# PROPERTIES OF SEPARABLE BANACH-VALUED MARTINGALES. 

by

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## Statement

Except where it is indicated, this thesis is my own work.

Robert J. Arnott.

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## INTRODUCTION

This thesis is conerned with the extension of the classical theory of martingales of real random variables as contained in Doob [4] to the abstract theory of martingales of random variables whose values lie in a real Banach space. Extensions of almost all the convergence theorems in [4] for discrete parameter martingales can be found in Chatterji [1] and [2], Scalora [10], Tulcea and Tulcea [12], and Driml and Hans [5].

In addition to extending the existing theory this thesis also attempts to further the correlation between the abstract and classical theories. To pursue this aim I follow much of the development of [4] and show how frequently its proofs can be abstracted in a straight-forward manner. To do this satisfactorily it has been necessary to define and use a type of measurability for a Banach-valued function analogous to the type of measurability for a real-valued function used in [4]. In chapters 2 and 3 I demonstrate the properties of such a measurable function and those of its conditional expectations.

As might be expected certain additional restrictions are frequently required in the extended theory. Two attitudes have been adopted towards the nature of these restrictions; either they are placed on the Banach space itself or on the martingale. [1], [2], [10], and [12] often require the Banach space to be reflexive but not necessarily separable, though in theorem 4 [12] reflexivity can be replaced by the condition that the space be separable and be the conjugate space of a Banach space. On the other hand Drim1 and Hans [5] tends to place restrictions on the martingale itself: however they always deal with separable Banach spaces.

The first of these attitudes is better suited to my purpose of correlating the abstract and classical theories. Because of this [5] is not regarded as a basic reference. On the other hand the method of proving the existence of the conditional expectation of a Bochner integrable function in [5] is more natural than that used in [10] and is similar to my own. The device is a common one in integration theory.

In Chapter 1 I have defined those terms whose definitions are Independent of concepts which occur later on in the paper. I have also included indexes of symbols and terms.

In Chapter 2 I define and develop two types of measurability one of which is essentially different from those in common use (Dunford and Schwartz [6] and Hille and Phillips [9]) but which bears a far closer resemblance to the type of measurability for a real-valued function used in [4]. The relationships among these types of measurability and those in [6] and [9] are discussed. A relationship between the concepts of integrability in [6] and [9] is also given together with those properties of Bochner integrable functions which are required in this thesis. In Chapter 3 I demonstrate the existence and properties of conditional expectations of measurable functions which are also Bochner integrable and define martingales of such functions. The development in Chapters 2 and 3 does not exploit the relevant work in Hans [8] and [10]; however it seems to me that my treatment is more basic and unified than the alternative of incorporating results in [8] and [10] piecemeal.

In Chapter 4 I extend the concept of a separable stochastic process and several theorems pertaining to it. And in Chapter 6 results in [4] on optional skipping and sampling are extended.

In Chapter 5 I deal with martingale convergence theorems. This chapter is comprised of three sections. Firstly, several theorems in §4 chapter VII [4] are extended. Next I extend the theorems in the first section to martingales with linear uncountable parameter sets. Finally, several theorems in $\$ 11$ chapter VII [4] on "continuous parameter" martingales are extended.

I have also included an appendix containing various results on real-valued integrable functions, $\sigma$-fields, and separable sets whose statements or proofs do not appear in those texts with which I am familiar.

The definitions, theorems, notations, and remarks of which the text is composed are numbered serially in a single system that proceeds by sections. Chapters 1 and 4 each have only one section, while chapters 2, 3, 5, and 6 are divided into several sections. Accordingly, the second numbered item in chapter 4, in this case a notation, is denoted by notation 4.2; and the third item (a theorem) in section 2 of chapter 5 is denoted by theorem 5.2.3. The theorems, definitions, and remarks in the appendix are denoted, for instance, theorem 1 of the appendix.

## CHAPTER 1

Preliminary Definitions and Notation

This chapter consists of the definitions, or references for the definitions, of basic concepts together with some basic notation.
[4], [6], [7], and [9] are used throughout this thesis as references not only for results but also for the definitions of many concepts.

## Notation 1.1:

(i) $\mathrm{I}^{+}$denotes the set of positive integers and $\mathrm{I}^{\boldsymbol{+}}$ the set of negative integers.
(ii) For each $n \varepsilon I^{+}$, the set

$$
\left.\left\{\mathrm{m} \varepsilon \mathrm{I}^{+}\right\} \leqq n\right\} \text { is denoted by } \mathrm{J}_{\mathrm{n}}
$$

(iii) $R$ denotes the real line.

Notation 1.2: Let $W$ be a non-empty abstract space.
(i) If $A$ is a subset of $W$ and if $C$ is a nonempty class of subsets of $W$, then the class $\{C \cap A \mid C \in \mathcal{C}\}$ is denoted by $C \cap A$.
(ii) If $A$ is a non-empty subset of $W$ and if $D$ is a non-empty class of subsets of $A$, then the minimal $\sigma$-field of subsets of $A$ (III.4.2[G]) which contains $D$ is denoted by $\sigma_{A}(D)$. That is, $\sigma_{A}(\mathcal{D})$ is the $\sigma$-field relative to $A$ which is generated by $\mathcal{D}$. $\sigma_{W}(\mathcal{D})$ is written $\sigma(D)$.
(iii) If $\tau$ is a topology for $W$, then $\sigma(\tau)$ is denoted by $3(W)$. $B(W)$ is the Borel field of ( $W, \tau$ ) and its elements are the Borel subsets of ( $W, \tau$ ).
(iv) If $A$ is a subset of $W$, then $\bar{A}$ denotes the closure of $A$ in ( $W, \tau$ ).
(v) If $A$ and $B$ are subsets of $W$, then $A \triangle B$ denotes the symmetric difference $(A \backslash B) \cup(B \backslash A)$ and occasionally $A \cap B$ is written $A B$.
(vi) If $\left\{\mathrm{H}_{\mathrm{n}}, \mathrm{n} \in \mathrm{I}^{+}\right\}$is a sequence of disjoint subsets of $W$, then if it is desirable to emphasize the disjointness $\bigcup_{n=1}^{\infty} H_{n}$ is written $\sum_{n=1}^{\infty} M_{n}$ and $M_{i} \cup M_{j}$ is written $M_{i}+M_{j}$.
(vii) If $A$ is a subset of $W$, then define

$$
I_{A}: x \varepsilon W \rightarrow\left(\begin{array}{lll}
1 & \text { if } & x \varepsilon A \\
0 & \text { if } & x \notin A
\end{array}\right)
$$

(viii) $\phi$ always denotes the empty set.

Notation 1.3: $(\Omega, \Gamma, P)$ is used to denote the underlying measure space, which is a complete probability space (pp.31, 73, and 191 Halmos [7]), in this thesis. An arbitrary element of $\Omega$ will be denoted by $w$.

Definition 1.4:
(i) An element $\Lambda$ of $\Gamma$ is said to be $P$-negligible if $P \Lambda=0$.
(ii) A $\sigma$-field of subsets of $\Omega$ which is contained in $\Gamma$ is called a $\sigma$-subfield of $\Gamma$.
(iii) $A \quad \sigma$-subfield $\Phi$ of $\Gamma$ is said to be complete if every subset of every $P$-negligible element of $\Phi$ is an element of $\Phi$.
(iv) A o-subfield $\Phi$ of $I$ is said to be P-complete if every P-negligible element of $\Gamma$ is an element of $\Phi$.
(v) If $\Phi$ is a $\sigma$-subfield of $\Gamma$, then the $\sigma$-subfield of $\Gamma$ which is generated by $\Phi$ and the class of $p$-negligible elements of $\Gamma$ is called the $P$-completion of $\Phi$.

## Notation 1.5:

(i) If $\Phi$ is a $\sigma$-subfield of $\Gamma$, then its P-completion is denoted by $\Phi^{\prime}$. The notation of a dash superscript on a $\sigma$-subfield of r will always denote its p -completion.
(ii) If $\Phi$ is a $\sigma$-subfield of $\Gamma$, then $P_{\Phi}$ denotes the restriction of $P$ to $\Phi$.

Remark 1.6:
(i) Clearly $\Gamma^{\prime}=\Gamma$ 。
(ii) If $\Phi$ is a $\sigma$-subfield of $\Gamma$, then $\Lambda^{\prime}$ is an element of $\Phi^{2}$ if, and only if, there exists an element $\Lambda$ of $\Phi$ such that

$$
P\left(\Lambda^{\prime} \Delta \Lambda\right)=0
$$

(iii) Plainly the smallest $P$-complete $\sigma$-subfield of $\Gamma$ which contains the $\sigma$-subfield $\Phi$ of $\Gamma$ is $\Phi^{\prime}$ 。
(iv) Clearly $\Phi^{\prime}$ is complete: however it is in general strictly larger than the smallest complete $\sigma$-subfield of $\Gamma$ which contains $\Phi$.

Notation 1.7: Let $X$ be a real Banach space (linear space over the reals with a norm $\|\|$ and let $\xi$ be a function on $\Omega$ into $X$.
(1) The additive identity of X is denoted by $\theta$.
(ii) $\quad\|x\|$ denotes the value of $\|\|$ at $x \in X$.
(iii) $||\xi||$ denotes the real-valued composite function

$$
\|\| \circ \xi: \Omega \rightarrow R .
$$

(iv) The conjugate space of $X$ (II.3.7 [6]) is denoted by $X *$ and an arbitrary element of $X *$ by $\xi^{*}$.
(v) $\left\langle\mathrm{X}, \xi^{*}\right\rangle$ denotes the value of $\xi^{*} \varepsilon \mathrm{X}^{*}$ at $\mathrm{X} \varepsilon \mathrm{X}$.
(vi) A norm is defined on $X^{*}$ by

$$
\|\xi *\|=1 . \text { u.b. }\left\{\left|<x, \xi^{*}\right\rangle||||x|| \leq 1\}\right.
$$

(vii) $\langle\xi, \xi \%>$ denotes the real-valued composite function

$$
\xi^{*} \circ \xi: \Omega \rightarrow R .
$$

(viii) $S_{r}(x)$ denotes an open sphere in $X$ with centre $x \in X$, and radius $r>0$. Similarly, $S_{r}[x]$ denotes a closed sphere in $X$ with centre $x \in X$ and radius $r \geq 0$.

Remark 1.8:
(i) All the Banach spaces which appear in this thesis are assumed implicitly to be real.
(ii) The definition of a B-subspace of a Banach space is given In definition 10 of the appendix.

Definition 1.9: Let $X$ be a Banach space and let $\xi$ be a function on $\Omega$ into $X$.
(i) If $\xi(\Omega)$ is a finite subset of $X$, then $\xi$ is said to be finitely-valued.
(ii) If $\xi(\Omega)$ is a countable subset of $X_{\text {, }}$ then $\xi$ is said to be countably-valued.

Definition 1.10: Let $X$ be a Banach space, ( $\Omega, \Gamma, P$ ) a complete probability space.
(i) If a proposition is valid outside of a P-negligible element of a $\sigma$-subfield $\Phi$ of $\Gamma$, then the proposition is said to be valid $P_{\Phi}$ a.e.. $P$ a.e. is written for $P_{\Phi}$, a.e. and $P_{\Gamma}$ a.e..
(ii) A function $\xi: \Omega \rightarrow X$ is said to be $P_{\Phi}$ a.e. separablyvalued if there exists a P-negligible element $\Lambda$ of $\Phi$ such that $\xi(\Omega \backslash \Lambda)$ is a separable subset of $X$.
(iii) Convergence everywhere or $P_{\Phi}$ a.e. signifies pointwise convergence everywhere or $\mathrm{P}_{\Phi}$ a.e. respectively.
(iv) Let $\xi: \Omega \rightarrow X$ and let $\left\{\xi_{n}{ }^{s} n \varepsilon I^{+}\right\}$be a family of functions on $\Omega$ to $X$. The sequence $\left\{\xi_{n}\right\}$ is said to converge to $\xi$ almost uniformly relative to $\Phi$ if to every $\varepsilon>0$ there is an element $\Lambda_{\varepsilon}$ of $\Phi$ such that

$$
P \Lambda_{\xi}<\varepsilon
$$

and $\left\{\xi_{n}\right\}$ converges uniformly to $\xi$ on $\Omega \backslash \Lambda_{\varepsilon^{\circ}}$

Definition 1.11: Let $X$ be a Banach space, $(\Omega, \Gamma, P)$ a complete probability space. If $\xi: \Omega \rightarrow \mathrm{X}$, then the P -equivalence class of $\xi$, denoted $[\xi]_{\Phi}$, is defined by

$$
[\xi]_{\Phi}=\left\{_{\rho}:\left.\Omega \rightarrow X\right|_{\rho}=\xi \quad P_{\Phi} \quad \text { a.e. }\right\}
$$

Write $[\xi]$ for $[\xi]_{\Phi^{\prime}}$ and for $[\xi]_{\Gamma}$.

Definition 1.12: The concepts of measurable and integrable real-valued functions used in this thesis are as in Halmos [7]. However, since a realvalued measurable function is always defined on $\Omega$, if it is measurable with respect to the measurable space $(\Omega, \Phi)$ where $\Phi$ is a $\sigma$-subfield of $\Gamma_{\text {, }}$ then the function is said to be measurable with respect to $\Phi$.

Notation 1.13: If $f$ is an integrable real-valued function, then $E(f)$ will frequently be used to denote $\int_{\Omega}$ fdr. Occasionally this notation will
be used in the generalized sense; that is, if $f$ is a real-valued measurable function, then $E(|f|)<\infty$ if, and only if, $f$ is integrable.

## CHAPTER 2

Measurability and Integrability

The object of this chapter is threefold; firstly, to define a type of measurability for a Banach-valued function analogous to the type of measurability for a real-valued function used in [4]; secondly, to demonstrate some of the relationships among several types of measurability: thirdly, to collate those results which are required in later chapters.

Two types of measurability are introduced in section 2.1 and some of the relationships between them are discussed. One of these types of measurability is effectively strong measurability with respect to a probability space which is not necessarily complete. There are also included in section 2.1 several miscellaneous definitions and results on measurability which are required in this thesis.

Theorem 2.2.1 demonstrates the equivalence under certain conditions of several types of measurability. It is then proved in theorem 2.2.3 that the concepts of integrability in [6] and [9] coincide if ( $\Omega, \Gamma, P$ ) is the underlying measure space. There then follow several miscellaneous definitions and theorems on integrable functions and on the convergence in measure of a sequence of measurable functions. Finally there are included several results on the uniform integrability of a family of integrable functions.

### 2.1 Measurability

Let $X$ be a Banach space, $(\Omega, \Gamma, P)$ a complete probability space, and $\Phi$ a $\sigma$-subfield of $\Gamma$.

Definition 2.1.1: A function $\xi: \Omega \rightarrow X$ is said to be $X$-measurable with respect to $\Phi$ if
(1) $\xi^{-1}(A) \varepsilon \Phi$ for every $A \varepsilon B(X)$ and (ii) $\quad \xi$ is $P_{\Phi}$ ace. separably-valued.

Notation 2.1.2: If a function $\xi$ is $X$-measurable with respect to $\Phi$, write $\xi$ is an X r.v.Ф. X riv. $\Gamma$ will be written Xr.v. . The plurals will be given no inflexion.

Definition 2.1.3: A function $\xi: \Omega \rightarrow \mathrm{X}$ is said to be SX -measurable with respect to $\Phi$ if there exists a sequence $\left\{\xi_{n}, n \in I^{+}\right\}$of finitelyvalued Xr.v. $\Phi$ which converges $P_{\Phi}$ are. to $\xi$.

Notation 2.1.4: If a function $\xi$ is SX-measurable with respect to $\Phi$, write $\xi$ is an SXr.v.ф. SXr.v. $\quad$ will be written SXr.v. . The plurals will be given no inflexion.

The relationships between the concepts of X -measurability and SX-measurability will now be summarized. One of these is proved directly below. The rest are separate theorems, stated and proved later in this section.

Summary 2.1.5:
(i) If $\xi$ is an Xr.v. $\Phi$, then $\xi$ is an SXr.v. $\Phi$.
(ii) $\quad \xi$ is an SXr.v. $\Phi^{\prime}$ if, and only if, $\xi$ is an Xr.v. $\Phi^{\prime}$. (iii) If $\xi$ is an Xr.v. $\Phi^{\prime}$ and if $\rho \varepsilon[\xi]$, then $\rho$ is an Xr.v.Ф'.
(iv) If $\xi$ is an Xr.v. $\Phi^{\prime}$, then there exists $\rho \varepsilon[\xi]$ such that $\rho$ is an Xr.v.Ф.
(v) If $\xi$ is an SXr.v. $\Phi$, then there exists $\rho \varepsilon[\xi]_{\Phi}$ such that $\rho$ is an Xr.v.థ.

Proof: Parts (i) and (ii) follow immediately from theorems 2.1.16 and 2.1.17. Part (iii) is implied by part (ii); nevertheless it will be proved independently.

If $\xi$ is an Xr.v. $\Phi^{\prime}$, then there exists a P-negligible element $\Lambda_{1}$ of $\Gamma$ such that $\xi\left(\Omega \backslash \Lambda_{1}\right)$ is separable. Also if $\rho \varepsilon[\xi]$, then there exists a P -negligible element $\Lambda_{2}$ of $\Gamma$ such that

$$
\rho=\xi \quad \text { on } \quad \Omega \backslash \Lambda_{2}
$$

Therefore, since it follows from theorem 7 of the appendix that $\rho\left(\Omega \backslash\left(\Lambda_{1} \cup \Lambda_{2}\right)\right)$ is separable, $\rho$ is $P$ a.e. separably-valued. Moreover if $A \varepsilon B(X)$, then

$$
\begin{aligned}
\rho^{-1}(A) & =\left[\rho^{-1}(A) \cap\left(\Omega \backslash \Lambda_{2}\right)\right] \cup\left[\rho^{-1}(A) \cap \Lambda_{2}\right] \\
& =\left[\xi^{-1}(A) \cap\left(\Omega \backslash \Lambda_{2}\right)\right] \cup\left[\rho^{-1}(A) \cap \Lambda_{2}\right] \\
& \varepsilon \Phi^{\prime} .
\end{aligned}
$$

Therefore $\rho$ is an Xr.v. $\Phi^{\text {' }}$.
Finally theorem 2.1. 18 and remark 2.1.12 are restatements of parts (iv) and (v) respectively.

Theorem 2.1.6: A function $\xi: \Omega \rightarrow X$ is an $X r . v . \Phi$ if, and only if,
(i) $\quad \xi^{-1}(F) \varepsilon \Phi$ for every closed set $F$ in $X$
and (ii) $\xi$ is $P_{\Phi}$ a.e. separably-valued.

Proof: Since $B(X)$ is generated by the norm topology for $X$ and since an inverse mapping preserves all set operations, this theorem is equivalent to the definition of an Xr.v.Ф.

Theorem 2.1.7: If $\xi$ is a finitely-valued $X r . v . \Phi^{\prime}$, then there exists a finitely-valued $\mathrm{Xr} . \mathrm{v} . \Phi \rho \mathrm{p}$ such that

$$
\rho=\xi \mathrm{P} \text { ave. . }
$$

Proof: By definition there exist $n \varepsilon I^{+}, \quad\left\{x_{i}, i \in J_{n}\right\} \subseteq X, \quad$ and $\left\{\Lambda_{i}^{\prime}, i \varepsilon J_{n}\right\} \subseteq \Phi^{\prime}$, where $\Lambda_{i}^{\prime} \cap \Lambda_{j}^{\prime}=\phi$ if $i \neq j$, such that

$$
\xi=\sum_{i=1}^{n} x_{i} I_{\Lambda_{i}^{\prime}}
$$

For each $i \in J_{n}$, remark $1.6(i i)$ implies that there exists $\Lambda_{i} \varepsilon \Phi$ such that $P\left(\Lambda_{i} \Delta \Lambda_{i}^{\gamma}\right)=0$. Therefore the function

$$
\rho=\sum_{i=1}^{n} x_{i} I_{i}
$$

$$
\text { where } M_{1}=\Lambda_{1} \text { and } M_{i}=\Lambda_{i} \backslash \bigcup_{j<i} \Lambda_{j} \text { for } i=2, \ldots, n
$$

is a finitely-valued Xr.v.Ф which equals $\xi$ P a.e..

Theorem 2.1.3: Let $\xi_{1}$ and $\xi_{2}$ be Xr.v.Ф. If $\Lambda$ is an element of $\Phi$ and $Y$ is a countable subset of $X$ such that

$$
\xi_{i}(\Omega \backslash \Lambda) \subseteq \overrightarrow{\mathbf{Y}} \quad \text { for } \quad i=1,2
$$

then for every $a>0$

$$
(\Omega \backslash \Lambda) \cap\left\{\omega \left|\left|\mid \xi_{1}(\omega)-\xi_{2}(\omega) \|<a\right\} \varepsilon \Phi .\right.\right.
$$

Proof: Put $Y$ into a sequence, $\left\{y_{n}, n \in I^{+}\right\}$say. Then for every a > 0

$$
\begin{gathered}
(\Omega \backslash \Lambda) \cap\left\{\omega\left|\left|\left|\xi_{1}(\omega)-\xi_{2}(\omega)\right|\right|<a\right\}=\right. \\
(\Omega \backslash \Lambda) \cap\left(\bigcup_{n \geq 2 i 1}\left\{\omega| | \mid \xi_{1}(\omega)-y_{i} \|<a-\frac{a}{n}\right\}\right. \\
\left.\cap\left\{\omega\left\|\mid \xi_{2}(\omega)-y_{i}\right\|<\frac{a}{n}\right\}\right)
\end{gathered}
$$

$$
\varepsilon \Phi
$$

Remark 2.1.9:
(i) Clearly there can always be found a P-negligible $\Lambda$ and a $Y$ which satisfy the conditions of theorem 2.1.8.
(ii) If $\xi_{1}$ and $\xi_{2}$ are Xr.v.Ф such that $\xi_{1}(\Omega)$ and $\xi_{2}(\Omega)$ are separable, then theorem 2.1 .8 implies that $\left\|\xi_{1}-\xi_{2}\right\|$ is an Rr.v.Ф.
(iii) If $\xi_{1}$ and $\xi_{2}$ are Xr.v. $\Phi^{\prime}$, then clearly it follows from theorem 2.1 .8 and remark 2.1 .9 (i) that $\left\|\xi_{1}-\xi_{2}\right\|$ is an $R r . v . \Phi^{\prime}$.

Theorem 2.1.10: If $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is a sequence of $X r . v . \phi$ such that $\xi_{n}(\Omega)$ is separable for every $n \varepsilon I^{+}$, then $M=\left\{\omega \mid \lim _{n \rightarrow \infty}(\omega)\right.$ exists $\} \varepsilon \Phi$.

Proof: It follows from remark 2.1.9 (ii) and theorem A 520 [7] that for each $m \in I^{+}$the real-valued function $g_{m}$ defined by

$$
g_{m}: \omega \varepsilon \Omega \rightarrow\binom{\lim _{n \rightarrow \infty}| | \xi_{n+m}(\omega)-\xi_{n}(\omega) \| \text { if the limit exists }}{a \neq 0 \text { otherwise }}
$$

is an Rr.v.Ф. Therefore, since

$$
\begin{aligned}
& M=\left\{\omega\left|\lim _{n \rightarrow \infty}\right|\left|\xi_{n+m}(\omega)-\xi_{n}(\omega)\right| \mid=0 \text { for every } m \varepsilon I^{+}\right\} \\
& M=\bigcap_{m=1}^{\infty} g_{m}^{-1}(\{0\})
\end{aligned}
$$

Theorem 2.1.11: If $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is a sequence of $X r . v . \Phi$ and if $N$ is any element of $\Phi$ such that

$$
N \subseteq\left\{\omega \mid \lim _{n \rightarrow \infty} \xi_{n}(\omega) \text { exists }\right\}
$$

then the function

$$
\rho_{N}: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{lll}
\lim _{n \rightarrow \infty}(\omega) & \text { if } \omega \in \mathbb{N} \\
b & \text { if } & \omega \notin \mathbb{N}
\end{array}\right)
$$

Where $b$ is any element of $X$, is an $X r . V_{0} \Phi_{\text {。 }}$

Proof: Let $F$ be any closed set in $X$. For every natural number r let

$$
F_{r}=\left\{x \mid x \in X \quad \text { and } \underset{y \in F}{\left.g .1 . b| | x-y| | \leqq \frac{1}{r}\right\}}\right.
$$

It follows that

$$
\rho_{N}^{-1}(F) \bigcap N=\bigcap_{r=1}^{\infty} \bigcup_{m} \bigcap_{n>m} \xi_{n}^{-1}\left(F_{r}\right) \cap N
$$

$\varepsilon \Phi$
and that $\rho_{N}^{-1}(F) \backslash N=\left(\begin{array}{cc}\Omega>N & \text { if } b \in F \\ \phi & \text { if } b \notin F\end{array}\right)$.
Hence if $F$ is any closed set in $X$, then

$$
\rho_{N}^{-1}(F) \varepsilon \Phi .
$$

It follows from theorem 8 of the appendix that $\rho_{M}$ is $P_{\Phi}$ a.e. separably-valued. Therefore, since $\rho_{\text {N }}$ satisfies the conditions of theorem 2.1.6, it is an $\mathrm{Xr} . \mathrm{v} . \Phi$.

Remark 2.1.12: Theorem 2.1.11 implies that if a sequence of Xr.v.Ф converges $P_{\Phi}$ a.e., then there exists an $\mathrm{Xr} . \mathrm{v} . \Phi$ to which it converges $P_{\Phi}$ a.e. -

The statements and proofs of theorems 2.1.13-2.1.16 are based on those of theorem II.1(2), proposition II.13, proposition II.14, and theorem II. 2 in Dinculeanu [3] respectively.

Theorem 2.1.13: If $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is a sequence of Xr.v. $\Phi$ which converges $P_{\Phi}$ a.e. to a function $\xi_{\text {, then }}\left\{\xi_{n}, n \in I^{+}\right\}$converges to $\xi$ almost uniformly relative to $\Phi$.

Proof: Given $\varepsilon>0$. We shall construct an element ${ }_{\xi}$ of $\Phi$ such that

$$
P\left(\Omega \backslash M_{\varepsilon}\right)<\varepsilon
$$

and

$$
\left\{\xi_{n}\right\} \text { converges uniformly to } \xi \text { on } M_{\varepsilon}
$$

By theorem 8 of the appendix $\Lambda$ of $\Phi$ and a countable subset $Y$ of $X$ with the following properties:
(a) $\quad \xi_{n}(\Omega \backslash \Lambda) \subseteq \bar{Y}$ for every $n \in I^{+}$,
and (b) $\left\{\xi_{n}\right\}$ converges to $\xi$ everywhere on $\Omega \backslash \Lambda$ 。
It is an immediate consequence of theorem 2.1.8 that

$$
M_{n, r}=(\Omega \backslash \Lambda) \cap\left(\bigcap_{p, q>n}\left\{\omega \left\lvert\,\left\|\xi_{p}(\omega)-\xi_{q}(\omega)\right\|<\frac{1}{r}\right.\right\}\right)
$$

$\varepsilon \Phi$ for every pair of natural numbers ( $n, r$ ).
For every $r$, (b) implies that $\left\{\mathrm{M}_{\mathrm{n}, \mathrm{r}}, \mathrm{n} \varepsilon \mathrm{I}^{+}\right\}$is monotone increasing as $n$ increases and that

$$
\Omega \backslash \Lambda=\bigcup_{n=1}^{\infty} n_{n, r} .
$$

Given $\varepsilon>0$. Then for each $r \varepsilon I^{+}$there exists $n_{r} \varepsilon I^{+}$such that

$$
P\left[(\Omega \backslash \Lambda) \backslash i_{j, r}\right]<\frac{\varepsilon}{2^{r}} \quad \text { for every } \quad j \geqq n_{r} .
$$

Define $M_{\varepsilon}=\bigcap_{r=1}^{\infty} M_{n_{r}} r$
$\varepsilon \Phi$.
This set clearly has the asserted properties.

Theorem 2.1.14: If $\xi$ is a countably-valued Xr.v.థ, then there exists a sequence $\left\{\xi_{n}, n \in I^{+}\right\}$of finitely-valued $\operatorname{Xr} . v . \Phi$ with the following properties:
(i) $\left\{\xi_{\mathfrak{n}}\right\}$ converges to $\xi$ everywhere,
and (ii) $\quad\left\|\xi_{n}\right\| \leqq\|\xi\|$ everywhere for every $n \in I^{+}$.

Proof: Put the elements of $\xi(\Omega)$ into a sequence, $\left\{X_{n}, n \in I^{+}\right\}$say. The sequence $\left\{\xi_{n}, n \in I^{+}\right\}$defined by

$$
\xi_{n}: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{llll}
\xi(\omega) & \text { if } \omega \varepsilon \bigcup_{i=1}^{n} \xi^{-1}\left(\left\{x_{i}\right\}\right) \\
\theta & \text { if } \omega \notin \bigcup_{i=1}^{n} \xi^{-1}\left(\left\{x_{i}\right\}\right)
\end{array}\right)
$$

for every $n \in I^{+}$, is a sequence which satisfies the requirements of the theorem.

Theorem 2.1.15: If $\xi$ is an Xr.v.Ф, then there exist a P-negligible element $\Lambda$ of $\Phi$ and a seçuence $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$of countably valued Xr.v. $\Phi$ with the following properties:
(i) $\left\{\xi_{\mathrm{n}}\right\}$ converges uniformly to $\xi$ on $\Omega \backslash \Lambda$,
and (ii) $\quad\left|\left|\xi_{n} \| \leqq||\xi||\right.\right.$ everywhere for every $n \in I^{+}$.

Proof: There exist a P-negligible element $\Lambda$ of $\Phi$ and a countable subset $Y$ of $X$ such that

$$
\xi(\Omega \backslash \Lambda) \subseteq \overline{\mathrm{Y}}
$$

Put the elements of $Y$ into a sequence, $\left\{y_{p}, p \varepsilon I^{+}\right\}$say. Clearly $M_{n, P}=(\Omega \backslash \Lambda) \cap \xi^{-1}\left(S_{\frac{1}{3 n}}\left[y_{p}\right]\right)$
$\varepsilon \Phi$ for every pair of natural numbers ( $n, p$ ).

For each $n \in I^{+}$, define a sequence $\left\{N_{n, p}, p \varepsilon I^{+}\right\}$of mutually disjoint elements of $\Phi$ by

$$
N_{n, 1}=M_{n, 1} \text { and } N_{n, p}=M_{n, p} \bigcup_{i=1}^{p-1} M_{n, i} \text { for } p=2,3, \ldots
$$

Clearly $\bigcup_{p=1}^{\infty} N_{n, p}=\Omega \backslash \Lambda$ for each $n \varepsilon I^{+}$.

$$
\text { Put } a_{n, p}=\underset{\omega \in \mathbb{N}_{n, p}}{g .1 . b}| | \xi(\omega) \| \text { for } n, p=1,2, \ldots
$$

For each $n \varepsilon I^{+}$, define the funcjtion

$$
\xi_{n}: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{ll}
y_{p} \prod_{n_{,} p} y_{p} \text { if }^{a_{p}} \omega \varepsilon \Lambda_{n_{,} p} \\
\theta & \text { if } \omega \in \Lambda
\end{array}\right)
$$

taking as a convention that $y_{p} \prod_{n_{p}, p}^{y_{p}} \|^{a_{n}}$ is be $\theta$ if $y=\theta$.
Clearly $\left\{\xi_{n}\right\}$ satisfies requirement (ii) of the theorem.
It follows from the construction of $\left\{\xi_{n}\right\}$ that for each $n, p \varepsilon I^{+}$, $\omega \in N_{n, p}$ implies that

$$
\left\|\xi_{n}(\omega)-\xi(\omega)\right\| \leqq \frac{2}{3 n} .
$$

Hence for each $n \varepsilon I^{+}$, the real function $\left\|\xi_{n}-\xi\right\|$ satisfies

$$
\| \xi_{n}-\xi| |<\frac{1}{n} \text { on } \bigcup_{p=1}^{\infty} N_{n, p}=\Omega \backslash \Lambda
$$

This establishes claim (i) of the theorem.

The next result uses most of the foregoing.

Theorem 2.1.16: If $\xi$ is an $\mathrm{Xr} . \mathrm{v} . \Phi$, then there exists a sequence $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$of finitely-valued Xr.v. $\Phi$ with the following properties:
(1) $\left\{\xi_{n}\right\}$ converges $P_{\Phi}$ a.e. to $\xi$,
and (ii) $\quad\left|\left|\xi_{n} \| \leqq||\xi||\right.\right.$ everywhere for every $n \varepsilon I^{+}$.

Proof: It follows from theorem 2.1 .15 and its proof that there exist a P-negligible element $\Lambda$ of $\Phi$ and a sequence $\left\{\rho_{n}, n \varepsilon I^{+}\right\}$of countably-valued $\mathrm{Xr} . \mathrm{v}. \mathrm{\Phi}$ such that

$$
\left\|\xi-\rho_{n}\right\|<\frac{1}{n} \text { on } \Omega \backslash \Lambda \text { for every } n \varepsilon I^{+}
$$

and $\left\|\rho_{n}\right\| \leqq\|\xi\|$ everywhere for every $n \varepsilon I^{+}$.
Theorem 2.1.14 implies that there exists, for each $n \varepsilon I^{+}$, a sequence $\left\{\rho_{n, p}, p \varepsilon I^{+}\right\}$of finitely-valued Xr.v. $\Phi$ such that

$$
\left\{\rho_{n, p}, p \varepsilon I^{+}\right\} \text {converges to } \rho_{n} \text { everywhere }
$$

and

$$
\left\|\rho_{n, p}\right\| \leqq\left\|\rho_{n}\right\| \text { everywhere for every } p \varepsilon I^{+}
$$

Accordingly theorem 2.1.13 implies that for every pair of natural numbers ( $n, m$ ) there exists an element $M_{n, m}$ of $\Phi$ such that $P\left(\Omega \backslash M_{n, m}\right)<\frac{1}{2^{n}} \cdot \frac{1}{m}$
and $\quad\left\{\rho_{n, p}, p \varepsilon I^{+}\right\}$converges uniformly to $\rho_{\Omega}$ on $M_{n, m}$ for every $m \in I^{+}$.

Let $M_{m}=\bigcap_{n=1}^{\infty} M_{n, m} \quad \varepsilon \Phi \quad$ for each $m \varepsilon I^{+}$.
Then for each $m \in I^{+}, P\left(\Omega \backslash l_{m}\right)<\frac{1}{m}$ and $\left\{\rho_{n, p}, p \in I^{+}\right\}$converges uniformly to $\rho_{\mathrm{n}}$ on $\mathrm{in}_{\mathrm{m}}$ for every $\mathrm{n} \varepsilon \mathrm{I}^{+}$.

$$
\text { Let } \mathbb{N}=\bigcup_{m=1}^{\infty} H_{m} \backslash \Lambda
$$

and

$$
N_{k}=\bigcup_{m=1}^{k} I_{m} \backslash \text { for each } k \in I^{+}
$$

Then $\Omega \backslash \mathbb{N}$ is a $P$-negligible element of $\Phi$. $\left\{\mathrm{TH}_{\mathrm{k}}, \mathrm{k} \varepsilon \mathrm{I}^{+}\right\}$is a monotone increasing sequence of elements of $\Phi_{\rho}$ and $\left\{\rho_{n, p}, p \in I^{+}\right\}$ converges uniformly to $\rho_{n}$ on $M_{k}$ for all $n, k \in I^{+}$. Accordingly, for each $n \varepsilon I^{+}$, there exists $p_{n}$ such that

$$
\left\|\rho_{n}(\omega)-\rho_{n_{,} p_{n}}(\omega)\right\|<\frac{1}{n} \text { for every } \omega \in N_{n} .
$$

And so

$$
\begin{aligned}
& \left\|\xi-\rho_{n, p_{n}}\right\| \\
& \quad \leqq\left\|\xi-\rho_{n}\right\|+\left\|\rho_{n}-\rho_{n, p_{n}}\right\| \\
& \quad<\frac{2}{n} \text { on } N_{n} \text { for every } n \varepsilon I^{+} .
\end{aligned}
$$

Denoting the diagonal sequence

$$
\left\{\rho_{n, P_{n}}, n \varepsilon I^{+}\right\} \quad \text { by }\left\{\xi_{n}, n \in I^{+}\right\}
$$

it follows that $\left\{\xi_{\mathrm{n}}\right\}$ is a sequence of finitely-valued $\mathrm{Xr} . \mathrm{v} . \Phi$ which
converges to $\xi$ on $\mathbb{N}$. That is, $\left\{\xi_{n}\right\}$ converges $P_{\Phi}$ abe. to $\xi$. Moreover $\left\|\rho_{n}\right\| \leqq\|\xi\|$ everywhere for every $n \varepsilon I^{+}$, and $\left\|\rho_{n, p}\right\| \leqq\left\|\rho_{n}\right\|$ everywhere for all $n_{, p} \| I^{+}$. Hence

$$
\left\|\xi_{n}\right\| \leqq\|\xi\| \text { everywhere for every } n \varepsilon I^{+}
$$

Therefore $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$satisfies the requirements of the theorem.

Theorem 2.1.17: If $\xi$ is an SXr.v. $\Phi^{\prime}$, then $\xi$ is an Xr.v. $\Phi^{\prime}$.

Proof: Since $\xi$ is an SXr.v. $\Phi^{\prime}$, there exist a sequence $\left\{\xi_{n}, n \in I^{\dagger}\right\}$ of finitely-vaiued Xr.v. $\phi^{\prime}$ and a P-negligible element $N$ of $\Phi^{\prime}$ such that $\left\{\xi_{n}\right\}$ converges everywhere to $\xi$ on $\Omega \backslash N$.

Theorem 2.1.11 implies that the function $\rho$ defined by

$$
\rho: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{lll}
\lim _{n \rightarrow \infty} \xi_{\mathrm{n}}(\omega) & \text { if } \omega \varepsilon \Omega \backslash N \\
\theta & \text { if } \omega \in \mathrm{N}
\end{array}\right)
$$

is an $X r \cdot v . \Phi^{\prime}$. Thus if $A \varepsilon \zeta(X)$,
then

$$
\begin{aligned}
\xi^{-1}(A) & =\left(\xi^{-1}(A) \backslash N\right) \cup\left(\xi^{-1}(A) \cap N\right) \\
& \left.=\rho^{-1}(A) \backslash N\right) \cup\left(\xi^{-1}(A) \cap N\right) \\
& \varepsilon \Phi^{\prime}
\end{aligned}
$$

Soreover theorem 8 of the appendix implies that $\xi$ is P a.e. semarablyvalued. Therefore $\xi$ is an $\mathrm{Xr} . \mathrm{v} . \Phi^{\prime}$.

Theorem 2.1.18: If. $\xi$ is an Xr.v. $\Phi^{\prime}$, then there exists $\rho \varepsilon$ [ $\xi$ ] such that $\rho$ is an Xr.v.Ф.

Proof: Theorem 2.1.16 implies that there exists a sequence $\left\{\xi_{n}, n \in I^{+}\right\}$of finitely-valued $X r . v . \Phi^{\prime}$ which converges $P$ a.e. to §. Accordingly it follows from theorem 2.1.7 that there exists a sequence $\left\{\rho_{n}, n \in I^{+}\right\}$of finitely-valued $X r . v . \Phi$ such that

$$
\rho_{n}=\xi_{n} P \text { a.e. for every } n \varepsilon I^{+}
$$

Clearly $\left\{\rho_{n}, n \in I^{+}\right\}$converges $P$ a.e. to $\xi$. And so theorem 2.1.10 implies that $i=\left\{\omega \mid \lim _{n \rightarrow \infty}(\omega)\right.$ exists $\}$ is an element of $\Phi$ which is of probability one. Therefore it follows from theorem 2.1.11 that the function $\rho$ defined by

$$
\rho: \omega \in \Omega \rightarrow\left(\begin{array}{lll}
\lim _{n \rightarrow \infty}(\omega) & \text { if } & \omega \varepsilon N \\
\theta & & \omega \notin \mathbb{N}
\end{array}\right]
$$

satisfies the requirements of the theorem.

Theorem 2.1.19: If $\xi$ is an Xr.v. , then, for any $\alpha \geqq 1,\|\xi\|^{\alpha}$ is an Rr.v.థ.

Proof: For $\alpha \geqq 1$ and $c \varepsilon R$, the set

$$
\begin{aligned}
\{\omega\||\mid \xi(\omega) \| \leqq c\} & =\left(\begin{array}{ll}
\phi & \text { if } \quad c<0 \\
\xi^{-1}(\{\theta\}) & \text { if } c=0 \\
\xi^{-1}\left(S_{1 / \alpha}^{[\theta])}\right. & \text { if } c>0
\end{array}\right] \\
\varepsilon \Phi . &
\end{aligned}
$$

Theorem 2.1.20: If $\xi$ is an $\mathrm{Xr} . \mathrm{v} . \Phi$, then there exist a separable B-subspace $L$ of $X$ and an $X r . v . \Phi \rho$ such that

$$
\rho=\xi \quad \mathrm{P}_{\Phi} \quad \text { a.e. }
$$

and

$$
\rho(\Omega) \subseteq L
$$

Proof: There exists a P-negligible element $\Lambda$ of $\Phi$ and a countable subset $Y$ of $X$ such that

$$
\xi(\Omega \backslash \Lambda) \subseteq \overline{\mathrm{Y}}
$$

Let $L$ be the closed linear manifold determined by $Y$. Then definition 10 and remark 11 (ii) of the appendix imply that $L$ is a separable $B$-subspace of $X$. Since $L \in B(X), \quad M=\Omega \backslash \xi^{-1}(L)$ is clearly a $P-$ negligible element of $\Phi$. Ioreover $\xi(\Omega \backslash M)=\xi \bar{\xi}^{1}(L)$

$$
\subseteq \mathrm{L}
$$

$$
\rho: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{llll}
\xi(\omega) & \text { if } & \omega \varepsilon \Omega \backslash M \\
\theta & \text { if } & \omega \varepsilon & M
\end{array}\right]
$$

are as asserted.

Notation 2.1.21: If $\xi$ is an Xr.v., then $\sigma\left(\xi^{-1}(A) \mid A \in B(X)\right)$ will be denoted by $B(\xi)$. Clearly if $\xi$ is an $\mathrm{Xr} . \mathrm{v} . \Phi$, then $B(\xi) \subseteq \Phi$.

Theorem 2.1.22: If $\xi$ is an Xr.v., then $\xi$ is an Xr.v.B( $\xi$ ).

Proof: Obviously if $A \varepsilon B(X)$, then

$$
\xi^{-1}(A) \varepsilon B(\xi)
$$

Accordingly it remains to be shown that $\xi$ is $P_{B(\xi)}$ a.e. separablyvalued. Since $\xi$ is an $\mathrm{Xr} . \mathrm{V}_{0}$, there exists a P-negligible element $\Lambda$ of $\Gamma$ such that $\xi(\Omega \backslash \Lambda)$ is separable. Let $Z=\overline{\xi(\Omega \backslash \Lambda)}$. Then $M=\Omega \backslash \xi^{-1}(Z)$ is clearly a $P$-negligible element of $B(\xi)$. Moreover $\quad \xi(\Omega \backslash M)=\xi \xi^{-1}(\Omega)$

$$
\subseteq \mathrm{Z}
$$

Therefore it follows from theorem 7 of the appendix that $\xi$ is $P_{B(\xi)}$ a.e. separably-valued.

Notation 2.1.23: If $\left\{\xi_{t}, t \varepsilon T\right\}$ is a family of Xr.v., then $\sigma\left(\xi_{t}^{-1}(A) \mid t \varepsilon T\right.$ and $\left.A \varepsilon B(X)\right)$
will be denoted by

$$
B\left(\xi_{t}, t \varepsilon T\right) .
$$

Clearly $B\left(\xi_{t}, t \in T\right)$ is a $\sigma$-subfield of $r$ and clearly $B\left(\xi_{t}\right) \subseteq B\left(\xi_{t}, t \in T\right)$ for every $t \varepsilon T$. Accordingly it follows from tieorem 2.1.22 that $\left\{\xi_{t}, t \varepsilon T\right\}$ is a family of Xr.v. $B\left(\xi_{t}, t \varepsilon T\right)$.

### 2.2 Integrability

In this section $I$ establish conditions under which results in [6] and [ 9$]$ can be transliterated into my nomenclature. This has permitted me to omit many proofs and simply to refer to the corresponding results in [6] and [9].

Let $X$ be a Banach space, $(\Omega, \Gamma, P)$ a complete probability space, and $\Phi$ a $\sigma$-subfield of r .

Theorem 2.2.1: The following three statements are equivalent:
(i) $\quad \xi$ is an Xr.v. $\Phi^{\prime}$.
(ii) $\quad \xi$ is $\mathrm{P}_{\Phi^{\prime}}$-measurable with respect to $\left(\Omega, \Phi^{\prime},{ }^{\prime}{ }_{\Phi^{\prime}}\right.$ )

## (III. 2.10 [6])

(iii) $\xi$ is strongly measurable with respect to $\left(\Omega, \Phi^{\prime}, P_{\Phi}{ }^{\prime}\right)$ (Definition 3.5.4 [9]).

Proof: III.6.9 [9] implies that (i) and (ii) are equivalent. And it follows from theorem 2.1 .15 that (i) implies (iii) and from remark 2.1 .12 and summary 2.1 .5 (iii) that (iii) implies (i).

It follows from theorem 2.2.1 that it is meaningful to say that an Xr.v. is Bochner integrable with respect to ( $\Omega, \Gamma, P$ ) (Definition 3.7.3 [9]). And similarly it is meaningful to say that an Xr.v. is P-integrable with respect to ( $\Omega, \Gamma, P$ ) (III. 2.17 [6]). It will be assumed in future that ( $\Omega, \Gamma, P$ ) is the underlying measure space for both types of integration.

Remark 2.2.2: Clearly a real-valued function is P-measurable with respect to ( $\Omega, \Gamma, P$ ) if, and only if, it is an Rr.v. and is P-integrable if, and only if, it is Lebesgue integrable (cf. §25 Halmos [7]).

Theorem 2.2.1 implies that the class of $\mathrm{Xr} . \mathrm{v}$., the class of functions which are $P$-measurable with respect to $(\Omega, \Gamma, P)$, and the class of functions which are strongly measurable with respect to ( $\Omega_{s} \Gamma, P$ ) are identical. Accordingly it remains to be shown that the concepts of Bochner integration and $P$-integration coincide for ( $\Omega, \Gamma, P$ ) and this will be done in the next theorem.

Theorem 2.2.3: Let $\xi$ be an Xr.v. .
(i) $\quad \xi$ is Bochner integrable if, and only if, $\xi$ is $P$ integrable.
(ii) If $\xi$ is Bochner integrable then (D.S) $\int_{\Lambda} \xi d P=$ ( $\left.B\right) \int_{\Lambda} \xi d P$
for every $\Lambda \in \Gamma$ where the integrals are those of [6] and [9] respectively.

Proof: The validity of (i) is an immediate consequence of remark 2.2.2, theorem 3.7.4 [9], and III.2.22 (a) [6]. (That is, that $\xi$ is Bochner (and p-) integrable if, and only if, $\|\xi\|$ is Lebesgue integrable.)
III. 2.13 [6] and definition 3.7.2 [9] imply that every finitelyvalued Xr.v. $\rho$ is Bochner (and $P$.) integrable and that

$$
\text { (D.S) } \quad \int_{\Lambda} \rho d P=(B) \iint_{\Lambda} \rho d P \text { for every } \Lambda \varepsilon \Gamma \text {. }
$$

Moreover theorem 2.1.16 implies that there exists a sequence $\left\{\xi_{n^{s}} n \varepsilon I^{+}\right\}$of finitely-valued $X r . v$. such that

$$
\left\{\xi_{n}\right\} \text { converges } P \text { a.e. to } \xi
$$

and

$$
\left\|\xi_{n}\right\| \leqq\|\xi\| \text { everywhere for every } n \varepsilon I^{+} .
$$

Accordingly it follows from III.2.22 (a) [6] and III.6.16 [6] that

$$
\lim _{\mathrm{m}^{+\infty}} \text { (D.S) } \int_{\Omega}\left\|\xi_{\mathrm{n}}\right\| d \mathrm{\xi} \| \mathrm{F}=0
$$

which implies that

$$
\lim _{n \rightarrow \infty}(D . S) \int_{\Lambda} \xi_{n} d P=\text { (D.S) } \int_{n} \xi d P \text { for every } \Lambda \in \Gamma \text {. }
$$

It also follows from theorem 3.7.9 [9] that

$$
\lim _{n \rightarrow \infty}(B) \int_{n} \xi_{n} d P=(B) \int_{n} \xi d P \text { for every } \Lambda \varepsilon \Gamma \text {. }
$$

Therefore, since

$$
\text { (D.S ) } \iint_{n} \xi_{n} d P=(B) \int \xi_{n} d P \text { for every } \Lambda \varepsilon \Gamma \text { and } n \varepsilon I^{+}
$$

it follows that

$$
\text { (D. S) } \int_{\Lambda} \xi d P=(B) \int_{\Lambda} \xi d P \text { for every } \Lambda \varepsilon \Gamma \text {. }
$$

Notation 2.2.4:
(i) If $\xi$ is a Bochner integrable $\mathrm{Xr} . \mathrm{v}$. and if $\xi$ is an Xr.v. $\Phi$ write $\xi$ is a BXr.v.Ф. BXr.v. $\overline{\text { will be written BXr.v. }}$
(ii) The integrals of a BXr.v. $\xi$ will be written thus:
$\int_{n} \xi \mathrm{~d} P$ for $\Lambda \varepsilon \Gamma$.
$\mathrm{E}(\xi) \quad$ rill frequently be used to denote $\int_{\Omega} \xi \mathrm{dP}$.
Theorems 2.2.1 and 2.2 .3 enable us to attribute properties of $P_{\Phi^{9}}{ }^{-}$ measurable functions and strongly measurable functions to Xr.v. and properties of P -integrable and Bochner integrable functions to $3 \mathrm{Xr} . \mathrm{v}$. . Those properties which will be used in this paper will be transliterated below. The reference for each result will be placed after its enumerator.

Remark 2.2.5:
(i) (Theorem 3.5.4 [9]). Let $g$ be an $R r . v . \Phi^{\prime}$ and let a and $b$ be elements of $R$. If $\xi$ and $\rho$ are $X r . v . \Phi^{\prime}$, then $a \xi+b \rho$ and
go are $\mathrm{Zr} . \mathrm{v}^{\prime} \Phi^{\prime}$.
(ii) (Definition 3.7.1 [9] and p.80 [9].) If $\xi$ is a BKr.v., then
$\left.<\int_{\Lambda} \xi \mathrm{dP}, \xi^{*}\right\rangle=\int_{\Lambda}\left\langle\xi, \xi^{*}\right\rangle \mathrm{dP}$ for every $\Lambda \varepsilon \Gamma$ and $\xi * \varepsilon \mathrm{X} \%$.
(iii) (Theorem 3.7.4 [9].) If $\xi$ is an Xr.v., then $\xi$ is a BXr.v. if, and only if, $E(\|\xi\|)<\infty$.
(iv) (Theorem 3.7.5 [9].) If $\left\{\xi_{i}, i \varepsilon J_{n}\right\}$ is a set of BXr.v. and if $\left\{a_{i}, i \in J_{n}\right\}$ is a subset of $R$, then

$$
\sum_{i=1}^{n} a_{i} \xi_{i} \text { is a BXr.v. }
$$

and

$$
\int_{\wedge} \sum_{i=1}^{n} a_{i} \xi_{i} d P=\sum_{i=1}^{n} a_{i} \int_{\hat{N}} \xi_{i} d P \quad \text { for every } \Lambda \varepsilon \Gamma .
$$

(v) (Theorem 3.7.6[9].) If $\xi$ is a $B X r . v .$, then

$$
\left\|\int_{\Lambda} \xi \mathrm{dP}\right\| \leqq \int_{\Lambda}\|\xi\| \mathrm{dP} \quad \text { for every } \Lambda \varepsilon \Gamma
$$

(vi) It follows from part (v) that if $\xi$ is a BKr.v. and if $\Lambda$ is a $P$-negligible element of $\Gamma$, then

$$
\int_{n} \xi d P=\theta
$$

(vii) (Theorem 3.7.7[9].) If $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is a family of BXr.v. such that

$$
\lim _{m, n \rightarrow \infty} E\left(\left\|\xi_{m}-\xi_{n}\right\|\right)=0,
$$

then there exists a BXr.v. $\xi$ such that

$$
\lim _{\mathfrak{n} \rightarrow \infty} \mathbb{E}\left(\left\|\rho-\xi_{\mathrm{n}}\right\|\right)=0 \text { if, and only if, } \rho \varepsilon[\xi] .
$$

Finally,

$$
\lim _{n \rightarrow \infty} \int_{\Lambda} \xi_{n} d P=\int_{\Lambda} \xi \mathrm{dP} \text { for every } \Lambda \varepsilon \Gamma .
$$

(viii) (III.6.16[6].) Let $1 \leqq p<\infty$ and let $\left\{\xi_{n}, n \in I^{+}\right\}$ be a sequence of $\mathrm{BXr} . \mathrm{v}$. converging Pa.e. to $\xi: \Omega \rightarrow \mathrm{X}$. Suppose that $E\left(\left\|\xi_{n}\right\|^{p}\right)<\infty$ for every $n \varepsilon I^{+}$and that there exists a BXr.v. $p$ such that

$$
E\left(\|\rho\| \|^{P}\right)<\infty
$$

and

$$
\left\|\xi_{\mathrm{n}}\right\| \leqq\|\rho\| \text { P are. for every } \mathrm{n} \varepsilon I^{+} .
$$

Then $\xi$ is a BXr.v. such that

$$
E\left(\|\xi\|^{p}\right)<\infty
$$

and $\lim _{\mathrm{n} \rightarrow \infty} \mathbb{E}\left(\left\|\xi-\xi_{\mathrm{n}}\right\|^{\mathrm{p}}\right)=0$.
It is sufficient to demand the existence of a nonnegative $B R r . v . \quad \rho$ such that

$$
E\left(g^{p}\right)<\infty
$$

and

$$
\left\|\xi_{\mathrm{n}}\right\| \leqq \mathrm{g} \quad \text { P ace. for every } \mathrm{n} \varepsilon \mathrm{I}^{+}
$$

Since if $x$ is any element of $X$ such that $\|x\|=1$, then $\rho$ can be taken to be gr.
(ix) (Theorem 3.7.10 [9].) Let $\left\{\mathrm{n}_{\mathrm{n}}, \mathrm{n} \varepsilon \mathrm{I}^{+}\right\}$be a set of disjoint elements of $\Gamma$. If $\xi$ is a BXr.v., then

$$
\int_{\Sigma M_{n}} \xi d P=\sum_{n=1}^{\infty} \int_{M_{n}} \xi d P .
$$

(x) (Theorem 3.7.11 [9].) If $\xi$ is a BXr.v., then
$\lambda(\Lambda)=\iint_{\Lambda} \xi \mathrm{dP}$ is a strongly absolutely continuous set function on $\Gamma$.
The next theorem is the extension of summaries 2.1 .5 (iii) and (iv). Its proof is trivial and so will not be given.

Theorem 2.2.6:
(i) If $\xi$ is a BXr.v. $\Phi^{\text { }}$ and if $\rho \varepsilon[\xi]$, then $\rho$ is a BXr.v. $\Phi^{\prime}$.
(ii) If $\xi$ is a BXr.v. $\Phi^{\text {i }}$, then there exists $\rho \varepsilon[\xi]$ such that $\rho$ is a $B X r . v . \Phi$.

The following theorem is an immediate consequence of theorems 2.1.20 and 2.2.6 (1) .

Theorem 2.2.7: If $\xi$ is a BXr.v. $\Phi$, then there exist a separable $B-$ subspace $L$ of $X$ and a $B X r . v . \Phi \rho$ such that

$$
\rho=\xi \mathrm{P}_{\Phi} \text { a.e. }
$$

and

$$
\rho(\Omega) \subseteq L .
$$

The following theorem is my result; however the proof is similar to that of lemma 3 [5].

Theorem 2.2.8: Let $\xi$ be a BXr.v.Ф. Then

$$
\int_{\Lambda} \xi d P=\theta \quad \text { for every } \quad \Lambda \varepsilon \Phi
$$

if, and only if,

$$
\xi=\theta \mathrm{P}_{\Phi} \text { ate. }
$$

Proof: Remark 2.2.5 (v) implies that if $\xi=\theta P_{\Phi}$ abe., then

$$
\int_{n} \xi \mathrm{dP}=\theta \quad \text { for every } \quad \Lambda \varepsilon \Gamma \geq \Phi
$$

So assume that $\int_{\Lambda} \xi d P=\theta$ for every $\Lambda \varepsilon \Phi$.
Then remark 2.2 .5 (ii) implies that

$$
\int_{\alpha}\langle\xi, \xi *\rangle \mathrm{dP}=0 \text { for every } \Lambda \varepsilon \Phi \text { and } \xi * \varepsilon \mathrm{X} *
$$

It follows from theorem 1 of the appendix that

$$
\left\langle\xi_{,} \xi_{k}^{*}\right\rangle=0 \quad P_{\Phi} \text { a.e. for every } \xi^{*} \varepsilon X^{*} .
$$

Clearly there exist a P-negligible element $\Lambda_{0}$ of $\Phi$ and a countable subset $Y$ of $X$ such that $\xi\left(\Omega \backslash M_{0}\right) \subseteq \bar{Y}$.

Accordingly theorem 9 of the appendix implies that there exists

$$
\left\{\xi_{n}^{*}, n \in I^{+}\right\} \subseteq X^{*}
$$

such that

$$
\left|\left|\xi(\omega) \|=\underset{\mathrm{n}}{1_{\mathrm{u}}} \mathbf{\text { u.b. }}\right|\left\langle\xi(\omega), \xi_{\mathrm{n}}^{*}\right\rangle \quad \text { for every } \omega \varepsilon \Omega \backslash \Lambda_{0}\right.
$$

Clearly for each $n \varepsilon I^{+}$, there exists a P-negligible element $\Lambda_{n}$ of $\Phi$ such that

$$
\langle\xi, \xi \underset{n}{*}\rangle=0 \text { on } \Omega \backslash \Lambda_{n}
$$

It follows that

$$
\|\xi(\omega)\|=0 \quad \text { for every } \quad \omega \varepsilon \Omega \backslash \bigcup_{n=0}^{\infty} \Lambda_{n^{\circ}}
$$

Clearly $\bigcup_{n=0}^{\infty} \Lambda_{n}$ is a P-negligible element of $\Phi$.
Therefore $\xi=\theta \mathrm{P}_{\Phi}$ a.e. .

Theorem 2.2.9: Let $\xi$ and $\rho$ be BXr.v. $\Phi^{\text { }}$. Then

$$
\int_{\Lambda} \xi \mathrm{dP}=\int_{\Lambda} \rho \mathrm{d} P \text { for every } \Lambda \varepsilon \Phi
$$

if, and only if,

$$
\xi=\rho \mathrm{P} \text { ate. }
$$

Proof: It follows from remark 2.2.5 (iv) that

$$
\int_{\Lambda} \xi d P=\int_{\Lambda} \rho d P \text { for every } \Lambda \varepsilon \Phi
$$

is equivalent to

$$
\int_{\alpha}(\xi-\rho) d P=\theta \quad \text { for every } \quad \Lambda \varepsilon \Phi .
$$

Also remarks 2.2 .5 (i) and (iv) imply that ( $\xi-\rho$ ) is a BXr.v. $\Phi^{\prime}$. And so it remains to be shown that

$$
\int_{n}(\xi-\rho) d P=\theta \quad \text { for every } \quad \Lambda \varepsilon \Phi
$$

implies that

$$
\int_{n}(\xi-\rho) d P=\theta \quad \text { for every } \quad \Lambda \varepsilon \Phi^{\prime}
$$

for then the conclusion of the theorem will follow from theorem 2.2.8.

Let $\mathbb{N}$ be any element of $\Phi^{8}$. Then there exists an element $M$ of $\Phi$ such that $N \backslash M$ and $M \backslash i$ are $P$-negligible elements of $\Gamma$. Accordingly it follows from remarks 2.2 .5 (vi) and (ix) that

$$
\begin{aligned}
\int_{\mathbb{N}}(\xi-\rho) \mathrm{dP} & =\int_{N+(M \backslash N)}(\xi-\rho) \mathrm{dP} \\
& =\int_{M+(\mathbb{N} \backslash M)}(\xi-\rho) \mathrm{dP} \\
& =\int_{M}(\xi \cdots \rho) \mathrm{dP} \\
& =\theta
\end{aligned}
$$

Therefore, since $\mathbb{N}$ was an arbitrary element of $\Phi^{\prime}$,

$$
\int_{\Lambda}(\xi-\rho) \mathrm{d} P=\theta \quad \text { for every } \quad \Lambda \varepsilon \Phi^{\prime}
$$

Remark 2.2.10: The device of considering the values of an integral over the elements of $\Phi$ instead of over the elements of $\Phi^{\top}$ will be used in future without comment.

Theorem 2.2.11: Let $L$ be a B-subspace of $X$. Then a function $\xi: \Omega \rightarrow \mathrm{L}$ is an Lr.v. $\Phi$ if, and only if, $\xi$ is an Xr.v.Ф. Moreover $\xi$ is a BLr.v. if, and only if, $\xi$ is a BKr.v.Ф.

Proof: Theorem 13 of the appendix states that the Borel field of the Banach space $L$ is identical with the class $B(X) \cap$. Since $L \varepsilon B(X), B(L) \subseteq B(X)$. Accordingly if $\xi$ is an Xr.v. $\Phi$, then $A \varepsilon B(L)$ implies that

$$
\xi^{-1}(\mathrm{~A}) \varepsilon \Phi
$$

Also if $\xi$ is an Lr.v.Ф, then $A \varepsilon B(X)$ implies that

$$
\begin{aligned}
\xi^{-1}(A) & =\xi^{-1}(A \cap L) \cup \xi^{-1}(A \cap L) \\
& =\xi^{-1}(A \cap L) \\
& \varepsilon \Phi
\end{aligned}
$$

Therefore it follows from remark 11 (i) of the appendix that $\xi$ is an Lr.v. $\Phi$ if, and only if, $\xi$ is an $\mathrm{Xr} . \mathrm{v} . \Phi$. The final statement of the theorem follows immediately from remark 2.2.5 (iii).

Definition 2.2.12: Let $M$ be an element of $\Gamma$ and let $\xi$ be an Xr.v. . $\xi$ is said to be Bochner Integrable on $M$ if, and only if, the function $\rho$ defined by

$$
\rho: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{lll}
\xi(\omega) & \text { if } & \omega \varepsilon M \\
\theta & \text { if } & \omega \nless M
\end{array}\right)
$$

is a BXr.v. . That is, $\xi$ is Bochner integrable on $X$ if, and only if, $\xi I_{M}$ is a BXr.v. .

Definition 2.2.13: (III.2.7[6].) A sequence $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$of Xr.v. is said to converge in measure to an $\mathrm{Xr} . \mathrm{v}, \xi$ if

$$
\lim _{n \rightarrow \infty} P\left\{\omega\| \| \xi(\omega)-\xi_{n}(\omega) \|>\varepsilon\right\}=0 .
$$

The convergence in measure of $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$to $\xi$ is denoted by

$$
\underset{n \rightarrow \infty}{p-1 i m} \xi_{n}=\xi
$$

$\xi$ is called the stochastic limit of $\left\{\xi_{n}, n \in I^{+}\right\}$.

Remark 2.2.14: Let $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$be a sequence of $\mathrm{Xr} . \mathrm{v}$. .
(i) (Theorem 3.5.1 [9].)

If there exists an Xr.v. $\xi$ such that $\left\{\xi_{n, n} n \in I^{+}\right\}$converges Pase. to $\xi$, then $\xi$ is also the stochastic limit of $\left\{\xi_{n}{ }^{n}{ }^{n} \varepsilon I^{+}\right\}$.
(ii) (III.6.4 [6] and III.6.3 [6].) If

$$
\underset{m, n \rightarrow \infty}{p-1 i m_{m}}\left(\xi_{n}-\xi_{n}\right)=0
$$

then there exist an $\mathrm{Xr} . \mathrm{v} . \xi$ and a subsequence $\left\{\xi_{n_{i}, i} \in I^{+}\right\}$of $\left\{\xi_{n^{\prime}} n \in I^{+}\right\}$such that

$$
\underset{n \rightarrow \infty}{p-1 i m} \xi_{n}=\xi
$$

and

$$
\lim _{i \rightarrow \infty} \xi_{n_{i}}=\xi \quad \text { P ace. }
$$

Let $Z$ be the set of all P-equivalence classes of $\mathrm{Xr} . \mathrm{v}$. and d the function $Z \times Z \rightarrow R$ defined by
$\mathrm{d}:([\xi],[\rho]) \varepsilon Z \times Z \rightarrow \mathrm{~g} \cdot 1 . \mathrm{b} \cdot\{\varepsilon \mid \mathrm{P}\{\omega| ||\xi(\omega)-\rho(\omega)| \mid \geqq \varepsilon\} \leqq \varepsilon\}$
where $\xi$ and $\rho$ are arbitrary elements of [ $\xi$ ] and [ $\rho$ ]. Clearly $d$ is well-defined. The proof of the next theorem is straight-forward and so will not be given.

Theorem 2.2.15: ( $\mathrm{Z}, \mathrm{d}$ ) is a complete metric space where the metric convergence is equivalent to the convergence in measure of arbitrary elements of the P-equivalence classes.

Theorem 2.2.16: Let $\left\{\xi_{\epsilon}, t \varepsilon T\right\}$ be a family of Xr.v. whose parameter set $T$ is a subset of the extended reals. Suppose that $t_{0}$ is a limit point of $T$ from the left (right) and that there exists an $\mathrm{Xr} \cdot \mathrm{v} \cdot \xi_{\mathrm{t}_{0}}\left(\xi_{\mathrm{t}_{0}+}\right)$ which is the stochastic limit of every sequence $\left\{\xi_{S_{n}{ }^{n}} \varepsilon I^{+}\right\}$whose parameter set $\left\{s_{n}, n \varepsilon I^{+}\right\}$is a subset of $T$ such that

$$
s_{n}<t_{0} \quad\left(s_{n}>t_{0}\right) \quad \text { for every } n \in I^{+}
$$

and

$$
\lim _{n \rightarrow \infty} s_{n}=t_{0}
$$

Then

$$
\left.\operatorname{p-lim}_{s \uparrow t_{0}} \xi_{s}=\xi_{t_{0}} \underset{s \nmid t_{0}}{(p-11 m} \xi_{s}=\xi_{t_{+}}\right)
$$

Proof: The proof will only be given for when $t_{0}$ is a limit point of $T$ frown the left, since the proof for when $t_{0}$ is a limit point of $T$ from the right is analogous.

For each $n \in I^{+}$, define

$$
I_{n}=\left(\begin{array}{lll}
\left(t_{0}-\frac{1}{n}, t_{0}\right) & \text { if } & t_{0} \\
\text { is finite } \\
\left(n, t_{0}\right) & \text { if } & t_{0}=+\infty
\end{array}\right)
$$

and, for any $\varepsilon>0$, define, for each $n \varepsilon I^{+}$,

$$
K_{n}=\underset{t \in I_{n} T}{1 . u_{0} .} P\left\{\omega\left\|\xi_{t_{-}}-\xi_{t}\right\|>\varepsilon\right\}
$$

Clearly for each $n \varepsilon I^{+}$, there exists on $S_{n} \in I_{n} T$ such that

$$
P\left\{\check { \omega } \left\|\left\|\xi_{t_{0}}-\xi_{s_{n}}\right\|>\varepsilon \geqq \frac{1}{2} K_{n} .\right.\right.
$$

Accordingly there exists $\left\{\xi_{S_{n}}, \mathfrak{n} \in I^{+}\right\}$such that

$$
\begin{aligned}
& s_{n}<t_{0} \text { for every } n \in I^{+}, \\
& \lim _{n \rightarrow \infty} s_{n}=t_{0},
\end{aligned}
$$

and

$$
P\left\{(\omega) \| \xi_{0}^{\left.--\xi_{s_{n}} \|>\varepsilon\right\}}<\frac{1}{2} K_{n} \text { for every } n \varepsilon I^{+}\right.
$$

And so the hypothesis of the theorem implies that $\lim _{n \rightarrow \infty} K_{n}=0$.
Therefore, since $\mathcal{\&}>0$ was arbitrary,

$$
\underset{s \uparrow t}{p-1 \operatorname{lm}_{s}} \xi_{t_{0^{-}}}
$$

Definition 2.2.17: A family $\left\{s_{t}, t \in T\right\}$ of nonnegative BRr.v. is said to be uniformly integrable if the following conditions are satisfied:
(i) $B\left(x_{t}\right)$ is uniformly bounded in $t$
and (ii) $\quad \lim _{P A \rightarrow 0} \int_{\Lambda} x_{t} d P=0$ uniformly in $t$.

That is, the set functions $\int_{A} x_{t} d P$ for $\Lambda \in \Gamma$ are uniformly absolutely continuous.

Remark 2.2.18: (p.629 [4].)
(i) It is sufficient for uniform integrability that $E\left(x_{t}^{\alpha}\right)$ be uniformly bounded in $t$ for some $\alpha>1$.
(ii) If $\left\{x_{n}, n \in I^{+}\right\}$is a sequence of nonnegative BRr.v. converging $P$ a.e. to $x$ with expectations converging to the finite limit $K$, then $E(x) \leqq K$. There is equality if, and only if, $\left\{x_{n}, n \varepsilon I^{+}\right\}$is uniformly integrable.

Definition 2.2.19: A family $\left\{\xi_{t}, t \in T\right\}$ of $B X r$.v. is said to be uniformly integrable if $\left\{\left|\mid \xi_{t} \|, t \in T\right\}\right.$ is uniformly integrable.

Theorem 2.2.20: (III.6.15 [6].) Let $\left\{\xi_{n}, n \in I^{+}\right\}$be a sequence of Xr.v. which converges $P$ ace. to an Xr.v. $\xi$. If, for some $\alpha \geqq 1, \quad \mathrm{E}\left(\left\|\xi_{\mathrm{n}}\right\|^{\alpha}\right)<\infty \quad$ for every $\mathrm{n} \varepsilon \mathrm{I}^{+}$, then

$$
E\left(\|\xi\|^{\alpha}\right)<\infty
$$

and

$$
\lim _{n \rightarrow \infty} E\left(\left\|\xi-\xi_{n}\right\|^{\alpha}\right)=0
$$

if, and only if,

$$
\lim _{P \Lambda \rightarrow 0} \int_{\wedge}\left\|\xi_{n}\right\|^{\alpha} d P=0 \quad \text { uniformly in } n
$$

The next theorem is a corollary of theorem 2.2.20.

Theorem 2.2.21: Let $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$be a sequence of $B X r . v$. which: converges Pace. to an Xr.v. $\xi$. If $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is uniformly integrable, then $\xi$ is a BXr.v. and $\lim _{\mathrm{n} \rightarrow \infty} \int_{\Lambda} \xi_{\mathrm{n}} \mathrm{dP}=\int_{\Lambda} \xi \mathrm{dP}$ for every $\Lambda \varepsilon \Gamma$.

## CHAPTER 3

## Conditional Expectations and X-martingales

In section 3.1 the concept of the conditional expectation of a BXr.v. relative to a $\sigma$-subfield of $\Gamma$ is defined. The definition in this thesis of a conditional expectation is different from the usual one in that a conditional expectation is a class of $B X r . v$. rather than any element of a class of BXr.v. . That is, the conditional expectations are elements of the separated space associated with the space of all BXr.v. .

In section 3.2 the concepts of an $X$-stochastic process and an X-martingale are defined. Several miscellaneous definitions and results concerning these concepts are also included in this section. Remark 3.2.6 (i) is equivalent to theorem 2.1 chapter III [10]. Theorem 3.2.7 and remark 3.2.8 are extensions of results contained in theorem 1.1 cinapter IV [10].

### 3.1 Conditional Expectations

Let $X$ be an arbitrary Banach space and ( $\Omega, \Gamma, P$ ) a complete probability space.

Definition 3.1.1: Let $\xi$ be a BXr.v. and let $\Phi$ be a $\sigma$-subfield of $\Gamma$. If an Xr.v. $\Phi^{\prime} \rho$ has the following property:
(1) $\quad \int_{\wedge} \rho \mathrm{dP}=\int_{\wedge} \xi \mathrm{dP}$ for every $\Lambda \varepsilon \Phi$
then it is said to be an element of the conditional expectation of $\xi$ relative to $\Phi$. If there exists an $\operatorname{Xr}, v . \Phi^{\prime} \rho$ satisfying (i), then it is said that the conditional expectation of $\xi$ relative to $\Phi$ exists.

Remark 3.1.2: Let $\xi$ be a BXr.v. whose conditional expectation relative to $\Phi$ exists.
(i) Clearly the relation 3.1 .1 (i) implies that every element of the conditional expectation of $\xi$ relative to $\Phi$ is a BXr.v. $\Phi^{\boldsymbol{\gamma}}$.
(ii) If $\rho_{1}$ and $\rho_{2}$ are any two elements of the conditional expectation of $\xi$ relative to $\Phi$, then theorem 2.2.9
implies that $\rho_{1}=\rho_{2} P$ a.e. And if $\left.\rho \varepsilon\left[\rho_{1}\right]=\emptyset_{2}\right]$, then theorems 2.2.6 (i) and 2.2.9 imply that $\rho$ is an element of the conditional expectation of $\xi$ relative to $\Phi$. Therefore, if the conditional expectation of a $B X r$.v. relative to a $\sigma$-subfield of $\Gamma$ exists, then it is a P -equivalence class.
(iii) If $g$ is a BRr.v., then clearly for any $\sigma$-subfleld $\Phi$ of $\Gamma$ the conditional expectation of $g$ relative to $\Phi$ exists.

Notation 3.1.3:
(1) Let $\xi$ be a BXr.v. and let if be an element of $\Gamma$. If the conditional expectation of $\xi$ relative to $\Phi$ exists, then this P-equivalence class is denoted by $[E(\xi \mid \Phi)]$. Since $I_{M}$ is a BRr.v.,
$\left[E\left(I_{M} \mid \Phi\right)\right]$ exists and is written $[P(M \mid \Phi)]$.
(ii) $E(\xi \mid \Phi)$ and $P(M \mid \Phi)$ are frequently used to denote arbitrarily chosen elements of $[E(\xi \mid \phi)]$ and $[P(M \mid \Phi)]$.
(iii) Let. $\xi$ be a BXr.v. and let $\left\{\xi_{t}, t \in T\right\}$ be a family of $X r . v$. where $T$ is a subset of the extended reals. If $\left[E\left(\xi \mid B\left(\xi_{t}, t \leqq t^{\gamma}\right)\right)\right]$ exists, then this P-equivalence class will be denoted by $\left[\mathbb{E}\left(\xi \mid \xi_{t}, t \leqq t^{\prime}\right)\right]$.

Remark 3.1.4:
(i) Let $\xi$ be a BXr.v. . If $\rho \varepsilon[\xi]$, then $[E(\xi \mid \Phi)]$ exists if, and only if, $[E(\rho \mid \Phi)]$ exists and if they exist, they are identical.
(ii) Clearly if $\xi$ is a BXr.v., then $[E(\xi \mid \Phi)]$ exists if, and only if, $\left[\mathbb{E}\left(\xi \mid \Phi^{\prime}\right)\right]$ exists and if they exist, they are identical.
(iii) If $\xi$ is a BXr.v. whose $[E(\xi \mid \Phi)]$ exists, then relation 3.1.1 (i) implies that

$$
E(E(\xi \mid \Phi))=E(\xi)
$$

(iv) If $\xi$ is a $B X r \cdot v \cdot \Phi^{\prime}$, then $[E(\xi \mid \Phi)]$ exists and is identical with [ $\xi]$.
(v) Accordingly if $\xi$ is a $B X r . v$. , then $[E(\xi \mid \Gamma)]$ exists and is identical with [ $\xi$ ].

Theorem 3.1.5: Let $\left\{a_{i}, i \in J_{N}\right\}$ be a subset of $R$ and let
$\left\{\xi_{i}, i \varepsilon J_{n}\right\}$ be a set of BXr.v. . If $\left[Z\left(\xi_{i} \mid \Phi\right)\right]$ exists for each $i \in J_{n}$, then

$$
\left[E\left(\sum_{i=1}^{n} a_{i} \xi_{i} \mid \Phi\right)\right]=\left[\sum_{i=1}^{n} a_{i} E\left(\xi_{i} \mid \Phi\right)\right]
$$

Proof: Remarks 2.2.5 (i) and (iv) imply that

$$
\sum_{i=1}^{n} a_{i} \xi_{i} \text { is a BXr.v: }
$$

and that

$$
\sum_{i=1}^{n} a_{i} E\left(\xi_{i} \mid \Phi\right) \text { is a BXr.v. } \Phi^{\prime}
$$

Accordingly remark 2.2.5 (iv) implies that

$$
\begin{aligned}
\int_{\Lambda}^{\sum_{i=1}^{n} a_{i} \xi_{i} d P} & =\sum_{i=1}^{n} a_{i} \int_{n} \xi_{i} d P \text { for every } \Lambda \varepsilon \Phi \\
& =\sum_{i=1}^{n} a_{i} \int_{\Lambda} E\left(\xi_{i} \mid \Phi\right) d P \text { for every } \Lambda \varepsilon \Phi \\
& =\int_{\Lambda} \sum_{i=1}^{n} a_{i} E\left(\xi_{i} \mid \Phi\right) d p \text { for every } \Lambda \varepsilon \Phi
\end{aligned}
$$

Theorem 3.1.6: If $x \in X$ and $: \Gamma$, then

$$
\left[E\left(I_{M} x \mid \Phi\right)\right]=[P(H \mid \Phi) x]
$$

Proof: Remark 2.2.5 (ii) implies that

$$
\begin{aligned}
& \int_{n}\left\langle E\left(I_{M} x \mid \Phi\right), \xi^{*}\right\rangle d P=\int_{n}\left\langle I_{M} X_{,} \xi^{*}\right\rangle d P \text { for every } \Lambda \varepsilon \Phi \text { and } \xi^{*} \varepsilon X^{*} \\
& =\left(\int_{\Lambda} I_{M} d P\right)\left\langle X, \xi^{*}\right\rangle \text { for every } \Lambda \varepsilon \Phi \text { and } \xi * \varepsilon X^{*} \\
& =\left(\int_{\Lambda} P(M \mid \Phi) d P\right)<x, \xi^{*}>\text { for every } \Lambda \varepsilon \Phi \text { and } \xi^{*} \varepsilon X * \\
& =\int_{\Lambda}\left\langle P(M \mid \Phi) \mathrm{x}, \xi^{*}\right\rangle \mathrm{dP} \text { for every } \Lambda \varepsilon \Phi \text { and } \xi^{*} \varepsilon \mathrm{X}^{*} .
\end{aligned}
$$

Since $P(M \mid \Phi)$ is a $B R \cdot v . \Phi^{\prime}$ and since

$$
\|P(M \mid \Phi) x\|=\mid P(M \mid \Phi)\|x\| \text { everywhere, }
$$

it follows from remarks 2.2 .5 (i) and (iii) that $P(M \mid \Phi) x$ is a BXr.v. $\Phi^{\text {s. }}$ Theorem 1 of the appendix then implies that

$$
\left\langle\mathbb{E}\left(I_{M} x \mid \Phi\right)-P(M \mid \Phi) x, \xi *>=0 \quad P \text { a.e. for every } \xi^{*} \varepsilon X^{*} .\right.
$$

Therefore the conclusion of the theorem follows from an application of theorem 9 of the appendix similar to that used in the proof of theorem 2.2.8.

Theorem 3.1.7: If $\xi$ is a $B X r . v$. whose $[\mathbb{E}(\xi \mid \Phi)]$ exists, then

$$
\|E(\xi \mid \Phi)\| \leqq \because(\|\xi\| \| \Phi) \quad \text { P ace. . }
$$

Proof: Theorem 2.2.7 and remark lr (i) of the appendix imply that there exist

$$
\xi^{\prime} \varepsilon[\xi] \text { and } \rho \varepsilon[\Xi(\xi \mid \Phi)]
$$

such that $\xi^{\prime}(\Omega)$ and $\rho(\Omega)$ are separable. Accordingly it follows from theorem 8 of the appendix that there exists a countable subset $Y$ of $X$ such that

$$
\xi^{\prime}(\Omega) \cup \rho(\Omega) \subseteq \bar{Y}
$$

Theorem 9 of the appendix then implies that there exists

$$
\left\{\xi_{\left.\mathfrak{n}^{*}, n \in I^{+}\right\} \subseteq X^{*}}\right.
$$

such that

$$
\left|\left|\xi^{\prime}(\omega)\right|\right|=\underset{\mathbf{n}}{1 . u . b .\left|\left\langle\xi^{\prime}(\omega), \xi_{\mathrm{n}}^{\xi}\right\rangle\right|} \text { for every } \omega \varepsilon \Omega
$$

and

$$
\| \rho(\omega)| |=\underset{\mathbf{n}}{1 . u . b . ~} \mid\left\langle\rho(\omega), \xi_{\mathrm{n}}^{*>}\right| \quad \text { for every } \omega \varepsilon \Omega .
$$

It follows from remarks 3.1.2 (ii) and 3.1.4 (i) that

$$
\int_{\Lambda} \rho d P=\int_{\Lambda} \xi^{\prime} d P \quad \text { for every } \Lambda \varepsilon \Phi
$$

Therefore remark 2.2.5 (ii) implies that

$$
\begin{aligned}
\left|\int_{n}\left\langle\rho, \xi_{n}^{*}\right\rangle \mathrm{dP}\right| & =\left|\int_{\Lambda}\left\langle\xi^{\prime}, \xi_{n}^{*}\right\rangle d P\right| \text { for every } \Lambda \varepsilon \Phi \text { and } n \varepsilon I^{+} \\
& \leqq \int_{n}\left|\left\langle\xi^{\prime}, \xi_{n}^{*}\right\rangle\right| d P \text { for every } \Lambda \varepsilon \Phi \text { and } n \varepsilon I^{+} \\
& \leqq \int_{n}| | \xi^{\prime} \| d P \text { for every } \Lambda \varepsilon \Phi \\
& =\int_{\Lambda} E\left(\left\|\xi^{\prime}\right\|| | \Phi\right) d P \text { for every } \Lambda \varepsilon \Phi .
\end{aligned}
$$

Hence theorem 2 of the appendix implies that

$$
\left|<\rho, \xi_{n}^{*}\right| \leqq E\left(| | \xi^{\prime}| | \mid \Phi\right) \quad \text { P a.e. for every } n \varepsilon I^{+} .
$$

And so $\|\rho\| \leqq E\left(\left\|\xi^{\prime}\right\| \Phi\right)$ P ace. . Therefore remark 3.1.2 (ii) implies that

$$
\|E(\xi \mid \Phi)\| \leqq E(\|\xi\| \| \Phi) \quad \text { P are. }
$$

The proof of the next theorem was derived from theorem 2.2(4) of chapter II [10]; however it is similar to that of 1 emma 8 [5].

Theorem 3.1.8: If there exist a sequence $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$of $B X r . v$. whose $\left[\mathrm{Z}\left(\xi_{\mathrm{n}} \mid \Phi\right)\right]^{\prime} \mathrm{s}$ exist and a nonnegative BRr.v. g with the following properties:
(i) $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$converges $P$ a.e. to a limit function $\xi$
and
(ii) $\quad\left\|\xi_{n}\right\| \leqq g \quad P$ ace. for every $n \varepsilon I^{+}$, then $\xi$ is a $B X r . v$. whose $[E(\xi \mid \Phi)]$ exists. Moreover
$\left\{E\left(\xi_{\mathbf{n}} \mid \Phi\right), \quad \mathrm{n} \varepsilon \mathrm{I}^{+}\right\}$converges P a.e. to $\mathrm{E}(\xi \mid \Phi)$.

Proof: Remark 2.2.5 (viii) implies that $\xi$ is a BXr.v. and that

$$
\lim _{n \rightarrow \infty} \int_{n} \xi_{n} d P=\int_{\hat{N}} \xi \mathrm{dP} \text { for every } \Lambda \varepsilon \Gamma
$$

Accordingly

$$
\left\|\xi_{m}-\xi_{n}\right\| \text { for all } m, n \varepsilon I^{+}
$$

and

$$
\left\|\xi-\xi_{n}\right\| \text { for every } n \varepsilon I^{+}
$$

are BRr.v. .
It follows from (i) that
$\left\{\left|\mid \xi-\xi_{n} \|, n \varepsilon I^{+}\right\}\right.$converges $P$ ace. to zero and from (ii) that

$$
\left\|\xi_{m}-\xi_{n}\right\| \leqq 2 g \quad \text { Pare. for all } m, n \varepsilon I^{+}
$$

and that

$$
\left\|\xi--\xi_{n}\right\| \leqq 2 g \quad \text { P a.e. for every } n \varepsilon I^{+}
$$

Accordingly $C_{5}$ p. 23 [4] implies that

$$
E\left(\left\|\xi-\xi_{n}\right\| \Phi\right) \rightarrow 0 \quad \text { P ale. }
$$

Also

$$
\begin{aligned}
E\left(\left\|\xi_{m}-\xi_{n}\right\||\mid \Phi)\right. & \leqq E\left(\left\|\xi-\xi_{m}\right\| \Phi\right) \\
& +E\left(\left\|\xi_{n} \xi_{n}\right\| \Phi\right) \quad \text { P a.e. for all } m, n \varepsilon I^{+}
\end{aligned}
$$

since clearly

$$
\begin{aligned}
& \int_{\alpha} E\left(\left\|\xi_{m}-\xi_{n}\right\| \|\right) d P=\int_{\alpha}\left\|\xi_{m}-\xi_{n}\right\| d P \text { for every } \Lambda \in \Phi \\
& \leqq \int\left\|\xi_{\mathrm{m}}-\xi\right\| \mathrm{dP}+\int_{n}\left\|\xi-\xi_{\mathrm{n}}\right\| \mathrm{dP} \\
& =\int_{n} E\left(\left\|\xi-\xi_{m}\right\| \mid \Phi\right) d P+\int_{n} E\left(\left\|\xi-\xi_{\mathrm{n}}\right\| \| \Phi\right) \mathrm{dP} \\
& =\int\left\{E\left(\left\|\xi-\xi_{m}\right\| \Phi\right)+E\left(\left\|\xi-\xi_{n}\right\| \mid \Phi\right)\right\} d P \\
& \text { for every } \Lambda \varepsilon \Phi \text { and all } m, n \in I^{+} \text {. }
\end{aligned}
$$

Therefore theorems 3.1.5 and 3.1.7 imply that

$$
\begin{aligned}
\| E\left(\xi_{m} \mid \Phi\right)-E\left(\xi_{n} \mid \Phi\right)| | & =\left\|E\left(\xi_{m}-\xi_{n} \mid \Phi\right)\right\| \text { P a.e. } \\
& \leqq E\left(\left\|\xi_{m}-\xi_{n}\right\|| | \Phi\right) \text { P a.e. } \\
& \leqq E\left(\left\|\xi-\xi_{m}\right\| \| \Phi\right)+E\left(\left\|\xi-\xi_{n}\right\| \| \Phi\right) \text { P a.e. } \\
& \rightarrow 0 \text { as } m, n \rightarrow \infty
\end{aligned}
$$

Hence remark 2.1.12 implies that there exists an Xr.v. $\Phi^{\prime} \rho$ which is a limit function of $\left\{E\left(\xi_{n} \mid \Phi\right), n \varepsilon I^{+}\right\}$. Moreover theorem 3.1.7 and (ii) above imply that

$$
\begin{aligned}
\left\|E\left(\xi_{n} \mid \Phi\right)\right\| & \leqq E\left(\left\|\xi_{n}\right\| \Phi\right) \text { P a.e. } \\
& \leqq E(g \mid \Phi) \text { P a.e. for every } n \varepsilon I^{+} .
\end{aligned}
$$

Therefore remark 2.2.5 (viii) implies that

$$
\rho \text { is a BXr.v. } \Phi^{8}
$$

and that

$$
\begin{aligned}
\int_{\wedge} \rho d P & =\lim _{n \rightarrow \infty} \int_{\wedge} E\left(\xi_{n} \mid \Phi\right) d P \text { for every } \Lambda \varepsilon \Phi \\
& =\lim _{n \rightarrow \infty} \int_{n} \xi_{n} d P \text { for every } \Lambda \varepsilon \Phi \\
& =\int_{n} \xi d P \quad \text { for every } \Lambda \varepsilon \Phi .
\end{aligned}
$$

Therefore $[E(\xi \mid \Phi)]$ exists and satisfies

$$
\lim _{\mathrm{n} \rightarrow \infty} E\left(\xi_{\mathrm{n}} \mid \Phi\right)=\mathbb{E}(\xi \mid \Phi) \quad \text { P a.e. }
$$

The following theorem is the existence theorem for conditional expectations. Its proof is similar to that of theorem 1 [5].

Theorem 3.1.9: If $\xi$ is a $B X r . v$. , then $[E(\xi \mid \Phi)]$ exists.

Proof: Theorem 2.1.16 implies that there exists $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$of finitely-valued Xr.v. with the following properties:
(i) $\left\{\xi_{\mathrm{n}}, \mathrm{n} \varepsilon \mathrm{I}^{+}\right\}$converges P a.e. to $\xi$ and (ii) $\quad\left\|\xi_{\mathrm{n}}\right\| \leqq\|\xi\|$ everywhere for every $n \varepsilon \mathrm{I}^{+}$.

It follows from theorems 3.1 .5 and 3.1.6 that $\left[E\left(\xi_{n} \mid \Phi\right)\right]$ exists for every $n \varepsilon I^{+}$. Therefore the conclusion of the theorem follows from theorem 3.1.8.

The following theorem contains extensions of (10.8) and theorem 8.1 chapter I [4]. Part (i) is equivalent to the corollary of theorem 2.3 chapter II [10], from the present context.

Theorem 3.1.10: Let $\Phi_{1}$ and $\Phi_{2}$ be $\sigma$-subfield of $\Gamma$ such that

$$
\Phi_{1}^{\prime} \subseteq \Phi_{2}^{\prime} .
$$

(i) If $\xi$ is a BXI.v., then

$$
\left[E\left(\xi \mid \Phi_{1}\right)\right]=\left[E\left(E\left(\xi \mid \Phi_{1}\right) \mid \Phi_{2}\right)\right]
$$

$$
=\left[E\left(E\left(\xi \mid \Phi_{2}\right) \mid \Phi_{1}\right)\right] .
$$

(ii) If $E\left(\xi \mid \Phi_{2}\right)$ is a $B X r . v . \Phi_{1}^{1}$, then

$$
\left[E\left(\xi \mid \Phi_{1}\right)\right]=\left[E\left(\xi \mid \Phi_{2}\right)\right]
$$

Proof:
(i) It follows from remark 3.1.4 (i) that these relations are well-defined. And so since $E\left(\xi \mid \Phi_{1}\right)$ is clearly a $B X r . v . \Phi_{2}^{\prime}$, remark
3.1.4 (iv) implies that $\left[E\left(E\left(\xi \mid \Phi_{1}\right) \mid \Phi_{2}\right)\right]=\left[E\left(\xi \mid \Phi_{1}\right)\right]$. Now let $\Lambda$ be any element of $\Phi_{1}^{\prime}$. Then

$$
\begin{aligned}
\int_{A} \mathrm{Z}\left(E\left(\xi \mid \Phi_{2}\right) \mid \Phi_{1}\right) \mathrm{dP} & =\int_{n} E\left(\xi \mid \Phi_{2}\right) \mathrm{dP} \\
& =\int_{n} \xi \mathrm{dP} \\
& =\int_{n} E\left(\xi \mid \Phi_{1}\right) \mathrm{dP} .
\end{aligned}
$$

Therefore, since $\Lambda$ was an arbitrary element of $\Phi_{1}^{\prime}$ and since both $E\left(E\left(\xi \mid \Phi_{2}\right) \mid \Phi_{1}\right)$ and $E\left(\xi \mid \Phi_{1}\right)$ are $B X r \cdot v . \Phi_{1}^{i}$, theorem 2.2.9 implies that

$$
E\left(E\left(\xi \mid \Phi_{2}\right) \mid \Phi_{1}\right)=E\left(\xi \mid \Phi_{1}\right) \quad \text { Pa.e. . }
$$

(ii) Remark 3.1.4 (iv) implies that

$$
\left[E\left(E\left(\xi \mid \Phi_{2}\right) \mid \Phi_{1}\right)\right]=\left[E\left(\xi \mid \Phi_{2}\right)\right]
$$

Therefore the desired result follows inmediately from part (i).

Theorem 3.1.11: If $\xi$ is a BXr.v., then there exists a separable B-subspace $L$ of $X$ such that for every $\sigma$-subfield $\Phi$ of $\Gamma$ there exists an element of $[E(\xi \mid \Phi)]$ whose values lie in $L$.

Proof: Theorem 2.2.7 implies that there exist a separable B-subspace $L$ of $X$ and a BXr.v. $\rho$ such that

$$
\rho=\xi P \text { a.e. }
$$

and

$$
\rho(\Omega) \subseteq L
$$

Theorem 2.2.11 implies that $\rho$ is also a BLr.v. and so it follows from theorem 2.1.16 that there exists a sequence $\left\{\rho_{n}{ }^{n} n \varepsilon I^{+}\right\}$of finitely-valued BLr.v. such that

$$
\left\{\rho_{n}, n \varepsilon I^{+}\right\} \text {converges } P \text { a.e. to } \rho
$$

and

$$
\left\|\rho_{n}\right\| \leqq\|\rho\| \text { everywhere for every } n \varepsilon I^{+}
$$

Clearly for each $n \varepsilon I^{+}, \rho_{n}$ is of the form $\sum_{i=1}^{m} I_{\Lambda_{i}} x_{i}$ where
$\left\{x_{i}, i \varepsilon J_{m}\right\} \subseteq L$ and $\left\{\Lambda_{i}, i \varepsilon J_{m}\right\}$ is a set of mutually disjoint elements of $\Gamma$. Theorem 2.2.11 implies that $\rho_{n}$ is a BXr.v. for every $n \varepsilon I^{+}$. Therefore, taking the conditional expectations of the $\rho_{n}$ 's as BXr.v., it follows from theorems 3.1 .5 and 3.1 .6 that, for any $\sigma$-subfield $\Phi$ of $\Gamma$, there exists for each $n \varepsilon I^{+}$an element of

$$
\left[E\left(\rho_{n} \mid \Phi\right)\right]=\left[\sum_{i=1}^{m} P\left(\Lambda_{i} \mid \Phi\right) x_{i}\right]
$$

whose values lie in $L$. Accordingly, since $L$ is a closed set in $X$, the conclusions of the theorem follow from theorems 3.1.8, 2.1.11, and 2.2.6 (i) and remarle 3.1.4 (i).

### 3.2 X-martingales.

Let $X$ be an arbitrary Eanach space and ( $\Omega, \Gamma, P$ ) a complete probability space.

Definition 3.2.1: A family of Xr.v. $\left\{\xi_{t}, t \varepsilon T\right\}$ is said to be an X-stochastic process (with parameter set $T$ ). If the $\xi_{t}{ }^{2} s$ are BXr.v., then $\left.{ }^{\{ } \xi_{t}, t \varepsilon T\right\}$ is said to be a $B X$-stochastic process.

Definition 3.2.2: Let $\left\{\xi_{t}, t \varepsilon T\right\}$ be an $X$-stochastic process. A function of $t \in T$ obtained by fixing $\omega$ in $\xi_{t}(\omega)$ and letting $t$ vary is called a sample function of the process. If there exists a P-negligible element $\Lambda$ of $\Gamma$ such that every sample function which corresponds to an $\omega \in \Omega \backslash \Lambda$ has a certain property, then it is said that almost all sample functions have that property.
iotation 3.2.3: Let $\left\{\xi_{t}, t \varepsilon T\right\}$ be an $X$ stochastic process with linear parameter set T. (That is, the parameter set is a subset of the extended reals.) For each $t \in T, B\left(\xi_{t^{\prime},} t^{\prime} \leqq t\right)$ will be denoted by $B_{t}$.

Definition 3.2.4: Let $\left\{\xi_{t}, t \in T\right\}$ be a $B X$-stochastic process with linear parameter set $T$ and let $\left\{\Phi_{t}, t \in T\right\}$ be a family of $\sigma$-subfields of $\Gamma$. $\left\{\xi_{t}, t \in T\right\}$ is said to be an $X$-martingale relative to the $\Phi_{t}$ 's if the following conditions are satisfied:
(i) $\Phi_{s} \subseteq \Phi_{t}$ for all $s, t \varepsilon T$ such that $s \leqq t$,
(ii) $B\left(\xi_{t}\right) \subseteq \Phi_{t}^{\prime}$ for every $t \varepsilon T$,
and (iii) $\left[\xi_{s}\right]=\left[E\left(\xi_{t} \mid \Phi_{s}\right)\right]$ for all $s, t \in T$ such that $s \leq t$. Such an $X$-martingale will be denoted by $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T\right\}$.

Notation 3.2.5: If $\left\{\xi_{t}, B_{t}, t \in T\right\}$ is an X-martingale, then it will be denoted by $\left\{\xi_{t}, t \varepsilon T\right\}$.

Bemark 3.2.6:
(1) If $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T\right\}$ is an $X$-martingale, then condition
3. 2.4 (iii) is equivalent to the demand that

$$
\int_{n} \xi_{s} d p=\int_{A} \xi_{t} d p \text { for every } \Lambda \varepsilon \Phi_{s}
$$

for all $s, t \in T$ such that $s \leqq t$.
(ii) If $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ is an $X$-martingale, then remark
3.2.6 (i) implies that

$$
E\left(\xi_{s}\right)=E\left(\xi_{t}\right) \text { for all } s, t \in T .
$$

(iii) If $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ is an $X$-martingale, then theorem
3.1.10 implies that

$$
\left\{\xi_{t}, t \varepsilon T\right\} \text { is an } X \text {-martingale. }
$$

Theorem 3.2.7: If $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ is an $X$-martingale, then $\left\{\left\|\xi_{t}\right\|, \Phi_{t}, t \in T\right\}$ is a real-valued semi-martingale (p. 294 [4]).

Proof: It follows from theorem 2.1.19, remark 2.2 .5 (iii), and condition 3.2.4 (ii) that

$$
B\left(\left\|\xi_{t}\right\|\right) \subseteq B\left(\xi_{t}\right) \subseteq \Phi_{t}^{\prime} \quad \text { for every } t \varepsilon T
$$

and that

$$
E\left(\left\|\xi_{t}\right\|\right)<\infty \quad \text { for every } t \in T .
$$

Condition 3.2.4 (iii) and theorem 3.1 .7 imply that

$$
\left\|\xi_{s}\right\| \leqq E\left(\left\|\xi_{t}\right\| \|_{s}\right) P \text { a.e. for all } s, t \in T \text { such that } s \leqq t
$$

Therefore the conclusion of the theorem follows from the definition of a real-valued semi-martingale.

Remark 3.2.8: Let $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ be an $X$-martingale.
(i) Theorem 3.2 .7 and theorem 3.1 (i) chapter VII [4] imply that

$$
E\left(\left\|\xi_{s}\right\|\right) \leqq R\left(\left\|\xi_{t}\right\|\right) \text { for all } s, t \varepsilon T \text { such that } s \leqq t
$$

(ii) If $t_{1} \& T$, then theorem 3.2 .7 and theorem 3.1 (iii) chapter VII [4] imply that $\left\{\xi_{t} t \leqq t_{1}\right\}$ is uniformly integrable.

For each $\alpha>1$, define a real function of the real variable $\lambda$ thus:

$$
f_{\alpha}: \lambda \varepsilon R \rightarrow\left(\begin{array}{lll}
0 & \text { if } & \lambda \leqq 0 \\
\lambda^{\alpha} & \text { if } & \lambda>0
\end{array}\right) \text {. }
$$

Clearly for each $\alpha>1, f_{\alpha}$ is monotone non-decreasing and convex. Therefore the following theorem is an immediate consequence of theorem 3.2 .7 and theorem 1.1 (i) chapter VII [4].

Theorem 3.2.9: Let $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T\right\}$ be an $X$-martingale. If, for some $\alpha>1$ and some $t_{0} \varepsilon T_{2} \quad E\left(\left\|\xi_{t_{0}}\right\|^{\alpha}\right)<\infty$, then $\left\{\left\|\xi_{t}\right\|^{\alpha}, \Phi_{t}, t \leqq t_{0}\right\}$
is a real-valued semi-martingale.
The next theorem follows from theorem 3.2.9 and theorems 3.1 (i) and (iii) chapter VII [4].

Theorem 3.2.10: Let $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T\right\}$ be an $X$-martingale. If, for some $\alpha>1$,

$$
E\left(\left\|\xi_{t}\right\|^{\alpha}\right)<\infty \quad \text { for every } t \varepsilon T
$$

then
(i) $\quad\left\{\left|\mid \xi_{t} \|^{\alpha}, \Phi_{t}, t \in T\right\} \quad\right.$ is a real-valued semi-martingale,
(ii) $E\left(\left\|\xi_{s}\right\|^{\alpha}\right) \leqq E\left(\left\|\xi_{t}\right\|^{\alpha}\right)$ for all $s, t \varepsilon T$ such that $s \leqq t$, and (iii) $\left\{\left|\mid \xi_{t} \|^{\alpha}, t \leqq t_{1}\right\}\right.$ is uniformly integrable for every $t_{1} \varepsilon T$.

## CHAPTER 4

## Separable X-stochastic Processes

This chapter extends the concept of a separable stochastic process (pp. 51 and 52 [4]) to that of a separable X-stochastic process where $X$ is a real Banach space.

The definition of an $X$-stochastic process, which is separable relative to a subclass of $B(X)$, is given. The main results of this chapter provide conditions under which sequential convergence of a family of Xr.v. can be replaced by ordinary convergence, and conditions for the existence of a separable modification of an $X$-martingale.

Let $X$ be a Banach space, let $\left\{\xi_{t}, t \varepsilon T\right\}$ be an X-stochastic process with linear parameter set $T$, let $Q$ be any subclass of $B(X)$, and let ( $\Omega, \mathrm{F}, \mathrm{P}$ ) be a complete probability space.

Notation 4.1: If $U \subseteq T$ and $\omega \varepsilon \Omega$, then the range of the sample function of the process at $\omega$, restricted to $U \subseteq T$,

$$
\left\{x \mid \xi_{u}(\omega)=x \text { for some } u \varepsilon U\right\}
$$

will be denoted by $(U ; \omega)$ with the convention that $(\phi ; \omega)=\phi$. The closure of $(U ; \omega)$ in $X$ will be denoted by $[U \div \omega$ ].

Notation 4. 2; If $U \subseteq T$ and $A \varepsilon B(X)$, then

$$
\left\{\omega \mid \xi_{\mathbf{u}}(\omega) \varepsilon A \quad \text { for every } u \varepsilon U\right\}
$$

will be denoted by $\{U ; A\}$ with the convention that $\{\phi ; A\}=\Omega$.

Definition 4.3: $\left\{\xi_{t}, t \in T\right\}$ is said to be separable relative to $Q$ if there exist a countable subset $S$ of $T$ and a P-negligible element $\Lambda$ of $\Gamma$ such that if $A$ is any element of $Q$ and if $I$ is any open interval with finite or infinite endpoints, then

$$
\{I S ; A\} \backslash\{I T ; A\} \subseteq \Lambda
$$

If $Q$ is the class $0_{0}$ of all closed spheres in $X$, then separable will be written instead of separable relative to $Q_{0}$.

Remark 4.4:
(i) If $\left\{\xi_{t}, t \in T\right\}$ is separable relative to $Q$,
then

$$
\{I T ; A\} \in \Gamma
$$

for every $A \in Q$ and every open interval $I$. If $S \subseteq T$ is a countable subset whose existence is guaranteed by definition 4.3 , then $S$ is necessarily dense in $T$.
(ii) Since the concept of separability has only been defined for X-stochastic processes whose parameter sets are linear, when it is written that an $X$-stochastic process is separable relative to $Q$ it will be assumed implicitly that its parameter set is linear.
(iii) If $Q_{1}$ and $Q_{2}$ are subclasses of $B(X)$ such that
$\mathrm{Q}_{1} \subseteq \mathrm{Q}_{2}$, then an X -stochastic process which is separable relative to $Q_{2}$ is necessarily separable relative to $Q_{1}$.
(iv) Let $T$ and $T^{\prime}$ be subsets of the extended reals such that

$$
T \subseteq T^{i} \text { and } T^{i} \backslash T \text { is countable. }
$$

If $\left\{\xi_{t}, t \in T^{p}\right\}$ is an $X$-stochastic process and if $\left\{\xi_{t}, t \in T\right\}$ is separable relative to $Q$, then clearly $\left\{\xi_{t}, t \varepsilon I^{\dagger}\right\}$ is separable relative to $Q$.

The following theorem will permit sequential convergence of $\mathrm{Xr} . \mathrm{v}$. to be replaced by oxdinary convergence in certain circumstances. It is an analogue of theorem 2.2 .16 to $P$ a.e. convergence. This theorem is effectively an extension of theorem 2.3 chapter II [4], to the present context; however its proof is independent of Doob's.

Theorem 4.5: Let $X$ be an arbitrary Banach space and let $\left\{\xi_{t}, t \varepsilon T\right\}$ be a separable X-stochastic process. Suppose that $\tau$ is a limit point of $T\{t \mid t>\tau\} \quad(T\{t \mid t<\tau\}$ ). If there exists an Xr.v. $\rho$ such that for any monotone decreasing (increasing) sequence $\left\{t_{n}, n \varepsilon I^{+}\right\}$in $T$, which converges to $\tau$,

$$
\lim _{n \rightarrow \infty} t_{n}=\rho \quad \text { Pa.e., }
$$

then

$$
\left.\lim _{t \downarrow \tau} t=\rho P \text { a.e. } \underset{t \uparrow \tau}{\left(\lim \xi_{t}\right.}=\rho P \text { a.e. }\right)
$$

Proof: The proof for the increasing case will be omitted since it is an analogue of the proof for the decreasing case. Since the process is separable, there exist a countable subset $S$ of $T$ and a $P$ negligible element $\Lambda$ of $\Gamma$ such that for every closed sphere $A$ in $X$ and every open interval I

$$
\{I S ; A\} \backslash\{I T ; A\} \subseteq \Lambda
$$

For each $n \varepsilon I^{+}$, define

$$
I_{n}=\left(\begin{array}{ll}
\left(\tau, \tau+\frac{1}{n}\right) & \text { if } \tau \text { is finite } \\
(\tau,-n) & \text { if } \tau=-\infty
\end{array}\right)
$$

Since $\tau$ is a limit point of $T\{t \mid t>\tau\}$, there exists a monotone decreasing sequence $\left\{s_{n}, n \in I^{+}\right\}$in $I_{P} S$ which converges to $\tau$. The hypothesis implies that $\lim _{n \rightarrow \infty} s_{n}=\rho P$ a.e. . Moreover, since $I_{P} S$ is a countable subset of $T$, it follows from remark 2.1 .9 (iii) that

$$
\left\{\left\|\rho-\xi_{s}\right\|, \quad s \varepsilon I_{1} S\right\}
$$

is a real separable stochastic process (pp. 46 and 51[4]). Accordingly theorem 2.3 chapter II [4] implies that

$$
\lim _{s \neq \tau}\left\|\rho-\xi_{s}\right\|=0 \quad \text { P a.e. }
$$

where $s \varepsilon I_{1} S$. Therefore there exists a $P$-negligible element $M$ of $\Gamma$ such that

$$
\lim _{\mathrm{s} \psi \tau}(\omega)=\rho(\omega) \text { for every } \omega \varepsilon \Omega \backslash \mathrm{M} \text { where }
$$

$\mathrm{s} \varepsilon \mathrm{I}_{\mathrm{l}} \mathrm{S}$. Clearly $\mathrm{it}=\Lambda U \mathrm{H}$ is P -negligible.
If $A \in Q_{0}$ (the class of closed spheres in $X$ ), then for each $\mathrm{n} \varepsilon \mathrm{I}^{+}$,

$$
A \supseteq\left(I_{n} T: \omega\right) \Leftrightarrow \omega \varepsilon\left\{I_{n} T_{;} A\right\}
$$

and

$$
A \supseteq\left(I_{n} S ; \omega\right) \Leftrightarrow \omega \varepsilon\left\{I_{n} S ; A\right\} .
$$

Also if $A \varepsilon Q_{0}$ and if $\omega \notin \Lambda$, then for each $n \in I^{+}$,

$$
\omega \in\left\{I_{n} S ; A\right\} \Leftrightarrow \omega \in\left\{I_{n} T: A\right\},
$$

by definition 4.3.
Accordingly if $A \varepsilon Q_{0}$ and $\omega \notin \Lambda$, then for each $\mathfrak{n} \varepsilon I^{+}$,

$$
\begin{equation*}
A \supseteq\left(I_{n} S ; \omega\right) \Leftrightarrow A \supseteq\left(I_{n} T \omega\right) . \tag{I}
\end{equation*}
$$

Plainly if $\omega \notin: A$, then the diameter of $\left[I_{n} S ; \omega\right]$ converges monotonically to zero since the hypothesis implies that $I_{n} S$ is nonempty for every $n \in I^{+}$. And so if $\omega \notin i j$, then there exists $m(\omega) \varepsilon I^{+}$ such that $\left[I_{n} S: \omega\right]$ is bounded for every $n \geqq m(\omega)$. Therefore $w \notin$ in and $n \geq m(\omega)$ imply that there exists $U_{n, \omega} \in Q_{0}$ which covers [ $I_{n} S ; \omega$ ] and whose radius equals the diameter of $\left[I_{n} S ; \omega\right.$ ]. It follows from (I) that if $\omega \notin \mathbb{N}$ and if $\mathrm{n} \geqq \mathrm{m}(\omega)$, then the diameter of $\left[I_{n} S ; \omega\right]$ is greater than or equal to half the diameter of $\left[I_{n} T ; \omega\right]$.

Hence if $\omega \notin H$, then $\left\{\left[I_{n} T: \omega\right], n \in I^{+}\right\}$is an infinite descending sequence of non-empty closed sets with diameters tending to zero. Therefore if $\omega \notin \mathbb{N}$, then theorem 12-C Simmons [11] implies that $\infty$ $\bigcap_{n=1}\left[I_{n} T ; \omega\right]$ consists of one and only one element. Clearly this element $\mathrm{n}=1$ must be $\rho(\omega)$. We have thus shown that

$$
\begin{aligned}
& \lim _{t \downarrow \tau} \xi_{t}(\omega)=\rho(\omega) \text { for every } \omega \in \Omega \backslash N, \\
& \text { as required. }
\end{aligned}
$$

The following theorem is an extension of 1 emma 2.1 chapter II [4]. Its proof is analogous to Doob's and so will not be given.

Theorem 4.6: Let $X$ be an arbitrary Banach space and let $\left\{\xi_{t}, t \in T\right\}$ be an $X$-stochastic process. To each element $A$ of $B(X)$ there corresponds a countable subset $S$ of $T$ such that for every $t \in T$

$$
\{S ; A\} \cap\left\{\omega \mid \xi_{t}(\omega) \notin A\right\}
$$

is a P -negligible element of $\Gamma$.
More generally, let $V_{0}$ be a countable subclass of $B(X)$ and let $V$ be the class of sets which are intersections of sequences of elements of $V_{0}$. Then there is a countable subset $S$ of $T$ such that to each $t \in T$ there corresponds a P-negligible element $\Lambda_{t}$ of $\Gamma$ such that

$$
\{S ; A\} \cap\left\{\omega \mid \xi_{t}(\omega) \notin A\right\} \subseteq \Lambda_{t} \text { for every } A \varepsilon V \text {. }
$$

The next theorem is an extension of theorem 2.4 chapter II [4]. Its proof can be accomplished analogously to Nob's and will not be given. However we note that when applying theorem 4.6 (lemma 2.1 chapter II [4]) $V_{0}$ is here taken to be $\left\{X \backslash G_{n}, n \in I^{+}\right\}$where $\left\{G_{n}, n \in I^{+}\right\}$is a countable open base for the topology of $X$. Although the present extension only pertains to a separable Banacil space, a further extension will be given in theorem 4.8 for a non-separable Banach space.

Theorem 4.7: Let $X$ be a separable Banach space and let $\left\{\xi_{t}, t \in T\right\}$ be an X-stochastic process with linear parameter set T. There is then an $X$-stochastic process $\left\{\tilde{\xi}_{t}, t \in T\right\}$ which is separable relative to the class of closed sets, with the property that

$$
P\left\{\omega \mid \tilde{\xi}_{t}(\omega)=\xi_{t}(\omega)\right\}=1 \quad \text { for every } t \in T
$$

Theorem 4.8: Let $X$ be an arbitrary Banach space and let $\left\{\xi_{t}, \Phi_{t} s \in T\right\}$ be an X-martingale. If there exists a BXr.v. $\rho$ such that

$$
\left[\xi_{t}\right]=\left[E\left(\rho \mid \xi_{t}\right)\right] \text { for every } t \varepsilon T
$$

then there exists an $X$-martingale

$$
\left\{\tilde{\xi}_{t}, \Phi_{t}, t \varepsilon T\right\}
$$

which is separable relative to the class of closed sets, with the property that

$$
p\left\{\omega \mid \tilde{\xi}_{t}(\omega)=\xi_{t}(\omega)\right\}=1 \quad \text { for every } \quad t \varepsilon \text { i. }
$$

Proof: Theorem 3.1.11 implies that there exists a separable Bsubspace $L$ of $X$ such that for each $t \in T, \xi_{t}^{\prime}$, a BKr.vo ${ }_{t}^{?}$ exists and has the following properties;

$$
\xi_{t}^{p}(\Omega) \subseteq \mathrm{L}
$$

and

$$
\xi_{t}^{\prime} \varepsilon\left[E\left(\rho \mid \Phi_{t}\right)\right] .
$$

It follows from theorem 2. 2.11 that $\left\{\xi_{t}^{\prime}, t \in T\right\}$ is an L-stochastic process. Accordingly theorem 4.7 implies that there exists an L-stochastic process $\left\{\tilde{\xi}_{t}, t \in T\right\}$ which is separable relative to the class of closed sets in $L$, with the property that

$$
P\left\{\omega \mid \tilde{\xi}_{t}(\omega)=\xi_{t}^{\prime}(\omega)\right\}=1 \quad \text { for every } \quad t \in \mathrm{~T} .
$$

Theorem 2.2.11 implies that $\left\{\tilde{\xi}_{t}, t \varepsilon T\right\}$ is an $X$-stochastic process. Clearly $\left[\tilde{\xi}_{t}\right]=\left[\xi_{t}\right]$ for every $t \in T$.
Therefore $\left\{\tilde{\xi}_{t}, \Phi_{t}, t \varepsilon T\right\}$ is an $X$-martingale such that $P\left\{\tilde{\xi}_{t}(\omega)=\xi_{t}(\omega)\right\}=1$ for every $t \& T$. All that remains to be shown is that
$\left\{\tilde{\xi}_{t} t \in T\right\}$ is separable relative to the class of closed sets in $X$. Since $\left\{\tilde{\xi}_{t}, t \varepsilon T\right\}$ is separable relative to the class of closed sets in $L$, there exist a countable subset $S$ of $T$ and $a$ P-negligible element $\Lambda$ of $\Gamma$ such that for every open interval I and every closed set $F$ in $L$

$$
\{I S ; F\} \backslash\{I T ; F\} \subseteq \Lambda
$$

Clearly if $H$ is any closed set in $S$, then $H \cap L$ is closed in $L$. Accordingly, since $\left\{\tilde{\xi}_{t}, t \varepsilon T\right\}$ is an $L$-stochastic process, if $H$ is any closed set in $X$ and if $I$ is any open interval, then

$$
\begin{aligned}
\{I S ; H\} \backslash\{I T ; H\} & =\{I S ; H \cap L\} \backslash\{I T ; H \cap L\} \\
& E \Lambda .
\end{aligned}
$$

Remark 4.9: Let $X$ be a Banach space and let $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ be an X-martingale.
(i) If $b=\underset{t \in T}{\text { l.u.b.t } \varepsilon T,}$ then $\xi_{b}$ clearly satisfies the
requirements of theorem 4.8 for $\rho$.
(ii) If $X$ is reflexive, then conditions will be given in theorem 5.2.1 under which the process satisfies the requirements of theorem 4.8.

X-martingale Convergence Theorems

This chapter is divided into three sections. The first gives extensions of theorems $4.1,4.2$, and 4.3 chapter VII [4], which are convergence theorems for discrete parameter martingales. The second extends the theorems of the first section to $X$-martingales with nondenumerable parameter sets. Alternatively theorems 5.2.1 (i) and 5.2.3 can be considered as extensions of (theorem 4.1) and (theorem 4.3) in section 11 of chapter VII [4]. The third section consists of extensions of theorems $11.1,11.2$, and 11.4 chapter VII [4], which are several results concerning continuous parameter martingales.

In section 5.1 it is necessary to use theorem 4 [12] and theorems 4 and 5[2]. If it is assumed that the probability space in [2] is complete and that the $\sigma$-fields in the $X$-martingale triples in [2] and [12] are P-complete, then it follows from theorem 2.2.1 and theorem 3 of the appendix that theorem 4 [12] and theorems 4 and 5[2] are equivalent to theorem 5.1 .1 (i), the second part of theorem 5.1.3, and the first part of theorem 5.1 .2 respectively. Since the only use that is made of the hypothesis that the Banach space is reflexive is the application of theorem 4 [12] in theorem 5.1.1 (i), it follows from a comment in [12] that this hypothesis can always be replaced by the hypothesis that the Banach space is separable and is the conjugate space of a Banach space.

$$
(\Omega, \Gamma, P) \text { denotes a complete probability space. }
$$

5.1 Discrete Parameter X-martingales

The first theorem of this section is an extension of theorem
4.1 chapter VII [4].

Theorem 5.1.1: Let $X$ be a reflexive Banach space, let $\left\{\xi_{n}, \Phi_{n}, n \in I^{+}\right\}$be an $X$-martingale, let $K$ be the (not necessarily finite) $\lim _{\mathfrak{n} \rightarrow \infty} E\left(\left\|\xi_{n}\right\|\right)$, and let $\Phi_{\infty}=\sigma\left(U_{n}\right)$.
(i) If $K<\infty$, then there exists a BXr.v. $\Phi_{\infty}^{\prime} \xi_{\infty}$ such
that

$$
\left\{\xi_{n}, n \varepsilon I^{\dagger}\right\} \text { converges } P \text { a.e. to } \xi_{\infty}
$$

and

$$
E\left(\left\|\xi_{\infty}\right\|\right) \leqq K
$$

(ii) The following conditions are equivalent:
(a) $K<\infty$ and $\left\{\xi_{n}, \Phi_{n}, n \in I^{+} \cup\{\infty\}\right\}$ is an X -martingale.
(b) $\quad\left\{\xi_{n^{, n}}, I^{+}\right\}$is uniformly integrable.
(c) $K<\infty$ and $E\left(\left\|\xi_{\infty}\right\|\right)=K$.
(d) $\quad K<\infty$ and $\lim _{n \rightarrow \infty} E\left(| | \xi_{\infty}-\xi_{n}| |\right)=0$.
(iii) If, for some $\alpha>1, \lim _{n \rightarrow \infty} E\left(\left\|\xi_{n}\right\|^{\alpha}\right)<\infty$, then the conditions
of (ii) are satisfied,

$$
E\left(\left\|\xi_{\infty}\right\|^{\alpha}\right)<\infty,
$$

and
$\lim _{n \rightarrow \infty} E\left(\left\|\xi_{\infty}-\xi_{n}\right\|^{\alpha}\right)=0$.

Conversely, if the conditions of (ii) are satisfied and if $E\left(\left\|\xi_{\infty}\right\|^{\alpha}\right)<\infty$ for some $\alpha>1$, then

$$
E\left(\left\|\xi_{n}\right\|^{\alpha}\right) \leqq E\left(\left\|\xi_{\infty}\right\|^{\alpha}\right) \text { for every } n \varepsilon I^{+} .
$$

Proof: Remark 3.2.8 (i) implies that

$$
\underset{n \in I^{+}}{\text {1. u.b } E\left(\left\|\xi_{n}\right\|\right)=\lim _{n \rightarrow \infty} E\left(\left\|\xi_{n}\right\|\right)}
$$

(i) This part follows immediately from theorem 4[12] and remark 2.2.18 (ii).
(ii) It follows from definition 2.2 .17 (i) that (b) implies that $K<\infty$. Thus statements (a) - (d) of (ii) either imply or suppose that $K<\infty$, so that $\xi_{\infty}$ is defined in each case. Remark 2.2.18 (ii) implies that (b) and (c) are equivalent, and theorem 2.2.20 implies that (b) and (d) are equivalent. It follows from remark 3.2 .8 (ii) that (a) implies (b) and so it remains to be shown that (b) implies (a).

If $\left\{\xi_{n}, n \in I^{+}\right\}$is uniformly integrable, then theorem 2.2.21 implies that

$$
\lim _{n \rightarrow \infty} \int_{n} \xi_{n} d P=\int_{n} \xi_{\infty} d P \text { for every } \Lambda \varepsilon \Gamma
$$

Hence, since for each $\mathrm{m} \varepsilon \mathrm{I}^{+}$

$$
\int_{\Lambda} \xi_{\mathrm{n}} \mathrm{dP}=\int_{\hat{N}} \xi_{\mathrm{m}} \mathrm{dP} \text { for every } \Lambda \varepsilon \Phi_{\mathrm{m}} \text { and } \mathrm{n} \geqq \mathrm{~m}_{9}
$$

clearly

$$
\int_{a} \xi_{\infty} \mathrm{dP}=\int_{a} \xi_{\mathrm{m}} \mathrm{dP} \text { for every } \Lambda \varepsilon \Phi_{\mathrm{m}} .
$$

Therefore, since $m$ was arbitrary and since $\xi_{\infty}$ is a $B X r \cdot v \cdot \Phi_{\infty}^{?}$,
(b) implies (a).
(iii) If, for some $\alpha>1, \lim _{n \rightarrow \infty} E\left(\left\|\xi_{n}\right\|^{\alpha}\right)<\infty$, then theorem 3.2.9 implies that

$$
E\left(\left\|\xi_{n}\right\|^{\alpha}\right)<\infty \quad \text { for every } n \varepsilon I^{+}
$$

Accordingly the limit relation above and theorems 3.2.9 and 3.2.10
(ii) imply that $E\left(\left\|\xi_{n}\right\|^{\alpha}\right)$ is uniformly bounded in $n$. It follows from remark 2.2.18 (i) that $\left\{\xi_{n}, n \varepsilon I^{+}\right\}$is uniformly integrable and so the conditions of (ii) are satisfied. Hence $\xi_{\infty}$ is defined. From theorem 3.2.10 (iii), condition (a) of (ii) implies that $\left\{\left\|\xi_{n}\right\|^{\alpha}{ }_{, n} n \varepsilon I^{+}\right\}$is uniformly integrable. Accordingly theorem 2.2.20 implies that

$$
E\left(\left|\mid \xi_{\infty} \|^{\alpha}\right)<\infty\right.
$$

and

$$
\lim _{n \rightarrow \infty} E\left(\left\|\xi_{\infty}-\xi_{n}\right\|^{\alpha}\right)=0
$$

Conversely, if the conditions of (ii) are satisfied and if
$F\left(\left\|\xi_{\infty}\right\|^{\alpha}\right)<\infty$ for some $\alpha>1$, then theorems 3.2.9 and 3.2.10 (ii) imply that $E\left(\left\|\xi_{n}\right\|^{\alpha}\right) \leqq \mathbb{Z}\left(\left\|\xi_{\infty}\right\|^{\alpha}\right)$ for every $n \in I^{+}$.

The next theorem extends theorem 4.2 chapter VII [4].

Theorem 5.1.2: Let $X$ be an arbitrary Banach space, let $\left\{\xi_{n}, \Phi_{n}, n \in I^{-n}\right\}$ be an $X$-martingale, and let

$$
\Phi_{-\infty}=\bigcap_{n} \Phi_{n} .
$$

Then there exists a BXr.v. $\Phi_{-\infty}{ }_{-\infty} \xi_{-\infty}$ such that $\left\{\xi_{n}, n \in I^{-}\right\}$converges $P$ a.e. to $\xi_{-\infty}$ and $\left\{\xi_{n}, \Phi_{n}, n \varepsilon I \ddot{ } \cup\{-\infty\}\right\}$ is an $X$-martingale. $\left\{\xi_{n^{\prime}}, n \in I^{-} \cup\{-\infty\}\right\}$ is uniformly integrable, and

$$
\begin{equation*}
E\left(\left\|\xi_{-\infty}\right\|\right)=\lim _{n \rightarrow-\infty} E\left(| | \xi_{n} \|\right) \leqq \cdots \leqq\left(\left|\left|\xi_{-2}\right|\right|\right) \leqq E\left(| | \xi_{-1} \|\right) . \tag{I}
\end{equation*}
$$

If, for some $\alpha \geqq 1, E\left(| | \xi_{-1} \|^{\alpha}\right)<\infty$, then
(II) $\quad \lim _{n \rightarrow-\infty} E\left(| | \xi_{-\infty} \ldots \xi_{n} \|^{\alpha}\right)=0$.

Proof: The existence of a BXr.v. $\Phi_{-\infty}^{\prime} \xi_{-\infty}$ such: that $\left\{\xi_{n}, n \in I^{-}\right\}$ converges P a.e. to $\xi_{-\infty}$ and $\left\{\xi_{\mathrm{n}}, \Phi_{\mathrm{n}}, \mathrm{n} \in \mathrm{I}^{-} \cup\{-\infty\}\right\}$ is an X martingale follows from theorem 5[2]. Remark 3.2.8 (ii) implies that $\left\{\xi_{n}, n \in I^{-} \cup\{-\infty\}\right\}$ is uniformly integrable. Accordingly remarks 2.2.18 (ii) and 3.2 .8 (i) and theorem 2.2 .20 imply (I), and (II) for $\alpha=1$.

If, for some $\alpha>1, E\left(\left\|\xi_{-1}\right\|^{\alpha}\right)<\infty$, then (II) for $\alpha>1$ follows from theorems 3.2.9, 3.2.10 (iii), and 2.2.20.

The next theorem is an extension of theorem 4.3 chapter VII [4].
It is an immediate consequence of theorem 5.1.2 and theorem 4 [2].

Theorem 5.1.3: Let $X$ be an arbitrary Banach space, let $\rho$ be a BXr.v. and let $\cdots \Phi_{-2} \subseteq \Phi_{-1} \subseteq \Phi_{1} \subseteq \Phi_{2} \cdots$ be $\sigma$-subfield of $\Gamma$. Let $\Phi_{-\infty}=\bigcap_{\mathbf{n}} \Phi_{\mathbf{n}}$ and $\Phi_{\infty}=\sigma\left(\bigcup_{\mathbf{n}} \Phi_{\mathbf{n}}\right)$. Then

$$
\lim _{n \rightarrow-\infty} E\left(\rho \mid \Phi_{n}\right)=E\left(\rho \mid \Phi_{-\infty}\right) P \text { a.e. }
$$

and

$$
\lim _{n \rightarrow \infty} E\left(\rho \mid \Phi_{n}\right)=E\left(\rho \mid \Phi_{\infty}\right) \quad \text { P are. }
$$

## 5. 2 Continuous Parameter X -martingales I

Theorem 5.2.1: Let $X$ be a reflexive Banach space and let $\left\{\xi_{t}, \Phi_{t}, t \in T\right\}$ be an $X$-martingale. Suppose that $b=1 . u . b, t \notin T$ where $b$ may teT be finite or infinite. Also let $K$ be the (not necessarily finite) $\lim _{t \rightarrow b} E\left(\left\|\xi_{t}\right\|\right)$ and let $\Phi_{b}=\sigma\left(U_{t} \Phi_{t}\right)$.
(1) If $K<\infty$, then there exists a $B X r \cdot v \cdot \Phi_{b}^{\prime} \xi_{b}$ with $E\left(\left\|\xi_{b}\right\|\right) \leqq K$, such that, if $\left\{s_{n^{n}}, n \in I^{+}\right\}$is a sequence in $T$, $\lim _{n \rightarrow \infty} s_{n}=b$ implies that

$$
\lim _{n \rightarrow \infty} \xi_{s_{n}}=\xi_{b} P \text { ace. }
$$

If the $\xi_{t}$ process is separable, this limit relation can be strengthened to the relation $\lim _{\mathrm{s} \rightarrow \mathrm{b}} \xi_{\mathrm{s}}=\xi_{\mathrm{b}}$ P a.e. .
(ii) The following conditions are equivalent:
(a) $T V<\infty$ and $\left\{\xi_{t}, \Phi_{t}, t \in T \cup\{b\}\right\}$ is an $X$-martingale.
(b) $\quad\left\{\xi_{t}, t \in T\right\}$ is uniformly integrable.
(c) $K<\infty$ and $E\left(\left\|\xi_{b}\right\|\right)=K$.
(d) $\quad K<\infty \quad$ and $\lim _{s \rightarrow b} \mathbb{E}\left(| | \xi_{b}--\xi_{s}| |\right)=0$.

If these conditions are satisfied and if the $\xi_{t}$ process is separable, then $\left\{\xi_{t}, t \in T \cup\{b\}\right\}$ is separable.
(iii) If, for some $\alpha>1, \lim _{t \rightarrow b} \mathbb{E}\left(| | \xi_{t} \|^{\alpha}\right)<\infty$, then the conditions of (ii) are satisfied,

$$
E\left(\left\|\xi_{b}\right\|^{\alpha}\right)<\infty
$$

and

$$
\lim _{s \rightarrow b} E\left(\left\|\xi_{b}-\xi_{s}\right\|^{\alpha}\right)=0
$$

Conversely, if the conditions of (ii) are satisfied and if $E\left(\left\|\xi_{b}\right\|^{\alpha}\right)<\infty$ for some $\alpha>1$, then $E\left(\left\|\xi_{t}\right\|^{\alpha}\right) \leqq E\left(\left\|\xi_{b}\right\|^{\alpha}\right)$ for every $t \varepsilon T$.

Proof:
(i) If $\left\{s_{n}\right\}$ is a sequence of parameter values which converges monotonely to $b$, then clearly $\left\{\xi_{S_{n}}, \Phi_{S_{n}}, n \in I^{+}\right\}$is an X-martingale.

Accordingly, since $\lim _{n \rightarrow \infty} E\left(\left\|\xi_{s_{n}}\right\|\right)=K<\infty$, theorem 5.1.1 (i) and theorems 3 (ii) and (iv) of the appendix imply that there exists a BXr.v. $\Phi_{b}^{\prime}$ $\xi_{b}$ such that $\left\{\xi_{s_{n}}, n \varepsilon I^{+}\right\}$converges $P$ a.e. to $\xi_{b}$ and

$$
\mathbb{E}\left(\left\|\xi_{b}\right\|\right) \leqq K .
$$

$\xi_{b}$ nust be independent of the monotone sequence $\left\{\mathrm{s}_{\mathrm{n}}\right\}$, neglecting values on P-negligible sets, because any two sequences $\left\{s_{n}\right\}$ can be combined into a single one which corresponds to a sequence of $\mathrm{BXr} . \mathrm{v}$. which converges $P$ a.e. . Moreover, the limit must also exist $p$ a.e. if the sequence $\left\{s_{n}\right\}$ is convergent to $b$, but is not necessarily monotone, because sucin a sequence can be reordered to be monotone.

Therefore $\xi_{b}$ satisfies the requirements of the theorem.
If the $\xi_{t}$ process is separable, then theorem 4.5 implies that
$\lim _{s \rightarrow b} \xi_{s}=\xi_{b}$ Pa.e..
(ii) It follows from definition 2.2.17 (i) that (b) implies that $K<\infty$. Thus statements (a) - (d) of (ii) either imply or suppose that $K<\infty$, so that $\xi_{b}$ is defined in each case. It follows from remark 3.2.8 (ii) that (a) implies (b). Let $\left\{\mathrm{s}_{\mathrm{n}}, \mathrm{n} \varepsilon \mathrm{I}^{+}\right\}$ be any monotone sequence of parameter values which converges to $b$. If (b) is valid, then $\left\{\xi_{s_{n}}, \mathrm{n} \varepsilon I^{+}\right\}$is uniformly integrable. Hence, theorem 5.1.1 (ti) mplies that $E\left(\left\|\xi_{b}\right\|\right)=K$ and so (b) implies (c).

If (c) is valid, then theorem 5.1 .1 (ii) implies that $\left\{\xi_{s_{n}}, \Phi_{s_{n}}, n \in I^{+} \cup\{\infty\}\right\}$ where $s_{\infty}=b$ is an $X$-martingale. Accordingly $\left\{\xi_{t}, \Phi_{t}, t \in T \cup\{b\}\right\}$ must be an $X$-martingale and so
(c) implies
(a). If
(d) holds, then $\lim _{n \rightarrow \infty} E\left(\left\|\xi_{b}-\xi_{s_{n}}\right\|\right)=0$ and
so (d) implies (c) by theorem 5.1.1 (ii). It remains to be shown that
(d) is implied by any one of the other three conditions.

For each $n \in I^{+}$, define

$$
I_{n}=\left(\begin{array}{lll}
\left(b-\frac{1}{n}, b\right) & \text { if } \quad b \text { is finite } \\
(n, b) & \text { if } & b=+\infty
\end{array}\right)
$$

and define $K_{n}=\underset{t \in I_{n} T}{1 . u . b} E\left(\left\|\xi_{b}-\xi_{t}\right\|\right)$. Assume that
(c)
is valid. Then $K_{n} \leqq 2 \mathrm{~K}<\infty$ for every $\mathrm{n} \varepsilon \mathrm{I}^{+}$. Clearly there exists $\left\{s_{n}, n \in I^{+}\right\}$such that for each $n \varepsilon I^{+}$,

$$
s_{n} \in I_{n} T
$$

and

$$
E\left(\left\|\xi_{b}-\xi_{s_{n}}\right\|\right) \geqq \frac{1}{2} K_{n} .
$$

$\left\{s_{n}, n \in I^{+}\right\}$can be reordered in such a way that it becomes a monotone sequence which converges to $b,\left\{s_{m}^{\gamma}, m \in I^{+}\right\}$say. It follows from theorem 5.1.1 (ii) that (c) implies that

$$
\lim _{\mathrm{m} \rightarrow \infty} \mathbb{E}\left(\left\|\xi_{\mathrm{b}}-\xi_{\mathrm{s}_{\mathrm{m}}^{\prime}}\right\|\right)=0
$$

Accordingly, for any $\varepsilon>0$, there exists $m(\varepsilon) \varepsilon I^{+}$such that

$$
\because\left(\left\|\xi_{\mathrm{b}}-\xi_{s_{m}^{\prime}}\right\|\right)<\varepsilon \text { for every } m \geqq m(\varepsilon)
$$

moreover there exists $n(\varepsilon) \varepsilon I^{+}$such that

$$
s_{\mathrm{n}} \geqq \mathrm{~s}_{\mathrm{m}(\varepsilon)}^{\ell} \text { for every } \mathrm{n} \geqq \mathrm{n}(\varepsilon)
$$

which implies that

$$
\frac{1}{2} z_{n}<\varepsilon \quad \text { for every } n \geqq n(\varepsilon)
$$

Therefore, since $\varepsilon>0$ was arbitrary,

$$
\lim _{n \rightarrow \infty} E_{n}=0
$$

Accordingly (c) implies (d).
The last statement in (ii) is an immediate consequence of
and remark 4.4 (iv).
(iii) If, for some $\alpha>1, \quad \lim _{t \rightarrow b} E\left(\left\|\xi_{t}\right\|^{\alpha}\right)<\infty$,
then theorem 3.2.9 implies that

$$
E\left(\left\|\xi_{t}\right\|^{\alpha}\right)<\infty \quad \text { for every } t \varepsilon T
$$

By theorem 3.2.10 (ii) and the limit relation above, $E\left(\left\|\xi_{t}\right\|^{\alpha}\right)$ is uniformly bounded in $t$. It follows from remark 2.2.18 (i) that
$\left\{\xi_{\epsilon}, t \in T\right\}$ is uniformly integrable and so the conditions of (ii) are satisfied. Hence $\xi_{b}$ is defined. It follows from theorem 3.2.10 (iii) that condition (a) of (ii) implies that $\left\{\left\|\xi_{t}\right\|^{\alpha}, t \varepsilon T\right\}$ is uniformly integrable and 30 if $\left\{s_{n}, n \varepsilon I^{+}\right\}$is any monotone sequence of parameter values which converges to $b$, then $\left\{\left\|\xi_{s_{n}}\right\|_{{ }_{\rho}}^{\alpha} n \in I^{+}\right\}$ is uniformly integrable. Accordingly theorem 2.2.20 implies that

$$
E\left(\left\|\xi_{b}\right\|^{\alpha}\right)<\infty
$$

and

$$
\lim _{n \rightarrow \infty} E\left(\left\|\xi_{b} \cdots \xi_{s}\right\|_{n}^{\alpha}\right)=0
$$

Therefore, since $\left\{s_{n}, n \in I^{+}\right\}$was arbitrary, it can be shown in the same way as in (ii) that

$$
\lim _{s \rightarrow b} \mathbb{Z}\left(| | \xi_{b}-\xi_{s}| |^{\alpha}\right)=0 .
$$

Conversely, if the conditions of (ii) are satisfied and if $E\left(\left\|\xi_{b}\right\|^{\alpha}\right)<\infty$ for some $\alpha>1$, then theorems 3.2 .9 and 3.2 .10 (ii) imply that

$$
E\left(\left\|\xi_{t}\right\|^{\alpha}\right) \leqq E\left(\left\|\xi_{b}\right\|^{\alpha}\right) \text { for every } t \varepsilon T .
$$

Theorem 5.2.2: Let $X$ be an arbitrary Banach space and let $\left\{\xi_{t}, \Phi_{t}, t \in \mathrm{~T}\right\}$ be an X-martingale. Suppose that $a=$ g.1.b.t $\notin \mathrm{T}$ where $a$ may be finite or infinite and let $\Phi_{a}=\bigcap_{t} \Phi_{t}$.
(i) Then there exists a BXr.v. $\Phi_{a}^{i} \xi_{a}$ such that, if $\left\{s_{n}, n \in I^{+}\right\}$is a sequence in $T, \quad \lim _{n \rightarrow \infty} s_{n}=a \quad$ implies that

$$
\lim _{n \rightarrow \infty} s_{n}=\xi_{a} \quad \text { P a.e.. }
$$

$\left\{\xi_{t}, \Phi_{t}, t \in T \cup\{a\}\right\}$ is an $X$-martingale.
(ii) If, for some $\alpha \geqq 1$ and some $t_{1} \varepsilon T$,

$$
E\left(\left\|\xi_{t_{1}}\right\|^{\alpha}\right)<\infty,
$$

then

$$
\lim _{s \rightarrow a} E\left(\left\|\xi_{a}-\xi_{s}\right\|^{\alpha}\right)=0 .
$$

If the $\xi_{t}$ process is separable, then

$$
\lim _{s \rightarrow a} \xi_{s}=\xi_{a} \text { Pa.e.s }
$$

and

$$
\left\{\xi_{t}, t \in T \cup\{a\}\right\} \text { is separable. }
$$

## Proof:

(i) This part follows from theorem 5.1.2 in the same way that theorem 5.2.1 (i) follows from theorem 5.1.1 (i).
(ii) If $\alpha=1$, then $E\left(\left\|\xi_{t}\right\|^{\alpha}\right)<\infty$ for every $t \varepsilon T$ and so remark 3.2 .8 (ii) implies that for any $t_{1} \in T$

$$
\left\{\xi_{t}, t \leqq t_{1}\right\} \text { is uniformly integrable. }
$$

If, for some $\alpha>1$ and some $t_{1} \varepsilon T_{\text {, }}$

$$
E\left(\left\|\xi_{t_{1}}\right\|^{\alpha}\right)<\infty,
$$

then theorem 3.2.9 implies twat

$$
E\left(\left\|\xi_{t}\right\|^{\alpha}\right)<\infty \text { for every } t \leq t_{1} .
$$

Accordingly it follows from theorem 3.2.10 (iii) that

$$
\left\{\left\|\xi_{t}\right\|^{\alpha}, t \leqq t_{1}\right\} \text { is uniformly integrable. And so }
$$

theorem 2.2.20 implies that if $\left\{s_{n}, \eta \varepsilon I^{+}\right\}$is any monotone sequence in $T$ which converges to $a$ and if, for some $\alpha \geqq 1$ and some $t_{1} \varepsilon T$,

$$
\mathrm{E}\left(\left\|\xi_{\mathrm{t}_{1}}\right\|^{\alpha}\right)<\infty,
$$

then

$$
\lim _{n \rightarrow \infty} 历\left(| | \xi_{a}-\xi_{s_{n}} \|^{\alpha}\right)=0
$$

These expectations may be undefined for $\alpha>1$ and $s_{n}>t_{1}$; however, for a sufficiently large $n^{\prime}, s_{n} \leqq t_{1}$ for every $n \geqq n^{\prime}$. It follows by a method similar to that used in theorem 5.2.1 (ii) that

$$
\lim _{s \rightarrow a} E\left(\left\|\xi_{a}-\xi_{s}\right\|^{\alpha}\right)=0 .
$$

The last statement of the theorem follows immediately from theorem 4.5 and remark 4.4 (iv).

Theorem 5.2.3: Let $X$ be an arbitrary Banach space, let $\rho$ be a BXr.v., and let $\left\{\Phi_{t}, t \in T\right\}$ be a family of $\sigma$-subfield of $\Gamma$ with linear parameter set $T$ and with

$$
\Phi_{s} \subseteq \Phi_{t} \text { for every } s, t \varepsilon T \text { such that } s \leqq t
$$

$$
\text { Let } a=\underset{\substack{\text { g.l.b.t. } \\ t \in T}}{ } \text { and } b=\underset{t \in T}{1_{t} . u . b . t .} \text { and define } \Phi_{a+}=\bigcap_{t} \Phi_{t}
$$ and $\Phi_{b-}=\sigma\left(U_{t}\right)$. Then an element $\rho_{t}$ of $\left[E\left(\rho \mid \Phi_{t}\right)\right]$ can be chosen for each $t \varepsilon I$ in such a way that

$$
\lim _{t \rightarrow a} \rho_{t}=E\left(\rho \mid \Phi_{a+}\right) \quad \text { pace. }
$$

and

$$
\lim _{t \rightarrow b} \rho_{t}=E\left(\rho \mid \Phi_{b-}\right) \quad P \text { a.e. }
$$

Proof: Theorem 3.1.10 implies that

$$
\left\{E\left(\rho \mid \Phi_{t}\right), \Phi_{t}, t \in T\right\} \text { is an } X \text {-martingale and so it follows }
$$

from theorem 4.8 that there exists a separable $X$-martingale
$\left\{\rho_{t}, \Phi_{t}, t \in T\right\}$ such that

$$
\rho_{t} \varepsilon\left[E\left(\rho \mid \Phi_{t}\right)\right] \text { for every } t \varepsilon T
$$

Theorem 5.1.3 and theorem 3 of the appendix imply that if $t$ goes to its limit along a sequence of values, then the limit equations are true
for any choice of elements. Accordingly it follows from theorem 4.5 that the $\rho_{t}$ process exhibited above satisfies the requirements of the theorem.

### 5.3 Continuous Parameter X-martingales II

The first theorem of this section is an extension of theorem 11.1 chapter VII [4]. Its proof is analogous with Doob's although his "metric" space is only a semi-metric space. Hovever, if the space ( $Z, d$ ) of theorem 2.2 .15 and the function $f: T \rightarrow Z$, defined by

$$
f: t \varepsilon T \rightarrow\left[\xi_{t}\right]
$$

are used, then the proof of theorem 5.3 .1 is reduced to Doob's and so will not be given.

Theorem 5.3.1: Let $X$ be an arbitrary Banach space, let $\left\{\xi_{t}, t \in T\right\}$ be an $X$-stochastic process with linear parameter set $T$, and let $T_{1}$ be a set of limit points of $T$. Suppose that, if $t \varepsilon T_{1}$, at least one of the stochastic limits
exists.

There is then an at most countable subset $T_{0}$ of $T_{1}$ such that, if $t \in T_{1} \backslash \mathrm{~T}_{0}$, then both stochastic limits $\xi_{t-}$ and $\xi_{t+}$ are defined, and

$$
\begin{aligned}
\xi_{t} & =\xi_{t+} \text { Pa.e. } \\
& =\xi_{t} \quad \text { Pa.e. if } \quad t \varepsilon T .
\end{aligned}
$$

Definition 5.3.2: Let $X$ be a Banach space and let $\left\{\xi_{t}, t \in T\right\}$ be an $X$-stochastic process with linear parameter set $T$. A point $t_{0} \varepsilon T$ is said to be a fixed point of discontinuity of the process if it is false that whenever $s_{n} \rightarrow t_{0}, \lim _{n \rightarrow \infty} \xi_{s_{n}}=\xi_{t_{0}}$ Pa.e.. If the process is separable, it follows that $t_{0}$ is a fixed point of discontinuity if, and only if, it is false that

$$
\lim _{s \rightarrow t_{0}} \xi_{s}=\xi_{t_{0}} \quad \text { P a.e. }
$$

The next theorem is an extension of theorem 11.2 chapter VII [4].

Theorem 5.3.3: Let $X$ be an arbitrary Banach space, let $\left\{\xi_{t}, t \in T\right\}$ be an $X$-martingale, and let $a$ and $b$ be respectively the minimum and maximum values of the closure of $T$. Define $T^{7}$ as the set of limit points of $T$, except that $b$ is to be excluded from $T^{1}$ unless $b \in T$.
(i) To each point $t \in T^{\prime}$ which is a limit point of $T$ from the left (right) there corresponds a BXr.v. $\xi_{t-}\left(\xi_{t+}\right)$ such that, if $s_{n} \rightarrow t$ with $s_{n}<t\left(s_{n}>t\right)$ and $s_{n} \varepsilon T$, then

$$
\lim _{n \rightarrow \infty} \xi_{s_{n}}=\xi_{t-}\left(\lim _{n \rightarrow \infty} \xi_{s_{n}}=\xi_{t+}\right) \text { P a.e. . }
$$

If the $\xi_{t}$ process is separable, these sequential 1 imits can be replaced by ordinary limits

$$
\lim _{s \uparrow t} \xi_{s}=\xi_{t-}\left(\lim _{s+t} \xi_{s}=\xi_{t+}\right) \text { P a.e.. }
$$

(ii) Except possibly for the points of an at most countable subset of $T^{\prime}$, for each $t \varepsilon T^{\text {; }}$ the following equation holds $P$ a.e. between as mahy of the three members as are defined:

$$
\xi_{t-}=\xi_{t}=\xi_{t+} .
$$

In particular, at most countably many parameter points are fixed points of discontinuity.

Proof: Let $t \varepsilon T^{r}$ be a limit point of $T$ from the left. Then there exists $t_{1} \varepsilon T$ such that $t \leqq t_{1}$. Let $\left\{s_{n}\right\}$ be a monotone increasing sequence which converges to $t$. For an arbitrary choice of elements of the conditional expectations theorem 5.1.3 implies that

$$
\lim _{n \rightarrow \infty} E\left(\xi_{t_{1}} \mid B_{s_{n}}\right)
$$

is equal $P$ a.e. to a BXr.v. . Therefore $\lim _{n \rightarrow \infty} \xi_{n}$
is equal P a.e. to a BXr.v. .

Accordingly part (i) follows by an argument similar to that in the proof of theorem 5.2.1 (i). If $t \varepsilon T^{\prime}$ is a limit point of $T$ from the right then part (i) follows in a similar manner. Part (ii) follows from remart 2.2.14 (i) and theorems 2.2.16 and 5.3.1.

Remark 5.3.4: If $X$ is reflexive and if 1.u.b. $E\left(\left\|\xi_{t}\right\|\right)<\infty$ in theorem 5.3.3, then $b$ can be allowed in $T^{\text {i }}$, even if it does not belong to T. The proof of theorem 5.3 .3 (i) would then follow from theorems 5.2.1 (i) and 5.2.2.

The next theorem is an extension of theorem 11.4 chapter VII [4]. It shows that it can be assumed, without loss of generality, that the parameter set of an $X$-martingale is an interval.

Theorem 5.3.5: Let $X$ be an arbitrary Banach space, $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T\right\}$ an X-martingale, and $I$ the closed interval whose endpoints are the maximum and minimum values of the closure of $T$, except that the righthand endpoint is to be excluded from $I$ unless this endpoint is in $T$. Then it is possible to define $\xi_{t}$ and $\Phi_{t}$ for every $t \varepsilon I \backslash T$ in such a way that $\left\{\xi_{t}, \Phi_{t}, t \in I\right\}$ is an $X$-martingale.

Proof: If $t \in I \backslash T$ and if $t$ is a limit point of $T$ from the right define

$$
\xi_{t}=\xi_{t+} \text { and } \Phi_{\mathbf{t}}=\bigcap_{s>t} \Phi_{\mathbf{s}}
$$

Theorem 5.2.2 implies that the process with the thus enlarged parameter set is an $X$-martingale. If $t \varepsilon I \backslash T$ implies that $t$ is a limit point of $T$ from the right, then we are finished, and so it will be assumed that this is not the case.

Let [ $c, d]$ be a non-degenerate closed interval whose endpoints but no other points lie in the closure of $T$. Then $\xi_{d}$ and $\Phi_{d}$ are already defined. Define

$$
\xi_{t}=\xi_{d} \text { and } \Phi_{t}=\Phi_{d} \text { for } t \varepsilon(c, d)
$$

and if $\xi_{c}$ and $\Phi_{d}$ are not already defined, then define $\xi_{c}=\xi_{\mathrm{d}}$ and $\Phi_{c}=\Phi_{d}$. Then $\left\{\xi_{t}, \Phi_{t}, t \in I\right\}$ is clearly an $X$-martingale.

Remark 5.3.6: If $X$ is reflexive and if $\left\{\xi_{t}, t \in T\right\}$ is uniformly integrable in theorem 5.3.5, then the maximum value of the closure of T need not be excluded from $I$ even if it does not belong to $T$.

## Chapter 6

Optional Skipping and Sampling

This chapter follows up a remark in [2] that "for reflexive spaces, all classical martingale convergence theorems including those involving stopping rules etc. (as in Doob [4]) can be extended ${ }^{\text {² }}$. Because they are merely special cases of results concerning optional sampling for general linear parameter sets, results concerning optional sampling for discrete parameter sets and optional stopping are not discussed here.

Extensions are given for results on optional skipping (pp.309-311 [4]) and on optional sampling (pp.365-379 [4]). The main results of the chapter are theorems 6.1 .6 and 6.2 .13 which give conditions under which the martingale properties are preserved under optional skipping and sampling. It has not been found necessary to impose the condition of reflexivity.

1. Optional Skipping

Let $X$ be a Banach space and $(\Omega, \Gamma, P)$ a complete probability space.

Let $\left\{\rho_{n}, \Phi_{n}, n \in I^{+}\right\}$denote a $B X-$ stochastic process $\left\{\rho_{n}, n \varepsilon I^{+}\right\}$ together with a sequence $\left\{\Phi_{n^{n}} n \varepsilon I^{+}\right\}$of $\sigma$-subfields of $\Gamma$.

Let $\left\{\rho_{n}, \Phi_{n}, n \in I^{+}\right\}$have the following properties:
[i] $\quad \Phi_{m} \subseteq \Phi_{n}$ if $m \leqq n$ for all $m, n \varepsilon I^{+}$,
[ii] $B\left(\rho_{n}\right) S \Phi_{n}^{\prime}$ for every $n \varepsilon I^{+}$,
and [iii] $\left[E\left(\rho_{n+1} \mid \Phi_{n}\right)\right]=[\theta]$ for every $n \varepsilon I^{+}$.
Let $\left\{m_{n}, n \in I^{+}\right\}$be a sequence of rev. taking on integral values and having the following properties:
[iv] $1<m_{1}<m_{2}<\ldots<\infty \quad$ Pase.
and $[v] \quad\left\{\omega \mid m_{j}(\omega)=k\right\} \varepsilon \Phi_{k-1}^{\prime}$ if $k \geqq j$ for every $j \varepsilon I^{+} \quad k \varepsilon I^{+} \backslash\{1\}$.

Notation 6.1.1: The following shorthand notation will be used in this section.

Define for all $n_{s} j \varepsilon I^{+}$,

$$
\begin{aligned}
& \Omega_{n}(j)=\left\{\omega \mid m_{n}(\omega)=j\right\}, \\
& \Omega_{n}(<j)=\left\{\omega \mid m_{n}(\omega)<j\right\}, \\
& \Omega_{n}(\leq j)=\left\{\omega \mid m_{n}(\omega) \leqq j\right\}, \\
& \Omega_{n}(>j)=\left\{\omega \mid m_{n}(\omega)>j\right\}, \\
& \Omega_{n}(\searrow j)=\left\{\omega \mid m_{n}(\omega) \geqq j\right\}
\end{aligned}
$$

and

For each $n \in I^{+}$, define

$$
\hat{\rho}_{\mathrm{n}}: \omega \varepsilon \Omega \rightarrow \rho_{\mathrm{m}_{\mathrm{n}}(\omega)}(\omega)
$$

Definition 6.1.2:
$\left\{\rho_{n}, \Phi_{n}, n \in I^{+}\right\}$is said to be transformed into $\left\{\hat{\rho}_{n}, n \varepsilon I^{+}\right\}$ by optional skipping.

Theorem 6.1.3:
$P \Omega_{i}(j)=0$ if $i \geq j$ for all $i, j \varepsilon I^{+}$.
That is, $P \Omega_{i}(\leq i)=0$ for every $i \varepsilon I^{+}$.

Proof: The proof will be by induction. It follows from [iv] that $P \Omega_{1}(\leq 1)=0$. Take any $n \varepsilon I^{+} \backslash\{1\}$ and assume that

$$
P \Omega_{\mathrm{n}}(\underline{\underline{n}})=0
$$

[iv] implies that

$$
\Omega_{n}(<j) \geq \Omega_{n+1}(j) \quad \text { P a.e. for every } j \varepsilon I^{+}
$$

Hence, since the induction assumption implies that

$$
\begin{aligned}
& P \Omega_{n}(<j)=0 \text { for every } j \in J_{n+1} \\
& P \Omega_{n+1}(j)=0 \text { for every } j \in J_{n+1} .
\end{aligned}
$$

Therefore the conclusion of the theorem follows by induction.

Remark 6.1.4: Theorem 6.1.3 implies that condition [v] can be replaced by

$$
\Omega_{j}(k) \varepsilon \Phi_{k-1}^{j} \text { for every } j \varepsilon I^{+}: k \varepsilon I^{+} \backslash\{1\}
$$

Clearly $\Omega_{j}(1) \varepsilon \Phi_{i}^{l}$ for all $i, j \varepsilon I^{+}$.

Theorem 6.1.5:
(i) $\quad\left\{\hat{\rho}_{n}, n \varepsilon I^{+}\right\}$is an $X$-stochastic process.
(ii) If $\mathbb{H}$ is an element of $B\left(\hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)$, then

$$
\left.M \cap \Omega_{n+1}(j) \varepsilon \Phi_{j-1}^{i} \text { for every } j \varepsilon I^{+} \backslash 1\right\}_{\%} n \varepsilon I^{+} .
$$

Proof:
(i) For any $n \varepsilon I^{+}$and any $A \varepsilon B(X)$,

$$
\begin{aligned}
\hat{\rho}_{n}^{-1}(A) & =\sum_{k=1}^{\infty}\left(\hat{\rho}_{n}^{-1}(A) \cap \Omega_{n}(k)\right) \\
& =\sum_{k=1}^{\infty}\left(\rho_{k}^{-1}(A) \cap \Omega_{n}(k)\right) \\
& \varepsilon \Gamma .
\end{aligned}
$$

For each $n \varepsilon I^{+}$, there exists a P-negligible element $\Lambda_{n}$ of $\Gamma$ such that $\rho_{n}\left(\Omega \backslash \Lambda_{n}\right)$ is separable. Clearly $\Lambda=\bigcup_{n=1}^{\infty} \Lambda_{n}$ is a P-negligible element of $\Gamma$. And so for each $n \varepsilon I^{+}$,

$$
\begin{aligned}
\hat{\rho}_{n}(\Omega \backslash \Lambda) & =\hat{\rho}_{n}\left(\sum_{k=1}^{\infty}(\Omega \backslash \Lambda) \cap \Omega_{n}(k)\right) \\
& =\bigcup_{k=1}^{\infty} \rho_{k}\left(\Omega_{n}(k) \backslash \Lambda\right)
\end{aligned}
$$

which is separable by theorems 7 and 8 of the appendix. Accordingly, for each $n \varepsilon I^{+}, \hat{\rho}_{n}$ is $P$ ace. separably-valued. Therefore $\left\{\hat{\rho}_{n}, n \in I^{+}\right\}$is an $X$-stochastic process.
(ii) If $i \in J_{n}$, then [iv] implies that

$$
P\left(\Omega_{i}(>j) \cap \Omega_{n+1}(j)\right)=0 \quad \text { for every } j \varepsilon I^{+}
$$

And so for any $A \in B(X), j \varepsilon I^{+} \backslash\{1\}$, and $i \in J_{n}$, $[i]$,
[ii], and remark 6.1.4 imply that

$$
\begin{aligned}
\hat{\rho}_{i}^{-1}(A) \cap \Omega_{n+1}(j) & =\sum_{k=1}^{\infty} \rho_{k}^{-1}(A) \cap \Omega_{i}(k) \cap \Omega_{n+1}(j) \\
& =\sum_{k=1}^{j-1} \rho_{k}^{-1}(A) \cap \Omega_{i}(k) \cap \Omega_{n+1}(j) \text { are. } \\
& \ddots \Phi_{j-1}^{\prime}
\end{aligned}
$$

Accordingly if $C_{n}=\left\{\hat{\rho}_{i}^{-1}(A) \mid i \varepsilon J_{n}\right.$ and $\left.A \varepsilon B(X)\right\}$, then $\sigma\left(C_{n} \cap \Omega_{n+1}(j)\right) \subseteq \Phi_{j-1}^{\prime}$ for every $j \varepsilon I^{+} \backslash\{1\}$. Moreover, since clearly $\Omega \varepsilon C_{n}$, theorem 4 of the appendix implies that for each $j \varepsilon I^{+}$

$$
\sigma\left(C_{n}\right) \cap \Omega_{n+1}(j) \subseteq \sigma\left(C_{n} \cap \Omega_{n+1}(j)\right)
$$

Therefore part (ii) follows from the observation that $\sigma\left(C_{n}\right)=B\left(\hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)$.

The following theorem is an extension of theorem 2.3 chapter VII [4]. Its proof is an extension of boob's.

Theorem 6.1.6: If $\left\{\hat{\rho}_{\mathrm{n}}, \mathrm{n} \in \mathrm{I}^{+}\right\}$is a BX -stochastic process, then
(i) $\left[E\left(\hat{\rho}_{n+1} \mid \hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)\right]=[\theta]$ for every $n \varepsilon I^{+}$.
(ii) $\left\{\hat{\rho}_{n}, n \varepsilon I^{+}\right\}$is a BX-stochastic process if either of the following conditions is satisfied.
$C_{1}$ : Each $\mathrm{m}_{\mathrm{j}}$ is bounded P a.e. .
$C_{2}$ : There is a finite number $K$ such that, for each $j \varepsilon I^{+}$, $E\left(\left\|\rho_{n+1}\right\| \Phi_{n}\right) \leqq K \quad$ Pare. on $\quad \Omega_{j}(\geq n)$.

Proof:
(i) If $M$ is an element of $B\left(\hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)$, then it follows from theorem 6.1.3 that

$$
M=\sum_{j=n+2}^{\infty} M \cap \Omega_{n+1}(j) \quad \text { P a.e. . }
$$

Moreover theorem 6.1.5 (ii) implies that

$$
M \cap \Omega_{n+1}(j) \varepsilon \Phi_{j-1}^{!} \quad \text { for every } \quad j \varepsilon I^{+} \backslash J_{n+1}
$$

Accordingly [iii] implies that
$\int_{M \cap \Omega_{n+1}(j)}^{\hat{\rho}_{n+1} d P}=\quad \int_{i} \rho_{\Omega_{n+1}(j)} d P$
$=\theta$ for every $j \varepsilon I^{+} \backslash J_{n+1}$.

And so it follows from remarks 2.2 .5 (vi) and (ix) that

$$
\begin{aligned}
\int_{M} \hat{\rho}_{n+1} d P= & \int_{\rho_{n+1} d P} \\
& \sum_{j=n+2}^{\infty} M \cap \Omega_{n+1}(j)
\end{aligned}
$$

$$
=\sum_{j=n+2}^{\infty} \int_{M \cap \Omega_{n+1}(j)}^{\hat{\rho}_{n+1} d P}
$$

$=\theta$.

Therefore, since $M$ was arbitrary in $B\left(\hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)$, theorem 2.2.8 implies that

$$
\left[E\left(\hat{\rho}_{n+1} \mid \hat{\rho}_{1}, \ldots, \hat{\rho}_{n}\right)\right]=[\theta] .
$$

(ii) The hypothesis of $C_{1}$ implies that for each $n \varepsilon I^{+}$, there exists an integer $N_{n}$ such that $m_{n} \leqq N_{n}$ Pare. .

And so

$$
\begin{aligned}
E\left(\left\|\hat{\rho}_{n}\right\|\right) & =\sum_{k=1}^{\infty} \int_{\Omega_{n}(k)}\left\|\rho_{k}\right\| d P \\
& =\sum_{k=1_{\Omega_{n}}}^{N_{n}} \int_{(k)}\left\|\rho_{k}\right\| d P \\
& \leqq \sum_{k=1}^{N_{n}} E\left(\left\|\rho_{k}\right\|\right) \\
& <\infty \text { for every } n \varepsilon I^{+}
\end{aligned}
$$

Therefore theorem 6.1.5 (i) implies that $\left\{\hat{\rho}_{n}, n \varepsilon I^{+}\right\}$is a BX-stochastic process.

Under $C_{2}$, for each $n \in I^{+}$,

$$
\begin{aligned}
E\left(\left\|\hat{\rho}_{n}\right\|\right) & =\sum_{k=1}^{\infty} \int_{\Omega_{n}}(k)\left\|\rho_{k}\right\| \mathrm{dP} \\
& =\int_{\Omega_{n}(1)}\left\|\rho_{1}\right\| \mathrm{dP}+\sum_{k=2}^{\infty} \int_{\Omega_{n}(k)} E\left(\left\|\rho_{k}\right\| \| \Phi_{k-1}\right) \mathrm{dP}
\end{aligned}
$$

$$
\leqq \sum_{k=2}^{\infty} K P \Omega_{n}(k)
$$

$$
=\mathrm{K}
$$

Therefore theorem 6.1.5 (i) implies that $\left\{\hat{\rho}_{n}, n \varepsilon I^{+}\right\}$is a BX-stochastic process.

## 2. Optional Sampling

Let $X$ be a Banach space, $(\Omega, \Gamma, P)$ a complete probability space, and $T(\xi)$ and $T(\tau)$ subsets of the extended reals. The following hypotheses are now made.
[i] $\left\{\xi_{t}, t \in T(\xi)\right\}$ is an $X$-stochastic process.
[ii] For each $t \in T(\xi)$, there exists a $\sigma$-subfield $\Phi_{t}$ of $\Gamma$ with the following properties:
(a) $\Phi_{s} \subseteq \Phi_{t}$ for every $s \varepsilon T(\xi)$ such that $s \leqq t$
and

$$
\text { (b) } \quad B\left(\xi_{t}\right) \subseteq \Phi_{t}^{\prime}
$$

[iii] Almost all sample functions of the $\xi_{t}$ process have limits from the right,

$$
\xi_{t+}=\lim _{s \downarrow t} \xi_{s} \text { for every } t \in T(\xi)
$$

[iv] $\left\{\tau_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ is an R-stochastic process with the following properties:
(a) for each $\alpha \in T(\tau), \quad \tau_{\alpha}(\Omega) \subseteq T(\xi)$,
(b) $\quad \tau_{\alpha}(\omega)$ is monotone non-decreasing in $\alpha$ for fixed $\omega$,
and (c) if $\alpha \in T(\tau)$, then

$$
\left\{\omega \mid \tau_{\alpha}(\omega) \leqq s\right\} \in \Phi_{s}^{\prime} \text { for every } s \in T(\xi)
$$

Notation 6.2.1: The following shorthand notation will be used in this section.

Define for all $\alpha, \beta \in T(\tau)$ and $s, t \in T(\xi)$,

$$
\begin{aligned}
& S_{\alpha}=\left\{t \varepsilon T(\xi) \mid P\left\{\omega \mid \tau_{\alpha}(\omega)=t\right\}>0\right\} \\
& \Omega_{\alpha}\left(S_{\alpha}\right)=\left\{\omega \mid \tau_{\alpha}(\omega) \varepsilon S_{\alpha}\right\}, \\
& \Omega_{\alpha}(s)=\left\{\omega \mid \tau_{\alpha}(\omega) \leqq s\right\} \\
& \Omega_{\alpha}(s)^{\prime}=\left\{\omega \mid \tau_{\alpha}(\omega)>s\right\}, \\
& \Omega_{\alpha}(s, t)=\left\{\omega \mid s<\tau_{\alpha}(\omega) \leqq t\right\} \\
& \Omega_{\alpha, \beta}(s)=\Omega_{\alpha}(s) \cap \Omega_{\beta}(s)^{\prime} .
\end{aligned}
$$

Clearly $S_{\alpha}$ is countable for each $\alpha \varepsilon T(\tau)$.

Consider, for each $\alpha \in \mathrm{T}(\tau)$, a $\hat{\xi}_{\alpha}: \Omega \rightarrow \mathrm{X}$ satisfying

$$
\hat{\xi}_{\alpha}(\omega)=\left(\begin{array}{ll}
\xi_{\left.\tau_{\alpha}(\omega)^{( }\right)} \quad \text { if } \omega \varepsilon \Omega_{\alpha}\left(S_{\alpha}\right) \\
\xi_{\tau_{\alpha}(\omega) f^{(\omega)}} & \text { if } \quad \omega \notin \Omega_{\alpha}\left(S_{\alpha}\right) \quad \text { and if such a limit exists }
\end{array}\right\}
$$

All the functions satisfying this are P-equivalent since they can only differ from each other either on the $P$-negligible set corresponding to those sample functions which may not have limits from the right at all $\mathrm{t} \varepsilon \mathrm{T}(\xi)$, or when $\tau_{\alpha}$ takes on (with probability zero) one of the at most countably many values in $\mathrm{T}(\xi)$ which are not limit points of $\mathrm{T}(\xi)$ from
the right. Since the choice, within P-equivalence, of a function is unimportant in what follows, it will be assumed that a particular one has been chosen for each $\alpha \varepsilon T(\tau)$. This function will be denoted by $\hat{\xi}_{\alpha}$.

Definition 6.2.2: $\left\{\xi_{t}, \Phi_{t}, t \varepsilon T(\xi)\right\}$ is said to be transformed into $\left\{\hat{\xi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ by optional sampling.

Theorem 6.2.3: $\left\{\hat{\xi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ is an X-stochastic process.

Proof: Let $\alpha \varepsilon T(\tau)$. For each $q \varepsilon I^{+}$, choose finitely many points $a(1, q)<a(2, q)<\ldots$ of $T(\xi)$ in such a way that every point of $[-q, q] T(\xi)$ is within distance $1 / q$ of some $a(j, q)$, that the infinite points of $T(\xi)$, if any, are $a(j, q)$ ' $s$, and that if $S_{\alpha}$ is not empty, then the first $q$ points of $S_{\alpha}$, enumerated in some order, are $a(j, q)$ 's. Define

$$
\hat{\xi}_{\alpha, q}: \omega \varepsilon \Omega \rightarrow\left(\begin{array}{lll}
\xi_{a(1, q)}(\omega) & \text { if } \omega \varepsilon \Omega_{\alpha}(a(1, q)) \\
\xi_{\left.a(j, q)^{( }\right)} & \text {if } \omega \varepsilon \Omega_{\alpha}(a(j-1, q), a(j, q)) \text { for } j>1 \\
\theta & \text { if } \omega \varepsilon \Omega_{\alpha}(\max a(j, q))^{i}
\end{array}\right)
$$

It follows from [iv] (c) that $\hat{\xi}_{\alpha, q}$ is a finitely-valued Xr.v. . In fact it is an Xr.v. $\Phi_{\max } a(j, q)^{\circ}$

Moreover if $\omega^{\prime} \varepsilon \Omega_{\alpha}\left(S_{\alpha}\right)$, then there exists $q^{\prime} \varepsilon I^{+}$such that

$$
\begin{aligned}
\hat{\xi}_{\alpha, q}\left(\omega^{q}\right) & =\xi_{\tau_{\alpha}}\left(\omega^{\prime}\right)\left(\omega^{\prime}\right) \\
& \left.=\hat{\xi}_{\alpha}\left(\omega^{\prime}\right) \quad \text { for every } q \varepsilon I^{+}\right\rangle_{q^{\prime}} .
\end{aligned}
$$

Therefore $\lim _{q \rightarrow \infty} \hat{\xi}_{\alpha, q}=\hat{\xi}_{\alpha}$ on $\Omega_{\alpha}\left(S_{\alpha}\right)$.
Define

Let $\omega^{\prime} \varepsilon M_{\alpha}$. Then there exists $q^{\gamma} \varepsilon I^{+}$such that

$$
\tau_{\alpha}\left(\omega^{\prime}\right) \varepsilon[-q, q] \text { for every } q \varepsilon I^{+} \backslash J_{q^{\prime}}
$$

Hence there exists a sequence $\left\{a\left(j_{q}, q\right), q \varepsilon I^{+} \backslash J_{q}\right\} \quad$ such that

$$
\omega^{\prime} \varepsilon \Omega_{\alpha}\left(a\left(j_{q}-1, q\right), a\left(j_{q}, q\right)\right) \text { for every } q \varepsilon I^{+} \backslash J_{q^{\prime}}
$$

and $\lim _{q \rightarrow \infty} a\left(j_{q}, q\right)=\tau_{\alpha}\left(\omega^{\prime}\right)$.
And so $\lim _{\mathrm{n} \rightarrow \infty} \hat{\xi}_{\alpha, \mathrm{q}}\left(\omega^{\mathrm{q}}\right)=\lim _{\mathrm{q} \rightarrow \infty} \xi_{\mathrm{a}}\left(\mathrm{j}_{\mathrm{q}}, \mathrm{q}\right)^{\left(\omega^{\mathrm{q}}\right)}$

$$
=\xi_{\tau_{\alpha}}\left(\omega^{\prime}\right)+\left(\omega^{q}\right)
$$

$$
=\hat{\xi}_{\alpha}\left(\omega^{q}\right)
$$

$$
\begin{aligned}
& M_{\alpha}=\left\{\omega| | \tau_{\alpha}(\omega) \mid<\infty, \tau_{\alpha}(\omega)\right. \text { is a right } \\
& \text { limit point of } T(\xi) \text {, and } \lim _{s \downarrow t} \xi_{s}(\omega) \\
& \text { exists for every } t \in T(\xi)\} \backslash \Omega_{\alpha}\left(S_{\alpha}\right) \text {. }
\end{aligned}
$$

That is, $\lim _{q \rightarrow \infty} \hat{\xi}_{\alpha, q}=\hat{\xi}_{\alpha}$ on $M_{\alpha}$.
Therefore, since $P\left(\Omega_{\alpha}\left(S_{\alpha}\right) \cup M_{\alpha}\right)=1$, it follows that $\hat{\xi}_{\alpha}$ is an $\operatorname{SXr} \cdot \mathrm{V}$. . Therefore, since $\alpha \in T(\tau)$ was arbitrary, it follows from summary 2.1 .5 (ii) that $\left\{\hat{\xi}_{\alpha}, \alpha \in T(\tau)\right\}$ is an $X$-stochastic process.

Remark 6.2.4: The notation developed in the proof of theorem 6.2.3 will be used hereafter without comment.
$\sigma$-fields will now be constructed with respect to which the $\hat{\xi}_{\alpha, q}$ 's and the $\hat{\xi}_{\alpha}$ 's are X-measurable. The construction depends only on [ii] (a) and [iv] and so is precisely the same as Doob's on p. 367 [4]. Fix $\alpha \in T(\tau)$. For each $q \varepsilon I^{+}$, let $\Phi_{\alpha, q}$ be the $\sigma$-field generated by the $P$-negligible elements of $\Gamma$ and by those of the form

$$
\Lambda \cap\left\{\omega \mid a<\tau_{\alpha}(\omega) \leqq b\right\}
$$

where $b \in T(\xi), \quad \Lambda \varepsilon \Phi_{b}, \quad a$ is not necessarily finite, no one of the first $q$ points of $S_{\alpha}$, enumerated in some order, is an interior point of ( $a, b$, and arctan $b-\arctan a<1 / q$. If $\mathrm{b}=-\infty \varepsilon \mathrm{T}(\xi)$, then we understand by the above

$$
\Lambda \cap\left\{\omega \mid \tau_{\alpha}(\omega)=-\infty\right\} \quad \text { where } \Lambda \varepsilon \Phi_{-\infty}
$$

Clearly $\left\{_{\alpha, q}, q \in I^{+}\right\}$is a monotone non-increasing sequence of

P-complete $\sigma$-subfield of $\Gamma$. And so if $\hat{\Phi}_{\alpha}$ is defined as $\bigcap_{q} \Phi_{\alpha, q}$ then it follows from theorem 3 (iii) of the appendix that $\hat{\Phi}_{\alpha}$ is a P-complete $\sigma$-subfield of $\Gamma$.

Since $\hat{\xi}_{\alpha, q}$ is plainly an Xr.v. $\Phi_{\alpha, r}$ for large enough $q$, it follows that $\hat{\xi}_{\alpha}$ is an Xr.v. $\Phi_{\alpha_{,} r}$ for every $r$. Therefore $\hat{\xi}_{\alpha}$ is an Xr.v. $\hat{\Phi}_{\alpha}$.

Moreover it is proved on pp. 367 and 368 [4] that $\left\{\hat{\Phi}_{\alpha}, \alpha \in T(\tau)\right\}$ is a monotone non-decreasing sequence of $\sigma$-subfield of $\Gamma$. Therefore the triple

$$
\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \varepsilon \mathrm{T}(\tau)\right\}
$$

has the following properties:
(a) $\quad\left\{\hat{\Phi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ is a monotone non-decreasing sequence of P-complete $\sigma$-subfield of $\Gamma$ which have been constructed from the elements of

$$
\left\{\Phi_{t}, t \in \mathrm{~T}(\xi)\right\} \quad \text { and } \quad\left\{\tau_{\alpha}, \alpha \varepsilon \mathrm{T}(\tau)\right\}
$$

and
(b) $\quad\left\{\hat{\xi}_{\alpha}, \alpha \in T(\tau)\right\}$ is an X-stochastic process such that

$$
B\left(\hat{\xi}_{\alpha}\right) \subseteq \hat{\Phi}_{\alpha} \quad \text { for every } \quad \alpha \varepsilon T(\tau)
$$

The following theorem is of major importance since it enables us to utilise results in [4] and in so doing it allows us to refer to the classical case for several proofs. Its proof is straight-forward and so will not be given.

## Theorem 6.2.5:

(i) If for every $t \in T(\xi), \quad \xi_{t}$ is replaced by $\left\|\xi_{t}\right\|$ in [i] - [iv], then the resulting hypotheses are equivalent to $\mathrm{OS}_{1}-\mathrm{OS}_{4}$ on p. 365 [4].
(ii) $\left\{\left\|\xi_{t}\right\|, t \varepsilon T(\xi)\right\}$ is transformed into $\left\{\left\|\hat{\xi}_{\alpha}\right\|, \alpha \varepsilon T(\tau)\right\}$ by optional sampling.
(iii) $\left\{\hat{\Phi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ is identical with the corresponding class in [4].

The next theorem is simply a restatement of lemma 11.1 chapter VII [4].

Theorem 6.2.6: If $\alpha \varepsilon T(\tau), s \varepsilon T(\xi)$, and $\Lambda \varepsilon \hat{\Phi}_{\alpha}$, then

$$
\Lambda \cap \Omega_{\alpha}(s) \varepsilon \Phi_{s}^{\prime}
$$

Remark 6.2.7: In the proof of lemma 11.1 chapter VII [4], Doob only considers the $\sigma$-fields that are generated by sets of the form

$$
\Lambda=M \cap\left\{\omega \mid c_{1}<\tau_{\alpha}(\omega) \leqq c_{2}\right\}
$$

where $c \in T(\xi), \quad c \leqq c_{2}$, and $M \varepsilon \Phi_{c}$, whereas $\Phi_{\alpha, q}$ is generated not only by these sets but also by the $P$--negligible elements of $\Gamma$. Moreover if $s \in T(\xi)$ and $s \geqq c_{2}$, then [iv] (c) implies that $\Lambda \cap \Omega_{\alpha}(s) \varepsilon \Phi_{s}^{\gamma}$ not $\Phi_{s}$ as Doob claims. Therefore it is necessary to have $\Phi_{S}^{\prime}$ rather than $\Phi_{S}$ in the statement of theorem 6.2.6; however Doob does not indicate that this is so. On the other hand it seems likely that when on p. 366 [4] he states that it is no restriction to assume that the elements of $\left\{\Phi_{t}, t \varepsilon T(\xi)\right\}$ are $P$-complete he is in fact indicating that this will be assumed in what follows.

It will be assumed hereafter that

$$
\left\{\xi_{t}, \Phi_{t}, t \in T(\xi)\right\}
$$

is an $X$-martingale satisfying [i], [ii], and [iii], and that it is transformed into

$$
\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \varepsilon \mathrm{T}(\tau)\right\}
$$

by

$$
\left\{\tau_{\alpha}, \alpha \in \mathrm{T}(\tau)\right\} \text { satisfying [iv]. }
$$

Remark 6.2.8: It follows from theorem 6.2.5 (i) and theorem 3.2.7 that $\left\{\left\|\xi_{t}\right\|, \Phi_{t}, t \varepsilon T(\xi)\right\}$ is a semi-martingale of non-negative BRr.v. which satisfies $\mathrm{OS}_{1}-\mathrm{OS}_{3}$ (p.365 [4]) and which dominates (p. 297 [4]) itself.

The next theorem is an extension of lemma 11.2 chapter VII [4] in so far as it treats $\mathrm{Xr} . \mathrm{v}$. instead of $\mathrm{Rr} . \mathrm{v}$; however it follows from definition 2.2.12, theorem 6.2.5, and remark 6.2 .8 that it is an immediate consequence of Nob's result.

Theorem 6.2.9: If $\alpha \varepsilon T(\tau)$ and $s \varepsilon T(\xi)$, then $\hat{\xi}_{\alpha}$ is integrable on $\Omega_{\alpha}(s)$ and the integrability is uniform in $\alpha_{0}$ If $s$ is an $a(j, q)$ for every $q \varepsilon I^{+}$, then $\left\{\hat{\xi}_{\alpha, q}, q \varepsilon I^{+}\right\}$is uniformly integrable on $\Omega_{\alpha}(s)$, and the degree of uniformity does not depend on $\alpha$ or on the choice of the $a(j, q)^{\prime} s$.

Remark 6.2.10: If $b=1$. u.b.t $\varepsilon T(\xi)$, then we can put $s=b$ in $t \in T(\xi)$
theorem 6.2.9. It follows that if $b=1 . u . b . t \varepsilon T(\xi)$, then $t \varepsilon T(\xi)$ $\left\{\hat{\xi}_{\alpha}, \alpha \in \mathrm{T}(\tau)\right\}$ is a uniformly integrable $B X$-stochastic process.

The following theorem is an extension of lemma 11.3 chapter VII [4]. Its proof is derived from Nob's.

Theorem 6.2.11: If $\alpha, \beta \in T(\tau), \alpha \leqq \beta, \quad s \varepsilon T(\xi)$, and $\Lambda \in \hat{\Phi}_{\alpha}$, then
(i) $\int \hat{\xi}_{\alpha} d P=\int \hat{\xi}_{\beta} d P \quad+\int \xi_{s} d P$

$$
\Lambda \cap \Omega_{\alpha}(s) \quad \Lambda \cap \Omega_{\beta}(s) \quad \Lambda \cap \Omega_{\alpha, \beta}(s)
$$

and
(ii) $\int \hat{\xi}_{\alpha} d P=\int \xi_{s} d P$

$$
\Lambda \cap \Omega_{\alpha}(s) \quad \Lambda \cap \Omega_{\alpha}(B)
$$

Proof: Choose $a(j, q)$ 's to match both $\tau_{\alpha}$ and $\tau_{\beta}$, so that both $\hat{\xi}_{\alpha, q}$ and $\hat{\xi}_{\beta, q}$ are now defined for every $q \in I^{+}$, so that

$$
\lim _{q \rightarrow \infty} \hat{\xi}_{\alpha, q}=\hat{\xi}_{\alpha} \text { P ace. }
$$

and

$$
\lim _{q \rightarrow \infty} \hat{\xi}_{\beta, q}=\hat{\xi}_{\beta} P \text { a.e., }
$$

and so that for each $q$, $s$ is some $a(j, q)$. Define for each $q \in I^{+}$,

$$
\begin{aligned}
& \Lambda_{1, q}=\Lambda \cap \Omega_{\alpha}(a(1, q)), \\
& \Lambda_{j, q}=\Lambda \cap \Omega_{\alpha}(a(j-1, q), a(j, q)) \text { for } j \varepsilon I^{+} \backslash\{1\}, \\
& \Lambda_{1,1, q}=\Lambda_{1, q} \cap \Omega_{\beta}(a(1, q)), \\
& \Lambda_{j, k, q}=\Lambda_{j, q} \cap \Omega_{\beta}(a(k-1, q), a(k, q)) \\
& \quad \text { for } j \varepsilon I^{+} ; k \varepsilon I^{+}\{1\} \text { such that } k \geqq j,
\end{aligned}
$$

and

$$
\begin{aligned}
M_{j, k, q} & =\Lambda_{j, q} \cap \Omega_{\beta}(a(k, q))^{\prime} \text { for } j, k \varepsilon I^{+} \text {such that } k \geqq j \\
& =\Lambda_{j, q} \backslash \bigcup_{r \leq k} \Lambda_{j, r}, q^{\circ}
\end{aligned}
$$

Clearly the following relationships are valid for those $j$ and $m$ for which the terms are defined.

$$
\Lambda_{j, j, q}+M_{j, j, q}=\Lambda_{j, q}
$$

and

$$
\Lambda_{j, j+m, q}+M_{j, j+m, q}=M_{j, j+(m-1), q}
$$

Also [iv] (b) plainly implies that for any $t \in T(\xi)$

$$
\Omega_{\alpha}(t) \geqslant \Omega_{\beta}(t)
$$

and

$$
\Omega_{\beta, \alpha}(t)=\phi .
$$

Moreover it follows from [ii] (a) and theorem 6.2.6 that for each $q \in \mathrm{I}^{+}$,

$$
\begin{aligned}
& \Lambda_{j, q} \varepsilon \Phi_{a(j, q)}^{p} \text { for } j \varepsilon I^{+}, \\
& \Lambda_{j, k, q} \Phi_{a(k, q)}^{\prime} \text { for } j, k \in I^{+} \text {such that } k \geqq j,
\end{aligned}
$$

and

$$
M_{j, k, q} \varepsilon \Phi_{a(k, q)}^{\prime} \text { for } j, k \varepsilon I^{+} \text {such that } k \geqq j
$$

Using the martingale property of the process it follows that for each $q, \alpha$, and $j$

$$
\begin{aligned}
\int_{\Lambda_{j, q}} \hat{\xi}_{\alpha, q} d P & =\int_{\Lambda_{j, q}} \xi_{a(j, q)} d P \\
& =\int_{\Lambda_{j, j, q}} \xi_{a(j, q)} d P+\int_{M_{j, j, q}} \xi_{a(j, q)} d P
\end{aligned}
$$

$$
\begin{aligned}
&=\int_{\Lambda_{j, j, q}} \xi_{a(j, q)^{d P}}+\int_{M_{j, j, q}} \xi_{a(j+1, q)^{d P}} \\
&=\int_{\Lambda_{j, j, q}} \xi_{a(j, q)^{d P}}+\int_{\Lambda_{j, j+1, q}} \xi_{a(j+1, q)^{d P}} \\
&+\int_{M_{j, j+1, q}} \xi_{a(j+1, q)^{d P}} .
\end{aligned}
$$

It can easily be shown by induction that for any integer $N \geqq \mathbf{j}$ such that $a(N, q)$ exists that

$$
\begin{aligned}
& \int_{\Lambda_{j, q}}^{\hat{\xi}_{\alpha, q} d P}=\sum_{k=j}^{N} \int_{\Lambda_{j, k, q}} \xi_{a(k, q)} d P+\int_{M_{j, N, q}} \xi_{a(H, q)} d P \\
& =\int \hat{\xi}_{\beta, q} \mathrm{dP}+\int \xi_{\mathrm{a}(\mathrm{~N}, \mathrm{q})} \mathrm{dP} . \\
& \Lambda_{j, q} \cap \Omega_{\beta}(a(N, q)) \quad M_{j, N, q}
\end{aligned}
$$

Choosing $N$ so that $a(N, q)=s$ and summing over $j \varepsilon J_{N}$ it follows that

$$
\int_{\Lambda \cap \Omega_{\alpha}(s)} \hat{\xi}_{\alpha, q} \mathrm{dP}=\int_{\Lambda \cap \Omega_{\beta}(s)} \hat{\xi}_{\beta, q} \mathrm{dP}+\quad \int \xi_{\mathrm{s}} \mathrm{dP} .
$$

According to theorem 6.2.9 the integrands are uniformly integrable in $q$ over the indicated integration sets. And so it follows from theorem 2.2.21 that when $q \rightarrow \infty$ the above relation becomes

$$
\begin{aligned}
& \int \hat{\xi}_{\alpha} \mathrm{dP}=\int \hat{\xi}_{\beta} \mathrm{dP}+\int \xi_{\mathrm{S}} \mathrm{dP} \\
& \Lambda \cap \Omega_{\alpha}(s) \\
& \Lambda \cap \Omega_{\beta}(s) \\
& \Lambda \cap \Omega_{\alpha, \beta}(s)
\end{aligned}
$$

which is (i).

For each $q \varepsilon I^{+}$, there exists an integer $N$ such that

$$
a(N, q)=s .
$$

Thus for each $j \varepsilon J_{N}$,

$$
\begin{aligned}
& \int_{\Lambda_{j, q}} \hat{\xi}_{\alpha, q} d P=\int_{\Lambda_{j, q}} \xi_{a(j, q)} d P \\
&=\int \xi_{a(N, q)} d P \\
&=\int \xi_{j, q} d P \\
& \Lambda_{j, q}
\end{aligned}
$$

Summing over $j \varepsilon J_{N}$ it follows that

$$
\int_{\Lambda \cap \Omega_{\alpha}(s) \hat{\xi}_{\alpha, q} d P=\int_{\Lambda \cap_{\alpha}(s)} \xi_{s} d P . .}
$$

It follows from theorems 6.2 .9 and 2.2 .21 that when $q \rightarrow \infty$ the above relation becomes

$$
\int \hat{\xi}_{\alpha} d P=\int \xi_{s} d P
$$

which is (ii).
The next theorem is an extension of theorem 11.6 chapter VII [4]. Its proof is derived from Nob's.

Theorem 6.2.12: If $b=1$. u.b.t $\varepsilon T(\xi)$, then $\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ is an X -mating 1 e with

$$
E\left(\hat{\xi}_{\alpha}\right)=E\left(\xi_{t}\right) \quad \text { for every } \alpha \varepsilon T(\tau) \text { and } t \varepsilon T(\xi)
$$

Proof: It follows from theorem 6.2.11 (i) that

$$
\int_{\Lambda} \hat{\xi}_{\alpha} \mathrm{dP}=\int_{\Lambda} \hat{\xi}_{\beta} \mathrm{dP} \text { for every } \Lambda \varepsilon \hat{\Phi}_{\alpha} \text { and } \alpha>\beta
$$

since [iv] (a) implies that

$$
\Omega_{\gamma}(b)=\Omega \text { for every } \gamma \in T(\tau) \text {. }
$$

Since this relation also implies that

$$
\hat{\xi}_{\gamma} \text { is a BXr.v. } \hat{\Phi}_{\gamma} \text { for every } \gamma \varepsilon T(\tau) \text {, it is }
$$

clear that $\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \varepsilon T(\tau)\right\}$ possesses the requisite properties of an X -martingale.

It is an immediate consequence of theorem 6.2.11 (ii) that

$$
E\left(\hat{\xi}_{\alpha}\right)=E\left(\xi_{t}\right) \quad \text { for every } \alpha \varepsilon T(\tau) \text { and } t \varepsilon T(\xi)
$$

since both the $\hat{\xi}_{\alpha}$ process and the $\xi_{t}$ process are $X$-martingales.

The following theorem is the major result of the section. It demonstrates conditions under which an $X$-martingale is transformed into an $X$-martingale by optional sampling. It is an extension of theorem 11.8 chapter VII [4] except that condition $C_{4}$ corresponds to condition $C_{4}^{\prime}$ of that theorem. The proof of part (i) is derived from Boob's.

Theorem 6.2.13: Suppose that 1.u.b.t $=b \notin T(\xi)$. $t \varepsilon T(\xi)$
(i) If
(I) $\quad E\left(\left\|\hat{\xi}_{\alpha}\right\|\right)<\infty \quad$ for every $\alpha \varepsilon T(\tau)$
and it
(II) $\underset{s \rightarrow b}{\lim \inf } \int_{\Omega_{\alpha}(s)^{\prime}}\left\|\xi_{s}\right\| d P=0$ for every $\alpha \in T(\tau)$,
then $\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \in T(\tau)\right\}$ is an $X$-martingale with
$E\left(\hat{\xi}_{\alpha}\right)=E\left(\xi_{t}\right)$ for every a $\varepsilon T(\tau)$ and $t \varepsilon T(\xi)$.
(ii) Each of the following conditions implies the validity of (I) and (II).
$C_{1}:\left\{\xi_{t}, t \in T(\xi)\right\}$ is uniformly integrable.
$\mathcal{C}_{2}:$ Each $\tau_{\alpha}$ is $P$ ace. bounded from above by a value in $T(\xi)$.
$C_{3}$ : There is a constant $K>0$ with the following properties.
$T(\xi)$ contains the integers $\geqq K$ (but not $b=\infty$ ). For each integer $\mathfrak{n} \geqq K$ and each $\alpha \varepsilon T(\tau)$,

$$
E\left(\left\|\xi_{n+1}\right\|-\left\|\xi_{n}\right\| \Phi_{n}\right)<\infty \quad \text { P ace. on } \quad \Omega_{\alpha}(n)^{\prime}
$$

Moreover $E\left(\left|\tau_{\alpha}\right|+\tau_{\alpha}\right)<\infty \quad$ for every $\alpha \varepsilon T(\tau)$.
$C_{4}$ : (I) is valid and there are a nonnegative $B R$ rv. $z$ and $a$ sequence $t_{1}<t_{2}<\ldots$ such that

$$
t_{n} \in T(\xi) \quad \text { for every } n \varepsilon I^{+}, \quad t_{n} \rightarrow b
$$

and

$$
\begin{aligned}
& P\left\{\omega\left|\left\|\xi_{t}(\omega)\right\| \geqq| | \xi_{t_{n}}(\omega) \|-z(\omega)\right\}=1\right. \\
& \text { for every } t \geqq t_{n} \text { and } n \varepsilon I^{+} .
\end{aligned}
$$

Proof:
(i) If $s \varepsilon T(\xi), \quad \alpha, \beta \varepsilon T(\tau), \alpha>\beta, \quad$ and $\Lambda \varepsilon \hat{\Phi}_{\alpha}$, then theorem 6.2.11 (i) implies that
(III)

$$
\begin{array}{cc}
\int \hat{\xi}_{\alpha} d P=\int \cap_{\Omega_{\alpha}(s)} \hat{\xi}_{\beta} d P+\quad \cap_{\Omega_{\beta}(s)} \quad \xi_{s} d P \cdot
\end{array}
$$

$\mathrm{b} \not \ddagger \mathrm{T}(\xi)$ implies that

$$
P_{\alpha}(s)^{i} \rightarrow 0 \text { as } s \rightarrow b
$$

and

$$
P \Omega_{\beta}(s)^{\prime} \rightarrow 0 \text { as } s \rightarrow b .
$$

Also, since (I) implies that $\hat{\xi}_{\alpha}$ and $\hat{\xi}_{\beta}$ are $B X r . v_{0}$, it follows from remark 2.2 .5 ( $x$ ) that the set functions $\int_{M} \hat{\xi}_{\alpha} d P$ and $\int_{M} \hat{\xi}_{\beta} d P$ are strongly absolutely continuous on $\Gamma$. Therefore it follows from remark 2.2 .5 (ix) that

$$
\begin{aligned}
& \lim _{s \rightarrow b} \int \hat{\xi}_{\alpha} d p=\int_{\alpha} \hat{\xi}_{\alpha} d p \\
& \quad \Lambda \Omega_{\alpha}(s)
\end{aligned}
$$

and

$$
\lim _{s \rightarrow b} \int_{\Lambda \cap \Omega_{\beta}(s)} \hat{\xi}_{\beta} d P=\int_{\alpha} \hat{\xi}_{\beta} d P
$$

And so it follows from (III) and (II) that

$$
\begin{aligned}
& \left\|\int_{\alpha} \hat{\xi}_{\alpha} \mathrm{dP}-\int_{\Lambda} \hat{\xi}_{\beta} \mathrm{dP}\right\|=\lim _{\mathrm{s} \rightarrow \mathrm{~b}}| | \int \hat{\xi}_{\alpha} \mathrm{dP} \quad-\quad \int \hat{\xi}_{\beta} \mathrm{dP} \|| | \begin{array}{l}
\Lambda \Omega_{\alpha}(\mathrm{s})
\end{array} \\
& \begin{array}{r}
=\underset{s \rightarrow b}{\liminf \|} \mid \int \hat{\xi}_{\alpha} \mathrm{dP} \\
\Lambda \cap \Omega_{\alpha}(\mathrm{s})
\end{array} \\
& =\underset{s \rightarrow b}{\lim \inf \|}\left\|\int \xi_{s} d P\right\| \\
& \Lambda \uparrow \Omega_{\alpha, \beta}(s) \\
& =0 .
\end{aligned}
$$

Hence $\int_{\alpha}^{\hat{\xi}_{\alpha}} \mathrm{dP}=\int_{\alpha}^{\hat{\xi}_{\beta}} \mathrm{dP}$ for every $\Lambda \varepsilon \hat{\Phi}_{\alpha}$ and $\alpha, \beta \in \mathrm{T}(\xi)$ such that $\alpha<\beta$.

It follows that $\left\{\hat{\xi}_{\alpha}, \hat{\Phi}_{\alpha}, \alpha \in T(\tau)\right\}$ possesses the requisite properties of an X -martingale.

Also theorem 6.2.11 (ii) implies that

$$
\begin{aligned}
& \int \hat{\xi}_{\alpha} \mathrm{dP}=\int \xi_{\mathrm{s}} \mathrm{dP} \text { for every } \mathrm{s} \varepsilon \mathrm{~T}(\xi) \text { and } \alpha \varepsilon \mathrm{T}(\tau) . \\
& \Omega_{\alpha}(\mathrm{s}) \\
& \Omega_{\alpha}(\mathrm{s})
\end{aligned}
$$

Accordingly

$$
\begin{aligned}
\left\|E\left(\hat{\xi}_{\alpha}\right)-E\left(\xi_{s}\right)\right\| & =\| \int \hat{\xi}_{\alpha} d P- \\
\Omega_{\alpha}(s)^{\prime} & \int \xi_{s} d P \| \\
& \Omega_{\alpha}(s)^{\prime} \\
& \leqq \int_{\alpha} \hat{\xi}_{\alpha} d P\|+\| \int \xi_{s} d P \| \\
\Omega_{\alpha}(s)^{\prime} & \Omega_{\alpha}(s)^{\prime}
\end{aligned}
$$

Clearly

$$
\begin{aligned}
& \lim _{\mathrm{s} \rightarrow \mathrm{~b}}| | \int \hat{\xi}_{\alpha} \mathrm{dP}| |=0 \\
& \Omega_{\alpha}(\mathrm{s})^{\prime}
\end{aligned}
$$

and

$$
\begin{gathered}
\lim \inf \left|\left|\int \xi_{\mathrm{s}} \mathrm{dP}\right|\right|=0 \\
\Omega_{\alpha}(\mathrm{s})^{\prime}
\end{gathered}
$$

Also plainly $\left\|E\left(\hat{\xi}_{\alpha}\right)-E\left(\xi_{s}\right)\right\|$ is a constant for every $s \varepsilon T(\xi)$ since the $\xi_{s}$ process is an $X$-martingale.

Accordingly

$$
\left\|E\left(\hat{\xi}_{\alpha}\right)-E\left(\xi_{s}\right)\right\|=0 \text { for every } s \varepsilon T(\xi) .
$$

Therefore $E\left(\hat{\xi}_{\alpha}\right)=E\left(\xi_{s}\right)$ for every $\alpha \varepsilon T(\tau)$ and $s \varepsilon T(\xi)$.
(ii) It follows from remark 6.2.8 that $C_{1}, C_{2}$, and $C_{3}$ are implied by the corresponding conditions of theorem 11.8 chapter VII [4], and that condition $C_{4}$ is implied by condition $C_{4}$ of theorem 11.8 chapter VII [4].

## Appendix

This appendix contains various results concerning BRr.v., $\sigma$-fields, and separable sets. These results have been collected here because they are required in the thesis, but are independent of its development.

Although several of these results are well-known, in different forms, it seemed desirable to restate and prove them here in the terminology of this thesis. The statements or proofs of the remainder are not contained in those references with which I am familiar. ( $\Omega, \Gamma, P$ ) denotes a complete probability space.

The first theorem of this chapter was derived from theorem $E$ § 25
Halmos [7].

Theorem Al: Let $\Phi$ be a $\sigma$-subfield of $\Gamma$ and let $f$ be a BRr.v. If

$$
\int_{n} \mathrm{fdP}=0 \text { for every } \Lambda \varepsilon \Phi,
$$

then

$$
f=0 P_{\Phi} \text { a.e. }
$$

Proof: Clearly $\Lambda_{1}=\{\omega \mid f(\omega)>0\} \varepsilon \Phi$
and

$$
\Lambda_{2}=\{\omega \mid f(\omega)<0\} \varepsilon \Phi
$$

and so

$$
\int_{\Lambda_{1}} \mathrm{fdP}=0
$$

and

$$
\int_{\Lambda_{2}}(-f) d P=0
$$

Accordingly it follows from theorem $D$ §25[7] that $\Lambda_{1}$ and $\Lambda_{2}$ are P-negligible. Therefore

$$
\mathrm{f}=0 \quad \mathrm{P}_{\Phi} \text { ane. . }
$$

Theorem A2: Let $\Phi$ be a $\alpha$-subfield of $\Gamma$ and let $f$ and $g$ be BR r.v. $\Phi$. If

$$
\int_{\Lambda} \mathrm{fdP} \geqslant\left|\int_{\Lambda} g \mathrm{gdP}\right| \text { for every } \Lambda \varepsilon \Phi,
$$

then

$$
f \geqq|g| P_{\Phi} \text { a.e. }
$$

Proof: Clearly the hypothesis of the theorem implies that

$$
\int_{\Lambda}(f-g) d P \geqq 0 \quad \text { for every } \Lambda \varepsilon \Phi
$$

and

$$
\int(f+g) \mathrm{dP} \geqq 0 \quad \text { for every } \Lambda \varepsilon \Phi .
$$

Accordingly it follows that

$$
\{\omega \mid f(\omega)<g(\omega)\} \text { and }\{\omega \mid f(\omega)<-g(\omega)\}
$$

are P-negligible elements of $\Phi$. Therefore $f \geqq|g| \quad P_{\Phi}$ ane..

Theorem A3: Let $\left\{\Phi_{t}, t \varepsilon T\right\}$ be a family of $\sigma$-subfield of $\Gamma$ with linear parameter set $T$ such that

$$
\Phi_{s} \subseteq \Phi_{t} \text { for all } s, t \varepsilon T \text { such that } s \leqq t
$$

Suppose that $a=$ g.l.b.t $\notin T$ and $b=1 . u . b . t \notin T$. $t \varepsilon T$ teT
(i) If $\left\{s_{n}, n \varepsilon I^{+}\right\}$is any sequence in $T$ which converges to $a$, then

$$
\bigcap_{t \in T} \Phi_{t}=\bigcap_{n=1}^{\infty} \Phi_{S_{n}}
$$

(ii) If $\left\{s_{n}, n \varepsilon I^{+}\right\}$is any sequence in $T$ which converges to b, then

$$
\bigcup_{t \in T} \Phi_{t}=\bigcup_{n=1}^{\infty} \Phi_{S_{n}}
$$

Therefore $\sigma\left(\bigcup_{t \in T} \Phi_{t}\right)=\sigma\left(\bigcup_{n=1}^{\infty} \Phi_{S_{n}}\right)$.

$$
\text { (iii) } \bigcap_{t \in I} \Phi_{t}^{\prime}=\left(\bigcap_{t \in T} \Phi_{t}\right)^{\beta}
$$

$$
\text { (iv) } \quad \sigma\left(\bigcup_{t \in T} \Phi_{t}^{\prime}\right)=\left(\sigma\left(\bigcup_{t \in T} \Phi_{t}\right)\right)^{\prime}
$$

Proof: The proofs of statements (i) and (ii) are straightforward and so will not be given.
(iii) Clearly $\bigcap_{t \in T} \Phi_{t}^{\prime} \geq\left(\bigcap_{t \in T} \Phi_{t}\right)^{\prime}$
and so only the opposite inclusion will be demonstrated. Let $\left\{s_{n}\right\}$ be a monotone decreasing sequence in $T$ which converges to $a$. It follows from part (i) that it is sufficient to prove that

$$
\bigcap_{n=1}^{\infty} \Phi_{S_{n}}^{\prime} \subseteq\left(\bigcap_{n=1}^{\infty} \Phi_{S_{n}}\right)^{\prime}
$$

If $\Lambda$ is any element of $\bigcap_{n=1}^{\infty} \Phi_{S_{n}}$, then, for each $n \varepsilon I^{+}$there exists and element $\Lambda_{n}$ of $\Phi_{S_{n}}$ such that

$$
P\left(\Lambda_{n} \Delta \Lambda\right)=0
$$

For each $n \varepsilon I^{+}$, define

$$
M_{n}=\bigcup_{m=n}^{\infty} \bigcap_{k=m}^{\infty} \Lambda_{k}
$$

Clearly $M_{n} \varepsilon \Phi_{S_{n}}$ for every $n \varepsilon I^{+}$, and $M_{m}=M_{n}$ for all mon $\varepsilon I^{+}$.
Therefore $M_{1} \prod_{n=1}^{\infty} \Phi_{s_{n}}$. Moreover, since

$$
\begin{gathered}
M_{1} \Delta \Lambda=\left[\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty}\left(\Lambda_{k} \backslash \Lambda\right)\right] \cup\left[\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty}\left(\Lambda \backslash \Lambda_{k}\right)\right], \\
P\left(M_{1} \Delta \Lambda\right)=0 .
\end{gathered}
$$

Hence $\Lambda \varepsilon\left(\bigcap_{n=1}^{\infty} \Phi_{s_{n}}\right)^{\prime \prime}$. Therefore

$$
\bigcap_{n=1}^{\infty} \Phi_{S_{n}} \subseteq\left(\bigcap_{n=1}^{\infty} \Phi_{S_{n}}\right)^{\prime}
$$

(iv) Clearly $\sigma\left(\bigcup_{t \in T} \Phi_{t}^{p}\right)=\left(\sigma\left(\bigcup_{t \in T} \Phi_{t}^{\prime}\right)\right)^{p}$
and so

$$
\sigma\left(\bigcup_{t \in T} \Phi_{t}^{\prime}\right) \supseteq\left(\sigma\left(\bigcup_{t \in T} \Phi_{t}\right)\right)^{\prime}
$$

Moreover, plainly

$$
\left(\sigma\left(\bigcup_{t \varepsilon T} \Phi_{t}\right)\right)^{\prime} \supseteq \bigcup_{t \varepsilon T} \Phi_{t}^{\prime}
$$

Accordingly it follows that

$$
\left(\sigma\left(\bigcup_{t \in T} \Phi_{t}\right)\right)^{\prime} \supseteq \sigma\left(\bigcup_{t \in T} \Phi_{t}^{\prime}\right) .
$$

Theorem A4: Let $W$ be an arbitrary non-empty space. If $A$ is a subset of $W$ and $C$ is a nonempty class of subsets of $W$, then

$$
\text { (i) } \quad \sigma(C) \cap A=\sigma_{A}(C \cap A)
$$

and

$$
\begin{equation*}
\sigma(C \cap A) \supseteq \sigma_{A}(C \cap A) \quad \text { if, and only if, } \quad A \varepsilon \sigma(C \cap A) \tag{ii}
\end{equation*}
$$

There is equality in (ii) if, and only if, $A=W$.

## Proof:

(i) Clearly $\sigma(C) \cap A$ is a $\sigma$-field relative to $A$ which contains $C \cap A$. Therefore $\sigma(C) \cap A \geq \sigma_{A}(C \cap A)$. Define $D$ as the class of subsets of $W$ such that $D \varepsilon D$ implies that

$$
D \cap A \varepsilon \sigma_{A}(C \cap A)
$$

Plainly $D$ is a $\sigma$-field which contains $C$. Hence $D \underline{(C) \text {. }}$ Therefore $\sigma(C) \cap A \subseteq D \cap A$

$$
\subseteq \sigma_{A}(C \cap A)
$$

(ii) The necessity of the condition is trivial and so is the necessary and sufficient condition for equality. So it remains to be shown that if $A \varepsilon \sigma(C \cap A)$, then

$$
\sigma(C \cap A) \geq \sigma_{A}(C \cap A)
$$

Part (i) implies that

$$
\begin{aligned}
\sigma(C \cap A) \cap A & =\sigma_{A}((C \cap A) \cap A) \\
& =\sigma_{A}(C \cap A)
\end{aligned}
$$

Moreover $A \varepsilon \sigma(C \cap A)$ implies that

$$
\sigma(C \cap A) \geq \sigma(C \cap A) \cap A .
$$

Therefore $\sigma(C \cap A) \geqq \sigma_{A}(C \cap A)$.

Theorem A5: Let $X$ be a Banach space and $Y$ a subset of $X$. Then $Y$ is separable if, and only if, $\overline{\mathrm{Y}}$ is separable.

Proof: The necessity is trivial. If $\overline{\mathrm{Y}}$ is separable, then there exists a countable subset $Z$ of $\bar{Y}$ such that

$$
\bar{Z}=\bar{Y} .
$$

Clearly if $Z \leq Y$, then $Y$ is separable and so the contrary will be assumed. Puit $Z \backslash Y$ into a sequence, $\left\{z_{n}\right\}$ say. For each $n \varepsilon I^{+}, z_{n} \varepsilon \bar{Y} \backslash Y$ and so for each pair of natural numbers ( $n, m$ ) there exists an element $v_{n, m}$ of $Y$ such that $\left\|z_{n}-v_{n, m}\right\|<\frac{1}{m}$. Let $V=(Z \cap Y) U \underset{n ; m}{\bigcup}\left\{v_{n, m}\right\}$. Plainly $V$ is a countable subset of $Y$ such that $\bar{V}=\bar{Y}$. Therefore $Y$ is separable.

Theorem A6: Let $X$ be a Banach space and let $F_{1}$ and $F_{2}$ be closed sets in $X$. If $F_{2} \subseteq F_{1}$ and if $F_{1}$ is separable, then $F_{2}$ is separable.

Proof: Since $F_{1}$ is separable, there exists a countable subset $Z$ of $F_{1}$ such that $\bar{Z}=F_{1}$. Put $Z$ into a sequence, $\left\{z_{n}\right\}$ say. Then for every pair of natural numbers $(n, m)$ such that $F_{2} \cap S_{1 / m}\left(z_{n}\right) \neq \phi$ choose an element $v_{n, m}$ of this intersection. Let $V$ be the set of all $\mathrm{v}_{\mathrm{n}, \mathrm{m}}$ 's which are defined. Clearly V is a countable subset of $F_{2}$ such that $\overline{\mathrm{V}}=\mathrm{F}_{2}$. Therefore $\mathrm{F}_{2}$ is separable.

The next theorem is a corollary of theorems 5 and 6 of the appendix.

Theorem A7: Let $X$ be a Banach space. Then a subset of a separable subset of $X$ is separable.

The proof of the next theorem is straight-forward and so will not be given.

Theorem A8: Let $X$ be a Banach space and $\left\{Y_{n}, n \in I^{+}\right\}$a family of separable subsets of $X$. If $\left\{Z_{n}, n \varepsilon I^{+}\right\}$is a family of countable subsets of $X$ such that, for each $n \varepsilon I^{+}$,

$$
Z_{n} \subseteq Y_{n}
$$

and

$$
\overline{\mathrm{Z}}_{\mathrm{n}}=\overline{\mathrm{Y}}_{\mathrm{n}},
$$

then

$$
\overline{\bigcup_{n} Z_{n}}=\overline{\bigcup_{n} Y_{n}}=\overline{\bigcup_{n} \bar{Y}_{n}}
$$

Moreover, since such a $\left\{Z_{n}, n \varepsilon I^{+}\right\}$always exists, $\bigcup_{n} Y_{n}$ and $\bigcup_{\mathrm{n}} \overline{\mathrm{Y}}_{\mathrm{n}}$ are always separable.

The proof of the next theorem was derived from that of theorem 2.8 .5 [9].

Theorem A9: Let $X$ be a Banach space and $Y$ a countable subset of $X$. Then there exists

$$
\left\{\xi_{\mathrm{n}}^{*}, \mathrm{n} \varepsilon \mathrm{I}^{+}\right\} \subseteq \mathrm{X}^{*}
$$

such that if $x \in \bar{Y}$, then

$$
\|x\| \underset{n \in I^{+}}{\operatorname{In}}\left|\left|<x, \xi_{n}^{*}>\right| .\right.
$$

Proof: Put $Y$ into a sequence, $\left\{y_{n}, n \in I^{+}\right\}$say. By theorem 2.7.4 [9], for each $n \varepsilon I^{+}$, there exists $\xi_{n}^{*} \varepsilon X *$ such that

$$
\left\langle y_{n^{\prime}} \xi_{n}^{*}\right\rangle=\left\|y_{n}\right\|
$$

and

$$
\left\|\xi_{n}^{*}\right\|=1
$$

Let $x$ be any element of $\overline{\mathrm{Y}}$. Then given $\varepsilon>0$ there exists $\mathrm{m} \varepsilon \mathrm{I}^{+}$ such that

$$
\varepsilon>\left|\left|x-y_{m}\|\geqq\| x\right|\right|-| | y_{m}\| \|
$$

That is, $\quad\left\|y_{m}\right\|>\|x\|-E$.
Accordingly

Since $\left|\left|x \| \geqslant\left|\left\langle x, \xi{ }_{n}^{*}\right\rangle\right|\right.\right.$ for every $n \in I^{+}$, it follows that

$$
\|x\|=\underset{n \in I^{+}}{1 . u . b .} \mid\left\langle x, \xi_{\mathrm{n}}^{*}\right\rangle .
$$

Let X be a Banach space and let its norm topology be denoted by ${ }^{\tau}{ }_{1}$. If $L$ is a closed linear manifold in $X$ (p. 36 [6]), then, since clearly its relative topology is identical with its norm topology, ${ }^{\tau} 2$ say, ( $L, \tau_{2}$ ) is a Banach space.

Definition A10: ( $L, \tau_{2}$ ) will be called a B-subspace of $X$.

Remark All:
(i) Clearly a subset $A$ of $L$ is separable in ( $L, \tau_{2}$ ) if, and only if, $A$ is separable in ( $X, \tau_{1}$ ).
(ii) If $L$ is determined by a countable set, then $L$ is separable (II.1.4 [6] and II.1.5 [6]).

Definition A12: $B(X) \cap L$ will be called the relative Borel field of $\left(L, \tau_{2}\right)$.

Theorem A13: The Borel field of ( $L, \tau_{2}$ ) is identical with the relative Borel field of $\left(\mathrm{L}, \tau_{2}\right)$. That is,

$$
\sigma_{L}\left(\tau_{2}\right)=B(X) \cap L
$$

Proof: Since the relative topology of $L$ is identical with $\tau_{2}$, it follows that

$$
\tau_{2}=\tau_{1} \cap \mathrm{I}_{1}
$$

Therefore, since by notation 1.2(iii)

$$
B(X)=\sigma\left(\tau_{1}\right),
$$

theorem 4 (i) of the appendix implies that

$$
\begin{aligned}
\sigma_{L}\left(\tau_{2}\right) & =\sigma_{L}\left(\tau_{1} \cap \mathrm{~L}\right) \\
& =\sigma\left(\tau_{1}\right) \cap \mathrm{L} \\
& =B(\mathrm{X}) \cap \mathrm{L}
\end{aligned}
$$

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