitrary. We can find an open $W \subset U$ with $\xi \in W$ and such that there are $C^2_c(U)$ functions $f_1,\ldots,f_k$ with $M \subset \bigcap_{j=1}^{k} \{x : f_j(x) = 0\}$ and with $(T_xM)^\perp$ being exactly the space spanned by $\operatorname{grad} f_1(x),\ldots,\operatorname{grad} f_k(x)$ for each $x \in M \cap W$. (One easily checks that such W and $f_1,\ldots,f_k$ exist.) Then (2) implies that

(3) $$p_S((T_xM)^\perp) = 0 \quad \text{for all} \quad (x,S) \in G_n(W) \cap \operatorname{spt} V .$$

But (3) says exactly that $S = T_xM$ for all $(x,S) \in G_n(W) \cap \operatorname{spt} V$, so that (since ξ was an arbitrary point of M), we have

(4) $$\int f(x,S)\,dV(x,S) = \int_{M\cap U} f(x,T_xM)\,d\mu_V(x) \ , \ f \in C_c(G_n(U)) .$$

On the other hand we know from monotonicity 40.2 that $\theta(x) \equiv \Theta^n(\mu_V,x)$ exists for all $x \in M \cap U$, and hence (since $\Theta^n(\mathcal{H}^n \llcorner M,x) = 1$ for each $x \in M$, by smoothness of M), we can use the differentiation theorem 4.7 to conclude from (4) that in fact

(5) $$\int f(x,S)\,dV(x,S) = \int_{M\cap U} f(x,T_xM)\,\theta(x)\,d\mathcal{H}^n(x) \ , \ f \in C_c(G_n(U)) ,$$

(so that $V = \underline{\underline{v}}(M,\theta)$ as required).

It thus remains only to prove that $\theta = \text{const.}$ on $M \cap U$. Since M is C^2 we can take $X \in \mathcal{K}(U,\mathbb{R}^{n+k})$ such that $X_x \in T_xM \quad \forall\ x \in M \cap U$. Then by (5) and 39.2 $\delta V(X) = 0$ is just the statement that $\int_{M\cap U} \operatorname{div} X \theta\, d\mathcal{H}^n = 0$, where div X is the classical divergence of $X|M$ in the usual sense of differential geometry. Using local coordinates (in some neighbourhood $\tilde{U} \subset \mathbb{R}^n$) this tells us that

$$\int_{\tilde{U}} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} \tilde{\theta}\, d\mathcal{L}^n = 0 \quad \text{if} \quad X_i \in C_c^1(\tilde{U}) \ , \ i=1,\ldots,n \ ,$$

where $\tilde{\theta}$ is θ expressed in terms of the local coordinates. In particular

$$\int_{\tilde{U}} \frac{\partial \zeta}{\partial x_i} \tilde{\theta}\, d\mathcal{L}^n = 0 \qquad \forall\ \zeta \in C_c(U) \ , \ i=1,\ldots,n$$

and it is then standard that $\tilde{\theta}$ = constant in $\tilde{U}$. Hence (since M is connected) θ is constant in M .

§42. VARIFOLD TANGENTS AND RECTIFIABILITY THEOREM

Let V be an n-varifold in U and let x be any point of U such that

42.1 $$\Theta^n(\mu_V, x) = \theta_0 \in (0,\infty) \quad \text{and} \quad \lim_{\rho\downarrow 0} \rho^{1-n} \|\delta V\|(B_\rho(x)) = 0 \ .$$

By definition of δV (in §39) and the compactness theorem 4.4 for Radon measures, we can select a sequence $\lambda_j \downarrow 0$ such that $\eta_{x,\lambda_j \#}V$ converges (in the sense of Radon measures) to a varifold C such that

$$C \text{ is stationary in } \mathbb{R}^{n+k}$$

and

(*) $$\frac{\mu_C(B_\rho(x))}{\omega_n \rho^n} \equiv \theta_0 \qquad \forall\ \rho > 0 \ .$$

Since $\delta C = 0$ we can use (*) together with the monotonicity formula 40.3 to conclude

$$\int_{G_n(B_\rho(0))} \frac{|p_{S^\perp}(x)|^2}{|x|^{n+2}} \, dC(x,S) = 0 \qquad \forall\ \rho > 0 \ ,$$

so that $p_{S^\perp}(x) = 0$ for C - a.e. $(x,S) \in G_n(\mathbb{R}^{n+k})$, and hence $p_{S^\perp}(x) = 0$ for *all* $(x,S) \in \operatorname{spt} C$ by continuity of $p_{S^\perp}(x)$ in (x,S) . Then by the same argument as in the proof of 19.3, except that we use 40.4 in place of 18.1, we deduce that μ_C satisfies

42.2 $$\lambda^{-n}\mu_C(\eta_{0,\lambda}(A)) = \mu_C(A) \ , \quad A \subset \mathbb{R}^{n+k} \ , \ \lambda > 0 \ .$$

We would *like* to prove the stronger result $\eta_{0,\lambda\#}C = C$ (which of course implies 42.2), but we are only able to do this in case $\Theta^n(\mu_C,x) > 0$ for μ_C - a.e. x (see 42.6 below). Whether or not $\eta_{0,\lambda\#}C = C$ without the additional hypothesis on $\Theta^n(\mu_C,\cdot)$ seems to be an open question.

42.3 DEFINITION Given V and x as in 42.1 we let $\operatorname{Var\,Tan}(V,x)$ ("the varifold tangent of V at x") be the collection of all $C = \lim \eta_{x,\lambda_j\#}V$ obtained as described above.

Notice that by the above discussion any $C \in \operatorname{Var\,Tan}(V,x)$ is stationary in $\mathbb{R}^{n+k}$ and satisfies 42.2.

The following *rectifiability theorem* is a central part of the theory of n-varifolds with locally bounded first variation.

42.4 THEOREM *Suppose* V *has locally bounded first variation in* U *and* $\Theta^n(\mu_V,x) > 0$ *for* μ_V -a.e. $x \in U$. *Then* V *is an* n-*rectifiable varifold.* (Thus $V = \underline{\underline{v}}(M,\theta)$, with M an $\mathcal{H}^n$-measurable countably n-rectifiable subset of U and θ a non-negative locally $\mathcal{H}^n$-integrable function on U .)

42.5 REMARK We are going to use Theorem 38.3. In fact we show that V has a tangent plane (in the sense of 38.1) at any point x where

(i) $\Theta^n(\mu_V,x) > 0$, (ii) $\eta_V^{(x)}$ (as in Lemma 38.4) exists, (iii) $\Theta^n(\mu_V,\cdot)$ is μ_V-approximately continuous at x , and (iv) $\|\delta V\|(B_\rho(x)) \le \Lambda(x)\mu_V(B_\rho(x))$ for $0<\rho<\rho_0 = \min\{1,\mathrm{dist}(x,\partial U)\}$. Since conditions (i)-(iv) all hold μ_V-a.e. in U (notice that (iii) holds μ_V-a.e. by virtue of the μ_V-measurability of $\Theta^n(\mu_V,\cdot)$ proved in 40.5), the required rectifiability of V will then follow from 38.3.

Before beginning the proof of 42.2 we give the following important corollary.

42.6 COROLLARY *Suppose* $x \in U$, 42.1 *holds, and* $\lim_{\rho\downarrow 0} \rho^{-n}\mu_V(\{y\in B_\rho(x) : \Theta^n(\mu_V,y) < 1\}) = 0$. *If* $C \in \mathrm{Var\,Tan}(V,x)$, *then* C *is rectifiable and*

$$(*) \qquad \eta_{0,\lambda\#}C = C \qquad \forall\ \lambda > 0 .$$

Proof From the hypothesis $\rho^{-n}\mu_V(\{y\in B_\rho(x) : \Theta^n(\mu_V,y) < 1\}) \to 0$ and the semi-continuity theorem 40.6, we have $\Theta^n(\mu_C,y) \ge 1$ for μ_C-a.e. $y \in \mathbb{R}^{n+k}$. Hence by Theorem 42.4 we have that C is n-rectifiable. On the other hand, since $\Theta^n(\mu_C,y) = \Theta^n(\mu_C,\lambda y)$ $\forall\ \lambda > 0$ (by 42.2), we can write $C = \underline{\underline{v}}(M,\theta)$ with $\eta_{0,\lambda}(M) = M$ $\forall\ \lambda > 0$ and $\theta(\lambda y) = \theta(y)$ $\forall\ \lambda > 0$, $y \in \mathbb{R}^{n+k}$. (Viz. simply set $\theta(y) = \Theta^n(\mu_C,y)$ and $M = \{y\in \mathbb{R}^{n+k} : \theta(y) > 0\}$.) It then trivially follows that $y \in T_yM$ whenever the approximate tangent space T_yM exists, and hence $\eta_{0,\lambda\#}C = C$ as required.

Proof of Theorem 42.2 Let x be as in 42.5(i)-(iv) and take $C \in \mathrm{Var\,Tan}(V,x)$. (We know $\mathrm{Var\,Tan}(V,x) \ne \emptyset$ because 42.5(i),(iv) imply 42.1.) Then C is stationary in $\mathbb{R}^{n+k}$ and

$$(1) \qquad \frac{\mu_C(B_\rho(0))}{\omega_n\rho^n} \equiv \theta_0 \qquad \forall\ \rho > 0 \qquad (\theta_0 = \Theta^n(\mu_V,x)) .$$

Also for any $y \in \mathbb{R}^{n+k}$ (using (1) and the monotonicity formula 40.2)

$$\frac{\mu_C(B_\rho(y))}{\omega_n\rho^n} \le \frac{\mu_C(B_R(y))}{\omega_n R^n}$$

$$\le \frac{\mu_C(B_{R+|y|}(0))}{\omega_n(R+|y|)^n}\,(1+|y|/R)^n$$

$$= \theta_0(1+|y|/R)^n \to \theta_0 \quad \text{as} \quad R \uparrow \infty .$$

That is (again using the monotonicity formula 40.2),

$$\text{(2)} \qquad \Theta^n(\mu_C, y) \le \frac{\mu_C(B_\rho(y))}{\omega_n\rho^n} \le \theta_0 \qquad \forall\, y \in \mathbb{R}^{n+k},\ \rho > 0 .$$

Now let $V_j = \eta_{x,\lambda_j\#}V$, where $\lambda_j \downarrow 0$ is such that $\lim \eta_{x,\lambda_j\#}V = C$ and where we are still assuming x is as in 42.5(i)-(iv).

From 42.5(iii) we have (with $\varepsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$)

$$\text{(3)} \qquad \Theta^n(\mu_V, y) \ge \theta_0 - \varepsilon(\rho) , \quad y \in G \cap B_\rho(x) ,$$

where $G \subset U$ is such that

$$\text{(4)} \qquad \mu_V(B_\rho(x) \sim G) \le \varepsilon(\rho)\rho^n , \quad \rho \text{ sufficiently small.}$$

Taking $\rho = \lambda_j$ we see that (3), (4) imply

$$\text{(3)}' \qquad \Theta^n(\mu_{V_j}, y) \le \theta_0 - \varepsilon_j , \quad y \in G_j \cap B_1(0)$$

with G_j such that

$$\text{(4)}' \qquad \mu_{V_j}(B_1(0) \sim G_j) \le \varepsilon_j ,$$

where $\varepsilon_j \to 0$ as $j \to \infty$. Thus, using (3)', (4)' and the semicontinuity result of 40.6, we obtain

(5) $$\Theta^n(\mu_C,y) \ge \theta_0 \quad \text{for } \mu_C\text{-a.e. } y \in \mathbb{R}^{n+k}$$

(and hence for *every* $y \in \operatorname{spt}\mu_C$ by 40.3). Then by combining (2) and (5) we have

(6) $$\Theta^n(\mu_C,y) \equiv \theta_0 \equiv \frac{\mu_C(B_\rho(y))}{\omega_n\rho^n} \qquad \forall\, y \in \operatorname{spt}\mu_C\,,\ \rho > 0\,.$$

Then by the monotonicity formula 40.3 (with $V = C$), we have

$$p_{S^\perp}(x-y) = 0 \quad \text{for } C\text{-a.e. } (x,S) \in G_n(\mathbb{R}^{n+k})\,.$$

Thus (using the continuity of $p_{S^\perp}(x-y)$ in (x,S)) we have

(7) $$x-y \in S \qquad \forall\, y \in \operatorname{spt}\mu_C \ \textit{and}\ \forall (x,S) \in \operatorname{spt} C\,.$$

In particular, choosing T such that $(0,T) \in \operatorname{spt} C$ (such T exists because $0 \in \operatorname{spt}\mu_C = \pi(\operatorname{spt} C)$), (7) implies $y \in T \quad \forall\, y \in \operatorname{spt}\mu_C$. Thus $\operatorname{spt}\mu_C \subset T$, and hence $C = \theta_0\underline{\underline{v}}(T)$ by the constancy theorem 41.1.

Thus we have shown that, for $x \in U$ such that 42.5(i), (iii), (iv) hold, each element of $\operatorname{Var}\operatorname{Tan}(V,x)$ has the form $\theta_0\underline{\underline{v}}(T)$, where T is an n-dimensional subspace of $\mathbb{R}^{n+k}$. On the other hand, since we are assuming (42.5(ii)) that $\eta_V^{(x)}$ exists, it follows that for continuous β on $G(n+k,n)$

(8) $$\lim_{\rho\downarrow 0} \frac{\int_{G_n(B_\rho(x))} \beta(S)\,dV(y,S)}{\mu_V(B_\rho(x))} = \int_{G(n+k,n)} \beta(S)\,d\eta_V^{(x)}(S)\,.$$

Now let $\theta_0\underline{\underline{v}}(T)$ be any such element of $\operatorname{Var}\operatorname{Tan}(V,x)$ and select $\lambda_j \downarrow 0$ so that $\lim \eta_{x,\lambda_j\#}V = \theta_0\underline{\underline{v}}(T)$. Then in particular

$$\lim_{j\to\infty} \frac{\int_{G_n(B_1(0))} \beta(S)\,dV_j(y,S)}{\mu_{V_j}(B_1(0))} = \beta(T)\,,$$

and hence (8) gives

$$\beta(T) = \int_{G(n+k,n)} \beta(S)\, d\eta_V^{(x)}(S) \;,$$

thus showing that $\theta_0 \underline{\underline{v}}(T)$ is the *unique* element of $\mathrm{Var\,Tan}(V,x)$. Thus $\lim_{\lambda\downarrow 0} \eta_{x,\lambda\#} V = \theta_0 \underline{\underline{v}}(T)$, so that T is the tangent space for V at x in the sense of 38.1. This completes the proof.

The following *compactness theorem* for rectifiable varifolds is now a direct consequence of the rectifiability theorem 42.4, the semi-continuity theorem 40.6, and the compactness theorem 4.4 for Radon measures, and its proof is left to the reader.

42.7 THEOREM *Suppose* $\{V_j\}$ *is a sequence of rectifiable* n*-varifolds in* U *which are locally of bounded first variation in* U,

$$\sup_{j\geq 1} (\mu_{V_j}(W) + \|\delta V_j\|(W)) < \infty \qquad \forall\, W \subset\subset U \,,$$

and $\Theta^n(\mu_{V_j}, x) \geq 1$ *on* $U \sim A_j$, *where* $\mu_{V_j}(A_j \cap W) \to 0$ *as* $j \to \infty$ $\forall\, W \subset\subset U$.

Then there is a subsequence $\{V_{j'}\}$ *and a rectifiable varifold* V *of locally bounded first variation in* U, *such that* $V_{j'} \to V$ *(in the sense of Radon measures on* $G_n(U)$*)* , $\Theta^n(\mu_V, x) \geq 1$ *for* μ_V-a.e. $x \in U$, *and* $\|\delta V\|(W) \leq \liminf_{j\to\infty} \|\delta V_j\|(W)$ *for each* $W \subset\subset U$.

42.8 REMARK An important additional result (also due to Allard [AW1]) is the *integral compactness theorem,* which asserts that if all the V_j in the above theorem are integer multiplicity, then V is also integer multiplicity. (Notice that in this case the hypothesis $\Theta^n(\mu_{V_j}, x) \geq 1$ on $U \sim A_j$ is automatically satisfied with an A_j such that $\mu_{V_j}(A_j) = 0$.)

Proof that V is integer multiplicity if the V_i are:

Let $W \subset\subset U$. We first assert that for μ_V-a.e. $x \in W$ there exists c (depending on x) such that

$$(1) \qquad \liminf \|\delta V_i\|(\bar{B}_\rho(x)) \le c\mu_V(\bar{B}_\rho(x)) \ , \ \rho < \min\{1, \mathrm{dist}(x, \partial U)\} \ .$$

Indeed otherwise $\exists$ a set $A \subset W$ with $\mu_V(A) > 0$ such that for each $j \ge 1$ and each $x \in A$ there are $\rho_x > 0$, $i_x \ge 1$ such that $\bar{B}_{\rho_x}(x) \subset W$ and

$$\mu_V(\bar{B}_{\rho_x}(x)) \le j^{-1}\|\delta V_i\|(\bar{B}_{\rho_x}(x)) \ , \ i \ge i_x \ .$$

By the Besicovitch covering lemma 4.6 we then have

$$\mu_V(A_i) \le cj^{-1}\|\delta V_\ell\|(W) \ , \ \ell \ge i \ ,$$

where $A_i = \{x \in A : i_x \le i\}$. Thus

$$\mu_V(A_i) \le cj^{-1} \limsup_{\ell \to \infty} \|\delta V_\ell\|(W) \ ,$$

and hence since $A_i \uparrow A$ as $i \uparrow \infty$ we have

$$\mu_V(A) \le cj^{-1}$$

for some $c \ (< \infty)$ independent of j. That is, $\mu_V(A) = 0$, a contradiction, and hence (1) holds. Since $\Theta^n(\mu_V, x)$ exists μ_V-a.e. $x \in U$, we in fact have from (1) that for μ_V-a.e. $x \in U$ there is a $c = c(x)$ such that

$$(2) \qquad \liminf \|\delta V_i\|(B_\rho(x)) \le c\rho^n \ , \ 0 < \rho < \min\{1, \mathrm{dist}(x, \partial U)\} \ .$$

Now since $V = \underline{\underline{v}}(M, \theta)$, it is also true that for μ_V-a.e. $\xi \in \mathrm{spt}\,\mu_V$ we have $\eta_{\xi,\lambda\#}V \to \theta_0 \underline{\underline{v}}(P)$ as $\lambda \downarrow 0$, where $P = T_\xi M$ and $\theta_0 = \theta(\xi)$. Then (because $V_i \to V$, and hence $\eta_{\xi,\lambda\#}V_i \to \eta_{\xi,\lambda\#}V$ for each fixed $\lambda > 0$), it follows that for μ_V-a.e. $\xi \in U$ we can select a sequence $\lambda_i \downarrow 0$ such that, with $W_i = \eta_{\xi,\lambda_i\#}V_i$,

(3) $$W_i \to \theta_0 \underline{\underline{v}}(P)$$

and (by (2)) for each $R > 0$

(4) $$\|\delta W_i\|(B_R(0)) \to 0 .$$

We claim that θ_0 must be an integer for any such ξ ; in fact for an arbitrary sequence $\{W_i\}$ of integer multiplicity varifolds in $\mathbb{R}^{n+k}$ satisfying (3), (4), we claim that θ_0 always has to be an ineger.

To see this, take (without loss of generality) $P = \mathbb{R}^n \times \{0\}$, let q be orthogonal projection onto $(\mathbb{R}^n \times \{0\})^\perp$, and note first that (3) implies

(5) $$p_{\mathbb{R}^n\#}(W_i \llcorner G_n\{x \in \mathbb{R}^{n+k} : |q(x)| < \varepsilon\}) \to \theta_0 \underline{\underline{v}}(\mathbb{R}^n)$$

for each fixed $\varepsilon > 0$. However by the mapping formula for varifolds (§15), we know that (5) says

(5)' $$\underline{\underline{v}}(\mathbb{R}^n, \psi_i) \to \theta_0 \underline{\underline{v}}(\mathbb{R}^n) ,$$

where

(6) $$\psi_i(x) = \sum_{y \in p^{-1}_{\mathbb{R}^n}(x) \cap \{z \in \mathbb{R}^{n+k} : |q(z)| < \varepsilon\}} \theta_i(y)$$

(θ_i = multiplicity function of W_i , so that ψ_i has values in $\mathbb{Z} \cap \{\infty\}$) . Notice that (5)' implies in particular that

(7) $$\int_{\mathbb{R}^n} f\psi_i \, dL^n \to \theta_0 \int_{\mathbb{R}^n} f dL^n \qquad \forall f \in C^0_c(\mathbb{R}^n) .$$

(i.e. measure-theoretic convergence of ψ_i to θ_0 .)

Now we claim that there are sets $A_i \subset B_1(0)$ such that

(8) $$\psi_i(x) \le \theta_0 + \varepsilon_i \quad \forall x \in B_1(0) \sim A_i \ , \ L^n(A_i) \to 0 \ , \quad \varepsilon_i \downarrow 0 ;$$

this will of course (when used in combination with (7)) imply that for any integer $N > \theta_0$, $\max\{\psi_i, N\}$ converges in $L^1(B_1(0))$ to θ_0, and, since $\max\{\psi_i, N\}$ is integer-valued, it then follows that θ_0 is an integer.

On the other hand (8) evidently follows by setting $W = W_i$ in the following lemma, so the proof is complete.

In this lemma, p, q denote orthogonal projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ and $\{0\} \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$ respectively.

42.9 LEMMA *For each* $\delta \in (0,1)$, $\Lambda \geq 1$, *there is* $\varepsilon = \varepsilon(\delta, \Lambda, n) \in (0, \delta^2)$ *such that if* W *is an integer multiplicity varifold in* $B_3(0)$ *with*

$$(*) \qquad \mu_W(B_3(0)) \leq \Lambda\ ,\quad \|\delta W\|(B_3(0)) < \varepsilon^2\ ,\quad \int_{B_3(0)} \|p_S - p\|\, dW(y,S) < \varepsilon^2\ ,$$

then there is a set $A \subset B_1^n(0)$ *such that* $L^n(A) < \delta$ *and*, $\forall\, x \in B_1(0) \sim A$,

$$\sum_{y \in p^{-1}(x) \cap \operatorname{spt}\mu_W \cap \{z: |q(z)| < \varepsilon\}} \Theta^n(\mu_W, y) \leq (1+\delta)\, \frac{\mu_W(B_2(x))}{\omega_n 2^n} + \delta\ .$$

42.10 REMARK It suffices to prove that for each fixed N there is $\delta_0 = \delta_0(N) \in (0,1)$ such that if $\delta \in (0, \delta_0)$ then $\exists\ \varepsilon = \varepsilon(n, \Lambda, N, \delta) \in (0, \delta^2)$ such that $(*)$ implies the existence of $A \subset B_1^n(0)$ with $L^n(A) < \delta$ and, for $x \in B_1^n(0) \sim A$ and distinct $y_1, \ldots, y_N \in p^{-1}(x) \cap \operatorname{spt}\mu_W \cap \{z: |q(z)| < \varepsilon\}$,

$$(**) \qquad \sum_{j=1}^{N} \Theta^n(\mu_W, y_j) \leq (1+\delta)\, \frac{\mu_W(B_2(x))}{\omega_n 2^n} + \delta\ .$$

Because this firstly implies an *a-priori* bound, depending only on n, k, Λ, on the number N of possible points y_j, and hence the lemma, as originally stated, then follows. (Notice that of course the validity of the lemma for small δ implies its validity for any larger δ.)

Proof of 42.9 By virtue of the above Remark, we need only prove (**). Let $\mu = \mu_W$, and consider the possibility that $y \in B_1(0)$ satisfies

(1) $$\|\delta W\|(B_\rho(y)) \le \varepsilon\mu(B_\rho(y)) \ , \ 0 < \rho < 1 \ ,$$

(2) $$\int_{B_\rho(y)} \|p_S - p\| dW(z,S) \le \varepsilon\rho^n \ , \ 0 < \rho < 1 \ .$$

Let

$$A_1 = \{y \in B_2(0) \cap \operatorname{spt} W : (1) \text{ fails for some } \rho \in (0,1)\}$$

$$A_2 = \{y \in B_2(0) \cap \operatorname{spt} W : (2) \text{ fails for some } \rho \in (0,1)\}.$$

Evidently if $y \in \operatorname{spt} \mu_W \cap B_2(0) \sim A_1$ then by the monotonicity formula 40.2 we have

(3) $$\frac{\mu(B_\rho(y))}{\omega_n\rho^n} \le e^\varepsilon \frac{\mu(B_1(y))}{\omega_n} \le c \ , \ 0 < \rho < 1 \ ,$$

$(c = c(\Lambda,n))$, while if $y \in A_2 \sim A_1$ we have (using (3))

(4) $$\int_{B_\rho(y)} \|p_S - p\| dW(z,S) \ge \varepsilon\rho_y^n \ge c\varepsilon\mu(B_{\rho_y}(y))$$

for some $\rho_y \in (0,1)$. If $y \in A_1$ then

(5) $$\mu(B_{\rho_y}(y)) \le \varepsilon^{-1}\|\delta W\|(B_{\rho_y}(y))$$

for some $\rho_y \in (0,1)$.

Since then $\{B_{\rho_y}(y)\}_{y \in A_1 \cup A_2}$ covers $A_1 \cup A_2$ we deduce from (4), (5) and the Besicovitch covering lemma 4.6 that

$$(6) \qquad \mu(A_1 \cup A_2) \le c\varepsilon^{-1}\left(\int_{B_3(0)} \|p_S - p\| dW(a,S) + \|\delta W\|(B_3(0))\right)$$

$$\le c\varepsilon$$

by the hypotheses on W.

Our aim now is to show (**) holds whenever $x \in B_1^n(0) \sim p(A_1 \cup A_2)$. In view of (6) this will establish the required result (with $A = p(A_1 \cup A_2)$). So let $x \in B_1^n(0) \sim p(A_1 \cup A_2)$. In view of the monotonicity formula 40.2 it evidently suffices (by translating and changing scale by a factor of 3/2) to assume that $x = 0 \in B_1^n(0) \sim p(A_1 \cup A_2))$. We shall subsequently assume this.

We first want to establish the two formulae, for $y \in B_1(0) \sim (A_1 \cup A_2)$ and $\tau > 0$:

$$(7) \qquad \Theta^n(\mu,y) \le e^{\varepsilon\sigma} \frac{\mu(U_\sigma^{2\tau}(y))}{\omega_n \sigma^n} + c\varepsilon\sigma/\tau\ , \quad 0 < \sigma < 1\ ,$$

and

$$(8) \qquad \frac{\mu(U_\sigma^{\tau}(y))}{\omega_n \sigma^n} \le e^{\varepsilon\rho} \frac{\mu(U_\rho^{2\tau}(y))}{\omega_n \rho^n} + c\varepsilon\rho/\tau\ , \quad 0 < \sigma < \rho \le 1\ ,$$

where

$$U_\sigma^\tau(y) = B_\sigma(y) \cap \{z \in \mathbb{R}^{n+k}: |q(z-y)| < \tau\}\ .$$

Indeed these two inequalities follow directly from 40.2 and 40.4. For example to establish (7) we note first that 40.2 gives (7) directly if $\tau \ge \sigma$, while if $\tau < \sigma$ then we first use 40.2 to give $\Theta^n(\mu,y) \le e^{\varepsilon\tau} \dfrac{\mu(B_\tau(y))}{\omega_n \tau^n}$ and then use 40.4 with h of the form $h(z) = f(|q(z-y)|)$, $f(t) \equiv 1$ for $t < \tau$ and $f(t) \equiv 0$ for $t > 2\tau$.

Since $|\nabla^S f(|q(z-y)|)| \le f'(|q(z-y)|)|p_S - p|$ (Cf. the computation in 19.5) we then deduce (by integrating in 40.4 from τ to σ and using (3))

$$\frac{\mu(B_\tau(y))}{\omega_n \tau^n} \le \frac{\mu(U_\sigma^{2\tau}(y))}{\omega_n \sigma^n} + c\varepsilon\sigma/\tau \ .$$

(8) is proved by simply integrating in 40.4 from σ to ρ (and using (3)).

Our aim now is to use (7) and (8) to establish

$$(9) \qquad \sum_{j=1}^{N} \frac{\mu(U_\sigma^\tau(y_j))}{\omega_n \sigma^n} \le (1+c\delta^2)\frac{\mu(B_2(0))}{\omega_n 2^n} + c\delta^2$$

with $c = c(n,k,N,\Lambda)$, provided $2\delta^2\sigma \le \tau \le \frac{1}{4}\min_{j\ne\ell}|y_j - y_\ell|$, $y_j \in \operatorname{spt}\mu \cap p^{-1}(0) \cap \{z : |q(z)| < \varepsilon\}$, $0 \notin p(A_1 \cup A_2)$. (In view of (7) this will prove the required result (**) for suitable $\delta_0(N)$.)

We proceed by induction on N . $N=1$ trivially follows from (8) by noting that $U_\rho^{2\tau}(y_1) \subset B_\rho(y_1)$ (by definition of $U_\rho^{2\tau}(y_1)$) and then using the monotonicity 40.2 together with the fact that $|y_1| < \varepsilon$. Thus assume $N \ge 2$ and that (9) has been established with any $M < N$ in place of N .

Let $y_1,\ldots,y_N$ be as in (9), and choose $\rho \in [\sigma,1)$ such that $\min_{j\ne\ell}|q(y_j) - q(y_\ell)| \left(= \min_{j\ne\ell}|y_j - y_\ell|\right) = 4\delta^2\rho$, and set $\tilde\tau = 2\delta^2\rho$ $(\ge 2\tau)$. Then

$$\begin{aligned}\frac{\mu(U_\sigma^\tau(y_j))}{\sigma^n} &\le \frac{\mu(U_\sigma^{\frac12\tilde\tau}(y_j))}{\sigma^n}\\ &\le e^{\varepsilon\rho}\frac{\mu(U_\rho^{\tilde\tau}(y_j))}{\rho^n} + c\varepsilon \quad \text{(by (8))} \ ,\end{aligned}$$

$c = c(n,k,\delta)$. Now since $\tilde\tau = \frac{1}{2}\min_{j\ne\ell}|q(y_j) - q(y_\ell)|$ we can select $\{z_1,\ldots,z_Q\} \subset \{y_1,\ldots,y_N\}$ $(Q \le N-1)$ and $\hat\tau \le c\tilde\tau$ such that $\hat\tau \ge 3\delta^2\rho$ and

$$\bigcup_{j=1}^{N} U_{\rho}^{\tilde{\tau}}(y_j) \subset \bigcup_{\ell=1}^{Q} U^{\hat{\tau}}_{\rho(1+c\delta^2)}(z_\ell) ,$$

where $c = c(N)$, and such that $\hat{\tau} \le \frac{1}{4} \min_{i \ne j} |z_i - z_j|$.

Since $c\delta^2 < 1/2$ for $\delta < \delta_0(N)$ (if $\delta_0(N)$ is chosen suitably) we then have $\hat{\tau} \ge 2\delta^2\tilde{\rho}$ and

$$\sum_{j=1}^{N} \frac{\mu(U_{\rho}^{\tilde{\tau}}(y_j))}{\rho^n} \le (1+c\delta^2) \sum_{j=1}^{Q} \frac{\mu(U_{\tilde{\rho}}^{\hat{\tau}}(z_j))}{\tilde{\rho}^n} ,$$

where $\tilde{\rho} = (1+c\delta^2)\rho$ and $c = c(N)$. Since $Q \le N-1$, the required result then follows by induction (choosing ε appropriately).

APPENDIX A

A GENERAL REGULARITY THEOREM

We here prove a useful general regularity theorem, which is essentially an abstraction of the "dimension reducing" argument of Federer [FH2]. There are a number of important applications of this general theorem in the text.

Let $P \geq n \geq 2$ and let F be a collection of functions $\phi = (\phi^1, \ldots, \phi^Q) : \mathbb{R}^P \to \mathbb{R}^Q$ ($Q = 1$ is an important case) such that each ϕ^j is locally H^n-integrable on $\mathbb{R}^P$. For $\phi \in F$, $y \in \mathbb{R}^P$ and $\lambda > 0$ we let $\phi_{y,\lambda}$ be defined by

$$\phi_{y,\lambda}(x) = \phi(y+\lambda x)\ ,\ x \in \mathbb{R}^P\ .$$

Also, for $\phi \in F$ and a given sequence $\{\phi_k\} \subset F$ we write $\phi_k \rightharpoonup \phi$ if $\int \phi_k f\, dH^n \to \int \phi f\, dH^n$ (in $\mathbb{R}^Q$) for each given $f \in C_c^0(\mathbb{R}^P)$.

We subsequently make the following 3 special assumptions concerning F:

A.1 (Closure under appropriate scaling and translation): *If* $|y| \leq 1-\lambda$, $0 < \lambda < 1$, *and if* $\phi \in F$, *then* $\phi_{y,\lambda} \in F$.

A.2 (Existence of homogeneous degree zero "tangent functions"): *If* $|y| < 1$, *if* $\{\lambda_k\} \downarrow 0$ *and if* $\phi \in F$, *then there is a subsequence* $\{\lambda_{k'}\}$ *and* $\psi \in F$ *such that* $\phi_{y,\lambda_{k'}} \rightharpoonup \psi$ *and* $\psi_{0,\lambda} = \psi$ *for each* $\lambda > 0$.

A.3 ("Singular set" hypotheses): *We assume there is a map*

$$\text{sing} : F \to C\ (\ = \text{set of closed subsets of } \mathbb{R}^P)$$

such that

(1) $\text{sing}\, \phi = \emptyset$ *if* $\phi \in F$ *is a constant multiple of the characteristic function of an* n*-dimensional subspace of* $\mathbb{R}^P$,

(2) *if* $|y| \le 1-\lambda$, $0 < \lambda < 1$, *then* $\mathrm{sing}\,\phi_{y,\lambda} = \lambda^{-1}(\mathrm{sing}\,\phi - y)$,

(3) *if* $\phi, \phi_k \in F$ *with* $\phi_k \to \phi$, *then for each* $\varepsilon > 0$ *there is a* $k(\varepsilon)$ *such that*

$$B_1(0) \cap \mathrm{sing}\,\phi_k \subset \{x \in \mathbb{R}^P : \mathrm{dist}(\mathrm{sing}\,\phi, x) < \varepsilon\} \qquad \forall\, k \ge k(\varepsilon) .$$

We can now state the main result of this section:

A.4 THEOREM *Subject to the notation and assumptions* A.1, A.2, A.3 *above, we have*

$$(*) \qquad \dim B_1(0) \cap \mathrm{sing}\,\phi \le n-1 \qquad \forall\, \phi \in F .$$

(Here "dim" is Hausdorff dimension, so that (*) means $H^{n-1+\alpha}(\mathrm{sing}\,\phi) = 0$ $\forall\, \alpha > 0$.)

In fact either $\mathrm{sing}\,\phi \cap B_1(0) = \emptyset$ *for every* $\phi \in F$ *or else there is an integer* $d \in [0, n-1]$ *such that*

$$\dim \mathrm{sing}\,\phi \cap B_1(0) \le d \qquad \forall\, \phi \in F$$

and such that there is some $\psi \in F$ *and a* d*-dimensional subspace* $L \subset \mathbb{R}^P$ *with*

$$(**) \qquad \psi_{y,\lambda} = \psi \qquad \forall\, y \in L ,\ \lambda > 0$$

and

$$\mathrm{sing}\,\psi = L .$$

If $d = 0$ *then* $\mathrm{sing}\,\phi \cap B_\rho(0)$ *is finite for each* $\phi \in F$ and each $\rho < 1$.

A.5 REMARK One readily checks that if L is an n-dimensional subspace of $\mathbb{R}^P$ and $\psi \in F$ satisfies (**), then ψ is exactly a constant multiple of the characteristic function of L (hence $\mathrm{sing}\,\psi = \emptyset$ by A.3(1)); otherwise we would have $P > n$ and $\psi \equiv \mathrm{const.} \ne 0$ on some $(n+1)$-dimensional half-space, thus contradicting the fact that ψ is locally $\mathcal{H}^n$-integrable on $\mathbb{R}^P$.

Proof of A.4 Assume $\operatorname{sing}\phi \cap B_1(0) \neq \emptyset$ for some $\phi \in \mathcal{F}$, and let $d = \sup\{\dim L : L$ is a d-dimensional subspace of $\mathbb{R}^P$ and there is $\phi \in \mathcal{F}$ with $\operatorname{sing}\phi \neq \emptyset$ and $\phi_{y,\lambda} = \phi \;\; \forall y \in L\,,\ \lambda > 0\}$. Then by Remark A.5 we have $d \le n-1$.

For a given $\phi \in \mathcal{F}$ and $y \in B_1(0)$ we let $T(\phi,y)$ be the set of $\psi \in \mathcal{F}$ with $\psi_{0,\lambda} = \psi \;\forall\lambda > 0$ and with $\lim \phi_{y,\lambda_k} = \psi$ for some sequence $\lambda_k \downarrow 0$. ($T(\phi,y) \neq \emptyset$ by assumption A.2).

Let $\ell \ge 0$ and let

$$\mathcal{F}^\ell = \{\phi \in \mathcal{F} : \mathcal{H}^\ell(\operatorname{sing}\phi \cap B_1(0)) > 0\} . \tag{1}$$

Our first task is to prove the implication

$$\phi \in \mathcal{F}^\ell \Rightarrow \exists\ \psi \in T(\phi,x) \cap \mathcal{F}^\ell \tag{2}$$

for $\mathcal{H}^\ell$ - a.e. $x \in \operatorname{sing}\phi \cap B_1(0)$.

To see this, let $\mathcal{H}^\ell_\delta$ be the "size δ approximation" of $\mathcal{H}^\ell$ as described in §2 and recall that $\mathcal{H}^\ell(A) > 0 \Leftrightarrow \mathcal{H}^\ell_\infty(A) > 0$, so that $\mathcal{F}^\ell = \{\phi \in \mathcal{F} : \mathcal{H}^\ell_\infty(\operatorname{sing}\phi \cap B_1(0)) > 0\}$. Also note that (by 3.6(2)), for any bounded subset A of $\mathbb{R}^P$,

$$\mathcal{H}^\ell_\infty(A) > 0 \Rightarrow \Theta^{*n}(\mathcal{H}^\ell_\infty \llcorner A, x) > 0 \quad \text{for } \mathcal{H}^\ell\text{- a.e. } x \in A . \tag{3}$$

Thus we see that if $\phi \in \mathcal{F}^\ell$ then for $\mathcal{H}^\ell$ - a.e. $x \in \operatorname{sing}\phi \cap B_1(0)$ we have $\Theta^{*\ell}(\mathcal{H}^\ell_\infty \llcorner \operatorname{sing}\phi, x) > 0$. For such x we thus have a sequence $\lambda_k \downarrow 0$ such that

$$(4) \qquad \lim_{k\to\infty} \frac{\mathcal{H}^{\ell}_{\infty}(\text{sing}\phi \cap B_{\lambda_k}(x))}{\lambda_k^{\ell}} > 0 ,$$

and by assumption A.2 there is a subsequence $\{\lambda_{k'}\}$ such that $\phi_{x,\lambda_{k'}} \rightharpoonup \psi \in T(\phi,x)$. If now $\mathcal{H}^{\ell}_{\infty}(\text{sing}\psi) = 0$, then for any $\varepsilon > 0$ we could find open balls $\{B_{\rho_j}(x_j)\}$ such that

$$(5) \qquad \text{sing}\,\psi \subset \bigcup_j B_{\rho_j}(x_j)$$

and

$$(6) \qquad \sum_j \omega_{\ell}\, \rho_j^{\ell} < \varepsilon$$

(by definition of $\mathcal{H}^{\ell}_{\infty}$). Now (5) in particular implies that $K \equiv \overline{B_1}(0) \sim \bigcup_j B_{\rho_j}(x_j)$ is a compact set with positive distance from $\text{sing}\,\psi$. Hence by assumption A.3(3) we have

$$(7) \qquad \text{sing}\,\phi_{x,\lambda_{k'}} \cap B_1(0) \subset \bigcup_j B_{\rho_j}(x_j)$$

for all sufficiently large k, and hence by (6)

$$\mathcal{H}^{\ell}_{\infty}(\text{sing}\,\phi_{x,\lambda_{k'}} \cap B_1(0)) < \varepsilon , \; k \geq k(\varepsilon) .$$

Thus since $\lambda_k^{-1}(\text{sing}\,\phi - x) = \text{sing}\,\phi_{x,\lambda_k}$ (by A.3(2)) we have

$$\lambda_{k'}^{-\ell}\, \mathcal{H}^{\ell}_{\infty}(\text{sing}\,\phi \cap B_{\lambda_{k'}}(x)) < \varepsilon$$

for all sufficiently large k, thus a contradiction for $\varepsilon < \lim_{k\to\infty} \lambda_k^{-\ell}\, \mathcal{H}^{\ell}_{\infty}(\text{sing}\,\phi \cap B_{\lambda_k}(x))$. (Such ε can be chosen by (4).)

We have therefore established the general implication (2). From now on take $\ell > d-1$ so that $F^{\ell} \neq \emptyset$ (which is automatic for $\ell \leq d$ by definition of d). By (2) there is $\phi \in F^{\ell}$ with $\phi_{0,\lambda} = \phi \quad \forall\, \lambda > 0$. Suppose also that there is a k-dimensional subspace $(k \geq 0)$ S of $\mathbb{R}^P$ such that $\phi_{y,\lambda} = \phi$ $\forall\, y \in S$, $\lambda > 0$. (Notice of course this is no additional restriction for ϕ in case $k = 0$.) Now if $k \geq d+1$ then, by definition of d, we can assert $\operatorname{sing}\phi = \emptyset$, thus contradicting the fact that $\phi \in F^{\ell}$. Therefore $0 \leq k \leq d$, and if $k \leq d-1$ $(< \ell)$, then $H^{\ell}(S) = 0$ and in particular

(8) $$\exists\, x \in B_1(0) \cap \operatorname{sing}\phi \sim S .$$

But by A.2 we can choose $\psi \in T(\phi, x)$. Since $\psi = \lim \phi_{x,\lambda_j}$ for some sequence $\lambda_j \downarrow 0$, we evidently have (since $\phi_{y+x,\lambda} = \phi_{x,\lambda} \quad \forall\, y \in S$, $\lambda > 0$)

(9) $$\psi_{y,1} = \lim \phi_{y+x,\lambda_j} = \lim \phi_{x,\lambda_j} = \psi \qquad \forall\, y \in S$$

and

(10) $$\psi_{\beta x,1} = \lim \phi_{x+\lambda_j \beta x,\lambda_j} = \psi \qquad \forall\, \beta \in \mathbb{R} .$$

(All limits in the weak sense described at the beginning of the section.) Thus $\psi_{z,\lambda} = \psi$ for each $\lambda > 0$ and each z in the $(k+1)$-dimensional subspace T of $\mathbb{R}^P$ spanned by S and x. Sing $\psi \neq \emptyset$ (by A.3(3)), hence by induction on k we can take $k = d-1$; i.e. $\dim T = d$, and hence $\operatorname{sing}\psi \supset T$ by A.3(2). On the other hand if $\exists\, \tilde{x} \in \operatorname{sing}\psi \sim T$ then we can repeat the above argument (beginning at (8)) with T in place of S and ψ in place of ϕ. This would then give a $(d+1)$-dimensional subspace $\tilde{T}$ and a $\tilde{\psi} \in F$ with $\operatorname{sing}\tilde{\psi} \supset \tilde{T}$, thus contradicting the definition of d. Therefore $\operatorname{sing}\phi = T$. Furthermore if $\ell > d$ then the above induction works up to $k = d$ and again therefore we would have a contradiction. Thus $\dim(B_1(0) \cap \operatorname{sing}\phi) \leq d \;\; \forall\, \phi \in F$.

Finally to prove the last claim of the theorem, we suppose that $d = 0$. Then we have already established that

$$(11) \qquad H^{\alpha}(\text{sing}\,\phi \cap B_1(0)) = 0 \quad \forall\ \alpha > 0\ ,\ \phi \in F\ .$$

If $\text{sing}\,\phi \cap B_\rho(0)$ is not finite, then we select $x \in \overline{B_\rho}(0)$ such that $x = \lim x_k$ for some sequence $x_k \in \text{sing}\,\phi \cap B_1(0) \sim \{x\}$. Then letting $\lambda_k = 2|x_k - x|$ we see from A.3(2) that there is a subsequence $\{\lambda_{k'}\}$ with $\phi_{x,\lambda_{k'}} \rightharpoonup \psi \in T(\phi, x)$ and $(x_{k'} - x)/|x_{k'} - x| \to \xi \in \partial B_1(0)$. Now by A.3(2), (3) we know that $\{\xi/2\} \cap \{0\} \subset \text{sing}\,\psi$ and, since $\psi_{0,\lambda} = \psi$, this (together with A.3(2)) gives $L_\xi \subset \text{sing}\,\psi$ where L_ξ is the ray determined by 0 and ξ. Then $H^1(\text{sing}\,\psi \cap B_1(0)) > 0$, thus contracting (11), because $\psi \in F$.

APPENDIX B

NON-EXISTENCE OF STABLE MINIMAL CONES, $2 \le n \le 6$.

Here we describe J. Simons [SJ] result on non-existence of n-dimensional stable minimal cones (previously established in case $n = 2,3$ by Fleming [F] and Almgren [A4] respectively). The proof here follows essentially Schoen-Simon-Yau [SSY], and is slightly cleaner than the original proof in [SJ].

Suppose to begin that $C \in \mathcal{D}_n(\mathbb{R}^{n+1})$ is a cone $(\eta_{0,\lambda\#}C = C)$ and C is integer multiplicity with $\partial C = 0$. If $\operatorname{sing} C \subset \{0\}$ and if C is minimizing in $\mathbb{R}^{n+1}$ then, writing $M = \operatorname{spt} C \sim \{0\}$ and taking M_t as in §9, we have $\frac{d}{dt}\mathcal{H}^n(M_t)\Big|_{t=0} = 0$ and $\frac{d^2}{dt^2}\mathcal{H}^n(M_t)\Big|_{t=0} \ge 0$. (This is clear because in fact $\mathcal{H}^n(M_t)$ takes its minimum value at $t = 0$, by virtue of our assumption that C is minimizing.) Notice that M is orientable, with orientation induced from C, and hence in particular we can deduce from 9.8 that

B.1 $$\int_M (|\nabla^M\zeta|^2 - \zeta^2|A|^2)\,d\mathcal{H}^n \ge 0$$

for any $\zeta \in C_c^1(M)$ (notice $0 \notin M$, so such ζ vanish in a neighbourhood of 0). Here A is the second fundamental form of M and $|A|$ is its length, as described in §7 and in 9.8.

The main result we need is given in the following theorem.

B.2 THEOREM *Suppose* $2 \le n \le 6$ *and* M *is an* n*-dimensional cone embedded in* $\mathbb{R}^{n+1}$ *with zero mean curvature* (*see* §7) *and with* $\bar{M} \sim M = \{0\}$, *and suppose that* M *is stable in the sense that* B.1 *holds. Then* $\bar{M}$ *is a hyperplane.*

(As explained above, the hypotheses are in particular satisfied if $M = \operatorname{spt} C \sim \{0\}$, with $C \in \mathcal{D}_n(\mathbb{R}^{n+1})$ a minimizing cone with $\partial C = 0$ and $\operatorname{sing} C \subset \{0\}$.)

B.3 REMARK Theorem B.2 is false for $n=7$; J. Simons [SJ] was the first to point out that the cone $M=\{(x^1,\ldots,x^8)\in\mathbb{R}^8 : \sum_{i=1}^{4}(x^i)^2=\sum_{i=5}^{8}(x^i)^2\}$ is a stable minimal cone. (Notice that M is the cone over the compact manifold $(\frac{1}{\sqrt{2}}S^3)\times(\frac{1}{\sqrt{2}}S^3)\subset S^7\subset\mathbb{R}^8$.) The fact that the mean curvature of M is zero is checked by direct computation. The fact that M is actually *stable* is checked as follows. First, by direct computation one checks that the second fundamental form A of M satisfies $|A|^2=6/|x|^2$.

On the other hand for a stationary hypersurface $M\subset\mathbb{R}^{n+1}$ the first variation formula 9.3 says $\int \operatorname{div}_M X\,dH^n=0$ if $\operatorname{spt}|X|$ is a compact subset of M. Taking $X_x=(\zeta^2/r^2)x$, $\zeta\in C_c^\infty(M)$, $r=|x|$, and computing as in §17, we get

$$(n-2)\int_M(\zeta^2/r^2)\,dH^n=-2\int_M\zeta r^{-2}x\cdot\nabla^M\zeta\,dH^n .$$

Using the Schwartz inequality on the right we get

$$\frac{(n-2)^2}{4}\int_M(\zeta^2/r^2)\,dH^n\le\int_M|\nabla^M\zeta|^2dH^n .$$

Thus we have stability for M (in the sense of B.1) whenever A satisfies $|x|^2|A|^2\le(n-2)^2/4$.

For the example above we have $n=7$ and $|x|^2\,|A|^2=6$, so that this inequality is satisfied, and the cone over $S^3\times S^3$ is stable as claimed. (Similarly the cone over $S^q\times S^q$ is stable for $q\ge 3$; i.e. when the dimension of the cone is ≥ 7.)

Before giving the proof of B.2 we need to derive the identity of J. Simons for the Laplacian of the length of the second fundamental form of a hypersurface (Lemma B.8 below).

The simple derivation here assumes the reader's familiarity with basic Riemannian geometry. (A completely elementary derivation, assuming no such background, is described in [G].)

For the moment let M be an arbitrary hypersurface in $\mathbb{R}^{n+1}$ (M not necessarily a cone, and not necessarily having zero mean curvature).

Let $\tau_1,\ldots,\tau_n$ be a locally defined family of smooth vector fields which, together with the unit normal ν of M, define an orthonormal basis for $\mathbb{R}^{n+1}$ at all points in some region of M.

The second fundamental form of M relative to the unit normal ν is the tensor $A = h_{ij}\tau_i \otimes \tau_j$, where $h_{ij} = \langle D_{\tau_j}\nu, \tau_i\rangle$. (Cf. §7.) Recall that we have

B.4 $$h_{ij} = h_{ji} ,$$

and, since the Riemann tensor of $\mathbb{R}^{n+1}$ is zero, we have the *Codazzi equations*

B.5 $$h_{ij,k} = h_{ik,j} , \quad i,j,k \in \{1,\ldots,n\} .$$

Here $h_{ij,k}$ denotes the covariant derivative of A with respect to τ_k; that is, $h_{ij,k}$ are such that $\nabla_{\tau_k} A = h_{ij,k}\tau_i \otimes \tau_j$.

We also have the *Gauss curvature* equations

B.6 $$R_{ijk\ell} = h_{i\ell}h_{jk} - h_{ik}h_{j\ell} ,$$

where $R = R_{ijk\ell}\, \tau_i \otimes \tau_j \otimes \tau_k \otimes \tau_\ell$ is the Riemann curvature tensor of M, and where we use the sign convention such that R_{ijji} $(i\neq j)$ are sectional curvatures of M ($= +1$ if $M = S^n$).

From the properties of R (in fact essentially by definition of R) we also have, for any 2-tensor $a_{ij}\,\tau_i \otimes \tau_j$,

$$a_{ij,k\ell} = a_{ij,\ell k} + a_{im}R_{mj\ell k} + a_{mj}R_{mi\ell k}$$

(where $a_{ij,k\ell}$ means $a_{ij,k,\ell}$ - i.e. the covariant derivative with respect to τ_ℓ of the tensor $a_{ij,k}\,\tau_i \otimes \tau_j \otimes \tau_k$). In particular

B.7 $$h_{ij,k\ell} = h_{ij,\ell k} + h_{im}R_{mj\ell k} + h_{mj}R_{mi\ell k}$$

$$= h_{ij,\ell k} + h_{im}[h_{m\ell}h_{jk} - h_{mk}h_{j\ell}] - h_{mj}[h_{i\ell}h_{mk} - h_{ik}h_{m\ell}]$$

by B.6.

B.8 LEMMA *In the notation above*

$$\Delta_M(\tfrac{1}{2}|A|^2) = \sum_{i,j,k} h^2_{ij,k} - |A|^4 + h_{ij}H_{,ij} + Hh_{mi}h_{mj}h_{ij}\ ,$$

where $H = h_{kk} = \text{trace } A$.

Proof We first compute $h_{ij,kk}$:

$$h_{ij,kk} = h_{ik,jk} \qquad \text{(by B.5)}$$

$$= h_{ki,jk} \qquad \text{(by B.4)}$$

$$= h_{ki,kj} + h_{km}[h_{mj}h_{ik} - h_{mk}h_{ij}]$$

$$- h_{mi}[h_{kj}h_{mk} - h_{kk}h_{mj}] \qquad \text{(by B.7)}$$

$$= h_{ki,kj} - \left(\sum_{m,k} h^2_{mk}\right)h_{ij} + h_{kk}h_{mi}h_{mj}$$

$$= h_{kk,ij} - \left(\sum_{m,k} h^2_{mk}\right)h_{ij} + h_{kk}h_{mi}h_{mj} \qquad \text{(by B.5)}$$

Now multiplying by h_{ij} we then get (since $h_{ij}h_{ij,kk} = \frac{1}{2}\left(\sum_{i,j} h^2_{ij}\right)_{,kk} - \sum_{i,j,k} h^2_{ij,k}$)

$$\frac{1}{2}\Big(\sum_{i,j} h_{ij}^2\Big)_{,kk} = \sum_{i,j,k} h_{ij,k}^2 - \Big(\sum_{i,j} h_{ij}^2\Big)^2 + h_{ij}H_{,ij} + Hh_{mi}h_{mj}h_{ij} ,$$

which is the required identity.

We now want to examine carefully the term $\sum_{i,j,k} h_{ij,k}^2$ appearing in the identity of B.8 in case M is a cone with vertex at 0 (i.e. $\eta_{0,\lambda}M = M$ $\forall\lambda > 0$). In particular we want to compare $\sum_{i,j,k} h_{ij,k}^2$ with $|\nabla^M|A||^2$ in this case. Since $|\nabla^M|A||^2 = \sum_{k=1}^{n} |A|^{-2}(h_{ij}h_{ij,k})^2$, we look at the difference

$$(*) \qquad D \equiv \sum_{i,j,k} h_{ij,k}^2 - \sum_{k=1}^{n} |A|^{-2}(h_{ij}h_{ij,k})^2 .$$

B.9 LEMMA *If* M *is a cone (not necessarily minimal) the quantity* D *defined in* (*) *satisfies*

$$D(x) \geq 2|x|^{-2}|A(x)|^2 , \ x \in M .$$

Proof Let $x \in M$ and select the frame $\tau_1,\ldots,\tau_n$ so that τ_n is radial $(x/|x|)$ along the ray ℓ_x through x, and so that (as vectors in $\mathbb{R}^{n+1}$) $\tau_1,\ldots,\tau_n$ are constant along ℓ_x. Then

$$(1) \qquad h_{nj} = h_{jn} = 0 \quad \text{along } \ell_x , \ j = 1,\ldots,n ,$$

and (since $h_{ij}(\lambda x) = \lambda^{-1}h_{ij}(x)$, $\lambda > 0$)

$$(2) \qquad h_{ij,n} = -r^{-1}h_{ij} \quad \text{along } \ell_x .$$

Rearranging the expression for D, we have

$$D = \frac{1}{2}\sum_{k=1}^{n}\sum_{i,j,r,s=1}^{n} |A|^{-2}(h_{rs}h_{ij,k} - h_{ij}h_{rs,k})^2 ,$$

as one easily checks by expanding the square on the right. Now since

$$\sum_{i,j,r,s=1}^{n} (\qquad)^2 \geq 4 \sum_{\substack{i,j,r=1 \\ s=n}}^{n-1} (\qquad)^2 ,$$

we thus have

$$D \geq 2|A|^{-2} \sum_{k=1}^{n} \sum_{i,j,r=1}^{n-1} (h_{ij}h_{rn,k})^2 .$$

By the Codazzi equations B.5 and (2) this gives

$$\begin{aligned} D &\geq 2r^{-2}|A|^{-2} \sum_{k=1}^{n} \sum_{i,j,r=1}^{n-1} h_{ij}^2 h_{rk}^2 \\ &= 2r^{-2}|A|^{-2}|A|^4 \qquad \text{(by (1))} \\ &= 2r^{-2}|A|^2 , \end{aligned}$$

as required.

Proof of B.2 Notice that so far we have not used the minimality of M (i.e. we have not used $H(=h_{kk}) = 0$). We now do set $H = 0$ in the above computations, thus giving (by B.8, B.9)

$$(1) \qquad \Delta_M(\tfrac{1}{2}|A|^2) + |A|^4 \geq 2r^{-2}|A|^2 + |\nabla|A||^2$$

for the minimal cone M. (Notice that $|A|$ is Lipschitz, and hence $|\nabla|A||$ makes sense $\mathcal{H}^n$ - a.e. in M.)

Our aim now is to use (1) in combination with the stability inequality B.1 to get a contradiction in case $2 \leq n \leq 6$.

Specifically, replace ζ by $\zeta|A|$ in B.1. This gives

$$\begin{aligned} (2) \qquad \int_m \zeta^2|A|^4 &\leq \int_M |\nabla(\zeta|A|)|^2 \\ &= \int_M (|\nabla\zeta|^2|A|^2 + \zeta^2|\nabla|A||^2) \\ &\quad + 2\int_M \zeta|A|\nabla\zeta\cdot\nabla|A| . \end{aligned}$$

Now

$$2\int_M \zeta|A|\nabla\zeta\cdot\nabla|A| = 2\int_M \zeta\nabla\zeta\cdot\nabla(\tfrac{1}{2}|A|^2)$$
$$= \int_M (\nabla\zeta^2)\cdot\nabla(\tfrac{1}{2}|A|^2)$$
$$= -\int_M \zeta^2\Delta_M(\tfrac{1}{2}|A|^2)$$
$$\leq \int_M (|A|^4\zeta^2 - 2r^{-2}\zeta^2|A|^2 + \zeta^2|\nabla|A||^2) \qquad \text{by (1)},$$

and hence (2) gives

$$2\int_M r^{-2}\zeta^2|A|^2 \leq \int_M |A|^2|\nabla\zeta|^2 \qquad \forall\ \zeta \in C_c^1(M)\ . \tag{3}$$

Now we claim that (3) is valid even if ζ does not have compact support on M, provided that ζ is locally Lipschitz and

$$\int_M r^{-2}\zeta^2|A|^2 < \infty\ . \tag{4}$$

(This is proved by applying (3) with $\zeta\gamma_\varepsilon$ in place of ζ, where γ_ε is such that $\gamma_\varepsilon(x) \equiv 1$ for $|x| \in (\varepsilon,\varepsilon^{-1})$, $|\nabla\gamma_\varepsilon(x)| \leq 3/|x|$ for all x, $\gamma_\varepsilon(x) = 0$ for $|x| < \varepsilon/2$ or $|x| > 2\varepsilon^{-1}$, and $0 \leq \gamma_\varepsilon \leq 1$ everywhere, then letting $\varepsilon \downarrow 0$ and using (4).)

Since M is a cone we can write

$$\int_M \phi(x)\,dH^n(x) = \int_0^\infty r^{n-1}\int_\Sigma \phi(r\omega)\,dH^{n-1}(\omega)\,dr \tag{5}$$

for any non-negative continuous ϕ on M, where $\Sigma = M \cap S^n$ is a compact $(n-1)$-dimensional submanifold. Since $|A(x)|^2 = r^{-2}|A(x/|x|)|^2$, we can now use (5) to check that $\zeta = r^{1+\varepsilon}r_1^{1-n/2-2\varepsilon}$, $r_1 = \max\{1,r\}$, is a valid choice to ensure (4), hence we may use this choice in (3). This is easily seen to give

$$(6) \qquad 2\int_M r^{2\varepsilon} r_1^{2-n-4\varepsilon} |A|^2 \le ((n/2)-2+\varepsilon)^2 \int_{M\cap\{r>1\}} |A|^2 r^{2-n-2\varepsilon}$$

$$+ (1+\varepsilon)^2 \int_{M\cap\{r<1\}} |A|^2 r^{2\varepsilon}$$

$$< \infty .$$

For $2 \le n \le 6$ we can choose ε such that $((n/2)-2+\varepsilon)^2 < 2$ and $(1+\varepsilon)^2 < 2$, hence (6) gives $|A|^2 \equiv 0$ on M as required.

REFERENCES

[A1] F.J. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Annals of Math. 27 (1968), 321-391.

[A2] ——————, *Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension 2*, Preprint, Princeton University (1984).

[A3] ——————, *The theory of varifolds*, Princeton notes, 1965.

[A4] ——————, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's Theorem*, Annals of Math. 84 (1966), 277-292.

[AW1] W.K. Allard, *First variation of a varifold*, Annals of Math. 95 (1972), 417-491.

[AW2] ——————, *First variation of a varifold-boundary behaviour*, Annals of Math. 101 (1975), 418-446.

[B] A.S. Besicovitch, *On the fundamental geometric properties of linearly measurable plane sets I, II, III*, Math. Ann. 98 (1927), 422-464; 115 (1938), 296-329; 116 (1939), 349-357.

[BG] E. Bombieri, E. Giusti, *Harnack's inequality for elliptic differential equations or minimal surfaces*, Invent. Math. 15 (1972), 24-46.

[DG] E. De Giorgi, *Frontiere orientate di misura minima*, Sem. Mat. Scuola Norm. Sup. Pisa (1961), 1-56.

[DJ] J. Duggan, *Regularity theorems for varifolds with mean curvature*, Ph.D. thesis, Australian National University, 1984.

[FH1] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin-Heidelberg-New York (1969).

[FH2] H. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. A.M.S. 76 (1970), 767-771.

[FF] H. Federer, W. Fleming, *Normal and integral currents*, Annals of Math. 72 (1960), 458-520.

[FW] W. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo 11 (1962), 69-90.

[G] E. Giusti, Minimal surfaces and functions of bounded variation, Notes on Pure Mathematics #10, A.N.U. (1977). (See also forthcoming book published by Birkhäuser.)

[GT] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order (2nd Edition), Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[HR] R. Hardt, *An introduction to Geometric Measure Theory*, Lecture Notes, Melbourne University (1979).

[HS] R. Hardt, L. Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Annals of Math. 110 (1979), 439-486.

[MS] J.H. Michael, L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbb{R}^n$, Comm. Pure Appl. Math. 26 (1973), 361-379.

[M] M.E. Munroe, Introduction to measure and integration, Addison-Wesley (1953).

[MM1] M. Miranda, *Sulle singolarità della frontiere minimali*, Rend. Sem. Mat. Padova (1967), 181-188.

[MM2] ———, *Frontiere minimali con ostacoli*, Ann. Univ. Ferrara Sez. VII (N.S.) 16 (1971), 29-37.

[PM] M. Protter, *Unique continuation for elliptic equations*, Trans. A.M.S. 95 (1960), 81-91.

[PJ] J. Pitts, Existence and Regularity of Minimal Surfaces on Riemannian manifolds, Princeton University Press (Mathematical Notes) (1981).

[RM] M. Ross, *Federer's Structure Theorem*. Research Report, Centre for Mathematical Analysis, Australian National University, 1984.

[Roy] H. Royden, Real Analysis (2nd Edition) Collier-Macmillan, New York (1968).

[R1] E. Reifenberg, *Solution of the Plateau problem for* m*-dimensional surfaces of varying topological type*, Acta Math. 104 (1960), 1-92.

[R2] ——————, *An epiperimetric inequality related to the analyticity of minimal surfaces. On the analyticity of Minimal surfaces.* Annals of Math. 80 (1961), 1-21.

[SS1] R. Schoen, L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. 34 (1981), 741-797.

[SS2] ————, *A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals*, Indiana Univ. Math. J. 31 (1982), 415-434.

[SJ] J. Simons, *Minimal varieties in Riemannian manifolds*, Annals of Math. 88 (1968), 62-105.

[SE] E. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton (1960).

[SL1] L. Simon, *Survey lectures on minimal submanifolds*, Annals of Math. Studies 103 (Ed. E. Bombieri) (1983), 3-52.

[SL2] ————, *Asymptotic behaviour of minimal surfaces and harmonic maps* (to appear).

[SL3] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Annals of Math. 118 (1983), 525-571.

[SSY] R. Schoen, L. Simon, S.-T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. 134 (1975), 276-288.

[SB] B. Solomon, Thesis, Princeton University (1982).

[SSA] R. Schoen, L. Simon (Part 1), F. Almgren (Part 2), *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals*, Acta. Math. 139 (1977), 217-265.

[WB] B. White, Thesis, Princeton University (1982).

ISBN 0 86784 429 9