ON MATRIX METHODS IN RING THEORY

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To my parents ...
PREFACE

The work for this thesis was done under the guidance of two supervisors: the late Professor Hanna Neumann and Dr Carl Christensen. I am indebted to Professor Neumann for her valuable supervision, for the Mathematics she taught me and for the seven genial years I spent in her department. I thank Dr Christensen for his help in writing the thesis.

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Except where otherwise stated, the results in this thesis are my own.

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A method for representing rings as matrix rings is used to investigate the structures of several well-known classes of rings. The general method is developed in Chapter 1.

In Chapter 2 nonsingular rings with essential socles are characterized by embeddings into products of full matrix rings over fields. This generalizes the known results in the case when the rings' identities are finite (sums of orthogonal idempotents).

The results are used in Chapter 3 to study nonsingular QF-3 rings with finite identities. In particular the structure of QF-3 rings whose identities are finite and whose (principal, finitely generated) ideals are projective is determined.

Chapter 4 is concerned with rings whose ideals are quasi-injective. It is shown that if such a ring is indecomposable and has more than one idempotent, then it is Artinian. The structure of these rings is then obtained.

In Chapter 5 the structure of left generalized uniserial rings is determined in terms of the structure of left uniserial rings. This generalizes the known results for (left and right) generalized uniserial algebras.
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CHAPTER 1

INTRODUCTION

This thesis is a study of several unrelated classes of rings, but is unified by a common approach to investigating the structures of the rings. This is a method for representing rings as matrix rings which is slightly more general than the standard methods. Matrix methods are such useful tools for determining ring structures that one of the main contributions of the thesis is the systematic use of such a method in quite different branches of ring theory.

Because the rings studied belong to different branches of ring theory, each of the following chapters has its own survey of the particular area with which it is concerned. These introductions are, necessarily, short but they give a general setting for the results in each chapter. The present chapter develops a general approach to matrix representation of rings and gives the details of arguments which are later stated in outline only. The subsequent chapters are independent of each other, except that Chapter 3 uses results from Chapter 2.

The following conventions, terminology and notation are used throughout the thesis. All exceptions to these are specifically mentioned, so there are, hopefully, no ambiguities. All rings have identities and all modules are unital. Modules, ideals and ring-theoretic properties are left ones, but the adjective left is sometimes added for emphasis, or to avoid possible doubt. A homomorphism between R-modules is an R-homomorphism. Homomorphisms are written on the right, except those between right modules which
are written on the left. Composition of homomorphisms is indicated by juxtaposition - on the right for homomorphisms written on the right, and on the left for those written on the left. The isomorphism symbol $\cong$ is usually not subscripted but its meaning is always clear. The same symbol is also used, on a few occasions, to mean isomorphic as rings. The symbol $\oplus$ means direct sum as modules.

If $M$ and $N$ are $R$-modules, then $\text{Hom}_R(M,N)$ and $\text{End}_R(M)$ denote, respectively, the group of $R$-homomorphisms from $M$ to $N$, and the ring of $R$-endomorphisms of $M$. If these have any extra structure, then the same symbol is, nevertheless, used to denote the more sophisticated object. Again, the subscript $R$ is usually omitted, if there is no likelihood of confusion. The group of row-finite $c \times d$-matrices over a ring $A$ is denoted by $M(A,c\times d)$.

The results and methods of this chapter are used constantly and, usually, without specific reference. Let $R$ be a ring, $L$ an ideal of $R$ and $e$ an idempotent in $R$. If $\phi:Re \rightarrow L$ is a homomorphism then $e\phi \in eL$, since $e(e\phi) = (e^2)\phi = e\phi$. Conversely, multiplying $Re$ on the right by an $x \in eL$ induces a homomorphism $Re \rightarrow L$. There is a natural, left $eRe$-module structure on $\text{Hom}(Re,L)$ such that if $x \in eRe$ and $\phi \in \text{Hom}(Re,L)$, then $x\phi$ is the homomorphism which maps $e$ onto $x(e\phi) = ex\phi$. Clearly, the function which takes $\phi \in \text{Hom}(Re,L)$ to $e\phi \in eL$ is an $eRe$-isomorphism from $\text{Hom}(Re,L)$ to $eL$. These two modules are often identified and, in particular, $eRe$ is usually considered to be the ring of $R$-endomorphisms of $Re$. 
1.1 DEFINITION A (left) R-module M is (left) faithful if, for every non-zero \( r \in R \), the product \( rM \) is non-zero. A submodule N of a module M is essential (in M), if \( N \cap X \neq 0 \) for every non-zero submodule X of M, and then M is an essential extension of N.

The following theorem is the fundamental result (in this thesis) on the representation of rings.

1.2 THEOREM

Let E be a right faithful, two-sided ideal in a ring R. Then there is an embedding \( \phi : R \to \text{End}(E_R) \) with the properties that \( E\phi \) is a faithful right ideal, and an essential left \( E\phi \)-submodule, of \( \text{End}(E_R) \). If \( r \in R \) then the endomorphism \( r\phi \) of E maps \( x \in E \) onto \( xr \).

Proof. As E is a right ideal, multiplying it on the right by an element of R induces an endomorphism of \( E_R \). Since E is right faithful, any such endomorphism which is induced by a non-zero element is non-zero. Therefore, the function \( \phi \) is a ring monomorphism. If \( b = b\phi \in E\phi \) and \( \phi \in \text{End}(E_R) \), then for \( x \in E \) the image \( x(b\phi) = (xb)\phi = x(b) = x\beta' \), where \( \beta' = (b\phi)\phi \in E\phi \). If \( b \notin \ker \phi \), then \( b\phi \neq 0 \). That is, \( E\phi \) is a faithful right ideal of \( \text{End}(E_R) \) and is, also, an essential left \( E\phi \)-submodule of \( \text{End}(E_R) \).

The utility of the above result depends on how much is known about the ideal E. The ring R is itself a particular case of E and is often the best ideal to use, especially since then the embedding is onto. *Finitely generated*

Assume that E is a direct sum of left ideals \( E_{ij} \), \( i \in I \), \( j \in J(i) \), whose indexing satisfies the relation \( E_{ij} = E_{st} \) if, and
only if, $i = s$, and let $N$ be the disjoint union of the $J(i)$. Then $\text{End}(E)$ is isomorphic to the ring of all row-finite $N \times N$ matrices whose entries at the place $(s,t)$, $s \in J(i)$, $t \in J(j)$, are elements of $\text{Hom}(E_{is}, E_{jt})$. It is clear that, for fixed $i \in I$, all the rings $\text{End}(E_{ij})$ are mutually isomorphic — isomorphic to $H_{ii}$, say. If $i, s \in I$, then it is also clear that the $(H_{ii}, H_{ss})$-bimodules $\text{Hom}(E_{ij}, E_{st})$ are isomorphic — isomorphic to $H_{is}$, say. Moreover, these isomorphisms can be picked in such a way that they commute with the multiplication (composition of homomorphisms) on $\cup \text{Hom}(E_{ij}, E_{st})$. By substituting the $H_{ij}$ into the above matrix representation of $\text{End}(E)$, it can be seen that $\text{End}(E)$ is isomorphic to the ring $H$ of all blocked row-finite $N \times N$ matrices whose $(i,j)^{th}$ block is a row-finite $J(i) \times J(j)$-matrix over $H_{ij}$. For simplicity of exposition, the rings $\text{End}(E)$ and $H$ are usually identified and so are $H_{is}$ and $\text{Hom}(E_{ij}, E_{st})$. If $E_{ij}$ is not finitely generated, then $\text{End}(E)$ is isomorphic to $H$ with its $(i,t)^{th}$ blocks replaced by blocks which may have infinite rows.

1.3 DEFINITION A matrix is finitary if it has only a finite number of non-zero entries. The matrix, in $H$, whose only non-zero entry is the element $\phi$ of $H_{ij}$ at the place $(s,t)$ in the $(i,j)^{th}$ block is denoted by $|\phi|_{ij}^{st}$. The idempotents $e_i$ are orthogonal if all products $e_i e_j$ are zero.

If $e$ and $f$ are orthogonal idempotents in a ring $R$, then $e + f$ is an idempotent and $R(e + f) = Re \oplus Rf$. In particular, $R = Re \oplus R(1-e)$.

Assume that each summand $E_{ij}$ of the ideal $E$ is generated by an idempotent $e_{ij}$ and that the $e_{ij}$ are orthogonal. If $\chi$ is the endomorphism of $E$ which is induced by right multiplication by $x \in e_{ij} Re_{pq}$, then $e_{ij} \chi = x$ and $e_{st} \chi = 0$, for $e_{st} \neq e_{ij}$. Therefore
χ corresponds to the matrix $|x|_{ij}^{st}$ of $H$. If $K \subseteq H$ is the image of $R$ under the ring monomorphism induced by $\phi$, then, clearly, $K$ contains the matrices $|a|_{k\ell}^{mn}$, and hence the finitary matrices, of $H$. Moreover, the identity of $H$ is in $K$, since the identity of $R$ induces the identity function on $E$.

Let $G$ be a subring of $H$ containing the finitary matrices and the identity, and consider the left ideal $F = \sum_{i,j} Gf_{ij}$, where $f_{ij} = |1|_{ij}$. If $x \in H$ then $f_{ij}x$ is the matrix which has the same entries as $x$, at the places $(j,\ell)$ in the $(i,k)^{th}$ blocks, and whose other entries are zeros. As $x$ is row-finite, this means that $f_{ij}x$ is a finitary matrix and is, therefore, in $F$. Hence, $F$ is a right ideal of $G$. If $x$ has a non-zero entry at the place $(s,t)$ of the $(i,j)^{th}$ block, then $f_{is}x \neq 0$, and so, $Fx \neq 0$. That is, $F$ is right faithful. It follows from the above discussion that $\text{End}(G_F)$ is isomorphic to the ring of all blocked row-finite $N \times N$-matrices whose $(i,j)^{th}$ block is a row-finite $J(i) \times J(j)$-matrix over $G_{ij} = \text{Hom}(Gf_{im}, Gf_{jn})$. But $G_{ij} = f_{ij}Gf_{ij} = H_{ij}$, therefore the rings $\text{End}(G_F)$ and $H$ are isomorphic. This proves the following theorem.

1.4 THEOREM

Let $R$ be a ring and $\{e_{ij} | i \in I, j \in J(1)\}$ a set of orthogonal idempotents of $R$ with the property that $\sum_{i,j} R_{ij}$ is a faithful right ideal. Then $R$ is isomorphic to a subring $K$ of the ring $H$ of all blocked row-finite matrices whose $(i,j)^{th}$ block is a row-finite $J(i) \times J(j)$-matrix over an additive abelian group $H_{ij}$. The groups $H_{ij}$ and the subring $K$ have the following properties.
(i) There is a partial, associative, multiplicative structure on \( \cup_{i,j} H_{ij} \) such that for every \( \phi \in H_{ij} \) and every \( \psi \in H_{jk} \) the product \( \phi \psi \) is defined and is in \( H_{ik} \).

(ii) Every \( H_{ii} \) is a ring and every \( H_{ij} \) is an \( (H_{ii}, H_{jj}) \)-bimodule.

(iii) The identity of \( H \) is in \( K \).

(iv) The finitary matrices of \( H \) are in \( K \).

(v) For each \( i \), the rings \( H_{ii} \) and \( \text{End}(R_{ij}) \) are isomorphic. For all \( i, j \), this makes \( \text{Hom}(R_{ij}, R_{lj}) \) into an \( (H_{ii}, H_{jj}) \)-bimodule isomorphic (as bimodules) to \( H_{ij} \). Under these isomorphisms the multiplication described in (i) corresponds to composition of homomorphisms.

Conversely, if \( K \) and \( H \) satisfy the above conditions down to, and including, condition (iv), then \( K \) has a set \( \{ f_{ij} | i \in I, j \in J(1) \} \) of orthogonal idempotents with the property that \( \sum_{i,j} K f_{ij} \) is a faithful right ideal of \( K \). Moreover, the rings \( H \) and \( \text{End}(\sum_{i,j} K f_{ij}) \) are isomorphic.

1.5 DEFINITION An idempotent is primitive if it is not the sum of two orthogonal idempotents. An idempotent is finite if it is a sum of (a finite number of) orthogonal idempotents. A ring is indecomposable if it is not a direct sum of two two-sided ideals.

Let \( R \) be an indecomposable ring with finite identity. Then \( R \) has a finite set \( \{ e_{ij} | 1 \leq i \leq n, 1 \leq j \leq v(i) \} \) of orthogonal primitive idempotents whose sum is the identity and which satisfy the relation \( R e_{ij} = R e_{st} \) if, and only if, \( i = s \). Since \( e_{ij} \) is primitive, \( R e_{ij} \) is indecomposable and, therefore, \( \text{End}(R e_{ij}) \) has a
unique idempotent, the identity. Two idempotents \( f, f' \) are said to be connected by idempotents \( f_1 = f, f_2, \ldots, f_m = f' \) if, for every \( i \), one of the products \( f_i R f_{i+1}, f_{i+1} R f_i \) is non-zero. If \( e \) is the sum of those \( e_{ij} \) which are connected by elements of \( \{e_{st}\} \) to \( e_{11} \) and if \( e_{pq} \) is not one of these, then \( e_{pq} \in R(1-e) \) and \( e_{pq} R e = 0 \). Therefore, \( (1-e)R e = 0 \) and, similarly, \( eR(1-e) = 0 \). Since \( R \) is indecomposable this means that \( e = 1 \). That is, every \( e_{ij} \) is connected by elements of \( \{e_{st}\} \) to \( e_{11} \). Combined with Theorem 1.4, this proves the following result.

1.6 COROLLARY

Let \( R \) be an indecomposable ring with a finite identity. Then there are positive integers \( n, v(1), \ldots, v(n) \) and additive abelian groups \( H_{ij}, 1 \leq i, j \leq n \), such that \( R \) is isomorphic to the ring \( H \) of all blocked matrices whose \( (i,j) \)-th block is a \( v(i) \times v(j) \)-matrix over \( H_{ij} \). The groups \( H_{ij} \) have the following properties.

(i) There is a partial, associative multiplication on \( \bigcup_{i,j} H_{ij} \) with the property that if \( \phi \in H_{ij} \) and \( \psi \in H_{jk} \), then \( \phi \psi \) is defined and is in \( H_{ik} \).

(ii) Every \( H_{ii} \) is a ring whose only idempotent is the identity and every \( H_{ij} \) is an \( (H_{ii}, H_{jj}) \)-bimodule.

(iii) For all \( s, t \) there is a sequence \( s = r(1), \ldots, r(m) = t \) with the property that for each \( r(i) \) either \( H_{r(i)r(i+1)} \neq 0 \) or \( H_{r(i+1)r(i)} \neq 0 \).

Conversely, every matrix ring which satisfies the above conditions is indecomposable and has a finite identity.
CHAPTER 2

NONSINGULAR RINGS WITH ESSENTIAL SOCLES

One of the most important recent advances in noncommutative ring theory has been the development of a theory of nonsingular rings and modules. This chapter is devoted to a study of a branch of this theory.

2.1 DEFINITION The singular submodule \( Z(M) \) of an \( R \)-module \( M \) is the submodule of all elements of \( M \) which are annihilated by essential ideals of \( R \). \( M \) is nonsingular if \( Z(M) = 0 \), and it is singular if \( Z(M) = M \). The left singular ideal \( Z_l(R) \) of a ring \( R \) is the ideal \( Z(R) \). The right singular ideal \( Z_r(R) \) of \( R \) is defined analogously. \( R \) is left (respectively, right) nonsingular if \( Z_l(R) = 0 \) (respectively, \( Z_r(R) = 0 \)). The socle \( S(M) \) of a module \( M \) is the sum of its simple (minimal) submodules. The left and right socles of a ring \( R \) are defined analogously and are denoted by \( S_l(R) \) and \( S_r(R) \), respectively. A module is uniform if all of its submodules are essential.

Nonsingular modules were introduced by R.E. Johnson [21] who showed that a nonsingular ring \( R \) can be embedded in a regular ring \( R' \) in such a way that \( R \) is an essential \( R \)-submodule of \( R' \). However, it was the appearance, a number of years later, of A.W. Goldie's famous papers [11] and [12] which inspired the widespread interest in the concept. Goldie showed that a ring \( R \) has a semi-simple Artinian ring as a classical quotient ring if, and only if, \( R \) is semi-prime (no nilpotent ideals), has maximum condition on annihilators, and has an essential ideal which is a direct sum of a finite number of
uniform ideals. It was the fact that these rings, often called semi-prime Goldie rings, are nonsingular which was responsible for the development of a theory of nonsingular rings. C. Faith and Y Utumi [9] improved Goldie's result by giving a "picture" of semi-prime Goldie rings. They showed that if a ring $R$ has a classical quotient ring which is isomorphic to the ring of all $n \times n$-matrices over a sfield $D$, then $R$ has a subring isomorphic to the ring of $n \times n$-matrices over an Ore domain $A$ which has $D$ as its ring of quotients. If the semi-prime condition is dropped then the results become weaker. V. Dlab [7], [8] showed that a ring is nonsingular and has an essential ideal which is a direct sum of uniform ideals if, and only if, its injective hull is a direct product of full, row-finite matrix rings each over a sfield. More generally, it can be shown that the injective hull of an arbitrary nonsingular ring is a self-injective regular ring. Johnson [22] had earlier shown that some of the rings studied by Dlab can be characterised by the property that they have certain full, finite matrix subrings.

Nonsingular Artinian rings were first studied by Goldie [13] who showed that a nonsingular generalized uniserial ring is a product of a finite number of full blocked triangular matrix rings, each over a sfield. The structure of nonsingular Artinian rings in which each ideal generated by a primitive idempotent is uniform was determined independently by R. Gordon [14] and R.R. Colby and E.A. Rutter Jr. [3]. The more general case when the ring is semi-primary was solved by A. Zaks [35]. Nonsingular rings with essential socles and finite identities were characterized
simultaneously, but independently, by Gordon [15] and the author [18].
The Artinian case was solved by the author in 1968 as part of his
undergraduate work at the Australian National University. The above-
mentioned structure theorems were obtained by representing the rings
in question as matrix rings.

This chapter is devoted to a study of nonsingular rings with
essential socles. Arbitrary nonsingular rings with essential socles
are characterized by the property that they have certain matrix
subrings. Unfortunately, this result is rather vague, but better
characterizations can be obtained under slightly stronger assumptions.
The chapter closes with a discussion of some module-theoretic
properties of nonsingular rings.

The following, rather trivial, well-known results are essential
for the discussion in this, and the following, chapter.

2.2 LEMMA

If \( N \) is an essential submodule of an \( R \)-module \( M \), then for any \( m \in M \)
there is an essential ideal \( E \) of \( R \) such that \( Em \subseteq N \).

Proof. Let \( E = \{ r \in R | rm \in N \} \) and let \( L \) be a non-zero ideal
of \( R \). If \( Lm = 0 \) then \( L \subseteq E \), if \( Lm \neq 0 \) then \( Lm \cap N \neq 0 \), and so
\( L \cap E \neq 0 \). Therefore \( E \) is essential in \( R \).

2.3 LEMMA

Let \( M, N \) be \( R \)-modules such that \( Z(N) = 0 \). If the kernel of a
homomorphism \( \phi : M \to N \) is essential, then \( \phi = 0 \). In particular, if
\( M \) is uniform then every non-zero \( \phi : M \to N \) is a monomorphism.

Proof. Assume that there is a non-zero homomorphism \( \phi : M \to N \nwhose kernel is essential. If \( m \in M \) is an element not in the kernel
of \( \phi \), then by Lemma 2.2 there is an essential ideal \( E \) of \( R \) with the property that \( E \mathfrak{m} \subseteq \ker \phi \). But this means that \( E(m \phi) = (Em) \phi = 0 \); a contradiction to the hypothesis that \( N \) is nonsingular. Therefore the kernel of every non-zero homomorphism \( \phi : M \to N \) is not essential.

From now on, let \( R \) be a nonsingular ring with essential socle. If \( M \) is a minimal ideal of \( R \) then there is a non-zero homomorphism \( \phi : R \to M \). By Lemma 2.3, \( \ker \phi \) is not essential in \( R \). As \( \ker \phi \) is a maximal ideal this means that it is a direct summand of \( R \). Therefore there is a, necessarily primitive, idempotent \( e \in R \) with the property that \( M = Re \). This proves the following lemma.

**2.4 Lemma**

*If \( M \) is a minimal ideal in the nonsingular ring \( R \) then there is a primitive idempotent \( e \in R \) for which \( Re = M \).*

Let the ideal \( E_0 \) be that part of the socle of \( R \) which is generated by idempotents, and let \( E_1 \) be an ideal maximal with respect to both not intersecting \( E_0 \) and the property that its sum with \( E_0 \) is a two-sided ideal. Since the socle of \( R \) is a two-sided ideal it is contained in \( E = E_0 \oplus E_1 \) and, therefore, \( E \) is a faithful right ideal. Hence it follows from Theorem 1.2 that there is an embedding of \( R \) into \( \text{End}(\mathcal{E})_R \). The image of \( R \) under this embedding is characterized by the following theorem.

**2.5 Theorem**

*Let \( R \) be a nonsingular ring with essential socle. Then there is a set \( I \) containing the integer 1 and, for each \( i \in I \), a set \( J(i) \) where \( J(1) = \{1\} \), such that \( R \) is isomorphic to a subring \( K \) of the matrix ring \( H \) of all blocked row-finite matrices whose \((i,j)\)th*
block is an arbitrary row-finite $J(1) \times J(j)$-matrix over an additive abelian group $H_{ij}$. The subring $K$ and the groups $H_{ij}$ have the following properties.

(i) Each $H_{ii}$ is a ring and each $H_{ij}$ is an $(H_{ii}, H_{jj})$-bimodule.

(ii) If $i \neq 1$ then $H_{ii}$ is a skewfield.

(iii) If for $i \neq 1$ the left $H_{ii}$-dimension of $H_{1i}$ is $b_i$, then

$$H_{1i} \subseteq \prod_{i \neq 1} M(H_{ii}, b_i \times b_i).$$

(iv) $\sum_{i \neq 1} H_{1i}$ is a faithful right $H_{11}$-module.

(v) If $i \neq j$ and $j \neq 1$ then $H_{ij} = 0$.

(vi) The identity of $H$ is in $K$.

(vii) For each $i \in I \setminus \{1\}$ and each $j \in J(i)$ there is a matrix $f_{ij} \in K$ whose non-zero entries are all in the $j^{th}$ column of the $(i,i)^{th}$ block and whose entry at the place $(j,j)$ is the identity of $H_{ii}$.

(viii) Each matrix $f_{ij} \cdot d_{ij}^{ik}$ is in $K f_{ik}$.

(ix) Each matrix $f_{ij} \cdot h_{ij}^{11}$ is in $K$.

(x) For each $i \in I \setminus \{1\}$, each $j \in J(i)$ and every non-zero element $x$ of $K$ whose non-zero entries are all in the $j^{th}$ column of the $(i,i)^{th}$ block, there is a matrix $y \in K$ with the property that $yx = f_{ij}$.

Conversely, if $K$ is a subring of a blocked matrix ring $H$ such that both are described by the above theorem, then $K$ is a nonsingular ring with essential socle.
Proof. As mentioned above, the isomorphism between $R$ and $K$ is obtained by applying Theorem 1.2. To do this, express $E_0$ as a direct sum $\bigoplus_{i \in I'} R_{i,j}$ of minimal ideals generated by primitive idempotents, which are indexed in such a way that $R_{i,j} \cong R_{s,t}$ if, and only if, $i = s$. Assume, moreover, that $I'$ does not contain the integer 1, and let $I = I' \cup \{1\}$, $J(1) = \{1\}$ and $E_{11} = E_1$. It follows from the discussion following the proof of Theorem 1.2 that $\text{End}(E)$ is isomorphic to a ring $H'$ of blocked matrices whose $(i,j)^{th}$ block, $i \neq 1$, is an arbitrary row-finite $J(i) \times J(j)$-matrix over $H_{i,j}$, which is isomorphic to $\text{Hom}_R(R_{i,s}, R_{j,t})$, and whose $(1,p)^{th}$ block is a $1 \times J(p)$-matrix over $H_{1,p}$, which is isomorphic to $\text{Hom}_R(E_{11}, R_{1,q})$.

As before, $\text{End}(E)$ and $H'$ are identified. In view of this representation and the fact that, by Theorem 1.2, $R$ is isomorphic to a subring $K$ of $H'$, it is sufficient to show that $K$ satisfies conditions (vi)-(x) and is contained in a subring $H$ of $H'$ which satisfies conditions (ii)-(v).

(ii) Each $R_{i,j}$ is a minimal ideal so, for $i \neq 1$, each $H_{i,i}$ is a sfield.

(iii) Clearly, $b_i$ is the cardinal of a set $s(i)$ of images of $R_{i,j}$ in $E_{11}$ which is maximal with respect to the sum $\sum_{s(i)} M$ being direct. Since every minimal ideal of $R$ is isomorphic to an $R_{i,j}$ (Lemma 2.4) the socle of $E_{11}$ is the direct sum $\bigoplus_{i \neq 1} \sum_{s(i)} M$. By Lemma 2.3, the restriction to $s(i)$ of a non-zero endomorphism of $E_{11}$ is non-zero. Therefore $H_{11}$ can be regarded as a subring of $\bigoplus_{i \neq 1} \sum_{s(i)} M(H_{i,i} \times b_i)$ which is isomorphic to $\text{End}(S(E_{11}))$. 

(iv) Each \( R_{ij} \) is a direct summand of \( R \), so \( S(E_{11}) = R(\sum_{i \neq 1} H_{1i}) \). As \( S(E_{11}) \) is not annihilated by non-zero endomorphisms of \( E_{11} \), the product \( (\sum_{i \neq 1} H_{1i})h \) is non-zero, for each non-zero \( h \in H_{11} \).

(v) If \( i \neq j \) and \( i, j \neq 1 \) then \( R_{is} \nparallel R_{jt} \) and, therefore, \( H_{ij} = 0 \). For non-zero \( x \in R_{pq} \) if \( E_{11}x \neq 0 \) then, since \( Rx = R_{pq} \) is projective, \( E_{11} = N \oplus M \) where \( Mx \neq 0 \) and \( M = Rx \). By definition of \( E_0 \), \( M \) is nilpotent and so \( Mx = 0 \): a contradiction. Hence \( E_{11}x = 0 \) for all \( x \in R_{pq} \) and, therefore, \( K \) is contained in the subring \( H \) of \( H' \) of all matrices whose \((1,p)\)th blocks, \( p \neq 1 \), are zero.

(vi) Since the identity endomorphism on \( E \) can be extended to the identity on \( R \), the identity of \( H \) is in \( K \).

(vii) The idempotent \( e_{ij} \) induces, by multiplication on the right, a homomorphism \( E \to R_{ij} \) which is the identity on \( R_{ij} \). Therefore \( e_{ij} \) is mapped, by the embedding of \( R \) into \( H \), onto a matrix \( f_{ij} \in K \) whose non-zero entries are all in the \( j \)th column of the \((i,i)\)th block and whose entry at the place \((j,j)\) is the identity of \( H_{11} \).

(viii) Every homomorphism \( d:R_{ij} \to R_{ik} \) can be extended to the endomorphism \( |d|_{ij}^{ik} \) of \( E \). Therefore \( R_{ij} \) is mapped by \( |d|_{ij}^{ik} \) into \( R_{ik} \), that is, \( x|d|_{ij}^{ik} \in Kf_{ik} \), for every \( x \in Kf_{ij} \).

(ix) A similar argument shows that for every \( h \in H_{11} \) the matrix \( f_{ij}h|_{ij}^{11} \) is in \( K \).

(x) The ideal \( E \) is mapped by \( x \) into \( R_{ij} \), so \( x(1-f_{ij}) \) annihilates \( E \). Hence Lemma 2.3 implies that \( x(1-f_{ij}) = 0 \) and so \( Kx \subseteq Kf_{ij} \). As \( Kf_{ij} \) is a minimal ideal, there is a matrix \( y \in K \) such that \( yx = f_{ij} \).
The ring $K$ has an identity (condition (vi)), so to prove the converse it is necessary to show only that the socle of $K$ is essential and that $K$ is nonsingular. The former is true if every non-zero principal ideal $Kx$ contains a minimal ideal. By $(x)$, $Kf_{ij}$ is a minimal ideal, so $Kx$ contains a minimal ideal isomorphic to $Kf_{ij}$, if $f_{ij}Kx \neq 0$. If $x$ has a non-zero entry at the place $(j,k)$ of the $(i,t)^{th}$ block, $i \neq 1$, then $f_{ij}x \neq 0$; and if $x = |h|^{11}$ then, by (iv), there is a $g \in H_{11}$, for some $i \neq 1$, such that $gh \neq 0$ and so $f_{ij}|g|^{11}_{ij}x \neq 0$. That is, $Kx$ always has a minimal ideal: therefore $S_{\mathcal{A}}(K)$ is essential in $K$. To show that $K$ is nonsingular it is sufficient to prove that $S_{\mathcal{A}}(K)x \neq 0$, for every non-zero $x \in K$. If $x \neq 0$, then one of the ideals $Kf_{ij}x$, $Kf_{ij}|h|_{ij}^{11}$ is non-zero. and so, since $Kf_{ij}$ and $Kf_{ij}|h|_{ij}^{11}$, $h \neq 0$, are minimal ideals, $S_{\mathcal{A}}(K)x \neq 0$: therefore $K$ is nonsingular.

Dorroh’s extension $A^*$ of a ring $A$ with characteristic $t$ is the unital ring on $\mathbb{Z}_t \times A$, $\mathbb{Z}_t$ being the integers modulo $t$, whose addition is component-wise and whose multiplication is given by $(m,x)(n,y) = (mn, my + nx + xy)$, for all $m, n \in \mathbb{Z}_t$, $x, y \in A$. It is easy to see that the isomorphic image $\{0\} \times A$ of $A$ is an essential ideal of $A^*$. Therefore, $A$ is a nonsingular ring with essential socle if, and only if, $A^*$ is. Consequently, the next result follows immediately from conditions (ii) and (iii) of Theorem 2.5.
2.6 COROLLARY

The characteristic of a nonsingular ring with essential socle (but not necessarily with identity) is not divisible by $p^2$, for any prime $p$.

2.7 COROLLARY

The ring $R$ of Theorem 2.5 is indecomposable if, and only if, for every proper subset $I_0 \subseteq I \setminus \{1\}$ and every idempotent $e \in H_{11}$ with the properties that $H_{ii}e = H_{ii}$, for $i \in I_0$, and $H_{ii}e = 0$, for $i \notin I_0$, the matrix $f(I_0, e)$ is not in $K$, where $f(I_0, e)$ is the matrix whose only non-zero entries are the identities of the corresponding $H_{ii}$ on the diagonals of the blocks $(i,i)$, $i \in I_0$, and $e$ in the block $(1,1)$.

Proof. If $R$ is decomposable then it has an idempotent $e$ such that $R$ is the sum of the non-zero two-sided ideals $Re$ and $R(1-e)$, that is, $R = eRe \oplus (1-e)R(1-e)$. Therefore each $e_{ij}$ is in $eRe$ or in $(1-e)R(1-e)$. If $M \approx Re_{ij}$ and $e_{ij} \in eRe$ then, since $e_{ij}M \neq 0$, $eM \neq 0$ and so $M \subseteq eRe$. Therefore there is a proper subset $I_0 \subseteq I \setminus \{1\}$ with the property that a minimal ideal is in $eRe$ if, and only if, it is isomorphic to an $Re_{ij}$, $i \in I_0$.

Conversely, if for a proper subset $I_0 \subseteq I \setminus \{1\}$ there is an idempotent $e \in R$ with the property that a minimal ideal is in $Re$ if, and only if, it is isomorphic to an $Re_{ij}$, $i \in I_0$, then $R$ is the sum of the non-zero two-sided ideals $Re$, $R(1-e)$ and so is decomposable. The ideals $Re$, $R(1-e)$ are two-sided since $eR(1-e) = (1-e)Re = 0$. For if $eR(1-e) \neq 0$ then there is a non-
zero homomorphism \( \phi : R \times e \to R(1-e) \) which, by Lemma 2.3, does not kill a minimal ideal in \( R \times e \). Therefore \( R(1-e) \) contains a minimal ideal isomorphic to a minimal ideal in \( R \times e \): a contradiction to the hypothesis on \( e \). Therefore \( e R(1-e) = 0 \) and, similarly, \( (1-e)R = 0 \). The ideals \( R \times e, R(1-e) \) are non-zero since \( I_0 \) is a proper subset of \( I \setminus \{1\} \) and so each contains an ideal \( R e_i \). It follows that \( R \) is indecomposable if, and only if, it does not have an idempotent \( e \) with the above properties. If \( R \) does contain such an idempotent \( e \), then because \( e \) is a left and a right identity for \( R e_i \), \( i \in I_0 \), and their homomorphic images in \( R \), \( e \) is mapped, by the embedding of \( R \) into \( H \), onto a matrix \( f(I_0, e') \), where \( e' \in H_{11} \) is an idempotent with the properties that 
\[ H_{11} e' = H_{11}, \text{ for } i \in I_0, \text{ and } H_{11} e' = 0, \text{ for } i \notin I_0. \]
Conversely, if \( K \) contains an idempotent \( f = f(I_0, e') \) then \( K = fKf \oplus (1-f)K(1-f) \) and so \( K \) is decomposable.

2.8 COROLLARY

The ideal \( E_0 = \sum_{i,j} R e_{ij} \) of the ring \( R \) of Theorem 2.5 is generated by a set of orthogonal primitive idempotents if, and only if, conditions (vii) - (x) of Theorem 2.5 can be replaced by the condition that all of the finitary matrices with zero in the block \((1,1)\) are in \( K \).

Proof. Necessity. Clearly it can be assumed that the idempotents \( e_{ij} \) are orthogonal. Therefore, their images, the idempotents \( f_{ij} \), in \( K \) are orthogonal, and so \( f_{ij} = [1]_{ij}^{11} \). Hence
conditions (viii) and (ix) merely state that the finitary matrices with zero in the block $(1,1)$ are in $K$. If $x \in K$ has non-zero entries only on the $j^{th}$ column of the $(i,i)^{th}$ block and the non-zero entry $d$ at the place $(k,j)$ then $|d^{-1}i_k|_{ij} x = |1|_{ij}$. So condition (x) is satisfied.

Sufficiency. It was just shown that the above condition implies conditions (vii) - (x), so it remains to show only that the sum of all minimal ideals of $K$ generated by idempotents is generated by a set of orthogonal primitive idempotents. It is clear that $\{ |1|_{ij} \}$ is such a generating set.

2.9 THEOREM

If $R$ is a nonsingular ring with essential socle and an essential ideal which is generated by a set of orthogonal primitive idempotents and which is also a right ideal, then there are disjoint sets $I_0$ and $I_1$ and sets $J(i)$, $i \in I = I_0 \cup I_1$, such that $R$ is isomorphic to a subring $K$ of the ring $H$ of all blocked row-finite matrices whose $(i,j)^{th}$ block is an arbitrary row-finite $J(i) \times J(j)$-matrix over an additive abelian group $H_{ij}$. The subring $K$ and the groups $H_{ij}$ have the following properties.

(i) Each $H_{ii}$ is a ring with a unique idempotent, the identity, and each $H_{ij}$ is an $(H_{ii}, H_{jj})$-bimodule. There is a partial, associative multiplication on $H_{ij}$ such that for every $\phi \in H_{ij}$ and every $\psi \in H_{jk}$ the product $\phi \psi$ is defined and is in $H_{ik}$.

(ii) If $i \in I_0$ then $H_{ii}$ is a sfield and $H_{ij}$ is a left vector space over $H_{ii}$ with dimension $b_{ij}$, say.
(iii) For all \( j, k \in I \), \( H_{jk} \subseteq \prod_{i \in I} M(H_{ii}, b_{ij} \times b_{ik}) \).

(iv) If \( i \neq j \) and \( j \in I_0 \) then \( H_{ij} = 0 \).

(v) For all \( j, k \in I \), if \( \phi \in H_{jk} \) and \( H_{ij} \phi = 0 \), for each \( i \in I_0 \), then \( \phi = 0 \).

(vi) The identity of \( H \) is in \( K \).

(vii) All the finitary matrices of \( H \) are in \( K \).

Conversely, every such matrix ring \( K \) is a nonsingular ring with essential socle and a set of orthogonal primitive idempotents which generates a left essential two-sided ideal.

Proof. Let \( E \) be the left essential two-sided ideal of \( R \) generated by orthogonal idempotents \( e_{ij}, i \in I, j \in J(i) \), which are indexed in the usual way. Lemma 2.4 implies that there is a subset \( I_0 \) of \( I \) with the properties that \( R_{ij}, i \in I_0 \), is a minimal ideal and every minimal ideal is isomorphic to an \( R_{ij}, i \in I_0 \). It follows from Theorems 1.4 and 2.5 that \( R \) can be represented as a matrix ring \( K \) which has all the above properties except (iii) and (v), so it remains to prove that \( K \) satisfies (iii) and (v).

(iii) The socle of \( R_{j\ell} \) is a direct sum of ideals \( L_{ij\ell}, i \in I_0 \), where \( L_{ij\ell} \) is the sum of all minimal ideals in \( R_{j\ell} \) which are isomorphic to an \( R_{is}, i \in I_0 \). As \( L_{ij\ell} \) is a sum of minimal ideals it can be expressed as a direct sum of \( b_{ij} \) (say) minimal ideals. By Lemma 2.3, different homomorphisms from \( R_{j\ell} \) to \( R_{km} \) have different restrictions to \( S(R_{j\ell}) \). Therefore \( \text{Hom}(R_{j\ell}, R_{km}) \cong H_{jk} \) can be
embedded in $\text{Hom}(S(Re_{j, l}^k), S(Re_{km})) = \prod_{i \in I_0} M(H_{ij}^{b_{ij} \times b_{ik}})$.

(v) If $\phi: Re_{j, l} \to Re_{km}$ is non-zero then, by Lemma 2.3, $L_{ij}^\phi \neq 0$ for some $i \in I_0$. Therefore there is a homomorphism $\psi: Re_{i, s} \to Re_{j, l}$ with the property that $\psi \phi \neq 0$.

The converse follows from Theorems 1.4 and 2.5.

When $R$ has a finite identity the above embedding provides a representation of the whole of $R$. Gordon [15] obtained the next result independently from, and at about the same time as the author [18], although Gordon states it only for semi-perfect rings.

2.10 Theorem [15, Theorem 5.1; 18, Theorem 1.1]

If $R$ is an indecomposable nonsingular ring with essential socle and a finite identity, then there are positive integers $m, n, m \leq n$, and positive integers $v(1), \ldots, v(n)$ such that $R$ is isomorphic to the ring $H$ of all blocked matrices whose $(i, j)^{th}$ block is an arbitrary $v(i) \times v(j)$-matrix over an additive abelian group $H_{ij}$. The groups $H_{ij}$ have the following properties.

(i) Each $H_{ii}$ is a ring with a unique idempotent, the identity, and each $H_{ij}$ is an $(H_{ii}, H_{jj})$-bimodule. There is a partial, associative multiplication on $\bigcup_{i, j} H_{ij}$ such that for each $\phi \in H_{ij}$ and each $\psi \in H_{jk}$ the product $\phi \psi$ is defined and is in $H_{ik}$.

(ii) If $m \leq i \leq n$ then $H_{ii}$ is a field and the left dimension of $H_{ij}$ over $H_{ii}$ is $b_{ij}$, say.
(iii) $H_{jk} \subseteq \prod_{m \leq i \leq n} M(H_{ii}, b_{ij} \times b_{ik})$.

(iv) If $1 \neq j$ and $m \leq j \leq n$ then $H_{ij} = 0$.

(v) If $\phi \in H_{jk}$ and $H_{ij}\phi = 0$ for every $i \in \{m, \ldots, n\}$ then $\phi = 0$.

(vi) If $s, t \in \{1, \ldots, n\}$ then there is a sequence $s = r(1), \ldots, r(\ell) = t$ of elements of $\{1, \ldots, n\}$ with the property that, for each $r(j)$, there is an $i \in \{m, \ldots, n\}$ such that both $H_{ir(j)} \neq 0$ and $H_{ir(j+1)} \neq 0$.

Conversely, every matrix ring $H$ satisfying the above conditions is an indecomposable nonsingular ring with essential socle and a finite identity.

Proof. Except for condition (vi) the theorem is an immediate corollary of Theorem 2.5 and Corollary 1.6. Condition (vi) is implied by, and implies, the indecomposability of $R$. For by Corollary 1.6, if $1 \leq s, t \leq n$, then there is a sequence $s = r(1), \ldots, r(\ell) = t$ of elements of $\{1, \ldots, n\}$ with the property that either $H_{pq} \neq 0$ or $H_{qp} \neq 0$, where $p = r(j)$ and $q = r(j+1)$.

Therefore condition (iii) implies that there is an $i \in \{m, \ldots, n\}$ such that both $H_{ir(j)} \neq 0$ and $H_{ir(j+1)} \neq 0$. It is easy to see that condition (vi) implies that $H$ is indecomposable.

2.11 Definition The (Jacobson) radical of a ring is the intersection of the rings maximal ideals. A ring is semi-primary if its Jacobson radical is nilpotent and if the ring modulo its radical is a semi-simple Artinian ring.
2.12 COROLLARY [35, (§4) Theorem 1.4]

If \( R \) is an indecomposable semi-primary ring with essential socle and with the property that each ideal generated by a primitive idempotent has a unique minimal submodule, then \( R \) is isomorphic to a matrix ring \( H \) described by Theorem 2.10 and the following additional properties.

(vii) \( m = n \).

(viii) Each \( H_{ii} \) is a sfield.

(ix) Each \( H_{ij} \subseteq H_{nn} \).

(x) Each \( H_{ni} = H_{nn} \).

(xi) If \( i \neq j \) and \( H_{ij} \neq 0 \), then \( H_{ji} = 0 \).

Conversely, the ring \( H \) is an indecomposable semi-primary nonsingular ring with essential socle and with the property that every ideal generated by a primitive idempotent has a unique minimal submodule.

Proof. The converse is obvious, and since \( R \) is isomorphic to a matrix ring \( H \) satisfying Theorem 2.10, it is necessary only to show that \( H \) enjoys properties (vii) - (xi).

(vii) If \( m < n \) then \( R \) has two non-isomorphic minimal ideals \( R_{e^1} \) and \( R_{e^2} \) generated by idempotents. By condition (vi) of Theorem 2.10 the sfields \( H_{nn} \) and \( H_{mm} \) can be joined by a sequence of \( H_{ij} \)'s. This means that there is a \( j \in \{1, \ldots, n\} \) such that both \( H_{mj} \neq 0 \) and \( H_{nj} \neq 0 \): a contradiction since \( R_{e^k} \) has only one minimal ideal. Therefore \( m = n \).
(ix) This is an immediate consequence of (vii) and (iii) of Theorem 2.10.

(x) As each $R_{ij}$ has a unique minimal submodule, $H_{ni}$ is a one-dimensional vector space over $H_{nn}$. But $H_{ni} \subseteq H_{nn}$ so $H_{ni} = H_{nn}$.

(viii) Since $R$ is semi-primary the ideal $R_{ij}$ has a unique maximal submodule $J_{ij}$ which is nilpotent. Therefore if $\phi$ is an endomorphism of $R_{ij}$ which is not an automorphism then $e_{ij}\phi \in e_{ij}J_{ij}$. But $e_{ij}R_{ij} = H_{ii} \subseteq H_{nn}$ and so $e_{ij}R_{ij}$ has no nilpotent elements, that is $e_{ij}J_{ij} = 0$. Therefore $\phi = 0$ and so $H_{ii}$ is a sfield.

(xi) If $\phi: R_{ik} \to R_{jl}$ and $\psi: R_{jl} \to R_{ik}$ are non-zero homomorphisms then, by (viii), $\phi \psi$ is an automorphism of $R_{ik}$. Therefore $\phi$, $\psi$ are both epimorphisms and, by Lemma 2.3, they are both monomorphisms. Therefore, if $H_{ij} \neq 0$ and $H_{ji} \neq 0$ then $i = j$.

2.13 DEFINITION A (left) T-ring is a ring whose non-zero (left) modules have non-zero socles.

J.S. Alin and E.P. Armendariz [1] have investigated the structure of T-rings whose singular simple modules are injective. When the ring has finite identity Theorem 2.10 provides a quick solution to this problem.

2.14 LEMMA [1, Theorem 1.1 (a)]

If the singular simple modules of a T-ring are injective then the ring is nonsingular.
Proof. Let $R$ be a $T$-ring whose singular simple modules are injective, and assume that $Z(R) = 0$. Since $S(R)$ is the intersection of all essential ideals of $R$, for any minimal ideal $M \subseteq S(R)$ the product $S(R)M = 0$. Therefore $M$ is nilpotent. But, by the hypothesis, $M$ is injective and so is a direct summand of $R$. This means that $M$ is generated by an idempotent: a contradiction to the nilpotence of $M$. Therefore $Z(R) = 0$.

2.15 Theorem

If $R$ is an indecomposable $T$-ring whose identity is finite and whose singular simple modules are injective, then there are positive integers $m, n, m \leq n$, and positive integers $v(1), \ldots, v(n)$ such that $R$ is isomorphic to the ring $H$ of all blocked matrices whose $(i,j)^{th}$ block is an arbitrary $v(i) \times v(j)$-matrix over an additive abelian group $H_{ij}$. The groups $H_{ij}$ have the following properties.

(i) Each $H_{ii}$ is a $s$-field.

(ii) If $i \neq j$ and $i \notin \{m, \ldots, n\}$, then $H_{ij} = 0$.

(iii) If $m \leq i \leq n$, then $H_{ij}$ is an $(H_{ii}, H_{jj})$-bivector space whose left dimension is $b_{ij}$, say.

(iv) For each $j$, $H_{jj} \subseteq \prod_{m \leq i \leq n} M(H_{ii}, b_{ij} \times b_{ij})$.

(v) If $s, t \in \{1, \ldots, n\}$, then there is a sequence $s = r(1), \ldots, r(p) = t$ of indices such that for each $r(j)$ there is an $i \in \{m, \ldots, n\}$ with the property that both $H_{ir(j)} \neq 0$ and $H_{ir(j+1)} \neq 0$. 


Conversely, the ring $H$ is an indecomposable $T$-ring whose singular simple modules are injective and whose identity is finite.

Proof. Let $\overline{R} = R/S_\lambda(R)$ and note that, since $S_\lambda(R)$ is a two-sided ideal of $R$, $\overline{R}$ is a ring whose ideals coincide with its $R$-submodules. Since $\overline{R}$ is a singular $R$-module, every minimal ideal of $\overline{R}$ is a direct summand and so is generated by an idempotent.

Let $e \in R$ be a primitive idempotent which is not in $S_\lambda(R)$. By hypothesis, $\overline{R}$ has essential socle, so it has a minimal ideal $M \subseteq \overline{R}e = Re/S(Re)$. If $M$ is generated by an idempotent $f$ then $ef \neq 0$, since $fe = f$ and $f^2 = f$. Therefore $M$ is generated by the idempotent $f_1 = ef \in eRe$. By Lemma 2.4, $eS_\lambda(R) = 0$ which implies that $eRe \cap S_\lambda(R) = 0$ and, therefore, $eRe$ and $eRe$ are isomorphic rings. Since $e$ is the only idempotent in $eRe$, $f_1 = e$. Therefore, $\overline{eRe}$ is a minimal ideal. It follows that $S(Re)$ is the unique maximal submodule of $Re$ and that $eRe$ is a sfield. Consequently, if $e_1 \in R$ is also a primitive idempotent, then $eRe_1 \neq 0$ if, and only if, $Re = Re_1$. This proves that $H$ satisfies conditions (i) and (ii). The rest of the theorem is the same as in Theorem 2.10.

Now consider the converse. It is clear that every factor module of $H$ has non-zero socle, hence every cyclic $H$-module has non-zero socle and, therefore, $H$ is a $T$-ring. Every singular simple $H$-module $M$ is isomorphic to $H|_{ij}/S(H|_{ij})$, for some $i \in \{1, \ldots, m-1\}$. Consequently, the only ideals which have $M$ as homomorphic images are the $H|_{ij}$, $1 \leq j \leq v(i)$, and their sums.
Hence every homomorphism from an ideal of $H$ to $M$ can be extended to a homomorphism from $H$ to $M$. Therefore $M$ is injective.

When $R$ has infinite identity the problem appears to be difficult. It is still true that if $e \in R$ is a primitive idempotent then $eRe$ is a sfield and $S(Re)$ is the unique maximal submodule of $Re$, but these properties are no longer sufficient to guarantee the converse.
CHAPTER 3

NONSINGULAR QF-3 RINGS WITH FINITE IDENTITIES

In this chapter the results of Chapter 2 are used to study nonsingular QF-3 rings. Finite dimensional QF-3 algebras were introduced by R.M. Thrall [34] as generalizations of T. Nakayama's Quasi-Frobenius algebras [29], [31]. Subsequently, many authors have studied these algebras, and the concept has been generalized, in a number of ways, to rings. The definition used here is that originally given by Thrall. Despite the large amount of work done on QF-3 rings, it is clear, from the fact that the structure of Quasi-Frobenius rings is not yet known, that it is too early to hope for a complete understanding of arbitrary QF-3 rings. However, under the additional hypothesis of being nonsingular, the rings become more tractable.

R.R. Colby and E.A. Rutter Jr. [5] showed that a nonsingular left, and right, QF-3 ring $R$ is embeddable in a semi-simple Artinian ring $S$ in such a way that both $R$ is essential in $S$, and $R^R$ is essential in $S^R$. Earlier, they [4] had obtained a similar result for semi-perfect QF-3 rings which are also partially PP (some principal ideals are projective) rings. These rings are also nonsingular. This generalized the corresponding result for semi-primary rings obtained by M. Harada [16], [17]. These authors showed that if the ring is also hereditary (all ideals are projective) then it is a product of a finite number of full blocked triangular matrix rings, each over a sfield. The last result had been obtained by H.Y. Mochizuki [25] for finite dimensional algebras. In this chapter, a representation of nonsingular QF-3 rings with finite
identities is obtained, and then used to classify special cases. These results generalize and strengthen those mentioned above.

3.1 DEFINITION A ring \( R \) is a (left) QF-3 ring if it has a faithful left module which is (isomorphic to) a direct summand of every faithful left \( R \)-module.

In this chapter all rings have finite identity, although some results are valid without this assumption. The following characterization of arbitrary QF-3 rings, due to Colby and Rutter Jr., is fundamental to the work in this chapter.

3.2 THEOREM \([5, \text{Theorem } 1]\)

If \( R \) is a QF-3 ring (not necessarily with finite identity), then it has a finite set of orthogonal primitive idempotents \( \{e_1, \ldots, e_k\} \) with the properties that \( \sum_{i=1}^{k} e_i \) is injective, has essential socle, and is the (up to isomorphism) unique minimal faithful \( R \)-module.

Conversely, if a ring \( R \) has a set \( \{e_1, \ldots, e_k\} \) of orthogonal primitive idempotents such that \( \sum_{i=1}^{n} e_i \) is a faithful, injective ideal with essential socle, then \( R \) is a QF-3 ring.

3.3 LEMMA

If \( R \) is a nonsingular QF-3 ring (not necessarily with finite identity), then \( R \) has essential socle and every minimal ideal of \( R \) is isomorphic to the socle of one of the ideals \( e_i \) of Theorem 3.2.

Proof. Let \( R = \sum_{i=1}^{k} e_i R_i \) be the minimal faithful ideal of \( R \) given by Theorem 3.2. If \( x \in R \) is non-zero, then \( xR \neq 0 \) and so there is a non-zero homomorphism \( \phi : Rx \to Re \) (given by \( rx\phi = rxa \), for some \( a \in Re \) satisfying \( xa \neq 0 \)). Since \( Re \) has essential socle,
there is a minimal ideal $M \subseteq R \phi$. Let $K = M \phi^{-1}$. Then $\ker\phi|_K$ is a maximal submodule of $K$ but is, by Lemma 2.3, not essential in $K$. Therefore $K = L \oplus \ker\phi|_K$ for some minimal ideal $L = M$. Therefore, every ideal $R$ contains a minimal ideal and so the socle of $R$ is essential. Since $M$ is isomorphic to an $S(R \phi^1)$, every minimal ideal of $R$ is isomorphic to the socle of an $R \phi^1$.

It is clear from Theorem 3.2 that a nonsingular QF-3 ring with finite identity is a direct product of a finite number of indecomposable nonsingular QF-3 rings, so it is sufficient to study only the indecomposable ones.

3.4 THEOREM

If $R$ is an indecomposable nonsingular QF-3 ring with a finite identity, then there are positive integers $k, m, n$ with $k \leq m \leq n$ and $k = n - m + 1$, and positive integers $v(1), \ldots, v(n)$ such that $R$ is isomorphic to the ring $H$ of all blocked matrices whose $(i,j)^{th}$ block is an arbitrary $v(i) \times v(j)$-matrix over an additive abelian group $H_{ij}$. The groups $H_{ij}$ have the following properties, where property $(t)'$ is the right dual of property $(t)$:

(i) Each $H_{ii}$ is a ring with a unique idempotent (the identity) and each $H_{ij}$ is an $(H_{ii}, H_{jj})$-bimodule.

(ii) If $m \leq i \leq n$ then $H_{ii}$ is a sfield.

(ii)' If $1 \leq i \leq k$ then $H_{ii}$ is a sfield.

(iii) If $1 \leq i \leq k$ and $j = n - i + 1$ then $H_{ii} = H_{jj} = H_{ji}$.

(iv) If for $m \leq i \leq n$ the left $H_{ii}$ dimension of $H_{ij}$ is $b_{ij}$, then

$$H_{st} \subseteq \prod_{m < i \leq n} M(H_{ii}, b_{is} \times b_{it}).$$
(v) If \(1 \leq i \leq k\) and \(j = n - i + 1\) then \(H_{si} = M(H_{jj}, b_{js} \times 1)\).

(vi) If \(i \neq j\) and \(m \leq j \leq n\) then \(H_{ij} = 0\).

(vi)' If \(i \neq j\) and \(1 \leq i \leq k\) then \(H_{ij} = 0\).

(vii) If \(\phi \in H_{st}\) then there is an \(i \in \{m, \ldots, n\}\) such that \(H_{is}\phi \neq 0\).

(viii) If \(1 \leq s, t \leq n\) then there is a sequence \(s = r(1), \ldots, r(e) = t\), \(1 \leq r(i) \leq n\), with the property that for each \(r(i)\) there is a \(j \in \{m, \ldots, n\}\) such that both \(H_{jr(i)} \neq 0\) and \(H_{jr(i+1)} \neq 0\).

Conversely, every matrix ring \(H\) with the above properties is an indecomposable nonsingular QF-3 ring with a finite identity.

Proof. Necessity. Let \(R\) be as in the theorem and let

\[
R = R_{11} \oplus \ldots \oplus R_{1l} \oplus \ldots \oplus R_{iv(i)} \oplus \ldots \oplus R_{nv(n)}
\]

be a decomposition of \(R\) such that the \(e_{ij}\) are orthogonal primitive idempotents with the property that \(R_{ij} = R_{st}\) if, and only if, \(i = s\). For simplicity of notation, denote \(e_{11}\) by \(e_1\). In view of Theorem 3.2, it can be assumed that for a positive integer \(k \leq n\) each \(R_i\), \(1 \leq i \leq k\), is an injective ideal and \(\sum \limits_{1}^{k} R_i\) is the minimal faithful \(R\)-module. It follows from Lemmas 2.4 and 3.3 that for a positive integer \(m \leq n\) each \(R_i\), \(m \leq i \leq n\), is a minimal ideal and every minimal ideal is isomorphic to one of these. Since the \(R_i\), \(1 \leq i \leq k\), are non-isomorphic injective ideals their socles are non-isomorphic minimal ideals. Therefore \(n-m = k-1\) and it can be assumed that the indexing of the \(R_i\) is done in such a way that if \(m \leq i \leq n\) and \(1 \leq j \leq k\) then \(R_i = S(Re_j)\) if, and only if, \(i = n - j + 1\). It follows from Theorem 2.10 that \(R\) is isomorphic to a blocked matrix ring \(H\) which satisfies all the conditions stated...
in the theorem except (ii)', (iii), (v), (vi)' and the inequality 
k \leq m. So it is sufficient to show that \( H \) also satisfies these 
conditions.

(ii)' If \( 1 \leq i \leq k \) then \( R_{ij} \) is an indecomposable, injective ideal 
whose socle is therefore, a minimal ideal. Since \( R_{ij} \) is nonsingular 
it follows from Lemma 2.3, that \( \text{End}(R_{ij}) = \text{End}(S(R_{ij})) \). Therefore 
\( H_{ij} \) is a sfield.

(iii) If \( 1 \leq i \leq k \) and \( j = n - i + 1 \) then \( R_{ij} = S(R_{ij}) \) and so \( H_{ji} \) 
is a one dimensional vector space over \( H_{jj} \). Moreover, it follows, 
from the fact that \( \text{End}(R_{ij}) = \text{End}(S(R_{ij})) \), that \( H_{ii} = H_{jj} \). If \( \phi \), 
\( \psi \in H_{ji} \) are non-zero then \( R_{ij} \phi = R_{ij} \psi \) and so there is a homomorphism 
\( \alpha: R_{ij} \phi \to R_{ij} \psi \) satisfying \( \phi = \psi \). Since \( R_{ij} \) is injective \( \alpha \) can be 
extended to an element of \( H_{ii} \). Therefore \( H_{ji} \) is a one dimensional 
right vector space over \( H_{ii} \). Since \( H_{ji} \) is a one dimensional left 
vector space over \( H_{jj} \) and since \( H_{ii} = H_{jj} \), it follows that \( H_{ii}, H_{jj} \) 
and \( H_{ji} \) can be identified.

(v) It follows from the indexing of the \( R_{ij} \) that if \( 1 \leq i \leq k \) 
then \( b_{ji} \neq 0 \) if, and only if, \( j = n - i + 1 \) and then \( b_{ji} = 1 \). As 
\( R_{ij} \) is injective, every homomorphism \( S(R_{ij}) \to R_{ij} \) can be extended 
to a homomorphism \( R_{i} \to R_{j} \). Therefore, by (iv), \( H_{ii} = M(H_{jj}, b_{js} \times 1) \) 
where \( j = n - i + 1 \).

(vi)' If \( 1 \leq i \leq k \) then, by Lemma 2.3, every non-zero homomorphism 
\( \phi: R_{ij} \to R_{ij} \) is a monomorphism. As \( R_{ij} \) is injective, \( R_{ij} \phi \) is a 
direct summand of \( R_{ij} \) and so \( R_{ij} \phi = R_{ij} \). Therefore, \( i = j \) and 
\( H_{ij} = 0 \) if \( i \neq j \).
k \leq m. If k > m then conditions (vi) and (vi)' are incompatible with condition (viii): a contradiction. Therefore k \leq m.

Sufficiency. Let H be a matrix ring with all the properties stated in the theorem. Then it follows from Theorem 2.10 that H is an indecomposable, nonsingular ring with essential socle and a finite identity. So it remains to show that H is a QF-3 ring. In view of Theorem 3.2, it is sufficient to prove that \( \sum_{i=1}^{k} H|1|_{ii} \) is a faithful, injective ideal.

For each \( i \in \{m, \ldots, n\} \), let \( e_{ii} \) denote the element \( |1|_{ii} \) of H. If \( \phi \in H_{st} \) is non-zero then, by (iv), \( \phi = \sum_{j=m}^{n} \phi_{ij} \), where \( \phi_{ij} \in M(H_{ii}, b_{is} \times b_{it}) \), and so one \( \phi_{ij} \), say \( \phi_{ij} \), is non-zero. Therefore \( \phi_{ij} \in M(H_{jj}, b_{jt} \times 1) \neq 0 \) and so, if \( i = m - j + 1 \) then it follows from (v) that \( |\phi|_{st}^{|tq|} H_{ij} \neq 0 \). It can, similarly, be shown that if \( x \) is a non-zero element of H then \( x(\sum_{i=1}^{k} H_{ij}) \neq 0 \). That is, \( \sum_{i=1}^{k} H_{ij} \) is faithful.

To show that \( H_{ij}, 1 \leq i \leq k \), is injective, it is sufficient to prove that for any ideal L of H every homomorphism \( L \to H_{ij} \) can be extended to a homomorphism \( H \to H_{ij} \). Since \( S(L) \) is essential in L and is a direct summand of \( S(H) \), it follows from Lemma 2.3 that it is necessary to show only that every homomorphism \( \phi: S(H) \to H_{ij} \) can be extended to a homomorphism \( \phi: H \to H_{ij} \). Let \( e_{st} \) denote the element \( |1|_{st} \) of H and let \( \phi_{st}: S(H) \to H_{ij} \) be the map which agrees with \( \phi \) on \( S(H_{st}) \) and kills the other \( S(H_{pq}) \). The image of \( \phi_{st} \) is in \( S(H_{ij}) = H_{ij} \), where \( j = n - i + 1 \), so \( \phi_{st} \) is...
determined by its action on \( L_j \), the sum of the images of \( \text{He}_j \) in \( \text{He}_{st} \). As \( L_j \) is a direct sum of \( b_j \) minimal ideals (Theorem 2.10), 
\[ \phi_{st} \] can be regarded as an \( f \in \mathcal{M}(b_{jj}, b_j \times 1) \). The matrix \( |f|_{st} \)
is in \( H \) (condition (v)) and induces, by multiplication on the right, a homomorphism \( \phi' : H \to \text{He}_i \) which agrees with \( \phi_{st} \). Consequently, 
\[ \phi = \sum_{s,t} \phi_{st} \] can be extended to a homomorphism \( \phi' = \sum_{s,t} \phi_{st} : H \to \text{He}_i \) and, therefore, \( \text{He}_i \) is injective. This proves the theorem.

3.6 COROLLARY

A nonsingular QF-3 ring (with finite identity) is right nonsingular and has essential right socle.

Proof. This is an immediate consequence of Theorem 3.4 and the right dual of Theorem 2.10.

3.7 THEOREM

Let \( R \) be an indecomposable, nonsingular ring which has a finite identity and is a left, and a right, QF-3 ring. Then \( S_x(R) \) and \( S_y(R) \) are direct sums of a finite number of minimal left, and right, ideals, respectively, and \( R \) is isomorphic to a matrix ring \( H \) which is described by Theorem 3.4 and the following additional property. (ix) If \( m \leq i \leq n \) then each \( H_{ij} \) has finite left dimension over \( H_{ii} \).

Conversely, if \( H \) is a matrix ring with the above properties, then \( H \) is an indecomposable, nonsingular ring which has a finite identity and is a left, and a right, QF-3 ring.
Proof. Let \( H \) be the representation of \( R \) afforded by Theorem 3.4 and let \( e_i \) denote the matrix \( |1^{i1}| \) of \( H \). It is clear that \( e_iH, 1 \leq i \leq k, \) are minimal right ideals and that every minimal right ideal is isomorphic to such an \( e_iH \). Since these \( e_iH \) are non-isomorphic, it follows from the right dual of Lemma 3.3 that every \( e_jH, m \leq j \leq n, \) is an injective right ideal. For \( 1 \leq i \leq k \) let \( c_{si} \) be the right dimension of \( H_{si} \) over \( H_{ii} \). Since right homomorphisms are written on the left, it follows from condition (v) and its right dual, that if \( 1 \leq i \leq k \) and \( j = n - i + 1 \) then

\[
H_{si} = M(H_{ij}^{j}, b_{js}^{j} \times 1) \text{ and } H_{js} = M(H_{ii}^{i}, c_{si}^{i} \times 1). \]

If \( b_{js} \) is infinite then, from the first equation, \( c_{si}^{i} > b_{js}^{j} \) and, from the second equation, \( b_{js}^{j} < c_{si}^{i} \): a contradiction. Consequently, \( b_{js}^{j} \) is finite, \( b_{js}^{j} = c_{si}^{i} \), and both \( S_\ell(R) \) and \( S_r(R) \) are finite dimensional. The converse to the theorem follows immediately from Theorem 3.4 and its right dual.

A completely analogous argument shows that even without the assumption that the identity of \( R \) is finite, the above theorem is still true. It is, of course, necessary to use a different representation of \( R \). If \( H' \) is the representation of \( R \) afforded by the decomposition \( R = Re_1 \oplus \ldots \oplus Re_k \oplus Re_m \oplus \ldots \oplus Re_n \oplus R(1-e), \)

where \( e = \sum_{i=1}^{k} e_i + \sum_{j=m}^{n} e_j \), then it can easily be shown that \( e_iH' \), \( 1 \leq i \leq k, \) is a minimal right ideal and that \( H' \) satisfies (an analogue of) condition (v). The rest of the proof is verbatim.
3.8 COROLLARY

A nonsingular QF-3 ring whose socle is finite dimensional is a right QF-3 ring.

3.9 DEFINITION A ring $R$ with finite identity is a (left) partially PP ring if its identity has a decomposition $1 = \sum e_i$ into orthogonal primitive idempotents with the property that for every non-zero $x \in e_j R e_j$ the ideal $Rx$ is projective. For simplicity, only such decompositions of the identity of a partially PP ring will be discussed. The ring $R$ is a (left) PP ring if its principal ideals are projective. $R$ is (left) (semi-) hereditary if its (finitely generated) ideals are projective. A module is locally cyclic if its finitely generated submodules are cyclic.

3.10 LEMMA

A partially PP ring is nonsingular.

Proof. Let $1 = \sum e_i$ be a decomposition, into orthogonal primitive idempotents, of the identity of a partially PP ring $R$ and let $x_{ij} \in e_i R e_j$ be non-zero. If $\lambda(x_{ij})$ is the left annihilator of $x_{ij}$, then $Rx_{ij} = R/\lambda(x_{ij})$ and, since $Rx_{ij}$ is projective, $\lambda(x_{ij})$ is a direct summand of $R$. That is, $\lambda(x_{ij})$ is not essential. Every $x \in R$ is a sum of $x_{ij}$; therefore, $R$ is nonsingular.

Let $R$ be an indecomposable QF-3 and partially PP ring, and consider the decomposition (3.5) and the representation $H$, afforded by Theorem 3.4, of $R$. The image of a non-zero homomorphism $\phi: R e_i \to R e_j$ is projective, so its kernel is a direct summand of $Re_i$ and, therefore, $\phi$ is a monomorphism. Therefore it follows from conditions (v) and (vii) that every $Re_i$ is isomorphic to a
submodule of an $R_j$, $1 \leq j \leq k$. Therefore, the socle of each $R_i$ is indecomposable, and since, by condition (viii), $k = 1$ and $m = n$, every $b_{ns} = 1$. That is, all $S(R_i)$ are isomorphic. It follows, from condition (v), that $H_{il} = H_{ni} = H_{nn}$, for all $i$, and, from condition (iv), that $H_{ij} \subseteq H_{nn}$, for all $i, j$.

An equivalence relation can be defined on the set $\{R_i\}$ by relating two ideals if, and only if, they contain isomorphic copies of each other. The resulting set of equivalences classes $\{[R_i]\}$ is partially ordered by the relation: $[R_i] \leq [R_j]$ if, and only if, $R_i$ is isomorphic to a submodule of $R_j$. This order has a unique minimal element, $[R_n]$, and a unique maximal element, $[R_1]$. If $[R_i]$ and $[R_j]$ have an equal number of classes smaller than themselves, then they are not comparable: that is, $e_1 R_j = e_j R_1 = 0$. It follows that there is a sequence $1 < h(1) < ... < h(t) < n$, $h(i) \in \{1, ..., n\}$, with the properties that if $i \leq h(s) < j$, then $[R_i] \nsubseteq [R_j]$; and if $h(s) < i, j \leq h(s+1)$ then different $[R_i]$ and $[R_j]$ are not comparable. This proves the necessity of the following characterization of $R$. The sufficiency is clear.

3.11 THEOREM

If $R$ is an indecomposable QF-3 and partially PP ring, then there are positive integers $n, v(1), ..., v(n), h(1), ..., h(t)$, with $1 < h(1) < ... < h(t) < n$, such that $R$ is isomorphic to a blocked matrix ring $H$ with the following properties. The $(i,j)^{th}$ block of an arbitrary element of $H$ is an arbitrary $v(i) \times v(j)$-matrix over an additive subgroup $H_{ij}$ of a field $D$. The groups $H_{ij}$ have the following properties.
(i) \( H_{i j}H_{j k} \subseteq H_{i k} \), for all \( i, j, k \).

(ii) Each \( H_{i i} \) is a ring.

(iii) \( H_{i i} = H_{i i}^{-1} = D_i \), for every \( i \).

(iv) If \( i \leq h(s) \) and \( j > h(s) \), then \( H_{i j} = 0 \).

(v) If \( h(s) < i, j \leq h(s+1) \) and \( H_{i j} \neq 0 \), then \( H_{j i} \neq 0 \).

Conversely, every matrix ring with the above properties is an indecomposable QF-3 and partially PP ring.

3.12 COROLLARY

A (left) QF-3 and (left) partially PP ring is a right QF-3 and a right partially PP ring.

Now let the ring \( H \), of Theorem 3.11, be a PP ring and, for \( i \neq j \), let \( a \in e_i H \) and \( b \in e_j H \) be non-zero elements such that \( Ha \cap Hb \neq 0 \).

Then \( H(a+b) = Ha + Hb \) and \( H(a+b) \) is a homomorphic image of \( e_i (H) \oplus e_j (H) \) under the function \( \phi \) which maps \( e_i \) onto \( a \) and \( e_j \) onto \( b \). If \( x \in e_i (H) \), \( y \in e_j (H) \) and \( x + y \in \ker \phi \), then \( x\phi = -y\phi \). Since \( H(a+b) \) is projective, \( \ker \phi \) is a direct summand of \( e_i (H) \oplus e_j (H) \) and is, therefore, generated by an idempotent \( f \). Since \( e_i (H) \oplus e_j (H) \) have only one idempotent each, \( f = t_1 e_i + r_1 e_j + t_2 e_j + r_2 e_j \), where \( e_i r_1 = 0 \), \( e_j r_2 = 0 \), \( t_k \in \{0,1\} \) and at least one \( t_k \), say \( t_1 \), is non-zero. This implies that the idempotent \( e_i + r_1 e_i \) is mapped by \( \phi \) into \( Hb \). Since \( e_i (H) \) is indecomposable, \( H(e_i + r_1 e_i) = Hb \subseteq Hb \). Similarly if \( t_2 = 1 \) then \( Ha \subseteq Hb \). Hence, one of the ideals \( Ha, Hb \) is contained in the other. Therefore, either there is a \( c \in e_i (H) \) such that \( a = cb \), or there is a \( c \in e_j (H) \) such that \( b = ca \).

This shows that if \( \alpha \in H_{i k} \) and \( \beta \in H_{j k} \), then either there is a \( \gamma \in H_{i j} \) such that \( \alpha = \gamma \beta \), or there is \( \gamma \in H_{j i} \) such that \( \beta = \gamma \alpha \),
but since $H_{ij} \nmid H_{ij}$, not both. If $H_{ij} = 0$ then $Hb \subseteq Ha$ and, by restricting $a, b$ to $H_{ij}$, it can be seen that $H_{ij} = D$. As $H_{ij} \nmid H_{ij}$ no element of $H_{ij}$ has an inverse in $H_{ij}$, so if $H_{ij} = D$ then $H_{ij} = 0$. This proves the first part of Theorem 3.13. Its converse can be checked by noting that if $a = \sum a_{kl}^{ij} + \cdots + a_{kl}^{ij}$, $b = \sum b_{ij}^{1j} + \cdots + b_{ij}^{1j}$ and $Ha \cap Hb \neq 0$, then it follows, from condition (ix) and the fact that all $H_{ij}$ are in $D$, that either there is a $\gamma \in H_{ij}$ such that each $a_{ij} = \gamma b_{ij}$, or there is a $\gamma \in H_{ij}$ such that each $b_{ij} = \gamma a_{ij}$. Hence one of the ideals $Ha, Hb$ is contained in the other. It follows that every principal ideal of $H$ is projective.

3.13 THEOREM

An indecomposable PP and QF-3 ring is isomorphic to a matrix ring $H$ satisfying Theorem 3.11 and the following additional conditions.

(vi) If $i > j$ then $H_{ij} \neq 0$.

(vii) If $h(s) < i, j \leq h(s+1)$ then $H_{ij} \neq 0$. If, moreover, $h(s+1) > h(s) + 1$ then $H_{ij} \neq D$.

(viii) If $i > h(s) \geq j$ then $H_{ij} = D$.

(ix) If $a \in H_{ik}$, $\beta \in H_{jk}$, then either $a \in H_{ij} \beta$ or $\beta \in H_{ij} a$.

Conversely, every matrix ring $H$ with the above properties is an indecomposable PP and QF-3 ring.

If the ring $H$, in the above theorem, is (semi-) hereditary, then it can be shown, by an argument similar to the preceding one,
that every indecomposable (finitely generated) ideal of \( H \) is a principal ideal. The rest of the proofs of the following theorems is also similar to the proof of the preceding result.

3.14 **THEOREM**

A ring is an indecomposable, semi-hereditary QF-3 ring if, and only if, it is isomorphic to a matrix ring \( H \) satisfying Theorem 3.13 and the following additional condition.

(xi) Each \( H_{ij} \) is a locally cyclic (left) \( H_{ii} \)-module.

3.15 **THEOREM**

A ring is an indecomposable, hereditary QF-3 ring if, and only if, it is isomorphic to the ring of all blocked, lower triangular, finite matrices over a field.

It follows, from Theorems 2.12 and 3.14, that a semi-primary, semi-hereditary QF-3 ring satisfies Theorem 3.15 and is, therefore, a hereditary ring.
Injective modules play a central role in ring theory, so it is natural to try to classify those rings which are injective (as modules over themselves). The structure of these rings is very complicated and, in general, little is known about them. The usual approach at simplifying a problem by imposing chain conditions does not seem to work, as is demonstrated by the fact that Quasi-Frobenius rings (Artinian self-injective rings) have been studied intensely for decades but are still a mystery. Even the structure of group algebras of finite groups over fields, a very special case of Quasi-Frobenius rings, is not known, despite the fact that these algebras are of fundamental importance to group representations and have, therefore, been subjected to a lot of research. Consequently it appears that new types of restrictions are needed to simplify the problem.

An obvious type of restriction is the stipulation that the rings be "very injective", as it were. The class of rings whose ideals are all injective is a realization of this concept, but it is too small since, it is well-known and easy to prove that, it consists of the semi-simple Artinian rings. However, those rings whose ideals are injective only with respect to homomorphisms between their own submodules seem to be quite promising. This class appears to be not too large and yet is not too small - it certainly contains non semi-simple rings (e.g. \( \mathbb{Z}/p^n\mathbb{Z} \), where \( \mathbb{Z} \) is the ring of integers and \( p \) is a prime). We devote this chapter to the study of such rings.
4.1 **DEFINITION** A module $M$ is *quasi-injective* if for every submodule $N$ of $M$ and every homomorphism $\phi : N \to M$ there is an endomorphism of $M$ which agrees with $\phi$ on $N$. A ring is a *(left)* Q-ring if all of its left ideals are quasi-injective.

The fact that a Q-ring is injective follows from the well-known result that an R-module $M$ is injective if every homomorphism from any ideal of $R$ can be extended to a homomorphism from $R$ to $M$. Q-rings were first studied in [20]. This paper is mainly concerned with showing that certain well-known self-injective rings are not Q-rings. Its contribution to the structure of Q-rings is limited: the main results obtained are that a semi-prime (square of non-zero ideal is non-zero) Q-ring is regular (every principal ideal is generated by an idempotent), and that the simple Artinian rings are the only prime (product of non-zero ideals is non-zero) Q-rings. Saad's paper [32] shows that a Quasi-Frobenius ring is a left Q-ring if, and only if, it is a right Q-ring. The only other published article to deal with the structure of Q-rings is [19] which is a slightly condensed version of this chapter. Saad has published an article [33] on rings whose proper homomorphic images are Q-rings but it does not contribute to the structure of Q-rings.

Using methods which depend on the existence of non-central idempotents we determine the structure of indecomposable Q-rings which have more than one idempotent. In the process we make some contributions to the general theory. Unfortunately this is still embryonic. Nevertheless it can be seen, from the fact that every commutative self-injective ring is a Q-ring (Lemma 4.2), that the
structure of an arbitrary Q-ring is very complicated.

We begin by showing that an indecomposable Q-ring with more than one idempotent has non-zero socle: this guarantees that it has primitive idempotents. Then we show that it cannot have an infinite number of ideals with the two properties that the sum of the ideals is direct, and that each ideal has a non-zero homomorphic image which intersects the ideal trivially. Using this we deduce that if at least one minimal ideal is injective then the ring is simple Artinian. If the ring is not simple Artinian then every indecomposable injective ideal has a unique proper submodule and the socle of the ring is again essential. Finally we prove that the ring is Artinian and represent it as a full ring of matrices.

We need the following elementary, but often useful, characterization of Q-rings.

4.2 LEMMA [20, Theorem 2.3]
A ring is a Q-ring if, and only if, it is self-injective and all of its essential ideals are two-sided.

Proof. It is well-known that a module is quasi-injective if, and only if, it is invariant under all endomorphisms of its injective hull [10, Proposition 3.1]. Since a Q-ring is self-injective and every endomorphism of a ring is realizable as right multiplication by an element of the ring, this means that every essential ideal of a Q-ring is a right ideal.

Conversely, let R be a self-injective ring whose essential ideals are two-sided, let L be an ideal of R and Re an injective hull, in R, of L. Then L ⊗ R (1-e) is a right ideal so
(L ⊗ R(l-e))eRe = LeRe is contained in L. Therefore L is invariant under all endomorphisms of Re so it is quasi-injective. Hence R is a Q-ring.

4.3 LEMMA

If e, f are orthogonal idempotents in a Q-ring R such that fRe ≠ 0 then fRe ⊆ S(Re). If Re and Rf are isomorphic then they are both (finite) sums of minimal ideals. An indecomposable Q-ring with more than one idempotent has non-zero socle.

Proof. Let L be an essential submodule of Re. Then L ⊗ R(l-e) is an essential ideal of R so, by Lemma 4.2, it is a right ideal. Hence, as f ∈ R(l-e) the product f.fRe = fRe is in L ⊗ R(l-e), which implies that fRe ⊆ L. Therefore the intersection of all essential submodules of Re contains fRe, which means that fRe ⊆ S(Re), since the intersection of the essential submodules of a module is the socle of the module. If Re ∼= Rf then S(Re) contains fRe which generates Re: hence Re = S(Re). If R is an indecomposable Q-ring with idempotent e which is not the identity, then either eR(l-e) ≠ 0 or (1-e)R ≠ 0. It follows from the first part of the lemma that the socle of R is non-zero.

4.4 LEMMA

If R is a Q-ring and \{e_i | i ∈ I\} is a set of mutually orthogonal idempotents then only a finite number of e_i have the property that Re_i has a non-zero homomorphic image in R(l-e_i).
Proof. If not, then by Lemma 4.3 there is an infinite set \( \{ \alpha_i : R e_i \to R(1-e_i) \} \) of homomorphisms whose images are simple. If \( \bigoplus_{i \in I'} R e_i \alpha_i \) is an infinite sum, then there is an infinite set \( I'' \) such that \( (\bigoplus_{i \in I'} R e_i) \cup R e_i \alpha_i = 0, \) for all \( i \in I'' \). If only finite sums of \( R e_i \alpha_i \) are direct, then there is an infinite set \( \{ R e_i \alpha_i \}_{i \in I'} \) of isomorphic \( R e_i \alpha_i \) and, therefore, it follows from the projectivity of \( R e_i \) that, for all \( i, j \in I' \), there is a map \( \beta_{ij} : R e_i \to R e_j \). In both cases, there is an infinite set \( \{ \phi_i : R e_i \to R(1-e_i) \}_{i \in I''} \) of homomorphisms satisfying the conditions that \( \bigoplus_{i \in I''} R e_i \phi_i \) is a direct sum and \( (\bigoplus_{i \in I''} R e_i) \cup R e_i \phi_i = 0, \) for all \( i \in I'' \). Let \( Re \) be an injective hull of \( \bigoplus_{i \in I''} R e_i \) in \( R \) and let \( \phi : \bigoplus_{i \in I''} R(1-e) \) be the sum of all the \( \phi_i, i \in I'' \). As \( R(1-e) \) is injective, \( \phi \) can be lifted to a homomorphism \( \phi : R e \to R(1-e) \). But this is a contradiction since, by Lemma 4.3, \( R e \phi \) is a finite sum of minimal ideals. This proves the lemma.

4.5 THEOREM

Let \( R \) be an indecomposable \( Q \)-ring and \( e \in R \) an idempotent. If \( R e \) is a minimal ideal then \( R \) is a simple Artinian ring.

Proof. By Lemma 4.4 the sum of all ideals isomorphic to \( R e \) is a finite direct sum. If \( R f \) is this sum, then \( f R(1-f) = 0 \). Since \( R e \) is projective it is a homomorphic image only of modules which contain an isomorphic copy of it: therefore \( (1-f)R f = 0 \). As \( R \) is indecomposable this means that it is a sum of a finite number of mutually isomorphic minimal ideals. Therefore \( R \) is a simple Artinian ring.
We now turn our attention to indecomposable Q-rings which have more than one idempotent but which, in view of Theorem 4.5, have no injective minimal ideals. For simplicity of expression we will refer to Q-rings with these three properties by the unimaginative special Q-rings. To proceed with our investigations we need the following two results. But first let us explain a few terms.

4.6 DEFINITION The Jacobson radical of a ring is the intersection of all the maximal ideals of the ring. A ring is local if it has only one maximal ideal. A ring is regular if each of its principal ideals is generated by an idempotent.

4.7 THEOREM [10, Theorem 5.1]
The Jacobson radical \( J \) of the ring of endomorphisms \( E \) of a quasi-injective module \( M \) is the set of all those endomorphisms whose kernels are essential in \( M \). The ring \( E/J \) is regular.

4.8 LEMMA [10, Proposition 5.8]
A quasi-injective module is indecomposable if, and only if, its ring of endomorphisms is local.

4.9 LEMMA
If \( e \) is a primitive idempotent in a special Q-ring \( R \), then \( eRe \) is a field and \( Re \) has a unique proper submodule, the set \( (1-e)Re \).

Proof. First we will show that \( (1-e)Re \neq 0 \). Assume that \( (1-e)Re = 0 \); then the ideals of \( eRe \) and the \( R \)-submodules of \( Re \) coincide and, as \( R \) is indecomposable, \( eR(1-e) \neq 0 \). By Lemma 4.8 the ring \( eRe \) has a unique maximal ideal, \( J \) say, which is not zero since \( R \) has no minimal ideals which are injective. Therefore there is an ideal \( L \subseteq J \) which has a simple factor module. As \( eRe \) is a local ring
and as every image in \( R(1-e) \) of \( R(e) \) is a minimal ideal (Lemma 4.3), this means that there is a homomorphism from \( L \) to \( R(1-e) \). By the injectivity of \( R(1-e) \) this map can be lifted to \( R(e) \) : a contradiction, since by Lemma 4.3 every homomorphism from \( R(e) \) to \( R(1-e) \) kills \( J \) and hence \( L \). Therefore \( (1-e)R(e) \neq 0 \).

By Theorem 4.7 every element of \( J \) is annihilated on the left by an essential submodule of \( R(e) \). Hence, by Lemma 4.3, \((1-e)R(e)J = 0\) and so \( J \) is an ideal of \( R \). But every non-zero submodule of \( R(e) \) must contain the minimal ideal \( S(R(e)) \supseteq (1-e)R(e) \); therefore \( J = 0 \). That is, \( eR(e) \) is a sfield. Hence every non-zero element of \( eR(e) \) generates \( R(e) \) and, as \((1-e)R(e) \) generates \( S(R(e)) \), the only submodule of \( R(e) \) is \( S(R(e)) = (1-e)R(e) \).

4.10 LEMMA

The socle of a special \( Q \)-ring is essential.

Proof. Let \( R \) be a \( Q \)-ring and let \( R(e) \) be an injective hull of \( S(R) \). If \( f_1, f_2 \in R(1-e) \) are orthogonal idempotents then, by Lemma 4.3, both \( f_1Rf_2 = 0 \) and \( f_2Rf_1 = 0 \); so as \( R \) is indecomposable, both products \( f_iR(e) \neq 0, i = 1, 2 \). As \( R(1-e) \) does not contain any minimal ideals it does not contain any primitive idempotents (Lemma 4.9). Therefore \( R(1-e) \) has an infinite set \( \{e_i\} \) of mutually orthogonal idempotents such that each \( R(e_i) \) has a homomorphic image in \( R(e) \) : a contradiction to Lemma 4.4. Therefore \( R(1-e) = 0 \) and \( S(R) \) is essential in \( R \).

We are now ready to prove the main theorem of this chapter. The following result about arbitrary self-injective rings will be used.
4.11 THEOREM [10, Theorem 5.6]

Let \( R \) be a self-injective ring with Jacobson radical \( J \). If \( f_1, \ldots, f_n \) are finitely many orthogonal idempotents of \( R/J \) then there are orthogonal idempotents \( e_1, \ldots, e_n \) of \( R \) such that the canonic epimorphism \( R \to R/J \) maps \( e_i \) onto \( f_i \).

4.12 THEOREM

An indecomposable \( Q \)-ring with more than one idempotent is Artinian.

Proof. In view of Theorem 4.5 we need consider only special \( Q \)-rings. Let \( R \) be a special \( Q \)-ring and assume it is not Artinian.

Then, by Lemmas 4.9 and 4.10, the set \( \{M_i | i \in I\} \) of minimal ideals of \( R \) is infinite. For each \( i \in I \) let \( R e_i \) be an injective hull in \( R \) of \( M_i \). It follows from Lemmas 4.3, 4.4 and 4.9 that only a finite number of \( R e_i \), say \( R e_1, \ldots, R e_n \), have proper homomorphic images in \( R \), and that these are minimal ideals, say \( M_s(1), \ldots, M_s(m) \).

If \( e \) is a finite idempotent then, since \( R e \) is isomorphic to a direct sum of a finite number of \( R e_i \), the only minimal ideals which can be homomorphic images of \( R e \) are the \( M_s(i) \).

Let \( e', e'' \) be infinite orthogonal idempotents whose sum is the identity and having the property that one of \( R e' \), \( R e'' \) contains all the ideals \( R e_1, \ldots, R e_n, M_s(1), \ldots, M_s(m) \). There are an infinite number of minimal ideals which are homomorphic images of one of \( R e' \), \( R e'' \) and are contained in the other. For assume that this is not true, and let \( R e_1 \) (respectively, \( R e_1' \)) be an injective hull in \( R e'' \) (respectively, \( R e' \)) of all images of \( R e' \) (respectively, \( R e'' \)) which are in \( R e'' \) (respectively, \( R e' \)). If the \( M_s(i) \) are in \( R e' \) and
if \( \text{Re} \subseteq \text{Re}' \) is an injective hull of \( \sum_{i=1}^{m} M_{s(i)} \), then, since both idempotents \( e''_1 + e \) and \( e'_1 \) are finite, the ideals \( R(e''_1 + e) \) and \( \text{Re}'_1 \) have no homomorphic images outside themselves (in \( R \)). Therefore the ideals \( R(e' + e' - e'' - e) \) and \( R(e'' + e'' + e - e') \) annihilate each other: a contradiction, since the ring

\[ R = R(e' + e' - e'' - e) \oplus R(e'' + e'' + e - e') \]

is indecomposable.

If the \( M_{s(i)} \) are in \( \text{Re}'' \) then we can again obtain a contradiction, by a completely analogous argument. Therefore one of the ideals \( \text{Re}'_1, \text{Re}'' \), say \( \text{Re}'_1 \), has an infinite number of homomorphic images in the other, \( \text{Re}'' \).

As all \( M_i \) are mutually non-isomorphic minimal ideals it follows that for every pair \((i, j) \in I \times I\) there is an element \( x \in \ell(M_j) \), the left annihilator of \( M_j \), with the property that \( xM_i \neq 0 \). By Theorem 4.7, there is an element \( y \in R \) such that \( yx \) is an idempotent modulo the Jacobson radical \( J \) of \( R \) having the property that \( x + J = ryx + J \), for some \( r \in R \). By Theorem 4.11 there is an idempotent \( f \in R \) such that \( f + J = yx + J \). Therefore \( f \in \ell(M_j) \) and \( fM_i \neq 0 \), that is, \( M_i \) is a homomorphic image of \( Rf \). This result enables us to pick out an infinite number of orthogonal idempotents with the property that each ideal generated by such an idempotent has an image outside itself, thus contradicting Lemma 4.4.

It was shown that the set \( I_1 = \{ i \in I | M_1 \subseteq \text{Re}'' \text{ and } e'M_i \neq 0 \} \) is infinite. Let \( j \in I_1 \) and let \( f_0 \in \ell(M_j) \cap \text{Re}' \) be an idempotent such that \( f_0 M_i \neq 0 \) for some \( i \in I_1 \). Then one of \( f_0, e' - f_0 \) does not annihilate, by left multiplication, an infinite number of
M^i, i ∈ I_1. Denote it by f'_1 and the other idempotent by f_1. By a completely analogous argument we can show that f'_1 is the sum of two orthogonal idempotents f_2, f'_2 with the property that an infinite number of M^i, i ∈ I_1, are homomorphic images of Rf'_2. As this procedure is clearly inductive, we can pick, for each positive integer n, an idempotent f_n ∈ Re' such that all the f_n are orthogonal and each Rf_n has a non-zero homomorphic image in Re'. But this is a contradiction to Lemma 4.4: therefore our original assumption is false. That is, R is Artinian.

To complete the determination of the structure of special Q-rings we now represent them as rings of matrices.

4.13 DEFINITION For an integer m ≥ 2, a field D, and a null D-algebra V whose left and right D-dimensions are both equal to one, let H(m,D,V) be the ring of all mxm matrices whose only non-zero entries are arbitrary elements of D along the diagonal and arbitrary elements of V at the places (2,1), ..., (m, m-1) and (1,m).

4.14 THEOREM Every special Q-ring is isomorphic to an H(m,D,V); conversely, every H(m,D,V) is a special Q-ring.

Proof. Let R be a special Q-ring; then, by Theorem 4.12 and Lemma 4.9, there is an integer m ≥ 2 and a set \( \{e_i \mid 1 \leq i \leq m\} \) of mutually orthogonal primitive idempotents such that

\[ R = \bigoplus_{i} Re_i.\]

As the only minimal ideals of R are the socles of the Re_i, it follows from Lemma 4.9 that every minimal ideal is the image of an Re_i. If a minimal ideal M is the image of an Re_i, then, as Re_i is projective it follows from Lemma 4.3 that M is not the
image of any other $\text{Re}_j$, that is, $e_j^M = 0$ for all $j \neq i$. Therefore each $\text{Re}_i$ determines uniquely that $\text{Re}_j$ whose socle is the image of $\text{Re}_i$ and that $\text{Re}_k$ which has $S(\text{Re}_i)$ as a factor module. As $R$ is indecomposable, a standard argument shows that we may assume the $e_i$ to be indexed in such a way that $S(\text{Re}_i)$ is the image of $\text{Re}_{i+1}$ if $1 \leq i \leq m-1$ and $S(\text{Re}_m)$ is the image of $\text{Re}_1$.

We know that $R = \text{Hom}_R(R, R)$ which is isomorphic to $H$, say, the ring of all $m \times m$ matrices $(\phi_{ij})$ where $\phi_{ij} \in \text{Hom}_R(\text{Re}_i, \text{Re}_j) = H_{ij}$. It follows from the preceding paragraph that the only non-zero entries of $H$ are along the diagonal and at the places $(2,1), \ldots, (m, m-1)$ and $(1, m)$. As each $\text{Re}_i$ is injective and $e_i \text{Re}_i$ is a sfield, it follows that $H_{ii} = \text{Hom}_R(S(\text{Re}_i), S(\text{Re}_i))$. If $S(\text{Re}_i)$ is the image of $\text{Re}_j$ then $H_{ij} = \text{Hom}_R(S(\text{Re}_i), S(\text{Re}_i))$, therefore all the sfields $H_{ii}$ are mutually isomorphic; $H_{ii} = D$, say, for all $i$. Moreover, each non-zero $H_{ij}$ is a one dimensional left vector space over $H_{ii}$ and a one dimensional right vector space of $H_{ij}$. Hence all the non-zero $H_{ij}$ are mutually isomorphic $D$-bivector spaces, all isomorphic to $V$, say. As $H_{ij}H_{jk} = 0$, the space $V$ is a null algebra. Therefore $R = H(m, D, V)$.

We now prove the converse: we show that $H = H(m, D, V)$ is a special $Q$-ring. First we show that $H$ is injective. For any $i$ let $e_i \in H$ be the matrix whose only non-zero entry is 1 at the place $(i, i)$. If $L \subseteq H$ is an ideal then $L = L_1 \oplus L_2$ where $L_1 \subseteq H_{e_i}$ and $L_2 \cap H_{e_i} = 0$. Any homomorphism from $L_1$ to $H_{e_i}$ can be lifted to an endomorphism of $H_{e_i}$. If there is a homomorphism $\phi$ from $L_2$ to $H_{e_i}$, then $L_2$ contains a direct summand isomorphic to $H_{e_j}$ ($j = i+1$, if
\( i \neq m; j = 1 \text{ if } i = m \), and \( \ker \phi \) is a complement to this summand. Consequently every homomorphism from \( L \) to \( H e_i \) can be lifted to \( H \), that is, \( H e_i \) is injective. Therefore, as \( H \) is a finite sum of injective ideals, it is itself injective. For any element \( x \in H \) it is clear that \( xH \subseteq S(H) + Hx \). Therefore every essential ideal of \( H \) is two-sided. Now we apply Lemma 4.2 to deduce that \( H \) is a Q-ring. Since \( H \) is indecomposable and not simple it follows from Theorem 4.5 that it is a special Q-ring.

It may be worthwhile to point out that the last two paragraphs of Theorem 4.12, combined with Lemma 4.3, are a proof of the following result about arbitrary Q-rings.

4.15 LEMMA

If \( e \) is an idempotent in a Q-ring \( R \) then \( Re \) has at most a finite number (possibly zero) of homomorphic images in \( R(1-e) \).

The author included the following conjecture as a Remark in [19], but while writing this thesis he was not able to reproduce the "proof".

Conjecture

Every Q-ring is a direct sum of a finite number of Q-rings which have more than one idempotent and a Q-ring whose idempotents are all central.

The conjecture is true if, and only if, every ideal in a Q-ring is contained in a unique smallest ideal generated by a central idempotent. This is equivalent to the condition that the intersection of all ideals which have, in the ring, a common homomorphic image disjoint from them is non-zero.
It is clear that Q-rings which satisfy the conjecture have this property. To show the converse, let L be an ideal in a Q-ring R, \( f \) a central idempotent such that \( L \subseteq R_f \), and let \( R_f, f_0 \) an idempotent, be an injective hull of L. As \( R_f \) is injective \( f_0 R_f \neq 0 \), which implies that \( f_0 f \neq 0 \). Since \( R_f = R_f f_0 f \otimes R_f (1-f) \) and \( L \subseteq R_f \) is essential in \( R_f \), the ideal \( R_f (1-f) \) is trivial. Therefore \( R_f \subseteq R_f \). It follows that the intersection of all ideals which are generated by central idempotents and contain L, coincides with its injective hull. Let \( R_e, e \) an idempotent, be this two-sided ideal.

If \( x \in R(1-e) \) then \( ex \in R_e \cap R(1-e) \), that is \( ex = 0 \). If \( (1-e) R_e \neq 0 \) then there is a minimal ideal \( M \subseteq R_e \) which is a homomorphic image of \( R(1-e) \) (Lemma 4.3). Moreover, for any central idempotent \( f \) with the property that \( R_e \leq R_f \), the product \( (1-e)f \neq 0 \). This means that each \( R_f \) contains an ideal \( R(1-e)f \) which has \( M \) as an image. By assumption, the intersection \( K \) of these ideals is non-zero. But as each \( R(1-e)f = R_f(1-e) \) is contained both in \( R(1-e) \) and in \( R_f \), the ideal \( K \) is contained both in \( R_e \) and \( R(1-e) \) : a contradiction. Therefore \( (1-e)R_e = 0 \) and \( e \) is a central idempotent, since \( R_e \) and \( R(1-e) \) annihilate each other. Hence \( R_e \) is the unique smallest ideal which contains L and is generated by a central idempotent.
CHAPTER 5

LEFT GENERALIZED UNISERIAL RINGS

The first step in obtaining a deep understanding of rings is to determine the structure of Artinian rings. Contrary to popular opinion, this has not been done. The difficulty is that an Artinian ring lies almost completely in its radical which, being nilpotent, is quite amorphous. One, if not the only, way of overcoming this difficulty is by finding a method of determining the structure of rings from their ideal lattices – a very important problem in itself. From this aspect, the simplest non-trivial rings are those which are, both as left and as right modules, direct sums of ideals each having a unique composition series. Such rings are called generalized uniserial.

To develop a theory of modules it is desirable, if not completely indispensable, to know which rings have the property that their modules are direct sums of cyclic modules. These rings, necessarily Artinian, were first studied by G. Köthe [23] in 1935 and are, therefore, called Köthe rings. Köthe showed that in the commutative case, generalized uniserial rings are the only Köthe rings. In 1940 T. Nakayama [30], [31] introduced generalized uniserial rings and showed that they form a proper subclass of the class of Köthe rings. He also showed that generalized uniserial rings are the only Köthe rings whose indecomposable left and right modules are homomorphic images of principal ideals generated by primitive idempotents. So it is clear that generalized uniserial rings play a special role in the theory of Köthe rings.

It is an indication of the scarcity of ring-theoretic tools that,
despite considerable effort (if the volume of literature is a measure), limited progress has been made in determining the structure of generalized uniserial rings - rings which are really very simple. Because the literature on the subject is extensive, it would be too space-consuming and, in view of the contributions of this chapter, rather futile to survey it. However, two results will be mentioned. I. Murase [26], [27], [28] has determined the structure of large classes of generalized uniserial algebras. He showed that his algebras have bases (over their ground sfields) which are closed under multiplication, and specified the multiplication on each basis. One can hardly do better than that. Unfortunately his methods are inapplicable to the general case. Murase's work depends on H. Kupisch's [24] result that if \( \{ R_e_1 | 1 \leq i \leq n \} \) is a complete set of non-isomorphic indecomposable non-nilpotent ideals in a generalized uniserial ring \( R \) with radical \( W \), then the \( e_i \) can be indexed in such a way that

\[
\frac{W_{e_i}}{W^2 e_i} = \frac{R_{e_{i+1}}}{W_{e_{i+1}}} \text{ when } 1 \leq i < n, \text{ and } \frac{W_{e_n}}{W^2 e_n} = \frac{R_{e_1}}{W_{e_1}} \text{ when } W_{e_n} \neq 0.
\]

Although the author has not read Kupisch's proof of this theorem (he does not read German), he believes that it is similar to that used to prove a more general result in this chapter.

Throughout the rest of this chapter all rings and modules are left Artinian.

5.1 DEFINITION A module is uniserial if it has a unique composition series. The length \( d(M) \) of a uniserial module \( M \) is the length of its composition series. A ring \( R \) is left uniserial if \( R^R \) is uniserial. \( R \) is left generalized uniserial if \( R^R \) is a direct sum
of uniserial modules. R is (generalized) uniserial if it is both left and right (generalized) uniserial.

The literature on left generalized uniserial rings contains little of significance so it will not be discussed. In this chapter the structure of (left) generalized uniserial rings is determined by representing the rings as matrices over (left) uniserial rings. The structure of these is intimately connected with that of sfields, so the two are here lumped together, in true ring-theoretic fashion, as important building blocks which are assumed to be known — because they are too difficult to study. The representations are obtained by deducing how the ideals fit together, and then constructing all rings with such ideal lattices. The methods used are quite general and are applicable to any class of Artinian rings whose ideal lattices can be ascertained. The chapter closes with a converse to Nakayama's theorem.

Throughout this chapter R is an indecomposable left generalized uniserial ring, and

\[ R = R_{01} \oplus \cdots \oplus R_{i_1} \oplus \cdots \oplus R_{i_v} = \cdots \oplus R_{i_1} \oplus \cdots \oplus R_{i_1} \oplus \cdots \oplus R_w v(w) \]

is a decomposition of R into uniserial left ideals having the property that \( R_{i_j} = R_{st} \) if, and only if, \( i = s \). For simplicity of notation, the idempotent \( e_{il} \) will be denoted by \( e_i \). The reason for beginning the indexing of the \( e_i \) at 0 will become apparent later. The Wedderburn radical of R is denoted by W, and for any R module M the quotient module \( M/WM \) is denoted by \( \overline{M} \). For proofs of the following well-known facts, as well as for other elementary facts about Artinian rings which are used without specific reference, the
reader is referred to [2] or [6]. Every simple $R$-module is isomorphic to an $Re_i$. The composition series of $Re_i$ is

$$Re_i = W^0e_i \supseteq W^1e_i \supseteq \ldots \supseteq W^te_i = 0,$$

where $t = d(Re_i)$. If $M$ is a uniserial $R$-module then $e_iM = M$, for some $e_i$, hence $M$ is a homomorphic image of $Re_i$ and so the composition series of $M$ is

$$M = W^0M \supseteq WM \supseteq \ldots \supseteq W^{d(M)}M = 0.$$  

If $M$ and $N$ are uniserial $R$-modules such that $M \cong N$ then one of $M$, $N$ is a homomorphic image of the other.

We now begin the proof of the main theorem. For each idempotent $f \in E = \{e_i | 0 \leq i \leq w\}$ define inductively a sequence

$$s(f) = f_0, \ldots, f_t, \ldots$$

of elements of $E$ by the rule that $f_0 = f$ and that, for $i \geq 0$, $f_{i+1}$ is the unique element of $E$ with the property that $Wf_i$ is a homomorphic image of $Rf_{i+1}$. The idempotent $f_{i+1}$ is unique in $E$, since if $Wf_i$ is a homomorphic image of both $Re_p$ and $Re_q$ then $Wf_i = \overline{Re_p} = \overline{Re_q}$, which implies that $Re_p = Re_q$ and so, by the definition of the $e_i$, $e_p = e_q$. The sequence $s(f)$ may be finite or infinite. It is finite if, and only if, $Wf_m = 0$ for some $m$. If $s(f)$ is infinite then there are integers $t \geq 0$ and $n > 0$ with the properties that all the idempotents $f_0, \ldots, f_t$ are distinct, and that $f_{t+i} = f_{t+j}$ if, and only if, $i, j$ are non-negative and congruent modulo $n+1$. In this case, it is convenient to re-index the sequence $s(f)$ so that $f_t$ becomes $f_0$. Therefore $s(f)$ is either the finite sequence $f_0, \ldots, f_m$, or the infinite sequence $f_{-t}, \ldots, f_0, \ldots, f_n, \ldots$ where, for non-negative $i, j$, $f_i = f_j$ if, and only if, $i \equiv j \text{ mod}(n+1)$. It follows that a non-zero $W^0f_i$ is a homomorphic image of $Rf_{i+1}$ but is not an image of any other $Re, e \in E$. 


It is clear that if two sequences \( s(e_i), s(e_j) \) have a common term then they have a common subsequence. Consequently the relation, between the sequences \( s(e_i) \), of having a common term is transitive. Let \( F \subseteq E \) be the set of \( e_i \) for which \( s(e_i) \) has a term in common with \( s(e_0) \), and let \( e \in E \) and \( f \in F \). If \( eRf \neq 0 \), then there is a non-zero homomorphism from \( Re \) to \( Rf \), and so one of the non-zero \( W^g f \) is a homomorphic image of \( Re \). Therefore \( e \) is a term of \( s(f) \), which means that \( s(e) \) and \( s(e_0) \) have a term in common: hence \( e \in F \).

If \( fRe \neq 0 \) then a similar argument shows that again \( e \in F \). It follows that if \( g = \sum_{f \in F} f \) then the ideals \( Rg \) and \( R(1-g) \) are two sided since they annihilate each other. As \( R \) is indecomposable, this means that \( Rg = R \), and therefore \( F = E \). Hence any two sequences \( s(e_i), s(e_j) \) have a common term. Consequently, either all the sequences \( s(e_i) \) are finite or they are all infinite.

Assume that all the sequences \( s(e_i) \) are infinite, and let \( s(f) \) and \( s(f') \) be any two. Then \( s(f) = f_{-t}, \ldots, f_{-1}, f_0, \ldots, f_0, f_{1}, \ldots, f_n, \ldots, f_n, \ldots \), where the terms \( f_i \) for \( i \leq -1 \), are distinct, and the terms \( f_0, \ldots, f_n \) repeat infinitely as a block (in that order); and \( s(f') = f_{-s}, \ldots, f_0, \ldots, f_{m}, f_0', \ldots, f_m', \ldots \), where the block \( f_0', \ldots, f_m' \) repeats and the \( f_i' \), for \( i \leq -1 \), are distinct. It can be seen that a sequence \( s(e) \), which is common to both \( s(f) \) and \( s(f') \), contains all the terms \( f_0, \ldots, f_n \) and \( f_0', \ldots, f_m' \); that therefore \( n = m \); and that \( f_0, \ldots, f_n \) is the sequence \( f_{j'}, f_{j'+1}, \ldots, f_{j'}, f_{j'+1}, \ldots, f_{j-1}, \ldots \), for some \( j \in [0, \ldots, m] \). Consequently, if \( s(e) \) is a maximal sequence common to both \( s(f) \) and \( s(f') \), then \( e \) is one of the mutually distinct idempotents \( f_i, f_i' \), \( i \geq 0 \).
To simplify the discussion we make the following definition, which is not intended to be appropriate in any wider context.

5.3 DEFINITION The ring \( R \) has the FS (respectively, IS) property if all sequences \( s(e_i) \) are finite (respectively, infinite).

We partition the index set \( \{0, \ldots, w\} \), and re-index the idempotents \( e_i \), so as to take into account how the \( s(e_i) \) fit together. If \( R \) has the IS property then the set \( \{0, \ldots, n\} \), \( n \leq w \), will be used to index those idempotents which form the block of repeating terms in each \( s(e) \), the indexing being such that \( \overline{We_i} = \overline{Re_{i+1}} \), where addition of subscripts is taken modulo \( n+1 \). In fact, the set \( \{0, \ldots, n\} \) will be regarded as an additive cyclic group. If \( R \) has the FS property then the set \( \{0, \ldots, n\} \) will be used to index any maximal sequence \( s(e) \), the indexing being such that \( \overline{We_i} = \overline{Re_{i+1}} \), for \( 0 \leq i < n \). The set \( \{n+1, \ldots, w\} \) is partitioned by defining, recursively, mutually disjoint subsets \( I(t(1), \ldots, t(s)) \) as follows. If \( t \in \{0, \ldots, n\} \) then \( I(t) = \{i|n+1 < i < w \text{ and } \overline{We_i} = \overline{Re_{t}}\} \). If the set \( I(t(1), \ldots, t(s)) \) has been defined, then for every element \( p \in I(t(1), \ldots, t(s)) \) there is a, possibly empty, set \( I(t(1), \ldots, t(s), p) = \{i|n+1 \leq i \leq w \text{ and } \overline{We_i} = \overline{Re_{p}}\} \). The sets \( I(t(1), \ldots, t(s)) \) are not only mutually disjoint but they are uniquely determined by the sequences \( t(1), \ldots, t(s) \). For if \( m \) is an element of both \( I(t(1), \ldots, t(s)) \) and \( I(p(1), \ldots, p(q)) \) then, \( \overline{We_m} = \overline{Re_{t(s)}} = \overline{Re_{p(q)}} \) which implies that \( I(t(1), \ldots, t(s)) = I(p(1), \ldots, p(q)) \) and that \( p(q) = t(s) \). This argument, applied consecutively to \( t(s) = p(q) = t(s-1) \), shows
that \( t(s) = p(q) \), \( t(s-1) = p(q-1) \), \ldots, \( t(1) = p(1) \) and that \( s = q \).

For \( m \in I(t(1), \ldots, t(s)) \), if the sequence \( s(e_m) \) is finite then
\[
s(e_m) = e_m, e_{t(1)}, \ldots, e_{t(1)}+1, \ldots, e_n ;
\]
and if \( s(e_m) \) is infinite then
\[
s(e_m) = e_m, e_{t(1)}, e_{t(1)}+1, \ldots, e_n, e_0, \ldots, e_n.
\]

It is clear that if \( W_{e_i} \) is a homomorphic image of \( R_{e_j} \) then
\[
d(R_{e_i}) \leq d(R_{e_j}) + 1.
\]
Consequently, if \( e_i, e_j \) are the \( p \)th and \( q \)th terms respectively, of a sequence \( s(e) \) and if \( q \geq p \), then
\[
d(R_{e_i}) \leq d(R_{e_j}) + (q-p).
\]
By considering the minimum values that the numbers \( q-p \) can have, we deduce the following relations between the lengths of the \( R_{e_i} \).

5.4 LEMMA

If \( R \) has the FS property then the following inequalities hold.

(a) If \( i, j \in \{0, \ldots, n\} \) then
\[
d(R_{e_i}) \leq d(R_{e_{i+j}}) + j,
\]
whenever \( i + j \leq n \), and so
\[
d(R_{e_i}) \leq n + 1 - i.
\]

(b) If \( i \in I(t(s), \ldots, t(1)) \) and \( j \in \{t(s), \ldots, n\} \) then
\[
d(R_{e_i}) \leq d(R_{e_{t(k)}}) + k,
\]
whenever \( 1 \leq k \leq s \)
\[
d(R_{e_i}) \leq d(R_{e_j}) + s + j - t(s)
\]
and so
\[
d(R_{e_i}) \leq n + 1 + s - t(s).
\]

5.5 LEMMA

If \( R \) has the IS property then the following inequalities hold.

(a) If \( i, j \in \{0, \ldots, n\} \) then
\[
d(R_{e_i}) - j \leq d(R_{e_{i+j}}) \leq d(R_{e_i}) + n + 1 - j
\]
where addition of subscripts is taken modulo \( n + 1 \).

(b) If \( i \in I(t(s), \ldots, t(1)) \) and \( j \in \{0, \ldots, n\} \) then
\[ d(\text{Re}_i) \leq d(\text{Re}_{t(k)}) + k, \]  
whenever \( 1 \leq k \leq s \)

\[ d(\text{Re}_i) \leq d(\text{Re}_j) + s + j - t(s), \]  
whenever \( j \geq t(s) \)

\[ d(\text{Re}_i) \leq d(\text{Re}_j) + n + 1 + s + t(s) - j, \]  
whenever \( j < t(s) \)

The structure of \( R \) is determined by using Corollary 1.6 to represent it as a ring of matrices. To do this, we need to know the structure of the groups \( \text{H}_{ij} = \text{Hom}_R(\text{Re}_i, \text{Re}_j) \) and the multiplication in \( \bigcup_{i,j} \text{H}_{ij} \). The \( \text{Re}_i \) are left uniserial modules so each \( \text{H}_{ij} \) is a left uniserial \( \text{H}_{ij} \)-module. This is proved by showing that for any two elements \( \phi, \psi \in \text{H}_{ij} \) one of the \( \text{H}_{ij} \)-submodules \( \text{H}_{ij} \phi \), \( \text{H}_{ij} \psi \) is contained in the other. If \( \text{Re}_i \psi \subseteq \text{Re}_i \phi \) then there is an \( x \in R \) such that \( e_i \psi = x(e_i \phi) = e_i x e_i \phi \). If \( \rho \in \text{H}_{ij} \) is an endomorphism of \( \text{Re}_i \) that is given by \( e_i \rho = e_i x e_i \), then \( e_i (\rho \phi) = (e_i x e_i) \phi = e_i \psi \) and so \( \rho \phi = \psi \). Therefore \( \text{H}_{ij} \psi \subseteq \text{H}_{ij} \phi \). Similarly, if \( \text{Re}_i \phi \subseteq \text{Re}_i \psi \) then \( \text{H}_{ij} \phi \subseteq \text{H}_{ij} \psi \). Note that this argument shows that \( \text{H}_{ij} \phi \) is the set of homomorphisms from \( \text{Re}_i \) to \( \text{Re}_i \phi \), and therefore the length of \( \text{H}_{ij} \) is equal to the number of distinct non-zero homomorphic images of \( \text{Re}_i \) which are in \( \text{Re}_j \). The composition series of the left \( \text{H}_{ij} \)-module \( \text{H}_{ij} \) is denoted by \( \text{H}_{ij} = \text{H}_{ij}^0 \supseteq \text{H}_{ij}^1 \supseteq \cdots \supseteq \text{H}_{ij}^{d(\text{H}_{ij})} = 0 \).

From the construction of the sequences \( s(e_i) \) it is clear that \( \text{H}_{ij} \neq 0 \) only if \( e_i \) is a term of \( s(e_j) \). If \( R \) has the FS property, then each \( \text{Re}_j \) has at most one submodule which is a homomorphic image of \( \text{Re}_i \) and therefore \( \text{H}_{ij}^1 = 0 \) for all \( i,j \). In this case, \( e_i \) is a term of \( s(e_j) \) only if: \( j \leq i \leq n \); \( i,j \geq n+1 \) and \( j \in I(...,i,...) \); or \( j \geq n+1 > i \) and \( j \in I(k,...) \) for some \( k \leq i \). If \( R \) has the IS property and \( i,j \geq n+1 \) then \( \text{H}_{ij} \neq 0 \) only if \( j \in I(...,i,...) \). If
If \( i \leq n \) then it can be shown, by an example, that all the \( H_{ij} \) may be non-zero. Moreover, if \( i,j,k \in \{0,\ldots,n\} \) then it follows, from Lemma 5.5 and the fact that \( \bar{W}^e_i = \bar{R}_j \) whenever \( j \equiv i+t \mod(n+1) \), that

\[
\begin{align*}
\min\{d(H_{ij}) - 1 \leq d(H_{kj}) \leq d(H_{ij}) + 1\}, \\
\min\{d(H_{ij}) - 1 \leq d(H_{ik}) \leq d(H_{ij}) + 1\}, \text{ and} \\
\min\{d(H_{ij}) - 1 \leq d(H_{kj}) \leq d(H_{ij}) + 1\}.
\end{align*}
\]

Still under the assumption that \( R \) has the IS property, let \( \sigma \) be the maximum \( d(H_{ij}) \), for \( i,j \in \{0,\ldots,n\} \). As each \( H_{ij} \) is a homomorphic image of \( H_{ii} \), \( \sigma = d(H_{kk}) \) for some \( k \). It follows from (5.6) that every \( d(H_{ik}) \) is either \( \sigma-1 \) or \( \sigma \) and that, therefore, each \( d(H_{ij}) \) is either \( \sigma-1 \) or \( \sigma \). Hence if \( i,j \in \{0,\ldots,n\} \) then \( d(H_{ij}) \in \{\sigma-2,\sigma-1,\sigma\} \). Consequently, if \( i \in \{0,\ldots,n\} \) and \( j \in \{n+1,\ldots,w\} \) then \( d(H_{ij}) \leq \sigma \) and in fact, it can be shown, by example, that \( d(H_{ij}) \) can take any non-negative value not greater than \( \sigma \).

There is a deeper relationship between the rings \( H_{ii} \) than merely between their lengths. For \( i,j \in \{0,\ldots,w\} \) let \( K = \bar{W}^e_j \) be that image of \( \bar{R}_i \) in \( \bar{R}_j \) which has the greatest length, and let \( G = \text{End}_\bar{R}(K) \). As \( \bar{R}_j \) is uniserial the function \( \psi \), which maps an endomorphism of \( \bar{R}_j \) onto its restriction to \( K \), is a ring homomorphism from \( H_{jj} \) to \( G \). If \( \phi \in \bar{H}_{jj} \) has trivial kernel (that is, if \( \phi \not\subseteq \bar{H}_{jj}^1 \)) then so has \( \phi \psi \) and hence \( \phi \psi \not\subseteq G^1 \). Therefore \( \psi \) induces an embedding of the sfield \( \bar{H}_{jj} \) into the sfield \( \bar{G} \).
Assume that $i, j \in \{0, \ldots, n\}$. It follows from the periodicity of the composition series of $R_i$ that $1 \leq t \leq n$. If $\phi \in H^p_{jj}$ then $R_t^i\phi$ has $d(H^p_{jj})-p$ distinct endomorphic images of $R_t^i$. Between any two consecutive endomorphic images of $R_t^i$ there is an endomorphic image of $K$, consequently $K\phi = K(\psi)$ has $d(H^p_{ij})-p = d(G)-p$ distinct homomorphic images of $K$ and, therefore, $\psi \in G^p$. In fact, if $\phi \in H^p_{jj}$ then $\phi \psi \in G^p$. Hence $\phi \in \text{ker} \psi$ if, and only if, $G^p = 0$.

As $d(G) = d(H^p_{ij}) \in \{d(H^p_{jj})-1, d(H^p_{ij})\}$ either $\psi$ is a monomorphism (when $d(H^p_{ij}) = d(H^p_{jj})$), or $\ker \psi$ is the socle of $H^p_{jj}$ (when $d(H^p_{ij}) = d(H^p_{jj})-1$). Therefore $\psi$ induces an embedding of $G_j = H^p_{jj}/H^p_{jj}$ into $G$. Because $K$ is a homomorphic image of $R_t^i$, its ring of endomorphisms $G$ is a factor ring of $H^p_{ij}$; hence $G = G_j = H^p_{jj}/H^p_{jj}$. Consequently, $\psi$ induces an embedding $\phi$ of $G_j$ into $G_j$ with the property that $L$ is a left ideal of $G_j$ if, and only if, $L\phi = G_j \phi \cap L'$ for a left ideal $L'$ of $G_j$.

Consider the case when $H^p_{ij}$ is a (right as well as a left) uniserial ring and has a proper ideal. Let $J$ be the largest homomorphic image, in $K$, of $R_t^i$. If $\pi : R_t^i \rightarrow J$ is the canonic projection and $\phi \in F = \text{End}(J)$, then $\pi \phi \in H^1_{jj}$. As $H^1_{jj}$ is a cyclic right $H^p_{jj}$ module, for every pair of elements $\phi, \psi \in F$ there is a $\beta \in H^p_{jj}$ such that $\pi \phi = \pi \psi \beta$, and so $\phi = \psi \beta$. The action of $\beta$ on $J$ is merely that of $\beta|_J$: therefore $F$ is a cyclic right module over its subring of restrictions to $J$ of endomorphisms of $R_t^i$. Hence every element of $F$ is such a restriction, that is, every endomorphism of $J$ can be extended to an endomorphism of $R_t^i$. Consequently, the ring homo-
morphism from $G$ to $F$, which maps an endomorphism of $K$ onto its restriction to $J$, is an epimorphism. As $d(F) = d(H_{jj})^{-1}$ and $d(G) \in \{d(H_{jj})^{-1}, d(H_{jj})\}$ it follows that either $F = G$ or $F = G/S(G)$.

In terms of $H_{ii}$ and $H_{jj}$, this means that $G_j = G_i$ when $d(H_{ij}) = d(H_{jj})^{-1}$, and $G_j/S(G_j) = G_i/S(G_i)$ when $d(H_{ij}) = d(H_{jj})$.

If, moreover, $G$ is uniserial then, from a similar argument it follows that, every endomorphism of every endomorphic image $K'$ of $K$ is the restriction to $K'$ of an element of $G$. Therefore every endomorphism of $K'$ can be extended to an endomorphism of $J$ and hence to an endomorphism of $Re$. Hence the ring homomorphism $\Psi$ is an epimorphism: so $G_j = G_i$.

The number of composition factors of $Re_j$ which are isomorphic to $Re_i$ is equal to $d(H_{ij})$, therefore $d(Re_j) = \sum_{i=0}^{\infty} d(H_{ij})$. Applying this formula to Lemmas 5.4 and 5.5, we obtain the following relations.

5.7 **LEMMA**

*If $R$ has the FS property then the following inequalities hold.*

(a) *If $i, j \in \{0, \ldots, n\}$ then*

$$\sum_{k=i}^{n} d(H_{ki}) \leq j-i + \sum_{k=j}^{n} d(H_{kj}), \text{ if } j \geq i$$

*and so*

$$\sum_{k=i}^{n} d(H_{ki}) \leq n+1 - i.$$
(b) If \( j \in \{0, \ldots, n\}, i \in I(t(1), \ldots, t(s)) \) and \( j \geq t(1) \) then
\[
\sum_{k=2}^{s} d(H_{t(k)}, i) + \sum_{k=t(1)}^{n} d(H_{k,i}) \leq s - p + \sum_{k=2}^{p} d(H_{t(k)}, t(p)) + \sum_{k=t(1)}^{n} d(H_{k,t(p)})
\]
when \( 2 \leq p \leq s \)
\[
\leq s - j + t(1) + \sum_{k=j}^{n} d(H_{k,j}), \text{ and so}
\leq n+1 + s - t(1).
\]

5.8 LEMMA

If \( R \) has the IS property then the following inequalities hold.

(a) If \( i,j \in \{0, \ldots, n\} \) then
\[
\sum_{k=0}^{n} d(H_{k,i}) - j \leq \sum_{k=0}^{n} d(H_{k,i+j}) \leq n+1 - j + \sum_{k=0}^{n} d(H_{k,i})
\]
where the subscript \( i+j \) is taken modulo \( (n+1) \).

(b) If \( i \in I(t(1), \ldots, t(s)) \) and \( j \in \{0, \ldots, n\} \) then
\[
\sum_{k=2}^{s} d(H_{t(k)}, i) + \sum_{k=0}^{n} d(H_{k,i}) \leq s - p + \sum_{k=2}^{p} d(H_{t(k)}, t(p)) + \sum_{k=0}^{n} d(H_{k,t(p)})
\]
when \( 2 \leq p \leq s \)
\[
\leq s + j - t(1) + \sum_{k=0}^{n} d(H_{k,j}), \text{ when } j \geq t(1)
\]
\[
\leq n+1 + s + t(1) - j + \sum_{k=0}^{n} d(H_{k,j}), \text{ when } j < t(1)
\]

We investigate the product \( H_{ij} H_{jk} \) when \( H_{ij}, H_{jk}, \) and \( H_{ik} \) are all non-zero - the other cases are trivial. First consider the case
when $i,j,k \in \{0,...,n\}$. In this case let $r$, $s$ and $t$ be the smallest integers such that $W^j e_j^r = E_e^i$, $W^k e_k^s = E_e^i$ and $W^e e_k^t = E_e^j$, respectively. It follows from previous discussion that $0 \leq r$, $s$, $t \leq n$.

If $\phi \in H^P_{ij}$ and $\psi \in H^Q_{jk}$ then, form the periodicity of the composition series of $Re_j$ and $Re_k$, it follows that $e_i^\phi \in W^b e_j^r$, where $b = r + p(n+1)$, and $e_j^\psi \in W^c e_k^s$, where $c = t + q(n+1)$; therefore $e_i^\phi e_j^\psi \in W^m e_k^t$, where $m = r + t + (p+q)(n+1)$. As $\phi \psi \in H^P_{ik}$ this means that either $r + t = s$ or $r + t = s + n + 1$. So if $s \geq t$ then $r + t = s$, and if $s < t$ then $r + t = s + n + 1$. Hence $\phi \psi \in H^P_{ik}$ if $s \geq t$, and $\phi \psi \in H^P_{ik} + (p+q)$ if $s < t$. Consequently, if $i,j,k \in \{0,...,n\}$, then

$$H^P_{ij} H^Q_{jk} = \begin{cases} H^P_{ik} & \text{if } j \leq i \leq k \\ H^P_{ij} H^Q_{jk} & \text{if } i < k < j \\ \end{cases}$$

This can be stated more simply as follows: if $k,s,t \in \{0,...,n\}$ and $i \equiv k+s \mod (n+1)$ and $j \equiv k+t \mod (n+1)$ then

$$H^P_{ij} H^Q_{jk} = \begin{cases} H^P_{ik} & \text{if } s \geq t \\ H^P_{ik} + (p+q) & \text{if } s < t. \\ \end{cases}$$

Now consider the case when $i,j \in \{0,...,n\}$ and $k \in \{n+1,...,w\}$. Then $k \in I(\lambda,\ldots)$, for some $\lambda \in \{0,...,n\}$, and,
by examining the sequence \( s(e_k) \), it can easily be seen that
\[
h^q_{jk} = h^q_{jj} \quad \text{for every } q \geq 0.
\]
Applying equations (5.9) to this formula, we get
\[
H^p h^q_{ij} h^q_{jk} = \begin{cases} 
H^{p+q}_{lk} & \text{if } \ell \leq j \leq i, \quad j \leq i \leq \ell, \quad i \leq \ell \leq j \\
H^{p+q+1}_{lk} & \text{if } j \leq \ell \leq i, \quad \ell \leq i \leq j, \quad i \leq j \leq \ell
\end{cases}
\]
(5.10)

Or more concisely, if \( i \equiv \ell + s \mod (n+1) \) and \( j \equiv \ell + t \mod (n+1) \)
where \( s, t \in \{0, \ldots, n\} \) then
\[
H^p h^q_{ij} h^q_{jk} = \begin{cases} 
H^{p+q}_{lk} & \text{if } s \geq t \\
H^{p+q+1}_{lk} & \text{if } s < t.
\end{cases}
\]
(5.10a)

Finally consider the case when both \( j, k \in \{n+1, \ldots, w\} \). In this case \( h^1_{jk} = 0 \), so an argument, similar to that used above, shows that \( H^p_{ij} h^q_{jk} = H^p_{ik} \).

Substituting into Corollary 1.6 the results of the preceding discussion shows that \( R \) is isomorphic to a matrix ring \( H \) as described by the following theorem, which is the main result in this chapter.

5.11 THEOREM

Let \( R \) be an indecomposable left generalized uniserial ring. Then there are non-negative integers \( w \) and \( n \leq w \); positive integers \( v(0), \ldots, v(w) \) and \( \sigma \); a partition...
\{(i(1), \ldots, i(t)) | 0 \leq i(1) \leq n, i(p) \in I(i(1), \ldots, i(p-1)) \text{ for } p \geq 2\} \text{ of } \{n+1, \ldots, w\}; \text{ and additive abelian groups } H_{ij}, 0 \leq i, j \leq w, \text{ such that } R \text{ is isomorphic to the ring } H \text{ of all blocked matrices whose } (i,j)^{th} \text{ block is an arbitrary } v(i) \times v(j) \text{ matrix over the group } H_{ij}. \text{ The groups } H_{ij} \text{ have the following properties.}

(i) There is a partial associative multiplicative structure on
\bigcup_{i,j} H_{ij} \text{ with the property that for any two elements } \phi \in H_{ij}, \psi \in H \text{ st the product } \phi \psi \text{ is defined if } j = s \text{ and then } \phi \psi \in H_{it}. \text{ (ii) Each } H_{ii} \text{ is a left uniserial ring and each } H_{ij} \text{ is a left uniserial } (H_{ii} - H_{jj})\text{-bimodule.}

(iii) If } i \in \{n+1, \ldots, w\} \text{ then } H_{ij} \neq 0 \text{ only if } j \in I(\ldots, i, \ldots).\text{ If } i \in \{1, \ldots, n\} \text{ and } j = i-1 \text{ then } H_{ij} \neq 0.\text{ If } j \in I(\ldots, i) \text{ then } H_{ij} \neq 0. \text{ (iv) If } H_{0n} = 0 \text{ then}
\text{ (a) } H_{0j} = 0 \text{ for each } j \neq 0, \text{ and}
\text{ (b) if } i \in \{1, \ldots, n\} \text{ then } H_{ij} \neq 0 \text{ only if } j \leq i \text{ or } j \in I(k, \ldots), \text{ for some } k \leq i. \text{ (v) If } H_{0n} = 0 \text{ then } H_{ij}^{1} = 0 \text{ for all } i, j. \text{ (vi) If } H_{0n} \neq 0 \text{ then the following conditions hold.}
\text{ (a) } H_{ij}^{\sigma} = 0 \text{ for all } i, j. \text{ (b) If } i \in \{n+1, \ldots, w\} \text{ then } H_{ij}^{1} = 0 \text{ for every } j. \text{ (c) If } i, j, k \in \{0, \ldots, n\} \text{ then } d(H_{i1}) \in \{\sigma-1, \sigma\},
d(H_{ij}) \in \{\sigma-2, \sigma-1, \sigma\}, \text{ and}
\[ \begin{align*} 
\text{d}(H_{ij}) - 1 & \leq \text{d}(H_{kj}) \leq \text{d}(H_{ij}) + 1, \\
\text{d}(H_{ij}) - 1 & \leq \text{d}(H_{ik}) \leq \text{d}(H_{ij}) + 1, \text{ and} \\
\text{d}(H_{ij}) - 1 & \leq \text{d}(H_{jk}) \leq \text{d}(H_{ij}) + 1. 
\end{align*} \]

(vii) If \( j, k \in \{0, \ldots, n\} \) then

\[
\sum_{i=0}^{n} \text{d}(H_{ij}) - k \leq \sum_{i=0}^{n} \text{d}(H_{i, j+k}) \leq n+1 - k + \sum_{i=0}^{n} \text{d}(H_{ij})
\]

where the subscript \( j+k \) is taken modulo \((n+1)\).

If \( j \in I(t(1), \ldots, t(s)) \) and \( k \in \{0, \ldots, n\} \) then

\[
\sum_{i=2}^{s} \text{d}(H_{t(i), j}) + \sum_{i=0}^{n} \text{d}(H_{ij}) \leq s-p + \sum_{i=2}^{p} \text{d}(H_{t(i), t(p)}) + \sum_{i=0}^{n} \text{d}(H_{i, t(p)}),
\]

for \( 2 \leq p \leq s \)

\[
\leq s+k-t(l) + \sum_{i=0}^{n} \text{d}(H_{ik}), \text{ if } k \geq t(l)
\]

\[
\leq n+1+s+t(l)-k + \sum_{i=0}^{n} \text{d}(H_{ik}), \text{ if } k < t(l)
\]

(viii) If \( H_{0n} = 0 \) then

\[
\sum_{i=j}^{n} \text{d}(H_{ij}) \leq k + \sum_{i=j+k}^{n} \text{d}(H_{i, j+k}) \leq n+1-j, \text{ when } j+k \leq n
\]

and \( \sum_{i=2}^{s} \text{d}(H_{t(i), j}) + \sum_{i=t(l)}^{n} \text{d}(H_{ij}) \leq n+1+s-t(l), \)

when \( j \in I(t(1), \ldots, t(s)) \).
(ix) (a) If \( i, j, k \in \{0, \ldots, n\} \) then

\[
H^p_{ij} H^q_{jk} = \begin{cases} 
H^{p+q}_{ik} & \text{when } k \leq j \leq i \\ \ j \leq i \leq k \\ \ i < k < j \\
H^{p+q+1}_{ik} & \text{when } j < k \leq i \\ \ i < j < k
\end{cases}
\]

(b) If \( i, j \in \{0, \ldots, n\} \) and \( k \in I(\ldots) \) then

\[
H^p_{ij} H^q_{jk} = \begin{cases} 
H^{p+q}_{ik} & \text{when } j \leq i \leq k \\ \ k \leq j \leq i \\ \ i < j < k \\
H^{p+q+1}_{ik} & \text{when } j < k \leq i \\ \ i < j < k
\end{cases}
\]

(ix) (c) If \( j, k \in \{n+1, \ldots, w\} \) then \( H^p_{ij} H^q_{jk} = H^p_{ik} \)

(x) If \( h_{ij} \neq 0 \) then \( H_{jj} \subseteq H_{ii} \).

(xi) If \( h_{0n} \neq 0 \) and \( i, j \in \{0, \ldots, n\} \) then

\[
G_j = H_{jj} / H_{jj} \subseteq H_{ii} / H_{jj} = G_1 \quad \text{and every left ideal of } G_j \text{ is the intersection of } G_j \text{ with a left ideal of } G_1. \quad \text{If moreover } H_{jj} \text{ is a uniserial ring, then } G_j = G_1 \text{ when } d(H_{ij}) = d(H_{jj}) - 1 \text{ and }
\[ G_j / S(G_j) = G_i / S(G_i) \text{ when } d(H_{ij}) = d(H_{ij}). \] If \( G_1 \) is also a uniserial ring then \( G_j = G_i \).

Conversely, if a matrix ring \( H \) has all the above properties, then it is an indecomposable left generalized uniserial ring.

Proof. It has already been shown that \( R \) is isomorphic to the matrix ring \( H \), so it remains only to show that the above description of \( H \) guarantees that \( H \) is an indecomposable left generalized uniserial ring. Denote by \( |\phi|_{1j}^{k\ell} \) the element of \( H \) whose only non-zero entry is the element \( \phi \in H_{\ell k} \) at the place \( (j, k) \) in the \((1, k)\)th block, let \( |H_{1k}^t| = \{ |\phi|_{1j}^{k\ell} \phi \in H_{1k}^t \} \) and, for simplicity, let \( e_i = |1|_{11} \). As \( H = \sum_{i,j} H|1|^{1j}_{ij} \), to show that \( H \) is a left generalized uniserial ring it is sufficient to show that each \( H|1|^{1j}_{ij} \) is a uniserial ideal. For each \( j \), \( H|1|^{1j}_{ij} \subseteq He_1 \); therefore it need only be shown that each \( He_1 \) is a uniserial ideal. To do this, it is sufficient to show that for any two elements \( x, y \in He_1 \) one of the ideals \( Hx, Hy \) is contained in the other.

Let \( \ell \in \{0, \ldots, n\} \) and consider the non-zero ideal \( Hx \subseteq He_1 \), where \( x = |\phi|_{j k}^{1j} \), \( \phi \in H_1^\alpha \setminus H_1^\alpha+1 \). Condition (iii) implies that \( 0 \leq j \leq n \) and therefore \( j = \ell + t \text{ mod.}(n+1) \), for some \( t \in \{0, \ldots, n\} \).

As \( H = \sum H_{ip}^{pq} \) and \( |H_{ip}^{pq}|_{jk}^{1j} = 0 \), if \( (p, q) \neq (j, k) \), it follows that \( Hx = \sum H_{ij}^{1j} x \). By condition (iii), \( H_{ij} \neq 0 \) only if \( 0 \leq i \leq n \); therefore \( Hx = \sum_{0 \leq i \leq n} H_{ij}^{1j} x \) which can be expressed as

\[ Hx = \sum_{0 \leq s \leq n} H_{s+j}^{1j} x \mid_{x+s, m} \] where the subscript \( s \) is taken
modulo \((n+1)\). Applying condition (ix) to this equation shows that

\[
H_x = \sum_{n>s>t} |H|^a_{\infty, s+m} + \sum_{0<s<t} |H|^a_{\infty, s+m} \quad \text{If if}
\]

If \(y = \psi_{p\infty} \), \(\psi \in H_{\infty}^a \setminus H_{\infty}^{a+1} \), is a non-zero element of \(H_{\infty}^{a+1} \), then a completely analogous argument proves that

\[
H_y = \sum_{n>s>t} |H|^a_{\infty, s+u} + \sum_{0<s<t} |H|^a_{\infty, s+u} \quad \text{where}
\]

If \(\alpha < \beta\) then clearly \(H_y \subseteq H_x\). If \(\alpha = \beta\) and \(t < t'\), then \(H_y \subseteq H_x\). Consequently, one of the ideals \(H_x, H_y\) is contained in the other. Now let \(z\) be any non-zero element of \(H_{\infty}^{a+1} \), then \(z = \sum z_{ij}\), where \(z_{ij} = |1|_{i,j}^a z\), and so \(H_z = \sum H_{z_{ij}}\).

By condition (iii), \(z_{ij} = 0\) if \(i > n+1\): therefore \(H_z = \sum_{0<i<n} H_{z_{ij}}\).

By the above result, all the \(H_{z_{ij}}\), \(0 \leq i \leq n\), are linearly ordered by inclusion, hence \(H_z\) is one of the ideals \(H_{z_{ij}}\). It follows that for any two elements \(z, z' \in H_{\infty}^{a+1}\), one of the ideals \(H_z, H_{z'}\) is contained in the other. Therefore \(H_{\infty}^{a+1}\) is a uniserial ideal.

By a completely analogous argument, it can be shown that if \(i \in \{n+1, \ldots, w\}\), then \(H_i\) is a uniserial ideal. Therefore \(H\) is a left generalized uniserial ring. It follows from condition (iii) that for any \(m \in \{0, \ldots, w\}\) there is a sequence \(n = r(1), \ldots, r(t) = m\) of elements of \(\{0, \ldots, w\}\) with the property that each \(H_{r(i), r(i+1)}\) is non-zero. Therefore \(H\) is indecomposable.

As an immediate application of this theorem we determine the structure of generalized uniserial rings. Assume that the ring \(R\) is a (right as well as a left) generalized uniserial ring, then it
follows that \( n = w \). For if \( j \neq k \), and both \( W_k \) and \( W_n \) are homomorphic images of \( R_k \), then \( e_1 W_k \cong e_1 W_n \) and \( e_1 W_k \cong e_1 W_n \), which implies that \( W_1 W_k \) contains homomorphic images of the non-isomorphic right ideals \( e_1 R_k \) and \( e_1 R_n \) : a contradiction to the assumption that \( e_1 R_k \) is a uniserial right ideal. By writing homomorphisms between right ideals on the left, it follows from an argument, analogous to that used in the proof of the above theorem, that \( H_{jj} = \text{Hom}_R(e_j R, e_j R) \) is a uniserial right, as well as a uniserial left, \( (H_{11}, H_{jj}) \)-bimodule. Since every left \( H_{11} \)-submodule of \( H_{jj} \) is a right \( H_{jj} \)-submodule of \( H_{jj} \) and conversely, the left and right submodules of \( H_{jj} \) coincide. Therefore the same symbol \( H_{jj}^P \) can be used to denote the \( p \)th largest left and right submodule \( (H_{11}^1)_{H_{jj}}^p = H_{jj} (H_{11}^1)^P \) of \( H_{jj} \), and the same symbol \( d(H_{jj}^P) \) can denote both the left and right lengths of \( H_{jj}^P \). Combining these facts and notations with the above theorem and its right hand analogue, produces the following characterization of generalized uniserial rings.

5.12 THEOREM

Let \( R \) be an indecomposable generalized uniserial ring. Then there is a non-negative integer \( n \), positive integers \( v(0), \ldots, v(n) \) and \( \sigma \), and additive abelian groups \( H_{ij} \), \( 0 \leq i, j \leq n \), such that \( R \) is isomorphic to the ring \( H \) of all blocked matrices whose \( (i,j)^{th} \) block is an arbitrary \( v(i) \times v(j) \) matrix over the group \( H_{ij} \). The groups \( H_{ij} \) have the following properties.
(i) There is a partial, associative, multiplicative structure on 
\( \bigcup_{i,j} H_{ij} \) with the property that, for any elements \( \phi \in H_{ij} \) and \( \psi \in H_{st} \), the product \( \phi \psi \) is defined if \( j = s \) and then \( \phi \psi \in H_{it} \).

(ii) Each \( H_{ii} \) is a uniserial ring, and each \( H_{ij} \) is a left uniserial and a right uniserial \( (H_{ii}, H_{jj}) \)-bimodule.

(iii) If \( 1 \leq i \leq n \) and \( j = i-1 \), then \( H_{ij} \neq 0 \).

(iv) If \( H_{0n} = 0 \) then \( H_{ij} = 0 \), whenever \( i < j \).

(v) If \( H_{0n} = 0 \) then every \( d(H_{ij}) \in \{0,1\} \).

(vi) If \( H_{0n} \neq 0 \) then every \( d(H_{ii}) \in \{\sigma-1, \sigma\} \), every \( d(H_{ij}) \in \{\sigma-2, \sigma-1, \sigma\} \), and for all \( i,j,k \),

\[
\begin{align*}
&d(H_{ij}) - 1 \leq d(H_{kj}) \leq d(H_{ij}) + 1, \\
&d(H_{ij}) - 1 \leq d(H_{ik}) \leq d(H_{ij}) + 1, \text{ and} \\
&d(H_{ij}) - 1 \leq d(H_{jk}) \leq d(H_{ij}) + 1.
\end{align*}
\]

(vii) For all \( j,k \),

\[
\sum_{i=0}^{n} d(H_{ij}) - k \leq \sum_{i=0}^{n} d(H_{i,j+k}) \leq n+1-k + \sum_{i=0}^{n} d(H_{ij})
\]

where the subscript \( j+k \) is taken modulo \( n+1 \).

(viii) If \( H_{0n} = 0 \) and \( j+k \leq n \), then

\[
\sum_{i=j}^{n} d(H_{ij}) \leq k + \sum_{i=j+k}^{n} d(H_{i,j+k}) \leq n+1 - j.
\]

(ix) If \( i,j,k \in \{0, \ldots, n\} \), then
\[
H^p_{ik} \quad \text{when} \quad j \leq i \leq k \\
H^q_{jk} \\
\left\{ \begin{array}{ll}
H^p_{ij} & \text{when} \quad j < k < i \\
H^p+q_{ik} & \text{when} \quad i < k < j \\
H^p+q+1_{ik} & \text{when} \quad j < k < i \\
H^p_{ij} & \text{when} \quad k < i < j \\
H^q_{jk} & \text{when} \quad i < j < k
\end{array} \right.
\]

\(\bar{H}_{ii} = \bar{H}_{jj}\) for all \(i, j\).

If \(H_{0n} \neq 0\) then \(H_{ii}/H_{ii} = H_{jj}/H_{jj}\), for all \(i, j\).

Conversely, the above matrix ring \(H\) is an indecomposable generalized uniserial ring.

As further clarification of the relationship between left generalized uniserial rings and generalized uniserial rings, note that the ring \(R\) of Theorem 5.11 is a generalized uniserial ring if \(n = w \neq 0\). For if \(n = w \neq 0\) then, for any \(t \geq 0\), at most one non-zero \(W^t e_j\) is a homomorphic image of \(Re_j\), that is, at most one \(e^W_t e_j\) is not contained in \(e^W_t + 1\). But this means that each \(e^W_t / e^W_t + 1\) is either zero or a simple right \(R\)-module. Therefore \(e_i^R\) is a uniserial right ideal, and so \(R\) is a generalized uniserial ring. In view of Theorem 5.12 this means that \(R\) is a generalized uniserial ring with more than one idempotent if, and only if, \(n = w \neq 0\).

The preceding two theorems would completely determine the structure of (left) generalized uniserial rings if the structure of (left) uniserial rings were known. Unfortunately the structure of
these rings is not known. In general, the structure of a (left) uniserial ring is inextricably connected with that of its factor sfield (the ring modulo its radical), which is an indication of the difficulty of characterising these rings. However, if a left uniserial ring is cleft, that is, if it is a (group) direct sum of its radical and a subring (necessarily a sfield), then it is easy to reduce the structure of the ring to that of its factor sfield.

5.13 DEFINITION For a sfield $D$, a ring homomorphism $\phi : D \rightarrow D$, and a left vector space $V = Dx_1 \oplus \cdots \oplus Dx_m$, disjoint from $D$, let $A(D,V,\phi)$ be the ring on $D \oplus V$ whose multiplication is given by

$$d_1x_1d_2x_2 = d_1(d_2^S)x_{s+t},$$

where $x_0$ is the identity of $D$, $\phi^0$ is the identity map on $D$, and $x_r = 0$ for $r > m$.

5.14 THEOREM Every $A(D,V,\phi)$ is a cleft, left uniserial ring, and every cleft, left uniserial ring is isomorphic to an $A(D,V,\phi)$. An $A(D,V,\phi)$ is uniserial if, and only if, $\phi$ is an epimorphism.

Proof. It is clear that $A(D,V,\phi)$ is a cleft, left uniserial ring, and that it is also a right uniserial ring if $\phi$ is an epimorphism.

Therefore it is necessary to prove only the converse. Let $R$ be a cleft left uniserial ring, then $R = D \oplus W$ where $D$ is a sfield and $W$ is the radical of $R$. If $x \in W \setminus W^2$ then $x^t \in W \setminus W^{t+1}$, and, since $D = R/W$, it follows that each $W^t = Dx^t \oplus \cdots \oplus Dx^n$, where $x^n \neq 0$ but $x^{n+1} = 0$. It is clear that the map $\phi : D \rightarrow D$, given by $xd = (d\phi)x$, is a ring homomorphism and that $x^Sd = (d^S)x^S$, for any $x^S$,
where \( \phi^0 \) is the identity map on \( D \). Therefore the multiplication in \( R \) is given by \( d_1 x^s d_2 x^t = d_1 (d_2 \phi^s) x^{s+t} \).

If \( R \) is right uniserial then \( Dx = xD \), since \( W/W^2 \) is a simple right module, and so \( \phi \) is an epimorphism.

The original interest in generalized uniserial rings was inspired by Nakayama's result [31] that their left and right modules are direct sums of uniserial submodules. The converse to this result is true under weaker assumptions. Let the ring \( R \) of Theorem 5.11 have the property that all of its left modules which are generated by two elements are direct sums of uniserial submodules, and let \( M \) be a uniserial left \( R \)-module whose length is maximal. If \( M \) is not injective, then in any injective hull \( H(M) \) of \( M \) there is an element \( x \) with the properties that \( x \not\in M \) and \( M \nsubseteq Rx \). Therefore \( Rx + M \) is a direct sum of at least two uniserial submodules and so its socle is not simple: a contradiction, since \( H(M) \) is indecomposable and so has simple socle. Therefore \( M \) is injective. It follows that every uniserial \( R/W^2 \)-module with length equal to two is injective.

If \( R \) has more than one idempotent and \( \overline{W}_e = \overline{W}_e \), then, by the above result, \( Re_i /W^2e_i \) and \( Re_j /W^2e_j \) are isomorphic \( R/W^2 \)-modules. It follows that \( \overline{Re}_i = \overline{Re}_j \) and so \( n = v \). Hence the remark following Theorem 5.12 implies that \( R \) is a generalized uniserial ring. If \( R \) is left uniserial, then it is left injective since it has maximal length as a uniserial left module. Consequently every endomorphism of each \( W^t \) can be extended to an endomorphism of \( R \) and, therefore, each \( W^t \) is a cyclic right ideal. Hence \( R \) is a uniserial ring.

Combined with the above result of Nakayama, this proves the following theorem.
5.15 THEOREM

If the two generator left modules of a ring $R$ are direct sums of uniserial submodules, then $R$ is a generalized uniserial ring and so every left, and every right, $R$-module is a direct sum of uniserial submodules.
REFERENCES


