MINIMAL GROUP PRESENTATIONS:

A COMPUTATIONAL APPROACH

by

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STATEMENT

The work contained in this thesis is my own except where otherwise stated.

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Abstract

The method of describing a group by means of generators and relations is an old one. A question which arises when using this method is what is the minimum number of relations required to describe a given group. Schur (1907) provided a lower bound for the number of relations required, in terms of a group invariant known as the Schur multiplicator. It would be interesting to know which finite groups have a presentation achieving the Schur bound on the number of relations required, but this remains an open question. It is known that the Schur bound is not achievable for some finite groups.

We consider the problem of finding a minimal presentations for a number of finite groups. In Chapter Three and Appendix A we give minimal presentations for the groups of order less than or equal to 84 and minimal presentations for some families of groups having composition length less than five and order greater than 84. Some of the techniques for working with finitely presented groups are illustrated by proving that the groups defined by some families of deficiency zero presentations are finite.

In Chapter Four, we give presentations for several finite groups having soluble length five and six, and deficiency zero presentations for two infinite families of finite groups, one family having soluble length six, and the other having soluble length five; we also give a deficiency one presentation for a finite preimage of a group having soluble length seven.

Finally, in Chapter Five we give some minimal presentations for some quasi-simple groups.

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Chapter One

Introduction and Background

Coxeter and Moser (1980) in the preface to *Generators and relations for discrete groups* state "When we began to consider the scope of this book, we envisaged a catalogue supplying at least one abstract definition for any finitely-generated group that the reader might propose. But we soon realized that more or less arbitrary restrictions are necessary, because interesting groups are so numerous." We will consider the problem of constructing a finite presentation for a given group (or family of groups) having the property that the number of defining relations is a minimum.

Let X be a set, let F_X be the free group freely generated by X, and let R be a set of words over $X \cup X^{-1}$. The normal closure of R in F_X will be denoted by $\overline{R}$.

The pair $(X;R)$ determine a group G, namely $F_X/\overline{R}$, and we will say that G is *presented* by the pair $(X;R)$ or that G has a *presentation* $(X;R)$. We will write $G = (X;R)$ to indicate that G is the group associated with the presentation $(X;R)$. If X is finite, G is *finitely generated*; if both X and R are finite, G is *finitely presented.* Every finite group is finitely presented. We will only be concerned with finitely presented groups. The elements of R are called *relators.* We will also describe a presentation by giving a set of equations of the form $u_1 = v_1, \ldots, u_n = v_n$, where the u_i, v_i are words over $X \cup X^{-1}$. This description corresponds to the presentation $(X; \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\})$. Equations of this form are called *relations* and we will also mix the two notations. In giving a presentation, we will often specify only the set of relators or relations. In such cases, the generating set X consists of the symbols occurring in the elements of R; we will not be concerned with the cases when a generator does not occur in any element of R.

The *deficiency of a presentation* is $|R| - |X|$. (Note that some authors take the negative of this to be the definition of deficiency.) The *deficiency of a group* is then defined as

$$\operatorname{def}(G) = \min_P(\operatorname{def}(P)),$$

where P ranges over all the presentations for G. A *minimal* presentation $(X;R)$ for a group G is one having deficiency equal to $\operatorname{def}(G)$.

If a group G is associated with a presentation having negative deficiency, then clearly the quotient of G by the derived group is infinite, and hence G is infinite. An interesting boundary case occurs when def(G) is zero. Schur (1907) provided a lower bound on the number of relations required to present a finite group, and hence a lower bound on the deficiency of a finite group. He introduced an important group invariant, $M(G)$, of a finite group G which has become known as the *Schur multiplicator*. Suppose that G is presented by $(X;R)$. We define $M(G)$ as

$$M(G) = (F_X' \cap \overline{R})/[F_X, \overline{R}],$$

where F_X' is the derived subgroup of F_X. (Here $[F_X, \overline{R}]$ is the group generated by elements of the form $[f,r] = f^{-1}r^{-1}fr$, where $f \in F_X$ and $r \in \overline{R}$.) Schur showed that if G is finite, then $M(G)$ is independent of the presentation used to define G, and that if G is finite and presented by $(X;R)$, then $|R| \geq |X| + \text{rank}(M(G))$. We will say that a finite group G is *Schur-efficient* (or has a *Schur-efficient presentation*) if $G = (X;R)$ and $|R| = |X| + \text{rank}(M(G))$. Swan (1965) showed that not all finite groups are Schur-efficient. A lower bound on the number of relations required to define a finite group $G = (X;R)$ is $\max(n(\overline{R}/\overline{R}'\overline{R}^p))$, where p ranges over all the primes dividing $|G|$, and $n(M)$ is the number of generators for the module M (see Gruenberg (1976)). It is still an open question whether this bound is achieved for all finite groups.

Deficiency zero presentations for finite groups having trivial Schur multiplicator, and, more generally, minimal presentations for groups have long been sought (see Neumann (1955)), and the problem of determining which finite groups are Schur-efficient is far from solved. Both Schur-efficient and minimal presentations are known for a small number of families of groups together with some individual results.

It is easy to see, as follows, that finite abelian groups are Schur-efficient. Let K be a finite abelian group. K is isomorphic to a direct product of cyclic groups

$$K \simeq \prod_{i=1}^{n} C_{d_i}$$

where the d_i are integers such that $d_1|d_2|\ldots|d_n$, and C_j is a cyclic group of order j. There are elements $z_1, z_2, \ldots, z_n$ of K which generate K and satisfy the relations

$$\begin{aligned} 1 &= [z_i, z_j] \ (1 \leq i < j \leq n), \\ &= z_1^{d_1} = z_2^{d_2} = \ldots = z_n^{d_n}. \end{aligned}$$

To see that this is a Schur-efficient presentation, we make use of the Schur-Künneth formula for the multiplicator of a direct product of two finite groups (see Beyl and Tappe (1982))

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H),$$

where $G \otimes H$ is the tensor product of G and H and $G \otimes H = G/G' \otimes H/H'$. For abelian groups A, B, C, $(A \times B) \otimes C = (A \otimes C) \times (B \otimes C)$ and for cyclic groups C_a, C_b of orders a and b,

$$C_a \otimes C_b = C_{(a,b)},$$

where (a, b) is the greatest common divisor of a and b. An inductive argument shows that

$$M(K) = \prod_{i=2}^{n} C_{d_{n-i}}^{(i-1)}.$$

(Here $C_k^{(n)}$ denotes the direct product of n copies of C_k.) This result was first proved by Schur (1907), and may also be found in Wiegold (1982). The rank of $M(K)$ is $n(n-1)/2$, so the presentation given above is a Schur-efficient one.

Neumann (1955) has shown that all groups of square-free order are Schur-efficient. This is a special case of a result showing that all metacyclic groups are Schur-efficient. A group G is *metacyclic* if it has a normal subgroup N such that both N and G/N are cyclic. G is generated by two elements x and y which satisfy the relations

$$x^m = 1, \; x^y = x^r, \; y^n = x^s,$$

where $r^n \equiv 1 \pmod{m}$ and $rs \equiv s \pmod{m}$. These relations show that the rank of the Schur multiplicator is at most one. Wamsley (1970) and Beyl (1973) have determined the Schur multiplicator for metacyclic groups. Beyl (1973) showed that

$$h = (m, r-1)(m, 1 + r + \ldots + r^{n-1})/m$$

is an integer, and that there is an integer l such that $s = lm/(m, r-1)$. The order of the Schur multiplicator is then (l, h). In the case that G is metacyclic and has trivial Schur multiplicator, they have shown that G has a Schur-efficient presentation with defining relations

$$y^n = x^s, \; [y, x^{-t}] = x^{(m,r-1)},$$

where $r^n \equiv 1 \pmod{m}$ and $s = m/(m, r-1)$. There are integers u and v such that $(m, r-1) = u(r-1) + vm$. Let w be the largest factor of m coprime to u, and let $t = u + ws$.

It is not known whether all finite nilpotent groups are Schur-efficient. The groups of order 2^n for $n \leq 6$ (Sag and Wamsley (1973)) are known to be Schur-efficient. Keane (1976) has given a list of Schur-efficient presentations for the groups of order $3^n, n \leq 6$. However, this list is incomplete. Robertson (1980) and Wiegold (1989) give families of two generator finite nilpotent groups having deficiency zero and unbounded nilpotency class. Johnson and Robertson (1978) in their survey of finite groups of deficiency zero give more examples of finite nilpotent groups having deficiency zero. All nilpotent groups known to them at that time to have deficiency zero have soluble length less than or equal to four. Havas and Newman (1983) give a number of examples of finite 2-groups (non-trivially) generated by four elements having five defining relations. The Golod-Šafarevič Theorem (in Johnson (1990, p.186)) states

Theorem. *If $G = (X; R)$ is a finite p-group with $|X| = d, |R| = r$ and d is minimal, then $r > d^2/4$.*

This result shows that the Havas-Newman result is the best possible for finite p-groups. No examples are known of finite groups having four generators and four defining relations. It is conjectured that no such examples exist.

Examples of infinite families of finite soluble non-nilpotent groups having Schur-efficient presentations are known, for example, see Johnson and Robertson (1978) and Kenne (1990). Johnson and Roberston (1978) observed that the derived length of all known finite soluble groups having deficiency zero was less than or equal to four, and asked whether the derived length of a finite soluble group having deficiency zero was bounded. This question remains unanswered. However, examples of finite soluble groups with deficiency zero having derived length five and six have been given by Kenne (1988, 1990), Newman and O'Brien (19xx) and A.Wegner (Newman, private communication). Further examples of such groups are given in Chapter Four.

A group G is *semi-simple* if it is perfect and $G/Z(G)$ is a direct product of nonabelian simple groups. A number of results about minimal or Schur-efficient presentations for semi-simple groups are known. Zassenhaus (1969) gave an Schur-efficient presentation for $PSL(2,p)$ for p a prime greater than 3. This result was extended by Sunday (1972), who proved

Theorem. *For p a prime greater than 3, $PSL(2,p)$ is generated by elements a and b satisfying the relations*

$$a^p = 1, b^2 = (ab)^3,$$
$$1 = (a^t b a^4 b)^2,$$

where $2t \equiv 1 \pmod p$.

The results of Zassenhaus and Sunday were incomplete, and have been corrected by Beyl (1986).

Campbell and Roberston (1980a) showed that $SL(2,p)$ is Schur-efficient by showing that $SL(2,p)$ is generated by elements x and y satisfying

$$x^2 = (xy)^3,$$
$$1 = (xy^4 xy^t)^2 y^p x^{2k},$$

where $t = (p+1)/2$ and k is the integer part of $p/3$.

In the case of the simple groups other than $PSL(2,p^n)$, p a prime, very much less is known. All the simple groups of order less than one million (with the exception of $PSL(3,5)$ and $PSp(4,4)$) are known to be efficient. See Campbell and Robertson (1982a,1984b), Kenne (1986), Jamali (1988), Campbell, Robertson and Williams (1989) and Jamali and Robertson (1989) for details of these presentations.

The other general result related to finite simple groups concerns direct powers. In the case of a direct power of a perfect group, the multiplicator of the power is simply the corresponding direct power of the multiplicator of the group. Wiegold has asked whether $\text{def}(G^n) \to \infty$ as $n \to \infty$ and suggested examining the deficiency of $PSL(2,5)^2$ and $SL(2,5)^2$ as an initial approach to the problem. Kenne (1983) showed that $PSL(2,5)^2$ is Schur-efficient. Campbell *et al.* (1986) showed that $SL(2,5)^2$ is Schur-efficient by giving a zero deficiency presentation for it. The only general result in this area is due to Campbell, Robertson and Williams (1990b):

Theorem. *For p a prime greater than 3, $G = PSL(2,p) \times PSL(2,p)$ is Schur-*

efficient. G is generated by a and b with defining relations

$$\begin{aligned} a^{3p} &= (ba^{-1})^2, \\ b^p &= (ab^{(p-1)/2}a^{-1}b^{-4})^2, \\ a^p &= (ba^{(p-1)/2}b^{-1}a^{-4})^2, \\ b^{p+1} &= a^{p-1}ba^{p-1}. \end{aligned}$$

If $p \equiv -1 (\mod 6)$ replace p by $-p$.

It is not known whether $SL(2,p) \times SL(2,p)$ has a deficiency zero presentation for all odd primes p. It is known to be Schur-efficient for $p = 5$ (Campbell *et al.* (1986)) and we give a deficiency zero presentation for $SL(2,7) \times SL(2,7)$ in Chapter Four.

Most of the above presentations have two generators and two relations. The first (non-trivial) presentations for finite groups having three generators and three relations were given by Mennicke (1959). Additional examples are given by Johnson and Roberston (1978) and Post (1978).

In subsequent chapters, we

- (Chapter Three) give minimal presentations for the groups of order less than or equal to 84; we also give minimal presentations for some families of groups having composition length less than five and order greater than 84;

- (Chapter Three) illustrate some of the techniques for working with finitely presented groups by proving the finiteness of a number of infinite families of finite groups of deficiency zero;

- (Chapter Four) give presentations for several finite groups having soluble length five and six, and deficiency zero presentations for two infinite families of finite groups, one family having soluble length five, and the other having soluble length six; we also give a deficiency one presentation for a finite preimage of a group having soluble length seven;

- (Chapter Five) give minimal presentations for various semi-simple groups, including a deficiency zero presentation for $SL(2,7) \times SL(2,7)$, which was previously not known to be Schur-efficient.

In dealing with finitely presented groups, one is faced with the general problem of manipulating words to show that pairs of words represent the same element. It would have been convenient if such manipulations were algorithmic. Unfortunately, Novikov (1955) and Boone (1955) have shown that the *word problem* is algorithmically unsolvable. That is, there is no uniform algorithm which will determine whether an arbitrary word in a finitely presented group defines the identity element. Similarly, there is no uniform algorithm to determine whether the two groups determined by the two arbitrary presentations $(X; R)$ and $(Y; S)$ are isomorphic. This is the *isomorphism problem.*

It can be difficult is to perform calculations within a (given) finitely presented group unless the presentation is particularly 'nice' in some way. Methods for addressing this problem usually involve finding a permutation or matrix representation for the group to perform calculations. With luck, such a representation will be faithful. These methods are most useful when the group being presented is a preimage of a known group. In the case when one has a totally unknown presentation to deal with, they do not prove to be very useful.

Notwithstanding the unsolvability results, methods have been developed for the manipulation of presentations, both by hand and by computer. One such method , which plays an important role here, is coset enumeration, first described by Todd and Coxeter (1936). The method described by Todd and Coxeter gives a systematic procedure for enumerating the cosets of a subgroup H of finite index in a group G, given the defining relations of G and a set of words (in the generators of G) generating H. We describe coset enumeration, and some techniques based on it below.

Computer implementations of various techniques have been used extensively to obtain some of the results in this thesis, and the question of the accuracy of such calculations compared to hand calculation arises. The majority of the results are based on the use of coset enumeration to show in each case that a particular finitely presented group is finite. This finiteness is used to establish that the group is in fact isomorphic to a 'known' group. The other calculations are checkable by hand. Many of the calculations in this thesis have been performed using the Cayley system (see Cannon (1984) and also below). Cayley output is provided of all coset enumerations so it is possible to check independently the enumerations.

Cannon *et al.* (1973), Neubüser (1982) and Havas (1991) provide descriptions of coset enumeration and details of various computer implementations of differing methods of coset definition. We will call this style of coset enumeration simple enumeration. The possibility of error exists when performing a hand coset enumeration, as does the possibility of an error in the implementation of coset enumeration. Such errors may lead to an incorrect index being calculated, and such errors have been reported in the literature (see Leech (1977)). The input for coset enumeration is two finite sets of words: the defining relations for a group and a set of subgroup generators. If the process terminates the output is the index of the subgroup in the given group, and a coset table over the subgroup. Little is known about the theoretical performance of various coset enumeration methods; there is no useful bound in terms of the length of the input and the index of the subgroup to the number of cosets which need to be defined for an enumeration to complete. In addition, Sims (Havas, private communication) has shown that there is no polynomial bound in terms of the maximum number of cosets for the number of coset tables which can be derived by simple coset enumeration procedures like those used in coset enumeration programs. A criticism of coset enumeration has been that any error in the implementation may not be obvious, and that it provides no mechanism for checking the result. Another criticism has been that considerable information about the structure of the group or subgroup is available in some sense during the enumeration and none of this information is available at the termination of the process. Such criticisms have prompted refinements to simple enumeration described by Leech (1977) and Neubüser (1982), in which the enumeration process is modified to provide also additional relations which hold in the group, together with a method for proving relations which hold in the group. The advantages claimed for these methods are that they provide a proof which may be checked by hand (or machine), and that such proofs may be (sometimes) generalized. We provide examples of such proofs in Chapter Three, where an incomplete coset table is used to derive information in an infinite group, and the proof is generalized to a family of finite groups. Leech provides several examples of the methods used, Havas (1976) uses similar methods to determine the structure of a Fibonacci group and we give a number of examples of use of the method below. All of the coset enumeration based proofs appear to suffer from the problem that they are 'unintuitive', difficult to generalize and provide little or no other useful information about the group structure. We give some examples of such proofs in Chapter Three. Arrell and Robertson (1984) describe a modified coset enumeration process which, through the use of Tietze transformations, is used to obtain a presentation for a subgroup in terms of a given set of generators. The process described provides presentations which are considerably shorter than those obtained by using the Reidemeister-Schreier method (which we describe in Chapter Two). Their method is based on the earlier work of Arrell *et al.* (1982) and McLain (1977).

Another coset enumeration based technique for the investigation of finitely presented groups is the low-index subgroup algorithm (see Dietze and Schaps (1974)). This algorithm accepts as input a finite set of words over some alphabet which is taken as the presentation for a group, and an integer n. The output of the algorithm is a list of all subgroups having index less than or equal to n. Each subgroup in the list is specified by its generators. The algorithm is easily modified to find the subgroups having index in a given range.

Other methods have been described by Knuth and Bendix (1970). In contrast to the techniques described above, Wos *et al.* (1979) describe an automated theorem proving system and use it to answer questions such as

- of the five axioms for a ternary Boolean algebra, which, if any, are dependent axioms?

- does there exist a finite semigroup that admits a nontrivial antiautomorphism but that does not admit any nontrivial involutions? If such semigroups exist, what is the order of the smallest?

Wos (1989) describes a theorem proving system, OTTER (a descendant of the 1979 system), which may also be used to prove theorems in algebraic structures (including groups). Examples are given of the use of these systems to prove results in group theory. However, the results proved are things such as "if a group has exponent two, the the group is abelian", which are not very difficult to prove 'manually'. The literature describing other aspects of logic programming, such as Prolog, also contains examples of theorem provers which are applied to finitely presented groups. However, these types of theorem proving systems have not had a great deal of use (or success) in group theory, partly because of the extreme generality of the language which is used to express axioms and deductive processes, and perhaps also partly because of the limited access that workers in finitely presented groups have had to such systems.

By far the most successful techniques are largely *ad-hoc* and are coset enumeration based, and involve methods such as trying to enumerate the cosets of G over the identity or some identified subgroup; similarly, working down a chain of subgroups, either found by using the low index subgroup algorithm or by some other means (working down the derived series for example). At each stage, Reidemeister-Schreier rewriting may occur, and Tietze or Nielsen transformations may be used to modify a presentation to obtain one which is more amenable to currently available coset enumeration implementations. Other techniques available are the integer matrix diagonalization, see Havas *et al.* (1979) for a description of the use of these tools.

Rutherford (1989) has implemented a collection of coset enumeration based techniques, together with a set of heuristics to apply them, and an application of these techniques is described in Robertson and Rutherford (1991).

Computer algebra systems designed specifically for modern algebra (group theory, ring theory, polynomial rings *etc.*) exist. Two such systems are Cayley, described by Cannon (1984), and the GAP system (Niemeyer *et al.* (1988)). Both systems provide similar functionality, although the underlying philosophies differ somewhat. Both provide the user (a mathematician) with an Algol or Pascal like language together with suitable primitives (or built in functions) to be able to express their problems (for example, both have functions allowing the user to define alternating groups, both have language constructs to allow the manipulation of sets of group elements represented as permutations, matrices or abstract words). The systems differ in that GAP is implemented as a small kernel of system dependent or time critical functions, together with a 'library' of algorithms expressed in the GAP language. The approach adopted in the Cayley system differs in that most of the algorithms are not implemented in the Cayley language, but in C (which is the implementation language for the entire system).

Chapter Two

Mathematical Preliminaries

In this chapter we give more results used in subsequent chapters, usually with reference. We will assume the notation and results given in Chapter One. Further general information may be found in the books of Johnson (1990), Magnus, Karrass and Solitar (1976) or Suzuki (1982).

A group G is a *semidirect product* of a normal subgroup N and a subgroup T if $G = NT$ and $N \cap T = 1$. The structure of G is determined by N, T and the action of T on N by conjugation. There is a map $\alpha : T \to Aut(N)$, with the action of elements of T defined by $n^t = \alpha(t)(n)$. Suppose that $N = (X; R)$ and $T = (Y; S)$; G has a presentation

$$(X \cup Y; R \cup S \cup \{y^{-1}xy = \theta_y(x)\}),$$

where the y range over the elements of Y, x range over the elements of X and θ_y is the automorphism of N corresponding to y. For example, if

$$N = (\{a, b\}; \{a^2, b^2, ab = ba\}),\ T = (\{c\}, \{c^2\}),$$

the direct product of N and T has a presentation with generating set $\{a, b, c\}$ and relators

$$a^2,\ b^2,\ a^{-1}b^{-1}ab,\ c^2,\ c^{-1}aca^{-1},\ c^{-1}bcb^{-1}.$$

(The direct product corresponds to the case when the action of each y is trivial.) Another example (with nontrivial action) is the case when c corresponds to the automorphism of N which maps a to b and b to a. In this case G has relators

$$a^2,\ b^2,\ a^{-1}b^{-1}ab,\ c^2,\ c^{-1}acb^{-1},\ c^{-1}bca^{-1}.$$

Both of these presentations define groups of order eight. The direct product is the elementary abelian group of order eight, and second group is isomorphic to the dihedral group of order eight.

If $G = (X; R)$, then every factor group H of G has a presentation $(X; S)$ with $R \subseteq S$. We call G a *preimage* of H.

Let $G = (X; R)$ and let H be a group, and φ a mapping from X to H. If for all $x \in X$, $r \in R$ the result of substituting $\varphi(x)$ for x in r yields the identity of H, φ extends to a homomorphism of G into H. If such an extension exists, it is unique. This result is called the substitution test by Johnson (1990, p.44).

(Tietze transformations). Let $G = (X, R)$, and let $w, r \in F_X$ and $r \in \overline{R} \backslash R$. If y is a symbol not in X then both $(X; R \cup \{r\})$ and $(X \cup \{y\}; R \cup \{y^{-1}w\})$ are isomorphic to $(X; R)$. These isomorphisms provide a means for adjusting a given presentation for a group to another presentation of the same group. A standard example of Tietze transformations shows that the von Dyck groups $D(l, m, n)$ and $D(n, m, l)$ are isomorphic. $D(l, m, n)$ is the group generated by x and y with relations

$$1 = x^l = y^m = (xy)^n.$$

This example appears in Johnson (1990, p.47). Introduce a new generator $a = xy$ so that $1 = a^n, x = ay^{-1}, (ay^{-1})^l = 1$ and the presentation reduces to

$$1 = a^n = (ay^{-1})^l = y^m.$$

Introducing another new generator $b = y^{-1}$, the presentation reduces to

$$1 = a^n = (ab)^l = b^m.$$

Programs implementing Tietze transformations have been written by a number of authors, and are described by Havas *et al.* (1984) and Rutherford (1989).

The method described as "systematic enumeration of cosets" was first described by Todd and Coxeter in 1936. It is a mechanical process which, given as input sets X, R defining a group, and a set Y of words over X which define a set of subgroup generators , produces as output the index of H (where H is generated by Y) in G if it is finite. We now give a brief description of coset enumeration.

We first describe the case when H is the trivial subgroup of G and Y is empty. If $X = \{x_1, \ldots, x_t\}$, for each relator $r = r_1 \ldots r_n \in R$ (where each r_i is of the form x_j or x_j^{-1}, where j depends on i), a table is drawn having $n+1$ columns with the number of rows being unspecified. The symbol 1 is entered in the first and last places of the first row of each table, with the other places remaining empty. Initially we have $|R|$ tables of the form

	r_1		r_2	$\ldots$	r_n	
1						1

An empty space next to some 1 is filled with the symbol 2. Suppose for definiteness that $r_1 = x_1$. and that the space immediately to the right of x_1 is chosen.

We record this *definition* $1x_1 = 2$ in another table, which we will call the *coset table*. The coset table consists of $2t$ columns, headed by $x_1, x_1^{-1}, x_2, \ldots, x_t^{-1}$, and has the same number of rows as each of the relator tables. Whenever a new symbol j is defined by an equation of the form $ix = j$, a new row labelled j is added to the coset table, and two new entries are made in the table: a j is entered at the meet of the i row and the x column, and an i entry is made at the meet of the j row and the x^{-1} column. We also add a new row to each of the relator tables, and insert the new symbol j in the first and last position of this new row. The symbols 1 and 2 correspond to the cosets H and Hx_1 of G. Having defined the symbol 2, a new row with 2 in the first and last positions is added to each of the relator tables, and a row labelled 2 is also added to the coset table. We also enter the symbol 2 in any relator table where the symbol 1 occurs to the immediate left of the symbol x_1, or a 1 to the immediate right of an occurrence of x_1^{-1}. This is called the *scan phase* of the enumeration. At this stage the tables are of the form

	r_1	r_2	$\ldots$	r_n	
1	2				1
2					2

and the coset table is

	x_1	x_1^{-1}	x_2	x_2^{-1}	$\ldots$
1	2				
2		1			

All spaces which may be filled by the symbol 2 are completed, and then the symbol 3 is entered in an empty space adjacent to a filled space (and the definition $ix = 3$ is recorded). All possible spaces which may be filled with the symbol 3 are completed, and then in a similar fashion, the symbol 4 is introduced. During the course of defining new symbols for entry into the relator tables, we may arrive at the situation when a row has one empty space remaining. For example, there is a relator table of the form

$\ldots$	r_p	r_{p+1}	$\ldots$
	i		k

where $r_p = x_m$, $r_{p+1} = x_n$, and we define the symbol j by $ix_m = j$. We obtain another piece of information, namely $kx_n^{-1} = j$, which we will call a *deduction.* It is also possible that the deduced information may be inconsistent with what is already known, and we give an example to illustrate this. Suppose that $R = \{x^2, x^3\}$, so that the group defined by R is trivial. Enumerating the cosets of the trivial subgroup, we define $1x = 2$. After this definition, we are able to deduce that $2x = 1$. Before processing the deduction $2x = 1$, the relator tables are

$$\begin{array}{ccccc} & x & & x & \\ 1 & & 2 & = & 1 \\ 2 & & 1 & & 2 \end{array}$$

and

$$\begin{array}{ccccccc} & x & & x & & x & \\ 1 & & 2 & & * & & 1 \\ 2 & & * & & 1 & & 2, \end{array}$$

where the $=$ indicates the place where a deduction was made. Note that the gaps marked $*$ may be filled by two different symbols; by processing the deduction $2x = 1$ applied to the middle x, the gap should be filled with a 1, if the deduction is applied to the rightmost x, the gap should be filled with a 2. We conclude that the symbols 1 and 2 represent the same coset, and replace all instances of 2 by 1 throughout the tables. At this stage, the tables are complete, and we conclude that the group has order one. The situation where a gap may be filled by several symbols is known as a coset *coincidence*, and in general the inconsistent information is of the form $ix = j$, $ix = n$ with $j \neq n$ for some i and x. We proceed as follows. We conclude that $j = n$ and replace the greater of j and n by the smaller throughout all the tables (both the relator tables and the coset table). This may yield further coincidences which are processed in the same way until all inconsistent information has been removed. All rows of the relator tables which begin with the larger of each coincident pair of symbols are then removed, and the remaining symbols are renamed to form a set of integers beginning with 1. If the group defined by the relations is finite, the process will terminate when there are no more empty spaces in the relator tables. However it is possible for the process not to terminate. In practice, when performing coset enumerations either by hand or by computer, a limit is placed on the number of rows allowed in the relator tables. Imposing such a limit guarantees termination. If the tables are complete, the number of rows in each relator table is equal to $|G|$.

The method described above extends easily to computing the finite index of a subgroup H specified by a set of generating words Y over $X \cup X^{-1}$. The modification involves adding another set of tables, one for each element of Y. These new tables are constructed in the same way as the relator tables , with the letters of y separating adjacent columns, except that they have only one row, beginning and ending with the symbol 1. The method then proceeds as above, with the subgroup generator tables being completed according to the same rules as the relator tables. As above, if a limit is imposed on the number of rows allowed in the relator tables, the process will terminate. If there are no more empty spaces in the relator and subgroup generator tables when the process terminates, the number of rows in each relator table is the index of H in G.

We now enumerate the cosets of a subgroup in an infinite family of groups defined by a parameterized presentation. Existing implementations of coset enumeration are unable to perform such enumerations. Let $G = (\{x, y\}; \{x^3 = y^{8n}, x^{y^2} = [x, y]\})$ where n is an integer and let H be the subgroup generated by y^2, xy^2x^{-1} and $xy^{-1}xy$. We will show that H has index 6 in G for all nonzero n. Assume that n is nonnegative (the case when n is negative is similar). We have three subgroup generator tables and two relator tables. Initially, each row starts and ends with a 1 and there are no other entries.

$$\begin{array}{ccccc} & y & & y & \\ 1 & & & & 1 \end{array},$$

$$\begin{array}{ccccccccc} & x & & y & & y & & x^{-1} & \\ 1 & & & & & & & & 1 \end{array}$$

and

$$\begin{array}{ccccccccc} & x & & y^{-1} & & x & & y & \\ 1 & & & & & & & & 1 \end{array}$$

are the subgroup generator tables, and we have the two relator tables:

$$\begin{array}{ccccccccccccccc} & \multicolumn{7}{c}{\overbrace{\hspace{12em}}^{8n}} & & & & & & & \\ & y & & y & & \cdots & & y & & y & & x^{-1} & & x^{-1} & & x^{-1} & \\ 1 & & & & & & & & & & & & & & & & 1 \end{array}$$

and

$$\begin{array}{ccccccccccccccc} & y^{-1} & & y^{-1} & & x & & y & & x^{-1} & & y & & x & \\ 1 & & & & & & & & & & & & & & 1 \end{array}.$$

We define $1y = 2$, adjoin a row starting and ending with a 2 but is otherwise empty to each of the relator tables and commence a scan. Note that the row of the y^2 subgroup generator table is complete and we deduce $2y = 1$. In the scan phase, we are able to fill in all but two entries of the first two rows of the $y^{8n}x^{-3}$ table. Following this definition-and-scan phase, the tables look like

$$\begin{array}{ccccc} & y & & y & \\ 1 & & 2 & = & 1 \end{array},$$

where the = sign denotes the place where a deduction occurred. The other subgroup generator tables are

	x		y		y		x^{-1}	
1								1 ,

and

	x		y^{-1}		x		y	
1						2		1 .

The relator tables are now

(the first four y columns are spanned by a brace labelled $8n$)

	y		y		$\dots$		y		y		x^{-1}		x^{-1}		x^{-1}	
1		2		1	$\dots$	1		2		1						1
2		1		2	$\dots$	2		1		2						2

and

	y^{-1}		y^{-1}		x		y		x^{-1}		y		x	
1		2		1										1
2		1		2										2 ,

and the coset table is

	x	x^{-1}	y	y^{-1}
1			2	2
2			1	1 .

Defining $1x = 3$ generates no deductions, nor does it give any coincidences. Defining $3y = 4$ next allows the xy^2x^{-1} table to be completed by giving the deduction $4y = 3$. We are also able to deduce that $4x = 2$ from the single row of the $xy^{-1}xy$ table. Performing a scan through the relator tables gives no coincidences and no further deductions. Following these definition-and-scan phases, the subgroup generators tables are

	y		y	
1		2		= 1 ,

	x		y		y		x^{-1}	
1		3		4		= 3		1 ,

and

	x		y^{-1}		x		y	
1		3		4		= 2		1 .

The relator tables are

			$8n$					
	y	y	…	y	y	x^{-1}	x^{-1}	x^{-1}
1	2	1	… 1	2	1		3	1
2	1	2	… 2	1	2	4		2
3	4	3	… 3	4	3	1		3
4	3	4	… 4	3	4		2	4

and

	y^{-1}	y^{-1}	x	y	x^{-1}	y	x
1	2	1	3	4			1
2	1	2			3	4	2
3	4	3			2	1	3
4	3	4	2	1			4 .

At this stage, one empty space remains in the first row of the $y^{8n}x^{-3}$ table. We complete this row by defining $1x^{-1} = 5$ and deduce that $5x^{-1} = 3$. We complete the second row of the $y^{8n}x^{-3}$ table by defining $4x^{-1} = 6$, and from the same row deducing that $6x^{-1} = 2$. During the scan phase we deduce that $6y = 5$ from the first row of the $y^{-2}xyx^{-1}yx$ table and that $5y = 6$ from the third row of this table. There are no coincidences and the tables are complete. Thus H has index six in G. The final relator tables are

			$8n$					
	y	y	…	y	y	x^{-1}	x^{-1}	x^{-1}
1	2	1	… 1	2	1	5 =	3	1
2	1	2	… 2	1	2	4	6 =	2
3	4	3	… 3	4	3	1	5	3
4	3	4	… 4	3	4	6	2	4
5	6	5	… 5	6	5	3	1	5
6	5	6	… 6	5	6	2	4	6

and

	y^{-1}	y^{-1}	x	y	x^{-1}	y	x
1	2	1	3	4	6 =	5	1
2	1	2	6	5	3	4	2
3	4	3	5 =	6	2	1	3
4	3	4	2	1	5	6	4
5	6	5	1	2	4	3	5
6	5	6	4	3	1	2	6 ,

and the coset table is

	x	x^{-1}	y	y^{-1}
1	3	5	2	2
2	6	4	1	1
3	5	1	4	4
4	2	6	3	3
5	1	3	6	6
6	4	2	5	5

.

The coset table provides a representation for G on the cosets of H. In this case, G is represented by the permutations $(1,3,5)(2,6,4)$ and $(1,2)(3,4)(5,6)$. (If H is trivial, or core-free, the coset table provides a faithful permutation representation of G.) Coset enumeration also provides sufficient information to show that H is a normal subgroup of G. The symbols $1,2,3,4,5,6$ represent the cosets of H in G and we may obtain a set of coset representatives by finding a set of words $w_1, \ldots, w_6$ such that $1w_i = i$. Each w_i is of the form $x_{i1}x_{i2}\ldots x_{in_i}$, where x_{ij} is x, y, x^{-1} or y^{-1}. Applying w_i to the symbol k means $(\ldots((kx_{i1})x_{i2})\ldots)x_{in_i}$, where the action of a generator (or its inverse) on a symbol is determined from the coset table. For example, the following table shows the result of applying the word $x^2y^{-1}xyx^{-1}$ to the symbol 1:

$$1 \xrightarrow{x} 3 \xrightarrow{x} 5 \xrightarrow{y^{-1}} 6 \xrightarrow{x} 4 \xrightarrow{y} 3 \xrightarrow{x^{-1}} 1\ .$$

To test whether H is a normal subgroup, form the conjugates of each of the subgroup generators by each of the group generators, and apply the resulting words to the symbol 1. If the result of each application is again 1, the subgroup is normal. In the example above, each of the conjugates of the subgroup generators applied to 1 results in 1, showing that H is a normal subgroup of G.

The above descriptions of coset enumeration do not specify fully the way to choose the next gap to define a coset, the method to use to perform a scan of the relator and subgroup generator tables, and the order in which to process coincidences. All of these factors influence the performance of any implementation of coset enumeration, and can affect considerably the number of cosets which are to be defined. The choices of appropriate strategy for performing coset enumeration have been studied and are reported by Cannon *et al.* (1973) and more recently by Havas (1991). Beetham and Campbell (1976) have described a variation of coset enumeration which uses the deduced information to provide a presentation for the subgroup H in terms of the elements of Y. Leech (1977) has described an extension of coset enumeration which "allows the user to deduce formal proofs of relations whose proofs are implicit in the working of the coset enumeration, and as far as possible to have these proofs derived by the computer itself." Havas (1976) has given a computer generated proof using methods similar to those described by Leech that the Fibonacci group $F(2,7)$ has order 29. We also give a number of examples of such proofs in Chapter Three, and provide illustrations of how such a machine generated proof may be generalised. As Leech remarks, "Rather frequently, especially in the more complicated cases involving many cosets, the formal proofs are long and far from perspicuous. Sometimes human editing can ease this and lead to perspicuous proofs. On other occasions it is likely that no such comprehensible proof is possible." Examples of coset enumeration based proofs are given in Chapter Three. Many of these are "far from perspicuous".

We now list without proof some results about the Schur multiplicator of a finite group G. These results may be found in Wiegold (1982) or Huppert (1967).

(Schur, 1904) Let G be a group and $Z(G)$ be the centre of G. If $G/Z(G)$ is finite, then the derived subgroup G' of G is finite.

This theorem is used extensively in later chapters to show that a group G with a finite central quotient and a finite abelian quotient is finite.

Schur (1904) also showed that if H is a central subgroup of G then $G' \cap H$ is isomorphic to a subgroup of $M(G/H)$.

A group G^* is a *covering group* of G if there is a subgroup A of G^* such that A is contained in both centre $Z(G^*)$ and in the derived subgroup of G^*, $A \simeq M(G)$ and $G \simeq G^*/A$. Schur showed that a group G has at least one covering group.

$M(G)$ is a finite group, whose elements have order dividing the order of G and $M(G) = 1$ if G is cyclic.

Let G be a finite nilpotent group, and let $S_1, \ldots, S_n$ be all the Sylow subgroups of G. Then $M(G) = M(S_1) \times \ldots \times M(S_n)$.

If F is a free group and $G = F/R$, then $(F' \cap R)/[F, R]$ is the torsion part of $R/[F, R]$.

Let S_n be the symmetric group of degree n and let A_n be the alternating group of degree n. Schur showed

$$M(S_n) = \begin{cases} 1, & \text{if } n \leq 3, \\ C_2 & \text{otherwise;} \end{cases}$$

and also that

$$M(A_n) = \begin{cases} 1, & \text{if } n \leq 3; \\ C_2, & \text{if } n \geq 4, n \neq 6, 7; \\ C_6, & n = 6, 7. \end{cases}$$

If G is finite, H is a group with a central subgroup A and $G \simeq H/A$, then $H' \cap A$ is isomorphic to a homomorphic image of $M(G)$. A pair of finite groups (H, A) is called a *defining pair* for the group G if $G \simeq H/A$ and A is contained in $H' \cap Z(H)$. The rank of $M(G)$ is greater than or equal to the rank of A.

Let G be of order p^n, p prime, and be generated by d generators and have r relators. Then

$$p^{d(d-1)/2} \leq |M(G)||G'| \leq p^{n(n-1)/2},$$

and

$$r \geq d(d+1)/2 - d(G'),$$

where $d(G')$ is the minimum number of generators required for G'. If $G/\Phi(G)$ has order p^f (here $\Phi(G)$ is the Frattini subgroup of G) and G' has order p^h, then

$$|M(G)| \leq p^{(f-1)h}|M(G/G')|,$$

and

$$|M(G)| \leq p^{(n-h)(n-h-1)/2}.$$

If G is an extra-special group of order p^{2n+1}, with $n > 1$, $M(G)$ is an elementary abelian group of order p^{2n^2-n-1}. Beyl and Tappe (1982) show that if G is non-abelian of order p^3 and p is an odd prime, $M(G) = C_p \times C_p$ if the exponent of G is p and $M(G)$ is trivial otherwise.

Coset enumeration also provides sufficient information to obtain the presentation of a subgroup. Let U be a transversal for H in G. A set of generators for H is given by $\{ux\varphi(ux)^{-1}\}$, where $u \in U, x \in X$ and $\varphi(y)$ is the coset representative of y in U. (See Theorem 2.7 of Magnus *et al.* (1976), p.89.) The following result is Theorem 2.8 of Magnus *et al.* (1976), and is due to Reidemeister. A set of relators for H is given by

$$\tau(uru^{-1}),$$

where the u range over U, $r \in R$, and τ is a mapping from H to H, which we define below. Let

$$W = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$$

where each e_i is either 1 or -1, and define a sequence of elements (relative to W) by

$$u_1 = 1, u_{i+1} = \varphi(u_i x_i)^{-1}, a_i = u_i x_i u_{i+1}^{-1} \text{ for } 1 \leq i \leq n.$$

Define $\tau(W)$ to be $\prod_{i=1}^{n} a_i$. The mapping τ is called a Reidemeister rewriting process.

A set of coset representatives for which any initial segment of a representative is again a representative is called a Schreier system. A Reidemeister rewriting process using a Schreier system is called a Reidemeister-Schreier rewriting process. In general, the Reidemeister-Schreier rewriting process gives a presentation which is not in terms of the originally specified subgroup generators. Beetham and Campbell (1976), McLain (1977) and Arrell and Robertson (1984) have given methods to obtain a presentation for H in terms of the original subgroup generators.

This method may be applied to the calculation of the Schur multiplicator, and is described by Johnson (1990). Let G_1 be the group with generating set X and relations $\{[x,r] | x \in X, r \in R\}$ and let H be the subgroup generated by R. This subgroup is central, and hence abelian. Since H is finitely generated, $H \simeq A \times T$, where A is free abelian and T is finite. The Schur multiplicator of G is precisely T.

Chapter Three

Some Interesting Groups of Low Order

Introduction

In this chapter we give minimal presentations for a number of interesting groups of 'low order'. In this context, 'low order' means that the composition length of the group is small (in most cases less than five and the order is less than 85), and a group is interesting if it is not metacyclic and it is not abelian. We also give some deficiency zero presentations for some infinite families of two generator groups which do not overlap with other known families.

Minimal presentations for the groups of order 36, 48, 54, 60, 72 and 84 are given in Appendix A.

These groups provide examples of the techniques used to perform calculations in finitely presented groups, and examples of the techniques used to prove theorems about finitely presented groups.

The groups are classified according to their composition length. Since groups of composition length less than three are either abelian or metacyclic, they are omitted, so are groups of square-free order.

Composition length three

We follow the notation of Neubüser (1967) in describing the soluble groups of composition length three. In the presentations given below, p and q denote distinct primes. The presentations for the groups of composition length three are

$(pq \times p)$ $$A^{pq} = B^p = [A, B] = 1$$

$(p^2 q)$ $$A^{p^2 q} = 1$$

$(D_{pq} \times p)$
$$\begin{aligned} 1 = A^p = B^q = C^p \\ = [A, B] = [A, C] = 1, \\ [B, C] = B^{x-1}, \end{aligned}$$

where $x \not\equiv 1 \pmod q$, $x^p \equiv 1 \pmod q$, $p | q-1$

(G_{p^2q})
$$\begin{aligned} 1 &= A^p = B^q,\ C^p = A, \\ 1 &= [A,B] = [A,C], \\ [B,C] &= B^{x-1}, \end{aligned}$$

where $x \not\equiv 1 \pmod q$, $x^p \equiv 1 \pmod q$, $p | q-1$

(H_{p^2q})
$$A^q = B^{p^2} = 1,\ [A,B] = A^{x-1},$$

where $x^p \not\equiv 1 \pmod q$, $x^{p^2} \equiv 1 \pmod q$,

(A_4)
$$A^2 = B^3 = (BA)^3 = 1$$

$(D_{pq} \times q)$
$$\begin{aligned} 1 &= A^q = B^q = C^p \\ &= [A,B] = [A,C], \\ [B,C] &= B^{x-1}, \end{aligned}$$

where $x \not\equiv 1 \pmod q$, $x^p \equiv 1 \pmod q$, $p | q-1$

(K_{pq^2})
$$\begin{aligned} A^q = B^q = C^p = [A,B] = 1, \\ [A,C] = A^{x-1},\ [B,C] = B^{x-1}, \end{aligned}$$

where $x \not\equiv 1 \pmod q$, $x^p \equiv 1 \pmod q$, $p | q-1$

$(L_{pq^2}(s))$
$$\begin{aligned} A^q = B^q = C^p = [A,B] = 1, \\ [A,C] = A^{x-1},\ [B,C] = B^{x^s-1}, \end{aligned}$$

where $x \not\equiv 1 \pmod q$, $x^p \equiv 1 \pmod q$, $s \not\equiv 0,\ 1 \pmod q$, $p | q-1$, $p > 2$, $q > 3$. There are $(p-1)/2$ non-isomorphic groups $L_{pq^2}(s)$. $L_{pq^2}(s) \simeq L_{pq^2}(s')$ if and only if $s \equiv s' \pmod p$ or $ss' \equiv 1 \pmod p$.

(M_{pq^2})
$$A^{q^2} = B^p = 1,\ [A,B] = A^{x-1},$$

where $x \not\equiv 1 \pmod{q^2}$, $x^p \equiv 1 \pmod{q^2}$, $p | q-1$.

(N_{pq^2})
$$\begin{aligned} 1 &= A^q = B^q = C^p \\ &= [A,B], \\ [A,C] &= A^{-1}B, \\ [B,C] &= A^{-1}B^{x^q+x-1}, \end{aligned}$$

where x is an element of order p in $GF(q^2)$, $p|q+1$, $p \neq 2$.

(p^3)
$$\begin{aligned} 1 &= A^p = B^p = C^p, \\ C &= [A,B], \\ 1 &= [A,C] = [B,C]. \end{aligned}$$

Of these groups, only $A_4, K_{p^2q}, K_{pq^2}, L_{pq^2}(s), N_{pq^2}$ and p^3 are interesting. Since the multiplicator of A_4 has order 2, the presentation given above is an efficient one.

The group G generated by a, b and c with defining relations

$$\begin{aligned} c^2 &= a^q b^q, \\ 1 &= [a,b], \\ a^{\alpha+1} &= c^{-1}a^{\alpha}c, \\ b^{\alpha+1} &= c^{-1}b^{\alpha}c, \end{aligned}$$

where $\alpha = (q-1)/2$ is clearly a preimage of K_{2q^2}. It can be shown to be isomorphic to K_{2q^2} as follows. Since c^2 is central,

$$\begin{aligned} a^{\alpha} &= (a^{\alpha})^{c^2} = (c^{-1}a^{\alpha}c)^c \\ &= c^{-1}a^{\alpha+1}c = (a^{\alpha})^c a^c \\ &= a^{\alpha+1}a^c, \end{aligned}$$

and so $a^c = a^{-1}$; from this we have $(a^{\alpha})^c = a^{-\alpha}$, but $(a^{\alpha})^c = a^{\alpha+1}$, so $a^{-\alpha} = a^{\alpha+1}$, *i.e.* $a^q = 1$.

We will show that the group K defined by the relations

$$\begin{aligned} 1 &= A^q = B^q = C^2, \\ X &= [A,B], \\ A^C &= A^{-1},\ B^C = B^{-1}, \\ 1 &= [A,X] = [B,X] = [C,X] \\ &= X^q, \end{aligned}$$

has order $2q^3$ and together with the group generated by X is a defining pair for K_{2q^2}. This shows that the multiplicator of K_{2q^2} has rank at least one. The presentation given above for G is thus a Schur-efficient presentation for K_{2q^2}.

Let $x = A, y = B$ and $z = X$. We use the modified coset enumeration method of Beetham and Campbell (1976) to find a presentation for the subgroup H generated by x, y and z. From the requirement that each of the subgroup generators fix coset 1, we deduce that $1A = x1, 1B = y1$ and $1X = z1$. Applying these deductions to the relators gives no further deductions and no coincidences. Defining $1C = 2$ gives the following deductions: $2C = 1$ from $C^2 = 1$, $2A = x^{-1}2$ from $C^{-1}ACA = 1$, $2B = y^{-1}2$ from $C^{-1}BCB = 1$ and $2X = z^{-1}2$ from $[C, X] = 1$. These deductions lead to no coincidences and result in a complete coset table, showing that H has index two in K. A presentation for H on generators x, y and z is

$$\begin{aligned} 1 &= x^q \\ &= y^q, \\ z &= [x, y], \\ 1 &= [x, z] = [y, z] \\ &= z^q. \end{aligned}$$

(The relations $C^2 = 1, A^C = A^{-1}$ and $B^C = B^{-1}$ lead to trivial relations in H.) This is clearly a presentation for a group of order q^3.

The groups $L_{pq^2}(s)$ have order greater than 100, and are not considered here. Of the groups N_{pq^2} only the smallest (with $p = 3, q = 5$) is of interest here, and it has a presentation

$$\begin{aligned} 1 &= A^5 = B^5 = C^3 \\ &= [A, B] \\ [A, C] &= A^{-1}B, \\ [B, C] &= A^{-1}B^3. \end{aligned}$$

Let K be the group with the following relations

$$\begin{aligned} 1 &= A^5\gamma^7 = B^5\beta\gamma^2 \\ &= C^3 \\ &= [A, B]\alpha^{-1} \\ &= [A, C]B^{-1}A\gamma \\ &= A^{-1}B^3[C, B] \end{aligned}$$

and α, β and γ are central. From the presentation, α is both in the derived subgroup of K and in the centre of K. Provided that α is not trivial, K is a central extension of N_{35^2}, and $K' \cap Z(K)$ is a homomorphic image of $M(N_{35^2})$. A coset enumeration shows that K has order 375, and thus the multiplicator has rank at least one. The group with presentation

$$(3 \cdot 0) \qquad (a, c; a^5 = (ca)^3, c^3 = (ca^{-1})^3 = 1),$$

is clearly a preimage of N_{35^2}. (Use Tietze transformations to eliminate the generator B.) Another coset enumeration shows that the order of the group defined by $(3 \cdot 0)$ is 75, and thus $(3 \cdot 0)$ is a minimal presentation for N_{35^2}.

The quaternion group Q_8 has a well known deficiency zero presentation:

$$(a, b; bab = a, aba = b),$$

for example, see Coxeter and Moser (1980).

Let G_p be the group generated by x and y with defining relations

$$\begin{aligned} 1 &= x^p = y^p \\ &= [[x, y], y] \\ &= [[x, y], x], \end{aligned}$$

where p is an odd prime. It is easy to verify that G_p is a preimage of the group p^3, for example, under the mapping $x \mapsto A, y \mapsto B$. The order of G_p divides p^3 and so G_p is isomorphic to p^3 and p^3 is presentable with a 2 generator, 4 relation presentation. Since the multiplicator of p^3 is elementary abelian of order p^2, we have an efficient presentation for p^3.

Composition length four

In this section, we consider some of the groups of order p^4, p odd, 2^3q, $q > 1$ and q odd. The groups of order p^4 were first classified by Hölder (1893), and Burnside (1911) has given presentations for these groups. The groups of order p^3q have been classified by Western (1899).

The groups of order 2^3q, with $q > 1$ and q odd

The only non-abelian and non-metacyclic groups of order $8q$ are the following:

$$(3 \cdot 1) \qquad 1 = \alpha^2 = \beta^2 = \gamma^2 = \delta^q$$

$$
\begin{aligned}
&= [\alpha,\beta] = [\alpha,\gamma] = [\alpha,\delta] \\
&= [\beta,\gamma] = [\beta,\delta], \\
\delta^{\gamma} &= \delta^{-1}.
\end{aligned}
$$

This group is (9) in Western's list.

(3 · 2)

$$
\begin{aligned}
1 &= \alpha^4 = \beta^2 = \gamma^q \\
&= [\gamma,\beta], \\
\alpha^{\beta} &= \alpha^{-1},\ \gamma^{\alpha} = \gamma^{-1}.
\end{aligned}
$$

This group is (11) in Western's list.

(3 · 3)

$$
\begin{aligned}
1 &= \alpha^4 = \beta^4 = \gamma^3, \\
\alpha^2 &= \beta^2, \\
\alpha^{-1} &= \alpha^{\beta},\ \alpha^{\gamma} = \beta, \\
\beta^{\gamma} &= \alpha\beta.
\end{aligned}
$$

This group is (17) in Western's list, and is isomorphic to $SL(2,3)$. The group defined by these relations has order at most 24, and there is an isomorphism defined by

$$
\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \beta \mapsto \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \gamma \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}.
$$

(3 · 4)

$$
\begin{aligned}
1 &= \alpha^4 = \beta^2 = \gamma^3, \\
\alpha^{-1} &= \alpha^{\beta},\ (\alpha^2)^{\gamma} = \beta, \\
\beta^{\gamma} &= \alpha^2\beta, \\
\gamma^{\alpha} &= \gamma^2\alpha^2\beta.
\end{aligned}
$$

This group is (18) in Western's list, and is isomorphic to the symmetric group S_4. The mapping $\alpha \mapsto (1,2,3,4), \beta \mapsto (1,2)(3,4), \gamma \mapsto (1,4,2)$ extends to a homomorphism to $(3 \cdot 4)$ onto S_4, and a coset enumeration shows that $(3 \cdot 4)$ has order 24, and so is isomorphic to S_4.

(3 · 5)

$$
\begin{aligned}
1 &= \alpha^2 = \beta^2 = \gamma^2 = \delta^7, \\
1 &= [\alpha,\beta] = [\alpha,\gamma] = [\beta,\gamma], \\
\alpha^{\delta} &= \beta,\ \beta^{\delta} = \gamma,\ \gamma^{\delta} = \alpha\beta.
\end{aligned}
$$

This group is (19) in Western's list.

The groups $SL(2,p)$, p an odd prime have zero deficiency (Campbell and Roberston (1980)), so $(3\cdot 3)$ is efficient. S_4 has long been known to be efficient, with defining relations

$$(3\cdot 6) \qquad a^4 = b^3 = (ab)^2 = 1,$$

for example, see Coxeter and Moser (1980).

The group $(3\cdot 5)$ of order 56 is a cyclically presented group with deficiency zero found by Johnson and Mawdesley (1975):

$$(3\cdot 7) \qquad \begin{aligned} x_6 = x_1x_4,\ x_3 = x_4x_1,\ x_2 = x_3x_6, \\ x_5 = x_6x_3,\ x_4 = x_5x_2,\ x_1 = x_2x_5, \end{aligned}$$

which may be simplified to a two generator, two relation presentation

$$(3\cdot 8) \qquad \begin{aligned} 1 &= x^2yxy^3 \\ &= y^2xyx^3, \end{aligned}$$

where $x = x_1$ and $y = x_4$. We have found another zero deficiency presentation

$$(3\cdot 9) \qquad \begin{aligned} a^2 &= b^7, \\ (abab^{-1})^2 &= abab^{-3}ab^2, \end{aligned}$$

where the correspondence between this presentation and $(3\cdot 8)$ is given by $a = x^{-1}y$ and $b = y^{-1}$. Coset enumeration shows that $(3\cdot 9)$ has order 56.

Let $W_9(q)$ be the group defined by $(3\cdot 1)$, let q be an odd prime, and let K_q be the group defined by the relations

$$\begin{aligned} 1 &= x^{2q} = y^2 = z^2 \\ &= (zy)^2 = (zx^{-1})^2 = yxyx^{-1}. \end{aligned}$$

We now show that K_q is isomorphic to $W_9(q)$. From the relations, it is clear that y commutes with both x and z, and also that $zxz = x^{-1}$. Thus every element of K_q may be written in the form $x^ry^sz^t$, with $0 \le r < 2q$, $0 \le s,t < 2$, so K_q has order dividing $8q$. There is a homomorphism of K_q onto $W_9(q)$ defined by $x \mapsto \alpha\delta, y \mapsto \beta, z \mapsto \gamma$, so K_q is isomorphic to $W_9(q)$. It is clear that $W_9(q)$ is the direct product of the subgroups $\langle\alpha,\beta\rangle$ and $\langle\gamma,\delta\rangle$, so that $M(W_9(q)) = M(\langle\alpha,\beta\rangle) \times M(\langle\gamma,\delta\rangle) \times (\langle\alpha,\beta\rangle \otimes \langle\gamma,\delta\rangle)$. Now, $M(\langle\alpha,\beta\rangle) = C_2$, and the tensor product is isomorphic to $C_2 \times C_2$, so the rank of $M(W_9(q))$ is at least three, but the presentation given above for K_q shows that it is no more than three, and that $W_9(q)$ is a family of Schur-efficient groups.

Let q be an odd integer greater than one, and let G_q be the group defined by

$$1 = x^2 = y^4 = (yx)^{q+1}(y^{-1}x)^{q-1}.$$

We show that G_q is isomorphic to the group $(3 \cdot 2)$ in Western's list of groups of order $8q$. Consider the group K generated by t, α, β and γ with defining relations

$$\begin{aligned} t &= \alpha^4, \\ 1 &= \beta^2 = \gamma^q \\ &= [\gamma, \beta], \\ \alpha^\beta &= \alpha^{-1},\ \gamma^\alpha = \gamma^{-1}, \\ 1 &= [t, \beta] = [t, \gamma]. \end{aligned}$$

We show below that this group, together with the subgroup generated by t is a defining pair for $(3 \cdot 2)$, and this shows that the multiplicator of $(3 \cdot 2)$ has rank at least one, and thus the groups defined by G_q have minimal presentations.

The quotient by the derived group of K is elementary abelian of order four. Let $a = \alpha^2$, $b = t$ and $c = \gamma$. It is easy to see that a, b and c generate the derived group of K. We use the method of Beetham and Campbell (1976) to obtain a presentation for K'. From the requirement that each of the subgroup generators fix coset 1, we obtain the deductions $1t = b1$ and $1\gamma = c1$. Defining $1\alpha = 2$, we deduce $2\alpha = a1$ from the subgroup generator tables, $2t^{-1} = a^{-2}2$ from the relation $t = \alpha^4$ and $2\gamma = ac^{-1}a2$ from the relation $\gamma^\alpha = \gamma$. At this stage, no further deductions are possible and no coincidences have been found. Defining $1\beta = 3$, we deduce $3\beta = 1$, $3\gamma = c3$ and $3t = b3$. No further deductions are possible at this stage, so define $2\beta = 4$. This gives the deductions $4\beta = 2$, $4\gamma = ac^{-1}a^{-1}4, 3\alpha = a^{-1}4, 4\alpha = 3$ and $4t^{-1} = a^2 4$. At this stage, the coset table is complete, and there are no coincidences, and no further deductions. This shows that K' has index four in K.

Rewriting the relations of K, we obtain the following presentation for K':

$$\begin{aligned} 1 &= a^2 b = c^q \\ &= [a, c] = [b, c] \\ &= a^4. \end{aligned}$$

This clearly defines a group of order $2^2 q$, and thus K has order $2^4 q$, and is therefore a covering group of $(3 \cdot 2)$.

In G_q we have

$$1 = (xy^{-1})^{q-3}(xy)^{q+1}xy^{-1}xy^{-1}$$

and

$$x = (yx)^2(y^{-1}x)^{q-1}(yx)^{q-2}y,$$

we have

$$\begin{aligned} 1 &= (xy^{-1})^{q-3}(xy)^{q+1} \cdot (yx)^2(y^{-1}x)^{q-1}(yx)^{q-2}y \cdot y^{-1}xy^{-1} \\ &= (xy^{-1})^{q-3}(xy)^{q+1}(yx)^2(y^{-1}x)^{q-1}(yx)^{q-3}. \end{aligned}$$

Conjugating this relation by $(xy^{-1})^{q-3}(xy)^{q-1}$ gives $1 = (xy)^2(yx)^2 = (xy^2)^2$. In particular, we find that y^2 is central.

Let $b = (xy)^q$, then

$$\begin{aligned} b^2 &= (xy)^q xy(xy)^{q-1} \\ &= (xy)^q(yx)^2(y^{-1}x)^{q-1}(yx)^{q-2}y^2(xy)^{q-1}, \end{aligned}$$

and as y^2 is central we obtain

$$\begin{aligned} b^2 &= (xy)^{q-2}xy^2x(y^{-1}x)^{q-1}(yx)^{q-2}(xy)^{q-1} \\ &= (xy)^{q-2}(y^{-1}x)^{q-1}(yx)^{q-2}(xy)^{q-1}y^2 \\ &= y^{-1}x(yx)^{q-2}(xy)^{q-1}y^2. \end{aligned}$$

Since q is odd and greater than one, we also have that

$$\begin{aligned} (yx)^{q-2}(xy)^{q-1} &= (yx)^{q-3}(xy)^{q-2} \\ &= (yx)^{q-4}(xy)^{q-3} \\ &= (yx)(yx)^2 \\ &= xy^3. \end{aligned}$$

So $b^2 = y^{-1}xxy^3y^2 = 1$. Let $Q = (xy)^{q-1}$, then $Q^q = 1$. Let $a = (yx)^{q-1}y^{-1}$; then

$$\begin{aligned} a^2 &= (yx)^{q-1}x(yx)^{q-2}y^{-1} \\ &= (yx)^{q-2}y(yx)^{q-2}y^{-1} \\ &= (yx)^{q-3}y(yx)^{q-3}y \\ &= (yx)^{q-4}y(yx)^{q-4}y^{-1} \\ &= y^2 \end{aligned}$$

so that a^2 is central and $a^4 = 1$. Also,

$$\begin{aligned} ba &= (xy)^q (yx)^{q-1} y^{-1} \\ &= (xy)^{q-1} (yx)^{q-2} y^2 y^{-1} \\ &= (xy)^{q-2} (yx)^{q-3} y^4 y^{-1} \\ &= (xy)^2 (yx) y^{2(q-2)} y^{-1} \\ &= xy (y^2)^{q-1} y^{-1} \\ &= xy^{2(q-1)} \\ &= x \end{aligned}$$

since q is odd, so that $(ba)^2 = 1$. Note that

$$a^{-1} Q a Q = y(xy^{-1})^{q-1} (xy)^{q-1} (yx)^{q-1} y^{-1} (xy)^{q-1},$$

and then

$$\begin{aligned} (xy)^{q-1}(yx)^{q-1} &= (xy)^{q-2} (yx)^{q-2} y^2 \\ &= (xy)^{q-3} (yx)^{q-3} y^4 \\ &= (y^2)^{q-1} \\ &= 1, \end{aligned}$$

so $a^{-1} Q a Q = y(xy^{-1})^{q-1} y^{-1} (xy)^{q-1}$, and by above, $a^{-1} Q a Q = 1$. We also show that $Qa^{-1} = y$.

$$\begin{aligned} Qa^{-1} &= (xy)^{q-1} y (xy^{-1})^{q-1} \\ &= (xy)^{q-2} y^{-1} (xy^{-1})^{q-2} y^2 \\ &= (xy)^{q-3} y^{-1} (xy^{-1})^{q-3} y^2 \\ &\vdots \\ &= y. \end{aligned}$$

The above calculations show that a, b and Q generate G_q, and that they satisfy the relations $a^4 = b^2 = Q^q = 1$, $Q^a = Q^{-1}$ and $a^b = a^{-1}$. Since b and Q are both powers of xy it is clear that they commute. This shows that G_q is a quotient of $(3 \cdot 2)$. To see that G_q is isomorphic to $(3 \cdot 2)$, it is easy to check that the mapping of G_q onto $(3 \cdot 2)$ given by $x \mapsto \beta\alpha$ and $y \mapsto \gamma\alpha^{-1}$ extends to a homomorphism of G_q onto $(3 \cdot 2)$.

The groups of order p^4

The following presentations for the groups of order p^4, p an odd prime are given in Burnside (1911):

(3 · 10) $$1 = P^{p^3} = Q^p, \ P^Q = P^{1+p^2}.$$

This group is (6) in Burnside's list.

(3 · 11)
$$1 = P^{p^2} = Q^p = R^p,$$
$$QP^p = R^{-1}QR,\ [P,Q] = [P,R] = 1.$$

This group is (7) in Burnside's list.

(3 · 12)
$$1 = P^{p^2} = Q^{p^2},\ P^Q = P^{1+p}.$$

This group is (8) in Burnside's list.

(3 · 13)
$$1 = P^{p^2} = Q^p = R^p,$$
$$P^{1+p} = P^R,\ [P,Q] = [Q,R] = 1.$$

This group is the direct product of $\langle Q \rangle$ and $\langle P, R \rangle$ and is (9) in Burnside's list.

(3 · 14)
$$1 = P^{p^2} = Q^p = R^p,$$
$$PQ = P^R,\ [P,Q] = [Q,R] = 1.$$

This group is (10) in Burnside's list.

(3 · 15)
$$1 = P^{p^2} = Q^p,$$
$$P^{1+p} = P^Q,\ [P,R] = Q,$$
$$1 = [Q,R],\ R^p = P^{\alpha p},$$

where there are three isomorphism types of groups with $\alpha = 0$, $\alpha = 1$ and α any nonsquare modulo p. These groups are (11) in Burnside's list.

(3 · 16)
$$1 = P^p = Q^p = R^p = S^p,$$
$$R^S = RP,\ Q^S = Q,$$
$$P^S = P,\ Q^R = Q,$$
$$P^R = P,\ P^Q = P.$$

This group is the direct product of $\langle Q \rangle$ and $\langle P, R, S \rangle$ and is (14) in Burnside's list.

(3 · 17)
$$1 = P^p = Q^p = R^p = S^p,$$
$$[R,S] = Q, [Q,S] = P, [P,S] = 1,$$
$$[Q,R] = [P,R] = [P,Q] = 1,$$

if $p > 3$, ((15) in Burnside's list), and if $p = 3$

(3 · 18)
$$1 = P^9 = Q^3 = R^3$$
$$= [P,Q],\ Q = [P,R],$$
$$[Q,R] = Q^{-1}P^{-3}Q.$$

This group is (16) in Burnside's list.

Of these groups, $(3\cdot 10)$ and $(3\cdot 12)$ are metacyclic. The groups defined by $(3\cdot 13)$ and $(3\cdot 16)$ are direct products. In the case $(3\cdot 13)$, the group $\langle P, R\rangle$ is the non-abelian metacyclic group of order p^3, and has an efficient presentation

$$P^{p^2} = R^p,\ P^R = P^{1+p},$$

and so a presentation for $\langle Q\rangle \times \langle P, R\rangle$ is

$$\begin{aligned} 1 &= Q^p,\ P^{p^2} = R^p,\ P^R = P^{1+p}, \\ 1 &= [P,Q] = [R,Q]. \end{aligned}$$

Since the multiplicator of $(3\cdot 13)$ has rank two, the presentation given above for $(3\cdot 13)$ is an efficient one.

In the case $(3\cdot 16)$, the group $\langle P, R, S\rangle$ is the non-abelian, non-metacyclic group of order p^3 and exponent p. It has an efficient presentation

$$\begin{aligned} 1 &= R^p = S^p = [S,[R,S]] \\ &= S^2RS^{-1}R^{-2}S^{-1}R, \end{aligned}$$

where $P = [R,S]$. The direct product then has a presentation

$$\begin{aligned} 1 &= R^p = S^p = [S,[R,S]] \\ &= S^2RS^{-1}R^{-2}S^{-1}R \\ &= Q^p = [Q,R] = [Q,S]. \end{aligned}$$

The rank of the multiplicator of this group is four, and so the presentation given above is an efficient one.

Let K be the group with defining relations

$$(3\cdot 20) \qquad \begin{aligned} 1 &= P^9\gamma = Q^3\beta = R^3 \\ &= [P,Q]\alpha^{-1} = [P,R]Q^{-1} \\ &= Q^{-1}P^{-3}Q[R,Q] = [P,\alpha] \\ &= [Q,\alpha] = [R,\alpha] = [P,\beta] \\ &= [Q,\beta] = [R,\beta] = [\alpha,\beta] \\ &= [P,\gamma] = [Q,\gamma] = [R,\gamma] \\ &= [\alpha,\gamma] = [\beta,\gamma]. \end{aligned}$$

Coset enumeration shows that this group has order 3^6, and directly from the presentation, the group generated by α, β and γ is seen to be contained in both the derived subgroup of K and the centre of K, and the quotient $K/\langle\alpha,\beta,\gamma\rangle$ is isomorphic to the group (3·18). A calculation with Cayley shows that $K' \cap Z(K)$ has order nine and exponent three, and so the multiplicator of $(3 \cdot 18)$ has rank at least two. Clearly $x = P$ and $y = R$ generate $(3 \cdot 18)$ and satisfy the relations

$$1 = y^3 = (xy)^3 = (xy^{-1})^3 = [x^3, y],$$

and coset enumeration verifies that the group defined by these relations has order 3^4, and thus this is a minimal presentation for $(3 \cdot 18)$.

Let $G_{p \cdot 14}$ be the group generated by a and b with the defining relations:

$$\begin{aligned} 1 &= a^{p^2} = b^p \\ &= [a,[a,b]] = [[a,b],b]. \end{aligned}$$

Let G be be group defined by $(3 \cdot 14)$. The order of $M(G/G')$ is p, and $\Phi(G)$ is $C_p \times C_p$, so that by the results of Chapter Two, the order of $M(G)$ is p^2. To see that the rank of $M(G)$ is two, consider the group G^* defined by

$$\begin{aligned} 1 &= P^{p^2} = Q^p = R^p, \\ PQ &= P^R,\ [P,Q] = \alpha,\ [Q,R] = \beta, \\ 1 &= [P,\alpha] = [P,\beta] \\ &= [Q,\alpha] = [Q,\beta] \\ &= [R,\alpha] = [R,\beta] \\ &= [\alpha,\beta] \\ &= \alpha^p = \beta^p \\ &= [P^p,Q] = [P^p,R] \\ &= [P^p,\alpha] = [P^p,\beta]. \end{aligned}$$

It is clear that G^* has order dividing p^6 (for example, by considering the lower central series). We show that G^* has order p^6. Let $x = P^p, y = Q, z = R, w = \alpha$ and H be the subgroup of G^* generated by x, y, z and w. We (again) use the method of Beetham and Campbell (1976) to show that H has index p in G^*, and to obtain a presentation for H. From the subgroup generators, we deduce that $1Q = y1, 1R = z1$ and $1\alpha = w1$. For $i = 1, 2, \ldots, p-1$, define coset $i+1$ by $iP = i+1$. We deduce that $pP = x1$. From the $\alpha^{-1}P^{-1}\alpha P$ relator table, we deduce that $p\alpha = xwx^{-1}p$ from the first row, and that $i\alpha^{-1} = w^{-1}i$ for $2 \le i \le p-1$. The p–th row is also complete, but yields no new information, nor does it give any coincidences. We deduce that $1\beta^{-1} = [z, y]1$ from the first row of the $\beta^{-1}[Q, R]$ table. (In this coset enumeration, we only use the first row of this table to obtain a deduction.) From the $\beta^{-1}P^{-1}\beta P$ table, we deduce that $p\beta = x[z, y]x^{-1}p$ from the first row, and that $i\beta^{-1} = [z, y]i$ for $2 \leq i \leq p-1$. The last row is again complete, but yields no other deductions, nor does it give any coicidences. Similarly, we deduce that $pQ^{-1} = xwy^{-1}x^{-1}p$ from the first row of the $\alpha^{-1}[P, Q]$ table, and that $iQ = yw^{i-1}i$ for $2 \leq i \leq p-1$. As above, the last row yields no coincidences or new deductions. Using the $Q^{-1}[P, R]$ table gives $pR^{-1} = xyz^{-1}x^{-1}p$ from the first row and $iR = z\prod_{j=0}^{i} yw^j i$, for $2 \leq i \leq p-1$. These words may be simplified by observing that w is a central elements of G^*, so that for $2 \leq i \leq p-1$ we have $iR = zy^{i-1}w^{s(i)}i$, where $s(i) = i(i-1)/2$. Similarly, the last row gives no other information. At this stage the coset table is complete, and applying the information in the coset table to the other relator tables gives no new deductions, nor does it give any coincidences, and thus H has index p in G^*.

A presentation for H is terms of x, y, z and w is

$$\begin{aligned} 1 &= x^p = y^p = w^p = z^p \\ &= [z, y]^p \\ &= [z, y][y, zy^{i-1}] \text{ for } i = 2, 3, \ldots, p \end{aligned}$$

and x, w and $[z, y]$ are central elements of H. From this presentation, it is easy to see that the quotient of H by the derived group is elementary abelian of order p^4, and hence G^* is of order at least p^5. The method above may be used to show that the subgroup of H generated by x, w and $[z, y]$ has index p^2 in H, and is elementary abelian of order p^3, showing that G^* has order p^6. A more direct way to see that H has order p^5 is to observe that there is a homomorphism φ from H onto the group $\Phi(1^5)$ of order p^5 (see James (1980)) which has defining relations:

$$\begin{aligned} 1 &= \alpha_0^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p \\ \alpha_2 &= [\alpha_1, \alpha_0] \end{aligned}$$

and α_2, α_3 and α_4 are central in $\Phi(1^5)$. The homomorphism φ is defined by

$$\begin{aligned} \varphi(x) &= \alpha_3 \\ \varphi(y) &= \alpha_4 \\ \varphi(z) &= \alpha_0 \\ \varphi(w) &= \alpha_1. \end{aligned}$$

It is also clear that α and β are both contained in $(G^*)' \cap Z(G^*)$. We have that $G \simeq G^*/\langle \alpha, \beta \rangle$, and this suffices to show that the rank of $M(G)$ is two. $G_{p\cdot 14}$ is a pre-image of $(3 \cdot 14)$, and so it remains to show that $G_{p\cdot 14}$ has order dividing p^4 to give an efficient presentation.

As $[a, b]$ is central,

$$1 = [a, b^p] = [a, b^{p-1}][a, b][a, b, b^{p-1}] = [a, b]^p.$$

This completes the proof for this case.

Let $G_{p\cdot 17}$ be the group generated by a and b with the defining relations

$$\begin{aligned} 1 = a^p &= (ab)^p \\ &= [b, a, b] \\ &= [b, a, a, a]; \end{aligned}$$

$G_{p\cdot 17}$ is a pre-image of $(3 \cdot 17)$. Using the notation of Havas and Richardson (1983), define $c_3 = [b, a], c_4 = [c_3, a], c_5 = [c_3, b], c_6 = [c_4, a], c_7 = [c_4, b], c_8 = [c_5, b]$. To show that $G_{p\cdot 17}$ has class at most three, it suffices to show that $c_6 = c_7 = c_8 = 1$ (see Havas and Richardson (1983)). Newman (private communication) has pointed out the result of Sims (1987), where a procedure based on string rewriting rules is described to verify the nilpotence of finitely presented groups. This procedure may also be used to perform calculations similar to those given below.

It is obvious that $c_6 = 1$. We also have $c_5 = [c_3, b] = [b, a, b] = 1$, so $c_8 = 1$, and it remains to show that $c_7 = 1$.

Consider

$$\begin{aligned} [b, a, ab] &= [b, a, b][b, a, a][b, a, a, b] \\ &= [b, a, a][b, a, a, b] \\ &= c_4 c_7 \end{aligned}$$

Also,

$$\begin{aligned} [b, a, ab] &= [b, a, ba[a, b]] \\ &= [b, a, [a, b]][b, a, ba]^{[a,b]} \\ &= [[b, a], [a, b]][b, a, ba]^{[a,b]} \\ &= [b, a, ba]^{[a,b]} \\ &= \{[b, a, a][b, a, b][b, a, b, a]\}^{[a,b]} \\ &= [b, a, a]^{[a,b]} \\ &= c_4^{c_3^{-1}} \end{aligned}$$

So $c_3c_4c_3^{-1} = c_4c_7$, so $c_7 = [c_4, c_3^{-1}]$. We also have

$$\begin{aligned} c_7 &= [c_4, b] \\ &= [c_4, (abc_3)a^{-1}] \\ &= [c_4, a^{-1}][c_4, abc_3]^{a^{-1}} \\ &= [c_4, abc_3]^{a^{-1}} (\text{since } [c_4, a] = 1) \\ &= ([c_4, bc_3][c_4, a]^{bc_3})^{a^{-1}} \\ &= [c_4, bc_3]^{a^{-1}} \\ &= ([c_4, c_3][c_4, b]^{c_3})^{a^{-1}} \\ &= ([c_4, c_3][c_7^{c_3}])^{a^{-1}} \\ &= a[c_4, c_3]c_3^{-1}c_7c_3a^{-1} \\ &= 1 \end{aligned}$$

because $c_7 = [c_4, c_3^{-1}]$, and so $G_{p\cdot 17}$ is of class at most three, and thus $G_{p\cdot 17}$ has order dividing p^4.

Let $G_{p\cdot 15}$ be the group defined by the relations

$$\begin{aligned} a^p &= [u, a, a], 1 = u^p, \\ 1 &= [u, a, u]. \end{aligned}$$

$G_{p\cdot 15}$ is a preimage of the the group defined by $(3\cdot 15)$ in the case when $\alpha = 0$. (The mapping $a \mapsto P, u \mapsto R$ extends to a homomorphism of $G_{p\cdot 15}$ onto $(3\cdot 15)$.) It is clear that $[u, a, a, a] = 1$ and also that $[u, a, u, a] = 1$.

It follows from the argument used above that $[u, a, a, u] = 1$, so $G_{p\cdot 15}$ is of class 3. Let $t = [u, a]$. We want to show that $|G_{p\cdot 15}| \leq p^4$.

Now, $[t, a] = a^p$ is central, so

$$\begin{aligned} 1 &= [t, a^p] \\ &= [t, a \cdot a^{p-1}] \\ &= [t, a^{p-1}][t, a][t, a, a^{p-1}] \\ &= [t, a^{p-1}][t, a] \text{ since } G_{p\cdot 15} \text{ is class 3} \\ &= [t, a^{p-2}][t, a]^2 \\ &= [t, a]^p \end{aligned}$$

Also, let $x = [t^{-1}, a]$, so that $x = a^{-p}$, so γ_2 is cyclic of order at most p, and $G_{p\cdot 15}$ has order dividing p^4. To complete the proof, from the presentation

(3 · 15), the derived group is of order p^2, and hence the multiplicator has order at most $p^{(4-2)(4-2-1)/2} = p$; also, the multiplicator has order at least p, and so the presentation given above is a minimal one. These bounds on the order of the multiplicator also apply in the cases when α is non-zero in the presentation (3 · 15). Let $G_{p\cdot 15a}$ be the group defined by

$$a^p = [u, a, a], a^{\alpha p} = u^p,$$
$$1 = [u, a, u],$$

where α is non-zero. It is clear that the mapping $a \mapsto P, u \mapsto R$ extends to a homomorphism of $G_{p\cdot 15a}$ onto (3 · 15). The argument used above to show that $G_{p\cdot 15}$ is of order p^4 and class 3 is also applicable in this case, and so the presentation given for $G_{p\cdot 15a}$ is a minimal one.

Some Other Groups

In this section we present a number of families of two generator, two relation groups which are shown to be finite, non-metacyclic, and are not wholly contained in the known families of finite deficiency zero groups.

The groups are presented as follows:

$$G_1(n) = (\{x, y\}; x^3 = y^{8n}, x^{y^2} = xx^y),$$
$$G_2(n) = (\{x, y\}; x^3 = y^{8n}, x^{y^2} = [x, y]),$$
$$G_3(n) = (\{x, y\}; x^2 = y^{-3}, x^{y^2} = x^n x^y),$$
$$G_4(n) = (\{x, y\}; x^2 = y^n, y^2 = xyxyx),$$

where n is a non-zero integer.

The structure of $G_1(n)$

Let n be a positive integer (we show below that $G_1(n) \simeq G_1(-n)$), let $G = G_1(n)$, and let A be the preimage of G with relations

$$x^{y^2} = xx^y, [x^3, y] = 1.$$

(A is an infinite group, with A/A' free abelian of rank 1.) We show that the relation $x^3 = 1$ holds in A, and hence in G. The proof below was produced by a computer program (which we have written, and is based on one implemented by Havas and Alford) which implements a method described by Leech (1977) to prove relations in groups. We have

$$\begin{aligned}
1 &= x^2yx^{-1}x^{-1}y^{-1}x \\
&= x^2yx^{-1}y^{-1}x^{-1}y^2xy^{-1}x^{-1}y^{-1}x \\
&= x^2yx^{-1}y^{-2}x^{-1}y^2xyxy^{-1}x^{-1}y^{-1}x \\
&= x^3y^{-1}x^{-2}y^2xyxy^{-1}x^{-1}y^{-1}x \\
&= x^3y^{-1}x^{-2}y^2xy^2x^{-1}y^{-3}x \\
&= x^3y^{-1}x^{-2}y^2x^2yx^{-1}yx^{-2}y^{-3}x \\
&= x^3y^{-1}x^{-2}yx^3yx^{-1}yx^{-1}yx^{-2}y^{-3}x \\
&= x^3y^{-1}x^{-2}yx^3yx^{-1}yx^{-1}yxy^{-1}x^{-3}y^{-2} \\
&= x^3y^{-1}x^{-2}yx^2y^3xy^{-1}x^{-3}y^{-2}x \\
&= x^3y^{-1}xyx^{-1}y^3xy^{-1}x^{-3}y^{-2}x \\
&= yx^3y^{-2}xyx^{-1}y^3xy^{-1}x^{-3}y^{-2}x \\
&= yx^3y^{-2}xyx^{-1}y^3xy^{-1}x^{-2}y^{-1}xy^{-1} \\
&= yx^3y^{-2}xyx^{-1}yxyx^{-3}y^{-1}xy^{-1} \\
&= yx^3y^{-2}xyx^{-1}yxyx^{-3}y^{-1}xy^{-1} \\
&= yx^3y^{-2}xyx^{-1}yx^{-1}y^{-1} \\
&= yx^3y^{-1},
\end{aligned}$$

and thus $x^3 = 1$ in A, and G.

Let $u = x$, $v = x^y$, and let $H = \langle u, v \rangle$. It is easy to see that H is a normal subgroup of G, and from the presentation of G, G/H is cyclic of order $8|n|$. By the above, it's obvious that $u^3 = 1$ holds in H; as v is a conjugate of u, and uv is a conjugate of v, the relations $v^3 = (uv)^3 = 1$ follow. Using the relation $x^{y^2} = xx^y$, we have $x^{y^2}(x^{-1})^y = x$, and conjugating by y^{-1} gives $x^y x^{-1} = x^{y^{-1}}$, *i.e.* $vu^{-1} = x^{y^{-1}}$, and the relation $(uv^{-1})^3 = 1$ follows.

This shows that the group 3^3 is a preimage of H, and so G has order dividing $2^3 3^3|n|$.

Let a and b be elements of order 3 which generate 3^3, and let K be the split extension of 3^3 by a cyclic group of order $8n$ generated by c, with the action of c given by $a^c = b$, $b^c = ab$. We show below that G is isomorphic to K. K is generated by a and c, has order $2^3 3^3 n$, and the following relations hold:

$$a^3 = c^{8n} \; (= 1),$$

and

$$a^{c^2} = b^c = ab.$$

It is easy to see that the mapping of G onto K given by $x \mapsto a$, $y \mapsto c$ extends to a homomorphism of G onto K, and this establishes the isomorphism. Observe also that $G_1(n) \cong G_1(-n)$.

The Structure of $G_2(n)$

Let $G = G_2(n)$. In this section we show that G is finite of order dividing $2^8 3|n|^3$ by using coset enumeration and the Reidemeister-Schreier algorithm (see Johnson (1990)).

Let H be the subgroup of G generated by $y^2, xy^2x^{-1}, xy^{-1}xy$. H is a normal subgroup of G (see Chapter Two), and a transversal for H in G is easily seen to be the set

$$\{1, y, x, yx^{-1}, x^{-1}, yx\}.$$

Applying the Reidemeister-Schreier algorithm then gives the following presentation for H

$$\begin{aligned}
1 &= cd^{-1}ag^{-1} \\
1 &= cg^{-1}bd^{-1} = ag^{-1}f^{-1}e \\
&= ef^{-1}g^{-1}b = ce^{-1}d^{-1}af \\
&= cfbd^{-1}e^{-1} \\
a &= b = c^{4n}, \\
1 &= a(de)^{-4n} = a(fg)^{-4n}, \\
b &= (ea)^{4n} = (gf)^{4n},
\end{aligned}$$

where $a = x^3, b = yx^3y^{-1}, c = y^2, d = xyxy^{-1}, e = (yx^{-1})^2$, $f = x^{-1}yx^{-1}y^{-1}$, and $g = (yx)^2$.

Eliminating b, f and g from the presentation, and using the fact that a is central, a simplified presentation for H is:

$$\begin{aligned}
1 &= [c^{-1}, ad^{-1}] = [ae, cd^{-1}] \\
a &= [d^{-1}, e^{-1}]
\end{aligned}$$

$$
\begin{aligned}
&= [ed, c] = c^{4n} \\
&= (edc^{-1}a^{-1}de^{-1})^{4n} \\
&= (a^{-1}dc^{-1}edc^{-1})^{4n} \\
&= (de)^{-4n}.
\end{aligned}
$$

Hence $H/H' = (\{c, d, e\}; c^{4n} = d^{4n} = e^{4n} = 1$, abelian $)$ and $|H/H'| = 2^6|n|^3$, and so $|G|$ is divisible by $2^7 3|n|^3$. Note that since $H/\langle a\rangle \cong H/H'$, we have $\langle a\rangle = H'$.

To complete the proof of the finiteness of G, we show that $x^6 = 1$ in G by considering the (infinite) preimage K of G defined by the relations

$$
\begin{aligned}
x^{y^2} &= [x, y], \\
1 &= [x^3, y].
\end{aligned}
$$

In K,

$$
\begin{aligned}
1 &= yx^{-1}yxy^{-2}x \\
&= yx^2yx^{-2}y^{-2}x \\
&= yx^2yx^{-1}y^{-2}xyx^{-1}y^{-1}x \\
&= yx^2y^{-1}xyx^{-1}y^{-1}xyx^{-1}y^{-1}x \\
&= x^3yx^{-1}y^{-1}xyx^{-1}y^{-1}xyx^{-1}y^{-1}x \\
&= x^3yx^{-1}y^{-1}xyx^{-1}y^{-1}xy^{-1}xy \\
&= x^3yx^{-1}y^{-1}xy^{-1}x^2y \\
&= x^3y^{-1}x^3y \\
&= x^6.
\end{aligned}
$$

Since $a = x^3$, H' is of order 2, and so G is finite of order dividing $2^8 3|n|^3$.

We now consider the structure of G/H and H. It is easy to see that G/H is isomorphic to the symmetric group on three letters. Let $\alpha = c, \beta = cd^{-1}$, and $\gamma = cd^{-1}e$; then $H = \langle \alpha, \beta, \gamma\rangle$ and a presentation of H in terms of α, β, γ is

$$
\begin{aligned}
1 &= \alpha^{8n} = \beta^{8n} = \gamma^{8n} \\
a &= (\alpha\gamma)^{-4n} = \beta^{4n} = \alpha^{4n} = [\gamma, \alpha], \\
1 &= [\gamma, \beta] = [\alpha, \beta] \\
&= [a, \alpha] = [a, \beta] = [a, \gamma] \\
&= a^2.
\end{aligned}
$$

The structure of $G_3(n)$

Let n be a non-zero integer, and $G = G_3(n)$, where

$$G_3(n) = (\{x, y\}, \{x^2 = y^{-3}, x^{y^2} = x^n x^y\}).$$

We show that G is finite, and of order $2^3 3|n|$ when n is odd. G/G' is of order $3|n|$, and cyclic when n is odd. If $|n|$ is even, x^n is a central element, and from the second relation we get $y^{-2}xy = x^n y^{-1}x$, and so $y^{-1}xy = x^{n+1}$ and thence $[x, y] = x^n$. This relation shows that the subgroup generated by x is normal, and so $G_3(n)$ is metacyclic. From now on, suppose that n is odd. The subgroup $\langle x^4 \rangle$ is normal and it may be seen by either coset enumeration, or directly, that $G/\langle x^4 \rangle$ is of order 24; thus G is finite, and we obtain an upper bound on the order of G, by showing that the relation $x^{4n} = 1$ holds . In G,

$$\begin{aligned}
1 &= yx^2y^2 \\
&= y^2x^{-n}y^{-2}xyxy^2 \\
&= y^2x^{-n}y^{-1}x^{-n}y^{-2}xy^2xy^2 \\
&= y^2x^{-n}y^{-1}x^{-n}y^{-2}xy^{-1}x^{-1}y^2 \\
&= y^2x^{-n}y^{-1}x^{n}y^{3n-2}xy^{-1}x^{-1}y^2 \\
&= y^2x^{-2n}y^{-2}xyx^{n-1}y^{3n-2}xy^{-1}x^{-1}y^2 \\
&= y^{3n}xyx^{n-1}y^{3n-2}xy^{-1}x^{-1}y^2 \\
&= y^{3n}xyx^{n-1}y^{3n-2}xy^{-1}x^{-1}y^{-1}x^{-2} \\
&= y^{3n}xyx^{n-1}y^{3n-2}x^{-n+1}y^{-2}xyx^{-2}y^{-1}x^{-2} \\
&= y^{3n}xyx^{n-1}y^{3n-2}x^{-n+1}y^{-2}xy^3x^{-2} \\
&= xyx^{n-1}y^{6n-2}x^{-n+1}y^{-2}xy^3x^{-2} \\
&= xyx^{n-1}y^{6n+1}x^{-n+1}y^{-2}x^{-1} \\
&= yx^{n-1}x^{-4n}yx^{-n+1}y^{-2} \\
&= x^{-4n}x^{n-1}yx^{-n+1}y^{-1} \\
&= x^{-4n}[x^{-n+1}, y^{-1}].
\end{aligned}$$

Since all even powers of x are central, $x^{4n} = 1$, and $|G|$ divides $2^3 3|n|$.

Let $Q = \langle a,b \rangle$ be the quaternion group, and $A = \langle \alpha \rangle$ be of order $3t$, t odd. Let H be the split extension of Q by A with the presentation

$$a^4 = 1, b^2 = a^2, \alpha^{3t} = 1,$$

$$a^\alpha = b, b^\alpha = b^{-1}a^{-1},$$

$$[a^2, \alpha] = [a^2, b] = [\alpha^3, a] = [\alpha^3, b] = 1.$$

Let $S = \alpha^2$, and $T = a^{-1}\alpha^\lambda$, where

$$\lambda = \begin{cases} -3(t+1)/2, & \text{if } t \equiv -1 (\text{mod } 4), \\ 3(t-1)/2, & \text{if } t \equiv 1 (\text{mod } 4). \end{cases}$$

Then α^λ is a central element of H, and we show that $H = \langle S, T \rangle$ and that S, T satisfy the relations $S^{-3} = T^2, T^{S^2} = T^t T^S$.

Suppose that $t \equiv -1(\text{mod } 4)$; then $\alpha = S\alpha^2$ and $\alpha^\lambda = S^\lambda a^{-2\lambda}$ as a^2 is central. Now, $-2\lambda = 3(t+1) \equiv 0(\text{mod } 4)$ and $a^{-2\lambda} = 1$; hence $\alpha^\lambda = S^\lambda$. We also have $a = \alpha^\lambda T^{-1} = S^\lambda$, and $\alpha = ST^2S^{-2\lambda}$. Thus S and T generate H. Also, $S^{-3} = \alpha^{-3}a^2$, $T^2 = (a^{-1}\alpha^\lambda)^2 = a^{-2}\alpha^{2\lambda} = a^2\alpha^{-3} = S^{-3}$, and so,

$$\begin{aligned} T^{S^2}(T^{-1})^S &= \alpha^{-2}a^{-1}\alpha^{\lambda+1}a^2\alpha^{-\lambda}a\alpha a^2 \\ &= \alpha^{-2}a^{-1}\alpha a\alpha \\ &= a^{-1}. \end{aligned}$$

Thus $T^t = a^{-t}\alpha^{\lambda t} = a^{-1}$, and $T^{S^2} = T^t T^S$.

For $t \equiv 1(\text{mod } 4)$, similar arguments show that S and T generate H, and that the two relations hold. Thus H is a quotient of G, and $|G| = 24|n|$ when n is odd.

The Structure of $G_4(n)$

Let H be the group defined by

$$(\{x, y\}; [x^2, y] = 1, y^2 = xyxyx);$$

then H is infinite since H/H' is the direct product of a group of order 3, and a free abelian group of rank 1. We show that H' is a quotient of Q_8.

Let $a = yxy^{-1}x^{-1}, b = xy^{-1}x^{-1}y$. From the second relation of H, and as x^2 is central, we have $a = y^{-1}x^3y$,and $b = yx^3y^{-1}$; thus $a^2 = b^2 = x^6$. Firstly we show that $\langle a, b \rangle = H'$.

It is obvious that $a^y = b$; $b^y = y^{-1}xy^{-1}x^{-1}y^2 = y^{-1}xy^{-1}x^{-1}xyxyx = y^{-1}x^2yx = x^3$, and, $ba = xy^{-1}x^{-1}yyxy^{-1}x^{-1} = xy^{-1}x^{-1}xyxyxxy^{-1}x^{-1} = x^3 = b^y$.

Before showing that x normalizes $\langle a, b\rangle$, we show that $a^4 = 1$:

$$\begin{aligned} a^2 &= yxy^{-1}x^{-1}yxy^{-1}x^{-1} = yxyy^{-2}x^{-1}yxy^{-1}x^{-1} \\ &= yxyx^{-1}y^{-1}x^{-1}y^{-1}x^{-1}x^{-1}yxy^{-1}x^{-1} \\ &= x^{-2}yxyx^{-1}y^{-2}x^{-1} \\ &= x^{-6}, \end{aligned}$$

and so $a^4 = x^{12} = 1$.

Similarly,

$$\begin{aligned} a^x &= x^{-1}yxy^{-1} = x^{-1}yxy^{-2}y \\ &= x^{-1}yxx^{-1}y^{-1}x^{-1}y^{-1}x^{-1}y \\ &= x^{-2}y^{-1}x^{-1}y = x^{-2}y^{-1}x^{-2}xy = x^{-4}yy^{-2}xy \\ &= x^{-4}y(xyxyx)^{-1}xy = x^{-4}yx^{-1}y^{-1}x^{-1} \\ &= x^8yx^{-1}y^{-1}x^{-1} \\ &= x^6yxy^{-1}x^{-1} \\ &= a^2a = a^{-1}, \end{aligned}$$

and,

$$\begin{aligned} b^x &= y^{-1}x^{-1}yx = x^{-2}y^{-1}xyx \\ &= x^{-2}yy^{-2}xyx \\ &= x^{-2}y(xyxyx)^{-1}xyx \\ &= x^{-2}yx^{-1}y^{-1} \\ &= x^{-2}yx^{-1}y^{-2}y \\ &= x^{-2}yx^{-1}(xyxyx)^{-1}y \\ &= x^{-4}x^{-1}y^{-1}x^{-1}y \\ &= x^8x^{-1}y^{-1}x^{-1}y \\ &= x^6xy^{-1}x^{-1}y \\ &= b^2b = b^{-1}. \end{aligned}$$

This shows that $\langle a, b\rangle$ is a normal subgroup of H; we now show that $\langle a, b\rangle$ is a quotient of Q_8 by showing that $ba = a^{-1}b$.

From above, $ba = x^3$, and

$$\begin{aligned}a^{-1}b &= y^{-1}x^{-3}yyx^3y^{-1}\\ &= y^{-1}x^{-3}xyxyxx^3y^{-1}\\ &= xyxxy^{-1}\\ &= x^3.\end{aligned}$$

Since $H/\langle a,b\rangle \cong H/H'$, $\langle a,b\rangle = H'$, and the proof is complete.

Applying the above result to $G = G_4(n)$ shows that since $|G/G'| \leq 3|n|$, $|G| \leq 2^3 3|n|$. Assume now that n is positive (the case when n is negative is similar). Observe that both y^n and y^3 are central elements of $G_4(n)$. When n is relatively prime to three, $G_4(n)$ is abelian of order $3|n|$. When n is divisible by three, there is a homomorphism φ of $G_4(n)$ onto the group of order $2^3 3n$ with defining relations

$$\begin{aligned}1 &= a^4 = Y^{6n},\\ a^2 &= b^2 = Y^{3n},\\ ba &= a^{-1}b,\\ a^Y &= b,\ b^Y = ba\end{aligned}$$

given by $\varphi(x) = Y^{-1}bY^{-n+1}$ and $\varphi(y) = Y$, showing that $|G_4(n)| = 2^3 3n$.

Chapter Four

Some New Efficient or Nearly Efficient Soluble Groups

Johnson and Robertson (1978) gave a survey of the finite groups of deficiency zero known up to 1976. They observed that all known finite soluble groups of deficiency zero had soluble length less than five, and conjectured that the soluble length of a finite group having deficiency zero is bounded. Subsequent work on groups of deficiency zero (Campbell and Robertson (1978,1980ab,1982ab,1984abc)), Campbell, Robertson and Thomas (1987abc,1988), Campbell and Thomas (1987), Campbell *et al.* (1986), Kenne (1983,1986), Mennicke and Neumann (1987), Neumann (1985,1987) and Robertson (1980,1982)) did not provide an example of a finite group of deficiency zero with soluble length five or greater.

The first example of a finite group of soluble length five having deficiency zero was given by Kenne (1988). Additional examples of individual groups of soluble length five having deficiency zero are given below. The first example of an infinite family of finite groups of soluble length five having deficiency zero is given in Theorem 1. Newman and O'Brien (19xx) have also given an example of a finite group of soluble length five having deficiency zero which is distinct from all the examples given in this chapter. Theorem 3 gives an infinite family of groups of soluble length six having deficiency zero, and Theorem 4 gives a further example of a finite group of soluble length six having deficiency zero.

The more general problem remains open. Parts of this chapter have appeared as Kenne (1988,1990). We also give an example of a deficiency one presentation for a finite preimage of a group of soluble length seven.

Soluble length five

In this section, we give a number of examples of finite groups having deficiency zero and soluble length five. The structure of these examples is quite similar (semi-direct products of extra special groups with relevant linear groups, and direct products of such groups with cyclic groups).

The following theorem exhibits (disjoint) families of finite groups of soluble length five having deficiency zero. We first establish some notation.

Let E_p be the non-abelian group generated by x and y having exponent p and order p^3 where p is a prime. E_p is presented by

$$(4\cdot 1)\qquad \begin{aligned} 1 &= x^p = y^p = [x,y]^p \\ &= [x,[x,y]] = [y,[x,y]]. \end{aligned}$$

There are automorphisms α, β of E_3 defined by

$$\begin{aligned} \alpha(x) &= y^{-1}x^{-1}, \\ \alpha(y) &= y^{-1}xy, \\ \beta(x) &= yxy, \\ \beta(y) &= x^{-1}y^{-1}. \end{aligned}$$

We now show that the group generated by α and β is isomorphic to $SL(2,3)$, the 2-dimensional special linear group over the field of three elements. Observe that $SL(2,3)$ is generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

and that A and B satisfy the following relations:

$$(4\cdot 2)\qquad \begin{aligned} 1 &= A^3 = B^4 = AB(AB^{-1})^2 \\ &= AB^2A^{-1}B^{-2}. \end{aligned}$$

A somewhat tedious calculation shows that α and β also satisfy the relations

$$(4\cdot 3)\qquad \begin{aligned} 1 &= \alpha^3 = \beta^4 = \alpha\beta(\alpha\beta^{-1})^2 \\ &= \alpha\beta^2\alpha^{-1}\beta^{-2} \end{aligned}$$

and so the order of $\langle \alpha, \beta \rangle$ divides 24. Since $\beta^2(x) = yx^{-1}y^{-1}$, we have that the order of β is four. This suffices to show that the order of $\langle \alpha, \beta \rangle$ is 24 since $SL(2,3)$ has a unique minimal normal subgroup of order two generated by B.

Let E_3S be the semi-direct product of E_3 and $SL(2,3)$ with the action given above. E_3S is a group of order 648 and soluble length five. We show that E_3S has a deficiency zero presentation. The direct product of E_3S with a cyclic group C_k of order k is presented by

$$
\begin{aligned}
1 &= x^3 = y^3 = [x,y]^3 \\
&= [x,[x,y]] = [y,[x,y]] \\
&= c^k = a^3 = b^4 \\
&= ab(ab^{-1})^2 \\
&= ab^2a^{-1}b^{-2}, \\
x^a &= y^{-1}x^{-1}, \\
y^a &= y^{-1}xy, \\
x^b &= yxy, \\
y^b &= x^{-1}y^{-1}, \\
1 &= [c,a] = [c,b] \\
&= [c,x] = [c,y].
\end{aligned}
\tag{4·4}
$$

Theorem 1. *Let G_k be the direct product of E_3S and the cyclic group of order k, where k is coprime to 12; G_k is a group of soluble length five having deficiency zero.*

We prove Theorem 1 by establishing the following lemmas:

Lemma 1a. *Let $G_1(\alpha)$ be defined by the relations*

$$
\begin{aligned}
(AB)^3 &= B^4, \\
1 &= A^3B^{\alpha}A^{-1}B^{\alpha+1}AB^{\alpha+2},
\end{aligned}
\tag{4·5}
$$

where $\alpha \equiv 1 (\mathrm{mod}\ 4)$; $G_1(\alpha)$ is isomorphic to the direct product $E_3S \times C_{|3\alpha+4|}$.

Lemma 1b. *Let $G_2(\alpha)$ be defined by the relations*

$$
\begin{aligned}
AB &= (B^3A^{-1})^2, \\
1 &= A^3B^{\alpha}AB^{\alpha-1}A^{-1}B^{\alpha-2},
\end{aligned}
\tag{4·6}
$$

where $\alpha \equiv 3 (\mathrm{mod}\ 4)$; $G_2(\alpha)$ is isomorphic to the direct product $E_3S \times C_{|3\alpha+2|}$.

Proof

The deficiency zero presentation for G_k when $k \equiv 5 (\mathrm{mod}\ 12)$ or $k \equiv 7 (\mathrm{mod}\ 12)$ is the presentation for $G_1(\alpha)$ where $k = |3\alpha + 4|$, and in the cases when $k \equiv 1 (\mathrm{mod}\ 12)$ or $k \equiv 11 (\mathrm{mod}\ 12)$, is the presentation for $G_2(\alpha)$, where $k = |3\alpha + 2|$.

We will first prove Lemma 1a. The quotient by the derived group of $G_1(\alpha)$ is cyclic of order $3|3\alpha + 4|$. To show that $G_1(\alpha)$ is finite, observe that B^4 is central and consider the quotient $G_1(\alpha)/\langle B^4 \rangle$ which is presentable with relations

$$(4 \cdot 7) \qquad \begin{aligned} 1 = (AB)^3 &= B^4, \\ &= A^3 B^\alpha A^{-1} B^{\alpha+1} A B^{\alpha+2}. \end{aligned}$$

Since B has order 4, this presentation defines a single group, and a coset enumeration using Cayley shows that this group has order 216 and soluble length four). This completes the finiteness proof for $G_1(\alpha)$.

Let $n = |3\alpha + 4|$. Since n is relatively prime to 3, there are integers u and v such that $3u + nv = 1$. Put $t = -u$, and note that there is a homomorphism θ_1 of $G_1(\alpha)$ onto $E_3 S \times C_{|3\alpha+4|}$ given by

$$\begin{aligned} \theta_1(A) &= ba^{-1}c^t, \\ \theta_1(B) &= b^{-1}yc^{-1}. \end{aligned}$$

To see that θ_1 is onto $E_3 \times C_{3|\alpha+4|}$, let $Q = \theta_1(x^p) = ba^{-1}$ and $R = Q^{-1}\theta_1((xy^{-1})^p) = Q^{-1}ba^{-1}y^{-1}b$. We then have that $b = QRQ^{-2}R^{-1}Q$ which shows that θ_1 is onto.

The proof of Lemma 1b is similar. Again, B^4 is a central element of $G_2(\alpha)$, and the quotient $G_2(\alpha)/\langle B^4 \rangle$ presented by

$$(4 \cdot 8) \qquad \begin{aligned} 1 &= B^4 \\ &= A^3 B^\alpha A B^{\alpha-1} A^{-1} B^{\alpha-2}, \\ AB &= (B^3 A^{-1})^2 \end{aligned}$$

defines a single group, not an infinite family. This group has order 216 and soluble length four. As the quotient to the derived group is of order $3|3\alpha + 2|$, $G_2(\alpha)$ is finite for all $\alpha \equiv 3 (\mathrm{mod}\ 4)$.

As for Lemma 1a, observe that there is a homomorphism θ_2 of $G_2(\alpha)$ onto $E_3S \times C_{|3\alpha+2|}$ given by

$$\theta_2(A) = abac^5,$$
$$\theta_2(B) = y^{-1}a^{-1}b^{-1}ax^{-1}c^3.$$

To see that θ_2 is onto $E_3 \times C_{|3\alpha+2|}$, let $V = \theta_2(A^p) = (aba)^p$ and $W = \theta_2(B^p) = (y^{-1}a^{-1}b^{-1}ax^{-1})^p$. Since $(aba)^6 = (y^{-1}a^{-1}b^{-1}ax^{-1})^{12} = 1$ and $p \equiv 1 (\mathrm{mod}\ 12)$ or $p \equiv -1 (\mathrm{mod}\ 12)$, we have that $V = aba$ or $V = (aba)^{-1}$, depending upon the value of p. There is a similar situation for W. In the case that $V = aba$ and $W = y^{-1}a^{-1}b^{-1}ax^{-1}$, we have that $y = V^{-1}W^2VW^2$, $a = VWV^{-2}W^{-1}$ and $b = VW^{-2}V^{-1}W^{-1}$, showing that θ_2 is onto. The other cases are similar.

This completes the proof of Theorem 1.

The groups defined in Theorem 1 are not the only finite groups having soluble length five and deficiency zero. Theorem 2 below provides further examples of such groups.

Let G_1 be the group generated by u and v with the defining relations

$$1 = uv^2(uv^{-1})^2 = (u^2v)^2u^{-1}vu^2(vuv)^{-1}, \tag{4·9}$$

let G_2 be the group generated by u and v with the defining relations

$$1 = u^2v^{-1}uv^2u^2v^{-2} = u^2v^{-2}u^{-1}v^{-3}uv^{-2}u^{-1}v^{-1}, \tag{4·10}$$

let G_3 be the group generated by u and v with the defining relations

$$u^2 = ((uv)^3)^v = (v^{-3})^{u^{-1}v^{-1}}, \tag{4·11}$$

let G_4 be the group generated by u and v with the defining relations

$$1 = u^3v^{-1}u^{-1}v^{-1}u^{-1}v^{-1} = u^3vu^2v^{-1}uv^{-3}, \tag{4·12}$$

let G_5 be the group generated by u and v with the defining relations

$$1 = u^3vu^{-1}vu^{-1}v \tag{4·13}$$
$$= u^3v^{-3}uvu^2v^{-1},$$

and, G_6 be the group generated by u and v with the defining relations

$$1 = u^2vu^{-1}vu^{-1}v^{-1}uv^{-2}$$
$$= u^2v^{-1}uv^{-1}uvu^{-1}v^2.$$

Theorem 2. *Each of the groups G_1, G_2, G_3, G_4, G_5 and G_6 is finite and has soluble length five.*

Proof:

Let E_5 be generated by x and y. It is straightforward to check that there are automorphisms α, β of E_5 defined by

$$\alpha(x) = xy^{-1}x^2,$$
$$\alpha(y) = x^2yx,$$
$$\beta(x) = x^{-1}y^2xy^{-1},$$
$$\beta(y) = x^{-1}.$$

A straightforward calculation shows that α and β satisfy the relations

$$1 = \alpha^3 = \beta^4 = \alpha\beta(\alpha\beta^{-1})^2 \tag{4·14}$$
$$= \alpha\beta^2\alpha^{-1}\beta^{-2}$$

and so the order of $\langle\alpha, \beta\rangle$ divides 24. Since $\beta^2(x) = xy^{-1}x^{-2}y$, we have that the order of β is four. This suffices to show that $\langle\alpha, \beta\rangle$ is 24 since $SL(2,3)$ has a unique minimal normal subgroup of order two generated by B.

Let E_5S be the semi-direct product of E_5 and $< \alpha, \beta >$, with the action of α and β as given above. E_5S has order 3000, and is of soluble length 5. It has the following defining relations

$$
\begin{aligned}
1 = x^5 = y^5 &= [x,y]^5 = [x,[x,y]] = [y,[x,y]] \\
&= \alpha^3 = \beta^4 = \alpha\beta(\alpha\beta^{-1})^2 \\
&= \alpha\beta^2\alpha^{-1}\beta^{-2}
\end{aligned}
$$

$$
\begin{aligned}
x^\alpha &= xy^{-1}x^2, \\
y^\alpha &= x^2yx, \\
x^\beta &= x^{-1}y^2xy^{-1}, \\
y^\beta &= x^{-1}.
\end{aligned} \tag{4 \cdot 15}
$$

Put $u = \alpha y^{-1}x^{-1}\beta\alpha$ and $v = x\alpha^{-1}y\alpha^{-1}\beta$. E_5S is generated by u and v because $\alpha = u^{-3}v^{-1}u^{-2}$, $\beta = v^4u^{-2}$, $x = u^{-1}v^{-2}uvu^{-2}v^{-1}u$ and $y = uv^4u^{-2}v$. Defining relations for E_5S in terms of u and v were constructed using Cannon's algorithm and were found to be

$$
\begin{aligned}
1 &= uv^2(uv^{-1})^2 \\
&= (uv)^3uv^{-2} \\
&= (u^2v)^2u^{-1}vu^2(vuv)^{-1}, \\
u^6 &= v^{-6}.
\end{aligned} \tag{4 \cdot 16}
$$

A coset enumeration shows that the preimage of E_5S generated by u and v with the defining relations

$$
\begin{aligned}
1 &= uv^2(uv^{-1})^2 \\
&= (u^2v)^2u^{-1}vu^2(vuv)^{-1},
\end{aligned} \tag{4 \cdot 17}
$$

has order 3000, and thus E_5S has a deficiency zero presentation. This completes the proof for G_1.

We now consider G_2. Let E_5SC be the direct product of E_5S with a cyclic group C of order 13 generated by c.

Let G_2 be the group generated by X and Y having the defining relations

$$(4 \cdot 18) \qquad \begin{aligned} 1 &= X^2Y^{-1}XY^2X^2Y^{-2} \\ &= X^2Y^{-2}X^{-1}Y^{-3}XY^{-2}X^{-1}Y^{-1}. \end{aligned}$$

Coset enumeration shows that G_2 has order 39000. There is a homomorphism $\gamma : G_2 \to E_5SC$ defined by

$$\begin{aligned} \gamma(X) &= x^2\beta x\alpha^{-1}c^6, \\ \gamma(Y) &= \alpha y\beta^2c^4, \end{aligned}$$

which is onto E_5SC because

$$\begin{aligned} x &= \gamma(X^4Y^{-1}X^2Yx^{-1}Y^{-1}), \\ y &= \gamma(YX^3YX^{-3}Y^{-2}), \\ \alpha &= \gamma(Y^{-1}X^{-1}YX^{-1}YX^{-1}Y^{-3}), \\ \beta &= \gamma(X^2Y^{-1}X^3YXY^{-1}X^{-1}), \\ c &= \gamma((XY)^4). \end{aligned}$$

Since E_5SC has soluble length five, this completes the proof for G_2.

Another example of a finite group of soluble length five having deficiency zero with a structure slightly different from those given above is shown below.

Let E_3S be the semi-direct product of E_3 and $SL(2,3)$ with the action given above. E_3S is a group of order 648 and soluble length five. We show that E_3S has a deficiency zero presentation. Let G_3 be the group generated by a and b subject to the defining relations

$$(4 \cdot 19) \qquad \begin{aligned} a^2 &= ((ab)^3)^b \\ &= (b^{-3})^{a^{-1}b^{-1}}. \end{aligned}$$

In a similar manner to that above, we show that G_3 is isomorphic to E_3S in two stages. A coset enumeration over the identity shows that G_3 has order

648, and to complete the proof, note that there is a homomorphism δ from G_3 onto E_3S defined by

$$\delta(a) = \beta^{-1},$$
$$\delta(b) = \beta^{-1}x\alpha^{-1}\beta^{-1}.$$

To see that δ is in fact *onto* E_3S, observe that $x = \delta(baba^{-1}b^{-2}a^{-1})$ and $y = \delta(a^2b^3)$. In the next section, we provide another proof of the finiteness of G_3 by exhibiting it as the derived group of a finite group of soluble length six and deficiency zero.

Using similar methods to those above (which are not recorded here), we show that G_4 and G_5 are direct products of E_3S with cyclic groups.

A presentation for the direct product of E_3S and a cyclic group of order five is given by

$$(4\cdot 20) \qquad \begin{aligned} 1 &= u^3v^{-1}u^{-1}v^{-1}u^{-1}v^{-1}, \\ &= u^3vu^2v^{-1}uv^{-3}, \end{aligned}$$

where $u = c^2a^{-1}b^2a^2b$ and $v = a^{-1}b^{-1}c^{-1}$. (The cyclic group of order five is generated by c; a and b are the generators of E_3S.)

A presentation for the direct product of E_3S and a cyclic group of order seven is given by

$$(4\cdot 21) \qquad \begin{aligned} 1 &= u^3vu^{-1}vu^{-1}v, \\ &= u^3v^{-3}uvu^2v^{-1}, \end{aligned}$$

where $u = ab^3c^2$ and $v = ab^{-1}ab^{-1}a^2c^{-3}$.

Let G be the group generated by x, y, α and β subject to the defining relations

$$
(4\cdot 22)\qquad
\begin{aligned}
1 &= x^5 = y^5 = [x,y] \\
&= \alpha^4 = (\alpha\beta)^2\alpha\beta^{-1} \\
&= [\alpha,\beta^2], \\
x^\alpha &= x^{-1}, x^\beta = xy, \\
y^\alpha &= x^2y^{-2}, y^\beta = x^2y^{-1}.
\end{aligned}
$$

G is the semidirect product of the elementary abelian group of order 25 with a group of order 96 having soluble length four. G has soluble length five.

Finally, let G_6 be the group generated by X and Y with the defining relations

$$
(4\cdot 23)\qquad
\begin{aligned}
1 &= X^2YX^{-1}YX^{-1}Y^{-1}XY^{-2} \\
&= X^2Y^{-1}XY^{-1}XYX^{-1}Y^2.
\end{aligned}
$$

There is a homomorphism $\omega : G_6 \to G$ defined by

$$
\begin{aligned}
\omega(X) &= x^{-1}\beta\alpha^{-1}\beta, \\
\omega(Y) &= x^2\beta.
\end{aligned}
$$

and ω is onto G because

$$
\begin{aligned}
x &= \omega(Y^2X^{-1}Y^{-2}X), \\
y &= \omega(X^{-4}), \\
\alpha &= \omega(Y^{-1}X^{-1}Y^3), \\
\beta &= \omega(XY^2X^3Y^{-1}).
\end{aligned}
$$

A coset enumeration shows that G_6 has order 2400 and is thus isomorphic to G.

In summary, Theorems 1 and 2 have provided deficiency zero presentations for the split extensions of E_3 and E_5 by $SL(2,3)$, and for some direct products of these groups by cyclic groups of odd order. An exceptional case is the split extension of $C_5 \times C_5$ by a group order 96.

Soluble length six

The following theorem exhibits an infinite family of finite groups of deficiency zero having soluble length six.

Theorem 3. *Let $G(k)$ be the group generated by a and b subject to the defining relations*

$$(4 \cdot 24) \qquad \begin{aligned} (ab)^2 &= b^6 \\ 1 &= a^4 b^{-1} a b^k a^{-1} b. \end{aligned}$$

If $k \equiv 3 \pmod 6$, $G(k)$ is a group of order $1296|k+8|$ and has soluble length six. $G(k)$ is isomorphic to a direct product $H \times C_{|k+8|}$, where H is a group of order 1296, and C_n is the cyclic group of order n.

Proof:

Observe that the quotient by the derived group is cyclic of order $2|k+8|$. Since $(ab)^2$ is a power of b, it commutes with b, and since it clearly commutes with ab, it is a central element of $G(k)$. To establish that $G(k)$ is finite, we appeal to a theorem of Schur (see Chapter Two). Consider the quotient $G^*(k) = G(k)/\langle (ab)^2 \rangle$. When $k \equiv 3 \pmod 6$, $G^*(k)$ has the defining relations

$$(4 \cdot 25) \qquad \begin{aligned} 1 &= (ab)^2 = b^6 \\ &= a^4 b^{-1} a b^3 a^{-1} b. \end{aligned}$$

A straightforward coset enumeration using Cayley shows that this quotient has order 1296, completing the finiteness proof for $G(k)$, $k \equiv 3 \pmod 6$.

Let $G = \langle x, y \rangle$ be the non-abelian group of order 27 with exponent 3; then G has defining relations

$$(4 \cdot 26) \qquad x^3 = y^3 = (xy)^3 = (xy^{-1})^3 = 1,$$

and it is easy to see that there are automorphisms α, β of G given by $\alpha(x) = x$, $\alpha(y) = yx^{-1}$, $\beta(x) = y$, $\beta(y) = x$.

A calculation using Cayley shows that α and β generate a group isomorphic to $GL(2,3)$, the two dimensional general linear group over the field of three elements, of soluble length four. Let H be the semidirect product of G and $\langle \alpha, \beta \rangle$, with the action of α and β given above. H is a group of order 1296 and soluble length 6.

Let $H(k)$ be the direct product of H with a cyclic group of order $|k+8|$ generated by g. A generating set for $H(k)$ consists of x, y, α, β and g. This generating set satisfies the relations

$$\begin{aligned} 1 &= x^3 = y^3 = (xy)^3 = (xy^{-1})^3 \\ &= \alpha^3 = \beta^2 = ((\beta\alpha)^2\beta\alpha^2)^2, \\ x^\alpha &= x, \ y^\alpha = yx^{-1}, \\ x^\beta &= y, \ y^\beta = x, \\ 1 &= g^{|k+8|} \\ &= [x,g] = [y,g] \\ &= [\alpha,g] = [\beta,g]. \end{aligned} \tag{4·27}$$

There is a homomorphism $\theta \colon G(k) \to H(k)$ defined by

$$\theta(a) = \beta\alpha^{-1}g^{-2},$$
$$\theta(b) = \beta x^{-1}\alpha^{-1}\beta\alpha g^{-1}.$$

If $u = \theta(a)$ and $v = \theta(b)$, then

$$\begin{aligned} xg^{-5} &= uv^{-1}u^3v^2u^{-2} \\ yg^{-7} &= (uv^{-1})^2uv^2uv^{-1} \\ \alpha g^{-5} &= u^2v^{-1}uvu^{-1}v \\ \beta g^{-3} &= vu^{-1}vu^2v^{-1}, \end{aligned}$$

and so $g^6 = (vu^{-1}vu^2v^{-1})^{-2}$. Since $g^{|k+8|}$ is trivial and $k \equiv 3 \pmod 6$, the order of g is relatively prime to 6, and so g is in the group generated by u and v. This establishes that θ is onto $H(k)$; thus $G(k)$ is a finite group with soluble length at least six.

Observe that $\ker(\theta)$ is contained in the derived group of $G(k)$. It is also contained in the centre of $G(k)$. To see this, the structure of $G^*(k)$ needs to be determined. In a manner similar to that above, there is a homomorphism $\theta_1 : G^*(k) \to H$ defined by

$$\theta_1(a) = \beta\alpha^{-1},$$
$$\theta_1(b) = \beta x^{-1}\alpha^{-1}\beta\alpha.$$

Similar computations to those above show that θ_1 is onto H, and since coset enumeration shows that $G^*(k)$ has order 1296, θ_1 is an isomorphism. This suffices to show that $\ker(\theta)$ is contained in the centre of $G(k)$. We have shown that $(G(k), \ker(\theta))$ is a defining pair for $H(k)$. By a result of Jones and Wiegold (Wiegold (1982)), $\ker(\theta)$ is an epimorphic image of the Schur multiplicator of $H(k)$, which is trivial, and so θ is an isomorphism. We have established that $G(k)$ is a finite group of soluble length six. This completes the proof of the theorem.

As a generalization of the family $G(k)$ described above, let $G(i,k)$ be the group generated by a and b subject to the defining relations:

$$(4 \cdot 28) \qquad \begin{aligned} (ab)^2 &= b^{i+1} \\ 1 &= a^4 b^{-1} a b^k a^{-1} b. \end{aligned}$$

Imposing similar conditions to those above, consider only the case when $i \equiv 5 \,(\mathrm{mod} 6)$ and $k \equiv 3 \,(\mathrm{mod} 6)$. In this case there is some computational evidence that $G(i,k)$ is a finite group of soluble length at least six. However, it is not known in general, whether $G(i,k)$ is a finite group. Observe that $G(5,k)$ defines the same family as $G(k)$.

In attempting to establish the finiteness of $G(i,k)$, the method used is similar to that above. Observe that the quotient to the derived group is cyclic of order $2|k+2i-2|$. Again, $(ab)^2$ is a central element of $G(i,k)$. To establish that $G(i,k)$ is finite, we want to appeal as above to a theorem of Schur and Baer. It remains to show that the quotient $G^*(i,k) = G(i,k)/ < (ab)^2 >$ is finite. When $k \equiv 3 \pmod 6$, $G^*(i,k)$ has a presentation

$$(4 \cdot 29) \qquad \begin{aligned} 1 = (ab)^2 &= b^{i+1} \\ &= a^4 b^{-1} a b^k a^{-1} b. \end{aligned}$$

For a given i, the order of b is bounded above, and there are only $|(i+1)/6|$ (possibly non-isomorphic) groups in the family $G^*(i,k)$.

For $5 \leq i \leq 47$ we have verified using the Cayley that $G^*(i,3)$ is finite of order 1296.

We now show that the group $G(-9)$ of order 1296 and soluble length six also has a deficiency zero presentation on a generating set where the orders of the generators are different from those above.

Let H_0 be the group $G(-9)$ described above; H_0 is a group of order 1296, soluble length 6 and nilpotent length 4.

Let $a_0 = \alpha^{-1}x\beta$ and $b_0 = x\beta\alpha^{-1}$. Since $x\beta = a_0^2 b_0 a_0 b_0^{-1}$, a_0 and b_0 generate H_0. Using an algorithm due to Cannon (1973), H_0 is found to have a presentation with the relations

$$(4\cdot 30)\qquad \begin{aligned} 1 &= (b_0 a_0^{-2})^2 \\ &= a_0^8 \\ &= b_0^8 \\ &= b_0^2 a_0^{-1} b_0 a_0 b_0^{-1} a_0^{-1} b_0 a_0^2. \end{aligned}$$

A moderately difficult coset enumeration by computer shows that the preimage of H_0 with a presentation on a_0 and b_0 with relations

$$(4\cdot 31)\qquad \begin{aligned} 1 &= (b_0 a_0^{-2})^2, \\ a_0^8 &= b_0^2 a_0^{-1} b_0 a_0 b_0^{-1} a_0^{-1} b_0 a_0^2 \end{aligned}$$

has order 1296, and thus is a deficiency zero presentation for H_0.

The subgroup of H_0 generated by $a_1 = a_0^2$ and $b_1 = a_0 b_0^{-1}$ has index 2, and is thus the derived group of H_0. A two generator, two relation presentation for $H_1 = \langle a_1, b_1 \rangle$ is obtained by using a modified form of coset enumeration (see Beetham and Campbell (1976)):

$$(4\cdot 32)\qquad \begin{aligned} H_1 = \langle a_1, b_1 | a_1^2 &= ((a_1 b_1)^3)^{b_1}, \\ &= (b_1^{-3})^{a_1^{-1} b_1^{-1}} \rangle, \end{aligned}$$

and so we also have a deficiency zero presentation for a finite group of soluble length five. This is the group G_3 defined in the previous section.

Continuing down the derived series of H_0, $H_1' = \langle a_1, b_1^3, b_1 a_1 b_1^{-1} \rangle$ is of order 216 and has soluble length four. Using computer implementations of the Reidemeister-Schreier algorithm and Tietze transformations, a deficiency zero presentation for H_1' of order 216 is obtained on generators $a_2 = a_1$ and $b_2 = b_1^{-1} a_1 b_1$ with relations

$$(4\cdot 33)\qquad \begin{aligned} 1 &= a_2^2 b_2^{-1} a_2 b_2^{-2} a_2^{-1} b_2 \\ &= a_2^3 b_2^{-1} a_2^2 b_2^{-1} a_2^{-2} b_2 a_2 b_2^{-1}. \end{aligned}$$

The following theorem provides another example of a finite group of deficiency zero having soluble length six. This group has a somewhat different structure from those given earlier in this section.

Theorem 4. *Let G be generated by x and y with the following defining relations:*

$$(4 \cdot 34) \qquad \begin{aligned} 1 &= x^4y^{-4} \\ &= x^2yxyxy^{-1}x^{-1}y^{-1}x^{-2}y. \end{aligned}$$

G is a group of order 312000 having soluble length six.

Proof:

Coset enumeration over the identity shows that G has order 312000. We determine the structure of G below. The derived group G' of G is generated by yx,xy, and $y^{-1}xy^2$ and has index eight in G. We will show that G' is isomorphic to G_2, of order 39000, defined in the previous section.

A set of Schreier generators for G' is

$$\begin{aligned} &a = yx, b = xy, \\ &c = a^x, d = b^{x^{-1}}, \\ &e = c^x, f = e^x, \\ &g = f^x, h = d^{x^{-1}}, \\ &i = x^8. \end{aligned}$$

and using the Reidemeister-Schreier algorithm, a presentation for G' is

$$\begin{aligned} 1 &= bghd = cefg = dfgh \\ &= efgh = a^2b^{-2}c = a^2c^{-2}e^{-1} \end{aligned}$$

$$
\begin{aligned}
&= ab^2d^{-2} = acefi^{-1}\\
(4\cdot 35) \qquad &= acei^{-1}b = aci^{-1}db\\
&= ai^{-1}hdb = bd^2h^{-2}\\
&= c^2e^{-2}f^{-1} = e^2f^{-2}i^{-2}g^{-1}\\
&= dh^2g^{-1}i^{-1}g^{-1}i^{-1}\\
&= f^2(i^{-1}g^{-1})^2i^{-1}h^{-1}i
\end{aligned}
$$

This presentation contains many redundant generators and relations. We now simplify the presentation. Using the first relation to eliminate $h = g^{-1}b^{-1}d^{-1}$ and simplifying the presentation, we obtain

$$
\begin{aligned}
1 &= bf^{-1}\\
&= ag^{-1}i^{-1} = ai^{-1}g^{-1}\\
&= bf^{-1}e^{-1}d = cefg\\
&= die^{-1}i^{-1} = a^2b^{-2}c\\
(4\cdot 36) \qquad &= a^2c^{-2}e^{-1} = ab^2d^{-2}\\
&= acei^{-1}b = b^2if^{-1}f^{-1}\\
&= c^2e^{-2}f^{-1} = aif^2e^{-2}\\
&= agd^{-2}b^{-1}d^{-1}i\\
&= bd^2(dbg)^2
\end{aligned}
$$

Next we eliminate f and simplify the presentation to obtain

$$
\begin{aligned}
1 &= i = de^{-1}\\
&= ag^{-1}i^{-1} = ai^{-1}g^{-1}\\
&= bgce = die^{-1}i^{-1}\\
(4\cdot 37) \qquad &= a^2b^{-2}c = a^2c^{-2}e^{-1}\\
&= ab^2d^{-2} = acei^{-1}b\\
&= be^2c^{-2} = aib^2e^{-1}e^{-1}\\
&= agd^{-2}b^{-1}d^{-1}i\\
&= bd^2dbgdbg
\end{aligned}
$$

Observe that we have now obtained a proof that $x^8 = 1$ holds in G. Eliminate i, e and g to obtain

$$(4\cdot 38)\qquad \begin{aligned} 1 &= acdb \\ &= a^2b^{-2}c \\ &= a^2c^{-2}d^{-1} \\ &= ab^2d^{-2} \\ &= bd^2c^{-2} \end{aligned}$$

Using the first relation to eliminate d

$$(4\cdot 39)\qquad \begin{aligned} 1 &= a^2b^{-2}c \\ &= a^2c^{-2}bac \\ &= ab^3acbac \\ &= acbacb^{-1}c^2b \end{aligned}$$

From the first relation, $ca^2 = b^2$. Substituting this into the second relation, we obtain $ab^2c^{-2}b = 1$. From the third relation, we have $acbac = b^{-3}a$. Substituting this into the fourth relation, we also obtain $ab^2c^{-2}b = 1$, and so we obtain a deficiency zero presentation on three generators.

$$(4\cdot 40)\qquad \begin{aligned} 1 &= a^2b^{-2}c \\ &= ab^2c^{-2}b \\ &= ab^2bacbac \end{aligned}$$

Finally, eliminate c to obtain

$$(4\cdot 41)\qquad \begin{aligned} 1 &= a^2b^{-1}ab^2a^2b^{-2} \\ &= a^2b^{-2}a^{-1}b^{-3}ab^{-2}a^{-1}b^{-1} \end{aligned}$$

This is a presentation for the group G_2 of order 39000 described in the previous section, and so G is a finite group of deficiency zero having soluble length six.

A further calculation with Cayley shows that G is isomorphic to a semidirect product of G_2 and C_8 generated by X. The action of X is given by

$$\begin{aligned} x^X &= y^{-1}xy^{-1}, y^X = x^{-2}y, \\ \alpha^X &= xy\alpha^{-1}\beta^{-1}, \\ \beta^X &= \alpha x\beta^{-1}\alpha^{-1}y, \\ c^X &= c^{-5}. \end{aligned}$$

The following theorem is due to A.Wegner (Newman, private communication), who has shown that there are two other non-isomorphic groups of order 1296 with soluble length six having deficiency zero.

Theorem 5. *Let G_1 be presentable with defining relations*

$$\begin{aligned} 1 &= ababab^{-2}a^{-1}b \\ &= a^2ba^{-2}ba^2b^{-2} \end{aligned} \tag{4·42}$$

and let G_2 be presentable with defining relations

$$\begin{aligned} 1 &= aba^{-1}ba^{-1}bab^{-2} \\ &= a^2ba^{-1}b^{-3}ab^2. \end{aligned} \tag{4·43}$$

G_1 and G_2 define non-isomorphic groups of order 1296 having soluble length six.

Proof:

(Due to A.Wegner.) Coset enumeration shows that both groups are of order 1296. The Cayley program and output given in Appendix C show that they are not isomorphic. Cayley is used to compute the conjugacy classes of both groups and since they have different numbers of elements of order three, the groups cannot be isomorphic. Note that neither group is isomorphic to $G(-9)$ of order 1296 and soluble length six defined in Theorem 3 above. This may be seen again by considering the class structure of $G(-9)$. This completes the proof of Theorem 5.

Soluble length seven

A calculation with Cayley shows that the permutations x and y below generate a group G of order $82944 (= 2^{10}3^4)$ having soluble length seven.

$$\begin{aligned} x = &(1,66,38,41,15,56,28,32)(2,65,37,42,16,55,27,31) \\ &(3,7,58,25,5,13,71,48)(4,8,57,26,6,14,72,47) \\ &(9,36,46,30)(10,35,45,29)(11,17,39,52,12,18,40,51) \\ &(19,21,64,53,62,43,24,70)(20,22,63,54,61,44,23,69) \\ &(33,68,50,59,34,67,49,60), \\ y = &(1,22)(2,21)(3,29)(4,30)(5,6)(7,62)(8,61)(9,10)(11,42)(12,41) \\ &(13,14)(15,35)(16,36)(17,66)(18,65)(19,28)(20,27)(23,49)(24,50) \\ &(25,52)(26,51)(31,48)(32,47)(33,54)(34,53)(37,38)(39,67)(40,68) \\ &(43,45)(44,46)(55,69)(56,70)(57,60)(58,59)(63,72)(64,71). \end{aligned}$$

These permutations satisfy the following relations (which were constructed by using Cannon's algorithm)

$$\begin{aligned} &y^2, x^8, (xy^{-1})^6, \\ &x^4yx^{-2}yx^{-2}yx^{-2}yxyx^{-1}yx^{-1}y, \\ &(x^2yx^{-2}yx^{-1}y)^3. \end{aligned} \tag{4·44}$$

A coset enumeration shows that these words suffice to present G, so we have a deficiency three presentation for G. A further coset enumeration shows that the relation $(xy^{-1})^6$ may be omitted, giving a deficiency two presentation for G. This immediately gives us a deficiency one presentation for the finite group G^* presented by

$$\begin{aligned} &x^8 = y^2, \\ &x^4yx^{-2}yx^{-2}yx^{-2}yxyx^{-1}yx^{-1}y, \\ &(x^2yx^{-2}yx^{-1}y)^3. \end{aligned} \tag{4·45}$$

To see that G^* is finite, we use the (by now familiar) theorem of Schur.

Chapter Five

Presentations for Semi-simple Groups and Their Products.

This chapter gives a number of minimal presentations for some semi-simple groups and their products.

The motivation for the following theorem comes from a question of Wiegold (1982). He asked what happens to def(G^n) as $n \to \infty$, where G^n is the direct power of n copies of G. This question remains unanswered in general, and Wiegold considered the simplest case, namely when G is a perfect group. In this case, M(G^n) is the direct power of n copies of M(G). Wiegold asked if $SL(2,5) \times SL(2,5)$ is Schur-efficient, and suggested that a suitable method for approaching this problem would be to consider whether $A_5 \times A_5$ is Schur-efficient or not. Parts of this chapter have appeared in Kenne (1983); (4) has independently been proved by Campbell, Robertson and Williams (1990a); (7) has been proved by Campbell and Robertson (1980a); (8) has been proved by Campbell,Robertson and Williams (1990b).

Theorem. *The following groups are Schur-efficient:*

(1) $A_5 \times A_5$,

(2) $A_5 \times A_6$,

(3) $A_5 \times A_7$,

(4) $A_5 \times A_5 \times A_5$,

(5) $A_5 \times SL(2,5)$,

(6) the direct product of two copies of $SL(2,5)$ *with amalgamated central subgroups,*

(7) $SL(2,p)$, *where* p *is an odd prime,*

(8) $SL(2,5) \times SL(2,5)$,

(9) $SL(2,7) \times SL(2,7)$.

(10) A_4^n for any n.

(11) $A_5 \times A_5 \times A_5 \times A_5$.

Proof:

(5) Let H be generated by x and y subject to the defining relations

$$(5 \cdot 1) \qquad \begin{aligned} 1 &= y^6 = x^4yx^{-1}y^{-3}x^{-1}y^{-1} \\ &= (xy^2)^2x^{-1}y^{-1}(xy^{-1})^2. \end{aligned}$$

It is easy to see that H is perfect. Coset enumeration shows that H has order 7200. Sandlöbes (1981) has determined all the perfect groups of order less than 10^4. There are two isomorphism classes of perfect groups of order 7200, namely $A_5 \times SL(2,5)$ and the direct product of two copies of $SL(2,5)$ with amalgamated central subgroups. To see that H is isomorphic to $A_5 \times SL(2,5)$, observe that $SL(2,5)$ is generated by u and v with the defining relations

$$u^5 = v^3 = (uv)^2,$$

and there is a homomorphism θ mapping H onto $SL(2,5)$ defined by

$$\theta(x) = cd$$
$$\theta(y) = d.$$

This completes the proof of (5).

(1) Let H^* be the group generated by x and y with defining relations

$$(5 \cdot 2) \qquad \begin{aligned} 1 &= x^{10} = y^6 \\ &= x^4yx^{-1}y^{-3}x^{-1}y^{-1} \\ &= (xy^2)^2x^{-1}y^{-1}(xy^{-1})^2. \end{aligned}$$

(Note that H^* is perfect.) A coset enumeration shows that H^* has order 3600 and is therefore $A_5 \times A_5$.

(2) Let $a = (1,3,5,4,6)(7,8,10)$ and $b = (2,3)(4,5)(8,11)(9,10)$. These permutations generate $A_5 \times A_6$ since $aba^{-3}ba^{-1}ba^6b = (1,2,3)$,

$ba^6ba^{-1}ba^3ba = (3,4,5)$ and $baba^{-2}baba^{-2}ba = (1,2)(3,4,5,6)$. Using a program to determine relations satisfied by these permutations, it is found that a and b satisfy the following relations

$$(5 \cdot 3) \qquad \begin{aligned} 1 &= a^{15} = b^2 \\ &= (ab)^5 = (a^3b)^4 \\ &= a^2ba^{-3}ba^7ba^{-3}b. \end{aligned}$$

A coset enumeration shows that the group generated by b has index 1440 in $\langle a, b\rangle$. This is sufficient to show the relations above provide a defining set for $A_5 \times A_6$. Another machine based coset enumeration shows that the group generated by b has index 1440 in the preimage obtained by omitting the relation $1 = (a^3b)^4$. This provides a two generator, four relation presentation for $A_5 \times A_6$. Since the multiplicator of this group is just the direct product of $M(A_5)$ and $M(A_6)$, this completes the proof of (2).

(3) Let $a = (1,5,3,2,4)(6,10,9,11,8)$ and $b = (1,3)(2,5)(6,10,7,9,12,11,8)$. These permutations generate a subgroup of $A_5 \times A_7$. Relations satisfied by these permutations are

$$(5 \cdot 4) \qquad \begin{aligned} 1 &= a^5 = b^{14} = ab^2a^{-1}ba^{-1}b^{-2}ab^{-1} \\ &= aba^{-1}bab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1} \\ &= [a, b^2]^2 \\ &= (a^2b^{-1})^2ab^4ab^{-1} \\ &= a^2ba^2b^{-1}a^{-1}b^{-2}a^{-1}b^{-2}a^{-1}b^{-1}. \end{aligned}$$

(These relations were constructed by using a program to find relations satisfied by a pair of permutations.) Coset enumeration shows that the subgroup generated by a has index 30240 in the group with the above defining relations. This shows that a and b generate the whole of $A_5 \times A_7$. Another coset enumeration shows that the group generated by a has index 30240 in the preimage with relations

$$
(5\cdot 5)\qquad
\begin{aligned}
1 &= a^5 \\
&= aba^{-1}bab^{-1}ab^{-1}a^{-1}b^{-1}ab^{-1} \\
&= (a^2b^{-1})^2ab^4ab^{-1} \\
&= a^2ba^2b^{-1}a^{-1}b^{-2}a^{-1}b^{-2}a^{-1}b^{-1}.
\end{aligned}
$$

This establishes that the preimage has order dividing 151200 ($= 60 \times 2520 = 5 \times 30240$), and thus $A_5 \times A_7$ has an efficient presentation.

(4) Let

$$
\begin{aligned}
a &= (1,3,2,4,5)(6,9,7)(11,12,13,14,15) \\
b &= (2,3)(4,5)(7,8)(9,10)(11,14)(12,15).
\end{aligned}
$$

These permutations generate a subgroup of $A_5 \times A_5 \times A_5$ and satisfy the relations

$$
(5\cdot 6)\qquad
\begin{aligned}
1 &= a^{15} = b^2 \\
&= a^4ba^{-1}ba^3baba^{-1}ba^{-1}ba^3bab \\
&= a^4ba^{-1}b(ab)^4(a^{-1}b)^2a^{-3}b \\
&= (a^3b)^2(a^2b)^4(a^3b)^2a^{-1}b.
\end{aligned}
$$

The group generated by a has index 14400 in $\langle a, b\rangle$, and so these relations define $A_5 \times A_5 \times A_5$.

Independently, Campbell, Robertson and Williams (1990a) have also shown $A_5 \times A_5 \times A_5$ to be Schur-efficient.

(6) B.H. Neumann (private communication) has asked whether the group K generated by x and y with defining relations

$$
(5\cdot 7)\qquad
\begin{aligned}
x^{10} &= y^6, \\
1 &= x^4yx^{-1}y^{-3}x^{-1}y^{-1} \\
&= (xy^2)^2x^{-1}y^{-1}(xy^{-1})^2
\end{aligned}
$$

is the other perfect group of order 7200 *i.e.* the direct product of two copies of $SL(2,5)$ with amalgamated central subgroups (denoted by P_4 by Sandlöbes

(1981)). Coset enumeration shows that K has order 7200. The normal subgroup lattice of K was calculated using Cayley, and it was found that K has no normal subgroup of order 60. This shows that K is isomorphic to P_4. To see that $(5 \cdot 7)$ is a Schur efficient presentation, observe that the direct product of two copies of $SL(2,5)$ is a covering group of K.

(7) See Campbell and Roberston (1980a) for details.

(8) See Campbell,Robertson and Williams (1990b) for details.

(9) $SL(2,7) \times SL(2,7)$ is generated by the matrices

$$x = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 5 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 6 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These matrices satisfy the following relations

$$(5 \cdot 8) \qquad \begin{aligned} 1 &= x^4 y x^{-3} y = x^{14} \\ &= x^3 y^2 x y^5 x y^2 \\ &= x^2 y^2 x^{-1} y^2 x^{-1} y^2 x^2 y^{-1} x^{-1} y^{-1} \\ &= (x^2 y^3)^3 = (x y^2 x^{-1} y^{-1})^3. \end{aligned}$$

Let G be the group generated by x and y subject to the above relations. G is a preimage of $SL(2,7) \times SL(2,7)$. A coset enumeration shows that the subgroup generated by x has index 8064 in G, showing that G is isomorphic to $SL(2,7) \times SL(2,7)$. Another coset enumeration over x shows that the last two relations may be omitted.

Another coset enumeration shows that the group generated by x and y with defining relations

$$(5 \cdot 9) \qquad \begin{aligned} 1 &= x^4 y x^{-3} y \\ &= x^3 y^2 x y^5 x y^2. \end{aligned}$$

also has order 112896, and so we have shown $SL(2,7) \times SL(2,7)$ to be Schur-efficient.

(10) Campbell (private communication) has reported this result.

(11) Campbell (private communication) has reported this result.

Appendix A
The groups of order 36, 48, 54, 60, 72 and 84

This appendix contains Cayley output showing minimal presentations for the interesting groups of order 36, 48, 54, 60, 72 and 84. Appendix B lists the groups of these orders as supplied with the Cayley system. These presentations were originally listed by Neubüser (1967). The numbering given here does not correspond to Neubüser's as the abelian groups are not included in the Cayley libraries. For each group, the correspondence between the generating set given here and the generating set in the Cayley library is given. A bound on the rank of the multiplicator for each group was determined by use of the Cayley function DARSTELLUNGSGRUPPE, which computes for its argument G a group K with the property that $G \simeq K/A$, where $A = K' \cap Z(K)$. The kernel A is a homomorphic image of $M(G)$, and if a presentation can be found having deficiency equal to the rank of A, that presentation is a Schur-efficient one. Presentations were constructed by using Cannon's algorithm (Cannon (1973)) to find a set of defining relations. If the deficiency of the presentation thus found is equal to the rank of A, the presentation is a Schur-efficient one. If the deficiency is greater, the presentation was manipulated to obtain a Schur-efficient one (in all cases except for the group g54n7 below). The manipulations were fairly simple – coset enumeration was used to check if a relator could be omitted from a presentation, or if the quotient of two relators could be formed to obtain a new presentation with reduced deficiency.

The group $g54n7$ has trivial Schur multiplicator (and minimally requires three generators). However, this group cannot have a deficiency zero presentation. Suppose that it has a deficiency zero presentation on n generators. The subgroup of index two (which is the non-abelian group of exponent three) would have a presentation on $2n-1$ generators and at most $2n$ relators. This group of order 27 is known to have deficiency two, which is inconsistent with having a deficiency one presentation. (This proof is due to Beyl (private communication).)

```
SUN/UNIX CAYLEY V3.7.3 Fri Jun 28 1991 11:16:10 STORAGE 200000
>"
>g36n3
> x=ab,y=d
>"
>g=free(x,y);
>g.relations:
```

```
>x^6,(y*x^-1)^3,y^3=y*x^2*y^-1*x^-2;
>print order(g);
36
>clear;
>"
>g36n4
> x=b,y=d
>"
>
>g=free(x,y);
>g.relations:
>x^2,y^9,y^3=(x*y)^3;
>print order(g);
36
>clear;
>"
>g36n5
> x=ac^-1,y=ab,z=adb
>"
>g=free(x,y,z);
>g.relations:
> z^2,(x*z^-1)^2,x*y*x^-1*y^-1=x^3*y^-3,(y*z^-1)^2=x^6;
>print order(g);
36
>clear;
>"
>g36n6
> x=b^-1 y=c^-1 z=d
>"
>g=free(x,y,z);
>g.relations:
> z^4=y^3,(x,y),(x*y^4)^z=x^2*y^5,(x^4*y^5)^z=x^5*y^7;
>print order(g);
36
>clear;
>"
>g36n9
> x=ad,y=bc
>"
>g=free(x,y);
>g.relations:
> x^6=(y*x)^2,y^6,(y*x^-1)^2;
>print order(g);
```

```
36
>clear;
>"
>g36n10
> x=ab,y=d
>"
>g=free(x,y);
>g.relations:
> x^3=y^2*x*y^-2*x,y^4,(y*x)^2=x^-1*y*x*y;
>print order(g);
36
>clear;
>bye;
END OF RUN.
0.550 SECONDS
SUN/UNIX CAYLEY V3.7.3 Sun Jun 23 1991 15:42:17 STORAGE 200000
>"
> g48n1
> x = D y = C z= E A^-1
>"
>g=free(x,y,z);
>g.relations:
>x^4, y^2, z^6,
>z*y*z^-1*y,
>z*x*z^-1*x,
>y*x*y*x^-1;
>print order(g);
48
>clear;
>"
> g48n2
> x = D y = E z = C A^-1
>"
>g=free(x,y,z);
>g.relations:
>z^6, (z,y), (z,x),
>x^2*y^-2,
>y*x*y*x^-1;
>print order(g);
48
>clear;
>"
> g48n3;
> x = C y = D C^-1 z = E A^-1
```

```
>"
>g=free(x,y,z);
>g.relations:
>z^6, (z,x), (y,x),
>z*y*z^-1*y,
>y^2*x^-2;
>print order(g);
48
>clear;
>"
> g48n4;
> x = D y = A^-1 E^-1
>"
>g=free(x,y);
>g.relations:
>x^2, y^12, y^2*x*y^-2*x,
>(y*x)^2*(y^-1*x)^2;
>print order(g);
48
>clear;
>"
> g48n10;
> x = A D^-1
>"
>g=free(x,b,c,e);
>g.relations:
>b^2, c^2, e^2,
>(b,x), (c,x), (e,x^3),
>x^e*x,
>(b,c), (b,e), (c,e);
>print order(g);
48
>clear;
>"
> g48n11;
> x = C D^-1
>"
>g=free(x,b,e);
>g.relations:
>b^4, e^2, (b,e),
>(b,x), (e,x^3),
>x^e*x;
>print order(g);
```

```
48
>clear;
>"
> g48n12;
> x = E y = B z = C D^-1
>"
>g=free(x,y,z);
>g.relations:
>x^4, y^2, z^6,
>x*y*x^-1*y,
>x*z*x^-1*z,
>y*z*y*z^-1;
>print order(g);
48
>clear;
>"
> g48n17;
> x = A E^-1 y = B C
>"
>g=free(x,y);
>g.relations:
>y^2, x^6,
>y*x*y*x^-1*y*x*y*x^-1,
>y*x^3*y*x^-3;
>print order(g);
48
>clear;
>"
> g48n18;
> x = B C E y = B D E
>"
>g=free(x,y);
>g.relations:
>(x*y^-1)^2, x^12,
>x^2*y^-2*x^-1*y;
>print order(g);
48
>clear;
>"
> g48n19;
> x = D y = E z = B C^-1
>"
>g=free(x,y,z);
>g.relations:
```

```
>x^2, y^2, z^6,
>x*z*x*z^-1,
>(y*z)^2, (x*y)^4;
>print order(g);
48
>clear;
>"
> g48n20;
> x = D y = E z = B C^-1
>"
>g=free(x,y,z);
>g.relations:
>x^4, y^2, z^6,
>(x*y)^2, (x,z),
>(y*z)^2;
>print order(g);
48
>clear;
>"
> g48n21;
> x = D y = E z = B C^-1
>"
>g=free(x,y,z);
>g.relations:
>z*y*z*y^-1, (z,x),
>y^2*x^-2, y*x*y*x^-1,
>z^6;
>print order(g);
48
>clear;
>"
> g48n22;
> x = E y = D C^-1
>"
>g=free(x,y);
>g.relations:
>x^4,
>y^-6*y^2*x*y^2*x^-1,
>y*x^2*y^-1*x^-2,
>(y*x)^2*(y^-1*x^-1)^2;
>print order(g);
48
>clear;
```

```
>"
> g48n23;
> x = E y = C^-1 D^-1 E
>"
>g=free(x,y);
>g.relations:
>x^2, y^4,
>y^2*x*y^-2*x,
>(y*x)^6*(y^-1*x)^6;
>print order(g);
48
>clear;
>"
> g48n24;
> x = E y = C^-1 D^-1
>"
>g=free(x,y);
>g.relations:
>x^4,
>y^2*x*y^2*x^-1,
>y^-5*x*y*x;
>print order(g);
48
>clear;
>"
> g48n26;
> x = E y = D B^-1 z = B C E
>"
>g=free(x,y,z);
>g.relations:
>x^2,
>(x*y)^2,
>y^-2*z^2,
>y*z*y^-1*z,
>(x*z)^3*(x^-1*z^-1)^3;
>print order(g);
48
>clear;
>"
> g48n29;
> x = B E^-1 y = D^-1 E^-1
>"
>g=free(x,y);
>g.relations:
```

```
>y^2*x*y^-1*x,
>y^2*x^-2*y^-1*x^-2;
>print order(g);
48
>clear;
>"
> g48n30;
> x = A E^-1 y = B C
>"
>g=free(x,y);
>g.relations:
>y*x^3*y*x^-3,
>y*x*y*x^2*y*x*y*x^-1;
>print order(g);
48
>clear;
>"
> g48n31;
> x = E y = A B E D
>"
>g=free(x,y);
>g.relations:
>x^2, y^4,
>y^2*x*y*x*y^-2*x*y*x,
>(y*x)^3*(y^-1*x)^3;
>print order(g);
48
>clear;
>"
> g48n32;
> x = D^-1 y = B C E^-1
>"
>g=free(x,y);
>g.relations:
>x^3,
>y^-4*y^2*x*y^-2*x^-1,
>(y*x)^2*(y^-1*x)^2;
>print order(g);
48
>clear;
>"
> g48n33;
> x = B^-1 D^-1
```

```
>"
>g=free(x,c,e);
>g.relations:
>x^12,
>x^c*x^-7,
>x^e*x^7,
>c^2, e^2, (c,e);
>print order(g);
48
>clear;
>"
> g48n34;
> x = B^-1 D^-1 y = C z = E
>"
>g=free(x,y,z);
>g.relations:
>y^4*z^-2,
>x^6*y^-2,
>(y,z),
>y*x*y*x^-1,
>(z*x^-1)^2*y^2;
>print order(g);
48
>clear;
>"
> g48n35;
> x = B^-1 D^-1
>"
>g=free(x,c,e);
>g.relations:
>x^12*c^-2,
>x^6*e^-2,
>x^c*x^-7,
>x^e*x^7,
>(c,e);
>print order(g);
48
>clear;
>"
> g48n36;
> x = B^-1 D^-1
>"
>g=free(x,c,e);
>g.relations:
```

```
>x^12*(c,e)^-1,
>c^-2*x^6,
>e^-2*x^6,
>x^c*x^-7,
>x^e*x^7;
>print order(g);
48
>clear;
>"
> g48n40;
> x = E B^-1 y = E D
>"
>g=free(x,y);
>g.relations:
>y^-2*x^6,
>(y*x^-2)^2,
>(y*x)^4*(y*x^-1)^4;
>print order(g);
48
>clear;
>"
> g48n41;
> x = E B^-1 y = D
>"
>g=free(x,y);
>g.relations:
>y^3=x*y*x,
>x^6=y^2*x*y^2*x^-1;
>print order(g);
48
>clear;
>"
> g48n42;
> x = B^-1 E^-1 y = D
>"
>g=free(x,y);
>g.relations:
>(y*x)^2,
>y^3*x^-2*y*x^4;
>print order(g);
48
>clear;
>"
```

```
> g48n43;
> x = B^-1 E^-1 y = D
>"
>g=free(x,y);
>g.relations:
>y^-3*x*y*x,
>y^2*x*y^2*x^-1*(y*x^3*y*x^-3)^-1;
>print order(g);
48
>clear;
>"
> g48n44;
> x = E C^-1 y = B E D
>"
>g=free(x,y);
>g.relations:
>y^2*x^-1*y^2*x^-1,
>y*x*y*(x*y*x)^-1;
>print order(g);
48
>clear;
>"
> g48n45;
> x = B E y = A D^-1
>"
>g=free(x,y);
>g.relations:
>y^-2*x*y*x,
>x^-3*y*x*y;
>print order(g);
48
>clear;
>"
> g48n46;
> x = E^-1 y = B z = C
>"
>g=free(x,y,z);
>g.relations:
>x^3, y^2, z^2,
>(z*y)^2,
>(z*x^-1)^3,
>(y*x^-1)^3,
>(z*x^-1*y*x)^2;
```

```
>print order(g);
48
>clear;
>"
> g48n47;
> x = B y = E^-1
>"
>g=free(x,y);
>g.relations:
>x^4*y^-3,
>(y*x)^3,
>(y*x^-1)^3;
>print order(g);
48
>clear;
>bye;
END OF RUN.
2.170 SECONDS
SUN/UNIX CAYLEY V3.7.3 Sun Jun 30 1991 10:48:50 STORAGE 200000
>"
>g54n1
> x = c^-1 y = bd^-1
>"
>g=free(x,y);
>g.relations:
>x^3, y^6, (x*y)^2*x*y^-2,
>x*y^2*(x*y^-1)^2;
>print order(g);
54
>clear;
>"
>g54n3
> x = a^-1c^-1 y = db^-1
>"
>g=free(x,y);
>g.relations:
>x^3, (x*y)^2*(x^-1*y^-1)^2, (x,y^2)*y^6;
>print order(g);
54
>clear;
>"
>g54n5
> x = abdc y = abcd z = abc
```

```
>"
>g=free(x,y,z);
>g.relations:
>z^3, x^2*y^-2,
>(x*z^-1)^2*x^2*z*x^-2*z^-1,
>(y*z^-1)^2*x*y*x*y^-1*x^-1*y^-1;
>print order(g);
54
>clear;
>"
>g54n7
> x = b^-1 y = c^-1 z = da^-1
>"
>g=free(x,y,z);
>g.relations:
>x^3*y*z*y*z^-1,
>y^3, x*z*x*z^-1,
>z^-2*(x,y);
>print order(g);
54
>clear;
>"
>g54n8
> x = b^-1 y = dac^-1
>"
>g=free(x,y);
>g.relations:
>x^3*y^6,
>x*y*x^-1*y*x*y^-2,
>x*y*x^-1*y^-2*x^-1*y;
>print order(g);
54
>clear;
>"
>g54n11
>
> x = ab^-1 y = ac^-1 z = bda w = bc
>"
>g=free(x,y,z,w);
>g.relations:
> z^2*x^3, y^3*(x,w), w^3*(y,w),
> (x,y), (x*z^-1)^2, (y*z^-1)^2,
> (z*w)^2;
```

```
>print order(g);
54
>clear;
>"
>g54n11
>
> x = abdc y = abcd z = abc
>"
>g=free(x,y,z);
>g.relations:
>x^2*(x*y)^3,
>(x*z)^2*z^9,
>y^2, (y*z)^2;
>print order(g);
54
>clear;
>bye;
END OF RUN.
2.059 SECONDS
SUN/UNIX CAYLEY V3.7.3 Fri Jun 28 1991 10:08:11 STORAGE 200000
>"
>g60n3;
> x=ab y=d
>"
>g=free(x,y);
>g.relations:
> y^3,(y*x^-1)^2*y=x^3,(y,x)(y,x^-1);
>print order(g);
60
>clear;
>"
>g60n8
> x=db^-1 y=ca^2d
>"
>g=free(x,y);
>g.relations:
> x^6=(y*x)^2,y^10,(y*x^-1)^2;
>print order(g);
60
>clear;
>"
>g60n11;
>x=ab y=b
```

```
>"
>g=free(x,y);
>g.relations:
> x^5,y^3,(x*y)^2;
>print order(g);
60
>clear;
>bye;
END OF RUN.
0.219 SECONDS
SUN/UNIX CAYLEY V3.7.3 Sun Jun 23 1991 17:35:08 STORAGE 200000
>"
>g72n5
>x = A C^-1 D^-1 y = E C^-1 z = A B D^-1
>"
>g=free(x,y,z);
>g.relations:
>(x*y^-1)^2,(x,z),y*z*y^-1*z,x^6,(x^2*y)^2,x*y^-1*z^2*x^-1*y;
>print order(g);
72
>clear;
>"
>g72n9
>x = E y = A B D E
>"
>g=free(x,y);
>g.relations:
>x^3=x*y*x^-1*y*x^-1*y^-1*x*y^-1,y^6,(x*y)^2*x*y^-2;
>print order(g);
72
>clear;
>"
>g72n10
>x = B E^-1 y = A C B^-1
>"
>g=free(x,y);
>g.relations:
>x^3*y^2,y^6,x*y*x^-1*y*x*y^-1*x^-1*y^-1;
>print order(g);
72
>clear;
>"
>g72n11
```

```
>x = A C^-1 y = E z = B D^-1
>"
>g=free(x,y,z);
>g.relations:
>y^2,(x*y^-1)^2,x*z*x^-1*z^-1,(y*z)^2,x^6,z^6;
>print order(g);
72
>clear;
>"
>g72n12
>x = C^-1 y = B C^-1 E^-1 z = D C^-1 E^-1
>"
>g=free(x,y,z);
>g.relations:
>y^4,y^2=z^2,x*z*x*z^-1=(y*z)^3*(y^-1*z^-1)^3,x^3=x*y*x*y^-1;
>print order(g);
72
>clear;
>"
>g72n13
>x = A C^-1 y = E B^-1 z = C D E
>"
>g=free(x,y,z);
>g.relations:
>z^2, x*y*x*y^-1,
>(x*z^-1)^2=x^3*y^-2,
>y^4=(y*z)^3*(y^-1*z)^3;
>print order(g);
72
>clear;
>"
>g72n14
> x=d^-1e^-1,y=ac,z=ad
>"
>g=free(x,y,z);
>g.relations:
>(y,z)=y^3*z^-3,x*y*x^-1*y,x*z*x^-1*z,x^4*y^-3;
>print order(g);
72
>clear;
>"
>g72n19
>x = E y = D B^-1 C^-1
```

```
>"
>g=free(x,y);
>g.relations:
>x^2,y^6,(x*y)^2*(x*y^-1)^2;
>print order(g);
72
>clear;
>"
>g72n22
>x = E y = B^-1 D^-1
>"
>g=free(x,y);
>g.relations:
>x^3,(x*y)^3,x*y^2*x^-1*y^-2;
>print order(g);
72
>clear;
>"
>g72n23
> x=be^-1,y=e^-1d
>"
>g=free(x,y);
>g.relations:
>x^3=y^3,x^2*y^-1*x*y*x*y^-1;
>print order(g);
72
>clear;
>"
>g72n24
>x = A^-1 D^-1 y = C E
>"
>g=free(x,y);
>g.relations:
>x^3,y^4,(x*y)^2*x^-1*y^-1*x^-1*y;
>print order(g);
72
>clear;
>"
>g72n26
> x=c^-1d^-1,y=aeb^-1
>"
>g=free(x,y);
>g.relations:
```

```
>x^2*y*x^2*y^-1,y^6,x^3*y*x^-3*y;
>print order(g);
72
>clear;
>"
>g72n27
>x = C^-1 D^-1 y = E B^-1
>"
>g=free(x,y);
>g.relations:
>(x*y)^2,(x*y^-1)^2,x^5*y^-5*x^-1*y;
>print order(g);
72
>clear;
>"
>g72n28
> x=e,y=adb
>"
>g=free(x,y);
>g.relations:
>y^2,x^4,(x*y^-1)^4=(x*y*x^-1*y)^3;
>print order(g);
72
>clear;
>"
>g72n29
>x = E y = A B E^-1
>"
>g=free(x,y);
>g.relations:
>x^3*y*x^-1*y,x*y^3*x*y^-1,x^3*y^-1*x^-1*y^-1*x^-1*y^-1;
>print order(g);
72
>clear;
>"
>g72n30
> x=de,y=cb^-1,z=adb^-1
>"
>g=free(x,y,z);
>g.relations:
>y^3=x*y*x^-1*y,(x*z)^2=z^6,x^4=y*z*y^-1*z^-1,(x*z^-1)^2;
>print order(g);
72
```

```
>clear;
>"
>g72n31
> x=b^-1,y=e,z=c^-1d^-1
>"
>g=free(x,y,z);
>g.relations:
>y^2,(y*z)^2,x^3=x*z*x^-1*z^-1,(x*y^-1)^2=z^12;
>print order(g);
72
>clear;
>"
>g72n32
>x = B^-1 y = E z = C^-1 D^-1
>"
>g=free(x,y,z);
>g.relations:
>x*y*x*y^-1, y*z*y^-1*z,
>x^3=y^4,
>(x,z)=y^2*z^-6;
>print order(g);
72
>clear;
>"
>g72n33
> x=bdc^-1,y=bcde
>"
>g=free(x,y);
>g.relations:
>x^2,y^18,(x*y^2)^2=(x*y)^2*(x*y^-1)^2;
>print order(g);
72
>clear;
>"
>g72n36
> x=dc^-1,y=abce
>"
>g=free(x,y);
>g.relations:
>x^6=x^2*y*x^2*y^-1,y^6=x*y^2*x^-1*y^2,(x*y)^2*(x^-1*y^-1)^2;
>print order(g);
72
>clear;
```

```
>"
>g72n37
> x=eb^-1,y=c^-1d^-1
>"
>g=free(x,y);
>g.relations:
>(x*y)^2=x^6,(x*y^-1)^2,y^12;
>print order(g);
72
>clear;
>"
>g72n38
> x=bec,y=bcd
>"
>g=free(x,y);
>g.relations:
>x^3*y*x^-3*y,x*y*x*y^-5;
>print order(g);
72
>clear;
>"
>g72n40
> x=ba^-1,y=ed^-1
>"
>g=free(x,y);
>g.relations:
>x^6=x^2*y*x^2*y^-1,y^6,x*y*x^-1*y^-2*x^-1*y;
>print order(g);
72
>clear;
>"
>g72n41
> x=e,y=bdb^-1
>"
>g=free(x,y);
>g.relations:
>y^2,(x*y*x^-1*y)^2,x^4=(x*y^-1)^6;
>print order(g);
72
>clear;
>"
>g72n42
> x=ae,y=ade^-1
```

```
>"
>g=free(x,y);
>g.relations:
>y^4,x^2*y^2*x^-1*y^-1*x^-1*y,x*y*x*y^-1*x*y*x^-1*y;
>print order(g);
72
>clear;
>"
>g72n43
>x = D^-1 y = B E z = C D E A
>"
>g=free(x,y,z);
>g.relations:
>y^2=x^3,z^2=(y*z)^3,x*y*z*x*z*x^-1*y,(x*y^-1)^4;
>print order(g);
72
>clear;
>"
>g72n44
> x=cd^-1,y=abe
>"
>g=free(x,y);
>g.relations:
>y^2,(x^3*y)^2=(x*y^-1)^4,x^9;
>print order(g);
72
>bye;
END OF RUN.
3.890 SECONDS
SUN/UNIX CAYLEY V3.7.3 Fri Jun 28 1991 12:05:39 STORAGE 200000
>"
>g84n3
> x=a b d y = a c d
>"
>g=free(x,y);
>g.relations:
>(x*y^-1)^2, x^2*y^-2*x^-1*y, x^21;
>print order(g);
84
>clear;
>"
>g84n10
> x=ad,y=bc
```

```
>"
>g=free(x,y);
>g.relations:
>x^14,(y*x)^2,y^6=(y*x^-1)^2;
>print order(g);
84
>clear;
>"
>g84n11
> x=bc y=d
>"
>g=free(x,y);
>g.relations:
>y^3,(x*y)^3,y^-1*x^2*y=x^4;
>print order(g);
84
>clear;
END OF RUN.
0.230 SECONDS
```

Appendix B

The groups of order 36, 48, 54, 60, 72 and 84 – the Cayley libraries

```
LIBRARY G36N1;
"Group of order 36:  number 1."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 * (a^-1) = c^3 = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G36N2;
"Group of order 36:  number 2."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 * (a^-1) = c^3 = d^2 * (b^-1) = (a, b) = (a, c) =
(a, d) = (b, c) = (b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G36N3;
"Group of order 36:  number 3."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^2 = d^3 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * b * c = (c, d) * b = 1;
FINISH;
LIBRARY G36N4;
"Group of order 36:  number 4."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^2 = d^3 * (a^-1) = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * b * c = (c, d) * b = 1;
FINISH;
LIBRARY G36N5;
"Group of order 36:  number 5."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G36N6;
"Group of order 36:  number 6."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G36N7;
"Group of order 36:  number 7."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) * b * (c^-1) = 1;
FINISH;
LIBRARY G36N8;
```

```
"Group of order 36:  number 8."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 * a = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) * b * (c^-1) = 1;
FINISH;
LIBRARY G36N9;
"Group of order 36:  number 9."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^2 = d^2 = (a, b) = (a, c) * (a^-1) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) = 1;
FINISH;
LIBRARY G36N10;
"Group of order 36:  number 10."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^2 = d^2 * c = (a, b) = (a, c) * (a^-1) =
(a, d) * a * (b^-1) = (b, c) * (b^-1) = (b, d) * a * b = (c, d) = 1;
FINISH;
LIBRARY G48N1;
"Group of order 48:  number 1."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^2 * b = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * b = 1;
FINISH;
LIBRARY G48N2;
"Group of order 48:  number 2."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^2 * b = e^2 * b = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * b = 1;
FINISH;
LIBRARY G48N3;
"Group of order 48:  number 3."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 * b = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * b = 1;
FINISH;
LIBRARY G48N4;
"Group of order 48:  number 4."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^2 = e^2 * c = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * b = 1;
FINISH;
LIBRARY G48N5;
"Group of order 48:  number 5."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^2 * b = e^2 * c = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * b = 1;
```

```
FINISH;
LIBRARY G48N6;
"Group of order 48:  number 6."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 * b = d^2 = e^2 * (c^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * b = 1;
FINISH;
LIBRARY G48N7;
"Group of order 48:  number 7."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 * b = d^2 * b * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * b = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N8;
"Group of order 48:  number 8."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 * b = d^2 * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * b = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N9;
"Group of order 48:  number 9."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 * b = d^2 * b * (c^-1) = e^2 * b = (a, b) =
(a, c) = (a, d) = (a, e) = (b, c) = (b, d) = (b, e) =
(c, d) = (c, e) * b = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N10;
"Group of order 48:  number 10."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) = (a, e) =
(b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N11;
"Group of order 48:  number 11."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N12;
"Group of order 48:  number 12."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^3 = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
```

```
LIBRARY G48N13;
"Group of order 48:  number 13."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^3 = e^2 * c = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N14;
"Group of order 48:  number 14."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * b = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N15;
"Group of order 48:  number 15."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^3 = e^2 * (b^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N16;
"Group of order 48:  number 16."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * (b^-1) = d^3 = e^2 * c = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) =
(c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N17;
"Group of order 48:  number 17."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^2 = e^3 = (a, b) = (a, c) = (a, d) = (a, e) =
(b, c) = (b, d) = (b, e) = (c, d) = (c, e) * c * d = (d, e) * c = 1;
FINISH;
LIBRARY G48N18;
"Group of order 48:  number 18."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^2 = e^3 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * c * d = (d, e) * c = 1;
FINISH;
LIBRARY G48N19;
"Group of order 48:  number 19."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N20;
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"Group of order 48:  number 20."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N21;
"Group of order 48:  number 21."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N22;
"Group of order 48:  number 22."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 = e^2 * b = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N23;
"Group of order 48:  number 23."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 * b = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N24;
"Group of order 48:  number 24."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 * b = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N25;
"Group of order 48:  number 25."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^2 * a = e^2 * b = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N26;
"Group of order 48:  number 26."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
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LIBRARY G48N27;
"Group of order 48:  number 27."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^2 = e^2 * (b^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N28;
"Group of order 48:  number 28."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^2 * b = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G48N29;
"Group of order 48:  number 29."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 * a = d^2 * a = e^3 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) * a =
(c, e) * a * (c^-1) * (d^-1) = (d, e) * a * (c^-1) = 1;
FINISH;
LIBRARY G48N30;
"Group of order 48:  number 30."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * a = d^2 * a = e^3 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) * a =
(c, e) * a * (c^-1) * (d^-1) = (d, e) * a * (c^-1) = 1;
FINISH;
LIBRARY G48N31;
"Group of order 48:  number 31."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * b * c = (b, e) * b * c =
(c, d) * b = (c, e) * b * c = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N32;
"Group of order 48:  number 32."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^3 = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * b * c = (b, e) * b * c =
(c, d) * b = (c, e) * b * c = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N33;
"Group of order 48:  number 33."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) * a = (b, d) = (b, e) = (c, d) =
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(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N34;
"Group of order 48:  number 34."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * a = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) * a = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N35;
"Group of order 48:  number 35."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^3 = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) * a = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N36;
"Group of order 48:  number 36."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * a = d^3 = e^2 * a = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) * a = (b, d) = (b, e) =
(c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N37;
"Group of order 48:  number 37."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * a * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) * (b^-1) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N38;
"Group of order 48:  number 38."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) * (b^-1) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N39;
"Group of order 48:  number 39."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * a * (c^-1) = e^2 * a = (a, b) =
(a, c) = (a, d) = (a, e) = (b, c) = (b, d) = (b, e) * (b^-1) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N40;
"Group of order 48:  number 40."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
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G.relations:  a^2 = b^3 = c^2 * a = d^2 * a * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) * (b^-1) = (b, e) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N41;
"Group of order 48:  number 41."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) * (b^-1) = (b, e) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N42;
"Group of order 48:  number 42."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * (c^-1) = e^2 * a = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) * (b^-1) = (b, e) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N43;
"Group of order 48:  number 43."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * a * (c^-1) = e^2 * a = (a, b) =
(a, c) = (a, d) = (a, e) = (b, c) = (b, d) * (b^-1) = (b, e) =
(c, d) = (c, e) * a = (d, e) * (c^-1) = 1;
FINISH;
LIBRARY G48N44;
"Group of order 48:  number 44."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * a = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) * a = (b, d) * a * (b^-1) * (c^-1) =
(b, e) * (b^-1) * (c^-1) = (c, d) * a * (b^-1) =
(c, e) * a * (b^-1) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N45;
"Group of order 48:  number 45."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 * a = d^3 = e^2 * a = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) * a = (b, d) * a * (b^-1) * (c^-1) =
(b, e) * (b^-1) * (c^-1) = (c, d) * a * (b^-1) =
(c, e) * a * (b^-1) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G48N46;
"Group of order 48:  number 46."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^2 = d^2 = e^3 = (a, b) = (a, c) = (a, d) =
(a, e) * a * b = (b, c) = (b, d) = (b, e) * a = (c, d) =
(c, e) * c * d = (d, e) * c = 1;
FINISH;
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LIBRARY G48N47;
"Group of order 48:  number 47."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^2 = d^2 * c = e^3 = (a, b) = (a, c) = (a, d) =
(a, e) * a * c = (b, c) = (b, d) = (b, e) * a * (b^-1) * (d^-1) =
(c, d) = (c, e) * a = (d, e) * a * (b^-1) * c = 1;
FINISH;
LIBRARY G54N1;
"Group of order 54:  number 1."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^3 = d^3 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (a^-1) = 1;
FINISH;
LIBRARY G54N2;
"Group of order 54:  number 2."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^3 = d^3 * (a^-1) = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) = (c, d) * (a^-1) = 1;
FINISH;
LIBRARY G54N3;
"Group of order 54:  number 3."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G54N4;
"Group of order 54:  number 4."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 * (a^-1) = c^3 = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G54N5;
"Group of order 54:  number 5."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G54N6;
"Group of order 54:  number 6."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 * (b^-1) = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) = (b, d) * (b^-1) = (c, d) * b * (c^-1) = 1;
FINISH;
LIBRARY G54N7;
"Group of order 54:  number 7."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
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G.relations:  a^3 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) * (a^-1) = (b, d) * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G54N8;
"Group of order 54:  number 8."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) * (a^-1) =
(b, c) * (a^-1) = (b, d) * (b^-1) = (c, d) = 1;
FINISH;
LIBRARY G54N9;
"Group of order 54:  number 9."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 * (a^-1) = d^2 = (a, b) = (a, c) =
(a, d) * (a^-1) = (b, c) * (a^-1) = (b, d) = (c, d) * a * (c^-1) = 1;
FINISH;
LIBRARY G54N10;
"Group of order 54:  number 10."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 = c^3 = d^2 = (a, b) = (a, c) = (a, d) * (a^-1) =
(b, c) = (b, d) * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G54N11;
"Group of order 54:  number 11."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^3 * (a^-1) = c^3 = d^2 = (a, b) = (a, c) =
(a, d) * (a^-1) = (b, c) = (b, d) * a * (b^-1) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G60N1;
"Group of order 60:  number 1."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^5 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G60N2;
"Group of order 60:  number 2."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^5 = c^3 = d^2 * a = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G60N3;
"Group of order 60:  number 3."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^5 = b^2 = c^2 = d^3 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b * c = (c, d) * b = 1;
FINISH;
LIBRARY G60N4;
"Group of order 60:  number 4."
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"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^5 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * c^2 = 1;
FINISH;
LIBRARY G60N5;
"Group of order 60:  number 5."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^3 = c^5 = d^2 * a = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * c^2 = 1;
FINISH;
LIBRARY G60N6;
"Group of order 60:  number 6."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^5 = c^2 = d^2 * c = (a, b) = (a, c) = (a, d) =
(b, c) * b^2 = (b, d) * (b^-1) = (c, d) = 1;
FINISH;
LIBRARY G60N7;
"Group of order 60:  number 7."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^5 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b^2 = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G60N8;
"Group of order 60:  number 8."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^5 = c^3 = d^2 * a = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b^2 = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G60N9;
"Group of order 60:  number 9."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^5 = b^3 = c^2 = d^2 = (a, b) = (a, c) * a^2 = (a, d) * a^2 =
(b, c) * (b^-1) = (b, d) = (c, d) = 1;
FINISH;
LIBRARY G60N10;
"Group of order 60:  number 10."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^5 = b^3 = c^2 = d^2 * c = (a, b) = (a, c) * a^2 =
(a, d) * (a^-1) = (b, c) = (b, d) * (b^-1) = (c, d) = 1;
FINISH;
LIBRARY G60N11;
"Group of order 60:  number 11."
"Group:  G; Generators:  a, b."
G: free(a, b);
G.relations:  a^2 = b^3 = (a * b)^5 = 1;
FINISH;
LIBRARY G72N1;
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"Group of order 72:  number 1."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * a = 1;
FINISH;
LIBRARY G72N2;
"Group of order 72:  number 2."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * a = 1;
FINISH;
LIBRARY G72N3;
"Group of order 72:  number 3."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 * a = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * a = 1;
FINISH;
LIBRARY G72N4;
"Group of order 72:  number 4."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 * a = e^2 * a = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * a = 1;
FINISH;
LIBRARY G72N5;
"Group of order 72:  number 5."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 = e^2 = (a, b) = (a, c) = (a, d) = (a, e) =
(b, c) = (b, d) = (b, e) = (c, d) = (c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N6;
"Group of order 72:  number 6."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N7;
"Group of order 72:  number 7."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N8;
"Group of order 72:  number 8."
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"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 = e^2 * b = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N9;
"Group of order 72:  number 9."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 = d^2 = e^3 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * c * d = (d, e) * c = 1;
FINISH;
LIBRARY G72N10;
"Group of order 72:  number 10."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 = d^2 = e^3 * (b^-1) = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * c * d = (d, e) * c = 1;
FINISH;
LIBRARY G72N11;
"Group of order 72:  number 11."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N12;
"Group of order 72:  number 12."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N13;
"Group of order 72:  number 13."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N14;
"Group of order 72:  number 14."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 = e^2 * b = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * (d^-1) = 1;
FINISH;
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LIBRARY G72N15;
"Group of order 72:  number 15."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 * (c^-1) = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * c * (d^-1) = 1;
FINISH;
LIBRARY G72N16;
"Group of order 72:  number 16."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 * (c^-1) = e^2 * (a^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * c * (d^-1) = 1;
FINISH;
LIBRARY G72N17;
"Group of order 72:  number 17."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * c * (d^-1) = 1;
FINISH;
LIBRARY G72N18;
"Group of order 72:  number 18."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 * a = c^3 = d^3 * (c^-1) = e^2 * (b^-1) = (a, b) =
(a, c) = (a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * c * (d^-1) = 1;
FINISH;
LIBRARY G72N19;
"Group of order 72:  number 19."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N20;
"Group of order 72:  number 20."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N21;
"Group of order 72:  number 21."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) =
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(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N22;
"Group of order 72:  number 22."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * a = e^3 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) = (c, d) * a =
(c, e) * a * (c^-1) * (d^-1) = (d, e) * a * (c^-1) = 1;
FINISH;
LIBRARY G72N23;
"Group of order 72:  number 23."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^2 * a = d^2 * a = e^3 * (b^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) = (c, d) * a =
(c, e) * a * (c^-1) * (d^-1) = (d, e) * a * (c^-1) = 1;
FINISH;
LIBRARY G72N24;
"Group of order 72:  number 24."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * b * c = (b, e) * b * c = (c, d) * b =
(c, e) * b * c = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N25;
"Group of order 72:  number 25."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) = 1;
FINISH;
LIBRARY G72N26;
"Group of order 72:  number 26."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) = 1;
FINISH;
LIBRARY G72N27;
"Group of order 72:  number 27."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) = 1;
FINISH;
LIBRARY G72N28;
"Group of order 72:  number 28."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
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G.relations:  a^2 = b^3 = c^3 = d^2 = e^2 * d = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) * b * (c^-1) =
(c, d) * (c^-1) = (c, e) * b * c = (d, e) = 1;
FINISH;
LIBRARY G72N29;
"Group of order 72:  number 29."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * (d^-1) = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) * (b^-1) = (b, e) * b * (c^-1) =
(c, d) * (c^-1) = (c, e) * b * c = (d, e) = 1;
FINISH;
LIBRARY G72N30;
"Group of order 72:  number 30."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N31;
"Group of order 72:  number 31."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N32;
"Group of order 72:  number 32."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N33;
"Group of order 72:  number 33."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * b * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N34;
"Group of order 72:  number 34."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 * a = e^2 = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * b * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N35;
"Group of order 72:  number 35."
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"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 * (b^-1) = d^2 * a = e^2 * a = (a, b) = (a, c) =
(a, d) = (a, e) = (b, c) = (b, d) = (b, e) * (b^-1) = (c, d) =
(c, e) * b * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N36;
"Group of order 72:  number 36."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N37;
"Group of order 72:  number 37."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N38;
"Group of order 72:  number 38."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^3 = c^3 = d^2 * a = e^2 * a = (a, b) = (a, c) = (a, d) =
(a, e) = (b, c) = (b, d) * (b^-1) = (b, e) = (c, d) =
(c, e) * (c^-1) = (d, e) * a = 1;
FINISH;
LIBRARY G72N39;
"Group of order 72:  number 39."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^3 = c^2 = d^2 * c = e^2 * (d^-1) = (a, b) =
(a, c) * (a^-1) = (a, d) * b = (a, e) * a * (b^-1) =
(b, c) * (b^-1) = (b, d) * a * (b^-1) = (b, e) * (a^-1) * (b^-1) =
(c, d) = (c, e) = (d, e) = 1;
FINISH;
LIBRARY G72N40;
"Group of order 72:  number 40."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) * (a^-1) = (b, c) = (b, d) * b * c = (b, e) =
(c, d) * b = (c, e) = (d, e) = 1;
FINISH;
LIBRARY G72N41;
"Group of order 72:  number 41."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^3 = c^2 = d^2 = e^2 * c = (a, b) = (a, c) * (a^-1) =
(a, d) * a * (b^-1) = (a, e) * a * b = (b, c) * (b^-1) =
(b, d) * (a^-1) * b = (b, e) * (a^-1) * b =
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(c, d) = (c, e) = (d, e) * c = 1;
FINISH;
LIBRARY G72N42;
"Group of order 72:  number 42."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^3 = c^2 = d^2 * c = e^2 * c = (a, b) = (a, c) * (a^-1) =
(a, d) * a * b = (a, e) * (a^-1) * (b^-1) = (b, c) * (b^-1) =
(b, d) * (a^-1) * b = (b, e) * (a^-1) = (c, d) =
(c, e) = (d, e) * c = 1;
FINISH;
LIBRARY G72N43;
"Group of order 72:  number 43."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^3 = b^2 = c^2 = d^3 = e^2 = (a, b) = (a, c) = (a, d) =
(a, e) * (a^-1) = (b, c) = (b, d) * b * c = (b, e) = (c, d) * b =
(c, e) * b = (d, e) * (d^-1) = 1;
FINISH;
LIBRARY G72N44;
"Group of order 72:  number 44."
"Group:  G; Generators:  a, b, c, d, e."
G: free(a, b, c, d, e);
G.relations:  a^2 = b^2 = c^3 = d^3 * (c^-1) = e^2 = (a, b) = (a, c) =
(a, d) * (a^-1) * (b^-1) = (a, e) * (a^-1) * (b^-1) = (b, c) =
(b, d) * a = (b, e) * a * b = (c, d) = (c, e) * (c^-1) =
(d, e) * c * (d^-1) = 1;
FINISH;
LIBRARY G84N1;
"Group of order 84:  number 1."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^7 = b^2 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N2;
"Group of order 84:  number 2."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^7 = b^2 = c^3 = d^2 * b = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N3;
"Group of order 84:  number 3."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^7 = b^2 = c^2 = d^3 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b * c = (c, d) * b = 1;
FINISH;
LIBRARY G84N4;
"Group of order 84:  number 4."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^7 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
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(b, d) = (c, d) * c^2 = 1;
FINISH;
LIBRARY G84N5;
"Group of order 84:  number 5."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^3 = b^2 = c^7 = d^2 * b = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * c^2 = 1;
FINISH;
LIBRARY G84N6;
"Group of order 84:  number 6."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^2 = c^7 = d^3 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N7;
"Group of order 84:  number 7."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^2 * a = c^7 = d^3 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N8;
"Group of order 84:  number 8."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^7 = c^3 = d^2 = (a, b) = (a, c) = (a, d) =
(b, c) * (b^-1) = (b, d) * b^2 = (c, d) = 1;
FINISH;
LIBRARY G84N9;
"Group of order 84:  number 9."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^7 = c^3 = d^2 * a = (a, b) = (a, c) = (a, d) =
(b, c) * (b^-1) = (b, d) * b^2 = (c, d) = 1;
FINISH;
LIBRARY G84N10;
"Group of order 84:  number 10."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^7 = c^3 = d^2 = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b^2 = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N11;
"Group of order 84:  number 11."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^7 = c^3 = d^2 * a = (a, b) = (a, c) = (a, d) = (b, c) =
(b, d) * b^2 = (c, d) * (c^-1) = 1;
FINISH;
LIBRARY G84N12;
"Group of order 84:  number 12."
"Group:  G; Generators:  a, b, c, d."
```

```
G: free(a, b, c, d);
G.relations:  a^7 = b^3 = c^2 = d^2 = (a, b) = (a, c) * a^2 = (a, d) = (b, c) =
(b, d) * (b^-1) = (c, d) = 1;
FINISH;
LIBRARY G84N13;
"Group of order 84:  number 13."
"Group:  G; Generators:  a, b, c, d."
G: free(a, b, c, d);
G.relations:  a^2 = b^2 = c^7 = d^3 = (a, b) = (a, c) = (a, d) * a * b = (b, c) =
(b, d) * a = (c, d) * (c^-1) = 1;
FINISH;
```

Appendix C

This appendix contains the output from running various Cayley programs on a number of presentations given in the text. The output is presented ordered by Chapter.

```
SUN/UNIX CAYLEY V3.8-531 Mon Jun 24 1991 18:38:02 STORAGE 200000
>g=free(a,c);
>" (3.0) "
>" A group of order 75 "
>g.relations:
>a^5=(c*a)^3,c^3=(c*a^-1)^3=1;
>print order(g);
75
>clear;
>
>
>" A covering group for the group of order 75 above "
>
>g=free(a,b,c,x,y,z);
>g.relations:
>A^5*z^7, B^5*y*z^2,
>C^3,
>x=(A,B),
>(A,C)*B^-1*A*z,
>A^-1*B^3*(C,B),
>(x,a),(x,b),(x,c),(x,y),(z,x),
>(y,a),(y,b),(y,c),(y,z),
>(z,a),(z,b),(z,c);
>print order(g);
375
>clear;
>
>g=free(a,b);
>" (3.9) "
>" Another deficiency zero presentation for the group of
> order 56 found by Johnson and Mawdesly "
>
>g.relations:
>a^2=b^7,
>(a*b*a*b^-1)^2=a*b*a*b^-3*a*b^2;
>print order(g);
56
>clear;
>
> " A covering group for the group 3.20 of order 81 "
>
>g=free(p,q,r,x,y,z);
>g.relations:
>P^9*z, Q^3*y, R^3,
>(P,Q)*x^-1,(P,R)*Q^-1,
>Q^-1*P^-3*Q*(R,Q),(P,x)
>,(Q,x),(R,x),(P,y)
>,(Q,y),(R,y),(x,y)
>,(P,z),(Q,z),(R,z)
>,(x,z),(y,z);
```

```
>print order(g);
729
>bye;
END OF RUN.
3.559 SECONDS
SUN/UNIX CAYLEY V3.8-531 Tue Jun 11 1991 13:15:00 STORAGE 2000000
>library g4p2;
Library module found as ./g4p2
>print g;print order(g); clear;
GROUP G
RELATORS :
X^3
Y^4
X Y X Y^-1 X Y^-1
X^-1 Y^-2 X Y^2
24
>library g4p7;
Library module found as ./g4p7
>print g;print order(g); clear;
GROUP G
RELATORS :
(A B)^3
B^4
A^3 B A^-1 B^2 A B^3
216
>library g4p8;
Library module found as ./g4p8
>print g;print order(g); clear;
GROUP G
RELATORS :
B^4
A^3 B^3 A B^2 A^-1 B
B^3 A^-1 B^3 A^-1 B^-1 A^-1
216
>library g4p9;
Library module found as ./g4p9
>print g;print order(g); clear;
GROUP G
RELATORS :
U V^2 U V^-1 U V^-1
U^2 V U^2 V U^-1 V U^2 V^-1 U^-1 V^-1
3000
>library g4p10;
Library module found as ./g4p10
>print g;print order(g); clear;
GROUP G
RELATORS :
U^2 V^-1 U V^2 U^2 V^-2
U^2 V^-2 U^-1 V^-3 U V^-2 U^-1 V^-1
39000
>library g4p11;
Library module found as ./g4p11
>print g;print order(g); clear;
GROUP G
```

```
RELATORS :
V^-1 U V U V U V^2 U^-2
U^-1 V^3 U
2880
>library g4p12;
Library module found as ./g4p12
>print g;print order(g); clear;
GROUP G
RELATORS :
U^3 V^-1 U^-1 V^-1 U^-1 V^-1
U^3 V U^2 V^-1 U V^-3
3240
>library g4p13;
Library module found as ./g4p13
>print g;print order(g); clear;
GROUP G
RELATORS :
U^3 V U^-1 V U^-1 V
U^3 V^-3 U V U^2 V^-1
4536
>library g4p14;
Library module found as ./g4p14
>print g;print order(g); clear;
GROUP G
RELATORS :
U^2 V U^-1 V U^-1 V^-1 U V^-2
U^2 V^-1 U V^-1 U V U^-1 V^2
2400
>bye;
END OF RUN.
43.420 SECONDS
```

```
SUN/UNIX CAYLEY V3.8-531 Tue Jun 11 1991 11:35:58 STORAGE 200000
>
> " A group of order 1296 and soluble length six "
> " This presentation is due to Wegner "
>
>g=free(a,b);
>g.relations:
>a*b*a*b*a*b^-2*a^-1*b,
>a^2*b*a^-2*b*a^2*b^-2;
>print order(g),derived series(g);
1296
SEQ(
GROUP OF ORDER 648 = 2^3 * 3^4 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^-1 B^-1 A B
A^-1 B A B^-1
,
GROUP OF ORDER 216 = 2^3 * 3^3 IS A SUBGROUP OF G
```

```
GENERATORS AS WORDS IN SUPERGROUP :
B A B^-1 A^-1 B^4
B^-1 A B A^-2 B A^-1
,
GROUP OF ORDER 54 = 2 * 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^-2 B^3
A B^-3 A^-1
B^-3
,
GROUP OF ORDER 27 = 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
B^-1 A^-2 B
A^2 B^-6
,
GROUP OF ORDER 3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
B^6
,
GROUP OF ORDER 1 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
IDENTITY
)
>print classes(g);
CONJUGACY CLASSES OF G
----------------------
[1] ORDER 1 LENGTH 1 REP IDENTITY
[2] ORDER 2 LENGTH 9 REP A^2 B^-3 A^2
[3] ORDER 2 LENGTH 108 REP B^-1 A^3 B
[4] ORDER 3 LENGTH 2 REP B^-6
[5] ORDER 3 LENGTH 24 REP B^-1 A^2 B
[6] ORDER 4 LENGTH 54 REP B^-1 A B^-1 A
[7] ORDER 6 LENGTH 18 REP B^-3
[8] ORDER 6 LENGTH 216 REP B^-1 A B
[9] ORDER 8 LENGTH 162 REP B^-1 A
[10] ORDER 8 LENGTH 162 REP A^-1 B A^-1 B A^-1 B
[11] ORDER 9 LENGTH 24 REP B^-2
[12] ORDER 9 LENGTH 24 REP B^-4
[13] ORDER 9 LENGTH 24 REP A^2 B A^-1 B^-3 A
[14] ORDER 9 LENGTH 144 REP B^2 A^-2
[15] ORDER 12 LENGTH 108 REP A^2 B^-1 A B^2 A
[16] ORDER 18 LENGTH 72 REP B^-1
[17] ORDER 18 LENGTH 72 REP B^-5
[18] ORDER 18 LENGTH 72 REP B^-7
```

```
>clear;
>
> " Another deficiency zero presentation for a group of "
> " soluble length six.  The number of elements of order "
> " three show that this group is not isomorphic to the "
> " group above; nor is it isomorphic to the group G(-9) "
> " of Theorem 3 of Chapter Four (below) "
>
>g=free(a,b);
>g.relations:
>a*b*a^-1*b*a^-1*b*a*b^-2,
>a^2*b*a^-1*b^-3*a*b^2;
>print order(g),derived series(g);
1296
SEQ(
GROUP OF ORDER 648 = 2^3 * 3^4 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^-1 B^-1 A B
A^-1 B A B^-1
,
GROUP OF ORDER 216 = 2^3 * 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^-1 B A B^-1 A^-1 B A
A^2 B^-1 A B^2 A^-1
,
GROUP OF ORDER 54 = 2 * 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^-2 B^-3
A B^3 A^-1
B^3
,
GROUP OF ORDER 27 = 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
B^-1 A^-2 B
A^2 B^6
,
GROUP OF ORDER 3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
B^-6
,
GROUP OF ORDER 1 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
IDENTITY
)
```

```
>print classes(g);
CONJUGACY CLASSES OF G
----------------------
[1] ORDER 1 LENGTH 1 REP IDENTITY
[2] ORDER 2 LENGTH 9 REP A^2 B^3 A^2
[3] ORDER 2 LENGTH 108 REP A^3
[4] ORDER 3 LENGTH 2 REP B^6
[5] ORDER 3 LENGTH 24 REP A^2
[6] ORDER 3 LENGTH 144 REP B A^2 B
[7] ORDER 4 LENGTH 54 REP A^2 B^-1 A^-1 B^-1 A
[8] ORDER 6 LENGTH 18 REP B^3
[9] ORDER 6 LENGTH 216 REP A
[10] ORDER 8 LENGTH 162 REP B^2 A^-1
[11] ORDER 8 LENGTH 162 REP B A B^-1 A^-1 B^-1 A B^-1
[12] ORDER 9 LENGTH 24 REP B^2
[13] ORDER 9 LENGTH 24 REP B^4
[14] ORDER 9 LENGTH 24 REP A^2 B^3 A^2 B^-1
[15] ORDER 12 LENGTH 108 REP A B A^-1 B
[16] ORDER 18 LENGTH 72 REP B
[17] ORDER 18 LENGTH 72 REP B^5
[18] ORDER 18 LENGTH 72 REP B^7
>clear;
>
> " G(-9) of Theorem 3, Chapter Four "
> " The class structure shows that this group is "
> " not isomorphic to either of the groups above "
>
>g=free(a,b);
>g.relations:
>(a*b)^2=b^6,
>a^4*b^-1*a*b^-9*a^-1*b;
>print order(g),derived series(g);
1296
SEQ(
GROUP OF ORDER 648 = 2^3 * 3^4 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
B A^2 B
A^-1 B A B^-1
,
GROUP OF ORDER 216 = 2^3 * 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^3 B^2 A^-2 B A^-1
B A^-1 B A^2 B^-1 A B^-1
,
```

```
GROUP OF ORDER 54 = 2 * 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^2 B A^-2 B A^-2 B
B A^-1 B^2 A^3 B
B A^3 B^2 A^-1 B
,
GROUP OF ORDER 27 = 3^3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
A^3 B^3 A
A^-1 B A B^-1 A B A^-1
,
GROUP OF ORDER 3 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
(A^2 B^-1)^3
,
GROUP OF ORDER 1 IS A SUBGROUP OF G
GENERATORS AS WORDS IN SUPERGROUP :
IDENTITY
)
>print classes(g);
CONJUGACY CLASSES OF G
----------------------
[1] ORDER 1 LENGTH 1 REP IDENTITY
[2] ORDER 2 LENGTH 9 REP A^4
[3] ORDER 2 LENGTH 108 REP A B
[4] ORDER 3 LENGTH 2 REP A^2 B^-1 A^2 B^-1 A^2 B^-1
[5] ORDER 3 LENGTH 24 REP B^-2
[6] ORDER 3 LENGTH 24 REP B^-1 A B^-2 A B^-1
[7] ORDER 3 LENGTH 24 REP B A^-1 B^2 A^-2 B A^-1
[8] ORDER 3 LENGTH 24 REP A B^-1 A^2 B^-1 A B^-1
[9] ORDER 4 LENGTH 54 REP A^2
[10] ORDER 6 LENGTH 18 REP A B^2 A^-1 B^2 A^2
[11] ORDER 6 LENGTH 72 REP B^-1
[12] ORDER 6 LENGTH 72 REP A B^-1 A B^-2
[13] ORDER 6 LENGTH 72 REP A^2 B^2
[14] ORDER 6 LENGTH 216 REP B^-1 A B^-1
[15] ORDER 8 LENGTH 162 REP A
[16] ORDER 8 LENGTH 162 REP A^-3
[17] ORDER 9 LENGTH 144 REP A^4 B^-1
[18] ORDER 12 LENGTH 108 REP A B^-1 A B^2
>clear;
>bye;
END OF RUN.
6.969 SECONDS
```

```
SUN/UNIX CAYLEY V3.7.3 Mon Jun 24 1991 18:52:47 STORAGE 200000
SUN/UNIX CAYLEY V3.7.3 Mon Jun 24 1991 18:52:47 STORAGE 4500000
>
> " A finite group of soluble length seven "
>
>g=perm(72);
>g.genera:
>x=(1,66,38,41,15,56,28,32)(2,65,37,42,16,55,27,31)
>(3,7,58,25,5,13,71,48)(4,8,57,26,6,14,72,47)
>(9,36,46,30)(10,35,45,29)(11,17,39,52,12,18,40,51)
>(19,21,64,53,62,43,24,70)(20,22,63,54,61,44,23,69)
>(33,68,50,59,34,67,49,60),
>y=(1,22)(2,21)(3,29)(4,30)(5,6)(7,62)(8,61)(9,10)(11,42)(12,41)
>(13,14)(15,35)(16,36)(17,66)(18,65)(19,28)(20,27)(23,49)(24,50)
>(25,52)(26,51)(31,48)(32,47)(33,54)(34,53)(37,38)(39,67)(40,68)
>(43,45)(44,46)(55,69)(56,70)(57,60)(58,59)(63,72)(64,71);
>print order(g);
82944
>ds= derived series(g);
>for i=1 to length(ds) do
> print i,order(ds[i]);
>end;
1 41472
2 13824
3 3456
4 1728
5 192
6 64
7 1
>clear;
>set printi=true;
>g=free(x,y);
>w1=x^8;w2=y^2;
>w3=x^4*y*(x^-2*y)^3*x*y*(x^-1*y)^2;
>w4=(x^2*y*x^-2*y*x^-1*y)^3;
>g.relations:
>w1,w2,w3,w4;
>ind= todd coxeter(g,[x]);
>print ind*8;
82944
>bye;
END OF RUN.
73.180 SECONDS
SUN/UNIX CAYLEY V3.7.3 Sat Jun 29 1991 15:02:09 STORAGE 200000
```

```
SUN/UNIX CAYLEY V3.7.3 Sat Jun 29 1991 15:02:09 STORAGE 6000000
>"
>Perfect group isomorphic to A5 times SL(2,5)
>"
>g=free(x,y);
>g.relations:
>y^6,
>x^4*y*x^-1*y^-3*x^-1*y^-1,
>(x*y^2)^2*x^-1*y^-1*(x*y^-1)^2;
>print order(g);
7200
>clear;
>"
>A5 times A5
>"
>h=free(x,y);
>h.relations:
>x^10,y^6
>,x^4*y*x^-1*y^-3*x^-1*y^-1
>,(x*y^2)^2*x^-1*y^-1*(x*y^-1)^2;
>print order(h);
3600
>clear;
>" A5 times A6
>"
>g=perm(11);
>g.genera:
>b=(2,3)(4,5)(8,11)(9,10),
>a =(1,3,5,4,6)(7,8,10);
>print order(g);
21600
>clear;
>h=free(a,b);
>h.relations:
> a^15 , b^2
>,(a*b)^5 , (a^3*b)^4
>,a^2*b*a^-3*b*a^7*b*a^-3*b;
>print todd coxeter(h,[b]);
10800
>clear;
>h=free(a,b);
>h.relations:
> a^15 , b^2
>,(a*b)^5
```

```
>,a^2*b*a^-3*b*a^7*b*a^-3*b;
>print todd coxeter(h,[b]);
10800
>clear;
>"
>A5 times A7
>"
>g=perm(12);
>g.genera:
>a=(1,5,3,2,4)(6,10,9,11,8),
>b=(1,3)(2,5)(6,10,7,9,12,11,8);
>print order(g);
151200
>clear;
>h=free(a,b);
>h.relations:
> a^5,b^14,a*b^2*a^-1*b*a^-1*b^-2*a*b^-1
>,a*b*a^-1*b*a*b^-1*a*b^-1*a^-1*b^-1*a*b^-1
>,(a,b^2)^2
>,(a^2*b^-1)^2*a*b^4*a*b^-1
>,a^2*b*a^2*b^-1*a^-1*b^-2*a^-1*b^-2*a^-1*b^-1;
>print todd coxeter(h,[a]);
30240
>clear;
>h=free(a,b);
>h.relations:
> a^5
>,a*b*a^-1*b*a*b^-1*a*b^-1*a^-1*b^-1*a*b^-1
>,(a^2*b^-1)^2*a*b^4*a*b^-1
>,a^2*b*a^2*b^-1*a^-1*b^-2*a^-1*b^-2*a^-1*b^-1;
>print todd coxeter(h,[a]);
30240
>clear;
>"
>A5 times A5 times A5
>"
>g=perm(15);
>g.genera:
>a=(1,3,2,4,5)(6,9,7)(11,12,13,14,15),
>b=(2,3)(4,5)(7,8)(9,10)(11,14)(12,15);
>print order(g);
216000
>clear;
>g=free(a,b);
```

```
>g.relations:
> a^15,b^2
>,a^4*b*a^-1*b*a^3*b*a*b*a^-1*b*a^-1*b*a^3*b*a*b
>,a^4*b*a^-1*b*(a*b)^4*(a^-1*b)^2*a^-3*b
>,(a^3*b)^2*(a^2*b)^4*(a^3*b)^2*a^-1*b;
>print todd coxeter(g,[a]);
14400
>clear;
>"
>Another perfect group of order 7200
>"
>g=free(x,y);
>g.relations:
>x^10=y^6,
> x^4*y*x^-1*y^-3*x^-1*y^-1
>,(x*y^2)^2*x^-1*y^-1*(x*y^-1)^2;
>print order(g);
7200
>print normal subgroups(g);
NORMAL SUBGROUPS OF G
---------------------
[2] ORDER 2 GENERATING CLASSES: [2]
UNION OF: [1] [2]
-
[3] ORDER 120 GENERATING CLASSES: [4]
MAXIMALS: [2]
UNION OF: [1] [2] [4] [8] [11] [12] [17] [22] [23]
[4] ORDER 120 GENERATING CLASSES: [5]
MAXIMALS: [2]
UNION OF: [1] [2] [5] [7] [9] [10] [18] [20] [21]
-
[5] ORDER 7200 GENERATING CLASSES: [3]
MAXIMALS: [3] [4]
UNION OF: [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12]
[13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23]
[24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34]
[35] [36] [37] [38] [39] [40] [41]
>clear;
>"
>SL(2,7) times SL(2,7)
>"
>g=matrix(4,gf(7));
>g.genera:
>x=mat(6,0,0,0:
```

```
> 5,6,0,0:
> 0,0,5,4:
> 0,0,5,0),
>y=mat(6,4,0,0:
> 4,4,0,0:
> 0,0,1,5:
> 0,0,0,1);
>print order(g);
112896
>
> print x^4*y*x^-3*y;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
> print x^14;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
> print x^3*y^2*x*y^5*x*y^2;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
> print x^2*y^2*x^-1*y^2*x^-1*y^2*x^2*y^-1*x^-1*y^-1;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
> print (x^2*y^3)^3;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
> print (x*y^2*x^-1*y^-1)^3;
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
>clear;
>g=free(x,y);
>g.relations:
>x^4*y*x^-3*y,
```

```
>x^14,
>x^3*y^2*x*y^5*x*y^2,
>x^2*y^2*x^-1*y^2*x^-1*y^2*x^2*y^-1*x^-1*y^-1,
>(x^2*y^3)^3,
>(x*y^2*x^-1*y^-1)^3;
>print todd coxeter(g,[x]);
8064
>clear;
>g=free(x,y);
>g.relations:
>x^4*y*x^-3*y,
>x^14,
>x^3*y^2*x*y^5*x*y^2,
>x^2*y^2*x^-1*y^2*x^-1*y^2*x^2*y^-1*x^-1*y^-1;
>ind = todd coxeter(g,[x]);
>print ind * 14;
112896
>clear;
>g=free(x,y);
>g.relations:
>x^4*y*x^-3*y,
>x^3*y^2*x*y^5*x*y^2;
>print order(g);
112896
>clear;
>bye;
END OF RUN.
338.929 SECONDS
```

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