FREQUENCY WEIGHTED OPTIMAL HANKEL-NORM APPROXIMATION OF SCALAR LINEAR SYSTEMS

by

GEOFFREY A. LATHAM

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Signature of Author ______________

Signature of Supervisor ______________

I hereby certify that except where stated, the work contained in this thesis is the result of original research and has not been submitted for a higher degree at any other University or Institution.

(Signed) G.A. Latham.

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ABSTRACT

This thesis addresses the problem of model reduction for scalar linear time-invariant systems via the use of the optimal Hankel-norm approximation problem. Frequency weighting is combined with optimality in the Hankel norm to obtain a frequency shaped approximation to a given linear system. This is accomplished by the solution of a modified optimal Hankel-norm approximation problem. Also presented is an error analysis for the frequency shaped approximation.

CONTENTS

CHAPTER 1

§1.1 INTRODUCTION

The approximation of high order linear system transfer functions plays an important part in many areas of electrical engineering. The desirability of replacing a high order model by one of reduced order is motivated primarily by the advantages of implementing, either in hardware or software, as simple a system as possible while still meeting the design specifications. To determine whether an approximation scheme satisfies any design requirement, the size of the error incurred in making the approximation should be explicitly known, while simple algorithms for determining the approximation would also be of distinct advantage.

It is precisely these features that have led to the keen investigation and application of the so called optimal Hankel-norm approximation problem [3,14,27,36] to model reduction of engineering systems. This technique, essentially an application from the theory of Hankel operators, has led, together with the theory of balanced realizations, to a renewed interest in model reduction in recent years. In particular, optimal Hankel-norm approximation has found wide application among the following:

(i) the reduction of high order plant models to allow the design of low order controllers [16],

(ii) the reduction of high order controllers designed using the full order plant model [27-29],

(iii) the reduction of filter designs,

(iv) the approximation of infinite dimensional systems [9], and

(v) spectral approximation [22].

Besides having a computationally cheap closed form solution and calculable error bounds in terms of the frequency response, the method gives an approximation which, as the name suggests, is optimal in a specific measure (the Hankel-norm) and so is the first model reduction technique to obey any strong optimality criterion. More importantly however, is that the resulting approximation is guaranteed to be realizable.

The object of this thesis is to show how to combine the additional feature of specifying the approximation accuracy with frequency, thereby shaping the error of the optimal Hankel-norm approximation. This is highly desirable since in many practical situations, it is possible to identify frequency regions where any approximating system should be particularly 'close' to the original system. For example such a region is represented by the unity gain crossover for a closed-loop control system. From an engineering viewpoint, the synthesis of optimal Hankel-norm approximation and specification of error size with frequency, combines the power of frequency domain and state space methods. It allows the incorporation of intuitive classical design ideas, which provide unmatched insight and insensitivity to small errors in a system description, with formal tractable mathematics. Like state-space methods, the frequency-weighted approximation scheme also extends easily to multivariable systems.

§1.2 REVIEW

The modern theory of Hankel operators has its beginnings in the algebraic theory of Hankel and Toeplitz matrices and forms which originated in the memoirs of G. Frobenius [11,12], who studied them in connection with problems of stability theory and related questions of localization of roots of polynomials, and the works of L. Kronecker [24,25].

In the first half of this century, the study of Hankel operators gained an important impetus from applications in function theory, probability theory and the problem of moments [35]; however the present intense interest in the theory can largely be attributed to the discovery of new applications for recent advances, chiefly in function theory, Gaussian processes and systems theory.

Important recent results include the works of Nehari [31], Hartman [18], Adamjan, Arov and Krein [1-4], Clark [7,8] and Peller and Khrushchev [32,33]. A complete overview of this work appears in Power [34] and Peller and Khrushchev [33], while the progress enjoyed by the algebraic theory is detailed in Iohvidov [20].

From the systems theoretic point of view, the most significant of this work applicable to model reduction is that of Adamjan, Arov and Krein [3] which appeared in English in 1971. This work went largely unnoticed (by systems theorists) until 1978 when P. Dewilde [23] pointed out its importance to the larger system theoretic community. In particular [3] is the source of the theory behind the optimal Hankel-norm approximation technique which has found so many fruitful engineering applications.

Using the theory of balanced realizations, Silverman and Bettayeb [36] made the existence results of [3] concrete and gave explicit algorithms for obtaining the optimal Hankel-norm approximation of any specified order to a given finite-dimensional scalar system. Kung and Lin [26] employed a polynomial approach that led to a simple generalised eigenvalue formulation of the optimal Hankel-norm problem and gave a fast matrix-fraction description algorithm for its solution. This approach was extended in [27] to obtain efficient algorithms for multi-variable systems while [28] uses a state-space approach to solve the same problem.

A different approach is that of Harshavardhana et al. [19] who with the implementation of one algorithm obtained solutions of all orders to the optimal Hankel-norm problem for a scalar system from the eigenvectors of a bilinear matrix combination of rearranged versions of the Hurwitz matrix for the characteristic polynomial of the system. An interesting application to model reduction is that of Jonckheere and Helton [22], where the order of a stochastic process power spectrum is reduced by finding the optimal Hankel-norm approximant of the phase of the outer spectral factor. The most complete treatment, known to the author, of the optimal Hankel-norm approximation problem for multivariable systems is that of Glover [14]. [14] uses results in balanced realizations, all-pass functions and the inertia of matrices to obtain all solutions by solving Lyapunov equations and for one class of solutions gives explicit error bounds in the supremum norm.

§1.3 APPROACH AND CONTRIBUTION OF THE THESIS

Underlying the work of this thesis is the considerable theory of Hankel operators and their applications to linear systems. From this large body of theory, §§1.4 and 1.5 assemble some well known results, including the main result of Adamjan, Arov and Krein about which the thesis is developed. §1.5 also introduces and defines an important set of input-output invariants of a linear system; the singular values, in a more general setting. Also presented there, is a new derivation of a weak bound on the first singular value of the optimal Hankel-norm approximant. In §1.6, we derive some shift inequalities for the singular values of rational functions that contain an all-pass factor. In particular, we partially extend a result of Glover [14], concerning the optimal Hankel-norm approximation of stable rational all-pass systems to the unstable case.

The work in Chapter 2 introduces and develops the central new idea of the thesis; combining frequency weighting and optimal Hankel-norm approximation. The basic idea involved is as follows: consider a series combination of two stable systems the second of which is to be approximated by a lower order system. We desire the approximation error to be determined by the magnitude of the frequency response of the first system. Available from the Adamjan Arov and Krein theory, is a procedure for obtaining the explicit form of an optimal approximating system (of reduced order) which we wish to utilise. The synthesis is performed thus: first devise a transformation that, when applied to the first, frequency weighting system, produces a completely unstable system with the same magnitude (with frequency), as the frequency weighting. Next, combine it with the second system to form a modified system. Apply now the optimal Hankel-norm approximation technique to the modified system. The modification is now "undone" by multiplying the resulting approximation by the image of the frequency weighting under the inverse of the original transformation. Finally, take the stable part of the result as the frequency-weighted optimal Hankel-norm approximation. This method is formally developed in §§2.1 to 2.4. We conclude the chapter with a practical example of the technique. A classical Butterworth filter is reduced in order by two, the approximation being frequency shaped by a simple second order system.

The core of the third chapter consists of a more thorough error analysis of the frequency weighted optimal Hankel-norm approximation theory presented in Chapter 2. Error bonds are derived for the incurred error in both the supremum (L^{∞}) and Hankel-norms so that frequency weighting may be compared with other candidate model reduction schemes. In particular, the bounds allow an easy comparison with those for direct optimal Hankel-norm approximation without frequency weighting.

The final chapter of the thesis presents a retrospective overview, followed by several suggestions for future research into the many new ideas that arose during the investigations of the thesis.

The appendix is supplemental to Chapter 2 in that it presents in part the same theory contained in that chapter but derived independently in continuous time with the use of integral operators.

In summary, the contribution of the thesis is first, to propose and develop the theory of a new method for combining the optimal Hankel-norm approximation technique with the analytical specification of the approximation accuracy with frequency, and secondly to present an error analysis and demonstrate the applicability of the method via a common example from circuit theory.

§1.4 SOME PRELIMINARY THEOREMS

As a preliminary, we assemble several important known results for Hankel operators on a Hilbert space. They will be used repeatedly in the sequel.

Let ℓ^2 denote the Hilbert space of two-way square summable complex sequences $w = \{\ldots w_2, w_1, w_1', w_2', \ldots\}$ and ℓ_+^2 the (sub) Hilbert space of one sided square summable complex sequences $y = \{y_1', y_2', \ldots\}$. ℓ^2 is endowed with the inner product $(v,w) = \sum_{j=1}^{\infty} (v_j \bar{w}_j + v_j' \overline{w_j'})$, where the bar denotes the complex conjugate. Let $\{e_1', e_2', \ldots\}$ denote the standard orthonormal basis on ℓ_+^2. Each operator Y on ℓ_+^2 has a representing matrix $Y_{ij} = (Ye_i', e_j')$ with respect to the standard basis. The quantity $\|Y\|$ will denote the induced operator norm on ℓ_+^2. The mapping determined by $f(e^{i\theta}) = \sum_{j=1}^{\infty} (f_j e^{-ij\theta} + f_j e^{i(j-1)\theta})$ is a Hilbert space isomorphism from ℓ^2 to L^2, the space of complex valued functions defined on the unit circle $C = \{z : z = e^{i\theta}, -\pi \leqslant \theta < \pi\}$ which are square Lebesgue integrable there.

Each sequence in ℓ_+^2 under the mapping $y(z) = \sum_{j=1}^{\infty} y_j' z^{j-1}$ determines a member of the Hardy class H^2, analytic in the open unit disk. This is also a Hilbert space isomorphism.

By L^∞ denote the space of all complex valued functions, f, on the unit circle with bounded L^∞-norm, ie $\| f \|_\infty = \text{ess} \sup_\theta |f(e^{i\theta})| < \infty$, and H^∞ the space of complex valued functions which are analytic and bounded in the open unit disk, $|z| < 1$. It can be seen that $L^\infty \subset L^2$ and $H^\infty \subset H^2$. We now define a Hankel operator on ℓ_+^2.

Definition 1.1 A Hankel operator Γ on ℓ_+^2 is an operator for which there exists a sequence $a_1, a_2, \ldots$ such that $\Gamma_{ij} = a_{i+j-1}$, $i, j = 1, 2, \ldots$.

The following result allows us to construct bounded Hankel operators.

Theorem 1.1 (Nehari [31]) The infinite Hankel matrix $(a_{i+j-1})_{i,j=1}^{\infty}$ determines a bounded linear operator Γ on ℓ_+^2 if and only if there exists an L^∞ function f such that $a_n = f_n$, where f_n is the n-th negative Fourier coefficient of f. That is, $f_n = \frac{1}{2\pi i}\int_C f(\xi)\xi^{n-1} d\xi$, $n = 1, 2, \ldots$. Moreover, given an $f \in L^\infty$, and if Γ is the Hankel operator with matrix elements $\Gamma_{ij} = (f_{i+j-1})_{i,j=1}^{\infty}$, then $\| \Gamma \| = \inf_h \|f+h\|_\infty$, $h \in H^\infty$, and there is a unique $h' \in H^\infty$ such that $\| \Gamma \| = \| f+h' \|_\infty$.

We will call f the symbol function of the bonded Hankel operator Γ and denote the operator and matrix, (terms which are used interchangeably), by $\Gamma(f)$. It will sometimes be convenient to omit the symbol function when no ambiguity can arise. Clearly for any given bounded Hankel operator, the symbol function is not unique because $\Gamma(f) = \Gamma(f+h)$ for all $h \in H^\infty$. Given an $f \in L^\infty$, the unique symbol function $f + h'$ for which $\| \Gamma(f) \| = \| f + h' \|_\infty$, $h' \in H^\infty$, is called the Nehari extension of f and we denote it by f_N.

Definition 1.2 Given an $f \in L^{\infty}$, the Hankel-norm of f is defined by, $\| f \|_H \triangleq \| \Gamma(f) \|$.

Next is a characterization of the compact Hankel operators.

Theorem 1.2 (Hartman [18]). The Hankel operator $\Gamma(f)$ is compact if and only if $f \in H^{\infty} + C$, where C denotes the space of continuous functions on the unit circle.

Finally, we characterise the finite rank Hankel operators.

Theorem 1.3 (Kronecker [13,25]). The infinite Hankel matrix $\Gamma_{ij} = (a_{i+j-1})_{i,j=1}^{\infty}$ is of finite rank n, if and only if $r(z) \triangleq \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots$ is a rational function of z and that in this case, n is the number of poles of $r(z)$, counted with multiplicity.

Throughout the thesis, we will be primarily concerned with Hankel matrices whose symbols are real rational functions that represent discrete-time system transfer functions. Theorem 1.1 then ensures that if, as we will assume throughout, the symbol functions have no poles on $|z| = 1$, the Hankel operators will be bounded. Theorem 1.2 guarantees the existence of bounded singular values (see §1.5) and Theorem 1.3 implies that the Hankel matrices will be of finite rank equal to the number of stable poles (counting multiplicity) of the symbol function. A pole or zero will be called stable if it lies in $|z| < 1$ and unstable if it lies in $|z| > 1$.

Remark: In view of the isomorphism between ℓ_+^2 and H^2 one can also define the Hankel operator on H^2 in terms of projections on L^2 as follows. For $g \in L^2$, define the Riesz projections $[\cdot]_+$ and $[\cdot]_-$ on L^2

by $[g]_+ = \sum_{j=1}^{\infty} \hat{g}_j z^{j-1}$ and $[g]_- = g-[g]_+$. The projection $[\cdot]_-$ selects the stable part of g and projects L^2 onto $L^2 \ominus H^2$. If $f \in L^\infty$ define the Hankel operator $\Gamma(f)$ on H^2 by $\Gamma(f)h \triangleq [fh]_- = [[f]_-h]_-$, $h \in H^2$. This definition, which is equivalent to Definition 1.1, often leads to simplified proofs involving Hankel and Toeplitz operators [33].

1.5 s-NUMBERS AND THE ADAMJAN, AROV AND KREIN THEOREM

An important set of parameters of a linear system, are its singular values which are the s-numbers of the associated Hankel operator. We now define these.

<u>Definition 1.3</u> The (k+1)-st s-number of the Hankel operator $\Gamma(f)$, denoted $\sigma_{k+1}(f)$ is defined by

$$\sigma_{k+1}(f) = \inf_L \| \Gamma(f)-L \| \quad k = 0, 1, 2 \ldots \tag{5.1}$$

where the infemum is taken over all bounded linear operators L of rank $\leq$ k .

It can be shown [17] that $\sigma_{k+1}(f)$ is the (k+1)-st eigenvalue of $(\overline{\Gamma(f)}\Gamma(f))^{\frac{1}{2}}$ and that the s-numbers therefore form a decreasing sequence for increasing k . Note that according to Definition 1.2, $\| f \|_H = \sigma_1(f)$.

<u>Definition 1.4</u> Let σ be an s-number of $\Gamma(f)$. A pair (ξ, η), $\xi, \eta \in \ell_+^2$ is called a Schmidt pair of $\Gamma(f)$ corresponding to σ if

$$\Gamma(f)\xi = \sigma\eta \tag{5.2}$$

and

$$\overline{\Gamma(f)}\eta = \sigma\xi$$

<u>Remark</u>. The name singular values (or Hankel singular values) will be reserved exclusively for the s-numbers of Hankel operators whose symbol is a real rational function.

Definition 1.5 A function $\phi(z)$, not necessarily stable, is called all-pass if $\overline{\phi(z)}\phi(z) = 1$ for all $|z| = 1$.

Important in determining the s-numbers of a given function are the Blaschke products. A rational all-pass function of the form $B(z) = \prod_{j=1}^{n} e^{i\alpha_j} \frac{z-a_j}{1-\bar{a}_j z}$ for some complex numbers $a_1 \ldots a_n$ and where $\alpha_j = \arg a_j$, is called a Blaschke product with n zeros. Note that the function $\overline{B}(z) \overset{\Delta}{=} \overline{B(\overline{z}^{-1})}$ satisfies $\overline{B}B = 1$ for all z and has zeros at the poles of B(z) and poles at the zeros of B(z).

The definition (5.1) may be regarded as an approximation measure that gives the smallest attainable distance from $\Gamma(f)$ using rank k operators. An important contribution of Adamjan, Arov and Krein is the proof that the infemum in (5.1) is attained by a unique infinite Hankel matrix of rank k. In addition, their constructive proof gave a unique symbol function with exactly k stable poles that is a distance $\sigma_{k+1}(f)$ from f in the L^∞-norm. This symbol function is called the k-th order optimal Hankel-norm approximation to f. We state their result in the following theorem.

Theorem 1.4 (Adamjan, Arov and Krein [3]). Given an $f \in L^\infty$, there exists a unique infinite Hankel matrix $\Gamma(x)$ of finite rank $k <$ rank $\Gamma(f)$ such that

$$\|\Gamma(f) - \Gamma(x)\| = \sigma_{k+1}(f) . \tag{5.3}$$

Further, if (ξ,η) is any Schmidt pair of $\Gamma(f)$ corresponding to $\sigma_{k+1}(f)$, then the symbol function $x(z)$ that is given by

$$x(z) = f(z) - \sigma_{k+1}(f)\frac{\eta_-(z)}{\xi_+(z)} \tag{5.4}$$

where
$$\eta_-(z) \overset{\Delta}{=} \sum_{i=1}^{\infty} \eta_i z^{-i}$$
$$\xi_+(z) \overset{\Delta}{=} \sum_{i=1}^{\infty} \xi_i z^{i-1} \tag{5.5}$$

is the unique function that attains the infemum in the expression

$$D_k(f) = \inf_g \| f - g\|_\infty \quad , \tag{5.6}$$

the infemum in (5.6) being taken over all functions g of the form $r + h$ where r is a strictly proper rational function with no more than k stable poles and $h \in H^\infty$. That is

$$\| f - x\|_\infty = D_k(f) \tag{5.7}$$

and $x(z)$ has exactly k stable poles (counted with multiplicity). Moreover, the functions $\eta_-(z)/\xi_+(z)$ have unit modulus on the unit circle (ie they are all-pass).

A more concise proof of this theorem than that which appeared in the original paper, may be found in [33] or [34]. When applying Theorem 1.4 to approximate system transfer functions, $f(z)$ is a stable real rational function. In this case, it can be shown [36] that the rational all-pass function $\eta_-/\xi_+(z)$ has in general $n+k$ stable poles and $n-k-1$ unstable poles where n is the number of poles of $f(z)$. Therefore to avoid retaining unstable poles in the approximation x, $[x]_-$ is taken as the approximating system to $f(z)$. The equality (5.3) then holds since $\Gamma(x) = \Gamma([x]_-)$ but of course (5.7) is no longer valid. $[x]_-$ will be called the stable k-th order optimal Hankel-norm approximation to $f(z)$ and $x(z)$ will be said to solve the optimal Hankel-norm approximation problem of order k.

Theorem 1.4 enables the derivation of several equivalent characterizations for the s-numbers of a Hankel operator which we will now prove.

<u>Lemma 1.5</u> [6,7] The s-number $\sigma_{k+1}(f)$ may be expressed as:

$$\sigma_{k+1}(f) = \inf_{\Lambda} \| \Gamma(f) - \Lambda \| \tag{5.8}$$

$$= \inf_{u} \| \Gamma(uf) \| \tag{5.9}$$

$$= \inf_{v} \| f + v \|_{\infty} \tag{5.10}$$

where the infema are respectively taken over,

Hankel operators Λ of rank k,

Blaschke products u with k zeros in $|z|<1$ and

rational functions v with k stable poles.

<u>Proof</u>. Starting with (5.9), we get by using Nehari's theorem (Theorem 1.1) and Kronecker's theorem (Theorem 1.3),

$$\inf_{u} \| \Gamma(uf) \| = \inf_{u} \inf_{h} \| uf + h \|_{\infty} \qquad h \in H^{\infty}$$

$$= \inf_{u} \inf_{h} \| f + \bar{u}h \|_{\infty}$$

$$= \inf_{r} \inf_{g} \| f + r + g \|_{\infty} \; ; \; g \in H^{\infty}, \; r \text{ regular rational with } k \text{ stable poles}$$

$$= \inf_{r} \| \Gamma(f) - \Gamma(r) \|$$

$$= \inf_{\Lambda} \| \Gamma(f) - \Lambda \| = \sigma_{k+1}(f) \; .$$

The last equality follows from Theorem 1.4.

Note that if $\sigma_{k+1}(f)$ is a repeated s-number, we need to take the infemum in Lemma 1.5 over Blaschke products (etc.) with at most k zeros in $|z|<1$.

The equalities (5.8) and (5.10) show that the symbol function x of (5.4) satisfies

$$\| f - x\|_\infty = \sigma_{k+1}(f) \tag{5.11}$$

and that therefore $D_k(f) = \sigma_{k+1}(f)$.

The explicit form (5.4) gives the following easy bound for $\| x\|_H = \sigma_1(x)$

Lemma 1.6 $\sigma_1(x)$ is bounded according to

$$\sigma_1(f) - \sigma_{k+1}(f) \leqslant \sigma_1(x) \leqslant \sigma_1(f) + \sigma_{k+1}(f) \ . \tag{5.12}$$

Proof Although (5.12) is a trivial consequence of the equality (5.3), we give a slightly more instructive proof of the upper bound. Denote $\sigma_{k+1}(f)$ by σ. According to (5.4), on $z = e^{it}$, $-\pi \leqslant t < \pi$, we have $|x|^2 = |f|^2 - 2|f|\sigma\cos(\theta(e^{it}) - \arg f(e^{it})) + \sigma^2$, where $\theta(e^{it}) = \arg \eta_-/\xi_+$. Thus $(|f| - \sigma)^2 \leqslant |x|^2 \leqslant (|f| + \sigma)^2$ so that $| \, \| f^{-1}\|_\infty - \sigma| \leqslant \| x\|_\infty \leqslant \| f\|_\infty + \sigma$. Because $\Gamma(x) = \Gamma(x+h)$ for all $h \in H^\infty$, we can replace f in Theorem 1.4 by its Nehari extension f_N and consider $\hat{x} = f_N - \sigma\eta_-/\xi_+$. But f_N is by uniqueness just the function $\sigma_1(f)\eta_{1-}/\xi_{1+}$ where (ξ_1, η_1) is any Schmidt pair corresponding to $\sigma_1(f)$. The above argument then gives for $\hat{x}$, $(\sigma_1(f) - \sigma)^2 \leqslant |\hat{x}|^2 \leqslant (\sigma_1(f) + \sigma)^2$. Clearly, $\sigma_1(x) = \sigma_1(\hat{x}) \leqslant \| \hat{x}\|_\infty$ so that the upper bound immediately follows.

§1.6 SYSTEMS WITH A RATIONAL ALL-PASS FACTOR

The method of introducing frequency weighting, to be described in Chapter 2, produces a symbol function which has a rational all-pass factor. Here we present some useful new results on the singular values of such symbols. In particular, we partially extend a result in Glover [14] on the singular values of rational all-pass systems.

Let $F(z) \in L^{\infty}$ be a stable rational function of degree n and $\phi(z)$ a rational all-pass function with m_1 stable poles and m_2 unstable poles. That is,

$$\phi(z) = \bar{B}_1 B_2 \tag{6.1}$$

where B_1 and B_2 are (modulo a factor $e^{i\alpha}$, α real) Blaschke products of orders m_1 and m_2 respectively. Assume also that m_2 does not exceed $n-1$ and that no pole-zero cancellations occur in the combination $F\phi$. Noting that $\sigma_i(F) = 0$ for $i > n$, we have the following Lemma.

Lemma 1.7. For F and ϕ as defined above,

$$\text{(i)}\quad \sigma_k(F\phi) \geq \sigma_{k+m_2}(F) \qquad k = 1 \ldots n-m_2 \tag{6.2}$$

$$\text{(ii)}\quad \sigma_{k+m_1}(F\phi) \leq \sigma_k(F) \qquad k = 1 \ldots n . \tag{6.3}$$

Proof Using the characterization (5.18), we have that $\sigma_k(F\phi) = \inf_u \| F\phi + u\|_\infty$, $k = 1 \ldots n+m_1$, where u is rational and has $\leq k-1$ stable poles. ϕ is all-pass so that $\sigma_k(F\phi) = \inf_u \| F + \bar{\phi} u\|_\infty$. According to (6.1), $\bar{\phi} = B_1\bar{B}_2$ has m_2 stable poles and $\bar{\phi}u$ has $\leq k-1+m_2$ stable poles so that $\sigma_k(F\phi) \geq \inf_v \| F + v\|_\infty = \sigma_{k+m_2}(F)$, $k = 1 \ldots n-m_2$, where v is rational and has $k-1+m_2$ stable poles. The inequality (6.2) is trivial for $k > n - m_2$. Similarly, $\sigma_k(F) = \inf_u \|F.1 + u\|_\infty = \inf_u \|F\phi\bar{\phi} + u\|_\infty = \inf_u \|F\phi + \phi u\|_\infty \geq \inf_v \| F\phi + v\|_\infty = \sigma_{k+m_1}(F\phi)$, where v has $\leq k+m_1-1$ stable poles.

We note in passing the following special cases of Lemma 1.7.

Corollary 1.8 (i) If $B_2 = 1$, ie $m_2 = 0$ and ϕ therefore has only unstable zeros, then $\sigma_k(F\phi) \geq \sigma_k(F)$, $k = 1 \ldots n$

(ii) If $B_1 = 1$, ie $m_1 = 0$ and ϕ therefore has only stable zeros, then $\sigma_k(F\phi) \leq \sigma_k(F)$, $k = 1 \ldots n$.

Combining (i) and (ii) of Lemma 1.7 gives bounds on a restricted set of the singular values of $F\phi$ as follows:

$$\sigma_k(F\phi) \leqslant \sigma_{k-m_1}(F) \qquad m_1 < k \leqslant m_1 + n$$
$$\sigma_k(F\phi) \leqslant \sigma_{k+m_2}(F) \qquad 1 < k \leqslant n - m_2 \qquad (6.4)$$

so that when $n > m_1 + m_2$,

$$\sigma_{k+m_2}(F) \leqslant \sigma_k(F\phi) \leqslant \sigma_{k-m_1}(F) \qquad m_1 \leqslant k \leqslant n - m_2.$$

Glover [14] has noted the following result for stable all-pass systems.

<u>Lemma 1.9</u> Let $\phi(z)$ be a real rational stable all-pass function with n zeros. Then

$$\sigma_i(\phi) = 1 \qquad i = 1 \ldots n . \qquad (6.5)$$

Lemma 1.9 implies that all optimal Hankel-norm approximants of a stable all-pass system are equidistant in the Hankel-norm from the original system and that this distance is equal to 1. It would therefore appear useless to approximate all-pass systems because the resulting error is the same size (in Hankel-norm) as the original system. Although Lemma 1.9 takes care of stable all-pass systems, it is not immediately clear that optimal Hankel-norm approximation of unstable all-pass systems will be successful, since Nehari's theorem implies only that the singular values will be $\leqslant 1$. Some of the singular values may therefore be suitably small. The following result shows that certain approximants of unstable all-pass systems are again equidistant from the original system.

<u>Lemma 1.10</u> Let $\phi(z)$ be a real rational all-pass function such that the number n, of stable poles exceeds the number m, of unstable poles. Then

$$\sigma_j(\phi) = 1 \qquad j = 1 \ldots n-m$$
$$\sigma_j(\phi) \leq 1 \qquad j = n-m+1 \ldots n \tag{6.6}$$

Proof ϕ may be written as in (6.1) where B_1 has m zeros and B_2 has $n > m$ zeros. By Nehari's theorem,

$$\sigma_j(\phi) \leq \| \phi \|_\infty = 1 \qquad j = 1 \ldots n . \tag{6.7}$$

According to (5.10) we also have

$$\sigma_j(\phi) = \inf_u \| B_1\overline{B_2} + u \|_\infty = \inf_u \| \overline{B_2} + \overline{B_1}\, u \|_\infty \geq \sigma_{j+m}(\overline{B_2}) , \qquad j = 1 \ldots n-m . \tag{6.8}$$

By Lemma 1.9, $\sigma_i(\overline{B_2}) = 1$ $i = 1 \ldots n$. Thus for $j = 1 \ldots n-m$ (6.8) and (6.7) give (6.6).

Remark. Lemma 1.10 implies that all optimal Hankel-norm approximants of order k such that $k \leq n-m-1$ will be a distance 1 from ϕ in the Hankel-norm.

Finally, we state the important special case $j = 1$ of Lemma 1.10 separately.

Corollary 1.11 If $\phi(z)$ is a real rational all-pass function with more stable than unstable poles, then

$$\| \Gamma(\phi) \| = \| \phi \|_H = 1 . \tag{6.9}$$

CHAPTER 2

§2.1 INTRODUCTION

This chapter presents a method of frequency shaping the error obtained by performing an optimal Hankel-norm approximation of a scalar, finite-dimensional, linear, time-invariant system. As was seen in Theorem 1.4, the optimal Hankel-norm approximation procedure finds a transfer function (or transfer function matrix) of prescribed order which approximates a given transfer function (or transfer function matrix) of greater order.

One motivation for frequency weighting comes from the desire to implement a reduced order approximating controller within a closed-loop control system. Suppose an LQG designed series compensator is to be used in a control system implementation. The compensator will have the same dimension as the plant model. For simplicity, it is desirable to approximate the compensator by one of the lowest order possible while maintaining an overall acceptable degradation in performance. The approximating compensator should be obtained in a way that takes account of the frequency characteristics of the plant model. For example in the plant stop band and at frequencies of high loop gain, the detailed shape of the approximating compensator is not so important, however around the unity gain crossover frequency, it is desirable to secure accurate approximation.

We show here how to modify the approximation method originally developed by Adamjan, Arov and Krein [3] to allow for frequency weighting. The means for introducing frequency weighting while preserving the closed form solvability of the optimal Hankel-norm approximation problem is not immediately apparent. A technique for doing this is described below.

§2.2 DISCRETE-TIME FREQUENCY WEIGHTED APPROXIMATION

We first establish some preliminary notation and results.

Definition 2.1 Let $G(z)$ be a strictly minimum phase, strictly stable, real rational scalar transfer function, ie $G(z)$ has no poles or zeros in $|z|\geqslant 1$, save for a possible zero at $z = \infty$. Let $r \geqslant 0$ be the smallest integer such that $\lim_{z\to\infty} z^r G(z)$ is nonzero. Then define the "tilde" operation by

$$\tilde{G}(z) \triangleq z^{-r} G(z^{-1}) \ . \tag{2.1}$$

It may be noted here that $\tilde{G}$ and $\tilde{G}^{-1}$ are analytic in the closed unit disc.

Given a rational transfer function with Laurent expansion

$$K(z) = \sum_{i=-\infty}^{\infty} k_i z^{-i} \tag{2.2}$$

which converges in some open region containing the unit circle, $|z| = 1$, then the Hankel operator $\Gamma(K) = \Gamma([K]_-)$ has matrix elements

$$\Gamma(K(z))_{ij} = k_{i+j-1} \qquad i, j = 1,2, \ldots \tag{2.3}$$

where $[K]_-$, the stable part of $K(z)$, is equal to the sum of those terms in the partial fraction expansion of $K(z)$ that have poles in $|z|<1$.

The tilde operation (2.1) has the desirable property of preserving the rank of the Hankel matrix $\Gamma(K)$ under the change $K \to K\tilde{G}$. This property is crucial and is proved in the following lemma.

Lemma 2.1 Let $F(z)$ be a strictly stable proper rational transfer function and let $G(z)$ satisfy the conditions of Definition 2.1 and have Laurent expansion $G(z) = \sum_{i=r}^{\infty} g_i z^{-i}$, $g_r \neq 0$, which converges in an open region containing the unit circle. Then

(i) rank $\Gamma(F(z))$ = rank $\Gamma(F(z)\tilde{G}(z))$ (2.4)

and

(ii) $\Gamma(F\tilde{G}) = \Gamma(F)T(\tilde{G}) = T'(\tilde{G})\Gamma(F)$ (2.5)

where $T(\tilde{G})$ is the infinite lower triangular Toeplitz matrix with elements

$$(T(\tilde{G}))_{ij} = \begin{cases} g_{i-j+r} & i \geq j \\ 0 & i < j \end{cases} \qquad i, j = 1, 2, \ldots$$

and $\Gamma(F)$ is as defined by (2.3). Here $T'(\tilde{G})$ denotes the transpose of $T(\tilde{G})$.

<u>Proof</u> (i) Let $\delta_-(K(z)) \overset{\Delta}{=} \delta([K(z)]_-)$ be the McMillan degree of the strictly stable part of K(z). Expanding by partial fractions, we divide $F\tilde{G}$ into its strictly stable and unstable parts.

We then have $\delta_-(F\tilde{G}) = \delta_-([F\tilde{G}]_+) + \delta_-([F\tilde{G}]_-) = \delta_-([F\tilde{G}]_-)$

Observe now that by Definition 2.1, G has no stable pole and no stable pole of F can be cancelled by a zero of $\tilde{G}$. Hence $\delta_-([F\tilde{G}]_-) = \delta_-(F)$, and so by Kronecker's theorem (Theorem 1.3), rank $\Gamma(F\tilde{G})$ = rank $\Gamma(F)$.

(ii) The proof of (2.5) is computational.

Let F(z) have Laurent expansion $F(z) = \sum_{j=0}^{\infty} f_j z^{-j}$. The expansion of $\tilde{G}(z)$ is $\tilde{G}(z) = \sum_{k=0}^{\infty} g_{k+r} z^k$. Then, $[F\tilde{G}(z)]_- = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} g_{k+r} f_{k+j} z^{-j}$,

so that $(\Gamma(F\tilde{G}))_{ij} = \sum_{k+0}^{\infty} g_{k+r} f_{i+j+k-1}$, i, j = 1,2, ...

The centre member of (2.5) is $(\Gamma(F)T(\tilde{G}))_{ij} = \sum_{k=1}^{\infty} f_{i+k-1} g_{k-j+r}$,

but $g_{k-j+r} = 0$ if $j > k$. Substituting, this gives

$(\Gamma(F)T(\tilde{G}))_{ij} = \sum_{k \geq j} f_{i+k-1}\, g_{k-j+r} = \sum_{k \geq 0} f_{i+j+k-1}\, g_{k+r}$, which proves the result.

To prove the second equality of (2.5) we take the transpose of $\Gamma(F\tilde{G}) = \Gamma(F)T(\tilde{G})$ and note that Hankel matrices are symmetric.

In Lemma 2.1 (i), we used the strictly minimum phase nature of G to ensure that no stable pole-zero cancellations occur in $F\tilde{G}$. In fact if we define $\tilde{G}$ for stable non minimum phase G exactly as in (2.1), then all that is required of G for (2.4) to hold is that $\tilde{G}$ have no zeros coinciding with stable poles of F. The strictly minimum phase property of G is however essential as it will allow us to use the following Lemma.

<u>Lemma 2.2</u> Let G(z) be strictly stable. Then $T(\tilde{G})$ is invertible if and only if G is strictly minimum phase and then $T(\tilde{G})^{-1} = T(\tilde{G}^{-1})$.

<u>Proof</u> This is a standard result and the proof may be found for example in [33,38].

We now restate the result of Theorem 1.4 for proper stable rational transfer functions K(z) of McMillan degree n, ie the rank of $\Gamma(K)$ is n. For any positive integer $k < n$, there exists a unique bounded Hankel matrix $\hat{\Gamma}$ of rank k such that

$$\| \Gamma(K) - \hat{\Gamma} \| = \sigma_{k+1}(K) \tag{2.6}$$

where $\sigma_{k+1}(K)$ is the (k+1)-st singular value of $\Gamma(K)$, (the singular values being ordered in descending magnitude). Further $\hat{\Gamma} = \Gamma(X)$, where X(z) is a rational function given by

$$X(z) = K(z) - \sigma_{k+1}(K)\phi(z) \tag{2.7}$$

and $\phi(z)$ is a rational all-pass function with exactly k stable poles. The function X(z) is the unique best L^∞ approximation to K(z) having exactly k stable poles (and possibly some unstable ones), that is

$$\| K(z) - X(z) \|_\infty = \sigma_{k+1}(K) \tag{2.8}$$

and $\delta_-(X) = k$.

Here the L^∞-norm is taken on the unit circle. The transfer function $[X(z)]_-$ can be regarded as an approximation to $K(z)$ which is stable and has precisely k poles.

To make use of the explicit solution afforded by (2.6) and (2.7) we set $K(z) = F\tilde{G}(z)$ where $G(z)$ and $F(z)$ are as in Definition 2.1 and Lemma 2.1 respectively. Using Lemmas 2.1 and 2.2, the difference $\Gamma(F\tilde{G}) - \Gamma(X)$ may be written as

$$(\Gamma(F) - \Gamma(X\tilde{G}^{-1}))T(\tilde{G}) \tag{2.9}$$

which satisfies

$$\| (\Gamma(F) - \Gamma(X\tilde{G}^{-1}))\ T(\tilde{G}) \| = \sigma_{k+1}(F\tilde{G}). \tag{2.10}$$

The bracketed expression in (2.10) suggests identifying $\Gamma(X\tilde{G}^{-1})$ as a rank k approximation to $\Gamma(F)$, the function $X\tilde{G}^{-1}(z)$ as an L^∞ approximation to $F(z)$, and the function $[X\tilde{G}^{-1}(z)]_-$ as a stable reduced order approximation to $F(z)$. Note now that $\Gamma(X\tilde{G}^{-1})$ although not an optimal Hankel-norm approximation to $\Gamma(F)$, has the appropriate rank and incorporates the weighting G in a manner that preserves the structure of the problem. Defining

$$W \overset{\Delta}{=} X\tilde{G}^{-1} = (F\tilde{G} - \sigma_{k+1}(F\tilde{G})\phi(z))\tilde{G}^{-1} = F - \sigma_{k+1}(F\tilde{G})\phi\tilde{G}^{-1} \tag{2.11}$$

we can take

$$\hat{W} = F - \sigma_{k+1}(F\tilde{G})[\phi\tilde{G}^{-1}]_- \tag{2.12}$$

as a stable k-th order approximation to $F(z)$.

§2.3 L^∞ ERRORS OF THE APPROXIMATION

With the approximation (2.11), which retains an unstable part, the error in the weighted gain FG is

$$FG - WG = \sigma_{k+1}(F\tilde{G})\phi\tilde{G}^{-1}G\ . \tag{3.1}$$

Because both ϕ and $\tilde{G}^{-1}G$ are all pass functions, the magnitude of the difference in (3.1) is constant over all frequencies, that is

$$|FG(e^{i\omega}) - WG(e^{i\omega})| = \sigma_{k+1}(F\tilde{G}) \tag{3.2}$$

for all real ω. The role played by G as the frequency weighting is clearly displayed in this formula. Using the stable k-th order approximation of (2.12), we obtain the lower bound

$$\| FG - \hat{W}G\|_{\infty} \geq \sigma_{k+1}(F\tilde{G}) \ . \tag{3.3}$$

Note that the actual weighting in (3.2) and (3.3) is determined by $|G|$, rather than $|G|$ and argG. ArgG is only important in that it leads to the minimum phase, strictly stable property of G on which the algorithm rests. One could thus conceive of an arbitrary weighting function, free of poles and zeros on $|z|=1$, and replace it by a strictly minimum phase strictly stable function with the same amplitude on $|z|=1$.

Although we have not presented an upper bound (see §3.3), it seems fortuitously characteristic of most examples, that the maximum error is near the lower bound. The error in F using the approximation (2.11) is, $F - W = \sigma_{k+1}(F\tilde{G})\phi\tilde{G}^{-1}$, giving a frequency dependent error of

$$|F(e^{i\omega}) - W(e^{i\omega})| = \sigma_{k+1}(F\tilde{G})\,|\tilde{G}^{-1}(e^{i\omega})| = \sigma_{k+1}(F\tilde{G})\,|G^{-1}(e^{i\omega})| . \tag{3.4}$$

A small error given by (3.4) at a particular frequency, does not however guarantee a small error in $|F - \hat{W}|$. This error is bounded below by

$$\| F - \hat{W}\|_{\infty} \geq \sigma_{k+1}(F\tilde{G})\,\| G\|_{\infty}^{-1} \ . \tag{3.5}$$

We postpone a derivation of an upper bound for these errors to Chapter 3 (§3.3).

Note finally that if P and C are plant and nominal series transfer functions respectively, the use of $G = P(1 + CP)^{-1}$ seems appropriate in considering the problem of controller approximation. For if $\Delta = C-\hat{C}$ represents the error between C and an approximation $\hat{C}$, C is stabilizing and C and $\hat{C}$ have the same number of unstable poles, then $\hat{C}$ will be stabilizing when $\| \Delta G\|_\infty \leq 1$.

§2.4 THE CONTINUOUS-TIME CASE

Frequency weighting in continuous time can be performed by using the method of §2.3 after transforming to discrete time via the bilinear transformation $s = (z+1)/(z-1)$. This is described in detail by Lin and Kung [29]. Note however that if G(s) is proper, strictly stable and minimum phase and $\lim_{s\to\infty} G(s)$ is finite and nonzero, then the obvious analogue to Definition 2.1 is

$$\tilde{G}(s) = G(-s) . \tag{4.1}$$

It is however not clear how to define G for strictly proper systems. Now an optimal Hankel-norm reduction can be performed on the system $K(s) = [F(s)G(-s)]_-$ where F(s) is strictly stable and proper of order n. Again $[\cdot]_-$ denotes the operation of taking the strictly stable part. The resulting k-th order approximation has the same form as (2.7). We again take

$$\hat{W} = F - \sigma_{k+1}(F\tilde{G})[\phi\tilde{G}^{-1}]_- \tag{4.2}$$

as the stable k-th order frequency weighted approximation to F(s). Similar error bounds to those of §2.3 apply with L^∞-norms taken on the $j\omega$ axis. In particular, the corresponding results to (3.3) and (3.4) are,

$$\| FG - \hat{W}G\|_\infty \geq \sigma_{k+1}(F\tilde{G}) \tag{4.3}$$

and

$$|F(j\omega) - W(j\omega)| = \sigma_{k+1}(F\tilde{G})\,|G(j\omega)|^{-1} \tag{4.4}$$

§2.5 AN EXAMPLE

As an example of the method developed in §§2.2-2.4, we apply the algorithms of Glover [14] to a continuous time example. It should be noted however that instead of the approximation (4.2), these algorithms take

$$W_1 = F - \sigma_{k+1}(F\tilde{G})[\phi\tilde{G}^{-1}]_- + D \tag{5.1}$$

as the stable k-th order approximation to F(s). The choice of the constant D is specific and is explained in [14]. It is chosen to reduce the final L^∞ error. Also see §3.3.

We take for F(s) a sixth order Butterworth filter with 3dB point at $\omega = 1.0$. The transfer function is $F(s) = Q^{-1}(s)$, where

$$Q(s) = s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 7.4641s^2 + 3.8637s + 1.$$

The weighting is given by the second order system

$$G_\alpha(s) = \frac{(s+1)^2}{s^2+2\alpha s+1} \tag{5.2}$$

where $\alpha \leq 1$ is chosen to vary $\| G_\alpha\|_\infty$.

Fig. 1 shows the magnitudes $|G_\alpha|$ plotted against angular frequency for two particular choices of α used in this example. Curve 1 has $\alpha = 0.1$ and attains an L^∞-norm of 10, and curve 2 has $\alpha = .01$ with a corresponding L^∞-norm of 100. The L^∞-norms are attained at $\omega = 1.0$.

For convenience of notation we denote by $E_\alpha(s)$ the error system given by $E_\alpha(s) = F(s) - W_1(s)$, where W_1 is given by (5.1) and the subscript α indicates that G_α is the frequency weighting used in (5.1). Note that with this notation, $E_{1.0}(s)$ is the error obtained by performing a Hankel-norm approximation of F(s) with no frequency weighting.

The singular values of the system F are:
$\overline{\sigma}_1 = .94707$, $\overline{\sigma}_2 = .70013$, $\overline{\sigma}_3 = .32544$, $\overline{\sigma}_4 = .08278$, $\overline{\sigma}_5 = .00113$, $\overline{\sigma}_6 = .00630$.
This set of singular values indicates $\overline{\sigma}_4$ as a natural cutoff ($\overline{\sigma}_4/\overline{\sigma}_5 \simeq 8$) and so we perform a fourth order frequency weighted optimal Hankel-norm approximation of F(s).

Example (i): $\alpha = 0.1$

With $\alpha = 0.1$, the singular values of the system K(s) = $[F(s)G_{0.1}(-s)]_-$ are: $\sigma_1 = 2.6790$, $\sigma_2 = 2.1589$, $\sigma_3 = .84239$, $\sigma_4 = .19287$, $\sigma_5 = .021903$, $\sigma_6 = .0011311$. Although each σ_i is larger than the corresponding $\overline{\sigma}_i$, the new set of singular values still indicates a fourth order approximation is appropriate, ($\sigma_4/\sigma_5 \simeq 9$). Fig. 2 shows the magnitude of the resulting error systems $E_{1.0}(s)$ and $E_{0.1}(s)$. We see that $|E_{0.1}(j\omega)|$ is significantly smaller than $|E_{1.0}(j\omega)|$ in a frequency band about $\omega = 1.0$ and is frequency shaped according $|G_{0.1}^{-1}(j\omega)|$, the oscillation near $\omega = 1.0$ being a typical consequence of neglecting the unstable part of the approximation in (2.11). Away from $\omega = 1.0$, the magnitude of $E_{0.1}(j\omega)$ exceeds significantly that of $E_{1.0}(j\omega)$, however this is a region where we can tolerate a less accurate approximation. The method of frequency weighting has thus performed better than the direct Hankel-norm approximation in the pass band of $G_{0.1}(s)$ and worse in the stop band of $G_{0.1}(s)$.

The magnitudes of the errors in the weighted gains, namely $|E_{1.0}(j\omega)G_{0.1}(j\omega)|$ and $|E_{0.1}(j\omega)G_{0.1}(j\omega)|$ appear in Fig. 3. We see that

$\| E_{0.1}G_{0.1}\|_\infty \simeq .031$ which as expected is close to the lower bound in (3.3), namely $\sigma_5 = .0219$. In fact the magnitude of the errors over all frequencies is close to this value and again in the pass band of $G_{0.1}(s)$ it is much less than the error obtained when $E_{1.0}(s)$ is taken in series with the same frequency weighting.

Example (ii): $\alpha = 0.01$

With $\alpha = .01$, the singular values of $K(s) = [F(s)G_{.01}(-s)]_-$ are $\sigma_1 = 3.6669$, $\sigma_2 = 2.7631$, $\sigma_3 = .94358$, $\sigma_4 = .22032$, $\sigma_5 = 024776$, $\sigma_6 = .001228$. Figure 4 shows the magnitudes of $E_{1.0}(s)$ and $E_{.01}(s)$. The shape of the curve 2 is similar to Fig. 2. Although the minimum error is significantly less than that for $E_{0.1}(s)$, the oscillation near $\omega = 1.0$ causes an improvement by only a factor of about 2 in the pass band of $G_{.01}(s)$. The frequency band in which $|E_{.01}(j\omega)| < |E_{1.0}(j\omega)|$ is less than the corresponding band in Fig. 2. The L^∞-norm $\|E_{.01}(j\omega)G_{.01}(j\omega)\|_\infty$ can be seen from Fig. 5 to be approximately 0.15 which is much greater than the lower bound in (3.3), here $\sigma_5 = .024226$. In the pass band of $G_{.01}(s)$, the magnitude of $E_{.01}(j\omega)G_{.01}(j\omega)$ is still much less than that for the direct Hankel-norm approximation used in series with $G_{.01}(s)$, (curve 1 of Fig. 5). The increase in the L^∞-norm of $E_\alpha(j\omega)G_\alpha(j\omega)$ for small α suggests that the L^∞-norm of the frequency weighting G (s) needs to be carefully chosen to obtain a good compromise in the size of the loop gain error and the error in the approximation to F(s).

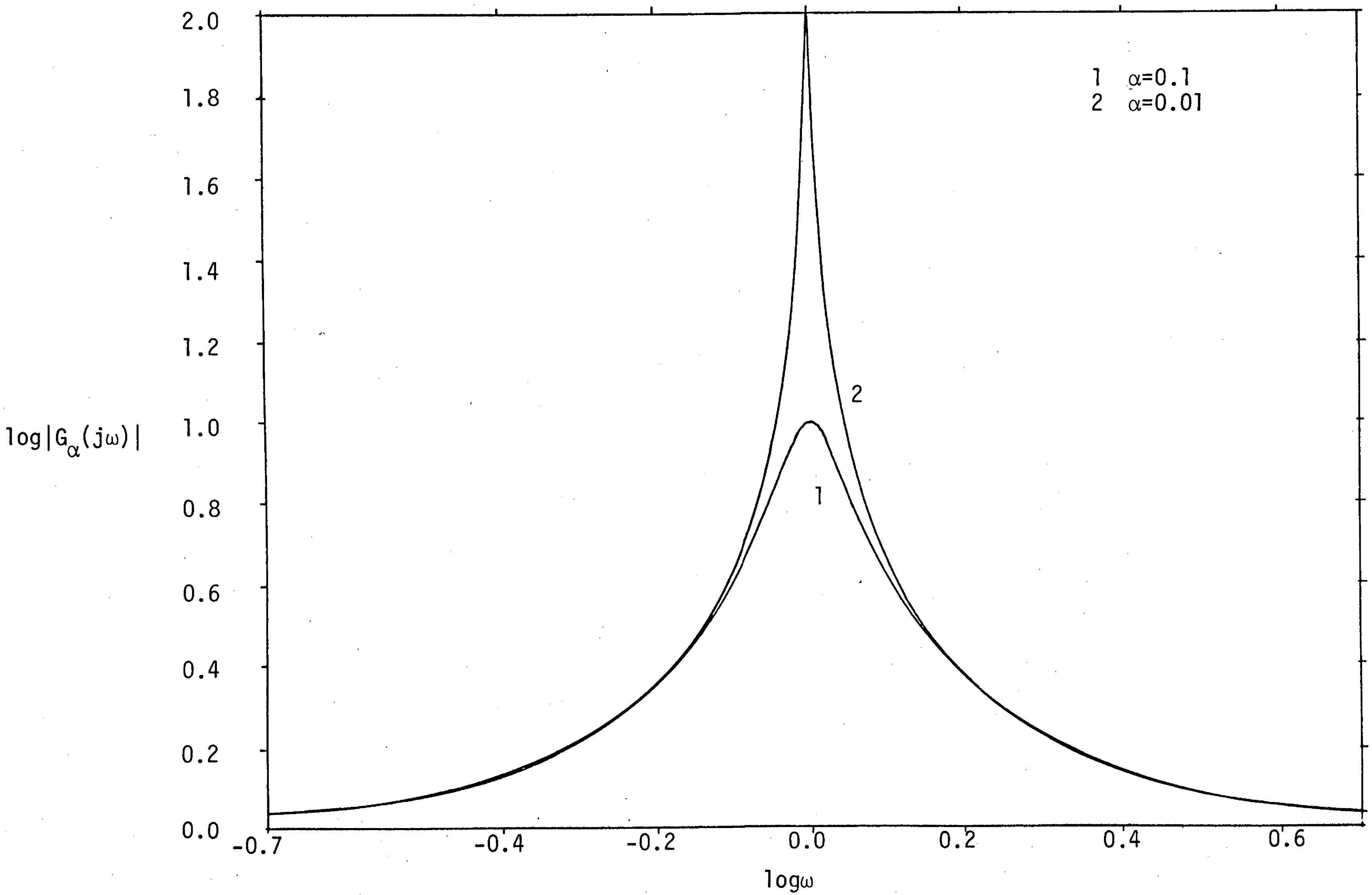

Fig 1. The magnitude of the frequency weighting.

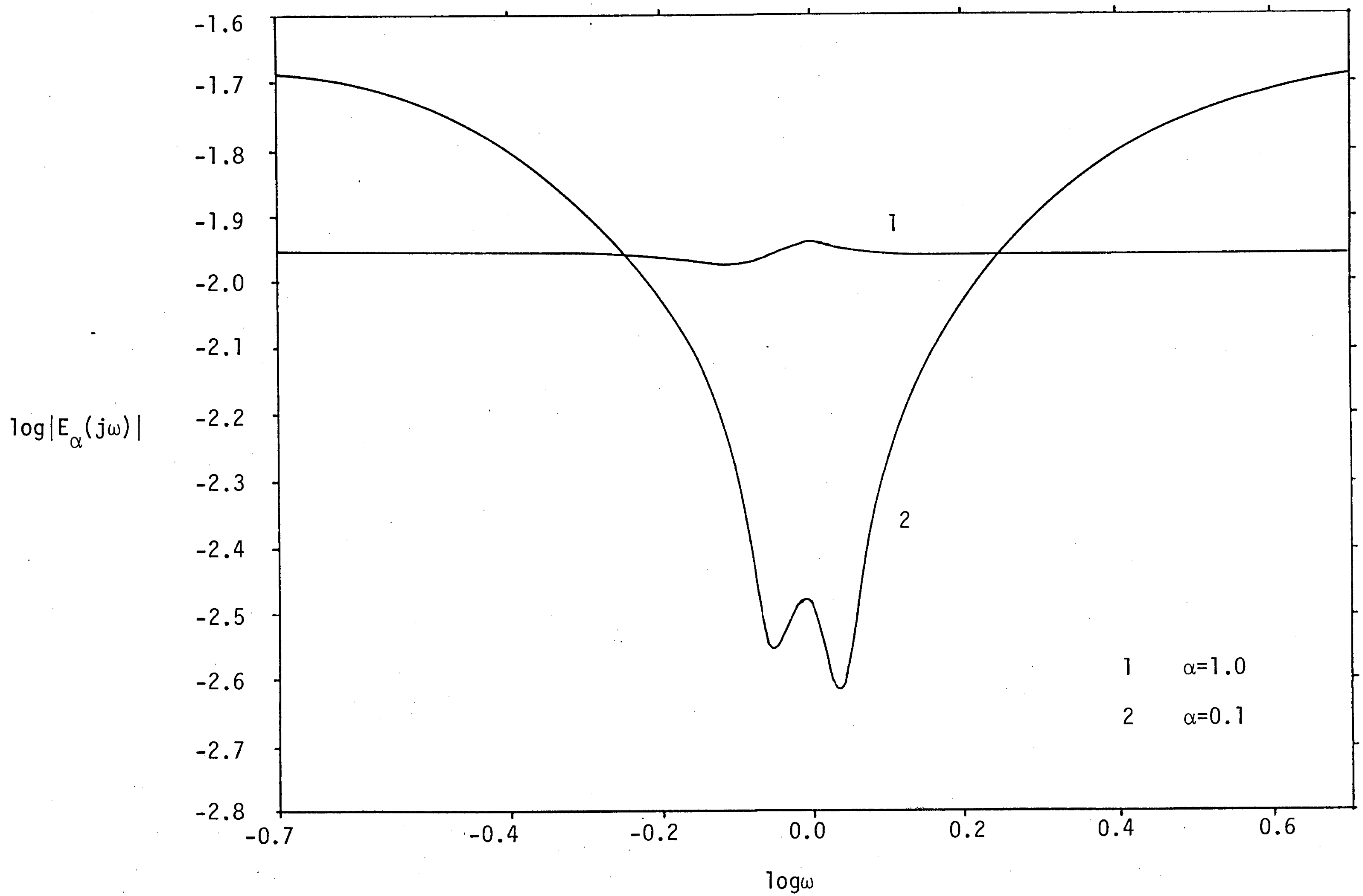

Fig 2. A comparison of the magnitudes of the errors for the unweighted and frequency weighted optimal Hankel-norm approximations.

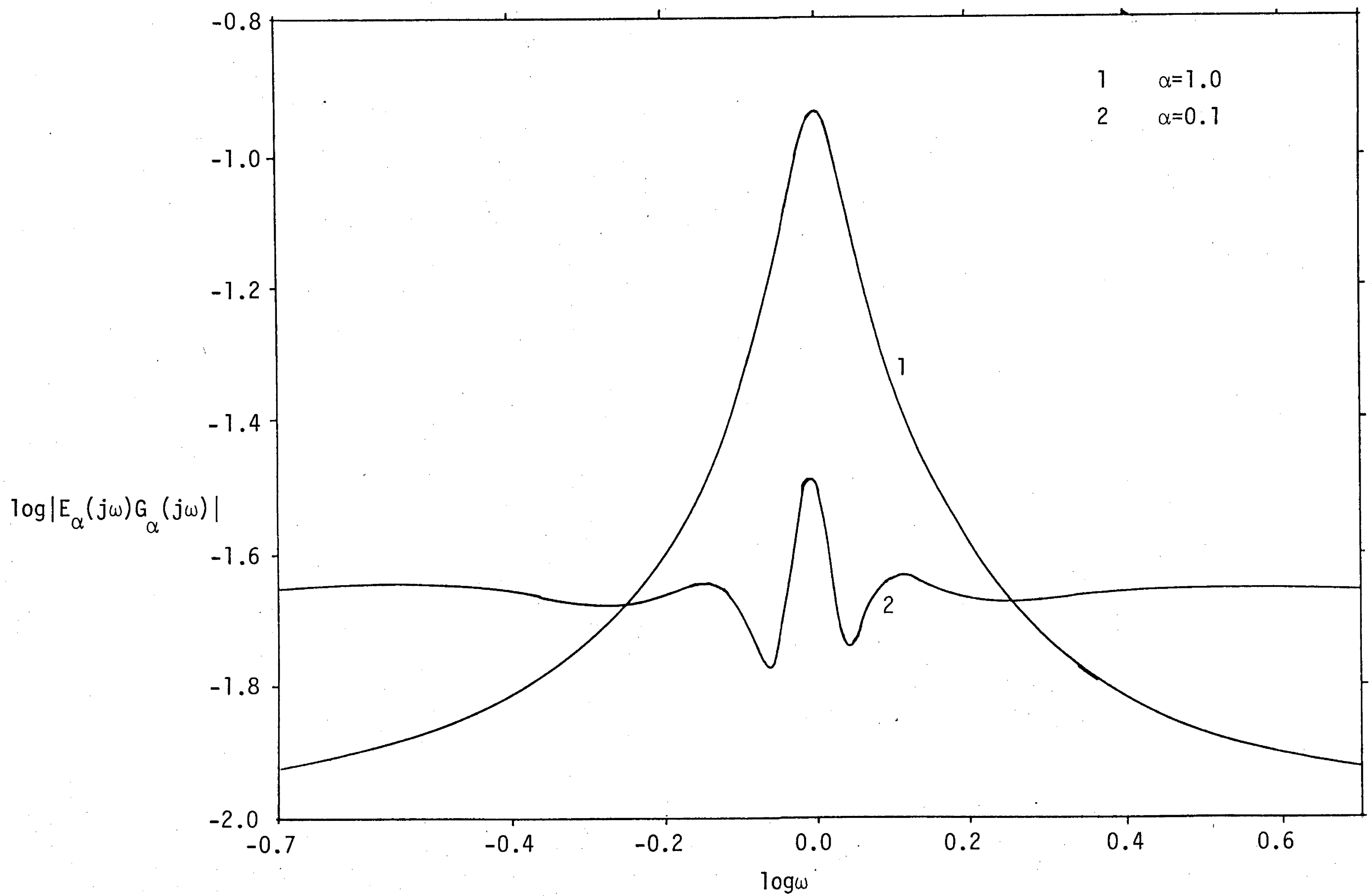

Fig 3. A comparison of the magnitudes of the weighted errors for the approxomations shown in Fig 2.

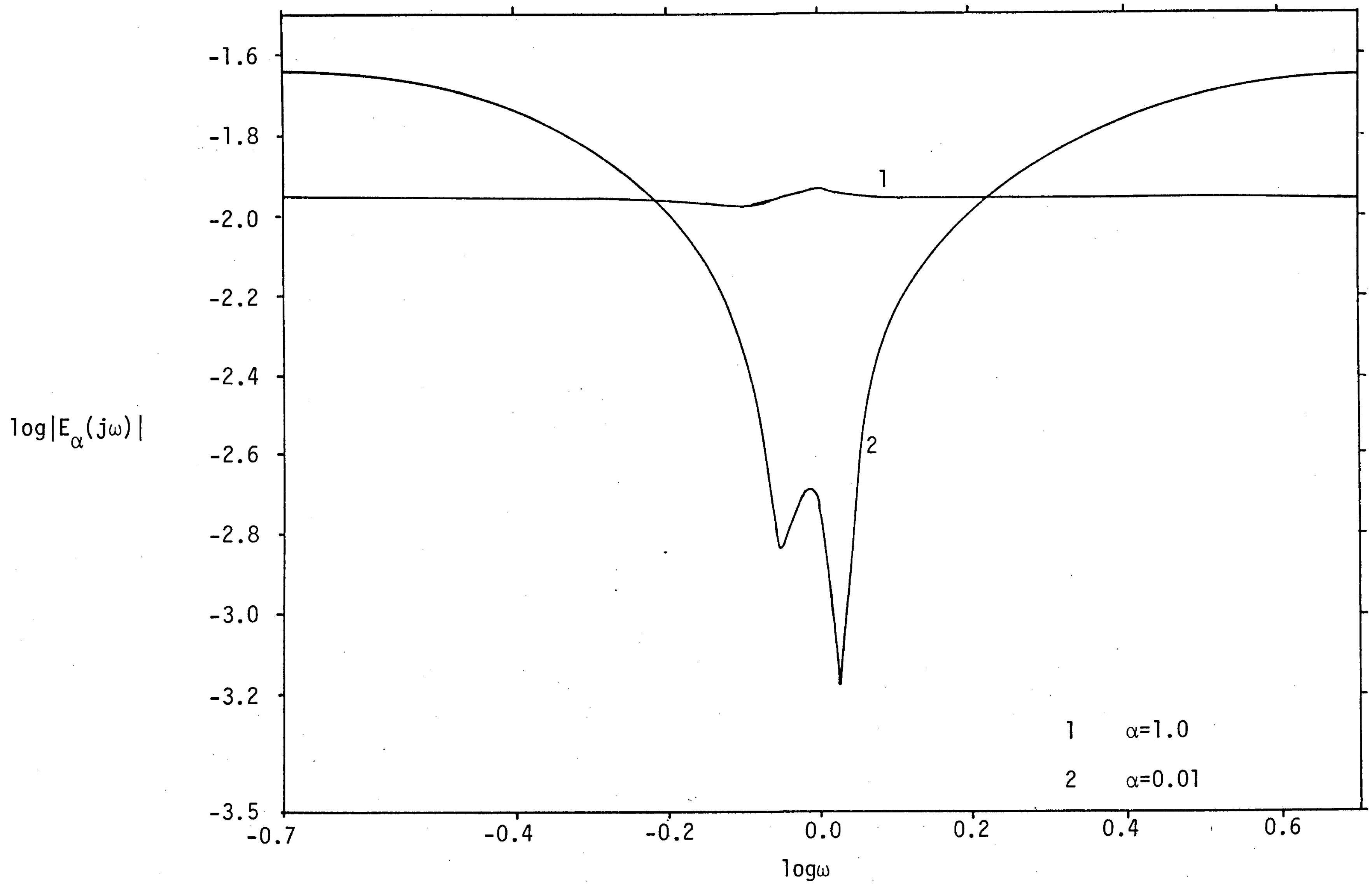

Fig 4. The magnitude of the error for a frequency weighting with L^{∞}-norm 10 times that in Fig 2.

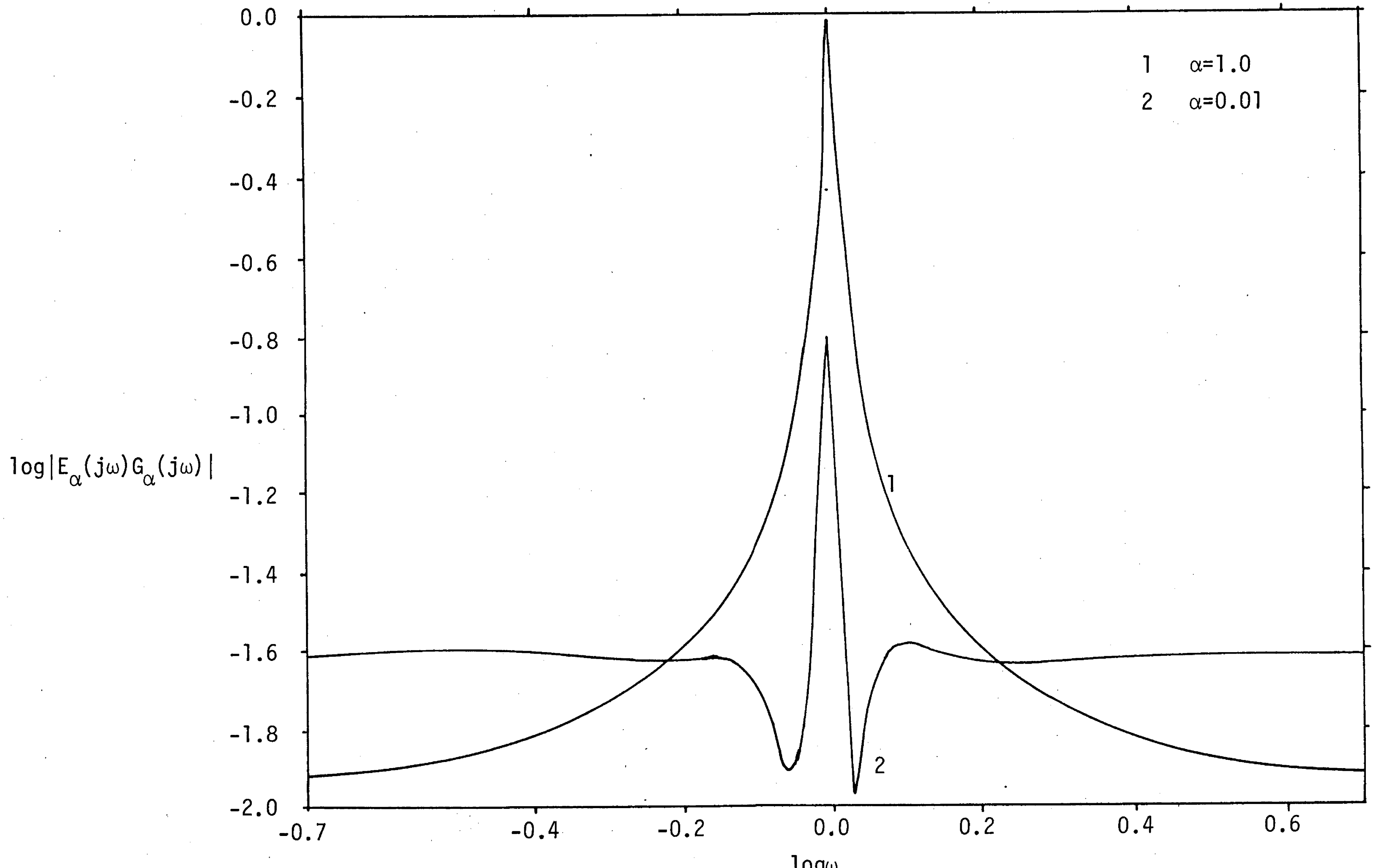

Fig 5. The magnitudes of the weighted errors corresponding to the approximations in Fig 4.

CHAPTER 3

§3.1 INTRODUCTION

The Hankel-norm approximation problem obtained by introducing frequency weighting in the manner described in §2.2, differs in several important respects from the direct Hankel-norm problem with no frequency shaping. Although the rank of the Hankel matrix $\Gamma(F\tilde{G})$ is preserved, the singular values are of course corrupted. This means that a new set of L^∞ and Hankel-norm error bounds will be obtained.

In this chapter, we explore by how much frequency weighting changes the singular values and derive L^∞ and Hankel-norm bounds for the error of the resulting stable approximating system. This will in part answer the following important questions:

(i) What sacrifice in accuracy is made in the resulting error in the L^∞ and Hankel-norm measures by using frequency weighting?

(ii) How are "closeness" in the L^∞ and Hankel-norms related?

(iii) What effect does the constant D of §2.5 have on the L^∞ and Hankel-norm bounds?

§3.2 KNOWN L^∞-NORM BOUNDS

The L^∞ error analysis will be based on the results of Glover [14]. Here, we present without full proof the relevant results in [14]. They relate the L^∞-norm of a stable transfer function to the Hankel singular-values of that function which are assumed for convenience to be non-repeated. First is a representation Lemma.

<u>Lemma 3.1</u> Let F(z) be a stable, rational transfer function of McMillan degree n and have Hankel singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_n$.

Then there exists a representation of F(z) as

$$F(z) = D_0 + \sigma_1 E_1(z) + \dots + \sigma_n E_n(z) \tag{3.1}$$

where

(1) $E_j(z)$ is all-pass and stable for all j

(2) For j = 1, ..., k

$$F_k(z) \triangleq D_0 + \sum_{j=1}^{k} \sigma_j E_j \tag{3.2}$$

has McMillan degree k.

This result is easily proved by recursing on the one-step optimal Hankel-norm approximation of F(z) while noting that in this case, the singular values are preserved and the all-pass functions η_-/ξ_+ (of §1.5) are stable.

Lemma 3.2 In addition to the properties of Lemma 3.1, let F(z) be such that $\lim_{z\to\infty} F(z) = 0$. Then

(1) $\| F\|_\infty \leqslant 2(\sigma_1 + \dots + \sigma_n)$ (3.3)

(2) there exists a constant D_0 such that

$$\| F - D_0\|_\infty \leqslant \sigma_1 + \dots + \sigma_n \tag{3.4}$$

and

$$\| D_0\|_\infty \leqslant \sigma_1 + \dots + \sigma_n \tag{3.5}$$

Proof (3.4) is obvious if D_0 is the constant in (3.1). (3.5) follows by setting z = ∞ in (3.1) and then (3.3) is immediate.

Although by definition, the constant term in the Laurent expansion of a transfer function plays no part in determining the optimal Hankel-norm approximation problem, it is possible to add a constant to the solution of this problem, in such a way that the overall L^∞-error is reduced. This is the theme of the following result.

<u>Lemma 3.3</u> Let X(z) be the optimal Hankel-norm approximation of degree k to the stable rational transfer function F(z) of degree n. Then there exists a constant D_0 such that:

$$\| F - X_- - D_0 \|_\infty < \sigma_{k+1}(F) + \sum_{i=k+2}^{n} \sigma_i(F) \tag{3.6}$$

and
$$\| D_0 \|_\infty \leq \sum_{k+1}^{n} \sigma_i(F) \tag{3.7}$$

where $X_- \triangleq [X]_-$.

§3.3 ERROR ANALYSIS

In this section, we use the results of §3.2 to derive upper and lower bounds on the L^∞-norm of the error for the frequency weighted approximation given in §2.3. We also present bounds for the norms of the associated Hankel matrices for the frequency weighted case.

The first result is a growth bound on the singular values of the system $F\tilde{G}$ in terms of those of the system F.

<u>Lemma 3.4</u> Let F(z) be a real proper rational and stable transfer function of degree n. Let G(z) be a stable and minimum phase proper transfer function. Then

$$\|G^{-1}\|_\infty^{-1} \leq \frac{\sigma_i(F\tilde{G})}{\sigma_i(F)} \leq \| G \|_\infty \qquad i = 1, \ldots, n \tag{3.8}$$

where $\tilde{G}$ is as in Definition 2.1, §2.2.

<u>Proof</u> Because G is real rational, $\|\tilde{G}\|_\infty = \|G\|_\infty$ and $\|\tilde{G}^{-1}\|_\infty = \|G^{-1}\|_\infty$. We also have that for any $K(z) \in L^\infty$, then $\|T(K)\| = \|K\|_\infty$ where T(K) is the Toeplitz matrix with symbol K(z) (see [5]). According to (5.10) of §1.5, $\|G\|_\infty \sigma_i(F) =$

$\|\tilde{G}\|_\infty \sigma_i(F) = \inf_L \|\Gamma(F) - L\| \; \|T(\tilde{G})\| \geqslant \inf_L \|(\Gamma(F) - L)T(\tilde{G})\|$.

Using Lemma 2.1, §2.2, this is $\|G\|_\infty \sigma_i(F) \geqslant \inf_L \|\Gamma(F\tilde{G}) - LT(\tilde{G})\|$

Again by Lemma 2.1, §2.2, $LT(\tilde{G})$ is Hankel and has rank i-1, hence

$\|G\|_\infty \sigma_i(F) \geqslant \inf_\Lambda \|\Gamma(F\tilde{G}) - \Lambda\| = \sigma_i(F\tilde{G})$

Similarly, $\|G^{-1}\|_\infty \sigma_i(F\tilde{G}) = \inf_L \|\Gamma(FG) - L\| \; \|T(\tilde{G}^{-1})\|$

$\geqslant \inf_L \|(\Gamma(F\tilde{G}) - L)T(\tilde{G}^{-1})\| \geqslant \inf_L \|\Gamma(F) - LT(\tilde{G}^{-1})\|$

$\geqslant \inf_\Lambda \|\Gamma(F) - \Lambda\| = \sigma_i(F)$.

It is also possible to relate the singular values of $F\tilde{G}$ to those of FG by using the results of §1.6. To do this, we note that $F\tilde{G} = FG\bar{B}$ where $B = G\tilde{G}^{-1}$ is all-pass. $F\tilde{G}$ is therefore a system with an all-pass factor. For simplicity, assume that B has $m < n$ stable poles and m unstable poles ie G has the same number of zeros and poles. The results of Lemma 1.7, §1.6, then translate as follows:

$\sigma_i(F\tilde{G}) \geqslant \sigma_{i+m}(FG)$ $\qquad i = 1 \ldots n$

and $\sigma_{i+m}(F\tilde{G}) \leqslant \sigma_i(FG)$ $\qquad i = 1 \ldots n-m$ $\qquad$ (3.9)

and $\sigma_{i+m}(FG) \leqslant \sigma_i(F\tilde{G}) \leqslant \sigma_{i-m}(FG)$ $\qquad i = m + 1 \ldots n$

<u>Remark</u> The inequalities (3.9) provide information about the relative merits, in terms of the L^∞ and Hankel-norm errors, of performing and frequency weighted approximation of F(z) or a direct optimal Hankel-norm approximation of FG.

For convenience, we recall the notation of §2.2 as follows:

(i) (ξ,η) is a Schmidt pair of $\Gamma(F\tilde{G})$ corresponding to $\sigma_{k+1}(F\tilde{G})$.

(ii) $X = F\tilde{G} - \tilde{\sigma}\phi$ (3.10)

is the k-th order of optimal Hankel-norm approximation of $F\tilde{G}$ where $\tilde{\sigma} \triangleq \sigma_{k+1}(F\tilde{G})$ and $\phi(z) = \eta_-(z)/\xi_+(z)$, and

(iii) $W = X\tilde{G}^{-1} = F - \tilde{\sigma}\phi\tilde{G}^{-1}$ (3.11)

is the frequency weighted approximation to the stable proper system $F(z)$.

We then have a bound on the Hankel matrices of $F(z)$ and $W(z)$ as follows.

Lemma 3.5 The Hankel matrices $\Gamma(F)$ and $\Gamma(W)$ satisfy,

$$\| G\|_\infty^{-1} \leq \frac{\| \Gamma(F) - \Gamma(W)\|}{\tilde{\sigma}} \leq \| G^{-1}\|_\infty \tag{3.12}$$

Proof. We have $\| \Gamma(F) - \Gamma(W)\| = \| \Gamma(\tilde{\sigma}\phi\tilde{G}^{-1})\| = \sigma_1(\tilde{\sigma}\phi\tilde{G}^{-1})$. Lemma 3.4 can be applied since $\tilde{G}^{-1} \in H^\infty$ so that $\sigma_1(\tilde{\sigma}\phi\tilde{G}^{-1}) \leq \| \tilde{G}^{-1}\|_\infty \tilde{\sigma}$ which is the upper bound. For the lower bound, consider $\Gamma(F\tilde{G}) - \Gamma(X)$.

$\Gamma(F\tilde{G}) - \Gamma(X) = (\Gamma(F) - \Gamma(X\tilde{G}^{-1}))T(\tilde{G}) = (\Gamma(F) - \Gamma(W))T(\tilde{G})$

where we have used Lemma 2.1, §2.2. Thus

$\| \Gamma(F\tilde{G}) - \Gamma(X)\| = \tilde{\sigma} \leq \| \Gamma(F) - \Gamma(W)\| \; \| T(\tilde{G})\|$. However

$\| T(\tilde{G})\| = \| \tilde{G}\|_\infty = \| G\|_\infty$.

Remark Using Lemma 3.4, (3.12) may be written as

$$\| G\|_\infty^{-1} \, \| G^{-1}\|_\infty^{-1} \leq \frac{\| \Gamma(F) - \Gamma(W)\|}{\sigma_{k+1}(F)} \leq \| G^{-1}\|_\infty \, \| G\|_\infty \tag{3.13}$$

which is in general a looser bound but involves the singular values of $\Gamma(F)$. Surprisingly, the weighted systems FG and WG have the same separation, in the Hankel-norm as the systems $F\tilde{G}$ and X. We prove this in the following Lemma.

Lemma 3.6 For all orders k of the approximation (3.10),

$$\| \Gamma(FG) - \Gamma(WG) \| = \tilde{\sigma} . \tag{3.14}$$

Proof We analyse the all-pass function $\phi\tilde{G}^{-1}G$.
F(z) has n stable poles. Let G(z) have m stable poles and l stable zeros (and only these). $\tilde{G}^{-1}G$ therefore has m stable poles and l unstable poles. ϕ has in general n+k stable and n-k-1 unstable poles. The difference in number of stable and unstable poles of $\phi\tilde{G}^{-1}G$ is therefore $(n+m+k) - (n-k-1+l) = 2k + m - l + 1$ which is always ≥ 1. Thus $\phi\tilde{G}^{-1}G$ is an all-pass function with more stable than unstable poles and so by Corollary 1.11, §1.6, $\sigma_1(\phi\tilde{G}^{-1}G) = 1$. This proves (3.14).

We now proceed to derive L^{∞}-norm error bounds for the approximations (3.10) and (3.11) and compare them with those in the Hankel-norm.

Denote as usual by $\hat{W}$, the quantity $F - \tilde{\sigma}[\phi\tilde{G}^{-1}]_-$. According to Lemma 3.1, $[\tilde{\sigma}\phi\tilde{G}^{-1}]_-$ has a representation as

$$[\tilde{\sigma}\phi\tilde{G}^{-1}]_- = \psi_0 + \overline{\sigma}_1\psi_1 + \ldots + \overline{\sigma}_{n+k}\psi_{n+k} \tag{3.15}$$

where the $\psi_j (j \geq 1)$ are all-pass and stable and we have accorded with the assumption that the $\overline{\sigma}_i$ are not repeated. In the same manner as §2.5, define a frequency weighted approximation W_0 to F by

$$W_0 = \hat{W} + \psi_0 \tag{3.16}$$

and the error system E_0 by

$$E_0 = F - W_0 \tag{3.17}$$

We then have the following bound for the L^∞-norm of E_0.

Lemma 3.7 For the error E_0 of (3.17),

$$\|G\|_\infty^{-1} \leq \frac{\|E_0\|_\infty}{\tilde{\sigma}} \leq (n+k)\|G^{-1}\|_\infty \tag{3.18}$$

where k is the degree of the approximant (3.10).

Proof We have that $\Gamma(F) - \Gamma(W_0) = \Gamma(F) - \Gamma(W) = \Gamma(F) - \Gamma(\hat{W})$

By Nehari's theorem,

$\|\Gamma(F) - \Gamma(W)\| = \|\Gamma(F) - \Gamma(W_0)\| \leq \|F-W_0\|_\infty = \|E_0\|_\infty$ so by Lemma 3.5, $\|E_0\|_\infty \geq \|G\|_\infty^{-1}\tilde{\sigma}$. By (3.15), $\|E_0\|_\infty = \|[\tilde{\sigma}\phi\tilde{G}^{-1}]_- - \psi_0\|_\infty = \|\tilde{\sigma}_1\psi_1 + \ldots + \bar{\sigma}_{n+k}\psi_{n+k}\|_\infty \leq \bar{\sigma}_1 + \ldots + \bar{\sigma}_{n+k}$. Again by Nehari's theorem, $\bar{\sigma}_j \overset{\Delta}{=} \sigma_j(\tilde{\sigma}\phi\tilde{G}^{-1}) \leq \|\Gamma(\tilde{\sigma}\phi\tilde{G}^{-1})\| \leq \tilde{\sigma}\|G^{-1}\|_\infty$ for j = 1 ... n+k. This gives the upper bound of (3.18).

Remark Lemma 3.4 may again be used to give the bounds (3.18) in terms of $\sigma_{k+1}(F)$ and (3.18) yields the upper bound $\|E_0G\|_\infty \leq (n+k)\tilde{\sigma}\|G\|_\infty\|G^{-1}\|_\infty$ for the weighted system E_0G.

Note here that the important consequence of Lemma 3.7 is to demonstrate the existence of a constant ψ_0 such that the upper bound (3.18) is accomplished. The lower bound is insensitive to the choice of the constant. Choosing the constant ψ_0 gives a suitably small L^∞ bound on E_0, however there is no guarantee that W_0 will be frequency shaped. For this reason, it is more suitable to choose the constant D in the expansion of the form (3.1) for $[F\tilde{G}]_-$ rather than ψ_0. This is in fact the type of choice that is made in the example of §2.5.

We now obtain the corresponding bound to (3.18) according to this choice of constant.

<u>Assumption</u>. We assume in the following that $\lim_{z\to\infty} \tilde{G}^{-1}(z) = 1$.

Let

$$W_1 = \hat{W} + D \tag{3.19}$$

be the frequency weighted approximation to F and

$$E = F - W_1 \tag{3.20}$$

the error system. The constant D is chosen so that according to Lemma 3.3,

$$\| [F\tilde{G}]_- - X_- - D\|_\infty \leq \sum_{k+1}^{n} \sigma_i(F\tilde{G}) \tag{3.21}$$

and

$$\| D\|_\infty \leq \sum_{k+1}^{n} \sigma_i(F\tilde{G}) \tag{3.22}$$

We formulate the L^∞-norm bounds for E and EG as a theorem.

<u>Theorem 3.8</u> The L^∞-norms of the error E and the weighted error EG are bounded according to

$$\| G\|_\infty^{-1}\tilde{\sigma} \leq \| E\|_\infty \leq K(n,k) \tag{3.23}$$

and

$$\| EG\|_\infty \leq \| G\|_\infty K(n,k) \tag{3.24}$$

where

$$K(n,k) = (2(n + k)\| G^{-1}\|_\infty+1)\ \tilde{\sigma} + \sum_{k+2}^{n} \sigma_i(F\tilde{G})$$

Proof Using (3.19) and (3.20), $W_1 = [X\tilde{G}^{-1}]_- + D + F(\infty)$ and $E = F - W_1 = F - [([F\tilde{G}]_- - \tilde{\sigma}\phi)\tilde{G}^{-1}]_- - D + F(\infty) = \tilde{\sigma}[\phi\tilde{G}^{-1}]_- - D$ so that $\| E\|_\infty \leq \| \tilde{\sigma}[\phi\tilde{G}^{-1}]_-\|_\infty + \| D\|_\infty$. Using Lemma 3.2, $\| \tilde{\sigma}[\phi\tilde{G}^{-1}]_-\|_\infty \leq 2 \sum_1^{n+k} \bar{\sigma}_i$, where $\bar{\sigma}_i$ are the singular values of $\tilde{\sigma}\phi\tilde{G}^{-1}$. As was shown earlier (Lemma 3.7), $\bar{\sigma}_i \leq \| G^{-1}\|_\infty\tilde{\sigma}$. Combining this with (3.22) gives $\| E\|_\infty \leq 2(n+k)\| G^{-1}\|_\infty\tilde{\sigma} + \sum_{k+1}^{n} \sigma_i(F\tilde{G})$ which is (3.23). The lower bound in (3.23) follows from Nehari's theorem and Lemma 3.5.

Remark If in fact $\lim_{z\to\infty} \tilde{G}^{-1} = \beta$ where $0 < \beta \neq 1$, the appropriate choice of constant is βD instead of D. In this case $K(n,k) = 2(n + k)\| G^{-1}\|_\infty\tilde{\sigma} + \beta \sum_{k+1}^{n} \sigma_i(F\tilde{G})$.

§3.4 CONCLUSION

In this chapter, we have derived weak bounds in the L^∞ and Hankel-norms for the frequency weighting scheme. Application of the bounds to the example in §2.5 shows that they are indeed weak. For example when $\| G\|_\infty = 10$, the singular values grow by a factor only at most 3. With $\| G^{-1}\|_\infty = 1$, (3.23) gives an upper bound of $20\tilde{\sigma}$ whereas the actual error has an L^∞-norm of approximately $\tilde{\sigma}$ (see Fig. 2), while the theoretical upper bound from (3.24) is $200\,\tilde{\sigma}$, compared to the actual weighted error L^∞-norm of approximately $2\,\tilde{\sigma}$.

Lemma 3.7 and Theorem 3.8 illustrates that in order to preserve the frequency weighting when choosing a constant D that reduces the final L^∞ error, a tradeoff has to be made. The L^∞ error will not be the smallest attainable for the choice of D given by (3.21) and (3.22), but as is evidenced in Fig. 2 and Fig. 4, the desired shaping is achieved.

The theory of §3.3 indicates that given a frequency weighting G with $\| G\|_\infty > 1$, some of the bounds will be small if $\| G^{-1}\|_\infty = 1$. Such a choice for the frequency weighting will in general be possible for simple weighting shapes or those that emphasise only narrow frequency bands. The bounds also indicate $\| G\|_\infty \; \| G^{-1}\|_\infty$ as an important parameter of the frequency weighting.

Although we have analysed the error for the scalar case, the results in §3.2 are valid for multivariable systems. Therefore a similar analysis can be used for frequency weighting of multivariable systems.

CHAPTER 4

§4.1 SYNOPSIS

In this thesis, we have presented a new model reduction technique for scalar linear dynamic systems. Further, we have given an error analysis and demonstrated the new methods' successful application via an example.

More precisely, we have developed a modification of the optimal Hankel-norm approximation technique that allows the use of classical intuitive design ideas through frequency weighting. Because the introduction of frequency weighting entails only the solution of a modified optimal Hankel-norm problem, much of the underlying theory already exists so that, for the error analysis in particular, we have used standard results and techniques throughout.

Because of the great intuitive appeal held by frequency domain design methods, the new approximation procedure represents an important bridge between frequency domain and state space ideas which should inevitably help the popularization of the still little known optimal Hankel-norm method. Its popularity is expected to be further increased by additional theoretical developments, especially for the multivariable problem, as more workers enter the field.

§4.2 FUTURE RESEARCH

The investigations of the thesis have uncovered several problems for future research. The following forms a collection of those that seem to the author to be the most tractable.

In any practical use of frequency weighting, it is clearly desirable to have the greatest computational simplicity. To this end, the frequency weighting G should be regarded as a design variable, a suitable G being chosen to achieve the required shape of error. We therefore envisage an iterative approach whereby more complicated features are added to G until the detail of additional shaping is no longer important to the final approximation. Special choices of G are possible that give the combination $F\tilde{G}$ added simplicity. One such choice, due to Glover, is to take the zeros of G equal to the poles of F, thereby giving an all-pass factor to $F\tilde{G}$. In this case, the one-step frequency weighted approximation has no unstable part, a fact which leads to an interesting multiplicative error analysis [15]. Other choices of G may also prove advantageous.

As was noted in §3.4, the error bounds of Chapter 3 are very weak. The good performance of the example of §2.5 suggests that tighter upper bounds may exist. For instance the inequality $\sigma_k(F\tilde{G}) \leq \sigma_i(G)\sigma_k(F)$, i = 1, 2, holds for this example. Two avenues are open; first to construct an example so that the upper bounds of §3.3 are attained or secondly to find directly, improved upper bounds.

It is known that when the optimal Hankel-norm approximation technique is used to approximate a controller that forms part of a stable closed loop system, the approximating closed loop system will not in general be stable. In fact it has been shown that the Hankel-norm is too weak a measure for studying stability of feedback systems [21]. Indeed, it is not reasonable to expect closed loop stability from any approximation technique that ignores the plant model. Consider however, the following situation. Let G be the plant model and F the controller in a stable closed loop system. Obtain a frequency weighted approximation W, of F that uses G, or its minimum phase equivalent, as the frequency weighting. The natural question that arises is whether the account taken of G allows a determination of the stability of the closed loop system employing G and W.

Although not widely used here, the definition of the Hankel operator as a projection (§§1.5 and A.2) is related closely to the theory of singular integrals [34]. One avenue of investigation would therefore be to reformulate the optimal Hankel-norm problem in terms of singular integrals, presumably giving a relation of the inhomogeneous Riemann problem [30], and then to solve this new problem. In view of the substantial theory of singular integral equations, this approach may well lead to new proofs or techniques.

In this thesis, we have not addressed the multivariable optimal Hankel-norm approximation problem. Its solution [14] is straightforward, the only significant difference from the scalar case being that the solution is not unique. Despite this, one solution can be chosen that has the smallest L^{∞} error bounds. Once a specific solution has been selected, it is a relatively straightforward matter to extend frequency weighting to multivariable systems and this work has been undertaken by Glover [15].

APPENDIX

§A.1 INTRODUCTION

As was apparent in Chapters 1 and 2, the theory of frequency weighted optimal Hankel-norm approximation is most conveniently developed in discrete time where the considerable tools of function theory are available. We then rely on a bilinear transform, that maps the unit disk conformally onto the left half plane, to obtain from a continuous time transfer function, an equivalent discrete time system, to which the discrete-time theory may be applied. This procedure has been expounded fully by Lin and Kung [29]. An alternative, is the direct approach of Glover [14] which uses the balanced realization of a given continuous-time system.

In §A.2 we demonstrate the validity of the frequency weighted approximation scheme directly in continuous time. This is done by proving the equivalent results to those of Chapter 2 by using an appropriate continuous time definition of the Hankel operator together with the Laplace transform.

§A.2 CONTINUOUS-TIME THEORY

We first establish some preliminary notation. Denote by Ω_+ (resp. Ω_-) the open half plane Re s < 0 (resp. Re s > 0). Let L^2_+ (resp. L^2_-) be the Lebesque space $L^2(R_+)$ (resp. $L^2(R_-)$ of real valued functions. The two-sided Laplace transform L of a function $u(t) \in L^2(R)$ is defined by

$$\mathcal{L}\{u(t);\ s\} = \int_{-\infty}^{\infty} e^{-st} u(t)\,dt \qquad (2.1)$$

and in general the integral will converge uniformly to an analytic function in some strip, $-\alpha_1 < \text{Re } s < \alpha_2$; $\alpha_1, \alpha_2 > 0$. The two-sided transform decomposes as

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_- \tag{2.2}$$

where for $u \in L^2_-$ and supp $u \subset R_-$

$$\mathcal{L}_-\{u(t);\ s\} \stackrel{\Delta}{=} \int_{-\infty}^{0} e^{-st}u(t)dt \tag{2.3}$$

is analytic for $s \in \Omega_+$, and for $v \in L^2_+$ and supp $v \subset R_+$

$$\mathcal{L}_+\{v(t);\ s\} \stackrel{\Delta}{=} \int_0^{\infty} e^{-st}v(t)dt \tag{2.4}$$

is the standard one-sided Laplace transform which is analytic for $s \in \Omega_-$. In what follows, the statement that $u \in L^2_-$ (resp. L^2_+) will mean that u is to be regarded as a function such that supp $u \subset R_-$ (resp. R_+). That is, we consider only the restriction of u.

The definitions (2.3) and (2.4) imply that formally

$$\mathcal{L}_-\{u(t);\ s\} = \mathcal{L}_+\{u(-t);\ -s\} . \tag{2.5}$$

We will also use the notations

$$\begin{aligned} \hat{v}(s) &\stackrel{\Delta}{=} V(s) \stackrel{\Delta}{=} \mathcal{L}_+\{v(t);\ s\} , \qquad v \in L^2_+ \\ \hat{u}(s) &\stackrel{\Delta}{=} U(s) \stackrel{\Delta}{=} \mathcal{L}_-\{u(t);\ s\} , \qquad u \in L^2_- \end{aligned} \tag{2.6}$$

where convenient. It will be obvious from the context which of $\mathcal{L}_+$ or $\mathcal{L}_-$ is meant by this notation. Finally, define $\tilde{V}(s)$ for $v \in L^2_+$ by

$$\tilde{V}(s) \stackrel{\Delta}{=} V(-s) \tag{2.7}$$

with V(s) as given by (2.6).

By the Schwarz symmetry principle, $\tilde{V}(s)$ is analytic for $s \in \Omega_+$. We now define the Hankel operator in continuous-time.

<u>Definition 2.1</u> The Hankel operator $\Gamma : L^2_- \to L^2_+$ is defined by:

$$(\Gamma u)(t) \triangleq \begin{cases} \int_0^\infty f(t+\tau)u(-\tau)\,d\tau & t > 0 \\ 0 & t \leq 0 . \end{cases} \tag{2.8}$$

where $u \in L^2_-$.

Nehari's theorem now states that Γ is a bounded operator on L^2_- if $f(t)$ is a generalised function coinciding for $t > 0$ with the Fourier transform of some function on the real axis. Here, we will be concerned with the sub-class of f's for which

$$F(s) = c + \frac{n(s)}{d(s)} \tag{2.9}$$

where c is a constant (possibly zero) and n/d is a real, strictly proper stable rational function. In the case that c is nonzero f(t) contains at worst a "Dirac Delta" at t = 0 and the Laplace transform is of course there interpreted in the generalised sense [37]. We will therefore be dealing with bounded Hankel operators.

<u>Definition 2.2</u> Let G(s) be of the form (2.9). Define the negative time convolution (n.t.c.) operator $T : L^2_- \to L^2_-$ by

$$(Tu)(t) \triangleq \begin{cases} \int_{-\infty}^0 g(-t+\tau)u(\tau)\,d\tau & t < 0 \\ 0 & t \geq 0 \end{cases} \tag{2.10}$$

where $u \in L^2_-$.

Note that according to (2.10) and by letting $v = Tu$,

$$v(t) = \int_t^0 g(\tau-t)u(\tau)d\tau \qquad t < 0$$

$$= \int_0^{-t} g(-t-\tau)u(-\tau)d\tau$$

or

$$v(-t) = \int_0^t g(t-\tau)u(-\tau)d\tau \ , \quad t > 0 \ . \qquad (2.11)$$

Thus if * denotes the usual convolution operation for $t > 0$ and R the reflection operator, ie $(Ru)(t) = u(-t)$, then (2.11) states that

$$Tu = R(g * Ru) \ . \qquad (2.12)$$

The definition (2.10) is therefore essentially the same as the usual positive-time convolution. Bearing this in mind, the following result is easily proved.

<u>Lemma 2.1</u> Let $u(t) \in L^2_-$ and $g(t)$ be as in Definition 2.2. Then the n.t.c. satisfies

$$(Tu)\hat{}(s) = \tilde{G}(s)U(s) \overset{\Delta}{=} G(-s)U(s) \ . \qquad (2.13)$$

<u>Proof</u> Applying the convolution theorem [10] for $\mathcal{L}_+$ to (2.11) gives, $\mathcal{L}_+\{v(-t); s\} = \mathcal{L}_+\{g(t); s\} \ \mathcal{L}_+\{u(-t); s\}$ and $\mathcal{L}_+\{v(-t); -s\} = \mathcal{L}_+\{g(t); -s\} \ \mathcal{L}_+\{u(-t); -s\}$. This by virtue of (2.5) is $\mathcal{L}_-\{v(t); s\} = \mathcal{L}_+\{g(t); -s\} \ \mathcal{L}_-\{u(t); s\}$ or $(Tu)\hat{}(s) = G(-s)U(s)$, which is (2.13).

Denote by $[\cdot]_-$ the operation of taking the strictly stable part of a rational function of the form (2.9). We now relate the Hankel operator (2.8) to $[\cdot]_-$ in the frequency domain.

Lemma 2.2 Let $u \in L^2_-$ and $f(t)$ be as determined by (2.9) and Definition 2.1. Then

$$(\Gamma u)\hat{} = [FU]_- \tag{2.14}$$

Proof We calculate directly $(\Gamma u)\hat{}$.

After a change of variable, (2.8) may be written, for $t > 0$, as

$$(\Gamma u)(t) = \int_t^\infty f(\tau)u(\tau-t)d\tau \ . \tag{2.15}$$

Thus

$$(\Gamma u)\hat{}(s) = \int_0^\infty e^{-st} \int_t^\infty f(\tau)u(\tau-t)d\tau \tag{2.16}$$

and is analytic for $s \in \Omega_-$, the integral conveying uniformly there. We can therefore reverse the order of integration in (2.16). Then

$$(\Gamma u)\hat{}(s) = \int_0^\tau e^{-st}dt\int_0^\infty f(\tau)u(t-\tau)d\tau \ .$$

Let $\lambda = t-\tau$ and $d\lambda = dt$, so that

$$(\Gamma u)\hat{}(s) = \int_{-\tau}^0 e^{-s(\tau+\lambda)}d\lambda \int_0^\infty f(\tau)u(\lambda)d\lambda$$

$$= \int_0^\infty e^{-s\tau}f(\tau)(\int_{-\tau}^0 e^{-s\lambda}u(\lambda)d\lambda) \ d\tau \ . \tag{2.17}$$

Letting $w(s,\tau) = \int_{-\tau}^0 e^{-s\lambda}u(\lambda)d\lambda$, (2.17) is

$$(\Gamma u)\hat{}(s) = \mathcal{L}_+\{f(t)w(s,t);\ s\} \ . \tag{2.18}$$

Now

$$W(s,p) = \int_0^\infty e^{-pt}w(s,t)dt$$

$$= \frac{e^{-pt}}{-p} \int_{-t}^0 e^{-s\lambda}u(\lambda)d\lambda]_0^\infty - \int_0^\infty \frac{e^{-pt}}{-p} e^{st}u(-t)dt$$

For $s \in \Omega_+$ and $p \in \Omega_-$ the first term vanishes to give

$$W(s,p) = \frac{1}{p} \int_0^\infty e^{-(p-s)t}u(-t)dt$$

$$= \frac{1}{p} \mathcal{L}_+\{u(-t);\ p-s\} \ , \qquad p-s \in \Omega_-$$

$$= \frac{1}{p} L_{-}\{u(t);\ s-p\}$$

$$= \frac{1}{p} U(s-p) \qquad\qquad \text{(by 2.5)}.$$

To (2.18) apply the convolution theorem [10] in the p-plane for the Laplace transform to give,

$$(\Gamma u)\hat{}(s) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{F(p)U(s-(s-p))\,dp}{s-p} \qquad s \in \Omega_{-}$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{F(p)U(p)}{p-s}\,dp\ . \tag{2.19}$$

We must have $\sigma = 0$ since $0 \leqslant \sigma \leqslant \text{Re } s - \text{Re } s$. Since we are concerned with rational functions of the form (2.9), (2.19) is

$$(\Gamma u)\hat{}(s) = -\sum_{p \in \Omega_{+}} \text{Res} \left[\frac{F(p)U(p)}{p-s}\right]$$

$$= [F(p)U(p)]_{-}\ . \tag{2.20}$$

(2.20) is easily seen by taking $F = \frac{\alpha}{p-q}$, $q \in \Omega_{+}$, and noting that $U(p)$ is analytic in Ω_{+} .

<u>Remark</u> (2.20) can also be deduced from (2.19) when we recognise the integral in (2.19) as the Riesz projection on $L^2(C)$, where C denotes the imaginary axis. If $K(p) \in L^2(C)$ and $s \in \Omega_{-}$ then

$$-\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K(p)\,dp}{p-s} = (P_{-}K)(s)$$

where $\qquad P_{-} : L^2(C) \to H^2_{-} \quad$ and $\quad P^2_{-} = P_{-}$.

Here H^2_{-} is the Hardy class on Ω_{-} .

Although according to (2.9), $F(p)U(p)$ may not belong to $L^2(C)$, the integral of the constant term in the integrand is however zero by Cauchy's theorem.

Consider now the stable transfer function

$$Q(s) = [F(s)\tilde{G}(s)]_- \ . \qquad (2.21)$$

Let S be the associated Hankel operator, ie $(Su)(t) = \int_0^\infty q(t+\tau)u(-\tau)d\tau$, $t > 0$. We then have the following analogue of Lemma 2.1(ii) of §2.2.

Lemma 2.3 S admits the factorization

$$S = \Gamma T \qquad (2.22)$$

where T is the n.t.c. for g(t) and Γ the Hankel operator with kernel f(t).

Proof Let $u \in L^2_-$. By Lemma 2.2 we have

$$(Su)\hat{} = [[F\tilde{G}]_- U]_- \qquad (2.23)$$

Because $[[F\tilde{G}]_+ U]_- = 0$, (2.23) is $(Su)\hat{} = [F\tilde{G}U]_- = [F(Tu)\hat{}]_-$ $= (\Gamma Tu)\hat{}$ as required.

The analogue of Lemma 2.2, §2.2 is also straightforward.

Lemma 2.4 If G(s) has, in addition to the requirements of Definition 2.2, $c \neq 0$ and is also minimum phase, ie all zeros are in Ω_+, then T is invertible and T^{-1} is the n.t.c. operator associated with $G^{-1}(s)$.

Proof G^{-1} is stable, hence define for $u \in L^2_-$, the n.t.c. $(Yu)(t) = \int_{-\infty}^{0} \mathcal{L}_+^{-1}\{G^{-1}(s); \tau - t\} u(\tau)d\tau$ for $t < 0$. Then by Lemma (2.1), $(Yu)\hat{} = (G^{-1})\tilde{}\, U$ so that $(YTu)\hat{} = (G^{-1})\tilde{}\,(Tu)\hat{} = (G^{-1})\tilde{}\,\tilde{G}U = U = \hat{u}$. The second last equality holds because $(G^{-1})\tilde{} = \tilde{G}^{-1}$. Similarly, $(TYu)\hat{} = \tilde{G}(Yu)\hat{} = G(G^{-1})\tilde{}\,U = \hat{u}$.

Lemmas 2.1 - 2.4 now enable a discussion of the operator S in (2.22) in the same way as in §2.2. We therefore take G(s) as the frequency weighting and form the system (2.21). G(s) is chosen so that Lemma 2.4 is valid. The k-th order optimal Hankel-norm approximant X(s) for Q(s) has an associated Hankel operator Λ such that for $u \in L^2_-$, $(\Lambda u)^\wedge = [XU]_-$. The difference $S-\Lambda$ may then be written as $(\Gamma - \Lambda T^{-1})T$ exactly as in the discrete-time case. This completes the continuous-time analogy.

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