A graphical calculus for shifted symmetric functions

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Declaration

I hereby declare that the material in this thesis is my own original work except where stated otherwise.

The material in Chapter 3 is included in the paper titled “Khovanov’s Heisenberg category, moments in free probability, and shifted symmetric functions” [KLM16]. The paper is based on work completed concurrently and independently by myself and Henry Kvinge, and extended and merged by the three authors. I have included here my own versions of the proofs of the main theorems where the difference of exposition was significant.

Stuart Mitchell

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I wish to first give my thanks to my supervisors Anthony Licata and James Borger. Both were ever available and flexible in accommodating my own work commitments. Their advice and patience has been invaluable and without which, this thesis would not have been possible.

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Finally, I would like to express my extreme gratitude for the patience and support of my wife Melissa Mitchell. During my most productive periods she took on more responsibility for our family than any one person can reasonably be asked to bear. It is my most sincere hope to return the favour to her soon.

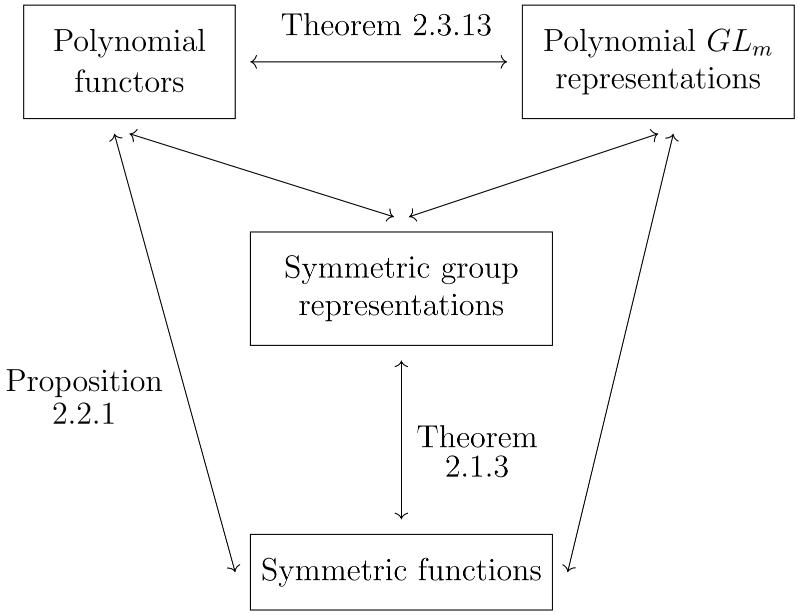
Abstract

The goal of this thesis is twofold. The first goal is to describe three categorifications of the algebra of symmetric functions and establish relationships between them all. The second goal is to establish an isomorphism between the centre of Khovanov's Heisenberg category [Kho14] and the algebra of shifted symmetric functions defined by Okounkov and Olshanski [OO97]. This isomorphism lends us a graphical description of some important bases of the algebra of shifted symmetric functions. Conversely, we are also able to describe some important generators of the centre of the Heisenberg category in the language of shifted symmetric functions. This turns out to be given in the language of free probability, in particular, the transition and co-transition measures of Kerov [Ker93, Ker00].

Contents

Introduction

In Chapter 1 we introduce the notations and key concepts upon which we will rely subsequently. The second chapter tells the story given by the diagram below, in which each arrow represents an equivalence of some kind.



In this chapter we establish an isomorphism of rings between the ring of symmetric functions and the graded ring formed by taking direct sum of the Grothendieck groups of the category of S_n representations as n varies, where the ring structure is given by the induction product. The structure of our proof follows Fulton [Ful97]. We introduce and classify polynomial functors. To this end we follow the construction of Macdonald [Mac95]. We also refer the reader to Friedlander and Suslin [FS97], who provide a more modern, category theoretic approach to their construction. In this section, we also establish an equivalence between the category of S_n representations and the category of polynomial functors homogeneous of degree n, and hence describe the category of polynomial functors as a categorification of the ring of symmetric functions.

Next, we describe the irreducible polynomial representations of GL_m and demonstrate that they are given by irreducible polynomial functors. We establish then that the category of polynomial GL_m representations is a categorification of the ring of symmetric polynomials in m variables. We then follow the stable category construction of Hong and Yacobi [HY13] to produce a category which describes polynomial GL_m representations for all values of m at once. We then show that this category is equivalent to the category of polynomial functors. This then establishes this stable category as yet another categorification of the ring of symmetric functions.

Finally, we discuss a multiplication on the ring of symmetric functions called plethysm, which is given by composition. We then describe the analogues in each of the categorifications we have discussed. Again we follow Macdonald [Mac95], however he does not cover the analogue for the category $\mathcal{M}$, the tower of sequences of polynomial representations of GL_m defined in §2.3.

The third and final chapter of this thesis contains the novel material. In [Kho14], Khovanov introduces a graphical calculus of oriented planar diagrams which we use to define a linear monoidal category $\mathcal{H}'$, designed to be a categorification of the Heisenberg algebra. We denote by $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ the endomorphism algebra of the monoidal unit in $\mathcal{H}'$. Diagramatically this is given by the algebra closed oriented planar diagrams, modulo the relations of his graphical calculus. Khovanov introduces two sets of generators for $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$: the clockwise curls $\{c_k\}_{k\geqslant 0}$ and the counterclockwise curls $\{\tilde{c}_k\}_{k\geqslant 2}$. He then establishes algebra isomorphisms

$$\mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \cong \mathbb{C}[c_0, c_1, c_2, \ldots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \ldots],$$

and gives a recursive relationship between the two sets of curls.

He then relates $\mathcal{H}'$ to representation theory by defining a sequence of functors $\mathcal{F}'_n$ from $\mathcal{H}'$ to bimodule categories for symmetric groups. A consequence of the existence of these functors is the existence of surjective algebra homomorphisms,

$$f_n^{\mathcal{H}'} : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow Z(\mathbb{C}[S_n]),$$

from $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ to the center of the group algebra of each symmetric group. Based in part on this, Khovanov suggests that there should be a close connection between $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and the asymptotic representation theory of symmetric groups. Furthermore, one might hope that $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ in fact gives a diagrammatic description of some algebra of pre-existing combinatorial interest.

The main goal of this chapter then, is to make precise the connection between $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and both the asymptotic representation theory of symmetric groups and algebraic combinatorics. This is achieved by establishing an isomorphism

$$\Psi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*,$$

where Λ^* is the *shifted symmetric functions* of Okounkov-Olshanski [OO97]. This is the content of theorem 3.1.3.) The algebra of shifted symmetric functions Λ^* is a deformation of the algebra of symmetric functions. As is the case for $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$, there are surjective algebra homomorphisms

$$f_n^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}[S_n]),$$

to the center of the group algebra of each symmetric group. The isomorphism $\Psi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow \Lambda^*$ is canonical, in that it intertwines the homomorphisms $f_n^{\mathcal{H}'}$ and $f_n^{\Lambda^*}$.

The remainder of the chapter is largely concerned with finding graphical descriptions of some important bases of Λ^* and conversely finding combinatorial descriptions of some important graphical bases of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$. The curl generators c_k and $\tilde{c}_k$ of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ can be described in the language of free probability. This was anticipated by Khovanov [Kho14], however the connection described between moments of the co-transition measure and the Boolean cumulants of the transition measure appears to be new. Table 1 provides a dictionary of these findings. Our last consideration is an involution common to both $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and Λ^*.

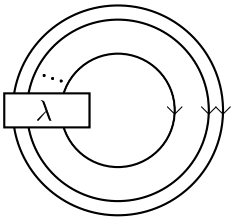
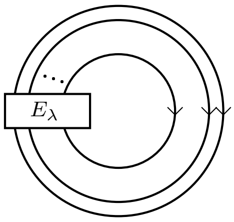
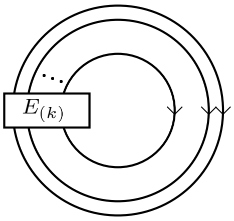
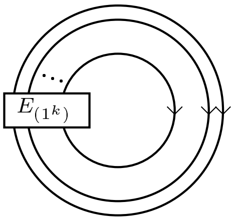
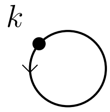
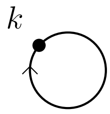
Λ^*	diagram in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$	page defined (object, diagram)
$p_\lambda^\#$		p. 20, p. 50
s_λ^*	$\frac{1}{\dim\lambda}$ 	p. 18 , p. 52
h_k^*		p. 20, p. 52
e_k^*		p. 20, p. 52
σ_k		p. 29, p. 22
$\widehat{b}_{k+2} = p_1^\# \widehat{\sigma}_k$		p. 29, p. 22

Table 1: A dictionary between Λ^* and diagrams in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$.

Chapter 1

Preliminaries

1.1 Partitions and Young's lattice

Following [Mac95], a *partition* is any sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots)$ of non-negative integers in weakly decreasing order, with only finitely many non-zero terms. The non-zero λ_i are called *parts*, and the number of parts the *length*, denoted $l(\lambda)$. The sum

$$|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$$

is called the *weight* of λ. If $|\lambda| = n$, we call λ a *partition of* n, and write $\lambda \vdash n$. Given two partitions μ and λ, we write $\mu \subset \lambda$ if for all $i = 1, 2, \ldots$, we have $\mu_i \leqslant \lambda_i$.

The *multiplicity*, $m_i = m_i(\lambda) = \text{card}\{j : \lambda_j = i\}$, is the number of times the part i appears in λ. At times we will find it convenient to write

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k}, \ldots)$$

to indicate the number of times each part occurs in the partition. Define

$$z_\lambda := \prod_{i \geqslant 1} i^{m_i(\lambda)} m_i(\lambda)! \in \mathbb{N}.$$

Let the set of all partitions of n be denoted by $\mathcal{P}_n$, and take

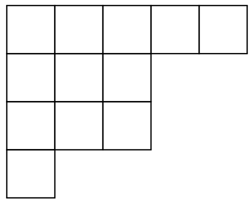
$$\mathcal{P} := \coprod_{n \geqslant 0} \mathcal{P}_n.$$

For $k \leqslant n$ we have an embedding $\phi_{k,n} : \mathcal{P}_k \hookrightarrow \mathcal{P}_n$ that sends $\lambda \vdash k$ to

$$(\lambda_1, \ldots, \lambda_{l(\lambda)}, 1, \ldots, 1).$$

We write $\overline{\lambda}$ for the partition obtained by removing all the parts equal to 1 from λ.

The *Young diagram* of a partition λ is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$. We adopt the convention that the i coordinate increases moving downwards, and the j coordinate increases moving from left to right. We represent the diagram pictorially with boxes for each point in the diagram. The diagram of $(5,3,3,1)$ is shown below.



The *conjugate* of a partition λ is the partition λ' such that

$$\lambda'_i = \text{card}\{j : \lambda_j \geqslant i\}.$$

Alternatively, it is the partition whose diagram is the reflection in the main diagonal. The conjugate of the partition $(5,3,3,1)$ above is readily seen from its diagram to be $(4,3,3,1,1)$.

The *content*, $c(i,j)$ of a node $(i,j) \in \lambda$, is defined to be $j - i$. We denote by A_λ the finite alphabet of contents of λ.

The *hook-length* of a partition λ at the point (i,j) is the quantity

$$h(i,j) := \lambda_i + \lambda'_j - i - j + 1.$$

This counts the number of cells to the right and directly below the cell, and the cell itself. The product of all the hook-lengths is then easily shown to be the product

$$H_\lambda := \frac{\prod_{i=1}^{l(\lambda)} (\lambda_i + l(\lambda) - i)!}{\prod_{1 \leqslant i < j \leqslant l(\lambda)} (\lambda_i - \lambda_j + j - i)}.$$

For any partition λ and any integer $1 \leqslant i \leqslant l(\lambda)+1$, let $\lambda^{(i)}$ denote the partition μ, if it exists, such that $\mu_j = \lambda_j$ for $j \neq i$, and $\mu_i = \lambda_i + 1$. Similarly, denote by $\lambda_{(i)}$ the partition ν, if it exists, such that $\nu_j = \lambda_j$ for $j \neq i$, and $\nu_i = \lambda_i - 1$.

Following [Ful97], we call a *numbering* or *filling* of the boxes of a Young diagram the result of placing elements of some alphabet, usually the numbers from 1 to n, where $\lambda \vdash n$. A (*semi-standard*) *Young tableau*, often lazily referred to as *tableau*, is a filling that is weakly increasing across each row, and strictly increasing down each column. A *standard tableau* is a tableau in which we further require strictness along the rows. An example of each is shown below for the *shape* $(4,2,2,1)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 5 \\ \hline 2 & 3 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 9 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

Tableau   Standard tableau

Define a *Young tabloid* to be an equivalence class of Young tableaux in which two tableaux are considered equivalent if they contain the same entries in the same rows. The tabloids of shape $(2, 1)$ are then

$$\left\{\begin{array}{cc}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{cc}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right\}, \left\{\begin{array}{cc}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{cc}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right\}, \left\{\begin{array}{cc}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{cc}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right\}.$$

Note that we omit column lines in our notation of each tableau in the equivalence class. We usually also denote the equivalence class with a representative of the class.

Let $\mathcal{Y}_n$ denote the set of Young diagrams with n boxes, and take

$$\mathcal{Y} := \coprod_{n \geqslant 0} \mathcal{Y}_n$$

to be the lattice of all Young diagrams, ordered by inclusion.

Let $\dim \lambda$ be the number of standard tableaux of shape λ. For $\lambda \vdash n$, this number is the same as the dimension of irreducible representation of S_n corresponding to λ. By setting $\dim \varnothing = 1$, we then have the recurrence relation

$$\dim \lambda = \sum_i \dim \lambda_{(i)}$$

as a direct consequence of the branching rule (Corollary 2.1.6).

The k-th *falling factorial power*, written $(x \downharpoonleft k)$, is defined by

$$(x \downharpoonleft k) := \begin{cases} x(x-1)\cdots(x-k+1), & k > 0, \\ 1, & k = 0. \end{cases}$$

To each partition $\mu \vdash k$ we then define a function f_μ on $\mathcal{Y}$ by

$$f_\mu(\lambda) := \begin{cases} (n \downharpoonleft k)\widehat{\chi}^\lambda_\mu, & n = |\lambda| \geqslant k, \\ 0, & |\lambda| < k, \end{cases}$$

where $\widehat{\chi}^\lambda_\mu$ is the normalised irreducible character defined below.

1.2 The symmetric group and its representations

Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$. S_n is generated by the elements $s_1, \ldots, s_{n-1}$, where s_i is the transposition $(i, i+1)$. For any $k \leqslant n$, there is an embedding $\iota_{k,n} : S_k \hookrightarrow S_n$, given by identifying S_k with the stabiliser of

$$\{k+1, \ldots, n\}.$$

Every permutation $\sigma \in S_n$ factorises uniquely as a product of disjoint cycles. If we order the disjoint cycles by length, we form a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weight n. The partition λ is called the *cycle-type* of σ. The cycle-types, and hence the partitions of n give the conjugacy classes of S_n. Given a partition $\lambda \vdash n$, we write C_λ for the conjugacy class of permutations of cycle-type λ.

Denote by $\mathbb{C}[S_n]$ the group algebra of S_n, and let $Z(\mathbb{C}[S_n])$ be its centre. We write

$$\mathrm{pr}_n : \mathbb{C}[S_{n+1}] \twoheadrightarrow \mathbb{C}[S_n]$$

for the orthogonal projection of $\mathbb{C}[S_{n+1}]$ onto $\mathbb{C}[S_n]$ defined by

$$\mathrm{pr}_n(g) = \begin{cases} g, & g \text{ fixes } n+1, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leqslant i \leqslant n$, the *Jucys-Murphy elements* J_i are defined by

$$J_i := \sum_{1 \leqslant j < i} (j, i).$$

Jucys [Juc74] showed that $Z(\mathbb{C}[S_n])$ is spanned by symmetric polynomials in the J_i. The set of elements

$$K_\lambda := \sum_{\sigma \in C_\lambda} \sigma$$

as λ ranges over all partitions of n give a basis for $Z(\mathbb{C}[S_n])$. When $\lambda \vdash k \leqslant n$, out of laziness we will write

$$C_\lambda := C_{\phi_{k,n}(\lambda)}, \quad z_\lambda := z_{\phi_{k,n}(\lambda)}, \quad K_\lambda := K_{\phi_{k,n}(\lambda)}.$$

The irreducible representations of S_n are indexed by partitions $\lambda \vdash n$. Let V_λ denote the irreducible representation of S_n corresponding to the partition λ, and write χ^λ for the corresponding irreducible character. For brevity we write χ^λ_μ for its value $\chi^\lambda(\sigma)$ at any permutation σ of cycle-type μ, and will conflate $\dim \lambda$ for $\dim V_\lambda$ (note that this number is the same as our previous designation of this notation anyway). The *normalised character*, is defined by

$$\widehat{\chi}^\lambda_\mu := \frac{\chi^\lambda_\mu}{\dim \lambda},$$

and the *central character* by

$$\theta^\lambda_\mu := \frac{H_\lambda}{z_\mu}\chi^\lambda_\mu = \frac{n!}{z_\mu}\widehat{\chi}^\lambda_\mu.$$

All characters can be extended by linearity to functions on $\mathbb{C}[S_n]$ in the following way: the application of the character χ on S_n to an element

$$\sum a_i g_i \in C[S_n],$$

is given by

$$\sum a_i \chi(g_i).$$

To each partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash k \leqslant n$ we assign an element

$$a_{\lambda,n} = \sum (i_1, \ldots, i_{\lambda_1})(i_{\lambda_1+1}, \ldots, i_{\lambda_1+\lambda_2}) \cdots (i_{k-\lambda_r+1}, \ldots i_k),$$

where the sum is taken over all distinct k-tuples $(i_1, \ldots, i_k)$ of elements drawn from $\{1, \ldots, n\}$. These elements have been the object of study in [KO94] and [IK99]. In particular, we have the result [KO94, Proposition 1]:

$$\widehat{\chi}^\lambda(a_{\mu,n}) = f_\mu(\lambda).$$

1.3 Polynomial functors

We closely follow here the construction and exposition given by Macdonald [Mac95, Chapter I Appendix A]. We refer the reader to Friedlander and Suslin [FS97] for a modern categorical approach to the subject. Let $\mathcal{V}$ denote the category whose objects are finite-dimensional vector spaces over the complex numbers, and whose morphisms are $\mathbb{C}$-linear maps. A *polynomial functor* is a functor $F : \mathcal{V} \to \mathcal{V}$ such that for each pair $V, W \in \mathcal{V}$, the mapping

$$F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$$

is a polynomial mapping. More precisely, if $(f_i)_{1 \leqslant i \leqslant r}$ is a collection of morphisms $V \to W$, and if $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, then $F(\lambda_1 f_1 + \ldots + \lambda_r f_r)$ is a polynomial function of the λ_i with coefficients in $\mathrm{Hom}(F(V), F(W))$, depending on the f_i.

If $F(\lambda_1 f_1 + \ldots + \lambda_r f_r)$ is a homogenous polynomial of degree n, for all choices of $f_1, \ldots, f_r$, then F is said to be *homogeneous of degree n*. The nth exterior power $\bigwedge^n$ and nth symmetric power Sym^n are both homogeneous of degree n.

We say that a polynomial functor F has *bounded degree* if for all sufficiently large n, $F_n = 0$ in the direct sum decomposition established below. We denote the category of polynomial functors of bounded degree by $\mathcal{F}$.

Proposition 1.3.1. *A polynomial functor $F : \mathcal{V} \to \mathcal{V}$ has a direct sum decomposition*

$$F = \bigoplus_{n \geqslant 0} F_n,$$

where each F_n is a homogeneous polynomial functor of degree n.

Proof. Let λ_V denote multiplication in V by the scalar $\lambda \in \mathbb{C}$. If F is a polynomial functor on $\mathcal{V}$, then $F(\lambda_V)$ is a polynomial function of λ with coefficients in $\mathrm{End}(F(V))$. That is, for each $n \geqslant 0$, we have

$$F(\lambda_V) = \sum_{n \geqslant 0} u_n(V)\lambda^n,$$

for some collection $u_n(V) \in \mathrm{End}(F(V))$.

Given that $F((\lambda\mu)_V) = F(\lambda_V \mu_V) = F(\lambda_V)F(\mu_V)$, it follows that

$$\sum_{n \geqslant 0} u_n(V)(\lambda\mu)^n = \left(\sum_{n \geqslant 0} u_n(V)\lambda^n\right)\left(\sum_{n \geqslant 0} u_n(V)\mu^n\right),$$

for all $\lambda, \mu \in \mathbb{C}$. We therefore have that $u_n(V)^2 = u_n(V)$, for all $n \geqslant 0$, and

$$u_m(V)u_n(V) = 0$$

whenever $m \neq n$. By taking $\lambda = 1$, we also find that

$$\sum_{n \geqslant 0} u_n(V) = F(1_V) = 1_{F(V)}.$$

If we take $F_n(V)$ to be the image of $u_n(V) : F(V) \to F(V)$, then the $u_n(V)$ determine a direct sum decomposition

$$F(V) = \bigoplus_{n \geqslant 0} F_n(V).$$

Since $F(V)$ is a finite dimensional complex vector space, it follows that all but a finite number of the summands $F_n(V)$ will be zero for any given space V.

If $f : V \to W$ is a $\mathbb{C}$-linear map, then $f\lambda_V = \lambda_W f$, for all $\lambda \in \mathbb{C}$. Therefore, $F(f)F(\lambda_V) = F(\lambda_W)F(f)$, and so for each $n \geqslant 0$ we have $F(f)u_n(V) = u_n(W)F(f)$, that is to say, each u_n is an endomorphism of the functor F. On restriction to $F_n(V)$, we therefore have a $\mathbb{C}$-linear map $F_n(f) : F_n(V) \to F_n(W)$, and hence each F_n is a polynomial functor homogeneous of degree n. As a consequence, the functor F has the required direct sum decomposition

$$F = \bigoplus_{n \geqslant 0} F_n$$

into polynomial functors homogeneous of degree n. □

Let $F : \mathcal{V} \to \mathcal{V}$ be a polynomial functor homogeneous of degree n, and let $W = V_1 \oplus \cdots \oplus V_n$, where each $V_i \in \mathcal{V}$. For each $1 \leqslant \alpha \leqslant n$ we have monomorphisms $i_\alpha : V_\alpha \to W$ and epimorphisms $p_\alpha : W \to V_\alpha$ which satisfy the relations

$$\begin{aligned} p_\alpha i_\alpha &= 1_{V_\alpha}, \\ p_\alpha i_\beta &= 0, \text{ if } \alpha \neq \beta, \text{ and} \\ \sum_\alpha i_\alpha p_\alpha &= 1_W. \end{aligned}$$

Given any composition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, let $(\lambda)_W$, or simply (λ) if the space is obvious from the context, denote the morphism $\sum_\alpha \lambda_\alpha i_\alpha p_\alpha : W \to W$, so that $(\lambda)_W$ acts as scalar multiplication by λ_α on the component V_α. Let $v(V_1, \ldots, V_n)$ be the coefficient of $\lambda_1 \cdots \lambda_n$ in $F((\lambda)_W)$. The *linearisation* of F is the functor L_F defined by taking $L_F(V_1, \ldots, V_n)$ to be the image of $v(V_1, \ldots, V_n)$. It is a direct summand of $F(W)$ and is homogeneous of degree one in each variable.

Let F be a polynomial functor homogeneous of degree n, and define

$$L_F^{(n)}(V) = L_F(V, \ldots, V).$$

For any $\pi \in S_n$, let $\pi : V^{\oplus n} \to V^{\oplus n}$ denote the morphism which permutes the summands of $V^{\oplus n}$. This is the map given by

$$\sum i_{\pi(\alpha)} p_\alpha.$$

Given any composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers, we have that

$$\pi(\lambda) = \sum_\alpha \lambda_\alpha i_{\pi(\alpha)} p_\alpha = (\pi\lambda)\pi,$$

where $\pi\lambda = (\lambda_{\pi^{-1}(1)}, \ldots, \lambda_{\pi^{-1}(n)})$. Applying the functor F therefore gives

$$F(\pi)F((\lambda)) = F((\pi\lambda))F(\pi).$$

Selecting the coefficient of $\lambda_1 \cdots \lambda_n$ on either side, it follows that

$$F(\pi)v = vF(\pi).$$

Let

$$j : L_F^{(n)}(V) \to F(V^{\oplus n}), \text{ and } q : F(V^{\oplus n}) \to L_F^{(n)}(V)$$

be the injection and projection, respectively, associated with the direct summand $L_F^{(n)}(V)$ of $F(V^{\oplus n})$, so that qj is the identity, and $jq = v$. We then define

$$\tilde{F}(\pi) = qF(\pi)j,$$

which gives an endomorphism of $L_F^{(n)}$. Further, if $\tau \in S_n$ is another permutation, then

$$\begin{aligned}\tilde{F}(\pi)\tilde{F}(\tau) &= qF(\pi)jqF(\tau)j \\ &= qjqF(\pi)F(\tau)j && \text{(since } jq = v) \\ &= qF(\pi\tau)j && \text{(since } qj = 1) \\ &= \tilde{F}(\pi\tau).\end{aligned}$$

We therefore have a representation of S_n given by $\pi \mapsto \tilde{F}(\pi)$ on the vector space $L_F^{(n)}(V)$, which is functorial in V.

The sequence of propositions which follows shows that this action of S_n determines the functor F up to isomorphism. In fact, we produce a functorial isomorphism of $F(V)$ onto the subspace of S_n-invariants of $L_F^{(n)}(V)$.

Proposition 1.3.2. *If $i = \sum i_\alpha : V \to V^{\oplus n}$, and $p = \sum p_\alpha : V^{\oplus n} \to V$, then*

$$vF(ip)v = \sum_{\pi \in S_n} F(\pi)v.$$

Proof. Let $f : V^{\oplus n} \to V^{\oplus n}$ be a linear transformation of the form

$$f = \sum_{\alpha,\beta} \xi_{\alpha\beta} i_\alpha p_\beta$$

where $\xi_{\alpha\beta} \in \mathbb{C}$. $F(f)$ is then a homogenous polynomial of degree n in the n^2 variables $\xi_{\alpha\beta}$, with coefficients in $\mathrm{End}(F(V^{\oplus n}))$, depending only on V and F. For each $\pi \in S_n$ let w_π denote the coefficient of $\xi_{\pi(1)1} \cdots \xi_{\pi(n)n}$ in $F(f)$.

Given that $v^2 = v$, and $F(\pi)v = vF(\pi)$, it follows that

$$F(\pi)v = vF(\pi)v.$$

Hence $F(\pi)v$ is the coefficient of $\lambda_1 \cdots \lambda_n \, \mu_1 \cdots \mu_n$ in

$$F((\lambda))F(\pi)F((\mu)) = F\left(\sum_\alpha \lambda_{\pi(\alpha)}\mu_\alpha i_{\pi(\alpha)} p_\alpha\right),$$

and therefore $F(\pi)v = w_\pi$.

We also have that $vF(ip)v$ is the coefficient of $\lambda_1 \cdots \lambda_n \, \mu_1 \cdots \mu_n$ in

$$F((\lambda))F(ip)F((\mu)) = F\left(\sum_{\alpha,\beta} \lambda_\alpha \mu_\beta i_\alpha p_\beta\right),$$

which is given by

$$\sum_{\pi \in S_n} w_\pi = \sum_{\pi \in S_n} F(\pi)v.$$

□

Proposition 1.3.3. *Define two natural transformations of functors*

$$\xi = qF(i) : F \to L_F^{(n)}, \text{ and } \eta = F(p)j : L_F^{(n)} \to F.$$

The composition $\eta\xi$ is scalar multiplication by $n!$, and $\xi\eta = \sum_{\pi\in S_n} \tilde{F}(\pi)$.

Proof. From their definition

$$\eta\xi = F(p)jqF(i) = F(p)vF(i),$$

which is the coefficient of $\lambda_1 \cdots \lambda_n$ in $F(p)F((\lambda))F(i)$. Given that $p(\lambda)i : V \to V$ is actually scalar multiplication by $\lambda_1 + \cdots + \lambda_n$, it follows that $F(p(\lambda)i)$ is scalar multiplication by $(\lambda_1 + \cdots + \lambda_n)^n$. The coefficient of $\lambda_1 \cdots \lambda_n$ is therefore $n!$.

Now the composition

$$\xi\eta = qF(i)F(p)j,$$

so by the previous proposition we have that

$$j\xi\eta q = vF(ip)v = \sum_{\pi\in S_n} F(\pi)v,$$

and therefore

$$\begin{aligned}\xi\eta &= qj\xi\eta qj \\ &= \sum_{\pi\in S_n} qF(\pi)vj \\ &= \sum_{\pi\in S_n} qF(\pi)jqj \\ &= \sum_{\pi\in S_n} qF(\pi)j \\ &= \sum_{\pi\in S_n} \tilde{F}(\pi).\end{aligned}$$

□

Proposition 1.3.4. *Let $L_F^{(n)}(V)^{S_n}$ denote the subspace of $L_F^{(n)}(V)$ that is invariant under the action of S_n, and take*

$$\iota : L_F^{(n)}(V)^{S_n} \to L_F^{(n)}(V), \text{ and } \rho : L_F^{(n)}(V) \to L_F^{(n)}(V)^{S_n}$$

to be the associated injection and projection. The natural transformations given by

$$\xi' = \rho\xi : F(V) \to L_F^{(n)}(V)^{S_n},$$

and

$$\eta' = \eta\iota : L_F^{(n)}(V)^{S_n} \to F(V)$$

are functorial isomorphisms such that the compositions $\xi'\eta'$ and $\eta'\xi'$ are both scalar multiplication by $n!$.

Proof. The map $\sigma = (n!)^{-1}\xi\eta$ is idempotent with image $L_F^{(n)}(V)^{S_n}$. We observe then that $\rho\iota = 1$, and $\iota\rho = \sigma$. By the previous proposition we therefore have that the composition

$$\begin{aligned}\eta'\xi' &= \eta\iota\rho\xi \\ &= (n!)^{-1}\eta\xi\eta\xi \\ &= n!.\end{aligned}$$

Since $\xi\eta = n!\sigma$, we have

$$\begin{aligned}\xi'\eta' &= n!\rho\sigma\iota \\ &= n!\rho\iota\rho\iota \\ &= n!.\end{aligned}$$

□

Let $\mathcal{V}^n$ denote the category of whose objects are sequences $V = (V_1, \ldots, V_n)$ of finite-dimensional complex vector spaces, and

$$\mathrm{Hom}(V, W) = \prod_{i=1}^{n} \mathrm{Hom}(V_i, W_i).$$

A functor $F : \mathcal{V}^n \to \mathcal{V}$ is said to be *polynomial* if

$$F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$$

is a polynomial mapping.

By 1.3.4 it follows that every homogenous polynomial functor F of degree n is of the form $V \mapsto L(V, \ldots, V)^{S_n}$, where $L : \mathcal{V}^n \to \mathcal{V}$ is homogenous of degree one in each variable. Our goal is then to find all such functors.

In light of 1.3.1, an irreducible polynomial functor will be homogeneous of some degree. We will show that the irreducible polynomial functors of degree n correspond to the irreducible representations of S_n, and are therefore indexed by partitions λ of n.

Proposition 1.3.5. *Let $F : \mathcal{V}^n \to \mathcal{V}$ be a polynomial functor and let $(\lambda) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. The functor F then has a direct sum decomposition*

$$F = \bigoplus_{m_1,\ldots,m_n} F_{m_1,\ldots,m_n}$$

such that $F_{m_1,\ldots,m_n}(\lambda_1, \ldots, \lambda_n) = \lambda_1^{m_1} \cdots \lambda_n^{m_n}$.

Proof. Let $V = (V_1, \ldots, V_n)$. Since F is polynomial, we have that

$$F((\lambda)_V) = \sum_{m_1,\ldots,m_n} u_{m_1,\ldots,m_n}(V_1, \ldots, V_n)\lambda_1^{m_1} \cdots \lambda_n^{m_n}.$$

If as in 1.3.1 we define $F_{m_1,\ldots,m_n}$ as the image of $u_{m_1,\ldots,m_n}(V_1, \ldots, V_n)$, then the result follows. □

Proposition 1.3.6. *There is a functorial isomorphism*

$$L(V) \cong L(\mathbb{C}) \otimes V,$$

where $L : \mathcal{V} \to \mathcal{V}$ is homogeneous of degree one.

Proof. For each $x \in V$, let $e(x) : \mathbb{C} \to V$ be the map $\lambda \mapsto \lambda x$. If W is another complex vector space, then we define

$$\psi_V : \mathrm{Hom}(L(V), W) \to \mathrm{Hom}(V, \mathrm{Hom}(L(\mathbb{C}), W))$$

by $\psi_V(f)(x) = f \circ L(e(x))$. Given that ψ_V is obviously functorial in V, it suffices to show that it is an isomorphism.

Since L must be additive, it follows that

$$L(V_1 \oplus V_2) \cong L(V_1) \oplus L(V_2)$$

Therefore, if ψ_{V_1} and ψ_{V_2} are isomorphisms, then so is $\psi_{V_1 \oplus V_2}$. Hence, it suffices to show that $\psi_{\mathbb{C}}$ is an isomorphism, which is clear. □

Proposition 1.3.7. *Let $L : \mathcal{V}^n \to \mathcal{V}$ be homogeneous and linear in each variable. Then there exists a functorial isomorphism*

$$L(V_1, \ldots, V_n) \cong L^{(n)}(\mathbb{C}) \otimes V_1 \otimes \cdots \otimes V_n,$$

where $L^{(n)}(\mathbb{C}) = L(\mathbb{C}, \ldots, \mathbb{C})$.

Proof. This is achieved by the repeated application of 1.3.6:

$$\begin{aligned} L(V_1, \ldots, V_n) &\cong L(V_1, \ldots, V_{n-1}, \mathbb{C}) \otimes V_n \\ &\cong L(V_1, \ldots, V_{n-2}, \mathbb{C}, \mathbb{C}) \otimes V_{n-1} \otimes V_n \\ &\vdots \\ &\cong L(\mathbb{C}, \ldots, \mathbb{C}) \otimes V_1 \otimes \cdots \otimes V_n. \end{aligned}$$

□

Lemma 1.3.8. *If F is a homogeneous polynomial functor of degree n, then there is an isomorphism of functors*

$$F(V) \cong (L_F^{(n)}(\mathbb{C}) \otimes V^{\otimes n})^{S_n}.$$

Proof. This is a direct consequence of 1.3.4 and 1.3.7. □

Theorem 1.3.9. *Let $\mathcal{F}_n$ be the category of homogenous polynomial functors of degree n. The functors $\alpha : \mathcal{F}_n \to \mathcal{S}_n$ and $\beta : \mathcal{S}_n \to \mathcal{F}_n$ defined by*

$$\alpha(F) = L_F^{(n)}(\mathbb{C}), \text{ and } \beta(M)(V) = (M \otimes V^{\otimes n})^{S_n}$$

constitute an equivalence of categories.

Proof. By 1.3.8 we have $\beta\alpha = 1_{\mathcal{F}_n}$. Let $M \in \mathcal{S}_n$, and suppose that $\beta(M) = F$, so that

$$F(V_1 \oplus \cdots \oplus V_n) = (M \otimes (V_1 \oplus \cdots \oplus V_n)^{\otimes n})^{S_n}.$$

The linearisation of F is then given by

$$L_F(V_1, \ldots, V_n) = \left(M \otimes \left(\bigoplus_{\pi \in S_n} V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)} \right) \right)^{S_n}.$$

We then have that the composition

$$\alpha\beta(M) = L_F^{(n)}(\mathbb{C}) \cong (M \otimes \mathbb{C}[S_n])^{S_n} \cong M.$$

□

Corollary 1.3.10. *The irreducible polynomial functors homogenous of degree n are indexed by partitions λ of n.*

Proof. Given the equivalence of categories in 1.3.9, the functors F_λ defined by

$$F_\lambda(V) = (V_\lambda \otimes V^{\otimes n})^{S_n},$$

exhaust the irreducible polynomial functors homogenous of degree n. □

Proposition 1.3.11. *The tensor product $V^{\otimes n}$ has the decomposition*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes F_\lambda(V).$$

Proof. If M and N are $\mathbb{C}[S_n]$-modules, then there is a canonical isomorphism

$$(M^* \otimes N)^{S_n} \cong \mathrm{Hom}_{\mathbb{C}[S_n]}(M, N),$$

where $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. Given that $V_\lambda \cong (V_\lambda)^*$, we therefore have that

$$F_\lambda(V) \cong \mathrm{Hom}_{\mathbb{C}[S_n]}(V_\lambda, V^{\otimes n}).$$

If we consider $V^{\otimes n}$ as a $\mathbb{C}[S_n]$-module, then its decomposition into isotypic components is given by

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathrm{Hom}_{\mathbb{C}[S_n]}(V_\lambda, V^{\otimes n}),$$

and the result follows. □

Proposition 1.3.12. *Let E and F be polynomial functors of degrees m and n, respectively, such that $E = \beta(M)$ for some S_m representation M, and $F = \beta(N)$ for some S_n representation N. The tensor product $E \otimes F : V \mapsto E(V) \otimes F(V)$ corresponds to $M * N = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(M \otimes N)$.*

Proof. From their definition we have that

$$\begin{aligned}(E \otimes F)(V) &= (M \otimes V^{\otimes m})^{S_m} \otimes (N \otimes V^{\otimes n})^{S_n} \\ &\cong (M \otimes N \otimes V^{\otimes m+n})^{S_m \times S_n} \\ &\cong (\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(M \otimes N) \otimes V^{\otimes m+n})^{S_{m+n}},\end{aligned}$$

where the final step is an application of Frobenius reciprocity. □

1.4 Shifted symmetric functions

We recall briefly the construction of the algebra of symmetric functions. We refer the reader to [Mac95] for a comprehensive coverage of the material. The algebra of symmetric polynomials Λ_n in the variables $x_1, \ldots, x_n$. It is a graded algebra, graded by the degree of the polynomials. The assignment $x_{n+1} = 0$ defines a morphism of graded algebras

$$\Lambda_{n+1} \to \Lambda_n$$

which we call the *stability condition*. The algebra of symmetric functions Λ is then defined as the projective limit

$$\Lambda := \varprojlim \Lambda_n$$

in the category of graded algebras. An element $f \in \Lambda$ is then a sequence $(f_n)_{n \geqslant 1}$ such that

(1) $f_n \in \Lambda_n$, $n \geqslant 1$,

(2) $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$,

(3) $\sup_n \deg f_n < \infty$.

Particular interest is given to the following algebraically independent generators of Λ:

- elementary symmetric functions $e_1, e_2, e_3, \ldots$,
- complete symmetric functions $h_1, h_2, h_3, \ldots$,
- power sum symmetric functions $p_1, p_2, p_3, \ldots$.

For any collection $\{f_k\}_{k \geqslant 1}$ equal to any of these sets of generators and any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we define

$$f_\lambda := f_{\lambda_1} \cdots f_{\lambda_k}.$$

Another important basis is provided by the Schur functions $\{s_\lambda\}_{\lambda \in \mathcal{P}}$.

We now define the algebra of *shifted symmetric polynomials* in the variables $x_1, \ldots, x_n$, denoted by Λ^*_n, be the algebra of polynomials that become symmetric in the change of variables

$$x'_i = x_i - i, \quad i = 1, \ldots, n.$$

It is important to note that this defines not a graded algebra but one that is filtered by degree. By analogy with Λ, setting $x_{n+1} = 0$ defines a homomorphism

$$\Lambda^*_{n+1} \to \Lambda^*_n,$$

and so we can take the projective limit

$$\Lambda^* := \varprojlim \Lambda^*_n$$

in the category of filtered algebras. We call Λ^* the algebra of *shifted symmetric functions*. We then have the following relationship between Λ and Λ^*:

Proposition 1.4.1. *The associated graded algebra* $\operatorname{gr} \Lambda^*$ *is canonically isomorphic to* Λ.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition with no more than n parts. The *shifted Schur polynomial* in n variables, indexed by μ is defined by the following ratio of $n \times n$ determinants:

$$s^*_\mu(x_1, \ldots, x_n) := \frac{\det[(x_i + n - i \downharpoonleft \mu_j + n - j)]}{\det[(x_i + n - i \downharpoonleft n - j)]},$$

where $1 \leqslant i, j \leqslant n$. If $l(\mu) > n$, then $s^*_\mu(x_1, \ldots, x_n) = 0$.

Okounkov and Olshanski [OO97] showed that the shifted Schur polynomials satisfy the stability condition

$$s^*_\mu(x_1, \ldots, x_n, 0) = s^*_\mu(x_1, \ldots, x_n),$$

and so the sequence $(s^*_\mu(x_1, \ldots, x_n))_{n \geqslant 1}$ defines an element s^*_μ of Λ^*, which we call the *shifted Schur function*. We also refer to them as s^*-*functions* for short. The shifted Schur functions form a linear basis in Λ^*.

Any shifted symmetric function $f \in \Lambda^*$ can be evaluated on an infinite sequence $(i_1, i_2, \ldots)$ so long as $i_k = 0$ for all sufficiently large k. We can then consider f as a function on partitions by taking $f(\lambda_1, \ldots, \lambda_{l(\lambda)}, 0, \ldots)$. Moreover, f is uniquely determined by its values on all partitions. Hence, we can regard the algebra Λ^* as an algebra of functions on the set of partitions.

The proofs of the following sequence of theorems due to Okounkov can also be found in [OO97]:

Theorem 1.4.2 (Vanishing Theorem). *We have*

$$\begin{aligned} s^*_\mu(\lambda) &= 0, \text{ unless } \mu \subset \lambda, \\ s^*_\mu(\mu) &= H_\mu. \end{aligned}$$

Theorem 1.4.3 (Characterisation Theorem I). *The function s^*_μ is the unique element of Λ^* such that $\deg s^*_\mu \leqslant |\mu|$ and*

$$s^*_\mu(\lambda) = \delta_{\mu\lambda} H_\mu$$

for all λ such that $|\lambda| \leqslant |\mu|$.

Theorem 1.4.4 (Characterisation Theorem I'). *Suppose $l(\mu) \leqslant n$. Then $s^*_\mu(x_1, \ldots, x_n)$ is the unique element of Λ^*_n such that $\deg s^*_\mu(x_1, \ldots, x_n) \leqslant |\mu|$ and*

$$s^*_\mu(\lambda) = \delta_{\mu\lambda} H_\mu$$

for all λ such that $|\lambda| \leqslant |\mu|$ and $l(\lambda) \leqslant n$.

Theorem 1.4.5 (Characterisation Theorem II). *The function s^*_μ is the unique element of Λ^* such that the highest term of of s^*_μ is the ordinary Schur function s_μ, and*

$$s^*_\mu(\lambda) = 0$$

for all λ such that $|\lambda| < |\mu|$.

The shifted analogues of the complete symmetric functions and the elementary symmetric functions are given by

$$h_k^* := s_{(k)}^*,$$
$$e_k^* := s_{(1^k)}^*.$$

We then have that Λ^* is the algebra of polynomials in the h_k^*, or the e_k^*.

There are numerous shifted analogues of the power sums

$$p_k = \sum_i x_i^k,$$

all of which have p_k as a leading term.

First we have the functions given by

$$p_k^* := \sum_i \left((x_i - i)^k - (-i)^k\right).$$

We then have two factorial analogues of the p_k^*:

$$\hat{p}_k := \sum_i \left((x_i - i \upharpoonright k) - (-i \upharpoonright k)\right),$$

where $(x \upharpoonright k) := x(x+1)\cdots(x+k-1)$ denotes the k-th *raising factorial power* of x, and

$$\check{p}_k := \sum_i \left((x_i - i + 1 \downharpoonleft k) - (-i + 1 \downharpoonleft k)\right).$$

The last, and possibly most important power sum analogue we describe here are the functions defined to mirror the identity

$$p_\mu = \sum_{\lambda \vdash |\mu|} \chi_\mu^\lambda s_\lambda.$$

The map $\varphi : \Lambda \to \Lambda^*$ given by

$$\varphi(s_\mu) = s_\mu^*$$

is a linear isomorphism. We then define

$$p_\mu^\# := \varphi(p_\mu),$$

so that

$$p_\mu^\# = \sum_{\lambda \vdash |\mu|} \chi_\mu^\lambda s_\lambda^*.$$

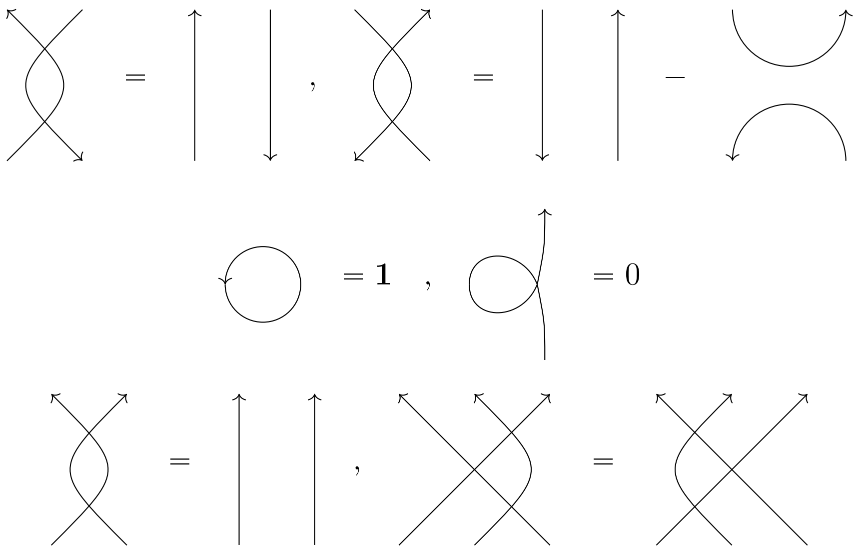
Considered as a function on partitions, we then have the following result, also found in [OO97]:

Proposition 1.4.6. *For $\mu \vdash k$ and $\lambda \vdash n$,*

$$p_\mu^\#(\lambda) = f_\mu(\lambda).$$

1.5 Khovanov's Heisenberg category

We define a $\mathbb{C}$-linear strict monoidal category $\mathcal{H}'$ generated by two objects Q_+ and Q_- as follows. An object of $\mathcal{H}'$ is a finite direct sum of tensor products $Q_{\epsilon_1}\otimes\cdots\otimes Q_{\epsilon_m}$, where $\epsilon_1,\ldots,\epsilon_m$ is a finite sequence of pluses and minuses. For the sake of brevity we denote this tensor product by Q_ϵ, where $\epsilon=\epsilon_1\cdots\epsilon_m$. The unit object, $\mathbf{1}$, then corresponds to the empty sequence $Q_\varnothing$. The space $\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$, for two sequences ϵ and ϵ' is the $\mathbb{C}$-module generated by planar diagrams modulo some local relations. The diagrams are oriented compact one-manifolds embedded in the strip $\mathbb{R}\times[0,1]$, modulo rel boundary isotopies. The endpoints of the one-manifolds are located at $\{1,\ldots,m\}\times\{0\}$ and $\{1,\ldots,n\}\times\{1\}$, where m and n are the lengths of ϵ and ϵ', respectively. Further, the orientation of the one-manifold at the endpoints must match the signs in the sequences ϵ and ϵ'. Triple intersections are not allowed. The composition of two morphisms is achieved by the natural glueing of the diagrams. A diagram with no endpoints is an endomorphism of $\mathbf{1}$. The local relations are as follows:



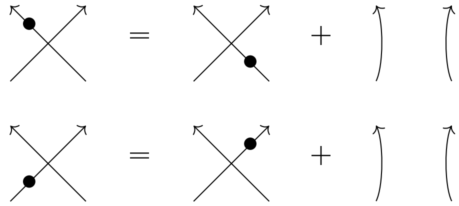
The relations in the first two rows are motivated by the Heisenberg relation $pq = qp + 1$, where p and q are the two generators of the Heisenberg algebra, that is, they imply that there is an isomorphism in $\mathcal{H}'$ of the form

$$Q_-Q_+ \cong Q_+Q_- \oplus \mathbf{1}\,.$$

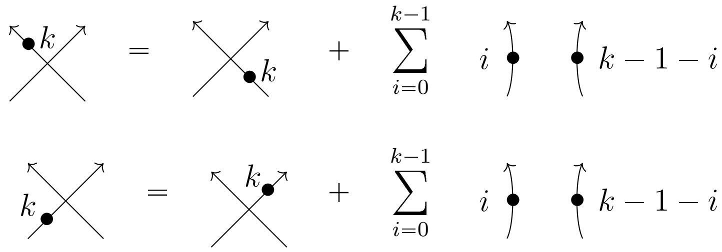
The relations in the third row are motivated by the symmetric group relations.

We find it convenient to denote a right curl by a dot on a strand, and a sequence of d right curls by a dot with a d drawn next to it. The local relations imply the following results regarding moving right curls across intersections:

Proposition 1.5.1. *A right curl can be moved across intersection points, according to the following relations:*



Proposition 1.5.2. *We can move k dots through an intersection according to the following relations:*

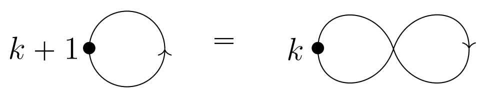


A closed diagram is an endomorphism of the unit object $\mathbf{1} \in \mathcal{H}'$. Let c_k denote a clockwise-oriented circle with k dots, and let $\tilde{c}_k$ denote an anticlockwise-oriented circle with k dots. Any $\tilde{c}_k$ can be written as a linear combination of products of clockwise circles. We already have that $\tilde{c}_0 = 1$, and $\tilde{c}_1 = 0$, since it is a figure eight and hence contains a left curl. For the rest we have:

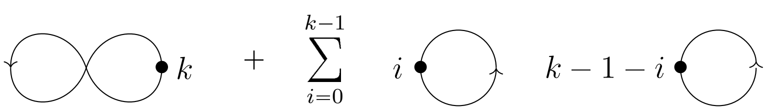
Proposition 1.5.3 ([Kho14], Proposition 2)**.** *For $k > 0$, we have*

$$\tilde{c}_{k+1} = \sum_{i=0}^{k-1} \tilde{c}_i c_{k-1-i}.$$

Proof. We begin by expanding a dot into a right curl:

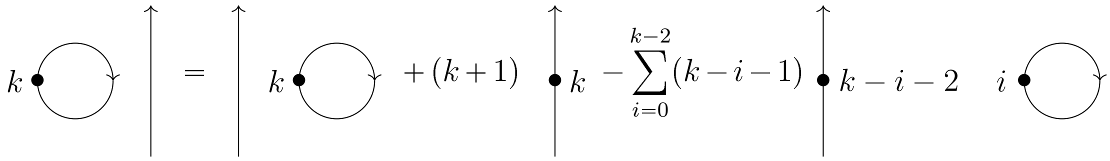


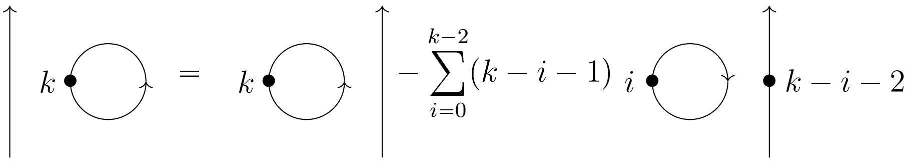
We then pull the k dots through the intersection to yield



The result then follows since the term on the left contains a left curl. □

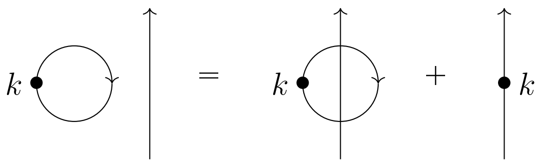
Proposition 1.5.4 (Bubble moves)**.** *We can move dotted circles past lines according to the following identities:*



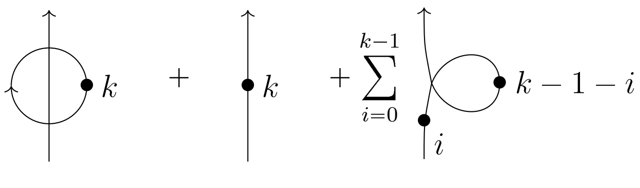


This result is given by Khovanov without proof. We provide a proof of the first relation, noting the second is analogous.

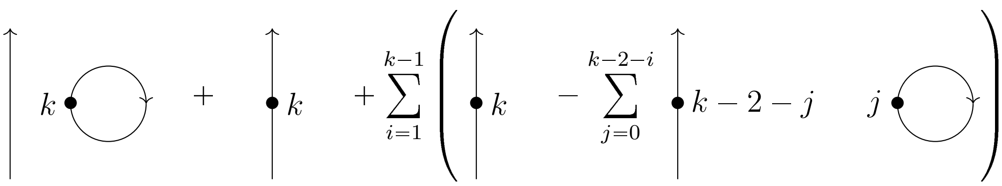
Proof. We begin by bringing the line halfway through the circle, which gives



We then pull the k dots through the intersection to yield



The next step is to untwine the double upwards crossing on the dotted circle in the left most term, and to pull the $k-1-i$ dots through the intersection in the rightmost term. We then have



and the result follows by counting the terms in the sums. □

The local relations allow us to convert any closed diagram into a linear combination of diagrams without crossings consisting of nested dotted circles. The proceeding two results show us how to split apart nested circles using bubble moves, and anticlockwise circles can be expressed as linear combinations of products of clockwise circles. It is then evident that $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ is a quotient of the polynomial algebra $\mathbb{C}[c_0, c_1, c_2, \ldots]$ in countably many variables via the map

$$\psi_0 : \mathbb{C}[c_0, c_1, c_2, \ldots] \to \mathrm{End}_{\mathcal{H}'}(\mathbf{1})$$

that by an abuse of notation identifies the formal variable c_k with the clockwise circle with k dots. It is a theorem of Khovanov [Kho14, Proposition 3] that ψ_0 is an isomorphism of algebras.

To simplify notation for bimodules, we use the following:

- (n) denotes $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_n])$-bimodule.
- $(n)_{n-1}$ denotes $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$-bimodule.
- ${}_{n-1}(n)$ denotes $\mathbb{C}[S_n]$ considered as a $(\mathbb{C}[S_{n-1}], \mathbb{C}[S_n])$-bimodule.

With a minor modification, we can interpret the graphical calculus of described above as giving bimodule maps between symmetric group representations. We label the regions of the strip $\mathbb{R} \times [0, 1]$ by non-negative integers, beginning with n in the rightmost region. An upwards oriented line separating two regions labelled n and $n + 1$,

$$n+1 \quad \uparrow \quad n$$

denotes the identity endomorphism of the induction functor

$$\mathrm{Ind}_n^{n+1} : \mathbb{C}[S_n]\text{-mod} \to \mathbb{C}[S_{n+1}]\text{-mod}.$$

This functor is given by tensoring with the bimodule $(n + 1)_n$.

Similarly, a downward oriented line separating regions labelled $n + 1$ and n,

$$n \quad \downarrow \quad n+1$$

denotes the identity endomorphism of the restriction functor

$$\mathrm{Res}_{n+1}^{n} : \mathbb{C}[S_{n+1}]\text{-mod} \to \mathbb{C}[S_n]\text{-mod}.$$

This functor is given by tensoring with the bimodule ${}_n(n + 1)$.

We follow the convention that a diagram is zero if it has a region labelled by a negative integer. We also have that the local relations with the rightmost region labelled n still hold. They encode the symmetric group relations $s_{n+1}^2 = 1$, and $s_{n+1}s_{n+2}s_{n+1} = s_{n+2}s_{n+1}s_{n+2}$, and the bimodule decomposition

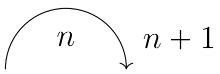
$$_n(n+1)_n \cong (n)_{n-1}(n) \oplus (n),$$

giving an isomorphism

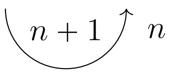
$$\mathrm{Res}^n_{n+1} \circ \mathrm{Ind}^{n+1}_n \cong \mathrm{Ind}^n_{n-1} \circ \mathrm{Res}^{n-1}_n \oplus \mathrm{Id}$$

of endofunctors in the category $\mathbb{C}[S_n]$-mod.

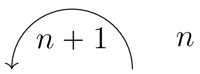
We have the following bimodule maps for the four U-turns, which for convenience we refer to as RCap, RCup, Lcap, and Lcup, respectively:



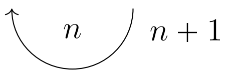
$, \quad (n+1)_n(n+1) \to (n+1), \quad g \otimes h \mapsto gh, \;\; g, h \in S_{n+1},$



$, \quad (n) \to {}_n(n+1)_n, \quad g \mapsto g, \;\; g \in S_n,$



$, \quad {}_n(n+1)_n \to (n), \quad g \mapsto \begin{cases} g, & g \in S_n \\ 0, & \text{otherwise} \end{cases},$

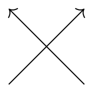


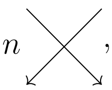
$, \quad (n+1) \to (n+1)_n(n+1),$

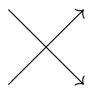
where the last map is determined by the condition that

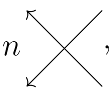
$$1_{n+1} \mapsto \sum_{i=1}^{n+1} s_i s_{i+1} \cdots s_n \otimes s_n \cdots s_{i+1} s_i.$$

We also have bimodule maps associated with the crossings, which we refer to as UCross, DCross, RCross, and LCross, respectively, given by:


n , $(n+2)_n \to (n+2)_n, \quad g \mapsto gs_{n+1}, \quad g \in S_{n+2},$

n 
, ${}_n(n+2) \to {}_n(n+2), \quad g \mapsto s_{n+1}g, \quad g \in S_{n+2},$


n , $(n)_{n-1}(n) \to {}_n(n+1)_n, \quad g \otimes h \mapsto gs_nh, \quad g,h \in S_n,$

n 
, ${}_n(n+1)_n \to (n)_{n-1}(n),$

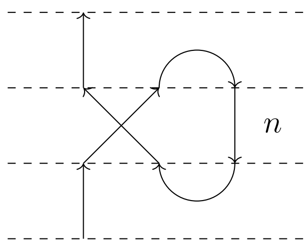
where the last map takes $g \in S_n$ to zero, and gs_nh to $g \otimes h$, for $g,h \in S_n$.

The right twist curl is then endowed with the following interpretation:

Proposition 1.5.5. *The right curl with the rightmost region labelled n is the endomorphism of $(n+1)_n$ which takes 1_n to the Jucys-Murphy element J_{n+1}.*

We provide here the proof omitted by Khovanov.

Proof. The right twist curl can be written as the composition of a cup, a crossing, and a cap:



Hence,

$$
\begin{aligned}
1_n &\mapsto \sum_{i=1}^{n} s_i \cdots s_{n-1} \otimes s_{n-1} \cdots s_i \\
&\mapsto \sum_{i=1}^{n} s_i \cdots s_{n-1} s_n \otimes s_{n-1} \cdots s_i \\
&\mapsto \sum_{i=1}^{n} s_i \cdots s_{n-1} s_n s_{n-1} \cdots s_i
\end{aligned}
$$

□

Let $\mathcal{S}'_n$ be the category whose objects are compositions of induction and restriction restriction functors, starting from the symmetric group S_n. For example, $\mathrm{Ind}_n^{n+1} \circ \mathrm{Ind}_{n-1}^n \circ \mathrm{Res}_n^{n-1}$ is an element of $\mathcal{S}'_n$. The morphisms are natural transformations of functors, and can be identified with homomorphisms of the corresponding bimodules. Let $\mathcal{S}'$ be the sum of the $\mathcal{S}'_n$ over $n \geqslant 0$.

To each $n \geqslant 0$ we then have a functor $\mathcal{F}'_n : \mathcal{H}' \to \mathcal{S}'_n$ which takes Q_ϵ to the corresponding composition of induction and restriction functors, matching the symbol $+$ with induction, and $-$ with restriction. For example,

$$\mathcal{F}'_n(Q_{-++}) = \mathrm{Res}_{n+2}^{n+1} \circ \mathrm{Ind}_{n+1}^{n+2} \circ \mathrm{Ind}_n^{n+1}.$$

If the string ϵ ends with at least $n+1$ more minuses than pluses, then $\mathcal{F}'_n(Q_\epsilon) = 0$. On morphisms, $\mathcal{F}'_n$ sends the diagram representing the morphism to the diagram with the rightmost region labelled n, viewed as a natural transformation between compositions of induction and restriction functors. Note that this functor is not monoidal, as $\mathcal{S}'_n$ does not have a monoidal structure matching that of $\mathcal{H}'$.

Applying the functor $\mathcal{F}'_n$ to the unit and its endomorphisms gives a homomorphism

$$f_n^{\mathcal{H}'} : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to Z(\mathbb{C}[S_n]).$$

Define $\psi_{0,n} : \mathbb{C}[c_0, c_1, \ldots] \to Z(\mathbb{C}[S_n])$ as the composition $f_n^{\mathcal{H}'} \circ \psi_0$. The following result is then a direct consequence of bimodule maps defined above:

Proposition 1.5.6. *We have*

$$\begin{aligned} c_{k,n} &:= \psi_{0,n}(c_k) = \sum_{i=1}^{n} s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i, \\ \tilde{c}_{k,n} &:= \psi_{0,n}(\tilde{c}_k) = \mathrm{pr}_n(J_{n+1}^k). \end{aligned}$$

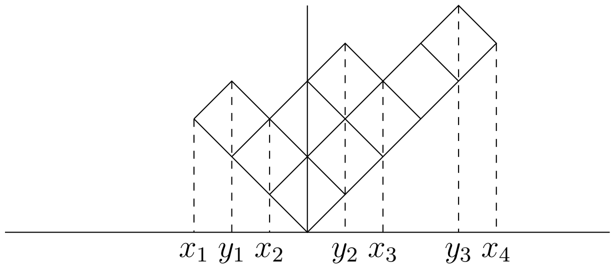
1.6 Free probability

With a rather analytical beginning in the study of operator algebras, the study of free probability has blossomed into an area with links to many fields. We focus mainly on the combinatorial aspects of the theory, in particular, the transition and co-transition distributions on Young's lattice. We refer the reader to [NS06] for a more detailed exposition of the combinatorial aspects of free probability theory.

We begin with a construction of Kerov's [Ker93, Ker00]. Two increasing sequences $\{y_1, \ldots, y_{d-1}\}$ and $\{x_1, \ldots, x_{d-1}, x_d\}$ are said to be *interlacing* if we have

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{d-1} < y_{d-1} < x_d.$$

The number $c = \sum x_k - \sum y_k$ is called the *centre* of the sequence. A Young diagram can be thought of as an interlacing sequence as illustrated below for the partition $(5, 3, 1)$.



To each Young diagram λ, with associated interlacing sequences $(x_i)_{1 \leqslant i \leqslant d}$ and $(y_i)_{1 \leqslant i \leqslant d-1}$ we have two probability measures (formally defined on the interval $[x_1, x_d]$). The first we call the *transition measure*, which is given by

$$\omega_\lambda = \sum_{k=1}^{d} \mu_k \delta_{x_k},$$

where

$$\mu_k := \prod_{i=1}^{k-1} \frac{x_k - y_i}{x_k - x_i} \prod_{j=k+1}^{d} \frac{x_k - y_{j-1}}{x_k - x_j},$$

and δ_x is the Dirac measure. The weights μ_k are called the *transition probabilities.* Similarly, we define the *co-transition measure* by

$$\widehat{\omega}_\lambda = \nu_k \delta_{y_k}$$

where

$$\nu_k := \frac{(x_d - y_k)(y_k - x_1)}{\sum_{i<j}(y_i - x_i)(x_j - y_{j-1})} \prod_{i=1}^{k-1} \frac{y_k - x_{i+1}}{y_k - y_i} \prod_{j=k+1}^{d} \frac{y_k - x_j}{y_k - y_j}.$$

The weights ν_k are called *co-transition probabilities.* The transition and co-transition measures so defined are a generalisation to a broader category of diagrams, not necessarily on an integer lattice and piecewise linear like a Young Diagram. The combinatorial definitions on Young diagrams given of the measures are given by

$$p_k(\lambda) := \mu_k = \frac{\dim \lambda^{(k)}}{|\lambda^{(k)}| \dim \lambda}, \quad \text{and} \quad q_k(\lambda) := \nu_k = \frac{\dim \lambda_{(k)}}{\dim \lambda},$$

recalling that $\lambda^{(k)}$ and $\lambda_{(k)}$ are the Young diagrams given by adding or removing a box, when possible, from the kth row.

The transition and co-transition measures are fundamental tools in the study of the asymptotic representation theory of symmetric groups and in the connection between asymptotic representation theory and free probability.

The r-th *moments* of the transition and co-transition distributions are given respectively by

$$\sigma_r(\lambda) := \sum_{k=1}^{d} \mu_k x_k^r, \quad \text{and} \quad \hat{\sigma}_r(\lambda) := \sum_{k=1}^{d-1} \nu_k y_k^r.$$

We write the moment generating series for the transition measure (resp. co-transition measure) as

$$\widehat{\mathcal{M}}_\lambda(z) := \sum_{k=0}^{\infty} \sigma_k(\lambda) z^{-k-1} \quad \text{and} \quad \widecheck{\mathcal{M}}_\lambda(z) := z - \sum_{k=0}^{\infty} |\lambda| \hat{\sigma}_k(\lambda) z^{-k-1}.$$

Note that we scale all coefficients of $\widecheck{\mathcal{M}}_\lambda(z)$ by $|\lambda|$ with the exception of the coefficient on z.

Lemma 1.6.1. *For $\lambda \in \mathcal{P}$*

$$\widehat{\mathcal{M}}_\lambda(z) = (\widecheck{\mathcal{M}}_\lambda(z))^{-1}.$$

Proof. This follows directly from equation (2.3) and Lemma 5.1, both found in [Ker00]. □

The *boolean cumulants* $\{\hat{b}_k(\lambda)\}_{k \geqslant 1}$ associated to ω_λ can be defined as the coefficients on the multiplicative inverse of $\widehat{\mathcal{M}}_\lambda(z)$,

$$\hat{\mathcal{B}}_\lambda(z) = z - \sum_{k=-1}^{\infty} \hat{b}_{k+2}(\lambda) z^{-k-1} = (\widehat{\mathcal{M}}_\lambda(z))^{-1}.$$

With Lemma 1.6.1 this definition immediately gives us the following fact.

Proposition 1.6.2. *Let $\lambda \in \mathcal{P}$ and $k \geqslant 0$, then $\hat{b}_1(\lambda) = 0$ and*

$$\hat{b}_{k+2}(\lambda) = |\lambda| \hat{\sigma}_k(\lambda).$$

There is a more algebraic approach to the transition measure due to Biane [Bia98]. In the context of probability theory, pr_n is sometimes known as the *conditional expectation.*

Proposition 1.6.3. *For $\lambda \vdash n$,*

$$\sigma_k(\lambda) = \widehat{\chi}^\lambda[\mathrm{pr}_n(J_{n+1}^k)]$$

and

$$\widehat{b}_{k+2}(\lambda) = |\lambda|\widehat{\sigma}_k(\lambda) = \widehat{\chi}^\lambda\Big(\sum_{i=1}^n s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i\Big).$$

Proof. The first statement appears in [Bia03] Section 4. A detailed proof is given in Theorem 9.23 of [HO07]. For the second statement, we note that since characters are class functions,

$$\widehat{\chi}^\lambda\Big(\sum_{i}^n s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i\Big) = |\lambda|\widehat{\chi}^\lambda(J_n^k).$$

As J_n eigenspaces, V_λ decomposes as

$$V_\lambda \cong \bigoplus_{i=1}^{d-1} V_{\lambda_{(i)}}$$

with $V_{\lambda_{(i)}}$ corresponding to eigenvalue b_i [VO04]. Hence,

$$|\lambda|\widehat{\chi}^\lambda(J_n^k) = |\lambda| \sum_{i=1}^{d-1} \frac{\dim(\lambda_{(i)}) b_i^k}{\dim(\lambda)} = |\lambda|\widehat{\sigma}_k(\lambda) = \widehat{b}_{k+2}(\lambda).$$

□

Chapter 2

Symmetric functions and representations

In this chapter we explore some categorifications of the algebra of symmetric functions and the connections between them. We begin by presenting the classically understood categorification of the algebra of symemtric functions given by considering symmetric group representations for all values of n at once. We then turn our attention to polynomial functors and polynomial representations of GL_m. Finally, we introduce plethysm of symmetric functions and describe the analogues of this construction in each of the categorifications.

2.1 Symmetric group representations

The section that follows owes much to Fulton's exposition [Ful97]. We denote the category of finite-dimensional $\mathbb{C}[S_n]$-modules by $\mathcal{S}_n$. It is an abelian category, so we can take the Grothendieck group $K(\mathcal{S}_n)$. The group $K(\mathcal{S}_n)$ is the quotient of the free abelian group on the set of isomorphism classes $[V]$ of all representations V of S_n by the subgroup generated by the relations $[V \oplus W] - [V] - [W]$. It is generated by the isomorphism classes of the irreducible modules V_λ, where λ is a partition of n. Since the representations of $\mathbb{C}[S_n]$ are semi-simple, the Grothendieck group of $\mathcal{S}_n$ coincides with the split Grothendieck group.

Taking $R_n = K(\mathcal{S}_n)$ and $R_0 = \mathbb{Z}$, we define

$$R = \bigoplus_{n \geqslant 0} R_n.$$

The *induction product* is the function $* : R_n \times R_m \to R_{n+m}$ given by

$$[V] * [W] = [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)].$$

Proposition 2.1.1. *With the induction product, R carries the structure of a graded commutative ring with unit.*

Proof. We recall that the induced representation is given by

$$\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W) = \mathbb{C}[S_{n+m}] \otimes_{\mathbb{C}[S_n] \otimes \mathbb{C}[S_m]} (V \otimes W).$$

The commutativity and associativity of the induction product are then direct consequences of the commutativity and associativity, up to isomorphism, of the tensor product. Distributivity follows from the fact that induction commutes with direct sums. □

We define an inner product $\langle \cdot, \cdot \rangle$ on R_n by requiring the isomorphism classes of the irreducible representations $[V_\lambda]$ to be an orthonormal basis. Let

$$V \cong \bigoplus_\lambda (V_\lambda)^{\oplus m_\lambda}, \text{ and } W \cong \bigoplus_\lambda (V_\lambda)^{\oplus n_\lambda}$$

be two representations of S_n, then $\langle [V], [W] \rangle = \sum_\lambda m_\lambda n_\lambda$. Given the orthonormality of the irreducible characters of a finite group, if χ_V is the character of V and χ_W is the character of W, then the inner product is also given by

$$\langle [V], [W] \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) \cdot \chi_W(\sigma^{-1}).$$

As before, let C_μ denote the conjugacy class of the cycle type μ, and let z_μ be the number of elements in the centraliser of a permutation of cycle type μ. Given that a permutation and its inverse are in the same conjugacy class, it follows that

$$\langle [V], [W] \rangle = \sum_\mu \frac{1}{z_\mu} \chi_V(C_\mu) \cdot \chi_W(C_\mu).$$

Denote by M_λ the representation associated with action of S_n on the set of tabloids of shape λ. We then have the following result:

Lemma 2.1.2. *The value of the character of M_λ on the conjugacy class of the cycle type μ is the coefficient of x^λ in p_μ.*

Proof. The trace of a permutation σ is the number of tabloids fixed by σ. If we express σ as a product of cycles, then a tabloid will be fixed by σ precisely when all elements of each cycle occur in the same row.

We can write the power sum p_μ as a sum of monomial symmetric functions

$$p_\mu = \prod_i (x_1^{\mu_i} + x_2^{\mu_i} + \cdots) = \sum_\nu c_{\nu,\mu} m_\nu.$$

The coefficients in the expansion of this product encode precisely the information we are seeking. The choice of the term $x_j^{\mu_i}$ encodes the choice of placing a cycle of length μ_i in the jth row of the tabloid. The coefficient of the x^λ term therefore counts precisely the number of ways the cycles of μ can be distributed among the rows of the λ-shaped tabloid, and hence counts the tabloids fixed by a permutation of cycle type μ. □

Theorem 2.1.3. *Define* $\varphi : \Lambda \to R$ *by* $\varphi(h_\lambda) = [M_\lambda]$. *The map* φ *is an isometric isomorphism of* Λ *with* R. *Further,* $\varphi(s_\lambda) = [V_\lambda]$.

Proof. The map φ takes the nth complete symmetric function h_n to the isomorphism class of the trivial representation $M_{(n)} = \mathbb{I}_n$. Since Λ is a polynomial ring in the h_n, to demonstrate that φ is a homomorphism it is sufficient to show that

$$[M_\lambda] = [M_{(\lambda_1)}] * [M_{(\lambda_2)}] * \cdots * [M_{\lambda_k}],$$

where $\lambda = (\lambda_1, \dots, \lambda_k)$. Let T be a tableau of shape λ. Let $R(T)$ be the subgroup of S_n which permutes the entries of each row of T among themselves. Now M_λ has as a basis elements of the form $\sigma\{T\}$ as σ ranges over the elements of $S_n/R(T)$, so it follows that M_λ is isomorphic to the induced representation of the trivial representation $\mathbb{I}$ from $R(T)$ to S_n, and given the natural isomorphism

$$R(T) \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k},$$

it follows that $[M_\lambda]$ is the required product. Since the modules $[M_\lambda]$ form a basis of R, it follows that φ is an isomorphism of $\mathbb{Z}$-algebras.

To prove the isometry, we first construct the inverse map ψ from R to Λ. The map φ above informs us the image of $[M_\lambda]$ should be h_λ, however, in light of 2.1.2, we will find it convenient to write

$$h_\lambda = \sum_\mu \frac{1}{z_\mu} c_{\lambda,\mu} p_\mu,$$

as the coefficients $c_{\lambda,\mu}$ give the value of the character of M_λ on C_μ. We can therefore define

$$\psi([V]) = \sum_\mu \frac{1}{z_\mu} \chi_V(C_\mu) p_\mu.$$

This gives us an additive homomorphism $\psi : R \to \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, but the composition $\psi \circ \varphi$ is the inclusion of Λ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows then that ψ is the inverse isomorphism of R onto Λ. We then show that φ is an isometry by showing that its inverse ψ is one. The inner product

$$\langle \psi([V]), \psi([W]) \rangle = \sum_{\lambda,\mu} \frac{1}{z_\lambda z_\mu} \chi_V(C_\lambda) \cdot \chi_W(C_\mu) \langle p_\lambda, p_\mu \rangle,$$

where the inner product on Λ defined by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

Given the values of the inner product on the power sums, the sum on the right is

$$\sum_\mu \frac{1}{z_\mu} \chi_V(C_\mu) \cdot \chi_W(C_\mu) = \langle [V], [W] \rangle.$$

The complete symmetric functions can be expressed in terms of the Schur functions in the following way

$$h_\lambda = s_\lambda + \sum_{\mu \rhd \lambda} K_{\lambda,\mu} s_\mu,$$

where the coefficients $K_{\lambda,\mu}$ are the Kostka numbers, and the modules M_λ decompose in the following way

$$M_\lambda \cong V_\lambda \oplus \left(\bigoplus_{\mu \rhd \lambda} (V_\mu)^{\oplus m_{\mu,\lambda}} \right),$$

where $m_{\mu,\lambda}$ is the multiplicity of V_μ in M_λ. By the definition of φ, we must therefore have integers $k_{\mu,\lambda}$ such that

$$\varphi(s_\lambda) = [V_\lambda] + \sum_{\mu \rhd \lambda} k_{\mu,\lambda} [V_\mu].$$

Since φ is an isometry

$$1 = \langle s_\lambda, s_\lambda \rangle = \langle \varphi(s_\lambda), \varphi(s_\lambda) \rangle = 1 + \sum_{\mu \rhd \lambda} k_{\mu,\lambda}^2,$$

and hence all the coefficients $k_{\mu,\lambda}$ must be zero. □

Corollary 2.1.4 (Young's rule). *$M_\lambda \cong V_\lambda \oplus \left(\bigoplus_{\mu \rhd \lambda} (V_\mu)^{K_{\lambda,\mu}} \right)$, where $K_{\lambda,\mu}$ is the Kostka number: the number of semi-standard tableaux of shape λ and weight μ.*

Corollary 2.1.5 (Littlewood-Richardson rule). *$[V_\lambda] * [V_\mu] = \sum_\nu c^\nu_{\lambda,\mu} [V_\nu]$, where the $c^\nu_{\lambda,\mu}$ are the Littlewood-Richardson coefficients: the coefficient of s_ν if the product $s_\lambda s_\mu$ is written in the Schur basis.*

Corollary 2.1.6 (Branching rule). *Let λ be a partition of n. The induced representation $\mathrm{Ind}_{S_n}^{S_{n+1}} V_\lambda$ is the direct sum of one copy of each of the modules $V_{\lambda'}$, where λ' is obtained from λ by adding one box.*

Proof. This is the special case of 2.1.5 where $\mu = (1)$, and the inclusion of $S_n \times S_1$ in S_{n+1} is the usual inclusion of S_n in S_{n+1}. □

Corollary 2.1.7. *By Frobenius reciprocity, the Branching rule is equivalent to saying that the restriction of V_λ from S_n to S_{n-1} is the sum of one copy of each of the modules $V_{\lambda'}$ where λ' is obtained from λ by removing one box.*

Corollary 2.1.8 (Frobenius character formula)**.** *The value of the character of V_λ on C_μ is given by the integer $d_{\lambda,\mu}$ in*

$$s_\lambda = \sum_\nu \frac{1}{z(\nu)} d_{\lambda,\nu} p_\nu.$$

Proof. From the definition of the inverse isomorphism ψ in 2.1.3, we have that $[V_\lambda]$ corresponds to both s_λ and the element

$$\sum_\nu \frac{1}{z(\nu)} \chi_{V_\lambda}(C(\nu)) p_\nu.$$

□

2.2 The characteristic map for polynomial functors

Given 1.3.1 and 1.3.9 it follows that $\mathcal{F}$ is abelian and semisimple, and further that

$$K(\mathcal{F}) \cong \bigoplus_{n \geqslant 0} K(\mathcal{F}_n) \cong \bigoplus_{n \geqslant 0} R_n.$$

The tensor product 1.3.12 defines a graded commutative ring structure on $K(\mathcal{F})$. Given its definition, it follows that this graded ring structure agrees with the one defined on R by the induction product. Hence, we can identify $K(\mathcal{F})$ with R as graded commutative rings.

Let F be a polynomial functor on $\mathcal{V}$. For each composition

$$\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$$

let (λ) denote the diagonal endomorphism of $\mathbb{C}^m$ with eigenvalues $\lambda_1, \ldots, \lambda_m$. The trace of $F((\lambda))$ is then a symmetric function of the $\lambda_1, \ldots, \lambda_m$, since for any $\pi \in S_m$ we have

$$\begin{aligned} \operatorname{tr} F(\pi\lambda) &= \operatorname{tr} F(\pi(\lambda)\pi^{-1}) && \text{(since } \pi(\lambda) = (\pi\lambda)\pi) \\ &= \operatorname{tr} F(\pi) F((\lambda)) F(\pi^{-1}) \\ &= \operatorname{tr} F(\lambda). \end{aligned}$$

Since trace is additive, we have a mapping $\chi_m : K(\mathcal{F}) \to \Lambda_m$ given by

$$\chi_m(F)(\lambda_1, \ldots, \lambda_m) = \operatorname{tr} F((\lambda)).$$

Further, since trace is multiplicative with respect to tensor products, χ_m is a homomorphism of graded rings. Let $\rho_{l,m} : \Lambda_l \to \Lambda_m$ $(l \geqslant m)$ be the restriction to Λ_l of the map $\mathbb{Z}[\lambda_1, \ldots, \lambda_l] \to \mathbb{Z}[\lambda_1, \ldots, \lambda_m]$ which sends each of $\lambda_{m+1}, \ldots, \lambda_l$ to zero. We therefore have that the composition

$$\rho_{l,m} \circ \chi_l = \chi_m.$$

The homomorphisms χ_m hence determine a homomorphism of graded rings

$$\chi : K(\mathcal{F}) \to \Lambda,$$

called the *characteristic map.*

Proposition 2.2.1. *The characteristic map coincides with the map* $\psi : R \to \Lambda$ *of 2.1.3.*

Proof. We first observe that $\chi(\bigwedge^n) = e_n$, which is also the character of the sign representation of S_n. Since the polynomial functor $\bigwedge^n$ corresponds to the sign representation under the equivalence of categories 1.3.9, the result follows from that fact that the e_n generate the ring Λ. □

Proposition 2.2.2. *If* $F_\lambda : \mathcal{V} \to \mathcal{V}$ *is the irreducible polynomial functor corresponding to the partition* λ*, then* $\chi(F_\lambda) = s_\lambda$*.*

Proof. By 1.3.10 the irreducible polynomial functors F_λ correspond to the irreducible modules V_λ, and hence the result follows from 2.2.1 and 2.1.3. □

2.3 Polynomial representations of GL_m

Let G be any group and let R be a matrix representation of degree d over $\mathbb{C}$, the representing matrices given by $R(g) = (R_{ij})$, where $g \in G$ and $1 \leqslant i, j \leqslant d$. The representation R determines d^2 functions $R_{ij} : G \to \mathbb{C}$, called the *matrix coefficients* of R.

Proposition 2.3.1. *Let* $R^{(1)}, R^{(2)}, \ldots, R^{(k)}$ *be a sequence of matrix representations over* $\mathbb{C}$ *of a group* G*. The following are then equivalent:*

(i) All the matrix coefficients $R^{(1)}_{ij}, R^{(2)}_{ij}, \ldots, R^{(n)}_{ij}$ *are linearly independent.*

(ii) The representations $R^{(1)}, R^{(2)}, \ldots, R^{(n)}$ *are irreducible and pairwise inequivalent.*

Proof. Suppose that some $R^{(k)}$ is a reducible representation, then $R^{(k)}$ is equivalent to a matrix representation such that some of the matrix coefficients are zero. Since the space of functions on G spanned by the matrix coefficients is the same, it follows then that the matrix coefficients $R_{ij}^{(k)}$ are linearly dependent over $\mathbb{C}$. Let us now suppose that some pair $R^{(k)}$ and $R^{(l)}$ are equivalent, irreducible representations. The matrix coefficients of $R^{(l)}$ are then linearly dependent on those of $R^{(k)}$. These two facts give us the implication $(i) \implies (ii)$.

For the reverse implication see Curtis and Reiner [CR62, (27.13), p. 184]. □

Let V be an m-dimensional complex vector space so that $GL(V)$ is identified with the group GL_m. Let $x_{ij} : GL_m \to \mathbb{C}$ $(1 \leqslant i, j \leqslant m)$ be the coordinate functions on GL_m, so that $x_{ij}(g)$ is the (i, j) element of the matrix $g \in GL_m$. Let

$$P = \bigoplus_{n \geqslant 0} P_n = \mathbb{C}[x_{ij} : 1 \leqslant i, j \leqslant m]$$

be the algebra of polynomial functions of GL_m, where P_n consists of the polynomials in the x_{ij} that are homogenous of degree n.

A matrix representation of GL_m is called *polynomial* if its matrix coefficients are polynomials in the x_{ij}. Let $\mathcal{M}_m$ denote the category of finite dimensional polynomial GL_m representations.

Theorem 2.3.2. *Let R^λ be the polynomial representation of GL_m in which an element $g \in GL_m$ acts as $F_\lambda(g)$ on $F_\lambda(V)$, where λ is a partition such that $l(\lambda) \leqslant m$. The representations R^λ are inequivalent irreducible polynomial representations of GL_m. Furthermore, every irreducible polynomial representation of GL_m is equivalent to some R^λ.*

Proof. By 2.2.2, the dimension of $F_\lambda(V)$ is given by taking $x_1 = \cdots = x_m = 1$ in the Schur polynomial $s_\lambda(x_1, \ldots, x_m)$. Given the well-known relation

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x_1, \ldots, x_m) s_\lambda(y_1, \ldots, y_m)$$

(see [CR62], p. 63), it follows that $d_n = \sum_{\lambda \vdash n} (\dim F_\lambda(V))^2$ is the coefficient of t^n in $(1 - t)^{-m^2}$, but this number is the same as $\dim_{\mathbb{C}} P_n$.

The decomposition $V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes F_\lambda(V)$ shows that the representation of GL_m on $V^{\otimes n}$ is the direct sum of $\dim V_\lambda$ copies of R^λ, for each partition λ of n such that $l(\lambda) \leqslant m$. The matrix coefficients of the representation $V^{\otimes n}$ are the degree n monomials in the coordinate functions x_{ij}, and hence span P_n. It follows that the coefficients R_{ij}^λ also span P_n. From the discussion above, the total number of these matrix coefficients is d_n, so the R_{ij}^λ form a $\mathbb{C}$-basis of P_n. By 2.3.1 it follows that the R^λ are irreducible and pairwise inequivalent.

Finally, suppose that R is a polynomial representation of GL_m. If R is reducible, then R is a direct sum of homogeneous polynomial representations. Hence, it is enough to assume that R is irreducible. If R is irreducible, its matrix coefficients R_{ij} are homogenous of some degree n. Therefore, by 2.3.1 R is equivalent to some R^λ. □

Corollary 2.3.3. *The character of the representation R^λ of GL_m is the Schur function s_λ. Further, a polynomial representation of GL_m is determined up to equivalence by its character.*

Proof. If f is a symmetric function and $x \in GL_m$, then we can regard f as a function of x by taking $f(x) = f(\xi_1, \ldots, \xi_m)$, where $\xi_1, \ldots, \xi_m$ are the eigenvalues of x.

If $x \in GL_m$ is a diagonal matrix and hence also if x is diagonalisable, then by 2.2.2

$$\operatorname{tr} F_\lambda(x) = s_\lambda(x).$$

Since the diagonalisable matrices are Zariski dense in GL_m, and both $s_\lambda(x)$ and $\operatorname{tr} F_\lambda(x)$ are polynomial functions of x, it follows that $s_\lambda(x) = \operatorname{tr} F_\lambda(x)$ for all $x \in GL_m$.

The final part of the proposition is a direct consequence of this, since the Schur functions s_λ where $l(\lambda) \leqslant m$ are linearly independent. □

Corollary 2.3.4. *If we endow $K(\mathcal{M}_m)$ with the commutative ring structure given by the tensor product of GL_m-modules, then we have an isomorphism of rings $K(\mathcal{M}_m) \cong \Lambda_m$.*

Let Z_m be the centre of GL_m, which consists of the scalar matrices. If V_m is a GL_m-module, let

$$V_m(k) = \{v \in V_m \ : \ z \cdot v = z^k v \text{ for all } z \in Z_m\}.$$

The module V_m then has the decomposition into weight spaces

$$V_m = \bigoplus_{k \in \mathbb{Z}} V_m(k)$$

under this action of the centre.

If a module $V_m = V_m(k)$ for some k, we say that V_m is *homogenous* of *degree* k. A polynomial representation of GL_m is then a direct sum of homogeneous representations of non-negative degrees. In particular, an irreducible GL_m-module R^λ is homogeneous of some degree $|\lambda|$. We denote the category of polynomial representations of GL_m of degree k by $\mathcal{M}_m(k)$.

For all $m \geqslant 1$, we embed GL_{m-1} in GL_m as the subgroup of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A \in GL_{m-1}$. Restriction to GL_{m-1} then defines a functor $\mathfrak{R} : \mathcal{M}_m \to \mathcal{M}_{m-1}$ given by $\mathfrak{R}(V_m) = \mathrm{Res}^{GL_m}_{GL_{m-1}}(V_m)$. Consider the embedding of GL_1 in GL_m as the subgroup of matrices of the form

$$\begin{pmatrix} I_{m-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

where I_{m-1} is the identity matrix in GL_{m-1}, and $\lambda \in \mathbb{C}$. The action of this copy of GL_1 on V_m then commutes with the action of GL_{m-1}. We can then decompose the functor $\mathfrak{R}$ into weight spaces

$$\mathfrak{R} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{R}_i$$

corresponding to this action of GL_1. In particular, $\mathfrak{R}_0(V_m) = V_m^{GL_1}$. The functor $\mathfrak{R}_0$ preserves degree, and hence defines a functor

$$\mathfrak{R}_0(k) : \mathcal{M}_m(k) \to \mathcal{M}_{m-1}(k).$$

Let $\widetilde{\mathcal{M}}$ denote the category whose objects are sequences

$$V = (V_m, \alpha_m)_{m \geqslant 0},$$

where $V_m \in \mathcal{M}_m$ and $\alpha_m : \mathfrak{R}_0(V_{m+1}) \to V_m$ is an isomorphism of GL_m-modules. By convention we take GL_0 to be the trivial group. If the maps α_m are obvious from the context, we will sometimes write $V = (V_m)_{m \geqslant 0}$ for an object of $\widetilde{\mathcal{M}}$.

A morphism $V \to W$ of objects in this category is given by a sequence $(f_m)_{m \geqslant 0}$ of morphisms $f_m : V_m \to W_m$, so that for sufficiently large m, the diagram

$$\begin{array}{ccc} \mathfrak{R}_0(V_{m+1}) & \xrightarrow{\mathfrak{R}_0(f_{m+1})} & \mathfrak{R}_0(W_{m+1}) \\ \downarrow{\scriptstyle \alpha_m} & & \downarrow{\scriptstyle \beta_m} \\ V_m & \xrightarrow{f_m} & W_m \end{array}$$

commutes.

An object $V = (V_m, \alpha_m)_{m \geqslant 0} \in \widetilde{\mathcal{M}}$ is *homogeneous* of *degree* k if every V_m is of degree k. Let $\mathcal{M}(k)$ denote the subcategory of $\widetilde{\mathcal{M}}$ consisting of objects of degree k.

Proposition 2.3.5. *Let* $V = (V_m, \alpha_m)_{m\geqslant 0}$ *and* $W = (W_m, \beta_m)_{m\geqslant 0}$ *be two objects of* $\widetilde{\mathcal{M}}$*, then* $V \otimes W = (V_m \otimes W_m, \alpha_m \otimes \beta_m)_{m\geqslant 0}$ *defines a monoidal structure on* $\widetilde{\mathcal{M}}$*.*

Proof. We simply need to check that $\mathfrak{R}_0(V_m \otimes W_m) \cong \mathfrak{R}_0(V_m) \otimes \mathfrak{R}_0(W_m)$. Under the action of GL_1 we have the decomposition into weight spaces

$$\bigoplus_{s\geqslant 0}(V_m \otimes W_m)(s) \cong \bigoplus_{k+l=s} V_m(k) \otimes W_m(l).$$

The GL_1-invariant subspace of $V_m \otimes W_m$ is $(V_m \otimes W_m)(0)$, which means we also require $k + l = 0$. Since V_m and W_m are polynomial, this forces k and l to also be zero, which gives the desired isomorphism. □

Hong and Yacobi [HY13] provide the following simple examples of objects living in $\widetilde{\mathcal{M}}$:

1. If $\mathbb{I}_m$ is the trivial representation of GL_m, then we have the obvious isomorphism $\mathfrak{R}_0(\mathbb{I}_{m+1}) \cong \mathbb{I}_m$. Hence they all glue together to form the object $\mathbb{I} = (\mathbb{I}_m)_{m\geqslant 0}$. This object plays the role of the unit object in $\widetilde{\mathcal{M}}$.

2. The standard representation $\mathbb{C}^m$ of GL_m is canonically isomorphic to $\mathfrak{R}_0(\mathbb{C}^{m+1})$, since the the GL_1-invariant vectors of $\mathbb{C}^{m+1}$ under the action described above are simply the vectors whose $(m + 1)$th component is zero. The standard representations hence glue together to produce the *standard* object $\mathrm{St} = (\mathbb{C}^m)_{m\geqslant 0}$.

3. For a non-negative integer r, we have the tensor product representation of GL_m given by $\bigotimes^r \mathbb{C}^m$. If we take $e_1, \dots, e_{m+1}$ to be the standard basis of $\mathbb{C}^{m+1}$, then the tensor product representation has vectors of the form

   $$e_{i_1} \otimes \cdots \otimes e_{i_r}$$

   as a basis. The GL_1-invariant vectors in the tensor product representation are therefore linear combinations of such basis vectors containing no instances of e_{m+1}. We therefore have an obvious isomorphism

   $$\mathfrak{R}_0\left(\bigotimes^r \mathbb{C}^{m+1}\right) \cong \bigotimes^r \mathbb{C}^m.$$

   The tensor product representations then glue together to an object which is canonically isomorphic to $\bigotimes^r \mathrm{St}$. The objects $\bigwedge^r \mathrm{St}$, and $\mathrm{Sym}^r \mathrm{St}$ are similarly defined.

4. Since the exterior algebra of a finite dimensional vector space is also finite dimensional, $\bigwedge \mathbb{C}^m$ is an object of $\mathcal{M}_m$. Again, from isomorphisms $\mathfrak{R}_0\left(\bigwedge \mathbb{C}^{m+1}\right) \cong \bigwedge \mathbb{C}^m$ we can glue together an object $\bigwedge \mathrm{St} \in \widetilde{\mathcal{M}}$.

For an object $V = (V_m)_{m\geqslant 0}$ in $\widetilde{\mathcal{M}}$, let

$$V(k) = (V_m(k))_{m\geqslant 0}$$

where each $V_m(k)$ is the weight space corresponding to the action of Z_m on V_m. We then have the potentially infinite direct sum in $\widetilde{\mathcal{M}}$:

$$V = \bigoplus V(k).$$

We say that an object is *compact* in $\widetilde{\mathcal{M}}$ if this direct sum is finite. Of the examples presented above, it follows that $\mathbb{I}$, St, $\bigotimes^r \mathrm{St}$, $\mathrm{Sym}^r \mathrm{St}$, and $\bigwedge^r \mathrm{St}$ are compact objects, while $\bigwedge \mathrm{St}$ is not.

Let $\mathcal{M}$ be the full subcategory of $\widetilde{\mathcal{M}}$ consisting of compact objects. We then have that $\mathcal{M}$ is the direct sum of categories

$$\mathcal{M} = \bigoplus_{k\geqslant 0} \mathcal{M}(k),$$

and that it is also a tensor category.

Proposition 2.3.6. *For $m \geqslant k$, the functor $\mathfrak{R}_0(k) : \mathcal{M}_{m+1}(k) \to \mathcal{M}_m(k)$ is an equivalence of categories.*

Proof. The irreducible representations in $\mathcal{M}_{m+1}(k)$ are the modules R^λ where $|\lambda| = k$. By 2.3.3, the restricted map $\chi_{m+1}(k) : \mathcal{M}_{m+1}(k) \to \Lambda^k_{m+1}$ defined by $R^\lambda \mapsto s_\lambda$ is a bijection. Let

$$\rho_{m+1,m} : \Lambda_{m+1} \to \Lambda_m$$

be the map defined by sending $s_\lambda(x_1, \ldots, x_{m+1})$ to $s_\lambda(x_1, \ldots, x_m)$ if $l(\lambda) \leqslant m$ and to zero otherwise. Upon restriction to Λ^k_{m+1}, this map produces an isomorphism

$$\rho^k_{m+1,m} : \Lambda^k_{m+1} \cong \Lambda^k_m$$

whenever $m \geqslant k$. It then follows that $\mathfrak{R}_0(k)$ is a bijection, and the result follows. □

Corollary 2.3.7. *Let Ψ_m denote the projection from $\mathcal{M}$ to $\mathcal{M}_m$, and let $\Psi_m(k)$ denote its restriction to $\mathcal{M}(k)$. For $m \geqslant k$, $\Psi_m(k) : \mathcal{M}(k) \to \mathcal{M}_m(k)$ is an equivalence of categories.*

Let $M_{m,n} = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, and let $\mathcal{O}(M_{m,n})$ denote the algebra of polynomials on $M_{m,n}$. There exists a natural action of $GL_m \times GL_n$ on $M_{m,n}$ given by

$$(g_1, g_2) \cdot M = g_1 M g_2^t.$$

On $\mathcal{O}(M_{m,n})$ this action becomes

$$((g_1, g_2) \cdot f)(M) = f(g_1^t M g_2).$$

Let $\mathfrak{I}_0 : \mathcal{M}_m \to \mathcal{M}_{m+1}$ be the functor defined by

$$V_m \mapsto (V_m \otimes \mathcal{O}(M_{m,m+1}))^{GL_m}.$$

Here the invariants are taken with respect to the tensor product action of GL_m on

$$V_m \otimes \mathcal{O}(M_{m,m+1}),$$

which commutes with the action of GL_{m+1} on $\mathcal{O}(M_{m,m+1})$. Hence, $\mathfrak{I}_0(V_m)$ is a polynomial GL_{m+1}-module.

Lemma 2.3.8. *The functor $\mathfrak{R}_0$ is left adjoint to $\mathfrak{I}_0$.*

Proof. We will show that for all $V_{m+1} \in \mathcal{M}_{m+1}$, and $W_m \in \mathcal{M}_m$ that

$$\mathrm{Hom}_{GL_m}(\mathfrak{R}_0(V_{m+1}), W_m) \cong \mathrm{Hom}_{GL_{m+1}}(V_{m+1}, \mathfrak{I}_0(W_m)).$$

Consider a map $f \in \mathrm{Hom}_{GL_m}(\mathfrak{R}_0(V_{m+1}), W_m)$. Given the definition of $\mathfrak{R}_0$ this lifts to a GL_m-equivariant map $f' : \mathfrak{R}(V_{m+1}) \to W_m$. Composing this with the map

$$W_m \to W_n \otimes \mathcal{O}(M_{m,m+1})$$

given by $w \mapsto w \otimes 1$ and restricting to GL_m invariants gives a GL_{m+1}-equivariant map $V_{m+1} \to \mathfrak{I}_0(W_m)$.

For the reverse map we send a GL_{m+1}-equivariant map $f : V_{m+1} \to \mathfrak{I}_0(W_m)$ to the map $R(V_{m+1}) \to W_n$ given by $v \mapsto f(v)(I_{m+1})$, where I_{m+1} is the $m \times (m+1)$ matrix with ones in the entries (i,i) for $1 \leqslant i \leqslant m$, and zero elsewhere. This descends to a GL_m-equivariant map $\mathfrak{R}_0(V_{m+1}) \to W_m$. □

Corollary 2.3.9. *For $m \geqslant k$,*

$$\mathfrak{I}_0(k) : \mathcal{M}_m(k) \to \mathcal{M}_{m+1}(k)$$

is an equivalence of categories.

Proof. By 2.3.8 and 2.3.6, $\mathfrak{I}_0(k)$ is adjoint to an equivalence of categories in the reverse direction. □

Proposition 2.3.10. *The functor*

$$\Psi : \bigoplus_{k\geqslant 0} \mathcal{M}(k) \to \bigoplus_{k\geqslant 0} \mathcal{M}_k(k),$$

defined by taking the direct sum of the projection functors $\Psi_k(k)$ is an equivalence of categories. The inverse morphism $\Psi^{-1} : \bigoplus_{k\geqslant 0} \mathcal{M}_k(k) \to \mathcal{M}$ is given by taking the direct sum of functors $\Psi_k^{-1} : \mathcal{M}_k(k) \to \mathcal{M}$ defined by $\Psi_k^{-1}(V_k) = (V_m)_{m\geqslant 0}$ where

$$V_m = \begin{cases} \mathfrak{I}_0^{m-k}(V_k), & m \geqslant k, \\ \mathfrak{R}_0^{k-m}(V_k), & m < k. \end{cases}$$

Proof. The first part of the proposition follows easily from 2.3.7, which implies that for all $m \geqslant k$ we have an equivalence of categories $\mathcal{M}_k(k) \to \mathcal{M}_m(k)$. To show Ψ^{-1} gives the inverse, we simply need to check that $\Psi^{-1}(V_k)$ is a well-defined object of $\mathcal{M}$. Clearly we have isomorphisms $\mathfrak{R}_0(V_{m+1}) \to V_m$ whenever $m < k$, so we need to check the case when $m \geqslant k$. This follows from 2.3.8 as we have $\mathfrak{R}_0(\mathfrak{I}_0(V_m)) = V_m$. □

Let Γ^k denote the kth divided power, that is $\Gamma^k(V) = \left(\bigotimes^k V\right)^{S_k}$, and let

$$\Gamma^{k,m} = \Gamma^k \circ \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, -).$$

The action of GL_m on $\mathbb{C}^m$ induces an action on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, -)$ and hence on $\Gamma^{m,k}$.

Lemma 2.3.11. *For any homogeneous polynomial functor F of degree k, and any $m > 0$,*

$$\mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F) = F(\mathbb{C}^m).$$

Proof. For any vector space V we have a natural $\mathbb{C}$-linear map

$$\theta_V : \Gamma^k(\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{C}^m, V)) \otimes F(\mathbb{C}^m) \to F(V),$$

which is functorial in V. This defines a natural transformation

$$\theta : F(\mathbb{C}^m) \otimes \Gamma^{k,m} \to F,$$

so by the tensor-hom adjunction we have a $\mathbb{C}$-linear map

$$F(\mathbb{C}^m) \to \mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F).$$

Conversely, for any map $\alpha : \Gamma^{k,m} \to F$, we can associate an element of $F(\mathbb{C}^m)$, namely the image $\alpha_{\mathbb{C}^m}(1_{\mathbb{C}^m} \otimes \cdots \otimes 1_{\mathbb{C}^m})$. It is clear that these two maps are mutually inverse. □

Lemma 2.3.12. *For any degree k polynomial functor F, we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_m}, F) \cong F(\mathbb{C}^m)^{k_1,\dots,k_m},$$

where $F(\mathbb{C}^m)^{k_1,\dots,k_m}$ is the weight space corresponding to the weight $(k_1,\dots,k_m)$.

Proof. Using the explicit formulae for the inclusions

$$\Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_m} \hookrightarrow \Gamma^{k,m} \text{ and } F(\mathbb{C}^m)^{k_1,\dots,k_m} \hookrightarrow F(\mathbb{C}^m),$$

it is a small chore to check that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_m}, F) & \longrightarrow & F(\mathbb{C}^m)^{k_1,\dots,k_m} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F) & \longrightarrow & F(\mathbb{C}^m) \end{array}$$

where the lower map sends $\alpha : \Gamma^{k,m} \to F$ to $\alpha_{\mathbb{C}^m}(1_{\mathbb{C}^m} \otimes \cdots \otimes 1_{\mathbb{C}^m})$ as in the proof of 2.3.11, and the upper map sends $\beta : \Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_m} \to F(\mathbb{C}^m)^{k_1,\cdots,k_m}$ to

$$\beta_{\mathbb{C}^m}(\underbrace{(e_1 \otimes \cdots \otimes e_1)}_{k_1} \otimes \cdots \otimes \underbrace{(e_m \otimes \cdots \otimes e_m)}_{k_m}).$$

□

Theorem 2.3.13. *The assignment*

$$F \mapsto (F(\mathbb{C}^m), F(\pi_m))_{m \geqslant 0},$$

where $\pi_m : \mathbb{C}^{m+1} \to \mathbb{C}^m$ is the natural GL_m-equivariant projection, defines an equivalence of categories

$$\Phi : \mathcal{F} \to \mathcal{M}.$$

Proof. We first need to check that $F(\pi_m)$ is an isomorphism. It is enough to check this in the case that F is of degree k. The action of GL_m induces on $\Gamma_{k,m}$ induces a GL_m-module structure on $\mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F)$, so by 2.3.11 we have a canonical isomorphism of GL_m-modules $\mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F) \cong F(\mathbb{C}^m)$. Hence, it is enough to show that $\mathfrak{R}_0(\mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F)) \cong \mathrm{Hom}_{\mathcal{F}_k}(\Gamma^{k,m-1}, F)$. Given that the functor $\Gamma^{k,m}$ decomposes canonically as

$$\Gamma^{k,m} = \bigoplus_{k_1+\cdots+k_m=k} \Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_m},$$

then by 2.3.12

$$\begin{aligned}\mathfrak{R}_0(\operatorname{Hom}_{\mathcal{F}_k}(\Gamma^{k,m}, F)) &\cong \bigoplus_{k_1+\cdots+k_{m-1}=k} \operatorname{Hom}_{\mathcal{F}_k}(\Gamma^{k_1} \otimes \cdots \otimes \Gamma^{k_{m-1}}, F)\\ &\cong \operatorname{Hom}_{\mathcal{F}_k}(\Gamma^{k,m-1}, F).\end{aligned}$$

We have then established that the assignment gives a well-defined object in $\mathcal{M}$, so we are left to show that this assignment constitutes an equivalence of categories. Now,

$$\mathcal{F} = \bigoplus_{k \geqslant 0} \mathcal{F}_k,$$

and since Φ clearly preserves degree, it is enough to show that $\Phi : \mathcal{F}_k \to \mathcal{M}(k)$ is an equivalence of categories for every $k \geqslant 0$. Let $\Phi_m(k) : \mathcal{F}_k \to \mathcal{M}_m(k)$ be the functor defined by $F \mapsto F(\mathbb{C}^m)$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_k & \xrightarrow{\Phi} & \mathcal{M}(k) \\ {\scriptstyle \Phi_m(k)}\downarrow & \swarrow {\scriptstyle \Psi_m(k)} & \\ \mathcal{M}_m(k) & & \end{array}$$

For $m \geqslant k$, by 2.3.7, $\Psi_m(k)$ is an equivalence of categories, hence we are left to show that $\Phi_m(k)$ is an equivalence of categories for $m \geqslant k$, but this is a simple consequence of 2.3.2. □

Corollary 2.3.14. *$\mathcal{M}$ is a categorification of the ring of symmetric functions.*

2.4 Plethysm

Once more we lean heavily on Macdonald [Mac95] to define another multiplication on Λ given by composition of symmetric functions, called *plethysm*. More formally, let $f, g \in \Lambda$, and write g as a sum of monomials:

$$g = \sum_{\alpha} u_\alpha x^\alpha.$$

Define variables y_i by setting

$$\prod (1 + y_i t) = \prod_{\alpha} (1 + x^\alpha t)^{u_\alpha}.$$

The composition or plethysm of f and g is then defined as

$$f \circ g = f(y_1, y_2, \ldots).$$

If $f \in \Lambda^m$ and $g \in \Lambda^n$, then $f \circ g \in \Lambda^{mn}$.

Proposition 2.4.1. *For each $g \in \Lambda$, the mapping $f \mapsto f \circ g$ is an endomorphism of Λ.*

Proof. From the definition of plethysm it is clear that plethysm respects sums and products, and hence it defines an endomorphism of Λ. □

Proposition 2.4.2. *If g is any symmetric function, then*

$$p_n \circ g = g \circ p_n$$

for any power sum p_n.

Proof. Since we can write any $g \in \Lambda$ as the sum

$$g = \sum_{\lambda} c_\lambda p_\lambda,$$

where $c_\lambda \in \mathbb{Q}$, by 2.4.1 it is enough to show that

$$p_\lambda \circ p_n = p_n \circ p_\lambda.$$

To this end, we first note that

$$p_m \circ p_n = p_m(x_1^n, x_2^n, \ldots) = p_{mn} = p_n(x_1^m, x_2^m, \ldots) = p_n \circ p_m.$$

We then have that

$$p_\lambda \circ p_n = p_\lambda(x_1^n, x_2^n, \ldots) = \prod_{i=1}^{l(\lambda)} p_{\lambda_i n} = \prod_{i=1}^{l(\lambda)} p_n(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots) = p_n \circ p_\lambda.$$

□

Proposition 2.4.3. *Plethysm is associative, that is for all $f, g, h \in \Lambda$*

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Proof. By 2.4.1 it is enough to demonstrate associativity when $f = p_m$ and $g = p_n$. By 2.4.2 we have

$$p_m \circ (p_n \circ h) = (h \circ p_n) \circ p_m = h(x_1^{mn}, x_2^{mn}, \ldots) = h \circ p_{mn} = (p_m \circ p_n) \circ h.$$

□

As in 1.3.9, let $E = \beta(M)$, and let $F = \beta(N)$, where M is an S_m-module, and N is an S_n-module. The composition of E and F is then given by

$$\begin{aligned}(E \circ F)(V) &= E((N \otimes V^{\otimes n})^{S_n}) \\ &= (M \otimes ((N \otimes V^{\otimes n})^{S_n})^{\otimes m})^{S_m} \\ &\cong (M \otimes (N^{\otimes m} \otimes V^{\otimes mn})^{S_n^m})^{S_m}.\end{aligned}$$

Now, the normaliser of

$$S_n^m := S_n \times \cdots \times S_n$$

in S_{mn} is the semi-direct product $S_n^m \times S_m$ in which S_m acts by permuting the factors of S_n^m. This is called the *wreath product* of S_n with S_m and is denoted by $S_n \sim S_m$. Using the fact that as subspaces of $L \otimes M$

$$(L \otimes M^H)^{G/H} = (L \otimes M)^G,$$

where L is a finite-dimensional $\mathbb{C}[G/H]$-module, and M is a finite-dimensional $\mathbb{C}[G]$-module, we find that

$$\begin{aligned}(E \circ F)(V) &\cong (M \otimes (N^{\otimes m} \otimes V^{\otimes mn}))^{S_n \sim S_m} \\ &\cong (\mathrm{Ind}_{S_n \sim S_m}^{S_{mn}}(M \otimes N^{\otimes m}) \otimes V^{\otimes mn})^{S^{mn}}.\end{aligned}$$

By 1.3.9 we can then define the *plethysm* of M and N as

$$M \circ N = \mathrm{Ind}_{S_n \sim S_m}^{S_{mn}}(M \otimes N^{\otimes m}).$$

Proposition 2.4.4. *For all S_m-modules M_1 and M_2 and S_n-modules N we have:*

(1) $(M_1 \oplus M_n) \circ N \cong (M_1 \circ N) \oplus (M_2 \circ N)$*, and*

(2) $(M_1 \circ N) * (M_2 \circ N) \cong (M_1 * M_n) \circ N$*.*

Proof. The first property is a simple consequence of the fact that induction commutes with direct sums. The second property is easily verified from the tensor product definition of induced modules. □

Corollary 2.4.5. *For polynomial functors E_1 and E_2 homogeneous of degree m, and a polynomial functor F homogeneous of degree n we have*

$$(E_1 \circ F) \otimes (E_2 \circ F) = (E_1 \otimes E_2) \circ F.$$

Proposition 2.4.6. *For any two polynomial functors E and F*

$$\chi(E \circ F) = \chi(E) \circ \chi(F).$$

Proof. If F is any polynomial functor, then the decomposition of 1.3.5 applied to the functor $F'(V_1, \dots, V_n) \mapsto F(V_1 \oplus \cdots \oplus V_n)$ shows that the eigenvalues of $F((\lambda))$ are monomials $\lambda_1^{m_1} \cdots \lambda_n^{m_n}$, with corresponding eigenspaces $F'_{m_1,\dots,m_n}(\mathbb{C}, \cdots, \mathbb{C})$, and therefore the character

$$\chi_n(F) = \sum_{m_1,\dots,m_n} \dim F'_{m_1,\dots,m_n}(\mathbb{C}, \dots, \mathbb{C}) x_1^{m_1} \cdots x_n^{m_n}.$$

The result then follows from the definition of plethysm for symmetric functions. □

Proposition 2.4.7. *The plethysm of two Schur functions is*

$$s_\lambda \circ s_\mu = \sum_{\nu \vdash |\lambda|\cdot|\mu|} a^\nu_{\lambda,\mu} s_\nu,$$

with coefficients $a^\nu_{\lambda,\mu} \geqslant 0$.

Proof. The plethysm of two irreducible polynomial functors F_λ and F_μ is a direct sum of irreducibles

$$F_\lambda \circ F_\mu = \bigoplus_{\nu \vdash |\lambda|\cdot|\mu|} a^\nu_{\lambda,\mu} F_\nu,$$

where each $a_{\lambda,\mu} \geqslant 0$. The result then follows from 2.4.6. □

In order to define the plethysm of two objects in $\mathcal{M}$ we first recall that we can compose the two representations $\rho : GL_m \to GL_k$ and $\rho' : GL_k \to GL_r$ to produce an r-dimensional representation of GL_m. Take two objects $V = (V_m, \alpha_m)_{m\geqslant 0}$ and $W = (W_m, \beta_m)_{m\geqslant 0}$ in $\mathcal{M}$. If $d_m = \dim W_m$, then we can compose the representation of GL_{d_m} on V_{d_m} with the representation of GL_m on W_m, to produce another representation of GL_m. We denote this composition by $V_{d_m} \circ W_m$. We can then define the plethysm of V and W in $\mathcal{M}$ as the object

$$V \circ W = (V_{d_m} \circ W_m, \alpha_{d_m})_{m\geqslant 0}.$$

By 2.3.13, there exist polynomial functors E and F such that

$$V_m = E(\mathbb{C}^m) \text{ and } \alpha_m = E(\pi_m),$$

and

$$W_m = F(\mathbb{C}^m) \text{ and } \beta_m = F(\pi_m),$$

for all $m \geqslant 0$. The plethysm of V and W can then equivalently be defined as

$$V \circ W = ((E \circ F)(\mathbb{C}^m), (E \circ F)(\pi_m))_{m\geqslant 0}.$$

Chapter 3

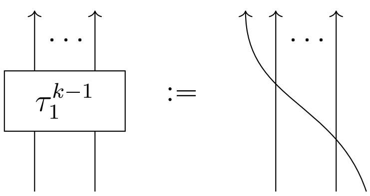
Diagramatics for shifted symmetric functions and free probability

The results in this chapter also appear in the paper coauthored with Kvinge and Licata [KLM16].

3.1 The main results

We begin by defining a new family of elements α_λ, where $\lambda \in \mathcal{P}$, whose image under the functor $\mathcal{F}'_n$ is the element $a_{\lambda,n}$. Since the value of $p^{\#}_\lambda$ considered as a function on partitions is $a_{\alpha,n}$ (see Proposition 1.4.6), this establishes our first, and as we will see, fruitful, connection between $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and Λ^*.

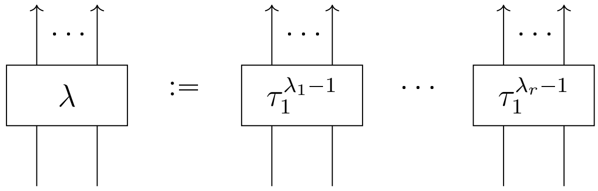
Our first construction is to make a diagram with k upwards oriented strands to represent a k-cycle:



Here the symbol τ_i^j denotes

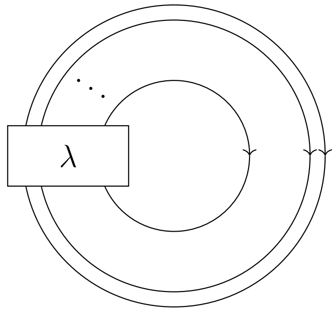
$$\tau_j^i := \begin{cases} s_j \cdots s_i, & 1 \leqslant j \leqslant i, \\ 1, & j = i+1 \\ 0, & \text{otherwise.} \end{cases}$$

To a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we then set



The elements α_λ are then defined by

$\alpha_\lambda \quad :=$



Proposition 3.1.1. *For a partition $\lambda \vdash k$, we have*

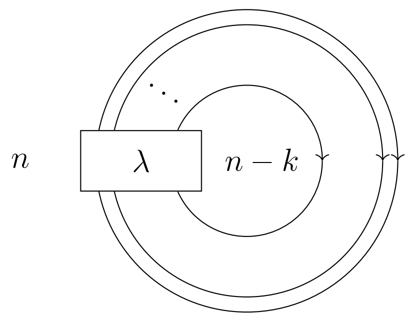
$$f_n^{\mathcal{H}'}(\alpha_\lambda) = \begin{cases} a_{\lambda,n}, & k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Recall that the elements $a_{\lambda,n}$ are defined by

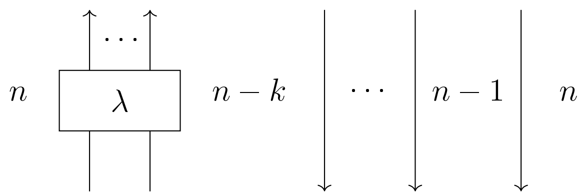
$$\sum (i_1, \ldots, i_{\lambda_1})(i_{\lambda_1+1}, \ldots, i_{\lambda_1+\lambda_2}) \cdots (i_{k-\lambda_r+1}, \ldots i_k),$$

where the sum is taken over all distinct k-tuples $(i_1, \ldots, i_k)$ of elements drawn from $\{1, \ldots, n\}$.

Consider the diagram



If $k > n$, then the centre of the diagram is negative, and the diagram is then zero. If $k \leqslant n$ then the diagram is the composition of a series of k RCups, the diagram



and a series of k LCaps to close it off. The result then follows from the definitions of the bimodule maps. □

Proposition 3.1.2. *The diagrams $\alpha_{(k)}$, $k \geqslant 1$ form an algebraically independent generating set of* $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$.

Proof. We first note that the diagram $\alpha_{(k)}$ can be redrawn as a dot below $k-2$ UCrosses, enclosed in LCups and Rcaps. By an exercise in pulling dots through intersections we find that

$$\alpha_{(k)} = c_{k-1} + \text{l.o.t.}.$$

The result then follows from the fact that the clockwise circles form an algebraically independent generating set of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$. □

We can consider the elements of $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ as functions on partitions in the following way. Denote by $\mathrm{Func}_{\mathcal{P}}(\mathbb{C})$ the algebra of functions $\mathcal{P} \to \mathbb{C}$. We define $\Phi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to \mathrm{Func}_{\mathcal{P}}(\mathbb{C})$ for $x \in \mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ and $\lambda \vdash n$ by

$$[\Phi(x)](\lambda) := (\widehat{\chi}^{\lambda} \circ f_n^{\mathcal{H}'})(x).$$

It is easily checked that this defines an algebra homomorphism. For the sake of convenience we will write $x(\lambda)$ in place of $[\Phi(x)](\lambda)$.

Theorem 3.1.3. *The map Φ induces an isomorphism $\Psi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \to \Lambda^*$ given by*

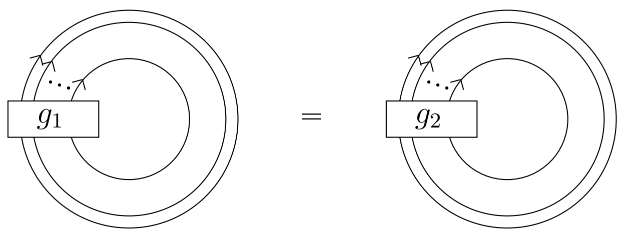
$$\alpha_\mu \mapsto p_\mu^{\#},$$

such that for each n the following diagram commutes:

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) & \xrightarrow{\quad\Psi\quad} & \Lambda^* \\ & {\scriptstyle f_n^{\mathcal{H}'}} \searrow \quad \swarrow {\scriptstyle f_n^{\Lambda^*}} & \\ & Z(\mathbb{C}[S_n]) & \end{array}$$

Proof. It follows from Proposition 3.1.1 that α_μ and $p_\mu^{\#}$ agree as functions on partitions. By Proposition 3.1.2 and the fact that $\{p_k^{\#}\}_{k \geqslant 1}$ is an algebraically independent generating set of Λ^*, the result follows. □

Lemma 3.1.4. *Suppose that $g_1, g_2 \in S_n$ are conjugate Then*



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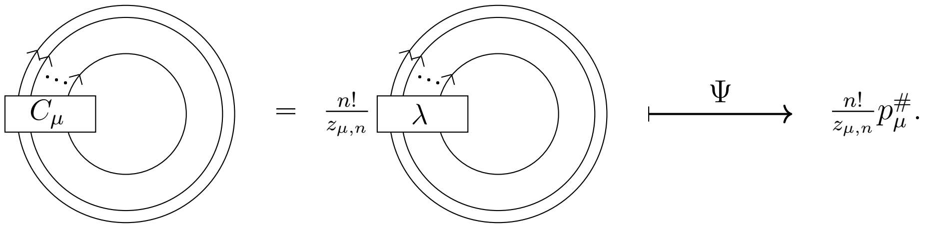
Proof. This is an easy diagrammatic argument which uses the fact that $g_1 = hg_2h^{-1}$ for some $h \in S_n$. Replacing g_1 by hg_2h^{-1}, we slide h around the diagram to cancel it with h^{-1}. □

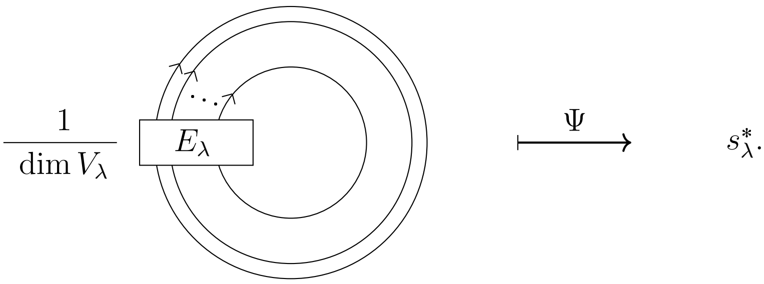
Theorem 3.1.3 and Lemma 3.1.4 then imply the following result.

Lemma 3.1.5. *For $\mu \vdash n$,*



For $\lambda \vdash n$ recall that E_λ is the Young idempotent associated to λ.

Theorem 3.1.6. *The isomorphism Ψ sends*



Proof. Recall that

$$\left(\frac{1}{\dim V_\lambda}\right) E_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{n!} C_\mu,$$

while

$$s^*_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_{\mu,n}} p^\#_\mu.$$

The result then follows from Lemma 3.1.5. □

In the proceeding sequence of results we have established some graphical realisations of some important bases of Λ^*. We now go the other way and describe two important generating sets of the centre Khovanov's Heisenberg category, $\tilde{c}_k$ and c_k, as elements of Λ^*. This description gives an explicit connection between $\mathcal{H}'$ and the transition and co-transition measures of Kerov.

Theorem 3.1.7. *The isomorphism Ψ sends:*

1. $\tilde{c}_k \mapsto \sigma_k \in \Lambda^*$,
2. $c_k \mapsto p_1^{\#}\widehat{\sigma}_k = \widehat{b}_{k+2} \in \Lambda^*$.

Proof. Let $\lambda \vdash n$, then from Proposition 1.5.6 and Proposition 1.6.3 we have

$$[\Psi(\tilde{c}_k)](\lambda) = \widehat{\chi}^\lambda(\mathrm{pr}_n(J_{n+1}^k)) = \sigma_k(\lambda)$$

and

$$[\Psi(c_k)](\lambda) = \widehat{\chi}^\lambda\Big(\sum_{i=1}^n s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i\Big) = p_1^{\#}(\lambda)\widehat{\sigma}_k(\lambda) = \widehat{b}_{k+2}(\lambda).$$

□

Remark 3.1.8. In [FH59], Farahat and Higman used the inductive structure of symmetric groups to construct a $\mathbb{C}$-algebra known as the Farahat-Higman algebra $\mathcal{FH}_\mathbb{C}$ (see also Example 24, Section I.7, [Mac95]). It follows from, for example [IK99], that there is an algebra isomorphism $\mathcal{FH}_\mathbb{C} \cong \Lambda^*$. So in principle all of the appearances of shifted symmetric functions in the previous sections could be rephrased in the language of the Farahat-Higman algebra.

3.2 Involutions on the centre of Khovanov's Heisenberg category

Khovanov [Kho14] describes three involutions on $\mathcal{H}'$. Only one of these acts non-trivially on $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$. We denote this involution by ξ, and it is defined on a diagram $D \in \mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon_2})$ by

$$\xi(D) := (-1)^{c(D)} D,$$

where $c(D)$ is the total number of crossings and dots in a diagram. Consequently, in $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$

$$c_k \overset{\xi}{\longmapsto} (-1)^k c_k,$$
$$\tilde{c}_k \overset{\xi}{\longmapsto} (-1)^k \tilde{c}_k,$$
$$\alpha_k \overset{\xi}{\longmapsto} (-1)^{k-1} \alpha_k.$$

Okounkov and Olshanski [OO97, §4] defined an involution $I : \Lambda^* \to \Lambda^*$ whose action on $f \in \Lambda^*$ is such that for $\lambda \in \mathcal{P}$

$$[I(f)](\lambda) = f(\lambda').$$

In particular,

$$I(s_\lambda^*) = s_{\lambda'}^*,$$
$$I(e_k^*) = h_k^*,$$
$$I(p_k^\#) = (-1)^{k-1} p_k^\#.$$

As a consequence of the isomorphism $\Psi : \mathrm{End}_{\mathcal{H}'}(\mathbf{1}) \cong \Lambda^*$ we therefore have the following proposition.

Proposition 3.2.1. *The involution ξ on $\mathrm{End}_{\mathcal{H}'}(\mathbf{1})$ coincides with the involution I on Λ^*.*

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