SOME REMARKS AND RESULTS

ON

RELATIONAL AND ALGEBRAIC STRUCTURES

by

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STATEMENT

Unless otherwise stated the results presented in this thesis are mine.

Ahmad Shafaat.

PREFACE

I learned the contents of this thesis as a post-graduate student in the Department of Mathematics, Institute of Advanced Studies, Australian National University. I am thankful to that University for granting me a Post-graduate Research Scholarship.

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CONTENTS

CHAPTER 0

§1 INTRODUCTION.

The concepts, remarks and results presented in this thesis are in four distinct directions in Model Theory and Universal Algebra. Since every chapter has a detailed introductory part of its own, in this Introduction we shall only make very brief remarks to point out to the directions taken up in the present work.

In Chapter 1 we deal with the following fundamental question: When should two classes of structures be regarded as 'essentially' the same? We define the concept of isotopic equivalence and make it seem plausible by an example that varieties of algebras can look fairly similar under isotopic equivalence.

In Chapter 2 we note that the form of sentences of an axiom system Σ, the categorical properties of the category $\underline{\underline{K}}(\Sigma)$ (of structures satisfying Σ and homomorphisms between them) and order-theoretic properties of certain partially ordered sets $L(\underline{\underline{V}}(\Sigma);Y)$ can have simple connections. Sometimes these connections can be close enough to provide characterizations of certain types of elementary classes of structures. Thus we characterize the following universal classes of algebras: negative assemblies, negationally defined assemblies, quasi-varieties, semi-varieties.

In Chapter 3, which is very closely related to §3 of Chapter 2, we give some properties of what were called quasi-free classes of structures by Mal'cev [17].

In Chapter 4 we continue with the application of some categorical concepts to Model Theory. We obtain a simple and general principle about categories of structures that admit free structures on all sets such that free structures on isomorphic sets are isomorphic.

Chapter 5 is concerned with constructions that obtain elementary (universal) classes (of structures) from elementary (universal) classes.

In Chapter 6 we define some languages and structures with non-finitary relations and operations for which many of the results of Model Theory (stated usually for finitary structures) can be generalized.

§2. NOTATION AND TERMINOLOGY

This work is nearly self-sufficient as far as notations are concerned —— few notations are used without explicitly mentioning the objects for which they stand. However, quite naturally, this is not so as far as terms are concerned —— we shall use some standard concepts from Universal Algebra, Logic, Model Theory, Lattice Theory, Category Theory, Set Theory etc. without explicit definitions.
Thus for example we shall not define a monomorphism or a language of the first order predicate calculus. For all such basic concepts used but not defined here we refer the reader to such standard works as [3], [4], [5], [6], [9], [20] and [24]. We remark that sometimes we shall define even some of the most standard and fundamental concepts.

This can be so, for example, when we use an unconventional notation or term.

We now set up a basic part of the notation and terminology employed in the present work. Terms and notations of more special nature will be defined as they arise.

By a predicate system we shall understand an ordered triplet $\langle P,\Omega,h\rangle$, where P, Ω are sets and h is a function from $P \cup \Omega$ into an infinite ordinal such that $h(\rho)$ is non-zero for all $\rho \in P$. For every $\zeta \in P \cup \Omega$ we refer to $h(\zeta)$ as the arity of ζ. The letter $\mathfrak{R}$ will always denote a predicate system. We write $P(\mathfrak{R})$, $\Omega(\mathfrak{R})$, $h(\mathfrak{R})$ for the first, second and third member of a predicate system $\mathfrak{R}$ respectively. A predicate system $\mathfrak{R}$ will be called restricted if the range of $h(\mathfrak{R})$ is the set of non-negative integers; otherwise $\mathfrak{R}$ is called unrestricted. Except in the last chapter we shall only deal with the restricted predicate systems without explicit mention.

We write $\Phi(\mathfrak{R})$ for the set of formulae of the first order language with:

$P(\mathfrak{R})$ as the set of relational symbols,

$\Omega(\mathfrak{R})$ as the set of operational symbols,

$[h(\mathfrak{R})](\zeta)$ as the arity of $\zeta \in P \cup \Omega$,

$X = \{x_1,\ldots,x_n,\ldots\}$ as the set of variables,

$\vee$ (or), $\wedge$ (and), $\sim$ (not), $\longrightarrow$ (implies),

$\longleftrightarrow$ (is equivalent to) as the connectives,

$\forall$ (for all), $\exists$ (there exists) as the quantifiers,

( ) (parentheses) as an auxiliary symbol ,

and

= as the symbol for equality.

In writing $\Phi(\mathfrak{R})$ we have not depicted the dependence of $\Phi(\mathfrak{R})$ on X and other sets of symbols because we shall assume these fixed.

For every $\varphi \in \Phi(\mathfrak{R})$ we write $[h(\mathfrak{R})](\varphi)$ for the number of free variables in φ .

For every non-negative integer n we write $\Phi_n(\mathfrak{R})$ for the set of those formulae in $\Phi(\mathfrak{R})$ that involve at most n free variables. Thus $\Phi_o(\mathfrak{R})$ is the set of all sentences in $\Phi(\mathfrak{R})$. We shall sometimes refer to members of $\Phi(\mathfrak{R})$ and $\Phi_o(\mathfrak{R})$ as $\mathfrak{R}$-formulae and $\mathfrak{R}$-sentences respectively.

By an $\mathfrak{R}$-structure we shall understand an ordered pair $\underline{A} = \langle A,\alpha\rangle$, where the carrier (Cf. [21]) A of $\underline{A}$ is a set (possibly empty) and the make α of $\underline{A}$ is a function from $P(\mathfrak{R}) \cup \Omega(\mathfrak{R})$ into the set of relations and operations over A such that $\alpha(\rho)$ is a relation of arity $[h(\mathfrak{R})](\rho)$ for all $\rho \in P$ and $\alpha(\omega)$ is an operation of arity $[h(\mathfrak{R})](\omega)$ for all $\omega \in \Omega$. The species of an $\mathfrak{R}$-structure is the predicate system $\mathfrak{R}$. An $\mathfrak{R}$-structure is called relational or operational according as $\Omega(\mathfrak{R}) = \emptyset$ or $P(\mathfrak{R}) = \emptyset$, where $\emptyset$ is the null set. Operational structures are also called algebras or algebraic structures. It is well known that algebras can be described as relational structures but due to their special nature and importance it is advantageous and much more convenient to include operational symbols in our languages. An important example of an $\mathfrak{R}$-algebra is the following. Let $P(\mathfrak{R}) = \emptyset$ and let Y be a set. We define $W_n(\mathfrak{R};Y)$ inductively as follows:

$$W_o(\mathfrak{R};Y) = Y ,$$

$$W_{n+1}(\mathfrak{R};Y) = \{w_1 \dots w_k \omega ;\ w_1,\dots,w_k \in W_n(\mathfrak{R};Y), \omega \in \Omega,\ k = [h(\mathfrak{R})](\omega)\} \cup \bigcup_{m=o}^{n} W_n(\mathfrak{R};Y) .$$

Let $W(\mathfrak{R};Y) = \bigcup_{n=o}^{\infty} W_n(\mathfrak{R};Y)$. We now define $\underline{W}(\mathfrak{R};Y)$ to be the $\mathfrak{R}$-algebra whose carrier is $W(\mathfrak{R};Y)$ and whose make α is defined by:

$$\langle w_1,\dots,w_k\rangle\alpha(\omega) = w_1\dots w_k\omega , \quad k = [h(\mathfrak{R})](\omega) , \ \omega \in \Omega .$$

The members of $W(\mathfrak{R};Y)$ are called $\mathfrak{R}$-<u>words</u> in Y, and the algebra $\underline{W}(\mathfrak{R};Y)$ is called the $\mathfrak{R}$-<u>word algebra</u> on Y. We observe that if w_1 , w_2 are $\mathfrak{R}$-words in X then $w_1 = w_2$ is an $\mathfrak{R}$-formula.

An $\mathfrak{R}$-structure $\underline{A} = \langle A,\alpha\rangle$ is called <u>trivial</u> if A is a singleton $\{a\}$, $\alpha(\rho)$ consists of the $[h(\mathfrak{R})](\rho)$-tuple $\langle a,\dots,a\rangle$ for all $\rho \in P$ and $\langle a,\dots,a\rangle\alpha(\omega) = a$ for all $\omega \in \Omega$. All trivial $\mathfrak{R}$-structures are isomorphic so that we can talk of 'the' trivial $\mathfrak{R}$-structure.

A structure $\underline{A} = \langle A,\alpha\rangle$ is said to be a structure <u>on</u> the set A .

Our notation for $\mathfrak{R}$-structures comes from [21] except that in [21] the role of h is not emphasized.

In our notation we can describe fundamental concepts like those of substructures, homomorphisms between structures, congruences over algebras etc. We shall not explicitly give this description in all cases but only give two examples. By an $\mathfrak{R}$-<u>homomorphism</u> between two $\mathfrak{R}$-structures $\underline{A}_1 = \langle A_1, \alpha_1 \rangle$ and $\underline{A}_2 = \langle A_2, \alpha_2 \rangle$ is understood a function $f : A_1 \longrightarrow A_2$ such that

$\langle a_1, \ldots, a_{h(\rho)} \rangle \in \alpha_1(\rho)$ implies $\langle f(a_1), \ldots, f(a_{h(\rho)}) \rangle \in \alpha_2(\rho)$ and

$\langle a_1, \ldots, a_{h(\omega)} \rangle \alpha_1(\omega) = a$ implies $\langle f(a_1), \ldots, f(a_{h(\omega)}) \rangle \alpha_2(\omega) = f(a)$

for all $a_1, \ldots, a_{h(\rho)}$, $a, a_1, \ldots, a_{h(\omega)} \in A_1$, $\rho \in P(\mathfrak{R})$, $\omega \in \Omega(\mathfrak{R})$, where we have abbreviated $h(\mathfrak{R})$ as h. Our second example is that of an $\mathfrak{R}'$-reduct of an $\mathfrak{R}$-structure. For predicate systems $\mathfrak{R}'$, $\mathfrak{R}$ we write $\mathfrak{R}' \leq \mathfrak{R}$ if $P(\mathfrak{R}') \subseteq P(\mathfrak{R})$, $\Omega(\mathfrak{R}') \subseteq \Omega(\mathfrak{R})$ and, $h(\mathfrak{R}')$ is the restriction of $h(\mathfrak{R})$ to $P(\mathfrak{R}') \cup \Omega(\mathfrak{R}')$. Let $\mathfrak{R}' \leq \mathfrak{R}$ and let $\underline{A}' = \langle A, \alpha' \rangle$ be an $\mathfrak{R}'$-structure and $\underline{A} = \langle A, \alpha \rangle$ be an $\mathfrak{R}$-structure. We shall say that $\underline{A}'$ is an $\mathfrak{R}$-reduct of $\underline{A}$ if α' is the restriction of α to $P(\mathfrak{R}') \cup \Omega(\mathfrak{R}')$.

Let φ be an $\mathfrak{R}$-formula with $x_{j_1}, \ldots, x_{j_n}$ as the free variables and let $\Sigma \subseteq \Phi_0(\mathfrak{R})$. Let $\underline{A} = \langle A, \alpha \rangle$ be an $\mathfrak{R}$-structure and let $a_1, \ldots, a_n \in A$. Then the phrases '$\underline{A}$ satisfies φ at $\langle x_{j_1}, \ldots, x_{j_n} \rangle = \langle a_1, \ldots, a_n \rangle$' and '$\underline{A}$ satisfies Σ' or '$\underline{A}$ is a model of Σ' or 'Σ holds in $\underline{A}$' have well-understood meanings that we shall not describe here (see, however, Chapter 6). Two $\mathfrak{R}$-structures are called <u>elementarily equivalent</u> if every $\mathfrak{R}$-sentence that holds in one of them also holds in the other.

We shall denote by $\mathcal{A}$ the class of all ordered pairs $\underline{A}$ such that $\underline{A}$ is an $\mathfrak{R}$-structure for some predicate system $\mathfrak{R}$. The class of all $\mathfrak{R}$-structures for a given $\mathfrak{R}$ will be denoted by $\underline{\underline{V}}(\mathfrak{R})$. For $\Sigma \subseteq \Phi_0(\mathfrak{R})$ we write $\underline{\underline{V}}(\Sigma)$ for the class of all models of Σ. For $\underline{\underline{V}} \subseteq \underline{\underline{V}}(\mathfrak{R})$ we write $\Sigma(\underline{\underline{V}})$ for the set of all $\sigma \in \Phi_0(\mathfrak{R})$ such that σ holds in every structure of $\underline{\underline{V}}$. A subclass $\underline{\underline{V}}$ of $\underline{\underline{V}}(\mathfrak{R})$ is said to be <u>definable by</u> $\Sigma \subseteq \Phi_0(\mathfrak{R})$ if $\underline{\underline{V}}$ is $\underline{\underline{V}}(\Sigma)$. Classes of the form $\underline{\underline{V}}(\Sigma)$ will be called <u>elementary</u>. An elementary class $\underline{\underline{V}}(\Sigma)$ is called <u>universal</u> if Σ is equivalent to a set of universal sentences, i.e., sentences of the form $\forall x_1,\ldots,x_n(\varphi)$, where φ is free of quantifiers and involves no variables other than $x_1,\ldots,x_n$.

We shall denote by $\underline{\underline{S}}$ the category of all functions and by $\underline{\underline{S}}_0$ the class of all sets. Note that $\underline{\underline{S}}_0$ is not contained in $\underline{\underline{S}}$. However if we agree to make no distinction between a set and the corresponding identity function we can say that $\underline{\underline{S}}_0 \subseteq \underline{\underline{S}}$ and that $\underline{\underline{S}}$ is the category of sets and functions. In this work we agree to do so.

By a <u>category of structures</u> we shall understand a category $\underline{\underline{K}}$ whose objects are structures. The category of structures consisting of all $\mathfrak{R}$-structures and $\mathfrak{R}$-homomorphisms will be denoted by $\underline{\underline{K}}(\mathfrak{R})$. The category $\underline{\underline{K}}(\mathfrak{R})$ is complete and cocomplete [20]. The trivial $\mathfrak{R}$-structure is the terminal or null object [20] of $\underline{\underline{K}}(\mathfrak{R})$ while the empty $\mathfrak{R}$-structure (i.e., the $\mathfrak{R}$-structure on the empty set) is the coterminal or conull object.

Morphisms in most categories of structures usually considered are functions but other types of morphisms could be useful for the study of structures. We give an example. By a <u>functopism</u> from a set Y_1 into another set Y_2 we shall understand a countable sequence $f^+ = \langle f_1, f_2, \ldots, f_n, \ldots \rangle$ of functions from Y_1 into Y_2. The product of two functopisms $f^+ : Y_1 \longrightarrow Y_2$, $g^+ : Y_2 \longrightarrow Y_3$ is defined to be the functopism $g^+f^+ = \langle g_1 f_1, \ldots, g_n f_n, \ldots \rangle$, where we have taken $f^+ = \langle f_1, \ldots, f_n, \ldots \rangle$, $g^+ = \langle g_1, \ldots, g_n, \ldots \rangle$. Denote by $\underline{\underline{S}}^+$ the category of all functopisms. By an $\mathfrak{R}$-<u>homotopism</u> from an $\mathfrak{R}$-structure $\langle A_1, \alpha_1 \rangle$ into another $\mathfrak{R}$-structure $\langle A_2, \alpha_2 \rangle$ we shall understand a functopism $f^+ = \langle f_1, \ldots, f_n, \ldots \rangle$ from A_1 into A_2 such that for all $a_1, \ldots, a_n, \ldots$ in A_1 if $\langle a_1, \ldots, a_k \rangle \in \alpha_1(\rho)$ then $\langle f_1(a_1), \ldots, f_k(a_k) \rangle \in \alpha_2(\rho)$ for $\rho \in P$, $k = [h(\mathfrak{R})](\rho)$ and if $\langle a_1, \ldots, a_k \rangle \alpha_1(\omega) = a_{k+1}$ then $\langle f_1(a_1), \ldots, f_k(a_k) \rangle \alpha_1(\omega) = f_{k+1}(a_{k+1})$ for $\omega \in \Omega$, $k = [h(\mathfrak{R})](\omega)$. The concept of $\mathfrak{R}$-homotopisms is a natural generalization of that of $\mathfrak{R}$-homomorphisms since an $\mathfrak{R}$-homomorphism f can be regarded as the $\mathfrak{R}$-homotopism $\langle f, \ldots, f, \ldots \rangle$. Write $\underline{\underline{K}}^+(\mathfrak{R})$ for the category of all $\mathfrak{R}$-structures and $\mathfrak{R}$-homotopisms.

For a subclass $\underline{\underline{V}}$ of $\underline{\underline{V}}(\mathfrak{R})$ we define $\underline{\underline{K}}(\underline{\underline{V}})$ to be the category of all structures in $\underline{\underline{V}}$ and all homomorphisms between them. If $\underline{\underline{V}} = \underline{\underline{V}}(\Sigma)$ for some $\Sigma \subseteq \Phi_0(\mathfrak{R})$ we write $\underline{\underline{K}}(\Sigma)$ for $\underline{\underline{K}}(\underline{\underline{V}})$. Define similarly $\underline{\underline{K}}^+(\underline{\underline{V}})$ to be the category of structures of $\underline{\underline{V}}$ and all homotopisms between them. Write $\underline{\underline{K}}^+(\Sigma)$ for $\underline{\underline{K}}^+(\underline{\underline{V}}(\Sigma))$.

Let $\underline{\underline{K}}$ be a category of structures whose morphisms are functions. We define the *forgetful functor* $F(\underline{\underline{K}};-) : \underline{\underline{K}} \longrightarrow \underline{\underline{S}}$ of $\underline{\underline{K}}$ to be the functor which takes structures to their carriers and morphisms to the corresponding functions. For a category of structures $\underline{\underline{K}}$ whose morphisms are functopisms we define $F^+(\underline{\underline{K}};-) : \underline{\underline{K}} \longrightarrow \underline{\underline{S}}^+$ to be the functor which takes structures to their carriers and morphisms to the corresponding functopisms. The functor $F^+(\underline{\underline{K}};-)$ will still be called the forgetful functor of $\underline{\underline{K}}$. Write $F(\mathfrak{R};-)$ and $F^+(\mathfrak{R};-)$ for $F(\underline{\underline{K}};-)$ and $F^+(\underline{\underline{K}}^+;-)$ respectively when $\underline{\underline{K}} = \underline{\underline{K}}(\mathfrak{R})$, $\underline{\underline{K}}^+ = \underline{\underline{K}}^+(\mathfrak{R})$.

If $P(\mathfrak{R}) = \emptyset$ we can define a functor $\underline{W}(\mathfrak{R};-) : \underline{\underline{S}} \longrightarrow \underline{\underline{K}}(\mathfrak{R})$ in the following natural way: For a set Y define $\underline{W}(\mathfrak{R};Y)$ to be the $\mathfrak{R}$-word algebra on Y (see page 5) and for a function $f : Y_1 \longrightarrow Y_2$ define $\underline{W}(\mathfrak{R};f)$ to be the $\mathfrak{R}$-homomorphism from $\underline{W}(\mathfrak{R};Y_1)$ into $\underline{W}(\mathfrak{R};Y_2)$ whose restriction to Y_1 is f .

The functors $F(\mathfrak{R};-)$ and $\underline{W}(\mathfrak{R};-)$ are related in an important way which is described by saying that $F(\mathfrak{R};-)$ and $\underline{W}(\mathfrak{R};-)$ form a pair of adjoint functors. See, for example, [20] for details on adjoint functors. Adjoint functors are useful in describing and studying 'freeness' (see Chapter 4).

We conclude this chapter by making some general remarks on the notation used in this thesis.

Let f be a function and let y be in Dom(f) (i.e., domain of f) . We shall usually write f(y) for the image of y under f . However at a few places we shall find it much more convenient to write yf for f(y) . For example we have already written (page 5) $\langle w_1, \ldots, w_k \rangle \alpha(\omega)$ for $[\alpha(\omega)](\langle w_1, \ldots, w_k \rangle)$. At all such places our meaning will be clear from the context. The composition of two functions $f : Y_1 \longrightarrow Y_2$, $g : Y_2 \longrightarrow Y_3$ will <u>always</u> be denoted by gf . We write Ran(f) for the range of f . If $Y' \subseteq \mathrm{Dom}(f)$ we write $f | Y'$ for the restriction of f to Y' . Similar notation holds for relations.

We shall quite closely follow the following convention about the use of different letters for denoting different types of mathematical objects. In adopting this convention we have kept in mind the suggestions put forward in [1, p. xiv]. Our convention is that we shall usually use:

A , B , C , X , Y , Z etc. for sets and corresponding small letters for their elements,

α, β, γ etc. and f , g , h etc. for functions,

k , ℓ , m , n etc. for ordinals,

φ , ψ , σ etc. for sets of formulae of a language,

$\underline{A}$, $\underline{B}$, $\underline{M}$, $\underline{I}$ etc. for structures

and

$\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$, $\underline{\underline{K}}$, $\underline{\underline{U}}$, $\underline{\underline{V}}$ etc. for classes and categories (of structures).

If in some context we talk about structures $\underline{A}$, $\underline{M}$, $\underline{I}$ etc. then in the same context we shall use, without explicit reference, A , M , I etc. for the corresponding carriers and α , μ , ι etc. for the corresponding makes.

CHAPTER 1.

SOME EQUIVALENCES BETWEEN CLASSES OF STRUCTURES

Algebraists tend to make no real distinction between the classes of Boolean algebras and Boolean rings although the algebraic structures in the two classes appear so different. This is because almost all information about Boolean rings can be translated, in a simple way, into that about Boolean algebras and vice versa. The relation between classes of Boolean algebras and Boolean rings, which we shall make precise under the term nomial equivalence, is particularly strong. Classes of structures may be related in other ways that could enable us to translate a considerable part of information about one class into that about the other. It is important to study such ways so that we may not have to do separate studies of the same or nearly the same class in different disguises. However not many equivalences between classes of structures have been studied. Perhaps the only two equivalences about which observations have been made are the rational and structural equivalences that we shall shortly define precisely. Roughly speaking, two classes of structures are rationally equivalent if first order statements about one class can be translated in a certain simple manner into first order statements about the other. On the other hand structural equivalence does not explicitly give a way of translating information about one class into that about the other. Instead, it arises from the realization that we can talk a great deal about a class of structures in terms of the homomorphisms between the structures, so that if a transformation of structures leaves homomorphisms unchanged then it must also leave unchanged something very essential about the structures. In the present chapter we also define homotopic equivalence using homotopisms in the way in which structural equivalence uses homomorphisms. A special case of homotopic equivalence is the isotopic equivalence under which every structure in any one of the equivalent classes is an isotope of some structure in any other. We give two varities of groupoids that are isotopically equivalent. One of these two varieties is the variety of abelian groups (regarded as groupoids with substraction as the binary operation).

An easy result of Mal'cev (see Theorem 1 below) shows that under certain conditions any two structurally equivalent classes are rationally equivalent so that we can find an explicit way of translating first order statements about one class into those about the other. In the case of the two isotopically equivalent varieties given here we can find a simple way of translating second order statements about one variety into those about the other. In general we do not know of any conditions under which for any two homotopically or isotopically equivalent classes we can find a way of translating second order statements holding for structures of one class into those holding for the structures of the other.

We now give precise definitions. Let $\mathfrak{R} = \langle P, \Omega, h \rangle$. For all $\underline{A} \in \underline{\underline{V}}(\mathfrak{R})$, $\varphi \in \Phi(\mathfrak{R})$ we write $R(\underline{A};\varphi)$ for the $h(\varphi)$-ary relation over A consisting of all $h(\varphi)$-tuples $\langle a_1, \ldots, a_{h(\varphi)} \rangle \in A^{h(\varphi)}$ such that φ holds in $\underline{A}$ at $\langle x_{j_1}, \ldots, x_{j_{h(\varphi)}} \rangle = \langle a_1, \ldots, a_{h(\varphi)} \rangle$, where $x_{j_1}, \ldots, x_{j_{h(\varphi)}}$ are the variables freely occurring in φ. Let $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathfrak{R}_1)$, $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathfrak{R}_2)$, $\mathfrak{R}_1 = \langle P_1, \Omega_1, h_1 \rangle$, $\mathfrak{R}_2 = \langle P_2, \Omega_2, h_2 \rangle$. We shall say that $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are <u>rationally equivalent</u> if there exist functions $f_1 : P_1 \cup \Omega_1 \longrightarrow \Phi(\mathfrak{R}_2)$ $f_2 : P_2 \cup \Omega_2 \longrightarrow \Phi(\mathfrak{R}_1)$ and $F : \underline{\underline{V}}_1 \longrightarrow \underline{\underline{V}}_2$ such that:

(i) F is one-to-one and onto and carrier $(\underline{A})$ = carrier $(F(\underline{A}))$ for all $\underline{A} \in \underline{\underline{V}}_1$.

(ii) For all $\rho \in P_j$, $j = 1,2$, the formula $f_j(\rho)$ involves $x_1, \ldots, x_{h_j(\rho)}$ as free variables and for all $\omega \in \Omega_j$ the formula $f_j(\omega)$ involves $x_1, \ldots, x_{h_j(\omega)+1}$ as free variables.

(iii) Let $\underline{A}_1 = \langle A, \alpha_1 \rangle \in \underline{\underline{V}}_1$, $\underline{A}_2 = F(\underline{A}_1) = \langle A, \alpha_2 \rangle \in \underline{\underline{V}}_2$ and let $\zeta_1 \in P_1 \cup \Omega_1$, $\zeta_2 \in P_2 \cup \Omega_2$. Then $\alpha_1(\zeta_1) = R(\underline{A}_2; f_1(\zeta_1))$, $\alpha_2(\zeta_2) = R(\underline{A}_1; f_2(\zeta_2))$.

We shall call $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ <u>nomially equivalent</u> if in addition to (i) - (iii) the following also holds:

(iv) For all $\omega \in \Omega_j$, $j = 1,2$, the formula $f_j(\omega)$ is of the form $w = x_{h_j(\omega)+1}$, where w is a word in $x_1, \ldots, x_{h_j(\omega)}$.

If $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are elementary classes, defined by $\Sigma_1 \subseteq \Phi_0(\mathfrak{R}_1)$,

$\Sigma_2 \subseteq \Phi_0(\mathfrak{R}_2)$ respectively, then the rational equivalence between $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ can be defined as a 'logical' relation between Σ_1 , Σ_2 in the following way. We can suppose, without loss of generality, that $(P_1 \cup \Omega_1) \cap (P_2 \cup \Omega_2) = \emptyset$. Write $\mathfrak{R}_1 \cup \mathfrak{R}_2 = \langle P_1 \cup P_2, \Omega_1 \cup \Omega_2, h_1 \cup h_2 \rangle$, where $h_1 \cup h_2 \mid P_1 \cup \Omega_1 = h_1$, $h_1 \cup h_2 \mid P_2 \cup \Omega_2 = h_2$. For $\zeta \in P_j \cup \Omega_j$, $j = 1,2$, we write $\varepsilon_{\mathfrak{R}_j}(\zeta)$ for the atomic $\mathfrak{R}_j$-formula $\zeta(x_1,\ldots,x_{h_j(\zeta)})$ or $x_1 \ldots x_{h_j(\zeta)}\zeta = x_{h_j(\zeta)+1}$ according as $\zeta \in P_j$ or $\zeta \in \Omega_j$. Now the two elementary classes $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are rationally equivalent if and only if there exist functions $f_1 : P_1 \cup \mathfrak{R}_1 \longrightarrow \Phi(\mathfrak{R}_1)$, $f_2 : P_2 \cup \Omega_2 \longrightarrow \Phi(\mathfrak{R}_2)$ satisfying (ii) such that

$$\Sigma_2 \cup \{\forall x_1,\ldots,x_k(\varepsilon_{\mathfrak{R}_1}(\zeta) \longleftrightarrow f_1(\zeta));\ \zeta \in P_1 \cup \Omega_1,\ k = h_1(\varepsilon_{\mathfrak{R}_1}(\zeta))\}$$

and

$$\Sigma_1 \cup \{\forall x_1,\ldots,x_k(\varepsilon_{\mathfrak{R}_2}(\zeta) \longleftrightarrow f_2(\zeta));\ \zeta \in P_2 \cup \Omega_2,\ k = h_2(\varepsilon_{\mathfrak{R}_2}(\zeta))\}$$

are equivalent as sets of $\mathfrak{R}_1 \cup \mathfrak{R}_2$-sentences.

The study of rational equivalence is closely related to what is called [1, pp. 170-176] the theory of definition.

A large part of the information about many important classes of structures can be presented as categorical properties of the corresponding category of homomorphisms. It is therefore natural to use these categories to define a concept of equivalence between classes of structures. In this direction the first concept that will strike anyone is that of <u>loose homomorphic equivalence</u>. We define two classes $\mathbf{V}_1$, $\mathbf{V}_2$ to be <u>loosely homomorphically equivalent</u> if the categories $\mathbf{K}(\mathbf{V}_1)$, $\mathbf{K}(\mathbf{V}_2)$ are equivalent [20, p. 32]. However, the loose homomorphic equivalence does not seem to connect the classes closely enough. A more appropriate concept is that of <u>structural equivalence</u> defined by Mal'cev [17] as follows. Two classes $\mathbf{V}_1$, $\mathbf{V}_2$ are called structurally equivalent if there is an isomorphism $F : \mathbf{K}(\mathbf{V}_1) \longrightarrow \mathbf{K}(\mathbf{V}_2)$ such that f, $F(f)$ are the same functions for all $f \in \mathbf{K}(\mathbf{V}_1)$. We can slightly relax the condition of this definition of structural equivalence and obtain <u>homomorphic equivalence</u>. We define two classes $\mathbf{V}_1$, $\mathbf{V}_2$ to be <u>homomorphically equivalent</u> if there exists an equivalence [20, p. 32] $F : \mathbf{K}(\mathbf{V}_1) \longrightarrow \mathbf{K}(\mathbf{V}_2)$ and one-to-one and onto functions $\eta(\underline{A}) : \text{carrier}\ (\underline{A}) \longrightarrow \text{carrier}\ (F(\underline{A}))$, $\underline{A} \in \mathbf{V}_1$, such that for all $f : \underline{A}_1 \longrightarrow \underline{A}_2$ in $\mathbf{K}(\underline{V})$ we have $\eta(\underline{A}_2)f = F(f)\eta(\underline{A}_1)$. There is another way of defining homomorphic equivalence. Let us define two functors $F_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}_1$, $F_2 : \mathbf{A}_2 \longrightarrow \mathbf{B}_2$ to be similar if there exist equivalences $H_1 : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$, $H_2 : \mathbf{B}_1 \longrightarrow \mathbf{B}_2$ such that F_2H_1, H_2F_1 are naturally equivalent [20, p. 59]. Now $\mathbf{V}_1$, $\mathbf{V}_2$ are homomorphically equivalent if and only if the forgetful functors $F(\mathbf{K}(\mathbf{V}_1);-) : \mathbf{K}(\mathbf{V}_1) \longrightarrow \mathbf{S}$ and $F(\mathbf{K}(\mathbf{V}_2);-) : \mathbf{K}(\mathbf{V}_2) \longrightarrow \mathbf{S}$ are similar.

The following result is a slight modification of Theorem 6 of [17] and is proved similarly.

THEOREM 1. Let $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ be two abstract classes of algebras of species $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively. Let $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ admit free algebras on all sets. Then $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are nomially equivalent if and only if $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are homomorphically equivalent.

Theorem 1 shows that under reasonably simple conditions homomorphic equivalence provides a very close connection between classes of algebras.

No simple conditions are known under which loose homomorphic equivalence implies rational equivalence. Nor do we know of two loosely homomorphically equivalent varieties that are not homomorphically equivalent.

The way we have used homomorphisms to define loose homomorphic equivalence and homomorphic equivalence can also be applied to other types of morphisms to get different concepts of equivalences. Thus if we consider homotopisms we arrive at <u>loose homotopic equivalence</u> and <u>homotopic equivalence.</u> The classes of loops and quasi-groups are homotopically equivalent - in fact they are <u>isotopically equivalent</u> in the sense that every structure in one class is an isotope (isomorphic in the category of homotopisms) to a structure in the other.

For homotopic equivalence we do not know of anything like Theorem 1 which can tell us how similar homotopically equivalent classes can be. We only know of an example that shows that varieties of algebras can be fairly closely connected by isotopic equivalence.

We now present this example.

Due to the special nature of the result that follows it will be much more convenient to use some simplified and usual notations different from those introduced in Chapter 0. Thus we shall write a groupoid as $\langle A,\omega\rangle$, where ω is a binary operation over A. In writing universal sentences we shall often omit the universal quantifier. In addition to $x_1, x_2, \ldots$ we shall also use $x, y, z, t, x_1, y_1, z_1, t_1$ etc. as variables.

THEOREM 2. The varieties of groupoids defined by the laws

(I) $\quad y = xt\omega xz\omega yz\omega\omega t\omega\omega$

and

(II) $\quad y = xxz\omega' yz\omega'\omega'\omega'$

are isotopically equivalent but not homomorphically equivalent.

The proof of the theorem is divided into the following four lemmas.

LEMMA 1. A groupoid satisfying (I) is a quasi-group.

PROOF. In (I) replace t by $y_2 z\omega$, y by y_1 and x by $xz\omega$. We get the law

$$y_1 = xz\omega y_2 z\omega\omega xz\omega z\omega y_1 z\omega\omega y_2 z\omega\omega .$$

If we write w for the word $xz\omega z\omega y_1 z\omega\omega y_2 z\omega\omega$ and u for the word $xw\omega$ then this last law together with (I) gives

$$y_2 = xw\omega xz\omega y_2 z\omega\omega w\omega \ = \ xw\omega y_1\omega \ = \ uy_1\omega \ .$$

Hence (I) implies that $\forall y_1, y_2 \ \exists t \quad (ty_1\omega = y_2)$. This can be expressed by saying that the equation $ty_1\omega = y_2$ has a solution in t in every groupoid satisfying (I) for all y_1 , y_2 . The equation $y_1 t\omega = y_2$ is also solvable. This can be seen as follows. We have just proved that we can find x , t_1 such that $xt_1\omega = y_1$. Then by (I)

$$y_1 xz\omega y_2 z\omega\omega t_1\omega\omega \ = \ y_2$$

and if we put $t = xz\omega y_2 z\omega\omega t_1\omega\omega$ we have $y_1 t\omega = y_2$.

We now prove the 'cancellation laws'. Let t_1 , y , t_2 be such that $t_1 y\omega = t_2 y\omega$. Then, by (I) ,

$$t_1 \ = \ xt\omega xy\omega t_1 y\omega\omega t\omega\omega \ = \ xt\omega xy\omega t_2 y\omega\omega t\omega\omega \ = \ t_2$$

and the right cancellation law holds. Next let t_1 , y , t_2 be such that $yt_1\omega = yt_2\omega$. Find x , z , y_1 , y_2 such that $y = xz\omega$, $t_1 = y_1 z\omega$, $t_2 = y_2 z\omega$. Then

$$\begin{aligned} y_1 \ &= \ xt\omega xz\omega y_1 z\omega\omega t\omega\omega \ = \ xt\omega yt_1\omega t\omega\omega \\ &= \ xt\omega yt_2\omega t\omega\omega \ = \ xt\omega xz\omega y_2 z\omega\omega t\omega\omega \ = \ y_2 \end{aligned}$$

so that $t_1 = t_2$ and the left cancellation law also holds. This completes the proof of the lemma.

<u>LEMMA 2</u>. The law (I) is equivalent to the following statement. There exists e such that:

$$(2.1) \qquad xe\omega e\omega = x$$

$$(2.2) \qquad xz\omega yz\omega\,\omega = xy\omega e\omega$$

$$(2.3) \qquad xe\omega xy\omega\omega = y$$

<u>PROOF</u>. (I) implies (2.1) -(2.3) :-

We have, by (I)

$$x = xt\omega xz\omega xz\omega\omega t\omega\omega .$$

By Lemma 1 $xz\omega$ can be any element t_1 for suitable z, Hence

$$(2.4) \qquad x = xt\omega t_1 t_1 \omega t\omega\omega , \quad \text{for all } x, t_1, t .$$

This gives

$$xx\omega = xx\omega t\omega xx\omega t\omega\omega$$

and if t is such that $xx\omega t\omega = y$ then

$$xx\omega = yy\omega = e , \quad \text{some fixed element.}$$

We can now write (2.4) in the form

$$x = xt\omega et\omega\omega .$$

In the last equation if t is taken equal to e we arrive at (2.1).

To prove (2.2) we note that

$$y = xt\omega xz_1\omega yz_1\omega\omega t\omega\omega$$

$$= xt\omega xz_2\omega yz_2\omega\omega t\omega\omega \,.$$

Hence by Lemma 1

$$xz_1\omega yz_1\omega\omega = xz_2\omega yz_2\omega\omega$$

so that $xz\omega yz\omega\omega = xy\omega yy\omega\omega = xy\omega e\omega$. This proves (2.2) .

By (I) , (2.1) and (2.2)

$$y = xe\omega xy\omega yy\omega\omega e\omega$$

$$= xe\omega xy\omega\omega \,.$$

This proves (2.3).

Now we show that (2.1) - (2.3) imply (I) .

We have

$$\begin{aligned} & xt\omega xz\omega yz\omega\omega t\omega\omega \\ &= xt\omega xy\omega e\omega t\omega\omega \quad , \quad \text{by } (2.2) \\ &= xe\omega xy\omega e\omega e\omega\omega \quad , \quad \text{by } (2.2) \\ &= xe\omega xy\omega\omega \quad , \quad \text{by } (2.1) \\ &= y \quad , \quad \text{by } (2.3) \,. \end{aligned}$$

Hence (I) holds and the lemma is proved.

LEMMA 3. A groupoid $\langle A,\omega\rangle$ satisfies (I) if and only if $\langle A,\omega'\rangle$ satisfies (II) and

$$(3.1) \qquad xy\omega = xy\omega'\gamma \ , \quad \text{for some } \gamma \text{ and all } x,y \in A$$

where $xy\omega' = xy\omega xx\omega\omega$ for all x , $y \in A$ and γ is an involution of $\langle A,\omega'\rangle$. (We recall that γ is called an involution if γ is an automorphism and γ^2 is the identity map.)

Further a groupoid $\langle A,\omega'\rangle$ satisfies (II) if and only if $\langle A,\omega\rangle$ satisfies (I) and

$$(3.2) \qquad xy\omega' = xy\omega xx\omega\omega \ , \quad \text{for all } x,y \in A \ ,$$

where ω is defined by (3.1) for some involution γ of $\langle A,\omega'\rangle$.

PROOF. Part 1. Let $\langle A,\omega'\rangle$ satisfy (II) and $\langle A,\omega\rangle$ be defined by (3.1). Then $\langle A,\omega\rangle$ satisfies (I) . For,

$$\begin{aligned} & xt\omega xz\omega yz\omega\omega t\omega\omega \\ &= xt\omega'\gamma xz\omega'\gamma yz\omega'\gamma\omega'\gamma t\omega'\gamma\omega'\gamma \ , \quad \text{by (3.1)} \\ &= xt\omega' xz\omega' yz\omega'\omega' t\omega'\omega' \ , \quad \text{since } \gamma \text{ is an involution} \\ &= y \ . \end{aligned}$$

In this last step we have used the result [10] that (II) characterizes abelian groups in terms of the operation of subtraction.

Part 2. Let $\langle A,\omega\rangle$ satisfy (I) and ω' be defined by (3.2). Then $\langle A,\omega'\rangle$ satisfies (II) . For, by the proof of Lemma 2 $xx\omega = e$ for some e and all x in A and hence

$$
\begin{aligned}
& xxz\omega'yz\omega'\omega'\omega' \\
&= xxz\omega e\omega yz\omega e\omega\omega e\omega\omega e\omega && \text{, by } (3.2) \\
&= xxz\omega yz\omega\omega e\omega e\omega\omega e\omega && \text{, by } (2.2) \\
&= xxz\omega yz\omega\omega e\omega && \text{, by } (2.1) \\
&= xe\omega xz\omega yz\omega\omega e\omega && \text{, by } (2.2) \\
&= y && \text{, by } (I)
\end{aligned}
$$

Part 3. Let $\langle A,\omega\rangle$ satisfy (I) and let $\langle A,\omega'\rangle$ be defined by (3.2). We show that (3.1) holds for some involution γ of $\langle A,\omega'\rangle$. Define γ by $x\gamma = xe\omega$ where e is the constant value of $xx\omega$ in $\langle A,\omega\rangle$. Then γ is an involution of $\langle A,\omega'\rangle$. For

$$
\begin{aligned}
xy\omega'\gamma &= xy\omega e\omega e\omega \\
&= xe\omega ye\omega\omega e\omega \quad \text{, by } (2.2) \\
&= x\gamma y\gamma\omega e\omega \\
&= x\gamma y\gamma\omega'
\end{aligned}
$$

and

$$x\gamma\gamma = xe\omega e\omega = x \quad \text{, by } (2.1) .$$

The involution γ satisfies (3.1); $xy\omega'\gamma = xy\omega e\omega e\omega = xy\omega$, by (2.1).

Part 4. Let $\langle A,\omega'\rangle$ satisfy (II) and $\langle A,\omega\rangle$ be defined by (3.1). We show that ω, ω' are related by (3.2). This is easy:

$$\begin{aligned} xy\omega x x\omega &= xy\omega'\gamma x x\omega'\gamma\omega'\gamma \\ &= xy\omega' x x\omega'\omega' \\ &= xy\omega' . \end{aligned}$$

Here we have used the fact that ω' is an operation of subxtraction in an abelian group and the assumption that γ is an involution of $\langle A,\omega'\rangle$.

The proof of the lemma is complete.

We observe that Part 3 of the proof of Lemma 3 shows that if $\langle A,\omega\rangle$ satisfies (I) and ω' is defined by 3.2 then $\langle A,\omega'\rangle$ is an isotope of $\langle A,\omega\rangle$. In view of this remark Lemma 3 shows that the varieties defined by (I) and (II) are isotopically equivalent. It is clear from Lemma 3 that every second order statement holding for groupoids satisfying (I) can be translated into a second order statement holding for groupoids satisfying (II) and vice versa.

LEMMA 4. The varieties defined by (I) and (II) are not homomorphically equivalent.

PROOF. Suppose that the lemma is not true. By Theorem 1 the varieties defined by (I) and (II) are nomially equivalent. This means that there exists an ω-word $w_{\omega'}(x,y)$ and an ω'-word $w_{\omega}(x,y)$ such that (I) together with $xy\omega' \longleftrightarrow w_{\omega'}(x,y)$ is equivalent to (II) together with $xy\omega \longleftrightarrow w_{\omega}(x,y)$. Using the familiar notation from **group theory** we write $w_{\omega}(x,y)$ as

$k_1x + k_2y$ where k_1 , k_2 are integers. Substituting $k_1x + k_2y$ for $xy\omega$ in (I) we find $k_1k_2 = \pm 1$. This shows that (I) implies either (II) or the law:

$$(\mathrm{II}^*) \qquad y = zy\omega zx\omega\omega x\omega .$$

However there exist groupoids that satisfy (I) but not (II) or (II^*) . An example is the groupoid consisting of four elements 0 , 1 , 2 , 3 in which the multiplication is defined by the table:

	0	1	2	3
0	0	3	2	1
1	3	0	2	1
2	2	1	0	3
3	1	2	3	0

This contradiction proves the lemma and Theorem 2.

Incidentally the law (I) has the interesting property that if $\langle A,\omega\rangle$ satisfies (I) then so does the dual groupoid $\langle A,\omega^*\rangle$ where $xy\omega^* = yx\omega$ for all $x,y \in A$. The law (II) does not have this property.

CHAPTER 2

CHARACTERIZATIONS OF SOME CLASSES OF ALGEBRAS

The nature of very many results in Algebra and Model Theory is somewhat like this —— classes $\underline{\underline{V}}(\Sigma)$ of structures are defined in terms of sets Σ of sentences of languages of some calculus (often first order predicate calculus) and then something syntactical, grammatical or logical about the sets Σ is connected with 'something structural' about the classes $\underline{\underline{V}}(\Sigma)$. An important observation made in this century is that this 'something structural' about $\underline{\underline{V}}(\Sigma)$ is almost always something categorical about some category associated with $\underline{\underline{V}}(\Sigma)$. We illustrate our remark about the nature of model-theoretic results by a couple of examples.

We can immediately mention Theorem 1 of Chapter 1 which relates nomial equivalence between Σ_1 , Σ_2 with the categorical concept of homomorphic equivalence between $\underline{\underline{V}}(\Sigma_1)$, $\underline{\underline{V}}(\Sigma_2)$. Before giving our second example we call $\underline{\underline{V}} \subseteq \underline{\underline{V}}(\mathcal{R})$ hereditary if $\underline{\underline{V}}$ contains all substructures of all of its structures. Now a result due to A. Robinson [24] states that $\underline{\underline{V}}(\Sigma)$ is hereditary if and only if Σ is equivalent to a set of universal sentences. Note that heredity is a categorical property of $\underline{\underline{V}}(\Sigma)$ and Robinson's result relates it to the form of sentences in Σ . In this connection one may profit by thinking of the following analogy. Compare Σ with a set of equations and $\underline{\underline{V}}(\Sigma)$ with the curve or surface defined by this set in a geometrical space. Many results in Geometry (after Descartes) relate analytical forms of equations with purely synthetic properties of the configurations defined by them. A simple example is this: a surface in three dimensional space is defined by an equation of the form $(x - a)^2 + (y - b)^2 + (z - c)^2 = d^2$ if and only if there is a point (a,b,c) in the space that is at a constant distant d from every point of the surface and every point at a distance d from (a,b,c) is on the surface. Note the anology between this result and Robinson's result mentioned above and some of the theorems that follow.

What is the significance of the geometrical and analogous model theoretic results we have exemplified above? Answer: They connect two <u>widely</u> different ways of talking about the same objects and hence enable us to use almost twice as much of our intuition as we would have used if there were only one way at our disposal. Geometry has exploited to a large extent Descartes' idea of connecting equations with configurations of points. Unfortunately the analogous study of the connections between syntactical forms of sentences and the categorical properties of the classes of structures defined by them has not been intensive enough to result in some deep general principles about structures. I feel that a more systematic and conscious study in this direction may prove useful in Mathematics.

In this chapter we consider sets Σ of first order sentences of some simple forms and give some categorical properties of $\underline{\underline{V}}(\Sigma)$.

The categorical properties of $\underline{\underline{V}}(\Sigma)$ considered are categorical statements about the corresponding category of homomorphisms.

In this chapter we fix arbitrarily a predicate system $\mathfrak{R} = \langle \phi, \Omega, h \rangle$.

§1. A GENERAL RESULT ON UNIVERSAL CLASSES OF ALGEBRAS.

The following result gives some of the most important properties of universal classes of algebras.

THEOREM 1. Let $\Sigma \subseteq \Phi(\mathfrak{R})$ be a set of universal sentences. Then the category $\underline{\underline{K}}(\Sigma)$ of all homomorphisms between $\mathfrak{R}$-structures in $\underline{\underline{V}}(\Sigma)$ has the following properties:

(1.1) For every monomorphism $\underline{A}_1 \longrightarrow \underline{A}_2$ in $\underline{\underline{K}}(\mathfrak{R})$ if $\underline{A}_2 \in \underline{\underline{K}}(\Sigma)$ then $\underline{A}_1 \in K(\Sigma)$.

(1.2) Inverse limits in $\underline{\underline{K}}(\mathfrak{R})$ of inverse systems [20] in $\underline{\underline{K}}(\Sigma)$ are in $\underline{\underline{K}}(\Sigma)$.

$(1.2)^*$ Direct limits in $\underline{\underline{K}}(\mathfrak{R})$ of direct systems [20] in $\underline{\underline{K}}(\Sigma)$ are in $\underline{\underline{K}}(\Sigma)$.

PROOF. The statement (1.1) is obvious.

To prove (1.2) let $\{f_{ji} : \underline{A}_i \longrightarrow \underline{A}_j\}_{i \leq j}$ be an inverse system in $\underline{\underline{K}}(\Sigma)$, where $i,j \in I$ and I is downward directed by the partial ordering $\leq$, that is, for all $i_1, i_2 \in I$ there exists $i_3 \in I$ such that $i_3 \leq i_1, i_2$. The inverse limit $\underline{A}$ of $\{\underline{A}_i \longrightarrow A_j\}_{i \leq j}$ is the subalgebra of the cartesian product $C\Pi_{i \in I}\, \underline{A}_i$ with carrier

$A = \{g \; ; \; g \in C\Pi_{i\in I} A_i \; , \; f_{ji}g(i) = g(j) \;\text{ for }\; i \leq j\}$. We verify that $\underline{A}$ satisfies Σ . Clearly Σ is equivalent to a set of sentences of the form

$$(1) \qquad \forall x_1,\dots,x_n[(v_1 \neq v_1' \vee \dots \vee v_\ell \neq v_\ell') \vee (w_1 = w_1' \vee \dots \vee w_m = w_m')]$$

where $v_1,\dots,v_p$, $v_1',\dots,v_p'$, $w_1,\dots,w_m$, $w_1',\dots w_m'$ are $\mathcal{R}$-words in $x_1,\dots,x_n$. Assume that (1) holds in $\underline{A}_i$ for all $i \in I$. Let $g_1,\dots,g_n$ be arbitrary elements of A . We have to show that the formula

$$(1') \qquad (v_1 \neq v_1' \vee \dots \vee v_\ell \neq v_\ell') \vee (w_1 = w_1' \vee \dots \vee w_m = w_m')$$

holds in $\underline{A}$ at $\langle x_1,\dots,x_n\rangle = \langle g_1,\dots,g_n\rangle$. Suppose otherwise. Then (1') fails to hold in $\underline{A}$ at $\langle x_1,\dots,x_n\rangle = \langle g_1,\dots,g_n\rangle$ if and only if $v_p(g_1(i),\dots,g_n(i)) = v_p'(g_1(i),\dots,g_n(i))$ for all p , i , $i \in I$, $1 \leq p \leq \ell$ and $w_q(g_1(i_q),\dots,g_n(i_q)) \neq w_q'(g_1(i_q),\dots g_n(i_q))$ for some $i_q \in I$ and all q , $1 \leq q \leq m$. Since I is downward directed we can find $j \in I$ such that $j \leq i_1,\dots,i_m$. Then for such a j we have that $v_p(g_1(j),\dots,g_n(j)) = v_p'(g_1(j),\dots,g_n(j))$ for all p , $1 \leq p \leq \ell$, and $w_q(g_1(j),\dots,g_n(j)) \neq w_q'(g_1(j),\dots,g_n(j))$ for all q , $1 \leq q \leq m$; this is because $f_{i_q j}$ is a homomorphism such that $f_{i_q j}g_k(j) = g_k(i_q)$ for all k , q , $1 \leq k \leq n$, $1 \leq q \leq m$. Hence $\underline{A}_j$ does not satisfy (i) at $\langle x_1,\dots,x_n\rangle = \langle g_1(j),\dots,g_n(j)\rangle$. This contradicts our assumption that (1) holds in $\underline{A}_i$ for all $i \in I$. The proof of (1.2) is therefore complete.

The proof of $(1.2)^*$, which is a categorical dual of (1.2), is not entirely 'dual' to the proof of (1.2). Let I be a set upward directed by the partial ordering $\leq$. Let $\{f_{ji} : \underline{A}_i \longrightarrow \underline{A}_j\}_{i \leq j, i,j \in I}$ be a direct system in $\underline{\underline{K}}(\Sigma)$. We shall assume that Ω does not contain o-ary operational symbols, the proof being essentially the same for the general case. The sum $\underline{S}$ in $\underline{\underline{K}}(\mathfrak{R})$ of $\{\underline{A}_i\}_{i \in I}$ can be constructed as follows. We can assume without loss of generality that the carriers A_i of the algebras $\underline{A}_i = \langle A_i, \alpha_i \rangle$ are disjoint. Let $S_o = \bigcup_{i \in I} A_i$,

$$S_1 = S_o \cup \{s_1, \ldots, s_{h(\omega)}\omega ;\ \omega \in \Omega,\ s_1, \ldots, s_{h(\omega)} \in S_o, \text{ but } s_1, \ldots, s_{h(\omega)} \text{ do not belong to the same } A_i\}$$

$$S_{n+1} = \{s_1, \ldots, s_{h(\omega)}\omega ;\ \omega \in \Omega,\ s_1, \ldots, s_{h(\omega)} \in S_n\} \cup S_n,$$

$$S = \bigcup_{n=o}^{\infty} S_n,$$

where $s_1 \ldots s_{h(\omega)}\omega$ is just a sequence. Clearly S is a set of $\mathfrak{R}$-words in S_o. We take S to be the carrier of $\underline{S}$ and define make $(\underline{S}) = \sigma$ by :

$$\langle s_1, \ldots, s_{h(\omega)} \rangle \sigma(\omega) = \langle s_1, \ldots, s_{h(\omega)} \rangle \alpha_i(\omega), \quad \text{if } i \in I,\ s_1, \ldots, s_{h(\omega)} \in A_i,$$

$$= s_1 \ldots s_{h(\omega)}\omega, \quad \text{otherwise.}$$

The algebra $\underline{S}$ so defined is the sum of $\{\underline{A}_i\}_{i\in I}$ in $\underline{\underline{K}}(\mathfrak{R})$. Over $\underline{S}$ we define a congruence κ as follows. For every set r of ordered pairs write $\mathrm{sym}(r)$ for the set $\{\langle s,t\rangle \; ; \; \langle t,s\rangle$ or $\langle s,t\rangle \in r\}$ and $\mathrm{tran}(r)$ for the set of ordered pairs $\langle s,t\rangle$ such that there exists a sequence $s_o,\dots,s_m$ satisfying $s_o = s$, $s_m = t$ and $\langle s_o,s_1\rangle,\dots,\langle s_{m-1},s_m\rangle \in r$. Let

$$r_o \;=\; \{\langle a_i, f_{ji}(a_i)\rangle \; ; \; i \in I \; , \;\; a_i \in A_i\} \; ,$$

$$\kappa_o \;=\; \mathrm{tran}(\mathrm{sym}(r_o))$$

$$r_{n+1} \;=\; \kappa_n \cup \{\langle\langle s_1,\dots,s_{h(\omega)}\rangle\sigma(\omega), \; \langle s_1',\dots,s_{h(\omega)}'\rangle\sigma(\omega)\rangle \; ; \; \omega \in \Omega, \; \langle s_1,s_1'\rangle,\dots,\langle s_{h(\omega)},s_{h(\omega)}'\rangle \in \kappa_n\} \; ,$$

$$\kappa_{n+1} \;=\; \mathrm{tran}(\mathrm{sym}(r_{n+1})) \; ,$$

$$\kappa \;=\; \textstyle\bigcup_{n=o}^{\infty} \kappa_n \; .$$

Thus κ is the least congruence over $\underline{S}$ containing r_o . We show that $\{\gamma_j : \underline{A}_j \longrightarrow \underline{S} \longrightarrow \underline{S}/\kappa\}_{j\in I}$ is a direct limit of $\{f_{ji} : \underline{A}_i \longrightarrow A_j\}_{i\leq j}$, where $\underline{A}_j \longrightarrow \underline{S}$ is the inclusion map and $\underline{S} \longrightarrow \underline{S}/\kappa$ is the canonical homomorphism which sends $s \in S$ to the equivalence class $[s]\kappa$ determined by s under κ . First note that $[a_i]\kappa = [f_{ji}(a_i)]\kappa$ for all $a_i \in A_i$ and hence $\gamma_j f_{ji} = \gamma_i$ for all i , j , $i \leq j$ in I . Next let $\{\delta_j : \underline{A}_j \longrightarrow \underline{M}\}_{j\in I}$ be another family of homomorphisms satisfying $\delta_j f_{ji} = \delta_i$. We have to show that there is a unique homomorphism ε such that $\varepsilon\gamma_j = \delta_j$ for all

j , $j \in I$. Let $\underline{M}_1$ be the subalgebra of $\underline{M}$ generated by the union of the images of the δ_j so that $\delta_j : \underline{A}_j \xrightarrow{\delta'_j} \underline{M}_1 \longrightarrow \underline{M}$, where $\underline{M}_1 \longrightarrow \underline{M}$ is the inclusion map and $\delta'_j f_{ji} = \delta'_j$ for all $i , j \in I$, $i \leq j$. Moreover, since $\underline{S}$ is the sum of $\{\underline{A}_j\}_{j \in I}$, the algebra $\underline{M}_1$ is isomorphic to $\underline{S}/\kappa'$ for some congruence κ' over $\underline{S}$. We can assume without loss of generality that $\underline{M}_1 = \underline{S}/\kappa'$ and $\delta'_j(a_j) = \delta_j(a_j) = [a_j]\kappa$ for all $j \in I$ and $a_j \in A_j$. By $\delta'_j j_{ji} = \delta'_i$ we conclude that $\langle a_i, j_{ji}(a_i)\rangle \in \kappa'$ for all i,j , a_i , $i \leq j$ in I and $a_i \in A_i$. Hence $r_o \subseteq \kappa'$, where r_o is one of the relations entering the definition of κ . Therefore $\kappa \subseteq \kappa'$ and the canonical homomorphism exists. Take ε to be the map $\underline{S}/\kappa \longrightarrow \underline{S}/\kappa' \longrightarrow \underline{M}$, where $\underline{S}/\kappa' \longrightarrow \underline{M}$ is the inclusion map. Then it is readily verified that $\varepsilon\gamma_j = \delta_j$ for all $j \in I$. The uniqueness of ε is also obvious. Hence $\underline{S}/\kappa$ together with the maps γ_j provides a direct limit in $\underline{\underline{K}}(\mathfrak{R})$ of $\{f_{ji}\}_{i \leq j}$.

Now we prove that $\underline{S}/\kappa$ is in $\underline{\underline{K}}(\Sigma)$. Before doing this we need to learn a few facts about the κ_n .

(1.3 The relation κ_o consists of ordered pairs $\langle a_i, a_j\rangle$, $a_i \in A_i$, $a_j \in A_j$ such that there exist a $k \in I$ satisfying $k \geq i,j$ and $f_{ki}(a_i) = f_{kj}(a_j)$.

Let $f_{ki}(a_i) = f_{kj}(a_j) = a_k$, say. Then $\langle a_i, a_k\rangle$, $\langle a_j, a_k\rangle \in r_o$, $\langle a_i, a_k\rangle$, $\langle a_k, a_j\rangle \in \mathrm{sym}(r_o)$, $\langle a_i, a_j\rangle \in \mathrm{tran}(\mathrm{sym}(r_o)) = \kappa_o$. Conversely let $\langle a_i, a_j\rangle \in \kappa_o$. Then if $\langle a_i, a_j\rangle \in \mathrm{sym}(r_o)$ we can take $k = j$ or i according as $\langle a_i, a_j\rangle \in r_o$ or $\langle a_j, a_i\rangle \in r_o$.

In the case when $\langle a_i, a_j\rangle \notin \mathrm{sym}(r_o)$ there exist $a_{i_o} \in A_{i_o}, \ldots, a_{i_m} \in A_{i_m}$ such that $a_{i_o} = a_i$, $a_{i_m} = a_j$ and $\langle a_{i_o}, a_{i_1}\rangle, \ldots, \langle a_{i_{m-1}}, a_{i_m}\rangle \in \mathrm{sym}(r_o)$. We can find $k_1, \ldots, k_m$ such that $f_{k_1 k_o}(a_{i_o}) = f_{k_1 i_1}(a_{i_1}), \ldots, f_{k_{m-1} i_{m-1}}(a_{i_{m-1}}) = f_{k_m i_m}(a_{i_m})$. Find $k \in I$ such that $k \geq k_1, \ldots, k_m$. Then $f_{ki}(a_i) = f_{kj}(a_j)$ because $f_{kk_1} f_{k_1 i_o} = f_{ki_o}, \ldots, f_{kk_m} f_{k_m i_m} = f_{ki_m}$. This proves (1.3).

(1.4) For every non-negative integer n the relation κ_n is a 'congruence over the partial subalgebra S_n ' of $\underline{S}$, that is, κ_n is an equivalence relation over S_n and if $\langle s_1, s_1'\rangle, \ldots, \langle s_{h(\omega)}, s_{h(\omega)}'\rangle \in \kappa_n$ and $s = \langle s_1, \ldots, s_{h(\omega)}\rangle \sigma(\omega) \in S_n$, $s' = \langle s_1', \ldots, s_{h(\omega)}'\rangle \sigma(\omega) \in S_n$ then $\langle s, s'\rangle \in \kappa_n$.

We use induction to prove (1.4). It follows directly from (1.3) and the definition of $\underline{S}$ that κ_o is a congruence over S_o. Assume (1.4) for all $n_1 \leq n$. The symmetry and reflexivity of κ_n implies the symmetry and reflexivity of r_{n+1} as may be directly seen by the definition of r_{n+1}. This shows that $\kappa_{n+1} = \mathrm{tran}(r_{n+1})$ and that κ_{n+1} is a reflexive relation over S_{n+1}. Transitivity of κ_{n+1} is immediate. For symmetry let $\langle s_o, s_{m+1}\rangle \in \kappa_{n+1}$ so that there exist $s_1, \ldots, s_m$ such that $\langle s_o, s_1\rangle, \ldots, \langle s_m, s_{m+1}\rangle = r_{n+1}$. But then $\langle s_{m+1}, s_m\rangle, \ldots, \langle s_1, s_o\rangle \in r_{n+1}$, since r_{n+1} is symmetric. Hence $\langle s_{m+1}, s_o\rangle \in \mathrm{tran}(r_{n+1}) = \kappa_{n+1}$. This proves that κ_{n+1} is an equivalence relation over S_{n+1}.

Before proving the congruence property for κ_{n+1} we need to show that the restriction of κ_{n+1} to S_n is κ_n. Let $\langle s,s'\rangle \in \kappa_{n+1}$, $s,s' \in S_n$. Let $s_o,\ldots,s_m$ be one of the shortest sequences such that $s_o = s$, $s_m = s'$, $\langle s_o,s_1\rangle,\ldots,\langle s_{m-1},s_m\rangle \in r_{n+1}$. Suppose $s_\ell \notin S_n$ for some ℓ, $1 < \ell < m$. Then we can write

$$s_{\ell-1} = \langle s_{\ell-1},\ldots,s_{\ell-1,h(\omega)}\rangle\sigma(\omega) ,$$

$$s_\ell = \langle s_{\ell,1},\ldots,s_{\ell,h(\omega)}\rangle\sigma(\omega) = \langle s'_{\ell,1},\ldots,s'_{\ell,h(\omega')}\rangle\sigma(\omega') ,$$

$$s_{\ell+1} = \langle s_{\ell+1,1},\ldots,s_{\ell+1,h(\omega')}\rangle\sigma(\omega') ,$$

where ω, $\omega' \in \Omega$ and $\langle s_{\ell-1,p},s_{\ell,p}\rangle \in \kappa_n$ for $1 \leq p \leq h(\omega)$ and $\langle s'_{\ell,p},s_{\ell+1,p}\rangle \in \kappa_n$ for $1 \leq p \leq h(\omega')$. This is immediate from the definition of r_{n+1} since $s_\ell \notin S_n$ and $\langle s_{\ell-1},s\rangle$, $\langle s_\ell,s_{\ell+1}\rangle \in r_{n+1}$. Now, by the definition of S, we find that $s_\ell \notin S_o$ and

$$s_\ell = \langle s_{\ell,1},\ldots,s_{\ell,h(\omega)}\rangle\sigma(\omega) = \langle s'_{\ell,1},\ldots,s'_{\ell,h(\omega')}\rangle\sigma(\omega')$$

together imply $\omega = \omega'$ and $s_{\ell,p} = s'_{\ell,p}$ for $1 \leq p \leq h(\omega)$. Thus $s_{\ell-1} = \langle s_{\ell-1,1},\ldots,s_{\ell-1,h(\omega)}\rangle\sigma(\omega)$, $s_{\ell+1} = \langle s_{\ell+1,1},\ldots,s_{\ell+1,h(\omega)}\rangle\sigma(\omega)$, and $\langle s_{\ell-1,p},s_{\ell+1,p}\rangle \in \kappa_n$ for $1 \leq p \leq h(\omega)$. Hence $\langle s_{\ell-1},s_{\ell+1}\rangle \in r_{n+1}$. This contradicts the assumption that $s_o,\ldots,s_m$ is one of the shortest sequences of a certain type. Hence $s_\ell \notin S_n$

is false for all ℓ , $1 < \ell < m$, so that $s_o, \ldots, s_m \in S_n$.
Again $\langle s_o, s_1 \rangle \in r_{n+1} - \kappa_n$ implies that

$$s_o = \langle s_{o,1}, \ldots, s_{o,h(\omega)} \rangle \sigma(\omega)$$

$$s_1 = \langle s_{1,1}, \ldots, s_{1,h(\omega)} \rangle \sigma(\omega)$$

for some $\omega \in \Omega$ and $\langle s_{o,1}, s_{1,1} \rangle, \ldots, \langle s_{o,h(\omega)}, s_{1,h(\omega)} \rangle \in \kappa_n$. Since s_o, s_1 have been shown to be in S_n we have, by the congruence property of κ_n , that $\langle s_o, s_1 \rangle \in \kappa_n$. Hence and similarly $\langle s_o, s_1 \rangle, \ldots, \langle s_{m-1}, s_m \rangle \in \kappa_n$ which implies that $\langle s_o, s_m \rangle = \langle s, s' \rangle \in \kappa_n$, as was to be proved.

We conclude the proof of (1.4) by showing that if $s, s' \in S_{n+1}$, $s = \langle s_1, \ldots, s_{h(\omega)} \rangle \sigma(\omega)$, $s' = \langle s'_1, \ldots, s'_{h(\omega)} \rangle \sigma(\omega)$ and $\langle s_1, s'_1 \rangle, \ldots, \langle s_{h(\omega)}, s'_{h(\omega)} \rangle \in \kappa_{n+1}$ then $\langle s, s' \rangle \in \kappa_{n+1}$. Clearly $s_1, \ldots, s_{h(\omega)}, s'_1, \ldots, s'_{h(\omega)} \in S_n$ and, by what we proved in the last paragraph, $\langle s_1, s'_1 \rangle, \ldots, \langle s_{h(\omega)}, s'_{h(\omega)} \rangle \in \kappa_n$. Hence $\langle s, s' \rangle \in r_{n+1} \subseteq \kappa_{n+1}$ and (1.4) is proved.

(1.5) The restriction of κ to S_n is κ_n for every non-negative

Let $\langle s, s' \rangle \in \kappa$, $s, s' \in S_n$. Let m be the least integer n . such that $\langle s, s' \rangle \in \kappa_m$. If $m > n$ then $s, s' \in S_{m-1}$ and since by the proof of (1.4) the restriction of κ_m to S_{m-1} is κ_{m-1} we conclude that $\langle s, s' \rangle \in \kappa_{m-1}$. This contradiction proves that $m \leq n$ and that $\langle s, s' \rangle \in \kappa_n$. Hence (1.5).

(1.6) For every finite subset $\{s_1,\dots,s_m\}$ of S there exist i, $a_1,\dots,a_m$ such that $i \in I$, $a_1,\dots,a_m \in A_i$ and $\langle s_1,a_1\rangle,\dots,\langle s_m,a_m\rangle \in \kappa$.

Again we use induction. Let $s_1,\dots,s_m \in S_o$ so that $s_1 \in A_{i_1},\dots,s_m \in A_{i_m}$ for some $i_1,\dots,i_m \in I$. Find $i \in I$ such that $i \geqq i_1,\dots,i_m$. Let $a_1 = f_{ii_1}(s_1),\dots,a_m = f_{ii_m}(s_m)$. Then $\langle s_1,a_1\rangle,\dots,\langle s_m,a_m\rangle \in \kappa$ and $a_1,\dots,a_m \in A_i$. Now assume (1.6) for all finite subsets of S_n and let $s_1,\dots,s_m \in S_{n+1}$. If $s_1 \in S_n$ we can find $i_1 \in I$, $a_1' \in A_i$, such that $\langle s_1,a_1'\rangle \in \kappa$. Otherwise we can write $s_1 = \langle s_{1,1},\dots,s_{1,h(\omega)}\rangle\sigma(\omega)$ for some $\omega \in \Omega$ and $s_{1,1},\dots,s_{1,h(\omega)} \in S_n$. By the induction hypothesis $\langle s_{1,1},a_{1,1}\rangle,\dots,\langle s_{1,h(\omega)},a_{1,h(\omega)}\rangle \in \kappa$ for some $i_1 \in I$, $a_{1,1},\dots,a_{1,h(\omega)} \in A_{i_1}$. Since κ is a congruence, $\langle s_1,a_1'\rangle \in \kappa$, where $a_1' = \langle a_{1,1},\dots,a_{1,h(\omega)}\rangle\sigma(\omega) \in A_{i_1}$. Similarly $\langle s_2,a_2'\rangle,\dots,\langle s_m,a_m'\rangle \in \kappa$ for some $a_2' \in A_{i_2},\dots,a_m' \in A_{i_m}$, $i_2,\dots,i_m \in I$. Find i such that $i \geqq i_1,\dots,i_m$ and let $a_1 = f_{ii_1}(a_1'),\dots,a_m = f_{ii_m}(a_m)$. Then $\langle s_1,a_1\rangle,\dots,\langle s_m,a_m\rangle \in \kappa$, $a_1,\dots,a_m \in A_i$. This completes the proof of (1.6).

We are now ready to prove that $\underline{S}/\kappa \in \underline{\underline{K}}(\Sigma)$.

Let the sentence

$$(1) \qquad \forall x_1,\dots,x_n[(v_1 \neq v_1' \vee \dots \vee v_\ell \neq v_\ell') \vee (w_1 = w_1' \vee \dots \vee w_m = w_m')]$$

hold in $\underline{A}_i$ for all $i \in I$. We show that (1) also holds in $\underline{S}/\kappa$. For this we need to show that for all $s_1,\ldots,s_n \in S$ either $\langle v_p(s_1,\ldots,s_n), v'_p(s_1,\ldots,s_n)\rangle \notin \kappa$ for some p, $1 \leq p \leq \ell$, or $\langle w_q(s_1,\ldots,s_n), w'_q(s_1,\ldots,s_n)\rangle \in \kappa$ for some q, $1 \leq q \leq m$. Suppose otherwise and let $s_1,\ldots,s_n \in S$ be such that $\langle v_p(s_1,\ldots,s_n), v'_p(s_1,\ldots,s_n)\rangle \in \kappa$ for all p, $1 \leq p \leq \ell$, and $\langle w_q(s_1,\ldots,s_n), w'_q(s_1,\ldots,s_n)\rangle \notin \kappa$ for all q, $1 \leq q \leq m$. Using (1.6) find i, $a_1,\ldots,a_n$ such that $i \in I$, $a_1,\ldots,a_n \in A_i$ and $\langle s_1,a_1\rangle,\ldots,\langle s_n,a_n\rangle \in \kappa$. Then
$\langle v_p(a_1,\ldots,a_n), v'_p(a_1,\ldots,a_n)\rangle \in \kappa$,
$\langle w_p(a_1,\ldots,a_n), w'_p(a_1,\ldots,a_n)\rangle \notin \kappa$ for all p, q, $1 \leq p \leq \ell$, $1 \leq q \leq m$. Write $b_p = v_p(a_1,\ldots,a_n)$, $b'_p = v'_p(a_1,\ldots,a_n)$ so that $b_p, b'_p \in A_i$, $\langle b_p,b'_p\rangle \in \kappa$ for all p, $1 \leq p \leq \ell$. By (1.5) and (1.3), for all p, $1 \leq p \leq \ell$, we can find $j_p \in I$ such that $f_{j_p i}(b_p) = f_{j_p i}(b'_p)$. Let $j \in I$ be such that $j \geq j_p$ for all p, $1 \leq p \leq \ell$. Then $f_{ji}(b_p) = f_{ji}(b'_p)$ for all p, $1 \leq p \leq \ell$. Let $c_1 = f_{ji}(a_1),\ldots,c_n = f_{ji}(a_n)$. Then, since f_{ji} is a homomorphism, we have

$$v_p(c_1,\ldots,c_n) = f_{ji}(b_p) = f_{ji}(b'_p) = v'_p(c_1,\ldots,c_n), \quad 1 \leq p \leq \ell,$$

and since (by (1.3)) $\langle c_1,a_1\rangle,\ldots,\langle c_n,a_n\rangle \in \kappa$, we have

$w_q(c_1,\ldots,c_n) \neq w'_q(c_1,\ldots,c_n)$, $1 \leq q \leq m$.

Since $c_1,\ldots,c_n \in A_j$ this contradicts our assumption that $\underline{A}_j$ satisfies (1) . This proves that every sentence of the form (1) that holds in every $\underline{A}_i$ also holds in $\underline{S}/\kappa$. In particular, $\underline{S}/\kappa \in \underline{\underline{K}}(\Sigma)$. This completes the proof of $(1.2)^*$ and the theorem.

For every set Y and subclass $\underline{\underline{V}}$ of $\underline{\underline{V}}(\mathfrak{R})$ we write $L(\underline{\underline{V}};Y)$ for the partly ordered set of congruences κ over $\underline{W}(\mathfrak{R};Y)$ such that $\underline{W}(\mathfrak{R};Y)/\kappa$ is isomorphic to an algebra in $\underline{\underline{V}}$. We write $L(\mathfrak{R};Y)$ for $L(\underline{\underline{V}};Y)$ when $\underline{\underline{V}} = \underline{\underline{V}}(\mathfrak{R})$. The following result is a consequence of Theorem 1.

<u>THEOREM 2.</u> For every universal class $\underline{\underline{V}}$ of $\mathfrak{R}$-algebras and every set Y the following conditions hold.

(2.1) The meet in $L(\mathfrak{R};Y)$ of a downward directed subset of $L(\underline{\underline{V}};Y)$ is in $L(\underline{\underline{V}};Y)$.

$(2.1)^*$ The join in $L(\mathfrak{R};Y)$ of an upward directed subset of $L(\underline{\underline{V}};Y)$ is in $L(\underline{\underline{V}};Y)$.

<u>PROOF.</u> Let I be a downward directed subset of $L(\underline{\underline{V}};Y)$. Consider the inverse system $\{f_{\kappa_2\kappa_1} : \underline{W}(\mathfrak{R};Y)/\kappa_1 \longrightarrow \underline{W}(\mathfrak{R};Y)/\kappa_2\}_{\kappa_1\subseteq\kappa_2,\kappa_1,\kappa_2\in I}$, where $f_{\kappa_2\kappa_1}$ is the canonical map which sends the equivalence class $[w]\kappa_1$ to $[w]\kappa_2$ for all $w \in W(\mathfrak{R};Y)$. Let $\kappa^* = \text{g.}\ell\text{.b.}(I)$. Then $\{f_{\kappa\kappa^*}\}_{\kappa\in I}$ is an inverse limit of $\{f_{\kappa_2\kappa_1}\}_{\kappa_1\subseteq\kappa_2,\kappa_1,\kappa_2\in I}$. For $\underline{W}(\mathfrak{R};Y)/\kappa^*$ is isomorphic to the subalgebra of the cartesian

product $C\Pi_{\kappa\in I}(\underline{W}(\mathcal{R};Y)/\kappa)$ defined by $\{g_w; g_w(\kappa) = [w]\kappa ;$ $w \in W(\mathcal{R};Y)\}, \kappa \in I\}$ which is an inverse limit of the inverse system under consideration, by the construction of inverse systems given in the proof of Theorem 1. In view of (1.1) and (1.2) of Theorem 1 this implies that $\underline{W}(\mathcal{R};Y)/\kappa^* \in \underline{\underline{V}}$ and hence $\kappa^* \in L(\underline{\underline{V}};Y)$. This proves (2.1).

The proof of $(2.1)^*$ is similar and is omitted.

Let us call a diagram in an arbitrary category mono (epi) if all of its morphisms are monomorphisms (epimorphisms). Then $(2.1)^*$ is clearly equivalent to $(1.2)^*$ (of Theorem 1) for epi direct systems. This special case of (1.2) has a simpler direct proof and was stated in [16]. The general property $(1.2)^*$ was given for some universal classes without detailed proof in [27] and [28].

§2. ASSEMBLIES OF ALGEBRAS.

A universal class $\underline{\underline{V}}(\Sigma)$ is called an assembly if Σ is equivalent to a set of sentences of the forms

(2) $$\forall x_1,\ldots,x_n(w_1 = w_1' \vee \ldots \vee w_m = w_m')$$

and

(3) $$\forall x_1,\ldots,x_n(w_1 \neq w_1' \vee \ldots \vee w_m \neq w_m') .$$

An assembly $\underline{\underline{V}}(\Sigma)$ is called positive or negative according as Σ is equivalent to a set of sentences of the form (2) or (3) only. An assembly definable by a set of equations or laws, i.e., sentences of the form

$$(4) \qquad \forall x_1, \ldots, x_n (w_1 \doteq w_1') \ ,$$

is called an *equationally defined assembly* or a *variety*. A negative assembly definable by a set of sentences of the form

$$(5) \qquad \forall x_1, \ldots, x_n (w_1 \neq w_1')$$

is called a *negationally defined* assembly; a sentence of the form (5) is called a *negation*. Negations arise quite commonly in Number Theory. The most well-known example of a number theoretic negation is the 'Fermat's negation' $\forall x_1, x_2, x_3 (x_1^n + x_2^n \neq x_3^n)$ which Fermat conjectured to hold for $n \geqq 3$ for the algebra of positive integers under ordinary addition and multiplication. Whether a systematic study of assemblies defined by equations and negations could be of any use to Diophantine Analysis is anybody's guess.

The following result characterizes negative assemblies in terms of the corresponding categories of homomorphisms.

THEOREM 3. A class $\mathbf{V}$ of $\mathfrak{R}$-algebras is a negative assembly if and only if:

(3.1) For every monomorphism $\mathbf{A}_1 \longrightarrow \mathbf{A}_2$ in $\mathbf{K}(\mathfrak{R})$ if $\mathbf{A}_2 \in \mathbf{K}(\mathbf{V})$ then $\mathbf{A}_1 \in \mathbf{K}(\mathbf{V})$.

(3.2) For every epimorphism $\mathbf{A}_1 \longrightarrow \mathbf{A}_2$ in $\mathbf{K}(\mathfrak{R})$ if $\mathbf{A}_2 \in \mathbf{K}(\mathbf{V})$ then $\mathbf{A}_1 \in \mathbf{K}(\mathbf{V})$.

(3.3) Direct limits in $\underline{\underline{K}}(\mathfrak{R})$ of direct systems in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$.

We prove Theorem 3 in the following form.

THEOREM 4. A class $\underline{\underline{V}}$ of $\mathfrak{R}$-algebras is a negative assembly if and only if:

(4.1) $\underline{A} \in \underline{\underline{V}}$ if and only if every finitely generated subalgebra of $\underline{A}$ is embeddable in an algebra of $\underline{\underline{V}}$.

(4.2) The join in $L(\mathfrak{R};X)$ of an upward directed subset of $L(\underline{\underline{V}};X)$ is in $L(\underline{\underline{V}};X)$.

(4.3) If $\kappa \subseteq \kappa_1$, $\kappa \in L(\mathfrak{R};X)$ and $\kappa_1 \in L(\underline{\underline{V}};X)$ then $\kappa \in L(\underline{\underline{V}};X)$.

PROOF. In view of Theorem 2 the 'only if' part of the present theorem is fairly straightforward. For the second part let Σ be the set of all sentences of the form

$$(3) \qquad \forall x_1, \ldots, x_n (w_1 \neq x_1' \vee \ldots \vee w_m \neq w_m') .$$

that hold in all the algebras in $\underline{\underline{V}}$. Let $\underline{A}$ be any $\mathfrak{R}$-algebra which satisfies Σ. By (4.1) we can assume that $\underline{A}$ is finitely generated and take $\underline{A}$ to be $\underline{W}(\mathfrak{R};X)/\kappa$ for some $\kappa \in L(\mathfrak{R};X)$; so that we have to show that $\kappa \in L(\underline{\underline{V}};X)$. Let $r = \{\langle w_1, w_1'\rangle, \ldots, \langle w_m, w_m'\rangle\} \subseteq \kappa$, where $w_1, \ldots, w_m$, $w_1', \ldots, w_m' \in W(\mathfrak{R};X)$. Then the sentence (3) does not hold in $\underline{A}$ and therefore does not belong to Σ. This implies that there is an algebra $\underline{A}_r \in \underline{\underline{V}}$ which does not satisfy (3). Clearly there is a finitely generated subalgebra $\underline{A}_r'$ of $\underline{A}_r$ which does not satisfy (3). By (4.1) the class $\underline{\underline{V}}$ is abstract so that we can find $\kappa'_r \in L(\underline{\underline{V}};X)$ such that $\underline{W}(\mathcal{R};X)/\kappa'_r$ is isomorphic to $\underline{A}'_r$ and therefore does not satisfy (3). This means that there exist $u_1,\ldots,u_n \in W(\mathcal{R};X)$ such that $\langle w_j(u_1,\ldots,u_n),\ w'_j(u_1,\ldots,u_n)\rangle \in \kappa'_r$ for $1 \leq j \leq m$. Let $\underline{A}''_r$ be the subalgebra of $\underline{A}'_r$ generated by $[u_1]\kappa'_r,\ldots,[u_n]\kappa'_r$. Then $\underline{A}''_r$ is isomorphic to $\underline{W}(\mathcal{R};X)/\kappa''_r$ for some $\kappa''_r \in L(\underline{\underline{V}};X)$ such that $\langle w_j(x_1,\ldots,x_n),\ w'_j(x_1,\ldots,x_n)\rangle \in \kappa''_r$ for all j, $1 \leq j \leq m$. Let κ_r be the least congruence over $\underline{W}(\mathcal{R};X)$ containing r. Then $\kappa_r \subseteq \kappa''_r$ and hence by (4.3) we have that $\kappa_r \in L(\underline{\underline{V}};X)$. The set $\{\kappa_r\ ;\ r$ is a finite subset of $\kappa\}$ is an upward directed subset of $L(\underline{\underline{V}};X)$ whose join in $L(\mathcal{R};X)$ is κ. Hence, by (4.2), $\kappa \in L(\underline{\underline{V}};X)$ and the proof of the theorem is complete.

Theorem 3 is now a direct consequence of Theorem 1, Theorem 2 and Theorem 4.

The following result about assemblies is obvious.

<u>THEOREM 5</u>. Let $\underline{\underline{V}}$ be an assembly. Then:

(5.1) For every monomorphism $\underline{A}_1 \longrightarrow \underline{A}_2$ in $\underline{\underline{K}}(\mathcal{R})$ if $A_2 \in \underline{\underline{K}}(\underline{\underline{V}})$ then $\underline{A}_1 \in \underline{\underline{K}}(\underline{\underline{V}})$.

(5.2) For epimorphisms $\underline{A}_1 \longrightarrow \underline{A}_2$ and $\underline{A}_2 \longrightarrow \underline{A}'_1$ in $\underline{\underline{K}}(\mathcal{R})$ if $A_1, A'_1 \in \underline{\underline{K}}(\underline{\underline{V}})$ then $\underline{A}_2 \in \underline{\underline{K}}(\underline{\underline{V}})$.

(5.3) Inverse (direct) limits of inverse (direct) systems in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$.

Whether (5.1), (5.2) and (5.3) characterise assemplies is not known. Nor do we know of any simple categorical characterization of positive assemblies.

For universal $\underline{\underline{V}}$ the partly ordered set $L(\underline{\underline{V}};Y)$ has maximal and minimal elements for every Y. This follows from (2.1), $(2.1)^*$ and Zorn's lemma. Birkhoff's theorem [2] about varieties shows that for positive assemblies $\underline{\underline{V}}$ the minimal elements of $L(\underline{\underline{V}};Y)$ collapse to a least element if and only if $\underline{\underline{V}}$ is equationally defined. However, it is not true that for negative assemblies $\underline{\underline{V}}$ the partly ordered set $L(\underline{\underline{V}};Y)$ has a greatest element for all sets Y if and only if $\underline{\underline{V}}$ is negatively defined. Indeed the only negative assembly $\underline{\underline{V}}$ for which $L(\underline{\underline{V}};Y)$ has a greatest element for all sets Y is the class $\underline{\underline{V}}(\mathcal{R})$ of all $\mathcal{R}$-algebras. To see this assume that Y is infinite. Let $\underline{A}$ be a finitely generated algebra in $\underline{\underline{V}}$ and w_1, w_2 be arbitrary words in Y. We can clearly find $\kappa_1, \kappa_2 \in L(\mathcal{R};Y)$ such that $\underline{W}(\mathcal{R};Y)/\kappa_1$, $\underline{W}(\mathcal{R};Y)/\kappa_2$ are isomorphic to $\underline{A}$ and $\langle w_1, y\rangle \in \kappa_1$, $\langle w_2, y\rangle \in \kappa_2$, where $y \in Y$ and y does not enter w_1 or w_2 . Since $\underline{A} \in \underline{\underline{V}}$ then $\kappa_1, \kappa_2 \in L(\underline{\underline{V}};Y)$. If $\bar{\kappa}$ is the greatest element of $L(\underline{\underline{V}};Y)$ then $\kappa_1, \kappa_2 \subseteq \bar{\kappa}$, so that $\langle w_1, y\rangle$, $\langle w_2, y\rangle \in \bar{\kappa}$ which implies that $\langle w_1, w_2\rangle \in \bar{\kappa}$. Since w_1, w_2 were arbitrary words of $W(\mathcal{R};Y)$ we see that $\bar{\kappa} = W(\mathcal{R};Y) \times W(\mathcal{R};Y)$. By (4,3), this implies that if $\underline{\underline{V}}$ is a negative assembly then $L(\underline{\underline{V}};Y) = L(\mathcal{R};Y)$ for all infinite Y .

But every $\mathfrak{R}$-algebra is isomorphic to an algebra of the form $\underline{W}(\mathfrak{R};Y)/\kappa$, where $\kappa \in L(\mathfrak{R};Y)$ and Y is infinite. Hence $\underline{\underline{V}} = \underline{\underline{V}}(\mathfrak{R})$.

The following result gives a necessary and sufficient condition for a negative assembly to be negationally defined.

<u>THEOREM 6</u>. A negative assembly $\underline{\underline{V}}$ is negationally defined if and only if:

(6.1) If the join κ in $L(\mathfrak{R};X)$ of a family $\{\kappa_i\}_{i\in I}$, $\kappa_i \in L(\underline{\underline{V}};X)$, coincides with the set-theoretic union $\bigcup_{i\in I} \kappa_i$ then $\kappa \in L(\underline{\underline{V}};X)$.

<u>PROOF</u>. Let $\underline{\underline{V}}$ be negationally defined and let the join κ in $L(\mathfrak{R};X)$ of $\{\kappa_i\}_{i\in I}$, $\kappa_i \in L(\underline{\underline{V}};X)$, coincide with $\bigcup_{i\in I} \kappa_i$. To show that $\kappa \in L(\underline{\underline{V}};X)$ we need to prove that every negation that holds in every algebra of $\underline{\underline{V}}$ also holds in $\underline{W}(\mathfrak{R};X)/\kappa$. Let $\forall x_1,\ldots,x_n(w \neq w')$ be such a negation. For all $u_1,\ldots,u_n \in W(\mathfrak{R};X)$ and all $i \in I$ we have that $\langle w(u_1,\ldots,u_n), w'(u_1,\ldots,u_n)\rangle \notin \kappa_i$. Since κ is the set-theoretic union of the κ_i we see that $\langle w(u_1,\ldots,u_n), w'(u_1,\ldots,u_n)\rangle \notin \kappa$ for all $u_1,\ldots,u_n \in W(\mathfrak{R};X)$. Hence $\forall x_1,\ldots,x_n(w \neq w')$ holds in $\underline{W}(\mathfrak{R};X)/\kappa$.

Conversely let (6.1) hold and let Σ be the set of all negations that hold in every algebra in $\underline{\underline{V}}$. Let $\underline{A} \in \underline{\underline{V}}(\mathfrak{R})$

and let $\underline{A}$ satisfy Σ. As in the proof of Theorem 4 we can take $\underline{A}$ to be $\underline{W}(\mathfrak{K};X)/\kappa$ for some $\kappa \in L(\mathfrak{K};X)$ and prove that for every $r \in \kappa$ the least congruence κ_r containing r is in $L(\underline{\underline{V}};X)$. But then κ is the set-theoretic union of $\{\kappa_r ; r \in \kappa\}$. Hence, by (6.1), $\kappa \in L(\underline{\underline{V}};X)$ and therefore $\underline{A} \in \underline{\underline{V}}$. Thus $\underline{\underline{V}}$ is negationally defined and the theorem is proved.

Condition (6.1) also enters a characterization of semi-varieties (see next section).

However, (6.1) is not an order-theoretic condition and therefore, perhaps, cannot be translated into a categorical property of $\underline{\underline{K}}(\underline{\underline{V}})$. We leave open the problem of characterizing negationally defined assemblies (and semi-varieties) in categorical terms. A solution of this problem may involve considering a category different from $\underline{\underline{K}}(\mathfrak{K})$.

The results, concepts and problems of this section form part of [28].

§3. QUASI-VARIETIES AND SEMI-VARIETIES.

A universal class $\underline{\underline{V}}(\Sigma)$, $\Sigma \subseteq \Phi_0(\mathfrak{K})$, is called a quasi-variety [6] if Σ is equivalent to a set of sentences of the form

$$(6) \qquad \forall x_1,\dots,x_n((w_1 = w_1' \wedge \dots \wedge w_m = w_m') \longrightarrow w = w'),$$

where $w_1,\dots,w_m,w_1',\dots,w_m',w,w'$ are $\mathfrak{K}$-words in $x_1,\dots,x_n$.

A sentence of the form (6) is called an <u>implication</u>. A universal class $\underline{\underline{V}}(\Sigma)$ is called a <u>semi-variety</u> if Σ is equivalent to a set of implications of the simpler form

$$(7) \qquad \forall x_1, \ldots, x_n (w_1 = w_1' \longrightarrow w = w') \ .$$

In this section we give characterizations of quasi-varieties and semi-varieties. Rather simple categorical characterizations of quasi-varieties are given but we are unable to characterize semi-varieties in a similar way.

We begin by defining an order-theoretic concept. Let L_1 be a subset of a partly ordered set L with $\leq$ as the partial order. By an L_1-cover of an element z of L we shall understand an element $\bar{z}$ of L_1 such that $z \leq \bar{z}$ and if $z < z'$ then $\bar{z} \not\geq z'$. Not every element of L may have an L_1-cover. If L_1 is a lattice then there is at most one L_1-cover of every element of L.

<u>THEOREM 7</u>. Let $\underline{\underline{V}}$ be abstract and hereditary. Then the following three conditions are equivalent.

(7.1) Every $\kappa \in L(\Re;Y)$ has an $L(\underline{\underline{V}};Y)$-cover for every set Y.

(7.2) The category $\underline{\underline{K}}(\underline{\underline{V}})$ is a coreflective [20] subcategory of $\underline{\underline{K}}(\Re)$.

(7.3) Cartesian products of families of algebras in $\underline{\underline{V}}$ are in $\underline{\underline{V}}$ and the trivial algebra is in $\underline{\underline{V}}$.

<u>PROOF</u>. Let (7.1) hold. For every $r \subseteq W(\Re;Y) \times W(\Re;Y)$ we write $\chi(\underline{\underline{V}};r)$ for the $L(\underline{\underline{V}};Y)$-cover of the least congruence over

$\underline{W}(\mathfrak{R};Y)$ containing r . We write $\chi(\mathfrak{R};r)$ for $\chi(\underline{\underline{V}};r)$ when $\underline{\underline{V}} = \underline{\underline{V}}(\mathfrak{R})$ so that $\chi(\mathfrak{R};r)$ is the least congruence over $\underline{W}(\mathfrak{R};Y)$ containing r . For every $\underline{A} = \langle A,\alpha\rangle \in \underline{\underline{K}}(\mathfrak{R})$ define
$R(\underline{A}) = \underline{W}(\mathfrak{R};A)/\chi(\underline{\underline{V}};\kappa(\underline{A}))$, where
$\kappa(\underline{A}) = \{\langle u_1,u_2\rangle \ ; \ u_1,u_2 \in W(\mathfrak{R};A) \ , \ u_1 = u_2 \text{ in } \underline{A}\}$. Since $\kappa(\underline{A}) \subseteq \chi(\underline{\underline{V}};\kappa(\underline{A}))$ there is an epimorphism $f(\underline{A}) : \underline{A} \longrightarrow R(\underline{A})$ which takes $a \in A$ to the equivalence class $[a]\chi(\underline{\underline{V}};\kappa(\underline{A}))$ determined by 'a' under $\chi(\underline{\underline{V}};\kappa(\underline{A}))$. Let $e : \underline{A} \longrightarrow \underline{A}'$ be any homomorphism in $K(\mathfrak{R})$ with $\underline{A}' \in \underline{\underline{V}}$. Then the function $g : R(\underline{A}) \longrightarrow \underline{A}'$ which takes $[a]\chi(\underline{\underline{V}};\kappa(\underline{A}))$ into $e(a)$, $a \in A$, is clearly such that the following diagram commutes:

$$\begin{tikzcd} \underline{A} \arrow[d, "e"'] \arrow[dr, "f(\underline{A})"] & \\ \underline{A}' & R(\underline{A}) \arrow[l, "g"] \end{tikzcd}$$

That g is a well-defined homomorphism can be seen as follows. Clearly by the abstractness and heredity of $\underline{\underline{V}}$ we can assume e to be onto, $\underline{A}'$ to be $\underline{W}(\mathfrak{R};A)/\kappa$ and $e(a) = [a]\kappa$ for some $\kappa \in L(\underline{\underline{V}};A)$. Then $\kappa(\underline{A}) \subseteq \kappa$ and by the definition of $\chi(\underline{\underline{V}};\kappa(\underline{A}))$ we have that $\chi(\underline{\underline{V}};\kappa(\underline{A})) \subseteq \kappa$. This shows that g is indeed an homomorphism. Further, it is clear that if e is given then g is the only map that makes the above diagram commute. Hence $R(\underline{A})$ together with the morphism $f(\underline{A})$ provides a coreflection of $\underline{A}$ in $\underline{\underline{K}}(\underline{\underline{V}})$. This proves that (7.1) implies (7.2).

Now assume (7.2). By Proposition 5.1 of [20] concerning full coreflective subcategories we see that products of families of objects in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$. But products in $\underline{\underline{K}}(\mathfrak{R})$ coincide with cartesian products and therefore the first part of (7.3) holds. Let $\underline{E}$ be the trivial $\mathfrak{R}$-algebra and let $R(\underline{E})$ together with the map $\underline{E} \longrightarrow R(\underline{E})$ provide a coreflection of $\underline{E}$. Since every homomorphism from $\underline{E}$ is a monomorphism $\underline{E}$ is a subalgebra of $R(\underline{E})$. But $R(\underline{E}) \in \underline{\underline{K}}(\underline{\underline{V}})$ and $\underline{\underline{V}}$ is hereditary. Hence $\underline{E} \in \underline{\underline{K}}(\underline{\underline{V}})$. This proves that (7.3) holds if (7.2) holds.

Finally assume (7.3). We first note that for all sets Y the meet in $L(\mathfrak{R};Y)$ of a subset of $L(\underline{\underline{V}};Y)$ is in $L(\underline{\underline{V}};Y)$. For let $\kappa_i \in L(\underline{\underline{V}};Y)$, where i ranges over an index set I, and let $\kappa = \bigcap_{i \in I} \kappa_i$. Then $W(\mathfrak{R};Y)/\kappa$ is isomorphic to a subalgebra of the cartesian product of $\{\underline{W}(\mathfrak{R};Y)/\kappa_i\}_{i\in I}$. By our assumption cartesian products of families of algebras in $\underline{\underline{V}}$ are in $\underline{\underline{V}}$. Hence, since $\underline{\underline{V}}$ is abstract and hereditary, $\underline{W}(\mathfrak{R};Y)/\kappa \in \underline{\underline{V}}$; so that $\kappa \in L(\underline{\underline{V}};Y)$. Next note that since $\underline{\underline{V}}$ contains the trivial algebra the greatest element $\bar{\kappa} = W(\mathfrak{R};Y) \times W(\mathfrak{R};Y)$ of $L(\mathfrak{R};Y)$ is in $L(\underline{\underline{V}};Y)$. Now let $\kappa \in L(\mathfrak{R};Y)$. Then the meet of the non-empty set $\{\kappa';\ \kappa \subseteq \kappa',\ \kappa' \in L(\underline{\underline{V}};Y)\}$ is the $L(\underline{\underline{V}};Y)$-cover of κ. This proves that (7.3) implies (7.1). The proof of the theorem is complete.

THEOREM 8. Let $\underline{\underline{V}} \subseteq \underline{\underline{V}}(\mathfrak{R})$. Then $\underline{\underline{V}}$ is a quasi-variety if and only if :

(8.1) For every monomorphism $\underline{A}_1 \longrightarrow \underline{A}_2$ in $\underline{\underline{K}}(\mathfrak{R})$ if $\underline{A}_2 \in \underline{\underline{K}}(\underline{\underline{V}})$ then $\underline{A}_1 \in \underline{\underline{K}}(\underline{\underline{V}})$.

(8.2) Direct limits in $\underline{\underline{K}}(\mathfrak{R})$ of direct systems in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$.

(8.3) The category $\underline{\underline{K}}(\underline{\underline{V}})$ is a coreflective subcategory of $\underline{\underline{K}}(\mathfrak{R})$.

PROOF. Let $\underline{\underline{V}}$ be a quasi-variety. Then (8.1), (8.2) follow from Theorem 1 and (8.3) follows from Theorem 7 since it is well-known (and easy to show) that quasi-varieties are abstract, hereditary and satisfy (7.3).

Conversely assume (8.1), (8.2) and (8.3). Theorem 5 of [16] tells us that (8.1), (8.2) and (7.3) imply that $\underline{\underline{V}}$ is universal. Hence by Theorem 7 we conclude that if $\underline{\underline{V}}$ satisfies (8.1), (8.2) and (8.3) then $\underline{\underline{V}}$ is universal. Now the corollary 4.4 on page 235 of [6] tells us that a universal class $\underline{\underline{V}}$ is a quasi-variety if and only if (7.3) holds. In view of the equivalence of (7.3) and (8.3) this completes the proof of Theorem 8.

A different proof of Theorem 8 is indicated in [27].

We can obtain two more characterizations of quasi-varieties by replacing (8.3) by (7.1) or (7.3).

We now give a characterization of semi-varieties.

THEOREM 9. A quasi-variety $\underline{\underline{V}}$ is a semi-variety if and only if:

(9.1) = (6.1) . If the join κ of $\{\kappa_i\}_{i\in I}$, $\kappa_i \in L(\underline{\underline{V}};X)$, coincides with the set-theoretic union of the κ_i then $\kappa \in L(\underline{\underline{V}};X)$.

PROOF. Let $\underline{\underline{V}}$ be a semi-variety defined by Σ and let $\kappa = \cup_{i\in I} \kappa_i$ be the join in $L(\mathfrak{R};X)$ of $\{u_i\}_{i\in I}$, $\kappa_i \in L(\underline{\underline{V}};X)$. We have to show that $\kappa \in L(\underline{\underline{V}};X)$. Let

(7) $$\forall x_1,\ldots,x_n(w_1 = w_1' \longrightarrow w = w')$$

be deducible from Σ . Let $u_1,\ldots,u_n \in W(\mathfrak{R};X)$ and $\langle w_1(u_1,\ldots,u_n), w_1'(u_1,\ldots,u_n)\rangle \in \kappa$. Then $\langle w_1(u_1,\ldots,u_n), w_1'(u_1,\ldots,u_n)\rangle \in \kappa_i$ for some $i \in I$, because $\kappa = \cup_{i\in I} \kappa_i$. Since $\underline{W}(\mathfrak{R};X)/\kappa_i$ satisfies (7) we have that $\langle w(u_1,\ldots,u_n), w'(u_1,\ldots,u_n)\rangle \in \kappa_i$ and hence $\langle w(u_1,\ldots,u_n), w'(u_1,\ldots,u_n)\rangle \in \kappa$; so that $\underline{W}(\mathfrak{R};X)/\kappa$ satisfies every sentence of the form (7) that is deducible from Σ. Hence $\underline{W}(\mathfrak{R};X)/\kappa \in \underline{\underline{V}}(\Sigma)$ or $\kappa \in L(\underline{\underline{V}};X)$.

Conversely let $\underline{\underline{V}}$ be a quasi-variety defined by Σ and let $\underline{\underline{V}}$ satisfy (9.1). Let Σ_1 be the set of all implications of the form (7) that are deducible from Σ and write $\underline{\underline{V}}_1 = \underline{\underline{V}}(\Sigma_1)$. We show that $\underline{\underline{V}}_1 = \underline{\underline{V}}$. But first we need to prove the following statement .

(9.2) The implication

(6) $$\forall x_1,\ldots,x_n((w_1 = w_1' \wedge \cdots \wedge w_m = w_m') \longrightarrow w = w')$$

is deducible from Σ if and only if $\langle w,w' \rangle \in \chi(\underline{\underline{V}};r)$, where $r = \{\langle w_1,w_1' \rangle,\dots,\langle w_m,w_m' \rangle\}$.

Assume that (6) is deducible from Σ . Then $\underline{W}(\mathcal{R};X)/\kappa$ satisfies (6) where $\kappa = \chi(\underline{\underline{V}};r)$. Since $r \subseteq \kappa$ this implies that $\langle w,w' \rangle \in \kappa$ and one part of (9.2) is proved.

Now let $\langle w,w' \rangle \in \chi(\underline{\underline{V}};r)$. We show that (6) holds in every algebra of $\underline{\underline{V}}$. Since $\underline{\underline{V}}$ is a universal class it is sufficient to show that $\underline{W}(\mathcal{R};X)/\kappa$ satisfies (6) for every $\kappa \in L(\underline{\underline{V}};X)$. Let $u_1,\dots,u_n$ be arbitrary elements of $W(\mathcal{R};X)$ such that $\langle w_i(u_1,\dots,u_n),\ w_i'(u_1,\dots,u_n) \rangle \in \kappa$ for $1 \leqq i \leqq m$. We have to show that $\langle w(u_1,\dots,u_n),\ w'(u_1,\dots,u_n) \rangle \in \kappa$. Let $\underline{A}$ be the subalgebra of $\underline{W}(\mathcal{R};X)/\kappa$ generated by $[u_1]\kappa,\dots,[u_n]\kappa$. Then $\underline{A}$ is isomorphic to an algebra $\underline{W}(\mathcal{R};X)/\kappa_1$ with an isomorphism $f : \underline{W}(\mathcal{R};X)/\kappa_1 \longrightarrow \underline{A}$ under which $[x_1]\kappa_1,\dots,[x_n]\kappa_1$ go to $[u_1]\kappa,\dots,[u_n]\kappa$ respectively. Then $\langle w_i,w_i' \rangle = \langle w_i(x_1,\dots,x_n),\ w_i'(x_1,\dots,x_n) \rangle \in \kappa_1$ for $1 \leqq i \leqq m$, so that $r \subseteq \kappa_1$. Since $\underline{\underline{V}}$ is abstract and hereditary $\kappa_1 \in L(\underline{\underline{V}};X)$. Hence $\chi(\underline{\underline{V}};r) \subseteq \kappa_1$ and therefore $\langle w(x_1,\dots,x_n),\ w'(x_1,\dots,x_n) \rangle \in \kappa_1$. Applying the isomorphism f we get $\langle w(u_1,\dots,u_n),\ w'(u_1,\dots,u_n) \rangle \in \kappa$. This completes the proof of (9.2).

We return to proving that $\underline{\underline{V}}_1 = \underline{\underline{V}}$. Recall that Σ_1 is deducible from Σ and therefore $\underline{\underline{V}}_1 = V(\Sigma_1) \supseteq \underline{\underline{V}} = \underline{\underline{V}}(\Sigma)$. This clearly implies that $L(\underline{\underline{V}}_1;X) \supseteq L(\underline{\underline{V}};X)$ and that

$\chi(\underline{\underline{V}};r) \supseteq \chi(\underline{\underline{V}}_1;r)$ for all $r \subseteq W(\mathfrak{R};X) \times W(\mathfrak{R};X)$. If $r = \{\langle w_1, w_1' \rangle\}$ then $\chi(\underline{\underline{V}};r) = \chi(\underline{\underline{V}}_1;r)$, for all $w_1, w_1' \in W(\mathfrak{R};X)$. For let $\langle w, w' \rangle \in \chi(\underline{\underline{V}};r)$. Then by (9.2) the sentence

$$\forall x_1, \ldots, x_n (w_1 = w_1' \longrightarrow w = w') \tag{7}$$

is deducible from Σ, where $x_1, \ldots, x_n$ are all the variables that occur in w, w', w_1 or w_1'. This means that (6) is in Σ_1 and therefore again applying (9.2) we see that $\langle w, w' \rangle \in \chi(\underline{\underline{V}}_1;r)$. Hence $\chi(\underline{\underline{V}};r) \subseteq \chi(\underline{\underline{V}}_1;r)$. The opposite inclusion is true anyway and therefore $\chi(\underline{\underline{V}};r) = \chi(\underline{\underline{V}}_1;r)$ when r is a singleton of the form $\{\langle w_1, w_1' \rangle\}$. We use this to prove that $\chi(\underline{\underline{V}};r) = \chi(\underline{\underline{V}}_1;r)$ for every subset r of $W(\mathfrak{R};X) \times W(\mathfrak{R};X)$. We can write $\chi(\underline{\underline{V}}_1;r) = \bigcup \chi(\underline{\underline{V}}_1;r')$, where $\bigcup$ extends over all r' of the form $\{\langle w_1, w_1' \rangle\}$, $\langle w_1, w_1' \rangle \in \chi(\underline{\underline{V}}_1;r)$. But $\chi(\underline{\underline{V}}_1;r') = \chi(\underline{\underline{V}};r')$ if $r' = \{\langle w_1, w_1' \rangle\}$. Hence $\chi(\underline{\underline{V}}_1;r)$ is a set-theoretic union of congruences in $L(\underline{\underline{V}};X)$. By (9.1) this means that $\chi(\underline{\underline{V}}_1;r) \in L(\underline{\underline{V}};X)$. Hence $\chi(\underline{\underline{V}}_1;r) \supseteq \chi(\underline{\underline{V}};r)$ and therefore $\chi(\underline{\underline{V}}_1;r) = \chi(\underline{\underline{V}};r)$. In view of (9.2) the equation $\chi(\underline{\underline{V}}_1;r) = \chi(\underline{\underline{V}};r)$ implies that every implication deducible from one of the sets Σ, Σ_1 is deducible from the other. Hence Σ, Σ_1 are equivalent and $\underline{\underline{V}} (= \underline{\underline{V}}_1)$ is a semi-variety. The proof of the theorem is complete.

Theorem 9 is proved in [28].

CHAPTER 3

GALAXIES OF ALGEBRAS.

The conditions of coreflectivity, abstractness and heredity entering the characterization of quasi-varieties given in §3 of Chapter 2 seem to form an interesting and fairly rich set of conditions. The classes of algebras satisfying these three conditions seem to deserve independent study. We give them a name - galaxy of algebras. More precisely, we first define a galaxy in a category $\underline{\underline{A}}$ to be a coreflective subcategory $\underline{\underline{A}}_1$ such that for every monomorphism $\underline{A} \longrightarrow \underline{A}'$ in $\underline{\underline{A}}$ if $\underline{A}' \in \underline{\underline{A}}_1$ then $\underline{A} \in \underline{\underline{A}}_1$. If $\underline{\underline{V}} \subseteq \underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathfrak{R})$ then $\underline{\underline{V}}$ is defined to be a galaxy in $\underline{\underline{V}}_1$ if $\underline{\underline{K}}(\underline{\underline{V}})$ is a galaxy in $\underline{\underline{K}}(\underline{\underline{V}}_1)$. A galaxy in $\underline{\underline{V}}(\mathfrak{R})$ will be simply called a galaxy of $\mathfrak{R}$-algebras. By Theorem 7 (Chap. 2) galaxies of $\mathfrak{R}$-algebras are precisely what were called 'quasi-free classes of algebras' in [17]. But our definition is conceptually different and lends itself to more general situations.

In this chapter we give some properties of galaxies of $\mathfrak{R}$-algebras. We shall assume $\mathfrak{R}$ to be arbitrarily fixed.

We begin by collecting some immediate consequences of our definitions.

<u>THEOREM 10</u>. Let $\underline{\underline{V}}_1, \underline{\underline{V}}_2, \underline{\underline{V}}_3 \subseteq \underline{\underline{V}}(\mathfrak{R})$. Then:

(0.1) If $\underline{\underline{V}}_1$ is a galaxy in $\underline{\underline{V}}_2$ and $\underline{\underline{V}}_2$ is a galaxy in $\underline{\underline{V}}_3$ then $\underline{\underline{V}}_1$ is a galaxy in $\underline{\underline{V}}_3$.

(0.2) If $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}_2$ and $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are galaxies in $\underline{\underline{V}}_3$ then $\underline{\underline{V}}_1$ is a galaxy in $\underline{\underline{V}}_2$.

(0.3) Let $\underline{\underline{V}}$ be a galaxy. Then $\underline{\underline{K}}(\underline{\underline{V}})$ is complete and cocomplete [20, p. 44] and limits in $\underline{\underline{K}}(\mathcal{R})$ of diagrams in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$. Moreover the partly ordered set $L(\underline{\underline{V}};Y)$ is a complete lattice for every set Y .

(0.4) If $\underline{\underline{V}}_1$ is a galaxy in $\underline{\underline{V}}_2$, $\underline{\underline{V}}_1$ contains an algebra generated by a set Y and $\underline{\underline{V}}_2$ admits a free algebra on Y then $\underline{\underline{V}}_1$ also admits a free algebra on Y .

<u>PROOF.</u> (0.1) and (0.2) follow easily from the definition of a galaxy. Categorical part of (0.3) follows from well-known [20, pp. 129-130] properties of coreflective subcategories in general. For the other part of (0.3) let $\kappa_i \in L(\underline{\underline{V}};Y)$; where i ranges over an index set. The $L(\underline{\underline{V}};Y)$-cover of $\bigcup_{i\in I} \kappa_i$ exists by Theorem 7 of Chap. 2 and is clearly the join of $\{\kappa_i\}_{i\in I}$ in $L(\underline{\underline{V}};Y)$. Moreover $\underline{W}(\mathcal{R};Y)/\bigcap_{i\in I} \kappa_i$ is a subalgebra of the cartesian product of $\{\underline{W}(\mathcal{R};Y)/\kappa_i\}_{i\in I}$ (Cf. proof of Theorem 7 of Chap. 2) and hence by (7.3) of Theorem 7 of Chap. 2 $\underline{W}(\mathcal{R};Y)/\kappa_{i\in I}\;{}_{i}$ is in $\underline{\underline{V}}$ which means that $\bigcap_{i\in I} \kappa_i \in L(\underline{\underline{V}};Y)$. Hence $L(\underline{\underline{V}};Y)$ is a complete lattice.

(0.4) can be proved as follows.

Let $F_2(Y)$ be the free algebra in $\underline{\underline{V}}_2$ on Y and let $F_1(Y)$, $f : F_2(Y) \longrightarrow F_1(Y)$ provide a coreflection of $F_2(Y)$ in $\underline{\underline{V}}_1$. By our assumption we can find $\underline{A} \in \underline{\underline{V}}_1$ such that $\underline{A}$ is generated by Y . Let $g : F_2(Y) \to \underline{A}$ be the unique homomorphism that extends the inclusion map from Y into $A = \text{carrier}(\underline{A})$. By the definition of coreflection there exists a unique map $e : F_1(Y) \longrightarrow \underline{A}$ such that the diagram

$$\begin{tikzcd}
F_2(Y) \arrow[d, "f"'] \arrow[dr, "g"] & \\
F_1(Y) \arrow[r, "e"'] & \underline{A}
\end{tikzcd}$$

commutes. Since $g|Y$ is one-to-one we see from the commutativity of the above diagram that $f\,|Y$ is also one-to-one. Hence we can assume that $Y \subseteq \text{carrier}\ (F_1(Y))$ and that f extends the inclusion map from Y into carrier $(F_1(Y))$. As in the proof of Theorem 7 of Chap. 2 we can further assume that f is onto. Since $F_2(Y)$ is generated by Y these assumptions imply that $F_1(Y)$ is also generated by Y . Let $\underline{A}_1$ be an arbitrary algebra in $\underline{\underline{V}}_1$. We complete the proof of (0.4) by showing that any function $d : Y \longrightarrow \underline{A}_1$ can be uniquely extended to a homomorphism $e_1 : F_1(Y) \longrightarrow \underline{A}_1$. Surely we can uniquely extend d to a homomorphism $g_1 : F_2(Y) \longrightarrow \underline{A}_1$. Let $e_1 : F_1(Y) \longrightarrow \underline{A}_1$ be the unique homomorphism which makes the diagram

$$\begin{tikzcd} F_2(Y) \arrow[d, "f"'] \arrow[dr, "g_1"] & \\ F_1(Y) \arrow[r, "e_1"'] & A_1 \end{tikzcd}$$

commute. Then it is clear that e_1 extends d. Let e_1^* also extend d. Then $g_1^* = e_1^* f$ is such that $g_1^* \mid Y = d$. Since $F_2(Y)$ is free in $\underline{\underline{V}}_2$ and $\underline{A}_1 \in \underline{\underline{V}}_1 \subseteq \underline{\underline{V}}_2$ this shows that $g_1^* = g_1$ which in turn implies that $e_1^* = e_1$ because the commutativity of the last diagram uniquely determines e_1 for given g_1. Hence $F_1(Y)$ is free in $\underline{\underline{V}}_1$. The proof of the theorem is complete.

Let $\underline{\underline{S}}$ be the category of sets and functions and let $\underline{\underline{P}}$ be the category of partly ordered sets and order preserving functions. Given $\underline{\underline{V}} \subseteq \underline{\underline{V}}(\mathfrak{R})$ we have already defined an object $L(\underline{\underline{V}};Y)$ of $\underline{\underline{P}}$ for every object Y of $\underline{\underline{S}}$. When $\underline{\underline{V}}$ is a galaxy (in $\underline{\underline{V}}(\mathfrak{R})$) we can also define a map $L(\underline{\underline{V}};f)$ of $\underline{\underline{P}}$ for every map $f : Y_1 \longrightarrow Y_2$ of $\underline{\underline{S}}$. For this let us define $[L(\underline{\underline{V}};f)](\kappa)$, for every $\kappa \in L(\underline{\underline{V}};Y_1)$, to be the $L(\underline{\underline{V}};Y_2)$-cover of

$\kappa^f = \{\langle \underline{f}(u_1), \underline{f}(u_2)\rangle;\ \langle u_1, u_2\rangle \in \kappa\}$, where $\underline{f} = \underline{W}(\mathfrak{R};f)$ is the unique homomorphism that extends f to $\underline{W}(\mathfrak{R};Y_1)$. The function

$L(\underline{\underline{V}};f) : L(\underline{\underline{V}};Y_1) \longrightarrow L(\underline{\underline{V}};Y_2)$ is clearly order preserving. We have thus defined, for every galaxy $\underline{\underline{V}}$, a function $L(\underline{\underline{V}};-) : \underline{\underline{S}} \longrightarrow \underline{\underline{P}}$.

In fact, as we shall now show, $L(\underline{\underline{V}};-)$ is a functor.

THEOREM .1. Let $\underline{\underline{V}}$ be a galaxy of $\mathfrak{R}$-algebras. The $L(\underline{\underline{V}};-)$ is a functor from $\underline{\underline{S}}$ into $\underline{\underline{P}}$.

PROOF. Let $f : Y_1 \longrightarrow Y_2$, $g : Y_2 \longrightarrow Y_3$ be arbitrary maps in $\underline{\underline{S}}$. We have to prove that $L(\underline{\underline{V}};gf) = L(\underline{\underline{V}};g)L(\underline{\underline{V}};f)$. For this it is convenient to introduce the following abbreviations. We write L_f and L_f^* for the functions $L(\underline{\underline{V}};f)$ and $L(\mathfrak{R};f) = L(\underline{V}(\mathfrak{R});-)$ respectively. For every set Y we write χ_Y for the function that takes $\kappa \in L(\mathfrak{R};Y)$ to the $L(\underline{\underline{V}};Y)$-cover of κ. We need some facts about the functions χ_Y , L_f , L_f^* .

For two maps $\lambda_1,\lambda_2 : \langle L_1,\leq\rangle \longrightarrow \langle L_2,\leq\rangle$ we write $\lambda_1 \leq \lambda_2$ if $\lambda_1(z) \leq \lambda_2(z)$ for all $z \in L_1$.

(.1.1) $$\chi_{Y_2}L_f^* \geq L_f^*\chi_{Y_1}$$

Let $\kappa \in L(\mathfrak{R};Y_1)$. Write $\kappa^+ = \chi_{Y_1}(\kappa)$, $\kappa^* = L_f^*(\kappa)$, $(\kappa^+)^* = L_f^*(\kappa^+)$ and $(\kappa^*)^+ = \chi_{Y_2}(\kappa^*)$. To prove (.1.1) we must show that $(\kappa^*)^+ \supseteq (\kappa^+)^*$. Let $\underline{W}(\mathfrak{R};Y_1)/\kappa \longrightarrow \underline{W}(\mathfrak{R};Y_2)/\kappa^*$ be the homomorphism under which the equivalence class $[u]\kappa$ goes upon $[\underline{f}(u)]\kappa^*$ for all $u \in W(\mathfrak{R};Y_1)$, where $\underline{f} = \underline{W}(\mathfrak{R};f)$. Let $\underline{W}(\mathfrak{R};Y_1)/\kappa \longrightarrow \underline{W}(\mathfrak{R};Y_1)/\kappa^+$ and $\underline{W}(\mathfrak{R};Y_2)/\kappa^* \longrightarrow \underline{W}(\mathfrak{R};Y_2)/(\kappa^*)^+$ be the canonical homomorphisms (that are well defined because, by definition, $\kappa \subseteq \kappa^+$, $\kappa^* \subseteq (\kappa^*)^+$). Now, by the proof of Theorem 7 of Chap. 2 $\underline{W}(\mathfrak{R};Y_1)/\kappa^+$ and $\underline{W}(\mathfrak{R};Y_1)/\kappa \longrightarrow \underline{W}(\mathfrak{R};Y_1)/\kappa^+$ provide a coreflection of

$\underline{W}(\mathcal{R};Y_1)/\kappa$ in $\underline{\underline{K}}(\underline{\underline{V}})$ and therefore there exists a (unique) homomorphism $\underline{W}(\mathcal{R};Y_1)/\kappa^+ \longrightarrow \underline{W}(\mathcal{R};Y_2)/(\kappa^*)^+$ such that the following diagram commutes:

$$\begin{tikzcd} \underline{W}(\mathcal{R};Y_1)/\kappa \arrow[r] \arrow[d] & \underline{W}(\mathcal{R};Y_2)/\kappa^* \arrow[d] \\ \underline{W}(\mathcal{R};Y_1)/\kappa^+ \arrow[r] & \underline{W}(\mathcal{R};Y_2)(\kappa^*)^+ \end{tikzcd}$$

From the commutativity of the above diagram and the definitions of the three maps $\underline{W}(\mathcal{R};Y_1)/\kappa \longrightarrow \underline{W}(\mathcal{R};Y_2)/\kappa^*$, $\underline{W}(\mathcal{R};Y_1)/\kappa \longrightarrow \underline{W}(\mathcal{R};Y_1)/\kappa^+$, $\underline{W}(\mathcal{R};Y_2)/\kappa^* \longrightarrow \underline{W}(\mathcal{R};Y_2)/(\kappa^*)^+$ we see that the fourth map $\underline{W}(\mathcal{R};Y_1)/\kappa^+ \longrightarrow \underline{W}(\mathcal{R};Y_2)/(\kappa^*)^+$ of our diagram must take $[u]\kappa^+$ upon $[\underline{f}(u)](\kappa^*)^+$ for all $u \in W(\mathcal{R};Y_1)$. This immediately shows that if $\langle u_1,u_2\rangle \in \kappa^+$ then $\langle \underline{f}(u_1),\underline{f}(u_2)\rangle \in (\kappa^*)^+$. Hence $(\kappa^+)^f = \{\langle \underline{f}(u_1),\underline{f}(u_2)\rangle ; \langle u_1,u_2\rangle \in \kappa^+\} \subseteq (\kappa^*)^+$. This implies that $(\kappa^+)^* \subseteq (\kappa^*)^+$, since, by definition, $(\kappa^+)^*$ is the least congruence over $\underline{W}(\mathcal{R};Y_2)$ containing $(\kappa^+)^f$. This completes the proof of (1.1).

(1.2) $$L^*_g L^*_f = L^*_{gf} \quad \text{and} \quad L_g L_f \geq L_{gf} .$$

For every function $e : Y \longrightarrow Z$ and set $r \subseteq W(\mathcal{R};Y) \times W(\mathcal{R};Y)$ we write r^e for the set $\{\langle \underline{e}(u_1),\underline{e}(u_2)\rangle ; \langle u_1,u_2\rangle \in r\}$, where $\underline{e} = \underline{W}(\mathcal{R};e)$. Let κ be an arbitrary congruence in $L(\mathcal{R};Y_1)$. Since $\underline{W}(\mathcal{R};-)$ is a functor we have that

$\underline{gf} = \underline{W}(\mathcal{R};g)\underline{W}(\mathcal{R};f) = \underline{W}(\mathcal{R};gf) = \underline{gf}$. From here it follows directly that $\kappa^{gf} = (\kappa^f)^g$. Since $\kappa^f \subseteq L_f^*(\kappa) \subseteq L_f(\kappa)$ this gives $\kappa^{gf} \subseteq (L_f^*(\kappa))^g \subseteq (L_f(\kappa))^g$. The inclusion $\kappa^{gf} \subseteq (L_f(\kappa))^g$ tells us that $L_{gf}^*(\kappa) \subseteq L_g^*L_f^*(\kappa)$ and the inclusion $\kappa^{gf} \subseteq (L_f(\kappa))^g$ gives $L_{gf}(\kappa) \subseteq L_gL_f(\kappa)$. We complete the proof of (1.2) by proving the inclusion $L_g^*L_f^*(\kappa) \subseteq L_{gf}^*(\kappa)$. For this we first make a trivial observation. Let Y be any set and let $r \subseteq W(\mathcal{R};Y) \times W(\mathcal{R};Y)$. Write κ_r for the least congruence over $\underline{W}(\mathcal{R};Y)$ containing r . Then $\langle v,v'\rangle \in \kappa_r$ if and only if the equality $v = v'$ is deducible † from $\mathcal{E}(r) = \{u = u' ; \langle u,u'\rangle \in r\}$. Moreover, if e is a function from Y and $v = v'$ is deducible from $\mathcal{E}(r)$ then $\underline{e}(v) = \underline{e}(v')$ is deducible from $\mathcal{E}(r^e)$, where as before $\underline{e} = \underline{W}(\mathcal{R};e)$ and $r^e = \{\langle \underline{e}(u), \underline{e}(u')\rangle ; \langle u,u'\rangle \in r\}$. Now let $\langle w,w'\rangle \in L_g^*L_f^*(\kappa)$. Then $w = w'$ is deducible from $\{\underline{g}(v) = \underline{g}(v'); \langle v,v'\rangle \in L_f^*(\kappa)\}$. Since $L_f^*(\kappa)$ is the least congruence containing κ^f then $\langle v,v'\rangle \in L_f^*(\kappa)$ if and only if $v = v'$ is deducible from $\{\underline{f}(u) = \underline{f}(u') ; \langle u,u'\rangle \in \kappa\}$. Hence for our

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† We can regard $v = v'$ as a sentence in the language obtained from $\Phi(\mathcal{R})$ by adding members of Y as constant symbols, that is, the language $\Phi(\mathcal{R}')$, where $\mathcal{R}' = \langle \emptyset, \Omega \cup Y , h'\rangle$, $h' \mid \Omega = h$, $h(Y) = 0$. Then by the 'deducibility of $v = v'$ from $\mathcal{E}(r)$ ' we mean the 'deducibility of $v = v'$ from $\mathcal{E}(r)$ in the language $\Phi(\mathcal{R}')$ '.

arbitrary member $\langle w,w'\rangle$ of $L_gL_f(\kappa)$ the equality $w = w'$ is deducible from $\{\underline{gf}(u) = \underline{gf}(u') \; ; \; \langle u,u'\rangle \in \kappa\}$. But we have already noted that $g\underline{f} = \underline{gf}$. Hence $w = w'$ is deducible from $\{g\underline{f}(u) = g\underline{f}(u'); \; \langle u,u'\rangle \in \kappa\}$, so that $\langle w,w'\rangle$ is in $L^*_{gf}(\kappa)$ (the least congruence containing κ^{gf}) . This proves that $L^*_gL^*_f(\kappa) \subseteq L^*_{gf}(\kappa)$ and the proof of (1.2) is complete.

The last fact which we need to know about the functions χ_Y , L_f , L^*_f etc. is an immediate consequence of the definitions of these functions and requires no proof.

$$(1.3) \qquad L_f = \chi_{Y_2} L^*_f \, .$$

It is now easy to prove that $L(\underline{\underline{V}};-)$ is a functor, i.e., $L_gL_f = L_{gf}$. By (1.1), (1.2), (1.3) and the obvious fact that $\chi^2_Y = \chi_Y$ for all sets Y we have

$$L_gL_f = \chi_{Y_3} L^*_g \chi_{Y_2} L^*_f \leq \chi_{Y_3} \chi_{Y_3} L^*_g L^*_f = \chi_{Y_3} L^*_{gf} = L_{gf} \leq L_g L_f \, .$$

Hence $L_gL_f = L_{gf}$ and the proof of the theorem is complete.

<u>THEOREM 2</u>. Let $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ be galaxies such that $\underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2$. Then there is a natural transformation from $L(\underline{\underline{V}}_1;-)$ to $L(\underline{\underline{V}}_2;-)$.

<u>PROOF</u>. Let $\eta_Y : L(\underline{\underline{V}}_1;Y) \longrightarrow L(\underline{\underline{V}}_2;Y)$ be the order preserving map under which $\kappa \in L(\underline{\underline{V}}_1;Y)$ goes to the $L(\underline{\underline{V}}_2;Y)$-cover of κ . We prove the theorem by showing that the diagram

$$\begin{tikzcd}
L(\underline{\underline{V}}_1;Y_1) \arrow[r, "L(\underline{\underline{V}}_1;f)"] \arrow[d, "\eta_{Y_1}"'] & L(\underline{\underline{V}}_1;Y_2) \arrow[d, "\eta_{Y_2}"] \\
L(\underline{\underline{V}}_2;Y_1) \arrow[r, "L(\underline{\underline{V}}_2;f)"'] & L(\underline{\underline{V}}_2;Y_2)
\end{tikzcd}$$

commutes for every $f : Y_1 \longrightarrow Y_2$ in $\underline{\underline{S}}$. For this we first introduce some abbreviations similar to those used in the proof of Theorem 1. We write L_f^*, $L_{1,f}$ and $L_{2,f}$ for the maps $L(\mathcal{R};f)$, $L(\underline{\underline{V}}_1;f)$ and $L(\underline{\underline{V}}_2;f)$ respectively, where f is an arbitrary map in $\underline{\underline{S}}$. For every set Y and $\ell = 1,2$ we write $\chi_{\ell,Y}$ for the order preserving function $L(\mathcal{R};Y) \longrightarrow L(\underline{\underline{V}}\ ;Y)$ which sends κ in $L(\mathcal{R};Y)$ to the $L(\underline{\underline{V}}_\ell;Y)$-cover of κ. It is clear that $\eta_Y = \chi_{2,Y} \mid L(\underline{\underline{V}}_1;Y)$ for all sets Y and that the commutativity of our diagram is equivalent to the equality $L_{2,f}\eta_{Y_1} = \eta_{Y_2}L_{1,f}$. Now for every $\kappa \in L(\underline{\underline{V}}_1;Y_1)$ we have

$$\begin{aligned}
L_{2,f}\eta_{Y_1}(\kappa) &= L_{2,f}\chi_{2,Y_1}(\kappa), && \text{since } \eta_{Y_1} = \chi_{2,Y_1} \mid L(\underline{\underline{V}}_1,Y_1) \\
&= \chi_{2,Y_2}L_f^*\chi_{2,Y_1}(\kappa), && \text{by } (1.3) \\
&\subseteq \chi_{2,Y_2}\chi_{2,Y_2}L_f^*(\kappa), && \text{by } (1.1) \\
&= \chi_{2,Y_2}L_f^*(\kappa), && \text{since } \chi_{2,Y_2}^2 = \chi_{2,Y_2} \\
&\subseteq \chi_{2,Y_2}L_f^*\chi_{2,Y_1}(\kappa), && \text{since } \chi_{2,Y_1}(\kappa) \supseteq \kappa \\
&= L_{2,f}\eta_{Y_1}(\kappa), && \text{by } (1.3).
\end{aligned}$$

Hence

$$(2.1) \qquad L_{2,f}\eta_{Y_1} = \chi_{2,Y_2}L_f^*$$

On the other hand for every $\kappa \in L(\underline{\underline{V}}_1;Y_1)$ we have

$$\begin{aligned} \eta_{Y_2}L_{1,f}(\kappa) &= \chi_{2,Y_2}L_{1,f}(\kappa) , && \text{since } \eta_{Y_2} = \chi_{2,Y_2} \mid L(\underline{\underline{V}}_1;Y_2) \\ &= \chi_{2,Y_2}\chi_{1,Y_2}L_f^*(\kappa) , && \text{by } (1.3) \\ &= \chi_{2,Y_2}L_f^* , && \text{since } \underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2 \text{ and therefore} \\ & && \chi_{2,Y_2}\chi_{1,Y_2} = \chi_{2,Y_2} . \end{aligned}$$

Hence

$$(2.2) \qquad \chi_{Y_2}L_{1,f} = \chi_{2,Y_2}L_f^* .$$

By comparing (.2.1) and (2.2) we find

$$\eta_{Y_2}L_{1,f} = L_{2,f}\eta_{Y_1} .$$

As mentioned earlier this is enough to prove the theorem.

THEOREM 3. Let $\underline{\underline{V}}$ be a galaxy of $\mathcal{R}$-algebras. Then $L(\underline{\underline{V}};f)$ preserves joins for all functions $f : Y_1 \longrightarrow Y_2$.

PROOF. We use the abbreviated notations introduced in the proof of Theorem 1. Let $\kappa_i \in L(\underline{\underline{V}};Y_1)$, where i ranges over an index set I , and let κ_1 , κ_2 be the joins of $\{\kappa_i\}_{i\in I}$, $\{L_f(\kappa_i)\}_{i\in I}$ in $L(\underline{\underline{V}};Y_1)$, $L(\underline{\underline{V}};Y_2)$ respectively. We have to show that $\kappa_2 = L_f(\kappa_1)$. Let κ_1^* , κ_2^* be the joins of $\{\kappa_i\}_{i\in I}$, $\{L_f^*(\kappa_i)\}_{i\in I}$

in $L(\mathcal{R};Y_1)$, $L(\mathcal{R};Y_2)$ respectively. Then $\kappa_1 = \chi_{Y_1}(\kappa_1^*)$, $\kappa_2 = \chi_{Y_2}(\kappa_2^*)$ (Cf. proof of Theorem 0). Before proceeding to the proof of $\kappa_2 = L_f(\kappa_1)$ we need to show the following:

$$(3.1) \qquad \kappa_2^* = L_f^*(\kappa_1^*) .$$

To prove (.3.1) we use the observation made in the proof of Theorem 1 (see page 59). If $\langle v,v'\rangle \in \kappa_1^*$ then $v = v'$ is deducible from $\{u = u' ; \langle u,u'\rangle \in \cup_{i\in I} \kappa_i\}$ and hence $\underline{f}(v) = \underline{f}(v')$ is deducible from $\{\underline{f}(u) = \underline{f}(u') ; \langle u,u'\rangle \in \cup_{i\in I} \kappa_i\}$. Thus the sets $\mathcal{E}_1 = \{\underline{f}(u) = \underline{f}(u') ; \langle u,u'\rangle \in \kappa_1^*\}$ and $\mathcal{E}_2 = \{\underline{f}(u) = \underline{f}(u') ; \langle u,u'\rangle \in \cup_{i\in I} \kappa_i\}$ are equivalent. But $\langle w,w'\rangle \in L_f^*(\kappa_1^*)$ if and only if $w = w'$ is deducible from $\mathcal{E}_1$ and $\langle w,w'\rangle \in \kappa_2^*$ if and only if $w = w'$ is deducible from $\mathcal{E}_2$. The equivalence of $\mathcal{E}_1$ and $\mathcal{E}_2$ now shows that $\kappa_2^* = L_f^*(\kappa_1^*)$ and (.3.1) is proved.

Now we can prove that $\kappa_2 = L_f(\kappa_1)$. We have

$$\begin{aligned}
L_f(\kappa_1) &= \chi_{Y_2} L_f^*(\kappa_1) , && \text{by (1.3)} \\
&= \chi_{Y_2} L_f \chi_{Y_1}(\kappa_1^*) \\
&\subseteq \chi_{Y_2} \chi_{Y_2} L_f^*(\kappa_1^*) , && \text{by (1.1)} \\
&= \chi_{Y_2} L_f^*(\kappa_1^*) && \text{since } \chi_{Y_2}^2 = \chi_{Y_2} \\
&= \chi_{Y_2}(\kappa_2^*) , && \text{by (.3.1)} \\
&= \kappa_2 .
\end{aligned}$$

Hence $L_f(\kappa_1) \subseteq \kappa_2$. On the other hand

$$\begin{aligned} L_f(\kappa_1) &= \chi_{Y_2} L_f^*(\kappa_1) \\ &\supseteq \chi_{Y_2} L_f^*(\kappa_1^*) , && \text{since } \kappa_1 \supseteq \kappa_1^* \\ &= \chi_{Y_2}(\kappa_2^*) , && \text{by } (3.1) \\ &= \kappa_2 . \end{aligned}$$

Hence $L_f(\kappa_1) \supseteq \kappa_2$, so that $\kappa_2 = L_f(\kappa_1)$. This completes the proof of the theorem.

We conclude this chapter by asking two questions that we have not been able to answer. We see from 0.3 that $L(\underline{\underline{V}};Y)$ is a complete lattice for galaxies $\underline{\underline{V}}$ and sets Y . Are there semi-varieties $\underline{\underline{V}}$, other than varieties, such that $L(\underline{\underline{V}};Y)$ is a complete sublattice of $L(\mathfrak{K};Y)$ for all sets Y ? We also learn from Theorem 3 that $L(\underline{\underline{V}};f)$ preserves joins for galaxies $\underline{\underline{V}}$ and functions f . For what quasi-varieties $\underline{\underline{V}}$ is the map $L(\underline{\underline{V}};f)$ a complete lattice homomorphism for all functions f ?

CHAPTER 4

A THEOREM ON RANKED CATEGORIES OF STRUCTURES

In the last two chapters we have given some applications of the categorical concept of coreflection to the study of algebraic structures. In the present short chapter we apply this important concept again and obtain a very general and simple principle about ranked categories of structures which says that all categories, that admit free algebras and are bigger than a ranked category, are themselves ranked.

We begin by describing what we mean by a **ranked category of structures.**

Let $\underline{\underline{K}}$ be an arbitrary category of structures. We shall assume in this chapter that morphisms of $\underline{\underline{K}}$ are functions although our considerations are applicable to other situations (e.g., to categories of structures whose morphisms are functopisms (see p. 8)). Categories of structures considered by Freyd in [1, pp. 107-120] satisfy our assumption about $\underline{\underline{K}}$.

We shall say that $\underline{\underline{K}}$ admits free structures if the forgetful functor $F(\underline{\underline{K}};-)$ from $\underline{\underline{K}}$ into $\underline{\underline{S}}$ has a left adjoint G . The structure $G(Y)$, determined within isomorphism by Y alone, may be called the free structure on Y . Freyd [1, pp. 107-120] has shown how this concept of 'freeness' translates the usual idea of 'freeness'.

Note that if morphisms of $\underline{\underline{K}}$ could be functopisms we could have defined freeness by considering the forgetful functor $F^{+}(K;-) : \underline{\underline{K}} \longrightarrow \underline{\underline{S}}^{+}$ defined on page 9 .

We shall say that $\underline{\underline{K}}$ is <u>ranked</u> if $\underline{\underline{K}}$ admits free structures and free structures on two sets are isomorphic if and only if the sets themselves are isomorphic (i.e., have the same cardinality). A class $\underline{\underline{V}}$ of $\mathfrak{R}$-structures is said to be ranked (to admit free structures) if $\underline{\underline{K}}(\underline{\underline{V}})$ is ranked (admits free structures). Many familiar algebraic systems like groupoids, loops, abelian groups, Boolean algebras form ranked classes. All quasi-varieties admit free algebraic structures but there are [11] quasi-varieties (in fact, varieties) that are not ranked. All non-trivial varieties of groups are ranked [22, p. 12]. In fact it is easy to deduce from the result just quoted that all classes of groups, other than the trivial variety, that admit free groups are ranked.

In this chapter we prove the following simple and general principle about ranked categories of structures.

<u>THEOREM 1</u>. Let $\underline{\underline{K}}_1$, $\underline{\underline{K}}_2$ be categories of structures that admit free structures. Let $\underline{\underline{K}}_1 \subseteq \underline{\underline{K}}_2$. Then $\underline{\underline{K}}_2$ is ranked if $\underline{\underline{K}}_1$ is ranked.

The above theorem is an immediate consequence of Theorem 2 below which is a purely categorical and more general result.

To state Theorem 2 we need few definitions. A functor

$F : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is called <u>ranked</u> if F has a left adjoint G such that for all $\underline{\underline{B}}_1 , \underline{\underline{B}}_2 \in B$ the objects $G(B_1), G(B_2)$ are isomorphic in $\underline{\underline{A}}$ if (and only if) B_1 , B_2 are isomorphic in $\underline{\underline{B}}$. For two functors $F_1 : \underline{\underline{A}}_1 \longrightarrow \underline{\underline{B}}_1$, $F_2 : \underline{\underline{A}}_2 \longrightarrow \underline{\underline{B}}_2$ we write $F_1 \leq F_2$ if $\underline{\underline{A}}_1 \subseteq \underline{\underline{A}}_2$, $\underline{\underline{B}}_1 \subseteq \underline{\underline{B}}_2$ and F_1 is the restriction of F_2 to $\underline{\underline{A}}_1$.

<u>THEOREM 2</u>. Let $F_1 : \underline{\underline{A}}_1 \longrightarrow \underline{\underline{B}}$, $F_2 : \underline{\underline{A}}_2 \longrightarrow \underline{\underline{B}}$ admit left adjoints. Let $F_1 \leq F_2$. Then F_2 is ranked if F_1 is ranked.

<u>PROOF</u>. Let $G_1 : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}_1$, $G_2 : \underline{\underline{B}} \longrightarrow \underline{\underline{A}}_2$ be the left adjoints of F_1 , F_2 respectively. By the defining property of adjoint functors, for all $A \in \underline{\underline{A}}_1$, $B \in \underline{\underline{B}}$ there exists a one-to-one onto function $\eta_1(B,A) : \mathrm{Hom}_1(G_1(B),A) \longrightarrow \mathrm{Hom}(B,F_1(A))$ such that for all morphisms $\alpha , \beta , \gamma , \delta$ the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A' \\ G_1(B) \arrow[u, "\gamma"] & G_1(B') \arrow[l, "G_1(\beta)"] \arrow[u, "\delta"'] \end{tikzcd}$$

commutes if and only if the diagram

$$\begin{tikzcd}
F_1(A) \arrow[rr, "F_1(\alpha)"] & & F_1(A') \\
B \arrow[u, "{[\eta_1(B,A)](\gamma)=\gamma_1}"] & & B' \arrow[ll, "\beta"] \arrow[u, "{\delta_1=[\eta_1(B',A')](\delta)}"']
\end{tikzcd}$$

commutes, where A , $A' \in \underline{\underline{A}}_1$, B , $B' \in \underline{\underline{B}}$ and $\mathrm{Hom}_1(G_1(B),A)$ is the set of all morphisms in $\underline{\underline{A}}_1$ from $G_1(B)$ to A while $\mathrm{Hom}(B,F_1(A))$ is the set of all morphisms in $\underline{\underline{B}}$ from B to $F_1(A)$. The functors F_2 , G_2 similarly determine a function $\eta_2(B,A)$ for all $B \in \underline{\underline{B}}$, $A \in \underline{\underline{A}}_2$. For every object $B \in \underline{\underline{B}}$ write $\pi_B : B \longrightarrow F_1G_1(B)$ for the image of the identity $1_{G_1(B)} : G_1(B) \longrightarrow G_1(B)$ under η_1'$B,G_1(B))$. Let $\tau_B : (G_2(B) \longrightarrow G_1(B)$ be the inverse image of π_B under $\eta_2(B,G_1(B))$. τ_B exists because $G_1(B) \in \underline{\underline{A}}_2$ and $F_1G_1(B) = F_2G_1(B)$ We show that $G_1(B)$ together with the map τ_B is a coreflection of $G_2(B)$ in $\underline{\underline{A}}_1$. Let $\alpha : G_2(B) \longrightarrow A$ be a map in $\underline{\underline{A}}_2$ and let $A \in \underline{\underline{A}}_1$. We have to show that we can find a unique map $\xi : G_1(B) \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
G_2(B) \arrow[d, "\tau_B"'] \arrow[dr, "\alpha"] & \\
G_1(B) \arrow[r, "\xi"'] & A
\end{tikzcd} \qquad (I)$$

commutes. To do this let $\beta : B \longrightarrow F_2(A)$ be the image of α under $\eta_2(B,A)$. Since $F_1 \leq F_2$ and $A \in \underline{\underline{A}}_1$ we have $F_2(A) = F_1(A)$. Hence β has an inverse image ξ, say, under $\eta_1(B,A)$. We show that ξ is the map we were looking for. Before doing this we collect the relations between different morphisms we are dealing with.

$$\begin{aligned}
(1.1) \qquad \pi_B &= [\eta_1(B,G_1(B))](1_{G_1(B)}) \\
\pi_B &= [\eta_2(B,G_1(B)](\tau_B) \\
\beta &= [\eta_2(B,A)](\alpha) \\
\beta &= [\eta_1(B,A)](\xi)
\end{aligned}$$

Now ξ is clearly in $\underline{\underline{A}}_1$. From the commutativity of the diagram

$$\begin{tikzcd}
G_1(B) \arrow[r, "\xi"] & A \\
G_1(B) \arrow[u, "1_{G_1(B)}"] & G_1(B) \arrow[l, "1_{G_1(B)}"] \arrow[u, "\xi"']
\end{tikzcd}$$

the defining property of adjoint functors mentioned above and (1.1), we see that the diagram

$$\begin{tikzcd}
F_1G_1(B) \arrow[r, "F_1(\xi)"] & F_1(A) \\
B \arrow[u, "\pi_B"] & B \arrow[l, "1_B"] \arrow[u, "\beta"']
\end{tikzcd} \tag{II}$$

commutes. Let α_1 be the map which makes the diagram

$$\begin{tikzcd}
G_1(B) \arrow[r, "\xi"] & A \\
G_2(B) \arrow[u, "\tau_B"] & G_2(B) \arrow[l, "1_{G_2(B)}"] \arrow[u, "\alpha_1"']
\end{tikzcd} \tag{III}$$

commute. Then

$$\begin{tikzcd}
F_2G_1(B) \arrow[r, "F_2(\xi)"] & F_2(A) \\
B \arrow[u, "\pi_B"] & B \arrow[l, "1_B"] \arrow[u, "\beta_1 = {[\eta_2(B,A)](\alpha_1)}"']
\end{tikzcd}$$

also commutes, by the definition of adjoint functors and the relation between the maps π_B , τ_B given in (1.1). Since $F_1 \leq F_2$ and $A, G_1(B)$, $\xi \in \underline{\underline{A}}_1$ we see that the diagram

$$\begin{tikzcd} F_1G_1(B) \arrow[r, "F_1(\xi)"] & F_1(A) \\ B \arrow[u, "\pi_B"] & B \arrow[l, "1_B"] \arrow[u, "\beta_1"'] \end{tikzcd} \qquad \text{(IV)}$$

is commutative. Comparing (IV) and (II) we find that $\beta = \beta_1$. Since $\beta_1 = [\eta_2(B,A)](\alpha_1)$, $\beta = [\eta_2(B,A)](\alpha)$ and $\eta_2(B,A)$ is one-to-one we conclude that $\alpha = \alpha_1$. Substituting α for α_1 in (III) we see that (I) commutes. To finish proving that $G_1(B)$ is a coreflection of $G_2(B)$ in $\underline{\underline{A}}_1$ we establish the uniqueness of the map $G_1(B) \longrightarrow A$ which makes (I) commute. Let $\xi' \in \underline{\underline{A}}_1$ be any map such that the diagram

$$\begin{tikzcd} G_1(B) \arrow[r, "\xi'"] & A \\ G_2(B) \arrow[u, "\tau_B"] & G_2(B) \arrow[l, "1_{G_2(B)}"] \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes. Then, as before, using the defining property of adjoint functors and (1.1) we see that

$$\begin{tikzcd}
F_2G_1(B) \arrow[r, "F_2(\xi')"] & F_2(A) \\
B \arrow[u, "\pi_B"] & B \arrow[u, "\beta"'] \arrow[l, "1_B"]
\end{tikzcd}$$

is commutative. Hence, since $F_1 \leq F_2$ and A , $G_1(B)$, $\xi' \in \underline{\underline{A}}_1$, the diagram

$$\begin{tikzcd}
F_1G_1(B) \arrow[r, "F_1(\xi')"] & F_1(A) \\
B \arrow[u, "\pi_B"] & B \arrow[u, "\beta"'] \arrow[l]
\end{tikzcd}$$

also commutes. This implies that

$$\begin{tikzcd}
G_1(B) \arrow[r, "\xi'"] & A \\
G_1(B) \arrow[u, "1_{G_1(B)}"] & G_1(B) \arrow[u, "\xi"'] \arrow[l, "1_{G_1(B)}"]
\end{tikzcd}$$

is commutative; so that $\xi = \xi'$ and it is proved that $G_1(B)$ is a coreflection of $G_2(B)$ in $\underline{\underline{A}}_1$.

We are now in a position to conclude the proof. Let $G_2(B_1)$, $G_2(B_2)$ be isomorphic in $\underline{\underline{A}}_2$ for some B_1 , $B_2 \in \underline{\underline{B}}$. Then $G_1(B_1)$, $G_1(B_2)$ are isomorphic in $\underline{\underline{A}}_1$ because coreflections of isomorphic objects are isomorphic. Since F_1 is ranked this implies that B_1 , B_2 are isomorphic in $\underline{\underline{B}}$. Hence F_2 is ranked and the theorem is proved.

Theorem 1' now follows from Theorem 2 by noting that $\underline{\underline{K}}$ is ranked if and only if $F(\underline{\underline{K}};-)$ is ranked.

Professor Grätzer has pointed out to me that the special case of Theorem 1', when $\underline{\underline{K}}_1$, $\underline{\underline{K}}_2$ are of the form $\underline{\underline{K}}(\underline{\underline{V}}_1)$, $\underline{\underline{K}}(\underline{\underline{V}}_2)$ for some classes $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ of $\mathfrak{R}$-algebras, can be directly deduced from Theorems 31.1 and 31.5 of his book [9].

CHAPTER 5

PRODUCTS OF RELATIONAL STRUCTURES

A very important and fundamental part of the study of mathematical objects of a certain type involves constructions or processes that obtain objects of that type from other objects of the same type. Elementary classes of structures form an important type of classes and it is worthwhile to know of ways of obtaining elementary classes from other elementary classes. Some such ways are known. For example let $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathcal{R})$ be elementary; then $\underline{\underline{V}}_1 \cap \underline{\underline{V}}_2$ is also elementary. A more sophisticated example is provided by a theorem of Vaught [31] which shows that the class $SC\Pi(\underline{\underline{V}})$ of all substructures of cartesian products of families of structures in $\underline{\underline{V}}$ is elementary (in fact universal) if $\underline{\underline{V}}$ is elementary.

In this chapter we give **some general types of constructions** of elementary classes and compact classes from compact classes. Our definitions, remarks and results generalize many of the concepts and results of [7], [12], [14], [19] **and** [31] and answer a question of Feferman (See Math. Reviews 32, (1966), 5512) for a certain type of ordinal products. Our last result (Theorem 6) suggests the possibility of a categorical study of at least some of the products of relational structures.

The background for our constructions is provided by [7] where a very general concept of 'generalized products' of relational structures is defined which comprehends many of the products arising in Mathematics. Roughly speaking, a generalized product of a family $\{\underline{A}_i\}_{i\in I}$ of relational structures is a structure $\underline{A}$ such that carrier $(\underline{A}) = A$ is the cartesian product $C\prod_{i\in I} A_i$ of the A_i = carrier $(\underline{A}_i)$ and the make of $\underline{A}$ not only depends on the makes of the $\underline{A}_i$ but also on the make of a structure on the power set $S(I)$ of I. In this chapter we first describe the products of [7] in a somewhat different set-up which involves not a structure on $S(I)$ but on I itself. Our description makes it possible to generalize the concept of 'generalized products' to what we shall call higher order products. The concept of the order of a product, which our description brings out, seems as important as that of the order of a language. Thus first order products (like first order languages) seem to distinguish themselves as specially important type of products. As we shall see, first order products share with the cartesian product the properties given by Makkai [14] and Vaught [31]. We observe that the cartesian product and the ordinal product are first order products as are the regular products of Mal'cev [19].

We now give precise descriptions.

Let $\mathfrak{R}_1 = \langle P_1,\emptyset,h_1\rangle$, $\mathfrak{R}_2 = \langle P_2,\emptyset,h_2\rangle$, $\mathfrak{R}_3 = \langle P_3,\emptyset,h_3\rangle$ be arbitrarily fixed predicate systems. For every $\varphi \in \Phi(\mathfrak{R}_1)$ we think of a new symbol ρ_φ which does not belong to any one of the sets P_1 , P_2 , P_3 . Write $\mathfrak{R}^+ = \langle P^+, \emptyset, h^+ \rangle$, where $P^+ = P_2 \cup \{\rho_\varphi ; \varphi \in \Phi(\mathfrak{R}_1)\}$ and h^+ coincides with h_2 on P_2 while $h^+(\rho_\varphi) = 1$ for all $\varphi \in \Phi(\mathfrak{R}_1)$.

By an $(\mathfrak{R}_1, \mathfrak{R}_2)$-<u>complex of structures</u> we shall understand an ordered pair $\Lambda = \langle \{\underline{A}_i\}_{i\in I}, \underline{I} \rangle$, where $\underline{I} = \langle I, \iota \rangle \in \underline{\underline{V}}(\mathfrak{R}_2)$, $\underline{A}_i = \langle A_i, \alpha_i \rangle \in \underline{\underline{V}}(\mathfrak{R}_1)$, $i \in I$. Denote by $\underline{\underline{C}}(\mathfrak{R}_1, \mathfrak{R}_2)$ the class of all $(\mathfrak{R}_1, \mathfrak{R}_2)$-complexes. Let $x_{j_1}, \ldots, x_{j_k} \in X$ and $f_1, \ldots, f_k \in C\Pi_{i \in I} A_i$. By the <u>graph of</u> Λ <u>at</u> $\langle x_{j_1}, \ldots, x_{j_k} \rangle = \langle f_1, \ldots, f_k \rangle$ we shall understand the $\mathfrak{R}^+$-structure $\langle I, \iota^+ \rangle$ defined as follows: $\iota^+ | P_2 = \iota$ and for all $\varphi \in \Phi(\mathfrak{R}_1)$ the unary relation $\iota^+(\rho_\varphi)$ is the set of all $i \in I$ such that $\forall x_{j_{k+1}}, \ldots, x_{j_n}(\varphi)$ holds in $\underline{A}_i$ at $\langle x_{j_1}, \ldots, x_{j_k} \rangle = \langle f_1(i), \ldots, f_k(i) \rangle$, where $x_{j_1}, \ldots, x_{j_n}$ are the variables that may occur freely in φ .

We are now in a position to define our products. Let $\tau : P_3 \longrightarrow \Phi_0(\mathfrak{R}^+)$ satisfy the requirement: For all $\rho \in P_3$ the formulae ψ such that ρ_ψ occurs in $\tau(\rho)$ together involve just $x_1, \ldots, x_{h_3(\rho)}$ as free variables. Now the <u>first order product of type</u> τ is defined to be the function $\Pi^\tau : \underline{\underline{C}}(\mathfrak{R}_1, \mathfrak{R}_2) \longrightarrow \underline{\underline{V}}(\mathfrak{R}_3)$ such that for $\Lambda = \langle \{\underline{A}_i\}_{i\in I}, \underline{I} \rangle$ the $\mathfrak{R}_3$-structure $\Pi^\tau(\Lambda)$ is $\langle A, \alpha \rangle$, where $A = C\Pi_{i\in I} A_i$, and for all $\rho \in P_3$ and $f_1, \ldots, f_{h_3(\rho)} \in A$ we have $\langle f_1, \ldots, f_{h_3(\rho)} \rangle \in \alpha(\rho)$ if and only if $\tau(\rho)$ holds in the graph of Λ at $\langle x_1, \ldots, x_{h_3(\rho)} \rangle = \langle f_1, \ldots, f_{h_3(\rho)} \rangle$.

We shall often write $\Pi^{\tau}\underline{I}(\underline{A}_i)$ for $\Pi^{\tau}(\Lambda)$.

For every positive integer n we can define <u>nth order products</u> by using nth order languages in a way exactly similar to the one we have used first order languages to define first order products. We omit the obvious details. Our interest here in this chapter is in the first order products only and we shall make only the following two remarks about higher order products. Firstly, it follows directly from the respective definitions that 'generalized products' of [7] are second order products but not every second order product is a 'generalized product'. Secondly, the Basic Theorem of [7] and therefore all the general results of [7] together with the analogues of Theorem 2 and Theorem 3 below hold for products of any order and can be proved similarly.

We now return to first order products and illustrate our definition of these products by describing cartesian, ordinal and regular products as first order products. Let $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 = \langle P,\emptyset,h\rangle$ and for every $\rho \in P$ let $\tau(\rho)$ be the sentence $\forall x_1 \rho_{\varphi}(x_1)$, where φ is the atomic formula $\rho(x_1,\dots,x_{h(\rho)})$. Then $\Pi^{\tau}\underline{I}(\underline{A}_i)$ is merely the cartesian product of the $\underline{A}_i$. For the case of ordinal products suppose that $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 = \langle P,\emptyset,h\rangle$ where P consists of a single binary relational symbol ρ so that $h(\rho) = 2$. This time take $\tau(\rho)$ to be the sentence

$$\exists x_1 (\rho_{\varphi_1}(x_1) \wedge \forall x_2 [\rho(x_2,x_1) \longrightarrow \rho_{\varphi_2}(x_2))] \text{ ,}$$

where φ_1 , φ_2 are the formulae $\rho(x_1, x_2)$ and $x_1 = x_2$ respectively. It is a simple matter to verify that $\Pi^{\tau}\underline{I}(\underline{A}_i)$ is the ordinal product of the $\underline{A}_i$ as defined in [7]. Finally, regular products of [19] are obtained as first order products if the function τ entering our definition is restricted in the following way: For every $\rho \in P_3$ the sentence $\tau(\rho)$ involves only the unary relational symbols of the form ρ_{φ} , $\varphi \in \Phi(\mathcal{R}_1)$, and (possibly) the symbol for equality $=$. In other somewhat less exact words, regular products are precisely those first order products that do not involve a structure on the set indexing the families of structures to which the products are to be applied; so that the image of $\langle\{\underline{A}_i\}_{i\in I}, \underline{I}\rangle$ under a regular product Π^{τ} is independent of the second member of the ordered pair. Observe that the cartesian product is a regular product while the ordinal product is not.

For classes $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathcal{R}_1)$, $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathcal{R}_2)$ we write $\underline{\underline{C}}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ for the class of $(\mathcal{R}_1, \mathcal{R}_2)$-complexes $\langle\{\underline{A}_i\}_{i\in I}, \underline{I}\rangle$, where $\underline{I} \in \underline{\underline{V}}_2$ and $\underline{A}_i \in \underline{\underline{V}}_1$ for all $i \in I$. Let us write $\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ for the class of $\mathcal{R}_3$-structures of the form $\Pi^{\tau}(\Lambda)$, where $\Lambda \in \underline{\underline{C}}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$. By $S\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ we shall denote the class of all $\mathcal{R}_3$-structures that are embeddable in a structure of $\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$.

We now give some properties of first order products. Our first result follows from the Basic Theorem of [7], since every first order product is a 'generalized product'. Before stating this result we bring out its contents. Let φ be an atomic formula of the form $\rho(x_{j_1},\ldots,x_{j_k})$, $k = h_3(\rho)$, $\rho \in P_3$. Write $\tau^+(\varphi)$ for the $\mathcal{R}^+$-**sentence** obtained from $\tau(\rho)$ by replacing every ρ_ψ that occurs in $\tau(\rho)$ by $\rho_{\psi'}$, where ψ' is obtained from ψ by substituting $x_{j_1},\ldots,x_{j_k}$ for $x_1,\ldots,x_k$ respectively.. Then our definition of Π^τ immediately tells us that given $\langle\{\underline{A}_i\}_{i\in I},\underline{I}\rangle \in \underline{\underline{C}}(\mathcal{R}_1,\mathcal{R}_2)$ and $f_1,\ldots,f_k \in C\Pi_{i\in I} A_i$ the atomic formula φ holds in $\Pi^\tau\underline{I}(\underline{A}_i)$ at $\langle x_{j_1},\ldots,x_{j_k}\rangle = \langle f_1,\ldots,f_k\rangle$ if and only if $\tau^+(\varphi)$ holds in the graph of $\langle\{\underline{A}_i\}_{i\in I},\underline{I}\rangle$ at $\langle x_{j_1},\ldots,x_{j_k}\rangle = \langle f_1,\ldots,f_k\rangle$. Now Theorem 1 below merely says that a sentence $\tau^+(\varphi)$ with the above property can be defined for all $\varphi \in \Phi(\mathcal{R}_3)$.

<u>THEOREM 1</u>. Let $\varphi \in \Phi(\mathcal{R}_3)$ and let $x_{j_1},\ldots,x_{j_k}$ be the variables freely occurring in φ . Then there exists a sentence $\tau^+(\varphi) \in \Phi(\mathcal{R}^+)$, determined effectively, such that:

(1.1) The formulae $\psi \in \Phi(\mathcal{R}_1)$ with ρ_ψ occurring in $\tau^+(\varphi)$ together involve only $x_{j_1},\ldots,x_{j_k}$ as free variables.

(1.2) Given $\Lambda = \langle\{\langle A_i,\alpha_i\rangle\}_{i\in I}, \langle I,\iota\rangle\} \in \underline{\underline{C}}(\mathcal{R}_1,\mathcal{R}_2)$ and $f_1,\ldots,f_k \in C\Pi_{i\in I} A_i$ the formula φ holds in $\Pi^\tau(\Lambda)$ at $\langle x_{j_1},\ldots,x_{j_k}\rangle = \langle f_1,\ldots,f_k\rangle$ if and only if $\tau^+(\varphi)$ holds in the graph of Λ at $\langle x_{j_1},\ldots,x_{j_k}\rangle = \langle f_1,\ldots,f_k\rangle$.

As mentioned earlier Theorem 1 is a specialization of the Basic Theorem of [7]. The following corollary is a specialization of Theorem 5.1 of [7].

COROLLARY 1. Let $\Lambda = \langle\{\underline{A}_i\}_{i\in I},\underline{I}\rangle$, $\Lambda' = \langle\{\underline{A}'_i\}_{i\in I},\underline{I}\rangle$ be $(\mathcal{R}_1,\mathcal{R}_2)$-complexes such that $\underline{A}_i$ and $\underline{A}'_i$ are elementarily equivalent for all $i \in I$. Then $\Pi^\tau(\Lambda)$ and $\Pi^\tau(\Lambda')$ are also elementarily equivalent.

Theorem 1 gives us a procedure for deciding whether a given $\mathcal{R}_3$-formula holds in $\Pi^\tau\underline{I}(\underline{A}_i)$ for given values of the variables involved. For $\mathcal{R}_3$-sentences this procedure can be simplified by using the following method of Mal'cev [19].

Let $\Lambda = \langle\{\underline{A}_i\}_{i\in I},I\} \in \underline{\underline{C}}(\mathcal{R}_1,\mathcal{R}_2)$. We can find $\Lambda' = \langle\{\underline{A}'_i\}_{i\in I},I\rangle \in \underline{\underline{C}}(\mathcal{R}_1,\mathcal{R}_2)$ such that the A'_i are pair-wise disjoint and $\underline{A}_i$, $\underline{A}'_i$ are isomorphic for all $i \in I$. In view of this we can assume that our typical $(\mathcal{R}_1,\mathcal{R}_2)$-complex $\Lambda = \langle\{\underline{A}_i\}_{i\in I},\underline{I}\rangle$ is such that the carriers A_i of the $\underline{A}_i$ are pair-wise disjoint. It is also no loss of generality to assume that the sets P_1 , P_2 , P_3 are pair-wise disjoint. Unless otherwise stated these assumptions will be tacit in all that follows.

Let ε be a new binary relational symbol and write $P^* = P_1 \cup P_2 \cup \{\varepsilon\}, \mathcal{R}^* = \langle P^*,h^*\rangle$, where $h^*|P_1 = h_1$, $h^*|P_2 = h_2$ and $h^*(\varepsilon) = 2$. Consider $\Lambda = \langle\{\underline{A}_i\}_{i\in I},\underline{I}\rangle \in \underline{\underline{C}}(\mathcal{R}_1,\mathcal{R}_2)$. Let $A = \bigcup_{i\in I} A_i$. We define an $\mathcal{R}$-structure $\underline{A} = \langle A,\alpha\rangle$ as follows. First of all $\alpha(\varepsilon)$ is the equivalence relation over A with the A_i as equivalence classes. Next, if $\rho \in P_1$ then $\alpha(\rho) = \bigcup_{i\in I} \alpha_i(\rho)$. Finally, for all $\rho \in P_2$ and $a_{i_1},\ldots,a_{i_k} \in A$, $k = h_2(\rho)$, if $a_{i_1} \in A_{i_1},\ldots,a_{i_k} \in A_{i_k}$

then $\langle a_{i_1},\ldots,a_{i_k}\rangle \in \alpha(\rho)$ if and only if $\langle i_1,\ldots,i_k\rangle \in \iota(\rho)$, where ι comes from Λ. We shall refer to the structure $\underline{A}$ as the index of Λ and denote it by $\mathrm{Ind}(\Lambda)$.

Let $\theta(x)$ be an $\mathfrak{R}^*$-formula with x as one of its free variables. Let ψ be another $\mathfrak{R}^*$-formula. We write $[\theta(x)]\psi$ for the result of relativising ψ with respect to $\theta(x)$. Semantically $[\theta(x)]\psi$ can be interpreted, roughly speaking, as saying that ' ψ holds in the substructure consisting of all x such that $\theta(x)$ '. For every $\sigma \in \Phi_0(\mathfrak{R}_3)$ we define $\tau^*(\sigma)$ to be the $\mathfrak{R}^*$-sentence obtained from $\tau^+(\sigma)$ by replacing every formula of the form $\rho_\psi(x_j)$ that occurs in $\tau^+(\sigma)$ by $[\theta(x)]\psi$, where $\theta(x)$ stands for $\varepsilon(x_j,x)$. The following result, which generalizes Theorem 4 of [19], is an immediate consequence of Theorem 1.

<u>THEOREM 2</u>. For all $\sigma \in \Phi_0(\mathfrak{R}_3)$ and $\Lambda \in \underline{\underline{C}}(\mathfrak{R}_1,\mathfrak{R}_2)$ the structure $\Pi^\tau(\Lambda)$ satisfies σ if and only if $\mathrm{Ind}(\Lambda)$ satisfies $\tau^*(\sigma)$

With Theorem 2 at our disposal we can now apply the method of Kogalovskij [12] to generalize the results of Makkai [14] and Vaught [31] on cartesian products to the first order products. For $\underline{\underline{V}}_1 \subset \underline{\underline{V}}(\mathfrak{R}_1)$, $\underline{\underline{V}}_2 \subset \underline{\underline{V}}(\mathfrak{R}_2)$ let $\mathrm{Ind}(\underline{\underline{V}}_1,\underline{\underline{V}}_2)$ denote the class of all $\mathfrak{R}^*$-structures of the form $\mathrm{Ind}(\Lambda)$, where $\Lambda = \langle\{\underline{A}_i\}_{i\in I},\underline{I}\}$, $\underline{I} \in \underline{\underline{V}}_2$, $\underline{A}_i \in \underline{\underline{V}}_1$. Compare the following result with Lemma 1 of [12].

THEOREM 3. If $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathfrak{R}_1)$, $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathfrak{R}_2)$ are elementary (universal) then $\mathrm{Ind}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ is also elementary (universal).

PROOF. Let Σ' , Σ'' , Σ_1 , Σ_2 be sets of $\mathfrak{R}^*$-sentences defined as follows: Σ' consists of the sentence that says that ε is an equivalence relation and of the sentences

$$\forall x_1,\dots,x_{h_2(\rho)},x_1',\dots,x_{h_2(\rho)}' [\varepsilon(x_1,x_1') \wedge \dots \wedge \varepsilon(x_{h_2(\rho)},x_{h_2(\rho)}') \longrightarrow \{\rho(x_1,\dots,x_{h_2(\rho)}) \longleftrightarrow \rho(x_1',\dots,x_{h_2(\rho)}')\}] ,$$

where $\rho \in P_2$; so that Σ' says that ' ε is a relation of equality for relations of P_2 '. Σ'' is the set of sentences

$$\forall x_1,\dots, x_{h_1(\rho)} [\rho(x_1,\dots,x_{h_1(\rho)}) \longrightarrow \varepsilon(x_1,x_2) \wedge \dots \wedge \varepsilon(x_1,x_{h_1(\rho)})] , \quad \rho \in P_1 ;$$

so that Σ'' says that relations in P_1 can hold between those elements only that are equal under ε '. Σ_1 is the set of sentences

$$\forall x_1 ([\theta(x)]\sigma) ,$$

where $\sigma \in \Sigma(\underline{\underline{V}}_1)$ and $\theta(x)$ stands for $\varepsilon(x_1,x)$; so that Σ_1 says that ' $\Sigma(\underline{\underline{V}}_1)$ holds in the substructures formed by equivalence classes under ε '.

Σ_2 is the set of sentences σ^{ε} , where $\sigma \in \Sigma(\underline{\underline{V}}_2)$ and σ^{ε} is obtained from σ by replacing every atomic formula of the form

$x_{j_1} = x_{j_2}$ occurring in σ by $\varepsilon(x_{j_1}, x_{j_2})$.

It is clear that $\Sigma = \Sigma' \cup \Sigma'' \cup \Sigma_1 \cup \Sigma_2$ defines $\mathrm{Ind}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$. Further Σ is equivalent to a set of universal sentences if $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are universal. This proves the theorem.

We can now state and prove the following generalization of the compactness theorems of [12] and [14]. We recall that a class $\underline{\underline{V}}$ of $\mathfrak{R}$-structures is called compact if for every $\Sigma \subseteq \Phi_0(\mathfrak{R})$ there is a model of Σ in $\underline{\underline{V}}$ provided that every finite subset of Σ has a model in $\underline{\underline{V}}$.

<u>THEOREM 4</u>. If $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathfrak{R}_1)$ is compact and $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathfrak{R}_2)$ is elementary then $\Pi^\tau(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ is compact.

<u>PROOF</u>. Let $\underline{\underline{V}}_1'$ be the elementary class defined by $\Sigma(\underline{\underline{V}}_1)$ so that $\underline{\underline{V}}_1'$ consists of all $\mathfrak{R}_1$-structures that are elementarily equivalent to structures of $\underline{\underline{V}}_1$. We first show that $\Pi^\tau(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$ is compact. Let every finite subset Σ' of $\Sigma \subseteq \Phi_0(\mathfrak{R}_3)$ have a model in $\Pi^\tau(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$; we want to show that Σ has a model in $\Pi^\tau(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$. By Theorem 2 we see that $\{\tau^*(\sigma); \sigma \in \Sigma'\}$ has a model in $\mathrm{Ind}(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$ for every finite $\Sigma' \subseteq \Sigma$. Since, by Theorem 3, $\mathrm{Ind}(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$ is compact this implies that $\{\tau^*(\Lambda); \sigma \in \Sigma\}$ has a model in $\mathrm{Ind}(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$, say, $\mathrm{Ind}(\Lambda)$, where $\Lambda = \langle \{\underline{A}_i\}_{i \in I}, \underline{I} \rangle$, $\underline{A}_i \in \underline{\underline{V}}_1'$, $\underline{I} \in \underline{\underline{V}}_2$. Another use of Theorem 2 now immediately shows that $\Pi^\tau(\Lambda)$ satisfies Σ. Since $\Pi^\tau(\Lambda) \in \Pi^\tau(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$ we conclude that $\Pi^\tau(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$ is

compact. Compactness of $\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ now follows by Corollary 1 which tells us that every structure in $\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ is elementarily equivalent to a structure in $\Pi^{\tau}(\underline{\underline{V}}_1', \underline{\underline{V}}_2)$. This completes the proof of the theorem.

We observe that the above theorem can also be proved by the method of Makkai [14] or of Onarov [23].

In Theorem 4 if we take $\underline{\underline{V}}_2 = \underline{\underline{V}}(\mathcal{R}_2)$ and take Π^{τ} to be a regular product we obtain the compactness result of [12] for regular products. We now apply Theorem 4 to a situation where the results of [12] are not applicable. Let $\underline{\underline{V}}_2$ be the class of all ordered sets and let $\underline{\underline{V}}_1$ be a compact class of binary relational structures. Then, by Theorem 4, $\Pi^{\mathrm{ord}}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ is compact where Π^{ord} is the ordinal product. We do not know what happens if $\underline{\underline{V}}_2$ is the class of all well-ordered sets but it seems reasonable to conjecture that in this case compactness of $\underline{\underline{V}}_1$ does not imply the compactness of $\Pi^{\mathrm{ord}}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$. These remarks are relevant to the question of Feferman mentioned in the introductory part of this chapter.

From Theorem 4 we obtain the following generalization of a theorem of Vaught [31].

<u>THEOREM 5</u>. If $\underline{\underline{V}}_1 \subseteq \underline{\underline{V}}(\mathcal{R}_1)$, $\underline{\underline{V}}_2 \subseteq \underline{\underline{V}}(\mathcal{R}_2)$ and $\underline{\underline{V}}_1$, $\underline{\underline{V}}_2$ are elementary then $S\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ is universal.

PROOF. We use the standard method of diagram. Let $\underline{\underline{V}}_3 = S\Pi^{\tau}(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ and let an $\mathfrak{R}_3$-structure $\underline{A} = \langle A, \alpha \rangle$ satisfy Σ. It is enough to show that $\underline{A} \in \underline{\underline{V}}_3$. For every $a \in A$ think of a new symbol ρ_a and let $P = \{\rho_a ; a \in A\}$, $\mathfrak{R}_1' = \langle P_1', \emptyset, h_1' \rangle$, $P_1' = P_1 \cup P$, $h_1' | P_1 = h_1$, $h_1'(P) = 1$, $\mathfrak{R}_3' = \langle P_3', \emptyset, h_3' \rangle$, $P' = P_3 \cup P$, $h_3' | P_3 = h_3$, $h_3'(P) = 1$. Let $\underline{A}' = \langle A, \alpha' \rangle$, where $\alpha' | P_3 = \alpha$ and $\alpha'(\rho_a) = \{a\}$ for $a \in A$. Write $\underline{\underline{V}}_1'$ for the class of all $\mathfrak{R}_1'$-structures $\underline{M}$ such that the $\mathfrak{R}_1$-reduct of $\underline{M}$ is in $\underline{\underline{V}}_1$ and $\underline{M}$ satisfies $\forall x_1 \rho(x_1)$ and $\forall x_1, x_2 [\rho(x_1) \wedge \rho(x_2) \longrightarrow x_1 = x_2]$ for all $\rho \in P$. Then $\underline{\underline{V}}_1'$ is elementary and every universal $\mathfrak{R}_1'$-sentence that holds in every structure of $\underline{\underline{V}}_3'$ also holds in $\underline{A}'$.

Let the diagram Δ of $\underline{A}$ be defined as the set of $\mathfrak{R}_3'$-sentences of one of the forms:

$$\exists x_1, \dots, x_k [\rho_{a_1}(x_1) \wedge \dots \wedge \rho_{a_k}(x_k) \wedge \rho(x_1, \dots, x_k)] ,$$
$$\rho \in P_3 , \quad k = h_3(\rho), \ \langle a_1, \dots, a_k \rangle \in \alpha(\rho) ,$$

$$\exists x_1, \dots, x_k [\rho_{a_1}(x_1) \wedge \dots \wedge \rho_{a_k}(x_k) \wedge \sim \rho(x_1, \dots, x_k)] ,$$
$$\rho \in P_3 , \ k = h_3(\rho) , \ \langle a_1, \dots, a_k \rangle \notin \alpha(\rho) .$$

Let τ' be a function from P_3' such that $\tau' | P_3 = \tau$ and $\tau'(\rho)$ is $\forall x_1 \rho_{\psi}(x_1)$ for all $\rho \in P$, where ψ stands for the formula $\rho(x_1)$. Thus the product $\Pi^{\tau'}$ behaves like Π^{τ} for relational symbols in P_3 and like the cartesian product for those in P.

From this it is immediate that $\underline{\underline{V}}_3' = \Pi^T(\underline{\underline{V}}_1, \underline{\underline{V}}_2)$ and hence that $\underline{A} \in \underline{\underline{V}}_3$ if and only if Δ has a model in $\underline{\underline{V}}_3'$. By Theorem 4, $\underline{\underline{V}}_3'$ is compact. This reduces the proof of the theorem to showing that every finite subset Δ' of Δ has a model. Let σ be the negation of the conjunction of the sentences in Δ' so that Δ' is equivalent to $\sim \sigma$. Since σ is a universal sentence that does not hold in $\underline{A}'$ we conclude that $\sigma \notin \Sigma'$; so that $\sim \sigma$ and hence Δ' has a model in $\underline{\underline{V}}_3'$. This proves our theorem.

Our last result is in an entirely different direction. Let $\Lambda = \langle \{\underline{A}_i\}_{i\in I}, \underline{I}\rangle$, $\Lambda' = \langle \{\underline{A}'_{i'}\}_{i'\in I'}, I'\rangle$ be $(\mathfrak{K}_1, \mathfrak{K}_2)$-complexes. By a <u>complex homomorphism</u> from Λ into Λ' we shall understand an ordered pair $\underset{\sim}{f} = \langle \{f_i\}_{i\in I}, f\rangle$, where $f_i : \underline{A}_i \longrightarrow \underline{A}'_{f(i)}$ is a homomorphism for $i \in I$ and $f : \underline{I} \longrightarrow I'$ is an isomorphism. If $\underset{\sim}{g}' = \langle \{g'_i\}_{i\in I'}, g'\rangle$ is another complex homomorphism from Λ' to Λ'' we define the product $\underset{\sim}{g}\,\underset{\sim}{f}$ to be the complex homomorphism $\langle \{g_{f(i)} f_i\}_{i\in I}, gf\rangle$. Denote by $\underline{\underline{K}}(\mathfrak{K}_1, \mathfrak{K}_2)$ the category of $(\mathfrak{K}_1, \mathfrak{K}_2)$-complexes and complex homomorphisms. With our typical complex homomorphism $\underset{\sim}{f}$ we can associate a function $C\Pi(\underset{\sim}{f}) : C\Pi_{i\in I} A_i \longrightarrow C\Pi_{i'\in I'} A'_{i'}$ as follows: For every $e \in C\Pi_{i\in I} A_i$ the image e' of e under $C\Pi(\underset{\sim}{f})$ is such that $e'(i') = f_i(e(i))$ for all i, i', $i \in I$, $i' \in I'$, $f(i) = i'$. The function $C\Pi(\underset{\sim}{f})$ is in fact a homomorphism from $C\Pi_{i\in I}(\underline{A}_i)$ into $C\Pi_{i'\in I'} \underline{A}'_{i'}$. We define a class of first order products that share the above property with the cartesian product.

Let $\mathfrak{R} = \langle P, \emptyset, h\rangle$ be any predicate system and let $P' \subseteq P$. An $\mathfrak{R}$-formula φ will be called <u>positive with respect to</u> P' if φ is equivalent to a formula of the form

$$Q_1 x_1 \ldots Q_n x_n [(q_{11}\varphi_{11} \vee \cdots \vee q_{1m_1}\varphi_{1m_1}) \wedge \cdots \wedge (q_{\ell 1}\varphi_{\ell 1} \vee \cdots \vee q_{\ell m_\ell}\varphi_{\ell m_\ell})]$$

where $Q_1, \ldots, Q_n$ are quantifiers and for j, k, $1 \leq j \leq \ell$, $1 \leq k \leq m_\ell$, $q_{jk} = \pm 1$, $+1\varphi_{jk}$ and $-1\varphi_{jk}$ stand for φ_{jk} and $\sim\varphi_{jk}$ respectively, φ_{jk} is an atomic formula and $q_{jk} = +1$ if φ_{jk} involves a relational symbol of P'. Now we define a <u>generalized cartesian product</u> to be a first order product Π^τ with τ restricted in the following way: For every $\rho \in P_3$ if ρ_ψ occurs in $\tau(\rho)$ then ψ is an atomic formula and $\tau(\rho)$ is positive in ρ_ψ. We can immediately mention the cartesian product and the ordinal product as examples of generalized cartesian products. For every first order product Π^τ we extend the domain of definition of Π^τ from $\underline{\underline{C}}(\mathfrak{R}_1, \mathfrak{R}_2)$ to $\underline{\underline{K}}(\mathfrak{R}_1, \mathfrak{R}_2)$ by defining $\Pi^\tau(\underset{\sim}{f}) = C\Pi(\underset{\sim}{f})$, for every complex homomorphism $\underset{\sim}{f} \in \underline{\underline{K}}(\mathfrak{R}_1, \mathfrak{R}_2)$. We denote this function from $\underline{\underline{K}}(\mathfrak{R}_1, \mathfrak{R}_2)$ by the same symbol Π^τ. The following theorem generalizes the property of the cartesian product noted earlier.

<u>THEOREM 6</u>. If Π^τ is a generalized cartesian product then Π^τ is a functor from $\underline{\underline{K}}(\mathfrak{R}_1, \mathfrak{R}_2)$ into $\underline{\underline{K}}(\mathfrak{R}_3)$.

PROOF. Let $\Lambda = \langle \{\underline{A}_i\}_{i\in I}, \underline{I}\rangle$, $\Lambda' = \langle \{\underline{A}'_{i'}\}_{i'\in I'}, \underline{I}'\rangle$ be $(\mathfrak{K}_1, \mathfrak{K}_2)$-complexes and let $\underset{\sim}{f} = \langle \{f_i\}_{i\in I}, f\rangle$ be a complex homomorphism from Λ to Λ' . Without loss of generality we can assume that f is an identity homomorphism so that $\Lambda' = \langle \{\underline{A}'_i\}_{i\in I}, \underline{I}\rangle$. Let $\Pi^\tau(\Lambda) = \langle A, \alpha\rangle$, $\Pi^\tau(\Lambda') = \langle A', \alpha'\rangle$ so that $A = C\Pi_{i\in I} A_i$, $A' = C\Pi_{i\in I} A'_i$. Let $\rho \in P_3$, $h_3(\rho) = k$, $e_1, \dots, e_k \in A$. We have to show that if $\langle e_1, \dots, e_k\rangle \in \alpha(\rho)$ then $\langle e'_1, \dots, e'_k\rangle \in \alpha'(\rho)$, where $e'_1, \dots, e'_k$ are the images under $\Pi^\tau(\underset{\sim}{f})$ of $e_1, \dots, e_k$ respectively. This is equivalent to showing that if $\tau(\rho)$ holds in the graph of Λ at $\langle x_1, \dots, x_k\rangle = \langle e_1, \dots, e_k\rangle$ then $\tau(\rho)$ also holds in the graph of Λ' at $\langle x_1, \dots, x_k\rangle = \langle e'_1, \dots, e'_k\rangle$. But $\tau(\rho)$ involves relational symbols in $P_o = P_2 \cup \{\rho_\psi ; \rho_\psi$ enters in $\tau(\rho)\}$ only. Hence we need to show only that if $\tau(\rho)$ holds in $\underline{I}_o = \langle I, \iota_o\rangle$ then $\tau(\rho)$ also holds in $I'_o = \langle I, \iota'_o\rangle$, where $\underline{I}_o$, $\underline{I}'_o$ are the $\mathfrak{K}_o$-reducts of the graphs of Λ , Λ' at $\langle x_1, \dots, x_k\rangle = \langle e_1, \dots, e_k\rangle$ and $\langle x_1, \dots, x_k\rangle = \langle e'_1, \dots, e'_k\rangle$ respectively and $\mathfrak{K}_o = \langle P_o, \emptyset, h^+|P_o\rangle$. Now by the definition of $\Pi^\tau(\underset{\sim}{f})$ and our assumption that f is an identity function we see that $e'_\ell(i) = f_i(e_\ell(i))$ for all i , ℓ , $i \in I$, $1 \le \ell \le k$. Since f_i is a homomorphism for all $i \in I$, this shows that if $i \in \iota_o(\rho_\psi)$ then $i \in \iota'_o(\rho_\psi)$ for all $i \in I$ and for all $\rho_\psi \in P_o$. Moreover f is an isomorphism from $\underline{I}$ onto $\underline{I}$ so that $\langle i_1, \dots, i_{h_2(\rho_2)}\rangle \in \iota_o(\rho_2)$ if and only if $\langle i_1, \dots, i_{h_2(\rho_2)}\rangle \in \iota'_o(\rho_2)$

for all $\rho_2 \in P_2$, $i_1, \ldots, i_{h_2(\rho_2)} \in I$. From this and our assumption that $\tau(\rho)$ is positive with respect to the unary relational symbols ρ_ψ it is now easy to see that $\tau(\rho)$ holds in $\underset{\sim}{I}'_0$ if $\tau(\rho)$ holds in $\underset{\sim}{I}_0$. This proves that $\Pi^\tau(\underset{\sim}{f}) : \Pi^\tau(\Lambda) \longrightarrow \Pi^\tau(\Lambda')$ is a homomorphism so that Π^τ is a function into $\underline{\underline{K}}(\mathfrak{R}_3)$. The functorial character of Π^τ is clear. This proves the theorem.

We conclude this chapter by raising a question. Let $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 = \mathfrak{R}$ and let $UC(\mathfrak{R})$ denote the set $\{\Sigma(\underline{\underline{V}}) ; \underline{\underline{V}} \subseteq \underline{\underline{V}}(\mathfrak{R})$ and $\underline{\underline{V}}$ is universal$\}$. We can refer to $UC(\mathfrak{R})$ as the 'set of universal classes of $\mathfrak{R}$-structures', although such a set does not exist in Gödel-Bernays set theory which is the basis of this work. We can define a unary operator $SC\Pi$ over $UC(\mathfrak{R})$ by: $SC\Pi(\Sigma) = \Sigma(SC\Pi(\underline{\underline{V}}(\Sigma)))$, where $SC\Pi(\underline{\underline{V}}(\Sigma))$ is the class of all substructures of cartesian products of families of structures in $\underline{\underline{V}}(\Sigma)$. We shall refer to $UC(\mathfrak{R})$ together with the unary operator $SC\Pi$ as <u>Vaught's algebra of universal classes</u>. Theorem 5 enables us to define binary operations $S\Pi^\tau$ over $UC(\mathfrak{R})$ as follows:

$$S\Pi^\tau(\Sigma_1, \Sigma_2) = \Sigma(S\Pi^\tau(\underline{\underline{V}}(\Sigma_1), \underline{\underline{V}}(\Sigma_2))) .$$

The algebra obtained in this way may be called the <u>first order algebra of universal classes</u>. Since $(S\Pi C)^2 = S\Pi C$ Vaught's algebra is not monogenic. We raise the following question which we have not been able to answer. Is the first order algebra monogenic?

The work described in the present chapter is in [29].

In this chapter as in the last two we restricted our attention to restricted predicate systems. The purpose of the next chapter is to see how far our results can be carried to unrestricted predicate systems.

CHAPTER 6

COMPACTNESS THEOREM FOR A MORE GENERAL TYPE OF LANGUAGES

With every restricted predicate system $\mathfrak{R}$ we associated in Chapter 0 a language $\Phi(\mathfrak{R})$ of the first order predicate calculus. One of the most important properties of first order languages is given by the compactness theorem which states [15] that a set Σ in such a language possesses a model if every finite subset of Σ possesses one. Many results in first order model theory come from this compactness theorem; Theorem 4 of Chapter 5 is a handy example. In this chapter we extend the definition of $\Phi(\mathfrak{R})$ to unrestricted $\mathfrak{R}$. More precisely, for every predicate system $\mathfrak{R}$ we obtain a language $\Phi'(\mathfrak{R})$ such that $\Phi'(\mathfrak{R})$ is essentially $\Phi(\mathfrak{R})$ when $\mathfrak{R}$ is restricted and prove the compactness theorem for $\Phi'(\mathfrak{R})$. This work was published in [26] in a different form.

We observe that compactness theorem has been proved for some other generalizations of first order languages. Thus Führken [8] has proved the compactness theorem for languages obtained from the first order languages by adding a new quantifier which is interpreted as 'there exists at least countable'.

We begin with the description of $\Phi'(\mathfrak{R})$. This description is very much similar to that of the first order languages which is well-known. We exploit this situation and occasionally sacrifice formality and explicitness for the sake of convenience.

Let $\mathcal{R} = \langle P, \Omega, h\rangle$ and let X be a set of cardinality equal to that of the range $\mathrm{Ran}(h)$ of h. For every $\mathcal{R}$ we assume the choice of X having been made. We need the following notations.

For sets Y, Z we write Z^Y for the set of all functions $f : Y \longrightarrow Z$ and $Z^{[Y]}$ for the set of functions $f : Y \longrightarrow Z$ such that $f(Y)$ is finite. A sequence is a function of the form $s : n \longrightarrow Z$, where n is an ordinal called the length of the sequence s. We shall often depict s as $\{z_m\}_{m \in n}$ or as $z_0 \ldots z_m \ldots,\ m \in n$, where $s(m) = z_m$. If y is a set and $z_0 \ldots z_m \ldots,\ m \in n$ is a sequence then $yz_0 \ldots z_m \ldots,\ m \in n$ and $z_0 \ldots z_m \ldots y,\ m \in n$ depict obvious sequences of length n or $n+1$ and $n + 1$ respectively. For every sequence $z_0 \ldots z_m \ldots$ and every set $\bigvee$ we write $\bigvee_{m \in n} z_m$ for the sequence depicted by $z_0 \bigvee \ldots \bigvee z_m \bigvee \ldots$ in the obvious way.

We turn to the description of the formulae of $\Phi'(\mathcal{R})$. By an <u>atomic</u> $\mathcal{R}$-<u>formula</u> we shall understand a sequence of the form $x_1 = x_2$, $\rho(\{x_m\}_{m \in h(\rho)})$ or $(\{x_m\}_{m \in h(\omega)})\omega = x_{h(\omega)}$, where $\rho \in P$, $\omega \in \Omega$ $x_1, x_2, x_m \in X$ for $m \in h(\rho)$ or $m \in h(\omega) + 1$. By a <u>quantifier-free</u> $\mathcal{R}$-<u>formula in normal form</u> we shall understand a sequence of the type $\bigvee_{m_1 \in n_1} \bigwedge_{m_2 \in n_2} \varphi_{m_1 m_2}$, where $n_1, n_2 \in \mathrm{Ran}(h)$ and $\varphi_{m_1 m_2} = \varphi'_{m_1 m_2}$ or $\sim \varphi'_{m_1 m_2}$ for some atomic $\mathcal{R}$-formula $\varphi'_{m_1 m_2}$. An $\mathcal{R}$-<u>formula in prenex normal form</u> is a sequence $\varphi = Q_0 x_0 \ldots Q_m x_m \ldots \varphi_1$, where φ_1 is a quantifier-free $\mathcal{R}$-formula in normal form called the <u>matrix</u> of φ, $x_m \in X$ are distinct and Q_m is either the 'universal quantifier' $\forall$ or the 'existential quantifier' $\exists$. The occurrences in φ of an atomic $\mathfrak{R}$-formula, a quantifier, a relational symbol (member of P) etc. is to be understood in the obvious way. The occurrence of the 'variables' x_m in the formula $\varphi = Q_o x_o \ldots Q_m x_m \ldots \varphi_1$ is by definition <u>bound</u> while all other variables (members of X) occurring in φ_1 are said to <u>occur freely</u> in φ . An $\mathfrak{R}$-formula φ in prenex normal form is called <u>finitary</u> if φ satisfies the following <u>finitary condition</u>. Let $e \in X^{[X]}$ and let φ_1^e denote the $\mathfrak{R}$-formula obtained from φ_1 by replacing every $x \in X$ occurring in φ_1 by $e(x)$, where φ_1 is the matrix of φ . Then there are only finitely many atomic $\mathfrak{R}$-formulae occurring in φ_1^e . Moreover there are only finitely many occurrences of $\exists$ in φ .

We define $\Phi'(\mathfrak{R})$ to be the set of all finitary $\mathfrak{R}$-formulae in prenex normal form. An example of an infinitely long formula in $\Phi'(\mathfrak{R})$ is the sequence

$$\forall x_o \, \forall x_1 \ldots \forall x_m \ldots \rho(x_o,x_1) \vee \ldots \vee \rho(x_o,x_m) \vee \ldots ,$$

where ρ is binary and m varies over an ordinal.

In the above description of $\Phi'(\mathfrak{R})$, which can be obviously completely formalized in Gödel-Bernays' set theory, we can regard $\vee$, $\sim$, (, $\forall$, $=$ etc as arbitrarily fixed sets independent of $\mathfrak{R}$. We also use $=$ to identify two objects as usual and the context will make our meaning clear. Similar remark holds for parenthesis (,) .

Clearly $\Phi'(\mathfrak{R})$ is essentially $\Phi(\mathfrak{R})$ when $\mathfrak{R}$ is restricted because then every formula of $\Phi(\mathfrak{R})$ is equivalent to a formula of $\Phi'(\mathfrak{R})$.

A <u>sentence</u> of $\Phi'(\mathfrak{R})$ is a formula of $\Phi'(\mathfrak{R})$ without free variables.

We now give the semantic interpretation of the formulae of $\Phi'(\mathfrak{R})$. By a <u>finitely supported</u> $\mathfrak{R}$-<u>structure</u> we shall understand a pair $\underline{A} = \langle A,\alpha\rangle$ where the <u>carrier</u> A of $\underline{A}$ is a set and the <u>make</u> α of $\underline{A}$ is a function from $P \cup \Omega$ such that $\alpha(\rho) \subseteq A^{[h(\rho)]}$ for all $\rho \in P$ and $\alpha(\omega)$ is a function from $A^{[h(\omega)]}$ into A for all $\omega \in \Omega$. Let $f \in A^{[X]}$, $\varphi \in \Phi'(\mathfrak{R})$ and let $\underline{A} = \langle A,\alpha\rangle$ be a finitely supported $\mathfrak{R}$-structure. We describe what we mean by saying that φ holds in $\underline{A}$ at f . We distinguish several cases.

(1) If φ is an atomic $\mathfrak{R}$-formula $x_1 = x_2$ then φ holds in $\underline{A}$ at f if and only if $f(x_1)$ is identical with $f(x_2)$.

(2) If φ is an atomic $\mathfrak{R}$-formula of the form $\rho(\{x_m\}_{m \in h(\rho)})$, $\rho \in P$, then φ holds in $\underline{A}$ at f if and only if the sequence $\{f(x_m)\}_{m \in h(\rho)}$ is in $\alpha(\rho)$.

(3) If φ is an atomic $\mathfrak{R}$-formula of the form $(\{x_m\}_{m \in h(\omega)})\omega = x_{h(\omega)}$, $\omega \in \Omega$, then φ holds in $\underline{A}$ at f if and only if the image under $\alpha(\omega)$ of $\{f(x_m)\}_{m \in h(\omega)}$ is $f(x_{h(\omega)})$.

(4) If $\varphi = \sim\varphi'$ for some atomic $\mathfrak{R}$-formula φ' then φ holds in $\underline{A}$ at f if and only if φ' does not hold in $\underline{A}$ at f .

(5) If $\varphi = \bigvee_{m_1 \in n_1} \bigwedge_{m_2 \in n_2} \varphi_{m_1 m_2}$, where $n_1, n_2 \in \mathrm{Ran}(h)$ and $\varphi_{m_1 m_2} = \varphi'_{m_1 m_2}$ or $\sim\varphi'_{m_1 m_2}$ for some atomic formula $\varphi'_{m_1 m_2}$, then

φ holds in $\underline{A}$ at f if and only if there exists $m_1 \in n_1$ such that $\varphi_{m_1 m_2}$ holds in $\underline{A}$ at f for all $m_2 \in n_2$.

(6) If $\varphi = \forall x_0 \dots \forall x_m \dots \varphi_1$ where φ_1 is quantifier-free then φ holds in $\underline{A}$ at f if and only if φ_1 holds in $\underline{A}$ at g for all $g \in A^{[X]}$ such that $f = g$ on the set of variables freely occurring in φ.

(7) Finally let φ be any formula of $\Phi'(\mathfrak{R})$ so that we can write φ in the form

$$\forall x_0 \dots \exists x_{m_1} \forall x_{m_1+1} \dots \exists x_{m_2} \forall x_{m_2+1} \dots \exists x_{m_p} \forall x_{m_{p+1}} \dots \varphi_1$$

where φ_1 is quantifier-free, $p = p(\varphi)$ is finite and $\exists$ occurs only with the 'm_kth variable' x_{m_k} for $1 \leq k \leq p$. With φ we associate arbitrary new sets $\omega_{\varphi,k}$, $1 \leq k \leq p(\varphi)$. We can refer to the sets $\omega_{\varphi,k}$ as the <u>Skolem operators</u> associated with φ. Let $h^o_\varphi(\omega_{\varphi,k})$ be the ordinal number of the set $m_k - \{m_1, \dots, m_{k-1}\}$ of ordinals (which is clearly well-ordered by $\in$). Let

$$\Omega^o = \Omega \cup \{\omega_{\varphi,k} \; ; \; \varphi \in \Phi'(\mathfrak{R}) \; , \; 1 \leq k \leq p(\varphi)\} .$$

Write $\mathfrak{R}^o = \langle P, \Omega^o, h^o \rangle$ where $\mathrm{Ran}(h^o) = \mathrm{Ran}(h)$, $h^o = h$ on $P \cap \Omega$ and $h^o(\omega_{\varphi,k}) = h^o_\varphi(\omega_{\varphi,k})$ for all φ, k such that $\varphi \in \Phi'(\mathfrak{R})$ and $1 \leq k \leq p(\varphi)$. Write $\varepsilon_{\varphi,k}$ for the atomic $\mathfrak{R}^o$-formula

$$(\{x_m\}_{m \in h^o(\omega_{\varphi,k})})\omega_{\varphi,k} = x_{m_k} \; , \quad \varphi \in \Phi'(\mathfrak{R}) \; , \quad 1 \leq k \leq p(\varphi) .$$

~~Let $\mathfrak{R}^o$ denote the formula~~

Let φ° denote the formula $\forall x_{0} \cdots \forall x_{m} \cdots \sim\varepsilon_{\varphi,1} \vee \cdots \vee \sim\varepsilon_{\varphi,p} \vee \varphi_1$. We shall refer to φ° as the <u>open form</u> of φ . Now we shall say that φ holds in $\underline{A}$ at f if and only if there exists a finitely supported $\mathfrak{R}^{\circ}$-structure $\underline{A}^{\circ} = \langle A, \alpha^{\circ} \rangle$ such that $\alpha^{\circ} = \alpha$ on $P \cup \Omega$ and φ° holds in $\underline{A}^{\circ}$ at f . (Note that φ° falls under the case taken care of by (6)).

We say that ' $\varphi \in \Phi'(\mathfrak{R})$ holds in $\underline{A}$ ' if φ holds in $\underline{A}$ at f for all $f \in A^{[X]}$.

A set Σ of sentences of $\Phi'(\mathfrak{R})$ is said to have a model if there exists a finitely supported $\mathfrak{R}$-structure $\underline{A}$ such that σ holds in $\underline{A}$ for all $\sigma \in \Sigma$.

We are now in a position to state and prove the result of this chapter.

<u>THEOREM 1</u>. A set Σ of sentences of $\Phi'(\mathfrak{R})$ has a model if every finite subset of Σ has one.

Proof. The theorem is trivially true if Σ is finite. We assume that the theorem is true for any Σ which can be so well-ordered as to have an ordinal number less than another ordinal number N (so that the cardinal number of Σ is less than N). Let N be the ordinal number of Σ under a well-ordering $<$. We prove the theorem for Σ ; by transfinite induction this is enough to prove the theorem. Assume then that every finite subset Σ' of Σ has a model. We have to prove that Σ has a model. We divide the proof of this in four parts. First two parts mainly set up the notations that we need to construct a model of Σ .

PART 1. Let Y be any set. We define $W(\mathfrak{R};Y)$ as follows:

$$\begin{aligned} W_m(\mathfrak{R};Y) &= Y , \quad \text{if } m = 0 \\ &= \bigcup_{m' \in m} W_{m'}(\mathfrak{R};Y) , \quad \text{if } m \text{ is a limit ordinal} \\ &= W_{m-1}(\mathfrak{R};Y) \cup \{(s)\omega;\ \omega \in \Omega,\ s \in (W_{m-1}(\mathfrak{R};Y))^{[h(\omega)]}\} , \\ &\qquad \text{if } m \text{ has a predecessor,} \end{aligned}$$

$$W(\mathfrak{R};Y) = \bigcup_{m \in n} W_m(\mathfrak{R};Y) ,$$

where n is the least ordinal containing the image of h . In the above definition of $W(\mathfrak{R};Y)$ we have written $(s)\omega$ for the sequence $(w_o \dots w_m \dots)\omega$, $m \in h(\rho)$, $w_m = s(m)$. We shall use similar abbreviations quite often in the rest of this proof. When $\mathfrak{R}$ is restricted $W(\mathfrak{R};Y)$ is the set of $\mathfrak{R}$-words in Y as defined in Chapter 0.

For every set Y we define $H(\mathfrak{R};Y)$ to be the set of sequences of the form $\rho(s)$, $\rho \in P$, $s \in Y^{[h(\rho)]}$; here again we have abbreviated $\rho(s)$ for the sequence $\rho(y_o \dots y_m \dots)$, $m \in h(\rho)$, $y_m \in s(m)$

Let κ be a binary relation over $W(\mathfrak{R};Y)$ and $f : H(\mathfrak{R};W(\mathfrak{R};Y)) \longrightarrow \{0,1\} = 2$ be a function such that:

(1.1) κ is an equivalence relation and if $s_1, s_2 \in (W(\mathfrak{R};Y))^{[h(\omega)]}$, $\omega \in \Omega$ and $\langle s_1(m), s_2(m)\rangle \in \kappa$ for all $m \in h(\omega)$ then $\langle (s_1)\omega, (s_2)\omega\rangle \in \kappa$.

(1.2) If $s_1, s_2 \in (W(\mathfrak{R};Y))^{[h(\rho)]}$, $\rho \in P$ and $\langle s_1(m), s_2(m)\rangle \in \kappa$ for all $m \in h(\rho)$ then $f(\rho(s_1)) = f(\rho(s_2))$.

Using properties (1.1) and (1.2) of κ , f we define a finitely supported $\mathfrak{R}$-structure $\underline{W}(\mathfrak{R};Y,\kappa,f)$ as follows.

For every $u \in W(\mathfrak{R};Y)$ write $[u]\kappa$ for the equivalence class determined by u under κ and let $W(\mathfrak{R};Y)/\kappa$ be the set of all such equivalence classes. For every $\rho \in P$ we define $\underset{\sim}{\rho}$ to be the set of $s \in (W(\mathfrak{R};Y)/\kappa)^{[h(\rho)]}$ such that $f(\rho(s_1)) = 1$, where $s_1 \in (W(\mathfrak{R};Y))^{[h(\rho)]}$ is such that $s(m) = [s_1(m)]\kappa$ for all $m \in h(\rho)$. For every ω we define $\underset{\sim}{\omega}$ to be the function from $(W(\mathfrak{R};Y)/\kappa)^{[h(\rho)]}$ into $W(\mathfrak{R};Y)/\kappa$ which maps $s \in (W(\mathfrak{R};Y)/\kappa)^{[h(\rho)]}$ upon $[(s_1)\omega]\kappa$, where as before $s_1 \in (W(\mathfrak{R};Y))^{[h(\rho)]}$ is such that $s(m) = [s_1(m)]\kappa$ for all $m \in h(\rho)$. By (1.1) the function $\underset{\sim}{\omega}$ is unambiguously defined, i.e., the image of s under $\underset{\sim}{\omega}$ as defined above does not depend on the choice of s_1 . We now write $\underline{W}(\mathfrak{R};Y,\kappa,f)$ for the finitely supported $\mathfrak{R}$-structure with $W(\mathfrak{R};Y)/\kappa$ as carrier and α as make where $\alpha(\rho) = \underset{\sim}{\rho}$, $\alpha(\omega) = \underset{\sim}{\omega}$ for all $\rho \in P$ and $\omega \in \Omega$. It is fairly clear that every finitely supported $\mathfrak{R}$-structure can be represented as a $\underline{W}(\mathfrak{R};Y,\kappa,f)$.

<u>PART 2</u>. For every $\sigma \in \Sigma$ let Σ_σ denote the 'closed initial segment' $\{\sigma_1;\ \sigma_1 \leq \sigma,\ \sigma \in \Sigma\}$, where $\leq$ is the well-ordering of Σ which we fixed in the beginning of the proof. Let Σ_σ° denote the set $\{\sigma_1^\circ;\ \sigma_1 \in \Sigma_\sigma\}$, where σ_1° is the open form of σ_1 . By our induction hypothesis and the assumption that the ordinal number N

of $(\Sigma,<)$ is a limit ordinal it follows that Σ_σ has a model for every $\sigma \in \Sigma$. Hence, by our semantic interpretation (case (7) on page 95), Σ_σ^o has a model $\underline{W}(\mathfrak{R}^o;Y_\sigma,\kappa_\sigma,f_\sigma)$, say, for every $\sigma \in \Sigma$. For every $\underline{W}(\mathfrak{R}^o,Y,\kappa, f)$ and $Y' \supseteq Y$ we can find κ', f' such that $\underline{W}(\mathfrak{R};Y',\kappa',f')$ is isomorphic to $\underline{W}(\mathfrak{R};Y,\kappa,f)$. Hence we can find a set Y such that Σ_σ^o has a model of the form $\underline{W}(\mathfrak{R}^o;Y,\kappa_\sigma,f_\sigma)$ for every $\sigma \in \Sigma$. We arbitrarily fix such a set Y. Since $\mathfrak{R}$ and hence $\mathfrak{R}^o$ is also arbitrarily fixed we can omit explicit mention of $\mathfrak{R}^o$, Y in different notations. Thus we write $W = W(\mathfrak{R}^o;Y)$, $\underline{W}(\kappa,f) = \underline{W}(\mathfrak{R}^o;Y,\kappa,f)$, $H = H(\mathfrak{R}^o;W)$.

In what follows U, V will <u>always</u> denote non-empty finite subsets of H. For every U we define U/P to be the smallest subset Z of W such that $U \subseteq H(\mathfrak{R}^o;Z)$. More explicitly $U/P = \{s(m);\ s \in W^{[h(\rho)]},\ m \in h(\rho),\ \rho(s) \in U\}$. We conclude this part of the proof by defining the following important set. For every U we define T_U to be the set of ordered pairs $\langle\kappa_U,f_U\rangle$ such that $\kappa_U = \kappa_\sigma|(U/P)$, $f_U = f_\sigma|U$ for some model $\underline{W}(\kappa_\sigma,f_\sigma)$ of Σ_σ^o where σ varies over Σ.

<u>PART 3</u>. In this part we prove the existence of a finitely supported $\mathfrak{R}^o$-structure $\underline{W}(\kappa,f)$ such that $\langle\kappa|(U/P),f|U\rangle \in T_U$ for every U. We do this by using an argument used in [21] to prove the embedding theorem given there. First note that T_U is a non-empty finite set for all U. Finiteness of T_U follows from the fact that there are only finitely many binary relations over the finite set U/P and only

finitely many functions from the finite set U into the finite set $\{0,1\}$. Suppose that T_U is empty so that for every $\tau = \langle \kappa_U, f_U \rangle$, $\kappa_U \subseteq U/P \times U/P$, $f_U : U \longrightarrow \{0,1\}$, we can find $\sigma_\tau \in \Sigma$ such that for every model $\underline{W}(\kappa_{\sigma_\tau}, f_{\sigma_\tau})$ of $\Sigma^0_{\sigma_\tau}$ it is false that $\kappa_{\sigma_\tau} \mid (U/P) = \kappa_U$ and $f_{\sigma_\tau} \mid U = f_U$. Let σ be the greatest of the finitely many σ_τ's in $(\Sigma, <)$. Let $\underline{W}(\kappa_\sigma, f_\sigma)$ be a model of Σ^0_σ and let $\tau = \langle \kappa_\sigma \mid (U/P), f_\sigma \mid U \rangle$. Then since $\sigma \geqq \sigma_\tau$, $\underline{W}(\kappa_\sigma, f_\sigma)$ is a model of $\Sigma^0_{\sigma_\tau} \subseteq \Sigma^0_\sigma$. This is a contradiction to the choice of σ_τ . Hence T_U is non-empty. Next, for $V \supseteq U$ we define $t_{V,U} : T_V \longrightarrow T_U$ to be the function which maps $\langle \kappa_V, f_V \rangle \in T_V$ upon $\langle \kappa_V \mid (U/P), f_V \mid U \rangle$; that this last pair is in T_U follows directly from the definition of the sets T_U . The sets T_U together with the functions $t_{V,U}$ form an inverse system of non-empty compact Hausdorff spaces (under the discrete topology on T_U). By a theorem of Steenrod [30] the inverse limit of such a system is non-empty so that we can find a family $\{\langle \kappa_U, f_U \rangle\}$, where U varies over the set of finite subsets of H , $\langle \kappa_U, f_U \rangle \in T_U$ and if $V \supseteq U$ then $\langle \kappa_U, f_U \rangle$ is the image under $t_{V,U}$ of $\langle \kappa_V, f_V \rangle$ for all finite subsets U , V of H . We choose one such family $\{\langle \kappa^*_U, f^*_U \rangle\}$ and define κ^*, f^* as follows. Let $u \in H$, $w_1, w_2 \in W$. Find U such that $u \in U$ and $w_1, w_2 \in U/P$. We set $f^*(u) = f^*_U(u)$ and put $\langle w_1, w_2 \rangle$ in κ^* if and only if $\langle w_1, w_2 \rangle \in \kappa^*_U$. It is easy to prove that κ^* , f^* are unambiguously defined. For this let V be another finite subset of H such that

$u \in V$ and $w_1, w_2 \in V/P$; we have to show that $f^*_V(u) = f^*_U(u)$ and $\langle w_1, w_2 \rangle \in \kappa^*_U$ if and only if $\langle w_1, w_2 \rangle \in \kappa^*_V$. Let $U' = U \cap V$, $U'' = U \cup V$ so that $u \in U'$, $w_1, w_2 \in U''/P$. By the choice of $\{\langle \kappa^*_U, f^*_U \rangle\}$ we see that $f^*_{U'} = f^*_U \; U' = f^*_V \; U'$ and hence $f^*_U(u) = f^*_{U'}(u) = f^*_V(u)$. Similarly $\kappa^*_{U''} \,|\, U = \kappa_U$, $\kappa^*_{U''} \,|\, V = \kappa^*_V$ and therefore $\langle w_1, w_2 \rangle \in \kappa^*_U$ if and only if $\langle w_1, w_2 \rangle \in \kappa^*_{U''}$ and $\langle w_1, w_2 \rangle \in \kappa^*_V$ if and only if $\langle w_1, w_2 \rangle \in \kappa^*_{U''}$. This completes the proof of the unambiguity of the definition of the pair $\langle \kappa^*, f^* \rangle$.. If we can show that κ^*, f^* satisfy (1.1) and (1.2) <u>respectively</u> then $\underline{W}(\kappa^*, f^*)$ would be clearly one of the $\mathfrak{R}_0$-structures we started to look for in this part. The proof of (1.1) and (1.2) for κ^*, f^* is easy.

Take **for** example (1.1) : κ^* is an equivalence relation over W because the restriction of κ^* to every finite subset of W is an equivalence relation. Moreover let $s_1, s_2 \in W^{[h(\omega)]}$, $\omega \in \Omega_0$, $\langle s_1(m), s_2(m) \rangle \in \kappa^*$ for all $m \in h(\omega)$; we have to show that $\langle (s_1)\omega, (s_2)\omega \rangle \in \kappa^*$. Take U such that $(s_1)\omega$, $(s_2)\omega$, $s_1(m)$, $s_2(m) \in U/P$ for all $m \in h(\omega)$. We can find such a U because the images of s_1, s_2 are finite. Since $\kappa \,|\, (U/P) = \kappa^*_U$ we see that $\langle s_1(m), s_2(m) \rangle \in \kappa^*_U$. But $\kappa^*_U = \kappa_\sigma \,|\, (U/P)$ for some model $\underline{W}(\kappa_\sigma, f_\sigma)$ of Σ^0_σ, where σ is any sentence in Σ. Now κ_σ, f_σ satisfy (1.1). Hence $\langle (s_1)\omega, (s_2)\omega \rangle \in \kappa_\sigma$ which gives $\langle (s_1)\omega, (s_2)\omega \rangle \in \kappa^*_U$. In view of $\kappa^* \,|\, (U/P) = \kappa^*_U$ we get $\langle (s_1)\omega, (s_2)\omega \rangle \in \kappa^*$. This proves (1.1) The proof of (1.2) is on similar lines.

<u>PART 4.</u> In this last part of the proof we show that $\underline{W}(\kappa^*,f^*)$ (as constructed in Part 3) is a model of Σ^o. By what has gone before this will obviously complete the proof of the theorem.

Let $\sigma \in \Sigma$ and let $\sigma^o = \forall x_1 \dots \forall x_m \dots \varphi$, where φ is a quantifier-free finitary $\mathfrak{R}^o$-formula. We have to show that φ holds in $\underline{W}(\kappa^*,f^*)$ at g for all $g \in (W/\kappa^*)^{[X]}$. This is clearly equivalent to showing that φ^e holds in $\underline{W}(\kappa^*,f^*)$ at g for all $e \in X^{[X]}$ and $g \in (W/\kappa^*)^{[X]}$, where φ^e denotes (see page 93) the formula obtained from φ by replacing $x \in X$ occurring in φ by $e(x)$ everywhere in φ. Let $e \in X^{[X]}$, $g \in (W/\kappa^*)^{[X]}$. By our finitary condition φ^e involves only finitely many atomic formulae, say, $\rho_1(s_1),\dots,\rho_\ell(s_\ell)$, $(s_1')\omega_1 = x_1',\dots,(s_k')\omega_k = x_k'$, $\varepsilon_1,\dots,\varepsilon_q$, where ℓ, k, q are finite, $\rho_i \in P$, $s_i \in X^{[h(\rho_i)]}$ for $1 \leq i \leq \ell$, $\omega_i \in \Omega^o$, $s_i' \in X^{[h(\omega_i)]}$ for $1 \leq i \leq k$ and ε_i is of the form $x_1 = x_2$, $x_1,x_2 \in X$ for $1 \leq i \leq q$. Let $g' \in W^{[X]}$ be such that $g(x) = [g'(x)]\kappa$ for all $x \in X$. Find a finite $U \subset H$ such that $\rho_i(g's_i) \in U$ for $1 \leq i \leq \ell$, $(g's_i')\omega_i \in U/P$ and $g'(x) \in U/P$ for every $x \in X$, where $g's_i'$, $g's_i$, as usual, denote the compositions of the functions involved; such a U exists because all our functions have finite images. By the definitions of κ^*, f^* find a model $\underline{W}(\kappa_\sigma,f_\sigma)$ of Σ_σ^o such that $\kappa^* \mid (U/P) = \kappa_U^* = \kappa_\sigma \mid (U/P)$ and $f^* \mid U = f_U^* = f_\sigma \mid U$. From this and the definition of $\underline{W}(\kappa,f)$ (page 98) we see that the function $g_\sigma \in (W/\kappa_\sigma)^{[X]}$ defined by $g_\sigma(x) = [g'(x)]\kappa_\sigma$ is such that any one of the atomic formulae $\rho_1(s_1),\ldots,\rho_\ell(s_\ell)$, $(s_1')\omega_1 = x_1',\ldots,(s_k')\omega_k = x_k'$. $\varepsilon_1,\ldots,\varepsilon_q$ holds in $\underline{W}(\kappa^*,f^*)$ at g if and only if it holds in $\underline{W}(\kappa_\sigma,f_\sigma)$ at g_σ. In view of our semantic interpretation this implies that φ^e holds in $\underline{W}(\kappa^*,f^*)$ at g if and only if φ^e holds in $\underline{W}(\kappa_\sigma,f_\sigma)$ at g_σ. But $\underline{W}(\kappa_\sigma,f_\sigma)$ satisfies σ^o. Hence φ^e holds in $\underline{W}(\kappa^*,f^*)$ at g. Since e, g, σ were arbitrary, this shows that σ^o holds in $\underline{W}(\kappa^*,f^*)$ for every $\sigma \in \Sigma$. This completes the proof of the theorem.

With the compact language $\Phi'(\mathfrak{R})$ at our disposal to talk about finitely supported $\mathfrak{R}$-structures we can now generalize to these structures many of the results of Model Theory that are stated for the restricted structures. In particular all the results on restricted structures, presented in this thesis, have generalizations to finitely supported structures. The details in the unrestricted case involve at most notational complications and we omit them.

REFERENCES

1. J.W. ADDISON, LEON HENKIN, ALFRED TARSKI (Editors): The Theory of Models. Proceedings of the 1963 International Symposium at Berkley. North Holland. Amsterdam. 1965.

2. GARRET BIRKHOFF: On the structure of abstract algebras. Proc. Cambridge Philos. Soc. 31, 433-454 (1935).

3. —————————— : Lattice Theory. American Math. Soc. Providence, R.I. 1967. New Edition.

4. R.H. BRUCK: A Survey of Binary Systems. Springer, Berlin-Göttingen-Heidelberg. 1958.

5. P.J. COHEN: Set Theory and the Continuum Hypothesis. W.A. Benjamin, Inc. New-York-Amsterdam. 1966.

6. P.M. COHN: Universal Algebra. Harper & Row and John Weatherhill, Inc. 1966.

7. S. FEFERMAN and R.L. VAUGHT: The first order properties of products of algebraic systems. Fund. Math. 47, 57-103 (1959).

8. ENNO GEBHARD FÜHRKEN: Countable first order languages with a generalized quantifier. J. Symb. Logic. 27, 479-480 (1962).

9. G. GRATZER: Universal Algebra. Van Nostrand. 1968.

10. G. HIGMAN and B.H. NEUMANN: Groups as groupoids with one law. Publ. Math. Debrecen. 2, 215-221 (1952).

11. BJARNI JONSON and ALFRED TARSKI: On two properties of free algebras. Math. Scand. 9, 95-101 (1961).

12. S.R. KOGALOVSKIJ: On algebraic constructions preserving compactness. [Russian] Sibirsk. Ž. 8 1202-1205 (1967).

13. ROGER G. LYNDON: Properties preserved under homomorphism. Pacific J. Math. 9, 143-154 (1959).

14. M. MAKKAI: A compactness result concerning direct products of models. Fund. Math. 57, 313-326 (1965).

15. A.I. MAL'CEV: On a general method for obtaining local theorems in group theory [Russian]. Notices of the Pedagogical Institute of Ivanovo, Physical-Mathematical Sciences. 1, 3-9 (1941).

16. ―――――――― : On certain classes of models [Russian]. Dokl. Akad. Nauk SSSR. 120, 245-248 (1958).

17. ―――――――― : Structural characteristic of certain classes of algebras [Russian]. Dokl. Akad. Nauk SSSR. 120, 29-32 (1958).

18. ―――――――― : Subdirect products of models [Russian]. Dokl. Akad. Nauk SSSR. (2) 109, 264-266 (1956).

19. ―――――――― : Regular products of models [Russian]. Izv. Akad. Nauk SSSR Ser. Mat. 23, 489-502 (1959).

20. BARRY MITCHELL: The Theory of Categories. Academic Press. New York. 1965.

21. B.H. NEUMANN: An embedding theorem for algebraic systems. Proc. London Math. Soc. (3) 4, 138-153 (1954).

22. HANNA NEUMANN: Varieties of Groups. Springer-Verlag. Berlin-Heidelberg-New York. 1967.

23. A.I. OMAROV: On compact classes of models [Russian]. Algebra i Logica Sem. 6, 49-60 (1967).

24. ABRAHAM ROBINSON: Introduction to Model Theory and the Metamathematics of Algebra. North Holland. Amsterdam. 1963.

25. ABRAHAM ROBINSON: A note on an embedding theorem for algebraic systems. J. London Math. Soc. 30, 249-252 (1955).

26. AHMAD SHAFAAT: Principle of localization for a more general type of languages. Proc. London Math. Soc. (3) 17, 629-643 (1967)

27. ——————— : On implicationally defined classes of algebras. J. London Math. Soc. (in print).

28. ——————— : Characterizations of some universal classes of algebras. J. London Math. Soc. (in print).

29. ——————— : On products of relational structures. Fund. Math. (in print).

30. N.E. STEENROD: Universal homological groups. American J. Math. 58, 661-701 (1936).

31. R.L. VAUGHT: The elementary character of two notions from general algebra. Essays on the Foundations of Mathematics. Magnes Press. Jerusalem. 1961.