LADDER INDICES AND OTHER TOPICS
IN RANDOM WALK

by

Ian Stewart McRae, B.Sc. (Hons) Monash

A thesis submitted for the degree
of Master of Science at
The Australian National University,
Canberra.

March 1972
Except where stated otherwise, the work included in this thesis is the author's own.
The author acknowledges the great assistance given to him by Dr. C.C. Heyde, and thanks Dr. Heyde for suggesting the topic and for many stimulating conversations. He also thanks his wife Roberta, to whom he dedicates this thesis, for her tolerance and understanding.
## CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER 1 - A LIMIT THEOREM FOR THE NUMBER OF LADDER STEPS OF A RANDOM WALK</td>
<td>3</td>
</tr>
<tr>
<td>CHAPTER 2 - FINITENESS OF THE EXPECTED FIRST POSITIVE STEP</td>
<td>10</td>
</tr>
<tr>
<td>2.1 Case of Zero Mean</td>
<td>10</td>
</tr>
<tr>
<td>2.2 Case of Positive Mean</td>
<td>25</td>
</tr>
<tr>
<td>CHAPTER 3 - ON THE APPROACH TO ZERO OF THE UNIFORM METRIC FOR CONVERGENCE TO A STABLE LAW</td>
<td>28</td>
</tr>
<tr>
<td>CHAPTER 4 - DRIFT OF A RANDOM WALK</td>
<td>36</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>45</td>
</tr>
</tbody>
</table>
INTRODUCTION

This thesis has as its central thread the study of drift and extremes of a random walk. This is studied in a number of ways - in terms of the number of ladder steps and the expected size of these ladder steps, as well as directly. Many of the results achieved are more general versions of known results (e.g. the generating random variables of random walks are treated in general or in terms of domains of attraction of stable laws rather than as random variables with finite variance).

In Chapter 1 we introduce the main variables which are used in the thesis, and prove a single theorem. The theorem describes the limiting distribution for the number of ladder steps of a random walk, and thus allows us to make statements about the limiting distribution of $\max_{0 \leq k \leq n} S_k$. In fact our theorem generalises a theorem of Heyde [18] to the case of general generating random variables, and also gives a converse. The limiting distribution of $\max_{0 \leq k \leq n} S_k$ which is derived from it is dependent upon the expected first positive step ($E[Z]$) of the random walk. Hence in Chapter 2 we go on to discuss the conditions under which $E[Z]$ is finite.

In Chapter 2 we conjecture and progress a long way towards proving that $E[Z]$ is finite if (i) the generating random variables of the random walk belong to the domain of normal attraction of stable law with index $\gamma$, and (ii) $1 - \lim_{n} n^{-1} \sum_{k=1}^{n} P(S_k \leq 0) = \gamma^{-1}$. Although we do not prove this result in general, we do establish it in the case of left continuous random walks. We also produce more general versions of the results of Spitzer [29] relating to the expected first positive step.
It is further shown in Chapter 2 that the expected value of the first positive step is largely dependent upon \( \sum_{n=1}^{\infty} n^{-1} \left[ P(S_n \leq 0) - \alpha \right] \) where \( \alpha = \lim_{n} n^{-1} \sum_{k=1}^{n} P(S_k \leq 0) \), and in Chapter 3 we discuss the global version of this series, viz. \( \sum_{n=1}^{\infty} n^{-1+\gamma} \sup_{x} \left| F_n(x) - G(x) \right| , \gamma \geq 0 \), where \( \lim_{n} F_n(x) = G(x) \) for all \( x \). Our results for this series, which have immediate application to the study of \( E[Z] \), treat the case of generating random variables which belong to the domain of normal attraction of a stable law.

In Chapter 4 we study the drift of random walks directly in terms of \( S_n/n \). We are able to obtain conditions for \( \lim_{n} S_n/n \) to be infinite almost surely (a.s.), although these conditions are not easily manipulated. We conjecture that \( \frac{P(X^+ > x)}{P(X^- > x)} \rightarrow \infty \) implies that \( S_n/n \rightarrow \infty \) a.s. to overcome this difficulty. Although we are unable to actually prove this highly plausible result, we do prove a number of corroborating lemmas. We conclude with a historical note on the main results obtained in this field by other authors.
CHAPTER 1
A LIMIT THEOREM FOR THE NUMBER OF
LADDER STEPS OF A RANDOM WALK

This chapter, which serves as an introduction to the terminology, notation and concepts of the later chapters, has as its main result a study of the limit behaviour of the number of ladder steps in a random walk. Here, as in the remainder of this thesis, we study the random walk described by partial sums $S_n = \sum_{i=1}^{n} X_i$ of random variables $X_1, X_2, \ldots$, which are independent and identically distributed. Our results extend Theorem 2 of Heyde [18] which applies to the very special situation where the $X_i$ are distributed according to a stable law, whereas our theorem treats the case of general $X_i$ and also provides a converse result.

We must now list the main definitions used throughout the thesis. We work largely in terms of "ladder steps", which are perhaps better described as "positive steps". We let $N_1, N_2, \ldots$, be the strong ascending ladder indices of $S_n$, defined iteratively by

$$N_1 = \min\{n : S_n > 0\}$$
$$N_{i+1} = \min\{n : S_n > S_{N_i}\}.$$ 

The strong ascending ladder steps $Z_1, Z_2, \ldots$, are then defined as $Z_i = S_{N_i}$ and $Z_{i+1} = S_{N_i+1} - S_{N_i}$. We also define $R_n$ as the number of strong ascending ladder steps in the first $n$ steps of the random walk (i.e. $R_n = \max\{k : N_k < n\}$). We denote the event "a strong ascending ladder step occurs" by $\delta$. Let $P_n$ be the number of positive terms in the sequence $S_1, S_2, \ldots, S_n$. We then define the indicator function $\delta_k$ by
\[ \delta_k = \begin{cases} 1 & \text{if } \delta \text{ occurs at the } k^{th} \text{ step} \\ 0 & \text{otherwise} \end{cases} \]

and Heyde [18], p.423 shows \( P(\delta_k = 1) = P(P_k = k) \).

Now, following standard renewal theory definitions we let

\[ u_0 = 1, \quad u_n = P(\delta_n = 1) = P(\delta \text{ occurs at step } n) \quad \text{for } n \geq 1 \]

and

\[ f_n = P(\delta_1 = 0, \ldots, \delta_{n-1} = 0, \delta_n = 1) = P(\delta \text{ first occurs at step } n) \quad \text{for } n \geq 1. \]

We also define \( U(t) = \sum_{n=0}^{\infty} u_n t^n \) and \( F(t) = \sum_{n=1}^{\infty} f_n t^n \) for \( 0 \leq t \leq 1 \). The standard identity of recurrent event theory (see for example Prabhu [25], p.186) gives

\[ U(t) = [1 - F(t)]^{-1} \quad (1-1) \]

for \( 0 \leq t < 1 \). For \( q_n = \sum_{r=n+1}^{\infty} f_r = P(\delta \text{ first occurs after step } n) \),

\[ Q(t) = \sum_{n=1}^{\infty} q_n t^n = (1 - t)^{-1} (1 - F(t)). \quad (1-2) \]

Also, using a result of Sparre-Andersen (see Spitzer [29], p.219) we have

\[ U(t) = \sum_{n=0}^{\infty} u_n t^n = \sum_{n=0}^{\infty} P(P_n = n) t^n \]

\[ = \exp \left\{ \sum_{k=1}^{\infty} \frac{t^k}{k} P(S_k > 0) \right\}. \quad (1-3) \]

The main theme of this thesis is the investigation of the positive motion of a random walk, and one way of doing so is as follows. Suppose \( S_0 = 0 \), then we have \( \max_{0 \leq k \leq n} S_k = Z_1 + Z_2 + \ldots + Z_{R_n} \).

\( R_n \) is non decreasing in \( n \), so that \( P_{\infty}^{R_n} (\text{i.e. } P(R_n \leq M) \to 0 \text{ as } n \to \infty \) for every \( M > 0 \) is equivalent to \( R_n \sim s. \to \infty \). This follows since

\[ P \left( U_{k=n}^{\infty} [R_k \leq M] \right) = P(R_n \leq M) \to 0 \quad (1-4) \]
and if the distribution of the recurrence time of \( \delta \) is proper (i.e. \( P(\delta \text{ occurs}) = 1 \)) we have

\[
P(R_n \geq k) = P(T_k \leq n)
\]

(1-5)

where \( T_k \) is the sum of \( k \) independent and identically distributed random variables each with the same distribution as the recurrence time of \( \delta \).

As \( \lim P(T_k \leq n) \) is clearly one, we deduce from this that \( R_n \xrightarrow{a.s.} \infty \) as \( n \to \infty \). As the \( Z_i \) are independent identically distributed random variables, we have thus shown that \( \limsup S_n = \infty \) with probability one.

Now \( P(\delta \text{ occurs}) = 1 \) if and only if \( \lim P(T^+ \leq n) = 1 \), which (from (1-1) and (1-3)) will happen if and only if \( \exp(\sum \frac{1}{k} P(S_k > 0)) \) is infinite.

These results provide a different approach to results of Spitzer [27] Theorem 4.1 in which it is shown that \( \limsup S_n = +\infty \) with probability one if and only if \( \sum \frac{1}{k} P(S_k > 0) = \infty \). The following theorem gives an answer to the problem of the existence of a proper limit law for \( R_n \) when suitably normed.

**Theorem 1-1** Let \( g_{\alpha}(x) \) be the distribution function corresponding to a stable law with characteristic function

\[
g_{\alpha}(t) = \exp\left\{ -|t|^{\alpha} \left( \cos(\pi \alpha/2) - i \sin(\pi \alpha/2) \sgn t \right) \Gamma(1 - \alpha) \right\}
\]

(1-6)

for some \( \alpha, 0 < \alpha < 2 \)

(i) If \( \lim n^{-1} \sum_{k=1}^{n} P(S_k > 0) \) exists and equals \( \alpha, 0 < \alpha < 1 \), then

\[
\lim_{n \to \infty} P(R_n \leq x/C_n) = 1 - G_{\alpha}(x^{-1/\alpha})
\]

(1-7)

where \( C_n = \Gamma(1 - \alpha) n^{-\alpha} L(n) \) and \( L(n) \) varies slowly at infinity, being asymptotic to

\[
\exp \left\{ \sum_{k=1}^{\infty} n^{-1}(1 - n^{-1})^k [P(S_k > 0) - \alpha] \right\}
\]

(ii) Conversely, if there exists a sequence of non-negative norming constants \( \{C_n\}_{n=1}^{\infty} \), such that \( \lim_{n \to \infty} P(R_n \leq x/C_n) = 1 - G_{\alpha}(x^{-1/\alpha}) \) with
0 < \alpha < 1 \text{ then } \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} P(S_k > 0) \text{ exists and equals } \alpha.

Proof of (i) Assume \( \lim_{n} n^{-1} [P(S_1 > 0) + \ldots + P(S_n > 0)] \) exists and equals \( \alpha \). Then from (1-3) we see

\[
U(t) = \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k [P(S_k > 0) - \alpha] \right\} \cdot \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k \alpha \right\}
\]

= \( h(t) (1 - t)^{-\alpha} \).

Now Heyde [17], p.546, shows that if

\[ L \left( \frac{1}{1-t} \right) = h(t) = \exp \left\{ k^{-1} t^k [P(S_k > 0) - \alpha] \right\}, \quad (1-8) \]

then \( L(u) \) is a slowly varying function as \( u \to \infty \). Substituting this into (1-2) we have

\[
Q(t) = (1 - t)^{-1} (1 - F(t))
\]

= \( (1 - t)^{-1} [U(t)]^{-1} \)

= \( (1 - t)^{\alpha-1} \left[ L \left( \frac{1}{1-t} \right) \right]^{-1} \)

and from the Tauberian theorem of Feller [11], p.423, (since \( q_n \) is monotonic decreasing) \( q_n \sim n^{\alpha} [\Gamma(1 - \alpha) L(n)]^{-1} \). Then by Theorem 7 of Feller [10], as \( 0 < \alpha < 1 \) and \( q_n \) gives the distribution of the recurrence time of the ladder steps, we have

\[
P(R_n \geq x/C_n) = G_n(x^{-1}/C_n) \quad (1-9)
\]

where \( C_n = q_n \sim n^{\alpha} [\Gamma(1 - \alpha) L(n)]^{-1} \) and thus will hold for \( C_n = n^{\alpha} [\Gamma(1 - \alpha) L(n)]^{-1} \) and part (i) of the theorem is proved.

If we try to extend this part of the theorem to the case \( \alpha = 1 \) (i.e. \( \lim_{n} n^{-1} \sum_{k=1}^{\infty} P(S_k > 0) = 1 \)) however, we are unable to obtain an explicit form for \( q_n \) by the methods used here. We therefore restrict \( \alpha \) to be less than unity.

Proof of (ii) We now prove the converse of (i). We assume that
there exists a sequence of non-negative numbers \( \{ C_n \}_{n=1}^{\infty} \) such that
\[
\lim_n P(R_n \leq x/C_n) = 1 - G_{\alpha^{-1}}(x), \quad 0 < \alpha < 1.
\]
Putting \( \mu_n = 0 \) and \( a_n = C_n \) for all \( n \) in Theorem 6 of Feller [10] we have, for some \( \delta, 0 < \delta < 2, \)
\[
\lim P(R_n \leq x/C_n) = 1 - G_{\alpha^{-1}}(x), \quad 0 < \alpha < 1.
\]
Putting \( p_n = 0 \) and \( n_n \to n_n \) we have, for some \( \delta, 0 < \delta < 2, \)
\[
1 - F(x) = x^{-\delta} h^*(x),
\]
where \( F(x) \) is the distribution function of the strong ascending ladder steps and \( h^*(x) \) varies slowly at infinity. Now if \( \delta \geq 1 \), from Theorem 7 of Feller [10], we have
\[
P \left( R_n \geq \frac{k}{\mu} - \frac{b_k}{\mu^{\delta+1/\delta}} x \right) = G_{\delta^{-1/\delta}}(x).
\]
However the theorem also states that this is the only possible non-normal limiting distribution of \( R_n \), so \( \lim_n P(R_n \leq x/C_n) = 1 - G_{\alpha^{-1}}(x) \) is not possible. Thus \( \delta \geq 1 \) gives a contradiction, so \( \delta < 1 \) and following a similar argument we see that \( \delta = \alpha \). As we are working in discrete time, we now have \( q_n = F(n) = n^{-\alpha} h^*(n) \). Furthermore as \( q_n \) is monotonic, Theorem 5 of Feller [11] p.423 gives
\[
Q(s) = \sum_{n=1}^{\infty} q_n s^n \sim \frac{1}{(1-s)^{1-\alpha}} h^* \left( \frac{1}{1-s} \right) \Gamma(1-\alpha)
\]
as \( s \to 1^- \). Then as before,
\[
U(s) = (1-s)^{-1} [Q(s)]^{-1} \sim (1-s)^{-\alpha} \left[ h^* \left( \frac{1}{1-s} \right) \Gamma(1-\alpha) \right]^{-1}
\]
and we showed (equation (1-3)) that
\[
U(s) \sim (1-s)^{-\alpha} \exp \left\{ \sum_{k=1}^{\infty} k^{-1} s^k [P(S_k > 0) - \alpha] \right\}
\]
so we have
\[
h^* \left( \frac{1}{1-s} \right) \Gamma(1-\alpha) \sim \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} s^k [P(S_k > 0) - \alpha] \right\}
\]
as \( s \to 1^- \). Defining \( k(t) = h^*(t) \Gamma(1-\alpha) \) we obtain
\[ k(t) \sim \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} \left( 1 - \frac{1}{t} \right)^k [P(S_k > 0) - \alpha] \right\} \]

which is slowly varying at \( \infty \). Now if we let \( U_k = \alpha - P(S_k > 0) \) and

\[ \delta(y) = \frac{1}{y} \sum_{k=1}^{\infty} \left( 1 - \frac{1}{y} \right)^{-1} U_k \]

we can represent \( k(t) \) by

\[ \exp \left\{ \int_{1}^{\infty} \frac{\delta(u)}{u} \, du \right\} . \]

As \( k(t) \) is slowly varying at infinity,

\[ k(tx)/k(t) \rightarrow 1 \text{ as } t \rightarrow \infty. \]

Now

\[ \log(k(ty)/k(y)) = \int_{1}^{yt} \frac{\delta(u)}{u} \, du - \int_{1}^{y} \frac{\delta(u)}{u} \, du \]

\[ = \int_{y}^{yt} \frac{\delta(u)}{u} \, du \]

\[ = \int_{1}^{t} \frac{\delta(xy)}{x} \, dx , \]

so since \( \log(k(ty)/k(y)) \rightarrow 0 \text{ as } y \rightarrow \infty \), we see \( \int_{1}^{t} \frac{\delta(xy)}{x} \, dx \rightarrow 0 \text{ as } y \rightarrow \infty \). We now readily show that \( \delta(x) \) is bounded for \( x \geq 1 \), since

\[ |\delta(x)| \leq \frac{1}{x} (1 + \alpha) \sum_{k=1}^{\infty} \left( 1 - \frac{1}{x} \right)^k \]

\[ = \frac{1}{x} (1 + \alpha) \left[ 1 - \left( 1 - \frac{1}{x} \right) \right]^{-1} = 1 + \alpha < 2 . \]

We can therefore apply the mean value theorem to give

\[ \int_{1}^{t} \frac{\delta(xy)}{x} \, dx = \delta(y \ t_0(y,t)) \int_{1}^{t} \frac{dx}{x} \]

\[ = \delta(y \ t_0(y,t)) \log t \]

\[ \rightarrow 0 \text{ as } y \rightarrow \infty \]

as we have shown \( \int_{1}^{t} \frac{\delta(xy)}{x} \, dx \rightarrow 0 \text{ as } y \rightarrow \infty \). Thus as \( 1 \leq t_0(y,t) \leq t \), \( \delta(t) \) is continuous and \( t \) is arbitrary, we have shown \( \delta(y) \rightarrow 0 \text{ as } y \rightarrow \infty \).

This means
0 = \lim_{y \to \infty} \delta(y) \\
= \lim_{y \to \infty} \left\{ y^{-1} \sum_{k=1}^{\infty} (1 - y^{-1})^{k-1} U_k \right\} \\
= \lim_{y \to \infty} \left\{ y^{-1} \sum_{k=1}^{\infty} (1 - y^{-1})^{k-1} \alpha - y^{-1} \sum_{k=1}^{\infty} (1 - y^{-1})^{k-1} P(S_k > 0) \right\} \\
= \alpha - \lim_{s \to 1} (1 - s) \sum_{k=1}^{\infty} s^{k-1} P(S_k > 0) \quad (1-12)

Hence, from Theorem 5 of Feller [11] p.423 we have
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P(S_k > 0) = \alpha and the theorem is proved. Once again this result applies only for \( \alpha < 1 \). If \( \alpha = 1 \), the results of Feller [10] do not apply to the conditions of our theorem. Although we have not done so, it may be possible to make an equivalent statement, for instance:

if

\[ P(N_k \geq \mu_n + a_n x) \to G_1(x^{-1}) \]

then \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P(S_k > 0) = 1 \), where \( \mu_n \) and \( a_n \) depend on whether \( \mathbb{E}[X] < \infty \). Work of this nature has been done by Doeblin [8].

Under appropriate conditions Theorem 1-1 leads to a proper limit law for \( \max_{0 \leq k \leq n} S_k \) when suitably normed. Suppose

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P(S_k > 0) \text{ exists and equals } \alpha, 0 < \alpha < 1 \].

If in addition \( \mathbb{E}[Z] < \infty \), we have

\[ C_n \max_{0 \leq k \leq n} S_k = R^{-1}_n (Z_1 + \ldots + Z_R) C_n R_n \]

and using Theorem 1-1, and a law of large numbers due to Richter (see [18] p.426)

\[ \lim_{n \to \infty} P(C_n \max_{0 \leq k \leq n} S_k \leq x) = 1 - G_\alpha((x E[Z])^{-1/\alpha}). \]

A study of the finiteness of \( \mathbb{E}[Z] \) is made in the following chapter.
CHAPTER 2
FINITENESS OF THE EXPECTED FIRST POSITIVE STEP

2.1 Case of Zero Mean

Throughout this section we consider the random walks generated by independent identically distributed random variables \( \{X_i\}_{i=1}^{\infty} \) with zero mean and partial sums \( \{S_n\}_{n=1}^{\infty} \). We now study the expected first positive steps of these random walks. We define

\[
\Delta = \sum_{n=1}^{\infty} \frac{1}{n} [P(S_n < 0) - \alpha] \quad (2-1)
\]

where

\[
\alpha = \lim_{n \to \infty} \frac{P(S_1 < 0) + P(S_2 < 0) + \ldots + P(S_n < 0)}{n}. \quad (2-2)
\]

This limit is assumed to exist in the cases which we study, although the converse to Theorem 1-1(i) would imply that \( \alpha \) need not exist, and Spitzer [27] Theorem 7.1 considers this possibility.

\( \Delta \) will be shown to be significant in the determination of whether \( E[Z] \) is finite. Later in this chapter, and in Chapter 3, we examine the behaviour of a more general form of \( \Delta \), viz.

\[
\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_x |F_n(x) - G(x)|, \quad \gamma \geq 0,
\]

where \( F_n(x) = P(S_n/B_n \leq x) \), \( B_n \) being chosen so that \( P(S_n/B_n \leq x) \to G(x) \) as \( n \to \infty \) for all \( x \). Any results on this generalised \( \Delta \) however, clearly have application to that defined in (2-1), since the existence of the real limit \( \lim_n P(S_n/B_n \leq x) \) implies the existence of the Cesaro limit.

Spitzer [28], [29] derived the first results in this field, showing in the case where the generating variables \( \{X_i\} \) had finite variance \( \sigma^2 \) (i.e. belonged to the domain of normal attraction of the normal law), that
\[ E[Z] = \frac{\alpha^2}{2} e^\Delta < \infty . \] (2-3)

In this case \( \alpha = \frac{1}{2} \) from application of the Central Limit Theorem.

Following the basic methods employed by Spitzer, we will derive several results under more general conditions, and apply these to prove (2-3).

**Theorem 2-1** If \( \{X_i\}_{i=1}^\infty \) are independent identically distributed random variables with mean zero and if \( \lim_{n \to \infty} \sum_{k=1}^{n} P(S_k < 0) = \alpha \), then

\[
E[Z] = \lim_{t \uparrow 1} F^*(t) G^*(t) \tag{2-4}
\]

where \( F^*(t) \) and \( G^*(t) \) are defined by

\[
F^*(t) = (1 - t)^\alpha \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k P(S_k \leq 0) \right\}
\]

\[
= \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k [P(S_k \leq 0) - \alpha] \right\} \tag{2-5}
\]

\[
G^*(t) = -(1 - t)^{1-\alpha} \sum_{k=1}^{\infty} k^{-1} t^k E[S_k, S_k \leq 0] \tag{2-6}
\]

**Proof** First note that \( E[A, B \in C] \) is defined as \( \int_{B \in C} A \ dP \). Let

\[ T = \min_n \{ n : 1 \leq n \leq \infty, S_n > 0 \} . \]

Now as \( E[Z] = E[S_T] \), we have

\[
E[Z] = \lim_{n \to \infty} \sum_{k=1}^{n} E[S_n, T = k]
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \{ E[S_k, T > k - 1] - E[S_k, T > k] \}
\]

\[
= - \lim_{n \to \infty} E[S_n, T > n] . \tag{2-7}
\]

Following Spitzer [28] we now define \( A_n = E[S_n, T > n] \) and
A(t) = \sum_{n=0}^{\infty} A_n t^n. Then

\[ E[Z] = - \lim_{n \to \infty} A_n = - \lim_{t \to 1} (1 - t) A(t) \quad (2-8) \]

from application of a standard Abelian Theorem (see for example Feller [11], p.423). Also from equation (6-8) of Theorem 6-1 of Spitzer [27] (which was incorrectly stated in the paper), we obtain

\[
\sum_{n=0}^{\infty} t^n \int_{T > n} \exp(-i\beta S_n) \, dP = \sum_{n=0}^{\infty} t^n E[\exp(-i\beta S_n), T > n]
\]

\[ = \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k [E(\exp(i\beta S_k)) - P(S_k > 0)] \right\} \]

\[ = \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k E[\exp(-i\beta S_k), S_k \leq 0] \right\} \]

as

\[
E[\exp(i\beta S_k^c)] = \int_{S_k \leq 0} \exp(-i\beta S_k) \, dP + \int_{S_k > 0} 1 \, dP
\]

\[ = E[\exp(-i\beta S_k, S_k \leq 0)] + P(S_k > 0). \]

Thus, putting \( \beta = -i\lambda \), we have

\[
\sum_{n=0}^{\infty} t^n E[\exp(\lambda S_n), T > n] = \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k E[\exp(\lambda S_k), S_k \leq 0] \right\}.
\]

To obtain a term of the form \( A(t) \), we now differentiate with respect to \( \lambda \), and put \( \lambda = 0 \). Thus we have

\[
\sum_{n=0}^{\infty} t^n E[S_n, T > n] = \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k P(S_k \leq 0) \right\}
\]

\[ \times \sum_{k=1}^{\infty} k^{-1} t^k E[S_k, S_k \leq 0]. \]

Equation (2-8) gives us
\[ E[Z] = - \lim_{t \uparrow 1} (1 - t) \sum_{n=0}^{\infty} t^n E[S_n, T > n] \]

\[ = - \lim_{t \uparrow 1} (1 - t) \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^k P(S_k \leq 0) \right\} \times \sum_{k=1}^{\infty} k^{-1} t^k E[S_k, S_k \leq 0] \]

\[ = \lim_{t \uparrow 1} F^*(t) G(t), \]

\( F^*(t) \) and \( G(t) \) being defined in the statement of the theorem. This completes the proof.

From this result we can immediately draw some conclusions about \( E[Z] \), as we know (mentioned in the previous chapter, and proved in Heyde [17]) that \( F^*(t) \) is slowly varying as \( t \to 1 \). We have

\[ E[Z] < \infty \text{ if and only if } \lim_{t \uparrow 1} \frac{F^*(t)}{G^*(t)} < \infty \text{ and from Feller [11], p.423 this holds if and only if } \sum_{k=1}^{\infty} k^{-1} E[S_k, S_k > 0] \sim n^{1-\alpha} \text{ as } n \to \infty. \]

Further, if \( \Delta \) is convergent, we have

\[ \lim_{t \uparrow 1} F(t) = \lim_{t \uparrow 1} \frac{F(1 - \frac{1}{n})}{G(t)} = e^\Delta, \text{ so } E[Z] < \infty \text{ if and only if } \sum_{k=1}^{\infty} k^{-1} E[S_k, S_k > 0] \sim n^{1-\alpha} \text{ constant.} \]

As an example, let \( X^+ \) be such that \( X^+_n/n^{1/r} \to \infty \) with probability one for some \( r, 1 < r < 2 \) (note that \( X^+ \) denotes \( \max(0,X) \)), but \( E[X_n^+] = 0 \). Then for any \( K > 0 \), \( E[S_k, S_k > 0]/k^{1/r} > K \) for \( k \) sufficiently large, say \( k > k_K \). Thus for \( n \) sufficiently large,

\[ \sum_{k=1}^{\infty} k^{-1} E[S_k, S_k > 0] \sim n^{1-\alpha} \times \text{constant.} \]

If \( 1/r > 1 - \alpha \),
\[ \lim_{n} \sum_{k=1}^{n} k^{-1} E[S_k^*, S_k > 0] / \text{const} \ n^{1-\alpha} \]

\[ > \lim_{n} [(\text{const}) \ n^{1/r-1-\alpha} - (\text{const}) \ n^{1/r-2-\alpha}] \]

\[ = \infty \]

so \[ \sum_{k=1}^{1} \frac{1}{k} E[S_k^*, S_k > 0] / \text{const} \ n^{1-\alpha} \] cannot occur. That means \( \frac{1}{r} > 1 - \alpha \) implies \( E[Z] = \infty \).

We now proceed to obtain more detailed results for \( F^*(t) \) and \( G^*(t) \), and hence \( E[Z] \), by making more restrictions on the form of \( X_i \).

**Theorem 2-2** If random variables \( \{X_i \}_{i=1}^{\infty} \) belong to the domain of attraction of a stable law with index \( \gamma \), \( 1 < \gamma < 2 \), then

\[ \lim_{t \to 1} G^*(t) = \infty \quad \text{if } 1 - \alpha < 1/\gamma \]

is indeterminate in general \( \text{if } 1 - \alpha = 1/\gamma \). \( \text{Eq. } (2-9) \)

The case \( 1 - \alpha > 1/\gamma \) does not arise.

**Proof** Our proof relies on a lemma which is a special case of Theorem 5 p.423 of Feller [11].

**Lemma 2-1** If \( \{C_n \}_{n=1}^{\infty} \) is a convergent sequence of positive numbers, and \( 0 < \delta < 1 \), then as \( n \to \infty \)

\[ (1 - t)^{1-\delta} \sum_{k=1}^{\infty} C_k t^k / n \delta \sim \Gamma(1 - \delta) \lim_{n} C_n \quad \text{Eq. } (2-10) \]

Applying this with \( C_n = n^{-1+\alpha} E[S_n^*, S_n > 0] \), we have

\[ G^*(t) = -(1 - t)^{1-\alpha} \sum_{k=1}^{\infty} k^{-1} t^k E[S_k^*, S_k \leq 0] \]

\[ = (1 - t)^{1-\alpha} \sum_{k=1}^{\infty} k^{-1} t^k E[S_k^*, S_k > 0] \]

\[ \sim \Gamma(1 - \alpha) \lim_{n} n^{-1+\alpha} E[S_n^*, S_n > 0] . \]
Thus,

$$\lim_{t \to 1} G^*(t) = \Gamma(1 - \alpha) \lim_{n \to \infty} n^{-1+\alpha} E[S_n, S_n > 0]$$

if \( \lim_{n \to \infty} n^{-1+\alpha} E[S_n, S_n > 0] < \infty \). We now proceed to prove the converse to this result, i.e. that \( \lim_{t \to 1} G^*(t) = \infty \) if \( \lim_{n \to \infty} n^{-1+\alpha} E[S_n, S_n > 0] = \infty \).

If this latter condition holds, then for any \( K > 0 \) there exists an \( N \) sufficiently large that for all \( n > N \), \( E[S_n, S_n > 0] > K n^{1-\alpha} \). Thus

$$\lim_{t \to 1} G^*(t) > \lim_{t \to 1} (1 - t)^{1-\alpha} \sum_{k=N}^{\infty} k^{-1} t^k k^{1-\alpha}$$

using Lemma 2-1. Thus as \( K \) is arbitrarily chosen, \( \lim_{t \to 1} G^*(t) = \infty \).

Therefore we see that \( \lim_{t \to 1} G^*(t) < \infty \) if and only if

$$\lim_{n \to \infty} n^{-1+\alpha} E[S_n, S_n > 0] < \infty.$$

We proceed via another lemma, which concerns the behaviour of \( E[S_n, S_n > 0] \) for generating variables belonging to the domain of attraction of an appropriate stable law.

**Lemma 2-2.** If random variables \( \{X_i\}_{i=1}^{\infty} \) belong to the domain of attraction of a stable law with distribution function \( V(x) \) and index \( \gamma \), \( 1 \leq \gamma \leq 2 \), then

$$\lim_{n \to \infty} \left[ L(n)^{n^{1/\gamma}} \right]^{-1} E[S_n, S_n > 0] < \infty \quad (2-11)$$

where \( L(n) \) varies slowly at infinity, and the distribution of \( S_n/n^{1/\gamma} \) \( L(n) \) converges to \( V(x) \).

**Proof.** Let \( F_n(x) = P(S_n \leq L(n)^{n^{1/\gamma}} x) \) where \( L(n) \) is chosen so that

$$\lim_{n \to \infty} P[S_n \leq L(n)^{n^{1/\gamma}} x] = \lim_{n \to \infty} F_n(x) = V(x) \quad (2-12)$$

For a proof that the norming takes this form, see Feller [11], pp.544,545. We will now show that
\[
\lim_n \int_0^\infty x \, dF_n(x) = \lim_n E[S_n/L(n)]^{1/\gamma}, \quad S_n \geq 0 \]
\[
= \int_0^\infty x \, dV(x) < \infty.
\]

We break up the limit condition as
\[
\left| \int_0^\infty x \, dF_n(x) - \int_0^\infty x \, dV(x) \right| \leq \left| \int_0^A x \, dF_n(x) - \int_0^A x \, dV(x) \right| + \int_A^\infty x \, dF_n(x) + \int_0^\infty x \, dV(x) \tag{2-13}
\]
for some \( A > 0 \). It is known (see Ibragimov and Linnik [21], p.174) that, for any random walk the generating random variables of which belong to a stable law of index \( \gamma \), there exists a \( K < \infty \) such that for all \( \eta > 0 \),
\[
E[|S_n/L(n)|^{\gamma-\eta}] \leq K < \infty. \tag{2-14}
\]
Then for \( \gamma-1 > \eta > 0 \),
\[
\int_A^\infty x \, dF_n(x) \leq A^{-(\gamma-1-\eta)} \int_A^\infty x^{\gamma-\eta} \, dF_n(x) \leq K A^{-(\gamma-1-\eta)}.
\]
We choose \( A \) so large that for some \( \delta > 0 \), \( K A^{-(\gamma-1-\eta)} < \delta/3 \) and \( \int_A^\infty x \, dV(x) < \delta/3 \) (as \( \int_{-\infty}^\infty |x| \, dV(x) < \infty \) the latter is clearly possible). Using this \( A \), there exist an \( N \) such that for all \( n > N \),
\[
\left| \int_0^A x \, dF_n(x) - \int_0^A x \, dV(x) \right| < \delta/3. \tag{2-14}
\]
(It can readily be shown that \( \int_0^A x \, dF_n(x) - \int_0^A x \, dV(x) \) using integration by parts and weak convergence of distribution functions.) Thus for any \( \delta \) and appropriately large \( n \),
\[
\left| \int_0^\infty x \, dF_n(x) - \int_0^\infty x \, dV(x) \right| \leq \left| \int_0^A x \, dF_n(x) - \int_0^A x \, dV(x) \right| + \int_A^\infty x \, dF_n(x) + \int_0^\infty x \, dV(x) < \delta, \tag{2-14}
\]
i.e.
\[
\lim_n [n^{1/\gamma} L(n)]^{-1} E[S_n/S_n^{\gamma} \geq 0] = E[X^*, X^* \geq 0] \text{ where } X^* \text{ is a
random variable which follows the stable law with distribution function \( V(x) \). As \( V(x) \) is of index \( \gamma > 1 \)

\[
\lim_n [L(n) n^{1/\gamma}]^{-1} E[S_n, S_n > 0] = E[X^*, X^* > 0] \\
\leq E[|X^*|] < \infty.
\]

Using the result of Lemma 2-2, we see that

\[
\lim_n n^{-1+\alpha} E[S_n, S_n > 0] = \lim_n n^{-1+\alpha+1/\gamma} L(n) [L(n) n^{1/\gamma}]^{-1} \\
\times E[S_n, S_n > 0] \\
= \begin{cases} \\
0 & \text{if } 1-\alpha > 1/\gamma \\
\lim_n L(n) E[X^*, X^* > 0] & \text{if } 1-\alpha = 1/\gamma \\
\infty & \text{if } 1-\alpha < 1/\gamma.
\end{cases}
\]

(This follows as \( L(n) \) is slowly varying and thus is dominated by \( n^\delta \) for any \( \delta > 0 \).)

The first of the cases listed above, however, does not arise. Chung-Teh [6] shows for \( X_1 \) belonging to the domain of attraction of the stable law with characteristic function

\[
\exp(-c|t|\gamma (1 + i\delta \sgn t \tan \pi \gamma/2)) \tag{2-15}
\]

with \( 1 \leq \gamma \leq 2, |\delta| \leq 1 \) and \( c > 0 \), that

\[
1 - \lim_n n^{-1} \sum_{k=1}^n P(S_k \leq 0) = 1 - \alpha \\
= \frac{1}{2} + [\pi \gamma]^{-1} \tan^{-1} [-\delta \tan \pi \gamma/2].
\]

However,

\[
\frac{1}{2} + [\pi \gamma]^{-1} \tan^{-1} [-\delta \tan \pi \gamma/2] \\
\leq \frac{1}{2} + [\pi \gamma]^{-1} [\pi - \pi \gamma/2] \\
= \frac{1}{\gamma}.
\]

Thus \( 1-\alpha \leq \frac{1}{\gamma} \), and Theorem 2-2 is proved, as we have shown

\[
\lim G^*(t) \sim \Gamma(1 - \alpha) \lim_n n^{-1+\alpha} E[S_n, S_n > 0]. \text{ We note here that if } \alpha \leq 1
\]
\begin{align*}
1 - \alpha &= \frac{1}{\gamma}, \\
\lim_{t \to \infty} G^*(t) &= \Gamma(1 - \alpha) \mathbb{E}[X^*, X^* > 0] \lim_{n \to \infty} L(n) < \infty,
\end{align*}

if and only if \( L(n) \sim \text{constant} \), so we have \( \lim_{t \to \infty} G^*(t) \) if and only if the generating variables of the random walk belong to the domain of normal attraction of a stable law. It should also be noted that for the condition \( 1 - \alpha = \frac{1}{\gamma} \) to hold, the generating random variables must belong to the domain of attraction of a stable law with \( \delta = 1 \), and this is a boundary case.

We now prove two corollaries, the first of which is of considerable interest in the later work in this chapter and in the next chapter, and the second of which is of some interest in its own right.

**Corollary 2-1** If \( \{X_i\}_{i=1}^\infty \) as described above also belong to the domain of normal attraction of the stable law, then \( \mathbb{E}[Z] < \infty \) if and only if \( 1 - \alpha = \frac{1}{\gamma} \) and \( \lim_{t \to \infty} F^*(t) < \infty \).

**Proof** We have actually noted this above, as belonging to the domain of normal attraction of a stable law means \( L(n) n^{1/\gamma} \sim k n^{1/\gamma} \) for some constant \( k \) (see Gnedenko and Kolmogorov [14], p.181). This clearly implies

\[
\mathbb{E}[Z] = \lim_{t \to \infty} F^*(t) G^*(t) = k \Gamma(1 - \alpha) \mathbb{E}[X^*, X^* > 0] \lim_{t \to \infty} F^*(t)
\]

and the corollary is proved.

**Corollary 2-2** If random variables \( \{X_i\}_{i=1}^\infty \) follow the stable law with characteristic function

\[
\exp\left(-c|t|^{\gamma}(1 + i \sgn t \tan \pi \gamma/2)\right)
\]

(i.e. the general function noted before with \( \delta = 1 \) if \( \gamma < 2 \)), then
If the \( X_i \) have any other stable distribution of index \( \gamma \), then \( E[Z] = \infty \).

**Proof**

As \( X_i \) follow the stable law, \( n^{-1/\gamma} S_n \) has the same distribution as \( X_i \) (this is shown in Feller [11], p.165) so \( P(S_n \leq 0) = P(X_i \leq 0) \) and

\[
E[Z] = \frac{1}{\gamma} \lim_{n \to \infty} \sum_{j=1}^{n} P(S_j \leq 0) = \frac{1}{\gamma} \lim_{n \to \infty} \sum_{j=1}^{n} P(X_i \leq 0). 
\]

This implies that

\[
F^*(t) = \exp \left\{ \sum_{k=1}^{\infty} t^{k} [P(S_k \leq 0) - \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} P(S_j \leq 0)] \right\} 
\]

\[
= \exp \left\{ \sum_{k=1}^{\infty} k^{-1} t^{k} [P(X_i \leq 0) - P(X_i \leq 0)] \right\} 
\]

\[
= 1
\]

and as \( L(n) \) is 1 and \( 1 - \alpha = \frac{1}{\gamma} \lim_{t \to 1} G^*(t) = \Gamma(1 - \alpha) E[X_i, X_i > 0] \), so that \( E[Z] = \lim_{t \to 1} G^*(t) F^*(t) = \Gamma(1 - \alpha) E[X_i, X_i > 0] \) as required. If \( \alpha \neq 1/\gamma \) (i.e. \( \delta \neq 1 \) and \( \gamma \neq 2 \), then from Theorem 2-1, we have \( E[Z] = \infty \). This completes the proof of Corollary 2-2.

As \( F^*(t) \) is a slowly varying function as \( t \to 1 \), we can write

\[
F^*(t) = M(1/1-t) \text{ with } M(n) \text{ slowly varying as } n \to \infty,
\]

and we have

\[
E[Z] = \lim_{t \to 1} M(1/1-t) G^*(t). \quad \text{Then from Theorem 2-2 we have}
\]

\[
E[Z] = \lim_{n \to \infty} n^{-1+\alpha+1/\gamma} L(n) M(n). [L(n) n^{1/\gamma}]^{-1} E[S_n, S_n > 0] \Gamma(1-\alpha)
\]

so we have proved the following theorem.

**Theorem 2-3**

Let \( \{X_i\}_{i=1}^{\infty} \) be independent identically distributed random variables with zero mean, which belong to the domain of attraction of a stable law with characteristic function

\[
\exp(-c|t|^\gamma (1 + i\delta \text{ sgn } \pi / 2)), \quad 1 \leq \gamma \leq 2, \quad |\delta| \leq 1.
\]

Then
\[E[Z] = \begin{cases} 
\infty & \text{if } \gamma < 2, \delta \neq 1 \\
\Gamma(1 - \alpha) E[X^*, X^* > 0] \lim_{n \to \infty} L(n) M(n) & \text{if } \gamma = 2 \text{ or } \gamma < 2, \delta = 1 
\end{cases}\]

where \(X^*\) has the distribution of the stable law to which \(X_i\) is attracted. The case \(\gamma = 2\) or \(\gamma < 2, \delta = 1\) is indeterminate in general.

We now examine the work of a number of authors on this topic, particularly with respect to the behaviour of \(F^*(t)\). Spitzer [28] and [29] dealt with the case \(E[X_i]^2 = \sigma^2 < \infty\) (i.e. the generating random variables belong to the domain of normal attraction of the normal law). In this case \(\alpha = \frac{1}{2}\), so \(1 - \alpha = \frac{1}{2} = 1/\gamma\), and \(E[Z]\) is finite if \(\lim_{t \to 1} F^*(t)\) is finite. Spitzer showed that in this finite variance case, \(\Delta = \sum_{n=1}^{\infty} n^{-1} t^n \left[ P(S_n \leq 0) - \frac{1}{2} \right] \) converged, and Rosén [26] showed that the series converged absolutely. Rosén's method, as with much of the later work, was based on the rediscovery of the Gil-Peleaz inversion formula

\[\frac{1}{2}[F(X^-) + F(X^+)] = \frac{1}{2} + \lim_{A \to 0} \lim_{B \to \infty} \int_A^B t^{-1} (e^{ixt} \phi(-t) + e^{-ixt} \phi(t)) \, dt\]

Baum and Katz [2] used similar methods to extend Rosén's result, and prove that if \(E[X] = 0\) and \(E[X]^2 + \delta < \infty, 0 \leq \delta < 1\), then

\[\sum_{n=1}^{\infty} n^{-1+\delta/2} |P(S_n \leq 0) - \frac{1}{2}| < \infty.\]

The form in which we are interested, \(\sum_{n=1}^{\infty} n^{-1} |P(S_n \leq 0) - \alpha|\) has not been studied extensively except in the case \(\alpha = \frac{1}{2}\). However, the more general form \(\Delta^* = \sum_{n=1}^{\infty} n^{-1} \sup_x |P(S_n \leq C_n x) - G(x)|\) has received a deal of attention, mainly from Heyde [15], [16], [17], [19] and [20]. First in [15] he treats the case \(C_n = n^p\) and \(G(x) = \frac{x}{2}\) for all \(x\). Under these circumstances, conditions are derived for \(\Delta^*\) to be finite in terms of bounds on the characteristic functions of the generating random variables (which need not be identically distributed in this case). A more relevant result is the corollary to this
Theorem, which shows for identically distributed, symmetric random variables with appropriate \( \beta \) and \( p \), \( \sum_{n=1}^{\infty} n^{-\beta} |P(S_n \leq n^p x) - \frac{1}{2}| < \infty \). In [16] Heyde gives a global generalisation of the Baum and Katz [2] result to prove that if the generating random variables of a random walk belong to the domain of normal attraction of the normal law, then
\[
\sum_{n=1}^{\infty} n^{-1+\epsilon/2} \sup_x |F_n(x) - \Phi(x)| < \infty, \quad 0 \leq \delta < 1, \text{ if and only if }
E[X_1^{2+\delta}] < \infty, \delta > 0, \text{ and } E[X_1^2 \log(1 + |X_1|)] < \infty, \delta = 0. \text{ In the case } \delta = 0,
Heyde [19] quotes a result of Friedman, Katz and Koopmans [12], which shows \( \Delta^* \) is finite with \( G(x) = \Phi(x) \), \( C_n = \sigma_n \sqrt{n} \), for appropriately defined \( \sigma_n \sim 1 \), if \( E[X_1^2] = 1 \). The next result of interest, derived by Heyde [20], shows that if \( \sum_{n=1}^{\infty} n^{-1} \sup_x |P(S_n \leq B_n x) - \Phi(x)| < \infty \) for any sequence \( \{B_n\} \) of normalising constants then \( X_1 \) belongs to the domain of normal attraction of the normal law (i.e. \( E[X_1^2] < \infty \)). This result, together with the example of Heyde [17], leads us to examine the conjecture of Heyde [17] that \( \sum_{n=1}^{\infty} n^{-1} |P(S_n \leq 0) - \alpha| < \infty \) if and only if \( X_1 \) belongs to the domain of normal attraction of the appropriate stable law. In the following theorem we prove this conjecture for the case of left continuous random walks. This result enables us to tell immediately whether \( E[Z] < \infty \). By definition a left continuous random walk is generated by a set of independent identically distributed random variables, such that \( P(X_1 = j) = p_j, j = 0, \pm 1, \pm 2, \ldots; \)
\[ p_j = 0, j < 0; \quad p_{-1} > 0 \text{ and } E[X_1] = \sum_{j=-1}^{\infty} j p_j = 0. \]
Following Heyde [17] we have that \( \sum_{n=1}^{\infty} n^{-1} [P(S_n \leq 0) - \alpha] \) converges if and only if
\[
\lim_{r \uparrow 1} [(1 - r)^{2-1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1}]^{\alpha} < \infty \text{ where }
\alpha = \lim_{r \uparrow 1} \left[ \sum_{n=1}^{\infty} u_n r^n \right]^{-1} \sum_{n=0}^{\infty} w_n r^n,
\]
\[ w_n = \sum_{j=1}^{\infty} p_j = \sum_{j=n}^{\infty} (j - n) p_j \text{ and } u_n = \sum_{j>n}^{\infty} p_j. \]
Theorem 2-4  For a left continuous random walk generated by random variables \( \{X_i\}_{i=1}^{\infty} \) which have zero mean and belong to the domain of attraction of the stable law of index \( \gamma, \gamma > 1 \), then
\[
\sum_{n=1}^{\infty} n^{-1} \left[ P(S_n \leq 0) - \lim_{n \to \infty} P(S_n \leq 0) \right] < \infty \text{ if and only if } \{X_i\} \text{ belong to the domain of normal attraction of the stable law.}
\]

Proof (i) Assume \( \{X_i\} \) belong to the domain of normal attraction of the stable law. Then
\[
P(X_j > k) = \sum_{j > k} p_j \sim \delta/k^\gamma \quad (2-16)
\]
(see for example Feller [11], p.547). Thus \( v_m = \sum_{j > m} p_j \sim \delta/m^\gamma \).

Now from Feller [11] p.273, as \( v_m \) varies regularly with exponent \(-\gamma\), \( Z_p(x) = \sum_{k=r}^{\infty} k^p v_k = n^{p+1} v_n |p - \gamma + 1|^{-1} \). Putting \( p = 0 \) gives
\[
\sum_{k=r}^{\infty} v_k \sim n v_n / \gamma - 1 \text{ as } \gamma > 1.
\]
Thus
\[
w_n = \sum_{m > n} \sum_{j > m} p_j = \sum_{m > n} v_m \\
\quad \sim n v_n / \gamma - 1 \\
\quad \sim n \delta/n^{\gamma}(\gamma - 1) \\
\quad = n^{1-\gamma}/\gamma - 1 .
\]

Now
\[
w_n = \sum_{j > n} (j - n) p_j = \sum_{j > n} j p_j - n \sum_{j > n} p_j \\
\quad = u_n - n v_n
\]
so that \( u_n - w_n \sim \delta/n^{\gamma-1} \). Further, as \( w_n \) and \( (u_n - w_n) \) are monotonic increasing functions, we may form
\[
u_n = (u_n - w_n) + w_n \sim [\delta/n^{\gamma-1}] + \delta/n^{\gamma-1} (\gamma - 1) \\
\quad = \delta \gamma/n^{\gamma-1} (\gamma - 1) \\
\quad \sim \gamma \delta \nu_n .
\]
As \( \gamma > 1 \), both \( \sum_{n=1}^{\infty} u_n r^n \) and \( \sum_{n=0}^{\infty} w_n r^n \) will converge for \( r < 1 \). From Feller [11] p.423, as \( u_n \) is monotonic, we have for \( \gamma < 2 \)

\[
\sum_{n=-1}^{\infty} u_n r^n \sim (1 - r)^{\gamma - 2} \Gamma(2 - \gamma) \frac{\delta}{\gamma} - 1
\]

and similarly

\[
\sum_{n=0}^{\infty} w_n r^n \sim (1 - r)^{\gamma - 2} \Gamma(2 - \gamma) \frac{\delta}{\gamma} - 1.
\]

Hence, \( \lim_{r \uparrow 1} \left[ \sum_{n=1}^{\infty} u_n r^n \right]^{-1} \sum_{n=0}^{\infty} w_n r^n = 1/\gamma \), so we have

\[
\lim_n P(S_n \leq 0) = \alpha = 1/\gamma.
\]

Now

\[
\lim_{r \uparrow 1} \left[ (1 - r)^{2 - 1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\alpha}
= \lim_{r \uparrow 1} \left[ (1 - r)^{2 - \gamma} \frac{r^{-1}}{\Gamma(2 - \gamma) \frac{\delta}{\gamma} - 1} \right]^{1/\gamma}
= \lim_{r \uparrow 1} \left[ r^{-1} \Gamma(2 - \gamma) \frac{\delta}{\gamma} - 1 \right]^{1/\gamma}
= \Gamma(2 - \gamma) \frac{\delta}{\gamma} - 1 < \infty
\]

as \( \gamma > 1 \). When \( \gamma = 2 \), \( \alpha = \frac{1}{2} \), this result does not hold as the theorem of Feller [11] p.423 is not applicable. However, as noted earlier in this chapter (see also Spitzer [29]), \( \sum_{n=1}^{\infty} n^{-1} [P(S_n < 0) - \frac{1}{2}] < \infty \) if \( E[X_1]^2 < \infty \).

(ii) Conversely, we assume that \( \lim_{r \uparrow 1} \left[ (1 - r)^{2 - 1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\alpha} < \infty \), where

\[
\lim_{r \uparrow 1} \left[ \sum_{n=-1}^{\infty} u_n r^n \right]^{-1} \sum_{n=0}^{\infty} w_n r^n = \alpha = \lim_n P(S_n \leq 0).
\]

Therefore there exists a \( \delta < \infty \) such that
\[
\lim_{r \uparrow 1} \left[ (1 - r)^{2-1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\alpha} = \delta ,
\]
so we have
\[
\sum_{n=0}^{\infty} w_n r^{n-1} \sim \frac{1}{\alpha} (1 - r)^{2+1/\alpha}
\] as \( r \uparrow 1 \), and clearly
\[
\sum_{n=0}^{\infty} w_n r^n \sim \sum_{n=0}^{\infty} w_n r^{n-1} \sim \frac{1}{\alpha} (1 - r)^{2+1/\alpha}.
\]

Again applying Feller [11] p.423, since \( w_n \) is monotonic for \( 1/\alpha < 2 \) we have \( w_n \sim n^{-1/\alpha} \delta^{1/\alpha}/\Gamma(2 - 1/\alpha) \). Now
\[
\lim_{r \uparrow 1} \left[ \frac{1}{\alpha} \sum_{n=1}^{\infty} u_n r^n \right]^{\alpha} \sum_{n=0}^{\infty} w_n r^n = \alpha,
\]
implies that
\[
\sum_{n=1}^{\infty} u_n r^n \sim \sum_{n=0}^{\infty} w_n r^{n/\alpha}
\]
\[
\sim \frac{1}{\alpha} \delta^{1/\alpha} (1 - r)^{2-1/\alpha}.
\]

as \( r \uparrow 1 \) and again applying Feller's theorem we have
\[ u_n \sim n^{-1/\alpha} \delta^{1/\alpha}/\Gamma(2 - 1/\alpha) \]. Hence we see
\[
\sum_{j>n} p_j \sim ((1/\alpha) - 1) n^{-1/\alpha} \delta^{1/\alpha}/\Gamma(2 - 1/\alpha)
\]
which implies
\[
\sum_{j>n} p_j \sim ((1/\alpha) - 1) n^{-1/\alpha} \delta^{1/\alpha}/\Gamma(2 - 1/\alpha)
\]
\[
= k n^{-1/\alpha} .
\]

This ensures that \( \alpha = 1/\gamma \) and the converse is proved for \( 1 > \alpha > \frac{1}{2} \). If however \( \alpha = \frac{1}{2} \),
\[
\lim_{r \uparrow 1} \left[ (1 - r)^{2-1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\frac{1}{2}} = \left[ \sum_{n=0}^{\infty} w_n \right]^{\frac{1}{2}} < \infty .
\]
Thus from Heyde [17],
\[
\sum_{j=0}^{\infty} j(j+1) p_j / 2 = \sum_{j=0}^{\infty} w_j < \infty ,
\]
but
\[
\text{Variance } (X_i) = \sum_{j=-1}^{\infty} j^2 p_j
\]
\[
= \sum_{j=0}^{\infty} j^2 p_j + p_{-1}
\]
\[
= \sum_{j=0}^{\infty} j^2 p_j + \sum_{j=1}^{\infty} j p_j + p_{-1}
\]
\[
= \sum_{j=0}^{\infty} j(j+1) p_j < \infty .
\]
Thus $X_i$ belongs to the domain of normal attraction of the normal law and our theorem is proved.

In terms of the earlier work in this chapter we have now shown that in the left continuous case $\lim_{t \uparrow} F^*(t) < \infty$ if and only if $X_i$ belongs to the domain of normal attraction of the appropriate stable law, in which case $\alpha = 1/\gamma$. Therefore $1-\alpha - 1/\gamma = 1 - 2/\gamma \leq 0$, with equality only when $\gamma = 2$. Thus if $\gamma < 2, E[Z] = \infty$, but if $\gamma = 2$, as $X_i$ belongs to the domain of normal attraction, $\lim_{t \uparrow} G(t) < \infty$ and we know $\lim_{t \uparrow} F^*(t) < \infty$, so $E[Z] < \infty$ (as was shown more directly by Spitzer [28] and [29]).

If our conjecture that belonging to the domain of normal attraction implies $\lim_{t \uparrow} F^*(t) < \infty$ proves to be true, then we have shown that $E[Z] < \infty$ if $\delta = 1$ ($\gamma \neq 2$) and the generating random variables belong to the domain of normal attraction of the appropriate stable law.

2.2 Case of Positive Mean

In this section we derive the explicit form of $E[Z]$ for random walks generated by independent identically distributed random variables $\{X_i\}_{i=1}^{\infty}$, under the condition $E[X_i] = \mu > 0$. We use mainly the methods of Feller [11], Chapter XII.

We consider sequences $T_i$ and $Z_i$ where

$$T_i = \min_n \left\{n, S_n > 0\right\},$$

$$T_{i+1} = \min_n \left\{n - T_i, S_n > \sum_{j=1}^{i} T_j\right\},$$

$$Z_i = \min_n \left\{n, S_n > 0\right\},$$

$$Z_{i+1} = \min_n \left\{S_n - \sum_{j=1}^{i} Z_j, S_n > \sum_{j=1}^{i} Z_j\right\}.$$
Thus we have \( \sum_{j=1}^{i} Z_j = S_{i-1} \). Using these definitions we will establish the following theorem.

**Theorem 2-5** If \( 0 < E[X_1] = \mu < \infty \), then

\[
E[Z] = \mu E[T_1] = \mu \exp \left\{ \sum_{k=1}^{\infty} k^{-1} P(S_k \leq 0) \right\}. \tag{2-17}
\]

**Proof** As \( \mu < \infty \), \( \lim_{n} S_n/n = \mu \) a.s., so taking the subsequence of \( S_n/n \) consisting of ladder steps we have

\[
\lim_{n} \frac{1}{i} \sum_{k=1}^{i} Z_k / \sum_{k=1}^{i} T_k = E[X_1].
\]

Now as \( \{Z_k\}_{k=1}^{\infty} \) and \( \{T_k\}_{k=1}^{\infty} \) are both independent and identically distributed sequences of random variables for all \( k \), we have

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} Z_k/n = E[Z_1] \quad \text{and} \quad \lim_{n} \frac{1}{n} \sum_{k=1}^{n} T_k/n = E[T_1].
\]

Thus we have

\[
E[Z_1] = E[X_1] E[T_1] = \mu E[T_1] \tag{2-18}
\]

which is effectively the Wald lemma of sequential analysis.

From Spitzer [28], Theorem 3-1 we have

\[
E \left[ X^T \right] e^{-\lambda Z_1} = 1 - \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} x^k E[\exp(-\lambda S_k), S_k > 0] \right\}. \tag{2-19}
\]

Putting \( \lambda = 0 \), we define

\[
H(x) = E \left[ x^T \right] = 1 - \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} x^k P(S_k > 0) \right\}.
\]

Thus

\[
E[T_1] = H'(1) = \lim_{x \rightarrow 1} \frac{1 - H(x)}{1 - x}
\]

\[
= \lim_{x \rightarrow 1} (1 - x)^{-1} \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} x^k P(S_k > 0) \right\}
\]

\[
= \lim_{x \rightarrow 1} \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} x^k P(S_k \leq 0) \right\}
\]

\[
= \exp \left\{ - \sum_{k=1}^{\infty} k^{-1} P(S_k \leq 0) \right\}.
\]
When $\mu > 0$, it is known (see for example Prabhu [25], p.207) that

$$\exp \left\{ \sum_{k=1}^{\infty} k^{-1} P(S_k \leq 0) \right\} < \infty,$$

so we have

$$E[Z] = E[Z_1] = \mu E[T_1]$$

$$= \mu \exp \left\{ \sum_{k=1}^{\infty} k^{-1} P(S_k \leq 0) \right\} < \infty$$

as required.
CHAPTER 3

ON THE APPROACH TO ZERO OF THE UNIFORM METRIC

FOR CONVERGENCE TO A STABLE LAW

As mentioned in Chapter 2, the logical extension of the function $\Delta$ studied there is the global version,

$$\sum_{n=1}^{\infty} n^{-5} \sup_x |P(S_n < B_n x) - V(x)|, \ 5 > 0,$$

where $B_n$ are such that

$$\lim_{n} P(S_n < B_n x) = V(x) \text{ for all } x.$$

As also noted in the previous chapter, the results of Heyde [17] and [20] lead us to conjecture that the domain of normal attraction of a stable law may be the domain of convergence of

$$\sum_{n=1}^{\infty} n^{-1} \sup_x |P(S_n \leq B_n x) - V(x)|.$$

The majority of this chapter is devoted to proving a theorem which attempts to generalise the results of Heyde [16] to the case of attraction to a general stable law, and at the same time incorporate the factor of normal attraction mentioned above.

We let $\{X_i\}_{i=1}^{\infty}$ be independent, identically distributed random variables with distribution function $F(x)$ and characteristic function $\phi(x)$. Let $F_n(x)$ be the distribution function of the normed partial sum $S_n/B_n$ which has characteristic function $\phi_n(t)$.

**Theorem 3-1** Let $\{X_i\}_{i=1}^{\infty}$ be random variables as described above with mean zero. Suppose that the $X_i$ belong to the domain of normal attraction of a stable law with index $\alpha, 1 < \alpha \leq 2$, which has distribution function $G(x)$ and characteristic function $g(t)$.

Then for $0 \leq \gamma < 1/\alpha$,

$$\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_x |F_n(x) - G(x)| < \infty \text{ if }$$

$$\int_{-B}^{B} |g(t) - \phi(t)|/t^{\alpha(1+\gamma)+1} \ dt < \infty \text{ for some } B > 0.$$
Proof We write
\[
\phi(t) = \exp(-|t|^\alpha (1 + i\beta \text{sgn } t \tan \pi \alpha/2 + \delta(t))) \tag{3-2}
\]
and now proceed to show that this is a valid representation of \(\phi(t)\).

As the \(X_i\) belong to the domain of normal attraction of the stable law, the norming on \(S_n\) is \(m n^{1/\alpha}\), with \(m\) a positive constant (see Feller [11], p.547). Without loss of generality we may take \(m = 1\). Thus we have
\[
\phi_n(t) = [\phi(t/n^{1/\alpha})]^n - g(t). \tag{3-3}
\]

Now for \(t\) in a neighbourhood of the origin in which \(\phi(t)\) has no zeros (such a neighbourhood exists since \(\phi(t)\) is continuous and \(\phi(0) = 1\)),
\[
[\phi(t/n^{1/\alpha})]^n = \exp(-|t|^\alpha [1 + i\beta \text{sgn } t \tan \pi \alpha/2 + \delta_n(t)]) , \tag{3-4}
\]
where \(\delta_n(t)\) is continuous and \(\delta_n(t) \to 0\) as \(n \to \infty\). From (3-4) we see that
\[
\phi(t/n^{1/\alpha}) = \exp(-|t/n^{1/\alpha}|^\alpha [1 + i\beta \text{sgn } t \tan \pi \alpha/2 + \delta_n(t)])
\]
and hence
\[
\phi(u) = \exp(-|u|^\alpha [1 + i\beta \text{sgn } u \tan \pi \alpha/2 + \delta_n(u n^{1/\alpha})]). \tag{3-5}
\]

Thus \(\delta_n(u n^{1/\alpha})\) is independent of \(n\), so we set \(\delta_n(u) = \delta_n(u n^{1/\alpha})\).

Substituting this transformation and retracing the above argument, we find
\[
[\phi(t/n^{1/\alpha})]^n = \exp(-|t|^\alpha [1 + i\beta \text{sgn } t \tan \pi \alpha/2 + \delta(t/n^{1/\alpha})])
- \exp(-|t|^\alpha [1 + i\beta \text{sgn } t \tan \pi \alpha/2]) \quad \text{as } n \to \infty.
\]

Thus \(\delta(t/n^{1/\alpha}) \to 0\) as \(n \to \infty\), i.e. \(\delta(u) \to 0\) as \(u \to 0\). Hence we have justified the use of the form of equation (3-2).

Next from Gnedenko and Kolmogorov [14], p.196, 197, we have

Lemma 3-1 If \(\int_{-\infty}^{\infty} |F_n(x) - G(x)|dx < \infty\) for all \(n\) and \(G'(x)\) exists and
is uniformly bounded in absolute value by $A$, then

$$\left| F_n(x) - G(x) \right| \leq L \int_{-T}^{T} \left| \phi_n(t) - g(t) \right| |t| \, dt + LA/T, \quad (3-6)$$

for all $x$, where $L$, $A$ and $T$ are positive constants.

Now under our conditions, if $X^*$ is a random variable having the stable distribution function $G(x)$, we have that $E[|X_1|]$ and $E[|X^*|]$ are both finite since $\alpha > 1$. Thus integrating by parts we may show

$$\int_{-\infty}^{\infty} |F_n(x) - G(x)| \, dx \leq \int_{-\infty}^{0} [F_n(x) + G(x)] \, dx$$

$$+ \int_{0}^{\infty} [1 - F_n(x) + 1 - G(x)] \, dx$$

$$= \int_{-\infty}^{\infty} |x| \, dF_n(x) + \int_{-\infty}^{\infty} |x| \, dG(x) < \infty.$$

It is also well known (see for example Feller [11], p.548) that $G'(x)$ exists and is bounded in absolute value, and hence the conditions of the lemma are met. We have

$$|\phi_n(t) - g(t)| = |\exp\left(-|t|^\alpha [1 + i\beta \sgn t \tan \pi \alpha/2 + \delta(t/n^{1/\alpha})]\right)$$

$$- \exp\left(-|t|^\alpha [1 + i\beta \sgn t \tan \pi \alpha/2]\right)|$$

$$= |\exp\left(-|t|^\alpha \delta(t/n^{1/\alpha})\right)| - 1|$$

$$\leq \exp\left(-|t|^\alpha \right) \left| |t|^\alpha \delta(t/n^{1/\alpha})\right| \exp\left(\left| |t|^\alpha \delta(t/n^{1/\alpha})\right|\right)$$

since $|\exp x - 1| \leq |x| \exp |x|.$

Now, since $\delta(t) \to 0$ as $t \to 0$, for any $k > 0$ we can choose $B$ such that $\max_{0 \leq t \leq B} |\delta(t)| \leq k$. We then define $T = Bn^{1/\alpha}$, and let

$$\Psi = \sum_{n=1}^{\infty} n^{-1+\gamma} \int_{-T}^{T} |t|^{-1} |\phi_n(t) - g(t)| \, dt$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\gamma} \int_{-T}^{T} |t|^{-1+\alpha \theta} \exp\left(-|t|^\alpha \right) \left| |t|^\alpha \delta(t/n^{1/\alpha})\right| \exp\left(\left| |t|^\alpha \delta(t/n^{1/\alpha})\right|\right) \, dt$$

$$\leq \sum_{n=1}^{\infty} n^{-1+\gamma} \int_{-T}^{T} |t|^{-\alpha - 1} \exp\left(-|t|^\alpha (1 - k)\right) \left| \delta(t/n^{1/\alpha})\right| \, dt.$$
Substituting \( u = t/n^{1/\alpha} \) gives

\[
\mathcal{Y} \leq \sum_{n=1}^{\infty} n^{-1+\gamma} \int_{-B}^{B} |u|^{1/\alpha - 1} \exp\left[-|u|^{1/\alpha} (1 - k)\right] |\delta(u)| n^{1/\alpha} \, du
\]

\[
= \sum_{n=1}^{\infty} \int_{-B}^{B} n^{\gamma} |u|^{\alpha - 1} \exp\left[-n|u|^\alpha (1 - k)\right] |\delta(u)| \, du
\]

\[
= \int_{-B}^{B} |u|^{\alpha - 1} |\delta(u)| \left[ \sum_{n=1}^{\infty} n^{\gamma} \exp\left[n|u|^\alpha (k - 1)\right] \right] \, du . \quad (3.7)
\]

Now applying the Tauberian Theorem of Feller [11] p.423, replacing the argument \( s \) by \( \exp\left[|u|^\alpha (k - 1)\right] \), we have

\[
\lim_{|u| \to 0} \left[1 - \exp\left[|u|^\alpha (k - 1)\right] \right]^{1+\gamma} \sum_{n=1}^{\infty} n^{\gamma} \exp\left[n|u|^\alpha (k - 1)\right] = \Gamma(1 + \gamma) .
\]

Thus for any \( K \) and \( |u| \) in an appropriate neighbourhood of zero,

\[
\sum_{n=1}^{\infty} n^{\gamma} \exp\left[n|u|^\alpha (k - 1)\right] \leq K \Gamma(1 + \gamma) \left[1 - \exp\left[|u|^\alpha (k - 1)\right]\right]^{-1+\gamma} . \quad (3.8)
\]

The application of the Tauberian theorem requires \( \exp\left[|u|^\alpha (k - 1)\right] < 1 \), i.e. \( k < 1 \) and \( |u| \neq 0 \). Thus we have

\[
\mathcal{Y} \leq \int_{-B}^{B} |u|^{\alpha - 1} |\delta(u)| K \Gamma(1 + \gamma) \left[1 - \exp\left[|u|^\alpha (k - 1)\right]\right]^{-1+\gamma} \, du .
\]

Since for small real \( z \), \( 1 - \exp[-z] \sim z \), we see that

\[
\mathcal{Y} \leq \int_{-B}^{B} |u|^{\alpha - 1} |\delta(u)| K \Gamma(1 + \gamma) \left[|u|^\alpha (k - 1)\right]^{-1+\gamma} \, du
\]

\[
= \int_{-B}^{B} |u|^{-1+\alpha \gamma} |\delta(u)| \, du \cdot K \Gamma(1 + \gamma)/(k - 1)^{1+\gamma} . \quad (3.9)
\]

Then from Lemma 3.1 we have

\[
\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_{x} |F_{n}(x) - G(x)| \leq L \mathcal{Y} + \sum_{n=1}^{\infty} n^{-1+\gamma} AL/T , \quad (3.10)
\]

where \( A \) and \( L \) are positive constants. As \( T = Bn^{1/\alpha} \),

\[
\sum_{n=1}^{\infty} AL/Tn^{1-\gamma} = (AL/B) \sum_{n=1}^{\infty} 1/n^{1-\gamma+1/\alpha} \quad \text{which is finite if } 1/\alpha > \gamma . \quad \text{Thus if } \gamma < 1/\alpha \text{ and } \int_{-B}^{B} |u|^{-1+\alpha \gamma} |\delta(u)| < \infty , \text{ from (3.10) we see that}
\]

\[
\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_{x} |F_{n}(x) - G(x)| < \infty . \quad \text{This result is a partial order.}
\]
generalisation of that of Lemma 1 of Heyde [16] from the case \( \alpha = 2 \), to the case \( 1 < \alpha \leq 2 \) (our result consists of a sufficient condition, not an equivalence). This is the form of Heyde's result, but we now convert it to a form which is more useful and meaningful for our purposes. We know that

\[
g(t) - \phi(t) = \exp[-|t|^\alpha [1 + i\beta \text{sgn } t \tan \frac{\pi \alpha}{2}]] (1 - \exp[-|t|^\alpha s(t)])
\]

so the condition

\[
\int_{-B}^{B} |t|^{-(1+\alpha\gamma)} |\delta(t)| \, dt < \infty \text{ becomes}
\]

\[
\int_{-B}^{B} |t|^{-(1+\alpha(1+\gamma))} |g(t) - \phi(t)| \, dt < \infty,
\]

and this can be further simplified to

\[
\int_{0}^{B} t^{-(1+\alpha(1+\gamma))} |g(t) - \phi(t)| \, dt < \infty.
\]

Since

\[
|\delta(t)| = |\delta(-t)| \quad \text{(as is readily shown, using } \phi(-t) = \overline{\phi(t)}). \]

Thus our theorem is proved.

Heyde [16] is able to show in the case of convergence to normality that the conditions \( E[|X_1|^2(1+\gamma)] < \infty \text{ if } 0 < \gamma < 1 \) or \( E[X_1^2 \log(1 + |X_1|)] < \infty \text{ if } \gamma = 0 \) imply the intermediate condition used, and hence that

\[
\sum_{n=1}^{\infty} n^{-1+\gamma} |P(S_n < \lfloor n E[X]^2 \rfloor^{1/2} x) - \phi(x)| < \infty.
\]

In the following lemma we derive similar results, but our results are somewhat restricted as some special properties of convergence to the normal distribution are not available to us.

**Lemma 3-2** Under the conditions of Theorem 3-1,

\[
\int_{0}^{B} t^{-(1+\alpha(1+\gamma))} |g(t) - \phi(t)| \, dt < \infty \text{ if}
\]

\[
\int_{-\infty}^{\infty} x^{\alpha(1+\gamma)-1} |G(x) - F(x)| \, dx < \infty \quad \text{for } \gamma < 2/\alpha - 1.
\]

**Proof** As \( \int_{-\infty}^{\infty} x \, dF(x) = 0 \) and \( \int_{-\infty}^{\infty} x \, dG(x) = 0 \),
$$g(t) - \phi(t) = \int_{-\infty}^{\infty} e^{itx} \ d[G(x) - F(x)]$$

$$= \int_{-\infty}^{\infty} (e^{itx} - itx) \ d[G(x) - F(x)]$$

$$= it \int_{-\infty}^{\infty} (e^{itx} - 1) [G(x) - F(x)] \ dx ,$$

using integration by parts. (In the case $\alpha = 2$, we may also introduce the term $t^2 x^2$ upon noting that $\int_{-\infty}^{\infty} x^2 \ dF(x) = \int_{-\infty}^{\infty} x^2 \ dG(x)$, and this is where Heyde [16] is able to extract the extra power of his result.)

Following this we see that

$$\int_{0}^{B} t^{-[1+\alpha(1+\gamma)]} |g(t) - \phi(t)| \ dt$$

$$= \int_{0}^{B} t^{-\alpha(1+\gamma)} | \int_{-\infty}^{\infty} [\exp(itx) - 1] [G(x) - F(x)] \ dx | \ dt$$

$$\leq \int_{-\infty}^{\infty} |G(x) - F(x)| \int_{0}^{B} t^{-\alpha(1+\gamma)} |\exp(itx) - 1| \ dt \ dx$$

and putting $u = tx$ in the inner integral gives

$$\int_{0}^{B} t^{-\alpha(1+\gamma)} |e^{itx} - 1| \ dt = \int_{0}^{B} |x| (x/u)^{\alpha(1+\gamma)} |\exp(iu) - 1| x^{-1} \ du$$

$$= x^{\alpha(1+\gamma)-1} \int_{0}^{B} |x| u^{-\alpha(1+\gamma)} |\exp(iu) - 1| \ du$$

$$\leq x^{\alpha(1+\gamma)-1} \left[ \int_{0}^{\delta} u^{-\alpha(1+\gamma)} |u| \ du + \int_{\delta}^{B} x^{\alpha(1+\gamma)-1} 2 \ du \right]$$

$$= x^{\alpha(1+\gamma)-1} \left[ k_1 + k_2 |x|^{1-\alpha(1+\gamma)} \right] \quad (3-11)$$

as $\gamma < 2/\alpha - 1$, $k_1$ and $k_2$ being finite constants dependent on the choice of $\delta$. $\delta > 0$ is chosen sufficiently small that $|\exp(iu) - 1| < |u|$ for $0 < u < \delta$. Thus we have

$$\int_{0}^{B} t^{-[1+\alpha(1+\gamma)]} |g(t) - \phi(t)| \ dt$$

$$\leq \int_{-\infty}^{\infty} |G(x) - F(x)| \left[ k_2 + k_1 |x|^{\alpha(1+\gamma)-1} \right] \ dx .$$
Now we have shown earlier that $\int_{-\infty}^{\infty} |G(x) - F(x)| \, dx < \infty$, so we see that our condition becomes $\int_{-\infty}^{\infty} |x|^{\alpha(1+\gamma)-1} |G(x) - F(x)| \, dx < \infty$ as required.

This result may be strengthened however when we are dealing with symmetric random variables. Under these circumstances,

$$g(t) - \phi(t) = \int_0^B \cos tx \, d[G(x) - F(x)] .$$

Following the argument of Lemma 3-2, we replace equation (3-11) by

$$\int_0^B t^{-\alpha(1+\gamma)} |\cos tx - 1| \, dt \leq \int_0^B t^{-\alpha(1+\gamma)} t^{2x^2/2!} \, dt$$

$$= x^2 \int_0^B t^{-\alpha(1+\gamma)/2!} \, dt$$

$$= cx^2$$

where $c$ is a positive constant. This integral will exist for $\alpha(1+\gamma) < 3$.

We can combine the results of Theorem 3-1 with Lemma 3-2 to give

**Theorem 3-2** Under the conditions of Theorem 3-1, and in addition $\alpha(1+\gamma) < 2$, we have $\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_x |F_n(x) - G(x)| < \infty$ if $\int_{-\infty}^{\infty} |x|^{\alpha(1+\gamma)-1} |G(x) - F(x)| \, dx < \infty$.

This result is highly relevant to the results of Chapter 2, as $\sum_{n=1}^{\infty} n^{-1+\gamma} \sup_x |F_n(x) - G(x)| < \infty$ implies the

$$\sum_{n=1}^{\infty} n^{-1+\gamma} |P(S_n < 0) - \alpha| < \infty,$$

where $\alpha = G(0)$. For example, with the aid of Corollary 2-1 we can deduce that if $\beta = 1$ in the characteristic function of the relevant stable law (see equation (3-1)), then $E[Z] < \infty$ if $\int_{-\infty}^{\infty} |x|^{\alpha-1} |G(x) - F(x)| \, dx < \infty$.

An interesting comparison is with Boonyasombut and Shapiro [3], p.248. They show, using a method based on accompanying infinitely divisible laws, as distinct from our direct approach, that if $g(t) = \exp(-|t|), \phi(t) = 1 - |t|$ for $|t| \leq 1$, then
\[ \sup_x |F_n(x) - G(x)| \leq c/x^{1/15}. \]

We have, as \( \alpha = 1 \),
\[ \int_0^B t^{-[\alpha(1+\gamma)+1]} |g(t) - \phi(t)| = \int_0^B t^{-[2+\gamma]} |\exp(-|t|) - 1 + |t|| dt. \]

Then, as \( |1 - t - \exp(-t)| \leq t^2/2! \) and \( \gamma < 1 \), we have
\[ \int_0^B t^{-[\alpha(1+\gamma)+1]} |g(t) - \phi(t)| dt \leq \int_0^B \left[ t^{2/2} t^{-[2+\gamma]} \right] dt \]
\[ = \frac{B^{1-\gamma}}{1-\gamma} < \infty. \]

Thus, \( \sum_{n=0}^{\infty} n^{-1+\gamma} \sup_x |F_n(x) - G(x)| < \infty \) for any \( \gamma < 1 \). The Boonyasombut and Shapiro result however implies this only if \( \gamma < 1/15 \).

It can readily be shown that \( 1/n^{1/15} \) is the smallest bound which can be derived from the Boonyasombut and Shapiro result, so our result is considerably stronger and indicates the advantage of the direct approach.
Having studied the drift of random walks somewhat obliquely in terms of the expected first positive step, we proceed in this chapter to look at the problem more directly. Specifically we investigate the behaviour of $S_n/n$ (for a random walk generated by a set of independent identically distributed random variables $\{X_i\}_{i=1}^{\infty}$) when $E[|X_1|] = \infty$ and $E[X_1^+] = E[X_1^-] = \infty$. If $E[|X_1|] < \infty$, $E[X_1] \neq 0$ or $E[|X_1|] = \infty$ and either $E[X_1^+] < \infty$ or $E[X_1^-] < \infty$ then the behaviour of $S_n/n$ is an immediate consequence of the strong law of large numbers.

Kesten [22] has shown that there are three possible limiting behaviours of $S_n/n$ under the condition $E[X_1^+] = E[X_1^-] = \infty$. Either

$$\lim_{n \to \infty} S_n/n = +\infty \text{ a.s.}, \quad \lim_{n \to \infty} S_n/n = -\infty \text{ a.s.}, \quad \lim_{n \to \infty} S_n/n = -\infty \text{ a.s.},$$

and all finite points are limit points of $S_n/n$. We attempt to derive conditions for $\lim_{n \to \infty} S_n/n = +\infty \text{ a.s.}$ Unless noted otherwise, throughout this chapter we let $\{X_i\}_{i=1}^{\infty}$ be a set of independent identically distributed random variables, with $E[|X_1|] = E[X_1^+] = E[X_1^-] = \infty$. We begin by examining the results of Derman and Robbins [7] and Baum [1]. Derman and Robbins proved

**Theorem 4-1** If $\{X_i\}_{i=1}^{\infty}$ have distribution function $F(x)$, and if for some constants $0 < \alpha < \beta < 1$, $c > 0$ we have $F(x) \leq 1 - c/x^\alpha$ for large positive $x$, and $\int_{-\infty}^{0} |x|^\beta dF(x) < \infty$ (i.e. $X_1^- \in L_\beta$), then

$$\lim_{n \to \infty} S_n/n = +\infty \text{ a.s.}$$

Derman and Robbins go on to conjecture that the conditions of the theorem may be simplified to $X_1^- \in L_\beta$, $X_1^+ \notin L_\alpha$, however Baum [1] has constructed a counter-example to this conjecture. Baum has also derived a set of conditions depending on one parameter only, which imply the
Theorem 4-2  If for some $a$, $0 < a < 1$, both $X_i^+ \in L$ and 
$t^a P(X_i^+ > t) \to \infty$ as $t \to \infty$, then $S_n/n \to \infty$ in probability.

The following Lemma, which gives a sufficient but not necessary condition for $\lim_{n} S_n/n = \infty$ a.s. can be related to both these theorems, and elucidates the reason for the failure of both Derman and Robbin's conjecture and Baum's Theorem to give almost sure convergence.

Lemma 4-1  If there exists a sequence of positive numbers $\{b_n\}_{n=1}^{\infty}$, such that $b_n/n \to \infty$, $\sum_{n=1}^{\infty} P(X_i^- > b_n) < \infty$ and $\lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^+ > 0$ a.s. then $\lim_{n} S_n/n = \infty$ a.s.

Proof  From Chow and Robbins [4], Lemma 2, we see that

$$\lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^- = 0 \text{ a.s.} \quad \text{If we let} \quad \lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^+ = \delta > 0, \text{then}$$

$$\lim_{n} b_n^{-1} S_n \geq \lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^+ - \lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^-$$

$$= \delta.$$

Hence $\lim_{n} S_n/n = \lim_{n} (b_n/n)(S_n/b_n) = \infty$ a.s., and the Lemma is proved.

As a special case of this lemma take $b_n = n^{1/\alpha}$, $0 < \alpha < 1$, with the condition on $X_i^-$ replaced by $X_i^- \in L$ (as, from Loève [23], p.242, $\sum_{n=1}^{\infty} P(X_i^- > n^{1/\alpha}) < \infty$ if and only if $E[X_i^-]^{\alpha} < \infty$). Thus we have the following lemma.

Lemma 4-2  If $E[X_i^-]^{\alpha} < \infty$, $0 < \alpha < 1$, and $\lim_{n} \sum_{i=1}^{n} X_i^+/n^{1/\alpha} > 0$, then $\lim_{n} S_n/n = \infty$ a.s.

We now show that Lemma 4-2 is implied by the condition of Derman and Robbins (although neither their conjecture nor Baum's conditions imply that $\lim_{n} \sum_{i=1}^{n} X_i^+/n^{1/\alpha} > 0$ a.s.). The conditions of Derman and Robbins are that $0 < \alpha < \beta < 1$, $F(x) \leq 1 - c/x^{\alpha}$ for $x > 0$ and $E[X_i^{\beta}] < \infty$. This last condition clearly satisfies the requirements
of Lemma 4-2 with regard to $X^+$, so it remains to show that
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{X_i^+}{n^{1/\beta}} > 0 \text{ a.s.}
\]
If this is not true, we have for all $\delta > 0$
\[
P \left( \sum_{i=1}^{n} \frac{X_i^+}{n^{1/\beta}} < \delta \text{ i.o.} \right) = 1 .
\] (4-1)

However,
\[
\sum_{n=1}^{\infty} P \left( \sum_{i=1}^{n} \frac{X_i^+}{n^{1/\beta}} < \delta n \right) \leq \\
\sum_{n=1}^{\infty} \left[ P(X_i^+ < \delta n^{1/\beta}) \right] \leq \\
\sum_{n=1}^{\infty} \left[ 1 - c/(\delta n^{1/\beta})^{\alpha} \right] \leq \\
\sum_{n=1}^{\infty} \left[ 1 - K/n^{1-\beta} \right]^{n}
\]
where $K = c/\delta^\alpha$ and $1-\delta = c/\beta$. Now $(1 - K/n^{1-\delta})^n \leq \exp(-K n^\delta)$ and for $\delta > 0$,\[
\exp(-K n^\delta) = \exp(-K(n^{\delta}/\log n) \log n) \leq \exp(-K \log n) = n^{-K}.
\]
Thus for suitably small $\delta$, $K$ will be greater than 1 and
\[
\sum_{n=1}^{\infty} P \left( \sum_{i=1}^{n} \frac{X_i^+}{n^{1/\beta}} < \delta n \right) < \sum_{n=1}^{\infty} n^{-K} < \infty, \text{ and by the Borel-Cantelli lemma}
\]
P \left( \sum_{i=1}^{n} \frac{X_i^+}{n^{1/\beta}} < \delta n \text{ i.o.} \right) = 0. \text{ This however contradicts (4-1), so we}
see that Lemma 4-2 contains the Derman and Robbins result.

We now prove a lemma which gives a condition for
\[
\lim_{n \to \infty} b_i^{-1} \sum_{i=1}^{n} X_i^+ > 0 \text{ a.s. which is better than that obtained directly}
\]
from the Borel-Cantelli lemma.

**Lemma 4-3** Let \( \{b_i\} \) denote a monotonic sequence of positive constants, then \( \lim_{n \to \infty} b_i^{-1} \sum_{i=1}^{n} X_i^+ > 0 \) if \( n P(X_i^+ > b_i) - \infty \text{ as } n \to \infty \) and
\[
\sum_{n=1}^{\infty} \exp(-n P(X_i^+ > b_i)) P(X_i^+ \geq b_{n+1} - b_n) < \infty. \text{ A Borel-Cantelli approach}
\]
in fact yields the same condition without the term \( P(X_i^+ \geq b_{n+1} - b_n) \).

**Proof** Instead of the Borel-Cantelli lemma we use the following result due to Barndorff-Neilsen, quoted in Chung [5] p.75. If \( E_n \) is a
sequence of events satisfying (a) \( \lim_{n} P(E_n) = 0 \) and
(b) \( \sum P(E_n \cap \bar{E}_{n+1}) < \infty \), then \( P(E_n \text{ i.o.}) = 0 \) (a bar denotes the complementary event). We are interested in the case \( E_n = \left\{ \sum_{i=1}^{n} X_i^+ < b_n \right\} \).

First we note that

\[
P\left( \sum_{i=1}^{n} X_i^+ < b_n \right) \leq \left[ P(X_1^+ < b_n) \right]^n
= \left[ 1 - n P(X^+ > b_n)/n \right]^n
= 0
\]
as \( n \to \infty \), since \( n P(X^+ > b_n) \to \infty \), so that (a) is satisfied. In the case of (b) we have

\[
\sum_{n=1}^{\infty} P\left( \sum_{i=1}^{n} X_i^+ < b_n, \sum_{i=1}^{n+1} X_i^+ \geq b_{n+1} \right)
\leq \sum_{n=1}^{\infty} P\left( \sum_{i=1}^{n} X_i^+ < b_n, X_{n+1}^+ \geq b_{n+1} - b_n \right)
= \sum_{n=1}^{\infty} P\left( \sum_{i=1}^{n} X_i^+ < b_n \right) P\left( X_{n+1}^+ \geq b_{n+1} - b_n \right)
\leq \sum_{n=1}^{\infty} \left[ 1 - n P(X_1^+ > b_n)/n \right]^n P\left( X_1^+ \geq b_{n+1} - b_n \right)
\leq \sum_{n=1}^{\infty} \exp\left\{ -n P(X_1^+ > b_n) \right\} P(X_1^+ \geq b_{n+1} - b_n).
\]

Thus, under the conditions of the lemma we can apply the Barndorff-Neilsen result to obtain \( P\left( \sum_{i=1}^{n} X_i^+ < b_n \text{ i.o.} \right) = 0 \), and hence

\[
\lim_{n} b_n^{-1} \sum_{i=1}^{n} X_i^+ \geq 1 \text{ a.s.}
\]
and our lemma is proved.

Combining Lemmas 4-1 and 4-3 we have the following result.

**Theorem 4-3** If there exists a sequence of positive constants \( \{b_n\} \), with \( b_n/n \to \infty \) as \( n \to \infty \) and \( \sum_{n=1}^{\infty} P(X_1^- > b_n) < \infty \), \( n P(X_1^+ > b_n) \to \infty \) as \( n \to \infty \) and \( \sum_{n=1}^{\infty} \exp\left\{ -n P(X_1^+ > b_n) \right\} P(X_1^+ \geq b_{n+1} - b_n) < \infty \), then

\[
\lim_{n} S_n/n = \infty \text{ a.s.}
\]

In the special case \( b_n = n^{1/\alpha} \) we have Baum's [1] result, with the added condition \( \sum_{n=1}^{\infty} \exp\left\{ -n P(X_1^+ > n^{1/\alpha}) \right\} P(X_1^+ > n^{1/\alpha-1}) \) strengthening
his result to one of almost sure divergence.

Theorem 4-3 however, as with most of the others mentioned in this chapter (except that of Derman and Robbins) has conditions which are extremely difficult to check. In fact even \( \lim_{n} b_{n}^{-1} \sum_{i=1}^{n} X_{i}^{+} > 0 \) a.s. is difficult to check, so we propose the following "theorem" which gives a much simpler condition, and prove a number of lemmas which lend credence to the conjecture without proving it.

"Theorem 4-4" For a sequence of independent, identically distributed random variables \( \{X_{i}\}_{i=1}^{\infty} \), with \( E[|X_{i}|] = E[X_{1}^{+}] = E[X_{1}^{-}] = \infty \), \( \frac{P(X_{i}^{+} > x)}{P(X_{i}^{-} > x)} \to \infty \) as \( x \to \infty \) implies that \( S_{n}/n \to \infty \) as \( n \to \infty \). (Note we assume \( X \overset{d}{=} X_{1} \).)

Proposition 4-1 In the counter-example derived by Baum [1], in which \( \lim_{n} S_{n}/n < \infty \) a.s., we have \( \lim_{x} \frac{P(X^{+} > x)}{P(X^{-} > x)} < \infty \).

Proof Baum's construction uses an increasing sequence of positive integers \( \{m_{j}\}_{j=1}^{\infty} \), so defined that \( P(X_{i}^{+} > m_{j}) = \mu_{j} = 1/jm_{j}^{\alpha} \). It also requires \( P\left( \sum_{i=1}^{n} X_{i}^{-}/n^{\alpha/\delta} > 1 \right) \to 0 \) as \( n \to \infty \). Now if \( \lim_{x} \frac{P(X_{i}^{+} > x)}{P(X_{i}^{-} > x)} = \infty \), then for any \( K > 0 \), \( \exists j_{0} \) such that for all \( j > j_{0} \),

\[
P(X_{i}^{+} > m_{j}) > K P(X_{i}^{-} > m_{j}).
\]

Now

\[
P\left( \sum_{i=1}^{n} X_{i}^{-} \leq n_{j}^{1/\delta} \right) \geq P\left( n \max_{i=1, \ldots, n} X_{i}^{-} \leq n_{j}^{1/\delta} \right) = \left[ P\left( X_{i}^{-} \leq n_{j}^{1-\delta}/\delta \right) \right]^{n_{j}}.
\]

If we choose \( n_{j} = m_{j}^{\alpha/\delta} \), since \( m_{j} \) is an increasing sequence, we have for suitably large \( j \),

\[
n_{j}^{1-\delta/\delta} = m_{j}^{1-\delta/\delta} > m_{j}.
\]
\[ P \left\{ \sum_{i=1}^{n} X_i^{-} \leq n \frac{1}{\delta} \right\} \geq P \left\{ X_1^{-} \leq m \right\}^{n_j} \]
\[ \geq P \left\{ X_1^{-} \leq m \right\} \]
\[ = \left[ 1 - P \left\{ X_1^{-} > m \right\} \right]^{n_j} \]
\[ \geq \left[ 1 - K^{-1} P \left\{ X_1^{+} > m \right\} \right]^{n_j} \]
\[ = \left[ 1 - (K j n_j)^{-1} \right]^{n_j} \]
\[ \rightarrow 1 \quad \text{as} \quad j \rightarrow \infty.\]

Thus we have contradicted one of Baum's conditions and \( \lim_{x \to \infty} \frac{P(X^+ > x)}{P(X^- > x)} < \infty \) as required.

We now prove two lemmas which relate our conjecture to the earlier condition \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ > 0 \) a.s.

**Lemma 4-4** If \( P(X_1^+ > b_n) > n^{-1}(1 + \delta) \log n \) for all \( n \), and some \( \delta > 0 \), then \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ > 0 \) a.s.

**Proof** If \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ = 0 \) a.s. then \( P \left( \sum_{i=1}^{n} X_i^+ < b_n \ i.o. \right) = 1, \)

which implies by the Borel-Cantelli lemma that

\[ \sum_{n=1}^{\infty} P \left( \sum_{i=1}^{n} X_i^+ < b_n \right) = \infty. \quad (4-2) \]

Consequently \( \sum_{n=1}^{\infty} [P(X_1^+ < b_n)]^{n} = \infty, \)

so \( \sum_{n=1}^{\infty} [1 - n P(X_1^+ > b_n)/n]^{n} = \infty \)

and hence \( \exp\left[ -n P(X_1^+ > b_n) \right] > 1/n^{1+\delta} \) for some \( \delta > 0 \). This in turn implies \( n P(X_1^+ > b_n) > -(1 + \delta)/\log n \) so

\[ P(X_1^+ > b_n) < n^{-1}(1 + \delta) \log n \quad (4-3) \]

and this contradicts our assumption, so \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ > 0 \) and the lemma is proved.

If we let \( \sum_{n=1}^{\infty} P(X_1^+ > b_n) < \infty \) and write \( K = \sup_{n} \{ n P(X_1^+ > b_n) \} \)

(which is clearly finite) then we have \( \frac{P(X_1^+ > b_n)}{P(X_1^+ > b_n)} > \frac{(1+\delta) \log n}{K} \), so
Lemma 4-5 If there exists a sequence of positive numbers \( \{b_n\}_{n=1}^\infty \) such that \( b_n/n \to \infty \), \( \sum_{n=1}^\infty \frac{P(X_1^+ > b_n)}{P(X_1^- > b_n)} < \infty \) and \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^n X_i^+ > 0 \) a.s., then
\[
\lim_{x \to \infty} \frac{P(X_1^+ > x)}{P(X_1^- > x)} = \infty.
\]

**Proof** If our conclusion is not true, then \( \lim_{x \to \infty} \frac{P(X_1^+ > x)}{P(X_1^- > x)} < \infty \), so there exists a sequence \( \{x_j\}_{j=1}^\infty \) with \( x_j \to \infty \) as \( j \to \infty \), such that \( \lim_{j \to \infty} \frac{P(X_1^+ > x_j)}{P(X_1^- > x_j)} < \infty \). Hence there must exist \( K < \infty \), such that for all \( j > j_K \),
\[
\frac{P(X_1^+ > x_j)}{P(X_1^- > x_j)} < K.
\]

We choose subsequences \( \{x_{j_k}\}_{k=1}^\infty \) of \( \{x_j\}_{j=1}^\infty \) and \( \{b_{n_k}\}_{k=1}^\infty \) of \( \{b_n\}_{n=1}^\infty \) such that for all \( k \),
\[
b_{n_k-1} < x_{j_k} < b_{n_k}.
\]

and so we have
\[
\sum_{k=1}^\infty P(X_1^+ > b_{n_k}) < \sum_{k=1}^\infty P(X_1^+ > x_{j_k}) < K \sum_{k=1}^\infty P(X_1^- > x_{j_k}) < K \sum_{k=1}^\infty P(X_1^- > b_{n_k}) < \infty.
\]

However, \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^n X_i^+ > 0 \) a.s. implies that \( \lim_{k \to \infty} b_{n_k}^{-1} \sum_{i=1}^n X_i^+ > 0 \) a.s., and hence from Chow and Robbins [4] we have \( \sum_{k=1}^\infty P(X_1^+ > b_{n_k}) = \infty \). Thus we have a contradiction and the lemma is proved.

Lemma 4-5 is also consistent with our conjecture, and we presume the converse does not hold, as there would appear to be cases where \( S_n/n \to \infty \) but \( \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^n X_i^+ = 0 \).

A number of authors, notably Feller [9], Miller [24], Stone [30] and Kesten [22] have contributed results related to the subject of this chapter.
1. Feller [9] proved a result which is closely related to the Borel-Cantelli lemma, and which underlies much of the work subsequent to his in this field (particularly that of Chow and Robbins [4]). Feller showed that for a sequence of positive numbers \( \{a_n\}_{n=1}^{\infty} \) such that \( a_n \uparrow \infty \), \( P(|S_n| > a_n \text{ i.o.}) = 0 \) or 1 according as 
\[
\sum_{n=1}^{\infty} P(|X_i| > a_n) \text{ is finite or infinite.}
\]
Fristedt [13] showed that the same result holds for \( \{a_n\} \) convex, but this does not generalise to our work.

2. Miller [24] gives a result for non-negative random variables whose distribution functions have regularly varying tails (i.e. distribution function \( F(x) = 1 - x^{-\alpha} L(x) \) where \( L(x) \) varies slowly at infinity). He proves that under these conditions, for \( \rho_n \) satisfying 
\[
\sum_{n=1}^{\infty} \rho_n^{\alpha} / n^2 L(\rho_n) < \infty,
\]
that if \( k_n = o(n) \) and \( A > 0 \), then 
\[
P[ \lim_{n \to \infty} S_n / A \rho_n^{\alpha} L(\rho_n) > 1 ] = 1.
\]
We may obtain a similar result following the argument of Lemma 4-4. This lemma shows (see equation (4-3)) that for any increasing sequence \( \{b_n\} \), 
\[
\lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ = 0 \text{ a.s.}
\]
is equivalent to 
\[
n P(X_i^+ > b_n) < (1 + \delta) \log n, \text{ so we have}
\]
\[
\sum_{n=1}^{\infty} \left[ n^2 P(X^+ > b_n) \right]^{-1} > (1 + \delta)^{-1} \sum_{n=1}^{\infty} \left[ n \log n \right]^{-1} = \infty.
\]
Following Miller's assumption \( P(X^+ > x) = x^{-\alpha} L(x) \), we thus have 
\[
\sum_{n=1}^{\infty} b_n^{\alpha} / n^2 L(b_n) = \infty.
\]
Reversing the argument we see 
\[
\sum_{n=1}^{\infty} b_n^{\alpha} / n^2 L(b_n) < \infty
\]
is equivalent to 
\[
\lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ > 0 \text{ a.s., which in turn implies}
\]
\[
P[ \lim_{n \to \infty} \sum_{i=1}^{n} X_i^+ / b_n = 1 ] = 1 \text{ if } \lim_{n \to \infty} b_n^{-1} \sum_{i=1}^{n} X_i^+ = \delta \text{ a.s.}
\]
Now as 
\[
n P(X_i^+ > b_n) > (1 + \delta) \log n, \text{ if } k_n = o(n) \text{ we have}
\]
\[
n P(X^+ > b_{k_n}) = (n/k_n) k_n P(X^+ > b_{k_n})
\]
\[
> (n/k_n) (1 + \delta) \log k_n
\]
\[
\to \infty \text{ as } n \to \infty.
\]
Thus \( n b^{-\alpha} L(b_{kn}) = n P(X^+ > b_{kn}) \to \infty \) as \( n \to \infty \), so Miller's result

\[
P(\lim_n \sum_{i=1}^{\infty} X_i/n \leq b_{kn}^{1-\alpha} L(b_{kn}) \geq 1) = 1
\]

is stronger than our result, \( P(\lim_n \sum_{i=1}^{\infty} X_i/n = b_{kn} \neq 0) = 1 \). It would appear therefore that the methods employed by Miller give better results for this specific problem than our more direct method.

3. Stone [30] showed that for any random walk, at least one of \( \lim_n S_n/n^{1/2} = -\infty \) a.s. and \( \lim_n S_n/n^{1/2} = +\infty \) a.s. must occur.

4. Kesten [22] proved a number of related results including a generalised form of Stone's result cited above. The most relevant of his results is that quoted in the introduction to this chapter describing the possible limiting behaviour of \( S_n/n \).
REFERENCES


