A CATEGORICAL APPROACH TO UNIVERSAL ALGEBRA

by

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A thesis submitted for the degree of Doctor of Philosophy

in the Australian National University

Canberra

January, 1970

The results presented in this thesis are mine except where otherwise stated.

R.F.C. Walters

PREFACE

The work of this thesis was made possible by a Commonwealth postgraduate award and an Australian National University research scholarship.

My supervisors, Professor Hanna Neumann and Dr S.S. Świerczkowski, have given me much help and advice for which I am most grateful, and I am also grateful to Professor B.H. Neumann and Professor S. MacLane for their encouragement.

I wish to thank Dr A. Shafaat and Mr N.H. Williams for many useful discussions.

The thesis was typed by Mrs F.A. Wickland and Miss T. Purdy and I appreciate their work.

Finally I wish to thank my wife, Margaret, for her multifarious efforts in the production of this thesis.

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CHAPTER 0

INTRODUCTION AND PRELIMINARIES

§0.1 Introduction.

There are two well-known ways of generalizing classical universal algebra to an abstract categorical setting. The most obvious approach is to generalize the concept of operation. In a category $\underline{\underline{A}}$ with products, an "n-ary operation" on an object A is defined to be a morphism from A^n to A . It is easy to see that with this definition the concepts of "algebra", "homomorphism" and "variety of algebras" can be defined categorically. This is the approach elaborated by Lawvere [16] and Linton [18]. A quite different generalization is suggested by the observation that the free algebras of a variety contain all the information about the variety. In the place of the free algebras we have the categorical notion of a "triple in $\underline{\underline{A}}$", and Eilenberg and Moore [6] have shown how to use this notion to define "algebra" and "homomorphism" in $\underline{\underline{A}}$.

In this thesis we explore a variation of the second approach which leads to a rather different development of universal algebra in a category. It is perhaps more suitable than the method of triples for simulating the classical situation where the operations are ranked.

In Chapter I we define and develop simple properties of the concept of a "device in a category" and the corresponding "category of algebras of a device". We also discuss the relationship between triples and devices and between their corresponding algebras. In the second chapter we develop some analogues of conventional results in universal algebra under various conditions on the category. For example, there is a theorem guaranteeing that the functor which forgets the structure of the algebras has a left adjoint. Another theorem gives conditions for recognizing when a functor is the forgetful functor corresponding to some device. This theorem is used to give a version of Birkhoff's Theorem on the subvarieties of a variety.

Each device in a category $\underline{\underline{A}}$ has associated with it a functor $X : I \to \underline{\underline{A}}$ whose domain is a set. We define a category $\underline{\underline{\text{Dev}}}\ (\underline{\underline{A}}, X)$ whose objects are all devices in $\underline{\underline{A}}$ with associated functor X, and we define a functor $U : \text{Dev}\ (\underline{\underline{A}}, X) \to \underline{\underline{A}}^I$. In Chapter III we prove under certain conditions on $\underline{\underline{A}}$ and X that U is the forgetful functor of some device in $\underline{\underline{A}}^I$. This chapter also includes descriptions of devices and their algebras in $\underline{\underline{\text{Sets}}}$.

The reader needs only a knowledge of elementary category theory, and of elementary universal algebra. He need not know about triples since they are defined in this chapter, and throughout the thesis no appeal is made to theorems about triples.

§0.2 The language.

The discussions of this thesis take place in a set theory with universes, as described for example in [4]. Throughout the thesis we consider two fixed universes M_1 and M_2 with $M_1 \in M_2$. The sets in M_1 are called small sets, and the category of all small sets is written as $\underline{\underline{\mathrm{Sets}}}$. A category defined in M_2 is called an M_2-category, while a category $\underline{\underline{A}}$, which has the property that for each pair A_1, A_2 of objects in $\underline{\underline{A}}$ the morphisms from A_1 to A_2 form a small set, is called a locally-small category. The category of all locally-small M_2-categories is written as $\underline{\underline{\mathrm{Cat}}}$, and the category of all small categories is called $\underline{\underline{\mathrm{Kit}}}$.

We assume that the reader is familiar with elementary category theory, as contained for example in the early chapters of [22], [7] or [26]. In the following sections we shall indicate our notations and recall some well-known definitions and results which are basic to the thesis.

§0.3 Some notation.

Categories will usually be denoted by symbols $\underline{\underline{A}},\underline{\underline{B}},\underline{\underline{C}},\ldots$; objects by $A,B,C,\ldots$ or $\underline{A},\underline{B},\underline{C},\ldots$; morphisms by $\alpha,\beta,\gamma,\ldots$ and $f,g,h,\ldots$; functors by $F,G,H,U,\ldots$; and natural transformations by $\eta,\mu,\lambda,\ldots$. The composition of morphisms $\alpha : A \to B$ and $\beta : B \to C$

will be written $\beta\alpha : A \to C$. If f is a function and a an element of its domain, f evaluated at a will be written fa or $f(a)$. The set of all $\underline{\underline{A}}$-morphisms from A_1 to A_2 is written $\underline{\underline{A}}(A_1,A_2)$. The set of objects of $\underline{\underline{A}}$ we write as obj $\underline{\underline{A}}$, and the opposite category as $\underline{\underline{A}}^{op}$. The category of functors from $\underline{\underline{B}}$ to $\underline{\underline{A}}$ is written $\underline{\underline{A}}^{\underline{\underline{B}}}$.

If η is a natural transformation from $G : \underline{\underline{A}} \to \underline{\underline{B}}$ to $H : \underline{\underline{A}} \to \underline{\underline{B}}$, and $F : \underline{\underline{C}} \to \underline{\underline{A}}$, $K : \underline{\underline{B}} \to \underline{\underline{D}}$, then $\eta F : GF \to HF$ is the natural transformation defined by $\eta F(C) = \eta(FC)$, and $K\eta : KG \to KH$ is the natural transformation defined by $K\eta(A) = K(\eta A)$.

A morphism $\alpha : A_1 \to A_2$ is a split epimorphism if there exists a morphism $\beta : A_2 \to A_1$ such that $\alpha\beta = 1 : A_2 \to A_2$. Dually we have the notion of split monomorphism.

<u>§0.4 Universal arrows and adjoint functors</u>.

0.4.1 DEFINITIONS Let $U : \underline{\underline{B}} \to \underline{\underline{A}}$ be a functor and A an object of $\underline{\underline{A}}$. Then a <u>universal arrow from</u> A <u>to</u> U is a pair $(\alpha : A \to UB, B)$ with B an object of $\underline{\underline{B}}$, such that for any $\underline{\underline{A}}$-morphism $\alpha_1 : A \to UB_1$, there is a unique $\underline{\underline{B}}$-morphism $\beta : B \to B_1$ such that $\alpha_1 = U\beta.\alpha$.

We often say in this situation that B is <u>free</u> on A , or that $\alpha : A \to UB$ <u>freely generates</u> B .

A <u>universal arrow from</u> U <u>to</u> A is a pair $(B , \alpha : UB \to A)$ such that for any morphism $\alpha_1 : UB_1 \to A$ there is a unique morphism

$\beta : B_1 \to B$ in $\underline{\underline{B}}$ such that $\alpha . U\beta = \alpha_1$.

0.4.2 DEFINITION A functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ "has a left adjoint" if for every A in $\underline{\underline{A}}$ there is a universal arrow from A to U .

If U has a left adjoint we can indeed define a functor $F : \underline{\underline{A}} \to \underline{\underline{B}}$ which we call a left adjoint of U . F is determined up to natural equivalence and we say that U is a right adjoint of F . Further, the pairs F,U so obtained can be characterized in other nice ways. For example, associated with F and U are two natural transformations $\eta : 1_{\underline{\underline{A}}} \to UF$ and $\varepsilon : FU \to 1_{\underline{\underline{B}}}$ satisfying $\varepsilon F . F\eta = 1_F$ and $U\varepsilon . \eta U = 1_U$. For an arbitrary pair of functors F,U the existence of such natural transformations guarantees that F is a left adjoint of U . Further, a functor $F : \underline{\underline{A}} \to \underline{\underline{B}}$ has a right adjoint if and only if to each B in $\underline{\underline{B}}$ there is a universal arrow from F to B .

0.4.3 DEFINITIONS Consider a functor $G : \underline{\underline{C}} \to \underline{\underline{A}}$, and consider the diagonal functor $d : \underline{\underline{A}} \to \underline{\underline{A}}^{\underline{\underline{C}}}$ defined by $dA : (\gamma : C_1 \to C_2) \rightsquigarrow (1 : A \to A)$ and $d\alpha(C) = \alpha$. We often confuse such a constant functor dA with the object A . Now if there is a universal arrow $(A , \lambda : dA \to G)$ from d to G , then we call the universal arrow, or just the natural transformation $\lambda : dA \to G$, a <u>limit</u> of G . Thus $\lambda : dA \to G$ is a limit of G if for every natural transformation $\lambda_1 : dA_1 \to G$ there is a unique morphism $\alpha : A_1 \to A$ such that $\lambda C . \alpha = \lambda_1 C$ for all

$C \in \underline{\underline{C}}$. A colimit of G is a universal arrow from G to d . A category $\underline{\underline{A}}$ is said to have $\underline{\underline{C}}$-limits if each functor $G : \underline{\underline{C}} \to \underline{\underline{A}}$ has a limit; that is, if $d : \underline{\underline{A}} \to \underline{\underline{A}}^{\underline{\underline{C}}}$ has a right adjoint. If for all small categories $\underline{\underline{C}}$ the category $\underline{\underline{A}}$ has $\underline{\underline{C}}$-limits, then $\underline{\underline{A}}$ is small complete. A functor $H : \underline{\underline{A}} \to \underline{\underline{B}}$ is said to preserve limits if whenever $\lambda : A \to G$ is a limit of a functor with codomain $\underline{\underline{A}}$, then $H\lambda$ is a limit of HG . The functor H creates limits if for each functor G with codomain $\underline{\underline{A}}$ and each limit $\mu : B \to HG$ of HG there exists a unique natural transformation $\lambda : A \to G$ such that $H\lambda = \mu$; and moreover λ is a limit of G .

We frequently use the following theorem of P. Freyd.

0.4.4 THEOREM Let $\underline{\underline{A}},\underline{\underline{B}}$ be locally small categories where $\underline{\underline{B}}$ is small complete. Then a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ has a left adjoint if and only if (1) U preserves small limits and (2) for each A in $\underline{\underline{A}}$ there is a small set S_A of objects in $\underline{\underline{B}}$ such that for any map $\alpha : A \to UB$ there is a B_1 in S_A , a map $\beta : B_1 \to B$, and a map $\alpha_1 : A \to UB_1$ such that $\alpha = U\beta.\alpha_1$.

Condition (2) is called the solution set condition for U , and S_A is called a solution set for A .

§0.5 2-categories.

0.5.1 DEFINITION (Bénabou [2]) A <u>2-category</u> $\underset{\sim}{A}$ is a set of elements $\alpha,\beta,\gamma,\ldots$ called 2-cells, equipped with two partial operations; (1) the weak composition whose effect is denoted $\alpha\cdot\beta$, and (2) the strong composition whose effect is denoted $\alpha * \beta$. The operations satisfy the following conditions.

(1) The 2-cells constitute a category with respect to the strong composition. We will denote this category by $\underset{\sim}{A}$. Its objects are called <u>vertices</u>.

(2) For each pair of vertices A,B the set $\underset{\sim}{A}(A,B)$ is a category with respect to the weak composition. The objects of this category are called <u>arrows</u>. The weak composition is not defined for 2-cells with different domain or codomain in $\underset{\sim}{A}$.

(3) If S is an arrow of $\underset{\sim}{A}(A,B)$ and T is an arrow of $\underset{\sim}{A}(B,C)$ then $T * S$ is an arrow of $\underset{\sim}{A}(A,C)$.

(4) $(\alpha * \beta)\cdot(\gamma * \delta) = (\alpha\cdot\gamma) * (\beta\cdot\delta)$ if both sides are defined.

For other ways of describing 2-categories see [10].

We can write a 2-cell σ as

$$\begin{tikzcd} A \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\sigma\downarrow"] & B \end{tikzcd}$$

where A and B are the domain and codomain of σ in $\underset{\sim}{A}$, and S and T are the domain and codomain of σ in $\underset{\sim}{A}(A,B)$.

The most familiar 2-category is the following nice structure on natural transformations. If $\eta : F \to G : \underline{\underline{A}} \to \underline{\underline{B}}$ and $\mu : G \to H$ are natural transformations then the weak composition of η and μ is just $\mu\eta$. If $\nu : K \to L : \underline{\underline{B}} \to \underline{\underline{C}}$ is a natural transformation then the strong composition of η and ν is $\nu * \eta = L\eta . \nu F = \nu G . K\eta$. The 2-category of all natural transformations between functors in $\underline{\underline{Cat}}$ is denoted Cat.

§0.6 Triples in a category.

0.6.1 DEFINITION (Godement [8]) A triple in a category $\underline{\underline{A}}$ consists of three things; a functor $T : \underline{\underline{A}} \to \underline{\underline{A}}$, and two natural transformations $\eta : 1_{\underline{\underline{A}}} \to T$ and $\mu : T^2 \to T$, which are required to satisfy three axioms; namely (1) $\mu . \eta T = 1_T$, (2) $\mu . T\eta = 1_T$, and (3) $\mu . T\mu = \mu . \mu T$. We denote the triple so defined as $\mathcal{T} = (T, \eta, \mu)$.

If we have an adjoint situation as described in 0.4.2 then $(UF, \eta, U\varepsilon F)$ is a triple in $\underline{\underline{A}}$, and in fact any triple may be so obtained from an adjoint situation.

0.6.2 DEFINITION (Eilenberg-Moore [6]) Let $\mathcal{T} = (T, \eta, \mu)$ be a triple in $\underline{\underline{A}}$. Then a $\mathcal{T}$-algebra is a pair (A, ξ) with A an object of $\underline{\underline{A}}$ and $\xi : TA \to A$, satisfying the requirements (1) $\xi . \eta A = 1_A$ and (2) $\xi . T\xi = \xi . \mu A$.

A $\mathcal{T}$-morphism from (A_1,ξ_1) to (A_2,ξ_2) is a morphism $\alpha : A_1 \to A_2$ with the property (1) $\xi_2.T\alpha = \alpha.\xi_1$.

$\mathcal{T}$-algebras and $\mathcal{T}$-morphisms with the obvious composition form a category $\underline{\underline{A}}^{\mathcal{T}}$. The functor which takes $\alpha : (A_1,\xi_1) \to (A_2,\xi_2)$ to $\alpha : A_1 \to A_2$ is written $U^{\mathcal{T}} : \underline{\underline{A}}^{\mathcal{T}} \to \underline{\underline{A}}$.

0.6.3 DEFINITION A functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ is <u>tripleable</u> if there exists a triple $\mathcal{T}$ in $\underline{\underline{A}}$ and an isomorphism of categories $K : \underline{\underline{B}} \to \underline{\underline{A}}^{\mathcal{T}}$ such that $U^{\mathcal{T}}K = U$.

There is another meaning of tripleable which requires only that K be an equivalence.

0.6.4 DEFINITION A <u>triple-map</u> from $\mathcal{T}_1$ to $\mathcal{T}_2$ is a natural transformation $\lambda : T_1 \to T_2$ such that $\lambda.\eta_1 = \eta_2$ and $\lambda.\mu_1 = \mu_2.(\lambda * \lambda)$

For some idea of the scope of recent literature on triples see [29].

<u>§0.7 Adequacy</u>.

0.7.1 DEFINITION A family of objects $X = (Xi;\ i \in I)$ in a category $\underline{\underline{A}}$ is called a <u>generating family</u> if for any two different maps $\alpha,\beta : A \to B$ in $\underline{\underline{A}}$ there is an $i \in I$ and a map $\gamma : Xi \to A$ such that

$\alpha\gamma \neq \beta\gamma$.

0.7.2 DEFINITION (Isbell [13]) A generating family $X = (Xi;\ i \in I)$ is said to be adequate in $\underline{\underline{A}}$ if, for any pair of objects A,B in $\underline{\underline{A}}$, and any family of functions $(fi;\ i \in I)$ with properties (1) $fi : \underline{\underline{A}}(Xi,A) \to \underline{\underline{A}}(Xi,B)$ and (2) for any $\varepsilon : Xi_1 \to Xi$, $\phi : Xi \to A$ it is true that $fi(\phi).\varepsilon = fi_1(\phi\varepsilon)$, then there exists a map $\alpha : A \to B$ such that $fi(-) = \alpha.(-)$ (all $i \in I$) .

In [13] the above property is called the left adequacy of the full subcategory of $\underline{\underline{A}}$ with objects $\{Xi;\ i \in I\}$.

§0.8 The classical notion of variety. (see [4], [9], [30] or [27])

0.8.1 DEFINITIONS To define abstract algebras we need a small set Ω of "formal operations" with a small set $a\omega$ assigned to each ω in Ω called the "arity" of ω . (In [4], [9] and [27] $a\omega$ is always a finite ordinal, and in [30], an ordinal.) Then an <u>Ω-algebra</u> $\underline{A}$ is a small set A and to each ω in Ω an operation $\omega_{\underline{A}} : A^{a\omega} \to A$.

A <u>homomorphism</u> from $\underline{A}$ to $\underline{B}$ is a map $\lambda : A \to B$ such that for all ω in Ω and all $\alpha : a\omega \to A$ we have $\lambda.\omega_{\underline{A}}(\alpha) = \omega_{\underline{B}}(\lambda\alpha)$.

All Ω-algebras and all homomorphisms form a category $\underline{\underline{\Omega\text{-Alg}}}$. There is a clear forgetful functor $U : \underline{\underline{\Omega\text{-Alg}}} \to \underline{\underline{\text{Sets}}}$ which has a left adjoint F .

Now an Ω-law in variables X is a pair of elements of UFX . An Ω-algebra $\underline{A}$ satisfies the law (t_1,t_2) if $\alpha t_1 = \alpha t_2$ for all homomorphisms $\alpha : FX \to \underline{A}$. Given Λ , a set of Ω-laws in variables X , then the set $\underline{\underline{V}} = \underline{\underline{V}}(\Lambda)$ of all algebras satisfying these laws is called a <u>variety</u>. There is an obvious category $\mathrm{Cat}(\underline{\underline{V}})$ and an obvious forgetful functor $U : \mathrm{Cat}(\underline{\underline{V}}) \to \underline{\underline{\mathrm{Sets}}}$ which again has a left adjoint.

0.8.2 REMARKS It has been recognized that this way of defining varieties of algebras is not completely satisfactory. For example, with this definition, the variety of groups defined in terms of multiplication, inverse, and identity, is different from the variety of groups defined in terms of right division and identity, where we would like them to be the same. The particular operations used to define a variety are less important than the "abstract clone of operations" of the variety, a concept we define in the next paragraph.

0.8.3 DEFINITIONS (P. Hall, see [4] p 132) Let S be a set. An <u>abstract S-clone</u> consists of the following data: an S-tuple of sets $(A_s;\ s \in S)$; specified elements $(d_{s_1,s}\ ;\ s_1 \in s)$ in A_s ; and a function which associates with each $\mu : s_1 \to A_s$ and each α in A_{s_1} an element denoted $\alpha\langle\mu\rangle$ in A_s . Using this last function we can define to any $\mu : s_1 \to A_s$ and any $\nu : s_2 \to A_{s_1}$ a function

$\nu\langle\mu\rangle : s_2 \to A_s$ by $\nu\langle\mu\rangle(t) = (\nu t)\langle\mu\rangle$ for all $t \in s_2$. Then the data are required to satisfy the following axioms where $\omega \in A_{s_2}$ and ν, μ are as above. (1) $\omega\langle\nu\rangle\langle\mu\rangle = \omega\langle\nu\langle\mu\rangle\rangle$. (2) $d_{s_3, s_1}\langle\mu\rangle = \mu(s_3)$.

A homomorphism from S-clone $\underline{A}$ to S-clone $\underline{A}'$ is a family of functions $(f_s : A_s \to A'_s ; s \in S)$ with the properties (1) $f_s(\alpha\langle\mu\rangle) = (f_{s_1}\alpha)\langle f_s.\mu\rangle$ and (2) $f_s d_{s_4,s} = d'_{s_4,s}$.

Now from any variety $\underline{\underline{V}}$ we can form an abstract M_1-clone of operations as follows. Let U be the forgetful functor for $\underline{\underline{V}}$, and F a left adjoint of U with associated natural transformation $\eta : 1 \to UF$. Then take the family of sets to be $(UFX ; X \in M_1)$; take the specified elements in UFX to be $(\eta X(x) ; x \in X)$; for $\mu : X \to UFY$ and w in UFX take $w\langle\mu\rangle$ to be $U\alpha(w)$ where $\alpha : FX \to FY$ is the homomorphism determined by $U\alpha.\eta X = \mu$.

A different left adjoint yields an isomorphic clone of operations. The name "clone of operations" arises since each element of the X-component of the clone induces an X-ary operation on every algebra of the variety.

0.8.4 REMARKS CONTINUED The operations in the clone of operations of $\underline{\underline{V}}$ may be thought of as all the operations "derived" from the defining operations, and it is our contention that two varieties should be considered essentially the same if they have isomorphic clones of operations.

Let us call functors $U_1 : \underline{\underline{A}}_1 \to \underline{\underline{B}}$ and $U_2 : \underline{\underline{A}}_2 \to \underline{\underline{B}}$ <u>strictly isomorphic</u> if there is an isomorphism $K : \underline{\underline{A}}_1 \to \underline{\underline{A}}_2$ such that $U_2K = U_1$. For any $\underline{\underline{\text{Sets}}}$-valued functor with a left adjoint we can derive an M_1-clone in much the same way as described in 0.8.3 above. A functor strictly isomorphic to $U : \text{Cat}(\underline{\underline{V}}) \to \underline{\underline{\text{Sets}}}$ yields a clone isomorphic to the clone of $\underline{\underline{V}}$. Thus we may regard any functor strictly isomorphic to $U : \text{Cat}(\underline{\underline{V}}) \to \underline{\underline{\text{Sets}}}$ as containing the essential information of $\underline{\underline{V}}$.

This discussion is contained implicitly in Linton [18] in terms of "theories" rather than clones of operations.

CHAPTER I

DEVICES IN A CATEGORY

As we remarked in §0.1, a triple in a category may be regarded as an analogue of "the set of all free algebras of a variety". Then the Eilenberg-Moore construction of the category of algebras of a triple amounts to the construction of the variety from its free algebras. In this chapter we define a notion, "device in a category", which is an abstraction of "a family of free algebras of a variety" and hence is a generalization of triple. There is a corresponding construction of "the category of algebras of a device" which in the special case of triples is equivalent to the construction of Eilenberg and Moore.

Also in this chapter we prove some simple properties of devices and their algebras. We define "morphisms of devices" and then, in the last section, we give an alternative global description of devices and their algebras.

The reader who would like to see devices in action in simulating conventional universal algebra might well read only the first two sections of this chapter before proceeding to Chapter II.

§1.1 Devices and the device construction.

1.1.1 DEFINITION Let $\underline{\underline{A}}$ be a category. A <u>device</u> $\mathcal{D}$ in $\underline{\underline{A}}$ consists of the following data:

(1) a family $X = (Xi; i \in I)$ of objects of $\underline{\underline{A}}$;

(2) a family $\eta = (\eta i \,; i \in I)$ of maps of $\underline{\underline{A}}$, where the domain of ηi is Xi and we write the codomain as Di;

(3) a family $\Delta = (\Delta(i_1,i_2); i_1,i_2 \in I)$ of sets $\Delta(i_1,i_2)$ of maps of $\underline{\underline{A}}$, where each map in $\Delta(i_1,i_2)$ has domain Di_1 and codomain Di_2.

The data are required to satisfy the following axioms for all i, i_1, i_2, i_3 in I.

The identity axiom: $1_{Di} \in \Delta(i,i)$.

The closure axiom: $\Delta(i_2,i_3).\Delta(i_1,i_2) \subseteq \Delta(i_1,i_3)$.

The universality axiom: for each $\mu : Xi_1 \to Di_2$ there is a unique $\delta \in \Delta(i_1,i_2)$ such that $\delta \circ \eta i_1 = \mu$.

We denote the device defined above by (X,η,Δ) or just (η,Δ). The set I is called the <u>index set</u>, and X is called the <u>object family</u> of $\mathcal{D}$. The special case when I is a set of objects of $\underline{\underline{A}}$ and $Xi = i$ for all $i \in I$ occurs frequently. In this case we say that $\mathcal{D}$ has a set of objects, and we identify I and X. We say that a device is <u>full</u> if it has a set of objects consisting of all the objects of $\underline{\underline{A}}$.

Unless otherwise indicated we adhere to the following convention. The device $\mathcal{D}$ will always be (η,Δ) with index set I and object family X. The map ηi will always have codomain Di. If $\mathcal{D}$ is decorated in any way, for example as $\mathcal{D}_1$ or $\mathcal{D}^*$, then we shall decorate the various parts of the device in the same way.

1.1.2 EXAMPLE Let $\underline{\underline{V}}$ be a variety of algebras, with forgetful functor U. Let $(Fi; i \in I)$ be any family of free algebras of $\underline{\underline{V}}$ with Fi freely generated by set Xi. Write the insertions of the free generating sets as $\eta i : Xi \to UFi$. Then, if we take $\Delta(i_1, i_2)$ to be the set of homomorphisms from Fi_1 to Fi_2, it is easy to see that (η, Δ) is a device in $\underline{\underline{Sets}}$.

More generally, let $U : \underline{\underline{B}} \to \underline{\underline{A}}$ be a functor and $S = ((\eta i, Fi); i \in I)$ a family of universal arrows to U. Then if we take $\Delta(i_1, i_2)$ to be $U\underline{\underline{B}}(Fi_1, Fi_2)$, clearly (η, Δ) is a device. From any $\underline{\underline{A}}$-valued functor with a left adjoint we can obtain a full device.

1.1.3 DEFINITIONS Let $\mathcal{D}$ be a device in $\underline{\underline{A}}$. Then a <u>$\mathcal{D}$-algebra</u> consists of an object A in $\underline{\underline{A}}$ and, to each i in I, a set Φi of morphisms from Di to A, satisfying the following axioms for all i_1, i_2 in I.

The closure axiom: $\Phi i_2 . \Delta(i_1, i_2) \subseteq \Phi i_1$.

The universality axiom: to each $\mu : Xi_1 \to A$ there is a unique $\phi \in \Phi i_1$ such that $\phi . \eta i_1 = \mu$.

We denote the $\mathcal{D}$-algebra defined above by (A, Φ). The object A is called the <u>carrier</u> and Φ is called the <u>structure</u> of the $\mathcal{D}$-algebra (A, Φ).

A <u>$\mathcal{D}$-morphism</u> from (A_1, Φ_1) to (A_2, Φ_2) is a morphism $\alpha : A_1 \to A_2$ such that $\alpha . \Phi_1 i \subseteq \Phi_2 i$ for all i in I. Then

$\mathcal{D}$-algebras and $\mathcal{D}$-morphisms form a category $\underline{\underline{A}}^{\mathcal{D}}$ if the composition of $\mathcal{D}$-morphisms is defined in the obvious way. There is a forgetful functor $U^{\mathcal{D}} : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}$ defined by:

$U^{\mathcal{D}} : (\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)) \rightsquigarrow (\alpha : A_1 \to A_2)$.

Notice that if $\underline{\underline{A}}$ and I are in Cat then so is $\underline{\underline{A}}^{\mathcal{D}}$.

The definitions of 1.1.1 and 1.1.3 appear in [31] with a slightly different notion of device.

1.1.4 EXAMPLE If $\mathcal{D}$ is the device obtained from a family of free algebras of variety $\underline{\underline{V}}$ as in 1.1.2, then given any algebra $\underline{A}$ of $\underline{\underline{V}}$ with underlying set A, we can form a $\mathcal{D}$-algebra with object A, and with maps Φi taken to be the homomorphisms from Fi to $\underline{A}$. A homomorphism from $\underline{A}_1$ to $\underline{A}_2$ clearly yields a $\mathcal{D}$-morphism from (A_1,Φ_1) to (A_2,Φ_2). These observations describe a functor from Cat($\underline{\underline{V}}$) to $\underline{\underline{A}}^{\mathcal{D}}$ which is an isomorphism provided we have taken a suitably large family of free algebras of $\underline{\underline{V}}$. A full description of devices and their algebras in the category of Sets is given later in Chapter III.

1.1.5 EXAMPLE Suppose X is any family of objects of $\underline{\underline{A}}$. Then take ηi to be 1_{Xi} and $\Delta(i_1,i_2)$ to be $\underline{\underline{A}}(Xi_1,Xi_2)$. This yields a device $\mathcal{D}$ which we call the <u>initial device</u> on X. It is easy to see that in this case $U^{\mathcal{D}} : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}$ is an isomorphism.

1.1.6 EXAMPLE Suppose $\underline{\underline{A}}$ has an initial object 0. Then if we take $X = \{0\}$ and $\eta(0) = 0 \to A$, and the maps from A to A to be just the identity map, then we have a device which can be described by the diagram $0 \to A \circlearrowleft 1$. Algebras are single morphisms $\alpha : A \to A_1$, and a morphism from α to $\beta : A \to A_2$ is a map $\gamma : A_1 \to A_2$ such that $\gamma \circ \alpha = \beta$. In other words, the category of algebras is the category of objects under A.

1.1.7 EXAMPLES In a category with a terminal object $\underline{1}$ there is also a very simple way of making a device with a single object. For any object A of $\underline{\underline{A}}$ the diagram $A \to \underline{1} \circlearrowleft 1$ describes a device. If $A \to \underline{1}$ is an epimorphism, the only possible structure on an object B is the set of all maps from $\underline{1}$ to B, and hence the category of algebras may be considered, by an abuse of language, to be a full subcategory of $\underline{\underline{A}}$. In $\underline{\underline{\text{Kit}}}$, if we take A to be the category with two objects and a single map between them, then the category of algebras is $\underline{\underline{\text{Sets}}}$. In the category of abelian groups, if we take A to be a cyclic group of order p (prime), then the algebras are the groups with no element of order p. We can enlarge this device to get just torsion-free groups in the following way. Let $I = \{2,3,5,\ldots,p,\ldots\}$ and $Xp = C_p$ (the prime cycle). Further let $\eta p = C_p \to \underline{1}$ and $\Delta(p,q) = \{1\}$.

This gives us a device with the torsion-free groups as algebras.

In $\underline{\underline{\mathrm{Kit}}}$ take X to be $\{\underline{\underline{A}}\}$ where $\underline{\underline{A}}$ is the category

$$\begin{tikzcd} A \arrow[loop left, "1"] \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[loop right, "1"] \end{tikzcd}$$

with $\alpha\beta = 1$, $\beta\alpha = 1$. Take $\eta\underline{\underline{A}}$ to be $G : \underline{\underline{A}} \to \underline{\underline{B}}$ where $\underline{\underline{B}}$ is the category generated by

$$\begin{tikzcd} A \arrow[loop left, "1"] \arrow[loop right, "\alpha"] \arrow[loop below, "\beta"] \end{tikzcd}$$

with again $\alpha\beta = 1$, $\beta\alpha = 1$, and $G\alpha = \alpha$, $G\beta = \beta$. Then, if the maps $\Delta(\underline{\underline{A}},\underline{\underline{A}})$ are taken to be all the endofunctors of $\underline{\underline{A}}$, we have a device in $\underline{\underline{\mathrm{Kit}}}$ whose algebras are the skeletal (small) categories.

1.1.8 EXAMPLE Consider the category $P(Y)$ of all subsets of a set Y (with inclusions as morphisms). Any closure operator J on Y yields a device with object set the finite subsets $X = \{X_1, X_2, \ldots\}$ of Y if ηX_1 is taken to be the map $X_1 \to JX_1$ and $\Delta(X_1, X_2)$ to be $P(Y)(JX_1, JX_2)$. An algebra is a subset A such that if $X \subseteq A$ then $JX \subseteq A$. Hence if J is an algebraic closure operator (see [4] page 45) the algebras are just the closed subsets. There is a precise correspondence between algebraic closure operators on Y and devices in $P(Y)$ with object set the finite subsets of Y. Of course, there is a well-known correspondence between closure operators on Y and triples in $P(Y)$.

§1.2 Devices and universal arrows.

As we remarked in 1.1.2 any family of universal arrows to a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ gives rise to a device in $\underline{\underline{A}}$ just as adjoint functors give rise to triples. In this section it is shown that all devices arise in this way, again in analogy with the triple-adjoint functor situation.

1.2.1 THEOREM Given device $\mathcal{D}$, then $F^{\mathcal{D}}i = (Di, \Delta(-,i))$ is a $\mathcal{D}$-algebra where $\Delta(-,i)(i_1) = \Delta(i_1,i)$. Further if (A,Φ) is any other algebra then the $\mathcal{D}$-morphisms from $F^{\mathcal{D}}i$ to (A,Φ) are precisely the morphisms in Φi.

PROOF Checking that $F^{\mathcal{D}}i$ is an algebra, we note that: (1) $\Delta(-,i)(i_1).\Delta(i_2,i_1) = \Delta(i_1,i).\Delta(i_2,i_1) \subseteq \Delta(i_2,i) = \Delta(-,i)(i_2)$; (2) given any $\mu : Xi_1 \to Di$ there is a unique δ in $\Delta(i_1,i) = \Delta(-,i)(i_1)$ such that $\delta.\eta i_1 = \mu$. Each ϕ in Φi is a $\mathcal{D}$-morphism since $\Phi i.\Delta(-,i)(i_1) = \Phi i.\Delta(i_1,i) \subseteq \Phi i$. To see that any $\mathcal{D}$-morphism $\alpha : F^{\mathcal{D}}i \to (A,\Phi)$ belongs to Φi, note that

$$\alpha = \alpha.1_{Di} \in \alpha.\Delta(-,i)(i) \subseteq \Phi i.$$

The notation $F^{\mathcal{D}}i$ will be used throughout this thesis for the algebra $(Di,\Delta(-,i))$.

1.2.2 THEOREM Consider device $\mathcal{D}$ in $\underline{\underline{A}}$. There is a family of universal arrows, indexed by I, to $U^{\mathcal{D}}$ such that the device obtained from these arrows as in 1.1.2 is just $\mathcal{D}$.

PROOF The i-th universal arrow is $(\eta i, F^{\mathcal{D}}i)$. It is a universal arrow since if $(A,\Phi) \in \underline{\underline{A}}^{\mathcal{D}}$ and μ is any map from Xi to A, then μ factors uniquely through ηi in Φi, and Φi, by 1.2.1, is $U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i,(A,\Phi))$. To see that the device obtained from these arrows is $\mathcal{D}$, note that by 1.2.1 $U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i,F^{\mathcal{D}}i_1) = \Delta(i,i_1)$.

We shall presently describe universal arrows to a much simpler functor which lead to $\mathcal{D}$.

1.2.3 DEFINITION Consider device $\mathcal{D}$ in $\underline{\underline{A}}$. The <u>Kleisli category of</u> $\mathcal{D}$, which we denote by $\underline{\underline{A}}_{\mathcal{D}}$, is the category constructed as follows: the objects of $\underline{\underline{A}}_{\mathcal{D}}$ are the ordered pairs $(i,Di)(i \in I)$; the set of morphisms in $\underline{\underline{A}}_{\mathcal{D}}$ from (i,Di) to (i_1,Di_1) is $\Delta(i,i_1)$; composition is the same as in $\underline{\underline{A}}$. That $\underline{\underline{A}}_{\mathcal{D}}$ is a category is an obvious consequence of the properties of device. (The name comes from the category defined by Kleisli in[15].)

There is a functor $U_{\mathcal{D}} : \underline{\underline{A}}_{\mathcal{D}} \to \underline{\underline{A}}$ defined by $U_{\mathcal{D}} : (\delta : (i,Di) \to (i_1,Di_1)) \rightsquigarrow (\delta : Di \to Di_1)$. We denote the object (i,Di) by $F_{\mathcal{D}}i$.

The definition of device leads immediately to the following result.

1.2.4 THEOREM There is a family of universal arrows to $U_{\mathcal{D}} : \underline{\underline{A}}_{\mathcal{D}} \to \underline{\underline{A}}$ such that the device obtained as in 1.1.2 from these arrows is $\mathcal{D}$.

The two functors $U^{\mathcal{D}}$ and $U_{\mathcal{D}}$ have a very special position among $\underline{\underline{A}}$-valued functors which have a family of universal arrows leading to the device $\mathcal{D}$. Before explaining this remark we prove a useful lemma.

1.2.5 LEMMA Let $\mathcal{D}$ be a device in $\underline{\underline{A}}$ and (A,Φ_1), (A,Φ_2) two $\mathcal{D}$-algebras. If for all i in the index set I of $\mathcal{D}$ we have $\Phi_1 i \subseteq \Phi_2 i$, then $\Phi_1 = \Phi_2$. That is, for maps α in $\underline{\underline{A}}^{\mathcal{D}}$ we have that $U^{\mathcal{D}}\alpha$ is an identity in $\underline{\underline{A}}$ if and only if α is an identity.

PROOF For any $\phi_2 \in \Phi_2 i$ we can find $\phi_1 \in \Phi_1 i$ such that $\phi_1 . \eta i = \phi_2 . \eta i$. But both ϕ_1 and ϕ_2 are in $\Phi_1 i$, so the uniqueness part of the universality condition for algebras gives us that $\phi_1 = \phi_2$ and hence $\Phi_1 = \Phi_2$.

1.2.6 THEOREM Consider functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and a family of universal arrows $((\eta i, Fi); i \in I)$ to U with associated device $\mathcal{D} = (\eta, \Delta)$. Then there is a unique functor $K : \underline{\underline{B}} \to \underline{\underline{A}}^{\mathcal{D}}$ such that $U^{\mathcal{D}}.K = U$ and $K(Fi) = F^{\mathcal{D}}i$ for all i in I. Further, there is a unique functor $K' : \underline{\underline{A}}_{\mathcal{D}} \to \underline{\underline{B}}$ such that $U.K' = U_{\mathcal{D}}$ and $K'(F_{\mathcal{D}}i) = Fi$ for all $i \in I$.

The situation is summarized in the following diagram

$$\begin{tikzcd}
& I \arrow[dl, "F_{\mathcal{D}}"'] \arrow[d, "F"] \arrow[dr, "F^{\mathcal{D}}"] & \\
\underline{\underline{A}}_{\mathcal{D}} \arrow[r, dashed, "K'"] \arrow[dr, "U_{\mathcal{D}}"'] & \underline{\underline{B}} \arrow[r, dashed, "K"] \arrow[d, "U"] & \underline{\underline{A}}^{\mathcal{D}} \arrow[dl, "U^{\mathcal{D}}"] \\
& \underline{\underline{A}} &
\end{tikzcd}$$

where the families $F_{\mathcal{D}}, F$ and $F^{\mathcal{D}}$ are here regarded as functors.

PROOF (a) Let $\Phi i = U\underline{\underline{B}}(Fi, B)$. Then K is defined by:

$$K : B \rightsquigarrow (UB, \Phi), \quad \text{and}$$

$$K : (\alpha : B_1 \to B_2) \rightsquigarrow (U\alpha : KB_1 \to KB_2).$$

Since the universal arrows to U yield $\mathcal{D}$, we must have $U\underline{\underline{B}}(Fi,Fi_1) = \Delta(i,i_1)$. Thus (UB,Φ) is an algebra, since

(1) $\Phi i . \Delta(i_1,i) = U\underline{\underline{B}}(Fi,B) . U\underline{\underline{B}}(Fi_1,Fi) \subseteq U\underline{\underline{B}}(Fi_1,B) = \Phi i_1$ and

(2) any $\mu : Xi \to UB$ factorizes uniquely through ηi in $U\underline{\underline{B}}(Fi,B)$. Similarly $U\alpha : KB_1 \to KB_2$ is a $\mathcal{D}$-morphism. That K is a functor is obvious, and K clearly takes Fi to $(Di,\Delta(-,i))$.

Consider another functor $K_1 : \underline{\underline{B}} \to \underline{\underline{A}}^{\mathcal{D}}$ with $U^{\mathcal{D}}K_1 = U$ and $K_1(Fi) = F^{\mathcal{D}}i$ for all $i \in I$. Then $K_1B = (UB,\Psi)$ for some structure Ψ. Further

$$U\underline{\underline{B}}(Fi,B) = U^{\mathcal{D}}K_1\underline{\underline{B}}(Fi,B) \subseteq U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i,(B,\Psi)) = \Psi i.$$

However, we have seen above that $(UB,U\underline{\underline{B}}(Fi,B))$ is a $\mathcal{D}$-algebra, and hence by Lemma 1.2.5 we have $\Psi i = U\underline{\underline{B}}(Fi,B)$. Thus K_1 agrees with K on objects. Since $U^{\mathcal{D}}K_1 = U^{\mathcal{D}}K$ and $U^{\mathcal{D}}$ is faithful, it is clear that K_1 also agrees with K on morphisms.

(b) The functor K' is defined by

$$K' : \begin{array}{c}(i,Di)\\ \big\downarrow{\scriptstyle\delta}\\ (i_1,Di,)\end{array} \qquad \rightsquigarrow \qquad \begin{array}{c}Fi\\ \big\downarrow{\scriptstyle\delta'}\\ Fi_1\end{array}$$

where δ' is determined by $U\delta'.\eta i = \delta.\eta i$. Clearly K' must act in this way to have the desired properties since $UK'\delta.\eta i = U_{\mathcal{D}}\delta.\eta i = \delta.\eta i$. Now $U\delta' \in \Delta(i,i_1)$ since the device associated with U is $\mathcal{D}$. Hence $U\delta' = \delta$ and so $U_{\mathcal{D}} = U.K'$.

§1.3 The enlargement of devices.

1.3.1 DEFINITIONS Given a device $\mathcal{D}$ with index set I, and a subset I_1 of I, then $\mathcal{D}|I_1 = (\eta|I_1, \Delta|I_1)$ is a device with index set I_1 if $\eta|I_1$ is defined by $\eta|I_1(i_1) = \eta(i_1)$ and $(\Delta|I_1)(i_1,i_1') = \Delta(i_1,i_1')$ for all i_1, i_1' in I_1. The device $\mathcal{D}|I_1$ is called the <u>restriction</u> of $\mathcal{D}$ to I_1, and $\mathcal{D}$ is called an <u>enlargement</u> of $\mathcal{D}|I_1$ to I.

There is an obvious functor $U(I_1) : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}^{\mathcal{D}|I_1}$ defined as follows:

$$U(I_1) : (A,\Phi) \rightsquigarrow (A,\Phi|I_1) \quad \text{and} \quad U(I_1) : \alpha \rightsquigarrow \alpha,$$

where $\Phi|I_1$ is a family indexed by I_1 and defined by $(\Phi|I_1)(i_1) = \Phi i_1$. Again, that $(A,\Phi|I_1)$ is a $\mathcal{D}|I_1$-algebra is clear, as is the functorial nature of $U(I_1)$.

If $U(I_1)$ is an isomorphism, then we say that $\mathcal{D}$ is an <u>inessential enlargement</u> of $\mathcal{D}|I_1$. Call two devices <u>equivalent</u> if there is a device which is an inessential enlargement of each of them. We will see in 1.3.3 that equivalence is indeed an equivalence relation.

1.3.2 THEOREM Let $\mathcal{D}$ be a device on $\underline{\underline{A}}$ with index set I, and let $S(I) = ((\eta i, F^{\mathcal{D}} i); i \in I)$ be the derived family of universal arrows to $U^{\mathcal{D}}$. Suppose $I_1 \supseteq I$, and there is a family $S(I_1)$ of universal arrows to $U^{\mathcal{D}}$ indexed by I_1, which, when restricted to I, is $S(I)$. Let $\mathcal{D}_1$ be the device associated with the family $S(I_1)$. Then $\mathcal{D}_1$ is an inessential enlargement of $\mathcal{D}$. Further any inessential enlargement of $\mathcal{D}$ may be obtained in this way.

PROOF Clearly $\mathcal{D}_1|I = \mathcal{D}$. We want to show that $U = U(I) : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}}$ is an isomorphism. Notice first that if $(\eta_1 i_1 : X_1 i_1 \to D_1 i_1, Fi_1)$ is the universal arrow corresponding to i_1 in $S(I_1)$ then $Fi_1 = (D_1 i_1, \Delta_1(-,i_1)|I)$ since $\Delta_1(i,i_1)$ is the set of morphisms from $F^{\mathcal{D}} i$ to Fi_1.

That U is faithful is clear. To see that U is onto objects, consider (A,Φ) in $\underline{\underline{A}}^{\mathcal{D}}$. Define Φ_1 by

$$\Phi_1 i_1 = U^{\mathcal{D}} \underline{\underline{A}}^{\mathcal{D}}(Fi_1, (A,\Phi)) \quad \text{for all} \quad i_1 \in I_1.$$

Notice that $\Phi_1|I = \Phi$, so that if $(A,\Phi_1) \in \underline{\underline{A}}^{\mathcal{D}_1}$ then certainly $U(A,\Phi_1) = (A,\Phi)$. But

$$\Phi_1 i_1 . \Delta_1(i_2,i_1) = U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,(A,\Phi)).U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_2,Fi_1)$$

$$\subseteq U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_2,(A,\Phi)) = \Phi_1\ i_2 .$$

Further given any $\mu : Xi_1 \to A$, the fact that $(\eta i_1, Fi_1)$ is a universal arrow enables us to factor μ uniquely through ηi_1 in $\Phi_1 i_1$.

To see that U is one-to-one on objects, consider (A,Ψ_1) in $\underline{\underline{A}}^{\mathcal{D}_1}$. It is sufficient to show for all i_1 in I_1 that $\Psi_1 i_1 = U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,U(A,\Psi_1))$. We know that $\Psi_1 i_1 . \Delta_1(i,i_1) \subseteq \Psi_1 i$. But the morphisms in $\Delta_1(i,i_1)$ (i in I) are the structure maps of the $\mathcal{D}$-algebra Fi_1 from Di, and so $\Psi_1 i_1$ consists of $\mathcal{D}$-morphisms from Fi_1 to $U(A,\Psi_1)$. That is $\Psi_1 i_1 \subseteq U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,U(A,\Psi_1))$. But as we have seen above $U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(F(-),U(A,\Psi_1))$ is a $\mathcal{D}_1$-structure for A, and hence by Lemma 1.2.5 we have that $\Psi_1 i_1 = U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,U(A,\Psi_1))$.

Finally, to see that $U : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}}$ is full, consider $\alpha : (A,\Phi) \to (B,\Psi)$ in $\underline{\underline{A}}^{\mathcal{D}}$. We need to show that $\alpha . U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,(A,\Phi)) \subseteq U^{\mathcal{D}}\underline{\underline{A}}^{\mathcal{D}}(Fi_1,(B,\Psi))$ which is clear.

The second part of the theorem is easy since if $\mathcal{D}_1$ is an inessential enlargement of $\mathcal{D}$ then $U^{\mathcal{D}_1}$ is strictly isomorphic to $U^{\mathcal{D}}$. Hence corresponding to universal arrows to $U^{\mathcal{D}_1}$ there are universal arrows to $U^{\mathcal{D}}$ which lead to the same device.

1.3.3 COROLLARY Devices $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\underline{\underline{A}}$ are equivalent if and only if $U^{\mathcal{D}_1}$ and $U^{\mathcal{D}_2}$ are strictly isomorphic.

PROOF If $U^{\mathcal{D}_1}$ and $U^{\mathcal{D}_2}$ are strictly isomorphic then, as we remarked at the end of the proof of 1.3.2, a family S_1 of universal arrows to $U^{\mathcal{D}_1}$ may be carried across via the isomorphism to a family S_2 of universal arrows to $U^{\mathcal{D}_2}$ with the same associated device as S_1. It is clear from this and Theorem 1.3.2 how to obtain a device $\mathcal{D}$ in $\underline{\underline{A}}$ which is a common inessential enlargement of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Next we give a theorem that allows us to reduce the object family of a device while still retaining an equivalent device.

1.3.4 THEOREM Consider device $\mathcal{D}$. If $I_1 \subseteq I$, and to each i in I there is an i_1 in I_1 and a split epimorphism $Xi_1 \to Xi$, then $\mathcal{D}|I_1$ is equivalent to $\mathcal{D}$.

PROOF To see that $U(I_1) : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}^{\mathcal{D}|I_1}$ is an isomorphism consider (A,Φ) in $\underline{\underline{A}}^{\mathcal{D}|I_1}$. Then if α is the promised epimorphism from $Xi_1 \to Xi$ $(i \notin I_1)$ for which there is a map β with $\alpha\beta = 1$, and Ψ is defined by $\Psi i = \Phi i_1 . D\beta$ and $\Psi|I_1 = \Phi$ then (A,Ψ) is a $\mathcal{D}$-algebra. Further it is the one and only $\mathcal{D}$-algebra going to (A,Φ) under $U(I_1)$. To see that (A,Ψ) is an algebra we first observe that

$$\Phi i_1.\Delta(i,i_1) = \Phi i_1.\Delta(i,i_1).D\alpha.D\beta \subseteq \Phi i_1.\Delta(i_1,i_1).D\beta = \Phi i_1.D\beta.$$

The closure condition follows easily from this. Consider $\mu : Xi \to A$. There exists a ϕ in Φi_1 such that $\phi.\eta i_1 = \mu\alpha$. Then $\phi.D\beta.\eta i = \phi.\eta i_1.\beta = \mu\alpha\beta = \mu$. Suppose for two maps ϕ_1,ϕ_2 in Φi_1 we have that $\phi_1.D\beta.\eta i = \phi_2.D\beta.\eta i$. Then

$$\phi_1.D\beta.D\alpha.\eta i_1 = \phi_1.D\beta.\eta i.\alpha = \phi_2.D\beta.\eta i.\alpha = \phi_2.D\beta.D\alpha.\eta i_1.$$

and hence $\phi_1.D\beta.D\alpha = \phi_2.D\beta.D\alpha$. It follows that

$$\phi_1.D\beta = \phi_1.D\beta.D\alpha.D\beta = \phi_2.D\beta.D\alpha.D\beta = \phi_2.D\beta.$$

Suppose (A,Ψ_1) is another algebra going to (A,Φ) under $U(I_1)$. Then certainly $\Phi i_1.D\beta \subseteq \Psi_1 i$. But, as we have seen, $\Phi i_1.D\beta$ is part of a structure for A, and hence $\Phi i_1.D\beta = \Psi_1 i$. It is easy to see that $U(I_1)$ is full, and hence is an isomorphism.

1.3.5 COROLLARY Given device $\mathcal{D}$, there is a subset I_1 of I such that for all i_1,i_1' in I_1, if $Xi_1 = Xi_1'$ then $i_1 = i_1'$, and further $\mathcal{D}$ is equivalent to $\mathcal{D}|I_1$.

1.3.6 REMARKS Consider device $\mathcal{D}$ and a bijection $\alpha : I_1 \to I$. If we define η_1 by $\eta_1 i_1 = \eta(\alpha i_1)$ and Δ_1 by $\Delta_1(i_1, i_1') = \Delta(\alpha i_1, \alpha i_1')$ then $\mathcal{D}_1$ so defined is a device which for most purposes is indistinguishable from $\mathcal{D}$. Any statement about $\mathcal{D}_1$ can be trivially translated into one about $\mathcal{D}$. If we are prepared to overlook the difference between these two devices then Corollary 1.3.5 states that any device is equivalent to a device with a set of objects.

Not all devices can be enlarged so that their object family includes all the objects of the category in which they are defined. For example consider the category described by the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & B \\ & C \end{tikzcd}$$

Then the diagram $A \xrightarrow{\alpha} B \circlearrowleft 1$ describes a device with one object A, which cannot be enlarged to the object C.

§1.4 Devices and triples.

As we remarked in §0.6, any triple in a category $\underline{\underline{A}}$ arises from an adjoint pair of functors; and, as we know, any adjoint pair leads to universal arrows from each object of $\underline{\underline{A}}$, and hence to a device with object set all the objects of $\underline{\underline{A}}$. Conversely any full device arises from an adjoint pair of functors and hence from a triple. This suggests a correspondence, which we make precise in the following theorem, between triples and full devices in $\underline{\underline{A}}$.

1.4.1 THEOREM Let $\mathcal{T} = (T,\eta,\mu)$ be a triple in $\underline{\underline{A}}$. If we define $\Delta(A_1,A_2)$ to be $\{\mu A_2.T\alpha;\ \alpha : A_1 \to TA_2\}$, then $\mathcal{D} = (\eta,\Delta)$ is a device with object set all the objects of $\underline{\underline{A}}$. Conversely, given such a device $\mathcal{D}$ in $\underline{\underline{A}}$ we can define a triple in $\underline{\underline{A}}$ as follows: the functor T is defined by the requirements (1) $TA = DA$ and (2) if $\alpha : A_1 \to A_2$ then $T\alpha \in \Delta(A_1,A_2)$ and $T\alpha.\eta A_1 = \eta A_2.\alpha$; the function η of the device $\mathcal{D}$ is then a natural transformation from $1_{\underline{\underline{A}}}$ to T; μ defined by $\mu A \in \Delta(DA,A)$ and $\mu A.\eta TA = 1_{TA}$ is natural from T^2 to T. These two processes, from triples to full devices, and from full devices to triples, are inverse to one another.

PROOF (a) To see that (η,Δ) (as defined in the first part of the statement) is a device, we observe the following facts. (1) $1_{TA} \varepsilon \Delta(A,A)$ since $\mu A \circ T\eta A = 1_{TA}$. (2) If $\delta_1 \varepsilon \Delta(A_1,A_2)$ and $\delta_2 \varepsilon \Delta(A_2,A_3)$ then $\delta_1 = \mu A_2 \circ T\alpha$ and $\delta_2 = \mu A_3 . T\beta$ for some α,β. Further,

$$\begin{aligned}\mu A_3 \circ T\beta . \mu A_2 . T\alpha &= \mu A_3 . \mu TA_3 . T^2\beta . T\alpha \\ &= \mu A_3 . T\mu A_3 \circ T^2\beta . T\alpha \\ &= \mu A_3 \circ T(\mu A_3 \circ T\beta . \alpha) \in \Delta(A_1,A_3).\end{aligned}$$

(3) Given $\alpha : A_1 \to TA_2$ then $(\mu A_2 \circ T\alpha) . \eta A_1 = \mu A_2 \circ \eta TA_2 \circ \alpha = \alpha$. Furthermore it is clear that α determines the map $\mu A_2 . T\alpha$ in $\Delta(A_1,A_2)$.

(b) To see that (T,η,μ) (as defined in the second part of the statement of 1.4.1) is a triple we need to consider the following facts. (1) T is well defined in view of the defining properties of device. It is clear then, by a well known argument, that T is a functor, and trivially that $\eta : 1 \to T$ is natural. (2) Again $\mu : T^2 \to T$ is well defined, by the definition of device. Further μ is natural since for each $\alpha : A_1 \to A_2$ we have

$$\mu A_2 . T^2\alpha \circ \eta TA_1 = \mu A_2 \circ \eta TA_2 . T\alpha = T\alpha = T\alpha . \mu A_1 \circ \eta TA_1,$$

and since both $\mu A_2.T^2\alpha$ and $T\alpha.\mu A_1$ belong to $\Delta(TA_1,A_2)$ we may cancel the ηTA_1. (3) That $\mu.\eta T = 1_T$ follows immediately from the definition of μ. (4) Since for each $A \in \underline{\underline{A}}$ we have

$$\mu A.T\eta A.\eta A = \mu A.\eta TA.\eta A = 1_{TA}.\eta A$$

and both $\mu A.T\eta A$ and 1_{TA} belong to $\Delta(A,A)$, we may cancel ηA to obtain $\mu.T\eta = 1_T$. (5) Finally, since for each object A in $\underline{\underline{A}}$

$$\mu A.T\mu A.\eta T^2 A = \mu A.\eta TA.\mu A = \mu A \qquad = \mu A.\mu TA.\eta T^2 A$$

and since both $\mu A.T\mu A$ and $\mu A.\mu TA$ belong to $\Delta(T^2A,A)$, we may cancel $\eta T^2 A$ to obtain $\mu.T\mu = \mu.\mu T$.

(c) To see that the two processes described in the theorem are inverse, suppose first that (T,η,μ) is taken by the first process to (η,Δ) and this device is taken by the second to (T_1,η_1,μ_1). We notice immediately that $\eta_1 = \eta$ and that T_1 on objects is the same as T on objects. Now $T_1\alpha$ for $\alpha : A_1 \to A_2$ is the unique map in $\Delta(A_1,A_2)$ satisfying $T_1\alpha.\eta A_1 = \eta A_2.\alpha$. Further $\Delta(A_1,A_2) = \{\mu A_2.T\beta;\ \beta : A_1 \to TA_2\}$, so

$$T\alpha = \mu A_2 \circ T\eta A_2 . T\alpha = \mu A_2 \circ T(\eta A_2 . \alpha) \ \varepsilon \ \Delta(A_1, A_2).$$

Since $T\alpha . \eta A_1 = \eta A_2 . \alpha$ we must have $T_1\alpha = T\alpha$. Finally $\mu_1 A$ is the unique map in $\Delta(TA, A)$ satisfying $\mu_1 A \circ \eta TA = 1_{TA}$. Clearly μA satisfies this equation and further $\mu A \circ T(1_{TA}) = \mu A \ \varepsilon \ \Delta(TA, A)$. So $(T_1, \eta_1, \mu_1) = (T, \eta, \mu)$.

Secondly, suppose that the second process takes the device (η, Δ) to the triple (T, η, μ) and the first takes this triple to (η_1, Δ_1). Again, it is clear that $\eta_1 = \eta$. Now $\Delta_1(A_1, A_2) = \{\mu A_2 . T\beta;\ \beta : A_1 \to A_2\}$ and $\mu A_2 . T\beta$ belongs to $\Delta(A_1, A_2)$. Hence $\Delta_1(A_1, A_2) \subseteq \Delta(A_1, A_2)$. Given $\delta \ \varepsilon \ \Delta(A_1, A_2)$ there exists a $\delta_1 \ \varepsilon \ \Delta_1(A_1, A_2)$ such that $\delta_1 . \eta A_1 = \delta . \eta A_1$. Since $\delta_1 \ \varepsilon \ \Delta(A_1, A_2)$ this implies that $\delta_1 = \delta$, and hence $\Delta_1(A_1, A_2) = \Delta(A_1, A_2)$.

Thus the two functions are inverse, and we may speak of the triple in $\underline{\underline{A}}$ corresponding to a full device, and vice versa.

1.4.2 THEOREM If $\mathcal{D}$ is the full device in $\underline{\underline{A}}$ corresponding to triple $\mathcal{T}$, then $U^{\mathcal{D}}$ is strictly isomorphic to $U^{\mathcal{T}}$.

PROOF We construct a pair of functors $L : \underline{\underline{A}}^{\mathcal{T}} \to \underline{\underline{A}}^{\mathcal{D}}$ and $L' : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}^{\mathcal{T}}$ such that $U^{\mathcal{D}}L = U^{\mathcal{T}}$, $L'L = 1$ and $LL' = 1$. The functor L is defined by:

$$L : (\alpha : (A_1, \phi_1) \to (A_2, \phi_2)) \rightsquigarrow (\alpha : (A_1, \Phi_1) \to (A_2, \Phi_2)),$$

where $\Phi_j A = \{\phi\circ T\gamma;\ \ \gamma : A \to A_j\}$ $(j = 1,2)$. The functor L' is defined by:

$$L' : (\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)) \rightsquigarrow (\alpha : (A_1,\phi_1) \to (A_2,\phi_2))$$

where ϕ_j is determined by $\phi_j \in \Phi_j A_j$ and $\phi_j . \eta A_j = 1_{A_j}$ $(j = 1,2)$.

To see that L is a functor, we need to check first that if $(A_1,\phi_1) \in \underline{\underline{A}}^{T}$, then $L(A_1,\phi_1) \in \underline{\underline{A}}^{\mathcal{D}}$. (1) Any $\delta \in \Delta(A',A)$ is of the form $\mu A\circ T\lambda$ for some $\lambda : A' \to TA$, and any $\psi \in \Phi_1 A$ is of the form $\phi . T\gamma$ for some $\gamma : A \to A_1$. Hence

$$\psi\circ\delta = \phi\circ T\gamma\circ\mu A . T\lambda = \phi . \mu A_1 . T^2\gamma . T\lambda = \phi . T\phi . T^2\gamma . T\lambda$$

and so $\Phi_1 A\circ\Delta(A',A) \subseteq \Phi_1 A'$. (2) For any $\nu : A \to A_1$ we have $\phi\circ T\nu . \eta A = \phi . \eta A_1\circ\nu = \nu$ and clearly $\phi . T\nu$ is unique in ΦA satisfying this equation.

We also need to check that if $\alpha : (A_1,\phi_1) \to (A_2,\phi_2)$ is in $\underline{\underline{A}}^{T}$ then $\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)$ is in $\underline{\underline{A}}^{\mathcal{D}}$. But notice that for any $\gamma : A \to A_1$ we have $\alpha . \phi_1\circ T\gamma = \phi_2\circ T\alpha . T\gamma$, and so $\alpha\circ\Phi_1(A) \subseteq \Phi_2(A)$.

Similarly, to see that L' is a functor, first consider $L'(A,\Phi) = (A,\phi)$. Clearly $\phi\circ\eta A = 1_A$. Further $\phi\circ T\phi . \eta TA =$ $\phi\circ\eta A . \phi = \phi = \phi\circ\mu A . \eta TA$ and we can cancel the morphism ηTA to complete the proof that (A,ϕ) belongs to $\underline{\underline{A}}^{T}$. Consider $\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)$ in $\underline{\underline{A}}^{D}$, and let $L'(\alpha) = \alpha : (A_1,\phi_1) \to (A_2,\phi_2)$. Then

$$\phi_2.T\alpha.\eta A_1 = \phi_2.\eta A_2.\alpha = \alpha = \alpha.\phi_1.\eta A_1.$$

We can cancel the morphism ηA_1 to see that $L'(\alpha)$ is a morphism in $\underline{\underline{A}}^{T}$.

Finally, let (A,ϕ_1) be the image of (A,ϕ) under $L'L$ and let (A,Φ_1) be the image of (A,Φ) under LL'. Then ϕ_1 is the unique map in $\{\phi.T\gamma;\ \gamma : A \to A\}$ satisfying $\phi_1.\eta A = 1_A$. The map ϕ clearly has these properties, so $(A,\phi_1) = (A,\phi)$. Further $\Phi_1 A_1 = \{\psi.T\gamma;\ \gamma : A_1 \to A\}$ where $\psi \in \Phi A$ and $\psi.\eta A = 1$, and so $\Phi_1 A_1 \subseteq \Phi A_1$. This is enough to prove that $(A,\Phi_1) = (A,\Phi)$. Hence L and L' are inverses.

1.4.3 REMARKS It is clear from 1.3.2 and 1.3.6 that a device $\mathcal{D}$ is equivalent to (the full device corresponding to) a triple if and only if $U^{\mathcal{D}}$ has a left adjoint. In fact, in most interesting cases $U^{\mathcal{D}}$ does have a left adjoint, so one might suppose that nothing is gained by discussing devices rather than triples. However many of the theorems and concepts of this thesis are not "preserved under equivalence" since they make essential use of properties of the families of objects of the devices involved. This might be considered the special advantage of our approach.

Notice that (the forgetful functor of) the Kleisli category of a full device, as defined in 1.2.3, is strictly isomorphic to (the forgetful functor of) the usual Kleisli category, defined in [15], of the corresponding triple. One could prove this by producing an explicit isomorphism, or by using the two comparison theorems, namely our Theorem 1.2.6 and the observation of Huber [12]. This second method of proof could have been employed for Theorem 1.4.2, using in this case the comparison theorem of [6], but we preferred to give the above explicit verification independent of [6].

§1.5 Morphisms of devices.

In this section we defined and discuss morphisms between devices with the same object family.

1.5.1 DEFINITIONS Consider devices $\mathcal{D}_1 = (\eta_1, \Delta_1)$ and $\mathcal{D}_2 = (\eta_2, \Delta_2)$, both with the same index set I and object family X. Then a morphism from $\mathcal{D}_1$ to $\mathcal{D}_2$ is a family of maps $(\lambda i;\ i \in I)$, where $\lambda i : D_1 i \to D_2 i$, satisfying for all $i, i_1 \in I$

(1) $\lambda i.\eta_1 i = \eta_2 i$, and

(2) $\lambda i_1.\Delta_1(i,i_1) \subseteq \Delta_2(i,i_1).\lambda i$.

The category of devices in $\underline{\underline{A}}$ with object family X, where the composition of morphisms is the obvious one, will be denoted by $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$, or just $\underline{\underline{\mathrm{Dev}}}(X)$ or $\underline{\underline{\mathrm{Dev}}}$ when these are unambiguous. Notice that if I is small and $\underline{\underline{A}}$ is in $\underline{\underline{\mathrm{Cat}}}$ then $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$ is in $\underline{\underline{\mathrm{Cat}}}$. There is a clear forgetful functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ defined by

$$U : (\lambda : (\eta_1,\Delta_1) \to (\eta_2,\Delta_2)) \rightsquigarrow (\lambda : D_1 \to D_2).$$

When X is the set of objects of $\underline{\underline{A}}$, we would like to see that morphisms correspond to morphisms of triples.

1.5.2 THEOREM If $\mathcal{D}_1,\mathcal{D}_2$ are the full devices corresponding to triples $\mathcal{T}_1,\mathcal{T}_2$, then $\lambda : D_1 \to D_2$ is a morphism from $\mathcal{D}_1$ to $\mathcal{D}_2$ if and only if it is a morphism from $\mathcal{T}_1$ to $\mathcal{T}_2$.

PROOF Consider morphism $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$. Now for any $\alpha : A_1 \to A_2$ in $\underline{\underline{A}}$ we have $\lambda A_2.D_1\alpha \in \Delta_2(A_1,A_2).\lambda A_1$; that is $\lambda A_2.D_1\alpha = \delta.\lambda A_1$ say, and hence

$$\delta . \eta_2 A_1 = \delta . \lambda A_1 . \eta_1 A_1 = \lambda A_2 . D_1 \alpha . \eta_1 A_1 = \lambda A_2 . \eta_1 A_2 . \alpha = \eta_2 A_2 . \alpha$$
$$= D_2 \alpha . \eta_2 A_1 .$$

It immediately follows that $D_2 \alpha = \delta$, and so $\lambda A_2 . D_1 \alpha = D_2 \alpha . \lambda A_1$. Hence λ is natural from T_1 to T_2. Further, for any A in $\underline{\underline{A}}$ we have

$$\mu_2 A . T_2 \lambda A . \lambda T_1 A . \eta_1 T_1 A = \mu_2 A . T_2 \lambda A . \eta_2 T_1 A = \mu_2 A . \eta_2 T_2 A . \lambda A = \lambda A .$$

Also $\lambda A . \mu_1 A . \eta_1 T_1 A = \lambda A$. Now there exists a morphism δ in $\Delta_2 (T_1 \; A, A)$ such that $\lambda A . \mu_1 A = \delta . \lambda T_1 A$. Then

$$\mu_2 A . T_2 \lambda A . \eta_2 T_1 A = \delta . \lambda T_1 A . \eta_1 T_1 A = \delta . \eta_2 T_1 A$$

and so $\mu_2 A . T_2 \lambda A . = \delta$. Hence as required $\mu_2 A . \lambda T_2 A . T_1 \lambda A = \mu_2 A . T_2 \lambda A . \lambda T_1 A = \delta . \lambda T_1 A = \lambda A . \mu_1 A$, and so λ is a triple morphism from $\mathcal{T}_1$ to $\mathcal{T}_2$.

Consider morphism $\lambda : \mathcal{T}_1 \to \mathcal{T}_2$. To see that $\lambda A_2 . \Delta_1 (A_1, A_2) \subseteq \Delta_2 (A_1, A_2) . \lambda A_1$ for all A_1, A_2 in $\underline{\underline{A}}$, remember that $\Delta_j (A_1, A_2) = \{ \mu_j A_2 . T_j \alpha ; \; \alpha : A_1 \to T_j A_2 \}$ $(j = 1,2)$. Then the inclusion follows from

$$\lambda A_2 . \mu_1 A_2 . T_1 \alpha = \mu_2 A_2 . T_2 \lambda A_2 . \lambda T_1 A_2 . T_1 \alpha = \mu_2 A_2 . T_2 \lambda A_2 . T_2 \alpha . \lambda A_1 \; .$$

1.5.3 DEFINITION The category $\underline{\underline{\mathrm{Alg}}}(\underline{\underline{A}},X)$ has as objects categories of algebras $\underline{\underline{A}}^{\mathcal{D}}$ together with specified devices $\mathcal{D}$ where the devices are required to have object family X.
A morphism from $\underline{\underline{A}}^{\mathcal{D}_1}$ to $\underline{\underline{A}}^{\mathcal{D}_2}$ (where the devices are understood) is a functor $G : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}_2}$ such that $U^{\mathcal{D}_2}G = U^{\mathcal{D}_1}$. Such functors are called algebraic functors.

1.5.4 THEOREM $(\underline{\underline{\mathrm{Alg}}}(\underline{\underline{A}},X))^{op} \cong \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$.

PROOF We define functors K_1 from categories to devices, and K_2 in the opposite direction. They are the obvious functions on objects, namely $K_1(\underline{\underline{A}}^{\mathcal{D}}) = \mathcal{D}$ and $K_2(\mathcal{D}) = \underline{\underline{A}}^{\mathcal{D}}$. Given an algebraic functor $G : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}_2}$, we form a morphism $K_1G = \lambda : \mathcal{D}_2 \to \mathcal{D}_1$ as follows: to each $Xi \in X$ we solve the diagram

$$\begin{tikzcd} Xi \arrow[r, "\eta_2 i"] \arrow[dr, "\eta_1 i"'] & D_2 i \arrow[d, dashed, "\lambda i"] \\ & D_1 i \end{tikzcd}$$

for λi, a structure map of $GF^{\mathcal{D}_1}i$. We check that λ so defined is a morphism of devices. The structure of $GF^{\mathcal{D}_1}i$ is $\underline{\underline{A}}^{\mathcal{D}_2}(F^{\mathcal{D}_2}(-),GF^{\mathcal{D}_1}i) = \Phi$, say, so clearly $\Delta_1(i_1,i).\lambda i_1 \subseteq \Phi i_1$.

Further, given any $\phi \in \Phi i_1$ we can solve the outer triangle of

$$\begin{tikzcd}
Xi_1 \arrow[r, "\eta_2 i_1"] \arrow[dr, "\phi.\eta_2 i_1"'] & D_2 i_1 \arrow[r, "\lambda i_1"] \arrow[d, "\phi"] & D_1 i_1 \arrow[dl, dashed, "\delta"] \\
& D_1 i &
\end{tikzcd}$$

for $\delta \in \Delta_1(i_1,i)$ and hence, by uniqueness, $\phi = \delta.\lambda i_1$ and $\Phi i_1 = \Delta_1(i_1,i).\lambda i_1$. Now, since $\lambda i \in \Phi i$, we have $\lambda i.\Delta_2(i_1,i) \subseteq \Phi i_1$, and so $\lambda i.\Delta_2(i_1,i) \subseteq \Delta_1(i_1,i).\lambda i_1$. It is easy to see that K_1 is functorial.

To define K_2 on morphisms consider $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$. Then $K_2\lambda$ is the functor $H : \underline{\underline{A}}^{\mathcal{D}_2} \to \underline{\underline{A}}^{\mathcal{D}_1}$ defined by

$$H : (\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)) \rightsquigarrow (\alpha : (A_1,\Phi_1\lambda) \to (A_2,\Phi_2\lambda)).$$

To check that $(A,\Phi\lambda)$ is a $\mathcal{D}_1$-algebra if (A,Φ) is a $\mathcal{D}_2$-algebra, note first that

$$\Phi i.\lambda i.\Delta_1(i_1,i) \subseteq \Phi i.\Delta_2(i_1,i).\lambda i_1 \subseteq \Phi i_1.\lambda i_1.$$

Further for any $\mu : Xi \to A$ we can solve for $\phi \in \Phi i$ in the outer triangle of

$$\begin{tikzcd}
Xi \arrow[rr, "\eta_2 i"] \arrow[dr, "\eta_1 i"] \arrow[ddr, "\mu"'] & & D_2 i \arrow[ddl, dashed, "\phi"] \\
& D_1 i \arrow[ur, "\lambda i"] & \\
& A &
\end{tikzcd}$$

and hence μ factors through $\eta_1 i$ in $\Phi i.\lambda i$. The uniqueness is clear.

Finally we shall prove that K_1 and K_2 are inverse to each other. On objects this is trivial. Consider $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ and suppose $K_1K_2\lambda = \lambda_1$. Then $\lambda_1 i$ is the structure map of $K_2\lambda(F^{\mathcal{D}_1} i)$ satisfying $\lambda_1 i.\eta_1 i = \eta_2 i$. But $K_2\lambda(F^{\mathcal{D}_1} i) = (D_1 i, \Delta(-,i).\lambda(-))$, and so λi is such a structure map. Consider algebraic functor $G : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}_2}$, and suppose $K_2K_1G = G_1$. We only have to check G_1 on objects since its action on morphisms is then determined. Now $G_1(A,\Phi) = (A,\Phi\lambda)$ where λi is in the structure of $GF^{\mathcal{D}_1} i$ and $\lambda i \circ \eta_2 i = \eta_1 i$. But an argument similar to that in the first paragraph of this proof establishes that the structure of $G(A,\Phi)$ is $\Phi\lambda$. Hence $G = G_1$.

Of course this theorem has as a special case the corresponding statement for triples given, for example, in [23] p.39.

The next theorem shows that every algebraic functor is the forgetful functor associated with some device.

1.5.5 THEOREM Given any algebraic functor $G : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}_2}$, there is device $\mathcal{D}_3$ in $\underline{\underline{A}}^{\mathcal{D}_2}$ with object family $F^{\mathcal{D}_2} = (F^{\mathcal{D}_2} i_2;\ i_2 \in I_2)$ such that $\underline{\underline{A}}^{\mathcal{D}_1}$ is isomorphic to $(\underline{\underline{A}}^{\mathcal{D}_2})^{\mathcal{D}_3}$, and G is strictly isomorphic to $U^{\mathcal{D}_3}$. Conversely, given any device $\mathcal{D}_3$ in $\underline{\underline{A}}^{\mathcal{D}_2}$ with object family $F^{\mathcal{D}_2}$, then there exists a device $\mathcal{D}_1$ in $\underline{\underline{A}}$ on X_2 such that $U^{\mathcal{D}_1}$ is strictly isomorphic to $U^{\mathcal{D}_2} U^{\mathcal{D}_3}$.

PROOF Consider the situation in the first part of the statement. By Theorem 1.5.4 there is a morphism $\lambda : \mathcal{D}_2 \to \mathcal{D}_1$ corresponding to G. Now we define a device $\mathcal{D}_3$ in $\underline{\underline{A}}^{\mathcal{D}_2}$ as follows: put $\eta_3 i : F^{\mathcal{D}_2} i \to GF^{\mathcal{D}_1} i = \lambda i$ for all i in the common index set I of $\mathcal{D}_1$ and $\mathcal{D}_2$, and take $\Delta_3(i,i_1) : GF^{\mathcal{D}_1} i \to GF^{\mathcal{D}_1} i_1$ to be $\Delta_1(i,i_1)$. To see that $\mathcal{D}_3$ is a device, notice that for any $\mathcal{D}_2$-morphism $\mu : F^{\mathcal{D}_2} i \to GF^{\mathcal{D}_1} i_1$, since $GF^{\mathcal{D}_1} i_1 = (D_1 i_1, \Delta_1(-,i_1)\lambda(-))$ is a $\mathcal{D}_2$-algebra, then, by Theorem 1.2.1, the morphism μ is of the form $\delta.\lambda i$ with δ in $\Delta_1(i,i_1)$. The uniqueness of δ follows by considering the equations

$$\delta.\eta_1 i = \delta.\lambda i.\eta_2 i = \mu.\eta_2 i$$

and the properties of device $\mathcal{D}_1$.

The isomorphism L from $(\underline{\underline{A}}^{\mathcal{D}_2})^{\mathcal{D}_3}$ to $\underline{\underline{A}}^{\mathcal{D}_1}$ is then defined by

$$L : ((A,\Phi),\Psi) \rightsquigarrow (A,\Psi), \quad \text{and}$$

$$L : \alpha \rightsquigarrow \alpha \quad .$$

To justify this definition we first show that if $((A,\Phi),\Psi)$ is in $(\underline{\underline{A}}^{\mathcal{D}_2})^{\mathcal{D}_3}$ then (A,Ψ) is in $\underline{\underline{A}}^{\mathcal{D}_1}$. The closure condition for algebras is clearly satisfied by (A,Ψ). Consider any $\mu : Xi \to A$. Then there is a $\phi \in \Phi i$ with $\phi.\eta_2 i = \mu$, and corresponding to the $\mathcal{D}_2$-morphism ϕ there is a $\psi \in \Psi i$ such that $\psi.\lambda i = \phi$. Hence there is a ψ in Ψi with $\psi.\eta_1 i = \mu$. The uniqueness of ψ is easily checked.

Notice that we have the inclusion $\Phi i \subseteq \Psi i.\lambda i$. Further, as we have seen in our proof of 1.5.4, λi is a $\mathcal{D}_2$-morphism from $F^{\mathcal{D}_2} i$ to $GF^{\mathcal{D}_1} i$ and, since Ψi consists of $\mathcal{D}_2$-morphisms from $GF^{\mathcal{D}_1} i$ to (A,Φ), we have that $\Psi i.\lambda i \subseteq \Phi i$ and hence $\Phi i = \Psi i.\lambda i$. In the opposite direction it is easy to check that if $(A,\Psi) \in \underline{\underline{A}}^{\mathcal{D}_1}$, then $((A,\Psi\lambda),\Psi) \in (\underline{\underline{A}}^{\mathcal{D}_2})^{\mathcal{D}_3}$. Hence $L' : \underline{\underline{A}}^{\mathcal{D}_1} \to (\underline{\underline{A}}^{\mathcal{D}_2})^{\mathcal{D}_3}$ defined by

$$L' : (A,\Psi) \rightsquigarrow ((A,\Psi\lambda),\Psi) \quad \text{and}$$

$$L' : \alpha \rightsquigarrow \alpha$$

is the inverse functor of L. Clearly $GL = U^{\mathcal{D}_3}$.

Now consider the situation in the second part of the theorem. We define device $\mathcal{D}_1$ with object family X_2 in $\underline{\underline{A}}$ as follows: put $\eta_1 i_2 : X_2 i_2 \to D_1 i_2 (= U^{\mathcal{D}_2} D_3 i_2)$ equal to $\eta_3 i_2 \circ \eta_2 i_2$, and put $\Delta_1(i_2, i_2') = \Delta_3(i_2, i_2')$. We need only check the universality condition. Given $\mu : X_2 i_2 \to D_1 i_2'$, consider the diagram

$$\begin{tikzcd}
X_2 i_2 \arrow[r, "\eta_2 i_2"] \arrow[dr, "\mu"'] & D_2 i_2 \arrow[r, "\eta_3 i_2"] \arrow[d, dashed, "\phi"'] & U^{\mathcal{D}_2} D_3 i_2 \arrow[dl, dashed, "\delta"] \\
& D_1 i_2' &
\end{tikzcd}$$

We can find $\phi : F^{\mathcal{D}_2} i_2 \to D_3 i_2'$ so that the left-hand triangle commutes. Then we can find $\delta \in \Delta_3(i_2, i_2')$ so that the right-hand triangle commutes and hence $\delta \cdot \eta_1 i_2 = \mu$. The uniqueness of δ is easily checked. Now it is not hard to see that $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$ are in the same situation here as in the first part of our proof, with η_3 here taking the place of λ. That is, that η_3 is a morphism from $\mathcal{D}_2$ to $\mathcal{D}_1$, and that $\mathcal{D}_3$ here has the properties which determined $\mathcal{D}_3$ in the earlier section. Hence we may use the proof of the first part of the theorem to conclude that $U^{\mathcal{D}_2} . U^{\mathcal{D}_3}$ is strictly isomorphic to $U^{\mathcal{D}_2} . G = U^{\mathcal{D}_1}$ where G is the algebraic functor corresponding to the device morphism η_3.

I would not expect the triple-theoretic version of the first part of this theorem to be true; namely, that algebraic functors between categories of algebras of triples are themselves tripleable. There are, of course, theorems in this direction, for example in [20]. As far as the second part of the theorem is concerned, it is certainly not true that the product of tripleable functors is tripleable.

§1.6 Operations.

We have not made any mention so far of the notion of operation, which is, of course, basic to classical universal algebra. In this section we introduce a generalized notion of operation which corresponds to the usual one in the case of $\underline{\underline{\text{Sets}}}$. We shall see that in terms of this notion some of the concepts introduced otherwise in this thesis have a more familiar appearance.

1.6.1 DEFINITIONS Let Z be a specified set of objects in $\underline{\underline{A}}$, and $\mathcal{D}$ a device in $\underline{\underline{A}}$. Then any map w from some object

Z_1 in Z to some object Di (i ε I) is called a formal operation of $\mathcal{D}$ or just an operation of $\mathcal{D}$. Then given a $\mathcal{D}$-algebra $(A,\Phi) = \underline{A}$ the operation corresponding to w in $\underline{A}$ is the map

$$w_{\underline{A}} : \underline{\underline{A}}(Xi,A) \to \underline{\underline{A}}(Z_1,A) \quad \text{defined by}$$

$$w_{\underline{A}} : \phi \circ \eta i \rightsquigarrow \phi w \quad \text{for all} \quad \phi \in \Phi i.$$

A formal operation w with codomain Di is called Xi-ary, and we write aw = Xi. The set Z is called the base of the operations of $\mathcal{D}$.

The idea is to keep Z fixed in the category $\underline{\underline{A}}$. For example, in Sets, to get the usual notion of operation, take Z = {1} where 1 is some singleton. Certainly then to each $w : 1 \to Di$ we get a conventional Xi-ary operation $w_{\underline{A}}$ in any algebra $\underline{A}$. Usually we shall need extra conditions on Z to have behaviour which simulates the usual behaviour of operations. We give here two examples of the use of operations.

1.6.2 THEOREM Suppose Z is a generating set in $\underline{\underline{A}}$. A map $\alpha : A \to B$ is a $\mathcal{D}$-morphism from $(A,\Phi) = \underline{A}$ to $(B,\Psi) = \underline{B}$ if and only if, for each i ε I, and each Xi-ary operation w of $\mathcal{D}$, and each $\mu : Xi \to A$, we have

$$\alpha \circ w_{\underline{A}}(\mu) = w_{\underline{B}}(\alpha\mu).$$

PROOF If α is a $\mathcal{D}$-morphism and $\mu = \phi \circ \eta i$ ($\phi \in \Phi i$) then certainly

$$\alpha \circ w_{\underline{A}}(\mu) = \alpha\phi w = w_{\underline{B}}(\alpha \circ \phi \circ \eta i) = w_{\underline{B}}(\alpha\mu)$$

since $\alpha\phi \in \Psi i$. Conversely, suppose α satisfies this equation for all w and all μ : $aw \to A$. Then, if for any $\phi \in \Phi i$ we have $\alpha \circ \phi \circ \eta i = \psi \eta i$ ($\psi \in \Psi i$), it follows that

$$\alpha\phi w = \alpha \circ w_{\underline{A}}(\phi \circ \eta i) = w_{\underline{B}}(\alpha \circ \phi \circ \eta i) = \psi \circ w .$$

Since this holds for any w, and Z is a generating set, we may cancel the w to get $\alpha \circ \Phi i \subseteq \Psi i$, and hence α is a $\mathcal{D}$-morphism.

1.6.3 THEOREM Suppose that Z is a generating set in $\underline{\underline{A}}$ and that $\mathcal{D}_1$ and $\mathcal{D}_2$ are devices in $\underline{\underline{A}}$ with index set I and with common family of objects X. Then a family of morphisms ($\lambda i : D_1 i \to D_2 i$; $i \in I$) is a morphism of devices if and only if for each $i, i_1 \in I$, (1) $\lambda i \circ \eta_1 i = \eta_2 i$ and (2) for each Xi_1-ary operation w of $\mathcal{D}$ and each $\mu : Xi_1 \to D_1 i$ we have

$$\lambda i \circ w_{F_1 i}(\mu) = (\lambda i_1 \circ w)_{F_2 i}(\lambda i \circ \mu),$$

where we have written $F^{\mathcal{D}_1} i$ as $F_1 i$ and $F^{\mathcal{D}_2} i$ as $F_2 i$.

PROOF If λ is a device morphism and $\mu = \delta \circ \eta_1 i_1$ where $\delta \in \Delta_1(i_1, i)$, and $\lambda i . \delta = \delta' . \lambda i_1$ for some $\delta' \in \Delta_2(i_1, i)$ then certainly

$$\lambda i \circ w_{F_1 i}(\mu) = \lambda i \circ \delta \circ w = \delta' \circ \lambda i_1 \circ w = (\lambda i_1 \circ w)_{F_2 i}(\delta' \circ \eta_2 i_1)$$

$$= (\lambda i_1 \circ w)_{F_2 i}(\lambda i \circ \delta \circ \eta_1 i_1) = (\lambda i_1 . w)_{F_2 i}(\lambda i . \mu).$$

Conversely, if λ satisfies this equation for all operations w in $\mathcal{D}$ and all appropriate maps $\mu : X i_1 \to D_1 i$, then for any $\delta \in \Delta_1(i_1, i)$ we have

$$\lambda i \circ \delta \circ w = \lambda i \circ w_{F_1 i}(\delta \circ \eta_1 i_1) = (\lambda i_1 \circ w)_{F_2 i}(\lambda i \circ \delta \circ \eta_1 i_1)$$

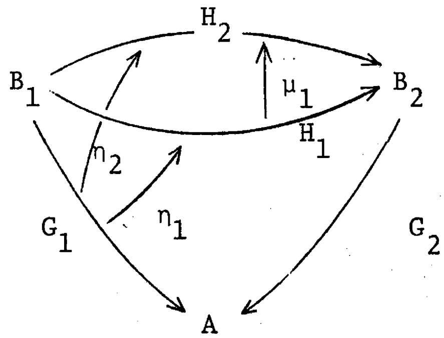
$$= (\lambda i_1 \circ w)_{F_2 i}(\delta' \circ \eta_2 i_1) = \delta' \circ \lambda i_1 \circ w$$

for some $\delta' \in \Delta_2(i_1, i)$ determined by $\lambda i . \delta . \eta_1 i_1 = \delta' . \eta_2 i_1$. Again, since this holds for all operations w, and Z is a generating set, we can cancel the w to obtain $\lambda i . \Delta_1(i_1, i) \subseteq \Delta_2(i_1, i) . \lambda i_1$.

§1.7 A global description of devices and algebras.

Essentially we have described devices and algebras set-theoretically and internally in a category $\underline{\underline{A}}$. In this section we give a categorical and external description of devices and algebras. Our field of operation is the 2-category, $\underset{\sim}{Cat}$, although some of our discussion applies to an arbitrary 2-category.

1.7.1 DEFINITIONS Let A be a vertex of 2-category $\underset{\sim}{A}$. Then the 2-category of vertices over A, denoted $\underset{\sim}{A}/A$, is defined as follows. Its vertices are arrows with codomain A; if $G_1 : B_1 \to A$ and $G_2 : B_2 \to A$ are two vertices of $\underset{\sim}{A}$, then an arrow $(\eta, H) : G_1 \to G_2$ is an arrow $H : B_1 \to B_2$ and a 2-cell $\eta : G_1 \to G_2H$. A 2-cell in $\underset{\sim}{A}/A$ from $(\eta_1,H_1) : G_1 \to G_2$ to $(\eta_2,H_2) : G_1 \to G_2$ is a 2-cell of $\underset{\sim}{A}$ $\mu : H_1 \to H_2$ such that $G_2\mu.\eta_1 = \eta_2$. If $\mu_1 : (\eta_1,H_1) \to (\eta_2,H_2) : G_1 \to G_2$ and $\mu_2 : (\lambda_1,K_1) \to (\lambda_2,K_2) : G_2 \to G_3$ are 2-cells in $\underset{\sim}{A}/A$, then their strong composition is defined to be $\mu_2*\mu_1 : (\lambda_1H_1\circ\eta_1,K_1H_1) \to (\lambda_2H_2\circ\eta_2,K_2H_2) : G_1 \to G_3$. If $\mu_3 : (\eta_2,H_2) \to (\eta_3,H_3) : G_1 \to G_2$ is another 2-cell in $\underset{\sim}{A}/A$ then the weak composition of μ_1 and μ_3 is defined to be $\mu_3\circ\mu_1 : (\eta_1,H_1) \to (\eta_3,H_3) : G_1 \to G_2$. We can visualize these 2-cells as diagrams



Strong composition amounts to pasting two of these together at a common vertical edge; weak composition amounts to pasting together at a common vertical face. (In the definition of $\underset{\sim}{A}/A$, we could just as easily have taken the arrows to be things like $(\eta,H) : G_1 \to G_2$, where $\eta : G_2H \to G_1$ rather than $\eta : G_1 \to G_2H$. However, we are not interested here in this alternative meaning for $\underset{\sim}{A}/A$.)

It is not hard to verify that $\underset{\sim}{A}/A$ is a 2-category. There is a corresponding definition of $A/\underset{\sim}{A}$, the 2-category of <u>vertices under</u> A.

Its vertices are arrows with domain A; its arrows from $G_1 : A \to B_1$ to $G_2 : A \to B_2$ are diagrams like

$$\begin{tikzcd} & A \arrow[dl, "G_1"'] \arrow[dr, "G_2"] \arrow[d, phantom, "\Downarrow\eta"] & \\ B_1 & & B_2 \arrow[ll, "H"] \end{tikzcd}$$

(Notice the direction of the arrow.) It is clear what the 2-cells and compositions must be. (Again there is an alternative meaning available for $A/\underset{\sim}{A}$; again we are not interested in this meaning here.)

Notice that a "triangle", that is, a diagram of the form

$$\begin{tikzcd} & A \arrow[dl, "G_1"'] \arrow[dr, "G_2"] \arrow[d, phantom, "\Downarrow\eta"] & \\ B & & C \arrow[ll, "G_3"] \end{tikzcd}$$

can be regarded both as an arrow in $A/\underset{\sim}{A}$ and as an arrow in $\underset{\sim}{A}/B$. Hence we shall use the non-committal notation $(\eta, G_1;\ G_2, G_3)$ for such a triangle. Unless we state otherwise, if T_1 and T_2 are triangles, then $T_2{*}T_1$ is the composite under the strong operation in $A/\underset{\sim}{A}$ where A is the common "apex" of T_1 and T_2.

The basic notion we need for our global description is the "initial triangle".

1.7.2 DEFINITION An <u>initial triangle</u> in a 2-category $\underset{\sim}{A}$ is a triangle

$$T = \begin{tikzcd} & I \arrow[dl, "X"'] \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\eta"] & \\ A & & B \arrow[ll, "U"] \end{tikzcd}$$

which when considered as an arrow in $\underset{\sim}{A}/A$ is initial in the category $\underset{\sim}{A}/A(X,U)$. The vertex I is called the apex of T.

1.7.3 EXAMPLE In $\underset{\sim}{\mathrm{Cat}}$ if we take I to be a set and $U : \underline{\underline{B}} \to \underline{\underline{A}}$ a functor with a family of universal arrows to it, indexed by I, namely $((Xi \xrightarrow{\eta i} UFi, Fi);\ i \in I)$, then $(\eta, X;\ F, U)$ is an initial triangle. Furthermore, any initial triangle in $\underset{\sim}{\mathrm{Cat}}$, with I being a set, is of this form. We will prove this below. This clearly allows us to attach to any initial triangle (with I a set) a device $\mathcal{D}$ in $\underline{\underline{A}}$.

PROOF Consider a triangle T constructed as above from universal arrows. Consider any other arrow $(\eta', F') : X \to U$ in $\underset{\sim}{\mathrm{Cat}}/\underline{\underline{A}}(X,U)$. Then to $\eta'i : Xi \to UF'i$, by the properties of the universal arrow $(\eta i : Xi \to UFi, Fi)$, there is a unique $\mathcal{E}i : Fi \to F'i$ such that $U\mathcal{E}i\,\eta i = \eta' i$. That is, there is a unique natural transformation $\mathcal{E} : F \to F'$ such that

$U\mathscr{E}\circ\eta = \eta'$. This is precisely the meaning of initial in $\underset{\sim}{\mathrm{Cat}}/\underline{\underline{A}}(X,U)$. The other statement in the example is now clear.

We next consider initial triangles in which the apex I is fixed as arrows in the 2-category $I/\underset{\sim}{A}$.

1.7.4 THEOREM Initial triangles with apex I form a category with respect to the strong composition in $I/\underset{\sim}{A}$. Further, if T_1 and T_2 are arrows of $I/\underset{\sim}{A}$, and both T_1 and $T_2{*}T_1$ are initial triangles, then so is T_2.

PROOF If $H : I \to A$ is a vertex of $I/\underset{\sim}{A}$ then it is easy to see that $(1_H, H;\ H, 1_A)$ is an initial triangle. Consider two initial triangles $T_1 = (\eta_1, G_1;\ G_2, H_1)$ and $T_2 = (\eta_2, G_2;\ G_3, H_2)$, where $G_1 : I \to A$, $G_2 : I \to B$ and $G_3 : I \to C$. Then $T_2{*}T_1$ is $(H_1\eta_2 \cdot \eta_1, G_1;\ G_3, H_1H_2)$. Let $(\mu, K) : G_1 \to H_1H_2$ be any arrow in $\underset{\sim}{A}/A(G_1, H_1H_2)$. By the fact that T_1 is an initial triangle, there is an $\mathscr{E} : G_2 \to H_2K$ such that $H_1\mathscr{E} \cdot \eta_1 = \mu$. Now $(\mathscr{E}, K) : G_2 \to H_2$ is an arrow in $\underset{\sim}{A}/B(G_2, H_2)$ and hence, since T_2 is an initial triangle, there is an $\mathscr{E}' : G_3 \to K$ such that $H_2\mathscr{E}' \cdot \eta_2 = \mathscr{E}$. Then

$$H_1H_2\mathscr{E}' \cdot H_1\eta_2 \cdot \eta_1 = H_1(H_2\mathscr{E}' \cdot \eta_2) \cdot \eta_1 = H_1\mathscr{E} \cdot \eta_1 = \mu.$$

The uniqueness of $\mathcal{E}'$ follows from the uniqueness at each stage.

For the proof of the second statement of the theorem consider triangles T_1 and T_2 as above, but assume only that T_1 and $T_2 * T_1$ are initial triangles. Then consider $(\mu_1, K_1) : G_2 \to H_2$ in $\underset{\sim}{A}/B(G_2, H_2)$. Clearly $(H_1\mu_1 \circ \eta_1, K_1) : G_1 \to H_1H_2$ is in $\underset{\sim}{A}/A(G_1, H_1H_2)$ and hence, since $T_2 * T_1$ is initial, there exists an $\mathcal{E} : G_3 \to K_1$ such that $H_1H_2\mathcal{E} . H_1\eta_2 \circ \eta_1 = H_1\mu_1 . \eta_1$; that is, such that $H_1(H_2\mathcal{E} \circ \eta_2) . \eta_1 = H_1\mu_1 . \eta_1$. By the uniqueness property of initial triangle T_1 we have $H_2\mathcal{E} . \eta_2 = \mu_1$. The uniqueness of $\mathcal{E}$ is clear.

1.7.5 DEFINITIONS Given an arrow $G : I \to A$ we define a category $\underline{\underline{It}}(G)$ as follows. Its objects are initial triangles like $T_1 = (\eta_1, G;\ H_1, K_1)$. A morphism from T_1 to $T_2 = (\eta_2, G;\ H_2, K_2)$ is a triangle $T = (1, H_1;\ H_2, K)$ such that $T * T_1 = T_2$. The composition of morphisms in $\underline{\underline{It}}(G)$ is also the strong composition in $I/\underset{\sim}{A}$.

In this category we want to consider the notion of "local initial object" and the dual notion of "local terminal object". By a local initial object in a category $\underline{\underline{A}}$ we mean an object A such that given any diagram

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & \\ B & C \arrow[l, "\beta"] \end{tikzcd}$$

there is just one map from A to C.

We say that $\underline{\underline{A}}$ has enough local initial (terminal) objects if to each B in $\underline{\underline{A}}$ there is a map $A \to B$ $(B \to A)$ where A is a local initial (terminal) object.

The local terminal objects in $\underline{\underline{It}}(G)$ are called <u>Kleisli triangles</u> or KL-triangles and the local initial objects are called <u>Eilenberg-Moore</u> <u>triangles</u> or EM-triangles.

1.7.6 LEMMA If $\underline{\underline{A}}$ has enough local initial objects then it is a union of disjoint subcategories in such a way that each local initial object is initial in the subcategory containing it.

PROOF Clear.

1.7.7 THEOREM If we define $\underline{\underline{It}}(G)$ in $\utilde{Cat}$ where the domain I of G is a set, then there are enough local initial and terminal objects. Further, each of the subcategories of $\underline{\underline{It}}(G)$ described in 1.7.6 consists of all the initial triangles corresponding to some device (see 1.7.3).

PROOF First we prove that if $T : T_1 \to T_2$ is a morphism in $\underline{\underline{It}}(G)$ then the device corresponding to T_1 is the same as the device corresponding to T_2. For let $T_1 = (\eta, G; H, K)$ and

and $T = (1_H, H;\ J, L)$ where $H = H : I \to \underline{\underline{B}}$ and $J = J : I \to \underline{\underline{C}}$. Then $T_2 = (\eta, G, J, KL)$ and the device corresponding to T_1 is $(\eta, K\underline{\underline{B}}(H-,H-))$, and that corresponding to T_2 is $(\eta, KL\underline{\underline{C}}(J-,J-))$. Clearly $KL\underline{\underline{C}}(Ji,Ji_1) \subseteq K\underline{\underline{B}}(Hi,Hi_1)$ $(i,i_1 \in I)$. Consider $\delta \in \underline{\underline{B}}(Hi,Hi_1)$. By Theorem 1.7.4 the triangle T is initial, and so there exists a morphism $\gamma : Ji \to Ji_1$ with $\delta = L\gamma . 1_{Hi} = L\gamma$. Hence the two devices are the same.

We next remark that Theorem 1.2.6 states that if $\mathcal{D}$ is a device in $\underline{\underline{A}}$, then the triangle $(\eta, X; F^{\mathcal{D}}, U^{\mathcal{D}})$ is a local initial object in $\underline{\underline{It}}(X)$ and $(\eta, X; F_{\mathcal{D}}, U_{\mathcal{D}})$ is a local terminal object. This is easily checked. In fact, Theorem 1.2.6 tells us more; since every initial triangle in $\underline{\underline{It}}(X)$ has a suitable device associated with it, we see that $\underline{\underline{It}}(X)$ has enough local initial and terminal objects.

1.7.8 THEOREM In $\underset{\sim}{Cat}$, Eilenberg-Moore triangles with apex I, a set, form a category with respect to the strong composition in $I/\underset{\sim}{A}$. Further if T_1 and $T_2 * T_1$ are EM-triangles, then so is T_2.

PROOF In view of Theorem 1.7.7 this is just a restatement of Theorem 1.5.5.

1.7.9 DEFINITIONS The category $\underline{\underline{EM}}(G)$, where G is the arrow $G : I \to A$ in $\underset{\sim}{A}$, has objects the EM-triangles in $\underline{\underline{It}}(G)$ and morphisms from T_1 to T_2 the triangles T with $T*T_1 = T_2$. The category $\underline{\underline{KL}}(G)$ has objects the KL-triangles in $\underline{\underline{It}}(G)$, and morphisms from $T_1 = (\eta_1,G;\ H_1,K_1)$ to $T_2 = (\eta_2,G;\ H_2,K_2)$ are triangles $T = (\mu,K_1;\ L,K_2)$ such that $T*T_1$ is T_2 where the composition is the strong composition in $\underset{\sim}{A}/A$.

1.7.10 THEOREM In $\underset{\sim}{Cat}$, if $X : I \to \underline{\underline{A}}$ is a functor with domain a set, then $\underline{\underline{KL}}(X)$ is equivalent to $\underline{\underline{EM}}(X)$.

PROOF The full subcategory $\underline{\underline{K}}$ of $\underline{\underline{KL}}(X)$ with objects being initial triangles $(\eta,X;\ F_{\mathcal{D}},U_{\mathcal{D}})$ is equivalent to $\underline{\underline{KL}}(X)$, and the full subcategory $\underline{\underline{E}}$ of $\underline{\underline{EM}}(X)$ with objects being $(\eta,X,F^{\mathcal{D}},U^{\mathcal{D}})$ is equivalent to $\underline{\underline{EM}}(X)$. If we show that $\underline{\underline{K}}$ is isomorphic to $\underline{\underline{Dev}}(\underline{\underline{A}},X)$ and $\underline{\underline{E}}^{op}$ is isomorphic to $\underline{\underline{Alg}}(\underline{\underline{A}},X)$, then by Theorem 1.5.4 we will have the required result.

The objects of $\underline{\underline{K}}$ and $\underline{\underline{Dev}}(\underline{\underline{A}},X)$ are in obvious one-to-one correspondence; namely, $\mathcal{D}_1 = (\eta_1,\Delta_1)$ corresponds to $T_{\mathcal{D}_1} = (\eta,X,F_{\mathcal{D}_1},U_{\mathcal{D}_1})$. Let λ be a morphism from $\mathcal{D}_1$ to $\mathcal{D}_2$. Then we can define a functor $H : \underline{\underline{A}}_{\mathcal{D}_1} \to \underline{\underline{A}}_{\mathcal{D}_2}$ as follows:

$$\begin{tikzcd} H : F_{\mathcal{D}_1} i \arrow[d, "\delta"'] & \rightsquigarrow & F_{\mathcal{D}_2} i \arrow[d, "\delta'"] \\ F_{\mathcal{D}_1} i_1 & & F_{\mathcal{D}_2} i_1 \end{tikzcd}$$

where $\lambda i_1.\delta = \delta'.\lambda i$. That there is such a δ' follows since λ is a morphism of devices. That δ' is uniquely determined by this equation follows from the observation that $\delta'.\lambda i.\eta_1 i = \delta'.\eta_2 i$ and the properties of devices. It is easy to check that H is functorial, and that λ is a natural transformation from $U_{\mathcal{D}_1}$ to $U_{\mathcal{D}_2}.H$. Clearly also $H.F_{\mathcal{D}_1} = F_{\mathcal{D}_2}$ and $\lambda F_{\mathcal{D}_1}.\eta_1 = \eta_2$. Hence $T_\lambda = (\lambda, U_{\mathcal{D}_1};\ H, U_{\mathcal{D}_2})$ is a morphism in $\underline{\underline{K}}$ from $T_{\mathcal{D}_1}$ to $T_{\mathcal{D}_2}$. The function defined by

$$\begin{tikzcd} \mathcal{D}_1 \arrow[d, "\lambda"'] & \rightsquigarrow & T_{\mathcal{D}_1} \arrow[d, "T_\lambda"] \\ \mathcal{D}_2 & & T_{\mathcal{D}_2} \end{tikzcd}$$

can be shown to be functorial, and, in fact, an isomorphism from $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}}, X)$ to $\underline{\underline{K}}$.

The objects of $\underline{\underline{\mathrm{Alg}}}(\underline{\underline{A}}, X)$ and $\underline{\underline{E}}$ are in obvious one-to-one correspondence; namely, $\underline{\underline{A}}^{\mathcal{D}}$ corresponds to $T^{\mathcal{D}} = (\eta, X; \; F^{\mathcal{D}}, U^{\mathcal{D}})$. Consider an algebraic functor $G : \underline{\underline{A}}^{\mathcal{D}_1} \to \underline{\underline{A}}^{\mathcal{D}_2}$. Since $T^{\mathcal{D}_2} = (\eta_2, X; F^{\mathcal{D}_2}, U^{\mathcal{D}_2})$ is an initial triangle, there is a (unique) natural transformation $\mathcal{E} : F^{\mathcal{D}_2} \longrightarrow G.F^{\mathcal{D}_1}$ such that $U^{\mathcal{D}_2}\mathcal{E}.\eta_2 = \eta_1$. From this we see that $T^G = (\mathcal{E}, F^{\mathcal{D}_2}; \; F^{\mathcal{D}_1}, G)$ is a morphism from $T^{\mathcal{D}_2}$ to $T^{\mathcal{D}_1}$. Again, the function from $\underline{\underline{\mathrm{Alg}}}(\underline{\underline{A}}, X)$ to $\underline{\underline{E}}^{\mathrm{op}}$ defined by:

$$\begin{tikzcd}
\underline{\underline{A}}^{\mathcal{D}_1} \arrow[d, "G"'] & & T^{\mathcal{D}_1} \\
\arrow[r, phantom, "\rightsquigarrow"] & & \\
\underline{\underline{A}}^{\mathcal{D}_2} & & T^{\mathcal{D}_2} \arrow[uu, "T^G"']
\end{tikzcd}$$

can be shown to be functorial and an isomorphism.

It is clear now that the use of the notion of Kleisli triangle instead of device, and Eilenberg-Moore triangle instead of category of algebras, leads to a global and more suggestive description of the basic notions of this thesis.

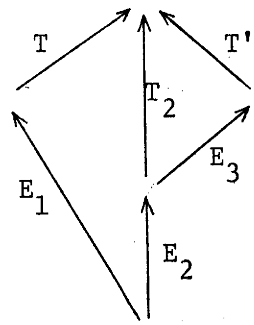
If we remove the restriction in the preceding discussion that I be a set, then we do not necessarily have that enough local initial objects exist.

In the final theorem of this section we give a "structure-semantics adjointness" theorem, which is a generalization of a theorem of Maranda [25]. For a very general discussion of structure-semantics adjointness theorems see Linton [21]. First we need a new category.

1.7.11 DEFINITION Let $G : I \to A$ be an arrow in $\underset{\sim}{A}$. Then the category $\underline{\underline{IT}}(G)$ is defined as follows: its objects are initial triangles of the form $T_1 = (\eta_1, G;\ H_1, K_1)$. A morphism from T_1 to $T_2 = (\eta_2, G, H_2, K_2)$ is a triangle $T = (\eta, H_1;\ H_2, K)$ such that $T{*}T_1 = T_2$.

1.7.12 THEOREM If $X : I \to \underline{\underline{A}}$ is a functor in $\underset{\sim\sim}{Cat}$ with domain a set, then $\underline{\underline{EM}}(X)$ is a coreflective subcategory of $\underline{\underline{IT}}(X)$.

PROOF Clearly $\underline{\underline{EM}}(X)$ is a full subcategory of $\underline{\underline{IT}}(X)$. Consider an object T_1 of $\underline{\underline{IT}}(X)$. By Theorem 1.7.7, there is an EM-triangle E_1 and a morphism $T : E_1 \to T_1$ in $\underline{\underline{It}}(X)$. We will show that this map T is the coreflection of T_1. Suppose $T_2 : E_2 \to T_1$ is any morphism in $\underline{\underline{IT}}(X)$ with domain an EM-triangle. Now again by Theorem 1.7.7, T_2 factors as $T'{*}E_3$ where T' is an $\underline{\underline{It}}$-morphism and E_3 is an EM-triangle. (By an $\underline{\underline{It}}$-morphism I mean a triangle whose first component is an identity natural transformation.) Hence we have the situation



Now both $T : E_1 \to T_1$ and $T' : E_3 * E_2 \to T_1$ are morphisms in $\underline{\underline{\mathrm{It}}}(X)$, and by Theorem 1.7.8, $E_3 * E_2$ is an EM-triangle. Hence there is an isomorphism $T'' : E_3 * E_2 \to E_1$ in $\underline{\underline{\mathrm{It}}}(X)$ with $T * T'' = T'$. Then $T'' * E_3 : E_2 \to E_1$ is a morphism in $\underline{\underline{\mathrm{IT}}}(X)$ with $T * (T'' * E_3) = T_2$. That $E = T'' * E_3$ is unique with this property follows since any other $E' : E_2 \to E_1$ with $T * E' = T * E$ is first of all an EM-triangle, by Theorem 1.7.8, and hence E and E' are isomorphic, say $E' = T_3 * E$. Clearly $T_3 * E * E_2 = E * E_2$, and hence $T_3 = 1$ and $E' = E$.

CHAPTER II

ALGEBRA WITH DEVICES

In this chapter we discuss some notions and results of conventional universal algebra in a category $\underline{\underline{A}}$ equipped with a device $\mathcal{D}$. In most of the chapter we need to impose conditions on $\underline{\underline{A}}$ and X ; that is, on the functor $X : I \to \underline{\underline{A}}$. We consider algebra in the "basic situation", the conditions for which are given in section 1, and algebra in the "elaborate situation", which is discussed in section 2. Our main adjoint functor theorem is in section 1. In section 3 we develop some theorems which help in recognizing when a functor is strictly isomorphic to $U^{\mathcal{D}} : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}$ for some device $\mathcal{D}$. In the final section of the chapter we give a version of a theorem of G. Birkhoff on the subvarieties of a variety.

§2.1 Algebra in the basic situation.

2.1.1 REMARK Suppose there is a morphism $\alpha : A_1 \to A_2$ in $\underline{\underline{A}}$ such that there are no morphisms from any objects in X to A_2 . Then clearly $\alpha : (A_1,\emptyset_1) \to (A_2,\emptyset_2)$ is a morphism in $\underline{\underline{A}}^{\mathcal{D}}$ if $\emptyset_1$ and $\emptyset_2$ are the empty structures (which are of course the only possible structures on A_1 and A_2). In other words, in regions of $\underline{\underline{A}}$ inaccessible to X the objects and morphisms are reproduced exactly in $\underline{\underline{A}}^{\mathcal{D}}$. Hence it would seem that the most natural conditions on $\underline{\underline{A}}$ and X would involve only the region of $\underline{\underline{A}}$ that can be reached from X . For example, in proving that $U^{\mathcal{D}}$ has a left adjoint one should not need any information about the remote regions of $\underline{\underline{A}}$. However in this thesis, for the sake of simpler presentation, our theorems are stated in terms of stronger, and less natural conditions.

The first result is true quite generally.

2.1.2 THEOREM The functor $U^{\mathcal{D}} : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}$ creates limits.

PROOF Consider a functor $L : \underline{\underline{C}} \to \underline{\underline{A}}^{\mathcal{D}}$ and let $\lambda : A \to U^{\mathcal{D}}L$ be a limit of $U^{\mathcal{D}}L$ in $\underline{\underline{A}}$ (where $A : \underline{\underline{C}} \to \underline{\underline{A}}$ is a constant functor). Let Φ_C be the structure of LC for C in $\underline{\underline{C}}$. Then we form an algebra (A,Φ) (confusing A with its image) by taking Φi to be all $\phi : Di \to A$ with $\lambda C.\phi \in \Phi_C i$ for all C in $\underline{\underline{C}}$. Clearly if (A,Φ) is a $\mathcal{D}$-algebra then the morphism λC is a $\mathcal{D}$-morphism from (A,Φ) to LC . We have that $\lambda C.\Phi i.\Delta(i_1,i) \subseteq \Phi_C i.\Delta(i_1,i) \subseteq \Phi_C i_1$ and hence $\Phi i.\Delta(i_1,i) \subseteq \Phi i_1$. Further, consider $\alpha : Xi \to A$. For each $C \in \underline{\underline{C}}$ there exists a $\phi C \in \Phi_C i$ such that $\phi C.\eta i = \lambda C.\alpha$. Let $\delta : (A_C,\Phi_C) \to (A_{C'},\Phi_{C'})$ be the image of a map in $\underline{\underline{C}}$ under L . Then $\delta.\phi C.\eta i = \delta.\lambda C.\alpha = \lambda C'.\alpha = \phi C'.\eta i$. We can cancel the map ηi to show that $\phi : Di \to U^{\mathcal{D}}L$ is natural, and hence there is a map $\gamma : Di \to A$ such that $\lambda C.\gamma = \phi C$ for all $C \in \underline{\underline{C}}$. Clearly we have that $\gamma \in \Phi i$ and $\gamma.\eta i = \alpha$. The uniqueness of γ with this property follows, since if

$\gamma'.\eta i = \alpha$ then $\lambda C.\gamma'.\eta i = \lambda C.\alpha = \phi C.\eta i$ and hence $\lambda C.\gamma' = \phi C$.

To see that $\lambda : (A,\Phi) \to L$ is a limit of L , consider any $\mu : (B,\Gamma) \to L$. Then there is a unique $\tau : B \to A$ such that $\mu C = \lambda C.\tau$ for all $C \in \underline{\underline{C}}$. This morphism τ is a $\mathcal{D}$-morphism from (B,Γ) to (A,Φ) since $\lambda C.\tau.\Gamma i = \mu C.\Gamma i \subseteq \Phi_C i$ and hence $\tau.\Gamma i \subseteq \Phi i$.

Suppose Φ' is another structure on A such that the maps λC $(C \in \underline{\underline{C}})$ are $\mathcal{D}$-morphisms. Then $\Phi' i \subseteq \Phi i$ $(i \in I)$, and so $\Phi' = \Phi$.

We introduce the five axioms for the basic situation (BS) in two parts, as we need to develop extra notions to express some of them. Those theorems which hold in the basic situation are indicated by writing THEOREM (BS). It will be obvious that many of them hold under much weaker conditions.

2.1.3 AXIOMS <u>BS(1)</u> $\underline{\underline{A}}$ is in $\underline{\underline{Cat}}$, and the index set I of X is a small set.

<u>BS(2)</u> $\underline{\underline{A}}$ is small complete.

<u>BS(3)</u> For any object A in $\underline{\underline{A}}$ there is a small representative set of monomorphisms with codomain A . That is, if $\{\alpha_j : A_j \to A \,;\, j \in J\}$ is the set of monomorphisms with codomain A , then there is a small set $K \subseteq J$ such that to each $j \in J$ there is a $k \in K$ and an isomorphism $\beta : A_j \to A_k$ such that $\alpha_k.\beta = \alpha_j$.

2.1.4 DEFINITION Let $(\alpha_k : A_k \to A \,;\, k \in K)$ be a family of morphisms with codomain A . Then let $\beta : B \to A$ be a monomorphism such that each α_k $(k \in K)$ factors through β ; that is, there exist maps γ_k $(k \in K)$ such that $\alpha_k = \beta . \gamma_k$. Let $\{\beta_j : B_j \to A \; ; \; j \in J\}$ be the set of all such monomorphisms β . Then the intersection of $\{\beta_j \; ; \; j \in J\}$ (determined up to isomorphism) is written as $\bigcup_{k \in K} \alpha_k : \bigcup_{k \in K} (A_k, \alpha_k) \to A$, and is called the union of the family $(\alpha_k \; ; \; k \in K)$. In the basic situation the union does exist since we can choose a small representative set S of morphisms in $\{\beta_j \; ; \; j \in J\}$. The intersection of S , which is a small limit, is then easily seen to be the intersection of $\{\beta_j \; ; \; j \in J\}$

2.1.5 THEOREM (BS) Consider families of morphisms $(\beta_j : B_j \to A \; ; \; j \in J)$ $(\gamma_k : C_k \to A \; ; \; k \in K)$. Then

(1) there is a unique morphism $q_{j_1} : B_{j_1} \to \bigcup_{j \in J} (B_j, \beta_j)$ such that $\bigcup_{j \in J} \beta_j . q_{j_1} = \beta_{j_1}$; further $\bigcup_{j \in J} q_j = 1$;

(2) if to each $j \in J$ there is a $k \in K$ such that β_j factors through γ_k , then there is a unique morphism $u : \bigcup_{j \in J} (B_j, \beta_j) \to \bigcup_{k \in K} (C_k, \gamma_k)$ such that $\bigcup_{k \in K} \gamma_k . u = \bigcup_{j \in J} \beta_j$;

(3) if β is monic then $\bigcup \beta = \beta$.

PROOF (1) β_k factors through each of the monomorphisms of which $\bigcup_{k \in K} \beta_k$ is the intersection. The second part is also straightforward.

(2) There are fewer monomorphisms through which each of $(\gamma_k \; ; \; k \in K)$ factors, than through which each of $(\beta_j \; ; \; j \in J)$ factors.

(3) Clear.

2.1.6 AXIOMS <u>BS(4)</u> There exists a small cardinal κ with the following property: for any family $(\alpha_k : A_k \to A \; ; \; k \in K)$, and for any morphism $\beta : Xi \to \bigcup_{k \in K} (A_k, \alpha_k)$ with $Xi \in X$, there is a subset $J \subseteq K$ with $|J| < \kappa$, and a morphism $\gamma : Xi \to \bigcup_{j \in J} (A_j, \alpha_j)$, such that β is the composition of γ and the canonical morphism $\bigcup_{j \in J} (A_j, \alpha_j) \to \bigcup_{k \in K} (A_k, \alpha_k)$ described in 2.1.5. If X has this property in $\underline{\underline{A}}$ we say that X is ranked, with rank $\leqslant \kappa$.

<u>BS(5)</u> For any small family $(A_k \; ; \; k \in K)$ of objects in $\underline{\underline{A}}$, consider all unions in $\underline{\underline{A}}$ of the form $\bigcup_{k \in K} (A_k, \alpha_k)$. There is a small representative set S of such objects. That is, any object of the form $\bigcup_{k \in K} (A_k, \alpha_k)$ is isomorphic to an object in S , and S is a small set.

It is clear that for many familiar categories $\underline{\underline{A}}$, any functor $X : I \to \underline{\underline{A}}$, with I a small set, satisfies the conditions of the basic situation. We shall discuss some examples in 2.1.19.

2.1.7 LEMMA (BS) If the diagrams $(j \in J)$

$$\begin{tikzcd} A \arrow[r, "\nu j"] \arrow[d, "\beta j"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

commute, $\bigcup_{j \in J} \nu j = 1_B$, and δ is a monomorphism, then there exists a $\lambda : B \to C$ such that $\delta.\lambda = \gamma$ and $\lambda.\nu j = \beta j$ $(j \in J)$.

PROOF Form the pullback diagram

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd} .$$

The morphism p is monic. For suppose that $pr_1 = pr_2$; then $\gamma.pr_k = \delta.qr_k$ $(k = 1,2)$ and, since δ is mono, $qr_1 = qr_2$. Hence by the universal property of pullback diagrams $r_1 = r_2$. All the morphisms νj factor through p , again by the pullback property, and since $1 : B \to B$ is the intersection of a set of morphisms including p , there is an $r : B \to E$ such that $p.r = 1$. Hence $\delta.(qr) = \gamma p.r = \gamma$, and so we may take λ to be qr .

2.1.8 REMARK In Sets and other familiar categories we have the useful factorization of morphisms into epimorphisms and monomorphisms. To take the place of this, in the basic situation we have that any map

$\alpha : A \to B$ factorizes as $A \overset{\beta}{\to} U(A,\alpha) \overset{U\alpha}{\to} B$ where β is the canonical map. The map $U\alpha$ is monic, and the map β has the property that $U\beta = 1$. Now Lemma 2.1.7 shows that any factorization of α into a map whose union is 1 and a monomorphism is essentially isomorphic to the above factorization. (Such a factorization we call a <u>standard factorization</u>.) For if the following diagram

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & U(A,\alpha) \arrow[d, "U\alpha"] \\ C \arrow[r, "\delta"] & B \end{tikzcd}$$

commutes, with $U\gamma = 1_C$ and δ monic, the lemma supplies maps $\lambda : C \to U(A,\alpha)$ and $\lambda' = U(A,\alpha) \to C$ which are clearly inverse, and $U\alpha.\lambda = \delta$, $\lambda\gamma = \beta$.

Other properties which maps whose union is 1 have in analogy with epimorphisms are:

(1) if α,β have union 1 then so does $\beta\alpha$.

(2) if $\beta\alpha$ has union 1 then so does β .

For a discussion of generalized epimorphisms see [14].

We next prove two useful lemmas.

2.1.9 LEMMA Consider a family of maps $(\alpha_j : A_j \to A \,;\; j \in J)$ and a map $\beta : A \to B$. Then

$$\bigcup_{j \in J} (\beta\alpha_j) = U(\beta. \bigcup_{j \in J} \alpha_j) .$$

PROOF Clearly any monomorphism in $\underline{\underline{A}}$ through which $\beta . \bigcup_{j \in J} \alpha_j$ factors, has the property that each $\beta\alpha_j$ factors through it. Conversely, consider a monomorphism $\gamma : C \to B$ through which each $\beta\alpha_j$ factors; say $\gamma\delta_j = \beta\alpha_j$. Further let $\alpha_{j_1} = \bigcup_{j \in J} \alpha_j . \varepsilon_{j_1}$. Then by Lemma 2.1.7, there exists a map $\lambda : \bigcup_{j \in J} (A_j, \alpha_j) \to C$ such that $\beta . \bigcup_{j \in J} \alpha_j = \gamma . \lambda$. So the monomorphisms through which $\beta . \bigcup_{j \in J} \alpha_j$ factors are the monomorphisms through which each of $\beta\alpha_j$ factors, which yields our result.

2.1.10 LEMMA If $\mathcal{D}$-morphism $\alpha : (A_1, \Phi_1) \to (A_3, \Phi_3)$ factorizes in $\underline{\underline{A}}$ as $\gamma\beta$, where $\gamma : (A_2, \Phi_2) \to (A_3, \Phi_3)$ is a $\mathcal{D}$-morphism and monic in $\underline{\underline{A}}$, then β is also a $\mathcal{D}$-morphism.

PROOF Certainly if $\phi_1 \in \Phi_1 i$ we have that $\gamma\beta\phi_1 \in \Phi_3 i$. Now there is a $\phi_2 \in \Phi_2 i$ such that $\phi_2 . \eta i = \beta\phi_1 . \eta i$. Then $\gamma\phi_2 . \eta i = \gamma\beta\phi_1 . \eta i$ and so $\gamma\phi_2 = \gamma\beta\phi_1$. Since γ is monic we deduce that $\beta . \Phi_1 i \subseteq \Phi_2 i$.

2.1.11 DEFINITION The map $\alpha : A \to B$ generates $\beta : (C, \Psi) \to (B, \Phi)$ in (B, Φ) if β is the intersection of all $\mathcal{D}$-morphisms which are monic in $\underline{\underline{A}}$, with codomain (B, Φ), and through which α factors.

We say that α generates (B, Φ) if it generates $1 : (B, \Phi) \to (B, \Phi)$.

Notice that in the basic situation, using BS(3) we can show, much as we did in 2.1.4, that given any $\alpha : A \to B$ and any $\mathcal{D}$-algebra (B,Φ) , there exists a β generated by α in (B,Φ) .

If α generates β , it follows, since $U^{\mathcal{D}}$ preserves intersections, that α factors through β , say $\alpha = \beta\gamma$. A simple consequence of the definition is that γ so defined generates (C,Ψ) .

In the next theorem we see another simple consequence of the definitions.

2.1.12 THEOREM (BS) If the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & (B,\Phi) \arrow[d, "\beta"] \\ (B,\Phi) \arrow[r, "\gamma"] & (C,\Psi) \end{tikzcd}$$

commutes and α generates (B,Φ) , then $\beta = \gamma$.

PROOF Consider the equalizer $\varepsilon : (E,\Gamma) \to (B,\Phi)$ in $\underline{\underline{A}}^{\mathcal{D}}$ of β and γ . Then ε is also an equalizer in $\underline{\underline{A}}$, and so α factors through ε . Since ε is monic and α generates (B,Φ) , there is a map $f : B \to E$ such that $\varepsilon f = 1$. Hence $\beta = \beta\varepsilon f = \gamma\varepsilon f = \gamma$.

2.1.13 CONSTRUCTION (BS). Given a map $\alpha : A \to (B,\Phi)$, we describe a construction for the $\mathcal{D}$-morphism generated by α . For each ordinal n we define a map $\alpha_n : A_n \to B$ as follows:

(1) $\alpha_0 = \cup\alpha$.

(2) If n is a limit ordinal then

$$\alpha_n = \bigcup_{m<n} \alpha_m .$$

(3) The map α_{n+1} is defined in terms of α_n as follows. Consider all $u : Xi \to A_n$ for all $i \in I$, and let ϕ_u be the map in Φi such that $\phi_u . \eta i = \alpha_n u$. Then α_{n+1} is the union of all the ϕ_u together with α_n .

If τ is an infinite cardinal, the successor of a cardinal greater than κ , write $\beta : C \to B$ for α_τ .

Then if we define Ψ by

$$\Psi i = \{\psi : Di \to C \ ; \ \beta\psi \in \Phi i\}$$

we claim that $\beta : (C,\Psi) \to (B,\Phi)$ is the $\mathcal{D}$-morphism generated by α .

PROOF First it is clear that

$$\beta . \Psi i_1 . \Delta(i,i_1) \subseteq \Phi i_1 . \Delta(i,i_1) \subseteq \Phi i$$

and so $\Psi i_1 . \Delta(i,i_1) \subseteq \Psi i$. Consider any $\mu : Xi \to C$. We know that $C = \bigcup_{n<\tau} (A_n, \alpha_n)$, and so, by BS(4) , that μ is the composition of a map $\lambda : Xi \to \bigcup_{k\in K} (A_k,\alpha_k)$ and the canonical map $\nu : \bigcup_{k\in K} (A_k,\alpha_k) \to C$, where $K \subseteq \tau$ and $|K| < \kappa$. Let m be the set-theoretical union of K . Clearly $m < \tau$.

Now it is easy to prove by induction that α_m factors through α_n for any ordinals $m \le n$. So if $m \in K$ we have that $\bigcup_{k\in K} (A_k,\alpha_k) = A_m$. If m is not in K then clearly it is a limit ordinal and $\bigcup_{k\in K} (A_k,\alpha_k) = \bigcup_{n<m} (A_n,\alpha_n) = A_m$. In both cases, $\lambda : Xi \to A_m$ and $\nu : A_m \to C$. Consider the following diagram:

$$\begin{tikzcd}
Xi \arrow[r, "\lambda"] \arrow[dd, "\eta i"'] & A_m \arrow[r, "\nu"] & C \arrow[dd, "\beta"] \\
& A_{m+1} \arrow[ur, "\gamma"] \arrow[dr, "\alpha_{m+1}"] & \\
Di \arrow[ur, "\delta"] \arrow[rr, "\phi"] & & B
\end{tikzcd}$$

We can find a $\phi \in \Phi i$ so that the outer square commutes. The map $\beta\nu$ is α_m by the defining property of the canonical map ν . Therefore by the definition of α_{m+1} , there exists a δ with $\alpha_{m+1}\delta = \phi$. Now since $\beta = \alpha_\tau$ and $m + 1 < \tau$, there exists a $\gamma : A_{m+1} \to C$ with $\beta\gamma = \alpha_{m+1}$. Hence

$$\beta\mu = \phi.\eta i = \alpha_{m+1}\delta.\eta i = \beta\gamma\delta.\eta i$$

and since β is monic, $\mu = \gamma\delta.\eta i$. It is clear that $\gamma\delta \in \Psi i$. The uniqueness is guaranteed by β being monic. Hence (C,Ψ) is a $\mathcal{D}$-algebra and β is a $\mathcal{D}$-morphism.

To see that α generates β in (B,Φ), consider any other $\mathcal{D}$-morphism $\varepsilon : (E,\Gamma) \to (B,\Phi)$, which is monic in $\underline{\underline{A}}$ and through which α factors. We wish to show that α_n factors through ε for all ordinals n, and hence that β factors through ε. By 2.1.7, α_0 factors through ε. The union of any set of maps, each of which factors through ε, also factors through ε. Suppose α_n factors through ε, then consider

$$\begin{tikzcd}
Xi \arrow[r, "u"] \arrow[d, "\eta i"'] & A_n \arrow[r, "\omega"] & E \arrow[d, "\varepsilon"] \\
Di \arrow[urr, dashed, "\gamma"] \arrow[rr, "\phi_u"] & & B
\end{tikzcd},$$

where $\varepsilon\omega = \alpha_n$ and the outer square commutes. The map α_{n+1} is the union of maps like ϕ_u together with α_n. Now we can find a γ in Γi such that $\gamma.\eta i = \omega u$ and hence that $\varepsilon\gamma = \phi_u$. That is, ϕ_u factors through ε, and hence so does α_{n+1}.

Now suppose $\beta_1 : (C_1,\Psi_1) \to (B,\Phi)$ is generated by α. Then β_1 factors through β in $\underline{\underline{A}}^{\mathcal{D}}$. But, by the last paragraph, β factors through β_1 in $\underline{\underline{A}}$. Since β and β_1 are monic in $\underline{\underline{A}}$, this is enough to show that they are isomorphic in $\underline{\underline{A}}^{\mathcal{D}}$, and hence β is generated by α.

2.1.14 THEOREM (BS) For any object A in $\underline{\underline{A}}$ there is a small representative set of $\mathcal{D}$-algebras generated by maps with domain A.

PROOF We prove this by considering the construction 2.1.13 and using

BS(1) and BS(5) . We prove that for each small ordinal n , as we vary the maps α with domain A and the structure on their codomains, there is a representative set of objects A_n . Then to each $A_\tau = C$, since I is a small set, there is at most a small set of different structures. Certainly, by BS(5) , our statement is true for A_0 . Suppose n is a small limit ordinal. Since $\{A_m ; m \in n\}$ is a small set, if we suppose that for each $m \in n$ there is a small set of possibilities for A_m , then, again by BS(5) , there is a small set of possibilities for A_n . Finally A_{n+1} is a union of a small set of maps with domains A_n and $(Di;\ i \in I)$ and hence, again by BS(5) , is one of a small set of possibilities.

2.1.15 THEOREM Consider two devices $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ respectively, and functors $G : \underline{\underline{A}}_2^{\mathcal{D}2} \to \underline{\underline{A}}_1^{\mathcal{D}1}$ and $H : \underline{\underline{A}}_2 \to \underline{\underline{A}}_1$ with $U^{\mathcal{D}1}G = HU^{\mathcal{D}2}$. If H has a left adjoint H' and X_2 and $\underline{\underline{A}}_2$ are in the basic situation, then G also has a left adjoint.

PROOF Since $\underline{\underline{A}}_2$ has small limits, so does $\underline{\underline{A}}_2^{\mathcal{D}2}$. Consider a $\underline{\underline{C}}$-limit $\lambda : \underline{A} \to L$, where λ is natural from $\underline{A} : \underline{\underline{C}} \to \underline{\underline{A}}_2^{\mathcal{D}2}$ to $L : \underline{\underline{C}} \to \underline{\underline{A}}_2^{\mathcal{D}2}$ and $\underline{A}$ is a constant functor. Then $HU^{\mathcal{D}2}\lambda : HU^{\mathcal{D}2}\underline{A} \to HU^{\mathcal{D}2}L$ is a limit in $\underline{\underline{A}}_1$. However, $HU^{\mathcal{D}2}\lambda = U^{\mathcal{D}1}G\lambda$. Hence, as we saw in our discussion of limits in 2.1.2, since $U^{\mathcal{D}1}G\lambda$ is a limit, we have that $G\lambda$ is a limit in $\underline{\underline{A}}_1^{\mathcal{D}1}$. That is, G preserves limits.

We only need the solution set condition to use Freyd's adjoint functor theorem. Consider (A_1,Φ_1) in $\underline{\underline{A}}_1^{\mathcal{D}_1}$. Then, by 2.1.14, there is a small representative set, S, of $\mathcal{D}_2$-algebras generated by $\underline{\underline{A}}_2$-maps with domain $H'A_1$. We take this set to be the solution set corresponding to (A_1,Φ_1). Consider any $\mathcal{D}_1$-morphism $\alpha : (A_1,\Phi_1) \to G(A_2,\Phi_2)$ where (A_2,Φ_2) is a $\mathcal{D}_2$-algebra. If $\eta' : 1 \to HH'$ is the natural transformation of the adjoint pair, H and H', then α factorizes in $\underline{\underline{A}}_1$ as $H\beta.\eta'A_1$ where $\beta : H'A_1 \to A_2$. Then in S there is an algebra (A_2',Φ_2') such that $\beta = \beta_2\beta_1$ where $\beta_2 : (A_2',\Phi_2') \to (A_2,\Phi_2)$ is a $\mathcal{D}_2$-morphism, monic in $\underline{\underline{A}}_2$. We have the diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha"] \arrow[d, "\eta' A_1"'] & HA_2 \\
HH'A_1 \arrow[ur, "H\beta"] \arrow[r, "H\beta_1"'] & HA_2' \arrow[u, "H\beta_2"']
\end{tikzcd}$$

Now since H has a left adjoint, $H\beta_2$ is monic. Further, $H\beta_2 = HU^{\mathcal{D}_2}\beta_2 = U^{\mathcal{D}_1}G\beta_2$. We can apply Lemma 2.1.10 to see that $H\beta_1.\eta'A_1$ is a $\mathcal{D}_1$-morphism from (A_1,Φ_1) to $G(A_2',\Phi_2')$. Hence S is a solution set.

2.1.16 COROLLARY (BS) The functor $U^{\mathcal{D}}$ has a left adjoint.

PROOF In 2.1.15 take $\underline{\underline{A}}_2 = \underline{\underline{A}}_1 = \underline{\underline{A}}$, $\mathcal{D}_2 = \mathcal{D}$ and $\mathcal{D}_1$ an initial device in $\underline{\underline{A}}$.

In order to obtain another application of Theorem 2.1.15 we need the following lemma.

2.1.17 LEMMA If $\mathcal{D}$ is a device in $\underline{\underline{A}}$ and $U^{\mathcal{D}}$ has a left adjoint, and $\underline{\underline{C}}$ is a small category, there is a device $\mathcal{D}_1$ in $\underline{\underline{A}}^{\underline{\underline{C}}}$ such that there is an isomorphism $K : (\underline{\underline{A}}^{\underline{\underline{C}}})^{\mathcal{D}1} \to (\underline{\underline{A}}^{\mathcal{D}})^{\underline{\underline{C}}}$. Further $U^{\mathcal{D}1}K^{-1}$ takes $L : \underline{\underline{C}} \to \underline{\underline{A}}^{\mathcal{D}}$ to $U^{\mathcal{D}}L$.

PROOF By Remark 1.4.3 we may consider a triple $\mathcal{T} = (T,\eta,\mu)$ in $\underline{\underline{A}}$ in the place of device $\mathcal{D}$. Then we can define a triple $\mathcal{T}_1 = (T_1,\eta_1,\mu_1)$ in $\underline{\underline{A}}^{\underline{\underline{C}}}$ as follows: T_1 takes $L : \underline{\underline{C}} \to \underline{\underline{A}}$ to TL ; T_1 takes $\alpha : L_1 \to L_2$ to $T\alpha$; $\eta_1(L) : L \to TL = \eta L$; $\mu_1(L) : T^2L \to TL = \mu L$. If (L,ξ) is a $\mathcal{T}_1$-algebra, the functor K mentioned above is defined at (L,ξ) by:

$$K(L,\xi) : \begin{array}{c} C_1 \\ \alpha \downarrow \\ C_2 \end{array} \rightsquigarrow \begin{array}{c} (LC_1,\xi C_1) \\ \downarrow L\alpha \\ (LC_2,\xi C_2) \end{array} .$$

If $\lambda : (L_1,\xi_1) \to (L_2,\xi_2)$ is a $\mathcal{T}_1$-morphism then $K\lambda$ is defined by $K\lambda(C) = \lambda C$.

It is straightforward to show that K is a functor with the required properties.

2.1.18 COROLLARY (BS) If $\underline{\underline{C}}$ is a small category and $\underline{\underline{A}}$ has $\underline{\underline{C}}$-colimits, then so does $\underline{\underline{A}}^{\mathcal{D}}$.

PROOF In 2.1.15 take $\underline{\underline{A}}_2 = \underline{\underline{A}}$, $\underline{\underline{A}}_1 = \underline{\underline{A}}^{\underline{\underline{C}}}$ and $\mathcal{D}_2 = \mathcal{D}$. Further take $\mathcal{D}_1$ to be the device provided by Lemma 2.1.17, and take G and H to be the diagonal functors (see §0.4).

Once we have proved that $U^{\mathcal{D}}$ has a left adjoint, we can use theorems available for triples; see for example the work of Linton [20]. However the results we have just given do not appear to be included in those of Linton.

2.1.19 COUNTEREXAMPLE There is a category $\underline{\underline{A}}$ in $\underline{\underline{\text{Cat}}}$ which is small complete, and a device $\mathcal{D}$ in $\underline{\underline{A}}$, such that $U^{\mathcal{D}}$ does not have a left adjoint.

PROOF Define $\underline{\underline{A}}$ as follows. The objects of $\underline{\underline{A}}$ are the small ordinals together with three other objects denoted X_1 , X_2 and DX_1 . From A_1 to A_2 of $\underline{\underline{A}}$, there is at most one morphism; the composition of morphisms is forced by this requirement. If m and n are ordinals then there is a morphism from m to n if and only if $m \geq n$. Further, from each of X_1 , X_2 , DX_1 to each ordinal, there is a morphism. The only other morphisms are identity morphisms, and morphisms from X_1 to DX_1 and X_1 to X_2 . The device $\mathcal{D}$ has a single object X_1 , $\eta(X_1) = X_1 \to DX_1$, and $\Delta(X_1,X_1) = \{1_{DX_1}\}$. To each of the following objects there is precisely one algebra with that object as underlying object: DX_1 , m (m an ordinal). These are all the algebras, and the morphisms between algebras are precisely the morphisms between the underlying objects. It is easy to check that there is no universal arrow from X_2 to $U^{\mathcal{D}}$; there is no map at all from X_2 to DX_1 , and for each map $X_2 \to m$ there is a map $X_2 \to n$ where $n \geq m$ but no corresponding map from m to n .

2.1.20 EXAMPLES We return to some of the concrete examples we gave in Chapter I. If X is any small family of objects in $\underline{\underline{\text{Sets}}}$, or in Cat $(\underline{\underline{V}})$ for any variety $\underline{\underline{V}}$, or in $\underline{\underline{\text{Kit}}}$, then we are in the basic situation. Thus in Example 1.1.4, $U^{\mathcal{D}}$ has a left adjoint. Further, in 1.1.7, the functor which embeds torsion free groups in the category of groups has a left adjoint, as do the two functors involving $\underline{\underline{\text{Kit}}}$. Again, we are in the basic situation for any device in the category P(Y) described in 1.1.8. This allows us to see that any device in P(Y) amounts to a closure operator on Y (since the axioms for a triple in P(Y) are the axioms for a closure operator on Y).

Finally in this section we give some more properties of the notion defined in 2.1.11.

2.1.21 THEOREM (BS) (1) Consider the following diagram:

$$\begin{tikzcd}
 & (B',\Phi') \arrow[r, "\beta"] & (B,\Phi) \arrow[d, "\gamma"] \\
A \arrow[ur, "\alpha"] \arrow[dr, "\varepsilon"'] & & \\
 & (C',\Psi') \arrow[r, "\delta"] & (C,\Psi)
\end{tikzcd} .$$

If this diagram commutes (in $\underline{\underline{A}}$), $\beta\alpha$ generates β in (B,Φ) and $\delta\varepsilon$ generates δ in (C,Ψ) , then there is a $\mathcal{D}$-morphism $\lambda : (B',\Phi') \to (C',\Psi')$ such that $\lambda\alpha = \varepsilon$ and $\delta\lambda = \gamma\beta$.

(2) Consider map $\alpha : A \to B$ and $\mathcal{D}$-morphism $\beta : (B,\Phi) \to (C,\Gamma)$. If α generates (B,Φ) and the union of β in $\underline{\underline{A}}$ is 1, then $\beta\alpha$ generates (C,Γ) .

(3) Consider maps $\alpha : A \to B$ and $\beta : B \to (C,\Gamma)$. If $\beta\alpha$ generates (C,Γ) then so does β .

(4) Consider $\alpha : A \to (B,\Phi)$, $\beta : (B,\Phi) \to (C,\Psi)$ where β is a $\mathcal{D}$-morphism monic in $\underline{\underline{A}}$. Then $\beta\alpha$ generates β in (C,Ψ) if and only if α generates (B,Φ) .

PROOF (1) We consider the construction 2.1.13. For each ordinal n , let $\beta_n : B_n \to B$ and $\delta_n : C_n \to C$ be the maps defined for that ordinal in 2.1.13, and let $\beta\alpha = \beta_n\alpha_n$, $\delta\varepsilon = \delta_n\varepsilon_n$. Now we want to prove that for each n there is a map $\lambda_n : B_n \to C_n$ such that $\lambda_n\alpha_n = \varepsilon_n$ and $\delta_n\lambda_n = \gamma\beta_n$. Lemma 2.1.7 assures us that this is true for $n = 0$. If n is a limit ordinal then $\bigcup(\gamma . \bigcup_{m<n} \beta_m) = \bigcup_{m<n} (\gamma\beta_m)$ $= \bigcup_{m<n} (\delta_m\lambda_m)$ (by Lemma 2.1.9). Now $\gamma \bigcup_{m<n} \beta_m = \gamma\beta_n$ factors through $\bigcup(\gamma . \bigcup_{m<n} \beta_m)$ and $\bigcup_{m<n} (\delta_m\lambda_m)$ factors through $\bigcup_{m<n} \delta_m = \delta_n$; so we have our assertion for n . Now β_{n+1} is the union of β_n and all maps ϕ in the structure of (B,Φ) for which $\phi.\eta i$ factors through β_n . For such maps ϕ , the maps $\gamma\phi$ are structure maps in Ψ such that

$\gamma\phi.\eta i$ factors through δ_n ; the maps $\gamma\phi$ go into the construction of δ_{n+1} . Hence it is clear that $\gamma\beta_{n+1}$ factors through δ_{n+1} . We use Lemma 2.1.10 to see that λ_τ is a $\mathcal{D}$-morphism.

(2) This is a simple application of (1).

(3) Clear.

(4) Suppose α generates (B,Φ) , and consider any $\gamma : (D,\Gamma) \to (C,\Psi)$, monic in $\underline{\underline{A}}$, through which $\beta\alpha$ factors. Then $\beta\alpha$ factors through the intersection of β and γ . Consider the diagram:

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr] & & (B,\Phi) \arrow[rr, "\beta"] & & (C,\Psi) \\
& {} \arrow[ur, "\delta"] \arrow[dr] & & & \\
& & (D,\Gamma) \arrow[uurr, "\gamma"'] & &
\end{tikzcd}$$

The $\mathcal{D}$-morphism δ must be an isomorphism since it is monic in $\underline{\underline{A}}$ and $1 : (B,\Phi) \to (B,\Phi)$ factors through it. This means that β factors through γ , and it follows that $\beta\alpha$ generates β .

§2.2 <u>Algebra in the elaborate situation</u>.

In this section we shall give some more axioms on X and $\underline{\underline{A}}$, which together with the axioms for the basic situation form the definition of the elaborate situation (ES).

2.2.1 DEFINITION A map $\alpha : A \to B$ in A is <u>X-surjective</u> if for any map $\beta : Xi \to B$ $(Xi \in X)$ there exists a map $\gamma : Xi \to A$ such that $\alpha\gamma = \beta$.

We will see in 2.2.8 that in the elaborate situation a map is X-surjective if and only if its union is 1. In the next few theorems we develop analogues of conventional results about surjective and injective homomorphisms.

2.2.2 AXIOM <u>ES(1)</u> The family X is adequate in $\underline{\underline{A}}$.

2.2.3 THEOREM (ES) Consider $\mathcal{D}$-morphism $\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)$.
(1) α is a monomorphism in $\underline{\underline{A}}^{\mathcal{D}}$ if and only if $\alpha : A_1 \to A_2$ is a monomorphism in $\underline{\underline{A}}$.
(2) α is an $F^{\mathcal{D}}$-surjection in $\underline{\underline{A}}^{\mathcal{D}}$ if and only if $\alpha : A_1 \to A_2$ is an X-surjection in $\underline{\underline{A}}$.

PROOF (1) Suppose α is a monomorphism in $\underline{\underline{A}}^{\mathcal{D}}$ and that $\alpha\gamma = \alpha\delta$ where $\gamma , \delta : B \to A_1$. If $\gamma \neq \delta$, there is a morphism ε from some Xi in X to B such that $\gamma\varepsilon \neq \delta\varepsilon$. There exist $\mathcal{D}$-morphisms $\phi , \psi : F^{\mathcal{D}}i \to (A_1,\Phi_1)$ such that $\phi.\eta i = \gamma\varepsilon$ and $\psi.\eta i = \delta\varepsilon$. Then $\alpha.\phi.\eta i = \alpha.\psi.\eta i$, and hence $\alpha\phi = \alpha\psi$. However it is clear that $\psi \neq \phi$, giving us a contradiction.
(2) If α is X-surjective, then consider any $\mathcal{D}$-morphism

$\beta : F^{\mathcal{D}}i \to (A_2,\Phi_2)$. Certainly there is a map $\gamma : Xi \to A_1$ such that $\alpha\gamma = \beta.\eta i$. Further, there is a $\phi \in \Phi_1 i$ such that $\phi.\eta i = \gamma$. Then $\alpha\phi.\eta i = \beta.\eta i$ and we may cancel the map ηi to obtain the required result. If α is $F^{\mathcal{D}}$-surjective and β is a map from Xi to A_2 , then $\beta = \phi.\eta i$ for some $\phi \in \Phi_2 i$, and there is a map $\phi' : F^{\mathcal{D}}i \to (A_1,\Phi_1)$ such that $\alpha\phi' = \phi$. Hence $\alpha\phi'.\eta i = \beta$.

2.2.4 THEOREM (ES) (1) X-surjective maps are epic in $\underline{\underline{A}}$. $F^{\mathcal{D}}$-surjective $\mathcal{D}$-morphisms are epic in $\underline{\underline{A}}^{\mathcal{D}}$.

(2) If $\alpha : A_1 \to (A_2,\Phi_2)$ is a monomorphism in $\underline{\underline{A}}$, then there is at most one structure on A_1 which makes α a $\mathcal{D}$-morphism.

(3) If $\alpha : (A_1,\Phi_1) \to A_2$ is X-surjective in $\underline{\underline{A}}$, then there is at most one structure on A_2 which makes α a $\mathcal{D}$-morphism.

PROOF (1) Consider X-surjective map $\alpha : A_1 \to A_2$ and map γ , δ such that $\gamma\alpha = \delta\alpha$. If $\gamma \neq \delta$ there is a map ε from some Xi in X to A_2 such that $\gamma\varepsilon \neq \delta\varepsilon$. But by the definition of X-surjective, there is a map $\mu : Xi \to A_1$ such that $\alpha\mu = \varepsilon$ and hence $\gamma\alpha\mu = \gamma\varepsilon \neq \delta\varepsilon = \delta\alpha\mu$, a contradiction. Since we have used only ES(1) here, the second part of (1) will follow when we prove in 2.2.5, without any further axioms, that $F^{\mathcal{D}}$ is adequate in $\underline{\underline{A}}^{\mathcal{D}}$.

(2) Suppose Φ_1 is a structure on A_1 which makes α a $\mathcal{D}$-morphism. Define Ψ_1 by $\Psi_1 i = \{\psi : Di \to A_1 ;\ \alpha\psi \in \Phi_2 i\}$. Then clearly

$\Phi_1 i \subseteq \Psi_1 i$. Given any $\mu : Xi \to A_1$ we can certainly factor μ through ηi in $\Phi_1 i$ and hence in $\Psi_1 i$. Suppose $\psi_1 . \eta i = \psi_2 . \eta i$ $(\psi_1, \psi_2 \in \Psi_1 i)$; then $\alpha\psi_1 . \eta i = \alpha\psi_2 . \eta i$ and hence $\alpha\psi_1 = \alpha\psi_2$. Since α is monic $\psi_1 = \psi_2$, and so any $\mu : Xi \to A_1$ factors uniquely through ηi in $\Psi_1 i$. We need only the closure property to see that $\Psi_1 i$ is a structure for A_1 and hence by Lemma 1.2.5 that $\Phi_1 = \Psi_1$. But $\alpha . \Psi_1 i . \Delta(i_1, i) \subseteq \Phi_2 i . \Delta(i_1, i) \subseteq \Phi_2 i_1$ and hence $\Psi_1 i . \Delta(i_1, i) \subseteq \Psi_1 i_1$.

Notice from our proof that there exists a suitable structure on A_1 if and only if to each $\phi \in \Phi_2 i$ we have that ϕ factors through α if and only if $\phi . \eta i$ factors through α .

(3) Suppose Φ_2 is a structure on A_2 which makes α a $\mathcal{D}$-morphism. Define Ψ_2 by $\Psi_2 i = \alpha . \Phi_1 i$. Clearly $\Psi_2 i \subseteq \Phi_2 i$. We will prove that Ψ_2 is a structure for A_2 and hence, by Lemma 1.2.5, that $\Phi_2 = \Psi_2$. The closure condition is trivial. Consider any $\mu : Xi \to A_2$. Then there is a map $\beta : Xi \to A_1$ such that $\alpha\beta = \mu$. Find ϕ in $\Phi_1 i$ with $\phi . \eta i = \beta$. Then $\alpha\phi$ in $\Psi_1 i$ has $\alpha\phi . \eta i = \mu$. The uniqueness part of the universality condition follows since $\Psi_2 \subseteq \Phi_2$.

Notice that there exists a suitable structure on A_2 if and only if the following condition holds: to each $\phi_1, \phi_2 \in \Phi_1 i$, we have that $\alpha\phi_1 = \alpha\phi_2$ if and only if $\alpha . \phi_1 . \eta i = \alpha_1 \phi_2 . \eta i$.

We prove next the promised result that $F^{\mathcal{D}}$ is adequate in $\underline{\underline{A}}^{\mathcal{D}}$.

2.2.5 THEOREM If X is adequate in $\underline{\underline{A}}$ then $F^{\mathcal{D}}$ is adequate in $\underline{\underline{A}}^{\mathcal{D}}$.

PROOF To see that $F^{\mathcal{D}}$ is a generating family in $\underline{\underline{A}}^{\mathcal{D}}$, consider two distinct $\mathcal{D}$-morphisms α , $\beta : (A_1,\Phi_1) \to (A_2,\Phi_2)$. There is a map γ from some Xi in X to A_1 such that $\alpha\gamma \neq \beta\gamma$. Now $\gamma = \phi.\eta i$ for some $\phi \in \Phi_1 i$, and so $\alpha\phi.\eta i \neq \beta\phi.\eta i$. Clearly this implies that $\alpha\phi \neq \beta\phi$.

To see that $F^{\mathcal{D}}$ is adequate in $\underline{\underline{A}}^{\mathcal{D}}$, consider a family of functions $fi : \underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i\ ,\ (A_1,\Phi_1)) \to \underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i\ ,\ (A_2,\Phi_2))$ with the property that for each $\delta \in \Delta(i_1,i)$ and each $\phi \in \underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i\ ,\ (A_1,\Phi_1))$ we have that $fi(\phi).\delta = fi_1(\phi\delta)$. Define $gi : \underline{\underline{A}}(Xi,A_1) \to \underline{\underline{A}}(Xi,A_2)$ by $gi : \phi.\eta i \rightsquigarrow fi(\phi).\eta i$ for each $\phi \in \Phi_1 i$. If $\mu : Xi_1 \to Xi$, and $\delta' \in \Delta(i_1,i)$ is defined by $\eta i.\mu = \delta'.\eta i_1$ then

$$\begin{aligned} gi(\phi.\eta i).\mu &= fi(\phi).\eta i.\mu = fi(\phi).\delta'.\eta i_1 = fi_1(\phi\delta').\eta i_1 \\ &= gi_1(\phi.\delta'.\eta i_1) = gi_1(\phi.\eta i.\mu) \ . \end{aligned}$$

Since X is adequate in $\underline{\underline{A}}$, there exists a map $\alpha : A_1 \to A_2$ such that $gi(\phi.\eta i) = \alpha.\phi.\eta i$; that is, $fi(\phi).\eta i = \alpha.\phi.\eta i$. Suppose $fi(\phi) \neq \alpha\phi$. Then for some i_2 in I there is a map $\psi.\eta i_2 : Xi_2 \to Di$ $(\psi \in \Delta(i_2,i))$ such that $fi(\phi).\psi.\eta i_2 \neq \alpha\phi\psi.\eta i_2$. But $fi(\phi).\psi.\eta i_2 = fi_2(\phi\psi).\eta i_2 = gi_2(\phi.\psi.\eta i_2) = \alpha\phi\psi.\eta i_2$, a contradiction. Clearly α is a $\mathcal{D}$-morphism with the required property.

Notice that Isbell in [13] discusses the adequacy of free algebras in a category of algebras. His result is not contained in 2.2.5.

2.2.6 THEOREM (ES) (1) Consider an X-surjection $\alpha : A \to B$ and a map $\beta : A \to C$. Then β factors through α if and only if, for each pair of maps μ_1, μ_2 with common domain in X, we have that $\alpha\mu_1 = \alpha\mu_2$ implies that $\beta\mu_1 = \beta\mu_2$.

(2) A map which is X-surjective and monic is an isomorphism.

PROOF (1) Consider the function $fi : \underline{\underline{A}}(Xi,B) \to \underline{\underline{A}}(Xi,C)$ defined by $fi : \alpha\mu \rightsquigarrow \beta\mu$ $(\mu : Xi \to A)$. This is clearly an unambiguous definition. Further, if $\varepsilon : Xi_1 \to Xi$, then $fi_1(\alpha\mu\varepsilon) = \beta\mu\varepsilon = fi(\alpha\mu).\varepsilon$. Hence there exists a map $\gamma : B \to C$ such that $\gamma\alpha\mu = \beta\mu$ for all $\mu : Xi \to A$. Since X is a generating family we have further that $\gamma\alpha = \beta$.

(2) Let $\alpha : A \to B$ be X-surjective and monic. Then apply (1) to α and 1_A. There is a map $\alpha' : B \to A$ such that $\alpha'\alpha = 1$, and hence, since α is epic, it follows that α is an isomorphism.

In the next theorem we develop results about factorization of maps into X-surjections and monomorphisms. First another axiom.

2.2.7 AXIOMS <u>ES(2)</u> Every map in $\underline{\underline{A}}$ is the composition of an X-surjection and a monomorphism.

2.2.8 THEOREM (ES) (1) If $\alpha = \gamma\beta$, and β is X-surjective, and γ is monic, then this factorization of α is a standard one (see 2.1.8), and hence two such factorizations are "isomorphic".

(2) Any $\mathcal{D}$-morphism has an essentially unique factorization into an $F^{\mathcal{D}}$-surjection and a $\mathcal{D}$-monomorphism.

(3) A map in $\underline{\underline{A}}^{\mathcal{D}}$ which is both $F^{\mathcal{D}}$-surjective and monic is an isomorphism.

PROOF (1) Suppose $\varepsilon\delta$ is a standard factorization of α . Then use Lemma 2.1.7 to supply a map λ such that $\gamma\lambda = \varepsilon$, $\lambda\delta = \beta$. Now it is clear that if the composition of two maps is X-surjective, then so is the second map. Thus λ is X-surjective. Further λ is monic since ε is monic, and hence, by 2.2.6 (2), λ is an isomorphism.

(2) Consider any $\mathcal{D}$-morphism $\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)$. Factor α into an X-surjection $\beta : A_1 \to A_3$ and a monomorphism $\gamma : A_3 \to A_2$. Define Φ_3 by $\Phi_3 i = \beta.\Phi_1 i$. To see that Φ_3 is a structure for A_3 we need only check the uniqueness part of the universality condition (see the proof of 2.2.4 (3)). Suppose $\beta\phi.\eta i = \beta\phi'.\eta i$ $(\phi , \phi' \in \Phi_1 i)$. Then $\gamma\beta\phi.\eta i = \gamma\beta\phi'.\eta i$; so $\gamma\beta\phi = \gamma\beta\phi'$ and hence $\beta\phi = \beta\phi'$. Clearly with this structure on A_3 both β and γ are $\mathcal{D}$-morphisms.

(3) Clear, since $F^{\mathcal{D}}$ is adequate in $A^{\mathcal{D}}$.

Next we give some properties of generation (2.1.11) in the elaborate situation.

2.2.9 THEOREM (ES) (1) A $\mathcal{D}$-morphism generates its codomain if and only if it is an $F^{\mathcal{D}}$-surjection.

(2) If $\alpha : A \to B$ is an X-surjection, and $\beta : B \to (C,\Phi)$ generates (C,Φ) then $\beta\alpha$ generates (C,Φ) .

(3) If (η_1,F_1) is a universal arrow to $U^{\mathcal{D}}$ then η_1 generates F_1 .

PROOF (1) If $\alpha : (A_1,\Phi_1) \to (A_2,\Phi_2)$ is $F^{\mathcal{D}}$-surjective then $U\alpha = 1$ in $\underline{\underline{A}}$, and since $U\alpha$ is the intersection of more maps than is the $\mathcal{D}$-morphism generated by α , clearly $U\alpha$ factors through this $\mathcal{D}$-morphism. Hence α generates (A_2,Φ_2) . The converse implication is also easy.

(2) Using 2.1.7 it is not hard to see that any $\mathcal{D}$-monomorphism which $\beta\alpha$ factors through, is factored through by β .

(3) Consider any $\mathcal{D}$-morphism $\alpha : (A,\Phi) \to F_1$, monic in $\underline{\underline{A}}$, which η_1 factors through in $\underline{\underline{A}}$; say $\eta_1 = \alpha\tau$. There is a $\mathcal{D}$-morphism $\beta : F_1 \to (A,\Phi)$ such that $\beta.\eta_1 = \tau$. Clearly $\alpha\beta = 1$ and hence $\alpha\beta\alpha = \alpha$. Since α is monic we have also $\beta\alpha = 1$, and hence α is an isomorphism.

Before introducing another axiom of the elaborate situation we show that BS(5) and BS(3) hold for $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ when X and $\underline{\underline{A}}$ are in the elaborate situation.

2.2.10 THEOREM (ES) (1) Consider a family of $\mathcal{D}$-morphisms

$(\alpha_j : (A_j,\Phi_j) \to (B,\Psi)\ ;\ j \in J)$. The union in $\underline{\underline{A}}^{\mathcal{D}}$ of this family is the $\mathcal{D}$-morphism generated by $\bigcup_{j\in J} \alpha_j$ in (B,Ψ) .

(2) BS(5) holds for $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$.

(3) BS(3) holds for $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$.

PROOF (1) Suppose that the union of the family in $A^{\mathcal{D}}$ is $\beta : (C,\Gamma) \to (B,\Psi)$, that the $\mathcal{D}$-morphism generated by $\bigcup_{j\in J} \alpha_j$ is $\varepsilon : (E,\Lambda) \to (B,\Psi)$, and that $\alpha_j = \beta\lambda_j = \varepsilon\nu_j$. Now $\bigcup_{j\in J} \alpha_j$, being the intersection of more maps than is β , factors through β . Clearly this implies that the $\mathcal{D}$-morphism ε generated by $\bigcup_{j\in J} \alpha_j$ factors through β in $\underline{\underline{A}}^{\mathcal{D}}$. But, by Lemma 2.1.10, each α_j factors through ε in $\underline{\underline{A}}^{\mathcal{D}}$, and hence β factors through ε in $\underline{\underline{A}}^{\mathcal{D}}$. Since both β and ε are monic in $\underline{\underline{A}}^{\mathcal{D}}$ this is sufficient to prove (1).

(2) Consider a small family $((A_j,\Phi_j)\ ;\ j \in J)$ of objects of $\underline{\underline{A}}^{\mathcal{D}}$. Then if we consider families of maps $(\alpha_j\ ;\ j \in J)$ in $\underline{\underline{A}}^{\mathcal{D}}$ where the domain of α_j is (A_j,Φ_j) we want to show that there is a small representative set of unions in $\underline{\underline{A}}^{\mathcal{D}}$ of such families. But by BS(5) there is a small representative set of unions in $\underline{\underline{A}}$ of such families. And to each such union, by Theorem 2.1.14, there is a small representative set of $\mathcal{D}$-algebras generated by maps with domain that union. However, as we have just seen in (1), any union in $\underline{\underline{A}}^{\mathcal{D}}$ is generated by a union in $\underline{\underline{A}}$, and so our result follows.

(3) Consider $(A,\Phi) \in \underline{\underline{A}}^{\mathcal{D}}$. Then there is a small representative set S of monomorphisms in $\underline{\underline{A}}$ with codomain A . Since there is at most one structure on the domain of each of these making it into a $\mathcal{D}$-morphism, we can select a subset of S to be a representative set for $\mathcal{D}$-monomorphisms with codomain (A,Φ) .

Next we give a further axiom of the elaborate situation.

2.2.11 AXIOM <u>ES(3)</u> There exists a small cardinal number $\kappa \geq 2$ which, in addition to having property BS(4), has the following property: given any family of maps $(\alpha_j\ ;\ j \in J)$ where $|J| < \kappa$, the domain of each α_j belongs to X ,and all of the maps have codomain A , then there exists an $Xi \in X$ and a map $\alpha : Xi \to A$ such that each α_j factors through α .

Using this condition and the factorization properties ES(2) and 2.2.6 (2), we can give a new and simpler construction for the algebra generated by a map.

2.2.12 CONSTRUCTION (ES) Consider a map $\alpha : A \to (B,\Phi)$. Consider all ϕ in Φi such that $\phi.\eta i$ factors through α . Let the union of all such ϕ for all $i \in I$ be $\beta : C \to B$, and define Ψ by $\Psi i = \{\psi : Di \to C\ ;\ \beta\psi \in \Phi i\}$. Then $\beta : (C,\Psi) \to (B,\Phi)$ is a $\mathcal{D}$-morphism, and is the morphism generated by α in (B,Φ) .

PROOF That β is a $\mathcal{D}$-morphism follows immediately if Ψ is a structure. From the proof of 2.2.4 (2) we need only consider the existence part of the universality condition. Consider $\mu : Xi \to C$. Then by BS(4) μ is the composition of a map $u : Xi \to \bigcup\limits_{j \in J} (Di_j, \phi_j)$ where $|J| < \kappa$, and a map $v : \bigcup\limits_{j \in J} (Di_j, \phi_j) \to C$. Further, by ES(3), there is an $i_1 \in I$ and a map $w : Xi_1 \to A$ such that if $\phi_j . \eta i_j = \alpha . \nu_j$, then each ν_j $(j \in J)$ factors through w . Hence if $\alpha w = \phi_1 . \eta i_1$, each ϕ_j factors through ϕ_1 , and so $\bigcup\limits_{j \in J} \phi_j$ factors through $U\phi_1$. Consider the diagram:

$$\begin{tikzcd}
 & Xi \arrow[rr, "\mu"] \arrow[dr] \arrow[ddl, "\eta i"'] \arrow[dd, dashed, "\varepsilon"'] & & C \arrow[r, "\beta"] & B \\
 & & \bigcup\limits_{j \in J} (Di_j, \phi_j) \arrow[ur] \arrow[dr] & & \\
Di \arrow[r, dashed, "\delta"] & Di_1 \arrow[rr, "\lambda"] & & U(Di_1, \phi_1) \arrow[uu, "\tau"'] &
\end{tikzcd}$$

where $\beta\tau = U\phi_1$ and $\beta\tau\lambda = \phi_1$. Since λ is X-surjective, it follows that the map $Xi \to \bigcup\limits_{j \in J} (Di_j, \phi_j) \to U(Di_1, \phi_1) = \lambda\varepsilon$ for some $\varepsilon : Xi \to Di_1$. The map $\delta \in \Delta\ (i, i_1)$ makes the left-hand triangle commute. Hence $\tau\lambda\delta . \eta i = \mu$ and $\beta\tau\lambda\delta \in \Phi i$.

Next, we wish to see that $\beta : (C,\Psi) \to (B,\Phi)$ is the intersection of all $\mathcal{D}$-monomorphisms through which α factors. First, to see that

α factors through β , factor α into an X-surjection $A \to L$ and a monomorphism $L \to B$, and form the intersection $E \to B = E \to L \to B$ $= E \to C \to B$ of $L \to B$ and β . Now if we can show that $E \to L$ is an X-surjection, then it is an isomorphism and $L \to B$ factors through $C \to B$, and hence so does α . Consider any $\mu : Xi \to L$. Then since $A \to L$ is an X-surjection, $Xi \to L = Xi \to A \to L$ for some map $Xi \to A$. Now by the construction of β , there is a map $Xi \to C$ such that $Xi \to A \to L \to B = Xi \to C \to B$. Hence by the defining property of intersections, $Xi \to A \to L = Xi \to L$ factors through $E \to L$, and hence $E \to L$ is an X-surjection.

Finally, consider any $\mathcal{D}$-monomorphism $\beta' : (C',\Psi') \to (B,\Phi)$ which α factors through ; say $\alpha = \beta' b$. Then consider $\phi \in \Phi i$ such that $\phi.\eta i$ factors through α ; say $\phi.\eta i = \alpha\tau$. Find $\psi' \in \Psi' i$ such that $\psi'.\eta i = b\tau$. Then $\beta'\psi'.\eta i = \alpha\tau = \phi.\eta i$, and so ϕ factors through β' . Hence β factors through β' (in $\underline{\underline{A}}^{\mathcal{D}}$ by Lemma 2.1.10) and this is enough to ensure our result.

Next we prove two useful lemmas.

2.2.13 LEMMA (ES) Suppose the map $\alpha : A \to (B,\Phi)$ generates (B,Φ) . Then any map $\mu : Xi \to (B,\Phi)$ $(i \in I)$ factors through some structure map $\phi_1 : Di_1 \to B$ which has the property that $\phi_1.\eta i_1$ factors through α .

PROOF Considering the construction 2.2.12 and BS(4) we see that μ factors into a map $u : Xi \to \bigcup_{j \in J} (Di_j, \phi_j)$ with $|J| < \kappa$, and a map $v : \bigcup_{j \in J} (Di_j, \phi_j) \to B$. The maps ϕ_j are in Φi_j and have the property that $\phi_j . \eta i_j$ factors through $\alpha : A \to B$. As in the proof of 2.2.12, there is a $\phi_1 : Di_1 \to B$ such that $\phi_1 . \eta i_1$ factors through α, and $\bigcup_{j \in J} \phi_j$ factors through $U\phi_1$. Consider the diagram:

$$\begin{tikzcd}
Xi \arrow[rr, "\mu"] \arrow[d, dashed] \arrow[drr] & & B \\
Di_1 \arrow[rr, "X\text{-surj've}"] & & U(Di_1, \phi_1) \arrow[u]
\end{tikzcd} .$$

Clearly μ factors through ϕ_1, and we have our result.

2.2.14 LEMMA (ES) Consider maps $\alpha : A \to B$ and $\beta : A \to C$ where α is an X-surjection. Then β factors through α if and only if for each map μ with domain in X and codomain A we have that $\beta\mu$ factors through $\alpha\mu$.

PROOF By 2.2.6 we need only show that if $\mu_1, \mu_2 : Xi \to A$ and $\alpha\mu_1 = \alpha\mu_2$ then $\beta\mu_1 = \beta\mu_2$. By ES(3) there is a map μ with domain in X such that $\mu_1 = \mu\varepsilon_1$ and $\mu_2 = \mu\varepsilon_2$ for some $\varepsilon_1, \varepsilon_2$. But by our assumption $\beta\mu = \gamma\alpha\mu$ for some γ. Hence

$$\beta\mu_1 = \beta\mu\varepsilon_1 = \gamma\alpha\mu\varepsilon_1 = \gamma\alpha\mu_1 = \gamma\alpha\mu_2 = \gamma\alpha\mu\varepsilon_2 = \beta\mu\varepsilon_2 = \beta\mu_2 .$$

Next in this section we introduce an axiom which enables us to mimic the classical theory of congruences. First we need two definitions.

2.2.15 DEFINITION A family of maps $(\alpha j \; ; \; j \in J)$ with common codomain A is called a <u>codiscriminating family</u> for A if, for each pair of maps $\beta \neq \gamma$ with domain A, there is a $j \in J$ such that $\beta.\alpha j \neq \gamma.\alpha j$.

I learnt of this notion from B.H. Neumann [27].

2.2.16 DEFINITION (ES) An <u>X-congruence</u> on A is a family $R = (Ri \; ; \; i \in I)$ of equivalence relations, where Ri is a relation on $\underline{\underline{A}}\;(Xi \; , \; A)$, which has the following additional properties. (1) If μ is any map in $\underline{\underline{A}}\;(Xi_1, Xi)$ and α, β are in $\underline{\underline{A}}\;(Xi, A)$ then $Ri\;(\alpha,\beta)$ implies that $Ri_1\;(\alpha\mu,\beta\mu)$.

(2) If $(\alpha j : Xi_j \to Xi \; ; \; j \in J)$ is a discriminating family for Xi, and $\beta,\gamma : Xi \to A$, then $Ri_j\;(\beta.\alpha j \; , \; \gamma.\alpha j)$ for all j in J implies $Ri\;(\beta,\gamma)$.

Corresponding to each map $\gamma : A \to B$ there is an obvious X-congruence R_γ on A defined by: $R_\gamma i\;(\alpha,\beta)$ if and only if $\gamma\alpha = \gamma\beta$.

2.2.17 AXIOM <u>ES(4)</u> To each X-congruence R on A there is an X-surjection γ with domain A with the property that $Ri\;(\alpha,\beta)$ if and only if $\gamma\alpha = \gamma\beta$; that is, $R = R_\gamma$.

2.2.18 THEOREM (ES) The axiom ES(4) holds for $F^{\mathcal{D}}$ and $A^{\mathcal{D}}$.

PROOF Consider an $F^{\mathcal{D}}$-congruence R_1 on (A,Φ) in $\underline{\underline{A}}^{\mathcal{D}}$. Define for each i in I a relation $R_2 i$ on $\underline{\underline{A}}$ (Xi,A) by: R_2 $(\phi_1.\eta i,\phi_2.\eta i)$ if and only if R_1 (ϕ_1,ϕ_2) $(\phi_1,\phi_2 \in \Phi i)$. Consider $\mu : Xi_1 \to Xi$. If $\delta.\eta i_1 = \eta i.\mu$ $(\delta \in \Delta\ (i_1,i))$ then $\phi_1.\eta i.\mu = \phi_1.\delta.\eta i_1$ and $\phi_2.\eta i.\mu = \phi_2.\delta.\eta i_1$. Since R_1 is an $F^{\mathcal{D}}$-congruence it follows that:

$$R_2 i\ (\phi_1.\eta i,\phi_2.\eta i) \Rightarrow R_1 i\ (\phi_1,\phi_2) \Rightarrow R_1 i_1\ (\phi_1\delta,\phi_2\delta)$$
$$\Rightarrow R_2 i_1\ (\phi_1.\eta i.\mu,\phi_2.\eta i.\mu) .$$

Given a codiscriminating family $(\alpha j : Xi_j \to Xi\ ;\ j \in J)$ for Xi in $\underline{\underline{A}}$, if we define $\delta j : F^{\mathcal{D}} i_j \to F^{\mathcal{D}} i$ by $\delta j.\eta i_j = \eta i.\alpha j$, then it is easy to see that $(\delta j\ ;\ j \in J)$ is a codiscriminating family for $F^{\mathcal{D}} i$ in $\underline{\underline{A}}^{\mathcal{D}}$. Hence $R_2 i_j$ $(\phi_1.\eta i.\alpha j,\phi_2.\eta i.\alpha j)$ for all $j \in J$ implies $R_2 i_j$ $(\phi_1.\delta j.\eta i_j,\phi_2.\delta j.\eta i_j)$ for all $j \in J$, which implies $R_1 i_j$ $(\phi_1.\delta j,\phi_2.\delta j)$ for all $j \in J$, and hence $R_1 i$ (ϕ_1,ϕ_2) . Thus R_2 is an X-congruence and there is an X-surjection $\gamma : A \to B$ such that $R_2 i$ $(\phi_1.\eta i,\phi_2.\eta i)$ if and only if $\gamma\phi_1.\eta i = \gamma\phi_2.\eta i$. Suppose $\gamma\phi_1.\eta i = \gamma\phi_2.\eta i$ but $\gamma\phi_1 \neq \gamma\phi_2$. Then for some $i_2 \in I$ there is a map $\delta_2.\eta i_2$ $(\delta_2 \in \Delta\ (i_2,i))$ such that $\gamma\phi_1\delta_2.\eta i_2 \neq \gamma\phi_2\delta_2.\eta i_2$. But $R_1 i_2$ $(\phi_1\delta_2,\phi_2\delta_2)$, and so $R_2 i_2$ $(\phi_1\delta_2.\eta i_2,\phi_2\delta_2.\eta i_2)$, a contradiction. Hence $R_1 i$ (ϕ_1,ϕ_2) if and only if $\gamma\phi_1 = \gamma\phi_2$. Further γ can be made into a $\mathcal{D}$-morphism by defining a structure Ψ on B as follows: $\Psi i = \gamma . \Phi i$. The only thing we need to check is that if $\gamma\phi_1 . \eta i = \gamma\phi_2 . \eta i$ $(\phi_1, \phi_2 \in \Phi i)$ then $\gamma\phi_1 = \gamma\phi_2$, and we have already proved this.

2.2.19 DEFINITION Consider an object A in $\underline{\underline{A}}$ and a family $R = (R_j \; ; \; j \in J)$ of families of binary relations indexed by I where $R_j i$ is a relation on $\underline{\underline{A}}$ (Xi, A) . The intersection $\bigcap_{j \in J} R_j$ of the families is defined by $(\bigcap_{j \in J} R_j)\, i\, (\alpha, \beta)$ if and only if $R_j i \; (\alpha, \beta)$ for all j in J . If $R_1 \cap R_2 = R_1$ then we write $R_1 \leq R_2$. This establishes a partial ordering on such families of binary relations.

2.2.20 THEOREM The intersection of any family of X-congruences on A is an X-congruence. Thus, given any family $R = (Ri \; ; \; i \in I)$ of binary relations, where Ri is a relation on $\underline{\underline{A}}$ (Xi, A) , there is a least X-congruence R' with $R' \geq R$.

PROOF Easy.

2.2.21 THEOREM The axiom ES(2) is implied by ES(1) together with ES(4).

PROOF Consider any map $\alpha : A \to B$. By ES(4) there is an X-surjection

β with $R_\beta = R_\alpha$. By 2.2.6 (1) α factors through β , say $\alpha = \gamma\beta$. Suppose $\gamma\varepsilon_1 = \gamma\varepsilon_2$. If $\varepsilon_1 \neq \varepsilon_2$ there is a map ν with domain in X such that $\varepsilon_1\nu \neq \varepsilon_2\nu$. Since β is X-surjective there exist maps μ_1, μ_2 such that $\varepsilon_1\nu = \beta\mu_1$ and $\varepsilon_2\nu = \beta\mu_2$. Then since $R_\beta = R_\alpha$ we must have $\gamma\varepsilon_1\nu = \gamma\beta\mu_1 = \alpha\mu_1 \neq \alpha\mu_2 = \gamma\beta\mu_2 = \gamma\varepsilon_2\nu$, and so $\gamma\varepsilon_1 \neq \gamma\varepsilon_2$, a contradiction. Hence γ is a monomorphism.

The final result of this section is that $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ are in the elaborate situation if X and $\underline{\underline{A}}$ are. First we give the final axiom of the elaborate situation.

2.2.22 AXIOM <u>ES(5)</u> There is a set J with $I \subseteq J$, and a family $Y = (Yj \ ; \ j \in J)$ of objects of $\underline{\underline{A}}$ with the following properties: $Y|I = X$; any composition $Xi \to Yj \to A$ factors as $Xi \to Yj \to Xi_1 \to A$ for some $i_1 \in I$, the second map being X-surjective; for each A in $\underline{\underline{A}}$ there is an X-surjection $Yj \to A$ for some $j \in J$. We call such a family Y an <u>extension</u> of X .

2.2.23 THEOREM (ES). (1) $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ satisfy ES(3).
(2) $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ satisfy BS(4).
(3) $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ satisfy ES(5).
(4) $F^{\mathcal{D}}$ and $\underline{\underline{A}}^{\mathcal{D}}$ are in the elaborate situation.

PROOF (1) Consider a family of $\mathcal{D}$-morphisms $(\phi_j : F^{\mathcal{D}}i_j \to (A,\Phi) \ ; \ j \in J)$

with $|J| < \kappa$. Then $(\phi_j . \eta i_j \ ; \ j \in J)$ is a family of morphisms in $\underline{\underline{A}}$ whose domains are in X . Hence there is a map $\alpha : Xi \to A$ for some $i \in I$ such that each $\phi_j . \eta i_j$ factors through α ; say $\phi_j . \eta i_j = \alpha . \beta_j$. Let $\delta_j . \eta i_j = \eta i . \beta_j$ $(\delta_j \in \Delta\ (i_j, i))$. Then if $\phi . \eta i = \alpha$ $(\phi \in \Phi i)$ we have $\phi_j = \phi . \delta_j$ as we require.

(2) Consider a family of $\mathcal{D}$-morphisms $(\alpha_j \ ; \ j \in J)$, with common codomain (B, Ψ) , and let the domain of α_j be (A_j, Φ_j) . Write $\bigcup_{j \in J} ((A_j, \Phi_j) \ , \ \alpha_j) = (A, \Phi)$. Then, consider a map $\beta : F^{\mathcal{D}} i \to (A, \Phi)$ for some $i \in I$. We know that there is a map $\gamma : \bigcup_{j \in J} (A_j, \alpha_j) \to A$ which generates (A, Φ) (see 2.2.10). Hence by Lemma 2.2.13, for some $i_1 \in I$ there is a map ϕ in Φi_1 such that $\beta . \eta i$ factors through ϕ , say $\phi\delta = \beta . \eta i$, and further $\phi . \eta i_1$ factors through γ , say $\gamma . \mu = \phi . \eta i_1$. Consider the diagram

$$\begin{tikzcd}
Xi \arrow[r, "\eta i"] \arrow[d, "\delta"'] & F^{\mathcal{D}} i \arrow[r, "\beta"] \arrow[d, dashed] & (A, \Phi) \\
F^{\mathcal{D}} i_1 \arrow[r, dashed] & \bigcup_{k \in K} (A_k, \Phi_k) \arrow[ur] & \\
Xi_1 \arrow[r] \arrow[u, "\eta i_1"] & \bigcup_{k \in K} A_k \arrow[r] \arrow[u] & \bigcup_{j \in J} A_j \arrow[uu, "\gamma"']
\end{tikzcd} .$$

The bottom edge is the factorization of μ provided by BS(4). The composite $\bigcup_{k \in K} A_k \to \bigcup_{j \in J} A_j \xrightarrow{\gamma} (A, \Phi) \to (B, \Psi)$ generates the union in $\underline{\underline{A}}^{\mathcal{D}}$ of $(\alpha_k \ ; \ k \in K)$, which factors through (A, Φ) . We may cancel the monomorphism $(A, \Phi) \to (B, \Psi)$ to see that the lower right-hand region commutes. Then we solve for structure map in the lower left square, and then for the structure map in the upper left square. Since the lower region commutes, the composite

$$Fi_1 \dashrightarrow \bigcup_{k\in K} (A_k,\Phi_k) \to (A,\Phi)$$

must be ϕ . Hence it follows that the upper right triangle commutes, as desired.

(3) Suppose $Y = (Y_j\ ;\ j\in J)$ is an extension of X . Then we will show that $G = (G_j\ ;\ j\in J)$ is an extension of $F^{\mathcal{D}}$ if there is a universal arrow of the form $(\eta_j : Y_j \to U^{\mathcal{D}}G_j, G_j)$ for each $j\in J$. Certainly by Corollary 2.1.16 such universal arrows exist, and so there is an extension G of $F^{\mathcal{D}}$. Consider a pair of $\mathcal{D}$-morphisms $F^{\mathcal{D}}i \to G_j \to (A,\Phi)$, and the diagram:

$$\begin{tikzcd}
Xi \arrow[r] \arrow[dr] & F^{\mathcal{D}}i \arrow[r] \arrow[d] & G_j \arrow[rr] \arrow[dr, dashed] & & (A,\Phi) \\
& F^{\mathcal{D}}i_1 \arrow[ur] & \arrow[r, phantom, "(1)"] & F^{\mathcal{D}}i_2 \arrow[ur, dashed] \arrow[r, phantom, "(2)"] & \\
& Xi_1 \arrow[u] \arrow[r] & Y_j \arrow[uu] \arrow[rr, "X\text{-surj'n}"] & & Xi_2 \arrow[ul] \arrow[uu]
\end{tikzcd}$$

The left-hand portion of the diagram is supplied from the map $Xi \to F^{\mathcal{D}}i \to G_j$ by Lemma 2.2.13 and the fact that $Y_j \to G_j$ generates G_j (see Theorem 2.2.9 (3)). By the property of extensions $Xi_1 \to Y_j \to G_j \to (A,\Phi)$ factors as $Xi_1 \to Y_j \to Xi_2 \to (A,\Phi)$. Solve for $\mathcal{D}$-morphisms in regions (1) and (2). It is easy to see by 2.1.20

and 2.2.9 that $G_j \to F^{\mathcal{D}}i_2$ is $F^{\mathcal{D}}$-surjective. Finally it is clear that $F^{\mathcal{D}}i \to G_j \to (A,\Phi) = F^{\mathcal{D}}i \to F^{\mathcal{D}}i_1 \to G_j \to F^{\mathcal{D}}i_2 \to (A,\Phi)$.

2.2.24 REMARK If we take X to be any small family of sets where the least cardinal greater than the cardinalities of the sets in X is an infinite regular cardinal, then X and $\underline{\underline{\text{Sets}}}$ satisfy the axioms for the elaborate situation. Theorem 2.2.23 provides further examples of the elaborate situation.

Suppose X and $\underline{\underline{A}}$ are in the elaborate situation and $\underline{\underline{A}}$ has an initial object which is in X . If n is a small set and we define a family $Y = (Y_j \; ; \; j \in I^n)$ of objects in $\underline{\underline{A}}^n$ by $Y_j(m) = X(jm)$ $(m \in n)$, then Y and $\underline{\underline{A}}^n$ are in the elaborate situation.

§2.3 Recognition Theorems.

In this section we develop conditions for recognizing when a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ is strictly isomorphic to a functor $U^{\mathcal{D}}$ for some device $\mathcal{D}$ in $\underline{\underline{A}}$. Of course, in the triple case there are several theorems of this type; notably Beck's theorem (for which the standard reference seems to be [1] but which is more easily found in [23]) and some theorems of Linton [19]. Other conditions occur in [5] and [28]. Our main interest is in recognizing functors isomorphic to $U^{\mathcal{D}}$ where the object family of $\mathcal{D}$ is small. First however we obtain some quite general theorems.

2.3.1 DEFINITION Consider a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and a family Y of objects in $\underline{\underline{B}}$. We say that $\underline{\underline{B}}$ is <u>saturated</u> with respect to Y and U if the following condition holds. Consider any $\alpha : A_1 \to A_2$ and any B_1, B_2 in $\underline{\underline{B}}$ with $UB_1 = A_1$ and $UB_2 = A_2$. If for all $\beta : Y_j \to B_1$ in $\underline{\underline{B}}$ with $Y_j \in Y$ there exists a $\beta' : Y_j \to B_2$ in $\underline{\underline{B}}$ such that $\alpha . U\beta = U\beta'$, then there is a $\beta'' : B_1 \to B_2$ in $\underline{\underline{B}}$ such that $\alpha = U\beta''$.

2.3.2 THEOREM Consider a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and families of objects X in $\underline{\underline{A}}$ and F in $\underline{\underline{B}}$ both indexed by I. Then if there are universal arrows $(\eta i : Xi \to UFi , Fi)$ for each $i \in I$, and if U is faithful, and if $\underline{\underline{B}}$ is saturated with respect to F and U, then $\underline{\underline{B}}$ is equivalent to a full subcategory of $\underline{\underline{A}}^{\mathcal{D}}$ for some device $\mathcal{D}$ with object family X (and U corresponds under the equivalence to the forgetful functor $U^{\mathcal{D}}$ restricted to this subcategory).

PROOF The basic tool in all these recognition theorems is the first part of the comparison theorem 1.2.6. From now on K will always mean the canonical functor $\underline{\underline{B}} \to \underline{\underline{A}}^{\mathcal{D}}$ with the properties (1) $U^{\mathcal{D}}.K = U$ and (2) $KF = F^{\mathcal{D}}$. The device $\mathcal{D}$ is that corresponding to the universal arrows $(\eta i, Fi)$. To prove our theorem we need only show that, under the given conditions, K is full and its image is a subcategory of $\underline{\underline{A}}$.

To see that the image of K is a category we only need to show that if $\beta_1 : B_1 \to B_2$ and $\beta_2 : B_3 \to B_4$ are in $\underline{\underline{B}}$, and $KB_2 = KB_3$ then there exists a $\beta : B_1 \to B_4$ in $\underline{\underline{B}}$ such that $U\beta = U\beta_2 . U\beta_1$. If $KB_2 = KB_3$ then $U\underline{\underline{B}}\ (Fi, B_2) = U\underline{\underline{B}}\ (Fi, B_3)$ for all $i \in I$. Consider $\phi : Fi \to B_2$ and the corresponding $\phi' : Fi \to B_3$ which has $U\phi' = U\phi$. Then $U\beta_2 . U\phi = U\beta_2 . U\phi' = U(\beta_2\phi')$. Hence there exists a $\beta_2' : B_2 \to B_4$ such that $U\beta_2' = U\beta_2$ and so $U(\beta_2'\beta_1) = U\beta_2' . U\beta_1 = U\beta_2 . U\beta_1$.

To see that K is full, consider any $\alpha : KB_1 \to KB_2$ in $\underline{\underline{A}}^{\mathcal{D}}$. If $\phi_1 : Fi \to B_1$ in $\underline{\underline{B}}$ then $U\phi_1 : F^{\mathcal{D}}i \to KB_1$ in $\underline{\underline{A}}^{\mathcal{D}}$, and so $\alpha . U\phi_1 : F^{\mathcal{D}}i \to KB_2$ in $\underline{\underline{A}}^{\mathcal{D}}$. That is, in $\underline{\underline{A}}$, $\alpha . U\phi_1 : Di \to UB_2$. By the universality of the arrow $(\eta i, Fi)$ there is a morphism $\phi_2 : Fi \to B_2$ such that $U\phi_2 . \eta i = \alpha . U\phi_1 . \eta i$. But $U\phi_2$ can be considered a morphism from $F^{\mathcal{D}}i$ to KB_2 in $\underline{\underline{A}}^{\mathcal{D}}$. Then by the universality of $(\eta i, F^{\mathcal{D}}i)$ this last equation means that $U\phi_2 = \alpha . U\phi_1$. Hence by the fact that $\underline{\underline{B}}$ is saturated with respect to F and U, there exists a map $\beta : B_1 \to B_2$ such that $U\beta = \alpha$ in $\underline{\underline{A}}$. Thus $K\beta = \alpha$ in $\underline{\underline{A}}^{\mathcal{D}}$.

2.3.3 DEFINITION Consider a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and a family S of universal arrows $((\eta i : Xi \to UFi, Fi)\ ;\ i \in I)$. Then a U,S-suitable X-surjection is an X-surjection $\alpha : UB \to A$ $(B \in \underline{\underline{B}}, A \in \underline{\underline{A}})$ in $\underline{\underline{A}}$ with the property that to each $\phi_1, \phi_2 : Fi \to B$ we have that $\alpha . U\phi_1 = \alpha . U\phi_2$ if and only if $\alpha . U\phi_1 . \eta i = \alpha . U\phi_2 . \eta i$.

This definition is motivated by the proof of 2.2.4 (3).

2.3.4 THEOREM Consider a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and a family X of objects of $\underline{\underline{A}}$ indexed by I. Suppose that for each A in $\underline{\underline{A}}$ there is an $i \in I$ with an X-surjection from Xi to A. Then U is strictly isomorphic to $U^{\mathcal{D}}$ for some device $\mathcal{D}$ with object family X if and only if the following conditions hold.

(1) There is a family S of universal arrows to U, namely $S = ((\eta i : Xi \to UFi, Fi) ; i \in I)$. The family $F = (Fi ; i \in I)$ is required to have the following additional properties.

(2) U is faithful, and $\underline{\underline{B}}$ is saturated with respect to F and U.

(3) Consider any $B \in \underline{\underline{B}}$ and any U,S-suitable X-surjection $\alpha : UB \to A$. Then there exists a unique map $\beta : B \to B_1$ in $\underline{\underline{B}}$ such that $U\beta = \alpha$.

PROOF The device $\mathcal{D}$ is that associated with the family of universal arrows. In the light of Theorem 2.3.2 we need only show that any object in $\underline{\underline{A}}^{\mathcal{D}}$ is the image of an object under K, and that K is one-one on objects. Consider $(A,\Phi) \in \underline{\underline{A}}^{\mathcal{D}}$. Let $\alpha : Xi \to A$ be an X-surjection assured by the conditions of the theorem. Then $\phi : Di \to A$ in Φi is an X-surjection if $\phi.\eta i = \alpha$, and hence $\phi : F^{\mathcal{D}}i \to (A,\Phi)$ is an $F^{\mathcal{D}}$-surjection. Consider maps $\mu_1,\mu_2 : Xi_1 \to Di$ for some $i_1 \in I$. If $\phi\mu_1 = \phi\mu_2$ then $\phi\delta_1 = \phi\delta_2$ where $\delta_1,\delta_2 \in \Delta\,(i_1,i)$ and $\delta_1.\eta i_1 = \mu_1, \delta_2.\eta i_1 = \mu_2$. Thus there exists a morphism $\beta : Fi \to B$ in $\underline{\underline{B}}$ such that $U\beta = \phi$. Further $K\beta = \phi : F^{\mathcal{D}}i \to (A,\Psi)$ for some structure Ψ . But we have seen in Theorem 2.2.4 (3) that there is at most one structure on A making ϕ a $\mathcal{D}$-morphism. Hence $KB = (A,\Phi)$.

Suppose B_1 is another object of $\underline{\underline{B}}$ with $KB_1 = (A,\Phi)$. Then since K is full, there is a map $\gamma : B \to B_1$ such that $K\gamma = 1$. This gives us another map $\gamma\beta : Fi \to B_1$ in $\underline{\underline{B}}$ which goes to $\phi : Di \to A$ under U , a contradiction if $B \neq B_1$.

Notice that even this simple theorem is quite helpful for proving that functors are tripleable. For example, it is not hard to check the conditions of this theorem when U is the forgetful functor from compact Hausdorff spaces to $\underline{\underline{\text{Sets}}}$ and X is all sets. We next prove a recognition theorem for the elaborate situation. But first, a definition.

2.3.5 DEFINITION Consider a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ and a family of universal arrows $S = ((\eta i : Xi \to UFi, Fi) \; ; \; i \in I)$. Then a U,S-suitable monomorphism is a monomorphism in $\underline{\underline{A}}$, $\alpha : A \to UB$ $(A \in \underline{\underline{A}}\ ,\ B \in \underline{\underline{B}})$ with the additional property that if ϕ is any morphism in $\underline{\underline{B}}$ from $Fi \to B$ then $U\phi$ factors through α if and only if $U\phi . \eta i$ factors through α .

For the motivation for this definition see the proof of 2.2.4 (2).

2.3.6 THEOREM (ES) Given X and $\underline{\underline{A}}$ in the elaborate situation, then a functor $U : \underline{\underline{B}} \to \underline{\underline{A}}$ is strictly isomorphic to $U^{\mathcal{D}}$ for some device $\mathcal{D}$ in $\underline{\underline{A}}$ with object family X if and only if the following conditions hold: (1), (2) and (3) of Theorem 2.3.4 and

(4) To any U,S-suitable monomorphism $\alpha : A \to UB$ there is a unique morphism $\beta : B_1 \to B$ in $\underline{\underline{B}}$ with $U\beta = \alpha$.

(5) $\underline{\underline{B}}$ has small products, and U preserves them.

PROOF Of course the device $\mathcal{D}$ is the one associated with the universal arrows $((\eta i, Fi) ; i \in I)$. Then in view of Theorem 2.3.2 we have only to show that the comparison functor $K : \underline{\underline{B}} \to \underline{\underline{A}}^{\mathcal{D}}$ is bijective on objects. Consider (A,Φ) in $\underline{\underline{A}}^{\mathcal{D}}$ and let $Y = (Y_j ; j \in J)$ be an extension (see 2.2.22) of X . Then there is an X-surjection $Y_j \to A$ for some $j \in J$. Consider all maps from Y_j to A and form the factorizations $Y_j \xrightarrow{\alpha_p} (A_p,\Phi_p) \xrightarrow{\beta_p} (A,\Phi)$ $(p \in P)$ where $\beta_p . \alpha_p$ generates β_p in (A,Φ) . Let Q be the subset of P for which the maps $\beta_q . \alpha_q$ $(q \in Q)$ factor as follows for some $i \in I$: $\beta_q . \alpha_q = Y_j \xrightarrow{\text{X-surj'n}} Xi \to A$. Now both sets P and Q are small, so we can form products $\prod_{p \in P} (A_p,\Phi_p)$ and $\prod_{q \in Q} (A_q,\Phi_q)$. Suppose that $\tau_1 : Y_j \to \prod_{p \in P} (A_p,\Phi_p)$ and $\tau_2 : Y_j \to \prod_{q \in Q} (A_q,\Phi_q)$ are the maps induced by the α's , and let $\tau_1 = Y_j \xrightarrow{\lambda_1} (A',\Phi') \xrightarrow{\mu_1} \prod_{p \in P} (A_p,\Phi_p)$ and $\tau_2 = Y_j \xrightarrow{\lambda_2} (A'',\Phi'') \xrightarrow{\mu_2} \prod_{q \in Q} (A_q,\Phi_q)$ where τ_k generates μ_k $(k = 1,2)$. We would like to show that (A',Φ') is isomorphic to (A'',Φ'') . Let us assume that this is true for the moment. Then, for each $q \in Q$ there is a map $Xi \to A_q$ (some $i \in I$) which generates (A_q,Φ_q) , and hence there is a corresponding

X-surjection $UFi \to A_q$ (see 2.2.9 (1)). This X-surjection being actually a $\mathcal{D}$-morphism is easily seen to be U,S-suitable, and hence there is map $Fi \to B$ in $\underline{\underline{B}}$ whose image under U is $Di \to A_q$. Clearly the image of this map $Fi \to B$ under K is $F^{\mathcal{D}}i \to (A_q, \Gamma_q)$ for some structure Γ_q which we may identify by 2.2.4 (3) as being Φ_q. Thus each (A_q, Φ_q) is the image of an object under K. If B_q goes to (A_q, Φ_q) under K, then consideration of the action of U and K on the family $(\prod_{q \in Q} B_q \to B_q \,;\, q \in Q)$ of projections, together with the knowledge that $U^{\mathcal{D}}$ creates limits, leads to the conclusion that at least one model of $\prod_{q \in Q} (A_q, \Phi_q)$ is in the image of K. Now since μ_2 is a $\mathcal{D}$-morphism, it is easy to check that $U^{\mathcal{D}}\mu_2 : A'' \to U \prod_{q \in Q} B_q$ is a U,S-suitable monomorphism and hence there is a $\underline{\underline{B}}$-morphism $B_1 \to \prod_{q \in Q} B_q$ with $U(B_1 \to \prod_{q \in Q} B_q) = \mu_2$. Clearly, by 2.2.4 (2), $KB_1 = (A'', \Phi'')$. Similarly (A', Φ'), being a subobject of $\prod_{q \in Q} (A_q, \Phi_q)$, is in the image of K. Now since among the maps $\beta_p \alpha_p$ there is an X-surjection, there is a map $A' \to A$ which is an X-surjection, and being a $\mathcal{D}$-morphism from (A', Φ') to (A, Φ) it is U,S-suitable. Hence (A, Φ) is in the image of K. Suppose $KB_1 = KB_2 = (A, \Phi)$. Then since K is full there is a map $\beta : B_1 \to B_2$ such that $K\beta = 1 : (A, \Phi) \to (A, \Phi)$. Then $1 : A \to UB_2$ is a U,S-suitable monomorphism and, since $U(1 : B_2 \to B_2) = 1 : A \to UB_2 = U\beta$, it follows that $B_1 = B_2$.

We still have one point to check, namely that (A', Φ') is isomorphic to (A'',Φ''). First, to see that there is a $\mathcal{D}$-morphism $(A',\Phi') \to (A'',\Phi'')$ such that $Y_j \xrightarrow{\lambda_1} (A',\Phi') \to (A'',\Phi'') = \lambda_2$, consider the diagrams:

$$\begin{tikzcd}
Yj \arrow[r, "\lambda_2"] \arrow[drr, "\lambda_1"'] & (A'',\Phi'') \arrow[rr, "\mu_2"] & & \prod\limits_{q \in Q} (A_q,\Phi_q) \arrow[r, "\text{proj'n}"] & (A_q,\Phi_q) \\
& & (A',\Phi') \arrow[ul, dashed, "g"'] \arrow[ur, dashed, "f"] \arrow[urr, "\text{proj'n}.\mu_1"'] & &
\end{tikzcd}$$

The outer triangle commutes for each corresponding pair of projections. We can find a $\mathcal{D}$-morphism f such that the right-hand triangles commute, and so $\mu_2.\lambda_2 = f.\lambda_1$. The $\mathcal{D}$-morphism g such that the left-hand triangle commutes, is supplied by 2.1.20 (1).

Now if we could factor each α_p $(p \in P)$ through λ_2 by a $\mathcal{D}$-morphism we could find a $\mathcal{D}$-morphism $(A'',\Phi'') \to (A',\Phi')$ with $Y_j \xrightarrow{\lambda_2} (A'',\Phi'') \to (A',\Phi') = \lambda_1$. For we would have diagrams

$$\begin{tikzcd}
Yj \arrow[r, "\lambda_1"] \arrow[drr, "\lambda_2"'] & (A',\Phi') \arrow[rr, "\mu_1"] & & \prod\limits_{p \in P} (A_p,\Phi_p) \arrow[r, "\text{proj'n}"] & (A_p,\Phi_p) \\
& & (A'',\Phi'') \arrow[urr] & &
\end{tikzcd}$$

commuting for each projection, and we could use exactly the same argument as that given above. We will prove that each α_p $(p \in P)$ does factor through λ_2.

Let Fj be a free $\mathcal{D}$-algebra generated by $\eta j : Y_j \to U^{\mathcal{D}}Fj$, and let ℓ_2 and a_p be $\mathcal{D}$-morphisms such that $\ell_2.\eta j = \lambda_2$ and $a_p.\eta j = \alpha_p$. All we need is that we can factor each a_p through ℓ_2 in $\underline{\underline{A}}^{\mathcal{D}}$. By 2.2.9 (1), ℓ_2 is an $F^{\mathcal{D}}$-surjection and so, since $F^{\mathcal{D}}$ and $A^{\mathcal{D}}$ are in the elaborate situation, by 2.2.14 we need only solve

$$\begin{tikzcd} F^{\mathcal{D}}i \arrow[r, "\delta"] \arrow[d, "\delta"'] & Fj \arrow[r, "\ell_2"] & (A'',\Phi'') \arrow[d, dashed] \\ Fj \arrow[rr, "a_p"] & & (A_p,\Phi_p) \end{tikzcd}$$

for each $i \in I$ and each $\delta : F^{\mathcal{D}}i \to Fj$. Now by Lemma 2.2.13, to the map $\delta.\eta i$ there corresponds an $i_1 \in I$ and a factorization $\delta = F^{\mathcal{D}}i \xrightarrow{\delta_1} F^{\mathcal{D}}i_1 \xrightarrow{\delta_2} Fj$ such that $\delta_2.\eta i_1$ factors through ηj; say $\delta_2.\eta i_1 = \eta j.d_2$. Now by the property of extensions,

$\alpha_p.d_2 : Xi_1 \to Y_j \to A_p = Xi_1 \xrightarrow{d_2} Y_j \xrightarrow{X\text{-surj'n}} Xi_2 \to A_p$ for some $i_2 \in I$. The algebra generated by $Xi_2 \to (A_p,\Phi_p)$ is easily seen to be (A_q,Φ_q) for some $q \in Q$ and so $Y_j \to Xi_2 \to (A_p,\Phi_p) = Y_j \xrightarrow{\alpha_q} (A_q,\Phi_q) \to (A_p,\Phi_p)$. Clearly it follows from this that $a_p.\eta j.d_2$ factors through $\ell_2.\eta j.d_2$ in $\underline{\underline{A}}^{\mathcal{D}}$. Hence $a_p.\delta_2.\eta i_1$ factors through $\ell_2.\delta_2.\eta i_1$, and so $a_p\delta = a_p\delta_2\delta_1$ factors through $\ell_2\delta = \ell_2\delta_2\delta_1$.

This is sufficient, as we have seen, to supply the map $(A'',\Phi'') \to (A',\Phi')$.

It is now easy, using the fact that λ_1 generates (A',Φ') and λ_2 generates (A'',Φ'') and Theorem 2.1.12, to show that the maps

$(A'',\Phi'') \to (A',\Phi')$ and $(A',\Phi') \to (A'',\Phi'')$ are inverse to each other.

Notice from the proof that the uniqueness parts of conditions (3) and (4) are not both needed; one is superfluous. In fact we can delete both of the uniqueness requirements if we add the condition:

(6) $U\beta = 1$ (β in $\underline{\underline{B}}$) implies that $\beta = 1$ in $\underline{\underline{B}}$.

§2.4 Birkhoff's Theorem.

The theorem of G. Birkhoff we are interested in states that a set of algebras is a subvariety of some variety if and only if it is closed under subalgebras, quotient algebras and small products (see [4] Theorem IV.3.1). Now, these conditions are similar to some of the conditions of Theorem 2.3.6, and in fact 2.3.6 is a direct generalization of Birkhoff's Theorem. Our version of Birkhoff's Theorem is in 2.4.4 and 2.4.6.

2.4.1 THEOREM (ES) Consider device $\mathcal{D}$ in $\underline{\underline{A}}$ for which the maps ηi $(i \in I)$ are X-surjective. Then $U^{\mathcal{D}}$ is a full embedding.

PROOF Let (A,Φ) be a $\mathcal{D}$-algebra. Then given $\mu : Xi \to A$ there is at most one ϕ in $\underline{\underline{A}}$ (Di,A) such that $\phi.\eta i = \mu$. Hence $\Phi i = \underline{\underline{A}}(Di,A)$. Clearly if (A_1,Φ_1) is also a $\mathcal{D}$-algebra, any morphism from

A to A_1 is a $\mathcal{D}$-morphism from (A,Φ) to (A_1,Φ_1) .

2.4.2 DEFINITIONS (ES) (1) A <u>variety</u> V of $\mathcal{D}$-algebras is the set of all $\mathcal{D}$-algebras in the image of a functor $U^{\mathcal{D}_1}$, where $\mathcal{D}_1$ is a device in $\underline{\underline{A}}^{\mathcal{D}}$ whose object family is $F^{\mathcal{D}}$ and whose maps $\eta_1 i$ are $F^{\mathcal{D}}$-surjections. The set V is also called a <u>subvariety</u> of $A^{\mathcal{D}}$.

(2) A subcategory $\underline{\underline{A}}_1$ of $\underline{\underline{A}}$ is said to be closed under subobjects, quotient objects and small products if the following conditions hold.

(a) If $A \to A_1$ is a monomorphism and $A_1 \in \underline{\underline{A}}_1$ then $A \in \underline{\underline{A}}_1$.

(b) If $A_1 \to A$ is an X-surjection and $A_1 \in \underline{\underline{A}}_1$ then $A \in \underline{\underline{A}}_1$.

(c) If $(A_j\ ;\ j \in J)$ is a small family of objects of $\underline{\underline{A}}_1$, then $\prod_{j \in J} A_j$ (calculated in $\underline{\underline{A}}$) is in $\underline{\underline{A}}_1$.

2.4.3 THEOREM (ES) Let $\underline{\underline{A}}_1$ be a full subcategory of $\underline{\underline{A}}$. Then the inclusion of the subcategory $\underline{\underline{A}}_1$ is strictly isomorphic to $U^{\mathcal{D}}$ for some device $\mathcal{D}$ on X , and each map ηi $(i \in I)$ of $\mathcal{D}$ is an X-surjection, if and only if $\underline{\underline{A}}_1$ is closed under subobjects, quotient objects and small products in $\underline{\underline{A}}$.

PROOF Clearly $\underline{\underline{A}}_1$ has limits and the inclusion functor preserves them. So we can hope to use Freyd's adjoint functor theorem to obtain a family of universal arrows from X to the inclusion of $\underline{\underline{A}}_1$. Given any map $A \to A_1$ $(A \in \underline{\underline{A}}\ ,\ A_1 \in \underline{\underline{A}}_1)$, factor it as $A \xrightarrow{\text{X-surj've}} A_1' \xrightarrow{\text{monic}} A_1$.

Then $A_1' \in \underline{\underline{A}}_1$, and by BS(5) we can find a small representative set of objects A_1' formed as above from A . This set is a solution set, and so we have the desired family $S = ((\eta i : Xi \to Di , Di) ; i \in I)$ of universal arrows, with associated device $\mathcal{D}$. Factor ηi as

$Xi \xrightarrow{\text{X-surj've}} A_1 \xrightarrow{\text{monic}} Di$. Then there exists a map $Di \to A_1$ such that $Xi \to A_1 = Xi \overset{\eta i}{\to} Di \to A_1$. Clearly $Di \to A_1 \xrightarrow{\text{monic}} Di = 1$ and so $A_1 \to Di$ is an isomorphism and ηi is X-surjective.

The conditions of Theorem 2.3.6 are now easily seen to be satisfied.

The converse implication is not difficult.

2.4.4 COROLLARY (ES) A set V of $\mathcal{D}$-algebras is a subvariety of $A^{\mathcal{D}}$ if and only if it is closed under subobjects, quotient objects, and small products in $\underline{\underline{A}}^{\mathcal{D}}$.

Manes [24] has a triple-theoretic version of this theorem.

To complete our treatment of Birkhoff's Theorem we will show that varieties of $\mathcal{D}$-algebras are setsof $\mathcal{D}$-algebras defined by "laws".

2.4.5 DEFINITIONS Consider device $\mathcal{D}$ in $\underline{\underline{A}}$. A <u>$\mathcal{D}$-law</u> is a pair (α , β) of maps where α and β have common domain in X and common codomain in $(Di ; i \in I)$. If $\alpha , \beta : Xi_1 \to Di_2$ we say that (α , β) is an (i_1,i_2)-law. A $\mathcal{D}$-algebra (A,Φ) <u>satisfies</u> the (i_1,i_2)-law (α , β) if for every ϕ in Φi_2 it is true that

$\phi\alpha = \phi\beta$.

2.4.6 THEOREM (ES) A set of $\mathcal{D}$-algebras is a subvariety of $\underline{\underline{A}}^{\mathcal{D}}$ if and only if it is the set of all $\mathcal{D}$-algebras satisfying some specified set of $\mathcal{D}$-laws.

To prove this theorem we need the following concept.

2.4.7 DEFINITIONS A family $(R_j \, ; \, j \in J)$ of X-congruences, where R_j is a congruence on A_j , is called a <u>fully-invariant family</u> for $(A_j \, ; \, j \in J)$ if for each j , $j_1 \in J$, each $\mu : A_j \to A_{j_1}$, and each α , $\beta : Xi \to A_j$ it is a fact that $R_j i(\alpha , \beta)$ implies $R_{j_1} i(\mu\alpha , \mu\beta)$.

Consider a family $(R^k \, ; \, k \in K)$ of families of X-congruences for $(A_j \, ; \, j \in J)$. Then $\bigcap_{k \in K} R^k$ is the family of X-congruences, indexed by j , defined by $(\bigcap_{k \in K} R^k)_j = \bigcap_{k \in K} (R^k{}_j)$. Again if $R^1 \cap R^2 = R^1$ we write $R^1 \leq R^2$.

2.4.8 THEOREM The family of X-congruences $\bigcap_{k \in K} R^k$ is fully invariant if each of the families R^k $(k \in K)$ is fully invariant.

PROOF Clear.

2.4.9 THEOREM Let $\mathcal{D}$ be a device in $\underline{\underline{A}}$. If a family

$R_i = (R_i i_1 \; ; \; i_1 \in I)$ of relations is defined by: $R_i i_1(\alpha , \beta)$ if $\alpha , \beta : Xi_1 \to Xi$ and $\eta i.\alpha = \eta i.\beta$, then $R = (R_i \; ; \; i \in I)$ is a fully invariant family of X-congruences. Conversely, any fully invariant family of X-congruences for X is obtained in this way from a device. In fact, the device $\mathcal{D}$ may be chosen so that each ηi is an X-surjection.

PROOF Each R_i $(i \in I)$ is clearly an X-congruence. Consider $\mu : Xi \to Xi_2$ and $\alpha , \beta : Xi_1 \to Xi$. Then $\eta i_2.\mu = \delta'.\eta i$ for some $\delta' \in \Delta_1 \; (i,i_2)$, and so $\eta i.\alpha = \eta i.\beta$ implies $\eta i_2.\mu.\alpha = \eta i_2.\mu.\beta$. Thus $R_i i_1(\alpha , \beta)$ implies $R_{i_2} i_1(\mu\alpha , \mu\beta)$.

Conversely, suppose $(R_i \; ; \; i \in I)$ is any fully invariant family of X-congruences. Then by ES(4) there are X-surjections $\eta i : Xi \to Di$ such that $R_i = R_{\eta i}$. Define $\Delta(i,i_2)$ to be $\underline{\underline{A}}$ (Di , Di_2). Then $\mathcal{D} = (\eta,\Delta)$ so defined is a device. For, consider $\mu : Xi \to Di_2$. Since ηi_2 is an X-surjection, there is a map $\nu : Xi \to Xi_2$ such that $\eta i_2.\nu = \mu$. Now if $\alpha , \beta : Xi_1 \to Xi$ then $R_i i_1(\alpha , \beta)$ implies $R_{i_2} i_1(\nu\alpha , \nu\beta)$, and hence $\eta i.\alpha = \eta i.\beta$ implies that $\mu\alpha = \mu\beta$. By 2.2.6 (1) there is a map $\delta : Di \to Di_2$ such that $\delta.\eta i = \mu$.

2.4.10 THEOREM Consider device $\mathcal{D}$ in $\underline{\underline{A}}$, and set Λ of $\mathcal{D}$-laws. There exists a fully invariant family R of $F^{\mathcal{D}}$-congruences for $F^{\mathcal{D}}$ such that $\mathcal{D}$-algebra (A,Φ) satisfies the laws in Λ if and only if for all i,i_1 in I and for all pairs of $\mathcal{D}$-morphisms $\delta_1,\delta_2 : F^{\mathcal{D}}i_1 \to F^{\mathcal{D}}i$

it is true that $R_i i_1(\delta_1,\delta_2)$ implies that $\phi\delta_1 = \phi\delta_2$ for all ϕ in Φi . Further every fully invariant family of $F^{\mathcal{D}}$-congruences stands in this relationship to some set of $\mathcal{D}$-laws.

PROOF Take R to be the intersection of the fully invariant families R' which have the following property: if $(\delta_1.\eta i_1,\delta_2.\eta i_1)$ is (i_1,i)-law in Λ and $\delta_1,\delta_2 \in \Delta(i_1,i)$, then $R'_i i_1(\delta_1,\delta_2)$. Suppose (A,Φ) satisfies the laws in Λ and consider the relation $R''_i i_1$ on $\underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i_1 , F^{\mathcal{D}}i)$ defined by: $R''_i i_1(\delta_1,\delta_2)$ if $\phi\delta_1 = \phi\delta_2$ for all $\phi \in \Phi i$. Then it is not hard to see that $R'' = (R''_i ; i \in I)$ is a fully invariant family of $F^{\mathcal{D}}$-congruences for $F^{\mathcal{D}}$. Further $R'' \geq R$ since R'' is one of the families of which R is the intersection. Hence $R_i i_1(\delta_1,\delta_2)$ implies that $\phi\delta_1 = \phi\delta_2$ for all ϕ in Φi . The other implication in the first part is simple.

If R is a fully invariant family of $F^{\mathcal{D}}$-congruences for $F^{\mathcal{D}}$, take the (i_1,i)-laws in Λ to be the pairs $(\delta_1.\eta i_1,\delta_2.\eta i_1)$ where $R_i i_1(\delta_1,\delta_2)$. The proof of the second part of the theorem is then clear.

2.4.11 PROOF OF 2.4.6 In view of 2.4.9 and 2.4.10 we need only prove the following statement. If $\mathcal{D}_1$ is a device in $\underline{\underline{A}}^{\mathcal{D}}$ which has object family $F^{\mathcal{D}}$ and whose maps $\eta_1 i$ are $F^{\mathcal{D}}$-surjections, and if R is the fully invariant family of $F^{\mathcal{D}}$-congruences associated with $\mathcal{D}_1$, then the $\mathcal{D}$-algebras in the image of $U^{\mathcal{D}_1}$ are precisely those algebras (A,Φ) for which $R_i i_1(\delta_1,\delta_2)$ implies $\phi\delta_1 = \phi\delta_2$ for all ϕ in Φi .

Consider an algebra (A,Φ) in the image of $U^{\mathcal{D}_1}$. Let $R_i i_1(\delta_1,\delta_2)$; that is $\eta_1 i.\delta_1 = \eta_1 i.\delta_2$. Now any $\phi : F^{\mathcal{D}} i \to (A,\Phi)$ factors through $\eta_1 i$; say $\phi = \phi'.\eta_1 i$. Hence $\phi.\delta_1 = \phi'.\eta_1 i.\delta_1 = \phi'.\eta_1 i.\delta_2 = \phi.\delta_2$.

Consider an algebra (A,Φ) for which $R_i i_1(\delta_1,\delta_2)$ implies $\phi\delta_1 = \phi\delta_2$ for all ϕ in Φi . That is $\eta_1 i.\delta_1 = \eta_1 i.\delta_2$ implies $\phi\delta_1 = \phi\delta_2$. By 2.2.6 (1) there is a map ϕ' such that $\phi'.\eta_1 i = \phi$. This is enough to ensure that (A,Φ) is in the image of $U^{\mathcal{D}_1}$.

CHAPTER III

THE CATEGORY OF DEVICES

The category of devices $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$ was defined in 1.5.1, as was the corresponding forgetful functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$. In this chapter we prove that under certain conditions the functor U is strictly isomorphic to $U^{\mathcal{D}}$ for some device $\mathcal{D}$ in $\underline{\underline{A}}^I$. Thus devices themselves may be regarded as algebras. In $\underline{\underline{\mathrm{Sets}}}$ the free devices are devices associated with the anarchic varieties; that is, the varieties of algebras satisfying no non-trivial laws. As we sketched in §0.8, the classical description of variety begins with a description of the anarchic varieties, and then laws are introduced. Using free devices we are able to mimic this procedure, and hence to show that in $\underline{\underline{\mathrm{Sets}}}$ the categories $\underline{\underline{\mathrm{Sets}}}^{\mathcal{D}}$ correspond to conventional varieties. We also discuss devices and their algebras over $\underline{\underline{\mathrm{Sets}}}^I$, and using this information we are able to show that the varieties of devices in $\underline{\underline{\mathrm{Sets}}}$ are the varieties of clones defined by Philip Hall.

§3.1 Devices as algebras.

The first thing we need is an adjoint to the functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$, and the first step to providing this is the following theorem.

3.1.1 THEOREM The functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ creates limits.

PROOF Consider a diagram $G : \underline{\underline{C}} \to \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$. Let $\lambda : D \to UG$ be a limit of UG in $\underline{\underline{A}}^I$. If E_i is the evaluation (at i) functor which takes any functor $L : I \to \underline{\underline{A}}$ to Li, then $\lambda_i : Di \to E_i\, UG$ is a limit in $\underline{\underline{A}}$ of $E_i UG$. If C is an object of $\underline{\underline{C}}$ write the image of C under G as $\mathcal{D}_C = (\eta_C, \Delta_C)$. We wish to make the objects Di into a device. Define ηi by $\lambda_i C.\eta i = \eta_C i$ $(C \in \underline{\underline{C}})$ and define $\Delta(i,i_1)$ by $\delta \in \Delta(i,i_1)$ if and only if there exist maps $\delta_C \in \Delta_C(i,i_1)$ for all $C \in \underline{\underline{C}}$ such that $\lambda_{i_1} C.\delta = \delta_C.\lambda_i C$. It is clear that if $\mathcal{D}$ so defined is a device, then it is the only device making λ into a device morphism. It is easy to check that, by the properties of limits, ηi is well defined, and further that $\Delta(i_1,i_2).\Delta(i,i_1) \subseteq \Delta(i,i_2)$ for all $i,i_1,i_2 \in I$. Given $\nu : Xi \to Di_1$, then for each C in $\underline{\underline{C}}$ we can solve for $\delta_C \in \Delta_C(i,i_1)$ in $\lambda_{i_1} C.\nu = \delta_C.\eta_C i$. Now if μ is a device morphism from $\mathcal{D}_C$ to $\mathcal{D}_{C'}$, which is the image under G of a map in $\underline{\underline{C}}$, then I claim that $\delta_{C'}.\mu i = \mu i_1.\delta_C$. Certainly there is a $\delta'_{C'}$ in $\Delta_{C'}(i,i_1)$ such that $\delta'_{C'}.\mu i = \mu i_1.\delta_C$, and this map has the property that:

$$\delta'_{C'}.\eta_{C'} i = \delta'_{C'}.\mu i.\eta_C i = \mu i_1.\delta_C.\eta_C i$$
$$= \mu i_1.\lambda_{i_1} C.\nu = \lambda_{i_1} C'.\nu$$

and hence $\delta'_{C'} = \delta_{C'}$. From this property of the maps δ_C $(C \in \underline{\underline{C}})$ we immediately derive that $\mu i_1 . \delta_C . \lambda_i C = \delta_{C'} . \lambda_i C'$. Hence there exists a unique $\delta \in \Delta(i, i_1)$ such that $\lambda_{i_1} C \circ \delta = \delta_C \circ \lambda_i C$ since λ_{i_1} is a limit and $(\delta_C . \lambda_i C;\ C \in \underline{\underline{C}})$ is a natural transformation from Di to E_{i_1} UG. This δ has the property that $\lambda_{i_1} C \circ \delta \circ \eta i = \delta_C . \lambda_i C \circ \eta i = \delta_C \circ \eta_C i = \lambda_{i_1} C . \nu$ for all C in $\underline{\underline{C}}$ and hence $\delta \circ \eta i = \nu$. The uniqueness of δ in $\Delta(i, i_1)$ with this property is easy. The definition of Δ immediately gives us that $\lambda C = (\lambda_i C;\ i \in I)$ is a device morphism from $\mathcal{D}$ to $\mathcal{D}_C$, and it is further clear that $\lambda = (\lambda C;\ C \in \underline{\underline{C}})$ is natural from $\mathcal{D}$ to G. We would like to show that λ is a limit for G. Consider a natural transformation $\lambda' : \mathcal{D}' \to G$. Then there is at most one device morphism $\mathcal{E} : \mathcal{D}' \to \mathcal{D}$ with $\lambda C . \mathcal{E} = \lambda' C$ for all $C \in \underline{\underline{C}}$, since if we apply U to this equation we get $\lambda C . \mathcal{E} = \lambda' C$ for all C in $\underline{\underline{C}}$, and $\mathcal{E}$ is determined by the universal property of λ. If we take $\mathcal{E}$ as determined by this equation $\lambda . \mathcal{E} = \lambda'$, then we need only check that $\mathcal{E}$ is a device morphism to see that λ is a limit. First $\lambda_i C . \mathcal{E} i \circ \eta' i = \lambda'_i C \circ \eta' i = \eta_C i = \lambda_i C \circ \eta i$ for all C in $\underline{\underline{C}}$, and hence $\mathcal{E} i \circ \eta' i = \eta i$. Secondly, consider $\delta' \in \Delta'(i, i_1)$. Then $\lambda_{i_1} C \circ \mathcal{E} i_1 \circ \delta' = \lambda'_{i_1} C \circ \delta' = \delta_C . \lambda'_i C$ for a uniquely determined δ_C

in $\Delta_C(i,i_1)$. Again it is easy to check that $(\delta C \circ \lambda_i C;\ C \in \underline{\underline{C}})$ is natural from Di to $E_{i_1}UG$, and so there exists a $\delta \in \Delta(i,i_1)$ with $\lambda_{i_1}C.\delta = \delta_C \circ \lambda_i C$. Then $\lambda_{i_1}C \circ \mathcal{E}i_1.\delta' = \delta_C.\lambda'_i C = \delta_C \circ \lambda_i C.\mathcal{E}i = \lambda_{i_1}C.\delta.\mathcal{E}i$ and hence we have our result.

3.1.2 THEOREM (BS) The functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}}X) \to \underline{\underline{A}}^I$ has a left adjoint.

PROOF Only the solution set condition needs to be checked before we can apply Freyd's theorem to obtain our result. Consider a morphism $\alpha : A \to D$ in $\underline{\underline{A}}^I$ where $D = U\mathcal{D}$. For each ordinal n we define a map $\alpha_n : A_n \to D$ as follows. The map α_0 is the union of α and η. At limit ordinal n, the map α_n is the union of all preceding α_m. The map $\alpha_{n+1}i$ is the union of $\alpha_n i$ and all maps $\delta.\alpha_n i_1$ where $\delta \in \Delta(i_1,i)$ and $\delta \circ \eta i_1$ factors through $\alpha_n i$. If τ is the same cardinal as that mentioned in 2.1.13, then we rewrite $\alpha_\tau i$ as $\lambda i : D_1 i \to Di$. It is clear that ηi factors through $\alpha_n i$ and hence through λi; say $\eta i = \lambda i.\eta_1 i$. We would like to make D_1 into a device and we do so by defining $\Delta_1(i,i_1)$ to be all those maps $\delta_1 : D_1 i \to D_1 i_1$ such that $\lambda i_1.\delta_1 = \delta.\lambda i$ for some

$\delta \in \Delta(i,i_1)$. We will prove that $\mathcal{D}_1 = (\eta_1,\Delta_1)$ is a device. Clearly $\Delta_1(i_2,i_3).\Delta_1(i_1,i_2) \subseteq \Delta_1(i_1,i_3)$ for all $i_1,i_2,i_3 \in I$. Consider any map $\mu : Xi \to D_1 i_1$. Since BS(4) holds and τ is a limit ordinal, μ factors as $Xi \to \bigcup_{k\in K} (A_k i_1,\alpha_k i_1) \to D_1 i_1$ where $|K| < \kappa$. Let m be the set-theoretical union of K. Clearly $m < \tau$. If $m \in K$ then it is easy to see that $\bigcup_{k\in K} (A_k i_1,\alpha_k i_1) = A_m i_1$ since all α_n $(n < m)$ factor through α_m. If m is not in K then clearly it is a limit ordinal and $\bigcup_{k\in K} (A_k,\alpha_k) = \bigcup_{n<m} (A_n,\alpha_n) = A_m$. In both cases $\lambda i_1.\mu$ factors through $\alpha_m i_1$. Let $\lambda i_1.\mu = \delta.\eta i$ where $\delta \in \Delta(i,i_1)$. Then if n is any ordinal with $m \leq n < \tau$, from the definition of α_{n+1} we have that $\delta.\alpha_n i$ factors through $\alpha_{n+1} i_1$ and hence through $\alpha_\tau i_1 = \lambda i_1$. Further, using Lemma 2.1.9, we see that $\delta. \bigcup_{m\leqslant n<\tau} (\alpha_n i) = \delta.\lambda i$ factors through λi_1; say $\delta.\lambda i = \lambda i_1.\delta_1$. Clearly $\delta_1 \in \Delta_1(i,i_1)$ and $\lambda i_1.\delta_1.\eta_1 i = \lambda i_1.\mu$. Since λi_1 is monic, $\delta_1.\eta_1 i = \mu$. Further it is clear from the definition of Δ_1 that $\lambda : \mathcal{D}_1 \to \mathcal{D}$ is a device morphism.

Now corresponding to $A \in \underline{\underline{A}}^I$ the various devices $\mathcal{D}_1$ obtained as above for different maps α with domain A make up a solution set, but not necessarily a small one. However, from our construction and BS(5) it is not hard to see that there is a small representative set of such devices $\mathcal{D}_1$ and this representative set is a satisfactory solution set, enabling us to use Freyd's theorem to obtain our result.

Now we have two theorems asserting that the functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ arises from a device; one in the basic situation and one in the elaborate situation. The first is

3.1.3 THEOREM (BS) The functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ is tripleable if $\underline{\underline{A}}$ has finite coproducts.

PROOF We apply our recognition theorem 2.3.4 to Y and $\underline{\underline{A}}^I$, where the family Y of objects in $\underline{\underline{A}}^I$ is the set of all objects in $\underline{\underline{A}}^I$. Then Y-surjections are split epimorphisms. Let F be a left adjoint to U, and consider a map $\lambda : U\mathcal{D}_1 \to U\mathcal{D}_2$ in $\underline{\underline{A}}^I$ which has the property that for all device morphisms $FA \to \mathcal{D}_1$ the composite $UFA \to U\mathcal{D}_1 \xrightarrow{\lambda} U\mathcal{D}_2$ is in the image of U. We can find a device morphism $\nu : FU\mathcal{D}_1 \to \mathcal{D}_1$ with a right inverse μ in $\underline{\underline{A}}^I$, namely that corresponding to the identity $U\mathcal{D}_1 \to U\mathcal{D}_1$ under the adjunction. Write $FU\mathcal{D}_1 = \mathcal{D}$. To see that λ is a device morphism consider the diagram:

$$\begin{tikzcd}
Di \arrow[r, bend left, "\nu i"] \arrow[d, dashed, "\delta''"'] & D_1 i \arrow[l, bend left, "\mu i"] \arrow[r, "\lambda i"] \arrow[d, "\delta"'] & D_2 i \arrow[d, dashed, "\delta'"'] \\
Di_1 \arrow[r, bend left, "\nu i_1"] & D_1 i_1 \arrow[l, bend left, "\mu i_1"] \arrow[r, "\lambda i_1"] & D_2 i_1
\end{tikzcd}$$

If δ is in $\Delta_1(i,i_1)$ we need a δ' in $\Delta_2(i,i_1)$ such that the right-hand square commutes. Choose δ'' in $\Delta(i,i_1)$ such that $\delta''\circ\eta i = \mu i_1.\delta.\nu i\circ\eta i = \mu i_1\circ\delta\circ\eta_1 i$. Then we have that $\nu i_1.\delta'' = \delta_1.\nu i$ for some $\delta_1 \in \Delta_1(i,i_1)$, and so $\delta_1\circ\eta_1 i = \delta_1\circ\nu i.\eta i = \nu i_1.\delta''.\eta i = \nu i_1.\mu i_1.\delta.\eta_1 i = \delta.\eta_1 i$. Hence $\delta_1 = \delta$ and $\nu i_1.\delta'' = \delta.\nu i$. Now, since $\lambda\nu$ is a device morphism, we have that $\lambda i_1.\nu i_1.\delta'' = \delta'\circ\lambda i.\nu i$ for some $\delta' \in \Delta_2(i,i_1)$, and hence

$$\lambda i_1.\delta = \lambda i_1.\delta.\nu i.\mu i = \lambda i_1.\nu i_1.\delta''.\mu i = \delta'.\lambda i.\nu i.\mu i = \delta'.\lambda i.$$

Further $\eta_2 i = \lambda i.\nu i.\eta i = \lambda i.\eta_1 i$ and hence $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ is a morphism of devices, and so $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$ is saturated with respect to $(FA;\ A \in \underline{\underline{A}})$ and U.

All that remains to be checked is condition (3) of 2.3.4. Consider a U,S-suitable map $\lambda : U\mathcal{D}_1 \to D_2$ in $\underline{\underline{A}}^I$ (where S is the set of universal arrows associated with the adjunction) and suppose λ has a right inverse λ'. Then we would like to show that $D_2 = U\mathcal{D}_2$ for some device $\mathcal{D}_2$, and $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ is a device morphism. Define η_2 by $\eta_2 i = \lambda i.\eta_1 i$ and Δ_2 by $\Delta_2(i,i_1) = \lambda i_1.\Delta_1(i,i_1).\lambda' i$ $(i,i_1 \in I)$. The closure property for devices clearly holds for $\mathcal{D}_2 = (\eta_2,\Delta_2)$ in view of the following fact: if $\delta \in \Delta_1(i,i_1)$ then $\lambda i_1.\delta.\lambda' i.\lambda i = \lambda i_1.\delta$. To see this consider the following diagram.

$$\begin{tikzcd}
 & D_1 i \arrow[dl, "\zeta i"'] \arrow[dr, shift right, "1"'] \arrow[dr, shift left, "\lambda' i.\lambda i"] & & \\
Di \arrow[rr, shift left, "\xi_1 i"] \arrow[rr, shift right, "\xi_2 i"'] \arrow[d, "\delta'"'] & & D_1 i \arrow[r, "\lambda i"] \arrow[d, "\delta"] & D_2 i \arrow[d, "\lambda i_1.\delta.\lambda' i"] \\
Di_1 \arrow[rr, "\xi_1 i_1"] & & D_1 i_1 \arrow[r, "\lambda i_1"] & D_2 i_1 \\
 & D_1 i_1 \arrow[ul, "\zeta i_1"] \arrow[ur, "1"'] & &
\end{tikzcd}$$

The device $\mathcal{D}$ is $FU\mathcal{D}_1$ and ζ is the canonical map $U\mathcal{D}_1 \to UFU\mathcal{D}_1$. The map ξ_1 is the device morphism with $\xi_1.\zeta = 1$, while ξ_2 is the device morphism with $\xi_2 i.\zeta i = \lambda' i.\lambda i$ (all $i \in I$). Define δ' in $\Delta(i,i_1)$ by $\delta'.\eta i = \zeta i_1.\delta.\eta_1 i$, and define δ_1, δ_2 in $\Delta_1(i,i_1)$ by $\xi_1 i_1.\delta' = \delta_1.\xi_1 i$ and $\xi_2 i_1.\delta' = \delta_2.\xi_2 i$. Then $\delta.\eta_1 i = \xi_1 i_1.\delta'.\eta i = \delta_1.\xi_1 i.\eta i = \delta_1.\eta_1 i$ and so $\delta_1 = \delta$. Further, since $\lambda i.\lambda' i.\lambda i = \lambda i.1$ we have $\lambda i.\xi_1 i = \lambda i.\xi_2 i$ and $\lambda i_1.\delta.\xi_1 i = \lambda i_1.\xi_1 i_1.\delta' = \lambda i_1.\xi_2 i_1.\delta' = \lambda i_1.\delta_2.\xi_2 i$. Hence

$$\lambda i_1.\delta = \lambda i_1.\delta.\xi_1 i.\zeta i = \lambda i_1.\delta_2.\xi_2 i.\zeta i = \lambda i_1.\delta_2 \lambda' i.\lambda i.$$

So $\lambda i_1.\delta.\xi_1 i = \lambda i_1.\delta_2.\lambda' i.\lambda i.\xi_1 i = \lambda i_1.\delta_2.\lambda' i.\lambda i.\xi_2 i = \lambda i_1.\delta.\xi_2 i$.

Finally $\lambda i_1.\delta = \lambda i_1 \circ \delta.\xi_1 i.\zeta i = \lambda i_1.\delta.\xi_2 i.\zeta i = \lambda i_1.\delta \circ \lambda' i.\lambda i$ as we required.

We can now continue to check that $\mathcal{D}_2$ is a device. Consider $\mu : Xi \to D_2 i_1$. Then $\lambda' i_1.\mu = \delta.\eta_1 i$ for some $\delta \in \Delta_1(i,i_1)$ and hence $\lambda i_1.\delta.\lambda' i.\eta_2 i = \lambda i_1.\delta.\lambda' i.\lambda i.\eta_1 i = \lambda i_1.\delta \circ \eta_1 i = \mu$. To check the uniqueness part of the universality condition, suppose $\lambda i_1 \circ \delta_1.\lambda' i.\eta_2 i = \lambda i_1.\delta_2.\lambda' i.\eta_2 i$ where $\delta_1, \delta_2 \in \Delta_1(i,i_1)$. Then $\lambda i_1.\delta_1.\eta_1 i = \lambda i_1.\delta_2.\eta_1 i$. Consider the following diagram when $i \neq i_1$:

$$
\begin{tikzcd}
 & D_1 i \arrow[dl, "\zeta i"'] \arrow[dr, "1"] & & \\
Di \arrow[rr, "\xi_k i"] \arrow[d, "\delta"'] & & D_1 i \arrow[r, "\lambda i"] \arrow[d, "\delta_k"] & D_2 i \\
Di_1 \arrow[rr, "\xi_k i_1"] & & D_1 i_1 \arrow[r, "\lambda i_1"] & D_2 i \\
 & Xi \arrow[ul, "\zeta i_1"] \arrow[ur, "\delta_k . \eta_1 i"'] & &
\end{tikzcd}
$$

The device $\mathcal{D}$ is FA where $Ai_1 = Xi$ and $Ai_2 = D_1 i_2$ for all other i_2 in I. Again ζ is the canonical map $\zeta : A \to UFA$. If we define $\alpha_1, \alpha_2 : A \to D_1$ by $\alpha_k i_1 = \delta_k.\eta_1 i$ $(k = 1,2)$ and $\alpha_k i_2 = 1$ for any other i_2 in I, then ξ_1, ξ_2 are the device morphisms defined by $\xi_k.\zeta = \alpha_k$ $(k = 1,2)$. Then since $\lambda.\xi_1.\zeta = \lambda \circ \xi_2 \circ \zeta$ it follows that $\lambda.\xi_1 = \lambda.\xi_2$. Define δ in $\Delta(i,i_1)$ by $\delta.\eta i = \zeta i$, and define

δ'_1, δ'_2 in $\Delta_1(i_1, i_2)$ by $\xi_k i_1 . \delta = \delta'_k . \xi_k i$ $(k = 1,2)$.
Then $\delta'_k . \eta_1 i = \delta'_k . \xi_k i . \eta i = \xi_k i_1 . \delta . \eta i = \xi_k i_1 . \zeta i = \delta_k . \eta_1 i$ and hence $\delta'_k = \delta_k$ $(k = 1,2)$. Hence $\lambda i_1 . \delta_1 =$ $\lambda i_1 . \delta_1 . \xi_1 i . \zeta i = \lambda i_1 . \xi_1 i_1 . \delta . \zeta i = \lambda i_1 . \xi_2 i_1 . \delta . \zeta i = \lambda i_1 . \delta_2$, and so $\lambda i_1 . \delta_1 . \lambda' i = \lambda i_1 . \delta_2 . \lambda' i$. When $i = i_1$ we need a different proof. Let $r_1 : D_1 i \to Z$, $r_2 : Xi \to Z$ be the canonical maps of the coproduct of $D_1 i$ and Xi. Then there are maps $q_1 i, q_2 i : Z \to D_1 i$ with the following properties:
$q_1 i . r_1 = 1$, $q_1 i . r_2 = \delta_1 . \eta_1 i$, $q_2 i . r_1 = 1$, $q_2 i . r_2 = \delta_2 . \eta_1 i$.
Consider the diagram:

$$\begin{tikzcd}
Di \arrow[rr, "\xi_k i"] \arrow[d, "\delta"'] & & D_1 i \arrow[d, shift right, "\delta_1"'] \arrow[d, shift left, "\delta_2"] & \\
Di \arrow[rr, "\xi_k i"] & & D_1 i \arrow[r, "\lambda i"] & D_2 i \\
& Z \arrow[ul, "\zeta i"] \arrow[ur, "q_k i"'] & &
\end{tikzcd}$$

The device $\mathcal{D}$ is FA where $Ai = Z$ and $Ai_2 = D_1 i_2$ for all other i_2 in I. Again ζ is the canonical map $A \to UFA$. If q_k $(k = 1,2)$ are defined as above at i, and elsewhere as 1, then ξ_k $(k = 1,2)$ are the device morphisms determined by $\xi_k . \zeta = q_k$. The map δ in $\Delta(i,i)$ satisfies $\delta . \eta i = \zeta i . r_2$.
Define δ'_1, δ'_2 in $\Delta_1(i,i)$ by $\delta'_k . \xi_k i = \xi_k i . \delta$ $(k = 1,2)$.

Then $\delta'_k \circ \eta_1 i = \delta'_k . \xi_k i \circ \eta i = \xi_k i . \delta . \eta i = \xi_k i . \zeta i . r_2 = q_k i . r_2$ $= \delta_k . \eta_1 i$. Hence $\delta_k = \delta'_k$ $(k = 1,2)$. Since $\lambda i \circ \xi_1 i . \zeta i = \lambda i . \xi_2 i . \zeta i$ we must have $\lambda i . \xi_1 i = \lambda i \circ \xi_2 i$ and hence $\lambda i \circ \delta_1 = \lambda i \circ \delta_1 . \xi_1 i . \zeta i \circ r_1 = \lambda i . \xi_1 i \circ \delta . \zeta i \circ r_1 =$ $\lambda i \circ \xi_2 i \circ \delta \circ \zeta i \circ r_1 = \lambda i . \delta_2 . \xi_2 i . \zeta i . r_1 = \lambda i . \delta_2$, as required.

We have shown that $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ is a device morphism which under U becomes $\lambda : U\mathcal{D}_1 \to D_2$. The uniqueness of the device $\mathcal{D}_2$ with this property is fairly clear. The maps $\lambda i_1 . \delta . \lambda' i$ with δ in $\Delta_1(i,i_1)$ must be in $\Delta_2(i,i_1)$ and we have shown that they suffice.

Next we give a theorem giving the more precise information available in the elaborate situation.

3.1.4 THEOREM (ES) Let J be all functions from I to I and let $Z = (Zj;\ j \in J)$ be the family of objects of $\underline{\underline{A}}^I$ with $Zj(i) = X(ji)$. Suppose $\underline{\underline{A}}$ contains an initial object 0 which is in X and to each pair Xi, Xi_1 of objects in X there is a model of their coproduct in X. Then $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ is strictly isomorphic to $U^{\mathcal{D}}$ for some device $\mathcal{D}$ with family of objects Z.

PROOF We intend to use recognition theorem 2.3.6. As we remarked in 2.2.24, Z and $\underline{\underline{A}}^I$ are in the elaborate situation. The functor $U : \underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X) \to \underline{\underline{A}}^I$ has a left adjoint and hence there exists a family $S = ((\eta j : Zj \to UFj, Fj);\ j \in J)$ of universal arrows from Z to U. To see that $\underline{\underline{\mathrm{Dev}}}(\underline{\underline{A}},X)$ is saturated with respect to $(Fj;\ j \in J)$ and U, consider a morphism $\lambda : U\mathcal{D}_1 \to U\mathcal{D}_2$ in $\underline{\underline{A}}^I$, and suppose that if $\xi : Fj \to \mathcal{D}_1$ is a device morphism then $\lambda\xi$ is a device morphism. We want to show that for any δ_1 in $\Delta_1(i,i_1)$ there exists a δ_2 in $\Delta_2(i,i_1)$ such that $\lambda i_1 . \delta_1 = \delta_2 . \lambda i$. Assume first that $i \neq i_1$. We will show that δ_2 defined by $\delta_2 \circ \eta_2 i = \lambda i_1 . \delta_1 . \eta_1 i$ has this property. Consider the diagram:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "\zeta i"'] \arrow[dr, "\tau_1"] & & \\
Di \arrow[rr, "\xi i"] \arrow[d, dashed, "\delta"'] & & D_1 i \arrow[r, "\lambda i"] \arrow[d, "\delta_1"'] & D_2 i \arrow[d, dashed, "\delta_2"] \\
Di_1 \arrow[rr, "\xi i_1"] & & D_1 i_1 \arrow[r, "\lambda i_1"] & D_2 i_1 \\
 & Xi \arrow[ul, "\zeta i_1"] \arrow[ur, "\delta_1 . \eta_1 i"'] & &
\end{tikzcd}$$

Since X is a generating set in $\underline{\underline{A}}$ it is sufficient to show for any map τ_1 with domain X_1 in X and codomain $D_1 i$ that $\lambda i_1 . \delta_1 \circ \tau_1 = \delta_2 . \lambda i \circ \tau_1$. Now we form the free device $\mathcal{D} = Fj$ where $Zj(i) = X_1$, $Zj(i_1) = Xi$ and $Zj(i_2) = 0$ for all other $i_2 \in I$. The device morphism ξ is defined by $\xi\zeta = \tau$ where

$\tau i = \tau_1$, $\tau i_1 = \delta_1 . \eta_1 i$ and $\tau i_2 = 0$ for other $i_2 \in I$. The map δ in $\Delta(i,i_1)$ is defined by $\delta . \eta i = \zeta i_1$. Then if $\xi i_1 . \delta = \delta' . \xi i$ we have

$$\delta' . \eta_1 i = \delta' . \xi i . \eta i = \xi i_1 . \delta . \eta i = \xi i_1 . \zeta i_1 = \delta_1 . \eta_1 i,$$

and so $\delta' = \delta_1$. Since $\lambda\xi$ is a device morphism, for some δ'' in $\Delta_2(i,i_1)$ we have $\lambda i_1 . \xi i_1 . \delta = \delta'' . \lambda i . \xi i$. But then $\lambda i_1 . \delta_1 . \eta_1 i = \lambda i_1 . \xi i_1 . \delta . \eta i = \delta'' . \lambda i . \xi i . \eta i = \delta'' . \eta_2 i$ and so $\delta'' = \delta_2$. Finally $\lambda i_1 . \delta_1 . \tau_1 = \lambda i_1 . \xi i_1 . \delta . \zeta i = \delta_2 . \lambda i . \xi i . \zeta i = \delta_2 . \lambda i . \tau_1$ as desired. If $i = i_1$ we have to vary the argument as we did in the proof of 3.1.3. Instead of taking the free device as indicated we find a map $\nu : X_3 \to D_1 i$ (with X_3 in X) through which both τ_1 and $\delta_1 . \eta_1 i$ factor. Then we form the free device on A_1 where $A_1 i = X_3$ and $A_1 i' = 0$ for other $i' \in I$. It is not hard to see that the above argument can be modified in this way to yield again the desired result.

To check condition (3) of 2.3.6 consider a Z-surjection $\lambda : U\mathcal{D}_1 \to D_2$ of the type considered in that condition where D_2 is not known to be in the image of U. To make D_2 into a device let $\Delta_2(i,i_1)$ be all the maps δ_2 such that there exists a δ_1 in $\Delta_1(i,i_1)$ with $\lambda i_1 . \delta_1 = \delta_2 . \lambda i$, and put $\eta_2 = \lambda\eta_1$. First we check that to each δ_1 in $\Delta(i,i_1)$ there is such a δ_2.

By 2.2.6(1) this will follow if to each pair of maps t,t' with common domain X' in X and with codomain $D_1 i$ we have that $\lambda i \circ t = \lambda i \circ t'$ implies that $\lambda i_1 \circ \delta_1 \circ t = \lambda i_1 \circ \delta_1 \circ t'$. Now assume $i \neq i_1$ and $\lambda i \circ t = \lambda i \circ t'$, and consider $\mathcal{D}' = FA'$ where $A'i = X'$, $A'i_1 = Xi$ and A' is trivial elsewhere. We define device morphisms ξ_1, ξ_2 by $\xi_k \cdot \zeta' = t_k$ $(k = 1,2)$ where $t_1 i = t$, $t_1 i_1 = \delta_1 \cdot \eta_1 i$, $t_2 i = t'$, $t_2 i_1 = \delta_1 \cdot \eta_1 i$ and both t_1 and t_2 are trivial elsewhere. Then again if we define δ in $\Delta(i,i_1)$ by $\delta \circ \eta i = \zeta' i_1$ it follows that $\xi_k i_1 \circ \delta = \delta_1 \cdot \xi_k i$ $(k = 1,2)$. Now since $\lambda \circ \xi_1 \cdot \zeta' = \lambda \cdot \xi_2 \cdot \zeta'$ we have that $\lambda \circ \xi_1 = \lambda \cdot \xi_2$. Thus $\lambda i_1 \cdot \delta_1 \cdot t = \lambda i_1 \circ \xi_1 i_1 \cdot \delta \circ \zeta' i = \lambda i_1 \cdot \xi_2 i_1 \circ \delta \circ \zeta' i = \lambda i_1 \circ \delta_2 \circ t'$ as required. If $i \neq i_1$ we can modify this proof.

Now it is clear that $\Delta_2(i_1,i_2) \cdot \Delta_2(i,i_1) \subseteq \Delta_2(i,i_2)$. Further, given $\mu : Xi \to D_2 i_1$, we can factor μ as $\lambda i_1 \cdot \mu'$ since λi_1 is X-surjective, and we can solve $\delta_1 \cdot \eta_1 i = \mu'$. Hence since $\lambda i_1 \circ \delta_1 = \delta_2 \circ \lambda i$ for some δ_2 in $\Delta_2(i,i_1)$ we have $\delta_2 \circ \eta_2 i = \delta_2 \cdot \lambda i \circ \eta_1 i = \lambda i_1 \circ \delta_1 \circ \eta_1 i = \mu$. The uniqueness part of the universality condition requires another argument using U,S-suitability similar to those given above. Hence (η_2, Δ_2) is a device and $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ is a device morphism. It is easy to see that (η_2, Δ_2) is the unique device on D_2 making λ a device morphism.

Finally we have to check condition (4) of 2.3.6. Suppose $\lambda : D_1 \to U\mathcal{D}_2$ is a U,S-suitable monomorphism. We define Δ_1 by $\delta_1 \in \Delta_1(i,i_1)$ if $\lambda i_1.\delta_1 \subseteq \Delta_2(i,i_1).\lambda i$. We define η_1 by $\lambda i.\eta_1 i = \eta_2 i$. To see that such an η_1 exists, notice that the free device on A_*, where $A_* i = 0$ for all $i \in I$, is (η_*,Δ_*) where $\eta_* i = 1 : Xi \to Xi$ and $\Delta_*(i,i_1) = \underline{\underline{A}}(Xi,Xi_1)$. Then $0 \to D_2 i$ factors through λi for each $i \in I$ and hence, by U,S-suitability, so does $\eta_2 i$. The only difficulty in showing that $\mathcal{D}_1 = (\eta_1,\Delta_1)$ is a device lies in the universality condition. Given $\mu: Xi \to D_1 i_1$ we can solve $\delta_2.\eta_2 i = \lambda i_1.\mu$ for δ_2 in $\Delta_2(i,i_1)$. Now assume $i \neq i_1$, and to each $k \in I$ form the free device $\mathcal{D}* = \mathcal{D}*_k$ on $A*$ where $A*i = Xk$, $A*i_1 = Xi$ and $A*$ is trivial elsewhere. To each $\tau : Xk \to D_1 i$ let $\xi* = \xi*_\tau$ be the device morphism such that $\xi*\zeta* = \nu$ where $\nu i = \lambda i.\tau$ and $\nu i_1 = \lambda i_1.\mu$. Define δ in $\Delta*(i,i_1)$ by $\delta.\eta*i = \zeta*i_1$. Then if $\xi*i_1.\delta = \delta'_2.\xi*i$ we have $\delta'_2.\eta_2 i = \xi*i_1.\delta.\eta i = \xi*i_1.\zeta*i_1 = \lambda i_1.\mu = \delta_2.\eta_2 i$ and hence $\xi*i_1.\delta = \delta_2.\xi*i$. Now since ν factors through λ, by U,S-suitability so does $\xi*$; say $\xi* = \lambda\lambda*$. Then we can define a function $f_k : \underline{\underline{A}}(X_k,D_1 i) \to \underline{\underline{A}}(X_k,D_1 i_1)$ by taking $f_k\tau = \lambda*i_1.\delta.\zeta*i$. If we can apply adequacy to the functions f_k $(k \in I)$ then there exists a map $\delta_1 : D_1 i \to D_1 i_1$ such that $\delta_1\tau = \lambda*i_1.\delta.\zeta*i$ for each τ and corresponding $\lambda*$, $\zeta*$. Hence $\lambda i_1.\delta_1\tau = \xi*i_1.\delta.\zeta*i = \delta_2.\xi*i.\zeta*i = \delta_2.\lambda i.\tau$, and we can cancel τ to get $\lambda i_1.\delta_1 = \delta_2.\lambda i$. Further $\lambda i_1.\delta_1.\eta_1 i =$

$\delta_2 \circ \eta_2 i = \lambda i_1 \circ \mu$ and since λi_1 is monic we have $\delta_1 . \eta_1 i = \mu$. So we have to prove for each $\mathscr{E} : Xk_1 \to Xk$ that $(f_k \tau).\mathscr{E} = f_{k_1}(\tau \mathscr{E})$. I omit this proof, as well as the proof when $i = i_1$.

It is clear that Δ_1, as we have defined it, is the only possible way of making D_1 into a device so that λ becomes a device morphism.

3.1.5 REMARKS It is clear from 2.2.24 how to find many examples where the conditions of Theorem 3.1.4 apply. An example of a family of objects in a category satisfying the conditions of 3.1.3 but not necessarily of 3.1.4 is any small family in $P(Y)$ (see 1.1.8).

Before we leave the elaborate situation for the concrete theorems of §3.2 we note some topics of interest not developed in this thesis.

One of the things suggested by the fact that devices are algebras is the possibility of classifying devices by the laws they satisfy. In $\underline{\underline{\text{Sets}}}$ there are existing results in this direction; see for example [17].

Given any device morphism $\lambda : \mathcal{D}_1 \to \mathcal{D}_2$ there are interesting morphisms from $\mathcal{D}_1$-algebras to $\mathcal{D}_2$-algebras defined as follows: $\alpha : A_1 \to A_2$ is a λ-morphism from (A_1, Φ_1) to (A_2, Φ_2) if for all $i \in I$ we have $\alpha . \Phi_1 i \subseteq \Phi_2 i . \lambda i$. One can treat isotopy in the theory of loops [3] in terms of these types of morphisms. In this connection it is of interest that $\underline{\underline{\text{Dev}}}(\underline{\underline{A}}, X)$ can be made into a 2-category via the isomorphism $\underline{\underline{\text{Dev}}}(\underline{\underline{A}}, X) \cong (\underline{\underline{\text{Alg}}}(\underline{\underline{A}}, X))^{op}$.

3.1.6 COUNTEREXAMPLES To indicate the limitations of 3.1.3 and 3.1.4 we give two counterexamples.

(1) It is not true that the category Triples of triples over Sets is tripleable over Sets^S where S = obj Sets. In fact the functor $U : \text{Triples} \to \text{Sets}^S$ does not even have a left adjoint. For suppose F is a left adjoint to U and consider FW where $WS = 1$ (a singleton) for each set S. Consider small ordinal n. Let T_n be the triple corresponding to the category of all Ω-algebras, where Ω has one operation of each arity $m \in n$. Using the operational interpretation of device morphism (1.6.3) one can show that for each n there is a triple morphism $\lambda_n : FW \to T_n$ where each component $\lambda_n S$ $(S \in S)$ is a surjection. But $|T_n 1|$ increases unboundedly as n increases through the small ordinals, and is $FW1$ cannot be a small set.

(2) It is not hard to see that U from the category of full devices in Finite Sets to Finite Sets also does not have a left adjoint. We can prove this by showing that to any finite ordinal n there is a triple T in Finite Sets which is "generated" by a single unary operation and which has $|T1| > n$.

§3.2 Devices and algebras in Sets^n.

By Sets^n we mean the functor category where n is a small set. We first discuss the free devices in Sets^n and their algebras

To do this we need an extension due to P.J. Higgins [11] of the conventional notion of algebra. (Our presentation of the extended notion is rather different from that of Higgins.)

Proofs in this section are sometimes omitted or only sketched.

3.2.1 DEFINITIONS Consider a small set n fixed throughout the following discussion. A formal n-operation is a set ω together with an n-tuple of sets $a\omega$ which called the arity of ω and a specified element $r\omega$ in n called the target of ω. Let Ω be a small set of formal n-operations. Then an Ω-algebra $\underline{A}$ is an n-tuple of sets $A = (A_m;\ m \in n)$ and to each ω in Ω an n-operation $\omega_{\underline{A}} : \underline{\underline{\mathrm{Sets}}}^n(a\omega, A) \to A_{r\omega}$.

A homomorphism from $\underline{A}$ to $\underline{B}$ is a morphism $\lambda : A \to B$ in $\underline{\underline{\mathrm{Sets}}}^n$ such that $\lambda.\omega_{\underline{A}}(\mu) = \omega_{\underline{B}}(\lambda\mu)$ for all $\mu \in \underline{\underline{\mathrm{Sets}}}^n(a\omega, A)$. Again we name the category of Ω-algebras and homomorphisms $\underline{\underline{\Omega\text{-Alg}}}$, and it is clear that by taking n to be a singleton we get the conventional notions as a special case.

It is straightforward to extend many of the results of conventional universal algebra to this new situation. In the next section we describe the free algebras in $\underline{\underline{\Omega\text{-Alg}}}$.

3.2.2 CONSTRUCTION Consider an object Y in $\underline{\underline{\mathrm{Sets}}}^n$. Then the free algebra in $\underline{\underline{\Omega\text{-Alg}}}$ on Y may be defined as follows. For each small ordinal k define sets $D_k m$ $(m \in n)$ by the following requirements.

(1) $D_0 m = Y_m$.

(2) At limit ordinals $D_k m$ is the union of all previous $D_{k'} m$.

(3) If μ is the graph of a morphism from $a\omega$ to D_k then the formal symbol $\omega[\mu]$ belongs to $D_{k+1}m$ whenever $r\omega = m$.

For sufficiently large small ordinals k the sets $D_k m$ $(m \in n)$ become simultaneously stationary, and it is the stationary value of $D_k m$ that we name Dm. For convenience if μ is the graph of a morphism which occurs in construction of $D = (Dm;\ m \in n)$ we attach to it the codomain D to make it a morphism in $\underline{\underline{\text{Sets}}}^n$. The canonical map $\eta : Y \to D$ is just the inclusion map.

Then D is the carrier of the free algebra F generated by η if we define the operations of F by: $\omega_F(\mu) = \omega[\mu]$ $(\mu : a\omega \to D)$.

We can define the length of an element in Dm as the least ordinal k for which the element is in $D_k m$. Many of our proofs are by induction on the length of an element.

3.2.3 THEOREM The obvious forgetful functor $U : \underline{\underline{\Omega\text{-Alg}}} \to \underline{\underline{\text{Sets}}}^n$ (where Ω is a set of formal n-operations) is strictly isomorphic to $U^{\mathcal{D}} : \underline{\underline{A}}^{\mathcal{D}} \to \underline{\underline{A}}$ for some device $\mathcal{D}$ in $\underline{\underline{\text{Sets}}}^n = \underline{\underline{A}}$.

PROOF Let X be a small family of sets containing the empty set and, to each $\omega \in \Omega$, $m \in n$, a set with cardinality greater than the cardinality of $a\omega(m)$. Further, suppose X satisfies the conditions described in 2.2.24. Let $Y = (Yj; j \in I^n)$ be the family of objects in $\underline{\underline{Sets}}^n$ defined in terms of X as in 2.2.24. Let Fj be the free algebra generated by $\eta j : Yj \to Dj$ as described in 3.2.2. Then the device $\mathcal{D}$ obtained from the universal arrows $((\eta j, Fj);\ j \in I^n)$ has the property stated in the theorem. To prove this we use our recognition theorem 2.3.6. Condition (2) is an immediate consequence of the definition of homomorphism. Condition (3) follows from the fact that congruences determine quotient algebras, and condition (4) from the fact that subsets closed under the operations determine subalgebras. It is clear how to construct products.

3.2.4 THEOREM Let Y and Ω be two functors from I to $\underline{\underline{Sets}}^n$. Then a free device in $\underline{\underline{Dev}}(\underline{\underline{Sets}}^n, Y)$ on Ω may be described as follows. Make Ω into a set of formal n-operations by specifying that $a\omega = Yi$ if $\omega \in \Omega i$, and $r\omega = m$ if $\omega \in \Omega i(m)$. Then the free device $\mathcal{D}$ is the device associated with the universal arrows from Y to $U : \underline{\underline{\Omega\text{-Alg}}} \to \text{Sets}^n$. The canonical map $\zeta : \Omega \to D$ is defined by $\zeta i(m) : \omega \rightsquigarrow \omega[\eta i]$.

PROOF Let $\mathcal{D}_1$ be any device in $\underline{\underline{Dev}}(\underline{\underline{Sets}}^n, Y)$. We wish to show that for any map $\alpha : \Omega \to D_1$ there is a unique device morphism $\lambda : \mathcal{D} \to \mathcal{D}_1$ such that $U\alpha.\zeta = \alpha$. This can be accomplished by considering the operational description of device morphism given in 1.6.3. Take as a base of operations the n objects $Z = \{Zm; m \varepsilon n\}$ of $\underline{\underline{Sets}}^n$ defined by $Zm(m) = \{\emptyset\}$, $Zm(m_1) = \emptyset$ if $m_1 \neq m$. Then the elements of the sets $Di(m)$ become formal operations in the sense of §1.6. Further, if $\omega \varepsilon \Omega i(m)$ and $\mu : Yi \to Di_1$ and Fi_1 is $F^{\mathcal{D}}i_1$ then $(\omega[\eta i])_{Fi_1}(\mu) = \omega[\mu]$, again in the sense of §1.6. Now by 1.6.3 the maps λi must have the following two properties: $\lambda i(m) : y \rightsquigarrow \eta_1 i(m)(y)$ $(y \varepsilon Yi(m))$ and $\lambda i(m) : \omega[\mu] \rightsquigarrow (\alpha i_1(m)(\omega))_{F_1 i}$ $(\lambda i.\mu)$ if $\mu : Yi_1 \to D_1 i$, $\omega \varepsilon \Omega i_1(m)$ and $F_1 i = F^{\mathcal{D}_1} i$. These two conditions clearly determine a unique family $(\lambda i; i \varepsilon I)$ of maps from D to D_1, with the property that $\lambda . \zeta = \alpha$. To see that the operational conditions of 1.6.3 are satisfied by this λ requires an induction argument which we omit.

With this information about free devices in $\underline{\underline{Sets}}$ we can now discuss, using §2.4, the algebras of small devices in $\underline{\underline{Sets}}$.

3.2.5 THEOREM Let $\mathcal{D}$ be a device with a small family of objects in $\underline{\underline{Sets}}$. Then there is a conventional variety of algebras $\underline{\underline{V}}$ such that the forgetful functor $U : Cat(\underline{\underline{V}}) \to \underline{\underline{Sets}}$ is strictly isomorphic to $U^{\mathcal{D}}$. Further, the forgetful functor of any conventional variety is strictly isomorphic to $U^{\mathcal{D}}$ for some such device $\mathcal{D}$.

PROOF Consider small device $\mathcal{D}$. Make an inessential enlargement $\mathcal{D}_1$ of $\mathcal{D}$ using 2.1.16 and 1.3.2 so that X_1 satisfies the conditions for X described in 2.2.24. Then there is a device morphism $\lambda : \mathcal{D}^* \to \mathcal{D}_1$ from the free device $\mathcal{D}^*$ on $\Omega = (D_1 i_1;\ i_1 \varepsilon I_1)$, and each λi_1 $(i_1 \varepsilon I_1)$ is a surjection. Now, by 1.5.4 and 1.5.5 there is a device $\mathcal{D}_2$ in $\underline{\underline{\text{Sets}}}^{\mathcal{D}^*}$ with $\eta_2 = \lambda$ and $U^{\mathcal{D}_1}$ is strictly isomorphic to $U^{\mathcal{D}^*}.U^{\mathcal{D}_2}$. By 2.4.1 the functor $U^{\mathcal{D}_3}$ is a full embedding, and so $U^{\mathcal{D}_1}$ is strictly isomorphic to $U^{\mathcal{D}^*}|\underline{\underline{A}}_1$ where $\underline{\underline{A}}_1$ is the image of $U^{\mathcal{D}_3}$. Now $U^{\mathcal{D}^*}$ is strictly isomorphic to $U : \underline{\underline{\Omega\text{-Alg}}} \to \underline{\underline{\text{Sets}}}$ for some Ω and we have seen in 2.4.6 that such categories $\underline{\underline{A}}_1$ consist of all algebras in $\underline{\underline{A}}^{\mathcal{D}^*}$ satisfying certain "laws". These "laws" are not quite the conventional laws described in §0.8, but it is easy to see that the difference is immaterial.

In the other direction consider a variety $\underline{\underline{V}}$ defined in terms of operations Ω and laws Λ. Let I be a set including the arities of the operations in Ω, the lengths of the laws in Λ, and satisfying the conditions on X described in 2.2.24. Then Ω may be regarded as a family $\Omega = (\Omega i;\ i \varepsilon I)$ where $\omega \varepsilon \Omega i$ if $a\omega = i$. Form the free device $\mathcal{D}^*$ on Ω in $\underline{\underline{\text{Dev}}}(\underline{\underline{\text{Sets}}}, I)$. Let $\mathcal{D}_1$ be the quotient device of $\mathcal{D}^*$ determined by laws Λ. Then clearly $U^{\mathcal{D}_1}$ is strictly isomorphic to $U : \text{Cat}(\underline{\underline{V}}) \to \underline{\underline{\text{Sets}}}$.

Notice that different "presentations" of a device in terms of a free device and laws yield different interpretations of the algebras. Our theorem is inadequate in that it does not show that all presentations yield interpretations.

Almost all the results in this thesis concern small devices. However the next two results are devoted to showing that any device in $\underline{\underline{\text{Sets}}}$ is equivalent to a triple.

3.2.6 THEOREM Suppose BS(2) and BS(3) hold for X and $\underline{\underline{A}}$, and let $\mathcal{D}$ be a device with object family X. Suppose further that $I_1 \supseteq I$ and there is a family X_1 of objects in $\underline{\underline{A}}$ indexed by I_1 such that to each $i_1 \in I_1$ there is an $i \in I$ and a split epimorphism $Xi \to Xi_1$. Then there is a device $\mathcal{D}_1$ on X_1 such that $\mathcal{D}_1|I$ is equivalent to $\mathcal{D}$.

PROOF For each i_1 in I_1 define $\eta_1 i_1 : X_1 i_1 \to D_1 i_1$ as follows. Choose a pair of maps $\alpha : Xi \to X_1 i_1$ and $\beta : X_1 i_1 \to Xi$ with $\alpha\beta = 1$. If $i_1 \in I$ take α and β to be 1. Let $\eta i.\beta$ factor as $X_1 i_1 \xrightarrow{\eta_1 i_1} F^{\mathcal{D}_1} i_1 \xrightarrow{D_1\beta} F^{\mathcal{D}} i$ where $\eta i.\beta$ generates $D_1\beta : F^{\mathcal{D}_1} i_1 \to F^{\mathcal{D}} i$. When $i_1 \in I$ take $D_1\beta$ to be 1. Define Δ_1 by $\Delta_1(i_1,i_2) = \underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}} i, F^{\mathcal{D}_1} i_2).D_1\beta$. We want to show that $\mathcal{D}_1$ so defined is a device. Clearly then $\mathcal{D}_1|I = \mathcal{D}$ and we can apply Theorem 1.3.4 to obtain the desired result. The closure property is immediate. Consider $\mu : Xi_1 \to D_1 i_2$. Let ϕ be the map $\underline{\underline{A}}^{\mathcal{D}}(F^{\mathcal{D}}i, F^{\mathcal{D}_1}i_2)$ such that $\phi.\eta i = \mu\alpha$. Then $\phi.D_1\beta.\eta_1 i_1 = \phi.\eta i.\beta = \mu\alpha\beta = \mu$. The uniqueness part of the universality condition follows from 2.1.12 and the fact $\eta_1 i_1$ generates $F^{\mathcal{D}_1}i_1$.

3.2.7 THEOREM Let $\mathcal{D}$ be a device in $\underline{\underline{\text{Sets}}}$. Then $U^{\mathcal{D}}$ is tripleable.

PROOF If the cardinalities of the objects in X are bounded, then by 1.3.4 we can omit all but a small set of objects from X and still retain an equivalent device. But then by 2.1.16 the functor $U^{\mathcal{D}}$ has a left adjoint.

If the cardinalities are unbounded then to each small set $S \neq \emptyset$ there is a split epimorphism $Xi \to S$ for some $i \in I$, and hence by 3.2.6 $\mathcal{D}$ can be enlarged inessentially to include all except possibly the empty set. Further it is possible to show by a special argument that if $\emptyset \to F^{\mathcal{D}}1$ generates $(B,\Psi) \to F^{\mathcal{D}}1$ then $(\emptyset \to B,(B,\Psi))$ is a universal arrow to $U^{\mathcal{D}}$.

The final result of this thesis is a theorem identifying small devices in $\underline{\underline{\text{Sets}}}$ with the clones of Philip Hall; a result presumably known by Hall when he defined clones.

3.2.8 THEOREM Let I be any set such that I and $\underline{\underline{Sets}}$ satisfy the conditions imposed on X and $\underline{\underline{A}}$ in Theorem 3.1.4. Then $U_1 : \underline{\underline{Dev}}(\underline{\underline{Sets}},I) \to \underline{\underline{Sets}}^I$ is strictly isomorphic to $U_2 : \underline{\underline{Cl}} \to \underline{\underline{Sets}}^I$ where $\underline{\underline{Cl}}$ is the category of abstract I-clones.

PROOF By Theorem 3.1.4 the functor U_1 is strictly isomorphic to $U^{\mathcal{D}}$ where $\mathcal{D} = (\zeta,\Delta')$ is a device on $J = I^I$. The nature of $\zeta j : j \to Dj$ is described in Theorem 3.2.4. Each Dj arises from a device $\mathcal{D}j = (\eta j,\Delta j)$ with object set I. In order to interpret the algebras of $\mathcal{D}$ we will present $\mathcal{D}$ as the quotient of a free device $\mathcal{D}^* = (\eta^*,\Delta^*)$ as follows. Let $\Omega = (\Omega j(i); i \in I, j \in J)$ be defined by: (1) if $ji = \emptyset$ for all $i \in I$ then $\Omega j(i) = i$; (2) if ji is trivial except when $i = i_1,i_2$ where $i_1 \neq i_2$, $ji_1 = i_1 \neq 0$ and $ji_2 = i_1$ then $\Omega j(i_2) = \{\omega\}$ (a special symbol) and $\Omega j(i) = \emptyset$ for all $i \neq i_2$; (3) if ji is trivial except when $i = i_1$ and $ji_1 = i_1 + i_1$ (a specified model of the coproduct), then $\Omega j(i_1) = \{\omega\}$ and $\Omega j(i) = \emptyset$ if $i \neq i_1$; (4) in all other cases $\Omega j(i) = \emptyset$ for all $i \in I$. The operations in cases (2) and (3) are very similar. For this reason from now on we omit mention of case (3). It will always be easy to put in extra details to include this case. Let $\mathcal{D}^* = (\eta^*,\Delta^*)$ be the free device. With family of objects I^I in $\underline{\underline{Sets}}^I$ generated by $\zeta^* : \Omega \to D^*$. Define the map $\alpha : \Omega \to D$ by (1) if $ji = \emptyset$ for all i then

$\Omega j(i) = i$ and $\alpha j(i) = \eta j(i)$; (2) if $ji_1 = i_1$, $ji_2 = i_1 \neq \emptyset$ and elsewhere j is trivial, then $\Omega j(i_2) = \{\omega\}$ and $\alpha j(i_2) : \omega \rightsquigarrow (\zeta j(i_1)(k))[\zeta j(i_2)]$, where k is a selected element in i_1. Then consider the device morphism $\lambda : \mathcal{D}^* \to \mathcal{D}$ defined by $\lambda \circ \zeta^* = \alpha$. Using the nature of α and the fact that $\lambda \circ \eta = \zeta$ one can show by an induction argument that each $\lambda j(i)$ is surjective. Hence we may apply Theorem 2.4.6 to interpret $\mathcal{D}$-algebras as $\mathcal{D}^*$-algebras satisfying certain laws.

The laws Λ are all the pairs of elements of $D^*j(i)$ for all $i \in I$, $j \in J$ which have a common image under $\lambda j(i)$. As we discussed in §2.4, if (t_1,t_2) is a law, where $t_1,t_2 \in D^*j(i)$, and $\delta \in \Delta^*(j,j_1)$ then $(\delta i(t_1),\delta i(t_2))$ is also a law. It is clear that any $\mathcal{D}^*$-algebra satisfying law (t_1,t_2) must also satisfy $(\delta i(t_1),\delta i(t_2))$. Now consider the set of laws Λ_1 defined as follows. (1) If $\omega \in \Omega j(i_2)$, $ji_2 = i_1$, k is the selected element of i_1 described earlier, $\mu : i_1 \to i_1$ is the constant function with value k, then $(\omega[\eta^*j],\omega[\nu]) \in \Lambda_1$ where $\nu \circ i_1 = \eta^*j(i_1) \circ \mu$ and $\nu i_2 = \eta^*j(i_2)$. (2) Suppose $\omega \in \Omega j(i_2)$ and $ji_2 = i_1$. Let $j_0 \in J$ be the map with $j_0 i = \emptyset$ for all $i \in I$, and $\mu_1 : i_1 \to i_1$ be a constant map whose image is k_1. Let δ_0 be the single map in $\Delta^*(j_0,j)$. Then $(\omega[\mu],\eta^*j(i_2)(k_1)) \in \Lambda_1$ where $\mu i_1 = \delta_0 \cdot \zeta^* j_0(i_1) \cdot \mu_1$ and $\mu i_2 = \eta^*j(i_2)$. (3) Suppose ω belongs to $\Omega j_1(i_3)$, $\Omega j_2(i_3)$ and $\Omega j_3(i_2)$ where $j_1 i_3 = i_1$, $j_2 i_3 = i_2$ and $j_3 i_2 = i_1$. Let

$j' : I \to I$ be the map which has $j'i_1 = i_1$, $j'i_2 = i_1$ and $j'i_3 = i_2$ and which is trivial elsewhere. For each $s \in i_1$ let $\alpha_s : i_2 \to i_1$ be the constant map with image s. Let $\beta_s : j_2 \to D^*j'$ be the map defined by $\beta_s i_2 = \eta^*j'(i_2)\alpha_s$ and $\beta_s i_3 = \eta^*j'(i_3)$. Let $\gamma : j_1 \to D^*j'$ be the map defined by $\gamma i_3(s) = \omega[\beta_s]$, $\gamma i_1 = \eta^*j'(i_1)$. Finally let $\mu : j_2 \to D^*j'$ be the map defined by $\mu i_3 = \eta^*j'(i_3)$ and μi_2 is the constant map with image $w[\nu]$ where $\nu : j_3 \to D^*j'$ has $\nu i_1 = \eta^*j'(i_1)$ and $\nu i_2 = \eta^*j'(i_2)$. Then $(\omega[\gamma],\omega[\mu])$ belongs to Λ_1.

It is not hard to prove that $\Lambda_1 \subseteq \Lambda$. Further the set of $\mathcal{D}^*$-algebras which satisfy the laws in Λ_1 may be interpreted via Theorem 3.2.3 as the set of all abstract I-clones. Thus only one thing remains to be proved; namely that any $\mathcal{D}^*$-algebra which satisfies Λ_1 must satisfy Λ. To see this consider the set Λ_2 of laws satisfied by all $\mathcal{D}^*$-algebras which satisfy Λ_1. Consider further the family of sets $N = (Nj(i);\ i \in I,\ j \in J)$, with $Nj(i) \subseteq D^*j(i)$, defined by (1) $ji \subseteq Nj(i)$; (2) if $\mu_2 : ji_2 \to D^*j_1(i_2)$ and the image of μ is contained in $Nj_1(i_1)$, and if $\mu_1 : ji_1 \to D^*j_1(i_1)$ is a constant map which factors through $\eta^*j_1(i_1)$, and further if $\omega \in \Omega j(i_2)$ and $ji_2 = i_1$, then $\omega[\mu] \in Nj_1(i_2)$ where $\mu : j \to D^*j_1$ is defined by $\mu i_1 = \mu_1$, $\mu i_2 = \mu_2$; and (3) if $j_0 \in J$ is defined by $j_0 i = \emptyset$ for all $i \in I$, δ_0 is the single map in $\Delta^*(j_0,j)$, and $k \in I$, then $\delta_0 \circ \zeta^*j_0(i)(k)$ is in $Nj(i)$. Now the following two facts can be proved by induction: (1) to every $w \varepsilon D^*j(i)$ there is a $w' \varepsilon Nj(i)$ such that $(w,w') \varepsilon \Lambda_2$; (2) if $(w_1,w_2) \varepsilon \Lambda$ and $w_1,w_2 \varepsilon Nj(i)$, then $w_1 = w_2$. These facts are sufficient to prove that $\Lambda_2 \supseteq \Lambda$.

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