Essays on Information and Markets

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This thesis consists of three chapters. Chapter 1 is forthcoming in Economic Theory, and Chapter 2 is a reproduction of (*Klishchuk*, 2015, Journal of Mathematical Economics). Except where otherwise indicated, these chapters are my own original work. Chapter 3, on the other hand, is an early report on the joint work with Professor Rabee Tourky. At this stage, my contribution there consists of Theorems 3–6, Lemmas 1, 2, and 14, as well as articulating the idea.

Bogdan Klishchuk
19 April 2017

To M Y.

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Abstract

This thesis provides new equilibrium existence results for markets with information asymmetries and relates them in a novel way to the role of economic intermediation. Under atomless uncertainty, single market linear price equilibria are known not to exist prevalently even when agents are risk averse expected utility maximizers. The notion of prevalence involves essentially picking an economy at random. Bypassing the nonexistence problem is one of the achievements of the nonlinear price decentralization theory. This thesis contributes by reconciling the nonlinear price decentralization theory to a large extent with certain competitive market structures. We do this in Chapter 1 by defining linear price equilibrium with multiple markets and establishing its existence. Each market has its own price vector (linear functional), and agents' involvement in various markets is heterogeneous. As a result, price differences across markets may prevail in equilibrium. We present an example in which single market linear price equilibrium does not exist but certain corresponding equilibrium with two markets does. Our equilibrium with multiple markets has a more standard economic interpretation than equilibrium with nonlinear prices used in nonlinear decentralization theory. Our framework can potentially accommodate even more nonlinearities if economic intermediaries are explicitly introduced into the model.

Despite the nonexistence problem, single market linear price equilibrium with infinitely many states is still known to exist under restrictive assumptions on the information structure. In Chapter 2, we introduce two new results on the existence of single market linear price (Radner) equilibrium with infinitely many states under economically meaningful conditions. Our first result requires that agents have independent information, while the second assumes that the total endowment of the economy is common knowledge.

In Chapter 3, we explore how economic agents can test the scope of their knowledge and, in particular, the informational content of equilibrium prices under asymmetric information. We show that one can go far in arguing that equilibrium prices tend to be fully revealing.

Contents

Chapter 1

Multiple Markets: New Perspective on Nonlinear Pricing*

1 Introduction

The theory of nonlinear price decentralization provides us with characterizations of Pareto efficiency and limiting core allocations where Walrasian linear prices fail. Our analysis in this paper suggests that the nonlinearities may often be interpreted in terms of linear prices that segment the economy into multiple markets.

In their approach to decentralization, *Aliprantis et al.* (2001) describe a nonlinear price by a family of personalized linear prices indexed by the set of agents. A commodity bundle is priced then at the maximum revenue from distributing the bundle among these individually price-taking agents.

We replace personalized prices with marketwise prices, carefully choosing competitive market structures which have room for linear pricing. We define markets by specifying their participants as well as traded commodity bundles and then segment the economy into multiple overlapping markets. We guarantee marketwise linearity by requiring that an agent either participates in a market fully or is not involved at all. The compatibility of market participation and the agents' heterogeneous consumption sets is of course ensured. As a matter of fact, the differences in otherwise

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exogenous consumption sets may be attributed to the agents' heterogeneous involvement in multiple markets. A standard approach in the literature (Section 2) is to model this situation as the agents' limited participation in a single market. But this entails failures of linear decentralization due to the nonexistence of single market linear price equilibrium necessary for the decentralization of the core. In fact, the nonexistence of single market equilibrium is known to be a prevalent situation in atomless economies with differential information (*Tourky and Yannelis*, 2003; *Podczeck et al.*, 2008). Nonlinear pricing is a way for the decentralization theory to bypass the nonexistence problem. Our approach, on the other hand, affords the existence of linear price equilibrium with multiple markets and a more standard economic interpretation.

An example, where single market linear equilibrium does not exist (Subsection 3.3.3) but equilibrium with two markets does (Subsection 3.3.2), clarifies our ideas. It contrasts nonlinear decentralization with our approach of multiple markets in all of Section 3. There multiple markets emerge not merely as a linear alternative to nonlinear pricing but also as its compatible reinterpretation. While all admissible trades are priced linearly in multiple markets, the induced pricing of commodity bundles agrees with the nonlinear pricing of *Aliprantis et al.* (2001). Sufficient conditions for such consistent segmentability into multiple markets in general are presented and related to the role of economic intermediation in Section 5. It requires a general model of multiple markets, which we develop before in Section 4. This framework can be additionally regarded as an independent model of multiple markets in their own right. The existence of equilibrium with multiple markets for our general model is established in Section 6.

2 Further Remarks on Related Literature

Ever since its inception, standard infinite dimensional theory of value has excluded most small consumption sets pertaining to limited market participation. The situation is especially tight with the lattice commodity spaces that motivated *Mas-Colell* (1986) and *Aliprantis and Brown* (1983). Limited market participation translates into heterogeneity of consumption sets and conflicts with the assumption of their equality initiated by *Mas-Colell* (1986).

One root of limited market participation lies in the agents' lack of information, as in the theoretical differential information economies introduced by *Radner* (1968). Infinite dimensional development of this finite dimensional prototype lagged behind the benchmark common knowledge theory of value in time and especially in scope. Only forty years later did *Podczeck and Yannelis* (2008) allow for infinitely many state independent commodities, but retain the assumption of a finite state space. Only slightly has this assumption been relaxed since then. Such a progress is made by *Hervés-Beloso et al.* (2009), but restricting all information asymmetries to the presence of private signals with only finitely many values. Besides, *Klishchuk* (2015) covers situations where agents have independent information, and also where the total endowment of the economy is common knowledge. Despite these limited equilibrium existence results, the nonexistence is prevalent in the sense of *Anderson and Zame* (2001) according to *Podczeck et al.* (2008). The problem arises because this literature is confined to single market linear price equilibria. We solve this nonexistence problem by modelling such situations as economies with multiple markets in contrast to nonlinear pricing of *Aliprantis et al.* (2001).

It is important to mention that the nonlinear decentralization theory of *Aliprantis et al.* (2001, 2005) also extends the theory of value beyond vector lattices. Without such structure, the nonexistence problem spreads even to the case of full market participation, as revealed by the example of *Aliprantis et al.* (2004b). Sufficient conditions for the existence of full participation linear price equilibrium in this context are provided by *Aliprantis et al.* (2004a, 2005).

A separate relevant observation is that information asymmetries often coexist with full market participation. This applies, for instance, to the model of *Correia-da Silva and Hervés-Beloso* (2009), Definitions 3.5 and 3.6 in *de Castro et al.* (2011), and a special case of *Angelopoulos and Koutsougeras* (2015).

A compelling sufficient condition for the existence of limited participation linear price equilibrium is seen in Remark 9 of *He and Yannelis* (2016). Their condition requires norm compactness of consumption sets, and we note that many sequence spaces admit appealing norm compact consumption sets (see *Wickstead*, 1975).

Finally, let us mention alternative approaches to nonlinear pricing present within decentralization theory. They allow dispensing with certain convexity assumptions according to *Chavas and Briec* (2012) and *Habte and Mordukhovich* (2011). The latter paper considers economies with public goods. Nonlinear and linear decentralization theory for them is studied at great economic generality by *Graziano* (2007).

3 Example

In the approach of *Aliprantis et al.* (2001), core allocations are decentralized as personalized equilibria. This example compares a personalized equilibrium and an equilibrium with multiple markets in an economy where single market linear decentralization fails. To the extent of featuring this failure (Subsection 3.3.3), the example is based on a sufficient condition and other ideas of *Tourky and Yannelis* (2003).

3.1 Economy

We consider a particular differential information economy in the framework of Section 9.2 in *Aliprantis et al.* (2001). Two agents face exogenous uncertainty. It is described by the probability space $([0,1], \Lambda, \lambda)$, where Λ is the σ-algebra of all Lebesgue measurable subsets of $[0,1]$, and λ is the Lebesgue measure on $\mathbb{R}$. Hereinafter, if x is an element of a vector space K and p is a linear functional on K, then the value of p at x is denoted by $p \cdot x$.

Agent 1 has full information represented by the σ-algebra Λ. The agent's consumption set X_1 is the positive cone of the commodity space $L_1([0,1], \Lambda, \lambda)$. Agent 2 has a coarser information (σ-algebra) Λ_2 generated by the family of intervals

$$I_n = \left[\frac{n}{n+1}, \frac{n+1}{n+2}\right)$$

with $n \in \{0, 1, 2, \dots\}$. The agent's consumption set X_2 is smaller and limited to Λ_2-measurable nonnegative commodity bundles, i.e.

$$X_2 = \{x \in X_1 : x \text{ is } \Lambda_2\text{-measurable}\}.$$

The description of the economy is completed below in Table 1, where $f_1 : [0,1] \to \mathbb{R}$ is defined by $f_1(s) = 1 - s$.

Table 1: Initial endowments and utility functions for Section 3

Agent	Endowment $\omega_i \in X_i$	Utility function $u_i : X_i \to \mathbb{R}$
1	$\omega_1(s) = s$	$u_1(x) = \int_0^1 f_1(s)\, x(s)\, ds$
2	$\omega_2(s) = 1$	$u_2(x) = \int_0^1 x(s)\, ds$

3.2 Multiple Markets versus Personalized Equilibria

To describe a typical nonlinear price, let $p = (p_1, p_2)$ be a vector of personalized prices both of which are continuous linear functionals on the commodity space. The induced possibly nonlinear price is the real function ψ_p that maps each nonnegative commodity bundle x to its value

$$\psi_p \cdot x = \sup \{p_1 \cdot y_1 + p_2 \cdot y_2 : y \in X_1 \times X_2, y_1 + y_2 \leq x\}.$$

Given an $(x_1, x_2) \in X_1 \times X_2$ which is an allocation (satisfies $x_1 + x_2 \leq \omega_1 + \omega_2$), the vector (x_1, x_2, p_1, p_2) is said to be a *personalized equilibrium* if

(a) for all i and $x \in X_i$, we have $u_i(x) > u_i(x_i) \implies \psi_p \cdot x > \psi_p \cdot x_i$,

(b) for all $\alpha \in \mathbb{R}^2_+$, we have $\psi_p \cdot (\alpha_1 \omega_1 + \alpha_2 \omega_2) \leq \alpha_1 \psi_p \cdot x_1 + \alpha_2 \psi_p \cdot x_2$, and

(c) $\psi_p \cdot (\omega_1 + \omega_2) > 0$.

In Subsection 3.3 we find such a personalized equilibrium (x_1, x_2, p_1, p_2), as given in (1) there, and describe it in terms of two markets. Any commodity bundle can be traded in market 1, but only fully informed agents participate in it. By contrast, market 2 accepts only Λ_2-measurable bundles but is open to all agents. Apart from these participation requirements, agents are only constrained by their initial endowments and marketwise prices. We find, in a sense, a market-clearing vector $p' = (p'_1, p'_2) \neq p$ of marketwise prices, which are continuous linear functionals on the commodity space as well. Agents sell their initial endowments in different markets so as to maximize revenue and make utility-maximizing purchases in those markets. Agent 1 maximizes revenue by selling a particular Λ_2-measurable bundle $\omega_{12} \in [0, \omega_1]$ in market 2 and the remainder $\omega_{11} = \omega_1 - \omega_{12}$ in market 1. Agent 2 participates and sells the initial endowment ω_2 only in market 2. The corresponding revenues and the marketwise prices determine the agents' budget sets. The consumption x_i of every agent i in the personalized equilibrium turns out to coincide with the agent's total utility-maximizing purchases in different markets. Thus, we obtain a competitive equilibrium description of the personalized equilibrium allocation (x_1, x_2). Since such allocations belong to the core (*Aliprantis et al.*, 2001, Theorem 7.5(3)), this example sets side by side the two approaches to decentralization.

Notice that $\psi_{p'}$ assigns to each nonnegative commodity bundle the maximum revenue from distributing the bundle among these two markets. Even though below we find $p' \neq p$, they are such that $\psi_{p'} = \psi_p$. Thus, we are able to reinterpret nonlinearities of ψ_p in terms of multiple markets.

3.3 Technicalities

Here we confirm our last series of statements as well as the failure of single market linear decentralization. We view prices p_i and p'_i as elements of $L_\infty([0,1], \Lambda, \lambda)$ so that corresponding values of a commodity bundle x are

$$p_i \cdot x = \int_0^1 p_i(s)\, x(s)\, ds$$

and $p'_i \cdot x$ defined analogously. A preliminary step is to decompose ω_1 into

$$\omega_{12} = \sum_{n=0}^{\infty} \frac{n}{n+1} \chi_{I_n} \in X_2 \text{ and } \omega_{11} = \omega_1 - \omega_{12} \in X_1,$$

as illustrated in Figure 1[1].

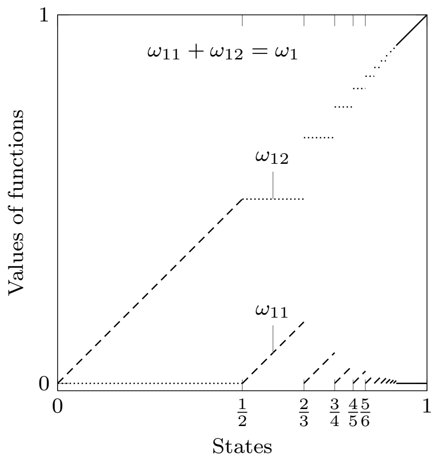

Figure 1: Decomposition of ω_1

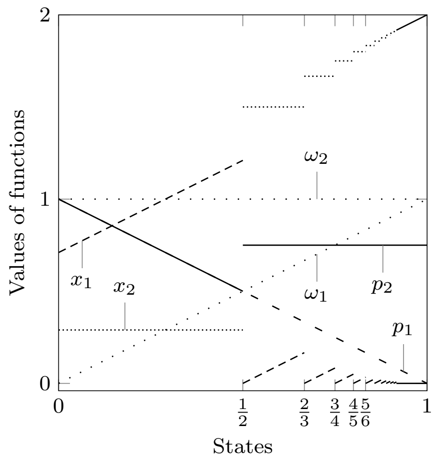

Figure 2: Personalized equilibrium

3.3.1 Personalized Equilibrium

We show in the next paragraph that a personalized equilibrium is obtained, as illustrated in Figure 2, by posing

$$x_1 = \omega_{11} + \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})}, \qquad x_2 = \omega_2 - \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} + \omega_{12},$$

[1]Figures 1–3 may be helpful rather as quick reminders of how various functions are defined in our example than as direct aids to reading.

$$p_1 = f_1, \qquad \text{and} \qquad p_2 = \chi_{[0,\frac{1}{2})} p_1 + \frac{3}{4}\chi_{[\frac{1}{2},1]}. \tag{1}$$

It is first useful to note that $4 - \pi^2/3 = 2\int_0^1 \omega_{12}(s)\,ds$ as confirmed below given that the sum $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ is just the Riemann zeta function evaluated at 2 (see, e.g., *Finch*, 2003):

$$\begin{aligned}
\int_0^1 \omega_{12}(s)\,ds &= \int_0^1 \sum_{n=0}^{\infty} \frac{n}{n+1}\chi_{I_n}(s)\,ds \\
&= \sum_{n=0}^{\infty} \int_{I_n} \frac{n}{n+1} ds \\
&= \sum_{n=0}^{\infty} \frac{n}{n+1} \int_{I_n} ds \\
&= \sum_{n=0}^{\infty} \frac{n}{n+1} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(n+1)^2}{(n+2)^2} - \frac{n^2}{(n+1)^2} - \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\sum_{n=0}^{N} \left(\frac{(n+1)^2}{(n+2)^2} - \frac{n^2}{(n+1)^2} \right) - \sum_{n=0}^{N} \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\frac{(N+1)^2}{(N+2)^2} - \sum_{n=0}^{N} \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\frac{(N+1)^2}{(N+2)^2} - \sum_{n=1}^{N} \frac{1}{n^2} + 1 - \frac{1}{(N+1)^2} - \frac{1}{(N+2)^2} \right) \\
&= 2 - \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 2 - \frac{1}{6}\pi^2.
\end{aligned}$$

Clearly, the pair $(x_1, x_2) \in X_1 \times X_2$ is an allocation. To see that condition (a) in the definition of personalized equilibrium is satisfied, notice that

$$\psi_p \cdot x \geq p_i \cdot x > p_i \cdot x_i = \psi_p \cdot x_i. \tag{2}$$

We verify condition (b) by demonstrating that for all $\alpha \in \mathbb{R}_+^2$ and $y \in X_1 \times X_2$ such that $y_1 + y_2 \leq \alpha_1\omega_1 + \alpha_2\omega_2$ we have $p_1 \cdot y_1 + p_2 \cdot y_2 \leq \alpha_1 p_1 \cdot x_1 + \alpha_2 p_2 \cdot x_2$:

$$\alpha_1 p_1 \cdot x_1 + \alpha_2 p_2 \cdot x_2 = \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 p_1 \cdot \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} + \alpha_2 p_2 \cdot x_2$$

$$= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 \frac{3}{8}\left(4 - \frac{1}{3}\pi^2\right) + \alpha_2 p_2 \cdot x_2$$
$$= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 \frac{3}{4}\int_0^1 \omega_{12}(s)\, ds + \alpha_2 p_2 \cdot x_2$$
$$= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 p_2 \cdot \omega_{12} + \alpha_2 p_2 \cdot \omega_2$$
$$= p_1 \cdot (\alpha_1 \omega_1 - \alpha_1 \omega_{12}) + p_2 \cdot (\alpha_2 \omega_2 + \alpha_1 \omega_{12})$$
$$\geq p_1 \cdot y_1 + p_2 \cdot y_2.$$

Finally, condition (c) is also satisfied as $\psi_p \cdot (\omega_1 + \omega_2) \geq p_2 \cdot \omega_2 > 0$.

3.3.2 Equilibrium with Multiple Markets

Let us describe the above personalized equilibrium in terms of two markets as outlined in the last two paragraphs of Subsection 3.2. Below we explain and illustrate in Figure 3 how marketwise prices $p'_1 = p_1$ and $p'_2 = (3/4)\,\chi_{[0,1]}$ clear the two markets. This idea is later developed into a general definition of equilibrium with multiple markets in Section 4.

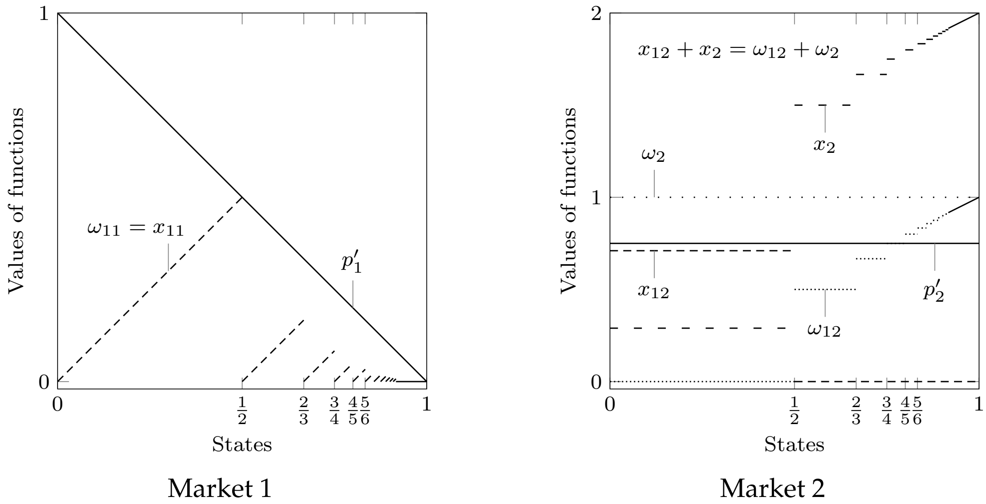

Figure 3: Equilibrium with multiple markets

First observe that all commodity bundles belonging to X_2 are at least as expensive in market 2 as in market 1. Next notice that $\omega_{12} = \sup(X_2 \cap [0, \omega_1])$. Thus we say that selling ω_{12} in market 2 and the remainder ω_{11} in market 1 maximizes revenue of agent 1. The maximum revenue of agent 2 is $p'_2 \cdot \omega_2$.

Let agent 1 buy

$$x_{11} = \omega_{11} \text{ in market 1 and } x_{12} = \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} \text{ in market 2,}$$

and agent 2 buy x_2 in market 2. Observe that both markets are cleared, i.e. $x_{11} = \omega_{11}$ and $x_{12} + x_2 = \omega_{12} + \omega_2$. Next we demonstrate how both agents' preferences are maximized subject to their budget constraints. These purchases are indeed affordable, as $p_1' \cdot x_{11} + p_2' \cdot x_{12} = p_1' \cdot \omega_{11} + p_2' \cdot \omega_{12}$ and $p_2' \cdot x_2 = p_2' \cdot \omega_2$. Moreover, agent 2 cannot afford commodity bundles $y \in X_2$ such that $u_2(y) > u_2(x_2)$, as $p_2' \cdot y > p_2' \cdot x_2 = p_2' \cdot \omega_2$. To see how agent 1's utility is also maximized, consider any $y_1 \in X_1$ and $y_2 \in X_2$ such that $u_1(y_1 + y_2) > u_1(x_{11} + x_{12})$. We have

$$\begin{aligned} p_1' \cdot y_1 + p_2' \cdot y_2 &\geq p_1' \cdot y_1 + p_1' \cdot y_2 = p_1' \cdot (y_1 + y_2) > p_1' \cdot (x_{11} + x_{12}) \\ &= p_1' \cdot x_{11} + p_1' \cdot x_{12} = p_1' \cdot x_{11} + p_2' \cdot x_{12} = p_1' \cdot \omega_{11} + p_2' \cdot \omega_{12}, \end{aligned}$$

meaning that agent 1 cannot afford a better consumption as well.

3.3.3 Failure of Single Market Linear Decentralization

We demonstrate the impossibility to decentralize the core allocation (x_1, x_2) by a single linear price. In a minimal sense, this allocation is said to be decentralized by a price (linear functional) q on the commodity space if $q \neq 0$ and, for all i and $x \in X_i$, we have

$$u_i(x) > u_i(x_i) \implies q \cdot x \geq q \cdot x_i.$$

Supposing the existence of such a q leads to the following contradiction. We start by observing that q must be positive and thus continuous. Using this property and the fact that $x_1(s) > 0$ a.e., we may assume without loss of generality that $q = f_1$. But this is indeed impossible given that $x_2 > 0$.

4 General Model

In the above example of two markets, all restrictions on an agent's participation in a market take the form of the agent's complete exclusion from the market. Whereas this market structure admits a competitive equilibrium, the alternative of a single market with partial participation does not in view of Subsection 3.3.3. That is why we cannot eschew embedding this requirement in our general model, which is presented next.

4.1 Mathematical Preliminaries

A *Banach lattice* is an ordered Banach space L such that every pair $x, y \in L$ has a supremum and an infimum ($\sup\{x, -x\}$ is written $|x|$), as well as satisfies

$$|x| \leq |y| \implies \|x\| \leq \|y\|.$$

For example, the commodity space $L_1([0,1], \Lambda, \lambda)$ in Section 3 is a Banach lattice. Every pair $x, y \in L$ defines the set $[x, y] = \{z \in L : x \leq z \leq y\}$. Sets of this form are called *order intervals*. A *vector sublattice* of a Banach lattice L is a vector subspace closed under pairwise suprema and infima taken in L. For instance, the space $L_1([0,1], \Lambda_2, \lambda)$ is a closed vector sublattice of the commodity space $L_1([0,1], \Lambda, \lambda)$. A Banach lattice is said to be *order complete* if every nonempty subset that is order bounded from above has a supremum. Order completeness is implied by weak compactness of order intervals (*Aliprantis and Border*, 2006, Theorem 9.22). In our example of a Banach lattice, order intervals are weakly compact.

4.2 Economy with Multiple Markets

A *market* is a pair (Z, J) consisting of a set Z of admissible trades and a set J of participants, e.g. $Z = L_1([0,1], \Lambda_2, \lambda)$ and $J = \{1, 2\}$. We consider an *economy* composed of a finite number of markets $(Z_1, J_1), (Z_2, J_2), \ldots, (Z_M, J_M)$. Participants of all markets form the set $I = \cup_{m=1}^{M} J_m$ of *agents*. The *trade spaces* $Z_1, Z_2, \ldots, Z_M$ are subspaces of a *commodity space* L. Their Cartesian product $\boldsymbol{L} = \prod_{m=1}^{M} Z_m$ is called the *market space*. Technically, we assume that

(a) L is a Banach lattice whose order intervals are weakly compact (see remarks preceding Theorem 2),

(b) each Z_m is a closed vector sublattice of L, and

(c) each J_m is nonempty and finite.

Agent i consumes nonnegative bundles from the markets in which she participates. We index these markets by the set $M_i = \{m \in \{1, 2, \ldots, M\} : i \in J_m\}$. Nonnegative bundles traded in market (Z_m, J_m) comprise the positive cone Z_m^+ of the trade space Z_m. The agent's *consumption set* is $X_i = \sum_{m \in M_i} Z_m^+$. The agent has a *consumption preference correspondence* $P_i : X_i \twoheadrightarrow X_i$, with $P_i(x)$ interpreted as the set of bundles strictly preferred to x. The *initial endowment* of the agent is a consumption bundle $\omega_i \in X_i \setminus \{0\}$. We let $\omega = \sum_{i \in I} \omega_i$ denote the total initial endowment.

4.3 Equilibrium

Agent i's *demand set* $\boldsymbol{X}_i$ is obtained by replacing the coordinate spaces of $\boldsymbol{L}$ having $i \in J_m$ with Z_m^+ and the remaining coordinate spaces of $\boldsymbol{L}$ with $\{0\} \subset \boldsymbol{L}$. An element $\boldsymbol{x} \in \boldsymbol{X}_i$ will usually represent the agent's purchases in different markets for the sake of consumption, with $\boldsymbol{x}_m$ interpreted as the bundle bought in the m^{th} market. There is a natural *consumption mapping* $c : \boldsymbol{L} \to L$ defined by $c(\boldsymbol{x}) = \sum_{m=1}^{M} \boldsymbol{x}_m$, and c_i denotes the restriction of c to $\boldsymbol{X}_i$. We define the induced *demand preference correspondence* $\boldsymbol{P}_i : \boldsymbol{X}_i \twoheadrightarrow \boldsymbol{X}_i$ by posing $\boldsymbol{P}_i(\boldsymbol{x}) = c_i^{-1}(P_i(c_i(\boldsymbol{x})))$.

Agent i's *supply set* $\boldsymbol{Y}_i = c_i^{-1}([0, \omega_i])$ captures how the agent's initial endowment can be sold in different markets. A coordinate $\boldsymbol{y}_m$ of an element $\boldsymbol{y} \in \boldsymbol{Y}_i$ is interpreted as the bundle sold in the m^{th} market.

The value of trades $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M) \in \boldsymbol{L}$ is given by a price system prevailing across markets. Formally, a *price system* is a continuous linear functional $\boldsymbol{p}$ on $\boldsymbol{L}$.

We define $\boldsymbol{X} = \prod_{i \in I} \boldsymbol{X}_i$ and $\boldsymbol{Y} = \prod_{i \in I} \boldsymbol{Y}_i$. An *allocation* is a vector $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{Y}$ such that $\sum_{i \in I} \boldsymbol{x}_i \leq \sum_{i \in I} \boldsymbol{y}_i$.

A *quasi-equilibrium* is a triple $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ consisting of an allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and a nonzero price system $\bar{\boldsymbol{p}}$ with the following properties for every agent $i \in I$:

(a) $\boldsymbol{y} \in \boldsymbol{Y}_i$ implies $\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$,

(b) $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, and

(c) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ implies $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$.

A quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is said to be an *equilibrium* if for every agent $i \in I$ and for all $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ we have $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} > \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$. A quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is said to be *nontrivial* if there exist an agent $i \in I$ and an $\boldsymbol{x} \in \boldsymbol{X}_i$ such that $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} < \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$.

5 General Perspective on Reinterpretation

Now we are ready for a general analysis of the ideas that in Section 3 allow us to reinterpret an example of nonlinear pricing in personalized equilibrium. Theorem 1 generalizes such reinterpretation under economically meaningful conditions, and we relate one of them to the role of economic intermediation.

We start by introducing nonlinear pricing independently of our general model of multiple markets (Section 4), and only keep the same *commodity space* L. *Agents* now collectively constitute a finite nonempty set $\hat{I}$. Their *consumption sets* $\hat{X}_i \subset L$, one for each agent $i \in \hat{I}$, enter this model directly as its primitives, in contrast to our more structural model of multiple markets. The *preference correspondence* $\hat{P}_i : \hat{X}_i \twoheadrightarrow \hat{X}_i$ of every agent $i \in \hat{I}$ is analogous to our consumption preference correspondences. Also analogously, the *initial endowment* of the agent is a consumption bundle $\hat{\omega}_i \in \hat{X}_i$. We define $\hat{X} = \prod_{i \in \hat{I}} \hat{X}_i$, let $\hat{\omega} = \sum_{i \in \hat{I}} \hat{\omega}_i$ stand for the total initial endowment, and associate with every commodity bundle $x \in L$ the set

$$\mathcal{A}(x) = \left\{ y \in \hat{X} : \sum_{i \in \hat{I}} y_i \leq x \right\}.$$

Technically, we assume for every agent $i \in \hat{I}$ that

(a) $\hat{X}_i$ is the positive cone of a closed vector sublattice $\hat{Z}_i$ of L, and

(b) $\hat{\omega}_i > 0$.

For our purpose it is convenient to suppose from the outset that a personalized equilibrium exists and then to analyze it. In this equilibrium personalized prices are also continuous linear functionals on the commodity space, i.e. they are elements of the topological dual L' of L. Let $\hat{p} \in (L')^{\hat{I}}$ be the vector of these equilibrium *personalized prices*. They induce a possibly *nonlinear price* $\psi_{\hat{p}} : \sum_{i \in \hat{I}} \hat{X}_i \to \mathbb{R}_+$ by assigning to a commodity bundle x in the domain the value

$$\psi_{\hat{p}} \cdot x = \sup \left\{ \sum_{i \in \hat{I}} \hat{p}_i \cdot y_i : y \in \mathcal{A}(x) \right\}.$$

Simultaneously, agents' choices determine the equilibrium allocation $\hat{x} \in \mathcal{A}(\hat{\omega})$ such that the following conditions of *Aliprantis et al.* (2001) hold:

(a) for all $i \in \hat{I}$, if $x \in \hat{P}_i(\hat{x}_i)$, then $\psi_{\hat{p}} \cdot x > \psi_{\hat{p}} \cdot \hat{x}_i$,

(b) for all $\alpha \in \mathbb{R}_+^{\hat{I}}$, we have $\psi_{\hat{p}} \cdot \sum_{i \in \hat{I}} \alpha_i \hat{\omega}_i \leq \sum_{i \in \hat{I}} \alpha_i \psi_{\hat{p}} \cdot \hat{x}_i$, and

(c) $\psi_{\hat{p}} \cdot \hat{\omega} > 0$.

Let us refer to this *personalized equilibrium* by the pair $(\hat{x}, \hat{p})$.

We say that the personalized equilibrium is *segmentable* if it can be modelled as an equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ of an economy with multiple markets in the sense that

(a) $I = \hat{I}$,

(b) for every agent $i \in I$, we have $X_i = \hat{X}_i$, $P_i = \hat{P}_i$, $\omega_i = \hat{\omega}_i$, as well as $c(\bar{\boldsymbol{x}}_i) = \hat{x}_i$, and

(c) for every commodity bundle x in the domain of the price $\psi_{\hat{p}}$, we have

$$\psi_{\hat{p}} \cdot x = \sup \left\{ \bar{\boldsymbol{p}} \cdot \boldsymbol{x} : \boldsymbol{x} \in c^{-1}([0, x]), \boldsymbol{x} \geq 0 \right\}. \tag{3}$$

Such segmentability is established in Theorem 1 under two conditions, which we next introduce and motivate.

Individual supportability. Looking carefully into the proofs of *Aliprantis et al.* (2001), one finds that their conclusions are stronger than those stated as theorems in the following sense. Their assumptions yield the existence of personalized equilibrium with personalized prices summarizing individual incomes and substitution attitudes (p. 44) via

$$\hat{p}_i \cdot \hat{x}_i = \psi_{\hat{p}} \cdot \hat{x}_i, \text{ and } x \in \hat{P}_i(\hat{x}_i) \implies \hat{p}_i \cdot x > \hat{p}_i \cdot \hat{x}_i. \tag{4}$$

This way, personalized prices are closer to Walrasian, as in the example of Section 3 in view of (2). We say that the personalized equilibrium $(\hat{x}, \hat{p})$ is *individually supporting* if conditions (4) hold for every agent $i \in \hat{I}$.

Bilateral feasibility. To motivate this concept, consider first the benchmark case in which all consumption sets coincide with the positive cone of L and there is no disposal, i.e.

$$\sum_{i \in \hat{I}} \hat{x}_i = \hat{\omega}.$$

Here $\hat{x}$ is bilaterally feasible in the sense that some $z \in \hat{X}^{\hat{I}}$ decomposes individually

$$\hat{\omega}_i = \sum_{j \in \hat{I}} z_{ij} \text{ and } \hat{x}_i = \sum_{j \in \hat{I}} z_{ji} \tag{5}$$

by the Riesz decomposition property (*Aliprantis and Tourky*, 2007, Theorem 1.54). A coordinate z_{ij} is interpreted as a consumption bundle sold by agent i to agent j. In the example of Section 3, the consumption set of agent 2 differs from the positive cone but there exist $z_{12} \in X_1 \cap X_2$, $z_{21} \in X_2 \cap X_1$, $z_{11} \in X_1$, and $z_{22} \in X_2$ with

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} z_{11} + z_{12} \\ z_{21} + z_{22} \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_{11} + z_{21} \\ z_{12} + z_{22} \end{pmatrix}.$$

While in general having $z_{ij} \in X_i \cap X_j$ only states that both the buyer and the seller can consume this traded bundle, it says more with differential information. In this context, the condition ensures that both agents are sufficiently informed to verify ex-post consequences of the ex-ante agreement z_{ij}, due to measurability. The following definition generalizes this property. We say that the personalized equilibrium $(\hat{x}, \hat{p})$ is *bilaterally feasible* if there exist vectors $z_{ij} \in \hat{X}_i \cap \hat{X}_j$, one for each pair $(i, j) \in \hat{I}^2$, satisfying (5) for all $i \in \hat{I}$.

Lest the indispensability of bilateral feasibility be a concern, a comforting observation is that a natural way forward suggests itself in that case. Bilaterally infeasible allocations require intermediation, and modelling intermediation explicitly may prove fruitful. In particular, it may be helpful for explaining the existence of intermediaries in the real world.

Theorem 1. *The personalized equilibrium $(\hat{x}, \hat{p})$ is segmentable if it is individually supporting and bilaterally feasible.*

Proof. We let markets be given by an arbitrary enumeration of the family

$$\left\{ \left(\hat{Z}_i \cap \hat{Z}_j, \{i, j\} \right) : i, j \in \hat{I} \right\}. \tag{6}$$

An allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ such that

$$\sum_{i \in I} \bar{\boldsymbol{x}}_i = \sum_{i \in I} \bar{\boldsymbol{y}}_i \tag{7}$$

is defined by applying to coordinates having $J_m = \{i, j\}$ the formulas $\bar{\boldsymbol{x}}_{im} = z_{ji}$ and $\bar{\boldsymbol{y}}_{im} = z_{ij}$. A price system $\bar{\boldsymbol{p}}$ is defined by considering the topological dual of each Z_m as a lattice, in which we take the supremum $\bar{\boldsymbol{p}}_m = \bigvee_{i \in J_m} (\hat{p}_i|_{Z_m})$, and letting

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m. \tag{8}$$

If $\boldsymbol{x} \geq 0$, then we have

$$\begin{aligned}\bar{\boldsymbol{p}} \cdot \boldsymbol{x} &= \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \sum_{m=1}^{M} \left(\bigvee_{i \in J_m} (\hat{p}_i|_{Z_m}) \right) \cdot \boldsymbol{x}_m \\ &\leq \sum_{m=1}^{M} \psi_{\hat{p}} \cdot \boldsymbol{x}_m \leq \psi_{\hat{p}} \cdot \sum_{m=1}^{M} \boldsymbol{x}_m = \psi_{\hat{p}} \cdot c(\boldsymbol{x}) . \end{aligned} \tag{9}$$

By this fact, the definition of supply set, condition (b) in the definition of personalized equilibrium, and individual supportability, all i and $\boldsymbol{y} \in \boldsymbol{Y}_i$ satisfy

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \psi_{\hat{p}} \cdot c(\boldsymbol{y}) \leq \psi_{\hat{p}} \cdot \hat{\omega}_i \leq \psi_{\hat{p}} \cdot \hat{x}_i = \hat{p}_i \cdot \hat{x}_i. \tag{10}$$

For all i and $\boldsymbol{x} \in \boldsymbol{X}_i$, we additionally obtain

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m \geq \sum_{m=1}^{M} \hat{p}_i \cdot \boldsymbol{x}_m = \hat{p}_i \cdot \sum_{m=1}^{M} \boldsymbol{x}_m = \hat{p}_i \cdot c(\boldsymbol{x}) , \tag{11}$$

and next verify conditions (a)–(c) in the definition of quasi-equilibrium.

(c) Combining (11), individual supportability, and (10) for $\boldsymbol{y} = \bar{\boldsymbol{y}}_i$ yields

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \hat{p}_i \cdot c(\boldsymbol{x}) > \hat{p}_i \cdot \hat{x}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i, \tag{12}$$

as required.

(b) Results (10) and (11) for $\boldsymbol{y} = \bar{\boldsymbol{y}}_i$ and $\boldsymbol{x} = \bar{\boldsymbol{x}}_i$ give us $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, while an application of equation (7) yields $\sum_{i \in I} (\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i) = 0$. It follows for each i that $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, establishing the equilibrium condition.

(a) Starting with the last condition, letting $\boldsymbol{x} = \bar{\boldsymbol{x}}_i$ in (11), and invoking (10), we obtain

$$\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \hat{p}_i \cdot c(\bar{\boldsymbol{x}}_i) = \hat{p}_i \cdot \hat{x}_i \geq \bar{\boldsymbol{p}} \cdot \boldsymbol{y},$$

as desired.

Due to the strict inequality in (12), the quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is actually an equilibrium. Now it only remains to verify formula (3). Firstly, every $y \in \mathcal{A}(x)$ defines an $\boldsymbol{x} \in c^{-1}([0, x])$ by matching coordinates so that $\boldsymbol{x}_m = y_i \geq 0$ if $J_m = \{i\}$ and $\boldsymbol{x}_m = 0$ otherwise. Since $\sum_{i \in I} \hat{p}_i \cdot y_i = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}$, the supremum on the right hand side of equation (3) is greater than or equal to $\psi_{\hat{p}} \cdot x$. The reverse inequality is obtained using (9) and calculating for every $\boldsymbol{x} \in c^{-1}([0, x])$ satisfying $\boldsymbol{x} \geq 0$ that

$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \leq \psi_{\hat{p}} \cdot c\left(\boldsymbol{x}\right) \leq \psi_{\hat{p}} \cdot x$. This yields (3) and completes the proof. □

6 General and Direct Equilibrium Existence Result

A corollary of Theorem 1 is equilibrium existence for economies with multiple markets fitted in a particular fashion to bilaterally feasible personalized equilibria. But operation (6) stands also as one way to model all other economies from the nonlinear decentralization theory in focus of Section 5 in terms of multiple markets. One refinement combines markets having identical trade spaces Z_m, letting their participants (the union of the corresponding sets J_m) redefine the set of participants. Thus our general existence result without feasibility restrictions in Theorem 2 establishes our model as a nonvacuous natural alternative to nonlinear pricing. Besides, our model accommodates economic situations with evident structures of multiple markets, e.g. the world economy with barriers to international trade. Closed economy examples are furnished by market exclusions via eligibility criteria in financial services, license or qualification requirements, or age restrictions.

One could be inclined to misjudge Theorem 2 by viewing each supply set $\boldsymbol{Y}_i$ as a production set of a firm owned privately by agent i. Indeed, our exchange economy with multiple markets may be viewed as a production economy in $\boldsymbol{L}$. Unfortunately, there is no result on the existence of production equilibria that we could apply directly. The reason is that consumption sets in our induced production economy may be thin in the sense that they need not coincide with the positive cone, which also may have an empty interior. Even though consumption sets are allowed to be thin in *Florenzano and Marakulin* (2001) and *Aliprantis et al.* (2006), their properness assumptions are too strong when consumption sets are actually thin. Nevertheless, it is convenient to keep this analogy with production economies in mind when proving the existence of equilibrium with multiple markets.

One technical assumption on the commodity space in the general model of Section 4 is weak compactness of order intervals. Even single market equilibrium existence theorems require this property partially but indispensably, as *Aliprantis et al.* (1987) show in their Example 5.7. They describe an economy which lacks this property but otherwise fits our framework with a single market and satisfies all our remaining existence conditions (Theorem 2). Simply missing weak compactness, this economy does not have an equilibrium, as the results of *Aliprantis et al.* (1987) let us see. When preferences have utility representations, a common alternative assumption is compactness of the individually rational utility set, used recently by *Xanthos* (2014).

The additional assumptions required by Theorem 2, which is stated below, are standard. For instance, Assumptions (a)–(d) are used by *Aliprantis et al.* (2001). Assumption (e) is satisfied if P_i has open values, which is also assumed by *Aliprantis et al.* (2001). Assumption (f) is a version of the properness condition introduced by *Tourky* (1998). We also remark that a nontrivial quasi-equilibrium is an equilibrium if irreducibility assumptions are satisfied (see *Florenzano*, 2003, Section 5.3.6).

Theorem 2. *There exists a nontrivial quasi-equilibrium if the following conditions hold for every agent $i \in I$:*

(a) *P_i is irreflexive, i.e. $x \notin P_i(x)$ for all $x \in X_i$;*

(b) *P_i is convex-valued, i.e. $P_i(x)$ is a convex set for all $x \in X_i$;*

(c) *P_i is monotone, i.e. $x + z \in P_i(x)$ for all $x \in X_i$ and $z \in X_i \setminus \{0\}$;*

(d) *P_i has weakly open lower sections, i.e. $\{z \in X_i : x \in P_i(z)\}$ is weakly open in X_i for all $x \in X_i$;*

(e) *P_i is "continuous" in the sense that $x \in X_i$, $z \in P_i(x)$, and $z' \in X_i$ implies $\alpha z + (1-\alpha) z' \in P_i(x)$ for some scalar $\alpha \in [0, 1)$;*

(f) *P_i is proper in the sense that there exists a convex-valued correspondence $\hat{P}_i : X_i \twoheadrightarrow L$ such that for all $x \in X_i$*

   (i) *$x + \omega_i$ is an interior point of $\hat{P}_i(x)$ and*

   (ii) *$\hat{P}_i(x) \cap X_i = P_i(x)$.*

Proof of Theorem 2

Lemma 1. *For every agent $i \in I$, the supply set $\boldsymbol{Y}_i$ has a supremum $\boldsymbol{u}_i$ in $\boldsymbol{L}$.*

Proof. Due to order completeness of L, the set $\boldsymbol{Y}_i$ has a supremum $\boldsymbol{u}_i \in L^M$. We prove the lemma by showing that $\boldsymbol{u}_i \in \boldsymbol{L}$.

For every m, let $Y_m = Z_m \cap [0, \omega_i]$. Notice that $\boldsymbol{u}_{im} = \sup Y_m \in L$ if $i \in J_m$ and $\boldsymbol{u}_{im} = 0$ otherwise. It suffices to show that $\boldsymbol{u}_{im} \in Z_m$ if $i \in J_m$. Observe that Y_m is directed by the order relation $\geq$ of L, making the identity function on Y_m a net in $[0, \omega_i]$. Since this order interval is weakly compact, the net has a subnet $\{y_\alpha\}_{\alpha \in D}$ which converges weakly to some $y \in [0, \omega_i]$. Since Z_m is closed and convex, it is weakly closed as well, and it follows that $y \in Z_m \cap [0, \omega_i]$. This also means that $y \leq \boldsymbol{u}_{im}$, and we complete the proof by demonstrating the reverse inequality $y \geq \boldsymbol{u}_{im}$. Let us denote the direction on D by $\succeq$. Consider another binary relation $\succsim$ on D defined by posing $\alpha \succsim \beta$ if and only if $\alpha \succeq \beta$ and $y_\alpha \geq y_\beta$. This binary relation $\succsim$ inherits reflexivity and transitivity from $\succeq$ and $\geq$. Thus $\succsim$ is also a direction provided that every pair $\alpha, \beta \in D$ has an upper bound in $(D, \succsim)$. Such an upper bound γ can be constructed as follows: let $z = y_\alpha \vee y_\beta \in Y_m$, using the definition of subnet take any $\delta_0 \in D$ such that $\delta \succeq \delta_0$ implies $y_\delta \geq z$, and let γ be an upper bound of $\{\alpha, \beta, \delta_0\}$ in $(D, \succeq)$. Now consider the net $\{y_\alpha\}_{\alpha \in (D, \succsim)}$, which also converges weakly to y. Moreover, this net is increasing, i.e. $\alpha \succsim \beta$ implies $y_\alpha \geq y_\beta$. Our last two observations reveal that $y = \sup \{y_\alpha : \alpha \in D\}$ (*Aliprantis and Tourky*, 2007, part (4) of Lemma 2.3). Now the definition of subnet yields for each $z \in Y_m$ some $\alpha \in D$ such that $y \geq y_\alpha \geq z$. This confirms that $y \geq \boldsymbol{u}_{im}$. □

For each i, let $\boldsymbol{u}_i \geq 0$ be the supremum given by Lemma 1. We define $\boldsymbol{u} = \sum_{i \in I} \boldsymbol{u}_i \in \boldsymbol{L}$, take the order interval $[-\boldsymbol{u}, \boldsymbol{u}]$ in $\boldsymbol{L}$, let $\boldsymbol{K} = \cup_{n=1}^{\infty} n\,[-\boldsymbol{u}, \boldsymbol{u}]$ be the principal ideal generated by $\boldsymbol{u}$ in $\boldsymbol{L}$, and $\boldsymbol{K}_+ = \{\boldsymbol{x} \in \boldsymbol{K} : \boldsymbol{x} \geq 0\}$.

At this stage, let us view $\boldsymbol{K}$ as the commodity space of the production economy which we now construct by restricting our economy with multiple markets. Agents in the production economy are the same as in the economy with multiple markets. For each i, the restricted demand set $\boldsymbol{X}_i \cap \boldsymbol{K}$ is viewed as the agent's consumption set, which is the domain of the agent's preference correspondence $\boldsymbol{x} \mapsto \boldsymbol{P}_i(\boldsymbol{x}) \cap \boldsymbol{K}$. The agent's initial endowment is $0 \in \boldsymbol{X}_i \cap \boldsymbol{K}$ but the agent possesses a private firm whose production set is $\boldsymbol{Y}_i \subset \boldsymbol{K}$. We let the price space be the topological dual of $(\boldsymbol{K}, \|\cdot\|_{\boldsymbol{u}})$, where the norm $\|\cdot\|_{\boldsymbol{u}}$ on $\boldsymbol{K}$ is defined by

$$\|\boldsymbol{x}\|_{\boldsymbol{u}} = \inf \{\alpha \in \mathbb{R}_{++} : \boldsymbol{x} \in \alpha\,[-\boldsymbol{u}, \boldsymbol{u}]\}$$

(making $\boldsymbol{u}$ an interior point of the positive cone $\boldsymbol{K}_+$). Conveniently, this production economy is a special case of the model in Chapter 5 of *Florenzano* (2003). We combine Propositions 5.2.3 and 5.3.1 there, whose assumptions are easy to verify noting that each i satisfies

$$\emptyset \neq c_i^{-1}(\omega_i) \subset (\boldsymbol{X}_i \cap \boldsymbol{K}) \setminus \{0\},$$

and first obtain a nontrivial quasi-equilibrium in $\boldsymbol{K}$. Namely, we are able to find an allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \boldsymbol{K}^I \times \boldsymbol{K}^I$ and a positive linear functional $\bar{\boldsymbol{q}}$ on $\boldsymbol{K}$ with

$$\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i > 0 \text{ for some } i \tag{13}$$

as well as the following properties for each i:

(a) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{K}$ implies $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$,

(b) $\boldsymbol{y} \in \boldsymbol{Y}_i$ implies $\bar{\boldsymbol{q}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$, and

(c) $\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$.

For each i, define the convex set $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i) = c^{-1}\left(\hat{P}_i(c(\bar{x}_i))\right)$. These sets inherit the following two properties from properness:

(i) elements of $\bar{\boldsymbol{x}}_i + c_i^{-1}(\omega_i)$ belong to the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, and

(ii) $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{X}_i = \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$.

We note that the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$ is convex and define an open convex set

$$\boldsymbol{V}_i = \left\{\alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) : \alpha \in \mathbb{R}_{++}, \boldsymbol{z} \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right)\right\} \subset \boldsymbol{L}.$$

Lemma 2. *For every agent $i \in I$, if we have $\boldsymbol{x} \in (\bar{\boldsymbol{x}}_i + \boldsymbol{V}_i) \cap \boldsymbol{X}_i \cap \boldsymbol{K}$, then $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i$.*

Proof. Write $\boldsymbol{x} = \bar{\boldsymbol{x}}_i + \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)$ for some $\boldsymbol{z} \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right) \cap \boldsymbol{K}$ and $\alpha \in \mathbb{R}_{++}$. Since $\bar{\boldsymbol{x}}_i$ belongs to the closure of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, it follows that $\bar{\boldsymbol{x}}_i + \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right)$ for all scalars $\beta \in (0, 1]$. On the other hand, taking positive parts on both sides of $\bar{\boldsymbol{x}}_i \geq -\alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)$ shows that $\bar{\boldsymbol{x}}_i \geq \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^-$. Let $\beta = \min\{1, \alpha\}$ and observe that

$$\begin{aligned} 0 &\leq \bar{\boldsymbol{x}}_i - \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^- \leq \bar{\boldsymbol{x}}_i - \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^- \\ &\leq \bar{\boldsymbol{x}}_i + \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right) \cap \boldsymbol{X}_i \cap \boldsymbol{K} \subset \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{K}. \end{aligned}$$

Now the restricted quasi-equilibrium properties of $\bar{\boldsymbol{x}}_i$ and $\bar{\boldsymbol{q}}$ yield $\bar{\boldsymbol{q}} \cdot (\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \geq 0$. It follows that $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i$ indeed. □

Lemma 3. *For every agent $i \in I$, there exist a price system $\boldsymbol{p}_i$ and a linear functional $\boldsymbol{p}'_i$ on $\boldsymbol{L}$ such that*

(a) $\boldsymbol{x} \in \bar{\boldsymbol{x}}_i + \boldsymbol{V}_i$ implies $\boldsymbol{p}_i \cdot \boldsymbol{x} \geq \boldsymbol{p}_i \cdot \bar{\boldsymbol{x}}_i$,

(b) $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$ implies $\boldsymbol{p}'_i \cdot \boldsymbol{x} \geq \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$, and

(c) $\boldsymbol{p}_i \cdot \boldsymbol{x} + \boldsymbol{p}'_i \cdot \boldsymbol{x} = \bar{\boldsymbol{q}} \cdot \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \boldsymbol{K}$.

Proof. This result follows from Lemma 2 and the Podczeck's extension lemma (*Podczeck*, 1996, Lemma 2). □

For every i, let $\boldsymbol{p}_i$ be a price system given by Lemma 3. For each m, this $\boldsymbol{p}_i$ induces a continuous linear functional $\boldsymbol{p}_{im}$ on Z_m by matching values so that $\boldsymbol{p}_{im} \cdot x = \boldsymbol{p}_i \cdot \boldsymbol{x}$ when $\boldsymbol{x}_m = x$ and $\boldsymbol{x}_n = 0$ for $n \neq m$. Since the topological dual of Z_m is a lattice, the supremum

$$\bar{\boldsymbol{p}}_m = \bigvee_{i \in J_m} \boldsymbol{p}_{im}$$

is also a continuous linear functional on Z_m. A price system $\bar{\boldsymbol{p}}$ is now defined by formula (8). For all i and $\boldsymbol{x} \in \boldsymbol{X}_i$, we have

$$\boldsymbol{p}_i \cdot \boldsymbol{x} \leq \bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m. \tag{14}$$

Lemma 4. *The following statements are true for every agent* $i \in I$*:*

(a) $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$ *implies* $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \boldsymbol{p}_i \cdot \boldsymbol{x}$, *and*

(b) $\bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) \leq \boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x})$ *for all* $\boldsymbol{x} \in \boldsymbol{X}_i$ *satisfying* $\bar{\boldsymbol{x}}_i \geq \boldsymbol{x}$.

Proof. (a) Since $\bar{\boldsymbol{x}}_i + \bar{\boldsymbol{x}}_i \in \boldsymbol{X}_i \cap \boldsymbol{K}$, part (b) of Lemma 3 yields $\boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i + \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \geq \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$, which implies that $\boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \geq 0$. Applying part (b) of Lemma 3 once again, we see that $\boldsymbol{p}'_i \cdot \boldsymbol{x} \geq \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \geq 0$ for all $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$. Combining this result with part (c) of Lemma 3, we obtain the desired conclusion.

(b) Since $\boldsymbol{x} \in \boldsymbol{K}$, part (b) of Lemma 3 yields $\boldsymbol{p}'_i \cdot \boldsymbol{x} \geq \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$. Now part (c) of Lemma 3 shows that $\boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) - \bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) = \boldsymbol{p}'_i \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) \geq 0$, completing the proof. □

Lemma 5. *For all* $\boldsymbol{x} \in \boldsymbol{K}_+$, *we have* $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \boldsymbol{x}$.

Proof. For each m, we write $\bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m$ to denote the value of $\bar{\boldsymbol{q}}$ at the point $\boldsymbol{y} \in \boldsymbol{K}$ with $\boldsymbol{y}_m = \boldsymbol{x}_m$ and $\boldsymbol{y}_n = 0$ for $n \neq m$. Since $\boldsymbol{x}_m \in Z_m^+$, the Riesz-Kantorovich formula yields

$$\begin{aligned} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m &= \left(\bigvee_{i \in J_m} \boldsymbol{p}_{im} \right) \cdot \boldsymbol{x}_m \\ &= \sup \left\{ \sum_{i \in J_m} \boldsymbol{p}_{im} \cdot z_i : z \in \left(Z_m^+ \right)^{J_m}, \sum_{i \in J_m} z_i = \boldsymbol{x}_m \right\}, \end{aligned} \tag{15}$$

and any z in this supremum has a corresponding $\boldsymbol{z} \in \prod_{i \in J_m} (\boldsymbol{X}_i \cap \boldsymbol{K})$ with $\boldsymbol{p}_i \cdot \boldsymbol{z}_i = \boldsymbol{p}_{im} \cdot z_i$ for all i and $\sum_{i \in J_m} \boldsymbol{z}_i = \boldsymbol{y}$. Using part (a) of Lemma 4, we obtain

$$\sum_{i \in J_m} \boldsymbol{p}_{im} \cdot z_i = \sum_{i \in J_m} \boldsymbol{p}_i \cdot \boldsymbol{z}_i \leq \sum_{i \in J_m} \bar{\boldsymbol{q}} \cdot \boldsymbol{z}_i = \bar{\boldsymbol{q}} \cdot \sum_{i \in J_m} \boldsymbol{z}_i = \bar{\boldsymbol{q}} \cdot \boldsymbol{y} = \bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m.$$

Since this is true for every z in formula (15), we conclude that $\bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m \geq \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m$. Observing that this applies to every m, we complete the proof by the calculation

$$\bar{\boldsymbol{q}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m \geq \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}.$$

□

Lemma 6. *The following statements are true for every agent $i \in I$:*

(a) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ *implies* $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$*,*

(b) $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$*, and*

(c) $\boldsymbol{y} \in \boldsymbol{Y}_i$ *implies* $\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$*.*

Proof. (a) Since $\boldsymbol{x}$ belongs to the closure of the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, part (a) of Lemma 3 implies that $\boldsymbol{p}_i \cdot \boldsymbol{x} \geq \boldsymbol{p}_i \cdot \bar{\boldsymbol{x}}_i$. Defining $\boldsymbol{x}' = \boldsymbol{x} \wedge \bar{\boldsymbol{x}}_i \in \boldsymbol{X}_i \cap \boldsymbol{K}$, we have $\boldsymbol{p}_i \cdot (\boldsymbol{x} - \boldsymbol{x}') \geq \boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}')$. Using this result, Lemma 5, the inequality in (14), and part (b) of Lemma 4, we calculate that

$$\begin{aligned} \bar{\boldsymbol{p}} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) + \bar{\boldsymbol{q}} \cdot \left(\bar{\boldsymbol{x}}_i - \boldsymbol{x}'\right) &\geq \bar{\boldsymbol{p}} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) + \bar{\boldsymbol{p}} \cdot \left(\bar{\boldsymbol{x}}_i - \boldsymbol{x}'\right) = \bar{\boldsymbol{p}} \cdot \left(\boldsymbol{x} - \boldsymbol{x}'\right) \\ &\geq \boldsymbol{p}_i \cdot \left(\boldsymbol{x} - \boldsymbol{x}'\right) \geq \boldsymbol{p}_i \cdot \left(\bar{\boldsymbol{x}}_i - \boldsymbol{x}'\right) = \bar{\boldsymbol{q}} \cdot \left(\bar{\boldsymbol{x}}_i - \boldsymbol{x}'\right). \end{aligned}$$

This yields $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$, as required.

(b) We first use Lemma 5, the restricted quasi-equilibrium properties, part (b) of Lemma 4, and (14) to obtain the inequality

$$\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i = \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}}_i \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i. \tag{16}$$

On the other hand, we utilize part (a) to deduce that $\bar{\boldsymbol{p}}$ is positive, and it follows that $\sum_{i \in I} (\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i - \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i) \geq 0$. Together these inequalities yield the desired result.

(c) Observing that $\boldsymbol{y} \in \boldsymbol{K}_+$, we combine Lemma 5, the restricted quasi-equilibrium properties, inequalities in (16), and part (b) of this lemma as follows:

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i.$$

This completes the proof. □

By property (13) and the right hand side of the equality sign in (16), there exists an agent i with $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i > 0$. This observation and Lemma 6 finalize the proof of the theorem.

Chapter 2

New Conditions for the Existence of Radner Equilibrium with Infinitely Many States*

1 Introduction

In 1968 *Radner* explored how far one could go in applying the theory of competitive equilibrium to the case of differentially informed agents. He tailored a way to model information asymmetries so that the standard notion of Walrasian equilibrium would apply. His economic model has subsequently become known as a differential information economy, and its Walrasian equilibrium is commonly referred to as Radner equilibrium (see Section 2). *Radner* concluded that standard theorems on the existence of Walrasian equilibrium continued to hold, but that referred only to the case of finitely many states of nature and finitely many state-independent commodities (available for consumption in each state), as infinite-dimensional equilibrium theory was only in its infancy back then.

It is now understood that the situation with infinite-dimensional commodity spaces is more subtle. *Podczeck and Yannelis* (2008) established the existence of Radner equilibrium with infinitely many state-independent commodities, but the case of infinitely many states of nature was left behind. *Tourky and Yannelis* (2003) and *Podczeck et al.* (2008) have discovered prevalent non-existence conditions peculiar to the case of infinitely many states even with preferences confined to risk averse expected utility. They utilize the notion of prevalence introduced by *Anderson and Zame* (2001). The basic idea behind this result is that in atomless differential information economies agents

*This research was inspired by two working papers (*Tourky and Yannelis*, 2003; *Podczeck et al.*, 2008) kindly provided to the author by his Ph.D. supervisor at the ANU, Professor Rabee Tourky.

with different priors and information seek to specialize their optimal consumption on a null event, and the duality condition characterizing the existence of equilibrium derived in *Aliprantis et al.* (2004a, 2005) cannot hold.

Radner equilibrium with infinitely many states is nevertheless known to exist under restrictive yet economically meaningful assumptions on the information structure. Such an assumption was found by *Hervés-Beloso et al.* (2009). They suppose that each agent observes a public and a private signal; the public signal may take infinitely many values, but private signals are restricted to take only finitely many values. We introduce two new economically meaningful conditions that guarantee the existence of Radner equilibrium with infinitely many states.

Our first condition (Section 3) requires that agents' information σ-algebras (or signals) are independent. If, in addition, there is only one commodity available for consumption in each state, then there exists a unique Radner equilibrium in which there is no trade. With more commodities per state, however, agents might be willing to trade, and the problem of existence becomes more challenging. In this more general scenario, we also make an assumption that limits the substitutability of one state-independent commodity by others.

Our second condition (Section 4) requires that the total endowment of the economy is common knowledge. This is the same as saying that the total endowment belongs to every agent's informationally constrained consumption set. We also make somewhat unusual assumptions on preferences, but we show that they are implied by standard assumptions if agents exhibit a degree of risk aversion. In particular, this risk aversion is satisfied in the standard case of expected utility with concave Bernoulli functions. We also give an example of non-expected utility preferences satisfying all our assumptions.

Let us briefly mention the importance of infinite state spaces. They arise naturally, for instance, if uncertainty is resolved sequentially over an infinite horizon (see *Shreve*, 2004). In addition, they are often utilized for the sake of mathematical convenience. For example, if one wants to work with continuously distributed random variables (say, agents' signals about the return of a risky asset), then the state space must be infinite. These considerations originally motivated *Bewley* (1972) and other authors to introduce infinite-dimensional commodity spaces into general equilibrium theory.

The existence of Radner equilibrium is an active area of research. Recent contributions include *Xanthos* (2014) and *Yoo* (2013).

2 Model

We have a finite set of *agents* $I = \{1, \ldots, m\}$ who face exogenous *uncertainty* described by a probability space $(S, \mathcal{F}, \mu)$. There is a finite nonempty set C of *state-independent commodities* available for consumption in each state. Agents are differentially informed, which simply means that they have heterogeneous ability to discern events in $\mathcal{F}$. Agent i can only discern events that belong to some sub-σ-algebra $\mathcal{F}_i$ of $\mathcal{F}$. If $\mathcal{G}$ is any sub-σ-algebra of $\mathcal{F}$, we denote the space $(L_1(S, \mathcal{G}, \mu|_{\mathcal{G}}))^C$ by $L_1(\mathcal{G})$ and its positive cone by $L_1^+(\mathcal{G})$. Notice that we can identify points $x \in L_1(\mathcal{G})$ and $y \in L_1(S, \mathcal{G}, \mu|_{\mathcal{G}}, \mathbb{R}^C)$ satisfying $x_c(s) = (y(s))_c$ for all $s \in S$ and $c \in C$. We take $L_1(\mathcal{F})$ as our *commodity space* and let $L_1^+(\mathcal{F}_i)$ be the informationally constrained *consumption set* of agent i. The agent has a *preference correspondence* $P_i : L_1^+(\mathcal{F}_i) \twoheadrightarrow L_1^+(\mathcal{F}_i)$ and an *initial endowment* $\omega_i \in L_1^+(\mathcal{F}_i)$. We let $\omega = \sum_{i=1}^m \omega_i$ denote the total initial endowment.

An *allocation* is a vector $x \in \prod_{i=1}^m L_1^+(\mathcal{F}_i)$ such that $\sum_{i=1}^m x_i = \omega$. A *price system* is an element of $(L_\infty(S, \mathcal{F}, \mu))^C$, the topological dual of $L_1(\mathcal{F})$. Given a price system p, the value of a commodity bundle $x \in L_1(\mathcal{F})$ is simply $p \cdot x = \sum_{c \in C} E(p_c x_c)$. An allocation x is said to be a *Radner equilibrium* if there exists a price system p such that, for all $i \in I$, we have $p \cdot x_i \le p \cdot \omega_i$, and $y \in P_i(x_i)$ implies $p \cdot y > p \cdot \omega_i$.

A remark about the interpretation of the probability measure μ is in order now. Technically, our model would not change if we replaced μ by another measure ν as soon as μ-null sets coincided with ν-null sets. This is because spaces of (equivalence classes of) μ-integrable and ν-integrable random variables are lattice isometric. But concepts of independence and risk aversion, which are of fundamental importance to the theory of choice under uncertainty and are of use in this paper, are not immune to such a change of measures. If two random variables are independent with respect to μ, they are not necessarily independent with respect to ν. If a preference relation is risk averse with respect to μ, it is not necessarily risk averse with respect to ν. To be able to interpret mathematical independence as a reflection of causal independence in the real world, we must suppose that μ is a "true" probability measure. On the other hand, the assumption of risk aversion with respect to μ is hard to justify unless μ is supposed to be our agents' common belief. So we can view μ as a "true" probability measure in Section 3, where we make use of the independence assumption, and as our agents' common belief in Section 4, where risk aversion comes into play.

We use expected values and conditional expectations extensively in the exposition of our results. If $x \in L_1(\mathcal{F})$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, the symbol $E(x)$ denotes the vector $a \in \mathbb{R}^C$ in which $a_c = E(x_c)$ for all $c \in C$, while the notation $E(x|\mathcal{G})$ stands for the element $y \in L_1(\mathcal{G})$ in which $y_c = E(x_c|\mathcal{G})$ for all $c \in C$.

We conclude this section by listing below standard assumptions that would typically be required for an existence proof even without information asymmetries, i.e. when $\mathcal{F}_i = \mathcal{F}_j$ for all $i, j \in I$, as in Section 9.1 of *Aliprantis et al.* (2001). Assumption (A6) is a version of the properness condition introduced by *Tourky* (1998). We will refer to these assumptions later.

(A) The following is true for every $i \in I$ and some $v \in \Pi_{j=1}^m L_1(\mathcal{F}_j)$ satisfying $\sum_{j=1}^m v_j \leq \omega$ and $v_j > 0$ for all $j \in I$.

(1) P_i is *irreflexive*, i.e. $x \notin P_i(x)$ for all $x \in L_1^+(\mathcal{F}_i)$.

(2) P_i is *convex-valued*, i.e. $P_i(x)$ is a convex set for all $x \in L_1^+(\mathcal{F}_i)$.

(3) P_i is *strictly monotone*, i.e. $x \in L_1^+(\mathcal{F}_i)$ implies $x + y \in P_i(x)$ for all $y \in L_1^+(\mathcal{F}_i) \setminus \{0\}$.

(4) P_i has *open values*, i.e. $P_i(x)$ is open in $L_1^+(\mathcal{F}_i)$, relative to a linear topology on $L_1(\mathcal{F})$, for all $x \in L_1^+(\mathcal{F}_i)$.

(5) P_i has *weakly open lower sections*, i.e. for every $z \in L_1^+(\mathcal{F}_i)$ the set $P_i^{-1}(z) = \{y \in L_1^+(\mathcal{F}_i) : z \in P_i(y)\}$ is weakly open in $L_1^+(\mathcal{F}_i)$.

(6) P_i is *proper* in the sense that there exists a convex-valued correspondence $\hat{P}_i : L_1^+(\mathcal{F}_i) \twoheadrightarrow L_1(\mathcal{F})$ such that for each $x \in L_1^+(\mathcal{F}_i)$

(i) $x + v_i$ is an interior point of $\hat{P}_i(x)$ and

(ii) $\hat{P}_i(x) \cap L_1^+(\mathcal{F}_i) = P_i(x)$.

3 Independent Information

In this section we establish the existence of Radner equilibrium when agents' information σ-algebras are independent. We consider the case of a single commodity per state separately first. In this scenario we obtain the existence of a unique Radner equilibrium in which there is no trade. This result is in accord with *Koutsougeras and Yannelis* (1993), who prove that only the initial allocation belongs to the private core when agents have independent information[1]. However, our main contribution is to show that the initial allocation can actually be supported by some price system. We cannot

[1]The author thanks Professor Nicholas Yannelis for bringing this result to his attention.

resort to the second welfare theorem or the converse part of the core equivalence theorem to obtain supporting prices. *Tourky and Yannelis* (2003) and *Podczeck et al.* (2008) demonstrate that these theorems do not hold for differential information economies with infinitely many states.

With a single commodity per state, we require only pairwise independence of agents' information. This assumption is expressed formally in (B) below.

Assumption (C), with a single commodity per state and monotonicity (A3), is satisfied whenever in the single-agent economy corresponding to each agent i we can find a Walrasian equilibrium. Technically, this assumption requires the strict separation of the sets $\{\omega_i\}$ and $P_i(\omega_i)$, which need not be convex, by a normalized continuous linear functional. This is implied by a wide variety of conditions on preferences. One set of sufficient conditions for Assumption (C) is given by Assumption (A) with v_i replaced by ω_i (*Aliprantis et al.*, 2001, Corollary 9.2).

Assumption (D) is used in the proof of uniqueness only. This monotonicity condition is implied by strict monotonicity as stated in (A3).

(B) *Pairwise independence:* $\mu\left(F_i \cap F_j\right) = \mu\left(F_i\right)\mu\left(F_j\right)$ for any choice of $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$, for all $i, j \in I$ such that $i \neq j$.

(C) *Separation:* there exist price systems p_i, for each $i \in I$, such that $y \in P_i(\omega_i)$ implies $p_i \cdot y > p_i \cdot \omega_i$, and $E(p_i) = E\left(p_j\right)$ for all $i, j \in I$.

(D) *Monotonicity:* for every $i \in I$ and for all scalars $\alpha > 0$, if $y \in L_1^+(\mathcal{F}_i)$ is such that $y_c(s) = \omega_{ic}(s) - \alpha$ for almost all $s \in S$ for all $c \in C$, then $\omega_i \in P_i(y)$.

Theorem 1. *Suppose that there is a single commodity per state, i.e. the set C is a singleton. If Assumptions (B) and (C) hold, then the initial allocation $(\omega_1, \ldots, \omega_m)$ is a Radner equilibrium. If, in addition, Assumption (D) holds, then this Radner equilibrium is unique.*

There is no trade because with independent information each agent's net trade must be constant across states (almost everywhere), for otherwise either the agent or the rest of the agents are unable to verify the trade. Since agreeing to a negative net trade contradicts individual rationality (with monotone preferences), and strictly positive net trades are infeasible, net trades must be zero in every individually rational allocation.

When the number of commodities per state is greater than one, then each agent's net trade in each commodity must be still constant across states. However, now a negative net trade in one commodity may be compensated with a positive net trade in another. In other words, the initial allocation need not be a Radner equilibrium. Technically, the normalization requirement in Assumption (C) makes it too strong.

In the case of many commodities per state, we will construct a Radner equilibrium from a personalized equilibrium, supported by a possibly non-linear value function, of *Aliprantis et al.* (2001). Let us define these concept precisely. An *allocation with free disposal* is a vector $x \in \Pi_{i=1}^{m} L_1^+ (\mathcal{F}_i)$ such that $\sum_{i=1}^{m} x_i \leq \omega$. A *personalized price system* is a vector $p = (p_1, \ldots, p_m)$, in which p_i is a price system for all $i \in I$. Every personalized price system p induces a *value function* $\psi_p : \sum_{i=1}^{m} L_1^+ (\mathcal{F}_i) \to \mathbb{R}_+$, which assigns to an element x of the domain the value

$$\psi_p \cdot x = \sup \left\{ \sum_{i=1}^{m} p_i \cdot y_i : y \in \Pi_{i=1}^{m} L_1^+ (\mathcal{F}_i) , \sum_{i=1}^{m} y_i \leq x \right\} .$$

An allocation with free disposal x is said to be a *personalized equilibrium* if there exists a personalized price system p such that

$$y \in P_i (x_i) \implies \psi_p \cdot y > \psi_p \cdot x_i \tag{1}$$

for all $i \in I$, and

$$\psi_p \cdot \sum_{i=1}^{m} \alpha_i \omega_i \leq \sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i \tag{2}$$

for all $\alpha \in \mathbb{R}_+^m$. Finally, it is convenient for us to define

$$\kappa_c = \operatorname{ess\,inf} \omega_c \text{ and } \kappa_{ic} = \operatorname{ess\,inf} \omega_{ic}$$

for every $i \in I$ and $c \in C$.

To ensure the existence of personalized equilibria, we need Assumption (E) stated below. It is an adaptation of assumption (A5) in *Aliprantis et al.* (2001) to our setting. We will utilize some of their results, which we present in the theorem following the statement of the assumption below. The assumption simply requires that the initial endowment of each agent is bounded away from zero in some commodity.

(E) *Boundedness:* for every $i \in I$, there exist a $c \in C$ such that $\kappa_{ic} > 0$.

Theorem 2 (*Aliprantis et al.*, 2001)**.** *If Assumptions (A) and (E) hold, then there exist a personalized equilibrium x and a personalized price system p such that, in addition to (1) and*

(2), the following two properties hold:

$$y \in P_i(x_i) \implies p_i \cdot y > p_i \cdot x_i = \psi_p \cdot x_i, \tag{3}$$

for all $i \in I$, and

$$\psi_p \cdot \omega = \sum_{i=1}^{m} p_i \cdot x_i. \tag{4}$$

We show that a personalized equilibrium x given by this theorem under independence is actually a Radner equilibrium once we limit the substitutability of one state-independent commodity by others. This is ensured by Assumption (F). We also require joint independence of agents' information, as stated in (B′).

(F) *Limited substitutability:* for every $i \in I$ and $c \in C$, if $x \in L_1^+(\mathcal{F}_i)$ is defined by letting $x_c(s) = \omega_{ic}(s) - \kappa_{ic}$ and, otherwise, $x_d(s) = \omega_{id}(s) + \lambda_d$ for some $\lambda_d \in [-\kappa_{id}, \kappa_d]$, then $\omega_i \in P_i(x)$.

(B′) *Independence:* $\mu\left(\bigcap_{i=1}^m F_i\right) = \Pi_{i=1}^m \mu(F_i)$ for all $(F_1, \ldots, F_m) \in \Pi_{i=1}^m \mathcal{F}_i$.

Theorem 3. *If Assumptions (A), (B′), and (F) hold, then there exists a Radner equilibrium.*

Let us present a simple example of Assumption (F) being satisfied. Consider a utility function $U_i : L_1^+(\mathcal{F}_i) \to \mathbb{R}$ defined by letting

$$U_i(x) = -\sum_{d \in C} \int_S e^{-\rho x_d} d\mu$$

for some scalar $\rho > 0$. Suppose that P_i derives from this utility function, i.e. $y \in P_i(x)$ if and only if $U_i(y) > U_i(x)$. Also, suppose that there exists an $F \in \mathcal{F}_i$ such that $\mu(F) > 0$ and $\omega_{id}(s) = \kappa_{id}$ for almost all $s \in S$ and for all $d \in C$. Since U_i is monotone, Assumption (F) holds if and only if it is satisfied when we let $\lambda_d = \kappa_d$ for all d. Consider the corresponding $x \in L_1^+(\mathcal{F}_i)$ for any $c \in C$. Observe that

$$\begin{aligned} U_i(x) &= -\left(\int_F e^{-\rho x_c} d\mu + \int_{S \setminus F} e^{-\rho x_c} d\mu + \sum_{d \neq c} \int_S e^{-\rho x_d} d\mu \right) \\ &\leq -\int_F e^{-\rho x_c} d\mu = -\mu(F). \end{aligned}$$

Also, notice that $U_i(\omega_i) \geq -\sum_{d \in C} e^{-\rho \kappa_{id}}$. Our last two observations show that we have $U_i(\omega_i) > U_i(x)$ if

$$\sum_{d \in C} e^{-\rho\kappa_{id}} < \mu(F).$$

This inequality is satisfied for all sufficiently large ρ provided that $\kappa_{id} > 0$ for all d. In other words, Assumption (F) holds for this particular agent i if $\kappa_{id} > 0$ for all d and ρ is sufficiently large, regardless of other agents' characteristics.

4 Common Knowledge of Total Endowment

In this section we prove that Radner equilibrium exists when the total endowment ω is common knowledge. Formally expressed in (G) below, this assumption holds if and only if ω belongs to the informationally constrained consumption set $L_1(\mathcal{F}_i)$ of every agent i.

(G) *Common knowledge:* ω_c is $\mathcal{F}_i$-measurable for all $i \in I$ and $c \in C$.

To get existence, we also need Assumptions (H1) – (H6), where we let $E(P_i(x)|\mathcal{G}) = \{E(y|\mathcal{G}) : y \in P_i(x)\}$, for all $x \in L_1^+(\mathcal{F}_i)$. These assumptions are somewhat unusual, but Proposition 1 shows that they are implied by standard assumptions listed in (A) provided that the risk aversion condition stated in (H7) holds. The term 'risk averse' is justified, because the conditional expectation operator preserves the mean and never increases the variance of a random variable (*Abramovich and Aliprantis*, 2002, Lemma 5.38).

(H) The following is true for every $i \in I$, some sub-σ-algebra $\mathcal{G}$ of $\bigcap_{j=1}^m \mathcal{F}_j$ such that ω_c is $\mathcal{G}$-measurable for all $c \in C$, and some $v \in (L_1(\mathcal{G}))^m$ satisfying $\sum_{j=1}^m v_j \leq \omega$ and $v_j > 0$ for all $j \in I$.

(1) P_i is *conditionally irreflexive*, i.e. $x \notin E(P_i(x)|\mathcal{G})$ for all $x \in L_1^+(\mathcal{G})$.

(2) P_i is *conditionally convex-valued*, i.e. $E(P_i(x)|\mathcal{G})$ is a convex set for all $x \in L_1^+(\mathcal{G})$.

(3) P_i is *conditionally strictly monotone*, i.e. $x \in L_1^+(\mathcal{G})$ implies $x + y \in E(P_i(x)|\mathcal{G})$ for all $y \in L_1^+(\mathcal{G}) \setminus \{0\}$.

(4) P_i has *open conditional values*, i.e. $E(P_i(x)|\mathcal{G})$ is open in $L_1^+(\mathcal{G})$, relative to a linear topology on $L_1(\mathcal{G})$, for all $x \in L_1^+(\mathcal{G})$.

(5) P_i has *weakly open conditional lower sections*, i.e. for every $x \in L_1^+(\mathcal{G})$ the set $\{y \in L_1^+(\mathcal{G}) : x \in E(P_i(y)|\mathcal{G})\}$ is weakly open in $L_1^+(\mathcal{G})$.

(6) P_i is *conditionally proper* in the sense that there exists a convex-valued correspondence $\tilde{P}_i : L_1^+(\mathcal{G}) \twoheadrightarrow L_1(\mathcal{G})$ such that for each $x \in L_1^+(\mathcal{G})$

(i) $x + v_i$ is an interior point of $\tilde{P}_i(x)$ and

(ii) $\tilde{P}_i(x) \cap L_1^+(\mathcal{G}) = E(P_i(x)|\mathcal{G})$.

(7) P_i is *risk averse* in the sense that $E(P_i(x)|\mathcal{G}) \subset P_i(x)$ for all $x \in L_1^+(\mathcal{G})$.

Proposition 1. *If Assumption (G) is satisfied, then the following statements are true:*

(1) (A1) and (H7) together imply (H1);

(2) (A2) implies (H2);

(3) (A3) implies (H3);

(4) (A4) and (H7) together imply (H4);

(5) (A5) implies (H5);

(6) (A6) with $v \in (L_1(\mathcal{G}))^m$ and (H7) together imply (H6).

Our approach in the following theorem is to find an equilibrium in the projection of our economy into a smaller commodity space, $L_1(\mathcal{G})$. The risk aversion assumption ensures that this projected economy is well-behaved, in the sense of meeting Assumptions (H1) – (H6), and has an equilibrium. We then show that this equilibrium is also a Radner equilibrium of the original economy.

Theorem 4. *If Assumptions (G), (H1) – (H6) hold and $\omega_i > 0$ for all $i \in I$, then there exists a Radner equilibrium.*

Examples

In this subsection we suppose that there is a single commodity per state, i.e. the set C is a singleton. We give examples of preferences that satisfy and violate our assumptions.

Assumption (H7) is satisfied for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}_i$ if P_i has an expected utility representation with a concave Bernoulli function $u_i : \mathbb{R} \to \mathbb{R}$, i.e. $y \in P_i(x)$ if and only if $E(u_i \circ y) > E(u_i \circ x)$. Indeed, Jensen's inequality yields $u_i \circ E(y|\mathcal{G}) \geq E(u_i \circ y|\mathcal{G})$, and consequently

$$E(u_i \circ E(y|\mathcal{G})) \geq E(E(u_i \circ y|\mathcal{G})) = E(u_i \circ y) > E(u_i \circ x). \tag{5}$$

The case of risk averse expected utility is important because even in such a simple setup equilibrium may fail to exist without the common knowledge assumption (*Tourky and Yannelis*, 2003; *Podczeck et al.*, 2008).

More generally, in the next paragraph we will show that Assumption (H7) holds if P_i can be represented by an implicitly separable utility function (*Epstein*, 1986; *Dekel*, 1986). Such a utility function $U_i : L_1^+(\mathcal{F}_i) \to \mathbb{R}$ is defined implicitly by

$$U_i(x) = E(v_i(\cdot, U_i(x)) \circ x) \tag{6}$$

with some $v_i : \mathbb{R}^2 \to \mathbb{R}$ such that $v_i(\cdot, \beta)$ is concave and strictly increasing for all $\beta \in U_i(L_1^+(\mathcal{F}_i))$ and such that $v_i(\alpha, \cdot)$ is decreasing for all $\alpha \in \mathbb{R}_+$. For instance, let $v_i(\alpha, \beta) = -e^{\alpha\beta}$. For each $x \in L_1^+(\mathcal{F}_i)$, the expected value $E(v_i(\cdot, \beta) \circ x)$ is continuous in β on $[-1, 0]$ by the Lebesgue dominated convergence theorem. Thus the intermediate value theorem gives us a $U_i(x) \in \mathbb{R}$ solving equation (6), and it is readily seen that this solution is unique. *This utility function cannot be always represented by expected utility maximization*, i.e. it may be impossible to find a Bernoulli function $u_i : \mathbb{R}_+ \to \mathbb{R}$ such that $U_i(x) > U_i(y)$ if and only if $E(u_i \circ x) > E(u_i \circ y)$. To see this, let $S = \{1, 2, 3\}$, $\mathcal{F}_i = 2^S$, and $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \frac{1}{3}$. In this case $U_i(x)$ solves

$$e^{x_1 U_i(x)} + e^{x_2 U_i(x)} + e^{x_3 U_i(x)} + 3U_i(x) = 0. \tag{7}$$

Differentiating the left-hand side of this equation with respect to $U_i(x)$ yields

$$x_1 e^{x_1 U_i(x)} + x_2 e^{x_2 U_i(x)} + x_3 e^{x_3 U_i(x)} + 3 > 0.$$

Thus the implicit function theorem allows us computing

$$\frac{\partial U_i(x)}{\partial x_3} = -\frac{U_i(x)\, e^{x_3 U_i(x)}}{x_1 e^{x_1 U_i(x)} + x_2 e^{x_2 U_i(x)} + x_3 e^{x_3 U_i(x)} + 3}$$

for all $x \in \mathbb{R}_+^3$. Let $y = (1, 1, 1) \in \mathbb{R}_+^3$ and notice that $U_i(y) = -e^{U_i(y)}$. It is easy to check that $-\frac{3}{5} < U_i(y) < -\frac{14}{25} < -\frac{5}{9}$. It follows that $e^{3U_i(y)} > e^{-\frac{9}{5}}$ and that $1 - 2e^{U_i(y)} + e^{-\frac{9}{5}} > 1 - 2e^{-\frac{5}{9}} + e^{-\frac{9}{5}} > 0$. Combining these inequalities yields $1 - 2e^{U_i(y)} + e^{3U_i(y)} > 0$, which implies

$$1 + e^{3U_i(y)} + e^{U_i(y)} + 3U_i(y) > e^{U_i(y)} + e^{U_i(y)} + e^{U_i(y)} + 3U_i(y) = 0.$$

Letting $z_0 = (0,3,1) \in \mathbb{R}^3_+$, the previous inequality and equation (7) imply $U_i(z_0) < U_i(y)$. Since for $z_1 = (1,3,1) \in \mathbb{R}^3_+$ we have $U_i(z_1) > U_i(y)$, the intermediate value theorem gives us a $\gamma \in (0,1)$ such that $z_\gamma = (\gamma,3,1) \in \mathbb{R}^3_+$ is indifferent to y, i.e. $U_i(z_\gamma) = U_i(y)$. We will show that in this case

$$\frac{\partial U_i(y)}{\partial x_3} = \frac{-U_i(y)\, e^{U_i(y)}}{2e^{U_i(y)} + e^{U_i(y)} + 3} \neq \frac{-U_i(y)\, e^{U_i(y)}}{\gamma e^{\gamma U_i(y)} + 3e^{3U_i(y)} + e^{U_i(y)} + 3} = \frac{\partial U_i(z_\gamma)}{\partial x_3},$$

which is inconsistent with the existence of an additively separable representation. Simply notice that $e^{U_i(y)} = -U_i(y) > \frac{14}{25} > 3e^{-\frac{42}{25}} > 3e^{3U_i(y)}$ and that $e^{U_i(y)} > \gamma e^{\gamma U_i(y)}$, because $\gamma e^{\gamma U_i(y)}$ is strictly increasing in γ on $[0,1]$.

If P_i can be represented by an implicitly separable utility function U_i as in (6), then $y \in P_i(x)$ if and only if

$$E\left(v_i(\cdot, U_i(x)) \circ y\right) > E\left(v_i(\cdot, U_i(x)) \circ x\right), \tag{8}$$

for all $x, y \in L_1^+(\mathcal{F}_i)$. Thus Assumption (H7) is implied by the concavity of $v_i(\cdot, U_i(x))$, as in (5), for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}_i$. Assumptions listed in (A) are also satisfied. (A1) holds by the existence of utility representation. (A2) is also implied by the concavity of $v_i(\cdot, U_i(x))$. (A3) is satisfied because $v_i(\cdot, U_i(x))$ is strictly increasing. (A4) holds because the left-hand side of (8) is continuous in y on $L_1^+(\mathcal{F}_i)$ (*Balder and Yannelis*, 1993, Corollary 2.11). (A5) is satisfied because the left-hand side of (8) is weakly upper semicontinuous in y on $L_1^+(\mathcal{F}_i)$ (*Balder and Yannelis*, 1993, Theorem 2.8). Finally, let us show that (A6) also holds for any $v_i \in L_1^+(\mathcal{F}_i)$ such that $v_i > 0$. Let $\beta \in \mathbb{R}_{++}$ be a supergradient of $v_i(\cdot, U_i(x))$ at zero. Now define a concave function $u : \mathbb{R} \to \mathbb{R}$ by letting $u(\alpha) = v_i(\alpha, U_i(x))$ if $\alpha \geq 0$ and $u(\alpha) = \beta\alpha + v_i(0, U_i(x))$ otherwise. A suitable correspondence $\hat{P}_i$ is obtained by letting $\hat{P}_i(x) = \{y \in L_1(\mathcal{F}) : E(u \circ y) > E(u \circ x)\}$.

We must admit that (H7) is a strong assumption, because it may fail when agents are subjective expected utility maximizers with priors $p_i \in L_\infty(S, \mathcal{F}, \mu)$, e.g. $y \in P_i(x)$ if and only if $E(p_i y) > E(p_i x)$. In this case even conditional irreflexivity (H1), which is implied by (H7) according to Proposition 1, may fail. To see this, suppose that $\mathcal{G} = \{\emptyset, S\}$ and we can pick an essentially bounded $p_i \in L_1^+(\mathcal{F}_i)$ having nonzero variance. Since $E\left(p_i^2\right) - E\left(p_i E(p_i|\mathcal{G})\right) = E\left(p_i^2\right) - \left(E(p_i)\right)^2 > 0$, we can find an $x \in L_1^+(\mathcal{F}_i)$ such that $E\left(p_i^2\right) > E(p_i x) > E\left(p_i E(p_i|\mathcal{G})\right)$, and thus $x \in E(P_i(x)|\mathcal{G})$.

5 Proofs

Proof of Theorem 3. First we show that

$$\operatorname{ess\,inf} y + \operatorname{ess\,inf} z = \operatorname{ess\,inf} (y+z), \tag{9}$$

whenever $y, z \in L_1(S, \mathcal{F}, \mu)$ are positive and independent (generating independent σ-algebras). Clearly, we have $\operatorname{ess\,inf} y + \operatorname{ess\,inf} z \leq \operatorname{ess\,inf}(y+z)$. We prove that $\operatorname{ess\,inf} y + \operatorname{ess\,inf} z \geq \operatorname{ess\,inf}(y+z)$ by demonstrating the following: for every scalar $\varepsilon > 0$, there exists an $F \in \mathcal{F}$ such that $\mu(F) > 0$ and $y(s) + z(s) \leq \operatorname{ess\,inf} y + \operatorname{ess\,inf} z + \varepsilon$ for almost all $s \in F$. We can find a set Y, belonging to the sub-σ-algebra $\mathcal{Y}$ of $\mathcal{F}$ generated by y, such that $\mu(Y) > 0$ and $y(s) \leq \operatorname{ess\,inf} y + \frac{\varepsilon}{2}$ for almost all $s \in Y$. Also, we can find a set Z, belonging to the sub-σ-algebra $\mathcal{Z}$ of $\mathcal{F}$ generated by z, such that $\mu(Z) > 0$ and $z(s) \leq \operatorname{ess\,inf} z + \frac{\varepsilon}{2}$ for almost all $s \in Z$. The independence implies that $\mu(Z \cap Y) = \mu(Z)\,\mu(Y) > 0$. Let $F = Z \cap Y$.

We also show that

$$\operatorname{ess\,inf} y \geq \operatorname{ess\,sup} z, \tag{10}$$

whenever $y, z \in L_1(S, \mathcal{F}, \mu)$ are independent and satisfy $y \geq z \geq 0$. Suppose, by way of contradiction, that $\operatorname{ess\,inf} y < \operatorname{ess\,sup} z \leq +\infty$. Pick an $\alpha \in (\operatorname{ess\,inf} y, \operatorname{ess\,sup} z)$. We can find a set Y, belonging to the sub-σ-algebra $\mathcal{Y}$ of $\mathcal{F}$ generated by y, such that $\mu(Y) > 0$ and $y(s) < \alpha$ for almost all $s \in Y$. Also, we can find a set Z, belonging to the sub-σ-algebra $\mathcal{Z}$ of $\mathcal{F}$ generated by z, such that $\mu(Z) > 0$ and $z(s) > \alpha$ for almost all $s \in Z$. Notice that $z(s) > \alpha > y(s)$ for almost all $s \in Z \cap Y$. Since $y \geq z$, it must be the case that $\mu(Z \cap Y) = 0 < \mu(Z)\,\mu(Y)$. This contradicts the independence of y and z.

Now we let L^+ denote the positive cone of $L = \sum_{i=1}^m L_1(\mathcal{F}_i)$ and argue that

$$L^+ \subset \sum_{i=1}^m L_1^+(\mathcal{F}_i). \tag{11}$$

Define $M = \left\{ y \in \prod_{i=1}^m L_1(\mathcal{F}_i) : \sum_{i=1}^m y_i \geq 0 \right\}$, and consider any $y \in M$. We have

$$\sum_{i=1}^m y_i^+ \geq \sum_{i=1}^m y_i^-. \tag{12}$$

Pick any $j \in I$. For every $c \in C$, we can find an $F_c \in \mathcal{F}_j$ such that $y_{jc}^-(s) > 0 = y_{jc}^+(s)$

for almost all $s \in F_c$ and $y^-_{jc}(s) = 0$ for almost all $s \in S \backslash F_c$. In view of (12), this observation shows that

$$\sum_{i \neq j} y^+_i \geq y^-_j . \tag{13}$$

Define $y^+_{-jc} = \sum_{i \neq j} y^+_{ic}$, for $c \in C$, and let $\mathcal{F}_J$ denote the smallest σ-algebra containing $\bigcup_{i \in J} \mathcal{F}_i$, for $J \subset I$. For all $k \in I$ and $J \subset I$, Assumption (B′) implies that $\mathcal{F}_k$ and $\mathcal{F}_{J \backslash \{k\}}$ are independent (*Skorokhod*, 2004, Corollary 3.1.1). It follows that $\sum_{i \in J \backslash \{k\}} y^+_{ic}$ and y^+_{kc} (or y^-_{kc}) are independent. Combining this observation with (9), (10), and (13), we see that

$$\sum_{i \neq j} \operatorname{ess\,inf} y^+_{ic} = \operatorname{ess\,inf} y^+_{-jc} \geq \operatorname{ess\,sup} y^-_{jc},$$

for all $c \in C$. Define a $z \in \Pi^m_{i=1} L_1(\mathcal{F}_i)$ by letting $z_{ic}(s) = y_{ic}(s) - \operatorname{ess\,inf} y^+_{ic}$ for $i \neq j$, and $z_{jc}(s) = y_{jc}(s) + \operatorname{ess\,inf} y^+_{-jc}$, for all $c \in C$ and $s \in S$. Letting $T_j y = z$ defines a transformation $T_j : M \to M$. This transformation satisfies

(i) $\sum^m_{i=1} z_i = \sum^m_{i=1} y_i$,

(ii) $z_j \geq 0$, and

(iii) $y_i \geq 0 \implies z_i \geq 0$, for all $i \in I$,

for all $y \in M$ and $z = T_j y$. Consider any $y \in M$ and notice that $z = T_1 T_2 \ldots T_m y \in \Pi^m_{i=1} L^+_1(\mathcal{F}_i)$ is such that $\sum^m_{i=1} z_i = \sum^m_{i=1} y_i$. This proves our claim.

Next we define $\mathcal{K} = \{\emptyset, S\}$ and argue that each $y \in L$ has a unique decomposition

$$y = E(y|\mathcal{K}) + \sum^m_{i=1} z_i, \tag{14}$$

such that $z_i \in L_1(\mathcal{F}_i)$ and $E(z_i) = 0$ for each i. A quick thought confirms that at least one such decomposition exists. So let us consider any two decompositions

$$y = E(y|\mathcal{K}) + \sum^m_{i=1} z'_i = E(y|\mathcal{K}) + \sum^m_{i=1} z''_i \tag{15}$$

with $z'_i, z''_i \in L_1(\mathcal{F}_i)$ and $E(z'_i) = E(z''_i) = 0$ for each i. Taking conditional expectations in (15) with respect to any $\mathcal{F}_j$ shows that $z'_j = z''_j$, because $E\left(z_{ic} \middle| \mathcal{F}_j\right)$ is equal to $E(z_{ic}) = 0$ almost everywhere for all $i \neq j$ and for all $c \in C$ due to Assumption (B′). This proves our claim.

Now notice that Assumption (E) holds, for otherwise Assumption (F) would yield $\omega_i \in P_i(\omega_i)$, which contradicts (A1). This means that we can use Theorem 2 to obtain a personalized equilibrium x and a personalized price system p satisfying (2), (3), and (4).

We proceed to demonstrate that x is an allocation, i.e. we have

$$\sum_{i=1}^{m} x_i = \omega. \tag{16}$$

Suppose, by way of contradiction, that x is not an allocation. Since x is an allocation with free disposal, we have $0 < \omega - \sum_{i=1}^{m} x_i \in L^+$. Now (11) yields a $y \in \Pi_{i=1}^{m} L_1^+(\mathcal{F}_i)$ such that

$$\sum_{i=1}^{m} y_i = \omega - \sum_{i=1}^{m} x_i > 0. \tag{17}$$

It must be the case that $y_j > 0$ for some j. Using (3) and Assumption (A3), we see that $p_j \cdot y_j > 0$. Since $y_j + \sum_{i=1}^{m} x_i \leq \omega$, it follows that

$$\psi_p \cdot \omega \geq p_j \cdot y_j + \sum_{i=1}^{m} p_i \cdot x_i > \sum_{i=1}^{m} p_i \cdot x_i,$$

which contradicts (4).

Our next step is to show that

$$\operatorname{ess\,inf} x_{ic} > 0 \tag{18}$$

for all $i \in I$ and $c \in C$. Suppose, by way of contradiction, that $\operatorname{ess\,inf} x_{ic} = 0$ for some i and c. Assumption (B′) implies that $E\left(x_{jd} \middle| \mathcal{F}_i\right)$ is equal to $E\left(x_{jd}\right)$ almost everywhere for all $j \neq i$ and $d \in C$. Now taking conditional expectations with respect to $\mathcal{F}_i$ on both sides of $\sum_{j=1}^{m} x_{jd} = \sum_{j=1}^{m} \omega_{jd}$ shows that

$$x_{id}(s) + \sum_{j \neq i} E\left(x_{jd}\right) = \omega_{id}(s) + \sum_{j \neq i} E\left(\omega_{jd}\right) \tag{19}$$

for almost all $s \in S$. Consequently, we have $x_{id}(s) = \omega_{id}(s) + \lambda_d$ for some $\lambda_d \in \mathbb{R}$. Now $\lambda_d \leq x_{id}(s) \leq \omega(s)$ reveals that $\lambda_d \leq \kappa_d$. On the other hand, we have $0 \leq \operatorname{ess\,inf} x_{id} = \operatorname{ess\,inf} \omega_{id} + \lambda_d$, which implies that $\lambda_d \geq -\kappa_{id}$. Since $\operatorname{ess\,inf} x_{ic} = 0$, we see that actually $\lambda_c = -\kappa_{ic}$. Our last three observations were meant to verify that x_i lies within the orbit of Assumption (F), which implies that $\omega_i \in P_i(x_i)$. This means that x

is not individually rational. However, it must be individually rational by Lemmas 6.2 and 6.4 of *Aliprantis et al.* (2001). This contradiction proves our claim.

Letting K be the subspace of all $y \in L_1(\mathcal{F})$ such that y_c is constant almost everywhere for all $c \in C$, we argue that

$$p_i \cdot y = p_j \cdot y \tag{20}$$

for all $y \in K$ and $i, j \in I$. Suppose, by way of contradiction, that $p_i \cdot y > p_j \cdot y$ for some $y \in K$ and $i, j \in I$. It must be the case that $E(p_{ic}y_c) > E(p_{jc}y_c)$ for some $c \in C$. Clearly, we have $y_c \neq 0$. We may assume that $y_c > 0$. Due to (18), we can pick a scalar $\alpha > 0$ such that $\alpha y_c < x_{jc}$. Now define an allocation z by letting $z_{ic} = x_{ic} + \alpha y_c$, $z_{jc} = x_{jc} - \alpha y_c$, and $z_{kd} = x_{kd}$ for $k \notin \{i, j\}$ and $d \neq c$. The fact that

$$\psi_p \cdot \omega \geq \sum_{k=1}^{m} p_k \cdot z_k > \sum_{k=1}^{m} p_k \cdot x_k$$

contradicts (4).

Now we define a linear functional q' on L by letting

$$q' \cdot y = p_1 \cdot E(y|\mathcal{K}) + \sum_{i=1}^{m} p_i \cdot z_i,$$

where z_i are uniquely chosen as in (14). Using (20), we see that

$$q' \cdot y = p_i \cdot E(y|\mathcal{K}) + p_i \cdot (y - E(y|\mathcal{K})) = p_i \cdot y \tag{21}$$

for all $i \in I$ and $y \in L_1(\mathcal{F}_i)$.

Next we show that q' is (weakly) continuous. Consider a net y^λ in L converging weakly to a point $y \in L$. Consider also the respective decompositions $y^\lambda = E\left(y^\lambda \middle| \mathcal{K}\right) + \sum_{i=1}^{m} z_i^\lambda$ such that $z_i^\lambda \in L_1(\mathcal{F}_i)$ and $E\left(z_i^\lambda\right) = 0$ for each i. Notice that $E\left(y_c^\lambda \middle| \mathcal{K}\right)$ is equal to $E\left(y_c^\lambda\right)$ almost everywhere, for all c and λ, and that $E\left(y_c^\lambda\right)$ converges to $E(y_c)$. These observations imply that $E\left(y^\lambda \middle| \mathcal{K}\right)$ converges weakly to $E(y|\mathcal{K})$. Consequently, the net $z^\lambda = \sum_{i=1}^{m} z_i^\lambda$ also converges weakly to some $z \in L$. Pick any $i \in I$ and a price system q such that q_c is $\mathcal{F}_i$-measurable for all $c \in C$. For every $j \neq i$, we have

$$E\left(q_c z_{jc}^\lambda\right) = E\left(E\left(q_c z_{jc}^\lambda \middle| \mathcal{F}_i\right)\right) = E\left(q_c E\left(z_{jc}^\lambda \middle| \mathcal{F}_i\right)\right) = 0,$$

since $E\left(z_{jc}^{\lambda}\middle|\mathcal{F}_i\right)$ is equal to $E\left(z_{jc}^{\lambda}\right)=0$ almost everywhere due to Assumption (B′). This implies that

$$q\cdot z^{\lambda}=\sum_{j=1}^{m}q\cdot z_j^{\lambda}=q\cdot z_i^{\lambda},$$

and it follows that z_i^{λ} converges weakly to $E\left(z|\mathcal{F}_i\right)$. Now we see that $y=E\left(y|\mathcal{K}\right)+\sum_{i=1}^{m}E\left(z|\mathcal{F}_i\right)$, and $q'\cdot y^{\lambda}=p_1\cdot E\left(y^{\lambda}\middle|\mathcal{K}\right)+\sum_{i=1}^{m}p_i\cdot z_i^{\lambda}$ converges to $q'\cdot y=p_1\cdot E\left(y|\mathcal{K}\right)+\sum_{i=1}^{m}p_i\cdot E\left(z|\mathcal{F}_i\right)$. This proves that q' is indeed continuous.

We obtain a price system q by taking any continuous extension of q' to all of $L_1\left(\mathcal{F}\right)$. Using (3) and (21), we see that

$$y\in P_i\left(x_i\right)\implies q\cdot y>q\cdot x_i, \tag{22}$$

for all $i\in I$.

We complete the proof by showing that

$$q\cdot x_i\leq q\cdot\omega_i,$$

for all $i\in I$. Suppose, by way of contradiction, that $q\cdot x_i>q\cdot\omega_i$ for some i. Using (21), (3), and (2), we also see that

$$q\cdot x_j=p_j\cdot x_j=\psi_p\cdot x_j\geq\psi_p\cdot\omega_j\geq p_j\cdot\omega_j=q\cdot\omega_j,$$

for all $j\in I$. It follows that $\sum_{j=1}^{m}q\cdot x_j>\sum_{j=1}^{m}q\cdot\omega_j$, which contradicts (16).

Proof of Theorem 1. Let p_i, for each i, be price systems given by Assumption (C). Consider the proof of Theorem 3 with $x=\left(\omega_1,\ldots,\omega_m\right)$ and $p=\left(p_1,\ldots,p_m\right)$. Notice that (20) holds because $E\left(p_i\right)=E\left(p_j\right)$ for all $i,j\in I$. This means that we can advance to obtain a price system q satisfying (22). Since $x_i=\omega_i$ for all i, we conclude that x is a Radner equilibrium.

To demonstrate uniqueness, consider an arbitrary Radner equilibrium x supported by a price system p. As in (19), we see that

$$x_i\left(s\right)+\sum_{j\neq i}E\left(x_j\right)=\omega_i\left(s\right)+\sum_{j\neq i}E\left(\omega_j\right)$$

for almost all $s\in S$ for all $i\in I$. Consequently, we have $x_i\left(s\right)=\omega_i\left(s\right)+E\left(x_i\right)-$

$E(\omega_i)$. It must be the case that $E(x_i) - E(\omega_i) \geq 0$, for otherwise Assumption (D) implies that $\omega_i \in P_i(x_i)$, and hence $p \cdot \omega_i > p \cdot \omega_i$. But if $E(x_i) - E(\omega_i) > 0$ for some i, then $E\left(\sum_{j=1}^m x_j\right) > E\left(\sum_{j=1}^m \omega_j\right)$, which is impossible. We conclude that $E(x_i) = E(\omega_i)$, and it follows that $x_i = \omega_i$ for all i.

Proof of Proposition 1. By Assumption (G), a suitable σ-algebra $\mathcal{G}$ exists, e.g. $\mathcal{G} = \bigcap_{j=1}^m \mathcal{F}_j$.

(4) Simply observe that $E(P_i(x)|\mathcal{G}) = P_i(x) \cap L_1^+(\mathcal{G})$.

(5) Similarly, observe that

$$\begin{aligned}\left\{y \in L_1^+(\mathcal{G}) : x \in E(P_i(y)|\mathcal{G})\right\} &= \bigcup_{z \in E(\cdot|\mathcal{G})^{-1}(x)} \left\{y \in L_1^+(\mathcal{G}) : z \in P_i(y)\right\} \\ &= \left(\bigcup_{z \in E(\cdot|\mathcal{G})^{-1}(x)} P_i^{-1}(z)\right) \cap L_1^+(\mathcal{G}).\end{aligned}$$

(6) Let $\tilde{P}_i(x) = \hat{P}_i(x) \cap L_1(\mathcal{G})$. Clearly, $\tilde{P}_i$ is convex-valued, and the point $x + v_i$ belongs to the interior of $\tilde{P}_i(x)$. Also, we have

$$\tilde{P}_i(x) \cap L_1^+(\mathcal{G}) = \hat{P}_i(x) \cap L_1^+(\mathcal{G}) = P_i(x) \cap L_1^+(\mathcal{G}) = E(P_i(x)|\mathcal{G}).$$

Proof of Theorem 4. We will first find an equilibrium in our economy projected into a smaller commodity space, $L_1(\mathcal{G})$. In this economy agents' consumption sets are identical and coincide with the positive cone $X = L_1^+(\mathcal{G})$ of the commodity space. Agent i's preference correspondence is $Q_i : X \twoheadrightarrow X$ defined by $Q_i(x) = E(P_i(x)|\mathcal{G})$, and the agent's initial endowment is $E(\omega_i|\mathcal{G})$. What we have just constructed is a classical Walrasian economy (see *Aliprantis et al.*, 2001, Section 9.1). Using Assumptions (H1) – (H6), utilizing the fact that $\omega_i > 0$ for all i, and invoking Corollary 9.2 of *Aliprantis et al.* (2001), we see that this economy has a Walrasian equilibrium $x \in X^m$. It is supported by a price system p such that p_c is $\mathcal{G}$-measurable for all $c \in C$. Consequently, we have

(i) $\sum_{i=1}^m x_i = \sum_{i=1}^m E(\omega_i|\mathcal{G}) = \omega$,

(ii) $p \cdot x_i = p \cdot E(\omega_i|\mathcal{G}) = p \cdot \omega_i$, for all i, and

(iii) $y \in Q_i(x_i)$ implies $p \cdot y > p \cdot E(\omega_i|\mathcal{G}) = p \cdot \omega_i$, for all i.

The allocation x is in fact a Radner equilibrium, because $y \in P_i(x_i)$ implies $E(y|\mathcal{G}) \in Q_i(x_i)$, and $p \cdot y = p \cdot E(y|\mathcal{G}) > p \cdot \omega_i$.

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Chapter 3

Seeming Genericity of Fully Revealing Equilibrium Pricing

With Rabee Tourky

When economic agents do not know how much they do not know, economic outcomes under uncertainty should arguably be viewed as uncertain themselves. In this paper, we envisage agents trying to test the scope of their knowledge, but we keep our analysis within the limits of standard economic models. Not only is this helpful for evaluating their consistency, but also the discussion benefits from being conducted in a familiar framework.

One source of information in markets lies in prevailing prices, and the arguments that prices reveal full information are not rare in the economics profession. We discuss how equilibrium prices may erroneously appear to reveal full information generically. Traders seek information about states of nature, which are modelled as real numbers. Equilibrium pricing is seemingly fully revealing in the sense that ex post traders have nontrivial information about every event represented by an interval, no matter how small. But full revelation is erroneous because there exist nonnegligible events which remain indiscernible by some traders. Moreover, the probability of such an event can be arbitrarily close to certainty.

1 Economic Model

We consider an equilibrium in a financial market for claims contingent on uncertain events given by a probability space (Ω, Σ, μ). Admissible trades in this market comprise some space $L^p(\mu)$ with $1 < p \leq \infty$, and the positive cone of this ordered vector space is denoted by $L^p_+(\mu)$. The price system here takes the form of a random variable $\bar{p}$ such that for every $x \in L^p(\mu)$ the product $\bar{p}x$ is integrable, and that integral represents the market value of x. This price system is observed by market traders who collectively constitute a finite nonempty set I. Being asymmetrically informed, they have heterogeneous capacities to discern events in Σ. Trader i can only discern events that belong to a particular sub-σ-algebra Σ_i of Σ such that the price system $\bar{p}$ is Σ_i-measurable (as it reveals information). Trader i's total information Σ_i shapes the consumption set of the trader into

$$X_i = \left\{x \in L_+^p(\mu) : x \text{ is } \Sigma_i\text{-measurable}\right\}.$$

Trader i has an associated demand $\bar{x}_i \in X_i$, and we allow for a degree of imbalance between the total demand $\sum_{i \in I} \bar{x}_i$ and the total initial endowment $\bar{e} \in L_+^p(\mu)$.

2 Testing Revelation: Imitative Analysis

We envisage a trading analyst trying to test the scope of information revelation by the price system. The revealed information is the σ-algebra $\sigma(\bar{p})$ induced by $\bar{p}$.

2.1 Big Gap Hypothesis

In a weak sense, incomplete revelation obtains when the revealed information $\sigma(\bar{p})$ is a strict subfamily of the σ-algebra Σ. As a first step, the analyst asks whether the revelation is incomplete in the stronger sense of a certain big gap between $\sigma(\bar{p})$ and Σ. Only if the answer is no can there be a hope for full revelation, and the big gap hypothesis is "easier" to reject.

For the test, the analyst takes a sample of nonnull events, which is a sequence $\{S_n\}$ in Σ such that $\mu(S_n) > 0$ for all n. Stating the analyst's big gap hypothesis requires comparing σ-algebras on the sampled events by certain means of the following definition.

Definition 1. Consider a probability space (Ω, Σ, μ) and a sub-σ-algebra Σ' of Σ. We say that Σ' is *μ-strictly coarser* than Σ if there is a Σ-measurable $f : \Omega \to \mathbb{R}_{++}$ such that for any Σ'-measurable $g : \Omega \to \mathbb{R}$ we have $g(\omega) \neq f(\omega)$ almost everywhere.

If the revealed information $\sigma(\bar{p})$ is μ-strictly coarser than Σ, then the revelation is incomplete, as confirmed by Lemma 1 below. Namely, the price system conceals many events of nonzero probability, and they constitute a lower bound for the gap between $\sigma(\bar{p})$ and Σ.

Lemma 1. *Consider a probability measure μ' on the measurable space (Ω, Σ) and an event E with $\mu(E) > 0$. If a sub-σ-algebra Σ' of Σ is μ'-strictly coarser than Σ, then some event $F \in \Sigma$ with $F \subset E$ and $\mu'(F) > 0$ does not belong to Σ'.*

(H1) For some term S_n of the sample $\{S_n\}$, the revealed information $\sigma(\bar{p})$ is $\mu(\cdot|S_n)$-strictly coarser than Σ.

Now the analyst formulates the big gap hypothesis in the form of (H1). The analyst examines this hypothesis under the assumption that some trader is still locally fully informed if the revelation is incomplete. We say that *trader i is fully informed locally on event $E \in \Sigma$ with $\mu(E) > 0$* if for every event $F \in \Sigma$ there is an event $G \in \Sigma_i$ such that $F \cap E = G \cap E$. The analyst also assumes that the locally fully informed trader has a local prior, which is a probability measure on (Ω, Σ) treating the event E as certain. To specify the space of prior densities, consider the conjugate exponent q of p with $q^{-1} + p^{-1} = 1$, the corresponding space $L^q(\mu(\cdot|E))$, its positive cone $L^q_+(\mu(\cdot|E))$, and let

$$\Delta_E = \left\{x \in L^q_+(\mu(\cdot|E)) : E(x) = 1\right\}.$$

Additionally, the analyst evaluates initial endowments and other random variables locally using the space $L^p(\mu(\cdot|E))$ and its positive cone $L^p_+(\mu(\cdot|E))$. The precise versions of the above and other assumptions of the analyst for testing hypothesis (H1) are listed below in (A9).

(A9) Every term S_n of the sample $\{S_n\}$ for which the revealed information $\sigma(\bar{p})$ is $\mu(\cdot|S_n)$-strictly coarser than Σ satisfies the following conditions:

(a) some trader $i \in I$ is fully informed locally on an event $E_n \in \Sigma$ such that $E_n \subset S_n$, $\mu(E_n) > 0$, and for $\mu(\cdot|E_n)$-almost all $\omega \in \Omega$ we have $\bar{p}(\omega) > 0$;

(b) the trader i has a local prior μ_{in}, some $j \in I \setminus \{i\}$ has a local prior μ_{jn}, and these priors have density functions $\rho_{in} \in \Delta_{E_n}$ and $\rho_{jn} \in \Delta_{E_n}$ respectively with respect to $\mu(\cdot|E_n)$;

(c) the traders i and j have concave continuously differentiable local Bernoulli functions u_{in} and u_{jn} respectively on $\mathbb{R}$ into itself such that for some $a_n, b_n \in \mathbb{R}_{++}$ every $\alpha \in \mathbb{R}_+$ satisfies

$$a_n \leq u'_{in}(\alpha) \leq b_n \text{ and } a_n \leq u'_{jn}(\alpha) \leq b_n;$$

(d) for each trader $k \in \{i, j\}$, the demand $\bar{x}_k$ is locally optimal in the sense that every $x \in X_k$ satisfies in $L^1(\mu(\cdot|E_n))$ the condition that

$$E((u_{kn} \circ x)\rho_{kn}) > E((u_{kn} \circ \bar{x}_k)\rho_{kn}) \implies E(\bar{p}x) \geq E(\bar{p}\bar{x}_k);$$

(e) the demand $\bar{x}_j$ of trader j is locally nonzero, i.e. nonzero in $L^p(\mu(\cdot|E_n))$;

(f) the market is locally cleared in the sense that the total demand $\sum_{k\in I} \bar{x}_k$ and the total initial endowment $\bar{e}$ are equal as elements of $L^p_+(\mu(\cdot|E_n))$;

(g) for each trader $k \in I \setminus \{i\}$, the information Σ_k is locally limited to what $\bar{p}$ reveals in the sense that for every $F \in \Sigma_k$ there is a $G \in \sigma(\bar{p})$ such that $F \cap E_n = G \cap E_n$;

(h) $\sigma(\bar{p})$ includes a countably infinite partition of Ω into events of nonzero conditional probability $\mu(\cdot|E_n)$;

(i) $\mu(\cdot|E_n)$ is separable.

Lacking information about traders, the analyst forms beliefs about negligible sets of priors and of initial endowments. If these vectors were finite dimensional, then a natural notion of negligibility would be that of zero Lebesgue measure. But the analyst faces infinite dimensions and is convinced by *Anderson and Zame* (2001) and references therein that there is no satisfactory infinite dimensional analogue of Lebesgue measure. Fortunately, these authors do develop an economically relevant infinite dimensional analogue of the notion of Lebesgue measure zero, building on *Hunt et al.* (1992). This notion is called shyness, and the analyst considers sets that are shy in the relevant parameter set $L^p_+(\mu(\cdot|E))$ or Δ_E negligible. We say that *hypothesis (H1) is false generically* if, for all terms S_n of $\{S_n\}$ for which $\sigma(\bar{p})$ is $\mu(\cdot|S_n)$-strictly coarser than Σ, (A9)(a–d) is true only if there are sets F and G such that:

(a) F is a shy subset of $\Delta^2_{E_n}$, and G is a shy subset of $L^p_+(\mu(\cdot|E_n))$;

(b) (A9)(e–i) implies that either $(\rho_{jn}, \rho_{in}) \in F$ or $\bar{e}$ as an element of $L^p_+(\mu(\cdot|E_n))$ belongs to G.

Theorem 1. *The big gap hypothesis (H1) is false generically.*

2.2 Seeming Full Revelation Hypothesis

Having rejected the big gap hypothesis, the analyst formulates another hypothesis and interprets it as essentially meaning full revelation. Here the analyst represents states of nature by real numbers, assuming that $\Omega = \mathbb{R}$ with Σ being the Borel σ-algebra on $\mathbb{R}$.

The new hypothesis takes the form of (H2), and the analyst's full revelation interpretation of this hypothesis begins with Lemma 2. By this lemma, hypothesis (H2) implies that every trader is fully informed locally on a subset of any interval of nonzero probability, no matter how small and where located. In this sense, all traders have nontrivial information about every event represented by an interval. The analyst finds it enough to infer full revelation.

(H2) For every interval S with $\mu(S) > 0$, the revealed information $\sigma(\bar{p})$ is not $\mu(\cdot|S)$-strictly coarser than Σ.

Lemma 2. *Consider a Borel probability measure μ on $\mathbb{R}$, an event E with $\mu(E) > 0$, and a sub-σ-algebra Σ' of the Borel σ-algebra. If Σ' is not $\mu(\cdot|E)$-strictly coarser than the Borel σ-algebra, then E has a Borel subset F such that $\mu(F) > 0$ and Σ' generates the Borel σ-algebra of F, i.e.*

$$\mathcal{B}_F = \left\{F \cap G : G \in \Sigma'\right\}.$$

The analyst affirms hypothesis (H2) by Theorem 2 and concludes that the price system $\bar{p}$ is generically fully revealing. We review the validity of this inference in the next section.

Theorem 2. *If $(\Omega, \Sigma) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then there is a sample $\{S_n\}$ in Σ with $\mu(S_n) > 0$ for all n, for which (H1) is false generically but is true unless the seeming full revelation hypothesis (H2) is true.*

3 Limitations of Countable Inference

The confirmation of the seeming full revelation hypothesis (H2) is not enough to conclude that the price system is fully revealing. Theorem 3 tells us that (H2) does not even ensure that there is no big gap between the revealed information $\sigma(\bar{p})$ and the full information, as seen by letting $\Sigma' = \sigma(\bar{p})$ in part (a). Moreover, it says that even a continuum of events of nonzero probability can remain unrevealed if the uncertainty is atomless. Further, this whole continuum of concealed events can be encountered even locally within any event E of nonzero probability. Even the conditional probability $\mu(\cdot|E)$ of these concealed events can be arbitrarily close to certainty.

Theorem 3. *Consider an atomless Borel probability measure μ on $\mathbb{R}$, an event E with $\mu(E) > 0$, and a real number $\alpha \in (0, \mu(E))$. There exists a sub-σ-algebra Σ' of the Borel σ-algebra such that:*

(a) for every real number $\beta \in (0, \alpha)$, some Borel subset F of E with $\mu(F) = \beta$ (event of probability β) does not belong to Σ', even though

(b) for every interval S with $\mu(S) > 0$, the σ-algebra Σ' is not $\mu(\cdot|S)$-strictly coarser than the Borel σ-algebra.

Next Theorem 4 shows that (H2) still does not ensure that there is no big information gap even if we assume that the price system reveals all information (Σ'') of any strictly uninformed trader. Here we think of $\Sigma'' \subset \Sigma' = \sigma(\bar{p})$.

Theorem 4. *Consider a Borel probability measure μ on $\mathbb{R}$ and a σ-algebra Σ'' which is included in and μ-strictly coarser than the Borel σ-algebra but such that the restricted probability space $(\mathbb{R}, \Sigma'', \mu|_{\Sigma''})$ is atomless. For every event $E \in \Sigma''$ with $\mu(E) > 0$, there exists a sub-σ-algebra Σ' of the Borel σ-algebra with $\Sigma'' \subset \Sigma'$ such that:*

(a) some Borel subset F of E with $\mu(F) > 0$ (event of nonzero probability) does not belong to Σ', even though

(b) for every interval S with $\mu(S) > 0$, the σ-algebra Σ' is not $\mu(\cdot|S)$-strictly coarser than the Borel σ-algebra.

Analogous conclusions for more general probability spaces are provided by Theorems 5 and 6 below. They tell us that the rejection of the big gap hypothesis based on such a countable inference is not enough to ensure that there is no big gap.

Theorem 5. *Let (Ω, Σ, μ) be a probability space with a sub-σ-algebra Σ'' of Σ such that the restriction $(\Omega, \Sigma'', \mu|_{\Sigma''})$ is atomless. Consider a sequence $\{S_n\}$ in Σ, an event $E \in \Sigma''$ with $\mu(E) > 0$, and a real number $\alpha \in (0, \mu(E))$. There exists a sub-σ-algebra Σ' of Σ such that:*

(a) for every real number $\beta \in (0, \alpha)$, some event $F \in \Sigma''$ with $F \subset E$ and $\mu(F) = \beta$ does not belong to Σ', even though

(b) for every index n with $\mu(S_n) > 0$, the σ-algebra Σ' is not $\mu(\cdot|S_n)$-strictly coarser than Σ.

Theorem 6. *Let (Ω, Σ, μ) be a probability space with a sub-σ-algebra Σ'' of Σ such that Σ'' is μ-strictly coarser than Σ and the restriction $(\Omega, \Sigma'', \mu|_{\Sigma''})$ is atomless. Consider a sequence $\{S_n\}$ in Σ and an event $E \in \Sigma''$ with $\mu(E) > 0$. There exists a σ-algebra Σ' with $\Sigma'' \subset \Sigma' \subset \Sigma$ such that:*

(a) *some event $F \in \Sigma$ with $F \subset E$ and $\mu(F) > 0$ does not belong to Σ', even though*

(b) *for every index n with $\mu(S_n) > 0$, the σ-algebra Σ' is not $\mu(\cdot|S_n)$-strictly coarser than Σ.*

4 New Facts about Shyness

Here we complement the results of *Anderson and Zame* (2001) with new facts about shyness. They are needed for our proofs and may be useful in other applications as well. The notation in this section is independent of the preceding ones.

Lemma 3. *Consider a Hausdorff topological vector space X, a completely metrizable convex subset C of X, and a universally measurable subset E of X such that $E \subset C$. Let T be a continuous linear projection on X. If $T(C)$ is a completely metrizable subset of C in the topology of $T(X)$ and $T(E)$ is both universally measurable in $T(X)$ and shy in $T(C)$, then E is shy in C.*

Proof. Consider any $c \in T(C)$, any $\delta \in \mathbb{R}_{++}$, and any neighborhood W of 0 in X. Since $T(E)$ is shy in $T(C)$ as a subset of $T(X)$, there is a regular Borel probability measure τ' on $T(X)$ with compact support such that

$$\operatorname{supp} \tau' \subset (\delta T(C) + (1-\delta)c) \cap ((W \cap T(X)) + c)$$

and every $x \in T(X)$ satisfies $\tau'(T(E) + x) = 0$. Define a Borel probability measure τ on X by $\tau(F) = \tau'(F \cap T(X))$. Since X is Hausdorff and τ inherits tightness from τ', the measure τ is itself regular (*Aliprantis and Border*, 2006, Theorem 12.4). Since $\operatorname{supp} \tau = \operatorname{supp} \tau'$, the support of τ is compact and

$$\operatorname{supp} \tau \subset (\delta C + (1-\delta)c) \cap (W + c).$$

Since every $x \in X$ satisfies

$$(E + x) \cap T(X) \subset T(E + x) \subset T(E) + T(x),$$

we have

$$0 \leq \tau(E + x) \leq \tau'(T(E) + T(x)) = 0.$$

We have proved that E is shy in C at $c \in C$, and thus E is shy in C. $\square$

Lemma 4. *Consider a real number $q \geq 1$, the space l^q, and its positive cone l^q_+. There exists a Borel probability measure τ on l^q with compact support such that:*

(a) $\operatorname{supp}\tau \subset l^q_+ \setminus \{0\}$*;*

(b) for every $x \in l^q$ and its set of smaller vectors $S(x) = \{y \in l^q : y \leq x\}$, we have $\tau(S(x)) = 0$.

Proof. Consider a bijection B on $\mathbb{N}$ onto the set

$$\bigcup_{i=1}^{\infty} \left(\{i\} \times \left\{0, 1, \ldots, 2^i - 1\right\}\right) \subset \mathbb{R}^2.$$

Now let $I = [0, 1)$, and define a Borel measurable function $\phi : I \to l^q$ by

$$\phi(\alpha)(n) = \begin{cases} 2^{-B_1(n)/q} & \text{if } \frac{B_2(n)}{2^{B_1(n)}} \leq \alpha < \frac{B_2(n)+1}{2^{B_1(n)}} \\ 0 & \text{otherwise} \end{cases}.$$

This function and the Lebesgue measure λ on $\mathbb{R}$ induce a Borel probability measure τ' on l^q_+ defined by $\tau'(E) = \lambda\left(\phi^{-1}(E)\right)$. Since l^q_+ is a Polish space, the measure τ' is tight (*Aliprantis and Border*, 2006, Theorem 12.7). It follows that there exists a compact subset F of $l^q_+ \setminus \{0\}$ such that $\tau'(F) > 0$. Now define a Borel probability measure τ on l^q by

$$\tau(E) = \frac{\tau'(E \cap F)}{\tau'(F)}.$$

The support of τ is a compact subset of $F \subset l^q_+ \setminus \{0\}$, establishing part (a).

Part (b) is proved by contradiction, supposing that some $x \in l^q$ satisfies $\tau(S(x)) > 0$. Observe that we must have $x \geq 0$, and consider the $y \in l^q_+$ defined by

$$y_n = \begin{cases} 2^{-B_1(n)/q} & \text{if } 2^{-B_1(n)/q} \leq x_n \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$\begin{aligned} \sum_{n=1}^{\infty} x_n^q &\geq \sum_{n=1}^{\infty} y_n^q \\ &= \sum_{i=1}^{\infty} 2^{-i} \left| \left\{ j \in \left\{0, 1, \ldots, 2^i - 1\right\} : 2^{-i/q} \leq x_{B^{-1}(i,j)} \right\} \right| \end{aligned}$$

$$= \sum_{i=1}^{\infty} \lambda \left(\bigcup \left\{ \left[\frac{j}{2^i}, \frac{j+1}{2^i} \right) : j \in \left\{ 0, 1, \ldots, 2^i - 1 \right\}, 2^{-i/q} \leq x_{B^{-1}(i,j)} \right\} \right)$$
$$\geq \sum_{i=1}^{\infty} \lambda \left(\phi^{-1} \left(S \left(x \right) \right) \right),$$

and it follows that $\lambda \left(\phi^{-1} \left(S \left(x \right) \right) \right) = 0$. This means that

$$\tau' \left(F \cap S \left(x \right) \right) = \lambda \left(\phi^{-1} \left(F \cap S \left(x \right) \right) \right) = 0,$$

yielding the contradictory conclusion that $\tau \left(S \left(x \right) \right) = 0$. □

Definition 2. A probability space (Ω, Σ, μ) is said to be *nontrivial* if the σ-algebra Σ includes a countably infinite partition of Ω into nonnull events.

Lemma 5. *Consider a nontrivial probability space (Ω, Σ, μ), real numbers $q \geq 1$ and $\varepsilon \in [0, 1)$, and let*

$$\Delta_\varepsilon = \left\{ x \in L^q \left(\mu \right) : x \geq \varepsilon \chi_\Omega, E \left(x \right) = 1 \right\}. \tag{1}$$

There exists a Borel probability measure τ on $L^q \left(\mu \right)$ with compact support such that:

(a) $\operatorname{supp} \tau \subset \Delta_\varepsilon$*;*

(b) for every $x \in L^q \left(\mu \right)$ and its set of smaller vectors $S \left(x \right) = \left\{ y \in L^q \left(\mu \right) : y \leq x \right\}$, we have $\tau \left(S \left(x \right) \right) = 0$.

Proof. Since the probability space is nontrivial, the σ-algebra Σ includes a partition $\mathcal{P} = \left\{ S_n \right\}_{n \in \mathbb{N}}$ of Ω such that $\mu \left(S_n \right) > 0$ for all n. This partition generates the sub-σ-algebra $\sigma \left(\mathcal{P} \right)$ of Σ, and we let $X = L^q \left(\Omega, \sigma \left(\mathcal{P} \right), \mu|_{\sigma(\mathcal{P})} \right)$. Now define functions $T : l^q \to L^q \left(\mu \right)$ with range X and $\phi : X_+ \setminus \left\{ 0 \right\} \to L^q \left(\mu \right)$ by

$$T \left(x \right) = \sum_{n=1}^{\infty} \frac{x_n}{\left(\mu \left(S_n \right) \right)^{\frac{1}{q}}} \chi_{S_n} \text{ and } \phi \left(x \right) = \varepsilon \chi_\Omega + \left(1 - \varepsilon \right) \frac{1}{E \left(x \right)} x.$$

Both T and ϕ are continuous, because the latter is defined by continuous vector operations and the former is a positive operator between Banach lattices (see *Aliprantis and Border*, 2006, Theorem 9.6). Thus, supplementing these functions with a Borel probability measure τ' on l^q chosen according to Lemma 4 defines a Borel measure τ on $L^q \left(\mu \right)$ by $\tau \left(E \right) = \tau' \left(T^{-1} \left(\phi^{-1} \left(E \right) \right) \right)$. This measure τ is indeed a probability measure because

$$1 \geq \tau\left(L^q\left(\mu\right)\right) \geq \tau'\left(\operatorname{supp}\tau'\right) = 1.$$

The support of τ is the compact subset $\phi\left(T\left(\operatorname{supp}\tau'\right)\right)$ of Δ_ε, which is what part (a) requires.

For part (b), pick an upper bound $k \in \mathbb{R}_+$ of the compact set $E\left(T\left(\operatorname{supp}\tau'\right)\right)$, let

$$z = E\left(\frac{k}{1-\varepsilon}\left(x - \varepsilon\chi_\Omega\right)\middle|\sigma\left(\mathcal{P}\right)\right),$$

and recall that there must be a $z' \in l^q$ such that $T\left(z'\right) = z$. The desired conclusion is established by the following calculation:

$$\begin{aligned}
\tau\left(S\left(x\right)\right) &= \tau'\left(T^{-1}\left(\phi^{-1}\left(S\left(x\right)\right)\right)\right) \\
&= \tau'\left(\operatorname{supp}\tau' \cap T^{-1}\left(\phi^{-1}\left(S\left(x\right)\right)\right)\right) \\
&= \tau'\left(\left\{y \in \operatorname{supp}\tau' : \phi\left(T\left(y\right)\right) \leq x\right\}\right) \\
&= \tau'\left(\left\{y \in \operatorname{supp}\tau' : T\left(y\right) \leq \frac{E\left(T\left(y\right)\right)}{1-\varepsilon}\left(x - \varepsilon\chi_\Omega\right)\right\}\right) \\
&\leq \tau'\left(\left\{y \in \operatorname{supp}\tau' : T\left(y\right) \leq \frac{k}{1-\varepsilon}\left(x - \varepsilon\chi_\Omega\right)\right\}\right) \\
&\leq \tau'\left(\left\{y \in \operatorname{supp}\tau' : T\left(y\right) \leq z\right\}\right) \\
&= \tau'\left(\left\{y \in \operatorname{supp}\tau' : T\left(y\right) \leq T\left(z'\right)\right\}\right) \\
&= \tau'\left(\left\{y \in \operatorname{supp}\tau' : y \leq z'\right\}\right) \\
&\leq \tau'\left(\left\{y \in l^q : y \leq z'\right\}\right) \\
&= 0.
\end{aligned}$$

□

Lemma 6. *Consider a nontrivial probability space (Ω, Σ, μ), real numbers $q \geq 1$, $r \in \mathbb{R}_{++}$, $\varepsilon \in [0, 1)$, and $\delta \in \mathbb{R}_{++}$, the ball $B_r = \left\{x \in L^q\left(\mu\right) : \|x\|_q < r\right\}$ and the set Δ_ε defined by (1). There exists a tight Borel probability measure τ on $L^q\left(\mu\right)$ with compact support such that*

$$\operatorname{supp}\tau \subset \left(B_r + \chi_\Omega\right) \cap \left(\delta\Delta_\varepsilon + \left(1-\delta\right)\chi_\Omega\right) \tag{2}$$

and part (b) of Lemma 5 holds.

Proof. Choose a Borel probability measure τ' on $L^q\left(\mu\right)$ according to Lemma 5. Since

$\operatorname{supp}\tau'$ is compact, it is norm bounded. It follows that there exists some $\alpha \in (0, \delta]$ such that $\alpha (\operatorname{supp}\tau' - \chi_\Omega) \subset B_r$. Define a continuous function $\phi : L^q(\mu) \to L^q(\mu)$ inducing a Borel probability measure τ on $L^q(\mu)$ by

$$\phi(x) = \alpha x + (1-\alpha)\chi_\Omega \text{ and } \tau(E) = \tau'\left(\phi^{-1}(E)\right).$$

The support of τ is the compact set $\phi(\operatorname{supp}\tau') \subset B_r + \chi_\Omega$, and condition (2) holds because the convexity of Δ_ε ensures that

$$\begin{aligned}\phi\left(\operatorname{supp}\tau'\right) &\subset \phi(\Delta_\varepsilon)\\ &= \alpha\Delta_\varepsilon + (1-\alpha)\chi_\Omega\\ &= \delta\left(\frac{\alpha}{\delta}\Delta_\varepsilon + \frac{\delta-\alpha}{\delta}\chi_\Omega\right) + (1-\delta)\chi_\Omega\\ &\subset \delta\Delta_\varepsilon + (1-\delta)\chi_\Omega.\end{aligned}$$

Since $\operatorname{supp}\tau$ is a Polish space, the restriction of τ to the Borel σ-algebra of $\operatorname{supp}\tau$ is a tight Borel probability measure (*Aliprantis and Border*, 2006, Theorem 12.7), and thus so is τ. Finally, it is true that $\tau(S(x)) = 0$ because

$$\phi^{-1}(S(x)) = \left\{y \in L^q(\mu) : y \le \frac{1}{\alpha}x - \frac{1-\alpha}{\alpha}\chi_\Omega\right\}.$$

□

Lemma 7. *Consider a nontrivial probability space* (Ω, Σ, μ)*, real numbers* $q \ge 1$ *and* $\varepsilon \in [0, 1)$*, and the set* Δ_ε *defined by (1). The set*

$$E = \left\{(x, y) \in \Delta_\varepsilon^2 : \text{there is a real number } k \text{ such that } x \le ky\right\}$$

is shy in Δ_ε^2*.*

Proof. By Facts 0 and 3 in *Anderson and Zame* (2001), it suffices to show that for every (nonzero) natural number n the closed set

$$E_n = \left\{(x, y) \in \Delta_\varepsilon^2 : x \le ny\right\}$$

is shy in Δ_ε^2 at $(\chi_\Omega, \chi_\Omega) \in \Delta_\varepsilon^2$. Fix any $\delta \in \mathbb{R}_{++}$ and any neighborhood W of 0 in $(L^q(\mu))^2$. Pick an $r \in \mathbb{R}_{++}$ such that the ball $B_r = \left\{x \in L^q(\mu) : \|x\|_q < r\right\}$ satisfies $B_r^2 \subset W$. Choose a tight Borel probability measure τ_1 on $L^q(\mu)$ according to Lemma 6, and let τ_2 be the (tight) Dirac measure on $L^q(\mu)$ at χ_Ω. The product measure $\tau_1 \otimes \tau_2$ extends to a regular Borel probability measure τ on $(L^q(\mu))^2$ (*Ressel*, 1977, Theorem 1). The support of τ is the compact set

$$\operatorname{supp}\tau = \operatorname{supp}\tau_1 \times \{\chi_\Omega\} \subset \delta\Delta_\varepsilon^2 + (1-\delta)(\chi_\Omega, \chi_\Omega)$$

with

$$\operatorname{supp}\tau \subset (B_r + \chi_\Omega) \times \{\chi_\Omega\} \subset B_r^2 + (\chi_\Omega, \chi_\Omega) \subset W + (\chi_\Omega, \chi_\Omega).$$

Now it only remains to show that every $(\bar{x}, \bar{y}) \in (L^q(\mu))^2$ satisfies

$$\tau(E_n + (\bar{x}, \bar{y})) = 0. \tag{3}$$

If $(x, y) \in E_n$ and $y + \bar{y} = \chi_\Omega$, then

$$x + \bar{x} \leq ny + \bar{x} = n\chi_\Omega - n\bar{y} + \bar{x}.$$

Thus, letting $S = \{x \in L^q(\mu) : x \leq n\chi_\Omega - n\bar{y} + \bar{x}\}$, we have

$$(E_n + (\bar{x}, \bar{y})) \cap (L^q(\mu) \times \{\chi_\Omega\}) \subset S \times \{\chi_\Omega\}.$$

Since $\operatorname{supp}\tau \subset L^q(\mu) \times \{\chi_\Omega\}$, condition (3) is verifies by the calculation

$$0 \leq \tau(E_n + (\bar{x}, \bar{y})) \leq \tau(S \times \{\chi_\Omega\}) = \tau_1(S) = 0.$$

□

Lemma 8. *Consider a probability space (Ω, Σ, μ), a sub-σ-algebra Σ' of Σ such that the restriction $(\Omega, \Sigma', \mu|_{\Sigma'})$ is nontrivial, real numbers $q \geq 1$ and $\varepsilon \in [0, 1)$, and the set Δ_ε defined by (1). The Borel subset*

$$E = \left\{(x, y) \in \Delta_\varepsilon^2 : \text{there is a real number } k \text{ such that } E\left(x \middle| \Sigma'\right) \leq kE\left(y \middle| \Sigma'\right)\right\}$$

of $(L^q(\mu))^2$ is shy in Δ_ε^2.

Proof. Let $\mu' = \mu|_{\Sigma'}$, and define

$$\Delta'_\varepsilon = \left\{x \in L^q\left(\mu'\right) : x \geq \varepsilon\chi_\Omega, E(x) = 1\right\},$$

$$E' = \left\{ (x,y) \in \left(\Delta'_\varepsilon\right)^2 : \text{there is a real number } k \text{ such that } x \le ky \right\},$$

as well as a continuous linear projection $T : (L^q(\mu))^2 \to (L^q(\mu))^2$ with range $(L^q(\mu'))^2$ by

$$T(x,y) = \left(E\left(x \middle| \Sigma'\right), E\left(y \middle| \Sigma'\right)\right).$$

By Lemma 7, the set $T(E) = E'$ is shy in $T\left(\Delta^2_\varepsilon\right) = (\Delta'_\varepsilon)^2 \subset \Delta^2_\varepsilon$ viewing $T\left(\Delta^2_\varepsilon\right)$ as a subset of $(L^q(\mu'))^2$. By Lemma 3, the Borel set E is shy in Δ^2_ε. □

Definition 3. Consider a probability space (Ω, Σ, μ), a sub-σ-algebra Σ' of Σ, and a function $f \in L^1(\mu)$. We say that Σ' is *μ-strictly dominated* by f if $f(\omega) > 0$ a.e. and, for any Σ'-measurable $g : \Omega \to \mathbb{R}$ with $g(\omega) \le f(\omega)$ a.e., we have $g(\omega) < f(\omega)$ almost everywhere.

Lemma 9. *Let (Ω, Σ, μ) be a probability space, let Σ' be a sub-σ-algebra of Σ, and suppose that Σ' is μ-strictly coarser than Σ. Let $1 \le p \le \infty$, let $\beta > 0$ be a real number, and let $B_\beta = \{x \in L^p(\mu) : x \ge \beta\chi_\Omega\}$. Then the set*

$$B = \left\{x \in B_\beta : \Sigma' \textit{ is } \mu\textit{-strictly dominated by } x\right\}$$

is $\|\cdot\|_p$-dense in B_β.

Proof. Since the (μ-equivalence classes) of Σ-measurable simple functions in B_β are $\|\cdot\|_p$-dense in B_β, it suffices to show that any such function is approximated, in the norm $\|\cdot\|_p$, by elements of B. Thus let $z = \sum_{i=0}^n \lambda_i \chi_{E_i}$ be a Σ-measurable simple function in B_β. We can assume that the family $\langle E_i \rangle_{i=0,\ldots,n}$ is a partition of Ω.

Recall that the hypothesis of Σ' being μ-strictly coarser than Σ means that we can select a Σ-measurable function $\hat{x} : \Omega \to \mathbb{R}_{++}$ such that given any Σ'-measurable function $g : \Omega \to \mathbb{R}$ we have $\hat{x}(\omega) \ne g(\omega)$ for almost all $\omega \in \Omega$. We may assume that $\hat{x}$ is bounded. (If necessary, select a homeomorphism $\phi : \mathbb{R}_{++} \to (0,1)$ and replace $\hat{x}$ by the composition $\phi \circ \hat{x}$). Then (identifying $\hat{x}$ with its μ-equivalence class) we have $\hat{x} \in L^p(\mu)$; in particular, $\hat{x} \gg 0$.

Fix any $\epsilon > 0$ and let $\hat{z} = z + \epsilon\hat{x}$. We claim that $\hat{z} \in B$. Note first that $\hat{z} \in B_\beta$ because $\hat{x} \ge 0$. Now let $g : \Omega \to \mathbb{R}$ be any Σ'-measurable function. Then for each $i = 0, \ldots, n$, the function $(1/\epsilon)(g - \lambda_i\chi_\Omega)$ is also Σ'-measurable, and hence we have $(1/\epsilon)(g - \lambda_i\chi_\Omega)(\omega) \ne \hat{x}(\omega)$ for almost all $\omega \in \Omega$ by the choice of $\hat{x}$. It follows that for each i, $g(\omega) \ne (\lambda_i\chi_{E_i} + \epsilon\hat{x}\chi_{E_i})(\omega)$ for almost all $\omega \in E_i$. Since $\langle E_i \rangle_{i=0,\ldots,n}$

is a partition of Ω, it follows from this that $g(\omega) \neq \hat{z}(\omega)$ for almost all $\omega \in \Omega$. In particular, were $g(\omega) \leq \hat{z}(\omega)$ for almost all $\omega \in \Omega$, we would have $g(\omega) < \hat{z}(\omega)$ for almost all $\omega \in \Omega$. Thus $\hat{z} \in B$, and as ϵ was arbitrary, the lemma is proved. □

Lemma 10. *Let (Ω, Σ, μ) be a separable probability space, and Σ' a sub-σ-algebra of Σ. Then the set*

$$U = \Big\{x \in L^0(\mu) : \textit{there is a } \Sigma'\textit{-measurable function } g : \Omega \to \mathbb{R} \\ \textit{such that } \mu(\{\omega \in \Omega : g(\omega) = x(\omega)\}) > 0\}$$

is universally measurable for the topology of convergence in measure on $L^0(\mu)$.

Proof. In this proof, all topological notion concerning $L^0(\mu)$ are with respect to the topology of convergence in measure on $L^0(\mu)$. Note first that since μ is a probability measure, $L^0(\mu)$ is completely metrizable. Also, since the probability space (Ω, Σ, μ) is separable, $L^0(\mu)$ is separable. Thus $L^0(\mu)$ is a Polish space. Let Y be the linear subspace of $L^0(\mu)$ consisting of the Σ'-measurable elements, endowed with the subspace topology. Note that Y is closed in $L^0(\mu)$. Thus Y is also a Polish space.

Let

$$H = \Big\{(x, y) \in L^0(\mu) \times Y : \mu(\{\omega \in \Omega : x(\omega) = y(\omega)\}) > 0\Big\}.$$

We claim that H is a Borel set in $L^0(\mu) \times Y$. To see this, let $\phi : L^0(\mu) \times Y \to L^0(\mu)$ be the mapping given by $\phi(x, y) = x - y, x \in L^0(\mu), y \in Y$, and let

$$C = \Big\{z \in L^0(\mu) : \mu(\{\omega \in \Omega : z(\omega) = 0\}) > 0\Big\}.$$

Then ϕ is continuous, since the topology of convergence in measure is a linear space topology. Also, C is a Borel set in $L^0(\mu)$. To see this latter fact, for each $n \in \mathbb{N} \setminus \{0\}$ let

$$C_n = \left\{z \in L^0(\mu) : \mu(\{\omega \in \Omega : z(\omega) = 0\}) \geq \frac{1}{n}\right\},$$

and note that C_n is closed for each n, and that $C = \cup_{n=1}^{\infty} C_n$. Now $H = \phi^{-1}(C)$, and it follows that H is a Borel set in $L^0(\mu) \times Y$.

Note that the set U from the statement of the lemma is just the image of H under the projection of $L^0(\mu) \times Y$ onto $L^0(\mu)$. Hence, since Y is a Polish space, the fact that H is a Borel set in $L^0(\mu) \times Y$ implies that U is universally measurable in $L^0(\mu)$

(*Fremlin*, 2000, 434X(d)). □

Lemma 11. *Let (Ω, Σ, μ) be a separable probability space, Σ' a sub-σ-algebra of Σ, and $1 \leq p \leq \infty$. Then the set*

$$V = \{x \in L^p(\mu) : \textit{there is a } \Sigma'\textit{-measurable function } g : \Omega \to \mathbb{R} \\ \textit{such that } \mu(\{\omega \in \Omega : g(\omega) = x(\omega)\}) > 0\}$$

is universally measurable for the norm-topology of $L^p(\mu)$.

Proof. The embedding of $L^p(\mu)$ into $L^0(\mu)$ is continuous for the norm-topology of $L^p(\mu)$ and the topology of convergence in measure on $L^0(\mu)$. Hence if $A \subset L^0(\mu)$ is universally measurable for the topology of convergence is measure, then $A \cap L^p(\mu)$ is universally measurable in $L^p(\mu)$ for the norm topology of $L^p(\mu)$. Thus the assertion of the lemma follows from the previous lemma. □

Lemma 12. *Let (Ω, Σ, μ) be a separable probability space and Σ' a sub-σ-algebra of Σ. Let $1 \leq p \leq \infty$, let $\beta > 0$ be a real number, and let $B_\beta = \{x \in L^p(\mu) : x \geq \beta\chi_\Omega\}$, and let*

$$W = \{x \in B_\beta : \textit{if } g : \Omega \to \mathbb{R} \textit{ is any } \Sigma'\textit{-measurable function}, \\ \textit{then } g(\omega) \neq x(\omega) \textit{ for almost all } \omega \in \Omega\}.$$

Then if W is non-empty, W is prevalent in B_β for the norm-topology of $L^p(\mu)$.

Proof. Suppose $W \neq \emptyset$ and let $E = B_\beta \backslash W$. We will show that E is shy in B_β. Note first by the previous lemma, E is universally measurable in $L^p(\mu)$. Let $v \in W$, let V be the linear subspace of $L^p(\mu)$ spanned by v, and let λ_V be Lebesgue measure on V. Note that $V \cap B_\beta$ is closed in V and that $\alpha v \in V \cap B_\beta$ for all real numbers $\alpha \geq 1$. Consequently $\lambda_V(V \cap B_\beta) > 0$.

According to Fact 6 in *Anderson and Zame* (2001), it now suffices to show that $(E + x) \cap V$ is countable for each $x \in L^p(\mu)$. Thus fix any $x \in L^p(\mu)$. Let $a, a' \in E$ with $a \neq a'$ and suppose $a + x \in V$ and $a' + x \in V$. Then by the definition of V, $a - a' = \alpha v$ for some $\alpha \neq 0$, and by the definition of E, there are Σ'-measurable functions g, g' from Ω to $\mathbb{R}$ and elements $S, S' \in \Sigma$, with $\mu(S) > 0$ and $\mu(S') > 0$, such that $g(\omega) = a(\omega)$ for almost all $w \in S$ and $g'(\omega) = a'(\omega)$ for almost all $\omega \in S'$. Then $g - g'$ is Σ'-measurable and $(g - g')(\omega) = \alpha v(\omega)$ for almost all $\omega \in S \cap S'$. Hence, by the choice of v, $\mu(S \cap S') = 0$. Thus we can attach elements $S_b \in \Sigma$ to the elements

$b \in E + x$ so that $\mu(S_b) > 0$ for each $b \in E + x$ and so that whenever $b, b' \in E + x$ with $b \neq b'$, then $\mu(S_b \cap S_{b'}) = 0$. This implies that $E + x$ must be countable. □

Lemma 13. *Consider a probability space (Ω, Σ, μ) such that the measure μ is separable, a μ-strictly coarser sub-σ-algebra Σ' of Σ, numbers $1 \leq p \leq \infty$ and $\beta \in \mathbb{R}_{++}$, and let*

$$B_\beta = \{x \in L^p(\mu) : x \geq \beta\chi_\Omega\}.$$

The set

$$A_\beta = \{x \in B_\beta : \Sigma' \text{ is } \mu\text{-strictly dominated by } x\} \tag{4}$$

is prevalent in B_β.

Proof. Consider the set W from the statement of the previous lemma. By what was remarked in the second paragraph of the proof of Lemma 9, the hypothesis that Σ' is μ-strictly coarser than Σ implies that W is non-empty. Hence, by the previous lemma, W is prevalent in B_β. Clearly, if $x \in \Omega$ and if $g : \Omega \to \mathbb{R}$ is any Σ'-measurable function with $g(\omega) \leq x(\omega)$ for almost all $\omega \in \Omega$, then $g(\omega) < x(\omega)$ for almost all $\omega \in \Omega$, then $g(\omega) < x(\omega)$ for almost all $\omega \in \Omega$. Thus $W \subset A_\beta$ and it follows that A_β is prevalent in A_β. □

Lemma 14. *Consider a probability space (Ω, Σ, μ) such that the measure μ is separable, a μ-strictly coarser sub-σ-algebra Σ' of Σ, any $p \in [1, \infty]$, the space $L^p(\mu)$, and its positive cone $L^p_+(\mu)$. For every real number $\beta \in \mathbb{R}_{++}$, consider the set A_β defined by* (4)*. The set $F = L^p_+(\mu) \backslash \bigcup_{\beta > 0} A_\beta$ is shy in $L^p_+(\mu)$.*

Proof. For every (nonzero) natural number n, the set $B_{1/n} \backslash A_{1/n}$ is shy in $B_{1/n}$ by Lemma 13. It follows that $B_{1/n} \backslash A_{1/n}$ is shy in the superset $L^p_+(\mu)$. Combining this observation with Fact 3 in *Anderson and Zame* (2001) and using their Theorem 3.3, we see that $\bigcup_{n=1}^{\infty} B_{1/n} \backslash A_{1/n}$ and $L^p_+(\mu) \backslash \bigcup_{\beta > 0} B_\beta$ are both shy in $L^p_+(\mu)$. Since the shy union of these two shy sets includes F, this set is indeed shy itself. □

5 Remaining Proofs

Proof of Lemma 1

Suppose, by way of contradiction, that every $F \in \Sigma$ with $F \subset E$ and $\mu'(F) > 0$ belongs to Σ'. Next choose a function f according to Definition 1, and pick a sequence $\{g_n\}$

of real simple functions converging pointwise to f. Let $G \in \Sigma$ be the union of all μ'-null inverse images of the elements of the ranges of the terms of the sequence $\{g_n \chi_F\}$, and observe that $\mu'(G) = 0$. Now $\left\{g_n \chi_{E \backslash G}\right\}$ is a sequence of Σ'-measurable simple functions, whose pointwise limit g is Σ'-measurable and also coincides with f on $E \backslash G$. This is a contradiction, because $\mu(E \backslash G) > 0$.

Proof of Theorem 1

The set

$$F = \left\{(x, y) \in \Delta^2_{E_n} : \text{there is a real number } k \text{ such that } E(x|\bar{p}) \leq kE(y|\bar{p})\right\}$$

is shy in $\Delta^2_{E_n}$ by Lemma 8. Since $\sigma(\bar{p})$ is $\mu(\cdot|E_n)$-strictly coarser than Σ, Lemma 14 yields a shy subset G of $L^p_+(\mu(\cdot|E_n))$.

We complete the proof by showing that $\bar{e} \notin G$ implies $(\rho_{jn}, \rho_{in}) \in F$. For all $k \in I \backslash \{i\}$, condition (A9)(g) implies that $\bar{x}_k$ as an element of $L^p_+(\mu(\cdot|E_n))$ is measurable with respect to the restriction of $\mu(\cdot|E_n)$ to $\sigma(\bar{p})$. It follows that $\sum_{k \neq i} \bar{x}_k$ is $\sigma(\bar{p})$-measurable. Since $\sigma(\bar{p})$ is $\mu(\cdot|E_n)$-strictly dominated by $\bar{e} \notin G$ and also $\bar{e} \geq \sum_{k \neq i} \bar{x}_k$ due to (A9)(f), for $\mu(\cdot|E_n)$-almost all $\omega \in \Omega$ we have $\bar{x}_i(\omega) > 0$. Combining this fact with (A9)(d–e) yields Lagrangian multipliers $\lambda_i, \lambda_j \in \mathbb{R}_{++}$ such that $\mu(\cdot|E_n)$-almost everywhere we have

$$u'_{in}(\bar{x}_i(\omega))\,\rho_{in}(\omega) = \lambda_i \bar{p}(\omega) \ \text{ and } u'_{jn}(\bar{x}_j(\omega))\,\rho_{jn}(\omega) \leq \lambda_j \bar{p}(\omega).$$

By (A9)(c), we have $a_n \rho_{jn} \leq \lambda_j \bar{p}$ and $\lambda_i \bar{p} \leq b_n \rho_{in}$. A combination of these inequalities produces

$$\rho_{jn} \leq \frac{\lambda_j b_n}{\lambda_i a_n} \rho_{in}, \text{ hence } E(\rho_{jn}|\bar{p}) \leq \frac{\lambda_j b_n}{\lambda_i a_n} E(\rho_{in}|\bar{p}).$$

This means that $(\rho_{jn}, \rho_{in}) \in F$ and completes the proof.

Proof of Lemma 2

By Definition 1, there exist a Σ'-measurable $g : \mathbb{R} \to \mathbb{R}$ and a Borel subset G of E such that $\mu(G) > 0$ and for all $\omega \in G$ we have $g(\omega) = |\omega|$. If $\mu(G \cap \mathbb{R}_+) > 0$, let $F = G \cap \mathbb{R}_+$, and otherwise let $F = G \backslash \mathbb{R}_+$. Every Borel set $B \in \mathcal{B}_F$ satisfies $B = F \cap g^{-1}(B)$ or $B = F \cap g^{-1}(-B)$.

Proof of Theorem 2

Let $\{S_n\}$ be any enumeration of the set of nonnull closed intervals with rational endpoints. By Theorem 1, hypothesis (H1) is false generically.

To prove that (H1) is true unless (H2) is true, suppose that the latter is false. This yields a nonnull interval S such that $\sigma(\bar{p})$ is $\mu(\cdot|S)$-strictly coarser than Σ. By Definition 1, the interval S is not a singleton, and the open interval with the same endpoints is also nonnull. It follows that some S_n is a subset of S, and thus $\sigma(\bar{p})$ is also $\mu(\cdot|S_n)$-strictly coarser than Σ, confirming (H1).

Proof of Theorem 5

Pick a real number $\theta > 1$ such that

$$\frac{\theta - 2}{\theta - 1} \geq \frac{\alpha}{\mu(E)}.$$

Next consider a sequence $\{E_n\}$ in Σ'' such that:

(i) $E_n = \emptyset$ if $\mu(S_n \backslash E) > 0$ or $\mu(S_n) = 0$, and

(ii) otherwise E_n is an event in Σ'' with

$$E_n \subset E,\ \mu(E_n \cap S_n) > 0, \text{and } \mu(E_n) = \frac{\mu(E)}{\theta^n},$$

which is possible in view of Theorems 10.52 and 10.23.7 in *Aliprantis and Border* (2006).

Define

$$C = E \Bigg\backslash \bigcup_{n=1}^{\infty} E_n \,.$$

Now let

$$\Sigma' = \{(A \backslash C) \cup B : A \in \Sigma, B = C \text{ or } B = \emptyset\}. \tag{5}$$

To prove part (b), consider any Σ-measurable $f : \Omega \to \mathbb{R}_{++}$, let $g = f\chi_{S_n \backslash C}$, and observe that $\mu(S_n \backslash C) > 0$.

Now it only remains to establish the validity of part (a). The first step is to calculate that

$$1-\sum_{n=1}^{\infty}\frac{1}{\theta^n}=1-\frac{1}{\theta}\sum_{n=1}^{\infty}\left(\frac{1}{\theta}\right)^{n-1}=1-\frac{1}{\theta}\frac{1}{1-1/\theta}=\frac{\theta-2}{\theta-1}$$

and

$$\mu(C)\geq\mu(E)-\sum_{n=1}^{\infty}\mu(E_n)\geq\mu(E)-\sum_{n=1}^{\infty}\frac{\mu(E)}{\theta^n}=\frac{\theta-2}{\theta-1}\mu(E)\geq\alpha.$$

Consequently, the event $C\in\Sigma''$ has a subset $F\in\Sigma''$ with $\mu(F)=\beta$. If this event $F\subset E$ were Σ'-measurable, then the definition (5) of Σ' would imply that $C\subset F$ and thus $C=F$, which would contradict $\mu(C)\geq\alpha>\mu(F)$.

Proof of Theorem 6

Start by picking any real number $\alpha\in(0,\mu(E))$, and then perform the steps given in the first paragraph of the proof of Theorem 5. Next let

$$\Sigma'=\left\{(A\cap C^c)\cup(B\cap C):A\in\Sigma,B\in\Sigma''\right\}. \tag{6}$$

Since (6) defines a finer σ-algebra than (5), part (b) follows as in the proof of Theorem 5.

Part (a) is proved by contradiction, supposing that every $F\in\Sigma$ with $F\subset E$ and $\mu(F)>0$ belongs to Σ'. In particular, every $F\in\Sigma$ with $F\subset C$ and $\mu(F)>0$ belongs to Σ'. By the definition (6) of Σ' and the fact that $C\in\Sigma''$, every such $F\in\Sigma$ with $F\subset C$ and $\mu(F)>0$ belongs to Σ''. This contradicts Lemma 1, because $\mu(C)\geq\alpha>0$.

Proof of Theorem 3

Apply Theorem 5 with Σ'' equal to the Borel σ-algebra and $\{S_n\}$ taken to be any enumeration of the set of nonnull closed intervals with rational endpoints. This yields a sub-σ-algebra Σ' of the Borel σ-algebra, and condition (a) holds. Finally, condition (b) is implied by part (b) of Theorem 5 for $S_n\subset S$, as in the second parapraph of the proof of Theorem 2.

Proof of Theorem 4

Apply Theorem 6 with $\{S_n\}$ taken to be any enumeration of the set of nonnull closed intervals with rational endpoints.

Bibliography

Abramovich, Y., and C. D. Aliprantis (2002), *An Invitation to Operator Theory*, Graduate Studies in Mathematics, American Mathematical Society.

Aliprantis, C. D., and K. C. Border (2006), *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer.

Aliprantis, C. D., and D. J. Brown (1983), Equilibria in markets with a Riesz space of commodities, *Journal of Mathematical Economics*, *11*(2), 189–207.

Aliprantis, C. D., and R. Tourky (2007), *Cones and Duality*, Graduate Studies in Mathematics, American Mathematical Society.

Aliprantis, C. D., D. J. Brown, and O. Burkinshaw (1987), Edgeworth equilibria, *Econometrica*, *55*(5), 1109–1137.

Aliprantis, C. D., R. Tourky, and N. C. Yannelis (2001), A theory of value with non-linear prices: Equilibrium analysis beyond vector lattices, *Journal of Economic Theory*, *100*(1), 22–72.

Aliprantis, C. D., M. Florenzano, and R. Tourky (2004a), General equilibrium analysis in ordered topological vector spaces, *Journal of Mathematical Economics*, *40*(3–4), 247–269.

Aliprantis, C. D., P. K. Monteiro, and R. Tourky (2004b), Non-marketed options, non-existence of equilibria, and non-linear prices, *Journal of Economic Theory*, *114*(2), 345–357.

Aliprantis, C. D., M. Florenzano, and R. Tourky (2005), Linear and non-linear price decentralization, *Journal of Economic Theory*, *121*(1), 51–74.

Aliprantis, C. D., M. Florenzano, and R. Tourky (2006), Production equilibria, *Journal of Mathematical Economics*, *42*(45), 406–421.

Anderson, R. M., and W. R. Zame (2001), Genericity with infinitely many parameters, *B.E. Journal of Theoretical Economics*, *1*(1), 1–64.

Angelopoulos, A., and L. C. Koutsougeras (2015), Value allocation under ambiguity, *Economic Theory*, *59*(1), 147–167.

Balder, E. J., and N. C. Yannelis (1993), On the continuity of expected utility, *Economic Theory*, *3*(4), 625–643.

Bewley, T. F. (1972), Existence of equilibria in economies with infinitely many commodities, *Journal of Economic Theory*, *4*(3), 514–540.

Chavas, J.-P., and W. Briec (2012), On economic efficiency under non-convexity, *Economic Theory*, *50*(3), 671–701.

Correia-da Silva, J., and C. Hervés-Beloso (2009), Prudent expectations equilibrium in economies with uncertain delivery, *Economic Theory*, *39*(1), 67–92.

de Castro, L. I., M. Pesce, and N. C. Yannelis (2011), Core and equilibria under ambiguity, *Economic Theory*, *48*(2-3), 519–548.

Dekel, E. (1986), An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom, *Journal of Economic Theory*, *40*(2), 304–318.

Epstein, L. G. (1986), Implicitly additive utility and the nature of optimal economic growth, *Journal of Mathematical Economics*, *15*(2), 111–128.

Finch, S. (2003), *Mathematical Constants*, Encyclopedia of Mathematics and its Applications, Cambridge University Press.

Florenzano, M. (2003), *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria*, Springer.

Florenzano, M., and V. M. Marakulin (2001), Production equilibria in vector lattices, *Economic Theory*, *17*(3), 577–598.

Fremlin, D. (2000), *Measure Theory, Volume 4*, Measure Theory, Torres Fremlin.

Graziano, M. G. (2007), Economies with public projects: Efficiency and decentralization, *International Economic Review*, *48*(3), 1037–1063.

Habte, A., and B. S. Mordukhovich (2011), Extended second welfare theorem for nonconvex economies with infinite commodities and public goods, *Advances in Mathematical Economics*, *14*, 93–126.

He, W., and N. C. Yannelis (2016), Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences, *Economic Theory*, *61*(3), 497–513.

Hervés-Beloso, C., V. F. Martins-da-Rocha, and P. K. Monteiro (2009), Equilibrium theory with asymmetric information and infinitely many states, *Economic Theory*, *38*(2), 295–320.

Hunt, B. R., T. Sauer, and J. A. Yorke (1992), Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces, *Bulletin (New Series) of the American Mathematical Society*, *27*(2), 217–238.

Klishchuk, B. (2015), New conditions for the existence of Radner equilibrium with infinitely many states, *Journal of Mathematical Economics*, *61*, 67–73.

Koutsougeras, L. C., and N. C. Yannelis (1993), Incentive compatibility and information superiority of the core of an economy with differential information, *Economic Theory*, *3*(2), 195–216.

Mas-Colell, A. (1986), The price equilibrium existence problem in topological vector lattices, *Econometrica*, *54*(5), 1039–1053.

Podczeck, K. (1996), Equilibria in vector lattices without ordered preferences or uniform properness, *Journal of Mathematical Economics*, *25*(4), 465–485.

Podczeck, K., and N. C. Yannelis (2008), Equilibrium theory with asymmetric information and with infinitely many commodities, *Journal of Economic Theory*, *141*(1), 152–183.

Podczeck, K., R. Tourky, and N. C. Yannelis (2008), Non-existence of Radner equilibrium already with countably infinitely many states, Working paper.

Radner, R. (1968), Competitive equilibrium under uncertainty, *Econometrica*, *36*(1), 31–58.

Ressel, P. (1977), Some continuity and measurability results on spaces of measures, *Mathematica Scandinavica*, *40*(1), 69–78.

Shreve, S. (2004), *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer Finance / Springer Finance Textbooks, Springer.

Skorokhod, A. (2004), *Basic Principles and Applications of Probability Theory*, Encyclopaedia of Mathematical Sciences, Springer.

Tourky, R. (1998), A new approach to the limit theorem on the core of an economy in vector lattices, *Journal of Economic Theory*, *78*(2), 321–328.

Tourky, R., and N. C. Yannelis (2003), Private expectations equilibrium, Working paper.

Wickstead, A. W. (1975), Compact subsets of partially ordered banach spaces, *Mathematische Annalen*, *212*(4), 271–284.

Xanthos, F. (2014), A note on the equilibrium theory of economies with asymmetric information, *Journal of Mathematical Economics*, *55*, 1–3.

Yoo, S. H. (2013), An alternative proof for the existence of Radner equilibria, *Mathematical Social Sciences*, *66*(1), 87–90.