The use of the Engel-Jacobson Theorem to prove (2.3.4) is not valid. Furthermore, because \( U_Y \) (having a unity) is a faithful \( U_Y \)-module, (2.3.4) is equivalent to the following more direct assertion.

The algebras \( \mathcal{N}_Y^+ \) are nilpotent.

Proof. It will be enough to consider the case of \( \mathcal{N}_Y^+ \).

In Kostant's lattice \( \mathcal{U}^{(Z)} \), let \( p^{(Z)} \) be the \( Z \)-subalgebra generated by all \( e^k_\alpha/k! \) with \( \alpha > 0 \), \( k > 0 \). Define a filtration

\[
p^{(Z)} = p^{(Z)}_1 \geq p^{(Z)}_2 \geq \ldots
\]

in \( p^{(Z)} \) as follows. Firstly, given a positive root \( \alpha \), let \( |\alpha| \) denote the number of fundamental roots (multiplicities counted) appearing in \( \alpha \) when written as sum of fundamental roots. Next, define the level of an associative product \( \prod_j e^{k(j)}_{\alpha(j)}/k(j)! \), with \( \alpha(j) > 0 \), \( k(j) > 0 \), to be the positive integer \( \sum_j k(j)|\alpha(j)| \). Finally, for \( i \geq 1 \), let \( p^{(Z)}_i \) be the \( Z \)-span of all monomials whose level is at least \( i \). Clearly

\[
p^{(Z)} = p^{(Z)}_1 \geq p^{(Z)}_2 \geq \ldots \quad p^{(Z)}_i p^{(Z)}_j \leq p^{(Z)}_{i+j}.
\]

Thus, the \( p^{(Z)}_i \) are ideals in \( p^{(Z)} \), and the chain is indeed a filtration.
Notice that the span of those standard basis monomials \( \prod_{\alpha > 0} e_\alpha^{k(\alpha)/k(\alpha)!} \) of level exactly \( i \), forms a \( \mathbb{Z} \)-complement for \( p^{(\mathbb{Z})}_{i+1} \) in \( p^{(\mathbb{Z})}_i \). This follows, using Kostant's Theorem, because the usual collection process, applied to an associative monomial in \( p^{(\mathbb{Z})}_i \), does not introduce monomials of a different level. In particular, \( \bigcap_i p^{(\mathbb{Z})}_i = \{0\} \). Note also that \( p^{(\mathbb{Z})}/p^{(\mathbb{Z})}_i \) is nilpotent.

Now tensor with the field \( K \), writing \( P = K \otimes p^{(\mathbb{Z})} \) and \( p^{(\mathbb{Z})}_i = K \otimes p^{(\mathbb{Z})}_i \). In view of the above basis property, we obtain the induced filtration

\[
P = P_1 \geq P_2 \geq \ldots, \quad \bigcap_i P_i = \{0\}, \quad P/P_i \text{ is nilpotent.}
\]

In short, \( P \) is residually nilpotent. Furthermore, \( N^+_{\gamma} \) is a finite-dimensional subalgebra of \( P \), and perforce nilpotent, as claimed. \( \square \)

Replace the italicized claim at the end of (4.1) (pages 73, 74) by the following (more complete) statement:

the strong polynomial for \( A_k \) contains all the information, dependent on the choice of \( A_k \), needed to compute the composition structure \( A_k \). (In practice, one needs to know the characters of the irreducible \( G \)-modules as well.)

The remark at the top of page 114 - namely, that to specify the submodule lattice of \( \text{Pol}^1_k \) it is enough to describe the \( M(\ell) \) - needs
amplification. Precisely, a finite distributive lattice can be recovered from the partially ordered set of its join-irreducibles. Thus, let $L$ be a finite distributive lattice, the poset of its join-irreducibles other than the least element, and write $2^\tilde{X}$ for the collection of anti-tone (that is, order-reversing) functions from $X$ to the Boolean lattice $\{0, 1\}$. This $2^\tilde{X}$ is itself a lattice under pointwise operations, and in fact (see Theorem 3 of Chapter III in the reference below),

$$L \cong 2^\tilde{X}.$$ 


Insert the following paragraph after (5.5.3).

In a paper whose details are given below, Alperin described the decomposition of all tensor products of irreducible modules for $\text{SL}(2, K)$, $K$ finite of characteristic 2, over an algebraically closed field of characteristic 2 (suppressing most of the details of his proofs). This easily yields that (5.5.3) is valid for all $k$ when $p = 2$.

### MODULE STRUCTURE IN CERTAIN ALGEBRAS

Michael G. Schooneveldt

### CORRIGENDA

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<td>10</td>
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<td>126</td>
<td>4</td>
<td>$I_{k+i} \otimes T_1$</td>
<td>$Z_i$</td>
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The results (3.4.3) and (3.4.4) need modification. The following is a corrected version:

(1) In the relatively free context, (3.4.3) and (3.4.4) (and their proofs) are correct.

(2) Take (3.4.3) in the weak case. If $(A, Y)$ is weakly relatively free, then one needs (typically) $(A, X)$ to satisfy the weak laws for $(A, Y)$ in order for the conclusion to hold. If $|Y| \leq |X|$, a straightforward argument shows $|Y| = |X|$, but it is by no means true that $(A, X) \cong (A, Y)$. For example, take $A = \text{Ext}(X)$ with $X = \{x, y, z\}$ and $Y = \{x+y z, y, z\}$.

(3) The conclusion of (3.4.4) is false (the example above will serve) but the comment at the end of the proof remains valid.

In the Dynkin-Specht-Wever Theorem on p. 63, one needs $k$ to be a unit in $R$ throughout.
MODULE STRUCTURE IN CERTAIN ALGEBRAS

by

Michael G. Schooneveldt

A thesis submitted to the
Australian National University
for the degree of
Doctor of Philosophy
June, 1980
STATEMENT

Except where otherwise stated, the work in this thesis is my own.

M.G. Schooneveldt
DEDICATION

For my parents ...

... with understanding and love
For ages now, the literature has abounded with various graded algebras whose homogeneous components can be treated as modules for the general linear group (by algebra automorphisms) and the general linear Lie algebra (by derivations). Most of these algebras are relatively free (for example, polynomial algebras) but the exterior algebra of a vector space instances one which is not. This paper is an attempt to treat these algebras in a uniform manner, with particular emphasis on the module structure of their components.

Aside from preliminaries, the thesis falls into three parts. The first gives an abstract definition of the relevant algebras; this involves a mild generalization of some concepts from Universal Algebra. The second introduces the two actions above, but treats them independently of each other. The final part brings the actions together by the process of Chevalley reduction; here, the components are treated as modules for certain distinguished subalgebras (first studied by J.E. Humphreys) of Kostant's algebra.

It will be found that, roughly speaking, information regarding composition structure is quite definitive (and algorithmically computable). We also examine the problem of decomposing the components of particular algebras (notably, the free Lie and Special Jordan algebras) in finite characteristic.
ACKNOWLEDGEMENTS

This paper was produced in the Department of Mathematics, Research School of Physical Sciences, Australian National University. Let me begin by thanking the ANU for granting me a scholarship to do the work. Through many uncertainties, the University has always given me the benefit of the doubt and for that I am grateful.

The Department of Mathematics is an excellent place in which to work. My fellow students, particularly Leon Sterling and Guan Aun How, were the mainsprings behind a series of "student seminars", a continual source of enjoyment. In the same way, Wednesday afternoons were organized by Dr M.F. Newman into occasions for airing views: the talks I gave helped clarify my own. Of others in the Department, I thank Dr R.W. Richardson for his continuing interest and B. Geary for maintaining a very high standard of typing. Leon Sterling also assisted with proof-reading.

And last! I owe a special debt of gratitude to my permanent supervisor, Dr L.G. Kovács. At first a bridge partner, then a supervisor, always a friend, Dr Kovács made many trenchant comments about this manuscript. Our conversations together extended over many hours and touched on numerous branches of algebra. Any merits this paper may contain are a compliment to his supervision; its defects are all my own.
CONTENTS

STATEMENT ......................................................... (i)
DEDICATION ....................................................... (ii)
ABSTRACT ......................................................... (iii)
ACKNOWLEDGEMENTS ................................................. (iv)
CONTENTS ......................................................... (v)
NOTATION ......................................... ............. (vii)

CHAPTER 1: INTRODUCTION ........................................ 1

CHAPTER 2: PRELIMINARY RESULTS ................................ 6
(2.1) Classical Representations of the General Linear Groups .. 6
(2.2) Complex Semisimple Lie Algebras and Chevalley Reduction .. 8
(2.3) Humphreys' Algebras ..................................... 12
(2.4) Specialization to the Type $A_1$ ........................ 20
(2.5) A Miscellany of Elementary Calculations ................ 21

CHAPTER 3: EXTENSION ALGEBRAS: FOUNDATIONS ............. 26
(3.1) The Definition of Strongly Graded Algebras .... ... 27
(3.2) The Definition and Structure of Weakly Relatively Free Algebras .... 32
(3.3) Introduction to Extension Algebras ..................... 45
(3.4) Elementary Properties of Extension Algebras .......... 51
(3.5) Changes to the Base Ring and Integral Extension Algebras .... 55
(3.6) Standard Examples of Integral Extension Algebras ........ 61

CHAPTER 4: THE COMPONENTS OF EXTENSION ALGEBRAS AS MODULES .... 66
(4.1) The Action of the General Linear Group ........... 67
(4.2) The Exterior and Polynomial Algebras (I) .... ... 74
(4.3) Examples of Composition Structure ............... 75
(4.4) Some Qualitative Remarks on the Structure of $A_k$ .... 78
(4.5) The Action of the General Linear Lie Algebra ....... 83
(4.6) The Exterior and Polynomial Algebras (II) ........ 88
NOTATION

$A_k$  homogeneous component of $A$ of total degree $k$

$A_{\lambda}, A(\lambda_1, \ldots, \lambda_r)$ homogeneous component of $A$ of strong degree $\lambda$

$\beta(a_1, \ldots, a_k)$ monomial in $a_1, \ldots, a_k$

$\mathbb{C}$ complex numbers

$D(r, k), d(r, \lambda)$ $R$-ranks of $A_k, A_{\lambda}$

$E(M)$ injective hull of the module $M$

$F_k$ minimal submodule of $A_k$ of highest weight

$F(X)$ nonzero functions $X \to \mathbb{N}$ of finite support

$F(X|L_L, M_W)$ relatively free algebra on $X$ determined by $L_L, M_W$

$F_\infty$ semisimple elements in $U_\gamma$

$L_L(A), L_W(A)$ verbal, weakly verbal ideals in $A$ determined by $L$

$L_R(L_L, M_W)$ verbal, weakly verbal ideals in $A$ determined by $L$

$L_L(F_\infty) + M_W(F_\infty)$ natural numbers

$\mathbb{N}$ rational numbers

$\mathbb{Q}$ rational numbers

$\text{REnd}(A)$ restricted endomorphisms of $A$

$\text{RHom}(A, B)$ restricted homomorphisms $A \to B$

$RX$ $R$-module generated by $X$

$s_k^{(r)}(x_1, \ldots, x_r), a_k^{(r)}(\zeta_1, \ldots, \zeta_r)$ strong polynomial for $A_k$
σM

u, u(κ)

u(ℤ)

u_Y

V^⊗k

V^⊗k±

V^*

X_∞

(A), χ(A)

χ_K, χ_λ

ℤ

⟨ ℤ ⟩_S

socle of the module M

Kostant's algebra

Kostant's lattice

γth Humphreys' algebra

k-fold tensor power of the vector space V

symmetric and skew-symmetric parts of V^⊗k

dual of the vector space V

alphabet

character, Brauer character of A_k

integers

S-module generated by the set ℤ
Our aim in this paper is to study the homogeneous components of certain graded algebras. Quite apart from their intrinsic interest, the algebras are important: they have occurred in applications beyond count. However, the literature does not appear to contain many attempts to study them coherently. In that regard, the work here is a small beginning. We raise more problems than we solve, and there is much in these pages that is folklore, but before all else, the discussion is systematic, and not ad hoc as seems to be the norm in this area.

Let us give a casual description of the algebras themselves. Two of their properties are immediate. The first we call strong gradedness (3.1). Thus, let $A$ be a graded algebra over a commutative ring $R$, and suppose $X$ is a generating set for $A$. A monomial is a product of elements of $X$; in general, monomials are not linearly independent. The essence of strong gradedness is that one can unambiguously count up the number of occurrences of a particular generator in any monomial. This extends naturally to linear combinations of monomials. It is a commonplace that polynomial, free associative, free Lie, and numerous other algebras have this property. Oddly enough, the tendency in the literature seems to be to focus attention on the gradedness in these algebras and to refer to their strong gradedness almost parenthetically. We prefer to coin a separate term at the outset.

The second outstanding property of the algebras is that they are weakly relatively free (3.2), that is, every function from $X$ to the degree one component of $A$ extends to an algebra endomorphism of $A$. Relatively free algebras have this: for them, every function $X \rightarrow A$ can be extended. The exterior algebra of a vector space is an instance of a weakly relatively
free algebra that is not relatively free. Again, the Lie algebra of a relatively free group is weakly relatively free, although here, it is not known whether the qualification "weakly" can be dropped.

Now, these two properties do not characterize the algebras in which we are interested. We have found it necessary to adopt a slightly more technical definition: see (3.3) for the details of this. The definition is strong enough to ensure a further property of the algebras, one of the most important. It is a fact (3.5) that the tensor product of one of the algebras with an extension of the underlying base ring $R$ retains the defining property. This motivates the name we give the algebras: we call them extension.

The homogeneous components of an extension algebra $A$ of finite rank $r$ are modules. The general linear group $GL(r, R)$ acts classically by linear substitutions, more precisely, by the algebra automorphisms afforded by the weakly relatively free property of $A$. On the other hand, the fact that the base ring can be extended is just sufficient to ensure that every function from $X$ to the degree one component of $A$ is the restriction of a unique derivation of $A$ (4.5). In consequence, the homogeneous components are also modules for the general linear Lie algebra $gl(r, R)$. Notwithstanding the length of preparatory work we do in leading up to these module structures, they remain the central focus of interest in this paper. To discuss them further here, we suppose for the rest of this introduction that $R$ is a field.

One facet of the work that we find personally satisfying is the conceptual framework for highlighting the close connections that exist between the strong gradedness in $A$ and its module structures. The actions of $GL(r, R)$ and $gl(r, R)$ are easily visualized. In the group case, one calculates the characters of the homogeneous components; over the Lie algebra, the weights and weight spaces are determined. At least in
characteristic $\infty$, the calculations provide arithmetic determinations of composition structure. The details are in Chapter 4.

Of course, one expects the actions of $\text{GL}(r, R)$ and $\mathfrak{gl}(r, R)$ to be related. For those extension algebras which have a natural realization over the integers (and we simply call them integral (3.5)), the connection is provided by Chevalley's process of reduction. Here, we are content to consider only the special linear case. The reduction process realizes the homogeneous components as modules for Kostant's algebra $\mathcal{U}$, and we tend to treat them in that light. Insofar as one is more interested in the action of $\text{SL}(r, R)$, that approach needs some justification. We reason as follows. Firstly, when the field $R$ is sufficiently large, the actions of $\mathcal{U}$ and $\text{SL}(r, R)$ are essentially equivalent; working with $\mathcal{U}$ allows a uniform treatment over arbitrary fields. Secondly, it seems unlikely that the complexities of the small field case can be adequately resolved without the preliminary study afforded by $\mathcal{U}$. Thirdly, direct calculation with $\mathcal{U}$ seems considerably easier than with $\text{SL}(r, R)$. This is illustrated in particular by the determination of the submodule lattices of the components of the rank 2 polynomial algebra in (5.4), a genuine simplification of the work of Carter and Cline [1] in the algebraic group setting. On the other hand, it does not pay to be too dogmatic: in the final section of this paper we have needed the equivalence mentioned above, and in consequence, the base field to have $p^2$ elements ($p$ the characteristic). There, the case $\text{GF}(p)$ remains open.

In finite characteristic, there is in Kostant's algebra a distinguished ascending chain of subalgebras. These were first studied by J.E. Humphreys in the context of algebraic groups over algebraically closed fields. We say something about their representations over arbitrary fields in (2.3), although much remains to be done. A module for a Humphreys' algebra is a
direct sum of weight spaces; needless to say, the weights and weight spaces in extension algebras are easy to compute. The discussion of the action of \( \mathcal{U} \) is sometimes facilitated by restricting to a "sufficiently large" Humphreys' algebra.

Most of our calculations with particular extension algebras are done in rank 2. The exterior and polynomial algebras run like a thread through the paper. Really, there is not too much to say regarding the exterior algebra, but we stop to say it anyway. On the other hand, the polynomial algebra remains a fascinating source of problems. We include a synopsis of some of the main basic results, together with the determination of the submodule lattice mentioned above. In the final sections of the paper, we turn to the free associative algebra (and its close relatives, the free Lie and special Jordan algebras). The work here centres around a conjecture in (5.5) which says, effectively, that the decompositions of the components into \( \mathcal{U} \)-indecomposables are completely determined by their composition structure. We find it remarkable that that may hold; however, we have only succeeded in proving a special case.

We make some general remarks. Firstly, apart from scalar action which is always written on the left, all our modules are right ones. Our only reason for doing this is a longstanding personal habit. Unfortunately, there seems to be a general tendency in the classification theory of complex Lie algebras to work with the left adjoint representation. In consequence, some of our equations of structure differ from those in the literature (to within a change of sign only). We have already corrected a number of errors stemming from this conversion from left to right modules; it is perhaps too much to hope we have caught them all. We emphasize too that all our modules are finite-dimensional as vector spaces. Next, we usually use algebraic notation for functions (\( a\varphi \) instead of \( \varphi(a) \)) but there are some
(reasonable) exceptions. Here, the context will always decide what is meant. Finally, we illustrate the numbering system in the paper. Thus, (3.2) is section 2 of Chapter 3; on the other hand, (3.2.5) is statement 5 (definition, example, ...) in (3.2). Most propositions are numbered without qualification. Occasionally, we dignify a result with the title "Theorem", mainly for emphasis, sometimes for some personal aesthetic quirk.

As we mentioned at the beginning of this introduction, this paper contains a certain amount of folklore. Some of that is rediscovery, stemming from the fact that we have been working in comparative isolation. Certainly, where we know of a reference we give it. On the other hand, there seems to be a genuine need in modern Mathematics for folklore to be written down: far too much information exists only in the minds of mathematicians. Perhaps that is the final justification for what is written here.
CHAPTER 2
PRELIMINARY RESULTS

This chapter serves two functions. On the one hand, it gives an outline of the basic theory on which the rest of the paper depends; on the other, it includes various results which will be needed explicitly. By and large, propositions not proved in the subsequent chapters receive a mention here, the exceptions being those results (almost all of them well-known) whose proper theoretical context only arises later.

Throughout the chapter, \( K \) is a field of characteristic \( p \leq \infty \).

(2.1) Classical Representations of the General Linear Groups

We begin the chapter by recalling some of the basics of the representation theory of the general linear groups \( \text{GL}(r, K) \), \( r \geq 2 \). General references for this section include James [1], Curtis and Reiner [1, §67] and Weyl [1].

Consider a vector space \( V \) over \( K \) of dimension \( r \geq 2 \). Write \( V^\otimes n \) for the \( n \)-fold tensor power of \( V \). The group \( G = \text{GL}(r, K) \) acts naturally on \( V \), and therefore acts on \( V^\otimes n \) via
\[
(v_1 \otimes \ldots \otimes v_n)g = (v_1g) \otimes \ldots \otimes (v_ng) , \quad v_i \in V , \quad g \in G .
\]
There is also an action of the symmetric group of degree \( n \), \( S_n \), by place permutations, namely,
\[
(v_1 \otimes \ldots \otimes v_n)\sigma = v_{1^{\sigma^{-1}}} \otimes \ldots \otimes v_{n^{\sigma^{-1}}} , \quad \sigma \in S_n .
\]
These two actions commute with each other. In the so-called Classical Case \( n < p \), the situation is particularly well-known, and we summarize a few facts from that case.

Assume then \( n < p \). Let \( \lambda \) be the \( G \)-module determined by the
partition $\lambda$ of $n$. Precisely, $W^\lambda \cong V^\otimes e_\lambda$, for any one of the (several) primitive idempotents $e_\lambda$ corresponding to $\lambda$ in the group algebra $KSn$.

(2.1.1). The $G$-module $W^\lambda$ is either zero or irreducible. It is zero if and only if $\lambda$ is a partition of $n$ into more than $r$ nonzero parts. As $G$-module, $V^\otimes_1$ is completely reducible, and every irreducible $G$-submodule is isomorphic to $W^\lambda$ for some $\lambda$. □

Following Carter, Lusztig [1], we refer to the nonzero $W^\lambda$ as Weyl modules. We shall have occasion to compute their tensor products, and we use the Littlewood-Richardson Rule for that purpose. Our description of the rule is a specialization of James [1, §16].

For the moment, let $n$ be any positive integer, and fix a partition $\mu$ of $n$. A sequence of nonnegative integers is of type $\mu$ if each positive integer $i$ occurs $\mu_i$ times in the sequence. In particular, if $\mu_s$ is the last nonzero term in $\mu$, then a sequence of type $\mu$ must consist of the integers $1, \ldots, s$ only.

Fix a sequence of type $\mu$. Each term in the sequence is said to be good or bad. Specifically, all occurrences of 1 in the sequence are good. Now, let $i$ be positive such that $(i+1)$ occurs in the sequence in position $l$. Then that occurrence of $(i+1)$ is good if and only if the number of good occurrences of $i$ preceding position $l$ is strictly greater than the number of good occurrences of $(i+1)$ preceding $l$. We call the sequence itself good if all its terms are good.

(2.1.2) EXAMPLE. The quality of the terms in the following sequence of type $(5, 3, 6)$ is indicated:

\[
\begin{array}{ccccccccccc}
3 & 1 & 2 & 2 & 3 & 3 & 1 & 3 & 1 & 1 & 2 & 3 & 3 & 1 \\
\times & \checkmark & \checkmark & \times & \checkmark & \times & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \times & \checkmark & \checkmark
\end{array}
\]

In the following statement, we use the terminology and notation in James [1, §3] for diagrams and nodes. We also assume $p = \infty$;
partial results can be had when \( p < \infty \), but we will not stop to elaborate them here.

(2.1.3) LITTLEWOOD-RICHARDSON RULE IN CHARACTERISTIC \( \infty \). Let \( k > l \geq 1 \) and suppose \( \lambda, \mu \) are partitions of \( k - l, l \) respectively. Write \( a_\nu \) for the multiplicity with which the Weyl module \( W^\nu \) appears as direct summand in the tensor product \( W^\lambda \otimes W^\mu \). Then \( a_\nu = 0 \) if any \( \lambda_i > \nu_i \), and when \( \lambda_i \leq \nu_i \) for all \( i \), \( a_\nu \) is given precisely as the number of ways of replacing the nodes of \([\nu] \setminus [\lambda]\) by integers such that

1. the numbers are non-decreasing along rows,
2. the numbers are strictly increasing down columns,
3. the sequence obtained by reading from right to left in successive rows is a good sequence of type \( \mu \). □

To complete the section, we recall the definitions of two distinguished submodules of the tensor power \( V^\otimes n \). Here, we do not assume \( n < p \). The modules are the symmetric and skew-symmetric parts, \( V_\pm^\otimes n \), defined by

\[
V_+^\otimes n = \left\{ z \in V_+^\otimes n \mid z \sigma = z, \sigma \in S_n \right\}, \\
V_-^\otimes n = \left\{ z \in V_-^\otimes n \mid z \sigma = (\text{sign } \sigma)z, \sigma \in S_n \right\}.
\]

When \( n < p \), these modules are respectively \( W^{(n)} \) and \( W^{(1^n)} \). In any case, \( V_+^\otimes n \) (\( V_-^\otimes n \)) has a basis of symmetric (skew-symmetric) tensors (formed from any basis for \( V \)), and we have

\[
\dim V_+^\otimes n = \binom{n + r - 1}{r - 1}, \quad \dim V_-^\otimes n = \binom{n}{r}.
\]

(2.2) Complex Semisimple Lie Algebras and Chevalley Reduction

This section, and the two following, form a natural progression. We start here by summarizing some of the representation theory of complex
semisimple Lie algebras and give a brief description of Chevalley's process of reduction modulo $p$. This leads into the next section, where we develop some information concerning Humphreys' algebras. As it happens, we only need the results of these two sections for Lie algebras of type $A_1$, but it costs nothing, and seems conceptually simpler, to work in full generality. Then, in (2.4), we specialize the discussion to the $A_1$ case. References for this section (2.2) include Humphreys [1] and Steinberg [1].

Let $L$ be a semisimple Lie algebra of rank $\rho$ over the field $\mathbb{C}$ of complex numbers, $H$ a Cartan subalgebra (necessarily abelian) and $\Phi$ the corresponding root system. We have the decomposition into root spaces

$$L = H \oplus \bigoplus_{\alpha \in \Phi} \mathbb{C} e_{\alpha}.$$  

Choose a basis $\{a_1, \ldots, a_{\rho}\}$ in $\Phi$ and a basis $\{h_1, \ldots, h_{\rho}\}$ in $H$ such that

$$[e_{-i}, e_{i}] = h_i, \quad 1 \leq i \leq \rho$$

where we write $e_{\pm i} = e_{\pm \alpha_i}$ for brevity.

Each (finite-dimensional) $L$-module $V$ is a direct sum of weight spaces with respect to $H$. There is a natural partial order in the set of all possible weights, namely, $\lambda > \mu$ if and only if $\lambda - \mu$ is a sum of positive roots. Consider specifically the case when $V$ is irreducible. Then, among the weights of $V$, there is a unique maximal weight with respect to this order. This highest weight, $\lambda$ say, is dominant integral (that is, $\lambda(h_i)$ is a non-negative integer for $1 \leq i \leq \rho$) and the corresponding weight space is one-dimensional, spanned by a maximal vector (a nonzero element $v^+$ with $v^+ e_{\alpha} = 0$ for $\alpha > 0$). In turn, the dominant integral linear functionals on $H$ index the isomorphism classes of irreducible $L$-modules. We recall in passing that two arbitrary $L$-modules have the same composition structure (and hence are isomorphic) if and only
if their weights are the same.

We now begin the discussion of Chevalley reduction. Assume specifically that the basis \( \{ e_\alpha | \alpha \in \Phi \} \cup \{ h_1, \ldots, h_\rho \} \) is a Chevalley basis for \( L \), and for the sake of definiteness, order it as

\[ e_\alpha < h_1 < \ldots < h_\rho < e_\beta, \quad \alpha > 0 > \beta \]

choosing any fixed orders for the positive (respectively negative) root vectors. Write \( U^{(\mathbb{C})} \) for the universal enveloping algebra of \( L \), and \( U^{(\mathbb{Z})} \) for Kostant's lattice. This \( U^{(\mathbb{Z})} \) is the \( \mathbb{Z} \)-subalgebra of \( U^{(\mathbb{C})} \) generated by the \( e_\alpha^k/k! \), \( k \geq 0 \), \( \alpha \in \Phi \). It has a \( \mathbb{Z} \)-basis consisting of the monomials

\[
\left[ \prod_{\alpha > 0} \left( e_\alpha^{k(\alpha)/k(\alpha)!} \right) \right] \left[ \prod_{i=1}^{\rho} \left( h_i^{i} \right) \right] \left[ \prod_{\alpha < 0} \left( e_\alpha^{m(\alpha)/m(\alpha)!} \right) \right]
\]

where the \( k(\alpha), m(\alpha) \) are non-negative integers and

\[
\left[ h_i \right] = \left( h(h-I) \ldots (h-l+1) \right)/l!.
\]

For the arbitrary but fixed field \( K \), we write \( U = U^{(K)} = K \otimes_{\mathbb{Z}} U^{(\mathbb{Z})} \), Kostant's algebra. Also, put

\[
J_\alpha^{(s)} = 1 \otimes e_\alpha^{s} / s! \quad \text{and} \quad h^{(l)} = 1 \otimes \left( h_i^{i} \right), \quad \text{with} \ s, l \geq 0.
\]

The \( J_\alpha^{(s)} \) generate \( U \) as \( K \)-algebra.

An admissible lattice in an \( L \)-module \( V \) is a \( \mathbb{Z} \)-submodule \( M \) of \( V \) with \( \mathbb{C} \otimes M \cong V \) as vector spaces and \( M \left[ e_\alpha^k/k! \right] \subseteq M \) for all \( \alpha \in \Phi \), \( k \geq 0 \). It is a fact that \( V \) always contains admissible lattices. When \( V \) is irreducible, generated by a maximal vector \( v^+ \), one calls the lattice \( M \) minimal if it is generated as \( U^{(\mathbb{Z})} \)-module by \( v^+ \). In any case, again tensoring with \( K \), \( \mathbb{V} = K \otimes M \) is a module for \( U \).

Let \( \mathbb{G} \) be the Chevalley group associated with \( L \) and the admissible lattice \( M \). Specifically, for \( \alpha \in \Phi \), \( s \geq 0 \), write \( J_\alpha^{(s)} \) for the
linear map representing $f_\alpha^{(s)}$ on $V$. For $a \in K$, the sum

$$E_\alpha(a) = \sum_{s \geq 0} a^s f_\alpha^{(s)}$$

is finite; indeed, it is an invertible linear transformation of determinant one. Then $G$ is the group generated by these $E_\alpha(a)$. In particular, we write $G_K = \langle E_\alpha(a) \mid \alpha \in \Phi, a \in K \rangle$ for the universal Chevalley group of $L$ over $K$. There is a unique homomorphism $G_K \to G$ satisfying $E_\alpha(a) \mapsto E_\alpha(a)$, and this realizes $V$ as $G_K$-module.

We describe explicitly the action of $U$ on tensor products. Suppose $M, N$ are admissible lattices in the $L$-modules $V, W$ respectively. Then $M \otimes N$ is an admissible lattice in $V \otimes W$, and tensoring with $K$, we may identify $V \otimes W$ with $V \otimes W$.

\[ (2.2.1) \] For $\alpha \in \Phi$, $s \geq 0$ and $v \in V$, $w \in W$,

$$ (v \otimes w) f_\alpha^{(s)} = \sum_{t=0}^{s} v f_\alpha^{(t)} \otimes w f_\alpha^{(s-t)}. $$

Now, treating $U^{(Z)}$ as admissible lattice in $U^{(C)}$, (2.2.1) defines an action of $U$ on the tensor product $U \otimes U$. Therefore, the tensor product of two free $U$-modules is realized as a $U$-module, and in turn, so too is the tensor product of two (abstract) $U$-modules. Similarly, the dual of a $U$-module is well-behaved:

\[ (2.2.2) \] Let $V$ be a $U$-module, $V^*$ its vector space dual. Then $V^*$ is also a $U$-module, with the action of $U$ satisfying

$$ v \left[ \theta f_\alpha^{(s)} \right] = (-1)^s \left[ v f_\alpha^{(s)} \right] \theta, $$

for all $v \in V$, $\theta \in V^*$, $\alpha \in \Phi$ and $s \geq 0$.

**Proof.** Let $\Delta$ be the function $\left\{ f_\alpha^{(s)} \mid \alpha \in \Phi, s \geq 0 \right\} \to U$ defined by $\Delta f_\alpha^{(s)} = (-1)^s f_\alpha^{(s)}$. It is enough to show $\Delta$ extends to an anti-homomorphism of algebras $U \to U$. For this, it suffices to show that the
map $\Delta : e^{s/\alpha} \mapsto (-1)^s e^{s/\alpha}$ extends to an anti-endomorphism of $U(\mathbb{Z})$.

However, $U(\mathbb{Z})$ is an admissible lattice in $U(\mathbb{C})$, and it is well-known that the $\mathbb{Z}$-dual of $U(\mathbb{Z})$ can be realized as an admissible lattice in $U(\mathbb{C})^\ast$. The action of $U(\mathbb{Z})$ on its $\mathbb{Z}$-dual is just that afforded by $\Delta$. Because $U(\mathbb{Z})$ is faithful as module over itself, the required extension of $\Delta$ exists. □

As usual, the $U$-module $V^\ast$ in (2.2.2) is called the contragredient of $V$.

(2.3) Humphreys' Algebras

This section studies the algebras first developed in Humphreys [2], [3]. Humphreys' starting point is a simply connected, semisimple algebraic group over an algebraically closed field of finite characteristic. For our part, we drop the assumption that the field be algebraically closed, and proceed independently of the theory of algebraic groups. References without an author's name in this section are to Humphreys' work.

We assume throughout that the characteristic $p$ of $K$ is finite. For $\gamma \geq 0$, let $U_\gamma$ be the subalgebra of Kostant's algebra $U$ generated by all $f_\alpha^{(s)}$ with $\alpha \in \Phi$ and $0 \leq s < p^\gamma$. We call $U_\gamma$ a Humphreys' algebra. Write $N_\gamma^+, H_\gamma, N_\gamma^-$ for the subalgebras (with 1) of $U$ generated by $\{f_\alpha^{(s)} \mid \alpha > 0, 0 \leq s < p^\gamma\}$, $\{h_i^{(n)} \mid 1 \leq i \leq \rho, 0 \leq n < p^\gamma\}$ and $\{f_\alpha^{(s)} \mid \alpha < 0, 0 \leq s < p^\gamma\}$.

(2.3.1). There is a basis for $U_\gamma$ consisting of all elements

$$
\begin{bmatrix}
\alpha > 0 & f_\alpha^{\rho(a)} \\
\overline{\rho} & h_i^{(n_i)} \\
\overline{\alpha < 0} & f_\alpha^{\overline{a}(a)}
\end{bmatrix}
$$
where \( 0 \leq s(\alpha), n_\alpha, t(\alpha) < p^\gamma \). The products in each segment form bases for \( N_\gamma^+, H_\gamma \) and \( N_\gamma^- \) respectively, and \( U_\gamma \cong N_\gamma^+ \otimes H_\gamma \otimes N_\gamma^- \) as vector spaces. In particular, \( U_\gamma \) has dimension \( p^{\gamma(p+1)} \), where \(|\Phi| = l\). □

The proof of (2.3.1) follows verbatim the argument in [2, 2.1] and we will not give it. Humphreys brings out some relevant facts which emerge from the proof, and we state these next.

(2.3.2). (1) For \( \alpha \in \Phi, s \geq 1 \), \( \left[ f_\alpha(s) \right]^p = 0 \), and for \( 1 \leq i \leq p \), \( n \geq 0 \), \( \left[ h_i^{(n)} \right]^p = h_i^{(n)} \).

(2) A \( K \)-algebra homomorphism \( H_\gamma \rightarrow K \) is completely determined by its action on the elements \( h_i^{(n)}, 1 \leq i \leq p, n = 1, p, \ldots, p^{\gamma-1} \), and these values all lie in \( GF(p) \). □

Suppose \( V \) is a \( U_\gamma \)-module. Because of (2.3.2) (1), the elements of \( H_\gamma \) have semisimple action on \( V \), and \( V \) is a direct sum of weight spaces. The weights themselves are \( K \)-algebra homomorphisms \( H_\gamma \rightarrow K \). Write \( X_\gamma \) for the collection of all such homomorphisms, the set of weights for \( U_\gamma \).

As usual, a nonzero \( v^+ \) in \( V \) is called maximal if \( v^+ f_\alpha(s) = 0 \) for all \( s \geq 1, \alpha > 0 \); should \( v^+ \) also be a weight vector, its weight is said to be high. This (convenient) terminology is not intended to imply any natural partial order in the set of weights for \( V \) (comparable to the complex case). In particular, despite the following classification of irreducible \( U_\gamma \)-modules, the weights for \( V \) do not determine its composition structure.

(2.3.3) THEOREM. (1) An irreducible \( U_\gamma \)-module \( V \) contains a maximal vector \( v^+ \). This \( v^+ \) is unique to within scalar multiples, and is also a weight vector. In particular, \( V \) has a unique high weight.
(2) Two irreducible $U_Y$-modules are isomorphic if and only if their high weights are the same.

(3) For each $\lambda \in X_Y$, there exists an irreducible $U_Y$-module with high weight $\lambda$.

(4) The irreducible $U_Y$-modules are all absolutely irreducible.

The proof of (2.3.3) follows a pattern which has become quite standard (cf. Humphreys' treatment in [2]). However, we take the trouble to give the argument, especially to justify the removal of the assumption that $K$ be algebraically closed.

We need some preliminary observations. Write $N_Y^{\pm}$ for the subalgebras without unity of $N_Y$ generated by their respective $f_\alpha^{(s)}$, $s \geq 1$. By (2.3.2) (1), the $f_\alpha^{(s)}$ are nilpotent operators on a $U_Y$-module. Therefore, by the Engel-Jacobson Theorem (see Jacobson [1, Theorem II.2.1]), or otherwise, one has the following.

(2.3.4) Each $U_Y$-module is annihilated by $(N_Y^{\pm})^t$ for some $t$ sufficiently large. □

We use (2.3.4) repeatedly. Next:

(2.3.5) For any $f_\alpha^{(s)}$ and any weight vector $v$ in a $U_Y$-module, $v_f^{(s)}$ is either zero or a weight vector.

Proof. Within the universal enveloping algebra $U^{(C)}$, we have, by induction on $s$,

$$e_\alpha^s h_n = \left[ h + sa(h) \right] e_\alpha^s$$

(cf. [1, 26.3, Lemma D]). Then, supposing $v$ to have weight $\lambda$, 
\[ v_{\alpha}^{f(s)}(n) = \left( 1 \otimes \left( e_{\alpha}^{s} / s! \right) \right)_{n}^{h} \]

\[ = v \left[ 1 \otimes \left( h + s\alpha(h) \right) \right] f_{\alpha}^{(s)} \]

\[ = \left[ \lambda \left( 1 \otimes \left( h + s\alpha(h) \right) \right) \right] \left[ v_{\alpha}^{f(s)} \right] \]

where the last step is justified using [1, 26.1]. Therefore, \( v_{\alpha}^{f(s)} \) is either zero or a weight vector. \( \square \)

We proceed to the proof of (1) in (2.3.3). By (2.3.4), the subspace \( V_0 \) of \( V \) annihilated by \( N_\gamma^+ \) is nonzero, so \( V \) contains maximal vectors. Furthermore, much as in the proof of (2.3.5), this \( V_0 \) is an \( H_\gamma \)-submodule of \( V \) and therefore is a direct sum of weight spaces. We claim \( V_0 \) lies in a single weight space of \( V \). For assume to the contrary that \( v_1^+, v_2^+ \) are maximal vectors of different weights. Because \( V \) is irreducible, it is generated as \( U_\gamma \)-module by \( v_1^+ \), and in consequence has a basis of elements \( v_1^+ u, \ u \in B \subseteq N_\gamma^- \) together with \( v_1^+ \) itself. So we may write

\[ v_2^+ = \xi v_1^+ + \sum_{u \in B} \alpha_u (v_1^+ u) \quad , \quad \xi, \alpha_u \in \mathbb{K} \]

Similarly

\[ v_1^+ = \xi' v_2^+ + \sum_{u \in B'} \alpha'_u (v_2^+ u) \quad , \quad \xi', \alpha'_u \in \mathbb{K} \]

By (2.3.5), and the fact that vectors of different weights are linearly independent, we have \( \xi = \xi' = 0 \). Then, substituting the second equation into the first, \( v_2^+ = v_2^+ u \) for some \( u \in N_\gamma^- \), contradicting (2.3.4).

We have shown then that \( V \) has a unique high weight. To complete the proof of (1), it remains to show \( \dim V_0 = 1 \), and it is convenient to defer the argument here until we come to (4) below.

For (2) in (2.3.3), suppose \( V_1, V_2 \) are irreducible \( U_\gamma \)-modules with
the same high weight $\lambda$. Write $W = V_1 \oplus V_2$, choose a maximal vector $v_i^+ \in V_i$ ($i = 1, 2$), and set $v^+ = (v_1^+, v_2^+)$. Let $W^+$ be the $U_\gamma$-submodule of $W$ generated by $v^+$. The natural projections $\pi_i : W^+ \rightarrow V_i$ are clearly surjections. To show $V_1 \cong W^+ \cong V_2$, it is enough to show they are injective. Indeed, embedding $V_1, V_2$ into $W$, ker $\pi_2 = W^+ \cap V_1$, and because $V_1$ is irreducible, generated by $v_1^+$, it is enough to show $v_1^+ \notin W^+$. Assume to the contrary, $v_1^+ = (v_1^+ + v_2^+) + u$ for some $u \in \overset{\circ}{\gamma}$. Then $v_1^+ = v_1^+ u$, contradicting (2.3.4), and the proof is complete.

There are a number of ways of constructing an irreducible $U_\gamma$-module of high weight $\lambda$. For example, let $B$ be the subalgebra of $U_\gamma$ generated by $\overset{\circ}{\gamma}$ and $\gamma$, and let $P = \langle v^+ \rangle_K$ be a one-dimensional $B$-module on which $B$ acts as $v^+ u = 0, \quad v^+ h = \lambda(h) v^+, \quad u \in \overset{\circ}{\gamma}, \quad h \in \gamma$.

Because $U_\gamma$ is a left $B$-module, we may form the tensor product $V = P \otimes_B U_\gamma$. This $V$, the $U_\gamma$-module "induced from $P"$, is generated by $v^+ \otimes 1$. Indeed, $U_\gamma$ is a free $B$-module, so $v^+ \otimes 1$ is a maximal vector of weight $\lambda$. Factoring out a maximal submodule of $V$ gives the required irreducible. This proves (3) in (2.3.3).

To prove (4), we begin with the observation that an irreducible $U_\gamma$-module $V$ is absolutely irreducible if and only if its space of maximal vectors is one-dimensional. Indeed, if $v_1^+, v_2^+$ are maximal vectors, then, by putting $V_1 = V = V_2$ in the proof of (2), we see that there is an automorphism of $V$ mapping $v_1^+ \rightarrow v_2^+$. However, $V$ is absolutely irreducible if and only if its only automorphisms are scalars, and so the
Now, let $\overline{K}$ be the algebraic closure of $K$ and set $U_{\gamma}(\overline{K}) = \overline{K} \otimes_K U_\gamma$. Then $U_{\gamma}(\overline{K})$ is the $\gamma$th Humphreys' algebra constructed with $\overline{K}$ instead of $K$, and we may embed $U_\gamma$ into $U_{\gamma}(\overline{K})$. In view of (2.3.2) (2), there is a natural identification of the set of weights for $U_{\gamma}(\overline{K})$ with $X_\gamma$, the weights for $U_\gamma$. Let $V$ be an irreducible $U_{\gamma}(\overline{K})$-module with high weight $\lambda$. By the observation in the previous paragraph, the space of maximal vectors in $V$ is one-dimensional, spanned by $v^+$ say. Write $W$ for the $U_\gamma$-submodule of $V$ generated by $v^+$. Any element $w$ of $W$ takes the form $w = kv^+ + v^+ u$ for $k \in K$, $u \in N_\gamma^\perp$. In particular, if $w$ is maximal with respect to $U_\gamma$, it is maximal with respect to $U_{\gamma}(\overline{K})$, and we are forced to have $\overline{k}v^+ = w = kv^+ + v^+ u$ for some $\overline{k} \in \overline{K}$. By (2.3.4), $k = \overline{k}$ and $v^+ u = 0$. This implies that the $U_\gamma$-maximal vectors in $W$ form a one-dimensional space. Since any proper, nonzero submodule of $W$ cannot contain $v^+$, and yet would have to contain maximal vectors, we conclude $W$ is $U_\gamma$-irreducible, with high weight $\lambda$. By the preceding paragraph, $W$ is absolutely irreducible, and since $\lambda$ is an arbitrary element of $X_\gamma$, (4) is proved, together with the assertion left unproved in (1). This completes the proof of (2.3.3).

Write $E_\lambda$ for the irreducible $U_\gamma$-module whose high weight is $\lambda$. The construction of $E_\lambda$ in the proof of (2.3.3) (2) has the advantage of being intrinsic to characteristic $p$, but there is an alternative approach, and we outline this to give a better overall picture of the representation theory of $U_\gamma$. 
Recall for the moment the original complex Lie algebra $L$. Let $V^\lambda$ be the irreducible $L$-module with highest weight $\lambda$. Choose a minimal admissible lattice in $V^\lambda$, and let $\overline{V}^\lambda$ be the corresponding $U$-module.

Now, $\lambda$ is dominant integral: assume specifically $0 \leq \lambda(h_i) < p^\gamma$ for $1 \leq i \leq \rho$. As $U_\gamma$-module by restriction, $\overline{V}^\lambda$ has a top composition factor whose high weight $\overline{\lambda}$ is computed from $\lambda$: indeed, $\overline{\lambda}\left(\frac{h_i(n)}{n}\right)$ is the reduction modulo $p$ of $\left\lfloor\frac{\lambda(h_i)}{n}\right\rfloor$. It is immediate that, allowing $\lambda$ to vary subject to the above restriction, we obtain $p^{\gamma p}$ distinct irreducibles for $U_\gamma$. However, by (2.3.2) (2), $|X_\gamma| \leq p^{\gamma p}$, and therefore we have obtained them all.

Now, in the previous paragraph, the top composition factor of $\overline{V}^\lambda$ is unique. Indeed, $\overline{V}^\lambda$ has a subspace $M$ which is at once the unique maximal $U_\gamma$-submodule and unique maximal $U$-submodule of $\overline{V}^\lambda$. This is proved when $K$ is an infinite field by paralleling the arguments in [4, 1.1 and 1.2] (see also [2, 2.3]); it then follows for arbitrary $K$ by (2.3.3) (4). In particular, the irreducible $U_\gamma$-modules are realized naturally as $G_K$-modules. In general, the problem of determining the structure of these irreducibles over $G_K$ appears to be a difficult one and we shall not pursue it here. Of course, $E^\lambda_\lambda$, $\lambda \in X_\gamma$, is always $G_K$-irreducible when $K$ has enough elements. This is a consequence of the following well-known statement (already used in the references above), which we number for future reference.

(2.3.6). Suppose $V$ is an $L$-module with $V\alpha^\lambda = \{0\}$ for some $t > 0$ and any $\alpha \in \Phi$. Assume $K$ has at least $t$ elements, and reduce $V$ via
an admissible lattice to $\overline{V}$. Then the $G_K$ and $U$-submodules of $\overline{V}$ coincide. In the same way, $G_K$ and $U$-homomorphisms coincide. □

We conclude the section with some remarks concerning the case $\gamma = 1$. Define $\overline{L} = K \otimes L(\mathbb{Z})$, where $L(\mathbb{Z})$ is the $\mathbb{Z}$-span of the original Chevalley basis for $L$. Thus, $\overline{L}$ is the classical Lie algebra over $K$ obtained from $L$. It is a Lie $p$-algebra. Humphreys attributes to Verma the observation that $U_1$ is the restricted universal enveloping algebra of $\overline{L}$. In particular, (2.3.3) affords a classification of the restricted irreducible $\overline{L}$-representations.

We recall that $U_1$ is a symmetric algebra (for example, [3, §2]). There is a one-one correspondence between the irreducible and principal indecomposable $U_1$-modules given by associating an irreducible with the principal indecomposable which has that irreducible as its unique top (and bottom) composition factor. Injective and projective $U_1$-modules coincide, and in addition, for any $U_1$-modules $M$ and $N$, with $M$ projective, $M \otimes N$ is projective.

One final remark. There is an action of the universal Chevalley group $G_K$ on $U_1$ by algebra automorphisms: this is afforded by the action of $U(\mathbb{Z})$ (via the adjoint representation) on the admissible lattice $L(\mathbb{Z})$ in $L$. This action of $G_K$ on $U_1$ is compatible with the actions of $G_K$ and $U_1$ on any module $\overline{V}$ given by the reduction process. Specifically, write $u^g$ for the image of $u \in U_1$ under $g \in G$. Then, for $v \in \overline{V}$, $v(u^g) = ([v^{-1}u]) g$. For example, if $W$ is a $U_1$-submodule of $\overline{V}$, then so too is $Wg$, $g \in G$.

In the context of algebraic groups, Humphreys has generalized the above
results to $U_\gamma$, $\gamma > 1$. However, at the time of writing, we are only ready to use the $\gamma = 1$ case.

(2.4) Specialization to the Type $A_r$

As we mentioned at the beginning of (2.2), our primary interest lies with the Lie algebras of type $A_r$, and we now consider some of the niceties of that case.

Begin with $L = \mathfrak{sl}(r, \mathbb{C})$, the Lie algebra of $r \times r$ matrices of trace zero over $\mathbb{C}$. We will always work with the standard basis for $L$. Specifically, for $1 \leq i, j \leq r$, define $e_{ij}$ to be the elementary matrix with $1$ in position $(ij)$ and zeros elsewhere, and set

$$h_i = e_{(i+1)(i+1)} - e_{ii}, \quad 1 \leq i < r.$$  

Then the standard basis is

$$\{e_{ij} \mid 1 \leq i \neq j \leq r\} \cup \{h_i \mid 1 \leq i < r\}.$$  

The subspace $H$ spanned by the $h_i$ is the collection of diagonal matrices in $L$ and is a Cartan subalgebra. The roots of $L$ with respect to $H$ are the functionals $\alpha_{ij}$:

$$\text{diag}(\lambda_1, \ldots, \lambda_r) \mapsto \lambda_j - \lambda_i \quad (i \neq j)$$  

and $e_{ij}$ is a root vector for $\alpha_{ij}$. We may assume specifically that $\{\alpha_{ij} \mid i < j\}$ is the set of positive roots. The rank $\rho$ of $L$ is $r - 1$.

The standard basis for $L$ is also a Chevalley basis, and we may write $L = \mathfrak{sl}(r, K) = K \otimes L^{(1)}$ for the $r \times r$ matrices of trace zero over $K$. In practice however, we drop the tensor signs in $\otimes e_{ij}$, $\otimes h_i$ and even write $L$ for $\mathcal{L}$, leaving the context to decide what is meant. It will be convenient to carry the "co-ordinate description" of $L$ through all our notation. Thus, putting $\alpha = \alpha_{ij}$, we write $f_{ij}(s)$ for $f_\alpha(s)$ and $E_{ij}(a)$ for $E_\alpha(a)$. The universal Chevalley group generated by the $E_{ij}(a)$ is
just $\text{SL}(r, K)$, and we have explicitly

$$E_{ij}(a) = I + ae_{ij}$$

where $I$ is the identity matrix. When $p < \infty$, $L$ as restricted Lie algebra has a $p$-map uniquely determined by $e_{ij}^{[p]} = 0$, $h_i^{[p]} = h_i$.

(2.5) A Miscellany of Elementary Calculations

This section contains an assortment of propositions, mostly arithmetical in nature, and all well-known.

To begin with, we make some remarks regarding symmetric polynomials. These occur quite naturally in calculating the composition structure of components of extension algebras (Chapter 4). We phrase our statements in terms of polynomials with integer coefficients.

Fix a set of indeterminates $z_1, \ldots, z_r$. We write $\zeta_1, \ldots, \zeta_r$ for the elementary symmetric functions in $z_1, \ldots, z_r$. Specifically

$$\zeta_1 = z_1 + \cdots + z_r,$$

$$\zeta_2 = z_1z_2 + z_1z_3 + \cdots + z_1z_r + \cdots + z_{r-1}z_r,$$

$$\vdots$$

$$\zeta_r = z_1z_2 \cdots z_r.$$

A symmetric polynomial in $z_1, \ldots, z_r$ is a polynomial in the $\zeta_i$. There are algorithms for actually performing the conversion, and for completeness we give one.

(2.5.1) ALGORITHM. Fix a symmetric polynomial $f(z_1, \ldots, z_r)$. The algorithm proceeds by computing sequences of (symmetric) polynomials $f_1, f_2, \ldots$ in the $z_i$ and (not necessarily symmetric) polynomials $g_0, g_1, \ldots$ in the $\zeta_i$. Thus, write $f_1 = f$, $g_0 = 0$ and for $k \geq 1$, ...
assume \( f_k, g_{k-1} \) are defined.

**Step (1).** Arrange the monomials \( z_1^{a_1} \ldots z_r^{a_r} \) appearing in \( f_k \) in decreasing left-to-right lexicographic order, that is, \( z_1^{a_1} \ldots z_r^{a_r} \) precedes \( z_1^{\beta_1} \ldots z_r^{\beta_r} \) if and only if the smallest \( i \) for which \( a_i - \beta_i \neq 0 \) has \( a_i - \beta_i > 0 \). Let \( nz_1^{a_1} \ldots z_r^{a_r} \) be the left-most term in \( f_k \) (\( n \in \mathbb{Z} \)). Because \( f_k \) is symmetric, we have perforce \( a_1 \geq a_2 \geq \ldots \geq a_r \).

**Step (2).** Write \( g_k = nz_1^{a_1} \ldots z_r^{a_r} \). When expanded, this polynomial has leading term \( n z_1^{a_1} \ldots z_r^{a_r} \). Define \( f_{k+1} = f_k - g_k \).

**Step (3).** If \( f_{k+1} \neq 0 \), replace \( f_k, g_{k-1} \) by \( f_{k+1}, g_k \) and revert to step (1). Otherwise, \( f_{k+1} = 0 \) and we are done: \( f = g_1 + g_2 + \ldots + g_k \).

The procedure terminates because, at step (2), the leading term in \( f_{k+1} \) is strictly less than the leading term in \( f_k \).

Also of interest to us are the \( h \)-polynomials \( h_1, h_2, \ldots \). These are defined by

\[
h_k = \sum_{\sum \dot{i}_1 + \ldots + \dot{i}_r = k} z_1^{\dot{i}_1} \ldots z_r^{\dot{i}_r}, \quad k \geq 1.
\]

They are symmetric. The most succinct method of writing \( h_k \) as a polynomial in the \( \zeta_j \) is probably by using formal power series. Given an indeterminate \( t \), define \( \Gamma(t) = \prod_{i=1}^r \left( 1 - tz_i \right) \). Putting \( \zeta_j = 0 \) for \( j > r \), we obtain
\[ \Gamma(t) = \sum_{j=0}^{\infty} (-1)^{j} \zeta_j t^j, \]
\[ \Gamma(t)^{-1} = \sum_{j=0}^{\infty} h_j t^j. \]

Then, using \( \Gamma(t)\Gamma(t)^{-1} = 1 \), we derive Wronski's Equalities:
\[ h_k - h_{k-1} \zeta_1 + \ldots + (-1)^{k-1} h_1 \zeta_{k-1} + (-1)^k \zeta_k = 0 , \quad k \geq 1. \]

These equalities give a recursive method of writing \( h_k \) as a polynomial
\[ h_k = H_k(\zeta_1, \ldots, \zeta_k) \]
in the \( \zeta_j \). The polynomial \( H_k \) is independent of \( r \). Furthermore, owing to the symmetry in Wronski's Equalities, we also have
\[ \zeta_k = H_k(h_1, \ldots, h_k). \]

We now change direction and list a few identities involving binomial coefficients. As usual, for integers \( m \) and \( n \), \( \binom{m}{n} \) is defined as follows
\[ \binom{m}{n} = \frac{m!}{n!(m-n)!} \quad \text{if } m \geq n \geq 0, \]
\[ \binom{m}{n} = (-1)^n \binom{n-m-1}{n} \quad \text{if } m < 0 < n, \]
\[ \binom{m}{n} = 0 \quad \text{otherwise}. \]

(2.5.2). Suppose \( p \) is a prime number.

1. Let \( m, n \) be non-negative integers, written to base \( p \) as
\[ m = m_0 + m_1 p + \ldots + m_s p^s, \quad n = n_0 + \ldots + n_s p^s \] where \( 0 \leq m_i, n_i < p \).

Then \( \binom{m}{n} \equiv \binom{m_0}{n_0} \ldots \binom{m_s}{n_s} \pmod{p} \).

2. Let \( m, n \) be arbitrary integers and let \( \alpha \) be a positive integer such that \( p^\alpha > n \). Then \( \binom{p^\alpha + m}{n} \equiv \binom{m}{n} \pmod{p} \).

\[ \square \]
(2.5.3). For all integers \( m, n, r \),
\[
\binom{m}{n} \binom{n-r}{r} = \binom{m}{r} \binom{m-n}{n-r} = \binom{m-n+r}{r} .
\]

Lastly, let \( A \) be an associative algebra with unity over the field \( K \). In a different direction yet again, we recall one or two facts regarding certain \( A \)-modules. Thus, let \( M \) be an \( A \)-module. The smallest injective module containing \( M \) is its injective hull, and we denote it \( E(M) \). Also, write \( oM \) for the socle of \( M \): this is the largest completely reducible submodule of \( M \).

(2.5.4). Let \( M, N \) be \( A \)-modules.

1. \( E(M \oplus N) \cong E(M) \oplus E(N) \).
2. \( E(M) \cong E(oM) \) and \( o(E(M)) = oM \).

Assume now that \( M, N \) contain submodules \( Z_M, Z_N \) respectively with \( \theta : Z_M \rightarrow Z_N \) some isomorphism. The sum of \( M \) and \( N \) with \( Z_M, Z_N \) amalgamated (via \( \theta \)) is just the quotient \( Q = (M \oplus N)/\Delta \) where \( \Delta \) is the diagonal submodule \( \Delta = \{ z-z\theta \mid z \in Z_M \} \). We shall refer to \( Q \) as an amalgam of \( M \) and \( N \). (There does not appear to be any generally accepted terminology here: we find ours convenient.) With this notation:

(2.5.5). (1) If \( Z_M \) is an absolutely irreducible \( A \)-module, then the isomorphism type of the amalgam \( Q \) is independent of \( \theta \).

2. Assume each of \( M \) and \( N \) has a unique minimal submodule, and that \( Z_M, Z_N \) are maximal in \( M, N \) respectively. If \( M \) is not isomorphic to \( N \), then \( Q \) has a unique minimal submodule.

Proof. (1) Suppose \( \psi \) is an isomorphism \( Z_M \rightarrow Z_N \). Then there exists a scalar \( k \) such that \( \theta = k\psi \). The map \( m + n \rightarrow m + kn \) for \( m \in M, n \in N \), is an automorphism of \( M \oplus N \), and it takes the diagonal \( \{ m-m\psi \mid m \in Z_M \} \) in \( Z_M \oplus Z_N \), to the diagonal \( \Delta \). Hence, the amalgam as
formed with \( \psi \) is isomorphic to \( Q \).

(2) Certainly, \( Q \) contains the minimal submodule
\[
(\Omega M \oplus \Delta)/\Delta = (\Delta \oplus \Omega N)/\Delta.
\]
Suppose \( U/\Delta \) is a minimal submodule of \( Q \), with \( U \neq \Omega M \oplus \Delta \). Then \( M \cap U = \{0\} = U \cap N \). Perforce \( M \oplus U = U \oplus N = M \oplus N \), and \( M \cong N \), as required. \( \square \)
CHAPTER 3
EXTENSION ALGEBRAS: FOUNDATIONS

This chapter begins the formal study of extension algebras. Section (3.1) contains a preliminary examination of gradedness in algebras: the main concept is that of strong gradedness. In (3.2), weakly relatively free algebras are defined and investigated. The results of this section closely parallel the existing theory of relatively free algebras. Strongly graded and weakly relatively free algebras are brought together in (3.3): we define the term extension algebra and give a sufficient condition for a weakly relatively free algebra to be extension. The next two sections provide some technical niceties. The results in (3.4) depend on the simple, but quite useful fact that extension algebras of finite rank are Hopf. On the other hand, (3.5) begins an examination of the effect on an extension algebra of changing its base ring. The investigation here motivates the definition of a distinguished class of extension algebras: the integral ones. Finally, a list of all those integral extension algebras of particular interest to this paper is given in (3.6), together with a summary of some of their well-known structural properties.

Throughout the chapter, $R$ is a commutative, associative ring with unity $1 \neq 0$. Thus, left and right (unital) $R$-modules coincide, and we do not distinguish them, although we shall consistently write $R$-action on the left. We work in the category of nonassociative $R$-algebras without unity. For example, homomorphisms of $R$-algebras need not preserve unities where these exist. The only reason for working in this category is that there are important algebras without unity which will arise (notably Lie algebras). If we ignore these, the chapter could equally well be developed within the category of algebras with unity, but the transition to that category is
easy, and we are satisfied to discuss only the one case.

(3.1) The Definition of Strongly Graded Algebras

A strongly graded algebra is nothing more than a graded algebra in the usual sense, but with a particularly well-behaved generating set. We begin the discussion of them by recalling some of the basic properties of graded algebras. Our treatment of gradedness is somewhat restricted, although adequate for our needs; for a more comprehensive abstract approach, see Greub [1, Chapter VI].

Let $A$ be an $R$-algebra. We say $A$ is graded if it has a decomposition $A = A_0 \oplus A_1 \oplus \ldots$ into $R$-submodules $A_i$, satisfying $A_i A_j \subseteq A_{i+j}$ for all $i, j$. The submodule $A_i$ is the homogeneous component of $A$ of degree $i$. For example, the zero of $A$ has degree $i$, for every $i$. If $A$ has a unity $1$, then it is immediate that $1 \in A_0$, so that the scalars in $A$ all have degree 0.

If $r$ is a positive integer, then $A$ is $r$-graded if, for each sequence $(\lambda_1, \ldots, \lambda_r)$ of non-negative integers, there is an $R$-submodule $A(\lambda_1, \ldots, \lambda_r)$ of $A$ such that

1. $A$ is the direct sum of the $A(\lambda_1, \ldots, \lambda_r)$ and
2. $A(\lambda_1, \ldots, \lambda_r) A(\mu_1, \ldots, \mu_r) \subseteq A(\nu_1, \ldots, \nu_r)$ where
   $$\nu_i = \lambda_i + \mu_i, \quad 1 \leq i \leq r.$$

In such an algebra, define $A_k$ for $k \geq 0$ to be the sum of all $A(\lambda_1, \ldots, \lambda_r)$ which satisfy $\lambda_1 + \ldots + \lambda_r = k$. Then $A = A_0 \oplus A_1 \oplus \ldots$ is a grading of $A$, and one speaks of the homogeneous component $A_k$ of total degree $k$ (whereas $A(\lambda_1, \ldots, \lambda_r)$ is the component with partial degrees $\lambda_1, \ldots, \lambda_r$). Thus, an $r$-graded algebra is a graded algebra
"with refinement".

In the above definitions, we shall also allow \( r \) to be countably infinite, denoted \( r = \infty \). In this case, it is understood that only finitely many \( \lambda_i \) in any sequence \( (\lambda_1, \lambda_2, \ldots) \) are nonzero.

It is not necessarily true that an \( r \)-graded algebra is generated by its elements of smallest (positive) degree. For example, let \( A = \mathbb{R}[x, y] \) be the algebra of polynomials in indeterminates \( x, y \). Introduce a 2-grading into \( A \) by defining the degree of a monomial \( x^i y^j \) to be \( i \) on \( x \) and \( 2j \) on \( y \), that is, we write \( A(i, 2j+1) = \{0\} \) and \( A(i, 2j) = \mathbb{R}x^i y^j \) for all \( i, j \geq 0 \). Then \( A \) is not generated by its elements of degree less than 2. Similarly, graded algebras are all 1-graded, so that the components \( A(...0, 1, 0, ...) \) of degree 1 of an arbitrary \( r \)-graded algebra \( A \), need not be cyclic \( \mathbb{R} \)-modules.

A strongly graded algebra with a countable generating set can be realized as an \( r \)-graded algebra without the inconveniences of the preceding paragraph. However, to make the most general definition of strongly graded, we require some notation.

If \( X \) is a non-empty set, let \( F(X) \) be the collection of all nonzero functions \( X \to \mathbb{N} \) of finite support, that is, \( \lambda \in F(X) \) if and only if \( \{x \in X \mid \lambda(x) \neq 0\} \) is finite and nonempty. For \( x \in X \), let \( \overline{x} \) be the characteristic function of \( X : \overline{x}(x) = 1 \) and \( \overline{x}(y) = 0 \) for \( x \neq y \in \overline{X} \). Under pointwise addition, \( F(X) \) is a commutative semigroup (indeed, with the identification of \( \overline{x} \) with \( x \), \( F(X) \) is the free commutative semigroup on \( X \)).

3.1.1 DEFINITION. A strongly graded algebra is a pair \((A, X)\) consisting of an \( \mathbb{R} \)-algebra \( A \) and a nonempty subset \( X \) of \( A \) which generates \( A \) as algebra. For each \( \lambda \in F(X) \), there is an \( \mathbb{R} \)-submodule \( A_\lambda \) of \( A \) such that the following properties hold:
(1) \( A \) is the direct sum of the \( A_\lambda, \lambda \in P(X) \);

(2) for all \( \lambda, \mu \in P(X) \), we have \( A_\lambda A_\mu \subseteq A_{\lambda+\mu} \);

(3) for each \( x \in X \), \( A_{\frac{x}{x}} \) is the cyclic \( R \)-module generated by \( x \).

We shall also say that \( A \) is strongly graded on \( X \).

For the rest of this section, let \( A \) be strongly graded on \( X \). For each integer \( k \geq 1 \), let \( A_k \) be the sum of all those \( A_\lambda \) for which \( \sum_{x \in X} \lambda(x) = k \). Clearly, \( A = A_1 \oplus A_2 \oplus \ldots \) is a grading of \( A \), and as in the \( r \)-graded case, we call \( A_k \) the homogeneous component of total degree \( k \). Note that the degree 0 component of \( A \) is actually zero because \( X \) generates \( A \) in the category of algebras without unity. In particular, a strongly graded algebra in our sense never has a unity.

When \( X \) is countable, say \( |X| = r \leq \infty \), then \( A \) is an \( r \)-graded algebra. Indeed, write \( X = \{x_1, x_2, \ldots\} \) and identify \( \lambda \in P(X) \) with its sequence of values \( (\lambda_1, \lambda_2, \ldots) \), where \( \lambda_i = \lambda(x_i) \), \( i \geq 1 \). Then we have \( A(\lambda_1, \lambda_2, \ldots) = A_\lambda \). This identification depends on the particular order chosen for the elements of \( X \), and for most of this chapter we avoid it. We do observe, however, that if \( (\ldots, 0, 1, 0, \ldots) \) is the sequence with 1 in the \( i \)th place and zeros elsewhere, then \( A(\ldots, 0, 1, 0, \ldots) = A_{\frac{x_i}{x_i}} = Rx_i \) is a cyclic \( R \)-module. Furthermore, \( A \) is generated by its elements of degree 1. Hence, as \( r \)-graded algebra, \( A \) does avoid the pathologies we mentioned earlier.

The degree one component \( A_1 \) of \( A \) is just the direct sum of the modules \( Rx, x \in X \). More generally, the \( k \)th component \( A_k \) is spanned over \( R \) by all monomials of degree \( k \). Here, a monomial of degree \( k \) is
a product of the generators in \( X \) containing \( k \) factors. In general, \( A \)
is not associative, and the bracketing in these monomials must be specified;we introduce a notation for them below. Now consider the component \( A_\lambda \) of\( A \) for some \( \lambda \in P(X) \). We see that \( A_\lambda \) is spanned by all those monomialsin which, for each \( x \in X \), the number of factors equal to \( x \) is just\( \lambda(x) \). We refer to \( A_\lambda \) as the strongly homogeneous component of \( A \) of

(strong) degree \( \lambda(x) \) in \( X \). Note that our discussion implies:

(3.1.2). As \( R \)-modules, \( A_\lambda \) and (when \( X \) is finite) \( A_k \), arefinitely-generated. □

We shall sometimes require a notation for monomials, and we adopt thatof Magnus, Karrass, Solitar [1, 5.2]. To begin with, a bracket arrangement
of weight \( n \) is a sequence of (matched) parentheses separated by \( n \)asterisks. The parentheses indicate the order in which a product is to beformed, and the asterisks are placeholders for elements of the underlyingalgebraical structure (in our case, an \( R \)-algebra). Formally, define theunique bracket arrangement \( \beta^1 \) of weight 1 by \( \beta^1 = (*) \). A bracketarrangement \( \beta^n \) of weight \( n \) is then defined recursively by \( \beta^n = (\beta^k \beta^l) \)where \( \beta^k, \beta^l \) are bracket arrangements of weights \( k, l \) satisfying\( k + l = n \). For example, there are two bracket arrangements of weight 3,

\[((**)(**))*\] and \[((**)(**)*)\]; adopting the usual convention of
dropping superfluous parentheses if no ambiguity can result, these may bewritten \((**)*\) and \(*(**)\). Now, if \( a \in A \), write \( \beta^1(a) = a \); and if\( a_1, \ldots, a_n \in A \) and \( \beta^n = (\beta^k \beta^l) \), define \( \beta^n(a_1, \ldots, a_n) \) recursively by

\[\beta^n(a_1, \ldots, a_n) = \beta^k(a_1, \ldots, a_k)\beta^l(a_{k+1}, \ldots, a_n)\]. Obviously,

\[\beta^n(a_1, \ldots, a_n)\] is a well-defined element of \( A \). In particular, if\( a_1, \ldots, a_n \) are elements of \( X \), then \( \beta^n(a_1, \ldots, a_n) \) is a monomial in
the sense of the foregoing paragraph. Bracket arrangements of weight \( n \) will be distinguished by subscripting \( \beta_1^n, \beta_2^n, \ldots \) and we often omit the superscript \( n \).

We now turn to consider free \( R \)-algebras. Fix a set \( Z \) with \(|Z| = |X|\). The free nonassociative \( R \)-algebra (without unity) on the set \( Z \) will be denoted \( F_R(Z|\emptyset) \) (this is a special case of the notation in the next section), or more simply \( F_R(Z) \), \( F(Z) \) or \( F \). For a proof of the existence of \( F \), see Chevalley [1, IV.5]. It is well-known, and indeed quite obvious, that \( F \) is strongly graded on \( Z \). In fact, if \( k > 0 \) and \( \beta_k^1, \ldots, \beta_s^k \) are all the bracket arrangements of weight \( k \), then \( F_k \) is a free \( R \)-module with a basis of monomials 
\[
\left\{ \beta_i^k(z_1, \ldots, z_k) \mid 1 \leq i \leq s, z_j \in Z \right\}.
\]
Nothing like this is true for the arbitrary strongly graded algebra \((A, X)\), although we shall see some similar results in (3.5).

Our algebra \( A \) is a homomorphic image of \( F \). Thus, let \( \theta : Z \rightarrow X \) be a bijection of sets, and extend \( \theta \) uniquely to an algebra surjection \( \overline{\theta} : F \rightarrow A \). If \( f \in \ker \overline{\theta} \), then \( f = f_\lambda^1 + \ldots + f_\lambda^t \), where \( f_\lambda^i \in F_\lambda^i \) is a strongly homogeneous component of \( f \). Then 
\[
0 = f \overline{\theta} = f_\lambda^1 \overline{\theta} + \ldots + f_\lambda^t \overline{\theta}
\]
where, by choice of \( \theta \), the \( f_\lambda^i \overline{\theta} \) are strongly homogeneous in \( A \). Thus, each \( f_\lambda^i \in \ker \overline{\theta} \). This shows that \( \ker \overline{\theta} \) is the direct sum of its strongly homogeneous components \( F_\lambda \cap \ker \overline{\theta} \), \( \lambda \in F(Z) \). This prompts the following definition.

(3.1.3) DEFINITION. If \((B, Y)\) is a strongly graded algebra, then an \( R \)-submodule \( J \) in \( B \) is strongly graded if \( J \) is the direct sum of its components \( J_\lambda = J \cap B_\lambda \), \( \lambda \in F(Y) \). Similarly, \( J \) is graded (without
Further qualification) if \( J = \bigoplus_{k \geq 1} J \cap B_k \).

Thus, we have realized \( A \) as a quotient of \( F \) by a strongly graded ideal, \( \ker \theta \). Conversely, if \( J \) is a strongly graded ideal in \( F \), then \( F/J \) is strongly graded on \( \{ n \cdot J \mid n \in \mathbb{Z} \} \). This gives a convenient characterization of strongly graded algebras which we shall often use; we make it a little more precise in the next section.

(3.2) The Definition and Structure of Weakly Relatively Free Algebras

If \( A \) is an \( R \)-algebra and \( X \) a subset of \( A \), one says that \( A \) is relatively free on \( X \) if every function \( X \to A \) extends to a unique endomorphism of \( A \). However, there are a number of algebras enjoying a similar but somewhat weaker property. Let us give an example.

(3.2.1) EXAMPLE. Let \( M \) be an \( R \)-module. The exterior algebra \( \text{Ext}(M) \) of \( M \) is that associative \( R \)-algebra satisfying

1. \( \text{Ext}(M) \) contains \( M \) as an \( R \)-submodule, and \( m^2 = 0 \) for \( m \in M \);

2. if \( A \) is an associative \( R \)-algebra and \( \theta : M \to A \) a homomorphism of \( R \)-modules satisfying \( (m\theta)^2 = 0 \) for \( m \in M \), then \( \theta \) has a unique extension to an algebra homomorphism \( \text{Ext}(M) \to A \).

In the case when \( M \) is a free \( R \)-module with finite basis \( B \), the structure of \( \text{Ext}(M) \) is well-known (see (3.6.5)). Here, we observe that every function \( B \to M \) extends to a unique endomorphism of \( \text{Ext}(M) \). However, \( \text{Ext}(M) \) is not relatively free on \( B \). This is because there exist elements \( e \in \text{Ext}(M) \) such that \( e^2 \neq 0 \), whereas no element of \( B \) has this property. \( \square \)

We use (3.2.1) to motivate the following generalization of "relatively
(3.2.2) DEFINITION. A weakly relatively free algebra is a pair \((A, X)\) consisting of an \(R\)-algebra \(A\) and a subset \(X\) of \(A\) which generates \(A\) as algebra, such that every function \(X \to RX\) extends to an endomorphism of \(A\). We shall also say \(A\) is weakly relatively free on \(X\).

In (3.2.2), the extending homomorphism is of course unique. Dr Kovács has pointed out that the converse of this is false: if \(A\) is an algebra with the property that every function \(X \to RX\) extends to a unique endomorphism, then \(A\) need not be generated by \(X\). Thus, take \(A = \mathbb{Q}\), the rational numbers, treated as \(Z\)-algebra with respect to the usual addition, and product \(ab = 0\) for \(a, b \in \mathbb{Q}\). As \(Z\)-module, \(\mathbb{Q}\) is injective, and every function \({1} \to \mathbb{Z}\) extends to a unique \(Z\)-algebra homomorphism \(\mathbb{Q} \to \mathbb{Q}\) (cf. Curtis and Reiner [1, §57]). However, \(X = \{1\}\) does not generate \(\mathbb{Q}\) as \(Z\)-algebra.

In the previous section, we had occasion to consider pairs \((A, X)\) where \(X\) generates \(A\), and the same situation emerges in (3.2.2). We therefore take the view that an arbitrary "algebra" is such a pair \((A, X)\), where the distinguished generating set is always implicit, even if unspecified. We digress to consider these pairs in a little more detail.

(3.2.3) DEFINITION. If \((A, X)\) and \((B, Y)\) are algebras, a homomorphism \(\theta : A \to B\) is restricted if \(X\theta \subseteq RY\). The collection of restricted homomorphisms \(A \to B\) will be denoted \(\text{RHom}_R(A, X; B, Y)\), or simply \(\text{RHom}(A, B)\); the restricted endomorphisms of \(A\) are \(\text{REnd}(A)\). We shall also refer to a function \(\theta : X \to B\) satisfying \(X\theta \subseteq RY\) as a restricted function.

For example, if \(\varphi\) is a homomorphism with domain \(A\), then \(\varphi\) is restricted when treated as a homomorphism \((A, X) \to (A\varphi, X\varphi)\).

The collection of pairs \((A, X)\), together with their restricted
homomorphisms, form a category, which we denote $\mathcal{AG}_R$ ($= R$-algebras with generating sets). We adopt a rather strong definition of isomorphism in $\mathcal{AG}_R$: algebras $(A, X)$ and $(B, Y)$ are isomorphic if there exists a bijective restricted homomorphism $\theta : A \to B$ satisfying $X\theta = Y$. We have already had an example of this isomorphism. In (3.1), we showed (essentially) that $(A, X)$ is strongly graded if and only if it is isomorphic to $\left( F(Z)/J, \{ z+J \mid z \in Z \} \right)$ for some strongly graded ideal $J$ in the free algebra $F(Z)$, where $|Z| = |X|$.

There exist free objects in $\mathcal{AG}_R$: these are the pairs $(F(Z), z)$. Here, the underlying functor to sets is $(A, X) \to RX$, acting on morphisms by restriction; its left adjoint is the functor $Z \to (F(Z), z)$. We have the following easy "projectivity" result for these free objects; as the proof suggests, it is too much to expect free objects to be projective in $\mathcal{AG}_R$.

(3.2.4). The free algebra $(F(Z), z)$ has the following property: whenever $(A, X)$ and $(B, Y)$ are algebras, together with restricted homomorphisms $\psi : F(Z) \to A$ and $\phi : B \to A$ satisfying $(RY)\phi = RX$, there exists a restricted homomorphism $\theta : F(Z) \to B$ such that $\psi = \theta \phi$.

Proof. Let $z \in Z$. Because $\psi$ is restricted, $z\psi \in RX$, and since $(RY)\phi = RX$, there exists $b \in RY$ such that $b\phi = z\psi$. Define $\theta : Z \to B$ by $z\theta = b$. Since $F(Z)$ is free, $\theta$ extends to a restricted homomorphism $F(Z) \to B$, also denoted $\theta$, and it is immediate $\psi = \theta \phi$.

We now begin a study of weakly relatively free algebras. It will emerge that almost every result obtained has an analogue in the theory of relatively free algebras: one need only drop the words weakly and restricted wherever they occur. We consider only the weak case in detail, but take the results for the relatively free situation as being justified.
simultaneously. For a general treatment of "relative freeness" see Cohn [1, Chapters III and IV].

Fix a countably infinite set \( X_\infty \), and form \( (F(X_\infty), X_\infty) \). One sometimes refers to \( X_\infty \) as the underlying alphabet. For brevity, we shall often denote \( F(X_\infty) \) as \( F_\infty \).

**(3.2.5) Definition.** If \( L \) is a nonempty subset of \( F_\infty \), then the \textbf{weakly verbal ideal} in an algebra \( (A, X) \) determined by \( L \) is the ideal of \( A \) generated by all \( L \psi, \psi \in \text{RHom}(F_\infty, A) \).

We shall denote this ideal by \( L_W(A) \). The \textbf{verbal ideal} determined by \( L \) in \( A \) is \( L_L(A) \). Here, \( W \) is an abbreviation for "weak law", \( L \) for "law", a notation which will become self-explanatory. To avoid repeatedly specifying that \( L \) be non-empty, we also define \( \varnothing_W(A) = \varnothing_L(A) = \{0\} \).

Note that \( \{0\}_W(A) = \{0\} \) and \( (F_\infty)_W(A) = A \); we shall refer to these two cases as the \textbf{extreme} ones.

The following proposition collects together some elementary results on weakly verbal ideals. We call a subset of \( A \) \textit{weakly invariant in} \( A \) if it is left invariant by all restricted endomorphisms of \( A \).

**(3.2.6).** \( L \) be a subset of \( F_\infty \) and \( (A, X) \) an algebra.

\[ (1) \ L_W(A) \text{ is weakly invariant in } A. \]

\[ (2) \text{ If } L \subseteq M \subseteq F_\infty, \text{ then } L_W(A) \subseteq M_W(A). \]

\[ (3) \text{ If } \phi \text{ is a homomorphism with domain } A, \text{ then } L_W(A\phi) = [L_W(A)]_\phi. \]

\[ (4) \ L_W(A) = [L_W(F_\infty)]_W(A), \text{ that is, } L \text{ and } L_W(F_\infty) \text{ determine the same weakly verbal ideal in } A. \]

\textbf{Proof.} (1) follows because the composite of restricted homomorphisms is restricted, and (2) is trivial. For (3), it is implicit that the generating set for \( A\phi \) is \( X\psi \). Then it is trivial that \( L_W(A\phi) \supseteq [L_W(A)]_\phi \). Conversely, let \( \psi \in \text{RHom}(F_\infty, A\phi) \). By (3.2.4), there
exists $\theta \in \text{RHom}(F_\infty, A)$ such that $\psi = \theta \psi$. Then $L\psi = (L\theta)\psi \subseteq [L_{W}(A)]\psi$, and (3) holds. Finally, $L_{W}(A) \subseteq [L_{W}(F_{\infty})]_{W}(A)$ by (2), and the reverse inclusion holds because $[L_{W}(F_{\infty})]_{W}(A)$ is the ideal of $A$ generated by all $L\theta \psi$ for $\theta \in \text{REnd}(F_{\infty})$, $\psi \in \text{RHom}(F_{\infty}, A)$. This gives (4). □

In particular, if $L \subseteq F_{\infty}$, then the weakly invariant ideal of $F_{\infty}$ generated by $L$ is just $L_{W}(F_{\infty})$. Thus, the weakly verbal ideals in $F_{\infty}$ are precisely the weakly invariant ones. More generally:

(3.2.7). An ideal in a free algebra is weakly verbal if and only if it is weakly invariant.

Proof. It remains to show that if $I$ is a weakly invariant ideal in the free algebra $F(Z)$, then $I = L_{W}(F(Z))$ for some $L \subseteq F_{\infty}$.

Suppose $f \in I$. When written as a linear combination of monomials, it involves only finitely many distinct free generators $x_1, \ldots, x_n \in Z$. Choose distinct $x_1, \ldots, x_n \in X_{\infty}$. There certainly exists a restricted homomorphism $\theta : F(Z) \rightarrow F_{\infty}$ such that $x_i \theta = x_i$, $1 \leq i \leq n$. We think of $\theta$ as "renaming the variables" in $f$. It is a straightforward matter to check that $I = L_{W}(F(Z))$, where $L$ is the subset of $F_{\infty}$ consisting of all $f\theta$, $f \in I$ and $\theta$ a renaming homomorphism. □

As in any theory of "relative freeness", the central concept is that of a "law".

(3 2.8) DEFINITION. If $L \subseteq F_{\infty}$, then the elements of $L$ are weak laws in an algebra $(A, X)$ if $L_{W}(A) = \{0\}$. We also say that $A$ satisfies $L$ weakly. If $M \subseteq F_{\infty}$, then $M$ is a weak consequence of $L$ if the elements of $M$ are weak laws in any algebra which satisfies $L$ weakly. Further, $L$ and $M$ are weakly equivalent if each is a weak consequence of the other.
We shall generally denote weak laws by capital letters $L, M, \ldots$ for emphasis. Thus, if $L \in F_\infty$, then $L$ is a weak law in $A$ if and only if $L$ is contained in the kernel of every restricted homomorphism $F_\infty \to A$.

(3.2.9). If $L, M \subseteq F_\infty$, then $M$ is a weak consequence of $L$ if and only if $M \subseteq \mathcal{L}_L(F_\infty)$. Also, $L$ and $M$ are weakly equivalent if and only if $\mathcal{L}_L(F_\infty) = \mathcal{L}_M(F_\infty)$.

Proof. Suppose $M \subseteq \mathcal{L}_L(F_\infty)$. For any algebra $A$, we have, by (3.2.6) (2) and (4), $M(A) \subseteq [\mathcal{L}_L(F_\infty)](A) = \mathcal{L}_L(A)$, so if $A$ satisfies $L$ weakly, so too does it satisfy $M$ weakly. Therefore, $M$ is a weak consequence of $L$.

Conversely, suppose $M$ is a weak consequence of $L$. By (3.2.6) (3), the algebra $A = F_\infty/\mathcal{L}_L(F_\infty)$ satisfies $L$ weakly. Then, a fortiori, $A$ satisfies $M$ weakly. But, again using (3.2.6) (3),

$$\{0\} = M(A) = [M(F_\infty) + \mathcal{L}_L(F_\infty)]/\mathcal{L}_L(F_\infty),$$

whence $M \subseteq \mathcal{L}_L(F_\infty)$, as required.

The second statement in (3.2.9) is now completely trivial. □

We pause in the general development to indicate a small anomaly in our terminology. "Weakly relatively free" generalizes "relatively free"; hence the use of "weakly". However, (3.2.9) indicates that "weak equivalence" is a stronger property than "equivalence". We tolerate this aberration, but mention it for clarity.

We are now in a position to develop the main theorem on weakly relatively free algebras: every weakly relatively free algebra is defined by a set of weak laws. To make this precise, we begin by considering a method of constructing weakly relatively free algebras.

Suppose $L$ is a subset of $F_\infty$. For some fixed nonempty set $Z$, write $I = \mathcal{L}_L(F(Z))$ and $A = F(Z)/I$. Let $\rho : F(Z) \to A$ be the natural
surjection. Then the algebra \((A, Z_p)\) is weakly relatively free; in fact, \((A, Z_p)\) satisfies the following stronger properties:

3.2.10. \(A\) satisfies \(L\) weakly.

3.2.11. If \((B, Y)\) is an algebra satisfying \(L\) weakly and \(\theta : Z_p \to B\) is a restricted function, then there exists a unique homomorphism \(\overline{\theta} : A \to B\) extending \(\theta\).

That (3.2.10) holds is an immediate consequence of (3.2.6) (3). For (3.2.11), suppose \((B, Y)\) and \(\theta\) are as given, and define \(\varphi = \rho\theta\). Then \(\varphi\) extends to a (restricted) homomorphism \(F(Z) \to B\), also denoted \(\varphi\). For \(L \in L\) and \(\psi \in \text{RHom}(F(Z), F(Z))\), \((L\psi)\varphi = 0\) because \(L\) is a weak law in \(B\). Thus \(\ker \varphi \supseteq I\), and \(\varphi\) induces a homomorphism \(\overline{\varphi} : A \to B\) satisfying \(\varphi = \rho \overline{\varphi}\). Then \(\overline{\theta}\) is the required extension of \(\theta\), and it is unique because \(Z_p\) generates \(A\).

There is another observation we wish to make regarding \((A, Z_p)\). Assume \(L^*_w(F_\infty) \neq F_\infty\). Then the function \(\rho|_Z\) is a bijection \(Z \to Z_p\). To see this, suppose \(z, z'\) are nonequal elements of \(Z\), and define \(\varphi \in \text{REnd}(F(Z))\) by \(z\varphi = z + z'\), \(z''\varphi = z''\) for \(z \neq z'' \in Z\). Then \((z-z')\varphi = z\). As \(I\) is weakly invariant, it follows that if \(z\varphi = z'\rho\), then \(z \in I\). This forces \(I = F(Z)\), and in turn, \(L^*_w(F_\infty) = F_\infty\), a contradiction. We have therefore shown \(|Z| = |Z_p|\). If \((A', Z')\) is any algebra satisfying (3.2.10) and (3.2.11), together with \(|Z| = |Z'|\), then by a quite standard "universality" argument, \((A', Z')\) is isomorphic to \((A, Z_p)\). Thus, \((A, Z_p)\) is completely determined by \(L\) and the cardinality of \(Z\). Note that when \(L^*_w(F_\infty) = F_\infty\), then \(A\) is just the zero algebra; in the other extreme case, when \(L = \emptyset\) or \(\{0\}\), \(A\) is \(F(Z)\) itself.

We now realize every weakly relatively free algebra in the form \((A, Z_p)\) above. Suppose \((C, X)\) is weakly relatively free with \(C \neq \{0\}\), and choose the set \(Z\) to satisfy \(|Z| = |X|\). Let \(\varphi : F(Z) \to C\) be the
surjection induced by some bijection $Z \rightarrow X$. We show $\ker \varphi$ is weakly verbal in $F(Z)$. Suppose $\theta \in \text{REnd}(F(Z))$. Because $(C, X)$ is weakly relatively free, we may define $\overline{\theta} \in \text{REnd}(C)$ by the requirement $x\overline{\theta} = x\varphi^{-1}\theta\varphi$ for $x \in X$. Here, $\varphi^{-1}$ is the inverse of $\varphi|_Z$. Thus, $\varphi\overline{\theta} = \theta\varphi$. Therefore, $(\ker \varphi)\theta\varphi = \{0\}$ and $(\ker \varphi)\theta \subseteq \ker \varphi$. By (3.2.7), $\ker \varphi$ is weakly verbal, as claimed. Choose $\overline{\theta} \in \text{End}(C)$ such that $L_{\overline{\theta}}(F(Z)) = \ker \varphi$; also, write $I = \ker \varphi$, $A = F(Z)/I$ and $\rho : F(Z) \rightarrow A$ canonical. Then $(C, X)$ is isomorphic to $(A, Zp)$, an algebra known to satisfy (3.2.10) and (3.2.11). For future reference, we mention in passing that we have also shown (effectively):

(3.2.12). If $\theta \in \text{REnd}(F(Z))$, then there exists $\overline{\theta} \in \text{REnd}(A)$ such that $\rho\overline{\theta} = \theta\rho$. □

We summarize the above discussion in the form of a theorem:

(3.2.13) THEOREM. Let $L$ be a subset of $F_{\infty}$ and $c$ a cardinal number. We allow $c = 0$ if and only if $L^*(F_{\infty}) = F_{\infty}$. Then there exists an algebra $(A, X)$, unique to within isomorphism, satisfying $|X| = c$ and in addition:

(1) $(A, X)$ satisfies $L$ weakly;

(2) if $(B, Y)$ is an algebra satisfying $L$ weakly, then every restricted function $X \rightarrow B$ extends to a unique homomorphism $A \rightarrow B$.

The algebra $(A, X)$ is weakly relatively free, and conversely, every weakly relatively free algebra can be realized in the form $(A, X)$ for suitable $L, c$. □

For the sake of completeness, we give the following results, the first a generalization of (3.2.7).

(3.2.14). An ideal in a weakly relatively free algebra is weakly verbal if and only if it is weakly invariant.
Proof. For the weakly relatively free algebra, we may take \((A, Zp)\), where \(A = F(Z)/I\), \(I\) is a weakly verbal ideal in \(F(Z)\) and \(\rho : F(Z) \rightarrow A\) is canonical. Let \(J\) be an ideal in \(F(Z)\) containing \(I\) such that \(J/I\) is weakly invariant in \(A\). We show \(J\) is weakly invariant in \(F(Z)\). Indeed, if \(0 \in \text{REnd}(F(Z))\), then by (3.2.12), there exists \(\bar{\rho} \in \text{REnd}(A)\) satisfying \(\rho \bar{\rho} = \theta \rho\). Thus \(J \theta \rho = (J \theta) \bar{\rho} \subseteq J \rho\), and this gives \(J \theta \subseteq J\), as claimed. By (3.2.7), \(J\) is weakly verbal, and by (3.2.6) (3), so too is \(J/I\).

The converse is a special case of (3.2.6) (1).

From (3.2.14), we immediately have:

(3.2.15). If \(I\) is a weakly verbal ideal in the free algebra \(F(Z)\), then there is a one-one correspondence between the weakly verbal ideals of \(F(Z)/I\) and the weakly verbal ideals of \(F(Z)\) which contain \(I\).

The algebra \((A, X)\) in (3.2.13) will be called the weakly relatively free algebra on \(X\) determined by the weak laws \(L\). We denote it \(F_R(X|L_w)\) or \(F(X|L_w)\). In the same way, the relatively free algebra on \(X\) determined by the laws \(L\) is \(F_R(X|L_L)\). The cardinality of \(X\) is called the rank of the algebra, and when there is no need to specify \(X\), we simply write \(F(c|L_w)\) and \(F(c|L_L)\). In practice, a weakly relatively free algebra can satisfy some distinguished law(s) as well as some weak law(s). For example, the exterior algebra in (3.2.1) satisfies the associative law \(x(yz) = (xy)z\) and the weak law \(x^2\). (Here, \(x, y, z\) are elements of \(X_\infty\).) To describe such algebras succinctly, a hybrid notation is useful, and we develop this next.

Let \(L, M\) be subsets of \(F_\infty\), and fix a free algebra \(F(Z)\). Write \(I = L_L(F(Z)) + M_W(F(Z))\) and \(A = F(Z)/I\). Suppose \(\rho : F(Z) \rightarrow A\) is canonical. Reasoning precisely as in the "pure" weakly relatively free
case, we see that the elements of $L$ are laws in $A$, the elements of $M$ are weak laws. We say simply $A$ satisfies $L_L$ and $M_W$. Furthermore, if $(B, Y)$ is an algebra satisfying $L_L$ and $M_W$, and $\theta : Zp \rightarrow B$ a restricted function, then $\theta$ has a unique extension to a homomorphism $A \rightarrow B$. Without further ado, we introduce $F_R(X|L_L, M_W)$, the weakly relatively free algebra on $X$ determined by $L_L$ and $M_W$. As before, its uniqueness is assured by a universality argument.

When does $F(X|L_L, M_W) = F(X|L'_L, M'_W)$? We say that $\{L'_L, M'_W\}$ is a consequence of $\{L_L, M_W\}$ when any algebra satisfying $L_L$ and $M_W$ also satisfies $L'_L$ and $M'_W$. Thus, the two algebras are the same when $\{L_L, M_W\}$ is equivalent to $\{L'_L, M'_W\}$. As (3.2.17) below indicates, this does not imply $L_L$ equivalent to $L'_L$ or $M_W$ weakly equivalent to $M'_W$. Because of this, one could make a case, in comparing $\{L_L, M_W\}$ to $\{L'_L, M'_W\}$, for using a term altogether different from "equivalence", but in practice, no real ambiguity results, and we will not stop to introduce such a term here.

We now give some examples of weakly relatively free algebras. These illustrate aspects of the preceding notation and theory.

(3.2.16) EXAMPLE. We give a list of important elements of $F_\infty$ and a list of some relatively free algebras. We say no more regarding these for the present, but they will recur frequently, and the notation we use will be standard to this paper. While the list may seem a little dry, it is convenient to have these elements and algebras in the one place.
Elements of $F_{\infty}$

<table>
<thead>
<tr>
<th>Name of Element</th>
<th>Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = xy - yx$</td>
<td>commutative.</td>
</tr>
<tr>
<td>$A = (xy)z - x(zy)$</td>
<td>associative</td>
</tr>
<tr>
<td>$A_1^* = (xx)y - x(xx)$</td>
<td>first alternative</td>
</tr>
<tr>
<td>$A_2^* = y(xx) - (yx)x$</td>
<td>second alternative</td>
</tr>
<tr>
<td>$J = x(ya) + a(xy) + y(ax)$</td>
<td>Jacobi</td>
</tr>
<tr>
<td>$J^* = [(xx)y]x - (xx)(yx)$</td>
<td>Jordan</td>
</tr>
<tr>
<td>$N_k = x_1x_2 \ldots x_k$</td>
<td>nilpotent of class $k$</td>
</tr>
<tr>
<td>$N_k^* = x^k$</td>
<td>nil of degree $k$</td>
</tr>
<tr>
<td>$E_k = \frac{yxx \ldots x}{k}$</td>
<td>Engel of degree $(k+1)$</td>
</tr>
</tbody>
</table>

(The last three unbracketed products are "left-normed": $N_k = N_{k-1}^* x_k$ and so on.)

Relatively Free Algebras

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Laws</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>nonassociative</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Ass</td>
<td>associative</td>
<td>$A_L$</td>
</tr>
<tr>
<td>Alt</td>
<td>alternative</td>
<td>$A_1^<em>, A_2^</em>$</td>
</tr>
<tr>
<td>Pol</td>
<td>polynomial</td>
<td>$C_L, A_L$</td>
</tr>
<tr>
<td>Lie</td>
<td>Lie</td>
<td>$N_2^<em>, J_L^</em>$</td>
</tr>
<tr>
<td>Jor</td>
<td>Jordan</td>
<td>$C_L, J_L^*$</td>
</tr>
</tbody>
</table>

For example, $\text{Jor}_R(c)$ is the free Jordan algebra of rank $c$ over $R$, given explicitly by $\text{Jor}_R(c) = F_R[c|C_L, J_L^*]$. Of course, any algebra satisfying $C_L$ and $J_L^*$ is called a Jordan algebra, and a similar remark applies to the other examples. $\Box$
(3.2.17) EXAMPLE. Let $M$ be a free $R$-module with a basis $B$. The symmetric algebra on $M$ is the associative algebra $\text{Sym}(M)$ satisfying

1. $\text{Sym}(M)$ contains $M$ as an $R$-submodule, and $mm' = m'm$ for $m, m' \in M$;
2. if $A$ is an associative algebra and $\theta : M \to A$ a homomorphism of $R$-modules satisfying $(m\theta)(m'\theta) = (m'\theta)(m\theta)$ for $m, m' \in M$, then $\theta$ has a unique extension to an algebra homomorphism $\text{Sym}(M) \to A$.

We see immediately that $\text{Sym}(M)$ is just $\mathbb{F}[B|A_L, C_W]$. However, an associative algebra is commutative if and only if the elements of a generating set commute. Therefore, $\{A_L, C_W\}$ is equivalent to $\{A_L, C_L\}$, and $\text{Sym}(M) = \mathbb{F}[B|A_L, C_L] = \text{Pol}(B)$. This gives a conceptual proof of the well-known fact that symmetric algebras of free $R$-modules are just polynomial algebras.

For completeness' sake, we also state formally that the familiar exterior algebra on $M$ is just $\mathbb{F}(B|A_L, N_{2W})$. Write this as $\text{Ext}(B)$ (suppressing the slightly different notation in (3.2.1)).

(3.2.18) EXAMPLE. In this example, we use the terminology and notation in Wall [2].

Let $G$ be a group with lower central series $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots$. Thus, $\gamma_{i+1}(G) = [\gamma_i(G), G]$, $i \geq 1$. It is well-known that this series is a filtration in $G$, that is, $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$ for all $i, j$. In particular, $\gamma_{i+1}(G)$ is normal in $G$ and the quotient $\gamma_i(G)/\gamma_{i+1}(G)$ is abelian. The Lie algebra $\gr G$ of $G$ with respect to this filtration is defined as follows. Write $\gamma_i(G)/\gamma_{i+1}(G)$ additively, and let $\gr G$ be the external direct sum $\gr G = \bigoplus_{i \geq 1} \gamma_i(G)/\gamma_{i+1}(G)$. Introduce a product into
gr \ G \ by \ setting \ [g_\gamma_{i+1}(G), \ h_\gamma_{j+1}(G)] = [g, \ h]_\gamma_{i+j+1}(G)

where \ g \in \gamma_i(G), \ h \in \gamma_j(G), \ and \ extending \ this \ bilinearly \ to \ the \ whole
of \ gr \ G. \ Then \ gr \ G \ becomes \ a \ graded \ Lie \ algebra \ over \ the \ integers \ \mathbb{Z}.
If \ X \ is \ a \ generating \ set \ for \ G, \ then \ \bar{X} = \{x + \gamma_2(G) \mid x \in X\} \ generates
gr \ G \ as \ Lie \ algebra.

We \ now \ introduce \ some \ elementary \ considerations \ from \ varietal \ group
theory \ (see \ for \ example \ Neumann [1]). \ Suppose \ L \ is \ the \ set \ of \ laws \ which
G \ satisfies. \ If \ L \in L, \ then \ we \ may \ abelianize \ it, \ that \ is, \ write \ L
additively \ and \ interpret \ the \ result \ as \ an \ element \ of \ the \ free \ algebra
F_{\mathbb{Z}}(X_\infty). \ It \ is \ easy \ to \ check \ that \ this \ abelianization \ is \ a \ law \ in \ gr \ G.
Now, \ the \ laws \ in \ L \ are \ equivalent \ to \ a \ subset \ of \ L, \ consisting \ of \ a
single \ exponent \ law \ x^r (r \geq 0) \ together \ with \ some \ commutator \ laws. \ The
abelianization \ of \ the \ commutator \ laws \ are \ all \ trivial. \ Abelianizing \ the
exponent \ law, \ we \ see \ that \ gr \ G \ is \ a \ Lie \ algebra \ for \ the \ ring \ of \ integers
modulo \ r, \ denoted \ \mathbb{Z}_r.

Suppose \ now \ G \ is \ in \ fact \ the \ relatively \ free \ group \ on \ the \ set \ X
with \ respect \ to \ the \ laws \ L.

(3.2.19). (gr \ G, \ \bar{X}) \ is \ a \ weakly \ relatively \ free \ algebra \ over \ \mathbb{Z}_r.

Proof. \ Let \ \theta \ be \ a \ restricted \ function \ \bar{X} \to \ gr \ G. \ Let
\rho : G \to G/\gamma_2(G) \ be \ canonical; \ we \ may \ define \ a \ function \ \varphi : X \to G
satisfying \ \varphi \rho = \rho \theta. \ Extend \ \varphi \ to \ a \ group \ endomorphism \ of \ G, \ also
denoted \ \varphi. \ Because \ \gamma_i(G) \ is \ a \ fully \ invariant \ subgroup \ of \ G, \ \varphi
induces \ a \ \mathbb{Z}_r-module \ endomorphism \ \overline{\varphi}_i \ of \ \gamma_i(G)/\gamma_{i+1}(G) \ by
\[ [g_\gamma_{i+1}(G)] \overline{\varphi}_i = (g_\varphi)_\gamma_{i+1}(G), \ g \in \gamma_i(G). \ \] The \ sum \ \overline{\varphi} = \overline{\varphi}_1 + \overline{\varphi}_2 + \ldots \ of
these \ \overline{\varphi}_i, \ defined \ pointwise \ on \ gr \ G, \ is \ then \ a \ \mathbb{Z}_r-endomorphism \ of
gr \ G. \ If \ g \in \gamma_i(G), \ h \in \gamma_j(G), \ then
\[ [g \gamma_{i+1}(G), h \gamma_{j+1}(G)] \overline{\theta} = [g, h] \gamma_{i+j+1}(G) \overline{\theta} = [g \gamma_{i+1}(G) \overline{\theta}, h \gamma_{j+1}(G) \overline{\theta}] . \]

Also, for \( x \in X \), \( (xp) \overline{\theta} = [x \gamma_2(G)] \overline{\theta}_1 = (xp) \gamma_2(G) = xp_0 = (xp) \theta \). Thus, \( \overline{\theta} \) is an algebra endomorphism extending \( \theta \). □

It is not known whether \( \text{gr} G \) in (3.2.19) is actually relatively free. (Wall [2] considers a similar question.) Nor does it appear an easy matter to determine the (weak) laws defining \( \text{gr} G \). For example, let \( G \) be a relatively free group of exponent \( p \) a prime, so that \( \text{gr} G \) is a GF(p)-algebra. The first significant result on \( \text{gr} G \) is that it satisfies the Engel law \( [E_{p-1}]_L \). (For example, Higman [1].) However, it is shown in Wall [1] that for \( p = 5, 7 \) or \( 11 \), \( \text{gr} G \) is not the relatively free Lie algebra determined by this law, and a determination of all the (weak) laws satisfied by \( \text{gr} G \) remains unsolved. □

We conclude the section with the following "finite basis problem"; we will make no attempt to solve this here.

(3.2.20). If a verbal ideal in \( F_\infty \) can be generated \textit{qua} verbal ideal by finitely many elements, can it be generated \textit{qua} weakly verbal ideal by finitely many?

(3.3) Introduction to Extension Algebras

We introduce this section with the definition of those algebras which are the central object of study in this paper. In the introduction, we made some preliminary remarks regarding these algebras.

For \( L, M \subseteq F_\infty \), we reserve the notation \( \Lambda_R(L_L, M_M) \) for the weakly verbal ideal \( [L_L(F_\infty), M_M(F_\infty)] \) in \( F_\infty \).

(3.3.1) DEFINITION. An extension algebra is a weakly relatively free algebra \( F_R(X|L_L, M_M) \) for which the ideals \( L_L(F_\infty) \) and \( M_M(F_\infty) \) are
In particular, $\Lambda(L_L, M_M)$ is strongly graded, and we obtain:

(3.3.2) An extension algebra is strongly graded.

**Proof.** We may realize the algebra as $F(Z)/I$ where 
$I = L_L(F(Z)) + M_M(F(Z))$. We seek to show $I$ is strongly graded in $F(Z)$.

To do so, use the technique of renaming variables in the proof of (3.2.7).

Suppose $f \in I$ involves the free generators $z_1, \ldots, z_n$ precisely, and choose distinct $x_1, \ldots, x_n \in X_f$. There exist restricted homomorphisms $\varepsilon : F_\infty \to F(Z)$ and $\delta : F(Z) \to F_\infty$ such that $x_i \varepsilon = z_i$, $z_i \delta = x_i$, $1 \leq i \leq n$. Then, as in the proof of (3.2.7), and using (3.2.6) (4), we have $f\delta \in \Lambda(L_L, M_M)$. Because $\Lambda(L_L, M_M)$ is strongly graded, $f\delta$ is a sum of strongly homogeneous elements of $\Lambda(L_L, M_M)$. Applying $\varepsilon$, we find $f = f\delta \varepsilon$ is a sum of strongly homogeneous elements of $I$, as required. □

It should be stressed that the property of being an extension algebra is not an isomorphism invariant, but rather, depends on the (weak) laws used to describe the algebra. Certainly, this situation is a little unsatisfactory. For example, it is not known whether (3.3.2) has a converse (that is, whether every strongly graded, weakly relatively free algebra can be realized as an extension algebra), and one can pose similar problems in relation to a number of later results. However, the definition in (3.3.1) is not so restrictive as to exclude any interesting algebras, and we are therefore prepared to adopt it.

(3.3.3) **Example.** Take $R = GF(2)$, and write $L = \{A, C, N_3\}$, $M = \{A, C, N_3^\perp\}$. Define $A(\sigma) = F(\sigma|L_L)$. This algebra is just the quotient $\text{Pol}(\sigma)/I(\sigma)$, where $I(\sigma)$ is the ideal of $\text{Pol}(\sigma)$ generated by all $\alpha^3$, $\alpha \in \text{Pol}(\sigma)$. It is immediate that $A(1) \cong F[1|L_L]$, and the algebra
F(1|L_L) is extension (cf. (3.3.6)). However, the description $A(1) = F(1|M_L)$ does not realize $A(1)$ as an extension algebra. In order to prove this, it is enough to show, by (3.3.2), that the ideal $I(2)$ is not strongly graded in $\text{Pol}(2)$.

Suppose $X = \{x, y\}$ is the free generating set for $\text{Pol}(2)$. Because $I = I(2)$ contains $x^3, y^3$ and $(x+y)^3$, we have $x^2y + xy^2 \in I$, and it is enough to show $x^2y \notin I$. However, by direct calculation, for any $a \in \text{Pol}(2)$, $a^3 = \delta (x^2y + xy^2) + \ldots$ for some scalar $\delta$. Therefore, the coefficients of $x^2y$ and $xy^2$ are equal in any element of $I$, and perforce $x^2y \notin I$, as required. □

Our main concern in this section is to give a sufficient condition for a weakly relatively free algebra to be extension. The condition is formulated in terms of the underlying base ring and the (weak) laws defining the algebra. It is nothing more than a mild generalization of the treatment in Jacobson [3, 1.6] for relatively free algebras over fields. As a preliminary, however, we make the observation that it is often possible to replace a (weak) law by a set of strongly homogeneous ones. The original form of this result is sometimes attributed to Kaplansky [1].

(3.3.4). Suppose $0 \neq L \in F_\infty$, and let $\mu$ be the maximum of the strong degrees of the components of $L$. If $\mu!$ is a unit in $R$, then $L$ is weakly equivalent to the set of its strongly homogeneous components.

Proof. Fix a free generator $x \in X_\infty$ appearing in $L$, and write $L = H_0 + H_1 + \ldots + H_k$ where $H_j$ is the sum of those strongly homogeneous components of $L$ with degree $j$ on $x$, $0 \leq j \leq k$. For each integer $0 \leq \lambda \leq k$, let $\theta_\lambda$ be the restricted endomorphism of $F_\infty$ satisfying $x^{\theta_\lambda} = \lambda x$ and $y^{\theta_\lambda} = y$, $x \neq y \in X_\infty$. Then we have
This system of equations has a Vandermonde determinant, and by assumption on $R$, the determinant is a unit. Hence, each $H_j$ is a weak consequence of $L$.

Repeating the argument with each $H_j$, it follows immediately that the strong components of $L$ are weak consequences of $L$, and this is enough for (3.3.4). □

In some cases, one can strengthen (3.3.4). For example, in view of its proof, it is enough to assume $R$ is a commutative $K$-algebra (with unity), where $K$ is a field with at least $(y+1)$ elements. That appears to be the best that can be done.

(3.3.5) EXAMPLE. Take $R = GF(2)$ and let $L = x^2 + x$. Suppose $L$ is equivalent to a set of homogeneous laws (we do not insist they be strongly homogeneous). Because endormophisms in $F_\infty$ never decrease degrees, the homogeneous component $x$ of $L$ is a consequence of $L$. Therefore, the only algebra satisfying $L$ is the zero algebra. But this is manifestly false: $GF(2)$ is already a nonzero algebra in which $L$ is a law. □

Because of the existence of examples like (3.3.5), we generally insist that all (weak) laws be strongly homogeneous. This also allows us, in the following result, to improve the condition on $R$ used in (3.3.4). In this theorem, the zero algebra is a trivial case, and we make the outright assumption for the rest of the section that this algebra does not appear. Note too that, analogously to (3.3.4), when $R$ is an algebra over a field $K$, we need only assume $|K| \geq y$.

(3.3.6) THEOREM. Let $L, M$ be sets of strongly homogeneous elements in $F_\infty$, and let $\mu$ be the maximum of the strong degrees of the elements of $L \cup M$. (We allow $\mu = \infty$ here.) If $k$ is a unit in $R$ for all integers
0 < k < \mu, then for any cardinal \( c > 0 \), \( F_R(\sigma|L, M_c) \) is an extension algebra.

Jacobson [3, I.6 (including Exercise 7)] proves (3.3.6) for relatively free algebras over fields containing \( \mu \) elements. Only a minor variation of his arguments is necessary to prove (3.3.6) itself. Notice that we cannot avoid some restriction on \( R \), as the rank two algebra in (3.3.3) indicates.

Recalling the list of relatively free algebras in (3.2.16), we see that, with the possible exception of \( \text{Jor} \), they are extension algebras over any ring. Further, the Jordan algebra is extension over all fields except GF(2) . (We shall see in (3.5.4) that this apparent exception is real.) Also, the exterior algebra, \( \text{Ext} \), is extension for all \( R \).

The proof of (3.3.6) proceeds via some lemmas; the ideas underlying these have been part of mathematical folklore for a long time, but for clarity, we work from first principles. For the moment, forget the restrictions on \( R \) in (3.3.6). We set some notation: \( L \) is a fixed, non-zero strongly homogeneous element of \( F^\infty \), and \( W \) is an infinite subset of \( \chi\infty \) not containing any of the free generators appearing in \( L \).

(3.3.7). Suppose \( x \in \chi\infty \) and that \( L \) has degree \( d \geq 1 \) on \( x \).
Choose distinct \( w_1, \ldots, w_s \) in \( W \) and let \( \varphi \) be the restricted endomorphism of \( F^\infty \) satisfying \( x\varphi = w_1 + \ldots + w_s \) and \( z\varphi = z \), \( x \neq z \in \chi\infty \). If \( (d-1)! \) is a unit in \( R \), then the strongly homogeneous components of \( L\varphi \) are weak consequences of \( L \).

Proof. For \( 1 \leq j \leq s \), let \( \varphi_j \) be the restricted endomorphism which satisfies \( x\varphi_j = w_j \) and \( z\varphi_j = z \), \( x \neq z \in \chi\infty \). Each \( L\varphi_j \) is a weak consequence of \( L \). Set \( M = L\varphi - \sum_{j=1}^{s} L\varphi_j \). This \( M \) is a weak consequence of \( L \), and it has degree at most \( (d-1) \) on each \( w_j \). By the argument in
(3.3.4), the strongly homogeneous components of $M$ are weak consequences of $M$. It follows that the strongly homogeneous components of $L\varphi$ are weak consequences of $L$. □

We may iterate (3.3.7):

(3.3.8). Suppose $L$ has degree at most $e \geq 1$ in any element of $X^\infty$. For each $x$ appearing in $L$, choose $s(x) \geq 1$ and $\psi_j^{(x)} \in W$, $1 \leq j \leq s(x)$. Assume that the $\psi_j^{(x)}$ are all distinct. Let $\psi$ be the restricted endomorphism of $F_{\infty}$ satisfying $x\psi = \psi_1^{(x)} + \ldots + \psi_{s(x)}^{(x)}$ if $x$ appears in $L$ and $x\psi = z$ otherwise. If $(e-1)!$ is a unit in $R$, then the strongly homogeneous components of $L\psi$ are weak consequences of $L$.

Proof. For $x$ appearing in $L$, define

$$\varphi(x) : x \mapsto \psi_1^{(x)} + \ldots + \psi_{s(x)}^{(x)} , \quad z \mapsto z (z \neq x) .$$

Then $\psi$ is the (commuting) product of these $\varphi(x)$, and the result follows by applying (3.3.7) repeatedly. □

Finally, (3.3.6) itself is an easy consequence of the following (double) statement.

(3.3.9). Again suppose that $L$ has degree at most $e \geq 1$ in any element of $X^\infty$ and that $(e-1)!$ is a unit in $R$. Then the strongly homogeneous components of any (weak) consequence of $L$ are themselves (weak) consequences of $L$.

Proof. Let $\varnothing$ be a (restricted) endomorphism of $F_{\infty}$. We show that $L\varnothing$ is a sum of strongly homogeneous (weak) consequences of $L$.

We may assume $L\varnothing \neq 0$, and therefore $x\varnothing \neq 0$ for each $x$ appearing in $L$. Specifically, write $x\varnothing = \sum_{j=1}^{s(x)} H_j^{(x)}$ where $s(x) \geq 1$ and the $H_j^{(x)}$ are the nonzero strongly homogeneous components of $x\varnothing$. Using the notation
in (3.3.8), define \( \chi \) to be the (restricted) endomorphism satisfying \( \omega_j^\prime \chi = H_j^\prime \) and \( a\chi = a \) for any \( a \) distinct from the \( \omega_j^\prime \). Then \( L\theta = (L\psi)\chi \). We may write \( L\psi \) as a sum \( L\psi = I_1 + \ldots + I_t \) of its strongly homogeneous components. By (3.3.8), each \( I_j \) is a (weak) consequence of \( L \). Furthermore, each \( I_j\chi \) is strongly homogeneous, and therefore the strongly homogeneous components of \( L\theta \) are sums of certain of the \( I_j\chi \). Thus, \( L\theta \) is a sum of strongly homogeneous (weak) consequences of \( L \), as claimed. \( \square \)

(3.4) Elementary Properties of Extension Algebras

In this short section, we prove some elementary results on extension algebras of finite rank. Some of these are quite useful, others only of intrinsic interest. They tend to illustrate the power of "gradedness" in weakly relatively free algebras, rather than relate specifically to extension algebras, but that degree of generality is more than we ever need.

Fix an extension algebra \((A, X)\). At the outset, we consider the degree one component \( A_1 \) of \( A \). This is just \( A_1 = \bigoplus_{x \in X} R x \). If the (weak) laws determining \( A \) have degree greater than 1, then \( X \) is a basis for \( A_1 \). In the general case, we can only assert that every function \( X \rightarrow A_1 \) extends to a (unique) endomorphism of \( A_1 \). An abstract module with this property may be called relatively free.

(3.4.1). Suppose \( M = RX \) is an \( R \)-module. Then \( M \) is relatively free on \( X \) if and only if the \( R x, x \in X \), are isomorphic \( R \)-modules, and in addition, \( M \) is their direct sum. In particular, writing \( \mathfrak{p} \) for the annihilator of such a module \( M \), the module is canonically a free \((R/\mathfrak{p})\)-module with basis \( X \).
Proof. Suppose $M$ is relatively free on $X$. Then, for $x_1, x_2 \in X$, there is an endomorphism mapping $x_1 \mapsto x_2$ and $x_2 \mapsto x_1$. Therefore, $Rx_1$ and $Rx_2$ have the same annihilator in $R$, and thus they are isomorphic.

Now suppose $r_1x_1 + \ldots + r_nx_n = 0$ for $r_i \in R$, $x_i \in X$. For $1 \leq i \leq n$, applying the endomorphism afforded by $x_i \mapsto x_i$, $x_j \mapsto 0$ ($j \neq i$), we obtain $r_ix_i = 0$. Hence the $Rx_i$ form a direct sum. This proves the first part of (3.4.1). The rest follows easily from the observation that $M = \bigoplus_{x \in X} Rx$ with the $Rx$ isomorphic if and only if $M$ is a free $(R/\mathfrak{m})$-module with basis $X$. □

Now let $\mathfrak{m}$ be the annihilator of $A_1$. Then $\mathfrak{m}$ is the annihilator of the whole of $A$, and $A$ is a faithful $(R/\mathfrak{m})$-algebra. As such, $A$ is weakly relatively free and strongly graded, with $\text{End}_R(A) = \text{End}_{R/\mathfrak{m}}(A)$.

(From the results of the next section, we also see that $A$ is an extension $(R/\mathfrak{m})$-algebra, but we shall not actually use that.)

Recall that an algebra is Hopf if every onto endomorphism is an automorphism.

(3.4.2). An extension algebra of finite rank is Hopf.

Proof. Let $(A, X)$ be the algebra, with $X = \{x_1, \ldots, x_n\}$, and treat $A$ as algebra for $R/\mathfrak{m}$. Let $\theta$ be an onto endomorphism of $A$, and write $y_i = x_i^\theta$, $1 \leq i \leq n$. Certainly, $\{y_1, \ldots, y_n\}$ generates $A$, and we may write, for suitable scalars $\alpha_{i,j}, \beta_{j,k} \in R/\mathfrak{m}$,

$$x_i = \sum_{j=1}^{n} \alpha_{i,j} y_j + \Delta_i, \quad 1 \leq i \leq n,$$

$$y_j = \sum_{k=1}^{n} \beta_{j,k} x_k + \Theta_j, \quad 1 \leq j \leq n,$$
where each $A_i, E_j$ has total degree greater than 1. Thus

$$x_i = \sum_{j,k} \overline{a}_{ij} \overline{e}_{jk} x_k + \text{terms of degree } > 1.$$ 

This implies that the matrix $\overline{B}_{jk}$ is invertible over $R/R^n$. Therefore, the endomorphism $\tau$ of $A$ induced by the restricted function $x_j \tau = y_j - E_j$, $1 \leq j \leq n$, is in fact an automorphism. Note that $x_j \theta = x_j \tau + E_j$, $1 \leq j \leq n$.

Now, to show $\theta$ is mono, assume to the contrary $0 \neq z \in \ker \theta$. Set $z = h + h'$ where $0 \neq h$ is homogeneous and $h'$ is of total degree greater than the degree of $h$. Then $0 = z\theta = h\theta + h'\theta = h\tau + h''$ where $h''$ has degree greater than the degree of $h\tau$. Perforce, $h\tau = 0$, a contradiction. □

As a corollary to (3.4.2):

(3.4.3). Suppose $(A, X)$ is an extension algebra of finite rank, and $Y$ a generating set for $A$. Assume either $|Y| \leq |X|$ or that $(A, Y)$ is weakly relatively free. Then $|Y| = |X|$ and $(A, Y)$ is an extension algebra isomorphic to $(A, X)$.

Proof. If $|Y| \leq |X|$, we may choose an onto endomorphism $\theta$ of $A$ with $X\theta = Y$. If $(A, Y)$ is weakly relatively free and $|Y| > |X|$, choose $\theta$ so that $Y\theta = X$. In either case, $\theta$ is an automorphism, and the result follows. □

This result (3.4.3) achieves a number of things. Firstly, it proves the invariance of (finite) rank for extension algebras. It is not known how far this generalizes. Secondly, it shows that the rank is actually the minimum number of generators an extension algebra can have. Thirdly, it partially removes the dependence of an extension algebra on its choice of generating set. This dependence can also be removed in the other direction, that of strong gradedness.
(3.4.4). Suppose \((A, X)\) is a torsion-free extension algebra of finite rank, and assume that \(A\) is also strongly graded on the subset \(Y\).
Then \(\|Y\| = \|X\|\), and \((A, Y)\) is an extension algebra isomorphic to \((A, X)\).

Proof. We are tacitly assuming that the triviality \(0 \in Y\) does not occur. Write \(X = \{x_1, \ldots, x_m\}\). For some finite subset \(\{y_1, \ldots, y_n\}\) of \(Y\) we have

\[
x_i = \sum_{j=1}^{n} \alpha_{ij}^j y_j + \Delta_i, \quad 1 \leq i \leq m,
\]

\[
y_j = \sum_{k=1}^{m} \beta_{jk}^k x_k + E_j, \quad 1 \leq j \leq n.
\]

Here, the \(\alpha_{ij}^j, \beta_{jk}^k\) are scalars. Also, \(\Delta_i\) is a linear combination of monomials in the \(y_j\) of total degree greater than one, and similarly, \(E_j\) has total degree greater than one in the \(x_k\). We see that \(\{y_1, \ldots, y_n\}\) generates \(A\), and because \((A, Y)\) is strongly graded, perforce \(Y = \{y_1, \ldots, y_n\}\). By (3.4.3), we may assume \(m \leq n\), and it is enough to prove \(m = n\).

From the two equations above, we have

\[
x_i = \sum_{j,k} \alpha_{ij}^j \beta_{jk}^k x_k + \text{terms of degree } > 1 \text{ in the } x_k,
\]

\[
y_j = \sum_{i,k} \beta_{ji}^i \alpha_{ik}^i y_k + \text{terms of degree } > 1 \text{ in the } y_k.
\]

Comparing strongly homogeneous components in these equations, and using \(A\) torsion-free, we see that the matrix \((\beta_{jk}^k)\) has a left and right inverse \((\alpha_{ij}^j)\) over \(R\). This forces \(m = n\), as required. \(\square\)

The argument in (3.4.4) really shows that if \((A, X)\) and \((A, Y)\) are strongly graded with \(A\) torsion-free and \(|X| < \infty\), then \(|Y| = |X|\). We
give an example to show the necessity of torsion-freeness.

(3.4.5) EXAMPLE. Let $R = \mathbb{Z}_6$, the ring of integers modulo 6. Take $X = \{x\}$ and let $A = \text{Pol}(X)/I$ where $I$ is the ideal of $\text{Pol}(X)$ generated by $x^3$. This $A$ has a description as an extension algebra (compare (3.3.3)) and is strongly graded on $X$. However, $A$ is also strongly graded on $\{2x, 3x\}$. □

(3.5) Changes to the Base Ring and Integral Extension Algebras

Extension algebras are quite well-behaved under changes to the underlying base ring. We propose to demonstrate this, and then use those anomalies which still remain to motivate the definition of a distinguished class of extension algebras: the "integral" ones.

Throughout the section, $S$ is a commutative, associative ring with unity, and $\rho : R \to S$ a fixed ring homomorphism. We insist that $\rho$ map the unity of $R$ to the unity of $S$. There are two prototypes for $S$ which will occur in this paper:

1. $S$ is an extension ring of $R$ (in the category of commutative rings with unity). Here, $\rho$ is the inclusion map $R \to S$.

2. $R = \mathbb{Z}$ and $S$ is any field (as in the process of Chevalley Reduction, Chapter 5). In this case, $\rho$ is the map $n \mapsto n \cdot 1$ with $n \in \mathbb{Z}$ and 1 the unity of $S$.

Even in the cases when $\rho$ is not an embedding $R \to S$, we still think of $S$ as an extension ring of $R$.

If $B$ is an $S$-algebra, then $B$ is also an $R$-algebra under the action $ra = \rho(r)a$ for $r \in R$, $a \in A$. In particular, $S$ is an $R$-algebra. Therefore, when $A$ is an algebra over $R$, we may form the tensor product $S \otimes_\rho A$, an algebra over $S$. For $X$ a subset of $A$, we adopt the usual notation $1 \otimes X = \{1 \otimes x \mid x \in X\}$. Then $1 \otimes X$ generates
$S \otimes A$ as $S$-algebra if $X$ generates $A$.

We continue to write $F_\infty$ for $F_R(X_\infty)$. The algebra $S \otimes F_\infty$ is the free algebra on $1 \otimes X$. There is a bijection $x \mapsto 1 \otimes x$, $x \in X_\infty$, and this extends to a unique $R$-algebra homomorphism $F_\infty \to S \otimes F_\infty$; we denote this map by $\rho$ also. Compare the following with Jacobson [3, I.6.5].

(3.5.1) THEOREM. Suppose $A = F_R(X|L_L, M_\mu)$ is an extension algebra, and assume $S \otimes A \neq \{0\}$. Then $S \otimes A$ is just $F_S(1 \otimes X | (L_\rho)_L, (M_\rho)_\mu)$, and is also extension. Further, the map $x \mapsto 1 \otimes x$, $x \in X$, is a bijection.

Given (3.5.1), there is a natural identification of $1 \otimes X$ with $X$ itself. Then, too, we can meaningfully write $(S \otimes A)_k = S \otimes A_k$ and $(S \otimes A)_\lambda = S \otimes A_\lambda$ for $k \geq 1$ and $\lambda \in F(X)$. In particular cases, (3.5.1) has a convenient paraphrase. For example, it yields that a free associative algebra remains free associative under extensions of the base ring, and a similar remark applies to the other extension algebras in (3.2.16). In order to prove the proposition, we need a lemma; we separate this out because we shall refer to it again.

(3.5.2). Suppose $L, M$ are subsets of $F_\infty$ with $L_L(F_\infty)$ and $M_\mu(F_\infty)$ strongly graded, and let $(A, X)$ be an algebra satisfying $L_L$ and $M_\mu$. Then the $R$-algebra $(S \otimes A, S(1 \otimes X))$ also satisfies $L_L$ and $M_\mu$.

Proof. It is obvious that $S(1 \otimes X)$ does indeed generate $S \otimes A$ as $R$-algebra. For the proof of the proposition, it will be enough to consider (say) the weak case. Thus, given a restricted homomorphism $\theta : F_\infty \to S \otimes A$, we seek to prove $I\theta = \{0\}$, where $I = M_\mu(F_\infty)$ is strongly graded.

Fix $0 \neq L \in I$. For a free generator $x$ appearing in $L$, write
\[ x^0 = \sum_{i=1}^{j(x)} s_i(x) \otimes a_i(x) \text{ with } a_i(x) \in X, \ s_i(x) \in S \text{ and } j(x) \geq 1. \] Also, as
ranges over the generators in \( L \), choose \( w_i(x) \), \( 1 \leq i \leq j(x) \) in \( X_\infty \), all distinct. Define restricted homomorphisms \( \psi : F_\infty \rightarrow F_\infty \), \( \chi : F_\infty \rightarrow A \) and \( \overline{\chi} : F_\infty \rightarrow S \otimes A \) by
\[ x^0 \psi = \sum_{i=1}^{j(x)} w_i(x), \]
\[ w_i(x) \chi = a_i(x), \]
\[ w_i(x) \overline{\chi} = s_i(x) \otimes a_i(x), \quad 1 \leq i \leq j(x), \ x \in L. \]
(Here, we may as well stipulate that each homomorphism annihilates any unspecified free generators.) We have \( \mathcal{L}^0 = (\mathcal{L}^0)^x = \sum_{l=1}^{t} I_l \overline{\chi} \), where the \( I_l \) are the strongly homogeneous components of \( \mathcal{L} \psi \). Suppose \( \eta \) is a monomial in \( I_l \) of degree \( d_i(x) \) on \( w_i(x) \). Then \( \eta \overline{\chi} = s \otimes (\eta \chi) \) where
\[ s = \prod_{i=x}^{j(x)} \left[ s_i(x)^{d_i(x)} \right]. \] Therefore, \( I_l \overline{\chi} = s \otimes (I_l \chi) \), and since \( I_l \in I \) and \( A \) satisfies \( I^\perp \), we have \( I_l \overline{\chi} = 0 \). Hence \( \mathcal{L}^0 = 0 \), and this completes the proof. \( \square \)

Proof of (3.5.1). To begin with, \( A_\perp \) is a direct summand in \( A \), so we may identify \( S \otimes A_\perp \) with \( S(1 \otimes X) \). Using (3.4.1), this module is relatively free on \( 1 \otimes X \). It cannot be zero by the assumption \( S \otimes A \neq \{0\} \). Hence, the map \( x \mapsto 1 \otimes x \) is indeed a bijection.

Write \( B = F_S(\overline{X}/(L\rho)^L, (M\rho)_\mu) \) , where to avoid ambiguities, we set \( \overline{X} = 1 \otimes X, \overline{x} = 1 \otimes x, \ x \in X \). We exhibit restricted \( S \)-algebra homomorphisms \( \theta : S \otimes A \rightarrow B \) and \( \phi : B \rightarrow S \otimes A \) which are mutually
inverse. This will prove \((S \otimes A, 1 \otimes X) \cong (B, \overline{X})\). On the one hand, the \(R\)-algebra \((B, S\overline{X})\) satisfies \(L_L\) and \(M_W\). Hence, the bijection \(x \mapsto \overline{x}\) extends uniquely to an \(R\)-algebra homomorphism \(A \rightarrow B\). In turn, this map affords an \(S\)-homomorphism \(\theta : S \otimes A \rightarrow B\) satisfying \((1 \otimes x)\theta = \overline{x}\). On the other hand, by (3.5.2), the \(R\)-algebra \([S \otimes A, S(1 \otimes X)]\) satisfies \(L_L\) and \(M_W\). This implies the \(S\)-algebra \((S \otimes A, 1 \otimes X)\) satisfies \((L_\theta)_L\) and \((M_\theta)_W\). Therefore, there exists a unique homomorphism \(\varphi : B \rightarrow S \otimes A\) with \(\overline{x}\varphi = 1 \otimes x\). Then \(\theta \varphi\) is the identity map on \(S \otimes A\), and \(\varphi \theta\) the identity on \(B\), as required.

To complete the proof of (3.5.1), it remains to show \(B\) is an extension algebra. We take \(I = (L_\theta)_L(S \otimes F_\infty)\) and show \(I\) is strongly graded in \(S \otimes F_\infty\); the same argument will prove the weak case. Define \(\overline{T}\) to be the \(S\)-submodule of \(S \otimes F_\infty\) generated by all \(s \otimes L\), for \(s \in S\), \(L \in L_L(F_\infty)\). This \(\overline{T}\) is an ideal in \(S \otimes F_\infty\), and we have the natural isomorphism of \(S\)-algebras

\[
(S \otimes F_\infty)/\overline{T} \cong S \otimes [F_\infty/L_L(F_\infty)] = S \otimes F_R(X_\infty|L_L).
\]

However, the preceding paragraph shows that the algebra on the right of the isomorphism is relatively free on \(1 \otimes X_\infty\), and it is certainly strongly graded on that set. This allows the identification of \(I\) with \(\overline{T}\), and \(I\) is strongly graded in \(S \otimes F_\infty\), as claimed. \(\square\)

That (3.5.1) holds for extension algebras is quite satisfactory. However, there are a few anomalies which still occur when the base ring is extended. We give some examples.

(3.5.3) **Example.** Take \(R = \mathbb{Z}\), and form \(A = F(1|A_L, N_4, T_L)\), where \(T = x(yx) + (xy)x\). The inclusion of the nilpotent law \(N_4\) ensures all products with more than three factors vanish, allowing some arguments to be simplified. By (3.3.6), \(A\) is an extension algebra. Obviously, \(A\) is not
torsion-free over \( R \). However, both \( F(1|A_L) \) and \( F(1|N_4L, T_L) \) have bases of monomials, and certainly are torsion-free. Here, then, is a case (cf. (3.2.17)) in which (weak) laws cannot be treated independently. More to the point, when \( S \) is a field, the dimensions of the strongly homogeneous components of \( S \otimes A \) depend on \( S \). For example, \( \text{GF}(3) \otimes A_3 = \{0\} \neq \text{GF}(2) \otimes A_3 \). □

(3.5.4) EXAMPLE. We remark that this example is particularly relevant in the light of (3.6.6).

Take \( R \) arbitrary, and let \( A = \text{Jor}(X) \) with \( X = \{x, y\} \). We have already noted that \( A \) is an extension algebra whenever \( R \) is a field not \( \text{GF}(2) \). However, \( A \) is not extension when \( R = \text{GF}(2) \) (and in turn, not extension when \( R = \mathbb{Z} \)). To see this, write \( B = F_R(X|C_L) \) and let \( J \) be the verbal ideal of \( B \) generated by \( J^* \). This \( J \) has a distinguished degree 4 component \( J_4 \), namely, the subspace of \( B \) spanned by all \( J^*\psi \), where \( J^* \) is interpreted as an element of \( B \), and \( \psi \in \text{REnd}(B) \). Taking \( R = \text{GF}(2) \), direct calculation (say, by substituting for various \( \psi \)) shows \( J_4 \) has dimension 6, with a basis consisting of

\[
[(xx)x]x + (xx)(xx), \quad [(xx)y]x + (xx)(yx),
\]

\[
[(xx)x]y + (xx)(xy) + [(yy)x]x + (yy)(xx)
\]

and the same three vectors with \( x, y \) interchanged. In particular, \( J \) is not strongly graded in \( B \).

Thus, it is possible for an algebra to be extension over all fields except precisely one, even when the coefficients in its laws are all \( \pm 1 \). □

Integral extension algebras highlight much that is desirable in an extension algebra.

(3.5.5) DEFINITION. An extension algebra \( A \) over a ring \( R \) is integral if \( A = R \otimes_{\mathbb{Z}} B \) for some torsion-free extension algebra \( B \) over \( \mathbb{Z} \).
Of course, an extension algebra can only be integral when it is possible to write its (weak) laws integrally, that is, to realize them as elements of $F_{\mathbb{Z}}(X_\infty)$. The study of these algebras begins in Chapter 5.

Here, we note that basic module theory, together with (3.1.2), implies:

(3.5.6). Suppose $A = R \otimes_{\mathbb{Z}} B$ is an integral extension algebra of finite rank. Then the strongly homogeneous components of $A$ are free $R$-modules of finite $R$-rank, and indeed, $\text{rank}_R A_\lambda = \text{rank}_{\mathbb{Z}} B_\lambda$ ($\lambda \in F(X)$) is independent of $R$. □

As (3.5.3) and (3.5.4) suggest, it can be a difficult matter to determine if an extension algebra is integral on the basis of its (weak) laws alone. However, there is a simple means of constructing integral extension algebras, which we outline briefly.

A groupoid is a nonempty set with a binary operation. On our alphabet $X_\infty$, we may construct the free groupoid $G(X_\infty)$, or simply $G_\infty$. Formally, this consists of monomials $\beta(x_1, \ldots, x_n)$ with $\beta$ a bracket arrangement of weight $n \geq 1$, and the $x_i \in X_\infty$. Multiplication is defined by juxtaposition, enclosing the multiplicands in parentheses to avoid ambiguity, if necessary. If $l_1, l_2$ are elements of $G_\infty$, we say that the pair $(l_1, l_2)$ is a law in a groupoid $G$ if $l_1 \theta = l_2 \theta$ for all groupoid homomorphisms $G_\infty \to G$. Then, if $L$ is a nonempty set of such pairs, and $X$ a nonempty set, there exists a unique groupoid $G(X|L)$ containing $X$ and satisfying the following properties.

(3.5.7). The elements of $L$ are laws in $G(X|L)$.

(3.5.8). If $G$ is a groupoid satisfying $L$, then every function $X \to G$ extends to a unique homomorphism $G(X|L) \to G$.

This groupoid $G(X|L)$, the relatively free groupoid on $X$ with respect to $L$, can be constructed from first principles. Alternatively,
we may apply the techniques of universal algebra (Cohn [1, Chapter IV]).

Now let $F^*(X|L)$ denote the free $R$-module with basis $G(X|L)$. Introduce a product into $F^*(X|L)$ by extending the product in $G(X|L)$ multilinearly. Then $F^*(X|L)$ becomes an $R$-algebra. Defining $G(X_0|\emptyset) = G_\infty$, we have in particular $F^*(X_0|\emptyset) = F_\infty$. This allows us to interpret the elements of $G_\infty$ as monomials in $F_\infty$, and we may define $L^* = \{z_1 - z_2 \mid \{z_1, z_2\} \in L\}$.

(3.5.9). Assume the elements of $L^*$ are multilinear. Then $F^*(X|L)$ is the relatively free algebra on $X$ with respect to the laws $L^*$. As such, $F^*(X|L)$ is an integral extension algebra.

The proof of (3.5.9) proceeds by familiar arguments, and we suppress the details.

(3.6) Standard Examples of Integral Extension Algebras

In this, the concluding section of this chapter, we list a number of integral extension algebras and state some of their well-known properties. With one exception, we have met the algebras before. The underlying (weakly) relatively free generating set for each algebra will be denoted $X$, and we assume $|X| = r < \infty$. Furthermore, because the algebras are all integral, we may meaningfully define $D(r, k) = \text{rank of the degree } k \text{ homogeneous component of the algebra under consideration;}$

$d(r, \lambda) = \text{rank of the strongly homogeneous component determined by } \lambda \in F(X)$.

It will be convenient to have a notation for multinomial coefficients. Precisely, for $\lambda \in F(X)$, we define $M[\lambda] = k! / \left[ \sum_{\lambda(x) > 0} \lambda(x)! \right]$ where

$k = \sum_{x \in X} \lambda(x)$. 
(3.6.1). FREE NON-ASSOCIATIVE ALGEBRA: $F(r|\emptyset)$.

We have seen that $F_k$ has a basis of monomials

$$\{\beta_i(a_1, \ldots, a_k) \mid 1 \leq i \leq s, a_j \in X\}$$

where $\beta_1, \ldots, \beta_s$ is a full list of bracket arrangements of weight $k$.

We have

$$D(r, k) = sr^k,$$

$$d(r, \lambda) = sM[\lambda].$$

By Magnus, Karrass and Solitar [1, 5.2, Problem 5], $s = \frac{1}{k-1} \left(\frac{2k-2}{k}\right)$.

We note in passing that we may also realize $F$ as a (vacuous) case of (3.5.9). □

(3.6.2). FREE ASSOCIATIVE ALGEBRA: $\text{Ass}(r) = F(r|A_L)$.

Applying (3.5.9), $\text{Ass}$ is an integral extension algebra obtained from the free semigroup on $X$. Thus, $\text{Ass}$ has a basis of associative monomials $a_1 \ldots a_k$, with $k \geq 1$ and $a_i \in X$. In particular:

$$D(r, k) = r^k,$$

$$d(r, \lambda) = M[\lambda].$$ □

(3.6.3). POLYNOMIAL ALGEBRA: $\text{Pol}(r) = F(r|C_L, A_L)$.

Again using (3.5.9), $\text{Pol}$ is integral, obtained from the free commutative semigroup on $X$. Write $X = \{x_1, \ldots, x_r\}$. Then $\text{Pol}$ has a basis of monomials $x_1^{\lambda_1} \ldots x_r^{\lambda_r}$ where $\lambda_i \geq 0$ (and not all $\lambda_i$ are zero). We find

$$D(r, k) = \binom{k+r-1}{r-1},$$

$$d(r, \lambda) = 1.$$ □
(3.6.4). FREE LIE ALGEBRA: $\text{Lie}(r) = F(r|J_L, N^*_{2L})$.

That $\text{Lie}$ is integral follows from the fact that it has a natural basis of monomials, the standard basic Lie elements. For their construction, see Magnus, Karrass and Solitar [1, Chapter 5]; that reference also contains proofs of the other assertions we make here. The dimensions of the components of $\text{Lie}$ are given by the celebrated Witt formulae:

$$D(r, k) = \frac{1}{k} \sum \mu(d)r^{k/d},$$

$$d(r, \lambda) = \frac{1}{k} \sum \mu(d)M[\lambda/d].$$

(In the second equation, $k = \sum \lambda(x)$, and when $d|\lambda(x)$ for each $x \in X$, $\lambda/d \in F(X)$ is defined pointwise.)

It is well-known that the algebra $\text{Ass}$ is a Lie algebra under the bracket product $[a, b] = ab - ba$. In turn, $\text{Lie}$ is realizable as the Lie subalgebra of $\text{Ass}$ generated by $X$. To emphasize this, we use bracket notation in the product for $\text{Lie}$. For $k \geq 1$, there is a natural surjection of $R$-modules $N^*_k : \text{Ass}_k \rightarrow \text{Lie}_k$ given by left-norming; precisely, $(d_1 \ldots d_k)N^*_k = [d_1^*, \ldots, d_k^*]$, $d_i \in X$. We recall the Dynkin-Specht-Wever Theorem: If $R$ is an integral domain and $z \in \text{Ass}_k$, then $z \in \text{Lie}_k$ if and only if $zN_k = kz$. In particular, whenever $k$ is a unit in $R$, $(1/k)N_k$ is an idempotent operator. □

(3.6.5). EXTERIOR ALGEBRA: $\text{Ext}(r) = F(r|A_L, N^*_{2L^*})$.

Write $X = \{x_1, \ldots, x_n\}$. There is a basis for $\text{Ext}$ of monomials $x_{i_1} \ldots x_{i_k}$, $i_1 < \ldots < i_k$, $k \geq 1$. In particular
\[ D(r, k) = \binom{r}{k}, \]
\[ d(r, \lambda) = \begin{cases} 1 & \lambda(x) \leq 1 \text{ for all } x \in X, \\ 0 & \text{otherwise.} \end{cases} \]

So \( \text{Ext} \) is a small algebra, of total \( R \)-rank \( 2^r - 1 \). \( \square \)

(3.6.6). FREE SPECIAL JORDAN ALGEBRA: \( \text{SJor}(r) \).

This algebra has not arisen before. To define it, recall that an associative algebra \( A \) is also a Jordan algebra under the Jordan product \( a \ast b = ab + ba \). Not every Jordan algebra arises via such a product; those which do are called special. The free special Jordan algebra on a set \( X \) is defined to be the Jordan subalgebra of \( \text{Ass}(X) \) generated by \( X \). We denote it \( \text{SJor} \). It is immediate that \( \text{SJor} \) is indeed relatively free, but the laws defining it are unknown (see Jacobson [3, I.1 and I.10]). Because the product \( a \ast b = ab + ba \) is bilinear, \( \text{SJor} \) is a strongly graded submodule of \( \text{Ass} \). In particular, \( \text{SJor} \) is an integral extension algebra. (See note added in proof, page 65.)

As far as we are aware, the dimensions of the components of \( \text{SJor} \) remain largely unknown: we summarize the work of Cohn in that regard (see Cohn [1, VII, after 7.5]). Assume specifically that \( R \) is a field of characteristic not 2, and that the rank \( r \leq 3 \). It is usual to take \( a \ast b = \frac{1}{2}(ab + ba) \) for the Jordan product in this context. For \( k \geq 1 \), we define a symmetry operator \( S_k : \text{Ass}_k \rightarrow \text{Ass}_k \); precisely, \( S_k \) is the \( R \)-module homomorphism which acts on monomials by

\[ (a_1 \ldots a_k)S_k = \frac{1}{2}(a_1 \ldots a_k + a_k a_1 \ldots a_1). \] (Our emphasis differs from Cohn's.) Nontrivially, the image of \( S_k \) is exactly \( \text{SJor}_k \). Therefore, \( \text{SJor}_k \) has a basis consisting of the distinct \( \eta S_k \), \( \eta \) a monomial in \( \text{Ass}_k \). Although Cohn does not do so explicitly, one may compute from this
that
\[ D(r, k) = \frac{1}{2} \left( r^{k+1/2} \right), \]
\[ d(r, \lambda) = \frac{1}{2} (M[\lambda] + \omega[\lambda]). \]

To define \( \omega[\lambda] \) here, firstly write \( \lambda/2 \) for the map \( x \mapsto \lfloor \lambda(x)/2 \rfloor \), \( x \in X \). Then
\[
\omega[\lambda] = \begin{cases} 
0 \text{ whenever } \sum_{x \in X} \lambda(x) = 1 \text{ or } \lambda(x) \text{ is odd for more than one } x, \\
M[\lambda/2] \text{ otherwise.}
\end{cases}
\]

We note that the operator \( S_{k} \) is idempotent. □

We have seen in (3.5.4) that \( \text{Jor} \) is not an integral extension algebra; we do not know whether the free alternative algebra, \( \text{Alt} \), is integral.

ADDED IN PROOF. As we go to press, the assertion that \( \text{SJor} \) is integral suddenly seems rather glib. Specifically, one readily checks that \( \text{SJor}_{\mathbb{Z}} \) is a torsion-free extension algebra. Then, assuming that \( R \) contains an element \( a \) with \( 2a = 1 \), it is reasonable to ask for an identification of \( R \otimes \text{SJor}_{\mathbb{Z}} \) with \( \text{SJor}_{R} \). However, we do not have an argument for that.

Notice that the references to \( \text{SJor} \) in Chapter 5 remain essentially intact. This is because, even if \( \text{SJor} \) fails to be integral, it is nevertheless a \( U \)-submodule of \( \text{Ass} \).
CHAPTER 4

THE COMPONENTS OF EXTENSION ALGEBRAS AS MODULES

The general linear group has a natural action, by automorphisms, on the homogeneous components of an extension algebra; the general linear Lie algebra acts by derivations. This chapter makes a preliminary study of the actions. The group is taken first, in (4.1)-(4.4). The emphasis here is on calculating the composition structure of the components, by means of a character formula; the problem is reduced to the numerical application of the Littlewood-Richardson Rule. The method is highlighted with various examples in (4.2), (4.3). In (4.4), we make some qualitative remarks regarding the decomposition of the components into direct sums of indecomposables.

One can give a comparable determination of composition factors for the derivation action of the general linear Lie algebra. However, we are content in this case to compute the weights and weight spaces of the components (4.5), and use the information so obtained to further the study of the polynomial algebra in (4.6).

As in the previous chapter, $\mathcal{R}$ is a commutative ring with unity. We shall in fact mostly take $\mathcal{R}$ to be a field, but we always state explicitly when we do this, writing $\mathcal{K}$ in place of $\mathcal{R}$ for emphasis. The algebra $(\mathcal{A}, \mathcal{X})$ is a fixed extension $\mathcal{R}$-algebra of finite rank $r \geq 2$. For $k$ a positive integer and $\lambda \in \mathcal{P}(\mathcal{X})$, we write (whenever it is meaningful to do so, specifically over a field)

$$D(r, k) = \text{rank}_{\mathcal{R}} \mathcal{A}_{\mathcal{X}},$$

$$d(r, \lambda) = \text{rank}_{\mathcal{R}} \mathcal{A}_{\lambda}.$$ 

However, we do not insist that $\mathcal{A}$ be integral so that this notation ignores the possible dependence of $D, d$ on the choice of $\mathcal{R}$. 
There is one last assumption to make: that the (weak) laws defining $A$ have degree greater than one. This ensures that the degree one component of $A$ is a free $R$-module with $X$ as basis. We make the assumption to avoid unimportant technicalities; it holds perforce when $A$ is an integral extension algebra or $R$ is a field.

(4.1) The Action of the General Linear Group

We consider in this section a group $G$ for which the degree one component $A_1$ of $A$ is a faithful module. Thus, $G$ may be regarded as a group of $R$-automorphisms of $A_1$. Because $A_1$ is a free $R$-module of rank $r$, $G$ can be embedded into $GL(r, R)$; conversely, every subgroup of $GL(r, R)$ has a faithful action on $A_1$. However, at least for the time being, we prefer to treat $G$ simply as an abstract group.

An element $g$ of $G$ determines a restricted function $X \rightarrow A$. Because $A$ is weakly relatively free, this function extends to an algebra endomorphism, which we denote $g_a$, of $A$. Then $g_a$ is an automorphism of $A$. In fact, we even have $(gh)_a = g_ah_a$ for $g, h \in G$, so that $A$ is a faithful $G$-module under the action $ag = ag_a$, $a \in A$. More to the point:

(4.1.1). For $g \in G$ and $a_1, \ldots, a_k \in X$, the action of $g$ on a monomial $\beta(a_1, \ldots, a_k)$ is given by

$$\beta(a_1, \ldots, a_k)g = \beta(a_1g, \ldots, a_kg).$$

In particular, each $A_k$ is a $G$-submodule of $A$. 

Regarding $G$ as a subgroup of $GL(r, R)$, the action in (4.1.1) is by "linear substitutions". Viewed in that light, the origins of the procedure go back to antiquity.

This section is concerned primarily with some character calculations.
over fields, but before proceeding with that, we make one more preliminary observation. We have seen that if $S$ is an extension ring of $R$ under a homomorphism $\rho : R \rightarrow S$, then $(S \otimes A, 1 \otimes X)$ is an extension algebra. Furthermore, $(S \otimes A)_k = S \otimes A_k$. There are then, two actions of $G$ on $S \otimes A_k$. The first is the induced action $(s \otimes a)g = s \otimes (ag)$ for $s \in S$, $a \in A_k$, $g \in G$. The second is obtained by algebra automorphisms: the induced action of $G$ on $S \otimes A_1$ extends to an action on $S \otimes A_k$ as per (4.1.1). (We should point out that $G$ does not necessarily act faithfully on $S \otimes A_1$; however, the action is faithful in the most important case when $S$ is a field extension of a field $R$.) It is straightforward to check that these two actions are the same, and we no longer distinguish them.

We now consider the case $R = K$ a field, and develop the promised character formulas. To begin with, we compute the character of $A_k$ for $k \geq 1$ - that is, the traces of the representing matrices - without regard to the characteristic of $K$. The resulting formula has long been known in special cases (Ext, Lie, Pol and others), so much so that the synthesis we offer here is perhaps overdue.

Fix $g \in G$, and write $\overline{g} = g_a|_{A_1}$, the linear map representing $g$ on $A_1$. Let $L$ be a field extension of $K$, large enough to include all the eigenvalues of $\overline{g}$, and write $B = L \otimes A$. Identifying $1 \otimes X$ with $X$ itself, we work for the moment over the extension algebra $(B, X)$. We shall find it convenient to order $X$ as $X = \{x_1, \ldots, x_r\}$, and as in (3.1), if $\lambda \in F(X)$, we identify $\lambda$ with $(\lambda_1, \ldots, \lambda_r)$ where $\lambda_i = \lambda(x_i)$, $1 \leq i \leq r$.

There is an ordered basis $Y = \{y_1, \ldots, y_r\}$ of $B_1$ with respect to
which the action of \( \bar{g} \) is lower triangular. Specifically, write

\[ y_{j}g = \alpha_{j}y_{j} + \text{(a linear combination of the } y_{j} \text{ with } j < i) \]

where the \( \alpha_{j} \) are the eigenvalues of \( \bar{g} \) in \( L \). Certainly, \((B, Y)\) is an extension algebra isomorphic to \((B, X)\). The degree \( k \) component of \((B, Y)\) is just the degree \( k \) component of \((B, X)\), and this is denoted \( B_{k} \) as usual. Furthermore, the isomorphism \((B, X) \rightarrow (B, Y)\) can be induced by the bijection \( x_{i} \mapsto y_{i}, 1 \leq i \leq r \), and in that case, \( B(\lambda_{1}, \ldots, \lambda_{r}) \) as formed in \((B, Y)\) has the same \( k \)-dimension as \( B(\lambda_{1}, \ldots, \lambda_{r}) \) in \((B, X)\). This dimension is just \( d(r, \lambda) \).

We wish to compute the action of \( g \) on \( B_{k} \). To this end, choose a basis for \( B_{k} \) consisting of monomials in \( Y \). Let \( \beta(a_{1}, \ldots, a_{k}) \) be a monomial in the basis, of degree \( \lambda_{i} \) on \( y_{i}, 1 \leq i \leq r \). Order the \( r \)-tuplets of non-negative integers in left-to-right lexicographic order.

Then we see that

\[ \beta(a_{1}, \ldots, a_{k})g = \alpha_{1} \cdots \alpha_{r} \beta(a_{1}, \ldots, a_{k}) + \Lambda \]

where \( \Lambda \) is a linear combination of monomials with strong degrees greater than \((\lambda_{1}, \ldots, \lambda_{r})\). It follows that our basis of monomials may be so ordered that \( g \) has lower triangular action on \( B_{k} \). In particular, the \( \alpha_{1} \cdots \alpha_{r} \) are the eigenvalues of the linear map representing \( g \) on \( A_{k} \).

Thus:

\[ (4.1.2). \text{ The character } \chi_{k}^{(A)} \text{ of } G \text{ on } A_{k} \text{ is given by } \]

\[ \chi_{k}^{(A)}(g) = \sum_{\lambda=(\lambda_{1}, \ldots, \lambda_{r})} d(r, \lambda) \alpha_{1}^{\lambda_{1}} \cdots \alpha_{r}^{\lambda_{r}}. \]
The formula in (4.1.2) has the disadvantage that it involves the eigenvalues of $g$, that is, it is not yet written as an equation over $K$. This can be circumvented as follows. First, we make a definition, to highlight the essential step in the method.

(4.1.3) DEFINITION. The $k$th strong polynomial in the indeterminates $z_1, \ldots, z_r$ for the extension algebra $(A, X)$ is the integral polynomial

\[ s_k^{(r)}(z_1, \ldots, z_r) = \sum_{\lambda = (\lambda_1, \ldots, \lambda_r)} d(r, \lambda) z_{\lambda_1} \ldots z_{\lambda_r}, \]

\[ k = \lambda_1 + \ldots + \lambda_r \]

Any permutation of $X$ induces an automorphism of the extension algebra $(A, X)$. Therefore, $s_k^{(r)}$ is a symmetric polynomial in its variables $z_1, \ldots, z_r$. With the notation of (2.5), write

\[ s_k^{(r)}(z_1, \ldots, z_r) = \sigma_k^{(r)}(\zeta_1, \ldots, \zeta_r) = \sum_{\lambda = (\lambda_1, \ldots, \lambda_r)} \rho(r, \lambda) \zeta_{\lambda_1} \ldots \zeta_{\lambda_r}, \]

\[ k = \lambda_1 + \ldots + \lambda_r \]

\[ \rho(r, \lambda) \in \mathbb{Z}. \]

To convert from $s_k$ to $\sigma_k$ in a particular instance, one may use the algorithm in (2.5.1). Now, setting $A = (\alpha_1, \ldots, \alpha_r)$ and

\[ \zeta(A) = \zeta(\alpha_1, \ldots, \alpha_r), \]

we have:

(4.1.4) THEOREM. The character $\chi_k^{(A)}$ is given by

\[ \chi_k^{(A)}(g) = \sum_{\lambda = (\lambda_1, \ldots, \lambda_r)} \rho(r, \lambda) \zeta(A)_{\lambda_1} \ldots \zeta(A)_{\lambda_r}. \]

This is the formula we are seeking. The field elements $\zeta(A)$ all lie in $K$; indeed, to within at most a change of sign, they are the coefficients of the characteristic equation for $g$. We may also note in passing that the formula does not depend in any essential way on the
original order chosen for $X$.

In the case when $K$ has characteristic $\infty$, the character in (4.1.4) is sufficient to determine the composition structure of $A_k$, at least in principle. This is an application of van der Waerden [1, §125], because the group algebra of $G$ is represented by a finite-dimensional algebra of linear transformations on $A_k$. However, we can do somewhat better than that.

Still assuming arbitrary characteristic for $K$, consider the exterior algebra $\text{Ext}(X)$. The degree one component $\text{Ext}_1$ is a vector space with $X$ as a basis, hence $\text{Ext}_1$ is a $G$-module isomorphic to $A_1$. Using the dimensions given in (3.6.5), we may specialize (4.1.2) to $\text{Ext}_l$, giving:

\[(4.1.5). \text{ For } 1 \leq l \leq r, \quad \chi^{(\text{Ext})}_l (g) = \zeta_l (A).\]

Therefore:

\[(4.1.6). \text{ The character } \chi_k^{(A)} = \frac{\sum_{\lambda=\lambda_1 \ldots \lambda_r} \rho (r, \lambda) \left[ \chi^{(\text{Ext})}_1 \right]^{\lambda_1} \cdots \left[ \chi^{(\text{Ext})}_r \right]^{\lambda_r}}{k = \lambda_1 + \ldots + \lambda_r}. \]

It follows that when $K$ has characteristic $\infty$, the composition factors of $A_k$ can be obtained, using (4.1.6), from the factors of tensor products of the $\text{Ext}_l$, $1 \leq l \leq r$. A word of caution is advisable here. We do not assert that $A_k$ has a chain of submodules with factors isomorphic to such tensor products. In fact, it is not really known to what extent the arithmetic formula in (4.1.6) can be used to reflect the submodule structure of $A_k$. This is because the coefficients in the strong polynomial $\sigma_k^{(r)}$ need not be non-negative.
(4.1.7) **EXAMPLE.** Take $A = \text{Lie}(2)$ and consider $\text{Lie}_4$. We have
$$r_1(0,4) = r_1(4,0) = 0, \quad r_1(3,1) = r_1(1,3) = 1, \quad r_1(2,2) = 1.$$ 
Thus
$$s_4(z_1, z_2) = z_1^3 z_2 + z_1^2 z_2 + z_1 z_2^3 = \zeta_1^2 \zeta_2 - \zeta_2^2 = s_4(\zeta_1, \zeta_2).$$

The equation in (4.1.6) indicates the importance of the exterior algebra. However, in order not to break the thread of the character calculations, we shall defer consideration of the algebra (and the tensor products of the $\text{Ext}_L$) to (4.2). In (4.3), we will compute some explicit instances of composition structure.

The formulae in (4.1.2) through (4.1.6) hold when $K$ has characteristic a prime, but the character is then no longer sufficient to determine the composition structure of $A_k$. To handle this case, it is necessary to calculate the Brauer character instead. The procedure is much as before, but some technical differences arise, and we proceed from first principles.

Suppose then $\text{char}(K) = p < \infty$. Let $L$ be the algebraic closure of $K$, and write $U(L)$ for the multiplicative group of roots of unity in $L$. Let $U(\mathbb{C})$ be the corresponding group in the field $\mathbb{C}$ of complex numbers. Choose a group isomorphism $\delta : U(L) \rightarrow U(\mathbb{C})$; all calculations are done with this choice of $\delta$ fixed. In order to allow a meaningful definition of Brauer character (and more to the point, to ensure that the Brauer character does determine composition structure), we also make the flat assumption that $G$ is a finite group. Write $B = L \otimes A$.

Let $g \in G$ be $p$-regular, represented as before on $A_\perp$ by $\overline{g}$. The eigenvalues $\alpha_1, \ldots, \alpha_r$ of $\overline{g}$ lie in $L$, and we may diagonalize the action of $\overline{g}$ on $B_\perp$: there exists a basis $\{y_1, \ldots, y_n\}$ of $B_\perp$ such that
73

\[ y_i g = \alpha_i y_i , \quad 1 \leq i \leq r . \]

Reasoning as in the modular case, we see that \( g \) acts diagonally on \( B_k \),
with eigenvalues \( \alpha_1 \ldots \alpha_r \), \( \lambda_1 + \ldots + \lambda_r = k \), and of course, these are roots of unity. Thus:

\[ (4.1.8). \text{ The Brauer character } \chi_k^{(A)} \text{ of } G \text{ on } A_k \text{ is given by} \]

\[ \chi_k^{(A)}(g) = \sum_{\lambda=(\lambda_1, \ldots, \lambda_r)} d(r, \lambda) \delta(\alpha_1)^{\lambda_1} \ldots \delta(\alpha_r)^{\lambda_r} . \quad \Box \]

This equation (4.1.8) is the same as (4.1.2) with the isomorphism \( \delta \) incorporated. We derive, in quick succession:

\[ (4.1.9). \text{ Writing } \delta(A) = (\delta(\alpha_1), \ldots, \delta(\alpha_r)), \]

\[ \chi_k^{(A)}(g) = \sum_{\lambda=(\lambda_1, \ldots, \lambda_r)} \rho(r, \lambda) \zeta_1^{\delta(A)}(\lambda_1) \ldots \zeta_r^{\delta(A)}(\lambda_r) . \quad \Box \]

\[ (4.1.10). \text{ For } 1 \leq \ell \leq r , \quad \chi_k^{(\text{Ext})}(g) = \zeta_\ell^{\delta(A)} . \quad \Box \]

\[ (4.1.11). \text{ The Brauer character } \chi_k^{(A)} \text{ is given by} \]

\[ \chi_k^{(A)} = \sum_{\lambda=(\lambda_1, \ldots, \lambda_r)} \rho(r, \lambda) \left[ \chi_1^{(\text{Ext})} \right]^{\lambda_1} \ldots \left[ \chi_r^{(\text{Ext})} \right]^{\lambda_r} . \quad \Box \]

Once again, the composition factors of \( A_k \) are composition factors of tensor products of the \( \text{Ext}_\ell \), \( 1 \leq \ell \leq r \).

As we mentioned at the outset, various particular cases of (4.1.2) have long been known, and we have certainly made no attempt to trace all such instances in the literature. Presumably, one of the most interesting features of the above calculations has been recognized before: the strong
polynomial for $A_k^n$ contains all the information necessary to compute the composition structure of $A_k^n$. Indeed, this is just one instance of a theme which will recur frequently.

(4.2) The Exterior and Polynomial Algebras (I)

We make no claims to originality in this section: the results presented are all well-established. However, there are two reasons for including them here. Firstly, we need a bridge to connect the discussion in (2.1) with the previous section (4.1); secondly, we propose to make an ongoing survey of the algebras $\text{Ext}$ and $\text{Pol}$, and the discussion here is a natural beginning. We work over a field $R = K$ of characteristic $p \leq \infty$.

To begin with, write $V$ for the degree one component $\text{Ass}_1$ of the free associative algebra. In (2.1), we denoted the $k$-fold tensor power of $V$ by $V^\otimes k$, $k \geq 1$. There is a natural isomorphism $V^\otimes k \to \text{Ass}_k$ of $G$-modules, given by $v_1 \otimes \ldots \otimes v_k \mapsto v_1 \ldots v_k$, $v_i \in V$. Allowing the symmetric group $S_k$ to act on $\text{Ass}_k$ by place permutations, this isomorphism also preserves $S_k$-action. Working inside $\text{Ass}_k$ instead of $V^\otimes k$, let $\text{Ass}_k^\pm$ be the symmetric and skew-symmetric parts of $\text{Ass}_k$. The following result is too well-known to need proof here.

(4.2.1). For $k < p$, $\text{Ass}_k^\pm$ is $G$-isomorphic to $\text{Pol}_k$ and $\text{Ass}_k^\pm$ to $\text{Ext}_k$. □

In particular, from the discussion in (2.1), the tensor products of the $\text{Ext}_k$ can be computed, when $p = \infty$, using the Littlewood-Richardson Rule. In this case, then, the calculation of the composition factors of extension algebras is solved.
While the algebra Ext arises naturally in the calculations in (4.1), one could use Pol just as well. We describe this duality between the two. The characters $\chi_k^{(\text{Ext})}$, $\chi_k^{(\text{Pol})}$ are given in (4.1.5), (4.1.10). Similarly, using the notation in (2.5), the following formulae are obtained for Pol.

\[(4.2.2)\] For $g \in G$, $\chi_k^{(\text{Pol})}(g) = h_k(\alpha_1, \ldots, \alpha_n)$. When $G$ is finite, $p < \infty$ and $g$ is $p$-regular, $\chi_k^{(\text{Pol})}(g) = h_k(\delta(\alpha_1), \ldots, \delta(\alpha_n)).$\]

With the understanding $\chi_k^{(\text{Ext})} = 0$ for $k > 0$, we therefore have

$$\chi_k^{(\text{Ext})} = H_k[\chi_1^{(\text{Ext})}, \ldots, \chi_k^{(\text{Ext})}] \quad \text{and} \quad \chi_k^{(\text{Pol})} = H_k[\chi_1^{(\text{Pol})}, \ldots, \chi_k^{(\text{Pol})}],$$

with similar equations for the Brauer character. The duality is quite visible. The only difference is that using Pol results in somewhat more prohibitive arithmetic.

The parallel between the two algebras is lost when one attempts to decompose their $k$th components in finite characteristic. The problem for Pol$_k$ is highly nontrivial. Glover [1] gives a complete solution when $r = 2$, $K = \text{GF}(p)$ and $G = \text{GL}(2, p)$ (and various subgroups of GL$(2, p)$); again, Pol$_k$ is SL$(r, K)$-indecomposable at least when $K$ has more than $k$ elements (cf. (2.3.6) and (5.4.1)). In sharp contrast, there is the well-known:

\[(4.2.3)\] For $1 \leq k \leq r$, Ext$_k$ is an irreducible SL$(r, K)$-module.\]

(4.3) Examples of Composition Structure

We illustrate the character calculations of the preceding sections in some more cases. The notation is the same as before.

(4.3.1) EXAMPLE. THE CLASSICAL CASE, $k < p$.

Take $G = \text{GL}(r, K)$. When $k < p \leq \infty$, $A_k$ is completely reducible as
**G-module.** To see this, let $F$ be the free algebra on $X$ and $\theta : F \to A$ the algebra surjection extending the identity map on $X$. This $\theta$ restricts to a surjection of $G$-modules $F_k \to A_k$. However, $F_k$ is $G$-isomorphic to a sum of copies of $\text{Ass}_k$, and by (2.1.1), $\text{Ass}_k$ is completely reducible. Thus, so too is $A_k$.

In this case, then, character calculations determine the structure of the module completely. $\square$

**Example.** We give a numerical instance.

Take $A = \text{S} \text{Jor}(2)$, the special Jordan algebra of rank 2, and let $k$ have characteristic $0$. Of course, $\text{S} \text{Jor}_k \cong \text{Pol}_k$ when $k \leq 2$. Using (3.6.6), the next two strong polynomials for $\text{S} \text{Jor}$ are

$$s_3(z_1, z_2) = z_1^3 + 2z_1z_2z_2 + z_2^3,$$

$$s_4(z_1, z_2) = z_1^4 + 2z_1^3z_2 + 4z_1^2z_2^2 + 2z_1z_2^3 + z_2^4.$$

Applying (2.5.1) (or otherwise), these give

$$\sigma_3(\zeta_1, \zeta_2) = \zeta_1^3 - \zeta_1\zeta_2,$$

$$\sigma_4(\zeta_1, \zeta_2) = \zeta_1^4 - 2\zeta_1^2\zeta_2 + 2\zeta_2^2.$$

By (4.1.6), we have the following equivalences for composition structure

$$\text{S} \text{Jor}_3 \cong \text{Ext}_1^3 - \text{Ext}_1 \otimes \text{Ext}_2,$$

$$\text{S} \text{Jor}_4 \cong \text{Ext}_1^4 - 2 \text{Ext}_1^2 \otimes \text{Ext}_2 + 2 \text{Ext}_2^2.$$

The Littlewood-Richardson Rule, applied repeatedly gives the following isomorphisms. (The zero $W^\lambda$ which occur have been deleted as they arose, using (2.1.1).)
\[ \text{Ext}_1 \otimes \text{Ext}_1 \cong \mathcal{W}^{(2)} \oplus \mathcal{W}^{(1^2)} , \]
\[ \text{Ext}_1 \otimes \text{Ext}_1 \cong \mathcal{W}^{(3)} \oplus 2\mathcal{W}^{(2,1)} , \]
\[ \text{Ext}_1 \otimes \text{Ext}_1 \cong \mathcal{W}^{(4)} \oplus 3\mathcal{W}^{(3,1)} \oplus 2\mathcal{W}^{(2)} , \]
\[ \text{Ext}_2 \otimes \text{Ext}_1 \cong \mathcal{W}^{(2,1)} , \]
\[ \text{Ext}_2 \otimes \text{Ext}_1 \cong \mathcal{W}^{(3,1)} \oplus \mathcal{W}^{(2^2)} , \]
\[ \text{Ext}_2 \otimes \text{Ext}_1 \cong \mathcal{W}^{(2^2)} . \]

We are left with
\[ \text{SJOR}_3 \cong \mathcal{W}^{(3)} \oplus \mathcal{W}^{(2,1)} , \]
\[ \text{SJOR}_4 \cong \mathcal{W}^{(4)} \oplus \mathcal{W}^{(3,1)} \oplus 2\mathcal{W}^{(2^2)} . \]

(4.3.3) EXAMPLE. It is not often one can obtain a closed formula for the multiplicity with which an irreducible appears as composition factor of \( A_k \). For completeness, we give the most famous example of such a calculation: that for the algebra \( \text{Lie}(K) \). In its original form, this goes back to Brandt [1]; the origins of the argument given here seem to be due to Wever [1], [2]. In the statement itself, \( S^\nu = e_{\tau} KS_k, e_{\tau} \) as in (2.1).

(4.3.4). Take \( p = \infty \). For any partition \( \nu \) of \( k \), let \( X^\nu, Y^\nu \) be respectively the characters of \( \text{GL}(r, K) \) and \( S_k \) determined by the modules \( \mathcal{W}^\nu, S^\nu \). Then the multiplicity \( m(\nu) \) with which \( \mathcal{W}^\nu \) appears in \( \text{Lie}_k \) is just
\[ m(\nu) = (1/k) \sum_{d \mid k} \mu(d) \psi^\nu(\sigma^{k/d}) \]
where \( \sigma \in S_k \) is a \( k \)-cycle.
Proof. We have

\[ \chi_k^{(\text{Lie})}(g) = \sum_\lambda \sum_{d \mid k} \mu(d) M[\lambda \alpha_1 \ldots \alpha_r] \text{ by (4.1.2)} \]

\[ = \frac{1}{k} \sum_\lambda \sum_{d \mid k} \mu(d) M[\lambda \alpha_1 \ldots \alpha_r] \text{ using (3.6.4)} \]

\[ = \frac{1}{k} \sum_\lambda \sum_{d \mid k} \mu(d) M[\lambda_1', \ldots, \lambda_r'] \left[ \alpha_1 \ldots \alpha_r \right]^d \]

interchanging the order of summation

\[ = \frac{1}{k} \sum_\lambda \sum_{d \mid k} \mu(d) \left[ \alpha_1^d + \ldots + \alpha_r^d \right]^{k/d} \]

by a multinomial expansion

where, as before, \( \bar{g} \) is the linear map representing \( g \) on \( A_1 \).

Now suppose \( \sigma \) is a \( k \)-cycle. For \( d \mid k \), \( \sigma^{k/d} \) is a product of \( k/d \)
disjoint \( d \)-cycles. By James [1, 26.6 (i)] (note that our notation differs
markedly from his)

\[ |\text{tr}(\bar{g}^d)|^{k/d} = \sum_\nu \psi^\nu(\sigma^{k/d}) \chi^\nu(g) . \]

Thus

\[ \chi_k^{(\text{Lie})}(g) = \frac{1}{k} \sum_{d \mid k} \sum_\nu \mu(d) \psi^\nu(\sigma^{k/d}) \chi^\nu(g) \]

\[ = \sum_\nu \left[ \frac{1}{k} \sum_{d \mid k} \mu(d) \psi^\nu(\sigma^{k/d}) \right] \chi^\nu(g) . \]

Hence, the multiplicity \( m(\nu) \) is as claimed. \( \square \)

(4.4) Some Qualitative Remarks on the Structure of \( A_k \)

Our aim in this section is to examine, on a purely qualitative basis,
some aspects of the \( G \)-structure of \( A_k \). We restrict ourselves to \( G \) a
finite group and $K$ a field of characteristic $p \leq \infty$. As the previous sections suggest, interest centres on the case $p < \infty$, but we do not assume that. The discussion is motivated by the work of Bryant and Kovács [1], [2]; indeed, we arrange our arguments in a manner only slightly different from theirs, and retain most of their notation.

Write $M$ for the subgroup of $G$ consisting of those elements with scalar action on $A_1$, and set $m = |M|$. Because $G$ acts faithfully on $A_1$, this $M$ is a cyclic, central subgroup, generated by some element $a$.

Let $\xi \in K$ satisfy $a\xi = \xi a$, $a \in A_1$; then $\xi$ is a primitive $m$th root of unity in $K$. The group $M$ has $m$ distinct isomorphism types $U_0, \ldots, U_{m-1}$ of irreducible modules over $K$, each one-dimensional, where $\sigma$ acts on $U_i$ as the scalar $\xi^i$, $0 \leq i \leq m-1$. The sum $U_0 \oplus \cdots \oplus U_{m-1}$ is then a regular $KM$-module, and in turn, the induced module $U^G = U \otimes_{KM} KG = U_0^G \oplus \cdots \oplus U_{m-1}^G$ is a regular for $G$. For each $i$, $\sigma$ acts as $\xi^i$ on the whole of $U_i^G$, so that the $U_i^G$ can have no composition factors in common. Furthermore, $\sigma$ acts on $A_k$ as the scalar $\xi^k$.

Therefore:

(4.4.1). For $k \geq 1$, an irreducible $KG$-module can appear as composition factor of at most one of the modules $A_k, \ldots, A_{k+m-1}$. □

For most of the important extension algebras, we can improve (4.4.1). Thus, we suppose for the rest of this section that $A$ is one of the algebras $F$, $Ass$, $Jor$, $Sjor$, $Lie$, $Alt$ or $Pol$. (It is trivial that the discussion we give fails for $Ext$.)

(4.4.2). For $k$ sufficiently large, $A_k \oplus \cdots \oplus A_{k+m-1}$ contains a regular $KG$-module.
Assume (4.4.2) for the moment. An irreducible $KG$-module occurs as unique minimal submodule of one, and (to within isomorphism) only one, principal indecomposable $KG$-module. Also, these principal indecomposables are injective. So we conclude the following statement.

(4.4.3). For $k$ sufficiently large, a principal indecomposable $KG$-module appears as a direct summand in exactly one of the modules $A_k, A_{k+1}, \ldots, A_{k+m-1}$.

In particular, $A_k$ decomposes when $k$ is large enough. This contrasts sharply with, for example, the action of $\text{sl}(2,K)$ on $\text{Pol}_k$ (see (4.4.6)).

Bryant and Kovács [1], [2] proved that (4.4.2) is satisfied for the algebras $\text{Ass}$ and $\text{Lie}$. We modify their treatment to include the other cases for $A$. In order for the discussion to proceed smoothly, we need to assume $K$ is an infinite field. This is justified by the following result (Bryant and Kovács [2, Corollary 2]) which we do not prove here.

(4.4.4). For any field extension $L$ of $K$ and any $KG$-module $V$, $V^L = L \otimes_K V$ contains a regular $LG$-module if and only if $V$ contains a regular $KG$-module.

Now, $U^G = U^G_0 \oplus \cdots \oplus U^G_{m-1}$ is a regular $KG$-module. Let $T$ be a full set of coset representatives of $M$ in $G$, with $1 \in T$. For $0 \leq i \leq m-1$, the $G$-module $U^G_i$ is characterized by the following two properties.

(4.4.5). The generator $\sigma$ of $M$ acts as the scalar $\xi^i$ on $U^G_i$.

(4.4.6). There is an element $u_i \in U^G_i$ such that $u_i T$ is a basis for $U^G_i$. 
Write \((n+1)\) for the index of \(M\) in \(G\). We claim:

\[\text{(4.4.7). For all } k \geq n, \text{ there is an element } v \in \text{Ass}_1 \text{ such that } v^k_T \text{ is linearly independent in } \text{Ass}_k.\]

It follows immediately from (4.4.7) that (4.4.2) holds for \(\text{Ass}\), and hence for \(F\) and \(\text{Alt}\), preimages of \(\text{Ass}\). If \(K\) has characteristic not 2, then \(v^k\) is an element of \(\text{Sjor} \subseteq \text{Ass}\), and (4.4.2) is satisfied by \(\text{Sjor}\) and its preimage \(\text{Jor}\). When \(\text{char } K = 2\), \(\text{Sjor}\) is nothing more than \(\text{Lie}\). The proof that \(\text{Lie}\) satisfies (4.4.2) (in any characteristic) lies only slightly deeper, and we defer the argument until the end of the section. Finally, \(v^k\) is also an element of the symmetric part \(\text{Ass}^+_k\) of \(\text{Ass}_k\). It is then easy to see, by using the duality statement in (5.2.1), that for some \(w \in \text{Pol}_k\), \(wT\) is linearly independent, and (4.4.2) holds for \(\text{Pol}\). Notice that, for all these algebras (except \(\text{Lie}\)), \(n\) is a lower bound for \(k\) in (4.4.2).

Proof of (4.4.7). We firstly show that we may choose \(v \in \text{Ass}_1\) such that \(\{v, v^T\}\) is linearly independent for all \(T \in T\). Indeed, for any \(g \in G \setminus M\), there are at most \(r\) eigenspaces of \(g\) in \(\text{Ass}_1\), and by the definition of \(M\) they are all proper. Thus, there can be no more than \(r mn\) eigenspaces of elements of \(G \setminus M\). Because \(K\) is an infinite field, \(\text{Ass}_1\) is not the set-theoretic union of a finite collection of proper subspaces. Therefore, there exists \(v \in \text{Ass}_1\) such that \(\{v, v^g\}\) is linearly independent, for all \(g \in G \setminus M\), proving a little more than the opening assertion.

Fix this element \(v\). Write \(T = \{1, \tau_1, \ldots, \tau_n\}\). For \(1 \leq i \leq n\), there is a \(K\)-linear map \(\theta_i : \text{Ass}_1 \to \text{Ass}_1\) satisfying \(v\theta_i = v\).
$\left(\nu \tau_{i}\right) \theta_{i} = 0$. Taking $k \geq n$, identify $\text{Ass}_{k}$ with the tensor power $\text{Ass}_{1}^{\otimes k}$, and write $\phi = \theta_{1} \otimes \ldots \otimes \theta_{n} \otimes 1 \otimes \ldots$, where the last $(k-n)$ factors are the identity map on $\text{Ass}_{1}$. Then $\nu^{k} \phi = \nu^{k}$ and $\left[\nu^{k} \tau_{i}\right] \phi = 0$.

Hence $\nu^{k}$ is not in the linear span of $\{\nu \tau_{i} \mid 1 \leq i \leq n\}$. This implies $\nu^{k} \tau$ is linearly independent, as required. □

It remains to prove (4.4.2) for $\text{Lie}(X)$. Here, a small modification of the proof of (4.4.7) is required. Choose $\nu, \theta_{i} (1 \leq i \leq n)$ as defined there. In addition, let $w$ be an element of $\text{Ass}_{1}$ such that $\{\nu, w\}$ is linearly independent and let $\psi$ be $K$-linear on $\text{Ass}_{1}$ such that $w \psi = w$, $\nu \psi = 0$. Embedding $\text{Lie}(X)$ into $\text{Ass}(X)$ as usual, the bracket product $k-1\quad z = [w, \nu, \ldots, \nu]$ has the well-known expansion

$$z = \sum_{s=0}^{k-1} (-1)^{s} \left(\begin{array}{c} k-1 \\ s \end{array}\right) \nu^{s} w \nu^{k-1-s}.$$

Take $k \geq 2n+1$, identify $\text{Ass}_{k}$ with $\text{Ass}_{1}^{\otimes k}$ as before, and consider the linear map

$$\phi = \psi \otimes \theta_{1} \otimes \theta_{1} \otimes \theta_{2} \otimes \theta_{2} \otimes \ldots \otimes \theta_{n} \otimes \theta_{n} \otimes 1 \otimes \ldots.$$

This $\phi$ annihilates all terms in the above summation except the $s = 0$ term. Therefore $aw \neq 0$. However, for $1 \leq j \leq n$, $\left[z \tau_{j}\right] \phi = 0$ because in each term in the summation for $\left[z \tau_{j}\right] \phi$, at least one occurrence of $\nu \tau_{j}$ is acted on by $\theta_{j}$. Thus $z$ does not depend linearly on $\{\tau_{j} \mid 1 \leq j \leq n\}$, and (4.4.2) follows. The lower bound of $(2n+1)$ for $k$ is a small improvement on the Bryant, Kovács bound of $3n$.

It would be nice to have a theorem giving sufficient conditions for an extension algebra to contain a regular $KG$-module: (4.4.2) provides some
evidence that such a proposition should exist.

(4.5) The Action of the General Linear Lie Algebra

The general linear Lie algebra is the Lie algebra \( gl(r, R) \) of all \( r \times r \) matrices over the ring \( R \). We turn now to consider the natural action of \( gl(r, R) \) on \( A \).

The starting-point is a property of extension algebras analogous to the characterization (3.2.2) of weakly relatively free algebras. To state it, recall that a derivation of an \( R \)-algebra \( C \) is an \( R \)-homomorphism \( D : C \to C \) satisfying \((ac')D = (aD)c' + a(c'D)\) for all \( a, c' \in C \). For the moment, forget the restrictions on \( A \) introduced in the preamble to this chapter.

(4.5.1) Theorem. For an arbitrary extension algebra \((A, X)\), every restricted function \( X \to A \) extends to a unique derivation of \( A \).

It seems rather nice in (4.5.1) to have the non-relatively free algebras (such as Ext) included under the same roof as the relatively free ones. Apart from that, we make no claims to originality in the theorem. Before giving the proof, we wish to point out that the result can fail for weakly relatively free algebras that are not extension.

(4.5.2) Example. Consider \( A = F[2|A_L, C^*_L, N^*_3] \) over \( R = GF(2) \). A small elaboration of the calculation in (3.3.3) shows that

\[
A = [Rx \oplus Ry] \oplus [Rx^2 \oplus Rxy \oplus Ry^2] \oplus Rx^2y \oplus Rx^2y^2.
\]

Hence, \( A \) is even a graded relatively free algebra. However, the map \( x \mapsto x + y, y \mapsto y \) does not extend to a derivation of \( A \). Indeed, such a derivation \( D \) would satisfy

\[
0 = x^3D = (x+y)x^2 + x(x+y)x + x^2(x+y) = x^2y \neq 0,
\]
a contradiction. □
Proof of (4.5.1). The proof uses the well-established trick from Jacobson [2, IV.6]. Let \( D = R \cdot 1 \oplus R \cdot t \) be a free \( R \)-module with basis \( \{1, t\} \), and introduce a multiplication into \( D \) by treating \( 1 \) as a unity and setting \( t^2 = 0 \). Thus, \( D \) becomes an associative \( R \)-algebra. Let \( C \) be an arbitrary \( R \)-algebra and consider the tensor product \( C \otimes_R D \); this is the algebra of dual numbers on \( C \). Every element of \( C \otimes D \) has a unique representation in the form \( c \otimes 1 + c' \otimes t \) for \( c, c' \in C \); hence, we may drop the tensor signs and regard \( C \) as embedded in \( C \otimes D \). Let \( \pi \) be the natural projection \( C \otimes D \to C \) defined by \( (c + c't)\pi = c \). The trick in Jacobson is to observe that the following holds.

(4.5.3). There is a one-one correspondence between derivations \( D \) of \( C \) and those algebra homomorphisms \( s : C \to C \otimes D \) which satisfy \( s\pi = 1_C \).
This correspondence is given explicitly by \( D \leftrightarrow s \) if and only if \( as = c + (cD)t , \ c \in C \).

The proof of (4.5.3) is a straight-forward computation, and we omit the details.

It follows from (4.5.3) that a derivation on \( C \) is completely determined by its action on a generating set for \( C \). More to the point, consider the case when \( C \) is our extension algebra \( A \). The \( R \)-algebra \( A \otimes D \) is generated by \( XD \) (that is, the \( D \)-submodule of \( A \otimes D \) generated by \( X \)), and to prove (4.5.1), it is certainly enough to show every restricted function \( X : A \otimes D \) extends to a homomorphism \( A \to A \otimes D \). However, this is an immediate consequence of (3.5.2). □

Although it will not be used explicitly in this paper, the relatively free analogue of (4.5.1) is perhaps worth stating separately anyway.

(4.5.4) THEOREM. If the extension algebra \( (A, X) \) is relatively free, every function \( X : A \) extends to a unique derivation of \( A \). □
We now revert to the particular extension algebra \((A, X)\) described at
the beginning of the chapter. Let \(L\) be a Lie algebra which acts faithfully
on \(A_1\). An element \(l \in L\) determines a restricted function \(X \to A\), and
this extends to a unique derivation \(l^\sharp_d\) of \(A\). We see that
\[(rl + sm)^d = rl^d + sm^d\] for \(r, s \in R\) and \(l, m \in L\). Furthermore, writing
\([l, m]\) for the Lie product in \(L\), and \([l^d, m^d] = l^d m^d - m^d l^d\) for the
bracket product of the maps \(l^d, m^d\), we have \([l, m]^d = [l^d, m^d]\). Hence,
\(A\) is a faithful \(L\)-module. A simple induction gives the following
statement.

\((4.5.5).\) The action of \(l \in L\) on a degree \(k\) monomial
\(\beta(a_1, \ldots, a_k), a_i \in X\), is given by
\[\beta(a_1,\ldots,a_k)^d = \sum_{i=1}^{k} \beta(a_1,\ldots,a_i l,\ldots,a_k)\, .\]
In particular, \(A_k\) is an \(L\)-module. □

Consider now the case \(L = \text{sl}(r, R)\), the \(r \times r\) matrices of trace
zero. Shortly, we shall restrict to \(R\) a field, but there is one technical
lemma which we need (both here and in Chapter 5) which we state for \(R\)
arbitrary. For the rest of this section, and the whole of the next, write
\(X = \{x_1, \ldots, x_r\}\) and for \(\lambda \in F(X)\), \(\lambda = (\lambda_1, \ldots, \lambda_r)\) as usual. We let
\(L\) act on \(A_1\) precisely as
\[x_i^d = \sum_{j=1}^{r} \alpha_{ij} x_j^d, \quad 1 \leq i \leq r\, , \quad A = (\alpha_{ij})\, .\]
As in \((2.4)\), \(e_{ij}\) is the matrix with zeros everywhere except for a 1 in
position \((ij)\) \((i \neq j)\).

\((4.5.6).\) Let \(g_{ij}^d\) be the restriction of the derivation \((e_{ij})^d\) to
\(A_k\), and suppose \(\eta\) is a monomial in \(A_k\) of degree \(\lambda_i\) on \(x_i\). For
\( s \geq 0 \), let \( N^{(s)} = N^{(s)}(\eta) \) be the sum of all those monomials obtained from \( \eta \) by formally replacing \( s \) occurrences of \( x_i \) by \( x_j \). Then
\[
\eta g^{s}_{ij} = s! N^{(s)}.
\]
(Here, we conventionally put \( N^{(s)} = 0 \) if \( s > \lambda_i \).)

**Proof.** The argument is an exercise in combinatorics. By (4.5.5), the action of \( g_{ij} \) on a monomial \( \eta' \) gives a sum of monomials, each obtained from \( \eta' \) by replacing one occurrence of an \( x_i \) by an \( x_j \). This certainly shows \( \eta g_{ij} = N^{(1)} \), and we may assume \( \eta g^{s-1}_{ij} = (s-1)! N^{(s-1)} \) for \( s > 1 \).

Applying \( g_{ij} \), each monomial in \( N^{(s-1)} \) gives rise to a monomial in \( N^{(s)} \); conversely, a monomial in \( N^{(s)} \) is clearly obtained from exactly \( s \) monomials in \( N^{(s-1)} \). Thus \( \eta g^{s}_{ij} = s(s-1)! N^{(s)} \), as required. \( \square \)

We have an immediate application for (4.5.6). For the rest of the chapter, except where otherwise specified, we shall take \( R = K \) a field of characteristic \( p \neq 0 \), and we freely use the notation in (2.4).

(4.5.7). If \( p < \infty \), then \( A_k \) is a module for \( L \) as restricted Lie algebra.

**Proof.** Fix a monomial \( \eta \) of degree \( \lambda_i \) on \( x_i \), \( 1 \leq i \leq r \). From (4.5.5), we have, for \( 1 \leq j \leq r-1 \), \( \eta h_j = (\lambda_{j+1} - \lambda_j) \eta \), and certainly
\[ [h_j, h_d]^P = (h_d) \eta. \] Also, by (4.5.6), \[ [e_{ij}, h_d]^P = 0. \] This is all we need. \( \square \)

Our primary objective now is to examine the weights of \( A_k \) with respect to the abelian subalgebra \( H \) of \( L \). This calculation can be done for other algebras (for example, for the diagonal matrices in \( \text{gl}(r, K) \) it...
even proceeds more smoothly), but we prefer to consider the \( \text{sl} \)-case in keeping with our approach to Chapter 5.

We identify a linear functional \( \mu : H \to K \) with the sequence \( \{\mu(h_1), \ldots, \mu(h_{r-1})\} \). Given a monomial \( \eta \) in the strongly homogeneous component \( A^*_\lambda \) of \( A_k \), we have \( \eta h_j = (\lambda_{j+1} - \lambda_j)\eta \), \( 1 \leq j \leq r-1 \).

Therefore:

(4.5.8) THEOREM. The weights of \( A_k^* \) are the sequences

\[
((\lambda_2 - \lambda_1) \cdot 1, \ldots, (\lambda_r - \lambda_{r-1}) \cdot 1)
\]

determined by the (nonzero) strongly homogeneous components \( A^*_\lambda \). The weight spaces are sums of such components. Two components \( A^*_\lambda \) and \( A^*_\lambda^* \) belong to the same weight if and only if

\[
\lambda_i - \lambda_{i+1} \equiv \lambda^*_i - \lambda^*_{i+1} \pmod{p} \quad \text{for} \quad 1 \leq i \leq r-1.
\]

We can usually improve (4.5.8).

(4.5.9). If \( p \nmid r \), then the strongly homogeneous components \( A^*_\lambda \) and \( A^*_\lambda^* \) belong to the same weight if and only if \( \lambda_i \equiv \lambda^*_i \pmod{p} \), \( 1 \leq i \leq r \).

Proof. If \( A^*_\lambda, A^*_\lambda^* \) belong to the same weight, then

\[
\lambda_i - \lambda_{i+1} \equiv \lambda^*_i - \lambda^*_{i+1} \pmod{p}, \quad 1 \leq i \leq r-1.
\]

These congruences give

\[
\lambda_i - \lambda_r \equiv \lambda^*_i - \lambda^*_r, \quad 2 \leq i \leq r-1.
\]

Adding, and using

\[
k = \lambda_1 + \ldots + \lambda_r = \lambda^*_1 + \ldots + \lambda^*_r,
\]

we obtain \( r\lambda_1 \equiv r\lambda^*_1 \). Thus, when

\( p \nmid r \), \( \lambda_1 \equiv \lambda^*_1 \) and the rest follows. \( \square \)

There is an interesting consequence of (4.5.8) and (4.5.9). If \( W \) is an \( L \)-submodule of \( A_k^* \), then \( W \) has a decomposition into weight spaces, namely, the (nonzero) \( W \cap [A_k^*]_{\mu} \), where \( [A_k^*]_{\mu} \) is the weight space of \( A_k^* \) determined by the weight \( \mu \). Therefore:

(4.5.10). If \( p \nmid r \), any submodule of \( A_k^* \) has a basis of elements
which are strongly homogeneous (mod p). □

We make some miscellaneous remarks regarding the above calculations. Firstly, when \( p = \infty \), the weights determine the composition structure of \( A^*_k \). Less familiarly, this remains true when \( p \mid r \) and \( k < p < \infty \): the procedure is the same as in (5.3). However, we will not stop to detail specific examples here. Also in (5.3), we strengthen and generalize (4.5.8)-(4.5.10) for integral extension algebras. Finally, (4.5.10) fails when \( p \mid r \) (see (4.6.6)).

To conclude this section, we consider the effect of changing the base ring. We do not insist that \( R \) be a field here. Let \( S \) be an extension ring of \( R \) under a homomorphism \( \rho : R \to S \). If \( L \) is again an arbitrary Lie algebra acting faithfully on \( A^*_1 \), then by (3.5.2), \( S \otimes L \) is a Lie algebra. This algebra has an induced action on \( S \otimes A^*_k \), namely,

\[
(s \otimes a)(s' \otimes l) = ss' \otimes al \quad \text{for} \quad s, s' \in S, \quad a \in A^*_k, \quad l \in L.
\]

On the other hand, there is the derivation action on \( S \otimes A^*_k \) afforded by the induced action of \( S \otimes L \) on \( S \otimes A^*_1 \). As in the group case, these two actions are the same, and we no longer distinguish them. We also note in passing that \( S \otimes L \) is easily recognized in special cases:

\[
S \otimes \mathfrak{gl}(r, R) = \mathfrak{gl}(r, S) \quad \text{and} \quad S \otimes \mathfrak{sl}(r, R) = \mathfrak{sl}(r, S).
\]

(4.6) The Exterior and Polynomial Algebras (II)

We continue in this section the discussion begun in (4.2). The emphasis now is on the components \( \text{Ext}^*_k \) and \( \text{Pol}_k \) as modules for \( L = \mathfrak{sl}(r, K) \). The notation is the same as in the previous section (4.5).

As in the group case (\textit{cf.} (4.2.3)), the exterior algebra is the easiest to analyze; this result hardly needs proof here.
(4.6.1). For $1 \leq k \leq r$, $\text{Ext}_k$ is an irreducible $L$-module.

We now restrict ourselves to $p < \infty$, and consider the polynomial algebra. We shall need repeatedly the explicit action of $L$ on monomials in $\text{Pol}_k$. This is, by (4.5.5), given as follows.

(4.6.2). Take $0 < i, j \leq r$ with $i \neq j$, and $0 \leq a + b \leq k$ with $0 \neq a = sp + t, 0 \leq t < p$. Then $e_{ij}$ acts on a monomial

$$\ldots x_i^a \ldots x_j^b \ldots$$

as

$$\left(\ldots x_i^a \ldots x_j^b \ldots \right) e_{ij} = t \left(\ldots x_i^{a+1} \ldots x_j^{b+1} \ldots \right)$$

where the unspecified parts of the monomial are unchanged.

We use the notation $\text{Pol}_0$ for the one-dimensional trivial $L$-module (on which $L$ acts as zero). [Of course, if we had allowed our algebras to have a unity in Chapter 3, there would be no need to separately define $\text{Pol}_0$ here.)

Consider firstly the rank 2 case. Here, the information is quite definitive. Let $\sigma \text{Pol}_k$ be the socle of $\text{Pol}_k$, that is, the largest completely reducible submodule of $\text{Pol}_k$. Put $k = \lambda p + \mu$, $0 \leq \mu < p$, $\lambda \geq 0$.

(4.6.3) THEOREM. When $r = 2$,

(1) $\sigma \text{Pol}_k \cong [\text{Pol}_\mu]^{(\lambda+1)}$ and $\text{Pol}_k/\sigma \text{Pol}_k \cong [\text{Pol}_{p-2}]^{\lambda}$,

(2) $\text{Pol}_k$ is indecomposable, except when $\lambda > 0$ and $\mu = p - 1$, in which case it is completely reducible, a sum of copies of $\text{Pol}_{p-1}$.

Notice that when $\mu = (p-1)$ in (1), it is understood that $\text{Pol}_{p-1} = \{0\}$, so that the second part of (2) is a particular case of (1).

Also, recall that, for $\delta < p$, $\text{Pol}_\delta$ is irreducible (for example, (5.4.2)).
We prove (1) first. To simplify notation, we specify monomials by writing their degrees on $x_1$ explicitly, leaving the degrees on $x_2$ understood. (However, these degrees are relevant in some of the assertions.)

Given $0 \leq \lambda' \leq \lambda$, define $X(\lambda')$ to be the subspace of $\text{Pol}_k^\mu$ spanned by $\{x_1^{\lambda'+\mu'}x_2^\mu : 0 \leq \mu' \leq \mu\}$. Visibly, $X(\lambda')$ is an $L$-submodule of $\text{Pol}_k^\mu$, isomorphic to $\text{Pol}_\mu^\mu$. The sum $X$ of these $X(\lambda')$ is direct, and contained in $\sigma \text{Pol}_k^\mu$. Suppose conversely, $Z$ is an irreducible submodule of $\text{Pol}_k^\mu$ not contained in $X$. Then, by definition of $X$, there is an element $z = x_1^{\lambda'+\mu'}x_2^\mu + \ldots$ in $Z$, where $0 \leq \lambda' < \lambda$, $\mu < \mu' \leq p-1$, and the unspecified part of $z$ consists of a linear combination of monomials, each with some degree $\lambda''p + \mu'' \neq \lambda'p + \mu'$, $\mu'' \leq \mu'$, on $x_1$.

Then $z x_{12}^{\mu'-\mu}$ is a nonzero element of $Z \cap X$, a contradiction. Hence $\sigma \text{Pol}_k^\mu = X$. Finally, the quotient $\text{Pol}_k^\mu/X$ is spanned by $\{x_1^{\lambda'+\mu'}x_2^\mu + x : 0 \leq \lambda' < \lambda, \mu' > \mu\}$. The subspace spanned by those elements with a given $\lambda'$ is a submodule isomorphic to $\text{Pol}_{p-\mu-2}^\mu$. Thus, $\text{Pol}_k^\mu/X$ is a direct sum of copies of $\text{Pol}_{p-\mu-2}^\mu$. This completes the proof of (1).

Consider now the action of the matrix $e_{12}$. Let $G = \langle g \rangle$ be a cyclic group of order $p$. By (4.5.7), $\text{Pol}_k^\mu$ is a module for $G$, where $zg = z + ze_{12}$, $z \in \text{Pol}_k^\mu$.

(4.6.4). Take $0 \leq \mu < p-1$. As $G$-module, $\text{Pol}_k^\mu \cong [\text{Pol}_{p-1}^\mu]^G \oplus \text{Pol}_\mu^\mu$. Furthermore, the modules $\text{Pol}_\nu^\mu$, $0 \leq \nu \leq p-1$, are a full set of pairwise non-isomorphic indecomposables for $G$, with $\text{Pol}_{p-1}^\mu$ a regular $KG$-module.

Proof. We may decompose $\text{Pol}_k^\mu$ as $G$-module thus
\[
\text{Pol}_{k} = \left\langle x_1^{\lambda p+j}x_2^\mu \mid 0 \leq j \leq \mu \right\rangle_k \oplus \left\langle \sum_{i=0}^{\lambda-1} \left\langle x_1^{i \lambda p+j}x_2^\mu \mid 0 \leq j \leq p-1 \right\rangle_k \right\rangle
\]

and this is of the form specified in the first part of (4.6.4).

Now, it is well-known that there are \( p \) distinct isomorphism types of indecomposable \( G \)-modules. They are distinguished (and determined) by the Jordan forms of the linear maps representing \( g \). In particular, \( g \) has one-dimensional fixed-point space in each. However, the matrix of \( g \) on \( \text{Pol}_\nu, \ 0 \leq \nu \leq p-1 \), takes the form

\[
\begin{pmatrix}
1 & \nu \\
1 & \nu - 1 \\
& \ddots \\
& & 1 \\
& & & 1
\end{pmatrix}
\]

and thus \((g-1)\) has rank \( \nu \). Therefore, \( g \) has one-dimensional fixed-point space in \( \text{Pol}_\nu \), and perforce, the \( \text{Pol}_\nu \) (distinguished by their dimensions) are a full set of \( G \)-indecomposables. In particular, \( \text{Pol}_{p-1} \), the only one of dimension \( p \), must be a regular.

(The argument using fixed-point spaces mimics the same approach used in Glover \([1, (3.2)]\) for the \( p \)-cycle \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \).) \( \square \)

We recall one further fact regarding the indecomposables for \( G \): each is uniserial. Indeed, for \( 0 \leq \nu \leq p-1 \), \( \text{Pol}_\nu \) has the unique chain of submodules

\[
\text{Pol}_\nu > \text{Pol}_\nu(g-1) > \ldots > \text{Pol}_\nu(g-1)^\nu > \{0\}
\]

where \( \text{Pol}_\nu(g-1)^i \cong \text{Pol}_{\nu-i} \), \( 0 \leq i \leq \nu \).

(4.6.5). If \( \nu < p-1 \), then no \( L \)-submodule of \( \text{Pol}_k \) is injective as \( G \)-module.

The indecomposability statement in (4.6.3) follows immediately from
(4.6.5). For suppose $\text{Pol}_k^L = A \oplus B$ with $A, B$ nonzero $L$-submodules. By (4.6.4), one of $A$ or $B$ is a direct sum of $KG$-regulars, hence injective, contradicting (4.6.5).

**Proof of (4.6.5).** Assume $A$ is an $L$-submodule of $\text{Pol}_k^L$, injective as $G$-module. We firstly derive a contradiction in the case when $p > 2$.

By (4.5.10), there is a nonzero element $z$ of $A$ in the form

$$z = x_1^{i p + \sigma} x_2^* + \sum_{j > i} \alpha_j x_1^{j p + \sigma} x_2^*, \quad \alpha_j \in K, \ 0 \leq \sigma \leq p - 1.$$ 

Choose $z$ so that, in lexicographic order, the pair $(i, \sigma)$ is as large as possible. Write $S$ for the subspace of $\text{Pol}_k^L$ spanned by all

$$x_1^{i p + \tau} x_2^* + \sum_{j > i} \alpha_j x_1^{j p + \tau} x_2^*, \quad 0 \leq \tau \leq \sigma.$$ 

Then $S$ is a $G$-submodule of $A$, isomorphic to $\text{Pol}_G^L$. By assumption on $A$, there is an injective hull of $S$ also contained in $A$. Now assume that $\sigma < p - 1$. From the remark just before (4.6.5), there exists an element $z_1 \in A$ such that $z_1(g - 1)$ is a scalar multiple of $z$. We may suppose

$$z_1 \equiv x_1^{i p + \sigma + 1} x_2^* + \ldots \pmod{\Gamma}$$

where $\Gamma$ is the fixed-point space of $g$. Applying (4.5.10) again, $A$ must contain

$$z'_1 = x_1^{i p + \sigma + 1} x_2^* + \ldots$$

This contradicts the choice of $(i, \sigma)$. Therefore, we may not assume $\sigma < p - 1$, and we have $z = x_1^{i p + p - 1} x_2^* + \ldots$. Now, applying $e_{21}$ to $z$, a nonzero element of $A$ is obtained, and its form also contradicts the choice of $(i, \sigma)$. This completes the proof when $p > 2$.

When $p = 2$, the argument is degenerately simple. Since $A$ is injective as $G$-module, it certainly contains an element
\[ z = x_1^{2i+1} x_2^* + \sum_{j \neq i} \alpha_j x_1^{2j+1} x_2^* + z_0 \]

where \( z_0 \in \Gamma \), the fixed-point space of \( g \). Choose such a \( z \) with \( i \) largest possible. Apply \( e_{21} \). Because \( k \) is now even, \( e_{21} \) annihilates \( z_0 \), and then, using the fact that \( A \) is \( G \)-injective, we contradict the choice of \( i \), much as above. \( \square \)

The argument in (4.6.5) uses the basis property in. (4.5.10) (although its use can be avoided). We note that the property fails when \( p | r \).

(4.6.6) EXAMPLE. Take \( r = p = 2 \). In \( \text{Pol}_k \), the following \( L \)-submodule will do:

\[ \left\langle x_1^{4} x_2^{2}, x_1^{2} x_2^{4}, x_1^{3} x_2^{2}, x_1^{2} x_2^{3}, x_1^{2} x_2^{2}, x_1^{2} x_2^{2} \right\rangle_k \]. \( \square \)

It seems very likely that when the rank \( r \) exceeds 2, \( \text{Pol}_k \) is indecomposable. Ad hoc arguments show this for all small ranks and degrees (including \( p | r \) and certain \( k > (p-1)r \)). However, the best general statement we have been able to make is the following.

(4.6.7). Take \( r > 2 \). If \( p | r \) and \( k \leq (p-1)r \), then \( \text{Pol}_k \) is indecomposable.

Proof. The assumption on \( k \) ensures that there exists an element \( z = x_1^{\lambda_1} \ldots x_r^{\lambda_r} \) in \( \text{Pol}_k \) with each \( \lambda_i \leq p-1 \). The weight space corresponding to the weight \( (\lambda_2 - \lambda_1, \ldots, \lambda_r - \lambda_{r-1}) \) is then one-dimensional, with \( z \) as basis. So we need only prove \( z \) generates \( \text{Pol}_k \) as \( L \)-module.

Let \( T \) be the module generated by \( z \). We show that a monomial \( x_1^{\mu_1} \ldots x_r^{\mu_r} \) is an element of \( T \). Indeed, there is nothing to prove if \( (\mu_1, \ldots, \mu_r) = (\lambda_1, \ldots, \lambda_r) \). Otherwise some \( \mu_i > \lambda_i \). Repeated applications of the \( e_{ji} \) (\( j \neq i \)) to \( z \) shows that \( T \) contains an element
\[ x_1 \cdots x_\ell \cdots x_r \] where each \( \nu_j \leq (p-1) \). If
\[ (\nu_1, \ldots, \nu_\ell, \ldots, \nu_r) = (\mu_1, \ldots, \mu_r) \] we are done. Otherwise, some
\[ \mu_j > \nu_j, \] and we continue the argument with \( x_1 \cdots x_\ell \cdots x_r \) replacing \( z \) (keeping the exponent on \( x_\ell \) fixed). Eventually, we arrive at
\[ x_1 \cdots x_r \in T. \] \( \Box \)
CHAPTER 5

CHEVALLEY REDUCTION IN INTEGRAL EXTENSION ALGEBRAS

We examine in this chapter some aspects of Chevalley reduction in integral extension algebras. The chapter opens with a description of the reduction process itself in (5.1). By way of illustration, we use this immediately in (5.2) to continue the study of $\text{Ext}$ and $\text{Pol}$; the discussion here (as in so much of this paper) is by no means completely original, but the results are intrinsically very interesting and will serve us well in later sections. In (5.3), we use Humphreys' algebras for the first time, calculating the weights and weight spaces of the homogeneous components. We apply the discussion to $\text{Pol}$ in (5.4). The next two sections are linked. Working in rank 2, we state a conjecture in (5.5) which has a direct bearing on computing the structure of $\text{Ass}$, $\text{Lie}$ and $\text{SJor}$. In (5.6), we prove a case of the conjecture.

Notation throughout this chapter will be progressive, beginning with the constructions in (5.1). We introduce one blanket convention immediately. Suppose there is given a sequence of modules $M_1, M_2, \ldots$ for an algebra. Then we take $M_0$ to be the one-dimensional trivial module for the algebra, and define $M_k$ to be the zero module for any negative integer $k$.

(5.1) Description of the Reduction Process

We begin with a torsion-free extension algebra of finite rank $r$ (at least 2) over the integers $\mathbb{Z}$, denoting it $(A^{(\mathbb{Z})}, X)$. Put $X = \{x_1, \ldots, x_p\}$ as usual. For any field $K$, write $A^{(K)} = K \otimes A^{(\mathbb{Z})}$, an
integral extension algebra over $K$. Also, put $L^{(Z)} = \mathfrak{sl}(r, Z)$,
$L^{(C)} = \mathfrak{sl}(r, C)$. Now, for $k \geq 1$, $A^{(Z)}_k$ is an $L^{(Z)}$-module under the
derivation action induced from

$$x_i A = \sum_{j=1}^{r} \alpha_{ij} x_j, \quad 1 \leq i \leq r, \quad A = (\alpha_{ij}) \in L^{(Z)}.$$ 

In addition, $A^{(C)}_k$ is a module for $L^{(C)}$ by restricting the derivation
action of $L^{(C)}$, and we may then embed $A^{(Z)}_k$ into $A^{(C)}_k$ as $L^{(Z)}$-modules.

Chevalley reduction begins with the following statement.

(5.1.1). For $k \geq 1$, $A^{(Z)}_k$ is an admissible lattice in $A^{(C)}_k$.

Proof. We must show $A^{(Z)}_k$ is invariant under all $s \mathfrak{e}_{ij}/s!$, $s \geq 1$,
$i \neq j$. However, for any monomial $\eta$ in $A^{(Z)}_k$, (4.5.6) gives

$$\eta e_{ij}^s = s! N_{ij}^{(s)}(\eta),$$

and the result is immediate. □

Not surprisingly, we cannot expect $A^{(Z)}_k$ in (5.1.1) to decompose as a
direct sum of admissible lattices afforded by the irreducible direct
summands of $A^{(C)}_k$. A striking instance of how this fails will arise in
(5.5).

Fix a field $K$ of characteristic $p \leq \infty$. We shall write $A^{(K)}_k$ for
$A^{(Z)}_k$, and drop the tensor signs throughout. We keep parity with (4.1.1)
and (4.5.5) by stating explicitly the action of Kostant's algebra $U$ on
$A^{(K)}_k$. In practice, this formula is very easy to apply; one need only recall
the verbal description of $N_{ij}^{(s)}(\eta)$ given in (4.5.6).

(5.1.2). For $1 \leq i \neq j \leq r$ and $s \geq 0$, $f_{ij}^{(s)}$ acts on a monomial
\[ \eta \text{ in } A_k \text{ by} \]
\[ n^{(s)}_{i,j} = N^{(s)}_{i,j}(\eta). \]

There is an alternative description of the action of \( U \) which we find intuitively satisfying, and we therefore give it also.

(5.1.3). For \( a, b \in A, i \neq j \) and \( s \geq 1 \),
\[ (ab)f^{(s)}_{i,j} = \sum_{t=0}^{s} \left[ a_{i,j}(t) \right] \left[ b_{i,j}(s-t) \right]. \]

Proof. It is enough to prove this for monomials \( a, b \). Then \( ab \) is a monomial (possibly zero) and we seek to prove
\[ N^{(s)}_{i,j}(ab) = \sum_{t=0}^{s} \left[ N_{i,j}(t) (a) \right] \left[ N_{i,j}(s-t) (b) \right]. \]

Using the definition of the \( N \)'s, this is quite obvious. \( \square \)

For example, given \( k, l \geq 1 \), there exists a natural linear transformation \( A_k \otimes A_l \to A_{k+l} \) obtained by forgetting the tensor sign:
\( a \otimes b \mapsto ab \). By (2.2.1) and (5.1.3), we see (directly) that this map is a \( U \)-homomorphism.

The action of \( U \) restricts to an action of the classical algebra \( L = \mathfrak{sl}(r, K) \), and we have seen at the end of (4.5) that this coincides with the usual action by derivations. Furthermore, the action of the universal group \( G = \text{SL}(r, K) \) afforded by \( U \) coincides with the automorphism action of (4.1). We check this. For \( i \neq j, a \in K \), and \( \eta \) a degree \( k \) monomial, the automorphism of \( A^{(K)} \) afforded by \( E_{i,j}(a) \) replaces each occurrence of \( x_i \) in \( \eta \) by \( x_i + ax_j \). Expanding \( \eta E_{i,j}(a) \) bilinearly, we obtain
\[ \eta E_{i,j}(a) = \eta + aN_{i,j}^{(1)} + a^2 N_{i,j}^{(2)} + \ldots \]
\[ = \eta + a\eta_{i,j}^{(1)} + a^2 \eta_{i,j}^{(2)} + \ldots \]
as required. Thus, for integral extension algebras, the module structures of Chapter 4 are included in the present context.

(5.2) The Exterior and Polynomial Algebras (III)

There are some miscellaneous results regarding $\text{Ext}$ and $\text{Pol}$ which we develop in this section. They will be of use later, but for the moment, we may simply regard them as particular instances of the action of $U$ described in generality in the previous section.

The first result is a duality theorem. This may be viewed as a generalization of (4.2.1). It has appeared in the literature in various contexts, but we do not have a convenient reference, and in any case, it does not seem to be as well-known as (4.2.1) itself; we therefore append a proof. Notice that, while we state it using $U$-action, the result (and its proof) holds for any group or Lie algebra acting on $\text{Ass}$, and in (4.4) we have already used it in that form.

Write $V = \text{Ass}$ and for $k \geq 1$, let $V_k$ be the $U$-module $[V^*]^\otimes k$. The symmetric and skew-symmetric parts $V^+_k$ of $V_k$ are $U$-submodules.

(5.2.1) THEOREM. For $k \geq 1$, $V^+_k$ are respectively contragredient to $\text{Pol}_k$, $\text{Ext}_k$.

Proof. It will be enough to give the argument for $V^+_k$ and $\text{Pol}_k$, the other case being similar.

Let $\phi$ be the algebra surjection $\text{Ass} \rightarrow \text{Pol}$ induced by the identity map on $X$. This $\phi$ restricts to a $U$-homomorphism $\text{Ass}_k \rightarrow \text{Pol}_k$, again denoted by $\phi$ and also onto. Taking contragredients, there is an embedding $\phi^* : \text{Pol}_k^* \rightarrow \text{Ass}_k^*$. 
Now, the pair \((\text{Ass}_k, V_k)\) is a dual pair of \(U\)-modules via the isomorphisms \(\text{Ass}_k \cong V_k^* \) and \(V_k = [V^*] \cong [V_k^*]^*\). Write \(\langle \ , \ \rangle\) for the scalar product in the pair. We seek to show that the isomorphism \(\text{Ass}_k^* \cong V_k\) restricts to an isomorphism \(\text{Pol}_k^* \cong V_k^+\). However, \(\text{Pol}_k^*\) is just the annihilator of \(\ker \varphi\) in the dual pair \((\text{Ass}_k, \text{Ass}_k^*)\). Because \(\text{Pol}_k^*\) and \(V_k^+\) have the same dimension, we need only prove \(\langle \ker \varphi, V_k^+ \rangle = 0\).

To see this last assertion, we show firstly that \(\langle a \sigma, \theta \rangle = \langle a, \theta \sigma^{-1} \rangle\) for \(a \in \text{Ass}_k\), \(\theta \in V_k\) and \(\sigma \in S_k\), the symmetric group. Indeed, for any \(a_1, \ldots, a_k \in V\), \(\theta_1, \ldots, \theta_k \in V^*\), we have

\[
\langle [a_1 \ldots a_k] \sigma, \theta_1 \otimes \ldots \otimes \theta_k \rangle = \langle a_{1\sigma^{-1}} \ldots a_k \sigma^{-1}, \theta_1 \otimes \ldots \otimes \theta_k \rangle
\]

\[
= \langle a_{1\sigma^{-1}} \theta_1 \ldots a_k \sigma^{-1} \theta_k \rangle
\]

\[
= \langle a_1 \theta_1 \sigma \ldots a_k \theta_k \rangle
\]

\[
= \langle a_1 \ldots a_k, (\theta_1 \otimes \ldots \otimes \theta_k) \sigma^{-1} \rangle,
\]

as claimed. Now, \(\ker \varphi\) is spanned by all \(a - a \sigma\) for \(a \in \text{Ass}_k\), \(\sigma \in S_k\). So we have, for any \(\theta \in V_k^+\),

\[
\langle a - a \sigma, \theta \rangle = \langle a, \theta \rangle - \langle a \sigma, \theta \rangle = \langle a, \theta \rangle - \langle a, \theta \sigma^{-1} \rangle = 0.
\]

Thus, \(\langle \ker \varphi, V_k^+ \rangle = 0\), and the proof of (5.2.1) is complete. \(\Box\)

Of particular interest in (5.2.1) is the rank 2 case: we do a quick analysis. Taking \(r = 2\), consider \(V = \text{Ass}_1\) as module for the full Lie algebra \(\mathfrak{gl}(2, K)\). For \(n\) an integer, let \(T^{(n)}\) be the one-dimensional module on which a matrix \(A\) acts as \(n(\text{tr} A)\). Write \(T = T^{(1)}\). Thus, when \(n\) is positive, \(T^{(n)}\) is the tensor power \(T^\otimes n\), and \(T^{(n)}, T^{(-n)}\) are mutually contragredient. Because we always have
\[
(\text{tr } A)I - A^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} A \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

there is a \( \mathfrak{gl}(2, K) \)-isomorphism \( T \otimes V^* \cong V \). Now,
\[
V_k \cong \left[ T^{(-1)} \otimes V \right] \otimes_k \cong T^{(-k)} \otimes \text{Ass}_k,
\]
and this restricts to an isomorphism \( V^+_k \cong T^{(-k)} \otimes \text{Ass}^+_k \). By (5.2.1), \( \text{Ass}^+_k \cong T^{(k)} \otimes \text{Pol}^+_k \). Hence, \( \text{Ass}^+_k \) and \( \text{Pol}^+_k \) are contragredient precisely when the characteristic \( p \) of \( K \) divides \( k \).

Now restrict to \( \mathfrak{sl}(2, K) \). Then \( T^{(k)} \) becomes a trivial module. Indeed, \( f_{i,j}^{(s)} \) annihilates \( V \) and \( V^* \) whenever \( s > 1 \), so we even have \( V \cong V^* \) as \( U \)-modules. So (5.2.1) gives the following proposition.

(5.2.2). When \( r = 2 \), \( \text{Ass}^+_k \) is the contragredient of \( \text{Pol}^+_k \) as \( U \)-modules.

The discussion above (and its analogue for groups) shows that (5.2.2) does not hold for any group or Lie algebra acting on \( \text{Ass}^+_1 \), in sharp contrast to (5.2.1). Furthermore, the matrix equation on which it was based is a decidedly rank 2 phenomenon, and we cannot expect (5.2.2) to hold when \( r > 2 \). Finally, the analogue of (5.2.2) for \( \text{Ext} \) is, of course, largely trivial.

The final proposition which we wish to develop in this section also relates to the structure of the polynomial algebra in rank 2. In its original form, it was proved for the group \( \text{GL}(2, p) \) with \( K = \text{GF}(p) \) in Glover [1, §5]. Glover attributes part of the result to Wall (unpublished). Our argument reflects the nature of Chevalley reduction, but the underlying functions are the same as in Glover.

(5.2.3) THEOREM. Take \( r = 2 \) and fix \( k, l \geq 1 \).

(1) There is an exact sequence of \( U \)-modules and \( U \)-homomorphisms
(2) When $p \nmid k + 1$, the sequence

$$0 \to \text{Pol}_{k-1} \otimes \text{Pol}_{l-1} \xrightarrow{\theta_{k,l}^{k-1,k}} \text{Pol}_k \otimes \text{Pol}_l \xrightarrow{\varphi_{k,l}^{k,l}} \text{Pol}_{k+l} \to 0.$$ 

Proof of (5.2.3) (1). We firstly prove this when $K = \mathbb{C}$, in which case only the action of $L = \mathfrak{sl}(2, \mathbb{C})$ need be considered.

The map $\varphi : \text{Pol}_k \otimes \text{Pol}_l \to \text{Pol}_{k+l}$ is just the function which forgets the tensor sign. This is clearly onto, and (cf. after (5.1.3)) it admits $L$. Now define $\theta : \text{Pol}_{k-1} \otimes \text{Pol}_{l-1} \to \text{Pol}_k \otimes \text{Pol}_l$ by

$$(u \otimes v)\theta = ux_1 \otimes vx_2 - ux_2 \otimes vx_1, \quad u \in \text{Pol}_{k-1}, \; v \in \text{Pol}_{l-1}.$$ 

This is a well-defined linear map. To see that it admits $L$, we consider the action of $e_{12}$, the case of $e_{21}$ being similar. We have

$$[(u \otimes v)\theta]_{e_{12}} = [ux_1 \otimes vx_2 - ux_2 \otimes vx_1]_{e_{12}}$$

$$= (ue_{12})x_1 \otimes vx_2 + ux_2 \otimes vx_1 + (ve_{12})x_1 - (ue_{12})x_2 \otimes vx_2$$

$$= [(ue_{12})x_1 \otimes vx_2 - (ue_{12})x_2 \otimes vx_1]$$

$$+ [ux_1 \otimes (ve_{12})x_2 - ux_2 \otimes (ve_{12})x_1]$$

$$= [ue_{12} \otimes v] + [u \otimes ve_{12}]$$

$$= [(u \otimes v)e_{12}] \theta$$

as claimed.

Next, we show $\theta$ is injective. Indeed, take $z \in \ker \theta$, writing it uniquely in the form $z = \sum_{i=0}^{k-1} x_1^{i}x_2^{k-1-i} \otimes v_i$, $v_i \in \text{Pol}_{l-1}$. Then
This representation for \( z_0 \) is unique also, and we derive successively

\[ v_0 = v_1 = \ldots = 0 \]

as required.

Finally, it is immediate that \( \Theta \Phi = 0 \), and by a dimension count,

\[ \text{im} \Theta = \ker \Phi \].

Thus (5.2.3) (1) is proved when \( K = \mathbb{C} \).

To prove (5.2.3) (1) in general, observe that the complex sequence restricts to one of admissible lattices

\[ 0 \to \text{Pol}_k^Z \otimes \text{Pol}_{k-1}^Z \xrightarrow{\Theta} \text{Pol}_k^Z \otimes \text{Pol}_{l-1}^Z \xrightarrow{\varphi} \text{Pol}_k^Z \otimes \text{Pol}_l^Z \to 0 \]

where we have kept the same symbols for the restrictions of \( \Theta \) and \( \varphi \). Certainly \( \Theta \) is injective, \( \varphi \) is surjective and \( \text{im} \Theta \subseteq \ker \varphi \). Tensoring with an arbitrary field \( K \), we obtain (by a dimension count, as in the case \( K = \mathbb{C} \)), the required exact sequence. (Notice that we do not need to prove that the integral sequence is exact.) \( \square \)

**Proof of (5.2.3) (2).** Again take the case \( K = \mathbb{C} \) first. Consider the map \( \delta : \text{Pol}_{k+1} \to \text{Pol}_k \otimes \text{Pol}_1 \) defined by

\[ \delta : x_1 x_2 \mapsto i x_1 x_2 \otimes x_1 + (k+1-i) x_1 x_2 \otimes x_2, \quad 0 \leq i \leq k+1 \]

(with the natural interpretation when \( i = 0 \) or \( k+1 \)). We show that \( \delta \) admits \( L \), and as in (5.2.3) (1), the action of \( e_{12} \) will be enough. We have

\[
\left[ \begin{array}{c}
ix_1 x_2 \\
ix_1 x_2
\end{array} \right] e_{12} \delta = 
\left[ \begin{array}{c}
i x_1 x_2 \\
i x_1 x_2
\end{array} \right] \delta = 
\left[ \begin{array}{c}
i x_1 x_2 \\
i x_1 x_2
\end{array} \right] \delta = 
\left[ \begin{array}{c}
i x_1 x_2 \\
i x_1 x_2
\end{array} \right] \delta
\]

However
\[
\begin{bmatrix}
i x_1^{i+1-k} x_2^1
\end{bmatrix} = \begin{bmatrix}
i x_1^{i-1-k} x_2^1 + (k+1-i) x_1^i x_2^1
\end{bmatrix} e_{12}
\]

\[
= i(i-1) x_1^{i-2} x_2^{k-i+2} \otimes x_1^1 + i x_1^{i-1} x_2^{k-i+1} \otimes x_2^1
\]

\[
+ i(k+1-i) x_1^{i-1} x_2^{k-i+1} \otimes x_2^2
\]

\[
= \begin{bmatrix}
i x_1^{i+1-k} x_2^1
\end{bmatrix} e_{12} \delta
\]

as required.

Now, the composite \( \delta \circ (k+1) \cdot 1 \), where \( 1 \) is the identity map on \( \text{Pol}_{k+1} \). Furthermore, \( \delta \) restricts to a \( U^{(Z)} \)-homomorphism

\[
\text{Pol}_{k+1}^{(Z)} \rightarrow \text{Pol}_{k}^{(Z)} \otimes \text{Pol}_{l}^{(Z)}
\]

Tensoring with a field \( K \) whose characteristic \( p \) does not divide \( (k+1) \), we see that (5.2.3) (2) is proved. \( \square \)

We make a few remarks regarding (5.2.3). Firstly, the map \( \delta \) in the proof of (2) is closely related to differentiation: indeed, that \( \delta \) admits \( L \) is essentially an expression of the product rule for differentiation (see Glover [1, Proof of (5.2)]). Secondly, take \( K = \mathbb{C} \) once again. It is well-known that the modules \( \text{Pol}_k^l \), \( k \geq 0 \), are a full set of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \)-modules. Also, (5.2.3) gives, by a simple induction, \( \text{Pol}_k^l \otimes \text{Pol}_k^l \cong \text{Pol}_{k+l}^l \oplus \text{Pol}_{k+l-2}^l \oplus \ldots \oplus \text{Pol}_{k-l}^l \) for \( k \geq l \). This expression, then, is the Clebsch-Gordon Formula (see, for example, Humphreys [1, Exercises 7.4 and 22.7]).

(5.3) The Components as Modules for Humphreys' Algebras

In this section, we treat \( A_k \) as module for a Humphreys' algebra. The main object of study is the decomposition of \( A_k \) into weight spaces, this being the promised generalization of (4.5.8). We assume for the rest of this
paper that the characteristic $p$ is finite.

Choose a positive integer $\gamma$ such that $k < p^\gamma$. We may regard $A_k$ as $U_\gamma$-module by restricting the action of $U$. Furthermore, by (5.1.2), $A_k^{(e)} = \{0\}$ for $s > k$. In general, this does not imply $A_k$ is annihilated by $U \setminus U_\gamma$, but in view of (2.3.1), only the ideal of $U$ generated by all $h_i^{(n)}$, $n \geq p^\gamma$, can have nonzero action, and we shall see the essential effect of that action shortly. Here, we remark that there is no real loss of structure in working over $U_\gamma$; the advantage in doing so is that $U_\gamma$ is a well-behaved, finite-dimensional algebra.

(5.3.1) EXAMPLE. Let us say a $U$-module $M$ has level $l$ if $l$ is the largest non-negative integer for which $M_{i,j}^{(l)} \neq \{0\}$ for some $i \neq j$. Extension algebras exhibit a variety of levels. Thus, in each of $F_k$, $Ass_k$, $Pol_k$ and $Sjor_k$ (the last for $p \neq 2$), the (left-normed) monomial $x^k$ is non-zero for any $x \in X$. We have $x_{12}^{k-1} = x_2^k$. Hence, each of these components has level $k$. For $Lie_k$, the product $[y^x \ldots x] \neq 0$ for $x, y$ nonequal in $X$, so that $Lie_k$ has level $k - 2$ if $k \geq 2$ (and level 1 if $k = 1$). Finally, $Ext_k$ has level 1 for $1 \leq k < r$, while $Ext_r$ has level 0. □

The subalgebra $H_\gamma$ of $U_\gamma$ acts on a monomial $\eta$ in $A_k$ by

$$\eta h_i^{(n)} = \binom{\lambda_i + 1 - \lambda_i}{n} \eta, \quad 1 \leq i < r, \quad 0 \leq n < p^\gamma$$

where $\eta$ has degree $\lambda_j$ on $x_j$, $1 \leq j \leq r$.

(5.3.2) THEOREM. Choose $\gamma$ at least 2 such that $k < p^{\gamma-1}$. The
weight spaces of \( A_k \) with respect to \( H \) are the strongly homogeneous components of \( A_k \). Specifically, the strong component \( A_\lambda \) is the weight space of the weight \( \alpha_\lambda \) defined by \( \alpha_\lambda : h^{(\lambda)}_n \rightarrow \binom{\lambda_{i+1} - \lambda_i}{n} \), where the binomial coefficient is reduced modulo \( p \). In particular, any \( U_Y \)-submodule of \( A_k \) is strongly graded.

Proof. We need to prove the following arithmetic assertion: if
\[
\lambda_1 + \ldots + \lambda_r = k = \lambda'_1 + \ldots + \lambda'_r \quad \text{and} \quad \binom{\lambda_{i+1} - \lambda_i}{n} \equiv \binom{\lambda'_{i+1} - \lambda'_i}{n} \pmod{p}
\]
for all \( 1 \leq i < r \) and \( 0 \leq n < p^Y \), then \( \lambda_i = \lambda'_i \), \( 1 \leq i \leq r \).

Fix \( i \) with \( 1 \leq i < r \), and write \( a = \lambda_{i+1} - \lambda_i \), \( b = \lambda'_{i+1} - \lambda'_i \).

It will be enough to show \( a = b \). Consider the cases which can arise.

Suppose firstly \( a, b \geq 0 \). Writing \( a, b \) to base \( p \), and successively putting \( n = 1, p, \ldots, p^{Y-1} \) into the congruence \( \binom{a}{n} \equiv \binom{b}{n} \), (2.5.2) (1) gives \( a = b \). Next, when \( a, b \leq 0 \), (2.5.2) (2) implies \( p^Y + a = p^Y + b \) and \( a = b \). Finally, suppose (say) \( a < 0 \) and \( b > 0 \). Then we obtain \( a + p^Y = b \). However, \(|a|, |b| \leq k < p^{Y-1} \), and this is a contradiction. This completes the proof. □

Of course, treating \( A_k \) as \( U_Y \)-module for arbitrary \( Y \), the weight spaces are sums of strongly homogeneous components; from the proof of (5.3.2), we see that increasing \( Y \) causes these sums to split.

(5.3.3) EXAMPLE. Take \( r = p = 2 \) and suppose \( 2^{Y-1} \leq k < 2^Y \). Over \( H \), the weight spaces of \( \mathsf{Ass}_k \) are the components \( \mathsf{Ass}(\lambda_1, \lambda_2) \) with \( \lambda_1, \lambda_2 < 2^{Y-1} \), together with all the sums.
Ass(λ₁, λ₂) ⊕ Ass(λ₁, -2Y-1, λ₂+2Y-1)

with λ₁ ≥ 2Y-1. So the choice of γ in (5.3.2) is necessary to obtain a complete splitting. □

Fix γ as in (5.3.2). We may then order the weights αₗ by their strong degrees λ: precisely, define αₗ > αₘ if and only if (lexicographically) λ < μ. (Notice especially the reversal of order here.) With respect to this ordering, the highest weight space, Aₚ say, is perforce spanned by maximal vectors. Therefore, the irreducible Eₚ with high weight αₚ (in the sense of (2.3.3)) is a composition factor of Aₚ.

Insofar as any section of Aₚ (that is, any quotient of a submodule) inherits the strong grading, we see that the weights of Aₚ determine its composition structure over Uγ (and thus over U). Indeed, as in (4.1), the kth strong polynomial for A contains all the information needed.

In order to calculate composition factors in practice, one needs the weights (with their multiplicities) of the irreducible Uγ-modules. In the (easiest) case of rank 2, we compute these irreducibles and their weights in a simple, self-contained manner in the next section. There too we illustrate the procedure above.

It must be conceded that we do not have a full understanding of this lexicographic order on weights: it seems to impose rather severe restrictions on the composition factors of the Aₚ. This is evidenced by (5.3.4) below. This result illustrates that, although the order is defined "locally", it is in fact independent of the choice of Aₚ. We omit the proof, a straightforward arithmetic manipulation. Also, in the statement itself, B is any integral extension algebra of rank r.
(5.3.4). Suppose $k, l$ are positive integers with $k, l < p^{Y-1}$, and let $\alpha$ be a weight in both $A_k$ and $B_l$. Write $\alpha = \alpha_\lambda = \alpha_\mu$ for strong degrees $\lambda, \mu$ in $A_k, B_l$ respectively. Then, for $1 \leq i \leq r$,

$$\lambda_i - \mu_i = (k-l)/r.$$  In particular, for any other weight $\beta$ in both $A_k$ and $B_l$, $\alpha > \beta$ in $A_k$ if and only if $\alpha > \beta$ in $B_l$. □

For example, (5.3.4) shows that if $E$ is a composition factor of $A_k$, then $E$ can only appear as factor of those $A_l$ with $l \equiv k \pmod{r}$. We may compare this with the periodicity statement in (4.4.1). Note that, because $A_k$ is a homomorphic image of a sum of copies of $Ass_k$, the statement may also be interpreted as a property of tensor powers.

We give an example to show that weights do not determine composition structure in general.

(5.3.5) EXAMPLE. Take $r = 2$, $p = 5$ and consider the modules $Pol_0$, $Pol_3$ and $Pol_4$ (where $Pol_0$, by our convention at the beginning of the chapter, is the one-dimensional trivial module). Each of these modules is irreducible over $U_1$. By (4.5.9), the weight spaces for $Pol_3$ and $Pol_4$ with respect to $H_1$ are the strongly homogeneous components. The weights themselves can be identified with integers (reduced modulo 5). We list them in increasing lexicographic order:

<table>
<thead>
<tr>
<th></th>
<th>Pol₀</th>
<th>Pol₃</th>
<th>Pol₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus $Pol_4$ has the same weights as $Pol_0 \oplus Pol_3$. Also,
lexicographically, \( 2 < 1 \) in \( \text{Pol}_3 \) while \( 2 > 1 \) in \( \text{Pol}_4 \). □

We conclude the section by introducing some notation. As above, let \( \alpha_\pi \) be the highest weight in \( A_k \); thus, the irreducible \( E_{\alpha_\pi} \) occurs as a composition factor of \( A_k \) with multiplicity the strong dimension \( d(r, \pi) = \dim A_\pi \). For all the important integral extension algebras (\( \text{Ass}, \text{Pol}, \text{Lie}, \text{Ext} \), and \( \text{Sjor} \)), this strong dimension \( d(r, \pi) \) is 1. The submodule \( A_\pi \) generates is then the unique minimal submodule of \( A_k \) of highest weight. We shall denote it \( F_k^{(A)} \). The module is easy to recognize in particular cases, and will be at the cornerstone of the subsequent discussion.

Consistent with the philosophy expressed in Chapter 1, we shall continue to treat \( A_k \) as \( U \)-module in the following sections. We restrict to \( U_\gamma \) in cases of particular interest, or when we need to order weights.

(5.4) The Exterior and Polynomial Algebras (IV)

The heading of this section is a slight misnomer, insofar as there is little more that we wish to say regarding \( \text{Ext} \). Thus, by (5.3.1), there is no real action on \( \text{Ext} \) apart from that of \( U_1 \). This makes it clear that the irreducibility statements in (4.2.3) and (4.6.1) are different facets of a single fact: for \( 1 \leq k \leq r \), \( \text{Ext}_k \) is an irreducible module for the Humphreys' algebra \( U_1 \), with maximal vector \( x_{r-k+1} \ldots x_r \) and high weight \( h_i \mapsto \begin{cases} 0, & i < r - k, \\ 1, & i = r - k, \\ 0, & r - k + 1 \leq i < r. \end{cases} \)

As usual, \( \text{Pol} \) is more interesting. We shall write \( F_k = F_k^{(\text{Pol})} \) for the unique minimal submodule of \( \text{Pol}_k \) of highest lexicographic weight. As
The submodule $F_κ$ is generated by $x_κ^k$.

(5.4.1). The submodule $F_κ$ is the unique minimal $U$-submodule of $\text{Pol}_κ$; in particular, $\text{Pol}_κ$ is indecomposable.

Proof. A nonzero submodule of $\text{Pol}_κ$ is strongly graded (by (5.3.2)), and the strongly homogeneous components of $\text{Pol}_κ$ are all one-dimensional. So the submodule contains a monomial $\lambda_1^{r_1} \ldots \lambda_r^{r_r}$. Applying $f_{1r} \ldots f_{(r-1)r}$ to this monomial, we see that the submodule contains $x_κ^k$, and hence all of $F_κ$. □

We may view (5.4.1) in a different light. Let $M$ be that minimal admissible lattice in $\text{Pol}_κ$ generated by the maximal vector $x_κ^k$. It is obvious that the lattice $\text{Pol}_κ^{(\mathbb{C})}$ is maximal among those admissible lattices $N$ with the property $N \cap x_κ^k = \mathbb{Z}x_κ^k$. Then (5.4.1) follows on general principles (see, for example, Humphreys [1, 27.3] and the discussion just prior to (2.3.6)).

In the rank 2 case, somewhat more can be said than in (5.4.1); the following theorem has been known for ages when $γ = 1$.

(5.4.2) Theorem. Take $r = 2$, $γ ≥ 1$. The modules $F_0, F_1, \ldots, F_{pγ-1}$ are a full set of pairwise non-isomorphic, irreducible $U_γ$-modules. Explicitly, $F_κ$ has a basis of monomials

$\left\{x_1^{k-i} x_2^{i} \mid 0 ≤ i ≤ k, \binom{k}{i} \equiv 0 \pmod{p}\right\}$. Furthermore, each $F_κ$ is self-contragredient.

Proof. The modules $F_0, F_1, \ldots, F_{pγ-1}$ are irreducible by (5.4.1),
distinguished by their high weights (even more visibly, their levels, in the
sense of (5.3.1)), and are \( p^\gamma \) in number. By the classification theory in
(2.3), they are a full set of irreducibles.

Write \( V \) for the span of the conjectured basis for \( F_k \). We show
firstly that \( V \) is a \( U_\gamma \)-submodule. By symmetry, it will suffice to show
\( V \) is closed under all \( f_{12}^{(s)} \), \( 1 \leq s < p^\gamma \). Indeed, suppose \( 0 \leq i \leq k \)
with \( \binom{k}{i} \not\equiv 0 \pmod{p} \). We have \( \left( x_1 x_2 \right)^{f_{12}^{(s)}} = \binom{i}{s} x_1^{i-s} x_2^{k-i+s} \). If
\( \binom{i}{s} \equiv 0 \), then we are done. Otherwise, \( \binom{k}{i} \binom{i}{s} \not\equiv 0 \), and by (2.5.3),
\( \binom{k}{i-s} \not\equiv 0 \). By definition, \( x_1^{i-s} x_2^{k-i+s} \in V \), as required. Now, observe that
\( x_2^{k-i} = \binom{k}{i} x_1^{i-k} \) for any \( i \). Thus, \( F_k \) contains \( V \), and since \( F_k \) is
irreducible, \( F_k = V \), showing that \( F_k \) has the specified set as a basis.

It remains to show that the \( F_k \) are self-contragredient. For this,
observe that the natural algebra surjection \( \text{Ass} \rightarrow \text{Pol} \) restricts to a
surjection of \( U_\gamma \)-modules \( \text{Ass}_k^+ \rightarrow F_k \). Thus, \( \text{Ass}_k^+ \) has a top composition
factor isomorphic to \( F_k \). Taking duals, and applying (5.2.2), we see that
\( F_k \) is self-contragredient. \( \square \)

Our next objective is to compute the submodule lattice of \( \text{Pol}_k \) in
rank 2. Working in the context of algebraic groups, Carter and Cline [1]
have calculated this lattice (see also Cline [1]). However, the description
we give seems substantially simpler than theirs. Our approach is to give a
single arithmetic lemma, and then to interpret it in the context of the
lattice itself. The positive integer \( k \) will be fixed throughout the
discussion: we do not indicate any dependence on \( k \) in our notation.

Given an arbitrary non-negative integer \( l \), we write \( l \) to base \( p \)
as \( l = i_0 + i_1p + i_2p^2 + \ldots \) where \( i_0 = 0 \) for all \( i \) with \( p^i > l \).

Whenever \( 0 \leq l \leq k \), we may define a sequence \( \Delta_l = \{ \delta_i(l) \}_{i \geq 0} \) of elements of \{0, 1\} by the requirement

\[
(k - l)_i = k_i - l_i + p\delta_{i+1} - \delta_i, \quad i \geq 0.
\]

This equation formally expresses the subtraction of \( l \) from \( k \) in base \( p \). We always have \( \delta_0(l) = 0 \). We partially order the sequences \( \Delta_l \) by defining \( \Delta_l \leq \Delta_m \) if and only if \( \delta_i(l) \leq \delta_i(m) \) for all \( i \geq 0 \).

(5.4.3). Suppose \( l, m \) are integers with \( 0 \leq l, m \leq k \). If

\[
\left( \begin{array}{c} m \\ k - l \end{array} \right) \equiv 0 \pmod{p},
\]

then \( \Delta_l \leq \Delta_m \). Conversely, if \( \Delta_l \leq \Delta_m \), then there exists an integer \( \alpha \) with \( 0 \leq \alpha \leq k \) such that

\[
\left( \begin{array}{c} m \\ k - \alpha \end{array} \right) \equiv 0 \pmod{p}.
\]

Proof. Suppose firstly \( \left( \begin{array}{c} m \\ k - l \end{array} \right) \equiv 0 \pmod{p} \). Then we have, for every \( i \geq 0 \),

\[
m_i \geq (k - l)_i \Rightarrow m_i \geq k_i - l_i + p\delta_{i+1} - \delta_i
\]

\[
\Rightarrow -p\delta_{i+1} \geq (k_i - m_i) - l_i - \delta_i
\]

\[
\Rightarrow p\left[ \delta_{i+1} - \delta_i \right] \geq (k - m)_i - l_i + \left[ \delta_{i+1} - \delta_i \right].
\]

This last inequality implies, by induction on \( i \), that \( \delta_i(m) \geq \delta_i(l) \). Thus \( \Delta_l \leq \Delta_m \).

Conversely, suppose \( \delta_i(l) \leq \delta_i(m) \) for all \( i \). We seek an integer \( \alpha \) for which \( \left( \begin{array}{c} m \\ k - \alpha \end{array} \right) \equiv 0 \pmod{p} \). To motivate its definition, assume \( \alpha \) is given as required. Certainly, then, \( m_i \geq \left( \alpha - (k - m) \right)_i \) and \( \alpha_i \geq (k - l)_i \). From \{0, 1\}, choose the sequence \{\alpha_i\} to satisfy
\[
(e-(k-m))_i = e_i - (k-m)_i + p\alpha_{i+1} - \alpha_i.
\]

Then we have
\[
m_i + (k-m)_i - p\alpha_{i+1} + \alpha_i \geq \alpha_i \geq (k-l)_i.
\]

The outer inequality imposes restrictions on the \(\alpha_i\). Choosing the \(\alpha_i\) to be smallest possible subject to these restrictions, we then choose \(\sigma_i\) as large as possible to satisfy the inner inequalities.

Precisely, by reverse induction on \(i\), define \(\alpha_i\) as follows. If \(p^i > k\), then \(\alpha_i = 0\). Then, assuming \(\alpha_{i+1}\) is defined, set
\[
\alpha_i = \begin{cases} 
1 & \text{if } (k-l)_i - (k-m)_i + m_i + p\alpha_{i+1} > 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Note explicitly that
\[
(k-l)_i - (k-m)_i + m_i = -p\left[\delta_{i+1}(m) - \delta_{i+1}(l)\right] + \left[\delta_i(m) - \delta_i(l)\right] - l_i.
\]

We claim that, with the \(\alpha_i\) so defined, \(\alpha_i \leq \delta_i(m) - \delta_i(l)\). Indeed, we have for any \(i\),
\[
\alpha_i > \delta_i(m) - \delta_i(l) \Rightarrow \alpha_i = 1 \quad \text{and} \quad \delta_i(m) = \delta_i(l) \quad \text{because} \quad \delta_i(m) \geq \delta_i(l)
\]
\[
\Rightarrow -p\left[\delta_{i+1}(m) - \delta_{i+1}(l)\right] + p\alpha_{i+1} > l_i \quad \text{by definition of} \quad \alpha_i
\]
\[
\Rightarrow \alpha_{i+1} > \delta_{i+1}(m) - \delta_{i+1}(l),
\]
so the claim follows by reverse induction.

Now define \(\sigma_i = \min[p-1, m_i + (k-m)_i - p\alpha_{i+1} + \alpha_i]\). We show firstly that \(\sigma_i \geq (k-l)_i\). This ensures in particular that \(\sigma_i \geq 0\), so the \(\sigma_i\) are digits in base \(p\) (that is, \(0 \leq \sigma_i \leq p-1\)). Indeed, we have
\[ a_i < (k-l)_i \Rightarrow m_i + (k-m)_i - p\alpha_{i+1} + \alpha_i < (k-l)_i \] by definition of \( a_i \)

\[ = -p\alpha_{i+1} + \alpha_i < (k-l)_i - (k-m)_i - m_i \]

\[ = p\left[ \delta_{i+1}^{(m)} - \delta_{i+1}^{(l)} - \alpha_{i+1} \right] - \left[ \delta_{i}^{(m)} - \delta_{i}^{(l)} \right] - \alpha_i < -l_i , \]

and this contradicts the upper bound on the \( \alpha_i \) determined above.

Write \( c = c_0 + c_1 p + c_2 p^2 + \ldots \). It only remains to verify that

\( c \geq k - m \) and \( (c-(k-m))_i = c_i - (k-m)_i + p\alpha_{i+1} - \alpha_i \). By the definition of "carries" in base \( p \) arithmetic, this is equivalent to showing

\( c_i - (k-m)_i - \alpha_i < 0 \) if and only if \( \alpha_{i+1} = 1 \). In the case when

\[ c_i = m_i + (k-m)_i - p\alpha_{i+1} + \alpha_i , \]

this is immediate. Otherwise,

\[ c_i = p - 1 < m_i + (k-m)_i - p\alpha_{i+1} + \alpha_i \]. In this case, \( \alpha_{i+1} = 1 \) certainly gives \( c_i - (k-m)_i - \alpha_i < 0 \), and conversely

\[ c_i - (k-m)_i - \alpha_i < 0 \Rightarrow \alpha_i > p - 1 - (k-m)_i \]

\[ \Rightarrow \alpha_i = 1 , \ (k-m)_i = p - 1 \]

\[ \Rightarrow (k-l)_i - (p-1) - m_i + p\alpha_{i+1} > 0 \] by definition of \( \alpha_i \)

\[ \Rightarrow \alpha_{i+1} = 1 . \]

Thus, the proof is complete. \( \square \)

We are now in a position to describe the submodule lattice of \( Pol_k \)

(when \( r = 2 \)). In the proof of (5.4.1), we saw that every \( U \)-submodule of \( Pol_k \) has a basis of monomials. Consider in particular the monomial submodules, that is, those generated by a single monomial. We define

\[ M(t) = \left\{ \left( \frac{t_{x_1}^{k-t} x_2^{k-t}}{x_1 x_2^{k-t}} \right)_U \right\}, \ 0 \leq t \leq k \]. This \( M(t) \) has a unique maximal submodule

(namely, the sum of those submodules whose monomial bases avoid \( x_1 x_2^{k-t} \)).

It follows that the distinct \( M(t) \) are exactly the join-irreducible
elements in the submodule lattice of $\text{Pol}_k$. Thus, to specify the lattice, it is surely enough to describe these $M(t)$. This is the content of the following theorem.

(5.4.4) THEOREM. Take $r=2$. Given $0 \leq s, t \leq k$, $M(s) \subseteq M(t)$ if and only if $\Delta_s \subseteq \Delta_t$.

Proof. We have, in view of the standard basis for Kostant's algebra,

$$M(t) = \left\{ x_1^{t-a} x_2^{k-t+a} f \left( \frac{b}{a} \right) \mid a, b, n \geq 0 \right\}_K$$

$$= \left\{ x_1^{t-a} x_2^{k-t+a} f \left( \frac{b}{a} \right) \mid a, b, n \geq 0 \text{ and } \left( \frac{t}{a} \right) \neq 0 \right\}_K$$

$$= \left\{ x_1^{t-a+b} x_2^{k-t+a-b} \mid a, b, n \geq 0 \text{ and } \left( \frac{t}{a} \right) \left( \frac{k-t+a}{b} \right) \neq 0 \right\}_K$$

$$= \left\{ x_1^s x_2^{k-s} \mid \text{for some } c, \left( \frac{t}{k-c} \right) \left( \frac{c}{k-s} \right) \neq 0 \right\}_K.$$ 

The theorem now follows from (5.4.3). □

Using (5.4.4), we are able to give a very succinct description of the composition factors of $\text{Pol}_k$. We first make a definition.

(5.4.5) DEFINITION. A sequence $\Delta = \{ \delta_i \}_{i \geq 0}$ of elements of $\{0, 1\}$ is permissible if the following conditions are satisfied:

$$\delta_0 = 0,$$

$$\delta_i = 0 \quad \text{for} \quad p^i > k,$$

$$\delta_i \leq \delta_{i+1} \quad \text{for} \quad p^i < k, \quad k_i = 0,$$

$$\delta_i \geq \delta_{i+1} \quad \text{for} \quad p^i < k, \quad k_i = p - 1.$$ 

Given the representation of $k$ to base $p$, the permissible sequences can be listed very quickly. It is straightforward to check that, for $0 \leq \ell \leq k$, $\Delta_\ell$ is a permissible sequence. Conversely, fix a permissible
sequence $\Delta = \{\delta_i\}_{i \geq 0}$: we realize $\Delta$ in the form $\Delta_i$ for suitable $l$.

Indeed, define $l$ by setting

$$l = \sum_{i \geq 0} \ell_i^i, \quad l_i = \delta_{i+1}^{-1}(k_i+1-\delta_i).$$

Notice that $0 \leq l_i \leq p-1$ for all $i$, the right-most inequality holding because $\Delta$ is permissible. We claim that $k \geq l$ and $\delta_i^{(L)} = \delta_i^{(L)}$. For this, it is enough to prove $k_i - l_i - \delta_i < 0$ if and only if $\delta_{i+1} = 1$.

We have

$$k_i - l_i - \delta_i = k_i - \delta_{i+1}(k_i+1-\delta_i) - \delta_i.$$

If $\delta_{i+1} = 0$, then $k_i - l_i - \delta_i = k_i - \delta_i \geq 0$ because $\Delta$ is permissible; if $\delta_{i+1} = 1$, then $k_i - l_i - \delta_i = -l < 0$. Hence the claim follows, and we have $\Delta = \Delta_i$. This integer $l$ associated with $\Delta$ is a distinguished one, and we write $l = \lambda(\Delta)$. We observe that we even have $k \geq 2l$ (for this, one quickly checks $(k-l)_i \geq l_i$). Furthermore, if $\Delta'$ is a permissible sequence, then $l = \lambda(\Delta')$ if and only if $\Delta = \Delta'$. (5.4.6) THEOREM. Take $r = 2$. For $m \geq 0$, the irreducible $U$-module $F_m$ is a composition factor of $Pol_k$ if and only if $l = (k-m)/2$ is a non-negative integer satisfying $l = \lambda(\Delta_i)$. In particular, the composition factors of $Pol_k$ are listed as all $F_m$, $m = k - 2\lambda(\Delta)$, as $\Delta$ ranges over the permissible sequences.

Proof. Treat $Pol_k$ as $U_\gamma$-module where $\gamma$ is large: we may then order weights lexicographically. Each composition factor of $Pol_k$ occurs with multiplicity one. From the discussion preceding (5.4.4), these factors can all be obtained by taking the top composition factors of the $M(t)$, $0 \leq t \leq k$. In turn, using (5.4.4) itself, we may identify the top factor
of $M(t)$ by determining that monomial $x_1^{k-t} x_2^{l}$ which is lexicographically least such that $\Delta_l = \Delta_t$.

Indeed, we define $l = l_0 + p l_1 + \ldots$ digit by digit. Specifically, write $l_i = 0$ if $k_i \geq t_i + \delta_i^{(t)}$ and $l_i = k_i + 1 - \delta_i^{(t)}$ if $k_i < t_i + \delta_i^{(t)}$. It is clear that $0 \leq l_i \leq p-1$. Also, $k_i - l_i - \delta_i^{(t)} < 0$ if and only if $\delta_i^{(t)} = 1$, whence $\Delta_l = \Delta_t$, and certainly $l$ is least with this property. It is immediate that the two choices for $l_i$ can be incorporated into the single equation $l_i = \delta_i^{(l)} \left( k_i + 1 - \delta_i^{(l)} \right)$. The sequence $\Delta_l$ is permissible and $l = \lambda(\Delta_l)$.

The weight of the monomial $x_1^{l} x_2^{k-l}$ is $\binom{\frac{h_1}{n}}{n} \mapsto \binom{k-2l}{n}$, $n \geq 0$, and this is the high weight of the irreducible $F_{k-2l}$. Thus, the composition factors of $Pol_k$ all arise from permissible sequences. Conversely, every permissible sequence obviously gives rise to a composition factor, and the proof of (5.4.6) is complete. □

Later, we shall need the following two examples of the above results.

(5.4.7) EXAMPLE. (1) Take $\gamma > 1$ and suppose $k \equiv -1 \pmod{p^\gamma}$, $k < p^{\gamma+1}$. Visibly, $\Delta_t = \{0, 0, \ldots\}$ for $0 \leq t \leq k$. Hence $Pol_k$ is irreducible.

(2) Suppose $p - 1 < k < p^2$ and $k \not\equiv -1 \pmod{p}$. The permissible sequences for this $k$ are just $\{0, 0, 0, \ldots\}$ and $\{0, 1, 0, \ldots\}$. So $Pol_k$ has composition factors $F_k$ and $F_l$ where $l = k - 2(k_0+1)$. □

When the rank $r$ is greater than 2, it is clear that one can always
write down the submodule lattice of any given $\text{Pol}_k$ by first listing the monomial submodules. However, we have not attempted to formalize the process. Nor do we determine the composition factors of these $\text{Pol}_k$.

We mention one final lemma. It is applied in (5.6), and we state it using the notation there.

(5.4.8). Take $r = 2$. Suppose $0 \leq k = \lambda p + \mu < p^2$ with $0 \leq \lambda, \mu < p$. As $U$-module, the subspace $F_k$ of $\text{Pol}_k$ is isomorphic to a sum of $(\lambda+1)$ copies of $\text{Pol}_\mu = F_\mu$.

Proof. Compare the basis of $F_k$ given in (5.4.2) (2) with the explicit description of $X = \sigma \text{Pol}_k$ in the proof of (4.6.3) (1).

(5.5) The Associative, Lie and Special Jordan Algebras

We now begin an examination of the three algebras in the title of this section. The discussion we give is only a small beginning in what appears to be a very interesting area.

The starting-point in the investigation is the following well-known observation. Recall the definition of $F_k^{(\text{Ass})}$ at the end of (5.3); for brevity, we put $T_k = F_k^{(\text{Ass})}$. Thus, $T_k$ is the $U$-submodule of $\text{Ass}_k$ generated by $x_r^k$.

(5.5.1). The submodule $T_k$ is the symmetric part $\text{Ass}_k^+$ of $\text{Ass}_k$.

Proof. We certainly have $x_r^k \in T_k \cap \text{Ass}_k^+$, so $T_k \subseteq \text{Ass}_k^+$. On the other hand, $\text{Ass}_k^+$ has a basis of symmetric tensors. Suppose $z$ is one such: this $z$ is the sum of all degree $k$ monomials with some fixed
strong degree \((\lambda_1, \ldots, \lambda_n)\). Then \(z = x_r^{f_r(\lambda_1)} \cdots f_r(r-1) \in T_k\).

Therefore, we also have \(T_k \supset \text{Ass}^+_k\). □

Thus, \(T_k\) is realized as a Weyl module (Carter, Lusztig [1, 3.2]); however, we proceed independently of that fact. Although we have stated (5.5.1) in full generality, it is in fact the last time we shall work in arbitrary rank: for the rest of this paper, assume that the rank \(r\) is 2. The structure of \(T_k\) is known by the duality statement (5.2.2) and the results of (5.4). For ease of reference, we dualize the exact sequences in (5.2.3) also:

\((5.5.2).\) Fix \(k, l \geq 0\).

1. There is an exact sequence of \(U\)-modules and \(U\)-homomorphisms
   \[0 \to T_{k+1} \to T_k \otimes T_l \to T_{k-1} \otimes T_{l-1} \to 0.\]

2. When \(p | k + 1\), the sequence
   \[0 \to T_{k+1} \to T_k \otimes T_l \to T_{k-1} \to 0\]
   splits. □

We remarked in (5.1) that the admissible lattice \(A_k^{(\mathbb{Z})}\) in \(A_k^{(\mathbb{C})}\) need not respect the decomposition of \(A_k^{(\mathbb{C})}\) into irreducibles. We have in \(T_k\) an instance of how that can fail. Thus, we shall see in (5.6) that \(T_k\) is not in general a direct summand in \(\text{Ass}^+_k\); however, the symmetric part of \(\text{Ass}^{(\mathbb{C})}_k\) is a direct summand. For all that, \(T_k\) is the unique minimal submodule of \(\text{Ass}^+_k\) of highest weight, so that, in any decomposition of \(\text{Ass}^+_k\) into indecomposables, exactly one summand contains \(T_k\). We let \(I_k\) denote this unique (to within isomorphism) indecomposable direct summand.

Interest now centres on the tensor product \(I_k \otimes T_1\) (where we note
This tensor product embeds as direct summand in $\text{Ass}_{k+1}$, and it contains a copy of $T_{k+1}$ by (5.5.2). A fortiori, $I_k \otimes T_1$ contains a copy of $I_{k+1}$. We suggest:

(5.5.3) **Conjecture.** The tensor product $I_k \otimes T_1$ splits as a direct sum of $I_{k+1}$ together with various $I_l$, $l \leq k$.

In practice, the difficulty with proving (5.5.3) is that the definition of $I_k$ is not very enlightening structurally. To illustrate the conjecture, consider the case $k < p - 1$. For any $l < p - 1$, $T_l$ is irreducible and $\text{Ass}_l$ is completely reducible. Thus, (5.5.3) holds by (5.5.2) (2). In the next section, we prove a case of (5.5.3) for $p - 1 \leq k < p^2 - 1$. For the moment, let us indicate the significance of the conjecture itself.

Iterating (5.5.3), we see that $\text{Ass}_k$ is a direct sum of certain $I_l$, $l \leq k$. Thus, any direct summand $M$ in $\text{Ass}_k$ is a sum of such $I_l$.

Recall the lexicographic order on the set of weights in $\text{Ass}_k$. We claim that the summands in $M$ with highest possible weight are precisely the copies of $I_l$ with $l$ largest possible. To see this, note that the maximal vector $x^l_{\pi}$ in any $T_l$, $l < k$, has weight, $\pi$, say, associated with the homogeneous component of strong degree $(0, \ldots, 0, l)$. Use (5.3.4): if a copy of $I_l$ is a direct summand in $M$, then the highest weight the copy can contribute to the weights of $M$ is $\pi$, and the corresponding strongly homogeneous component has strong degree $\left[\left(\frac{k-l}{r}\right), \ldots, \left(\frac{k-l}{r}\right)+l\right]$. Now, for any $l' < l$, we have $\left[\left(\frac{k-l}{r}\right), \ldots, \left(\frac{k-l}{r}\right)+l\right] < \left[\left(\frac{k-l'}{r}\right), \ldots, \left(\frac{k-l'}{r}\right)+l'\right]$, and the claim follows. Therefore, given (5.5.3), the composition structure of $M$
determines its decomposition into indecomposables; indeed, the weights of $M$ are sufficient.

There are two distinguished direct summands of $\text{Ass}^*_k$ to which one would seek to apply the preceding paragraph.

(5.5.4) EXAMPLES. (1) If $p|k$, then $\text{Lie}^*_k$ is a direct summand in $\text{Ass}^*_k$. Indeed, the left-norming operator in (3.6.4) visibly admits $U$.

(2) For $p \neq 2$, $S\text{Jor}^*_k$ is a direct summand in $\text{Ass}^*_k$. Here, we observe that the symmetry operator in (3.6.6) admits $U$. □

It should be pointed out that (5.5.4) (1) fails when $p|k$; it is convenient to interpolate an example here.

(5.5.5) EXAMPLE. Take $p = 2$. Set $g = E^{12}_1(1)$ and let $G$ be the cyclic group (of order 2) generated by $g$. This $G$ acts faithfully on $\text{Ass}^*_1$ by $x_1g = x_1 + x_2$, $x_2g = x_2$. Thus, $\text{Ass}^*_1$ is a regular $G$-module, and every $\text{Ass}^*_k$ is $G$-projective. However, we claim that $\text{Lie}^*_k$ has odd dimension whenever $k = 2q$ for (say) $q$ an odd prime. In that case, $\text{Lie}^*_k$ cannot be projective, and therefore is not a direct summand in $\text{Ass}^*_k$.

Indeed,

$$V_1 = (1/2q)\sum_{d|k} \mu(d)2^{k/d} = (1/2q)[2^k-2^2-2^2+2]$$

and this is odd.

The calculations about to come in (5.6) begin with some particular cases of the following result. It affords some instances of when $T_k = T_k$.

(5.5.6). Suppose $k \equiv -1 \pmod{p^\gamma}$ and $k < p^{\gamma+1}$. Then $T_k$ is a $U$-direct summand in $\text{Ass}^*_k$. 
Proof. We exhibit a complement for $T_k$. Let $R$ be the subspace of $\operatorname{Ass}_k$ spanned by all differences $\eta_1 - \eta_2$ of monomials $\eta_1, \eta_2$ with the same strong degree. Visibly, $R$ is a $U$-submodule. Also, the strongly homogeneous component of $R$ with strong degree $(\lambda_1, \lambda_2)$ has dimension $\binom{k}{\lambda_1} - 1$, so $\dim R = 2^k - (k+1)$. However, $T_k$ is irreducible (see (5.4.7)), and certainly $\frac{k}{x_2} \notin R$. Hence, $R \cap T_k = \{0\}$, and by a dimension count, $\operatorname{Ass}_k = T_k \oplus R$, as required. \hfill \Box

(5.6) Example: A Special Case of the Conjecture

To illustrate the ideas of the previous section, we prove a special case of (5.5.3): $p - 1 \leq k < p^2 - 1$, $|K| \geq p^2$.

We re-iterate that the rank $r$ is 2. Write $L = \operatorname{sl}(2, K)$, $G = \operatorname{SL}(2, K)$ and $\Gamma = \operatorname{SL}(2, p)$. Before coming to the main theorem of this section, we need to develop some preliminary information regarding $L$, $G$, and $\Gamma$. During the first part of this discussion, $K$ is arbitrary.

The principal indecomposable modules (PIMs for short) for both $L$ and the group algebra $\mathbb{K}\Gamma$ have been known for some time. Working in the context of algebraic groups over algebraically closed fields, Humphreys [5], [6, §§8, 10] has shown that the PIMs for $L$ can be realized as $G$-modules and then, by restriction, the PIMs for $\Gamma$ can be obtained. Similar arguments show that to be true for arbitrary $K$, but we will not stop to give them here. For our purposes, it is enough to observe that the PIMs for $L$ over $K$ have the same structure as over an algebraically closed field in view of (2.3.3) (4). Similarly, $K$ is a splitting field for $\Gamma$ (Glover [1]; that reference also gives an independent determination of the PIMs for $\Gamma$). We summarize the facts that we need regarding these PIMs.
Consider the $U_1$-modules $F_i = \text{Pol}_i$, $0 \leq i \leq p-1$, in (5.4). These are a full set of irreducibles for $L$, and simultaneously, a full set for $\Gamma$. Write $Q_i, R_i$ respectively for the principal indecomposable $L$ and $\Gamma$-modules which have $F_i$ as their unique minimal submodule.

(5.6.1). (1) The PIMs have the following dimensions:

\[
\dim Q_0 = 2p, \quad \dim R_0 = p,
\]
\[
\dim Q_i = \dim R_i = 2p, \quad 0 < i < p-1,
\]
\[
\dim Q_{p-1} = \dim R_{p-1} = p.
\]

(2) For $0 \leq i < p-1$, $Q_i$ has composition factors $F_i, F_{p-i-2}$, each with multiplicity 2. On the other hand, $R_i$ has factors $F_i$ (with multiplicity 2) and in addition $F_{p-3}$ (once) when $i = 0$, $F_{p-i-1}$ and $F_{p-i-3}$ (each once) when $0 < i < p-2$, and $F_1$ (once) when $i = p-2$. □

A special instance of (5.6.1) needs to be singled out. The irreducible $F_{p-1}$ is the Steinberg module: this is projective (and injective) both as $L$-module and as $\Gamma$-module.

We now assume specifically $|K| \geq p^2$, $p - 1 \leq k < p^2$. These assumptions ensure, by (2.3.6) and (5.3.1), that $G$ and $U$-structure coincide; this will allow effective use of the natural action of $G$ on $U_1$ via the adjoint representation (see the end of (2.3)). We lean heavily, and often without comment, on the other properties of $U_1$ mentioned in (2.3), and on the properties of injective hulls in (2.5).

Next, we collect together a miscellany of facts regarding $T_k$. Set $k = \lambda p + \mu$, $0 \leq \lambda, \mu < p$ (with $\mu = p - 1$ if $\lambda = 0$). Write $E[T_k]$ for the $L$-injective hull, and $\sigma T_k$ for the $L$-socle, of $T_k$. 
(5.6.2). (1) Take \( \mu = p - 1 \). As \( U \)-module, \( T_k \cong F_k \) is irreducible; as \( L \)-module, \( T_k \cong F_{p-1}^{\otimes(\lambda+1)} \) is \( L \)-injective.

(2) Take \( \mu \neq p - 1 \). As \( U \)-module, \( T_k \) is a 2-step uniserial, with unique minimal submodule isomorphic to \( F_{k-2(\mu+1)} \) and quotient isomorphic to \( F_k \). This unique minimal as \( L \)-module is \( \sigma T_k \cong F_{p-\mu-2}^{\otimes \lambda} \). We have \( T_k/\sigma T_k \cong F_{p-\mu-2}^{\otimes \lambda} \). Furthermore, \( E(T_k) \cong F_{p-\mu-2}^{\otimes \lambda} \) and has dimension \( 2\lambda p \).

Proof. This has all been seen before and is just a matter of collecting pieces. We simply list the relevant statements: (5.4.7), (5.4.8), (5.6.1) and (taking contragredients) (4.6.3). □

As we mentioned in (5.5), the definition of \( I_k \) does not lend itself readily to direct calculations. One symptom of this is the next proposition, a rather technical lemma which provides the key step for the proof of the main theorem, (5.6.5) below. To ease the overall description, we first make a definition.

(5.6.3) DEFINITION. Suppose \( I \) is a \( U \)-submodule of some \( \text{Ass}_l \), \( l < p^2 \), and take \( \mu \neq p - 1 \). We say \( I \) has type \( k \) if it satisfies the following conditions:

(1) \( I \) is self-contragredient;

(2) \( I \) is \( L \)-injective;

(3) \( I \) contains a \( U \)-submodule \( X \) with \( X \cong T_k \) and \( I/X \cong T_{k-2(\mu+1)} \).

It is easy to compute the submodule lattice of a module \( I \) of type \( k \). Thus, by (1) and (3) in the definition, \( I \) contains a copy, \( Y \) say, of \( \text{Pol}_{k-2(\mu+1)} \). By (2) and a dimension count, \( X \) and \( Y \) must intersect
in their (common) unique minimal submodule, and the following lattice clearly results:

\[
\begin{array}{c}
\text{Pol}_{k-2(\mu+1)} \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\text{Pol}_{k-2(\mu+1)} \\
\end{array}
\]

where the section \( H \) is either \( \{0\} \) or \( F_{k-2p} \) according as \( k < 2p \) or \( k \geq 2p \). We see, too, that \( I \) is \( U \)-indecomposable and, as \( L \)-module, is an injective hull of \( T_k \).

Naturally, it will emerge that \( I_k \) itself has type \( k \); indeed, it is the only such module.

(5.6.4). Take \( \mu \neq p - 1 \). Two modules of type \( k \) are \( U \)-isomorphic.

Proof. Let \( I, J \) have type \( k \), and write \( M_I, M_J \) for their unique maximal \( U \)-submodules. We may view \( M_I \) as the sum of the modules \( T_k \) and \( \text{Pol}_{k-2(\mu+1)} \) with their isomorphic unique minimal submodules amalgamated.

By (2.5.5) (1), their exists an isomorphism \( \theta : M_I \to M_J \). Now form \( Q = (I \oplus J)/\Delta \), the amalgam of \( I, J \) with respect to \( \theta \), where \( \Delta = \{m-m\theta \mid m \in M_I \} \). To show \( I \cong J \), we need only prove, by (2.5.5) (2), that \( Q \) is \( U \)-decomposable.

Because \( I \) is \( L \)-injective, we certainly have a decomposition \( Q = I' \oplus V \) where \( I' \) is a \( U \)-submodule isomorphic to \( I \), and \( V \) is an \( L \)-submodule. In view of (5.6.2) and the lattice structures for \( I \) and \( J \), this \( V \) can only be a sum of copies of \( F_{\mu-2} \), and therefore the
L-socle, $S = \sigma I' \oplus V$ of $Q$, is such a sum. Because $G$ acts on $U_1$ via the adjoint representation, $S$ is invariant under $G$, hence is a $U$-submodule, and its only composition factor is $F_{k-2(\mu+1)}$, with multiplicity 2. We show that $S$ is a completely reducible $U$-module; this will provide a splitting $Q = I' \oplus V'$ into $U$-submodules, where $V' \cong F_{k-2(\mu+1)}$, and the proof of (5.6.4) will be complete.

To show that $S$ is completely reducible, we borrow an argument from the theory of algebraic groups (see Humphreys [2, 4.1]). Thus, treat $I$ and $J$ as $U_\gamma$-modules where $\gamma$ is large. By (5.3.4), the lexicographic orders on the weights of $I$ and $J$ are compatible, and the weights of $S$ inherit this order. In particular, the highest weight $\pi$ of $S$ has two-dimensional weight space $\langle v_1, v_2 \rangle_K$ say. The $U$-module generated by $v_2$ involves the weight $\pi$ with multiplicity one. Perforce, $S = \langle v_1 \rangle_U \oplus \langle v_2 \rangle_U$ and is completely reducible, as claimed. □

We are now in a position to give the main result in this section. Recall that our objective is to compute the tensor product $I_k \otimes T_1$. However, we have seen in (5.5.6) that if $\mu = p - 1$, then $I_k = T_k$. Therefore, it will be enough to take $k$ with $k < p^2 - p$ and $\mu = p - 1$, and calculate the tensor products $I_{k+i} \otimes T_1$, $0 \leq i \leq p-1$.

(5.6.5) THEOREM. Assume $k = \lambda p + p - 1$, $0 \leq \lambda < p - 1$, and let $0 \leq i \leq p - 1$. The tensor product $I_{k+i} \otimes T_1$ is given explicitly as

$$I_{k+i} \otimes T_1 \cong \begin{cases} I_{k+1}, & i = 0, \\ I_k \oplus I_k \oplus I_{k+2}, & i = 1 \ (\text{with } p > 2), \\ I_{k+i-1} \oplus I_{k+i+1}, & 1 < i < p - 1, \\ I_{k-p} \oplus I_{k+p-2} \oplus I_{k+p}, & i = p - 1. \end{cases}$$
where \( I_{k-p} = \{0\} \) if \( k = p - 1 \). In addition, for \( i > 0 \), \( I_{k+i} \) has type \( k + i \).

**Proof.** Write \( Z_i = I_{k+i} \otimes T_1 \). We show by induction on \( i \) that \( I_{k+i} \otimes T_1 \) splits as claimed and that, when \( i < p - 1 \), \( I_{k+i+1} \) has type \( k + i + 1 \).

To start the induction, take \( i = 0 \) first. Because \( I_k = T_k \), it is \( L \)-injective and self-contragredient, and hence so too is \( Z_0 \). By (5.5.2) (1), \( Z_0 \) contains a submodule isomorphic to \( T_{k+1} \) with quotient isomorphic to \( T_{k-1} \). Thus, \( Z_0 \) has type \( k + 1 \). In particular, \( Z_0 \) is \( U \)-indecomposable, and perforce \( Z_0 \cong I_{k+1} \).

Now take \( i > 0 \). By the inductive assumption, \( I_{k+i} \) has type \( k + i \). Therefore, by (5.5.2) (2), there is an exact sequence

\[
0 \to T_{k+i+1} \oplus T_{k+i-1} \to Z_i \to T_{k-i-1} \oplus T_{k-i+1} \to 0.
\]

Write \( A, B \) for the submodules of \( Z_i \) given by this sequence:

\[
A \cong T_{k+i+1}, \quad B \cong T_{k+i-1}.
\]

Because \( Z_i \) is \( L \)-injective, it contains an injective hull \( \overline{A} \otimes \overline{B} \) for \( A \oplus B \), where \( \overline{A}, \overline{B} \) are injective hulls for \( A, B \) respectively. The essential idea now is to compare \( \overline{A} \otimes \overline{B} \) to \( Z_i \), the result depending on the choice of \( i \). It is perhaps clearer to take the possibilities out of order.

**Case (1).** \( 1 < i < p - 1 \).

By the dimension count in (5.6.2) (2), we find \( Z_i = \overline{A} \oplus \overline{B} \). Given \( g \in G \), \( Z_i = \overline{A}g \oplus \overline{B}g \). Therefore, \( \overline{A}g \) is an \( L \)-injective hull of \( A \).

However, \( \overline{A} \) (and so also \( \overline{A}g \)) has no \( L \)-composition factors in common with \( \overline{B} \). This forces \( \overline{A} = \overline{A}g \). Hence \( \overline{A} \), and similarly \( \overline{B} \), are \( G \)-submodules, and therefore invariant under \( U \). By the exact sequence (5.6.6),
and by examining \( L \)-composition structure again, this forces \( \overline{A}\slash A \cong T_{k-i-1} \), \( \overline{B}\slash B \cong T_{k-i+1} \). We see too that \( \overline{A} \) and \( \overline{B} \) are self-contragredient. Thus, \( \overline{A}, \overline{B} \) have types \( k+i+1, k+i-1 \) respectively. Then \( \overline{A} \cong I_{k+i+1} \), and by the inductive assumption, \( \overline{B} \cong I_{k+i-1} \). Thus, the inductive step is complete in this case.

Case (2). \( i = p-1 \).

Here, \( A \cong T_{k+p} \), \( B \cong T_{k+p-2} \). Also, taking duals in (5.6.6), \( Z_{p-1} \) contains a copy \( C \) of \( T_{k-p} \) (taking \( C = \{0\} \) if \( k = p-1 \)). The sum \( A + B + C \) is direct, and taking injective hulls, a dimension count gives \( Z_{p-1} = A \oplus \overline{B} \oplus C \). As in Case (1), \( \overline{B} \) is \( U \)-invariant, of type \( k+p-2 \), hence isomorphic to \( I_{k+p-2} \). Of course, \( I_{k+p} = T_{k+p} \). Thus, \( Z_{p-1} = I_{k+p} \oplus I_{k+p-2} \oplus I_{k-p} \), as required.

Case (3). \( i = 1 \) (with \( p > 2 \)).

In this case, begin by writing
\[
Z_1 = I_{k+1} \otimes T_1 \cong T_k \otimes T_1 \otimes T_1 \cong T_k \oplus \left( T_k \otimes T_2 \right),
\]
using the case \( i = 0 \) together with (5.5.2) (2). Now, again using (5.5.2) (twice), duality implies \( T_k \otimes T_2 \) contains a copy of \( T_k \). Thus, \( Z_1 \) contains a submodule \( B' \oplus B'' \cong T_k^{\oplus 2} \). The submodule \( A \) cannot intersect \( B' \oplus B'' \), and we find \( Z_1 = \overline{A} \oplus B' \oplus B'' \), where \( \overline{A} \) has type \( k+2 \) as \( U \)-module. Thus \( \overline{A} \cong I_{k+2} \) and this completes the proof. \( \square \)

From the discussion in the previous section, (5.6.5) implies that the decomposition of \( \text{Lie}_k \) and \( \text{Sjor}_k \) into \( U \)-indecomposables (with \( k < p^2 \), and the restrictions in (5.5.4) in force) can be written down using only their composition structure together with the composition structure of the
We give a numerical instance of this.

(5.6.7) **EXAMPLE.** Take $A = \text{Lie}_k$, $p = 3$, $k = 7$. Note that $3 | 7$, so $\text{Lie}_k$ is a direct summand in $\text{Ass}_k$ (see (5.5.4) (1)). Also, $7 < 3^2$.

The 7th strong polynomial for $A$ is

$$s_7(z_1, z_2) = z_1^6 + 3z_2^5 z_1 + 5z_1^4 z_2 + 5z_1^3 z_2^3 + 3z_1^2 z_2^5 + z_1 z_2^6.$$  

Using the procedure outlined in (5.3) together with the results of (5.4), we find that $A_7$ has composition factors $F_5$ once, $F_3$ twice and $F_1$ four times. Of these, $F_5$ has highest lexicographic weight, and therefore $I_5$ is a direct summand in $A_7$. Of course, $I_5 = F_5$. Therefore, the composition factors of $A_7$ not accounted for by $I_5$ are $F_3$ (twice) and $F_1$ (four times). Of these, $F_3$ has highest weight, so $I_3$ occurs as a direct summand of $A_7$ with multiplicity 2. Because the composition factors of $I_3$ are $F_3$ and $F_1$ (twice), we are left with

$$A_7 \cong I_5 \oplus I_3^{\oplus 2}.$$  

We make some remarks regarding one final structural problem, namely, the determination of the $I_k$ as $\Gamma$-modules. Clearly, this allows the structure of $\text{Lie}_k$ and $S\text{Jor}_k$ over the group algebra $GF(p)\Gamma$ to be computed. We mention two approaches to calculating $I_k|\Gamma$. First, the tensor products $R_i \otimes T_1$, $0 \leq i \leq p-1$, can be calculated from the information in (5.6.1), and using (5.6.5), one may develop recursive formulae for the structure of $I_k$. Alternatively, $I_k$ (for $k \geq p - 1$) is the $\Gamma$-injective hull of its $\Gamma$-socle. Taking duals realizes $I_k$ as the $\Gamma$-injective hull of the Frattini quotient of $\text{Pol}_k$, and Glover [1, §6] gives some explicit formulae for such quotients. Neither of these approaches
affords interesting structural information, however, and we will not elaborate them here.

(5.6.8) EXAMPLE. For $0 \leq i \leq p - 1$, $I_{p+i-1} \cong Q_{p-i-1}$ as $L$-modules. For $0 \leq i \leq p - 2$, $I_{p+i-1} \cong R_{p-i-1}$ as $\Gamma$-modules, while

$I_{2p} \cong R_0 \oplus R_{p-1}$. This restates the connection between the PIMs for $L$ and $\Gamma$ mentioned at the start of the section. □

In order to extend (5.6.5) to the range $p^{\gamma-1} - 1 \leq k < p^{\gamma} - 1$, one presumably needs to replace $L$ by the Humphreys' algebra $U_{\gamma-1}$ and to take $|K| \geq p^{\gamma}$. Here, a somewhat more uniform approach is afforded by taking $K$ algebraically closed at the outset, and working within the context of algebraic groups. That framework could also provide a better description of the lexicographic order on weights in (5.3). However, there seems to be some intrinsic interest in working over more general fields for as long as possible, and we are quite content not to have pursued the algebraic group context here.
REFERENCES

BRANDT, A.J.

BRYANT, R.M. and KOVÁCS, L.G.

CARTER, R. and CLINE, E.

CARTER, R. and LUSZTIG, G.

CHEVALLÉY, C.

CLINE, E.

COHN, P.M.

CURTIS, C.W. and REINER, I.
GLOVER, D.J.  

GREUB, W.  

HIGMAN, G.  

HUMPHREYS, J.E.  

JACOBSON, N.  
    Colloquium Publications, 39, American Mathematical Society, Rhode
    Island, 1968.

JAMES, G.D.

[1] The Representation Theory of the Symmetric Groups, Lecture Notes in
    Mathematics, 682, Springer-Verlag, Berlin, Heidelberg, New York,
    1978.

KAPLANSKY, I.

    575-580.

MAGNUS, W. and KARRASS, A. and SOLITAR, D.

[1] Combinatorial Group Theory, Pure and Applied Mathematics, 13, Wiley-

NEUMANN, H.

[1] Varieties of Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete,

STEINBERG, R.

[1] "Lectures on Chevalley Groups", Mimeographed Lecture Notes, Department
    of Mathematics, Yale University, New Haven, 1968.

van der WAERDEN, B.L.


WALL, G.E.

    Conf. Theory of Groups (ed. by Newman, M.F.), Lecture Notes in
    Mathematics, 372, 667-690, Springer-Verlag, Berlin, Heidelberg,

    Lecture Notes in Mathematics, 697, 137-173, Springer-Verlag,
WEVER, F.


WEYL, H.