On the Distribution of Queueing Times
for
Queues with Two Servers
by
N.M.H. Smith

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Preface

The work of this Thesis arose from a very natural, although really quite accidental drawing together of ideas from two completely separate fields of academic endeavour.

On one side, the Probabilitists had progressed quite well with most of those classes of queueing problems which could be reduced to equivalent random walks on the line, but only a very limited success had been achieved with those queues of the GI/G/C class which were such that the equivalent walks were of necessity over multivariate real space. Moreover, such progress as appeared to have been made was embodied in the incomplete and apparently very difficult works of Pollaczek.

On the other hand, this century has seen much progress in a branch of analysis which was little explored before the end of the nineteenth and which is, it seems, still too new to be taught to undergraduates or, to labour a point, undergraduate probabilists. Thus it was for me quite shocking to discover the existence of that vast and profoundly useful structure which is Multivariate Complex Analysis and to begin to dimly see that therein might lie a key to the general Multidimensional Random Walk.

I can only regret that my limitations in ability, time and training have not permitted me to do more in this Thesis, than merely solve one or two of the more elementary problems from that class which it appears, may now be accessible.
However I may at least hope that if this work shall be judged to have any lasting merit, this will be found in the fact that I have proven that one may usefully draw these two fields together. If so, my debt to Professor Hille is very great for he really suggested it.
Acknowledgements

Many people have helped me both directly and indirectly in the preparation of this Thesis but I should particularly like to thank the following:

(i) My Supervisor, Professor P.A.P. Moran for his many kindnesses and most valuable and diligent criticisms.

(ii) Dr. D. Vere-Jones who first introduced me to the Wiener-Hopf method and later deduced the ergodic limiting bivariate distribution for the M/M/2 Queue by probabilistic means, thereby verifying some of the results.

(iii) Drs. N.E. Day, C. Heathcote and P. Mandl for useful advice, stimulation and copies of papers at various times.

(iv) Emeritus Professor E. Hille who in a most brief yet useful conversation drew my attention to the works of Salomon Brochner and thence indirectly to the apparatus which is Multivariate Complex Analysis.

(v) Mrs. B. Cranstone, Mrs. W. Hunt and Mrs. K. Strickland for the accurate typing of some most illegible drafts.

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Certification

Except at those points where I have specifically indicated otherwise or where I have used a suitably named Theorem, the work of this Thesis is the result of my own efforts aided only by the numerous criticisms of my supervisors.

N.M.H. Smith
Initially we consider first come first served queues with two or more servers wherein the intervals between the successive arrivals are independently and identically distributed. The customers' service times are similarly distributed.

The method is to define an embedded Markov chain on the moments just before each arrival and thence recurrences which relate the state of the system just before the \((n+1)\)th arrival to that which existed just before the \(n\)th. The problem is then specialized to that for two servers and these probability recurrences are used to develop a relationship between the bivariate Laplace transformations of the distribution functions which arise.

The problem is reduced to the solution of a single integral equation for the Laplace transformation of the ergodic limiting distribution function by the definition of two compensation functions. These steps are analogous to the well known probability "sweeping up" operations for one server queues.

This integral equation is reduced to a functional equation for the case where the interarrival and service time distributions are both composed of any integral numbers of exponential stages. This equation is solved in principle for all finite numbers of such stages, and in detail when the service time distribution has one or two stages and the interarrival time distribution any number.

The results are checked against a known result for one case and a set of simulation results for another. The agreement is satisfactory.

Of particular note are the curious loci of certain singularities.

The Thesis also discusses a number of important intermediate results which suggest that the classical Wiener-Hopf method for a unidimensional integral equation may generalize to a useful multidimensional result. We conclude with a section which outlines the proofs of a more general Theorem which explores this possibility.
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Summary

The work was taken up from the following background;

1. Except for such special cases as GI/M/C and M/D/C, there existed no complete results for the queues of the GI/G/C class. Partial analytic results appeared to be available, however, from the incomplete work of Pollaczek, but nobody had been able to carry out the apparently finite number of algebraic operations required to obtain a specific result for any case where the Laplace Transformation of the Service Time Distribution was rational.

2. Almost innumerable papers exist on queues of the GI/G/1 class based on a wide variety of methods. However, Kingman has shown in [9], that all of these methods were in essence, identical. Furthermore, he had related all of these to Wendel's paper on the Theory of Projective Operators.

3. Kingman had also noticed a certain inherent weaknesses in the probabilistic basis of the Pollaczek method for the GI/G/C queue which only arises if C>2. See Kingman [9] paragraph 12.

4. During the discussion of a paper by Pollaczek [15], Weiss had suggested that it might be better to try to apply the notions of the Wiener-Hopf method to queues of the GI/G/C class rather than to proceed as it seemed to him Pollaczek was doing, backwards from an assumed form of solution.

5. I was introduced to the traditional Wiener-Hopf apparatus in 1966 and had, purely as an exercise, reinterpreted the classical "sweeping up" operation (See Kingman [9]) used for
the solution of GI/G/1 class queues in terms of Compensation Functions and immediately saw that it would be easy to find rules with which to construct the multidimensional compensation functions for multi-server queues.

The following connected sequence of investigation was put in hand:

1. The verification of Kingman's comment on Pollaczek's work.
2. The reformulation of the problem in such a way that any ambiguity if it existed, was overcome.
3. To apply to that formulation those perhaps more pedestrian analytic methods based on the use of Compensation Functions rather than the multidimensional Complex Integrals of unit steps at the origins on the real lines, in order to "sweep up" the probabilities associated with negative variables.

Remarks:

(i) Trivially, the two processes are equivalent, if they are correct, but one may reasonably expect that if one constructs a compensation function (or functions) in accordance with some pre-deduced rules, one may then be able to determine the analyticity and continuity properties of this function (or those functions) with the aid of those rules. Also, it is clear that since these are rules for "sweeping up" probabilities they must be unique.

(ii) Pollaczek's latent ambiguity arises from his use of symmetry as the basis for probabilistic (and thus Laplace Transfor-
mation space) iterations. Naturally, this leads him straight into C simultaneous integral or functional equations for the C server queue.

(iii) Furthermore one is then left with these simultaneous equations to solve and all that one knows is that the ergodic limiting probabilistic solution must, if it exists at all, be a C dimensional symmetric distribution function (with C dimensional symmetry) if the queue has C servers.

4. To obtain an equation or equations applicable to the simplest general multi-server queue, namely GI/G/2 which would be as closely analogous to the Wiener-Hopf - W.L. Smith equation for GI/G/1 as might be possible.

5. Obtain the equations to be solved for the ergodic limiting distributions in some of the simpler cases.

6. To solve these and verify the solutions by some independent means.

7. To consider the general implications of the work and any additional work which appears necessary.

The following thesis is the connected record of this work, arranged in seventeen Sections and two Appendices. For reasons of continuity, the verification of Kingman's comment on, and my own comments on the Pollaczek method are placed in Appendix II, but the remainder has been presented in the order in which it was carried out. Thus after Section 1 which is a glossary of the main notation we find:

1. Section 2 which gives a rigorous derivation of the two
random walks which correspond with the embedded Markov chain of the queue GI/G/2, and the recurrences which define these walks.

2. Section 3 outlines the alternative recurrences for the equivalent walks for the queue GI/G/C in terms of two ordered set of variables $V_1 \geq V_2 \geq \ldots \geq V_{C-1} \geq V_C$ where $V_C^+ = \max (0, V_C) = W$ where $W$ is the Waiting Time, and $V_1^{(+)} \geq V_2^{(+)} \geq \ldots \geq V_{C-1}^{(+)} \geq V_C^{(+)} = W$.

3. Section 4 moves the problem presented by the queue GI/G/2 from bivariate probability space to the space of Laplace Transformations in the bivariate complex domain, introduces the necessary compensation functions and culminates in a single general equation which is analogous to the well known Wiener-Hopf equation for the ergodic one server queue.

4. Sections 5 and 6 establish a large number of necessary properties of bivariate and multivariate Laplace Transforms of Distribution Functions. These Sections also introduce a number of important definitions and given Theorems about analytic functions of which, clearly, the most important is that of Hartogs.

5. Section 7 derives the form of the general functional equation for the Laplace Transformations of the ergodic limiting distributions which arise for suitably loaded queues of the class $E_K/E_L/2$ for all $K, L = 1, 2, \ldots$.
6. The short Section 8 summarizes the functional equations for the queues M/M/2, E^K/M/2, E_2/E_2/2, and E^K/E_2/2.

7. In Section 9 we show that for all finite integral K and L, the Centre Function C(\phi, \theta) is a bivariate polynomial very much analogous to the polynomial which arises in the Solution of E^K/E_L/1.

Note: We also outline in various remarks an alternative proof based on the principle of Analytic Completion between tubes, that all queues of the GI/E_L/2 class will have C(\phi, \theta) bivariate entire for all L = 1, 2, ...

8. Section 10 reviews and extends the properties of the Laplace Transformations \chi^*(\phi, \theta) and \chi^*_-(\phi, \theta).

9. Section 11 describes the principle of the method which may be used to solve all the functional equations which may arise from queues of the E^K/E_L/2 class for all K, L = 1, 2, ...

10. Section 12 contains a first application of these principles and gives the solution of E^K/M/2 for all K = 1, 2, ... and verifies the correctness of this for the simplest case, namely M/M/2.

11. Section 13 repeats these processes for the rather more complicated system E^K/E_2/2 for all K = 1, 2, ..., gives the form of solution, and reduces the problem to that of solving six simultaneous linear equations - the number six being entirely an outcome of the E_2 Service Time Distribution and satisfying two auxiliary equations, one of which
defines some of the coefficients of the linear equations whilst the other defines the probability of the queue being empty at the moment just before the arrival of an arbitrary (i.e. typical) customer.

These equations are summarized in Section 14.

12. In Section 15 we compute the theoretical result for the queue $E_2/E_2/2$ for a mean load per server ($\rho$) of 0.75 and compare this with the pooled and individual results of ten independent computer simulation runs each with 10,000 arrivals.

The result of this comparison is quite pleasing. Apparently this analytic method works.

13. In Section 16 we show why the random walk of the GI/G/2 queue gives rise to Laplace Transformations which have singularities with rather interesting loci and interpret this result probabilistically by showing that the traditional notion of first come first served queue operation is strictly equivalent to what we there call "Queuelet Operation". This Section ends with some suggestions and conjectures regarding approximations, starting approximations and possible bounds.

14. Section 17 presents somewhat abbreviated proofs of Theorems which suggest that the functional and integral equations for all queues of the GI/G/2 class may be so rearranged that a bivariate entire Centre Function always results.
Glossary of Notation

This section contains the definitions of the main notation used in this thesis.

1.1 Latin Symbols

\( \bar{a} \) is the time interval between the arrival of the \((n-1)\)th customer and that of the nth.

\( A(a) \) is the probability that any interarrival interval is less than or equal to \( a \).

\( \bar{a} \) is the mean interarrival time.

\( a_{n_1}, n_2 \) is a general coefficient in the bivariate power series \( f(z_1, z_2) \) discussed in paragraph (5.1.5).

\( a_{n_1}, \ldots, a_{n_m} \) is a generalization of \( a_{n_1}, n_2 \) used in the definition of a multivariate analytic function.

\( \varphi_1(\alpha), \ldots, \varphi_m(\alpha) \) is a point in \( m \) dimensional complex space. It is the \( \alpha \)th such point and is a centre point of the \( \alpha \)th element of a continuation process which also defines a Singular Chain.

\( Am(\cdot) \) used to indicate the Amplitude of the variable placed within the parenthesis.

\( B(s) \) is the probability distribution for the customer service times. That is \( B(s) = P(S \leq s) \)

\( \mathbb{C}^n \) Complex space in \( n \) dimensions. See also \( E_{2n} \) which is the equivalent Euclidean 2n dimensional space.

\( C(\phi, \theta) \) is the "Centre Function" of the functional equation of the \( E_k/E_L/2 \) queue. See equation (7.17).
$C_{KL}(\phi, \theta)$ whence $C_{11}(\phi, \theta)$, $C_{K1}(\phi, \theta)$, $C_{K2}(\phi, \theta)$ etc. for which see Section 8, are variants on $C(\phi, \theta)$ wherein the numbers $K$ and $L$ of the $E_K/E_L/2$ queue appear explicitly.

$C_l(\omega)$ for $l = 0, 1, \ldots, L-1$ is the function defined by the substitution of the $l^{th}$ branch out of $L$ branches of $f(\omega)$ in the relationship:

$$C(f(\omega), \omega - f(\omega)) = M(f(\omega), \omega - f(\omega))$$

Also written as $C_l$ for $l = 0, 1, \ldots, L-1$ for short.

$C(\omega)$ A column vector formed from the $C_l(\omega)$ for all $l = 0, 1, \ldots, L-1$. See equation (11.7.4).

$\tilde{C}(0, \theta)$ A complex valued function of the complex variable defined by equation (13.13.5).

$\mathcal{D}$ denotes a domain. Domains are defined by Definition (5.5.1.).

$D(z^0, r)$ denotes a polydisc in multidimensional complex space.

Polydisc is defined by Definition (5.1.6).

$D_\varepsilon$ denotes an elementary disc of the $D(z^0, r)$ class such that all radii $r < \varepsilon$ where $\varepsilon$ is arbitrarily small. See Definition (5.1.7) where this is used.

d is used to locate a common discontinuity of $A(a)$ and $B(s)$. See discussion after equation (6.3.4).

$\mathcal{F}_-$ denotes the primary domain of analyticity of the Laplace Transformation $\chi^*_-(\phi, \theta)$ in the bivariate complex space of $\phi$ and $\theta$.

$\mathcal{F}_+$ similar to $\mathcal{F}_-$ but relates to $\chi^*_+(\phi, \theta)$. See Lemma (6.11).
$D_\theta$ denotes differentiation with respect to $\theta$.

$D_\phi$ " " " " $\phi$.

$D_z$ " " " " $z$.

See proof of Lemma (6.21) and Section 7.

$D(\phi,\theta)$ denotes a complex valued function of the complex variables $\phi$ and $\theta$. See equation (9.3.1) et seq.

$d^{(+)}_m(\theta)$ denotes for each $m = 0,1,\ldots,K$ an analytic function of all finite $\theta$ s.t. $\text{Re}(\theta)>0$. See Theorem (9.3.3).

$d_{mj}$ denotes the $j^{th}$ of the $m^{th}$ set of complex valued coefficients in a power series representation of $D(\phi,\theta)$. See equation (9.3.21).

$d_m(\theta)$ denotes for each $m = 0,1,\ldots,K$ the $d^{(+)}_m(\theta)$ defined above after we have shown these to be analytic for all finite $\theta$. See equation (9.6.18).

$E_n$ denotes $2n$ dimensional Euclidean space.

$E_K$ denotes an Erlang distribution with $K$ exponential stages.

That is a Gamma (or Pearson III) or a $\chi^2$ distribution with $2K$ degrees of freedom.

$E_L$ denotes a similar distribution with $L$ exponential phases.

$E$ denotes a real variable used in the proof of Lemma (6.5(i)).

See Result 1 and equations (6.5.1.1) and (6.5.1.4).

$E$ also denotes a real variable used in the proof of Lemma (6.5).

$E(\cdot|\cdot)$ denotes the Laplace Transformation of the measure of the real variable or variables within the parenthesis with respect to the given complex variables.
\( E(\theta) \) denotes a complex valued function of \( \theta \) used in the proof of Theorem (9.3.2). See equation (9.3.9).

\( e \) and \( \lambda \). Except where confusion might arise, \( e \) is used to denote the base of the Naperian logarithms. However where \( e \) is also used as a real variable the symbol \( \lambda \) is substituted as the Naperian base.

**Note:** Originally it was intended to use \( \lambda \) as the Naperian base throughout to avoid any confusion. However a problem arose and this would now necessitate extensive retyping.

\( e \) denotes a real variable used in the proof of Lemma (6.5(i)).

\( \hat{e} \) denotes a real variable in the proof of Lemma (6.5).

\( \hat{e}^{(n)} \) denotes a real variable at the moment just after the \( n^{th} \) arrival at the queue. See proof of Lemma (6.5).

\( e^{(n)} \) denotes a real variable at the moment just before the \( n^{th} \) arrival at the queue. See proof of Lemma (6.5).

\( F(w) \) denotes Probability \( \{W < w\} \) where \( W \) is the waiting time of an arbitrary customer.

\( f(z_1, z_2) \) denotes a complex valued analytic function of the complex variables \( z_1 \) and \( z_2 \). See Definition (5.1.5).

\( f(z_1, \ldots, z_n) \) denotes the complex valued analytic function of \( n \) complex variables \( z_1, z_2, \ldots, z_n \).

\( F \) and \( f \) denote real variables used in \( R(f) = \text{Prob} \{F < f\} \) in the proof of Lemma (6.5).

**Note:** \( f \) and \( F \) are also used in the proof of Result 2 for this Lemma.
\( f_m(\omega) \) denotes for \( m = 0,1,\ldots,L \), the \( m^{th} \) of a set of polynomials in \( \omega \) which arise when \( C(v-\mu,\mu+\omega-v) \) is written down succinctly.

\( f_L(\omega) \) denotes a polynomial in \( \omega \), related to \( f_L(\omega) \) by the equation \( (\lambda-\omega)^K f_L(\omega) = f_L(\omega) \) which can be shown to hold for any integral \( K \) and \( L \). See Sections 12 and 13.

\( G(v_1,v_2) \) denotes the joint probability distribution of the variables \( v_1 \) and \( v_2 \).

\( g_j(\omega) \) denotes for \( j = 1,2,\ldots,m \) the \( j^{th} \) of \( m \) entire functions of \( \omega \).

\( \xi \) denotes a real variable used in the proof of Lemma (6.5).

See equation (6.5.1.19).

\( g \) denotes a real variable used to define \( \text{Prob}\{G<g\} \) in proof of Lemma (6.5).

\( G \) denotes a real variable used in the statement and proof of Result 2 employed to prove Lemma (6.5).

\( G_j(\omega) \) denotes: \( D_z \sum_{j=0}^{L-1} (\chi^*_+(z-\mu,\mu+\omega-z)) \)

\( \chi \)

\( \chi \) denotes a column vector of the \( G_j(\omega) \) for \( j = 0,1,\ldots,L-1 \).

See equation (11.7.4).

\( H(j) \) denotes for \( j = 0,1,2,\ldots \) the set of limits

\[ \lim_{\text{Re}(\omega) \to \infty} \{ D_z \sum_{j=0}^{L-1} (\chi^*_+(z-\mu,\mu+\omega-z)) \} \]

\( (n)H(j) \) denotes the analogous limits taken with respect to \( \chi^*_+(z-\mu,\mu+\omega-z) \) where the \( n \) refers to the number of the arrival.
$h_i(\omega)$ denotes for $i = 1, 2$ two rational functions of $\omega$ used to define the solution of the $E_k/M/2$ queue.

$H_i(\omega)$ denotes for $i = 1, 2$ two polynomials in $\omega$ used to define the functions $G_1(\omega)$ and $G_2(\omega)$ used in the solution of the $E_k/E_2/2$ queue.

$h$ denotes the mean service time $\bar{s}$ for the $M/M/2$ queue in order to be consistent with A.K. Erlang's notation.

$\text{Im}(\cdot)$ denotes Imaginary Part of $\cdot$.

$I(\theta, \theta_0)$ denotes a Laplace-Stieltjes (L.S.) integral

$$E(\mathcal{L}^{-}(\theta-\theta_0)y^{\theta_0}y)$$

with respect to a distribution $F(y)$ where $\theta_0$ is an arbitrarily chosen centre point s.t. $\text{Re}(\theta_0) > 0$. See proof of Theorem (5.3.3).

$I_-(\phi, \theta)$ denotes a L.S. integral with respect to the set function $\chi_-(x,y)$ used in the proof of Lemma (6.12).

$(\mu) I_+(-\mu, \mu x^2)$ denotes a L.S. integral with respect to the distribution $\chi_+(x,y)$ defined by equation (6.16.3).

$\mathcal{J}(\theta)$ denotes a L.S. integral used in the proof of Theorem (5.3.2).

$\mathcal{J}(\phi, \theta)$ denotes a L.S. integral over a bivariate Cartesian space with respect to a measure $\mathcal{M}(dxXdy)$.

$\mathcal{J}(z_1, \ldots, z_n)$ denotes the $n$ dimensional generalization of $(\phi, \theta)$.

$I_+(\phi, \theta)$ denotes the L.S. integral of the set function $\chi_-(x,y)$ over the real space s.t. $x \in (-\infty, 0]$, $y \in (0, \infty)$. 
\( i \) denotes a general integer variable used as an index.

Also used to denote \( \sqrt{-1} \) and the unit of imaginaries

where this is necessary and no confusion will arise.

Also used as a subscript.

\( j \) denotes a general index.

\( j_0(\omega), j_1(\omega), j_2(\omega) \) denote three rational functions of \( \omega \).

See Section 13 and equation (13.8.6).

\( J_+(\theta) \) denotes a Laplace Integral. First used in the proof

of Lemma (6.3).

\( J_-(\phi, \theta) \) denotes a Laplace Integral taken over the region

s.t. \( x \in (-\infty, 0) \) and \( y \in (-\infty, 0] \). See proof of Lemma (6.6).

\( (n)J_-(\phi, \theta) \) denotes a similar Laplace Integral where the set

function is \( (n)\chi_-(x,y) \) rather than \( \chi_-(x,y) \).

\( J_-(\theta) \) denotes a Laplace Integral over the region such that

\( x \in (-\infty, 0) \) \( y \in (-\infty, \infty) \) if \( \Re(\phi) \rightarrow -\infty \).

\( J \) denotes the first of \( L \) roots of unity. Thus \( J^1 \) denotes

the \( l^{th} \) such root.

\( k_{10}, k_{11}, k_{12} \) denote the three coefficients of the polynomial

\( H_1(\omega) \). See equation (13.8.2).

\( K \) denotes the number of exponential stages in an Erlang \( K \),

distribution.

\( l \) is an index which identifies the \( l^{th} \) root out of \( L \) roots,

of unity.
L denotes the number of exponential stages in an Erlang L,
   distribution.

$L_1^{(n)}$ and $L_2^{(n)}$ denote the maximum and second maximum of the set
   of departure times for all customers entering the queueing
   system before the $n^{th}$.

$\mathcal{M}(dx,dy)$ denotes a measure defined for all subsets of a
   bivariate Cartesian product set.

$[\mathcal{M}(\omega)]$ denotes an L by L square matrix of functions of
   which are the coefficients of the $M_1(\omega)$ for $l = 0,1,...,L-1$.
   See Section 11.

$M_1(\omega)$ denotes the $l^{th}$ out of L, functions used in Section 11.

$M(\phi,\theta)$ denotes a bivariate function defined by equations
   (7.19) and (7.20).

$\mathcal{M}(\theta)$ denotes a function used as an upper bound in the proof
   of Theorem (9.3).

$m$ denotes a general index variable as in $d_m(\theta)$. Also used
   to denote a measure.

$m^*(\omega)$ denotes a Laplace Transformation of the measure $m$ with
   respect to $\omega$.

$(n)^m_1^*(\omega)$ denotes the $m^*(\omega)$ which may be defined immediately
   before the $n^{th}$ arrival at the queue.

$m(\phi)$ denotes a part of the limit defined by equation (9.7.38).

$n$ denotes an index variable generally associated with stages or
   conditions existing immediately before the $n^{th}$ arrival at
   the queue.
n! denotes factorial n. n is now used as an index unrelated to the number of arrivals at the queue.

P(*) generally denotes the probability of the event or condition within the parentheses.

\( \hat{P}_m \) and \( \bar{P}_m \) denote probabilities of there having been m arrivals since the queue was last empty for two different terminal conditions.

P(y/x) denotes Prob\{Y<y given x\}

\( \wp \) denotes an elementary disc centred on the point \( \omega=\lambda \) which punctures the complex plane in \( \omega \).

P(0), P(1) and P(2) respectively denote the probabilities of emptiness (no servers busy), one server busy and two servers busy for the two server queue.

p denotes a general complex variable used in Section 4.

Also used as a probability.

p(j/x) denotes the conditional probability of j stages for \( j = 1, 2, ..., L \) when given x.

Q(e) denotes Prob \{E\leq e\} for \( e \in [0, \infty) \)

\( Q^*(\theta) \) denotes the L.T.w.r. to \( \theta \) of Q(e).

\( \sim \) Q(e) denotes a probability distribution for \( e \in (-\infty, \infty) \).

\( \sim^* \) (\( \theta \)) denotes the L.T. of \( \sim \) Q(e).

\( Q_+^*(\theta) \) denotes the L.T. of \( \sim \) Q(e) for \( e \in (0, \infty) \).

\( Q_-^*(\theta) \) denotes the L.T. of \( \sim \) Q(e) for \( e \in (-\infty, 0] \).

Q(\( \hat{\theta} \)) denotes a distribution for \( \hat{\theta} \in [0, \infty) \)
$(n)Q(\theta/x)$ denotes the conditional distribution of $\theta$ given $x$

at the moment just before the $n^{th}$ arrival.

$(n)Q_x^*(\theta)$ and $(n)Q_x^*(\theta)$ denote the L.T's of $(n)Q(\theta/x)$ and

$(n)Q(\theta/x)$ with respect to $\theta$ respectively.

$q(j,\phi)$ denotes:

$$-\int_0^\infty p(j/x)e^{-\phi x}d\Pi(x)$$

$q$ denotes a real constant.

$\text{Re}(\cdot)$ denotes the real part of the variable in the parentheses.

$R_n$ denotes the radius of convergence of the $n^{th}$ element of

a continuation chain in univariate complex space. Also

used to denote $n$ dimensional real space.

$R_j^{(n)}$ denotes the $j^{th}$ radius of convergence for $j = 1, 2, \ldots, n$

for the $n^{th}$ element of a continuation chain in $n$ dimensional

complex space.

denotes a region.

$R$ denotes a real number.

$\mathcal{R}$ denotes the portion of boundary included in

$R(f)$ denotes the probability distribution $\text{Prob}\{F < f\}$.

$R^*(\theta)$ denotes the L.T. of $R(f)$.

$\mathcal{R}_-$ and $\mathcal{R}_+$ denote the respective first regions where the

L.T's $\chi_-^*(\phi, \theta)$ and $\chi_+^*(\phi, \theta)$ are continuous.

$R(\phi, \theta)$ a function of the complex variables $\phi$ and $\theta$ defined

by equation (7.20)
$R_1$ and $R_2$ denote $r_1 + r_2$ and $r_1 r_2$ respectively. Used in the solution of $E_k/E_2/2$.

$r_i$ denotes for $i = 1, 2, \ldots, L$ a set of complex numbers which locate singularities. Also used as $r_j$ to indicate the $j^{th}$ radius of a polydisc.

$s_n$ denotes the service time of the $n^{th}$ customer to arrive.

$s_u$ denotes a singular chain in univariate complex space.

$s_m$ denotes a singular chain in multivariate complex space.

$\mathcal{S}$ denotes a two dimensional set.

$(n, \mathcal{S})$ denotes the two dimensional support set for $(n) \chi_+(x, y)$

$s_0$, $s_1$, $s_2$, and $s_3$ are a set of four constants used in the definition of $\chi_+^*(\phi, \theta)$ for the $E_k/E_2/2$ queue.

$s$ denotes the mean service time.

$\bar{s}$ denotes a constant used in the definition of the $\chi_+^*(\phi, \theta)$ of the $E_k/M/2$ queue.

$t_n$ denotes the absolute time at which the $n^{th}$ customer starts service.

$T(g)$ denotes an arbitrary probability distribution $\text{Prob}\{G \leq g\}$ for all $g \in [0, \infty)$

Note: $t$ and $T$ are also used in specific references to Students "$T$" Test in Section 15.

$u_n$ denotes the sum $x_n + y_n$. See Section 2.

$v_1(n)$ and $v_2(n)$ are variables which describe the delay state of the two server queue at the moment just before the $n^{th}$ customer arrives.
$v_1(n), \ldots, v_c(n)$ are a set of variables analogous to the $v_1(n)$ and $v_2(n)$ of the two server queue which describe the delay state of the C server queue at the moment just before the $n^{th}$ arrival.

$v$ denotes a complex variable. Generally used as in equation (11.2).

$W$ denotes a waiting time.

$W_n$ denotes the waiting time of the $n^{th}$ customer.

$W(n)$ denotes the waiting (or since the server became free if $< 0$) time for the $n^{th}$ customer at the GI/G/1 queue.

$W(f,g)$ denotes a bivariate probability distribution.

$W^*(\theta)$ denotes the L.T. $E(\epsilon^{-\theta \max(0,F-G)})$ with respect to $W(f,g)$.

$x^n$ one of two real variables used to describe the state 
Second Markov Chain at the moment just before the $n^{th}$ arrival.

$x$ denotes a real continuous variable used in the description 
of the random walk shown in Figure 2.1.

$x_n$ and $\hat{x}_n$ are also real continuous variables used in this description.

$x(n)$ denotes a real variable used in the description of the 
Second Markov Process. The $(n)$ refers to the observation 
of $x$ at the moment just before the $n^{th}$ arrival.

$\hat{x}(n)$ denotes a real variable immediately after the $n^{th}$ arrival has entered the system. See proof of Lemma (6.5).
\( y(n) \) denotes a real variable at the moment just before the \( \text{th} \) \( n \) arrival.

\( y \) denotes a real variable of the walk shown in Figure 2.1.
Also used generally in the description of the probability
spaces for the two server queue.

\( y_n \) and \( \bar{y}_n \) are variables similar to \( y \) used in the description
of the walk shown in Figure 2.1.

\( z \) denotes a complex variable. Frequently used in the manner
established by equation (7.14).

1.2 Greek Symbols

\( \alpha^*(P) \) denotes the Laplace Stieltjes Transformation (L.S.T.)
of the distribution \( A(a) \) with respect to the complex
variable \( p \).

\( \breve{\alpha}(u) \) denotes a probability density function such that;
\[
A(a) = \int_0^a \breve{\alpha}(u) \, du
\]

\( \alpha \) is also used as a general complex variable as in \( D_{\alpha} \) where
\( D \) refers to differentiation.

\( \beta^*(p) \) denotes the L.S.T. of the distribution \( B(s) \) with respect
to the complex variable \( p \).

\( \beta^*(q,\phi) \) denotes the L.S.T. of the distribution \( B(s) \) when the
integration ceases when \( s = q \) and with respect to the complex
variable \( \phi \).

\( \beta^*(L,W,p) \) denotes the L.T. of the distribution \( B(s) \) if \( B(s) \)
is an \( E_L \) distribution and the integration ceases when
\( s = p \) and with respect to the complex variable \( \omega \).
ß is also used as a general complex variable as in D, where the D refers to differentiation.

\( (n)\gamma(x) \) denotes the distribution of \( x^{(n)} \) where \( x^{(n)} \in (-\infty, \infty) \) is the real variable of the Lindley (i.e. GI/G/1) walk at the moment just before the \( n^{\text{th}} \) arrival.

\( (n)\tilde{\gamma} \) denotes a concentration of probability at zero.

\( (n)\gamma^*(z) \) denotes the L.S.T. of \( (n)\gamma(x) \) with respect to the complex variable \( z \) and over all \( x \in (-\infty, \infty) \).

\( (n)\gamma^+_{\pm}(z) \) denotes the L.S.T. of \( (n)\gamma(x) \) integrated for all \( x \in \mathbb{R} \) and w.r.t. to \( z \). \( \gamma^+(z) \) denotes the L.S.T. of \( (n)\gamma(x) \) with respect to the complex variable \( z \) and over all \( x \in (-\infty, 0) \).

\( (n)\delta^+_{\pm}(z) \) denotes the sum;

\[ (n)\gamma^+_{\pm}(z) + (n)\gamma^+(0) \]

which is the L.S.T. of a proper distribution \( (n)\delta^+_{\pm}(x) \) for all \( x \in [0, \infty) \).

\( (n)\delta^+_{\pm}(z) \) denotes the difference;

\[ (n)\gamma^+_{\pm}(z) - (n)\gamma^+(0) \]

which is the L.S.T. of a bounded set function \( (n)\gamma_{\pm}(x) \) which has zero total variation for all \( x \in (-\infty, 0] \).

\( \delta^+_{\pm}(z) \) and \( \delta^+_{\pm}(z) \) denote related functions defined by equations (16.2.5) and (16.1.14) respectively.

\( \varepsilon \) and \( \mathfrak{E} \) denote inclusion and exclusion respectively. \( \varepsilon \) is also used for an arbitrarily small quantity of radius.
\( \eta \) denotes an angle, i.e. an amplitude in the complex plane.

\( \theta \) denotes a complex variable.

\((n) \pi(x, y)\) denotes the distribution of \(x\) and \(y\) immediately before the \(n^{th}\) arrival. \(\pi(x, y)\) is the ergodic limiting form, if it exists.

\((n) \pi^*(\phi, \theta)\) denotes the L.S.T. of \((n) \pi(x, y)\)

\((n) \pi_0\) denotes the probability that the queue is empty just before the \(n^{th}\) arrival.

\(\pi_0\) is the ergodic limiting value just before an arrival, if this exists.

\((n) \pi_1(y)\) denotes the probability of the event that at the moment before the \(n^{th}\) arrival at the two server queue, one server is free and the other has a residual service time of at most \(y\).

\(\pi_1(y)\) is the ergodic limiting form, if this exists.

\(\pi_1^*(\theta)\) denotes the Laplace Transformation of \(\pi_1(y)\) with respect to \(\theta\).

\(\tilde{\pi}_1(y)\) denotes any aggregation of probability inherent in \(\pi(x, y)\) which lies on the line \(y \in (0, \infty)\) when \(x = 0\).

\(\tilde{\pi}_1^*(\theta)\) denotes the L.T. of \(\tilde{\pi}_1(y)\).

\((n) \Pi(x)\) denotes the otherwise unconditional probability that \(x^{(n)} \leq x(n)\) at the moment just before the \(n^{th}\) arrival.

\(\rho\) denotes the mean traffic load per server.

\(\bar{\rho}\) denotes the mean of a set of mean traffic loads per server.
σ denotes a real constant. Also used to denote the standard deviation of a distribution as say σ₁₀,000 in Section 15.

\( \tau_n \) denotes the absolute time at which the \( n \)th customer leaves the queue.

\( \phi \) denotes a complex variable.

\( (n)\chi_+ (x,y) \) denotes a distribution at the moment just before the \( n \)th arrival and with support set such that \( 0 \leq x \leq y < \infty \).

\( \chi_+ (x,y) \) is the ergodic limiting form, if this exists.

\( (n)\chi_- (x,y) \) denotes a bivariate set function over the space \( \mathcal{E}(\infty, 0] \times [0, \infty) \).

\( \chi_- (x,y) \) is the ergodic limiting form, if this exists.

\( (n)\chi_+^\ast (\phi, \theta) \) denotes the L.S.T. of \( (n)\chi_+ (x,y) \)

\( \chi_+^\ast (\phi, \theta) \) denotes the L.S.T. of \( \chi_+^\ast (x,y) \)

\( (n)\chi_-^\ast (\phi, \theta) \) and \( \chi_-^\ast (\phi, \theta) \) denote the L.S.T.'s of \( (n)\chi_- (x,y) \) and \( \chi_- (x,y) \) respectively.

\( (n)\chi (x,y) \) denotes the sum;

\[ (n)\chi_+ (x,y) + (n)\chi_- (x,y) = (n)\pi (x,y) \]

The ergodic limiting form is \( \chi (x,y) \) and;

\( \chi^\ast (\phi, \theta) \) denotes its L.S.T. which also satisfies

\[ \chi^\ast (\phi, \theta) = \chi_+^\ast (\phi, \theta) + \chi_-^\ast (\phi, \theta) \]

\( (n)\chi_+ (V_1^+, V_2^+, \ldots, V_c^+) \) is the distribution for the C server queue analogous to \( (n)\chi_+ (x,y) \).

\( \chi_+ (V_1^+, V_2^+, \ldots, V_c^+) \) is the analogous ergodic limit, if this exists.
\( \chi^*(\phi, \theta) \) denotes a portion of \( \chi^*(\phi, \theta) \) obtained by integrating

the L.T. of \( \chi(x, y) \) only over the region such that \(-\infty < x < y < 0\).  

\( \chi^+(\phi, \theta) \) denotes the portion of \( \chi^*(\phi, \theta) \) obtained by integrating

the L.T. of \( \chi(x, y) \) only over the region such that \( x \in (-\infty, 0] \)

and \( y \in (0, \infty) \).  Thus,

\[
\chi^*(\phi, \theta) = \chi^-(\phi, \theta) + \chi^+(\phi, \theta)
\]

\((n)\chi^-(\phi, \theta)\) and \((n)\chi^+(\phi, \theta)\) are analogous L.S.T's of \((n)\chi(x, y)\)

integrated over the same support sets.

\((n)\chi^M(x)\) denotes the marginal distribution of \( x \) at the

moment before the \( n^{th} \) arrival.

\( \chi^M(x) \) denotes the ergodic limiting form of \((n)\chi^M(x)\)

if this exists.

\( \chi^+(\phi) \) denotes \( \chi^+(\phi, \theta) \) when \( \theta = 0 \) and is thus the L.S.T. of the

ergodic limiting waiting time distribution for the two

server queue.

\( \psi(x) \) denotes a distribution on the positive half line.

\( \psi^*(\theta) \) denotes the L.S.T. of \( \psi(x) \) with respect to \( \theta \).

\( \psi_{-\mu} \) denotes a set of exceptional points where \( \chi^+(\phi, \theta) \) is not

bivariate analytic if \( \phi = -\mu \).

\( \omega \) denotes a complex variable.

\( \Omega^*(\phi, \theta) \) denotes a sum of functions for the ergodic queue.

See equation (6.4.1)

\( \Omega^*(x, y) \) denotes the distribution for which \( \Omega^*(\phi, \theta) \) is the

Laplace Transformation.
\( \Omega_+^{(M)} (x) \) denotes the marginal distribution of \( x \) derived from \( \Omega_+^{(M)} (x,y) \).

\( \Omega_+^{(M)} (x) \) denotes the distribution analogous to \( \Omega_+^{(M)} (x) \) which holds at the moment before the \( n^{th} \) arrival.

\( \lambda \) denotes a real constant used to specify the stage rate for the \( E_K \) distribution of interarrival times for the \( E_K/E_L/2 \) queue.

\( \mu \) denotes a real constant which specifies the service rate for each stage of the \( E_L \) service time distribution of the \( E_K/E_L/2 \) queue.
2. Two Alternative Markov Processes for GI/G/2.

We here derive the recurrence which defines the basic embedded vector Markov chain for GI/G/2. This result is similar to but not identical with that quoted by Kiefer and Wolfowitz (7) and Pollaczek (14, 15). We shall refer to this as the First Markov Chain and from this derive a Second Markov Chain which defines what Pollaczek calls the sequence of "last ends" which is also used by Kiefer and Wolfowitz. Both chains define the same distribution function for the waiting time \( w_n \) experienced by the \( n^{th} \) arriving customer, but the second leads to a simpler analysis. We shall also use ordered variables, whereas Pollaczek does not.

2.1 Assumptions.

We have two servers operating independently and in simple parallel serving a single queue where the customers arrive singly, and are served in the order of their arrival. Successive interarrival intervals are assumed to be independent and identically distributed. The service times of the individual customers are also assumed to be independent of both the individual customer or the server and to be identically distributed. This is the system considered as GI/G/2 in the Kendall notation, see [5,6].

Let \( S_n \) be the service time of the \( n^{th} \) customer to arrive. As defined above \( S_n \) is independent of both \( n \) and which of the two servers handles the \( n^{th} \) customer.

Let \( t_n \) be the absolute time at which the \( n^{th} \) customer starts service.
Let \( T_n \) be the absolute time at which the \( n^{th} \) customer leaves the system. Thus:

\[
T_n = t_n + S_n.
\]

Let \( a_n \) be the time interval between the moment of arrival of the \((n-1)^{th}\) customer and that of the \( n^{th} \).

Remark: There appears to be no mathematical restriction which would prevent the customer interarrival time distribution from having a saltus at zero. Hence "multiple arrivals" should be permissible in this very restricted sense. We shall not deal with such cases in this thesis.

### 2.2 Development of the First Markov Process

This first Markov process uses the method of ancillary variables to describe strictly (i.e. exactly) the state of the system just before the moments of arrival of the customers. The second which may be simply derived from it gives exactly the same delay distribution but offers a simpler analysis since it avoids certain ambiguities. For this reason both references (7) and (14) use the second process as shall we. However, neither (7) nor (14) gives a derivation of either process. We therefore give the following derivation both for its intrinsic interest and to aid our understanding of the random walk which derives from the second process. This is rather important in view of our later comments on Pollaczek's work.

For two servers the earliest time at which the \( n^{th} \) customer can be served if he is already present and waiting for service is

\[
t_n = \text{Second max} \{T_0, T_1, T_2, \ldots, T_{n-1}\} \quad (2.1)
\]
since he can be served as soon as at least one of the two servers becomes free and this happens when the second last of the previous customers to depart leaves the system.

Therefore

\[ t_n = \max\left[ \sum_{i=1}^{n} a_i, \text{2nd max}\{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \right] \]  \hspace{1cm} (2.2)

if \( \sum_{i=1}^{n} a_i \) is the time at which the \( n^{th} \) customer arrives.

Remark: By this convention the zero \( th \) customer arrived and commenced service at \( t_0 = 0 \).

We now introduce the variables:

\[ L_1(n) = \max\{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \]  \hspace{1cm} (2.3)

\[ L_2(n) = \text{second max}\{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \]

and use (2.3) in (2.2) to obtain:

\[ t_n = \max\{ \sum_{i=1}^{n} a_i, L_2(n) \} \]  \hspace{1cm} (2.4)

Let us now define:

\[ V_2(n) = L_2(n) - \sum_{i=1}^{n} a_i \]  \hspace{1cm} (2.5)

and

\[ V_1(n) = L_1(n) - \sum_{i=1}^{n} a_i \]  \hspace{1cm} (2.6)
Then the waiting time of the $n^{th}$ customer is:

$$w_n = t_n - \sum_{i=1}^{n} a_i$$

$$= \max\{0, L_2(n) - \sum_{i=1}^{n} a_i\}$$

$$= \max\{0, v_2(n)\}$$

$$= \{v_2(n)\}^+.$$ (2.7)

Consider now the $(n+1)^{th}$ customer to arrive. By analogy with (2.3) and (2.4) we define:

$$v_2(n+1) = L_2(n+1) - \sum_{i=1}^{n+1} a_i$$ (2.8)

$$v_1(n+1) = L_1(n+1) - \sum_{i=1}^{n+1} a_i$$ (2.9)

Also:

$$L_1(n+1) = \max\{\tau_0, \tau_1, \ldots, \tau_n\}$$

$$= \max\{\tau_0, \tau_1, \ldots, \tau_{n-1}, t_n + S_n\}$$

$$= \max\{\tau_0, \tau_1, \ldots, \tau_{n-1}, \sum_{i=1}^{n} a_i + w_n + S_n\}$$

$$= \max\{\tau_0, \tau_1, \ldots, \tau_{n-1}, \{v_2(n)\}^+ + S_n + \sum_{i=1}^{n} a_i\}$$ (2.10)
Similarly:

\[ L_2(n+1) = \text{Second max}\{\tau_1, \tau_2, \ldots, \tau_{n-1}, \{v_2(n)\}^+ + S_n + \sum_{i=1}^{n} a_i\} \] (2.11)

However equations (2.10) and (2.11) contain the same sequence of terms as equation (2.3) except that a new term has been added namely

\[ \{\{v_2(n)\}^+ + S_n + \sum_{i=1}^{n} a_i\} \]

Therefore:

\[ v_2(n+1) = L_2(n+1) - \sum_{i=1}^{n+1} a_i \]

\[ = \text{second max}\{L_1(n), L_2(n), \{v_2(n)\}^+ + S_n + \sum_{i=1}^{n} a_i\} - \sum_{i=1}^{n+1} a_i \] (2.12)

since terms smaller than \( L_2(n) \) cannot affect the outcome.

Therefore:

\[ v_2(n+1) = \text{second max}\{v_1(n), v_2(n), \{v_2(n)\}^+ + S_n\} - a_{n+1} \]

by use of (2.5)

\[ = \text{second max}\{v_1(n), \{v_2(n)\}^+ + S_n\} - a_{n+1} \] (2.13)

because \( \{v_2(n)\}^+ \geq v_2(n) \).

Similarly we may show that:

\[ v_1(n+1) = \max\{v_1(n), \{v_2(n)\}^+ + S_n\} - a_{n+1} \] (2.14)

whence it is clear that if ancillary variables \( v_1(n) \) and \( v_2(n) \)
are known for any \( n \), expressions (2.13) and (2.14) define a Markov chain for \( V_1^{(n+1)} \) and \( V_2^{(n+1)} \). Thus if this time homogeneous chain has a limiting joint distribution \( G(v_1, v_2) \) for the variable \( V_1^{(n)} \) and \( V_2^{(n)} \) as \( n \to \infty \), we may define the limiting distribution for the waiting time by:

\[
F(w) = P(W \leq w)
\]

\[
= F(v_2^+)
\]

by \( w = \max\{0, v_2\} = \{v_2\}^+ \)

\[
= G(v_1 = \infty, v_2)
\]

if \( v_2 > 0 \)

\[
= G(v_1 = \infty, 0)
\]

if \( v_2 \leq 0 \)

(2.15)

Note: Necessary and sufficient conditions for the existence of the limiting joint distribution of \( V_1 \) and \( V_2 \) were first given by Kiefer and Wolfowitz in [7].

2.3 The Second Markov Chain.

Consider the following recurrence analogous to (2.14) and (2.13):
\[ y(n+1) = \max\{\{y(n)\}^+, \{x(n)\}^+ + S_n\} - a_{n+1} \quad (2.16) \]

\[ x(n+1) = \min\{\{y(n)\}^+, \{x(n)\}^+ + S_n\} - a_{n+1} \quad (2.17) \]

to define the variables \( x(n) \) and \( y(n) \) for all \( n = 0, 1, 2, \ldots \).

Clearly these equations define a Markov chain which is very similar to that described by equations (2.13) and (2.14). To see that both define the same distribution function \( w_n \), the waiting time of the \( n \)th customer to arrive, we show that:

\[ \{x(n)\}^+ = \{v_2(n)\}^+ \]

for all \( n \).

Write

\[ \{y(n+1)\}^+ = \max\{\{y(n)\}^+, \{x(n)\}^+ + S_n\} - a_{n+1}^+ \]

\[ \{x(n+1)\}^+ = \min\{\{y(n)\}^+, \{x(n)\}^+ + S_n\} - a_{n+1}^+ \]

from (2.16) and (2.17).

Write

\[ \{v_1(n+1)\}^+ = \max\{v_1(n), \{v_2(n)\}^+ + S_n\} - a_{n+1}^+ \]

\[ \{v_2(n+1)\}^+ = \min\{v_1(n), \{v_2(n)\}^+ + S_n\} - a_{n+1}^+ \]

from (2.13) and (2.14).

Note that:

\[ v_1(n) \leq \{v_1(n)\}^+. \]
We postulate:

\[ x(n)_+ = v_2(n)_+ \]

\[ y(n)_+ = v_1(n)_+ \]

whence

\[ v_1(n) < y(n)_+ \text{ if } v_1(n) < 0 \text{ but } v_1(n) = y(n)_+ \text{ if } v_1(n) > 0, \]

and show that:

\[ x(n+1)_+ = v_2(n+1)_+ \]

\[ y(n+1)_+ = v_1(n+1)_+ \]

for all \( n = 0, 1, 2, \ldots \).

Proof: If \( v_1(n) < 0 \)

\[ v_1(n+1)_+ = \{v_2(n)_+ + s_n - a_{n+1}\}_+ = y(n+1)_+ \]

\[ v_2(n+1)_+ = 0 = x(n+1)_+ \]

by inspection.

If \( v_1(n) > 0 \) the expressions also give identical results and the induction is completed.

Therefore

\[ w_n = \max\{0, v_2(n)\} = \max\{0, x(n)\} \quad (2.18) \]

for all \( n \) and both the chains give the same delay distribution as asserted.
However the second chain defines distributions which gather rather more of the probability associated with zero waiting delay into an aggregation (i.e., a saltus) at the point \( X^+ = Y^+ = 0 \) and this is more convenient because it enlarges the region of convergence of one of the Laplace integrals used during the solution of these problems.

Finally we note that since both chains consider the states of the system at the moments just before the arrival of each customer \( w_n \) or in the limit as \( n \to \infty \), if it exists, \( w \) is the delay actually experienced by the customers. It is not therefore say, the Virtual Waiting Time of Tákacs.

Remarks: 1. Although most of the analysis which follows will be concerned with bivariate Laplace Transformation space, it will be from time to time useful to consider the bivariate probability space of the equivalent continuous time random walk and we note that the recurrences defined by (2.16) and (2.17) can be interpreted in the following way.

Suppose we have a particle undergoing a random walk in the half quadrant \( 0 \leq x \leq y < \infty \) and subject to the following rules:

(i) At time \( t_n + 0 \) the particle is projected for the \( n^{th} \) time from some initial position \( (x_n, y_n) \) horizontally and with constant velocity in such a manner that if it does not encounter the "mirror" barrier which is the line \( y = x \) it reaches the point \( y_n, x_n \) where \( y_n = y_n \) and \( x_n = x_n + S_n \).
(ii) If it encounters the mirror at \( y = x \) i.e. if \( S_n + x_n > y_n \) it reflects upwards a distance \( d_n = S_n + x_n - y_n \) and reaches \( (x_n', y_n') \) where \( x_n' = y_n \) and \( y_n' = S_n + x_n \).

Thus this reflection is equivalent to the exchanging of the variables \( x \) and \( y \) by the max, min operations of the recurrence.

(iii) The particle leaves the point \( (x_n, y_n) \) and drifts to the point \( x_{n+1}, y_{n+1} \) where

\[
\begin{align*}
x_{n+1} &= x_n - a_{n+1} \\
y_{n+1} &= y_n - a_{n+1}
\end{align*}
\]

provided \( x_{n+1} > 0 \).

(iv) Should the particle encounter the "sticky" barrier which is the line \( x = 0 \) it becomes captive and slides down this line at constant velocity to the point

\[
\begin{align*}
x_{n+1} &= 0 \\
y_{n+1} &= y_n - a_{n+1}
\end{align*}
\]

unless:

(v) The particle becomes trapped at the point \( x = y = 0 \) where it rests motionless until time \( t_{n+1} = 0 \) when the process repeats.

Thus diagrammatically the random walk is
where the numbers correspond with the stages. Naturally the queueing recurrence is arrived at by observing this random walk just once per cycle i.e., at the instants just before each successive horizontal projection.

(vi) It will also be observed that if the probability of the particle ever reaching the "sticky" barrier (or the origin) is very very small this random walk will very nearly follow the rule:

$$u_{n+1} = u_n + S_n - 2a_n$$

where,

$$u_n = x_n + y_n$$

and this simple observation leads to several interesting
consequences which are touched on in later sections.

2. Following Lindley [11] we note that equations (2.16) and
(2.17) mean that the joint distribution of \( y^{(n+1)} \) and \( x^{(n+1)} \)
is identical with that of the \( \max \{ \} \) and \( \min \{ \} \) of the
right hand sides. See also Section 4 where this is explored
in more detail.
3. The Analogous Markov Processes for GI/G/C.

Consider that we now have C servers and define

\[
V_1(n) > V_2(n) > \cdots > V_C(n)
\]

where the \(V_i(n)\), \(i = 1, 2, \ldots, C\) are real variables which will obey a recurrence to be defined below. Now note that

\[
\tau_n = \max \{ \sum_{i=1}^{n} a_i, \text{th max}\{\tau_0, \tau_1, \ldots, \tau_{n-1}\} \}
\]

where there are C servers, by analogy with (2.2).

Therefore we may extend the previous derivation to obtain the following analogous results:

\[
V_1(n+1) = \max\{V_1(n), V_2(n), \ldots, V_C(n)\} + S_n - a_{n+1}
\]

\[
= \max\{V_1(n), \{V_C(n)\}^+ + S_n\} - a_{n+1}
\]

\[
v_2(n+1) = \text{second max}\{V_1(n), V_2(n), \ldots, V_C(n)\} + S_n - a_{n+1}
\]

\[
= \text{median}\{V_1(n), V_2(n), \ldots, V_C(n)\} + S_n - a_{n+1}
\]

\[
v_x(n+1) = \text{median}\{V_{x-1}(n), V_x(n), \ldots, V_C(n)\} + S_n - a_{n+1}
\]

\[
v_C(n+1) = \min\{V_C(n), \{V_C(n)\}^+ + S_n\} - a_{n+1}
\]

This is once again a Markov process which may be altered by the replacement of some or all of the variables from \(V_1(n)\) to \(V_{C-1}(n)\) by \(\{V_1(n)\}^+\) etc., to \(\{V_C(n)\}^+\) without altering the distribution of \(w_n\) where,

\[
w_n = \max\{0, V_C(n)\}
\]
although the joint distribution of the $V_1^{(n)}$ to $V_{C-1}^{(n)}$ is quite extensively altered.

Note: The above processes use strictly ordered variables with

$$\infty > V_1^{(n)} > V_2^{(n)} > V_3^{(n)} \ldots > V_C^{(n)} > -\infty \quad (3.7)$$

whereas Pollaczek's treatment [15] does not use any such fixed ordering of the variables. The fixed ordering appears to lead to some simplifications in the analysis, and it is the ordering (or sorting) process which in effect generates the more important singularities.
4. **Laplace Stieltjes Transforms.**

To proceed further from equations (2.16) and (2.17) we define the Bivariate Distribution Function of the variables \( x^{(n)}, y^{(n)} \) to be;

\[
(n)\pi(x, y) = P(x^{(n)} \leq x, y^{(n)} \leq y)
\]  

(4.1)

for all \( n = 1, 2, \ldots \) and where

\(-\infty < x, y < \infty\)

We now define;

\[
\int_0^\infty e^{-pa} dA(a)
\]

where \( A(a) = P(A \leq a) \)

\[
= \int_0^\infty e^{-pa} \alpha(a) da
\]

if the density function \( \alpha(a) \) exists

\[= \alpha^*(p)\]  

(4.2)

Also define;

\[
\int_0^\infty e^{-ps} dB(s)
\]

where \( B(s) = P(S \leq s) \)

\[
= \int_0^\infty e^{-ps} \beta(s) ds
\]

if the density function \( \beta(s) \) exists

\[= \beta^*(p)\]  

(4.3)
We now reconsider equations (2.16) and (2.17) namely;

\[ y(n+1) = \max\{\{y(n)\}^+, \{x(n)\}^+ + s_n\} - a_{n+1} \quad (2.16) \]

\[ x(n+1) = \min\{\{y(n)\}^+, \{x(n)\}^+ + s_n\} - a_{n+1} \quad (2.17) \]

and use the previous definitions to write;

\[ E(e^{-\theta y(n+1)} - \phi x(n+1)) \]

\[ = (n+1)_{\pi^*}(\phi, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\phi x(n+1) - \theta y(n+1)} d(n+1) \pi(x,y) \]

\[ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-\theta \max\{\{y(n)\}^+, \{x(n)\}^+ + s\} - \phi \min\{\{y(n)\}^+, \{x(n)\}^+ + s\}} \]

\[ \underbrace{\int_{0}^{\infty} e^{(\theta + \phi) a} dB(s) dA(a) d(n) \pi(x,y)}_{(4.4)} \]

directly from (2.16) and (2.17) after noting that \( B(s) \) and \( A(a) \) are independent of \( n \) where;

\[ (n)_{\pi^*}(\phi, \theta) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-\phi x(n) - \theta y(n)} d(n) \pi(x,y) \quad (4.5) \]

is the Laplace Stieltjes Transform of the Bivariate Distribution Function \((n)_{\pi}(x,y)\) which may exist only for wholly imaginary \( \phi \) and \( \theta \). The extension to domains is considered later.

Equation (4.4) may be rewritten by subdividing the ranges of integration in the following manner;
\[(n+1)_{\pi}(\phi, \theta) = \alpha(-\theta + \phi)\]

\[
\left\{\begin{array}{l}
y(n) - x(n) \\
y(n) - x(n)e^{-\theta y(n)} - \phi s - \phi x(n) dB(s) d(n) \pi(x, y) \\
y(n) - x(n)e^{-\theta y(n)} - \phi s - \phi x(n) dB(s) d(n) \pi(x, y) \\
y(n) - x(n)e^{-\theta y(n)} - \phi s dB(s) d(n) \pi(x, y) \\
y(n) - x(n)e^{-\theta s} \phi y(n) dB(s) d(n) \pi(x, y) \\
y(n) - x(n)e^{-\theta s} dB(s) d(n) \pi(x, y)
\end{array}\right.
\]

Equation (4.6)

The simplification of (4.6) which we shall use may be more readily understood if we reconsider the analytic method often used for the one server queue. In our notation, Lindley's equation, (see [11]) reads;

$$w(n+1) = \max\{0,\{W(n)^+ + S_n - a_{n+1}\}$$

but if we define say $x(n)$ such that $-\infty < x(n) < \infty$, $x(n)$ real for all $n = 0,1,2,...$ we obtain;

$$x(n+1) = \{x(n)^+ + S_n - a_{n+1}\}$$

whence given a complex variable $z$ and noting that the distributions of $S_n$ and $a_{n+1}$ are for all $n = 0,1,2,...$ just the distributions $B(s)$ and $A(a)$ and thus independent of $n$, we obtain;

$$E(e^{-zx(n+1)}) = (n+1)\gamma^*(z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-z[\{x(n)^+ + s-a\}]} dA(a) dB(s) d(n)\gamma(x)$$

where $(n)\gamma(x)$ is the Distribution of $x(n)$ and $(n)\gamma^*(z)$ is its bilateral Laplace Stieltjes Transformation with respect to $z$ which may of course only exist for $z$ wholly imaginary.

Thus;

$$(n+1)\gamma^*(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-z(s-a)} dA(a) dB(s) d(n)\gamma(x)$$

$$+ \int_{0-}^{\infty} \int_{0-}^{\infty} e^{-z(x(n)^+ + s-a)} dA(a) dB(s) d(n)\gamma(x)$$

$$= \beta*(z)\alpha*(-z)\int_{-\infty}^{0} d(n)\gamma(x) + \int_{0-}^{\infty} e^{-zx(n)} d(n)\gamma(x)$$

(4.10)
Now consider the ordinary bilateral Laplace-Stieltjes Transformation of a variable \( v(n) \) where \(-\infty < v(n) < \infty\).

We may write:

\[
(n)\gamma^*(z) = \int_{-\infty}^{\infty} e^{-zv} d(n)\gamma(v)
\]  

(4.11)

and if we define

\[
(n)\gamma^*_-(z) = \int_{-\infty}^{0} e^{-zv} d(n)\gamma(v)
\]  

(4.12)

and,

\[
(n)\gamma^*_+(z) = \int_{0}^{\infty} e^{-zv} d(n)\gamma(v)
\]  

(4.13)

we note that

\[
\int_{-\infty}^{0} d(n)\gamma(v) = (n)\gamma^*_-(0)
\]  

(4.14)

Therefore we may use (4.14) in (4.10) to obtain;

\[
(n+1)\gamma^*(z) = \beta^*(z)\alpha^*(-z)\{ (n)\gamma^*_-(0) + (n)\gamma^*_+(z) \}
\]  

(4.15)

whence if we define new functions for all \( n \), by;

\[
(n)\delta^*_+(z) = (n)\gamma^*_+(z) + (n)\gamma^*_-(0)
\]  

(4.16)

and

\[
(n)\delta^*_-(z) = (n)\gamma^*_+(z) - (n)\gamma^*_-(0)
\]  

we obtain by use of these in (4.15);

\[
(n+1)\gamma^*(z) = \beta^*(z)\alpha^*(-z)(n)\delta^*_+(z)
\]  

(4.17)

Thus if the queue is ergodic and it can be shown that,
\[ \lim_{n \to \infty} \{n+1\} \delta^*_n(z) = \delta^*_n(z) = \lim_{n \to \infty} \{n\} \delta^*_n(z) > 0 \]

one obtains;

\[ (1-\beta^*(z)\alpha^*(-z))\delta^*_n(z) = -\delta^*_n(z) \quad (4.18) \]

Equations (4.17) and (4.18) are the standard "Wiener-Hopf" starting points for the extensions of the domains of analyticity by factorization and continuation arguments and thence the determination of the distributions \((n)\delta(w)\) and \(\delta(v)\) in the limiting case, but those simple processes are of no great interest for the moment. The important concept is the essential uniqueness of the constant \((n)\gamma^*(0)\) which results from the operation of "sweeping up" all the probability from the negative half line and placing it in a saltus at \(w = 0\) since this makes \((n)\delta^*_n(z)\) the Laplace Stieltjes Transformation of a proper distribution which will satisfy (4.17) and which thus defines a \((n)\gamma^*(z)\) which satisfies (4.15). The reader might also refer to the comments of Kingman [9] on this point.

We now apply these ideas to equation (4.6) as follows. Define;

\[ \beta^*(q,\phi) = \int_0^q e^{-\phi s} dB(s) \quad (4.19) \]

\[ \beta^*(\phi) = \int_0^\infty e^{-\phi s} dB(s) \]

and use these in \((4.6)\) to obtain;
\[(n+1)\pi^*(\phi, \theta) = \alpha^*(-\theta + \phi) + \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\theta y(n) - \phi x(n)} \beta*(y(n) - x(n), \phi) d(n) \pi(x, y) \]

\[
\quad + \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\phi y(n) - \theta x(n)} \{\beta*(\theta) - \beta*(y(n) - x(n), \theta)\} d(n) \pi(x, y) \\
\quad + \beta*(\theta) \int_0^{\infty} \int_{-\infty}^{0} d(n) \pi(x, y) \\
\quad + \int_0^{\infty} \int_{-\infty}^{0} e^{-\theta y(n)} \beta*(y(n), \phi) d(n) \pi(x, y) \\
\quad + \int_{-\infty}^{0} \int_0^{\infty} e^{-\phi y(n)} \{\beta*(\theta) - \beta*(y(n), \theta)\} d(n) \pi(x, y) \]

Equation (4.20)
We now note that only those two integrals for which \( x(n) > 0 \) contain such terms as \( e^{-\phi x(n)} \) and we therefore define:

\[
(n)\pi_0 = \int_{x(n) = -\infty}^{0-} \int_{y(n) = x(n)}^{0} d(n)\pi(x,y)
\]

(4.21)

= Total probability that both servers are simultaneously free at the moment just before the \( n \)th arrival and

\[
(n)\pi_1(y) = \int_{x(n) = -\infty}^{0-} d(n)\pi(x,y)
\]

(4.22)

= Total probability that one server is free and the other has a residual service time \( \leq y \) at the moment just before the \( n \)th arrival.

We now note that

\[
(n)\pi_0 + \int_{y(n) = 0}^{\infty} d(n)\pi_1(y) = P(W_n = \{x(n)\}^+ = 0)
\]

= Probability that the \( n \)th arriving customer does not wait.

Remark: The use of the subscripts 0 and 1 in \( \pi_0 \) and \( \pi_1(y) \) reminds us that these probabilities are associated with the states no servers busy and one server busy respectively just before an arrival.

We now use (4.21) and (4.22) in (4.20) to obtain;
\[(n+1)\pi^*(\phi, \theta) = \alpha*(-\theta+\phi) \left\{ \int_{x(n) = 0^+}^{\infty} \int_{y(n) = x(n)}^{\infty} e^{-\theta y(n)} e^{\phi x(n)} \beta^*(y(n) - x(n), \theta) \right. \]
\[\times \left[ e^{-\phi y(n)} e^{\theta x(n)} \beta^*(y(n) - x(n), \theta) \right] d(n)\pi(x, y) \]
\[+ \beta^*(\theta) \int_{x(n) = 0^+}^{\infty} \int_{y(n) = x(n)}^{\infty} e^{\phi y(n)} e^{\theta x(n)} d(n)\pi(x, y) \]
\[+ (n)\pi_0 \beta^*(\theta) \]
\[+ \int_{y(n) = 0^+}^{\infty} [e^{\theta y(n)} \beta^*(y(n), \phi) - e^{\phi y(n)} \beta^*(y(n), \theta)] d(n)\pi_1(y) \]
\[+ \beta^*(\theta) \int_{y(n) = 0^+}^{\infty} e^{\phi y(n)} d(n)\pi_1(y) \]
\[\text{Equation (4.23)}\]
We now define the distribution;

\[(n)\chi_+(x,y) = (n)\pi_0 + (n)\pi_1(y) + \int_{0^+} \int_{0^+} d(n)\pi(x',y') \]

if \(x(n) \geq 0, y(n) \geq x(n)\)

= 0

if \(x(n) < 0, -\infty < y(n) < \infty\)

and the set function;

\[(n)\chi_-(x,y) = \int_{-\infty} \int_{-\infty} d(n)\pi(x',y') - (n)\pi_1(y) - \pi_0^{(n)} \]

if \(x(n) \leq 0, -\infty < y(n) < \infty\)

= 0

if \(x(n) > 0, -\infty < y(n) < \infty\)

and with these form;

\[(n)\chi(x,y) = (n)\chi_+(x,y) + (n)\chi_-(x,y) = (n)\pi(x,y) \]

which recovers the original \((n)\pi(x,y)\) after the addition. We note that both \((n)\chi_+(x,y)\) and \((n)\chi(x,y)\) (= \((n)\pi(x,y)\)) are proper distribution functions on \(0 \leq x \leq y < \infty\) and \(-\infty < x \leq y < \infty\) respectively, whilst \((n)\chi_-(x,y)\) is merely a bounded set function and that these properties have been obtained by the use of the necessarily unique compensation functions \((n)\pi_0\) and \((n)\pi_1(y)\) for each \(n\).

The parallel between this procedure and that used in the classical one server Wiener-Hopf method will be obvious. This
is also the essential unambiguous projective operation for a two server queue. Analogous operations are clearly possible for queues with several servers.

From (4.2.4) define;

\[
\int \int_{0-0-} e^{-\phi x(n) - \theta y(n)} d(n) \chi_+(x,y) = \chi^*_+(\phi, \theta) \tag{4.27}
\]

\[
\int \int_{0+0+} e^{-\phi x(n)_y(n)} d(n) \chi(x,y) = \pi_0 + \pi_1^*(\theta) + \int \int_{0+0+} e^{-\phi x(n) - \theta y(n)} d(n) \chi(x,y) \tag{4.28}
\]

\[
\int \int_{0+0+} e^{-\phi x(n) - \theta y(n)} d(n) \pi(x,y) = \pi_0 + \pi_1^*(\theta) + \int \int_{0+0+} e^{-\phi x(n) - \theta y(n)} d(n) \pi(x,y) \tag{4.29}
\]

where

\[
\pi_1^*(\theta) = \int_{0+} e^{-\theta y(n)} d(n) \pi_1(y) \tag{4.30}
\]

and from (4.25) define;

\[
\int \int_{-\infty-\infty} e^{-\phi x(n) - \theta y(n)} d(n) \chi_-(x,y) = \chi^*_-(\phi, \theta) \tag{4.31}
\]

\[
\int \int_{-\infty-\infty} e^{-\phi x(n) - \theta y(n)} d(n) \chi_-(x,y) = \pi_0 - \pi_1^*(\theta) + \int \int_{-\infty-\infty} e^{-\phi x(n) - \theta y(n)} d(n) \chi_-(x,y) \tag{4.32}
\]

\[
\int \int_{-\infty-\infty} e^{-\phi x(n) - \theta y(n)} d(n) \pi(x,y) = \pi_0 - \pi_1^*(\theta) + \int \int_{-\infty-\infty} e^{-\phi x(n) - \theta y(n)} d(n) \pi(x,y) \tag{4.33}
\]

Gather the following terms in (4.23);

\[
\beta^*(\theta) \int_{x(n)=0+}^{\infty} y(n) \int_{x(n)=x(n)}^{\infty} e^{-\phi y(n) - \theta x(n)} d(n) \pi(x,y) + \beta^*(\theta) \int_{y(n)=0+}^{\infty} e^{-\phi y(n)} d(n) \pi_1(y) + \beta^*(\theta) \pi_0
\]
and observe with the aid of equations (4.29) and (4.30) (after exchanging the roles of \( \phi \) and \( \theta \)) that the sum of these three terms may be written as

\[
\beta^*(\theta)\chi^*_+(\theta, \phi) \tag{4.34}
\]

Similarly gather the remaining terms in (4.23) namely:

\[
x(n)\int_{0+}^{\infty} y(n)\int_{0+}^{\infty} \left[ e^{-\theta y(n)} - \phi x(n) \beta^*(y(n) - x(n), \phi) \right] d(n) \pi(x, y) \\
+ \int_{0+}^{\infty} y(n)\left[ e^{-\theta y(n)} - \phi y(n) - \theta x(n) \beta^*(y(n) - x(n), \theta) \right] d(n) \pi_1(y)
\]

and use (4.29) and (4.30) to obtain this sum in the form;

\[
x(n)\int_{0-}^{\infty} y(n)\int_{x(n)}^{\infty} \left[ e^{-\theta y(n)} - \phi x(n) \beta^*(y(n) - x(n), \phi) \right] d(n) \chi^*_+(x, y) \tag{4.35}
\]

Add (4.34) and (4.35) to obtain:

\[
(n+1) \pi^*(\phi, \theta) = (n+1) \chi^*(\phi, \theta)
\]

by the equivalence (4.26)

\[
= \alpha^*(-\theta+\phi) \left[ \beta^*(\theta)(n) \chi^*_+(\theta, \phi) \\
+ \int_0^{\infty} \int_{x(n)}^{\infty} \left[ e^{-\theta y(n)} - \phi y(n) - \theta x(n) \beta^*(y(n) - x(n), \theta) \right] d(n) \chi^*_+(x, y) \right]
\]

equivalent to (4.23) \tag{4.36}
Excursus:

In [7] Kiefer and Wolfowitz derived the necessary and sufficient conditions for the ergodicity of the multi-server queue. They showed that if \((n)\chi_+(v_1^+, v_2^+, \ldots, v_c^+)\) were the joint distribution function of the variables \(v_1^+, v_2^+, \ldots, v_c^+\) which may be defined from equations (3.2) to (3.7), just before the moment of the \(n^{th}\) arrival, then it would have the property.

\[
\lim_{n \to \infty} \{(n)\chi_+(v_1^+, v_2^+, \ldots, v_c^+)\} = \chi_+(v_1^+, v_2^+, \ldots, v_c^+) > 0
\]

\[
0 \leq v_c^+ \leq v_{c-1}^+ \leq \ldots \leq v_1^+ < \infty
\]

iff

\[
\lim_{n \to \infty} \{(n)\chi_+(\infty, \infty, \ldots, \infty, v_c^+)\} = \chi_+(\infty, \infty, \ldots, \infty, v_c^+) > 0
\]

\[
0 \leq v_c^+ < \infty
\]

were true. Conversely they showed that if the queue were not ergodic then

\[
\lim_{n \to \infty} \{(n)\chi_+(\infty, \infty, \ldots, \infty, v_c^+)\} = 0
\]

for any \(v_c^+ < \infty\) and that under these conditions;

\[
\lim_{n \to \infty} \{(n)\chi_+(v_1^+, v_2^+, \ldots, v_{c-1}^+, v_c^+)\} = 0
\]

also for all \(v_c^+ \leq v_{c-1}^+ \leq \ldots \leq v_1^+ < \infty\).
Their condition is now well known and may be simply stated.

If $\bar{S} = \text{Mean Service Time} < \infty$

and

$$\rho < 1$$  \hspace{1cm} (4.41)

where

$$\rho = \text{Mean Load per Server}$$  \hspace{1cm} (4.42)

$$= \frac{\bar{S}}{c.\bar{a}}$$

where

$$c = \text{Number of Servers}$$

and

$$\bar{a} = \text{Mean Inter-arrival Time}$$

the queue may be ergodic, whereas if $\rho \geq 1$ it is not.

Remark: The ergodicity of the two server queue was investigated by analytic methods and the foregoing condition found to be necessary for the existence of a non-zero limiting transform $X_f(\phi, \theta)$. It was also found that if;

$$\lim_{n \to \infty} (n)X_f(\phi, \theta) = 0$$

for all finite $\phi$ and $\theta$, this corresponded with the result that;

$$\lim_{n \to \infty} x^{(n)} + y^{(n)} = \infty$$

whilst

$$y^{(n)} - x^{(n)} < \infty$$

for all $n$. 
However, the possibility of periodicities such that no limiting behaviour might exist proved difficult although consideration of the equation,

\[ \chi^*(\delta, \delta) = \alpha^*(-2\delta)\beta^*(\delta)\chi^*_+ (\delta, \delta) \]

which may be obtained as a limiting form of (4.36) as \( n \to \infty \) and with \( \phi = \theta = \delta \) suggests that W.L. Smith's observation that if either \( A(a) \) or \( B(s) \) were a continuous distribution and \( \rho < 1 \), the GI/G/1 queue possessed an ergodic limiting distribution, may generalize easily in this case also. See [16].

Consequently this work is not included in the thesis although the corollaries to many of the Lemmas given in Section 6 give results which would be required for such an investigation.

For the purpose of this thesis we hereafter assume that the following limits exist for appropriate \(-\infty < x \leq y < \infty\)

\[
\lim_{n \to \infty} \{ (n) \chi_+ (x,y) \} = \chi_+ (x,y) \tag{4.43}
\]

\[
\lim_{n \to \infty} \{ (n) \chi_-(x,y) \} = \chi_- (x,y) \tag{4.44}
\]

\[
\lim_{n \to \infty} \{ (n) \pi_1 (y) \} = \pi_1 (y) \tag{4.45}
\]

and

\[
\lim_{n \to \infty} \{ (n) \pi_0 \} = \pi_0 \tag{4.46}
\]

and that in particular;
\[ \chi_+(x,y) \geq \pi_0 > 0 \]  \hspace{1cm} (4.47)

for all \( 0 \leq x \leq y < \infty \)

We also assume that the corresponding Laplace Transformations \( \chi^+_x(\phi, \theta) \), \( \chi^*_x(\phi, \theta) \) and \( \pi^*_1(\theta) \) also exist for some appropriate domains \( D_+(\phi, \theta) \), \( D_-(\phi, \theta) \) and \( D_1(\theta) \) in \( \phi \) and \( \theta \), whatever these may be.

We may write a limiting form for equation (4.26) as follows;

\[ \chi^*(\phi, \theta) = \alpha^*(-\phi+\phi) \left\{ \beta^*(\theta) \chi^+_x(\theta, \phi) + \int_0^{\infty} \int_{0^-}^{\infty} \left[ e^{-\theta y-\phi x} \beta^*(y-x, \phi) \right] \right. \\
\left. \left[ -e^{-\phi y-\theta x} \beta^*(y-x, \theta) \right] d\chi_+(x,y) \right\} \]  \hspace{1cm} (4.48)

which appears to be about the simplest general equation which can be written down for the queue \( G1/G/2 \).
5. Preliminary Definitions and Theorems.

Before proceeding we require a number of preliminary lemmas. The following definitions and theorems will assist in the statement and proof of these.

5.1 Definitions.

5.1.1 A Domain is an open arcwise and simply connected set in \( \mathbb{C}^n \) (an \( n \) dimensional complex space with \( n \geq 1 \), for most of our applications).

5.1.2 A Region is another set, not necessarily a domain, plus some portion of its edge set.

5.1.3 A Closed Set is an entirely closed region.

Remark: Generally the regions and closed sets which we will use will be both simply and arcwise connected. Thus the open interior will generally be a domain and any exception to this principle will be specifically mentioned.

5.1.4 Re (.) denotes Real Part of (.)

Im (.) denotes Imaginary Part of (.)

5.1.5 Bivariate Analytic Function.

A function \( f(z_1,z_2) \) is said to be analytic (strictly bivariate analytic) in both complex variables if it may be written in the form of a power series;

\[
f(z_1,z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1n_2} \left(z_1 - z_1^0\right)^{n_1} \left(z_2 - z_2^0\right)^{n_2}
\]

when centred on the point \( (z_1^0,z_2^0) \) and if this power series is convergent (or absolutely convergent) in some neighbourhood of that point.
Remarks: 1. Clearly $f(z_1, z_2)$ will be analytic in a domain $D$ if it can be written as a power series centred on every point $(z_1^0, z_2^0) \in D$ and convergent in a neighbourhood of each such point.

2. If $f(z_1, z_2)$ is analytic at a point $(z_1^0, z_2^0)$ all its derivatives and partial derivatives will exist at that point. This is obvious from the series definition.

3. An analytic function of $n$ complex variables may be written in the form:

$$f(z) = f(z_1, \ldots, z_j, \ldots, z_n) = \sum_{m_1 \ldots m_j \ldots m_n} a_{m_1 \ldots m_n} (z-z_1^0)^{m_1} \ldots (z_n-z_n^0)^{m_n}.$$

centred on the point $(z_1^0, \ldots, z_n^0)$ and this series will converge in some neighbourhood of the point.

5.1.6. Polydisc.

A convenient neighbourhood of a point such as $(z_1^0, \ldots, z_n^0)$ is the polydisc;

$$D(z^0, r_j) : |z_j - z_j^0| < r_j \text{ for all } j = 1, 2, \ldots, n$$

which generalizes the familiar concept of the disc.

Note: Bochner and Martin [1] refer to these as polycylinders but later authors such as Hormander [3] use this more reasonable name.
5.1.7. **Singular Chain (Univariate Case).**

Let $f(z)$ be a function of a complex variable $z$ defined by the equivalence class of its elements $\{f_\delta\}$ (where these elements are convergent power series representations centred on the sequence of points $\{\delta_n\}$) and write;

$$S_n : \{f(z, \delta_n), \ |z-\delta_n| < R_n \ \text{for all} \ n = 1, 2, \ldots \}$$

Then $S_n$ is said to be a singular chain of $f(z)$ if

1. $S_n$ is a chain
2. $\lim \{\delta_n\} = \delta_0$ exists

and

3. $\lim \{R_n\} = 0$

whence $\delta_0$ is the end point of $S$

**Remarks:** 1. Two singular chains are equivalent iff the corresponding classes $[f, D_\varepsilon]$ and $[g, D_\varepsilon]$ are identical for sufficiently small values of $\varepsilon$ where $D_\varepsilon$ is the disc $|z-\delta_0| < \varepsilon$ and $S_\varepsilon$ is the subset of $S$ which has centres which lie in $D_\varepsilon$.

This implies that $g$ and $f$ have the same endpoint $\delta_0$ and $g \sim f$ modulo $D_\varepsilon$.

2. A finite point $z = \delta_\varepsilon$ is a singular point of an analytic function $f(z)$ iff it is the endpoint of a singular chain made up of elements of $f(z)$. Two singular chains which have the same endpoint determine the same singular point iff the chains are equivalent.
3. These definitions from Hille [2], Vol. II, Chapter 10.

5.1.8. **Singular Chain (Multivariate Case)**

Let \( f(z_1, z_2, \ldots, z_m) \) be a function of the complex variables \( (z_1, z_2, \ldots, z_m) \) defined by the equivalence class of its elements \( \{f_{\alpha}\} \) where each element is a convergent power series in \( (z_1, z_2, \ldots, z_m) \) centred on a point \( (\alpha_1, \alpha_2, \ldots, \alpha_m) \) and we write

\[
\mathcal{S}_M = \{ f(z_1, z_j, z_m, \alpha_1(n), \alpha_j(n), \alpha_m(n)) ; |(z_j - \alpha_j(n))| < R_j(n) \}
\]

for all \( n = 1, 2, \ldots \)

and all \( j = 1, 2, \ldots, m \)

Then \( \mathcal{S}_M \) is a singular chain of the multivariate function \( f(z_1, \ldots, z_m) \) if

1. \( \mathcal{S}_M \) is a chain.

2. \( \lim_{n \to \infty} \{ \alpha_1(n), \ldots, \alpha_j(n), \ldots, \alpha_m(n) \} = \alpha_1(0), \ldots, \alpha_j(0), \ldots, \alpha_m(0) \)

exists and

3. \( \lim_{n \to \infty} R_j(n) = 0 \) for at least one of the \( j = 1, 2, \ldots, m \)

whence \( (\alpha_1(0), \ldots, \alpha_m(0)) \) is the endpoint of \( \mathcal{S} \).

**Remarks:**

1. Clearly (5.1.8) is an obvious extension of (5.1.7).

2. If we substitute \( z_j = g_j(\omega) \) where

(i) \( \omega \) is a complex variable

and

(ii) \( g_j(\omega) \) is an entire function of \( \omega \), for each

\( j = 1, 2, \ldots, m \), then the resulting function of \( \omega \) namely
\[ F(\omega) = f(q_1(\omega), q_2(\omega), \ldots, g_j(\omega), \ldots, q_m(\omega)) \]

will be analytic in \( \omega \) in the domain \( D \) where

\[ D = D_1(\omega) \cap D_2(\omega) \cap \ldots \cap D_j(\omega) \cap \ldots \cap D_m(\omega) \]

where \( D_j(\omega) \) is the inverse mapping of the disc

\[ |g_j(\omega) - g_j(\omega_0)| < R_j = R_j(g_j(\omega_0)) \]

into the \( \omega \) plane. Clearly therefore;

\( D \) is empty if any \( R_j = 0 \).

5.2 Standard Theorems.

The following are standard theorems from the theory of one or more complex variables which are given without proofs. References are given however.

5.2.1. Osgood's Lemma

If a function \( f(z) = f(z_1, \ldots, z_j, \ldots, z_n) \) with all \( z_j \) complex is continuous in a domain \( D \) and if at every point in \( D \) it is analytic in each variable, then \( f(z) \) is analytic (i.e. multivariate analytic) in \( D \).

Reference: Theorem II, Chapter 2 Bochner and Martin [1].

5.2.2. Osgood's Theorem.

If a function \( f(z) \) in a domain \( D \) of \( \mathbb{E}^{2n} \) is analytic in each complex variable \( z_j, \ j = 1, 2, \ldots, n \) (when all the others are complex constants) and if it is bounded in all variables, then it is analytic in all variables.
Note: $E_{2n}$ is the Euclidean $2n$ dimensional space which corresponds in this situation, with $C^n$.

Reference: Bochner and Martin [1], Chapter 7, Lemma 3.

5.2.3. Hartogs' Theorem.

If $f(z)$ is defined in a domain $D$ in $E_{2n}$ and $f$ has the property that for every point $(\hat{a}_1, \ldots, \hat{a}_n)$ of $D$ each of the functions

$$f(\hat{a}_1, \ldots, \hat{a}_{j-1}, z_j, \hat{a}_{j+1}, \ldots, \hat{a}_n) \text{ for all } j = 1, 2, \ldots, n$$

is analytic in the single variable $z_j$ in the neighbourhood of the point $a_j$; then $f$ is analytic in $(z_1, \ldots, z_n)$ throughout $D$.

Remark: This rather remarkable result largely subsumes the results of Osgood. It is to be found as Theorem 4, Chapter 7. Bochner and Martin [1].

5.3 Preliminary Theorems.

Theorem 5.3.1. The bivariate Laplace-kernel $e^{-\phi x - \theta y}$ is a bivariate entire function for all finite $x$ and $y$.

Proof: The result follows directly from the well known properties of the complex exponential function.

Theorem 5.3.2. The Lebesque Integral

$$I(\theta) = \int_{0-}^{\infty} e^{-\theta y} dF(y) \quad (5.3.2.1)$$

where $F(y)$ is a distribution function defined on the half line $[0, \infty)$ is continuous in $\theta$ at or toward all points in the region $\mathcal{R}$.
where
\[ R : \{ \text{All } \theta \text{ s.t. } \Re(\theta) \geq 0, |\theta| < \infty \} \]  

(5.3.2.2.)

if it be understood that if \( \theta_0 \) is an edge point of \( R \), the continuity shall only be defined to \( \theta_0 \) through points in \( R \) and not through points outside \( R \).

Remark: A region of this type is often referred to as a half strip or multivariate space. However a region such as \(-a \leq \Re(\alpha) \leq b \) \(-\infty < \Im(\theta) < \infty \) is referred to as a closed tube since its base is closed.

Proof: If \( \Re(\theta) > 0 \), the modulus of the kernel is bounded that is
\[ |e^{-\theta y}| \leq 1 \]  

(5.3.2.3)

for all \( y \in [0, \infty) \).

Hence the kernel is bounded in modulus by an integrable function for all \( \theta \in R \) whilst it is also continuous in \( \theta \). Hence by bounded convergence, \( I(\theta) \) is continuous in \( \theta \) for all \( \theta \in R \) as asserted.

Theorem 5.3.3. The Lebesgue integral;
\[ I(\theta) = \int_{0^-}^{\infty} e^{-\theta y} dF(y) \]  

(5.3.3:I)

When \( F(y) \) is a distribution function defined in \([0, \infty)\) is analytic at all points \( \theta_0 \) such that \( \theta_0 \in D \) where \( D \) is the domain.*

\[ D: \{ \text{all } \theta \text{ s.t. } \Re(\theta) > 0, |\theta| < \infty \} \]  

(5.3.3.2)

* An open tube or half plane.
Proof: By the proof of Theorem 5.3.2;

\[ \int_{0-}^{\infty} |e^{-\theta y}|dF(y) \leq 1 \] (5.3.3.3)

if \( \text{Re}(\theta) \geq 0 \) and \(-\infty < \text{Im}(\theta) < \infty\).

Hence for some \( R \) sufficiently large and \( \text{Re}(\theta) \geq 0 \).

\[ \int_{R}^{\infty} |e^{-\theta y}|dF(y) < \varepsilon \] (5.3.3.4)

where \( \varepsilon > 0 \) is arbitrarily small.

Hence we may write;

\[ I((\theta, \theta_0)) = \int_{0-}^{\infty} e^{-(\theta-\theta_0)y}e^{-\theta_0 y}dF(y) \] (5.3.3.5)

which will exist for any \( \theta_0 \) s.t. \( \text{Re}(\theta_0) \geq 0 \)

and any \( \theta \) s.t. \( |\theta - \theta_0| \leq \text{Re}(\theta_0) \) and thence;

\[ |I(\theta, \theta_0)| \leq \int_{0-}^{\infty} |e^{-(\theta-\theta_0)y}||e^{-\theta_0 y}|dF(y) \] (5.3.3.6)

\[ = \lim_{R \to \infty} \int_{R}^{\infty} e^{-(\theta-\theta_0)y}||e^{-\theta_0 y}|dF(y)| \] (5.3.3.7)

\[ \leq \lim_{R \to \infty} \int_{R}^{\infty} \sum_{n=0}^{\infty} \frac{(|\theta_0 - \theta|^n)}{n!}|y^n||e^{-\theta_0 y}|dF(y) \] (5.3.3.8)

\[ = \lim_{R \to \infty} \sum_{n=0}^{\infty} \frac{(|\theta_0 - \theta)|n}{n!} \left( \lim_{R \to \infty} \int_{0-}^{\infty} y^n|e^{-\theta_0 y}|dF(y) \right) \] (5.3.3.9)

\[ \leq \sum_{n=0}^{\infty} \frac{(|\theta_0 - \theta)|n}{n!} \lim_{R \to \infty} \int_{0-}^{\infty} \frac{n!}{(\text{Re}(\theta_0))^n} dF(y) \] (5.3.3.10)

\[ = \sum_{n=0}^{\infty} \frac{(|\theta_0 - \theta)|n}{(\text{Re}(\theta_0))^n} \] (5.3.3.11)
since
\[ y^n |e^{-\theta_0 y}| \leq y^n e^{-\text{Re}(\theta_0) y} \leq \frac{n!}{(\text{Re}(\theta_0))^n} \]

(5.3.3.12) (5.3.3.13)

and if \( \text{Re}(\theta_0) > 0 \)

\[ = n! \lim_{R \to \infty} \int_0^R \text{Re}(\theta_0)^{-n} dF(y) \]

\[ = n! \int_0^\infty \text{Re}(\theta_0)^{-n} dF(y) \]

which is finite.

Hence,
\[ |I(\theta, \theta_0)| \leq \sum_{n=0}^{\infty} \left| \frac{\theta - \theta_0}{\text{Re}(\theta_0)} \right|^n \]

which is finite if \( |\theta - \theta_0| < \text{Re}(\theta_0) \) by comparison with a geometric series. Hence the power series \( I(\theta, \theta_0) \) has a non-zero radius of convergence if \( \text{Re}(\theta_0) > 0 \). Thus if \( \text{Re}(\theta_0) > 0 \) the power series \( I(\theta, \theta_0) \) defines an analytic function.

However \( \theta_0 \) was an arbitrary point in the half plane s.t. \( \text{Re}(\theta) > 0 \). Hence we may choose any suitable sequence of centre points \( \{\theta_0^i\} \) such that \( \text{Re}(\theta_0^i) > 0 \) for all \( i = 1, 2, \ldots \), and such that the discs of convergence overlap and also cover the finite half plane. Clearly this is an equivalence class of power series which defines a unique function analytic in the infinite half plane s.t. \( \text{Re}(\theta) > 0 \). However this is in the domain \( D \) defined by equation (5.3.3.2)
and this proves the assertion.

Remarks: 1. A converse result is obviously true for integrals defined on \( y \in (-\infty, 0] \) where these are convergent if \( \text{Re}(\theta) \leq 0 \) whence the regions \( R \) and domains \( D \) are

\[
R = \{ \theta \text{ s.t. } \text{Re}(\theta) \leq 0 ; -\infty < \text{Im}(\theta) < \infty \} \\
D = \{ \theta \text{ s.t. } \text{Re}(\theta) < 0 ; -\infty < \text{Im}(\theta) < \infty \}
\]  

(5.3.3.14)

2. We have simply shown that there are no singular chains which end at points such that \( \text{Re}(\theta) > 0 \). More is true however if \( F(y) \) is a monotone function of \( y \).

Thus if the integral (5.3.2.1) diverges when \( \theta = \theta_0 \) and converges for all finite \( \theta \) s.t. \( \text{Re}(\theta) > \text{Re}(\theta_0) \) then \( \theta_0 \) is real and is a singular point the Laplace transformation \( F^*(\theta) \). See for example Widder [18] Chapter 2 §5.

3. An alternative proof of Theorem 5.3.3 is given in Widder [18] Chapter 2, wherein all derivatives of \( F^*(\theta) \) w.r. to \( \theta \) are shown to exist if \( \text{Re}(\theta) > 0 \).

Theorem 5.3.4. Given a two dimensional set \( S \) (generally this will be a bivariate Cartesian product set) a measure \( M \) defined for all subsets of \( S \) and a region \( R \) in \( C_2 \) such that

\[
|\phi| < \infty, \ |\theta| < \infty \\
\]

and

\[
|e^{-\phi x - \theta y}| \leq 1 
\]

(5.3.4.1)

if the point \((\phi, \theta) \in R\) for all points \((x, y) \in S\)
and if
\[ \int_S M(dx \times dy) < \infty \quad (5.3.4.2) \]
then
\[ I(\phi, \theta) = \int_S e^{-\phi x - \theta y} M(dx \times dy) \quad (5.3.4.3) \]
is:

(i) Continuous in \( \theta \) when \( \phi \) is a constant at any point \((\phi, \theta) \in \mathbb{R} \) provided that the continuity is defined only between points in \( R'(\phi_0) \) where \( R'(\phi_0) \) is the image of \( R \) in the complex plane of \( \theta \) when \( \phi = \phi_0 \) a complex constant and

(ii) Continuous in \( \phi \) when \( \theta \) is constant at every point \((\phi, \theta) \in \mathbb{R} \) etc.

Proof: Consider assertion (i). Clearly if \( \phi \) is constant and (5.3.4.1) and (5.3.4.2) are true we see that the kernel \( e^{-\phi x - \theta y} \) is bounded in modulus by an integrable function whilst it is also continuous in \( \theta \). Hence this assertion is really a generalized restatement of Theorem (5.3.2) the proof of which may be repeated in principle here. Hence the assertion is true.

Assertion (ii) is obviously then true by exchange of variables.

Theorem 5.3.5. Given a two dimensional set \( S \) a measure \( M \) defined for all subsets of \( S \), a region \( R \) in \( \mathbb{C}_2 \) and a domain \( D \subseteq R \) such that;
\[ |\phi| < \infty \quad |\theta| < \infty \]
\[ |e^{-\phi x - \theta y}| \leq 1 \]

if the point \((\phi, \theta) \in R\) (5.3.5.1)

for all points \((x, y) \in S\) and if

\[ \int_S M \, (dx \times dy) < \infty \] (5.3.5.2)

then

\[ I(\phi, \theta) = \int_S e^{-\phi x - \theta y} M(dx \times dy) \] (5.3.5.3)

is

(i) Analytic is \(\phi\) in a neighbourhood of any \(\phi = \phi_0\)

when \(\theta = \theta_0\) a complex constant if the point \((\phi_0, \theta_0) \in D\) and

(ii) Analytic in \(\theta\) in a neighbourhood of any \(\theta = \theta_0\)

when \(\phi = \phi_0\) etc...

when

\[ D = R - \overline{R} \] (5.3.5.4)

where \(\overline{R}\) is the boundary (if any) included in \(R\).

Remark: If \(R\) is not such that \(R - \overline{R}\) is a domain we may decompose \(R\) into such disjoint regions (or regions and domains) as have this property and apply the theorem to each in turn.

Proof: This theorem is a two dimensional generalization of theorem (5.3.3) wherein \(S\) may also incorporate both positive and negative \(x\) and \(y\). However the requirement of statement
(5.3.5.1) is such that we may prove both the above assertions by obvious repetitions of the arguments used to prove theorem (5.3.3).

Hence the assertion is true.

Remark: It seems clear that if we write:

$$I(z_1, \ldots, z_n) = \int_S e^{-z_1 x_1 - \ldots - z_n x_n} M(dx_1 \times dx_2 \times \ldots \times dx_n)$$

for an n dimensional set $S$ and $n$ complex variables $z_1 \ldots z_n$ then if:

$$|z_j| < \infty \quad j = 1, \ldots, n$$

$$|e^{-z_1 x_1 - \ldots - z_n x_n}| \leq 1$$

for all $(z_1 \ldots z_n) \in R$

for all $(x_1 \ldots x_n) \in S$

and if;

$$\int_S M(dx_1 \times dx_2 \times \ldots \times dx_n) < \infty$$

and we define

$$D = R - \overline{R}$$

as a domain in $C_n$, the previous Theorems will easily generalize to $n$ dimensions.

Theorem 5.3.6. $I(\phi, \theta)$ defined by equation (5.3.4.3) is analytic (i.e. bivariate analytic) within a neighborhood of all points $(\phi, \theta) \in D$ where
\[ D = R - R \]

and

\[ R \] and \[ R \] have been previously defined.

**Proof:** The result follows immediately from Hartog's Theorem and Theorem 5.3.5.

**Remarks 1.** Theorems 5.3.5 and 5.3.6 assert that if the bivariate defining integral (5.3.5.3) exists for all points \((\phi, \theta) \in R\) then \(I(\phi, \theta)\) is bivariate analytic in \(D = R - R\)

where \(R\) is the boundary of \(R\)

However a little more is true as we may show by a re-examination of the proof of theorem 5.3.5.

Suppose \(I(\phi, \theta)\) is analytic in \(D\) and continuous in \(R\)

and suppose furthermore \(\exists \phi_0\) such that all points \((\phi_0, \theta) \in R\) for all \(\theta \in R_{\phi_0}\) where \(R_{\phi_0}\) is the projection of \(R\) into the plane of \(\theta\) when \(\phi = \phi_0\). Then clearly \(I(\phi_0, \theta)\) cannot be bivariate analytic for any \(\theta \in R_{\phi_0}\) (although it is bivariate continuous) but we can show that \(\exists \alpha;\)

\[ D_{\phi_0} = R_{\phi_0} - R_{\phi_0} \]

such that \(I(\phi_0, \theta)\) is univariate analytic in \(\theta\) if \(\theta \in D_{\phi_0}\)
Proof: Proceeding from equation (5.3.5.3) which read;

$$I(\phi, \theta) = \int e^{-\phi x - \theta y} M(dx \times dy)$$

where this integral exists for all \((\phi, \theta) \in R\), we choose a \(\phi_0\) such that all \((\phi_0, \theta) \in \bar{R}\) if \((\phi_0, \theta) \in R\) [Examples are given to illustrate how this may happen below] and note that;

$$I(\phi_0, \theta) = \int e^{-\phi_0 x - \theta y} M(dx \times dy)$$

will continue to exist for all \(\theta\) such that \((\phi, \theta) \in R\).

Designate this region \(R_{\phi_0}\) and define;

$$D_{\phi_0} = R_{\phi_0} - \bar{R}_{\phi_0}$$

where \(\bar{R}_{\phi_0}\) is the boundary of \(R_{\phi_0}\). Now note that if

\(\theta \in D_{\phi_0}\), \(I(\phi_0, \theta)\) and all its derivatives w.r. to \(\theta\) will exist by Theorem 5.3.3. Hence \(I(\phi_0, \theta)\) is an analytic function in \(\theta\) alone for \(\theta \in D_{\phi_0}\) and continuous in \(\phi\) in \(R_{\phi_0}\).

Hence the assertion is true.

2. An equivalent result is obviously true for all points \((\phi, \theta_0) \in \bar{R}\) where \(\phi \in R_{\theta_0}\) ensures continuity and \(\phi \in D_{\theta_0} = R_{\theta_0} - \bar{R}_{\theta_0}\) analyticity.

3. Equivalent results appear to be true for many dimensions.

Examples: (1) Suppose \(S\) is the space \(x \in [0, \infty), y \in [0, \infty)\) whence
if

\[ |e^{-\phi x - \theta y}| < 1 \]

\[ \text{Re}(\phi) > 0, \quad \text{Re}(\theta) > 0, \]

\[ -\infty < \text{Im}(\phi), \text{Im}(\theta) < \infty \]

and

(i) \( I(\phi, \theta) \) is analytic if

\[ \text{Re}(\theta) > 0, \quad \text{Re}(\phi) > 0; \quad -\infty < \text{Im}(\phi), \text{Im}(\theta) < \infty \]

and

(ii) \( I(\phi, \theta) \) is continuous if

\[ \text{Re}(\theta) > 0, \quad \text{Re}(\phi) > 0; \quad -\infty < \text{Im}(\phi), \text{Im}(\theta) > \infty. \]

However if \( \text{Re}(\phi_0) = 0 \) and \( -\infty < \text{Im}(\phi_0) < \infty \), \( I(\phi_0, \theta) \) exists if \( \text{Re}(\theta) > 0 \).

Hence by use of theorem 5.3.3 \( I(\phi_0, \theta) \) is analytic in \( \theta \) if \( \text{Re}(\theta) > 0 \) and continuous in \( \theta \) if \( \text{Re}(\theta) > 0 \).

2. See proof of Corollary 2 to Lemma 6.13 where this property is used to prove a very useful result. See also Corollary 2 to Lemma 6.2 and Corollary 2 to Lemma 6.10.

4. Theorems 5.3.2 and 5.3.3 are also to be found in Widder [18].


This section contains a number of preliminary lemmata which establish the essential analyticity and continuity properties for the Laplace Transformations which enter into the problems in the later Sections.
Lemma 6.1. The Laplace Transformation $\chi_+(\phi, \theta)$ is continuous in $\theta$ when $\phi$ is constant and in $\phi$ when $\theta$ is constant if $(\phi, \theta) \in \mathcal{R}$ where

$$\mathcal{R} = \left\{ \text{All } (\phi, \theta) \text{ s.t. } \begin{array}{ll}
\Re(\theta) \geq 0 & \text{if } \Re(\phi) > 0, \\
\Re(\theta + \phi) \geq 0 & \text{if } \Re(\phi) \leq 0 \\
\text{and } |\phi| < \infty, |\theta| < \infty
\end{array} \right\} \tag{6.1.1}$$

Proof:

$$\chi_+(\phi, \theta) = \int_0^\infty \int_0^\infty e^{-\phi x - \theta y} d\chi_+(x,y) \tag{6.1.2}$$

from whence noting that

$$d\chi_+(x,y) > 0$$

if $0 \leq x \leq y < \infty$ and,

$$d\chi_+(x,y) = 0$$

elsewhere, from the definitions of $\pi(x,y)$ and $\chi_+(x,y)$ given in equations (4.1), (4.24) and (4.43), we define the set $S$ to be all points $(x,y)$ such that

$$(x,y) \in S \tag{6.1.3}$$

if $0 \leq x \leq y < \infty$.

Hence if $(\phi, \theta) \in \mathcal{R}$ and $(x,y) \in S$ we find

$$|e^{-\phi x - \theta y}| \leq |e^{-\Re(\phi) x - \Re(\theta) y}| \leq |e^{-\Re(\theta + \phi) x - \Re(\theta) (y-x)}| \leq 1 \tag{6.1.4}$$

and we may apply Theorem (5.3.4) whence the result follows immediately.
Corollary 1. The Lemma is true for \( (n)\chi_{\phi}(\phi, \theta) \) for all \( n = 0, 1, 2, \ldots \).

Proof: Each of the distributions \( (n)\chi_{\phi}(x, y) \) for \( n = 0, 1, 2, \ldots \) has a support set \( (n)S \) such that

\[ (n)S \subseteq S \] (6.1.5)

Hence the preceding definition of \( R \) is sufficient to permit the use of Theorem (5.3.4) in the above way, for all \( n \). Hence the assertion is true.

Remark: Equation (6.1.5) uses \( (n)S \subseteq S \) since a rather conventional assumption would be

\[ (0)S = (x_0, y_0) \] (6.1.6)

whence \( (0)\chi_{\phi}(\phi, \theta) \) would be entire.

Lemma 6.2. The Laplace Transformation \( \chi_{\phi}(\phi, \theta) \) is bivariate analytic in \( \phi \) and \( \theta \) at all points \( (\phi_0, \theta_0) \in D \) where;

\[
D: \begin{cases} 
\text{All } (\phi, \theta) \text{ s.t. } \text{Re}(\theta) > 0 \text{ if } \text{Re}(\phi) > 0 \\
\quad \text{Re}(\theta + \phi) > 0 \text{ if } \text{Re}(\phi) \leq 0 \\
\quad \text{and } |\phi| < \infty, |\theta| < \infty 
\end{cases} \] (6.2.1)

Proof: Define \( D' = R - \overline{R} \) (6.2.2)

where \( R \) is defined by equation (6.1.1) and \( \overline{R} \) is the boundary included in \( R \). Note that;

\[ D' = D \]

where \( D \) is defined by equation (6.2.1). Apply Theorems (5.3.5) and (5.3.6) directly and the result follows immediately.
Corollary 1. \((n)\chi_+^*(\phi,0)\) is analytic in at least the same domain for all \(n = 0,1,2,\ldots\).

Proof: This follows directly from the proof of Corollary 1 to Lemma 6.1 since;

\[
(n)_D = (n)_D \sum_{\mathcal{S} \subseteq \mathcal{S}}
\]
and Theorems (5.3.5) and (5.3.6) may be used directly.

Corollary 2. If \(\theta = 0\) and \(\phi\) is finite and s.t. \(\text{Re}(\phi) > 0, \chi_+^*(\phi,0)\) is analytic in \(\phi\) alone.

Proof: Consider the integral

\[
\int_0^\infty \int_0^\infty e^{-\phi x} d\chi_+(x,y)
\]

and note that if \(\text{Re}(\phi) \geq 0\)

\[
\int_0^\infty \int_0^\infty |e^{-\phi x}| d\chi_+(x,y) \leq 1.
\]

Hence by Fubinis Theorem we may consider;

\[
\int_0^\infty e^{-\phi x} d_x \int_0^\infty d_y \chi_+(x,y)
= \int_0^\infty e^{-\phi x} d\psi(x) = \psi*(\phi)
\]

where \(\psi(x)\) is a distribution function on the positive half line. Hence by Theorems (5.3.2) and (5.3.3) \(\psi*(\phi)\) is analytic if \(\text{Re}(\phi) > 0\) and continuous within the domain such that \(\text{Re}(\phi) \geq 0, |\phi| < \infty\) in both cases. However \(\psi*(\phi) = \chi_+^*(\phi,0)\) by the equivalence of the defining integrals and the assertion is true.
Corollary 3. A similar result is true for $(n)\psi^*(\phi)$ for each $n = 1, 2, ...$ where $(n)\psi^*(\phi) = \int_{0-}^{\infty} e^{-\phi x} dx \int_{0-}^{\infty} (n)\chi_+(x, y) dx$.  

Proof. Obvious repetition of the above method.

Remarks: 1. The method used to prove Corollary 4 to Lemma 6.13 may also be applied to $\chi^*_+(\phi, \theta)$ for all constant $\phi$ finite s.t. $\text{Re}(\phi) > 0$ and all finite $\theta$ s.t. $\text{Re}(\theta) > -\sigma$. See Corollary 4 to Lemma 6.13 for this notation but we shall not do this here since an alternative proof is given in Section 9.

2. The domains $D$ and $(n)D$ for $n = 1, 2, ...$ defined above are the first or most obvious domains where $\chi^*_+(\phi, \theta)$ and $(n)\chi^*_+(\phi, \theta)$ are analytic respectively. Much of what follows will be concerned with deducing the existence and nature of other domains of analyticity. Therefore we may from time to time, refer to these more obvious domains as the first domains of analyticity.

Definition 6.3.1.

$$\pi^*_1(\theta) = \int_{0+}^{\infty} e^{-\theta y} d\pi_1(y)$$

where;

$$\pi_1(y) = \lim_{n \to \infty}(n)\pi_1(y)$$

for which see equation (4.45).
Definition 6.3.3.

\[ \tilde{\pi}_1(\theta) = \int_{0^+}^{\infty} e^{-\theta y} d\tilde{\pi}_1(y) \quad (6.3.3) \]

where, \( \tilde{\pi}_1(y) \) is the distribution of any measure which may be concentrated on the line \( 0 \leq y < \infty, \ x = 0 \) in the \((x,y)\) plane, by a discontinuity (or discontinuities) of the distribution \( \pi(x,y) \). Generally \( \tilde{\pi}_1(y) = 0 \) for all \( y \) but the following discussion shows how \( \tilde{\pi}_1(y) > 0 \) could arise in some specific cases.

**Excursus. The Analogue of \( \tilde{\pi}_1(y) \) for the One Server Queue.**

Equations 4.16 read:

\[
\begin{align*}
(n)\delta_+^*(z) &= (n)\gamma_+^*(z) + (n)\gamma_-^*(0) \\
(n)\delta_-^*(z) &= (n)\gamma_+^*(z) - (n)\gamma_-^*(0)
\end{align*}
\quad (4.16)
\]

whence since \( (n)\delta_+^*(z) \) may be shown to be analytic in \( \text{Re}(z) > 0 \) and continuous in \( \text{Re}(z) \geq 0 \) for all \( |\text{Im}(z)| < \infty \), whilst \( (n)\delta_-^*(z) \) may be shown analytic in \( \text{Re}(z) < 0 \) and continuous in \( \text{Re}(z) \leq 0 \) if \( |\text{Im}(z)| < \infty \), we may if \( \alpha^*(z) \cdot \beta^*(z) \) is suitable, use equation (4.17) to define an analytic continuation by means of the Schwartz Continuation Principle. One might then be moved to incorrectly suppose that the uniqueness of \( (n)\gamma_+^*(0) \) implied that:

\[
\lim_{\text{Re}(z) \to \infty} \{ (n)\delta_+^*(z) \} = \lim_{\text{Re}(z) \to -\infty} \{ (n)\delta_-^*(z) \} = (n)\gamma_+^*(0) \quad (6.3.4)
\]
for all \( n = 0,1,2,\ldots \). However this assertion is not necessarily true unless either \( A(a) \) or \( B(s) \) is absolutely continuous, or both are, since if both \( A(a) \) and \( B(s) \) have discontinuities the recurrence

\[
x^{(n+1)} = \{x^{(n)}\}^+ + s_{n} - a_{n+1}
\]

\(-\infty < x^{(n)}, x^{(n+1)} < \infty\)

may produce a distribution \( (n+1)\gamma(x) \) for \( x^{(n+1)} \) which has a discontinuity at \( x^{(n+1)} = 0 \).

**Note.** This merely requires that \( A(a) \) and \( B(s) \) each have a discontinuity at any arbitrary \( a = s = d \) whence it is clear that since the distribution of \( \{x^{(n)}\}^+ \) will possess a discontinuity at zero (unless there is no probability associated with negative values of \( x^{(n)} \) which could only happen for infinite \( n \) and an overloaded queue), the distribution of \( x^{(n+1)} \) will also have a discontinuity at zero.

Assume that \( (n+1)\gamma(x) \) does possess a discontinuity at \( x^{(n+1)} = 0 \) as postulated, we find;

\[
\lim_{\text{Re}(z)\to\infty} \{ (n+1)\delta^+_x(z) \} = (n+1)\gamma^+(0) + (n+1)\gamma^-(0) \quad (6.3.5)
\]

where

\[
(n+1)\gamma^- = d_x (n+1)\gamma(x) \big|_{x=0} > 0 \quad (6.3.6)
\]

since the L.H.S. of (6.3.4) must include all the probability associated with the condition "the \( (n+1)^{th} \) arrival does not
"Clearly if either \( A(a) \) or \( B(s) \) is an absolutely continuous distribution (or both are), the convolution implied by the difference \( S_n - a_{n+1} \) is absolutely continuous. Hence the convolution implied by equation (4.8) makes \( (n+1)\gamma(x) \) an absolutely continuous distribution function whence \( (n+1)\gamma = 0 \).

**Remarks:**

1. The equivalent results for the two server queue appear to be as follows:

   (i) If \( A(a) \) and \( B(s) \) are both absolutely continuous distribution functions, all the \( (n)\pi(x,y) \) are absolutely continuous distributions for \( n > 0 \) \( n = 1, 2, \ldots \) and \( -\infty < x < y < \infty \).

   (ii) If either \( A(a) \) or \( B(s) \) is absolutely continuous but not both the \( (n)\pi(x,y) \) are not necessarily absolutely continuous and the space \( -\infty < x < y < \infty \) will contain sets of Lebesgue measure zero which have non-zero probabilities associated therewith, and

   (iii) If either \( A(a) \) or \( B(s) \) is absolutely continuous all the \( (n)\pi(x,y) \) will be such that the line \( 0 < y < \infty \), \( x = 0 \) will be of probability measure zero.

   That is, it appears that \( \tilde{\pi}_1(y) = 0 \) for all \( 0 < y < \infty \) if either \( A(a) \) or \( B(s) \) is absolutely continuous or both are.
2. We shall merely prove that $\tilde{\pi}^1(y) = 0$ for all $0 \leq y < \infty$ if $A(a)$ is an absolutely continuous distribution function in Lemma 6.4.

**Lemma 6.3.** \[ \lim_{\text{Re}(\phi) \to \infty} \{ \chi_+^\phi(\phi, \theta) \} = \pi_0 + \pi_1^\phi(\theta) + \tilde{\pi}_1^\phi(\theta) \] (6.3.7)

for all finite $\theta$.

**Proof:** Write

$$\phi = R + i\phi$$

where $R \geq 0$ is real and consider

$$J_+'(\theta) = \lim_{R \to \infty} \int_{-\infty}^{\infty} \int_{0-} e^{-Rx-i\phi_1 x-\theta y} d\chi_+(x,y) \quad (6.3.8)$$

$$= \lim_{R \to \infty} \int_{0-} \left[ (e^{-Re(\theta)(y-x)} e^{-Re(\theta)(y-x)} e^{-i(\text{Im}(\theta)y+\phi_1 x)} ) \right] d\chi_+(x,y) \quad (6.3.9)$$

However if $|\theta| < \infty$ then $\text{Re}(\theta) < \infty$ and we may consider firstly the domain wherein $0 < \text{Re}(\theta) < R$, $|\text{Im}(\theta)| < \infty$, for finite arbitrarily large $R$. Then if

$$J_+''(\theta) = J_+''(\theta) + J_+''(\theta) \quad (6.3.10)$$

where

$$J_+''(\theta) = \lim_{R \to \infty} \int_{0-} e^{0+ -Rx-i\phi_1 x-\theta y} d\chi_+(x,y) \quad (6.3.11)$$

and
\[ J_+''(\theta) = \lim_{R \to \infty} \int_0^\infty \int_{0+}^{\infty} e^{-(R+\Re(\theta))x} \left\{ (e^{-(\Re(\theta)(y-x))} \cdot i^{\Re(\theta)(y-x)} d\chi_+(x,y) ight\} (6.3.12) \]

both \( J_+'(\theta) \) and \( J_+''(\theta) \) will exist for all \( R \) and be analytic functions of \( \theta \) for \( \Re(\theta) > 0 \) by Theorem (6.3.5) since;

\[ |e^{-(R+\Re(\theta))x}| |e^{-\Re(\theta)(y-x)}| |e^{-i(\Im(\theta)y+\phi x)}| \leq 1 \]

for all \( 0 \leq x \leq y < \infty \) if \( R > 0, \Re(\theta) > 0, |\phi| < \infty, |\theta| < \infty. \)

Hence:

\[ J_+''(\theta) = \int_0^\infty e^{\theta y} d\chi_+(0,y) \]

\[ = \pi_0 + \int_0^\infty e^{\theta y} d\pi_1(y) + \int_0^\infty e^{-\theta y} d\bar{\pi}_1(y) \]

\[ = \pi_0 + \pi_1^\#(\theta) + \bar{\pi}_1^\#(\theta) \quad (6.3.13) \]

whilst

\[ J_+''(\theta) = 0 \]

for all finite \( \theta \) since

\[ \lim_{R \to \infty} |e^{-(R+\Re(\theta))x}| = 0 \]

for all \( x > 0 \) if \( |\theta| > \Re(\theta) \), is finite. Here the lemma is true as asserted.
Lemma 6.4. If \( A(a) \) is an absolutely continuous distribution function, the distribution \( \chi(x,y) = \pi(x,y) \) defined by the inversion of the Laplace Transformation \( \chi^*(\phi,\theta) \) defined by equation (4.38) namely;

\[
\chi^*(\phi,\theta) = \alpha^*(-\phi+\phi) \left\{ \begin{array}{ll}
\beta^*(\theta)\chi^*_+ (\theta,\phi) \\
+ \int_0^\infty \int e^{-\theta y - \phi x} \beta^*(y-x,\theta) \\
\quad \cdot \left( -e^{-\phi y - \theta x} \beta^*(y-x,\theta) \right) d\chi^*_+(x,y) \end{array} \right.
\]

say, is such that the line \( 0 < y < \infty \) \( x = 0 \) is a set of measure zero, for all distributions \( B(s) \) and \( \chi^*_+(x,y) \).

Proof: Since \( A(a) \) is absolutely continuous by assumption, we may write;

\[
A(a) = \int_0^a \tilde{\alpha}(u) du \tag{6.4.2}
\]

where \( \tilde{\alpha}(u) \) is a density function, whence noting that (6.4.1) is equivalent to a convolution wherein the variable \( a \) is subtracted from both \( x \) and \( y \) (since \( \alpha^*(-\phi+\phi) \) is the L.P.T. of a degenerate distribution on the line \( x = y = -a \) \( 0 < a < \infty \) by use of the Inversion Theorem, or by consideration of equations (2.16) and (2.17)) we may write;

\[
\chi(x,y) = \int_0^\infty \Omega^*_+(x+a,y+a) \tilde{\alpha}(a) da \tag{6.4.3}
\]

where \( \Omega^*_+(x,y) \) is the (for the moment arbitrary) bivariate distribution function which results from the inversion of
We now define the marginal distributions

\[
\chi^M(x) = \int_0^\infty dy \chi(x, y) = \int_x^\infty dy \chi(x, y)
\]  

(6.4.4)

since \(d\chi(x, y) = 0\) if \(y < x\), and

\[
\Omega^M_+(x) = \int_x^\infty dy \Omega_+(x, y)
\]  

(6.4.5)

by the same procedure and use these in 6.4.3 to obtain

\[
\chi^M(x) = \int_0^\infty \Omega^M_+(x+a) \tilde{a}(a) da
\]  

(6.4.6)

where we have reversed the order of integration since

\[|\Omega_+(x+a, y+a)| |\tilde{a}(a)|\] is integrable and Fubini's Theorem may be used.

We now note that (6.4.6) is the univariate convolution of an absolutely continuous distribution and a distribution which may have any system of discontinuities whatsoever. \(\chi^M(x)\) is therefore absolutely continuous see for example, Lukacs [12], Theorem (3.3.2).

However if \(\chi^M(x)\) is absolutely continuous (or merely continuous) \(\chi(x, y)\) can contain no (finite) concentrations of probability in points or along lines such that \(x\) is constant since if it had any of these the marginal distribution would possess a discontinuity at each corresponding point.
Remarks: There may, of course, be concentrations of probability on other lines and in particular lines such as \( y = x + c \) where \( c > 0 \) since these will not generate discontinuities in \( \chi^{(\mathcal{M})}(x) \).

Thus if \( A(a) \) is absolutely continuous \( \pi_1(y) = 0 \) for all \( y \in [0,\infty) \).

Corollary 1. The above result is true for \( (n)\pi_1(y) \) for all \( n = 0,1,2,\ldots \) where \( (n)\pi_1(y) \) holds at the moment just before the \( n \)th arrival.

Proof: For each \( n \), we may define an \( (n)\Omega_+(x,y) \) and corresponding \( (n)\Omega_*(\phi,\theta) \) and from the equation

\[
(n+1)\chi^{*}(\phi,\theta) = a^{*}(-\theta+\phi)(n)\Omega^{*}(\phi,\theta)
\]

(6.4.7)

which arises from (4.36), repeat the previous proof in an obvious way. Hence the assertion is true.

Lemma 6.5(i). If the queue commences operation from any arbitrary finite fixed point \( (x(0)^+,y(0)^+) \), and the distribution \( B(s) \) is such that its Laplace Transformation \( \beta^{*}(\theta) \) is analytic for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) (where \( \sigma > 0 \)) then the Laplace Transformations \( (n)\pi_1^*(\theta) \) and \( (n)\pi_1^*(\theta) \) (if the latter is not zero), exist for all \( n = 1,2,\ldots \) and are analytic for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \).
(ii) Furthermore if the distributions $\mathbf{A}(a)$ and $\mathbf{B}(s)$ are such that queue attains an ergodic limiting distribution $\pi(x,y)$, the limiting Laplace Transformations $\tilde{\pi}_1^*(\theta)$ (if $\pi_1^*(\theta) > 0$) and $\pi_1^*(\theta)$ will be analytic for all finite $\theta$ s.t. $\text{Re}(\theta) > -\sigma$ provided that $\beta^*(\theta)$ is analytic in this domain.

The proof requires the following preliminary results.

**Result 1.** Let the observed variables $E$, $S$ and $F$ have distributions as follows:

$$P(E \leq e) = Q(e) \quad e \in [0,\infty) \quad (6.5.1.1)$$
$$P(S \leq s) = B(s) \quad s \in [0,\infty) \quad (6.5.1.2)$$
$$P(F \leq f) = R(f) \quad f \in [0,\infty) \quad (6.5.1.3)$$

then the relationships

$$E = |S-F| \quad (6.5.1.4)$$

and

$$Q^*(\theta) = \int_0^\infty e^{-\theta e} dQ(e) \quad (6.5.1.5)$$

exist and define an analytic function for all finite $\theta$ s.t. $\text{Re}(\theta) > -\sigma$ if:

$$\beta^*(\theta) = \int_0^\infty e^{-\theta s} dB(s) \quad (6.5.1.6)$$

and

$$R^*(\theta) = \int_0^\infty e^{-\theta f} dR(f) \quad (6.5.1.7)$$

and both analytic in this domain, and provided that $S$ and
F are independent.

**Proof:** Consider;

\[ E = S - F \quad (6.5.1.8) \]

to define the distribution

\[ \tilde{Q}(e) = P(\tilde{E} \leq e) \quad (6.5.1.9) \]

when \( e \in (-\infty, \infty) \) whence, the Laplace Transformation;

\[ \tilde{Q}^*(\theta) = B^*(\theta)R^*(-\theta) \quad (6.5.1.10) \]

and this will be analytic if \( \theta \) is finite and

\[ -\sigma < \text{Re}(\theta) < \sigma. \]

Hence we may write;

\[ \tilde{Q}^*(\theta) = \tilde{Q}^*_+(\theta) + \tilde{Q}^*_-(\theta) \quad (6.5.1.11) \]

by subdivision of the defining integral where \( \tilde{Q}^*_+(\theta) \) exists if \( \text{Re}(\theta) > -\sigma \) and \( \theta \) is finite and \( \tilde{Q}^*_-(\theta) \) exists if \( \text{Re}(\theta) < \sigma \) and \( \theta \) is finite.

Consider now the operation

\[ E = |\tilde{E}| \quad (6.5.1.12) \]

to define the distribution

\[ Q(e) = P(E \leq e) \]
\[ = \tilde{Q}(e) - \tilde{Q}(e) \quad (6.5.1.13) \]

Hence,
\[ Q^*(\theta) = \int_{0}^{\infty} e^{-\theta e} dq(e) \]
\[ = \int_{0}^{\infty} e^{-\theta e} d\tilde{q}_+(e) - \int_{0}^{\infty} e^{-\theta e} d\tilde{q}_-(e) \]  
(6.5.1.14)

by use of (6.5.1.13)
\[ = Q^*+(\theta) + \int_{-\infty}^{0} e^{\theta e} d\tilde{q}_-(e) \]  
(6.5.1.15)
\[ = Q^*+(\theta) + Q^*(-\theta) \]  
(6.5.1.16)

where the first is clearly analytic if \( \theta \) is finite and \( \text{Re}(\theta) > -\sigma \) whilst the second is also analytic in this domain (by virtue of the reversed sign of \( \theta \)). Thus the assertion is true.

**Remarks:** 1. This result may be extended as follows:

Write
\[ \tilde{Q}(e) = \int_{0-}^{\infty} B(e+f)dR(f) \]  
(6.5.1.17)
\[ \geq B(e) \]  
(6.5.1.18)

Whence as \( e \to \infty \) the upper tail of the distribution \( \tilde{Q}(e) \) is bounded below by the distribution \( B(e) \).

Similarly we may write;
\[ \tilde{G} = -\tilde{E} = F - S \]  
(6.5.1.19)

whence;
\[ P(G \leq g) = \int_{0-}^{\infty} R(g+s)dB(s) \]  
(6.5.1.20)
\[ \geq R(g) \]  
(6.5.1.21)
Hence by reversal of signs

\[ Q(e) \leq R(e) \quad (6.5.1.22) \]

and the left hand tail of the distribution \( Q(e) \) is bounded above by the distribution of \( R(e) \).

2. The result is of course consistent with the notion that \( Q^*(\theta) \) derived its singularities from \( \Phi^*(\theta) \) whilst \( Q^*(\theta) \) derived its singularities from \( R^*(-\theta) \).

3. Clearly, if the distribution \( R(f) \) be such that \( R^*(\theta) \) is analytic in a domain \( D' \) which is larger than that previously used the result will continue to be true. Thus the result is still true even if \( R(\theta) \) is entire.

Result 2. The relationship

\[ E = \max(0,F-G) \quad (6.5.2.1) \]

defines a distribution

\[ P(E \leq \hat{e}) = Q(\hat{e}) \quad (6.5.2.2) \]

such that;

\[ Q^*(\theta) = \int_0^\infty e^{-\theta \hat{e}} dQ(\hat{e}) \quad (6.5.2.3) \]

is analytic for all finite \( \theta \) such that \( \text{Re}(\theta) > -\sigma \) if:

(i) \( P(F \leq f) = R(f) \) be any distribution on \([0,\infty)\) such that its Laplace Transformation \( R^*(\theta) \) has this domain of analyticity.
(ii) \( P(G \leq g) = T(g) \) be any distribution on \([0, \infty)\) whilst

(iii) \( G \) is not necessarily independent of \( F \).

**Proof:** For \( f, g \in [0, \infty) \) we define the bivariate probability distribution

\[
\hat{W}(f, g) = P(F < f, G < g)
\]  

(6.5.2.4)

and thence;

\[
\hat{W}^*(\theta) = E(e^{-\theta \{\max(0, F-G)\}})
\]  

(6.5.2.5)

\[
= \int_0^\infty \int_0^\infty e^{-\theta \{\max(0, f-g)\}} \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.6)

\[
= \int_0^\infty \int_0^\infty \hat{W}(f, g) \, df \, dg + \int_0^\infty \int_0^f e^{-\theta (f-g)} \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.7)

and this clearly exists if \( \theta \) is finite and \( \Re(\theta) \geq 0 \)

\[
\leq \int_0^\infty \int_0^\infty \hat{W}(f, g) \, df \, dg + \int_0^f \int_0^\infty e^{\Re(\theta) (f-g)} \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.8)

if \( \Re(\theta) < 0 \).

\[
\leq 1 + \int_0^\infty \int_0^\infty e^{\Re(\theta) f} \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.9)

\[
\leq 1 + \int_0^\infty e^{\Re(\theta) f} \int_0^\infty \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.10)

But

\[
\int_0^\infty e^{-\theta f} \int_0^\infty \hat{W}(f, g) \, df \, dg
\]  

(6.5.2.11)
which exists and defines an analytic function if \( \text{Re}(\theta) > -\sigma \).

Hence,

\[
\int_0^\infty |\text{Re}(\theta)| f_{\text{d}f} \int_0^\infty \text{d} \hat{W}(f,g)
\]

exists if \( -\sigma < \text{Re}(\theta) < 0 \).

Hence the Laplace integral;

\[
E(\mathcal{L}^{-\theta}(\max(0,F-G)))
\]

exists for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) whence by standard results, see for example Widder [18], the integral defines a function analytic in this domain.

Hence the result is true.

Proof of Lemma (6.5) (Part (i)).

Consider the two dimensional recurrences defined by equations (2.16) and (2.17) which were:

\[
y(n+1) = \max(y(n)^+,x(n)^+ + S_n) - a_{n+1} \tag{6.5.3.1}
\]

\[
x(n+1) = \min(y(n)^+,x(n)^+ + S_n) - a_{n+1} \tag{6.5.3.2}
\]

and;

\[
y(n+1)^+ = \max(0,y(n+1)) \tag{6.5.3.3}
\]

\[
x(n+1)^+ = \max(0,x(n+1)) \tag{6.5.3.4}
\]

for all \( n = 0,1,2,\ldots \).
We show that these are equivalent to the following set which are more convenient for our purposes;

\[ e(n+1) = |e(n) - S_n| \geq 0 \quad (6.5.3.5) \]
\[ ^\wedge(n) = x(n) + \min(S_n, e(n)) \geq 0 \quad (6.5.3.6) \]
\[ x(n+1) = x(n) - a_{n+1} \quad (6.5.3.7) \]
\[ e(n+1) = \max(0, (e(n+1) + \min(0, x(n+1)))) \geq 0 \quad (6.5.3.8) \]
\[ y(n+1) = x(n+1) + e(n+1) \quad (6.5.3.9) \]

where \( y(n+1) \) and \( x(n+1) \) are defined by (6.5.3.3) and (6.5.3.4) respectively.

Consider the possible transitions of the random walk of the two server queue between the times \( t_n = 0 \) and \( t_{n+1} = 0 \); that is, between the times immediately before the \( n^{th} \) and \( (n+1)^{th} \) arrivals. We define for any \( n = 0, 1, 2, \ldots \),

\[ e(n) = y(n) - x(n) \quad (6.5.3.10) \]

and show that this definition plus equations (6.5.3.1) to (6.5.3.4) leads to an equivalent recurrence using equations (6.5.3.5) to (6.5.3.9).

Firstly if \( S_n > e(n) \) then by use of (6.5.3.1) and (6.5.3.2)

\[ y(n+1) = x(n) + S_n - a_{n+1} \]
\[ x(n+1) = y(n) - a_{n+1} \]
Hence

\[ y(n+1) - x(n+1) = s_n - e(n) \quad (6.5.3.11) \]

by use of (6.5.3.10). However if \( s_n \leq e(n) \) then by use of (6.5.3.1) and (6.5.3.2) we have;

\[ y(n+1) = y(n) + - a_{n+1} \]

\[ x(n+1) = x(n) + s_n - a_{n+1}. \]

Hence;

\[ y(n+1) - x(n+1) = e(n) - s_n \quad (6.5.3.12) \]

by use of (6.5.3.10).

Thus in either of these cases;

\[ y(n+1) - x(n+1) = |e(n) - s_n| \quad (6.5.3.13) \]

and we may write (6.5.3.5) to define \( ^\wedge e(n+1) \) for all \( n = 0,1,2, \ldots \).

Clearly,

\[ 0 \leq e(n+1) = y(n+1) + - x(n+1) + \leq \hat{e}(n+1) = y(n+1) - x(n+1) \quad (6.5.3.14) \]

and

\[ e(n+1) = \hat{e}(n+1) \quad (6.5.3.15) \]

iff \( x(n+1) \geq 0 \)

However the use of (6.5.3.6) and (6.5.3.7) in (6.5.3.8) gives;

\[ e(n+1) = \max(0, (\hat{e}(n+1) + \min(0, (x(n) + + \min(s_n, e(n) - a_{n+1}))) \quad (6.5.3.16) \]
and we consider four mutually exclusive and exhaustive situations all of which lead to the result (6.5.3.10) in the form:

\[ e(n+1) = y(n+1) + x(n+1) \]  \hspace{1cm} (6.5.3.17)

Cases are:

(i) \( S_n > e(n); \quad x(n) > a_{n+1} \) whence by (6.5.3.16)

\[ e(n+1) = \max(0, (S_n - e(n))) = S_n - e(n) \]

\[ = e(n+1) \]

by (6.5.3.5)

\[ = y(n+1) + x(n+1) \]  \hspace{1cm} (6.5.3.18)

by (6.5.3.15) since \( x(n+1) \geq 0 \).

(ii) \( S_n > e(n); \quad x(n) < a_{n+1} \)

\[ e(n+1) = \max(0, S_n - e(n) + x(n) + e(n) - a_{n+1}) \]

by (6.5.3.16)

\[ = \max(0, y(n+1)) \]

\[ = y(n+1) + \]

by (6.5.3.3)

\[ = y(n+1) + x(n+1) \]  \hspace{1cm} (6.5.3.19)

since,

\[ x(n+1) = \max(0, x(n) + e(n) - a_{n+1}) \]

\[ = \max(0, x(n) - a_{n+1}) \]

\[ = 0 \]
by use of (6.5.3.2) and the assumption \( a_{n+1} > x(n) \)

(iii) \( S_n < e(n); \; x(n) > a_{n+1} \)

and

(iv) \( S_n < e(n); \; x(n) < a_{n+1} \)

for which the proofs also follow in an obvious way. That is;

(iii) \( \Rightarrow e(n+1) = x(n+1) = y(n+1) - x(n+1) > x(n+1) > 0 \) (6.5.3.20)

since \( x(n+1) > 0 \)

(iv) \( \Rightarrow e(n+1) = y(n+1) = y(n+1) - x(n+1) > x(n+1) > 0 \) (6.5.3.21)

since \( x(n+1) < 0 \)

Hence in all situations the equations (6.5.3.5, 6.7, and 8) and the definition (6.5.3.10) are equivalent for all \( n = 0, 1, 2, \ldots \) and all that remains to be shown is that (6.5.3.2) and (6.5.3.7) are equivalent. By use of (6.5.3.6) and (6.5.3.7) we find that;

(i) If \( S_n < e(n) \)

then

\[
x(n+1) = x(n) + S_n - a_{n+1} \quad (6.5.3.22)
\]

which is (6.5.3.2).

(ii) If \( S_n > e(n) \)

then

\[
x(n+1) = x(n) + S_n - a_{n+1} \quad (6.5.3.23)
\]
which is also (6.5.3.2).

Therefore the two sets of recurrences are equivalent for all \( n = 0, 1, 2, \ldots \).

We note also that the values taken by the variables \( \hat{e}(n) \) and \( e(n) \) are clearly conditional on \( x(n) \) but we now show that if the distribution \( B(s) \) is such that its Laplace Transformation \( \hat{B}(\theta) \) is analytic for all finite \( \theta \) such that \( \Re(\theta) > -\sigma \), the conditional distributions \( (n)\hat{Q}(e/x) \) and \( (n)Q(e/x) \) of the variables \( \hat{e}(n) \) and \( e(n) \) are also such that their Laplace Transformations \( (n)\hat{Q}^*(\theta) \) and \( (n)Q^*(\theta) \) also have this property for any \( x(n) \in (-\infty, \infty) \) and any \( n = 1, 2, \ldots \), provided that the queue starts operating at some finite initial point \( (x(0)^+, y(0)^+) \).

**Proof:** If at time \( t_0 = 0 \) the queue starts at the point \( (x(0)^+, y(0)^+) \) where \( x(0)^+ \) and \( y(0)^+ \) are finite then \( e(0) = y(0)^+ - x(0)^+ \) is a given finite constant. Then the operation

\[
\hat{e}(1) = |e(0) - s_0|
\]  

(6.5.3.24)
defines a distribution \( (1)\hat{Q}(e/x) \) conditional on \( x(1) \) which is such that its Laplace Transformation \( (1)\hat{Q}^*(\theta) \) is analytic wherever \( \hat{B}(\theta) \) is analytic since \( \hat{B}(\theta) \) is entire and we may use the obvious extension of Result 1.
(See the Remark at the end of the proof of Result 1.)

Hence \( \hat{Q}_x^*(\theta) \) is analytic for all finite \( \theta \) s.t. \( \Re(\theta) > -\sigma \) and this must also be true for any \( x^{(1)} \in (-\infty, \infty) \) which may arise.

Also;

\[
e^{(1)} = \max(0, \hat{e}^{(1)} + \min(0, x^{(1)}))
\]
defines a distribution \( \hat{Q}(e/x) \) which has the property that \( \hat{Q}^*_x(\theta) \) is analytic for at least all finite \( \theta \) s.t. \( \Re(\theta) > -\sigma \) and for all \( x^{(1)} \in (-\infty, \infty) \) since Result 2 may be applied directly.

Hence the distribution \( \hat{Q}(e/x) \) has the desired property for any \( x^{(1)} \in (-\infty, \infty) \).

We may now repeat the above reasoning to show that if for any \( n = 1, 2, \ldots \) \( \hat{Q}^{(n)}(e/x) \) has this property for any \( x^{(n)} \in (-\infty, \infty) \) then \( \hat{Q}^{(n+1)}(e/x) \) and \( Q^{(n+1)}(e/x) \) will also have it for any \( x^{(n+1)} \in (-\infty, \infty) \). Hence an induction exists and we have shown that for any finite integral \( n \) and any \( x^{(n)} \in (-\infty, \infty) \) both \( \hat{Q}^{(n)}(e/x) \) and \( Q^{(n)}(e/x) \) have the property that their Laplace Transformations \( \hat{Q}^*_x(\theta) \) and \( Q^*_x(\theta) \) are analytic for at least all finite \( \theta \) such that \( \Re(\theta) > -\sigma \).

However \( Q^{(n)}(e/x) \) for \( x \leq 0 \) is the conditional
distribution of \(y^{(n)^+}\) and we may write:

\[
0^- \int_{-\infty}^0 d_{x}^{(n)} \Pi(x) (n) Q(e/x) = (n) \pi_0 + (n) \pi_1(e) + (n) \pi_o
\]

and

\[
0^+ \int_{-\infty}^0 d_{x}^{(n)} \Pi(x) (n) Q(e/x) = (n) \pi_1(e) + (n) \pi_1(e)
\]

where

\[
(n) \Pi(x) = \int \int_{-\infty}^x d(n) \pi(x',y)
\]

and is the unconditional probability that \(x^{(n)} < x^{(n)}\) at the moment just before the \(n^{th}\) arrival and

\[
(n) \pi_0, (n) \pi_1(y) \text{ and } (n) \pi_1(y)
\]

have been defined by equations (4.21), (4.22) and the obvious extension of (6.3.1) respectively.

Thus we may write;

\[
(n) \pi_0 + (n) \pi_1(y) = \int_{-\infty}^0 d(n) \Pi(x) (n) Q(y/x)
\]

whence,

\[
(n) \pi_0 + (n) \pi_1(\theta) = \int_{0^-}^\infty \theta y dy \int_{-\infty}^0 d_x^{(n)} \Pi(x) (n) Q(y/x)
\]

However we have shown that for all finite \(n\) and for all \(x^{(n)} \in (-\infty, \infty)\)
is analytic if \( \theta \) is finite and \( \Re(\theta) > -\sigma \). Hence by use of a standard theorem (see for example (Widder [18] Chapter 2 theorem (2.2)) we find;

\[
(n) \xi^{(\infty/x)} - (n) \xi^{(y/x)} = o(\xi^{-\sigma y}) \quad \text{as } y \to \infty \quad (6.5.3.30)
\]

uniformly for all \( x \in (-\infty, \infty) \).

Hence;

\[
\int_{-\infty}^{0} \{ (n) \xi^{(y/x)} - (n) \xi^{(\infty/x)} \} d(n) \Pi(x) = o(\xi^{-\sigma y}) \quad \text{as } y \to \infty
\]

(6.5.3.31)

for all \( n = 1, 2, \ldots \) in view of the uniform behaviour of

\( (n) \xi^{x}(\theta) \) over \( x \) and the finiteness of \( (n) \Pi(0) \).

Thus the integral (6.5.3.28) exists for all finite integral \( n \) and defines an analytic function of \( \theta \) for all finite \( \theta \) such that \( \Re(\theta) > -\sigma \).

Similarly if;

\[
(n)_{\Pi}(0) = \int_{0-}^{0+} d(n) \Pi(x) > 0
\]

(6.5.3.32)

for any \( n = 1, 2, \ldots \)

we may write
\[(n) \hat{\pi}_1(\theta) = \int_{0^+}^{\infty} e^{-\theta y} dy \quad (n) \zeta(y/\theta)\]

and this will also be analytic if \(\theta\) is finite and s.t. \(\text{Re}(\theta) > -\sigma\).

Hence the assertion of the lemma is true for all finite integral \(n\).

If the queue is ergodic, then its states (or rather suitable* sets of states) are positive recurrent. Thus the queue empty state in which \(x(+) = y(+) = 0\) will be positive recurrent and there will exist a proper distribution \(\hat{P}_m\) for all \(m = 1, 2, \ldots\) where

\[
\hat{P}_m = \text{Prob}\{m \text{ arrivals since one found the queue empty and the next which so found it.}\}
\]

Note:* Clearly if the state space is continuous, points cannot be recurrent but such sets as \(x \in [x_1, x_2]\) for all \(y \in [y_1, y_2]\) \(x_2 > x_1, y_2 > y_1\) may be.

Consider an arbitrary arrival at the ergodic queue which finds the queue empty. Clearly \(\hat{P}_m\) also specifies probabilities for the numbers of arrivals which have taken place since the queue was last empty. If however the arrival finds at least one of the servers busy, then there will exist another distribution \(\bar{P}_m\) for \(m = 1, 2\), which specifies how many arrivals have taken place since the queue was last empty. Also it is clear that,
for all \( m = 1,2, \ldots \), since the queue has not yet "had time" to become empty again at the moment of this arrival.

Hence for any arbitrary arrival which may or may not find the queue empty there exists the distribution \( P_m \) for \( m = 1,2, \ldots \) such that

\[
P_m > \hat{P}_m
\]

for all \( m = 1,2, \ldots \) which specifies how many arrivals have taken place since the queue was last empty.

However the essential property of the Laplace Transformation of any residual service time distribution which this \( m \)th arrival since last emptyness may encounter has already been determined for all finite \( m \) since the preceding results may be obtained if \( x^{(0)} + = y^{(0)} + = 0 \) (whence \( e^{(0)} = 0 \)). Thus we have shown that uniformly for all \( m = 1,2, \ldots \)

\[
\pi_k^n(0) \quad \text{and} \quad \pi_k^n(0)
\]

are both analytic functions for all finite \( \theta \) s.t. \( \Re(\theta) > -\sigma \) if \( \beta^*(\theta) \) is analytic in this domain whilst the existence of the distributions \( P_m \) ensures that this is sufficient. Hence for the ergodic queue, the limits of

\[
\pi_k^n(0) \quad \text{and} \quad \pi_k^n(0)
\]

as \( n \to \infty \) namely,

\[
\pi_k^*(0) \quad \text{and} \quad \pi_k^*(0)
\]
(if \( \pi_1'(\theta) > 0 \)) exist and are analytic functions for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) provided that \( \beta^* (\theta) \) is analytic in this domain.

Hence the whole of the lemma is true.

Remarks: 1. One may use a very different method to show that if the distribution \( B(s) \) is a finite mixture of distributions composed of finite numbers of stages (with possibly different stage means), and an absolutely discontinuous distribution, and such that \( \beta^* (\theta) \) is analytic for all finite \( \theta \) such that \( \text{Re}(\theta) > -\sigma \), then \( \pi_1^* (\theta) \) and if \( \pi_1^* (\theta) > 0 \) it also, will be analytic functions for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) but without specifically assuming* that the queue is ergodic. This result is curious but is consistent with either.

\[
\begin{align*}
(1) \quad & \lim_{n \to \infty} (n) \pi_1^* (\theta) = 0 \\
& \lim_{n \to \infty} (n) \pi_1^* (\theta) = 0
\end{align*}
\]

if the queue diverges or

(ii) that the functions have no limit if the queue becomes periodic.

Note: * Other than in the sense that if the queue is not ergodic, the whole discussion lacks motivation.

Lemma 6.6 The Laplace Transformation \( \chi^*_-(\phi, \theta) \) defined below is continuous in \( \theta \) when \( \phi \) is constant, and in \( \phi \) when \( \theta \) is constant, in the closed region \( R \) where
\[ R : \begin{cases} (\phi, \theta) \text{ s.t. } \text{Re}(\phi + \theta) \leq 0, \text{Re}(\phi) \leq 0 \\ -\infty < \text{Im}(\theta) < \infty, -\infty < \text{Im}(\phi) < \infty \end{cases} \] (6.6.1)

**Definition:** \( \chi^*_-(\phi, \theta) = \int_{0+}^{0+} \int_{x=-\infty}^{y=x} e^{-\phi x - \theta y} d\chi_-(x, y) \) (6.6.2)

where \( \chi_-(x, y) \) is defined by equation (4.44). The transforms \( (n)\chi^*_-(\phi, \theta) \) may also be defined from \( (n)\chi_-(x, y) \) for \( n = 0, 1, 2, \ldots \) in the same manner.

**Proof:** We note that \( \chi_-(x, y) \) has for its support the set \( S \) where;

\[ S = \{(x, y) \text{ s.t. } -\infty < x < y < 0\} \] (6.6.3)

whence for all \( (\phi, \theta) \in R \)

\[ |e^{-\phi x - \theta y}| \leq 1 \] (6.6.4)

whilst;

\[ \int_{-\infty}^{x} \int_{y=x}^{0+} d\chi_-(x, y) = \pi_0 \leq 1 \] (6.6.5)

and we apply Theorem (5.3.4) to the integral;

\[ J_-(\phi, \theta) = \int_{x=-\infty}^{0-} \int_{y=x}^{0+} e^{-\phi x - \theta y} d\chi_-(x, y) \] (6.6.6)

to show that this is continuous. The asserted result then follows by noting that;

\[ \chi^*_-(\phi, \theta) = J_-(\phi, \theta) - \pi_0 \] (6.6.7)

from the definition of \( (n)\pi_0 \) and thence \( \pi_0 \) for which see (4.21) and (4.46).
Corollary 1. The result is true for all \( \chi^*_\phi (\phi, \theta) \).

Proof: Define an \( \chi^*_\phi (x, y) \) and an \( \pi_0 \leq 1 \) and an \( J_{\phi, \theta} \) and note that \( \pi \subseteq \pi \) for all \( n \) when \( S \) is defined as above. Hence the above proof may be repeated for any \( n = 0, 1, 2, \ldots \) and the assertion is true.

Lemma 6.7. The Laplace Transformation \( \chi^*_\phi (\phi, \theta) \) is analytic in both \( \phi \) and \( \theta \) in the domain \( D \) where

\[
D = \{ \text{All finite} \ \phi \text{ and } \theta \text{ s.t.} \ \Re(\phi) < 0 \text{ and} \ \Re(\theta + \phi) < 0 \} \tag{6.7.1}
\]

Proof: Consider \( J_{\phi, \theta} \) of equation (6.6.6) and define the domain

\[
D' = R - \bar{R} \tag{6.7.2}
\]

where \( \bar{R} \) is the set of edge points of the \( R \) defined by equation (6.6.1), note that

\[
D' = D \tag{6.7.3}
\]

apply theorems (5.3.5) and (5.3.6) and observe that \( J_{\phi, \theta} \) is a bivariate analytic function whence \( \chi^*_\phi (\phi, \theta) \) is also analytic, in \( D \).

Corollary 1. The result is true for \( \chi^*_\phi (\phi, \theta) \) for any \( n = 0, 1, 2, \ldots \).
Proof: The result follows in an obvious way from the proof of Corollary 1 of Lemma 6.6.

Corollary 2. If $\theta = \theta_0$ be a finite constant s.t. $\Re(\theta_0) = 0$ then $\chi^*_{-}(\phi, \theta_0)$ is univariate analytic in $\phi$ for all finite $\phi$ s.t. $\Re(\phi) < 0$ and continuous for all finite $\phi$ s.t. $\Re(\phi) \leq 0$.

Proof: Consider the integral which defines $J_{-}(\phi, \theta_0)$ [Equation 6.6.6] and note that if $\Re(\theta_0) = 0$ this integral exists if $\Re(\phi) \leq 0$. Hence the result follows immediately by the Corollaries to Theorems 5.3.2 and 5.3.3 (or see Widder [18] Chapter 2).

Corollary 3. A similar result holds which extends Corollary 2 to all $(n)\chi^*_{-}(\phi, \theta_0)$ for $n = 1, 2, \ldots$ if $\Re(\theta_0) = 0$.

Proof: Define an $(n)J_{-}(\phi, \theta_0)$ and repeat the preceding reasoning in the obvious way.

Lemma 6.8. The asymptotic limit;

$$\lim_{\Re(\phi) \to \infty} \{\chi^*_{-}(\phi, \theta)\} = -\pi_0 \quad \text{for all finite } \theta. \quad (6.8.1)$$

Proof: There are two steps. We first show that this limit exists and is an analytic function for all finite $\theta$. We then show that it is $-\pi_0$. 
Write

\[ \chi^{*} (\phi, \theta) = J_{-}(\phi, \theta) - \pi_0 \]  

(6.8.2)

by use of (6.6.7)

By lemma 6.7 \( \chi^{*} (\phi, \theta) \) is certainly analytic if \( \text{Re}(\phi) = -R \) where \( R > 0 \) and

\[ \text{Re}(\theta) \leq |\theta| < R \]  

(6.8.3)

Consider the integral

\[ J_{-}(\phi, \theta) = \int_{-\infty}^{0} \int_{x}^{0+} e^{-\phi x - \theta y} d\chi_{-}(x,y) \]  

(6.8.4)

whence if \( \text{Re}(\phi) = -R, \text{Im}(\phi) = \phi_i \) and \( i \) is used as the unit of imaginaries;

\[ |J_{-}(-R+i\phi_i, \theta)| \leq \int_{-\infty}^{0} \int_{x}^{0+} e^{-R|x|} e^{\text{Re}(\theta)} |y| d\chi_{-}(x,y) \]  

(6.9.5)

since \( x < 0, y \leq 0 \).

However the support of the integral (6.8.4) is such that \( |y| \leq |x| \) and

\[ |J_{-}(-R+i\phi_i, \theta)| \leq \int_{-\infty}^{0} \int_{x}^{0+} e^{-R(|x|-|y|)} e^{(\text{Re}(\theta)-R)|y|} d\chi_{-}(x,y) \]

(6.8.6)

\[ < \infty \]  

(6.8.7)

if \( \text{Re}(\theta) \leq R \) and \( R \in [0, \infty) \).
Hence the integral exists for all \( R \), whence by standard theorems (see for example Widder [18] Chapter 2), the limit;

\[
\lim_{\text{Re}(\phi) \to -\infty} \{ \chi^{x_-}_\phi(\phi, \theta) \}
\]

exists and defines an analytic function of \( \theta \) if \( |\theta| < -\text{Re}(\phi) \) and thus in the limit, if \( |\theta| < \infty \).

If, however, \( \text{Re}(\theta) \leq 0 \);

\[
|J_\phi^{x_-}(-R+i\phi, \theta)| \leq \int_0^{0-} \int_{-\infty}^{0+} e^{-R|x|} |\text{Re}(\theta)| y d\chi_{x-}(x, y) (6.8.8)
\]

\[
< \int_0^{0-} \int_{-\infty}^{0+} e^{-R|x|} d\chi_{x-}(x, y) (6.8.9)
\]

since,

\[
|e^{-\text{Re}(\theta)| y|} \leq 1 (6.8.10)
\]

if \( \text{Re}(\theta) \leq 0 \) because \( y \leq 0 \).

Hence,

\[
\lim_{R \to \infty} |J_\phi^{x_-}(-R+i\phi, \theta)| = 0 (6.8.11)
\]

for all finite \( \theta \) s.t. \( \text{Re}(\theta) \leq 0 \)

since

\[
e^{-R|x|} \to 0 (6.8.12)
\]

as \( R \to \infty \) for all \( |x| > 0 \).
Hence $J_{-\phi}(\theta)$ is an analytic function of $\theta$ if
$\Re(\theta) \leq |\theta| \leq R$ for all $R$ which converges to zero as
$R \to \infty$ for all such $\theta$ also s.t. $\Re(\theta) < 0$.

Hence

$$\lim_{\Re(\phi) \to -\infty} \{\|J_{-\phi}(\theta)\|\} = 0 \quad (6.8.13)$$

for all finite $\theta$ and the assertion is true.

**Corollary 1.**

$$\lim_{\Re(\phi) \to -\infty} \{(n)\chi_{-\phi}(\theta)\} = -(n)\pi_0 \quad (6.8.14)$$

for all finite $\theta$, where

$$(n)\chi_{-\phi}(\theta)$$

and $$(n)\pi_0$$

are defined in the Corollaries

to Lemmas 6.6 and 6.7.

**Proof:** We shall follow the method of proof of Corollary 1
to Lemma 6.6 and note that $(n)S \subseteq S$ whence $(n)J_{-\phi}(\theta)$
is analytic in $(n)D \supseteq D$ where $D$ has been defined by
equation (6.7.1). We then repeat the proof of Lemma 6.8
with the obvious minor changes and the assertion is clearly
true.

**Lemma 6.9.**

If

$$\int_{0^-}^{0^+} \int_{-\infty}^0 d\chi_-(x,y) = 0 \quad (6.9.1)$$
whence the line

\[ y = 0 \quad x \in (-\infty, 0) \]

is a set probability measure zero, then;

\[
\lim_{\text{Re}(\phi) \to -\infty} \{ \chi^*(\phi, \theta) \} = -\pi
\]  \hspace{1cm} (6.9.2)

at least for all finite \( \phi \) s.t. \( \text{Re}(\phi) \leq 0 \).

**Proof:** Write

\[ \theta = -R + i\theta_1 \]

and assume \( \text{Re}(\phi) \leq 0 \). Then by use of definition (6.6.6) and the above we may write;

\[
|J_\perp(\phi, -R+i\theta_1)| \leq \int_{-\infty}^{0-} \int_0^0 |e^{-\phi x}| e^{-R|y|} d\chi(x,y)
\]

\[
+ \int_{-\infty}^{0+} \int_0^0 |e^{-\phi x}| d\chi(x,y)
\]  \hspace{1cm} (6.9.3)

for all \( R \). However if \( \text{Re}(\phi) \leq 0 \)

\[
|e^{-\phi x}| \leq e^{-\text{Re}(\phi)|x|} \leq 1
\]  \hspace{1cm} (6.9.4)

since \( x < 0 \) in (6.9.3). Hence,

\[
\lim_{R \to \infty} |J_\perp(\phi, -R+i\theta_1)| \leq \int_{-\infty}^{0-} \int_0^0 d\chi(x,y)
\]

\[
= 0
\]  \hspace{1cm} (6.9.5)

by use of (6.9.1) for all \( \phi \) s.t. \( \text{Re}(\phi) < 0 \).

Hence the assertion is true.
Remark: It remains to be shown that if, for example, $A(a)$ and $B(s)$ are both continuous distributions, the line $y = 0, x \in (-\infty, 0)$ is a set of probability measure zero.

**Definition 6.10.**

$$\chi_{\pm}^{*(\phi, \theta)} = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\phi x - \theta y} d\chi_{\pm}(x, y)$$  \hspace{1cm} (6.10.1)

by use of equations (4.25) and (4.35), (4.36) and (4.37).

Recollect also the definition of $\chi_{\pm}^{*(\phi, \theta)}$ to include $-\pi_0$.

**Lemma 6.10.**

$\chi_{\pm}^{*(\phi, \theta)}$ is continuous in the region $R$ defined below and analytic in $D \subset R$ also defined below where

$$R: \{(\phi, \theta) \text{ s.t. } \text{Re}(\theta) \geq 0, \text{Re}(\phi) \leq 0, -\infty < \text{Re}(\phi), \text{Re}(\theta) < \infty\}$$  \hspace{1cm} (6.10.3)

$$D: \{(\phi, \theta) \text{ s.t. } \text{Re}(\theta) > 0, \text{Re}(\phi) < 0, -\infty < \text{Re}(\phi), \text{Re}(\theta) < \infty\}$$  \hspace{1cm} (6.10.4)

**Proof:** By Theorem (5.3.2), $\pi_1^{\tau}(\theta)$ is continuous for all $\theta$ s.t. $\text{Re}(\theta) \geq 0$ and by Theorem (5.3.3) $\pi_1^{\tau}(\theta)$ is analytic if $\text{Re}(\theta) > 0$. Hence it suffices to show that the integral in (6.10.2) has the desired properties. The support set of this integral is $S$ where
\[ S: \{(x,y) \text{ s.t. } -\infty < x < 0, 0 < y < \infty\} \quad (6.10.5) \]

whence for all \((\phi, \theta) \in \mathbb{R}\)

\[ |e^{-\phi x - \theta y}| \leq 1 \quad (6.10.6) \]

for all \((x,y) \in S\).

We may therefore apply Theorems (5.3.4), (5.3.5) and (5.3.6) to this integral since;

\[
\int_{-\infty}^{0-} \int_{0+}^{\infty} d\chi_-(x,y) = \pi_1 \leq 1 \quad (6.10.7)
\]

and the desired results follow at once.

**Corollary 1.** The above Lemma is true for \(\chi_{-+}^n(\phi, \theta)\) for all \(n = 1, 2\).

**Proof:** For each \(n\) we may write;

\[
(\chi_{-+}^n(\phi, \theta) = \int_{-\infty}^{0-} \int_{0+}^{\infty} e^{-\phi x - \theta y} d(n)\chi_-(x,y) - (n)\pi_1^\phi(\theta) \quad (6.10.8)
\]

by use of our previous definitions for \(\chi_-(x,y)\) and \(\pi_1^\phi(\theta)\).

We also note by use of Theorems (5.3.2) and (5.3.3) that each \(\pi_1^\phi(\theta)\) is analytic if \(\text{Re}(\theta) > 0\) and continuous if \(\text{Re}(\theta) \geq 0\) and naturally for all \(\phi\). Hence it suffices to apply Theorems (5.3.4, 5.3.5) and (5.3.6) to
the integral in (6.10.8) for each \( n = 1,2,\ldots \). However, the support set of the integral \((n)S = S\) as defined above for all \( n = 1,2,\ldots \), \((n)\chi_{-}(x,y)\) is a positive measure for all \( x < 0 \) and \( n = 1,2,\ldots \) and

\[
\int_{-\infty}^{0} \int_{0}^{\infty} d(n)\chi_{-}(x,y) = (n)\pi_{1} \leq 1 \tag{6.10.9}
\]

by the definition of \((n)\pi_{1}\). We may therefore re-employ the previous reasoning directly for any \( n = 1,2,\ldots \) and the assertion is true.

---

**Corollary 2.** If \( \theta_{0} \) be any finite complex constant such that \( \text{Re}(\theta_{0}) = 0 \) then \( \chi^{*}_{+}(\phi,\theta_{0}) \) is univariate analytic in \( \phi \) at least for all finite \( \phi \) s.t. \( \text{Re}(\phi) < 0 \).

**Proof:** Clearly if \( \theta = \theta_{0} \) a constant s.t. \( \text{Re}(\theta_{0}) \geq 0 \) then \( \pi_{1}^{*}(\theta_{0}) \) is a finite constant which cannot influence the domain of analyticity of \( \chi^{*}_{+}(\phi,\theta_{0}) \) in \( \phi \) since \( \pi_{1}(y) \) is a measure distributed on the positive half line of \( y \) and such that \( \pi_{1}^{*}(\theta) \) is analytic for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) if \( \beta^{*}(\theta) \) is analytic in this half plane. (See Lemma 6.5 where this is proven.)

Hence we need consider only the integral;

\[
I_{-+}(\phi,\theta_{0}) = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\phi x - \theta y} d\chi_{-}(x,y) \tag{6.10.10}
\]
\[
\begin{align*}
&< \int_0^\infty \int_{0^+} e^{-x\Re(\phi) - y\Re(\theta_0)} d\chi_-(x,y) \tag{6.10.11} \\
\quad = \int_0^\infty \int_{0^+} e^{-x\Re(\phi)} d\pi(x,y) \tag{6.10.12} \\
&\text{if } \Re(\theta_0) = 0 \\
\quad < \int_0^\infty \int_{0^+} d\pi(x,y) \tag{6.10.13} \\
&\text{if } \Re(\phi) \leq 0. \\
\quad \leq 1 \tag{6.10.14}
\end{align*}
\]

since \( \pi(x,y) \) is a distribution.

Hence \( I_{-\phi}(\phi,\theta_0) \) exists at least for all finite \( \phi \)
s.t. \( \Re(\phi) \leq 0. \)

Hence by the Corollaries to Theorems 5.3.2 and 5.3.3, this
integral defines a continuous function if \( \phi \) is finite and
\( \Re(\phi) \leq 0 \) which is also analytic if \( \phi \) is finite and
\( \Re(\phi) < 0. \)

However;

\[
\chi_{-\phi}(\phi,\theta_0) = \pi_1^*(\theta_0) + I_{-\phi}(\phi,\theta_0) \tag{6.10.15}
\]

whence it is clear that if \( \Re(\theta_0) = 0, \chi_{-\phi}(\phi,\theta_0) \) is
analytic in \( \phi \) for all finite \( \phi \) such that \( \Re(\phi) < 0 \)
and continuous from within and within the region where \( \phi \) is
finite and \( \Re(\phi) \leq 0. \)

**Corollary 3.** A similar result to that of Corollary 2 is
true for \( (n)\chi_{-\phi}(\phi,\theta_0) \) for any \( n = 1,2,\ldots \).
**Proof:** Define for some typical \( n \), \((n)I_+^{\star}(\phi, \theta_0)\) and 
\((n)\pi^{\star}_1(\theta)\) in terms of \((n)\pi(x, y)\) and 
\((n)\pi^{\star}_1(y)\) and repeat the reasoning used to prove Corollary 2.

**Definition 6.11.**

\[ \chi_1^{\star}(\phi, \theta) = \chi_{- -}^{\star}(\phi, \theta) + \chi_{- +}^{\star}(\phi, \theta) \]  
(6.11.1)

**Lemma 6.11.** The Laplace Transform \( \chi_1^{\star}(\phi, \theta) \) is continuous in \( R \) and analytic in \( D \) where \( R \) and \( D \) are as follows.

\[ R : \{ (\phi, \theta) \text{ s.t. } \text{Re}(\theta) \geq 0, \text{Re}(\theta+\phi) < 0, -\infty < \text{Im}(\phi), \text{Im}(\theta) < \infty \} \]  
(6.11.2)

\[ D : \{ (\phi, \theta) \text{ s.t. } \text{Re}(\theta) > 0, \text{Re}(\theta+\phi) < 0, -\infty < \text{Im}(\phi), \text{Im}(\theta) < \infty \} \]  
(6.11.3)

**Proof:** Clearly in view of (6.11.1), \( \chi_1^{\star}(\phi, \theta) \) will be continuous in at least the intersection of the regions of continuity of \( \chi_{- -}^{\star}(\phi, \theta) \) and \( \chi_{- +}^{\star}(\phi, \theta) \) and analytic in the intersection of the domains of analyticity of these. However by use of Lemmas 6.6, 6.7 and 6.10

\[ R = R_{- -} \cap R_{- +} \]  
(6.11.4)

\[ D = D_{- -} \cap D_{- +} \]  
(6.11.5)

where the subscripts \(- -\) and \(- +\) identify the corresponding transforms. Hence the assertion is true.
Corollary 1. The above Lemma is true for all $(n)\chi^*(\phi, \theta)$.

Proof: By Corollary 1 to Lemmas 6.6, 6.7 and 6.10 all the
$(n)\chi^*- (\phi, \theta)$ and $(n)\chi^+ (\phi, \theta)$ for $n = 1, 2, \ldots$ are continuous and analytic in such regions or domains that

$$(n)_R = (n)_{R-} \cap (n)_{R+} \supseteq R \quad (6.11.6)$$

and

$$(n)_{D} = (n)_{D-} \cap (n)_{D+} \supseteq D \quad (6.11.7)$$

for all $n = 1, 2, \ldots$, where $R$ and $D$ are defined by 6.11.2 and 6.11.3 respectively.

Corollary 2. If $\theta = \theta_0$ be a finite constant and such that $Re(\theta_0) = 0$, then $\chi^*(\phi, \theta_0)$ is univariate analytic in $\phi$ for all finite $\phi$ such that $Re(\phi) < 0$.

Proof:

$$\chi^*(\phi, \theta_0) = \chi^*(\phi, \theta_0) + \chi^+ (\phi, \theta_0) \quad (6.11.8)$$

However by Corollary 2 to Lemma 6.7, $\chi^*(\phi, \theta_0)$ is analytic in $\phi$ if $\phi$ is finite and $Re(\phi) < 0$ whilst by Corollary 2 to Lemma 6.10 $\chi^+ (\phi, \theta_0)$ is also analytic in $\phi$ if $Re(\phi) < 0$ and $|\phi| < \infty$. Hence the result is true.
Remark: Obviously \( \chi^*(\phi, \theta_0) \) is continuous in \( \phi \) at least if \( |\phi| < \infty \) and \( \text{Re}(\phi) \leq 0 \) since the two defining integrals both converge when \( \text{Re}(\phi) = 0 \).

**Corollary 3.** A similar result to that given by Corollary 2 is true for each \( (n)\chi^*(\phi, \theta_0) \) for all \( n = 1, 2, \ldots \).

**Proof:** Write

\[
(n)\chi^*(\phi, \theta_0) = (n)\chi^*_{-}(\phi, \theta_0) + (n)\chi^*_{+}(\phi, \theta_0) \quad (6.11.9)
\]

and repeat the previous reasoning but proceeding from Corollaries 3 to Lemmas 6.7 and 6.10.

**Corollary 4.** If the distributions \( \beta(s) \) be such that its Laplace Transformation \( \beta^*(\theta) \) be analytic for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) \((\sigma > 0)\), then \( (n)\chi^*(\phi_0, \theta) \) is univariate analytic in \( \theta \) for all finite \( \theta \) such that \( \text{Re}(\theta) > -\sigma \) and \( \text{Re}(\theta + \phi_0) < 0 \), where \( \phi_0 \) is a constant such that \( \text{Re}(\phi_0) < 0 \).

Furthermore if the queue is ergodic then the limit \( \chi^*(\phi_0, \theta) \) exists and is an analytic function for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \).

**Remark:** This result is placed as a Corollary to Lemma 6.11 since it modifies and in a sense extends the domain of analyticity defined there. However it is in many ways a
consequence of the argument used to prove Lemma 6.5,

**Proof:** It was shown in the proof of Lemma 6.5 that if \( R(s) \) were such that \( R(\theta) \) was analytic for all finite \( \theta \) such that \( \Re(\theta) > -\sigma \) \( \sigma > 0 \) and real, then \( \exists \) two variables \( e(n) = y(n) - x(n) \) and \( \hat{e}(n) = y(n) - x(n) \) with distributions \( e(n) Q(e/x) \) and \( \hat{e}(n) Q(e/x) \) such that their Laplace Transformations \( e(n) Q*(\theta) \) and \( \hat{e}(n) Q*(\theta) \) were also analytic for all finite \( \theta \) s.t. \( \Re(\theta) > -\sigma \) for any \( n = 1, 2, \ldots \) and any \( x(n) \in (-\infty, \infty) \) and the properties of \( e(n) Q*(\theta) \) were used to establish the analyticity \( e(n) \pi(\theta) \) and when it exists \( e(n) \pi(\theta) \), for all finite \( \theta \) such that \( \Re(\theta) > -\sigma \).

We now use the properties of \( e(n) Q*(\theta) \) to prove this corollary. Consider the integral:

\[
(n) I_-(\phi, \theta) = \int_{x=-\infty}^{0-} \int_{y=x}^{\infty} \mathcal{L}^{-\phi x - \theta y} d(n) \pi(x, y) \quad (6.11.10)
\]

for any \( n = 1, 2, \ldots \)

\[
= \int_{x=-\infty}^{0-} \int_{\hat{\theta}=0}^{\infty} \mathcal{L}^{-\hat{\theta} e x - \theta e x} d(n) \pi(x, x + e) \quad (6.11.11)
\]

where \( \hat{e} = y(n) - x(n) \)

\[
= \int_{\infty}^{0} \int_{0}^{\infty} \mathcal{L}^{-\theta + \hat{\theta} e x - \hat{\theta} e x} d \{\pi(x), d \hat{Q}(e/x)\} \quad (6.11.12)
\]

where

\[
\pi(x) = \int_{\infty}^{x} d\pi(x', \infty) \quad (6.11.13)
\]
\begin{equation}
\int_{-\infty}^{0} \int_{0}^{\infty} e^{-\theta e} d_x \{\Pi(x)d^{\hat{\phi}}(e/x)\} = 0 \quad \text{if} \quad \Re(\theta + \phi) < 0.
\end{equation}

However equation (6.11.14) is of exactly the same type as equation (6.5.3.28) whence by a repetition of the reasoning used there we find;

\begin{equation}
\int_{-\infty}^{0} \{\hat{n}(e/x) - \hat{n}(\infty/x)\} d_x \Pi(x) = o(e^{-\sigma e}) \quad e \to \infty
\end{equation}

for all $n = 1, 2, \ldots$ .

Thus if $|\theta| < \infty$, $\Re(\theta) > -\sigma$, $|\phi| < \infty$ and $\Re(\theta + \phi) < 0$, we may write,

\begin{equation}
(n)(\phi_0, \theta) \leq \int_{0}^{\infty} e^{-\theta e} d_x \{\int_{0}^{\infty} Q(e/x) d_x \Pi(x)\}
\end{equation}

since the R.H.S. will exist. Thus by standard theorems, (see for example Widder [18] Chapter 2), $(n)(\phi, \theta)$ is an analytic function of $\theta$ for constant $\phi$ if $|\theta| < \infty$ and $-\sigma < \Re(\theta) < -\Re(\phi)$. Hence for any constant $\phi_0$ such that $\Re(\phi_0) < 0$ and any $n = 1, 2, \ldots$

\begin{equation}
(n)(\phi_0, \theta) = (n)(\phi_0, \theta) + (n)_{\Pi}(\theta)
\end{equation}

is an analytic function of $\theta$ if $|\theta| < \infty$ and $\Re(\theta) > -\sigma$ and $\Re(\phi_0 + \theta) < 0$ since;
(i) By Lemma 6.5, \( (n) \pi_1^*(\theta) \) and \( (n) \pi_0^*(\theta) \) (\( n > 0 \)) are analytic if \( |\theta| < \infty \) and \( \text{Re}(\theta) > -\sigma \).

(ii) \( (n) \pi_0 \leq 1 \) is a constant.

(iii) \( (n) I_-(\phi_0, \theta) \) has been shown to be analytic in \( \theta \) if \( |\theta| < \infty \) and \(-\sigma < \text{Re}(\theta) \leq 0 < -\text{Re}(\phi), \) if \( \text{Re}(\phi) < 0, \) and

(iv) By Lemma 6.11, \( (n) \chi_+^*(\phi_0, \theta) \) is analytic if \( \text{Re}(\theta) > 0, \text{Re}(\theta+\phi_0) < 0 \) and \( |\theta|, |\theta| < \infty. \) Thus \( (n) \chi_+^*(\phi_0, \theta) \) may be continued in \( \theta \) over the specified domain by ordinary Continuation. Hence the assertion of the Corollary is true for any \( n = 1, 2, \ldots \).

**Corollary 6.**

We may also show that if the queue is ergodic, the limit:

\[
\lim_{n \to \infty} \{ (n) \chi_+^*(\phi, \theta) \} = \chi_+^*(\phi, \theta)
\]

exists and is an analytic function of at least for all finite \( \phi \) and \( \theta \) such that \( \text{Re}(\theta) > -\sigma, \text{Re}(\phi) < 0 \) and \( \text{Re}(\theta+\phi) < 0. \)

Since this result is obtained by a virtual repetition of the arguments used to prove a near equivalent result in Lemma 6.5, we give only an outline. Thus;

(i) Observe that the foregoing demonstration of the analyticity of \( (n) \chi_+^*(\phi_0, \theta) \) in \( \theta \) for any \( n = 1, 2, \ldots \) and constant \( \phi_0 \) such that \( |\phi_0| < \infty \) and \( \text{Re}(\phi_0) < 0 \) if
Re(\theta + \phi_0) < 0, Re(\theta) > -\sigma \text{ and } |\theta| < \infty \text{ holds even if } x(0)^+ = y(0)^+ = e(0) = 0.

(ii) Note that if the queue is ergodic the state \( x^+ = y^+ = 0 \) is recurrent whence the distribution \( P_m \) exists for \( m = 1,2,... \) as previously asserted in the proof of Lemma 6.5.

(iii) Again remark that for all \( m = 1,2,... \) the \((m)\chi_*(\phi_0,\theta)\) all have the same domain of analyticity, which is the desired domain.

(iv) As before, observe that since the distribution \( P_m \) exists the domain of analyticity of the Laplace Transformation of the distribution \( \chi_-(x,y) \) which an arbitrary arrival would possibly encounter is only that of \((m)\chi_*(\phi,\theta)\) for any \( m = 1,2,... \). Hence it must be the asserted domain and the claim is true.

Remarks: 1. We may also show that if \( B(s) \) is a mixture of distributions composed of stages and a discontinuous distribution and such that its Laplace Transformation \( B^*(\theta) \) is analytic if \( |\theta| < \infty \) and \( Re(\theta) > -\sigma \), then for all \( n = 1,2,... \) \((n)\chi_+^*(\phi_0,\theta)\) and the limit \( \chi_+^*(\phi_0,\theta) \) if this exists, are analytic in \( \theta \) for constant \( \phi_0 \) s.t. \( |\phi| < \infty \) and \( Re(\theta + \phi_0) < 0 \) if \( |\theta| < \infty \) and \( Re(\theta) > -\sigma \),
This proof uses the method of stages and certain properties of the biassed sampling of the distributions composed of stages but makes no specific assumptions about ergodicity other than that it is meaningful to assume that a limiting distribution exists.

Lemma 6.12:

\[
\lim_{\Re(\phi) \to -\infty} \chi^*(\phi, \theta) \equiv -\pi_0 - \pi^*(\theta)
\]

(6.12.1)

at least for all finite \( \theta \) s.t. \( \Re(\theta) > 0 \).

Proof: Write \( \phi = -R + i\phi_1 \) and

\[
J_-(\theta) = \lim_{R \to \infty} \int_{-\infty}^{0-} \int_{-\infty}^{\infty} e^{(R-i\phi_1)x-\theta y} d\chi_-(x,y) \quad (6.12.2)
\]

\[
= \lim_{R \to \infty} \int_{-\infty}^{0-} \int_{-\infty}^{\infty} e^{(R-i\phi_1)x-\theta y} d\chi_-(x,y) - \pi_0 - \pi^*(\theta)
\]

(6.12.3)

Consider;

\[
|I_-(R+i\phi_1, \theta)| = \left| \int_{-\infty}^{0-} \int_{-\infty}^{\infty} e^{(R-i\phi_1)x-\theta y} d\chi_-(x,y) \right| \quad (6.12.4)
\]

\[
\leq \int_{-\infty}^{0-} \int_{-\infty}^{\infty} e^{-(R-\Re(\theta))} |x-\Re(\theta)(y-x)| d\chi_-(x,y)
\]

(6.12.5)

for all \(-\infty < \phi_1, \Im(\theta) < \infty\).

However \( \chi^*(\phi, \theta) \) is an analytic function for all \( \phi \) and \( \theta \) such that \( \Re(\phi) < 0, \Re(\theta) > 0 \) and \( \Re(\theta+\phi) < 0 \) by Lemma 6.11. Hence \( I_-(\phi, \theta) \) is also analytic in this domain since \( \pi^*(\theta) \) is analytic if \( \Re(\theta) > 0 \) by Lemma 6.5.
Therefore \( I_{-(R+i\phi)} \) is analytic if \( \text{Re}(\theta) > 0 \) and 
\(|\theta| < R \) for all \( R \). Also if \( |\theta| < \infty \), then

\[
\lim_{R \to \infty} \{e^{-(R-\text{Re}(\theta))|x|} \} = 0 \quad (6.12.6)
\]

for all \(|x| > 0 \) and

\[
\lim_{R \to \infty} \{|I_{-(R+i\phi)}|\} = 0 \quad (6.12.7)
\]

if \( \text{Re}(\theta) > 0 \) since \( y-x \geq 0 \) in the region of integration.

Hence (6.12.1) is true as asserted at least if \( |\theta| < \infty \) and \( \text{Re}(\theta) > 0 \).

**Lemma 6.13.**

\[
\chi^*_+(0,\theta) = \chi^*_-(0,\theta) \quad (6.13.1)
\]

for all finite \( \theta \) s.t. these functions are analytic.

**Proof:**

\[
\chi^*(\phi,\theta) = \chi^*_-(\phi,\theta) + \chi^*_+(\phi,\theta) \quad (6.13.2)
\]

by Definition 6.11.

Hence it suffices to show that;

\[
\chi^*_+(0,\theta) = 0 \quad (6.13.3)
\]

for all finite \( \theta \)

Consider the integral;

\[
\chi^*_+(0,\theta) = \int_0^\infty \int_{-\infty}^{0+} e^{-\theta y} d\chi_-(x,y) \quad (6.13.4)
\]
\[
\begin{align*}
&= \int_{0+}^{0} \int_{\infty}^{-\infty} e^{-\theta y} d\chi(x,y) - \int_{0+}^{\infty} e^{-\theta y} d\pi_1(y) \quad (6.13.5)
\end{align*}
\]

from the definitions of \( \chi(x,y) \) and \( \pi_1(y) \) given in Section 4.

Choose any finite \( \theta_0 \) s.t. \( \text{Re}(\theta_0) > 0 \) and note that both integrals in (6.13.5) exist and define analytic functions of \( \theta \) if \( |\theta - \theta_0| < \text{Re}(\theta_0) \) by Theorem (5.3.3). However the existence of the integral:

\[
\int_{-\infty}^{\infty} \int_{0+}^{0} |e^{-\theta y}| d\chi(x,y) \quad (6.13.6)
\]

if \( \text{Re}(\theta) > 0 \) implies that,

\[
\int_{x=-\infty}^{0} \int_{y=0+}^{\infty} e^{-\theta y} d\chi(x,y) = \int_{y=0+}^{\infty} \int_{y=0+}^{0} e^{-\theta y} d\chi(x,y) \quad (6.13.7)
\]

\[
= \int_{0+}^{\infty} e^{-\theta y} d\pi_1(y) \quad (6.13.8)
\]

by use of Fubini's Theorem and the definition of \( \pi_1(y) \) given in Section 4.

Hence it is clear that if \( \theta \) is finite and s.t. \( \text{Re}(\theta) > 0 \);

\[
\chi_{k+}^n(0, \theta) = 0 \quad (6.13.10)
\]

Hence this function is zero over the whole plane in \( \theta \), and it is meaningful to write;
\[ \chi^*_+(0,\theta) = \chi^*_-(0,\theta) \]

as asserted wherever these may be otherwise shown to be analytic.

**Corollary 1.**

\[ (n)\chi^*_+(0,\theta) = (n)\chi^*_-(0,\theta) \quad (6.13.11) \]

for all \( n = 1, 2, \ldots \).

**Proof:** The preceding arguments may be repeated using the definition of \( (n)\chi_-(x,y) \) and \( (n)\pi_1(y) \) as given in Section 4.

**Corollary 2.** \( \chi^*(0,\theta) \) is continuous in \( \mathbb{R} \) and analytic in \( D \) where

\[ R = \{ \theta \text{ s.t. } \Re(\theta) \leq 0, |\theta| < \infty \} \quad (6.13.12) \]

\[ D = \{ \theta \text{ s.t. } \Re(\theta) < 0, |\theta| < \infty \} \quad (6.13.13) \]

**Proof:** We have just shown that \( \chi^*_+(0,\theta) = \chi^*_-(0,\theta) \) for all finite \( \theta \) s.t. these exist. Thus;

\[ \chi^*_-(0,\theta) = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\theta y} d\chi_-(x,y) \quad (6.13.14) \]

for all finite \( \theta \).

However if \( \Re(\theta) \leq 0 \) and \(-\infty < \Im(\theta) < \infty \) then,
\[
\int_{-\infty}^{0+} \int_{-\infty}^{0+} e^{-\theta y} dx_-(x,y)\]

Hence by Fubini's Theorem;

\[
\chi_-(0, \theta) = \int_{-\infty}^{0+} e^{-\theta y} dy \int_{-\infty}^{0+} dx_-(x,y)
\]  \hspace{1cm} (6.13.15)

However, \( \int_{-\infty}^{0+} dx_-(x,y) \) is a \( \psi \) function on \(-\infty < y < 0\) and we may apply the Corollaries to Theorems 5.3.2 and 5.3.3 immediately to obtain the asserted result.

**Corollary 3.** A similar result is true for \( (n)\chi_-(0, \theta) \) for all \( n = 1, 2, \ldots \).

**Proof:** We may repeat the above reasoning identically for any \( n = 1, 2, \ldots \) and \( (n)\chi_*(0, \theta) \) by use of Corollary 1, etc.

**Lemma 6.14:** The Laplace Transformation \( \chi^*_+(-\mu, \mu+\omega) \) is analytic in the domain \( D \) and continuous in the region \( R \) if \( \mu > 0 \) is a real constant where

\[
R = \{ \text{all finite } \omega \text{ s.t. } \Re(\omega) > 0 \} \hspace{1cm} (6.14.1)
\]

\[
D = \{ \text{all finite } \omega \text{ s.t. } \Re(\omega) > 0 \} \hspace{1cm} (6.14.2)
\]

**Proof:** The transform \( \chi^*_+ (\phi, \theta) \) is analytic in \( D' \) and continuous in \( R' \) where;

\[
R' = \{ \text{all finite } \phi, \theta \text{ s.t. } \Re(\theta) \geq 0, \Re(\theta + \phi) \geq 0 \} \hspace{1cm} (6.14.3)
\]
and
\[ D' = \{ \text{all finite } \phi, \theta \text{ s.t. } \Re(\theta) > 0, \Re(\theta + \phi) > 0 \} \]
(6.14.4)
by Lemmas (6.1) and (6.2).

Continuity is assured by noting that if \(|\omega| < \infty, \Re(\omega) > 0\) the point \((-\mu, \mu + \omega) \in R'\) for all \(\mu > 0\).

Similarly analyticity in \(D\) is shown by noting that if \(|\omega| < \infty, \Re(\omega) > 0\) whence \(\omega \in D\), the point \((-\mu, \mu + \omega) \in D'\) for all \(\mu > 0\).

**Corollary 1.** \((n)\chi^*_\mu(-\mu, \mu + \omega)\) is continuous in \(R\) and analytic in \(D\) for any \(n = 0, 1, 2, \ldots\) where \(R\) and \(D\) are defined as above.

**Proof:** By Corollary 1 to Lemmas 6.1 and 6.2 all \((n)\chi^*_\mu(\phi, \theta)\) are analytic and continuous in \(D'\) and \(R'\) as defined above for any \(n = 1, 2, \ldots\) whence preceding proof may be repeated for any \(n\). Hence the assertion is true.

**Lemma 6.15.** The Laplace Transform \(\chi^*_\mu(-\mu, \mu + \theta + \phi)\) is analytic in \(D\) and continuous in \(R\) where;

\[ D = \{ (\phi, \theta) \text{ s.t. } \Re(\theta + \phi) > 0, |\phi| < \infty, |\theta| < \infty \} \quad (6.15.1) \]

\[ R = \{ (\phi, \theta) \text{ s.t. } \Re(\theta + \phi) > 0, |\phi| < \infty, |\theta| < \infty \} \quad (6.15.2) \]

**Proof:** Write \(\theta + \phi = \omega\) note that if \(|\phi| < \infty, |\theta| < \infty\) then \(|\omega| = |\theta + \phi| < \infty\), and use Lemma (6.14). The result
follows immediately.

**Corollary 1.**\( \chi^*_\pm(-\mu,\mu+\theta+\phi) \) is continuous in \( R \) and analytic in \( D \) for any \( n = 0,1,2,... \) where \( R \) and \( D \) are defined above.

**Proof:** Use Corollary 1 to Lemma 6.14, write \( \theta+\phi = \omega \), note that if \( |\phi| < \infty \) and \( |\theta| < \infty \) then \( |\omega| = |\theta+\phi| < \infty \) and the result follows at once.

**Remark:** The analyticity referred to in Lemma (6.15) is clearly bivariate analyticity since the substitution \( \omega = \theta+\phi \) into a function analytic in \( \omega \) defines all partial derivatives with respect to \( \theta \) and \( \phi \) at all points where the function is analytic in \( \omega \).

**Lemma 6.16.**

\[
\lim_{\text{Re}(\omega) \to \infty} \{ \chi^*_\pm(-\mu,\mu+\omega) \} = \pi_0 \quad (6.16.1)
\]

if \( \mu > 0 \) and real.

**Proof:** Although an independent proof is easily constructed, this result follows as a direct consequence of Lemma 6.19. Therefore see Corollary 2 to that Lemma.

**Corollary 1.**

\[
\lim_{\text{Re}(\omega) \to \infty} \{ (n) \chi^*_\pm(-\mu,\mu+\omega) \} = (n) \pi_0 \quad (6.16.2)
\]

if \( \mu > 0 \) and real.
Lemma 6.17. The Laplace Transformation \( \chi^*(z-\mu, \mu+\omega-z) \) is bivariate analytic in \( z \) and \( \omega \) in the domain \( D \) and continuous in both \( z \) and \( \omega \) in the region \( R \) where

\[
R: \{(z, \omega) \text{ s.t. } \Re(\omega) \geq 0, \Re(z) \leq \Re(\omega)+\mu, |\omega| < \infty, |z| < \infty\}
\]

\[
(6.17.1)
\]

\[
D: \{(z, \omega) \text{ s.t. } \Re(\omega) > 0, \Re(z) < \Re(\omega)+\mu, |\omega| < \infty, |z| < \infty\}
\]

\[
(6.17.2)
\]

if \( \mu > 0 \) is a real constant.

Proof: The analyticity of \( \chi^*(z-\mu, \mu+\omega-z) \) in \( D \) arises from that of \( \chi^*(\phi, \theta) \) in \( D' \) where:

\[
D' = \{(\phi, \theta) \text{ s.t. } \Re(\theta+\phi) > 0 \text{ if } \Re(\phi) \leq 0 \text{ and } \Re(\theta) > 0 \text{ if } \Re(\phi) > 0 \text{ and } |\phi| < \infty, |\theta| < \infty\}
\]

\[
(6.17.3)
\]

from Lemma 6.2, since if \( \Re(z) \leq \mu, \Re(\phi) = \Re(z-\mu) \leq 0 \) and \( \Re(\theta+\phi) = \Re(\omega) > 0 \) is necessary, or if \( \Re(z) > \mu, \Re(\phi) = \Re(z-\mu) > 0 \) and \( \Re(\theta) = \Re(\mu+\omega-z) > 0 \) is necessary, whence \( \Re(z) < \Re(\mu+\omega) \) is necessary. However, these constraints define \( D \) as defined in (6.17.2) above. Further, the constraints are also sufficient since the substitutions \( \phi = z - \mu \) and \( \theta = \mu + \omega - z \) permit all derivatives and partial derivatives with respect to \( \omega \) and \( z \) to be well defined for all \( (z, \omega) \in D \) since \( D \) corresponds with \( D' \) where all derivatives and partial derivatives with respect to \( \phi \) and \( \theta \) are well defined. Continuity follows in the same manner from the definitions of a \( R' \) s.t. \( \chi^*(\phi, \theta) \) shall be continuous. See Lemma 6.1. Hence the assertion is true.
Corollary 1. The Laplace Transformations \( (n)X^\pm(z-\mu, \mu+\omega-z) \) are analytic in \( D \) and continuous in \( R \) defined above for all \( n = 0,1,2,\ldots \).

Proof: We use Corollaries to Lemmas (6.1) and (6.2) to define \( (n)_R \) and \( (n)_D \) where

\[
(n)_R \supseteq R
\]  

(6.17.4)

and

\[
(n)_D \supseteq D
\]  

(6.17.5)

for all \( n = 0,1,2,\ldots \) where \( R \) and \( D \) are defined by equations (6.17.1) and (6.17.2) and the \( (n)X^\pm(\phi, \theta) \) are analytic if \( (\phi, \theta) \in (n)_D \) and the \( (n)X^\pm(\phi, \theta) \) are continuous if \( (\phi, \theta) \in (n)_R \).

Hence we may repeat the above arguments for any \( n = 0,1,2,\ldots \) and the desired results follow in an obvious way.

Lemma 6.18. The asymptotic limit:

\[
\lim_{\Re(z) \to -\infty} \{X^\pm(z-\mu, \mu+\omega-z)\} = \pi^+ m(\omega) \]  

(6.18.1)

may be defined for all finite \( \omega \) s.t. \( \Re(\omega) \geq 0 \) where

\[
m(\omega) = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \int_{\infty}^{x-} e^{-\omega x} dX_+(x,y) \]  

(6.18.2)

where \( \mu > 0 \) and real.
Remarks: 1. If the integral;

\[ m = \int_0^\infty \int_{x^-}^{x^+} d\chi_+(x,y) \quad (6.18.3) \]

is zero, then the distributions \( \pi(x,y) = \chi(x,y) \) and \( \chi_+(x,y) \) are such that the line \( 0 < x = y < \infty \) is a set of probability measure zero and iff this is true then

\[ m(\omega) = 0 \]

for all \( \omega \).

2. It remains to be shown that \( m = 0 \) in certain specific cases dealt with in later sections.

3. We note the obvious requirements that \( m = 1 - \pi_\omega \leq 1 \) and that if \( \text{Re}(\omega) > 0, \ |m(\omega)| < 1 \) whilst

\[
\lim_{\text{Re}(\omega) \to \infty} \{m(\omega)\} = 0
\]

is clearly implied by Lemma (6.16).

Proof: Write;

\[
\chi_+(z-\mu,\mu+\omega-z) = \int \int e^{(\mu-z)x+(z-\mu-\omega)y} d\chi_+(x,y) = \int \int d\chi_+(x,y)
\]

\[ + \int \int e^{(\mu-z)x+(z-\mu-\omega)y} d\chi_+(x,y) \]

\[ + \int \int e^{(\mu-z)x+(z-\mu-\omega)y} d\chi_+(x,y) \]

\[ + \int \int e^{(\mu-z)x+(z-\mu-\omega)y} d\chi_+(x,y) \]
where \( e > o \) but small. However, as \( e \to o \); \( \chi^+_{\mu}(z-\mu,\mu+\omega-z) \)
approaches
\[
\pi o + \int_{o}^{\infty} e^{-\omega x}d\chi^+_{\mu}(x,x) + I^+_\mu(z,\omega)
\]
\[=
\pi o + m(\omega) + I^+_\mu(z,\omega) \quad (6.18.5)
\]
where,
\[
I^+_\mu(z,\omega) = \int_{o}^{\infty} \int_{o}^{\infty} e^{(\mu-z)x+(z-\mu-\omega)y}d\chi^+_{\mu}(x,y) \quad (6.18.6)
\]
and \( I^+_\mu(z,\omega) \) is certainly analytic for all finite \( \omega \) such that \( \text{Re}(\omega) > o \) and \( z \) such that \( \text{Re}(z) < \text{Re}(\omega)+\mu \) by Lemma (6.17). However,
\[
|I^+_\mu(z,\omega)| \leq \int_{o}^{\infty} \int_{o}^{\infty} |e^{(\mu-y-x)}||e^{-\omega y}||e^{z(y-x)}|d\chi^+_{\mu}(x,y) \quad (6.18.7)
\]
\[
\leq \int_{o}^{\infty} \int_{o}^{\infty} e^{\text{Re}(z)(y-x)}d\chi^+_{\mu}(x,y) \quad (6.18.8)
\]
if \( \text{Re}(\omega) \geq o \), since \( \mu > o \), \( y - x > o \) and \( y > o \), whence,
\[
\begin{align*}
|e^{\mu(y-x)}| &= e^{\mu(y-x)} \leq 1 \\
|e^{-\omega y}| &\leq 1 \\
|e^{z(y-x)}| &\leq e^{\text{Re}(z)(y-x)}
\end{align*} \quad (6.18.9)
\]
Hence if \( \text{Re}(z) = -R \) where \( R \) is real and arbitrarily large finite then;
\[
\lim_{\text{Re}(z) \to -\infty} |I^+_\mu(z,\omega)| \leq \lim_{R \to \infty} \int_{o}^{\infty} \int_{o}^{\infty} e^{-R(y-x)}d\chi^+_{\mu}(x,y) \}
\]
\[= 0 \quad (6.18.10)
\]
since \( y - x > o \).
Hence,

$$\lim_{\text{Re}(z) \to -\infty} \{\chi_+^x(z-u, u+w-z)\} = \pi_0 + m(\omega)$$  \hspace{1cm} (6.18.11)

for all $|\omega| < \infty$ and $\text{Re}(\omega) \geq 0$ as asserted.

**Corollary 1.** The limit:

$$\lim_{\text{Re}(z) \to -\infty} \{\chi_+^x(z-u, u+w-z)\} = \chi_+^x(z, x)$$  \hspace{1cm} (6.18.12)

where

$$(n)m(\omega) = \int_{-\infty}^{\infty} e^{-\omega x} d_x (n)\chi_+(x, x)$$  \hspace{1cm} (6.18.13)

if $|\omega| < \infty$ and $\text{Re}(\omega) \geq 0$, for any $n = 1, 2, 3, \ldots$.

**Proof:** Use the Corollary to Lemma 6.17 in lieu of 6.17 itself and repeat identically the above reasoning for any $n = 1, 2, \ldots$ by defining an $(n)I_+(z, \omega)$ analogous to the $I_+(z, \omega)$ of (6.18.6), in terms of $(n)\chi_+(x, y)$.

**Lemma 6.19.**

$$\lim_{\text{Re}(\theta) \to \infty} \{\chi_+^x(\phi, \theta)\} = \pi_0$$  \hspace{1cm} (6.19.1)

for all finite $\phi$.

**Proof:** By Lemma 6.2 $\chi_+^x(\phi, \theta)$ is analytic for all finite $\phi$ and $\theta$ such that $\text{Re}(\theta) > 0$ if $\text{Re}(\phi) > 0$ and $\text{Re}(\theta + \phi) > 0$ if $\text{Re}(\phi) < 0$. Hence we may choose any finite $\phi$ and write;
\[ \chi^*_+(\phi, R+i\theta_i) = \pi_0 + I_+(\phi, R+i\theta_i) \] (6.19.2)

where

\[ \theta = R+i\theta_i \text{ with } R > 0 \]

\[ I_+(\phi, \theta) = \int_0^\infty \int_{-\infty}^\infty e^{-\phi x - \theta y} d\chi_+(x,y) \] (6.19.3)

and

\[ -R < \text{Re}(\phi) < \infty \]

Clearly \( I_+(\phi, R+i\theta_i) \) and \( \chi^*_+(\phi, R+i\theta_i) \) are both analytic functions if \( \text{Re}(\phi) > -R \). Now consider;

\[ |I_+(\phi, R+i\theta_i)| \leq \int_0^\infty \int_{-\infty}^\infty e^{-\text{Re}(\phi)x - R y} d\chi_+(x,y) \] (6.19.4)

since \( x, y > 0 \)

\[ \leq \int_0^\infty \int_{-\infty}^\infty e^{-R(y-x)-(\text{Re}(\phi)+R)x} d\chi_+(x,y) \] (6.19.5)

However, for any finite \( \phi \) such that \( \text{Re}(\phi) > -R \)

\[ \lim_{R \to \infty} |e^{-(\text{Re}(\phi)+R)x}| = o(1) \]

for all \( x > 0 \) whilst

\[ \lim_{R \to \infty} e^{-R(y-x)} = o(1) \] (6.19.6)

for all \( y-x > 0 \)

Hence,

\[ \lim_{R \to \infty} |I_+(\phi, R+i\theta_i)| = o \] (6.19.7)

for all finite \( \phi \) and Lemma is true as asserted.
Corollary 1.

\[ \lim_{\text{Re}(\theta) \to \infty} \left\{ \chi^+_n(\phi, \theta) \right\} = (n) \pi_0 \]

for all \( n = 1, 2, \ldots \).

**Proof:** By the Corollary to Lemma 6.2 \( (n) \chi^+_n(\phi, \theta) \) is analytic in \( (n) D \supseteq D \) (where \( \chi^+_n(\phi, \theta) \) is analytic in \( D \)). We define an \( (n) I_+(\phi, \theta) \) by use of \( (n) \chi^+_n(\phi, \theta) \) in equation (6.19.3) and repeat identically the same reasoning to obtain the equivalent result, as asserted.

Corollary 2. Lemma 6.16 is true as asserted.

**Proof:** Write;

\[ \phi = -\mu \]

\[ \theta = \mu + \omega \]

in \( \chi^+_n(\phi, \theta) \) and take;

\[ \lim_{\text{Re}(\omega) \to \infty} \left\{ \chi^+_n(-\mu, \mu + \omega) \right\} \]

and the result follows immediately.

Corollary 3. Corollary to Lemma 6.16 is true as asserted.

**Proof:** Use Corollary 1 to this Lemma and repeat the reasoning used to prove Corollary 2 to this Lemma in the identical way.

**Definition:**

The following definition will be used in subsequent discussion.
Singularity Locus.

The individual singularities of multivariate complex functions are not isolated points but have loci which may be thought of as relationships between the complex variables such that wherever any one of these is satisfied the function is singular. We shall refer to such relationships as singularity loci and also note that they need not be continuous but may possess various types of discontinuities.

Lemma 6.20. (i) If the Laplace Transformation \( X^*(\mu, \omega + \phi) \) has any singularities, then these are points in the half-plane s.t. \( \text{Re}(\omega) < 0 \) and \( |\omega| < \infty \).

Proof: By Lemma 6.14 \( X^*(\mu, \omega + \phi) \) is analytic if \( \text{Re}(\omega) > 0 \) and \( |\omega| < \infty \). Hence any singularities in the finite plane must lie in \( \text{Re}(\omega) < 0 \) if these exist at all.

Lemma 6.20. (ii) The singularity loci of the Laplace Transformation \( X^*(\mu, \omega + \phi) \) are hyperplanes of the form:

\[
\theta + \phi = -r_i \quad (6.20.1)
\]

Where,

\( \text{Re}(r_i) > 0 \) and \( |r_i| < \infty \)

for all \( i = 1, 2, \ldots \).

Proof: Write the singular points of \( X^*(\mu, \omega + \phi) \) as \( \omega = -\{r_i\} \) \( i = 1, 2, \ldots \) and substitute \( \theta + \phi = \omega \).
Remarks. 1. These particular Singularity Loci are possibly continuous since $\theta + \phi$ is continuous.

2. These hyperplanes divide the space $C^2 (\equiv E^4$ here) into a system of disjoint domains. If we define a two dimensional basis space on the real parts of $\phi$ and $\theta$ (i.e., a $\text{Re}(\phi)$, $\text{Re}(\theta)$ plane), these hyperplanes project as a system of parallel straight lines inclined at $135^\circ$ to the $\text{Re}(\phi)$ axis. We also note that the spaces between these lines are the bases of a system of open tubes within each of which $\chi^*(\mu, \mu + \theta + \phi)$ is again analytic. However, it is known that if a function is analytic in a tube that tube must be convex — i.e., it must have a convex basis. (See for example, Bochner & Martin [1] Chapter 6). Thus if a system of open tube bases which are separated by common boundary lines (or Hyperplanes) which cover the whole of the basis plane (or space) these boundary lines (or Hyperplanes) must be straight (or first order in all variables) and contain all the singularities. Thus $\chi^*(\mu, \mu + \theta + \phi)$ is a very simple example of this sort of behaviour.

3. If such singularities are s.t. they may all be removed one at a time in some arbitrary order, the bases of the tubes which result must still be convex open areas. Thus if the system of tube bases cover the whole plane (or space) each of the singularity loci must be a straight line (or other first order function) which bisects the plane (or space.)
4. If we have multivariate Laplace-Fourier integral of a measure, in a complex space $\mathbb{C}_n$ it appears that since

$$|z^{-\sum_1^i \theta_1 x_1}| = z^{-\sum_1^i \Re(\theta_1)x_1}$$

the domain wherein the defining integral converges must be a closed convex tube, whence the corresponding domain of analyticity is the corresponding open convex tube. If in addition;

(i) the singularities are all removable

or (ii) the singularities are all points in the finite plane of any one variable when each of the others is a complex constant, then the basis space may be covered by a system of necessarily convex tubes separated by linear hyperplanes, each of which bisect the basis space and the space $\mathbb{C}_n$.

We have, therefore, some prospect for the simplification of all classes of multivariate probability problems since the singularities are quite likely to be of this particularly simple form.

Lemma 6.21. (i) The singularity loci of the functions

$$D_z^j \{x_+^j(z-\mu,\mu+\theta-\phi-z)\} \quad \text{for all finite } j = 1, 2, \ldots \quad \text{are at } z=0$$

those of,

$$x_+^j(-\mu,\mu+\theta+\phi),$$

where

$$D_z^j \equiv \frac{\partial^j x}{\partial z^j}$$
Proof: Consider firstly $\chi_+^*(\phi, \theta)$: By Lemma (6.2) this is analytic if $\text{Re}(\theta) > 0$ or if $\text{Re}(\theta + \phi) > 0$ when $\text{Re}(\phi) < 0$ and $-\infty < \text{Im}(\phi), \text{Im}(\theta) < \infty$. We cannot therefore choose any finite $\phi_0$ such that $\chi_+^*(\phi_0, \theta)$ is never bivariate analytic for any $\theta$. We may therefore fix $\phi = -\mu$ and consider $\chi_+^*(-\mu, \theta)$ as a function of $\theta$ only. Clearly there may be points in the $\theta$ plane such that $\chi_+^*(\phi, \theta) = -\mu$ is not bivariate analytic if $\theta$ takes any one of these values. Let the set of all such points be designated $\psi - \mu$. Then if $\theta \notin \psi - \mu$, $\chi_+^*(\phi, \theta) = -\mu$ is a bivariate analytic function and

$$\left(\frac{\partial^i \chi(\phi, \theta)}{\partial \phi^i \partial \theta^{j-i}}\right)_{\phi = -\mu}$$

exists for all $i = 0, 1, 2, \ldots, j$ and $j = 0, 1, 2, \ldots$

Thus if $\theta \notin \psi - \mu$,

$$(D_\phi - D_\theta)^j \{\chi_+^*(\phi, \theta)\}_{\phi = -\mu}$$

exists for all finite $j = 0, 1, 2, \ldots$ where;

$$D_\phi = \frac{\partial \chi}{\partial \phi} \quad D_\theta = \frac{\partial \chi}{\partial \theta}$$

and

$$(D_\phi - D_\theta)^j = \frac{j}{i^{j-1}} \binom{j}{i} (-1)^j D_\phi^i D_\theta^{j-i} \quad (6.21.2)$$

by the usual rules for the combination of linear differential operators. Consider now;

$$D_z^j \{\chi_+^*(p(z), q(z))\} = 0 \quad (6.21.2)$$

where

$$P(z) = z - \mu$$

$$q(z) = \mu + \theta^1 + \phi^1 - z$$

These functions will certainly exist if the sum $\theta^1 + \phi^1$ is such that $\chi_+^*(p(z), q(z))$ is bivariate analytic when $z = 0$. 
However we may write (6.21.2) in the form,

\[ \sum_{i=1}^{j} \left( \frac{\partial^{j} X_{\pm}(p,q)}{\partial p^{i} \partial q^{j-i}} \right) \left( \frac{\partial p}{\partial z} \right)^{i} \left( \frac{\partial q}{\partial z} \right)^{j-i} \]  

(6.21.3)

by the usual rules for the differentiation of a function of a function and after noting that since \( p \) and \( q \) are both linear in \( z \) all terms the form \( \frac{\partial^{2} p}{\partial z^{2}} \) are all zero.

However,

\[ \frac{\partial p}{\partial z} = 1, \quad \frac{\partial q}{\partial z} = -1 \]

whence; by comparison of (6.21.3) and (6.21.1) we note that;

\[ (D_{\phi} - D_{\theta})^{j}\{X_{\pm}(\phi, \theta)\} = -\mu \]

\[ \theta = \mu + \theta^{\prime} + \phi^{\prime} \]

\[ = D_{z}^{j}\{X_{\pm}(z - \mu, \mu + \theta^{\prime} + \phi^{\prime} - z)\} \]  

for all \( j = 0, 1, 2, \ldots \).

Thus we have shown that \( X_{\pm}(-\mu, \mu + \theta + \phi) \) and

\[ D_{z}^{j}\{X_{\pm}(z - \mu, \mu + \theta + \phi - z)\} \]  

for \( j = 1, 2, \ldots \) exist and are analytic except at a set of values of \( \theta + \phi \) where they are not defined. Clearly these shared exceptional values must be the \( r_{1} \) of equation (6.20.1) and the assertion is true.

**Remark:** We have not said that \( \exists \) only a finite number of singular points or that these are necessarily isolated, merely that the set \( \psi_{-\mu} \) exists and that if \( \theta \in \psi_{-\mu} \), then, \( \text{Re}(\theta) \leq \mu \).
Corollary 1. The singularities of the 
\[ D_z \{ \chi_+^*(z-\mu, \mu+\theta+\phi-z) \} \big|_{z=0} \] for \( j = 1, 2, \ldots \) all lie in 
Re(\theta+\phi) \leq 0. Hence all these functions are analytic in the domain s.t. Re(\theta+\phi) > 0, |\phi|, |\theta| < \infty.

Proof: This result follows immediately from the domain of analyticity of \( \chi_+^*(-\mu, \mu+\theta+\phi) \) defined by lemma (6.15).

Corollary 2. The result (6.21(i)) is true for any \( n = 1, 2, \ldots \) in the sense that;

\[ D_z \{ (n)^\chi_+^*(z-\mu, \mu+\theta+\phi) \} \big|_{z=0} \text{ has for any } j = 1, 2, \ldots \]

only the same singularity loci as

\[ (n)^\chi_+^*(-\mu, \mu+\theta+\phi) \]

Proof: By Corollaries 1 to Lemmas (6.2) and (6.14) we may define the set \{\( n \}_{r_i} \} for \( i = 1, 2, \ldots \) to be the set of singular points of \( (n)^\chi_+^*(-\mu, \mu+\theta+\phi) \) and repeat the previous reasoning to relate these to \( (n)^\psi_\mu \) where this is the set of points in the \( \theta \) plane such that \( (n)^\chi_+^*(\phi, \theta) \) is not bivariate analytic when \( \phi = -\mu \).

Lemma 6.21 (ii) For all \( \mu > 0 \) and \( j = 1, 2, \ldots \),

\[ D_z \{ \chi_+^*(z-\mu, \mu+\theta+\phi-z) \} \big|_{z=0} \] is continuous in the region \( R \) where;

\[ R = \{ (\phi, \theta) \text{ s.t. } \Re(\theta+\phi) > 0 \text{ and } |\phi| < \infty, |\theta| < \infty \} \]
Proof: Since we have shown \( D_z^j \{ \chi_+(z-\mu, \mu+\theta+\phi-z) \} \) is analytic for all \( j = 1, 2, \ldots \), if \( \text{Re}(\theta+\phi) > 0 \), it suffices to show that these derivatives maintain continuity as \( \text{Re}(\theta+\phi) \downarrow 0 \).

If the kernel of the R.H.S. is absolutely integrable we may write:

\[
D_z^j \{ \chi_+(z-\mu, \mu+\theta+\phi-z) \}_{z=0} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y-x)^j e^{-\mu(y-x)} e^{-(\theta+\phi)y} d\chi_+(x,y)
\]

(6.21.4)

by differentiation under the integral sign. This is clearly justified if \( \text{Re}(\theta+\phi) > 0 \) and \( \mu > 0 \). However we now show that:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(y-x)^j e^{-\mu(y-x)} e^{-(\theta+\phi)y}| d\chi_+(x,y)
\]

(6.21.5)

exists for all \( \text{Re}(\theta+\phi) > 0 \)

since:

\[
|(y-x)^j e^{-\mu(y-x)} e^{-(\theta+\phi)y}|
\]

(6.21.6)

\[
< \frac{j!}{\mu^j} e^{-\text{Re}(\theta+\phi)y}
\]

which is finite for all \( \mu > 0 \), finite \( j \) and \( \text{Re}(\theta+\phi) > 0 \), because \( y \geq 0 \) in the integral.

Hence by application of Theorem (5.3.4), to the R.H.S. of (6.21.5) we see that all the \( D_z^j \{ \chi_+(z-\mu, \mu+\theta+\phi-z) \} \) are continuous functions for all finite integral \( j \) if \( \text{Re}(\theta+\phi) > 0 \) and \(-\infty < \text{Im}(\theta), \text{Im}(\phi) < \infty \) whence the Lemma is true as asserted.
Corollary 1. The equivalent result is true for all:

$$D_z \chi_j^{(n)}(z-\mu, \mu+\theta+\phi-z) \mid_{z=0}$$

for all $n = 1, 2, 3...$ and $j = 1, 2,...$

Proof: We may repeat the preceding reasoning for each $n = 1, 2,...$ and $(n) \chi_j(x,y)$.

Remarks: 1. The above results regarding the locations of singularity loci do not imply that the actual singularities are necessarily of the same type or order. Thus, for example, differentiation generally raises the order of a pole but does not relocate it.

2. We note that singularity loci of the type $\theta + \phi = \{-r_i\}$ for $i = 1, 2,...$ are for each $i$, in a sense "Almost Everywhere Pervasive" since one cannot find a finite $\theta = \theta_o$ say, such that one cannot determine a corresponding $\phi$ for each $i = 1, 2,...$ (Call this $\phi_{oi}$ where $\theta_o + \phi_{oi} = -r_i$), unless the locus has points of interruption of some kind.

Lemma 6.22

$$\lim_{\text{Re}(\omega) \to \infty} \{D_z \chi_j(z-\mu, \mu+\omega-z)\} \mid_{z=0} = H(j) \quad (6.22.1)$$

where,

$H(j) = 0$ if $j = 1, 2,...$

$= \pi_o$ if $j = 0$
Proof: By Lemma (6.16) \( H(0) = \pi_o \) as asserted and it remains to show that \( H(j) = 0 \) if \( j = 1, 2, \ldots \). However if \( \text{Re}(\omega) > 0 \) \( \int_D \{ \chi^*_+(z, \mu, \mu+\omega-z) \} z=0 \) is analytic for all \( j = 1, 2, \ldots \) and we may consider:

\[
D_z \{ \chi^*_+(z, \mu, \mu+\omega-z) \} z=0 = D_z \{ \pi_o \} + \int_D \{ \int_0^\infty e^{-\phi(x-\theta y) \xi}(x, y) \} z=0
\]

where \( \phi = z-\mu, \Theta = z+\omega-z \) and \( z^* = \max\{z, 0^+\} \) (6.22.2)

\[
= \int_0^\infty \int_0^\infty \int_0^\infty (y-x)^j e^{-\mu(y-x)} e^{-\omega y} d\chi_+(x, y) (6.22.3)
\]

since \( \pi_o \) is constant, for all \( j = 1, 2, \ldots \) whence by

\[
\lim_{\text{Re}(\omega) \to \infty} \{ D_z \{ \chi^*_+(z, \mu, \mu+\omega-z) \} \} z=0 \]

\[
\leq \lim_{\text{Re}(\omega) \to \infty} \{ \int_0^\infty \int_0^\infty \int_0^\infty (y-x)^j e^{-\mu(y-x)} e^{-\omega y} d\chi_+(x, y) \} (6.22.4)
\]

\[
\leq \lim_{\text{Re}(\omega) \to \infty} \{ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 e^{-\text{Re}(\omega)y} d\chi_+(x, y) \} (6.22.5)
\]

since \( \mu > 0, y-x > 0, y > 0 \) and by use of \( \omega = \theta+\phi \) in equation (6.21.6)

\[
= 0 (6.22.6)
\]

since,

\[
\lim_{\text{Re}(\omega) \to \infty} \{ e^{-\text{Re}(\omega)y} \} = 0 (6.22.7)
\]

for all \( y > 0 \)

Hence the R.H.S. of (6.22.3) is zero for all \( j = 1, 2, \ldots \)

and the Lemma is true as asserted.
**Corollary 1.** For each \( n = 1, 2, \ldots \)

\[
\lim_{\text{Re}(\omega) \to \infty} \{ D_z \mathcal{J} \{ (n) \chi(z, z + \omega - \mu) \} \} \big|_{z=0} = (n)_H(j)
\]

where

\[
(n)_H(j) = \begin{cases} 
(n)_{\pi_0} & \text{if } j = 0 \\
0 & \text{if } j = 1, 2,
\end{cases}
\]

**Proof:** If \( j = 0 \) Corollary 1 to Lemma (6.16) suffices. For \( j = 1, 2, \ldots \) we may repeat the preceding argument with the obvious minor changes.
7. The Ergodic Functional Equation for the $E^{|L|^2}$ Queue

For $L>1$, $K>1$, this functional equation is sufficiently general to subsume all those for the cases which we propose to investigate initially. We therefore give its derivation in full.

The interarrival time density function exists and is

$$\alpha(s) = \frac{(\lambda a)^{K-1}}{(K-1)!} e^{-\lambda a} da \text{ for } K = 1, 2, ... \quad (7.1)$$

where $\lambda$ is the mean arrival rate for each interarrival stage, whence

$$\tilde{\alpha}_1 = \frac{K}{\lambda} \text{ for } K \text{ stages} \quad (7.2)$$

Similarly the service time density function is,

$$\beta(s) = \frac{(\mu s)^{L-1}}{(L-1)!} e^{-\mu s} ds \quad (7.3)$$

whence

$$\tilde{\beta} = \frac{L}{\mu} \quad (7.4)$$

for $L$ stages where $\mu$ is the mean departure rate per service stage and $L = 1, 2, ...$

From (7.1) and (7.3) we obtain

$$\alpha^*(p) = \left(\frac{\lambda}{\lambda + p}\right)^K \quad (7.5)$$

$$\beta^*(p) = \left(\frac{\mu}{\mu + p}\right)^L \quad (7.6)$$

$$\beta^*(L, \tilde{\omega}, p) = \int_0^{\tilde{\omega}} e^{-ps} \frac{(\mu s)^{L-1}}{(L-1)!} e^{-\mu s} ds$$

$$= \frac{\mu}{\mu + p} \left\{ \beta^*(L-1, \tilde{\omega}, p) - \frac{(\tilde{\omega}^{L-1}}{(L-1)!} e^{-(\mu + p)\tilde{\omega}} \right\} \quad (7.7)$$

by integration by parts, and
\[ \beta^*(0,\tilde{\omega},p) = 1 \quad (7.8) \]

for all \(\tilde{\omega},p\) by equation (7.7) and direct evaluation of \(\beta^*(1,\tilde{\omega},p)\) 
and \(\beta^*(2,\tilde{\omega},p)\) etc.

We may now rewrite (4.48) in the form,

\[ \chi^*(\phi,\theta) = \alpha^* \left( -\theta + \phi \right) \beta^*(\theta) \chi^* (0,\phi) + I(\phi,\theta) - I(\theta,\phi) \quad (7.9) \]

where

\[ I(\phi,\theta) = \iiint_{0-} e^{-\phi x - \theta y} \beta^*(L,y-x,\phi) dx dy \quad (7.10) \]

\[ = \frac{\mu}{\mu + \phi} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\phi x - \theta y} \beta^*(L-1,y-x,\phi) dx dy \]

\[ - \int_{0}^{\infty} \int_{0}^{\infty} \{ \mu(y-x) \}^{L-1} \frac{e^{-\phi x - \theta y - (\mu+\phi)(y-x)}}{(L-1)!} dx dy \]

\[ = \left( \frac{\mu}{\mu + \phi} \right)^{L} \chi^* (\phi,\theta) - \sum_{j=0}^{\infty} \left( \mu(y-x) \right)^{L-1} \frac{e^{-\phi x - (\mu+\phi)y}}{j!} \]

by induction. \(I(\theta,\phi)\) may be obtained by permutation.

Consider now,

\[ \int_{0}^{\infty} \int_{0}^{\infty} (y-x)^{j} e^{-\alpha x - \beta y} \chi^* (x,y) = (D_{\alpha} - D_{\beta})^{j} \chi^* (\alpha,\beta) \quad (7.12) \]

where the operator \(D_{\alpha}\) is defined to mean;

\[ D_{\alpha} \{ \chi^* (\alpha,\beta) \} = \frac{d}{d\alpha} \left( \int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha x - \beta y} \chi^* (x,y) \right) \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha x - \beta y} \chi^* (x,y) \quad (7.13) \]

wherever this exists.

\(D_{\beta}\) is defined similarly, and \((D_{\alpha} - D_{\beta})^{j}\) may be expanded

by the Binomial Theorem and the resulting terms rearranged to

prove (7.12).
may be simplified by noting that:

\[
(C - D)^j \chi_*(\alpha, \beta) = D_z \{\chi_*(\alpha + z, \beta - z)\}_{z=0}^{j}
\quad (7.14)
\]

and this may be used in (7.11) to give,

\[
I(\phi, \theta) = \left(\frac{1}{\mu + \phi}\right)^L \chi_*(\phi, \theta) - \sum_{j=0}^{L-1} \left(\frac{1}{\mu + \phi}\right)^j D_z \{\chi_*(z - \mu, \mu + \phi - z)\}_{z=0}^{j}
\quad (7.15)
\]

wherever this exists.

Also;

\[
I(\theta, \phi) = \left(\frac{1}{\mu + \phi}\right)^L \chi_*(\theta, \phi) - \sum_{j=0}^{L-1} \left(\frac{1}{\mu + \phi}\right)^j D_z \{\chi_*(z - \mu, \mu + \phi - z)\}_{z=0}^{j}
\quad (7.16)
\]

wherever this exists.

The use of these substitutions in (7.13) and some rearrangement gives the general functional equation for the \(E_k \vert E_L \vert 2\) queue, namely;

\[
(\lambda - \theta - \phi)^K (\mu + \phi)^L \chi_*(\phi, \theta)
\]

\[
= C(\phi, \theta)
\quad (7.17)
\]

\[
= \left(\lambda^K \mu^L - (\lambda - \theta - \phi)^K (\mu + \phi)^L\right) \chi_*(\phi, \theta)
\]

\[
+ \lambda^K \mu^L \left[ \sum_{j=0}^{L-1} \left(\frac{1}{\mu + \phi}\right)^j D_z \{\chi_*(z - \mu, \mu + \phi - z)\}_{z=0}^{j} \right]
\]

\[
- \left(\mu + \theta\right)^L \sum_{j=0}^{L-1} \left(\frac{1}{\mu + \phi}\right)^j D_z \{\chi_*(z - \mu, \mu + \phi - z)\}_{z=0}^{j}
\]

where the "Centre Function" \(C(\phi, \theta)\) is as yet, completely undefined. A more convenient representation is;
\[(\lambda - \theta - \phi)^K L (\mu + \theta)^L \chi_-(\phi, \theta) = C(\phi, \theta)\]

\[= (\lambda^K L - (\lambda - \theta - \phi)^K L) (\mu + \theta)^L \chi_+(\phi, \theta)\]

\[+ M(\phi, \theta)\]

where,

\[M(\phi, \theta) = R(\phi, \theta) - R(\theta, \phi)\]

and,

\[R(\phi, \theta) = \lambda^K L (\mu + \phi)^L \sum_{j=0}^{L-1} \frac{(\mu + \theta)^j}{j!} D_z^j \chi_+(z - \mu, \mu + \theta + \phi - z)\]

and \(R(\theta, \phi)\) is defined by exchange of \(\theta\) and \(\phi\) in (7.20).
8. Ergodic Functional Equations for Queues M/M/2, E2/E2

These may be written down from (7.17) by inspection. Thus

for \( K=L=1 \) which corresponds with the queue M/M/2 we have;

\[
(\lambda - \theta - \phi)(\mu+\phi)(\mu+\theta) \chi_-^*(\phi, \theta) = c_{11}(\phi, \theta)
\]

\[
= (\lambda \mu - (\lambda - \phi)(\mu+\phi))(\mu+\theta)\chi_+^*(\phi, \theta)
\]

\[
+ \lambda \mu (\phi-\psi) \chi_+^*(-\mu, \mu+\theta+\phi)
\]

(8.1)

For \( E_2/E_2/2 \) we obtain;

\[
(\lambda - \theta - \phi)^2(\mu+\phi)^2(\mu+\theta)^2\chi_-^*(\phi, \theta) = c_{22}(\phi, \theta)
\]

\[
= (\lambda^2 \mu^2 - (\lambda - \phi)^2(\mu+\phi)^2(\mu+\theta)^2\chi_+^*(\phi, \theta)
\]

\[
+ \lambda^2 \mu^2 \left\{ (\mu+\phi)^2\chi_+^*(-\mu, \mu+\theta+\phi)+(\mu+\theta)D_z \{\chi_+^*(z-\mu, \mu+\theta+\phi-z)\} \right\}_{z=0}
\]

\[
- (\mu+\theta)^2\chi_+^*(-\mu, \mu+\theta+\phi)+(\mu+\phi)D_z \{\chi_+^*(z-\mu, \mu+\theta+\phi-z)\} \right\}_{z=0}
\]

(8.2)

For \( K \) unspecified and \( L=2 \) we obtain;

\[
(\lambda - \theta - \phi)^K(\mu+\phi)^2(\mu+\theta)^2\chi_-^*(\phi, \theta) = c_{K2}(\phi, \theta)
\]

\[
= (\lambda^K \mu^2 - (\lambda - \phi)^K(\mu+\phi)^2(\mu+\theta)^2\chi_+^*(\phi, \theta)
\]

\[
+ \lambda^K \mu^2 \left\{ (\mu+\phi)^2\chi_+^*(-\mu, \mu+\theta+\phi)+(\mu+\phi)D_z \{\chi_+^*(z-\mu, \mu+\theta+\phi-z)\} \right\}_{z=0}
\]

\[
- (\mu+\theta)^2\chi_+^*(-\mu, \mu+\theta+\phi)+(\mu+\theta)D_z \{\chi_+^*(z-\mu, \mu+\theta+\phi-z)\} \right\}_{z=0}
\]

(8.3)

where the notation \( c_{KL}(\phi, \theta) \) is used to designate the particular function in each case.
9. The Nature of $C(\phi, \theta)$

In this section we prove that $C(\phi, \theta)$ is a bivariate polynomial for all finite $K$ and $L$.

9.1 Preliminary Lemmata

Lemma 9.1.1 $M(\phi, \theta)$ is analytic in $D$ where

$$D = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta+\phi)>0 \} \quad (9.1.1)$$

and continuous in $R$ where;

$$R = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta+\phi)>0 \} \quad (9.1.2)$$

Proof: Consider $R(\phi, \theta)$ defined by equation (7.20). For any finite $L$, this is a finite sum of elements of the form;

$$\left(\mu+\theta\right)^j D_z^j \left\{ \chi_+^* (z-\mu, \mu+\theta+\phi-z) \right\}_{z=0} / j! \quad (9.1.3)$$

By use of Lemmas (6.21(i)) and (6.21(ii)) each of the $D_z^j \left\{ \chi_+^* (z-\mu, \mu+\theta+\phi-z) \right\}_{z=0}$ is analytic in $\theta+\phi$ and thus in both $\theta$ and $\phi$ within $D$ and continuous within $R$ for each $j = 0, 1, \ldots, L-1$. Hence $R(\phi, \theta)$ is a finite sum of functions which are analytic in $D$ and continuous in $R$. Thus $R(\phi, \theta)$ is analytic in $D$ and continuous in $R$.

Similarly $R(\theta, \phi)$ is continuous in $R$ and analytic in $D$ by exchange of variables. Hence;

$$M(\phi, \theta) = R(\phi, \theta) - R(\theta, \phi) \quad (9.1.4)$$

must be analytic in $D$ and continuous in $R$ and the lemma is true.

Lemma 9.1.2 $C(\phi, \theta)$ is analytic if $\text{Re}(\theta)>0$, $\text{Re}(\theta+\phi)>0$ and $\theta$ and $\phi$ are finite.

Proof: Consider the RHS of (7.17). Clearly this is analytic wherever both $\chi_+^* (\phi, \theta)$ and $M(\phi, \theta)$ are analytic. However, by Lemmas (6.2) and (9.1.1), $C(\phi, \theta)$ is analytic at least where
\( \chi_+^{*}(\phi, \theta) \) is analytic since \( D(9.1.1) \supset D(6.2) \) and \( D(6.2) \) is the asserted domain.

**Lemma 9.1.3** \( C(\phi, \theta) \) is continuous in the region such that 
\( \phi \) and \( \theta \) are finite, \( \text{Re}(\theta) \geq 0 \) and \( \text{Re}(\theta+\phi) \geq 0 \).

**Proof:** Consider the RHS of (7.17). Clearly \( C(\phi, \theta) \) is continuous wherever both \( \chi_+^{*}(\phi, \theta) \) and \( N(\phi, \theta) \) are continuous. However, 
\( R(9.1.1) \supset R(6.1) \) whilst \( R(6.1) \) is the asserted region. Hence the Lemma is true.

**Lemma 9.1.4** \( C(\phi, \theta) \) is analytic in the domain such that \( \phi \) and \( \theta \) are finite, \( \text{Re}(\theta) > 0 \) and \( \text{Re}(\theta+\phi) < 0 \).

**Proof:** Consider the LHS of (7.17). Clearly this is continuous wherever \( \chi_+^{*}(\phi, \theta) \) is continuous and analytic wherever \( \chi_-^{*}(\phi, \theta) \) is analytic. Hence by Lemma (6.11), \( C(\phi, \theta) \) is analytic in at least \( D(6.11) \) which is the asserted domain.

**Lemma 9.1.5** \( C(\phi, \theta) \) is continuous if \( \phi \) and \( \theta \) are finite and \( \text{Re}(\theta) > 0 \), \( \text{Re}(\theta+\phi) \leq 0 \).

**Proof:** Follows from the proof of Lemma (9.1.4) since \( R(6.11) \) is the asserted region.

9.2 Partial Analytic Continuation

**Theorem 9.2.1** \( C(\phi, \theta) \) is continuous in the region such that \( \phi \) and \( \theta \) are finite and \( \text{Re}(\theta) > 0 \) and analytic in the domain such that \( \phi \) and \( \theta \) are finite and \( \text{Re}(\theta) > 0 \).

**Proof:** It is more convenient to prove the analyticity first.

Consider any finite point \( (\phi_0, \theta_0) \) such that \( \text{Re}(\phi_0) < 0 \), \( \text{Re}(\theta_0) > 0 \) and \( \text{Re}(\phi_0 + \theta_0) = 0 \), and equation (7.17). Clearly if \( \theta_0 \) remains constant, then by Lemma (9.1.2) \( C(\phi, \theta_0) \) is analytic
in $\phi$ if $|\phi|<\infty$ and $\text{Re}(\phi)>-\text{Re}(\theta_0)$. Similarly by Lemma (9.1.3), $C(\phi, \theta_0)$ is continuous in $\phi$ if $|\phi|<\infty$ and $\text{Re}(\phi)>-\text{Re}(\theta_0)$.

However by Lemma (9.1.4), $C(\phi, \theta_0)$ is analytic in $\phi$ if $|\phi|<\infty$ and $\text{Re}(\phi)<-\text{Re}(\theta_0)$ whilst by Lemma (9.1.5), $C(\phi, \theta_0)$ is continuous in $\phi$ for all finite $\phi$ s.t. $\text{Re}(\phi)<-\text{Re}(\theta_0)$.

Hence by Schwartz Continuation in the plane of $\phi$ we deduce that $C(\phi, \theta_0)$ is analytic in $\phi$ if $|\phi|<\infty$ for any finite $\theta_0$ s.t. $\text{Re}(\theta_0)>0$.

Similarly if $\theta_0$ is finite complex constant such that $\text{Re}(\theta_0)<0$ and we choose finite points $(\phi_0, \theta_0)$ such that $\text{Re}(\phi_0+\theta_0)=0$, we may repeat the preceding arguments to show that $C(\phi_0, \theta)$ is analytic in $\theta$ if $|\theta|<0$ and $\text{Re}(\theta)>0$ for any finite $\phi_0$ s.t. $\text{Re}(\phi_0)<0$. It is also obvious that $C(\phi_0, \theta)$ is analytic in $\theta$ for all finite $\phi_0$ s.t. $\text{Re}(\phi_0)>0$ provided $|\theta|<\infty$ and $\text{Re}(\theta)>0$ since $\chi^*(\phi_0, \theta)$ and $M(\phi_0, \theta)$ are both analytic there. (See Lemma 9.1.2)

Thus $C(\phi, \theta)$ is univariate analytic in $\phi$ if $|\phi|<\infty$ provided $\theta$ is any constant such that $|\theta|<\infty$ and $\text{Re}(\theta)>0$ and $C(\phi, \theta)$ is univariate analytic in $\theta$ if $|\theta|<\infty$ and $\text{Re}(\theta)>0$ provided that $\phi$ is any constant s.t. $|\phi|<\infty$.

Hence by use of Hartog's Theorem (Th.5,3,2), $C(\phi, \theta)$ is bivariate analytic if $\phi$ and $\theta$ are finite and $\text{Re}(\theta)>0$.

To prove the continuity it clearly suffices to show that this exists from the domain where $\text{Re}(\theta)>0$ into the space where $\text{Re}(\theta)=0$, wherever $|\phi|, |\theta|<\infty$. However this follows directly from the continuity of $\chi^*(\phi, \theta)$ from the domain where $|\theta|<\infty$. 
and \( \text{Re}(\theta) > 0 \) to the space where \( \text{Re}(\theta) = 0, |\theta| < \infty \) for all finite \( \phi \) s.t. \( \text{Re}(\phi) > 0 \), the equivalent behaviour of \( \chi^*_-(\phi, \theta) \) if \( \text{Re}(\phi) < 0 \) and equation (7.17).

*Since \( M(\phi, \theta) \) is clearly continuous in both of these cases, if \( \text{Re}(\phi) > 0 \).

Hence the Lemma is true as asserted.

**Remark:** Clearly Theorem (9.2.1) is an example of an analytic continuation (in this case a Schwartz continuation) between two contiguous convex tubes (the bases of which are separated by a common and thus necessarily straight line in \( \text{Re}(\phi), \text{Re}(\theta) \) space) which produces a third tube which is also convex.

Thus there is (as yet) no obvious Analytic Completion

which may be used to further extend the domain of analyticity.

*See Bochner & Martin [1] for this notion.

**Theorem 9.2.2** \( C(\phi, 0) \) is analytic in \( \phi \) for all finite \( \phi \).

**Proof:** By Corollary 2 to Lemma (6.2), \( \chi^*_+(\phi, 0) \) is analytic for all finite \( \phi \) s.t. \( \text{Re}(\phi) > 0 \). It is also clear that \( \chi^*_+(\phi, 0) \) is continuous if \( \phi \) is finite and \( \text{Re}(\phi) \geq 0 \) by Lemma (6.1). Similarly by Lemma (9.1.1), \( M(\phi, 0) \) is continuous if \( \phi \) is finite and \( \text{Re}(\phi) > 0 \) and \( \text{Re}(\phi) > 0 \). Thus if \( \theta = 0 \) the RHS of (7.17) is analytic in \( \phi \) if \( |\phi| < \infty \) and \( \text{Re}(\phi) > 0 \) and continuous if \( |\phi| < \infty \) and \( \text{Re}(\phi) \geq 0 \).

Similarly by the properties of \( \chi^*_-(\phi, \theta) \) given in Lemma (6.11), the LHS of (7.17) is analytic in \( \phi \) if \( |\phi| < \infty \) and \( \text{Re}(\phi) < 0 \) and continuous if \( |\phi| < \infty \) and \( \text{Re}(\phi) < 0 \) if \( \theta = 0 \).

Hence by Schwartz continuation in \( \phi \), \( C(\phi, 0) \) is analytic in \( \phi \) if \( |\phi| < \infty \) and the assertion is true.
9.3 Representation Theorems

Theorem 9.3.1  If \( \theta_0 \) be any finite complex constant s.t. \( \Re(\theta_0) > 0 \) then we may write:

\[
C(\phi, \theta_0) = (\mu + \phi)^L D(\phi, \theta_0) \tag{9.3.1}
\]

where \( D(\phi, \theta) \) is bivariate analytic for all finite \( \phi \) and \( \theta \) s.t. \( \Re(\theta) > 0 \).

Proof: By Lemma (6.11), \( \chi_*(\phi, \theta) \) is analytic if \( \Re(\theta) > 0 \) and \( \Re(\theta + \phi) < 0 \) and \( \phi \) and \( \theta \) are finite. By Theorem (9.2.1), \( C(\phi, \theta) \) is analytic for all finite \( \phi \) and \( \theta \) s.t. \( \Re(\theta) > 0 \). Hence by use of the LHS of (7.17), \( \chi_*(\phi, \theta) \) must either be analytic where \( C(\phi, \theta) \) is now known to be analytic or it possesses at worst a pole of the form \( (\lambda - \theta - \phi)^{-K} \) in the finite domain s.t. \( \Re(\theta) > 0 \) for all finite \( \phi \) and \( \theta \). Thus the product:

\[
(\lambda - \theta - \phi)^K \chi_*(\phi, \theta) \tag{9.3.2}
\]

must be an analytic function if \( |\phi|, |\theta| < \infty \) and \( \Re(\theta) > 0 \). Hence the product:

\[
(\mu + \phi)^L (\lambda - \theta - \phi)^K \chi_*(\phi, \theta) \tag{9.3.3}
\]

must possess a zero of order \( L \) in the plane of \( \phi \) when \( \phi = -\mu \), at least for all finite \( \theta \) s.t. \( \Re(\theta) > 0 \). Also \( \chi_*(\phi, 0) < \infty \) if \( \Re(\phi) < 0 \), by Lemma (6.11). Hence (9.3.3) is also zero when \( \phi = -\mu \) if \( |\theta| < \infty \) and \( \Re(\theta) > 0 \). Hence the assertion of the theorem is true.

Theorem 9.3.2  If \( \theta_0 \) be any finite complex constant such that \( \Re(\theta_0) > 0 \), the \( D(\phi, \theta_0) \) is a polynomial in \( \phi \) and

\[
D(\phi, \theta_0) = 0(|\phi|^K) \tag{9.3.4}
\]

as \( |\phi| \to \infty \).
Proof: Clearly if \( \theta_0 \) be any finite constant such that \( \Re(\theta_0) > 0 \),
\( C(\phi, \theta_0) \) is analytic in \( \phi \) for all finite \( \phi \) by Theorems (9.2.1) and (9.2.2). Thus we may employ Louville's Theorem in the plane of \( \phi \) for any arbitrary suitable \( \theta_0 \). There are three situations, namely:

(i) \( -\pi/2 < \text{Am}(\phi) < \pi/2 \)

(ii) \( \text{Am}(\phi) = \pm \pi/2 \)

(iii) \( \begin{cases} -\pi \leq \text{Am}(\phi) < -\pi/2 \\ \pi/2 < \text{Am}(\phi) < \pi \end{cases} \)

Case (i). If as \( |\phi| \to \infty \), then for any finite \( \theta_0 \) s.t.
\( \Re(\theta_0) > 0 \)

\[
\lim_{|\phi| \to \infty} \{ C(\phi, \theta_0) \} \sim (-1)^{K+1} \phi^{K+L} (\mu+\theta_0)^L \epsilon_0 \left( \pi_0 + \pi_1 (\theta_0) + \pi_1^*(\theta_0) \right) \quad (9.3.5)
\]

Proof: By Lemma (6.3),

\[
\lim_{\Re(\phi) \to \infty} \{ \chi^*_+(\phi, \theta_0) \} = \pi_0 + \pi_1^*(\theta_0) + \pi_1^*(\theta_0) \quad (9.3.6)
\]

whilst by Lemma (6.22);

\[
\lim_{\Re(\theta_0 + \phi) \to \infty} \{ D_z \chi^*_+(z - \mu + \theta_0 + \phi - z) \}_{z=0} \quad (9.3.7)
\]

\[
= 0 \quad \text{if } j > 0, \text{ and integral}
\]

\[
= \pi_0 \quad \text{if } j = 0
\]

Thus by the use of these limits in the RHS of equation (7.17) we obtain (9.3.5). Hence the assertion of the theorem is true for case (i).

Case (ii) If as \( |\phi| \to \infty \), then for any finite \( \theta_0 \) s.t.
\( \Re(\theta_0) > 0 \);

\[
\lim_{|\phi| \to \infty} \{ C(\phi, \theta_0) \} \sim (-1)^{K+L} \phi^{K+L} (\mu+\theta_0)^L \epsilon_0 \quad (9.3.8)
\]
where,

$$|E(\theta_o)| \leq \int \int_{-\infty}^{\infty} e^{-\theta_o y} |d\chi_+(x,y)|$$  \hspace{1cm} (9.3.9)

**Proof:** Clearly if $\text{Re}(\theta_0 + \phi) > 0$ and $|\phi|, |\theta_o| \to \infty$ then;

$$0 \leq \xi^* \leq \chi_+(\mu, \mu + \theta_o + \phi) \leq 1$$  \hspace{1cm} (9.3.10)

by the properties of the defining integral. See proof of Lemma (6.14) for more details. Then;

$$|d_{\xi^*} \chi_+(z-y, y + \theta_o - z)_{\xi^*=0} \leq \frac{1}{\mu^j}$$  \hspace{1cm} (9.2.11)

for all $j = 1, 2, \ldots, L-1$

by the proof of Lemma (6.21(ii)).

Hence as $|\phi| \to \infty$ with $\text{Re}(\phi) = 0$ and $\theta_o$ finite;

$$M(\phi, \theta_o) \leq L^L \phi L \sim \tilde{M}(\theta_o) + o(|\phi|)$$  \hspace{1cm} (9.3.12)

where,

$$\tilde{M}(\theta_o) \leq \sum_{j=0}^{L-1} \left( \frac{\theta_o}{\mu} \right)^j$$  \hspace{1cm} (9.3.13)

by use of equations (7.19) and (7.20).

Also for any $\phi$ s.t. $\text{Re}(\phi) = 0$;

$$|\chi^*(\phi, \theta_o)| \leq \int \int_{-\infty}^{\infty} e^{-\theta_o y} |d\chi_+(x,y)|$$  \hspace{1cm} (9.3.14)

$$\leq \int \int_{-\infty}^{\infty} e^{-\theta_o y} |d\chi_+(x,y)|$$

$$< \infty$$

at least if $\theta_o$ is finite and $\text{Re}(\theta_o) > 0$.

Hence;

$$\lim_{|\phi| \to \infty} \{\chi^*(\phi, \theta_o)\} = E(\theta_o)$$  \hspace{1cm} (9.3.15)

$$\text{Re}(\phi) = 0$$
where $|E(\theta_0)|$ is bounded by (9.3.9) which exists. Thus by use of (9.3.12), (9.3.13) and (9.3.15) in the RHS of (7.17) we obtain (9.3.8) and:

$$D(\phi, \theta_0) = 0(|\phi|^K)$$  \hspace{1cm} (9.3.16)

as $|\phi| \to \infty$ with $\text{Re}(\phi) = 0$ if $|\theta_0| < \infty$ and $\text{Re}(\theta_0) > 0$.

Case (iii) If as $|\phi| \to \infty$, $\text{Re}(\phi) \to \infty$, then for all finite $\theta_0$ s.t. $\text{Re}(\theta_0) > 0$.

$$\text{Lim} \left\{ K(\phi, \theta_0) \right\} = 0(|\phi|^K)$$  \hspace{1cm} (9.3.17)

whence by use of this in the RHS of (7.17), we obtain (9.3.17).

Thus:

$$D(\phi, \theta_0) = 0(|\phi|^K)$$  \hspace{1cm} (9.3.19)

for all $-\pi < \text{Am}(\phi) < \pi$

and all finite $\theta_0$ s.t. $\text{Re}(\theta_0) > 0$ and by Louville's Theorem $D(\phi, \theta_0)$ is at most a polynomial in $\phi$ of order $K$ and the assertion is true.

**Theorem 9.3.3.** We may write:

$$D(\phi, \theta) = \sum_{m=0}^{K} d_m^{(\pm)}(\theta) \phi^m$$  \hspace{1cm} (9.3.20)

where for each $m = 0, 1, \ldots, K$, $d_m^{(\pm)}(\theta)$ is an analytic function for all finite $\theta$ s.t. $\text{Re}(\theta) > 0$. 
Proof: Clearly since $D(\phi, \theta)$ is bivariate analytic if $\phi$ and $\theta$
are finite and $\text{Re}(\theta) > 0$ we may write;

$$D(\phi, \theta) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} d_{mj} (\theta - \theta_0)^j \phi^m$$  \hspace{1cm} (9.3.21)

where $\theta_0$ is finite and s.t. $\text{Re}(\theta_0) > 0$, and this bivariate power series will certainly be convergent if

$$|\phi|^{<\infty}$$

$$|\theta - \theta_0| < r = \text{Re}(\theta_0)$$  \hspace{1cm} (9.3.22)

However,

$$D(\phi, \theta) = O(|\phi|^K)$$

for all $\theta$ s.t. (9.3.22) is convergent thus;

$$d_{mj} = 0$$  \hspace{1cm} (9.3.23)

for all $m \in [K+1, K+2, \ldots]$ and all $j \in [0, 1, \ldots]$.

Also we may continue $D(\phi, \theta)$ in an equivalence class of bivariate power series centred on the sequence of points $\{0, \theta_i\}$ with $\text{Re}(\theta_i) > 0$, for all $i = 1, 2, \ldots$ whence after equating the coefficients of the individual powers of $\phi$, it is apparent that this process is equivalent to the continuation of the series;

$$\sum_{j=0}^{\infty} d_{mj} (\theta - \theta_0)^j$$

for each $m = 0, 1, \ldots, K$, through the sequence of centres $\{\theta_i\}$ for all $i = 1, 2, \ldots$

Hence for each $m = 0, 1, \ldots, K$, there exists an equivalence class of power series which are convergent if centred on points within the domain in $\theta$ s.t. $|\theta|^{<\infty}$ and $\text{Re}(\theta) > 0$. However any
such equivalence class uniquely defines a function analytic in the domain, and we may write:

\[ d^{(+)}_m(\theta) \quad (9.3.24) \]

for \( m = 0,1, \ldots, K \) to denote the \( m \)th of these. Hence the representation is justified.

**Corollary.** For the \( E_K|E_L|2 \) queue \( \tilde{\eta}_1^*(\theta) = 0 \) for all \( \theta \) and all finite \( K \) and \( L \).

**Proof:** By use of Theorem (9.3.3) we may write;

\[ C(\phi,\theta_o) = (\mu+\phi)_o^L \sum_{m=0}^{K} d^{(+)}_m(\theta_o)\phi^m \quad (9.3.25) \]

where \( \theta_o \) is any finite \( \theta \) s.t. \( \text{Re}(\theta)>0 \). Hence by equation (9.3.5),

\[ d^{(+)}_K(\theta_o) = (-1)^{K+1}(\mu+\theta_o)_o^L[\pi_0+\pi_1^*(\theta_o)+\tilde{\eta}_1^*(\theta)] \quad (9.3.26) \]

whilst by use of (9.3.17) we obtain;

\[ d^{(+)}_K(\theta_o) = (-1)^{K+1}(\mu+\theta_o)_o^L[\pi_0+\pi_1^*(\theta_o)] \quad (9.3.27) \]

Hence if (9.3.26) and (9.3.27) are both true \( \tilde{\eta}_1^*(\theta) = 0 \) for all finite \( \theta_o \) s.t. \( \text{Re}(\theta)>0 \). Hence the measure \( \tilde{\eta}_1(\theta) = 0 \) for all \( \theta \) as asserted.

**Remarks:**

1. It will be obvious that if each of the \( d^{(+)}_m(\theta) \) used in equations (9.3.20) and (9.3.25) is analytic for all finite \( \theta \) then \( C(\phi,\theta) \) is an entire function.

2. Conversely if there exists a set of sequences of points \( \{\hat{\theta}_{mj}\} \) for \( m = 0,1, \ldots, K \) and \( j = 0,1, \ldots \) such that \( d^{(+)}_m(\theta) \) has singularities at \( \{\hat{\theta}_{mj}\} \) for \( j = 0,1, \ldots \), then the function \( D(\phi,\theta) \)
cannot be bivariate analytic since a bivariate power series representation cannot be based on any point \((\phi, \theta_{m_j})\) for any finite \(\phi\) and any \(\theta_{m_j}\) for \(m = 0, 1, \ldots, K\) and \(j = 0, 1, \ldots\) which will not have a zero radius of convergence in one or both of the variables. That is, all points \((\phi, \theta_{m_j})\) for \(m = 0, 1, \ldots, K\) and \(j = 0, 1, \ldots\) may be defined as end points of bivariate singular chains.

3. In a sense it is not obvious that the radius of convergence of a bivariate series representation must always be zero in \(\theta\) when centred on any point \((\phi, \theta_{m_j})\) for consider the function

\[
D(\phi, \theta) = d_0(\theta) + \phi d_1(\theta)
\]

when \(d_0(\theta)\) is entire and \(d_1(\theta)\) has a singular point \(\hat{\theta}\). Clearly \(D(0, \theta)\) is entire in \(\theta\). Thus a power series representation based on the points \((\phi, \theta_{m_j})\) will be convergent if either

\[
(1) \quad R_\phi = |\phi - \phi_o| = 0
\]

and \(R_\theta = |\theta - \hat{\theta}| < \infty\)

or \((ii)\) \(R_\phi = |\phi - \phi_o| \geq \varepsilon > 0\)

and \(R_\theta = |\theta - \hat{\theta}| = 0\)

but it is also clear that (ii) is the more natural definition.

4. If the \(\{\theta_{m_j}\}\) do, in fact, exist, then it is obvious that these completely specify the singular points of \(D(\phi, \theta)\) and thus \(C(\phi, \theta)\).

5. The remainder of this section will be concerned with the proof that no such points may exist. Two courses are open to us, namely:
(i) The use of the principle of the Permanence of the Functional Equation (See for example Hille [2] Vol.II) which we shall use or;

(ii) Direct operation on the transform $\chi^*_-(\phi, \theta)$ to show that the LHS of (7.17) is analytic if $|\phi|, |\phi|<\infty$ and $\text{Re}(\phi)<0$, followed by an Analytic Completion. See remarks after proof of Theorems (9.5) and (9.7) where this notion is explained.

See also Section 17.

Theorem 9.4 \[ C(f(\omega), \omega-f(\omega)) = M(f(\omega), \omega-f(\omega)) \] for all $\omega$ iff; \[ f(\omega) = \mu((\frac{1}{\lambda-\omega})^{K/L-1}) \] if $K/L \neq I$ an integer or \[ f(\omega) = \mu((\frac{1}{\lambda-\omega})^{i2\pi l/L-1}) \] if $K/L = I$ an integer and $l = 0, 1, \ldots, L-1$

and $i = \sqrt{-1}$

Proof: We prove only the result for $K/L \neq I$ since the other case follows in an obvious way.

Consider the expression;

$F(\phi, \theta) = (\lambda^K \mu^L - (\lambda-\phi)^K (\mu+\phi)^L (\mu+\theta)^L \chi^*_+(\phi, \theta)) \tag{9.4.4}$

and substitute

$\phi = f(\omega)$

$\theta = \omega - f(\omega) \tag{9.4.5}$

when $f(\omega)$ is defined by (9.4.2). We show that;
\[ F(f(\omega), \omega - f(\omega)) = 0 \quad (9.4.6) \]

for all \( \omega \) since;

(i) The multiplier;
\[ \lambda^K \mu^L - (\lambda - \omega)^K \mu^L \left(\frac{\lambda}{\lambda - \omega}\right)^K = 0 \quad (9.4.7) \]

for all \( \omega \).

(ii) The multiplier;
\[ (\omega - \mu)\left(\frac{\lambda}{\lambda - \omega}\right)^{K/L} \mu^L < \infty \]

in the finite plane of \( \omega \) except the point \( \omega = \lambda \).

(iii) There exists a domain \( D^1 \) such that for all \( \omega \in D^1 \)
\( \chi_+^*(f_j(\omega), \omega - f_j(\omega)) \) is an analytic function of \( \omega \) for any branch

of \( f(\omega) \), namely \( f_j(\omega) \) such that \( j = 0, 1, \ldots, L-1 \) where;
\[ D^1 = \{ \text{All finite } \omega \text{ s.t. } \text{Re}(\omega) \geq 2\lambda \} \quad (9.4.8) \]

since clearly;
\[ \left| \frac{\lambda}{\lambda - \omega} \right| < 1 \quad (9.4.9) \]

if \( \text{Re}(\omega) > 2\lambda \)

Hence if \( \text{Re}(\omega) > 2\lambda \) all branches of the function \( f(\omega) \) take

values which are in the closed disc.
\[ |f(\omega) + \mu| < \mu \quad (9.4.10) \]

by equation (9.4.2). Hence for the \( j^{\text{th}} \) branch;
\[ -2\mu < \text{Re}(f_j(\omega)) < 0 \quad (9.4.11) \]

for all \( j = 0, 1, \ldots, L-1 \)

Write;
\[ \phi_j = f_j(\omega) \]
\[ \theta_j = \omega - f_j(\omega) \quad (9.4.12) \]

to distinguish the individual branches whence;
by use of (9.4.11) if \( \text{Re}(\omega) > 2\lambda \) and for all \( j = 0, 1, \ldots, L-1 \).

Hence for all \( j = 0, 1, \ldots, L-1 \) and all \( \omega \in \mathcal{D}^1 \) all points \((\phi_j, \theta_j)\)
defined by (9.4.12), lie in the convex tube \( \tilde{D} \) where:

\[
\tilde{D} = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta) > 2\lambda, \text{Re}(\theta+\phi) > 2\lambda \} \quad (9.4.14)
\]
since \( \theta + \phi = \omega \). However by Lemma (6.2), \( \chi^*_+(\phi, \theta) \) is bivariate
analytic in tube \( D \) such that:

\[
D = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta) > 0 \text{ and } \text{Re}(\theta+\phi) > 0 \} \quad (9.4.15)
\]
and

\[
\tilde{D} \subset D
\]

Thus for all finite \( \theta + \phi \in \mathcal{D} \) and all branches \((j), \omega = \theta + \phi \) is s.t.
\( \omega \in \mathcal{D}^1 \) as defined by equation (9.4.8) and

\[
\chi^*_+(f(\omega), \omega-f(\omega)) < \infty \quad (9.4.17)
\]
since \( \chi^*_+(f_j(\omega), \omega-f_j(\omega)) \) is an analytic function there.

Hence for any \( j = 0, 1, \ldots, L-1 \).

\[
F(f_j(\omega), \omega-f_j(\omega)) = 0 \quad (9.4.18)
\]
for any \( \omega \in \mathcal{D}^1 \)

However, it is also clear that the product:

\[
(m+\omega-f_j(\omega))^L \chi^*_+(f(\omega), \omega-f(\omega)) \quad (9.4.19)
\]
is an holomorphic function for any \( \omega \in \mathcal{D}^1 \) and any \( j = 0, 1, \ldots, L-1 \)
since both \( F_j(\omega) \) and \( \chi^*_+(f_j(\omega), \omega-f_j(\omega)) \) are holomorphic there,
and may therefore be represented by an equivalence class of
convergent power series centred on points in \( \mathcal{D}^1 \). Thus for any
\( \omega \in \mathcal{D}^1 \) the function:

\[
F(f_j(\omega), \omega-f_j(\omega))
\]
is analytic for any \( j=0,1,\ldots,L-1 \) and may also be represented as an equivalence class of convergent power series centred on points in \( D^1 \). However all the coefficients of each of these power series are identically zero by virtue of (9.4.18), and the class may be continued to all \( \omega \).

Hence the Theorem is true in the extended plane of \( \omega \) as asserted by use of equation (9.4.4) in equation (7.18).

**Theorem 9.5** None of the \( d_m^{(+)}(\theta) \) defined by Theorem (9.3.3) has any singular points \( \hat{\theta} \) s.t. \( \Re(\hat{\theta})>-\mu \) for any \( m = 0,1,\ldots,K \).

**Proof:** If any one or more of the \( d_m^{(+)}(\theta) \) possessed a singular point \( \hat{\theta} \) s.t. \(-\mu<\Re(\hat{\theta})\leq 0 \) then \( C(\phi,\theta) \) could not be univariate analytic in \( \theta \) for constant \( \phi \), when \( \theta=\hat{\theta} \) except possibly on a set of exceptional points \( \phi=\{\phi_j\} \) for \( j = 1,2,\ldots \). Remark 3 after Theorem (9.3.3) refers.

However by Corollary 5 to Lemma (6.11), \( \chi_-(\phi,0) \) is analytic in \( \theta \) for all finite constant \( \phi \) at least if \( |\theta|<\infty \) and \( \phi \) is s.t. \( |\phi|<\infty, \Re(\phi)<0 \) and \(-\sigma<\Re(\theta)<-\Re(\phi) \) if \( \beta^*(\theta) \) is analytic whenever \( |\theta|<\infty \) and \( \Re(\theta)>\sigma \). Also it will be clear that for the GI|\( E_L |2 \) queue \( \sigma=\mu \). Thus \( C(\phi,\theta) \) is analytic in \( \theta \) for constant \( \phi \) at least if \( \Re(\phi)<0, \Re(\theta+\phi)<0, \Re(\theta)>-\mu \) and \( \theta \) and \( \phi \) are finite by use of the LHS of equation (7.17).

Thus none of the \( d_m^{(+)}(\theta) \) may possess any singular points in the domain where \( |\theta|<\infty \) and \( \Re(\theta)>-\mu \).

Thus by use of equation (9.3.20) the domain of bivariate analyticity of \( C(\phi,\theta) \) is at least that where \( |\phi|, |\theta|<\infty \) and \( \Re(\theta)>-\mu \).
Corollary: \( \chi_+^*(\phi, \theta) \) is bivariate analytic in at least \( D \) where

\[
D = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \Re(\phi+\theta)>0 \text{ and } \Re(\theta)>-\mu \}
\]

Proof: Since by Theorem (9.1.1), \( M(\phi, \theta) \) is bivariate analytic if \( |\phi|, |\theta|<\infty \) and \( \Re(\theta+\phi)>0 \) and by Theorem (9.5), \( C(\phi, \theta) \) is bivariate analytic at least if \( |\phi|, |\theta|<\infty \) and \( \Re(\theta)>-\mu \) as defined by equation (9.4.4) must be analytic if \( |\phi|, |\theta|<\infty \) and \( \Re(\theta)>-\mu \) by use of RHS of (7.17).

However we have shown that the zeroes of the algebraic function;

\[
\lambda^K \mu^L - (\lambda-\phi)^K (\mu+\phi)^L
\]

are found whenever \( \phi \) and \( \theta \) are such that,

\[
\phi = \mu((\frac{\lambda}{\lambda-\phi})^{K/L}-1)
\]

for all finite \( k \) and \( L \) and that for all \( \theta+\phi \) s.t. \( \Re(\theta+\phi)>2\lambda \) all of the branches of (9.5.2) may be made to cut a domain where \( \chi_+^*(\phi, \theta) \) is known to be analytic. [See the proof of Theorem (9.4) for these results.]

Therefore the multiplier;

\[
(\lambda^K \mu^L - (\lambda-\phi)^K (\mu+\phi)^L)(\mu+\phi)^L
\]

cannot nullify any singularities of \( \chi_+^*(\phi, \theta) \) in the domain where \( \phi \) and \( \theta \) are both finite and \( \Re(\theta)>-\mu \). Thus either \( \chi_+^*(\phi, \theta) \) has no singularities in this domain or it has singularities with loci of the form;

\[
\theta+\phi+r = 0
\]

where \( \Re(r)<0 \) since it must share these with \( M(\phi, \theta) \) in such a manner that cancellation takes place. Thus \( \chi_+^*(\phi, \theta) \) is at least
analytic if $|\phi|, |\theta|<\infty \operatorname{Re}(\theta)>-\mu$ and $\operatorname{Re}(\theta+\phi)>0$ which is the asserted domain.

Remarks: 1. It will be obvious that the method used to establish Corollary 5 to Lemma (6.11) might have been applied directly to $\chi^*_+(\phi, \theta)$ to establish the enlarged domain of analyticity from first principles.

2. Given that $\chi^*_-(\phi, \theta)$ is analytic in $\theta$ for all finite $\phi$ and $\theta$ s.t. $\operatorname{Re}(\phi)<0$ and $\operatorname{Re}(\theta)>-\mu$ one may extend the proof of Corollary 5 Lemma (6.11) to show that the corresponding Laplace Integral converges at all points in $\mathbb{R}$ where:

$$R = \{\text{All finite } \phi \text{ and } \theta \text{ s.t. } \operatorname{Re}(\theta+\phi)<0 \text{ and } \operatorname{Re}(\theta)>-\mu\} \quad (9.5.5)$$

Hence by standard theorems $\chi^*_-(\phi, \theta)$ is bivariate analytic in the domain $D$ where:

$$D = \{\text{All finite } \phi \text{ and } \theta \text{ s.t. } \operatorname{Re}(\theta+\phi)<0 \text{ and } \operatorname{Re}(\theta)>-\mu\} \quad (9.5.6)$$

3. Given the result (9.5.6) one may then establish the domain of (bivariate) analyticity of $C(\phi, \theta)$ by Schwartz-Hartog's continuation to be $D^1$ where:

$$D^1 = \left\{\text{All finite } \phi \text{ and } \theta \text{ such that} \right\}$$

\begin{align*}
(1) \quad &\text{If } \operatorname{Re}(\theta+\phi)<0, \operatorname{Re}(\theta)>-\mu \\
(2) \quad &\text{If } \operatorname{Re}(\theta+\phi)>0 \text{ and } \operatorname{Re}(\theta)>0
\end{align*} \quad (9.5.7)

4. However $D^1$ is a tube with a non-convex basis which may be immediately completed to $D^{11}$ where

$$D^{11} = \{\text{All finite } \phi \text{ and } \theta \text{ s.t. } \operatorname{Re}(\theta)>-\mu \} \quad (9.5.8)$$

by use of the following rather remarkable Theorem;

"Every tube has a uniquely determined largest analytic completion. It is the convex hull of the given tube"
Brochner & Martin [1] Ch. V Th.9. Clearly it should be emphasized that this Theorem is only true for Analytic Completion and that it may be possible to further continue a specific function to an even larger domain than the convex hull of the first given tube. However this will require additional information which is not needed in the case now under discussion.

5. The convex hull of a given tube may be obtained by constructing a tube on the convex completion of the basis of the original tube.

6. The preceding steps outline an alternative proof for Theorem (9.5) which may be extended to give an alternative method for the continuation of $C(\phi, \theta)$ to all finite $\phi$ and $\theta$. However this requires an additional result which may be deduced from the proof of Theorem (9.7).

**Theorem 9.6** None of the $d_{m}^{(+)}(\theta)$ may have any singular points in the finite half plane of $\theta$ s.t. $\text{Re}(\theta) < -\mu$, whence $C(\phi, \theta)$ is bivariate entire.

**Proof:** This is a contradiction argument based on the notion of the Permanence of the Functional Equation. It uses the following Lemma which is taken from Hille [2] Vol.II Chapter 10.

**Lemma**

"Suppose we have a function $F(Z, \omega_1, \omega_2)$ of three variables analytic in each of them. More precisely we assume the existence of 3 domains $D_0$, $D_1$, and $D_2$ such that for $Z \in D_0$, $\omega_1 \in D_1$ and $\omega_2 \in D_2$ the function $F(Z, \omega_1, \omega_2)$ and its first order partial
derivatives exist and are continuous. Let $a \in D_0$ and suppose there is a circular disc

$D: \{ z: |z-a| < R \}$ also in $D_0$ and two analytic functions 

$f_1(z)$ and $f_2(z)$ which are holomorphic in $D$ and which take on values confined to $D_1$ and $D_2$ respectively. That is,

$$f_1(D) \subseteq D_1, f_2(D) \subseteq D_2$$

then;

$$F(z, f_1(z), f_2(z))$$

is a holomorphic function of $z$ in $D$ for this function is well defined there and it has a unique derivative since $F(z, \omega_1, \omega_2)$ has partial derivatives and $f_1(z)$ and $f_2(z)$ are differentiable functions."

Consider each $d^{(+)}$ in turn and assemble the complete sequence $\{ \theta_j \}_{j=1,2,\ldots}$ of suspect singular points of $C(\phi, \theta)$. If this be not possible by virtue of some assumed or suspected impenetrable barrier we merely assemble the sequence of the accessible suspect points.

Consider the finite point $\hat{\theta}$ as typical and the relationship

$$C(\hat{f}(\omega), \omega - \hat{f}(\omega)) = M(\hat{f}(\omega), \omega - \hat{f}(\omega))$$

for all finite or infinite $\omega$ established by Theorem (9.4) iff;

$$f(\omega) = \mu((\frac{\lambda}{\lambda - \mu})^{K/L})^{-1}$$

if $K/L \neq 1$ an integer.

$$= \mu(\{\frac{\lambda}{\lambda - \omega}\}_{p}^{1 \leq l/L - 1})$$

where $\{\cdot\}$ denotes the principal branch, for all $z = 0, 1, \ldots, L-1$, if $K/L = 1$ an integer and $i = \sqrt{-1}$. 

Clearly we may write:
\[ \omega - \phi(\omega) = \hat{\theta} \quad (9.6.1) \]
whence by including all the branches we obtain:
\[ (\lambda - \omega)^K (\omega + \mu - \hat{\theta})^{L-K\mu \lambda} = 0 \quad (9.6.2) \]
and it is clear that the roots of this polynomial in \( \omega \) define
\( K+L \) values of \( \omega \) and then \( K+L \) values of
\[ \phi = \omega - \hat{\theta} \]
which we shall denote \( \{\phi_{jj}\} \) for \( jj = 1, 2, \ldots, K+L \).

Clearly then:
\[ C(\phi_{jj}, \hat{\theta}) = M(\phi_{jj}, \hat{\theta}) \quad (9.6.3) \]
for each \( jj = 1, 2, \ldots, K+L \) but \( C(\phi, \theta) \) may not be bivariate
analytic at any of these points in view of the existence of
the singular point \( \hat{\theta} \).

Consider the contours:
\[ |\lambda - \omega| = \lambda \quad (9.6.4) \]
whence,
\[ \omega = \lambda(1 + e^{i\eta}) \quad (9.6.5) \]
if \( 0 \leq \eta < 2\pi \)
and
\[ |\omega + \mu - \hat{\theta}| = \mu \quad (9.6.6) \]
whence;
\[ \omega = \mu(e^{i\eta} - 1) + \hat{\theta} \quad (9.6.7) \]
if \( 0 \leq \eta < 2\pi \)
By Rouche's Theorem the contour;
\[ \omega = \lambda(1 + e^{i\eta}) \]
contains \( K \) roots either in it or on it which we designate \( \{\omega_j\} \)
for \( j = 1, \ldots, K \). Similarly the contour;

\[
\omega = \mu (e^{i\eta} - 1) + \hat{\theta}
\]

contains \( L \) roots either in or on it which are characterized by \( \text{Re}(\omega) \leq 0 \).

We ignore the roots such that \( \text{Re}(\omega) \leq 0 \) and consider only the \( \{\omega_j\} \) for \( j = 1, 2, \ldots, K \) for which \( \text{Re}(\omega) > 0 \). We also note that

\[
(\text{if} \quad |e| < \infty)
\]

(9.6.8)

(9.6.9)

Hence since \( \text{Re}(\theta) < -\mu \), the roots \( \{\omega_j\} \) all lie in \( \text{Re}(\omega) > 0 \) for all \( j = 1, 2, \ldots, K \), whilst no \( \omega_j = \lambda \) and \( |\omega_j| \leq 2\lambda \) for all \( j = 1, 2, \ldots, K \).

Hence for any \( j = 1, 2, \ldots, K \),

\[
C(\omega_j - \theta, \hat{\theta}) = M(\omega_j - \theta, \hat{\theta})
\]

by use of (9.6.3)

However by Lemma (9.1.1), \( M(\phi, \theta) \) is a bivariate analytic function if \( \text{Re}(\theta + \phi) > 0 \) and \( \phi \) and \( \theta \) are both finite whence by use of this property in equation (9.6.10) we see that; \( C(\phi, \theta) \) must be analytic in both \( \phi \) and \( \theta \) at the point;

\[
\phi = \omega_j - \hat{\theta}
\]

\[
\theta = \hat{\theta}
\]

for each \( j = 1, 2, \ldots, K \), which is a contradiction.

Hence \( \hat{\theta} \) cannot exist as a singular point. Therefore none of the \( \{\hat{\theta}_j\} \) for \( j = 1, 2, \ldots \) may exist as singular points, there can be no barriers to the continuation of the \( d_m^{(\dagger)}(\theta) \) for any \( m = 0, 1, \ldots, K \) over the whole finite plane of \( \theta \). Thus for each
in $\mathbb{R}^m$, we may write;

\[ d_m(\theta) = d_m(\theta) \quad (9.6.18) \]

where each $d_m(\theta)$ is an entire function. Hence the assertion of
the Theorem is true.

**Theorem 9.7.** $C(\phi, \theta)$ is a Bivariate Polynomial.

**Proof:** Clearly it suffices to show that for all finite constant
$\phi_0$ in some domain, $C(\phi_0, \theta)$ is of finite order in $\theta$ as $|\theta| \to \infty$ for
all $0 < \text{Am}(\theta) < 2\pi$ where $\text{Am}(\cdot) = \text{Amplitude (\cdot)}$, whence the result
follows immediately by use of Louvillés Theorem.

There are 3 situations, namely

(i) $-\pi/2 < \text{Am}(\theta) < \pi/2$ whence $\text{Re}(\theta) \to \infty$ as $|\theta| \to \infty$

(ii) $\text{Am}(\theta) = \pm \pi/2$ " $\text{Re}(\theta) = 0$ as $|\theta| \to \infty$

(iii) $\begin{cases} -\pi < \text{Am}(\theta) < \pi/2 \\ \pi/2 < \text{Am}(\theta) < \pi \end{cases}$ " $\text{Re}(\theta) \to \infty$ as $|\theta| \to \infty$

and $\pi/2 < \text{Am}(\theta) < \pi$

**Case (i)** Consider the RHS of equation (7.17) with $\phi = \phi_0$ where
$\phi_0$ is any finite constant $\phi$ s.t. $\text{Re}(\phi) > 0$. Clearly by use of
Lemmas (6.2) and (9.1.1) $\chi^\dagger(\phi_0, \theta)$ and $M(\phi_0, \theta)$ are analytic in
$\theta$ if $|\theta| < \infty$ and $\text{Re}(\theta) > 0$. Moreover by Lemma (6.19)

\[ \d \lim_{\text{Re}(\theta) \to \infty} \{ \chi^\dagger(\phi_0, \theta) \} = \pi_o \quad (9.7.1) \]

if $|\phi_0| < \infty$

whilst the use of Lemma (6.16) in equation (7.19) gives;

\[ \d \lim_{\text{Re}(\theta) \to \infty} \{ M(\phi_0, \theta) \} \sim -\lambda \mu L \pi_o \theta^L + o(|\theta|^L) \quad (9.7.2) \]

Hence by use of (9.7.1) and (9.7.2) in (7.17) we obtain;

\[ \d \lim_{\text{Re}(\theta) \to \infty} \{ C(\phi_0, \theta) \} \sim (-1)^{K+1} \pi_o (\mu + \phi_0)^L \theta^{K+L} + o(|\theta|^{K+L}) \quad (9.7.3) \]
if $K>1$ for all finite $\phi_0$ s.t. $\text{Re}(\phi_0)>0$.

Thus for each $m=0,1,\ldots,K$;

$$d_n(\theta) = O(|\theta|^{K+L})$$

(9.7.4)
as $|\theta|\to\infty$ with $\text{Re}(\theta)\to\infty$.

Hence by use of equation (9.3.16) \[\text{Theorem (9.3.3)}\]

$$C(\phi,\theta) = O(|\theta|^{K+L})$$

(9.7.5)

if $\text{Re}(\theta)\to\infty$ as $|\theta|\to\infty$ for all $|\phi|<\infty$.

Case (ii) If $|\theta|\to\infty$, $\text{Re}(\theta) = 0$, consider the LHS of equation (7.17) for any finite $\phi_0$ s.t. $\text{Re}(\phi_0)<0$ and observe that;

$$\left|\chi_-(\phi_0,\theta)\right| = \left|\int_{0^+}^{\infty} \int_{x^+} e^{-\phi_0 x - \theta y} d\chi_-(x,y)\right|$$

(9.7.6)

$$\leq \int_{-\infty}^{0^-} \int_{x^+} e^{-\phi_0 x} |d\pi(x,y) + \pi_+ + |\pi_1^*(\theta)|$$

(9.7.8)

by use of the definitions of $\chi_-(x,y),\pi_+(y),\pi_1^*(\theta)$ and $\pi_0$ for which see Section 4, and,

$$|e^{-\theta y}| = e^{-\text{Re}(\theta) y} = 1$$

(9.7.9)

for all $y \in (-\infty, 0)$ if $\text{Re}(\theta)=0$. Thus

$$\left|\chi_-(\phi_0,\theta)\right| \leq \int_{-\infty}^{0^-} \int_{x^+} d\pi(x,y) + \pi_0 + |\pi_1^*(\theta)|$$

(9.7.10)

if $\text{Re}(\phi_0)<0$ since;

$$|e^{-\phi_0 x}| = e^{-\text{Re}(\phi_0) x} < 1$$

(9.7.11)

for all $x < 0$, and $\text{Re}(\phi_0)<0$. 
\[ |\pi_1^*(\theta)| = \int_{0^+}^{\infty} e^{-\theta y} d\pi_1(y) \quad (9.7.12) \]

\[ \leq \int_{0^+}^{\infty} d\pi_1(y) \quad (9.7.13) \]

\[ = \pi_1(\infty) \quad (9.7.14) \]

since \( \pi_1(0) = 0 \) and if \( \text{Re}(\theta) = 0 \) by \( (9.7.9) \).

Thus;

\[ |\chi_*(\phi, \theta)| \leq 2\pi(0, \infty) \quad (9.7.15) \]

by use of the definitions of \( \pi(x, y) \), \( \pi_0 \) and \( \pi_1(y) \), for all finite \( \phi \) s.t. \( \text{Re}(\phi) < 0 \) and all \( \theta \) s.t. \( \text{Re}(\theta) = 0 \).

Hence;

\[ \lim_{|\theta| \to \infty} \left\{ C(\phi_0, \theta) \right\} = \lim_{|\theta| \to \infty} \left\{ (\lambda - \theta - \phi_0)^K (\mu + \phi_0)^L (\mu + \theta)^L \chi_*(\phi_0, \theta) \right\} \quad (9.7.16) \]

\[ \nu(-1)^K (\mu + \phi_0)^L \Theta^{K+L} E(\phi_{0}) \quad (9.7.17) \]

where \( |E(\phi_0)| \leq 2\pi(0, \infty) \quad (9.7.18) \)

and again,

\[ C(\phi_0, \theta) = O(|\theta|^{K+L}) \quad (9.7.19) \]

as \( |\theta| \to \infty \) for all finite \( \phi_0 \) s.t. \( \text{Re}(\phi) < 0 \), if \( \text{Re}(\theta) = 0 \).

Case (iii) This requires the use of the LHS of equation \( (7.17) \) in the form,

\[ C(\phi_0, \theta) = (\lambda - \theta - \phi_0)^K (\mu + \phi_0)^L (\mu + \theta)^L \left[ \chi_*(\phi_0, \theta) + \chi_*(\phi_1, \theta) \right] \quad (9.7.20) \]

where \( |\phi_0| < \infty \) and \( \text{Re}(\phi_0) < 0 \), and the following Lemma.

Lemma (9.8) For any ergodic GI\( |E_L| \)2 queue the distribution \( \pi(x, y) \) must be such that;
\[
\lim_{|\theta| \to \infty} \{x^*_\theta(\phi, \theta)\} = 0 \tag{9.7.21}
\]
for all \(0 < \Delta m(\theta) < 2\pi\) and all finite \(\phi_o\) s.t. \(\text{Re}(\phi_o) \leq 0\).

**Proof:** Consider the subsequence of all arrivals at the ergodic queue such that these find \(x < 0, y > 0\). Choose an arbitrary \(\tilde{x} \in (0, \infty)\) and that subsequence of the first subsequence of all arrivals such that \(y > 0\) and \(x \in [-\tilde{x}, -\tilde{x} + dx)\). Clearly there exists a distribution conditional on \(y > 0\), \(\mathbb{P}(\tilde{x})\) which defines the probability that \(x \in [-\tilde{x}, 0)\).

It will also be clear that since \(y > 0\) and the service time distribution of a \(GI|E_L|2\) queue is composed of \(L\) identical stages (each with mean rate \(\mu\) in our notation), the residual service time distribution must be characterized by being composed of from 1 to \(L\) such stages and that for arrivals which find \(x \in [-\tilde{x}, -\tilde{x} + dx)\) there exists a conditional distribution \(p(j|x)\) on the integers \(j = 1, 2, \ldots, L\) which defines the probability that \(j\) stages remain to be completed. Thus obviously;

\[
\sum_{j=1}^{L} p(j|x) = 1 \tag{9.7.22}
\]

for all \(\tilde{x}\)

and

\[
\int_{0}^{\infty} p(j|x) \, d\mathbb{P}(\tilde{x}) = p(j) \tag{9.7.23}
\]

where \(p(j)\) is the probability that \(j\) stages remain to be completed by one server given that the other has been free for some unspecified period of time.
Consider the Laplace Transformation $E(e^{-\theta y})$ of the distribution $P(y|\bar{x})$ encountered by the sub-sequence of arrivals which finds $y>0$ and $x \in [-\bar{x}, \bar{x}+dx)$. Clearly this is of the form:

$$
L \frac{\mu^j}{\mu^j + \theta} \int P(j|\bar{x}) \, d\Pi(\bar{x})
$$

for any given $\bar{x} \in (0, \infty)$ by the properties of the Gamma Distribution and its Laplace Transform. However, for the GI|E|2 queue we have:

$$
- \frac{e^{-\theta y}d_y}{e^{\theta y}d_y} P(y|\bar{x}) = \frac{e^{-\theta y}d_y}{e^{\theta y}d_y} \pi(x,y)
$$

where $\pi(x,y)$ is as defined in Section 4. Thus we may write:

$$
\pi_1^*(\theta) = - \int \frac{L}{\mu + \theta} \sum_{j=1}^{\infty} \frac{e^{-\theta y}d_y}{e^{\theta y}d_y} p(j|\bar{x}) d\Pi(\bar{x})
$$

by integration overall $\bar{x} \in (0, \infty)$ since $\pi_1^*(\theta)$ is the Laplace Transformation of the residual service time distribution for any $x \in (-\infty, 0)$.

Clearly if $L<\infty$ and $\theta \neq -\mu$, the integral:

$$
- \int \frac{L}{\mu + \theta} \sum_{j=0}^{\infty} \frac{e^{-\theta y}d_y}{e^{\theta y}d_y} p(j|\bar{x}) d\Pi(\bar{x})
$$

exists and

$$
\pi_1^*(\theta) = \sum_{j=1}^{L} \frac{e^{-\theta y}d_y}{e^{\theta y}d_y} p(j)
$$

by use of (9.7.23) after reversing the orders of summation and integration.
Consider also the integral:

\[- \int_0^\infty \frac{1}{L} \left( \frac{1}{\mu+\theta} \right)^j \sum_{j=1}^{\infty} \mu p(j \mid \tilde{x}) e^{\phi \tilde{x}} \mathrm{d} \Pi(\tilde{x}) \]  

(9.7.29)

which is equivalent to

\[ E(e^{-\phi x-\theta y}) \]

for the measure \( \pi(x, y) \) in the region s.t. \( x \in (-\infty, 0) \) and \( y \in (0, \infty) \)

\[ = \chi_{-\infty}^*(\phi, \theta) + \pi_1^*(\theta) \]  

(9.7.31)

by the definitions of \( \chi_{-\infty}^*(\phi, \theta) \) and \( \pi_1^*(\theta) \)

However if \( \text{Re}(\phi) < 0 \) and \( \theta \neq -\mu \)

\[- \int_0^\infty \frac{1}{L} \left( \frac{1}{\mu+\theta} \right)^j \left| e^{\phi \tilde{x}} \right| p(j \mid \tilde{x}) \mathrm{d} \Pi(\tilde{x}) < \infty \]  

(9.7.32)

if \( L < \infty \) and we may write;

\[ |\chi_{-\infty}^*(\phi, \theta) + \pi_1^*(\theta)| \leq \frac{1}{L} \sum_{j=1}^{\infty} \left| \frac{1}{\mu+\theta} \right|^j |p(j)| \]  

(9.7.33)

if \( \theta \neq -\mu \)

by reversal of order of summation and integration and using;

\[ |e^{\phi \tilde{x}}| < 1 \]  

(9.7.34)

if \( \tilde{x} > 0 \) and \( \text{Re}(\phi) < 0 \).

Hence

\[ |\chi_{-\infty}^*(\phi, \theta)| \leq 2 |\pi_1^*(\theta)| \]  

(9.7.35)

wherever this exists

\[ \leq 2 \sum_{j=1}^{\infty} \left| \frac{1}{\mu+\theta} \right|^j |p(j)| \]  

(9.7.36)

for all \( \theta \) s.t. \( \theta \neq -\mu \) if \( |\phi| < \infty \) and \( \text{Re}(\phi) < 0 \).
Thus;
\[
\lim_{|\theta| \to \infty} \{ \chi^*_{\phi_o, \theta} \} = 0 \quad (9.7.37)
\]
for all finite \( \phi_o \) s.t. \( \text{Re}(\phi_o) < 0 \) and all \( 0 < \Delta m(\theta) < 2\pi_2 \) asserted.

Hence the preliminary lemma is true as asserted.

By Lemma (6.9)
\[
\lim_{\text{Re}(\theta) \to -\infty} \{ \chi^*_{\phi_o, \theta} \} = -\pi_o + \nu(\phi) \quad (9.7.38)
\]
where \( \nu(\phi) = 0 \) if the line \( x \in (-\infty, 0), y = 0 \) is a set of probability measure zero.

Thus by taking \( \lim \) \( \{ (9.7.20) \} \) and the use of \( (9.7.37) \) and \( (9.7.38) \) we obtain;
\[
\lim_{\text{Re}(\theta) \to -\infty} \{ C(\phi_o, \theta) \} \nu(-1)^{K+1} (1+\phi_o)^{L K+L} \{ \pi_o + \nu(\phi) \} \quad (9.7.39)
\]
for any finite \( \phi_o \) s.t. \( \text{Re}(\phi_o) \leq 0 \)
\[
= 0(|\theta|^{K+L}) \quad (9.7.40)
\]

Hence we have shown that if \( |\theta| \to \infty \) and for all \( 0 < \Delta m(\theta) < 2\pi_2 \);
\[
C(\phi_o, \theta) = 0(|\theta|^{K+L}) \quad (9.7.41)
\]
for all finite \( \phi_o \) s.t. \( \text{Re}(\phi_o) < 0 \).

Hence by Louvilles Theorem \( C(\phi_o, \theta) \) is at most a polynomial of order \( K+L \) in \( \theta \). Hence for each \( m = 0, 1, \ldots, K \)
\[
d_m(\theta) = 0(|\theta|^{K+L}) \quad (9.7.42)
\]
Hence
\[
C(\phi, \theta) = 0(|\theta|^{K+L}) \text{ as } |\theta| \to \infty
\]
for all finite \( \phi \) by use of Theorems (9.3.3) and (9.6).

Hence the assertion that \( C(\phi, \theta) \) is a bivariate polynomial is true for the \( E_k |E_L| 2 \) queue.
Remarks: 1. The above rather laboriously established result
is of course merely a first generalization of the very much
more easily established result which arises for the GI|G|1
queue with rational transforms.

However, it does provide the rather fascinating question
"What else might one expect?" and thence speculation as to
whether the now classical W.L. Smith - Wiener-Hopf result might
not be the unidimensional special case of the general characteris-
tic of all queues of the GI|G|C class.

2. If one reverses the order of summation and inte-
gration in the expression numbered (9.7.29) and restricts $\phi_0$
to be finite and such that $\text{Re}(\phi_0) \leq 0$ one obtains a sum of the
form;

$$\sum_{j=1}^{L} \left( \frac{1}{j + \theta} \right) q_{j, \phi_0}$$

where for each $j = 1, ..., L$, the $q_{j, \phi_0}$ will exist if

$|\phi| < \infty$ and $\text{Re}(\phi_0) \leq 0$.

Thus the only possible singularities of the function
$\chi_{\phi_0}(\phi_0, \theta) + \pi_1(\theta)$ in the plane of $\theta$ (for constant $\phi_0$) are
poles of at most order $L$ at the point $\theta = -\mu$

Also if $\phi_0 = 0$ we obtain;

$$\pi_1^*(\theta) = \sum_{j=1}^{L} \left( \frac{1}{j + \theta} \right) q_{j, 0}$$

by use of equation (9.7.26). Thus we have shown that the
product

$(\mu + \theta)^L \chi_{\phi_0}(\phi_0, \theta)$
is analytic in $\theta$ for all finite $\theta$ at least if $\phi_o$ is a finite constant such that $\text{Re}(\phi_o) < 0$.

Also $\left(\mu+\phi_o\right)^L \chi_{-+}(\phi, \phi_o)$ is analytic in $\phi$ for constant $\theta_o$ s.t. $|\theta_o| < \infty$ since integrals of the type;

$$\int_0^\infty \sum_{j=1}^L \mu^j (\mu+\theta)^{L-j} p(j|x) e^{\phi_o x} d\Pi(x)$$

will exist for all finite $\phi_o$ s.t. $\text{Re}(\phi_o) < 0$, and thus define an analytic function of $\phi$ in the domain where $|\phi| < \infty$ and $\text{Re}(\phi) < 0$. Hence by Hartog's Theorem;

$$\left(\mu+\phi\right)^L \chi_{-+}(\phi, \theta)$$

is a bivariate analytic function if $|\phi|, |\theta| < \infty$ and $\text{Re}(\phi) < 0$.

However it has already been shown in Lemma (6.7) that $\chi_{-+}(\phi, \theta)$ is analytic if $|\phi|, |\theta| < \infty, \text{Re}(\phi) < 0$ and $\text{Re}(\theta + \phi) < 0$.

Thus the LHS of equation (7.17) and thus $C(\phi, \theta)$ is clearly analytic if $|\phi|, |\theta| < \infty, \text{Re}(\phi) < 0$ and $\text{Re}(\theta + \phi) < 0$.

However by Theorem (9.2.1), $C(\phi, \theta)$ is analytic if $|\theta|, |\phi| < \infty$ and $\text{Re}(\theta) < 0$. Hence by ordinary continuation in each variable in turn and the use of Hartog's Theorem, $C(\phi, \theta)$ is analytic in the domain such that $|\phi|, |\theta| < \infty$ and (i) $\text{Re}(\theta) > 0$ for all finite $\phi$ and (ii) $\text{Re}(\phi) < 0$ for all finite $\theta$.

But this is a tube with a non-convex basis which therefore admits of an immediate Analytic Completion to the whole finite basis plane such that $-\infty < \text{Re}(\phi), \text{Re}(\theta) < \infty$. 
Hence we have an alternative and in some ways much less tedious proof of Theorem (9.6) which does not depend on the finiteness of $K$.

3. This method of continuation may have advantages if an when one seeks to establish the properties of the functional or integral equations for more general queues of the $GI|G|C$ class for $C \geq 2$. 
10. The Nature of $\chi^*_+(\phi, \theta)$ and $\chi^*_-(\phi, \theta)$

**Theorem 10.1** For any finite $K$ and $L$,

$$\chi^*_-(\phi, \theta) = \frac{D(\phi, \theta)}{(\lambda-\delta-\phi)^K(\mu+\theta)^L} \tag{10.1.1}$$

**Proof:** Obvious from the definition of $C(\phi, \theta)$ and equation (7.17).

**Corollary.** For finite $K$ and $L$, $\chi^*_-(\phi, \theta)$ is rational and may have no other poles.

**Proof:** Obvious since $D(\phi, \theta)$ is a polynomial.

**Theorem 10.2.** $(\lambda-\delta-\phi)^K$ is a pole of $\chi^*_-(\phi, \theta)$.

**Proof:** Write

$$\theta = \phi = \frac{\lambda}{2}$$

then

$$M(\frac{\lambda}{2}, \frac{\lambda}{2}) = R(\frac{\lambda}{2}, \frac{\lambda}{2}) - R(\frac{\lambda}{2}, \frac{\lambda}{2}) \tag{10.2.1}$$

$$= 0 \tag{10.2.2}$$

since $R(\phi, \theta)$ exists if $|\phi|, |\theta| <^\infty$ and $\Re(\theta+\phi) > 0$, and by use of equation (7.19).

Hence;

$$C(\frac{\lambda}{2}, \frac{\lambda}{2}) = \lambda^K \mu^L (\mu+\frac{\lambda}{2})^L \chi^*_+(\frac{\lambda}{2}, \frac{\lambda}{2})$$

by equation (7.17) (RHS)

$$\geq \lambda^K \mu^L (\mu+\frac{\lambda}{2})^L \pi_0 \tag{10.2.4}$$

since

$$\chi^*_+(\frac{\lambda}{2}, \frac{\lambda}{2}) \geq \pi_0 \tag{10.2.5}$$
and
\[ C(\lambda/2, \lambda/2) > 0 \] 
(10.2.6)
since
\[ \pi_0 > 0 \] 
(10.2.7)
for an ergodic queue.

However if \((\lambda - \theta - \phi)^K\) were not a pole of \(\chi^*_-(\phi, \theta)\) then,
\[ C(\lambda/2, \lambda/2) = 0 \]
by LHS of (7.17).

Hence a pole exists there and is of the order \(K\) as asserted.

**Theorem 10.3.** Both \(\chi^*_+(\phi, \theta)\) and \(\chi^*_-(\phi, \theta)\) possess a pole due to the factor \((\mu + \theta)^{-L}\).

**Proof:** We first show that either or both must possess such a pole. Consider the following rearrangement of (7.17);
\[
M(\phi, \theta) = (\lambda - \theta - \phi)^K(\mu + \phi)^L(\mu + \theta)^L \chi^*_-(\phi, \theta) \\
- (\lambda^L - (\lambda - \theta - \phi)^K(\mu + \phi)^L(\mu + \theta)^L \chi^*_+(\phi, \theta) \) \] 
(10.3.1)

If either \(\chi^*_+(\phi, \theta)\) or \(\chi^*_-(\phi, \theta)\) possessed a pole of the form \((\mu + \theta)^{-L+1}\) with \(l > 0\) then by (10.3.1) both must possess it since \(M(\phi, \theta)\) may only possess singularities with loci of the form \(\theta + \phi + r = 0\) where \(Re(r) > 0\). See Lemma (6.20) et seq for proof of this.

However if both possessed it the factor \((\mu + \theta)^L\) which appears in both LHS and RHS of (7.17) would be insufficient to nullify it and \(C(\phi, \theta)\) could not be an entire function.

Hence the maximum order of any pole in \(\theta\) for all \(\phi\), which may
be shared by $\chi_+^*(\phi, \theta)$ and $\chi_-^*(\phi, \theta)$ is $(\mu+\theta)^L$.

If neither $\chi_+^*(\phi, \theta)$ nor $\chi_-^*(\phi, \theta)$ possessed this pole then by (10.3.1);

$$M(\phi, -\mu) = 0$$ (10.3.2)

at least for all finite $\phi$.

However,

$$\lim_{\Re(\phi) \to \infty} \{\chi_+^*(-\mu, \mu+\theta+\phi)\} \geq \pi_0$$ (10.3.3)

for all finite $\phi$ by Lemma (6.16),

and;

$$\pi_0 > 0$$ (10.3.4)

for the ergodic queue.

Also by Lemma (6.22);

$$\lim_{\Re(\phi) \to \infty} \{D_z \{\chi_+^*(z-\mu, \mu+\theta+\phi-z)\} \} = 0$$ (10.3.5)

for all $j = 1, 2, \ldots, L$ and all finite $\theta$.

Thus;

$$M(\phi, -\theta) \geq \lambda^L \mu^L \pi_0 \phi^L$$ (10.3.6)

as $\Re(\phi) \to \infty$. Thus,

$$M(\phi, -\mu) > 0$$ (10.3.7)

as $\Re(\phi) \to \infty$, which contradicts (10.3.2).

Hence either $\chi_+^*(\phi, \theta)$ or $\chi_-^*(\phi, \theta)$ or both have this pole.

Suppose $\chi_+^*(\phi, \theta)$ possesses the pole $(\mu+\theta)^{-L}$ then

$$C(\phi, \theta)^n - (-1)^K \lambda^{K+L} (\mu+\theta)^L \chi_+^*(\phi, \theta) + o(\phi^L)$$

as $|\phi| \to \infty$ (10.3.8)

by the RHS of equation (7.17)

$\neq 0$
if $\theta = -\mu$

by assumption

$$= 0(|\phi|^{K+L})$$

(10.3.9)

as $|\phi| \to \infty$ for all finite constant $\theta$. Thus by use of the

LHS of equation (7.17) and as $|\phi| \to \infty$.

$$C(\phi, \theta) \wedge (-1)^{K+1} K^{K+L} (\mu+\theta)^L \chi^*(\phi, \theta)$$

as $|\phi| \to \infty$ for all finite $\theta$.

$\neq 0$ when $\theta = -\mu$.

Therefore it is necessary that $\chi^*(\phi, \theta)$ also possess this pole.

Similarly it is obvious that if $\chi^*(\phi, \theta)$ possess the pole so

must $\chi^*_L(\phi, \theta)$.

Therefore both transforms possess this pole and the Theorem

is true as asserted.

Remark: It will be obvious from the foregoing remarks and the

proof of Theorem (9.7) and Lemmas (6.5) and (6.11), Corollary

(5), that this result is easily established for the $E^K E^L$ queue since both $\chi^*(\phi, \theta)$ and $\chi^*(\phi, \theta)$ contain $\pi_1^*(\theta)$ which

must possess this pole since $\pi_1(y)$ is a mixture of distributions with from 1 to $L$ stages.

Theorem 10.4. $\chi^*_L(\phi, \theta)$ has no singularities other than $(\mu+\theta)^{-L}$

and those which it shares with $M(\phi, \theta)$.

Proof: The polynomial

$$\lambda^K \mu^L - (\lambda-\phi)^K (\mu+\phi)^L$$

(10.4.1)

has already been investigated during the proof of Theorem (9.4)

where it was shown that all of the zeroes of this function
have loci which cut the domain wherein \( \chi^*_+ (\phi, \theta) \) is certainly analytic. Hence this polynomial cannot nullify any of the singularities of \( \chi^*_+ (\phi, \theta) \) if these exist. Therefore since \( C(\phi, \theta) \) is entire it is necessary that either \( \chi^*_+ (\phi, \theta) \) possess no singularities other than \( (\mu+\theta)^{-L} \) or that it possesses in addition to that only such singularities as may be removed by mutual annihilation against singularities of \( M(\phi, \theta) \). However all the singularities if these exist, of \( M(\phi, \theta) \) must have loci of the form

\[
\theta + \phi + r_1 = 0
\]

when \( \operatorname{Re}(r_1) \geq 0 \)

by Lemmas (6.20) and (6.21).

Also since singularities which mutually cancel must be of the same kind and possess identical locating constants, the assertion of the theorem is true.

Consider the equation;

\[
(\lambda - \theta - \phi)^K (\mu + \phi)^L (\mu + \theta)^L \chi_-(\phi, \theta)
\]

\[
= (\mu + \phi)^L \sum_{n=0}^{\infty} \phi^n d_n(\theta)
\]

\[
= (\lambda^K - (\lambda - \theta - \phi)^K (\mu + \phi)^L) (\mu + \theta)^L \chi_+(\phi, \theta)
\]

\[
+ \lambda^L \mu^L \left\{ (\mu + \phi)^L \sum_{j=0}^{L-1} \frac{(\mu + \theta)^j}{j!} D_z \{ \chi_+(z - \mu, \mu + \theta + \phi - z) \} \right\}
\]

where $K$ and $L$ are finite and;

\[
d_o(\theta) = (\mu + \theta)^L d_0(\theta)
\]

Substitute;

\[
\mu + \phi = v \\
\theta + \phi = \omega \\
\omega, \theta = \omega + \mu - v
\]

A more satisfactory form of equation results, namely;

\[
(\lambda - \omega)^K v^L (2\mu + \omega - v)^L \chi_-(v - \mu, \mu + \omega - v)
\]

\[
= v^L \sum_{m=0}^{L} v^m f_m(\omega)
\]

\[
= (\lambda^L - (\lambda - \omega)^K v^L) (2\mu + \omega - v)^L \chi_+(v - \mu, \mu + \omega - v)
\]
where the \( f_m(\omega) \) are also polynomials since the RHS of equation (11.3) is bivariate analytic (and a polynomial) in \( v \) and \( \omega \) since the RHS of equation (11.1) is a bivariate polynomial in \( \phi \) and \( \theta \) and the equations (11.2) define a non-singular finite order algebraic transformation, whilst;

\[
(\lambda^K \mu - (\lambda - \omega)^K \nu^L)(2 \mu + \omega - \nu) \chi_+(v - \mu, \mu + \omega - \nu)
\]

\[
\sim (-1)^{L+1}(\lambda - \omega)^K \nu^2L \pi_0
\]  

(11.3.1)

by use of Lemma (6.22). Also;

\[
M(v, \omega) = \lambda^K \mu^L \sqrt{v^L \sum_{j=0}^{L-1} \frac{(2 \mu + \omega - \nu)^j}{j!} D_z \{\chi_+(z - \mu, \mu + \omega - z)\}_{z=0}}
\]

\[
- (2 \mu + \omega - \nu)^L \sum_{j=0}^{L-1} \frac{\nu^j}{j!} D_z \{\chi_+(z - \mu, \mu + \omega - z)\}_{z=0}
\]

\[
\leq 2\lambda^K \nu^{2L-1} \nu^L - 1
\]

as \( v \to \infty \) for any finite \( \omega \).

Hence;

\[
M(v, \omega) = o(|v|^{2L-1})
\]  

(11.3.2)

as \( v \to \infty \) for all finite \( \omega \). However \( C(\phi, \theta) \) is a polynomial whence;

\[
C(v - \mu, \mu + \omega - \nu) = o(|v|^{2L})
\]  

(11.33.3)

by use of (11.3.1) and (11.3.2). Therefore the form,
\[ C(v-\mu, \mu+\omega-v) = \sum_{n=0}^{L} v^n f_n(\omega) \]  

(11.3.4)

is the obvious rearrangement of the original polynomial

\[ C(\phi, \theta) \] defined by equation (11.1). Write;

\[ e^{i2\pi l/L} = j^l \]  

(11.4)

where \( l = 0, 1, \ldots, L-1 \)

and \( i = \sqrt{-1} \)

whence the \( j^l \) are the \( L \) roots of unity. Write,

\[ C_1(\omega) = C_1 = C(v, \omega) \]  

(11.5.1)

when

\[ v = \mu \left\{ \left( \frac{\lambda}{\lambda-\omega} \right)^{K/L} \right\} p J^l \]  

(11.5.2)

where the notation \( \{ \} \) denotes the principal branch.

Consider equation (9.4.1) which may be rewritten as;

\[ C(f(\omega), \omega-f(\omega)) = M(f(\omega), \omega-f(\omega)) \]  

(11.6.1)

\[ f(\omega) = \mu \left\{ \left( \frac{\lambda}{\lambda-\omega} \right)^{K/L} \right\} p J^l \]  

for any \( l = 0, 1, \ldots, L-1 \)

in which form it may be written as \( L \) simultaneous equations

namely;

\[ C_1(\omega) = M_1(\omega) \]  

(11.6.2)

for \( l = 0, 1, \ldots, L-1 \)

where,

\[ M_1(\omega) = \lambda^{K/L} \sum_{j=0}^{L-1} \left\{ (2\mu+\omega-\mu)((\lambda/\lambda-\omega)^{K/L}) p J^l \right\} \sum_{j=0}^{L-1} \left\{ (\lambda/\lambda-\omega)^{K/L} p J^l \right\} D_z \sum_{j=0}^{L-1} \left\{ \chi^*_+(z-\mu, \mu+\omega-z) \right\} \]  

\[ M_1(\omega) = \lambda^{K/L} \sum_{j=0}^{L-1} \left\{ (2\mu+\omega-\mu)((\lambda/\lambda-\omega)^{K/L}) p J^l \right\} \sum_{j=0}^{L-1} \left\{ (\lambda/\lambda-\omega)^{K/L} p J^l \right\} D_z \sum_{j=0}^{L-1} \left\{ \chi^*_+(z-\mu, \mu+\omega-z) \right\} \]  

\[ M_1(\omega) = \lambda^{K/L} \sum_{j=0}^{L-1} \left\{ (2\mu+\omega-\mu)((\lambda/\lambda-\omega)^{K/L}) p J^l \right\} \sum_{j=0}^{L-1} \left\{ (\lambda/\lambda-\omega)^{K/L} p J^l \right\} D_z \sum_{j=0}^{L-1} \left\{ \chi^*_+(z-\mu, \mu+\omega-z) \right\} \]
Write:

\[ G_j(\omega) = D_z^j \{ \chi^*_+(z-\mu, \mu+\omega-z) \}_{z=0} \]  \hspace{1cm} (11.7.1)

for \( j = 0, 1, \ldots, L-1 \)

Form the column vector \( \chi(\omega) \) as follows:

\[
\begin{bmatrix}
G_0(\omega) \\
\vdots \\
G_{L-1}(\omega)
\end{bmatrix}
\]  \hspace{1cm} (11.7.2)

Form the square matrix \( [M(\omega)] \) as follows:

\[
[M(\omega)] = \lambda^{KL} \begin{bmatrix} \text{Coefficients of the } M_1(\omega) \end{bmatrix}
\]  \hspace{1cm} (11.7.3)

for all \( l = 0, 1, \ldots, L-1 \)

Form the column vector \( \chi(\omega) \) from the \( C_1(\omega) \)

\[
\begin{bmatrix}
C_0(\omega) \\
\vdots \\
C_{L-1}(\omega)
\end{bmatrix}
\]

Then (11.6.2) may be written in the form:

\[
[M(\omega)] \cdot \chi(\omega) = C_1(\omega)
\]  \hspace{1cm} (11.8)

As an example, consider the matrix for the \( E_k | E_2 | 2 \) queue

which is given on the next page.
\[
T^X = \left[ \{ (m) U \} \right]_{j=-1}^{\infty}
\]
Thus generally,

\[ G(\omega) = [M(\omega)]^{-1} C(\omega) \]  

(11.9)

provided the determinant \([M(\omega)] \neq 0\) for all \(\omega\). Hence if the polynomial \(C(v-\mu,\mu+w-v)\) can be determined \(C(\omega)\) and thence \(G(\omega)\) follow in an obvious way. The remaining problem is therefore to determine the \(f_\pi(\omega)\) of equation (11.3) and thence \(\chi^*(v-\mu,\mu+w-v)\).

The detailed properties of \([M(\omega)]\) and thence \([M(\omega)]^{-1}\)

are best elaborated for specific cases but some general observations may be made.

Consider the plane of \(\omega\) less an elementary disc around the point \(\omega = \lambda\). Denote this region \(P\). Then for all \(\omega \in P\),

\([M(\omega)]\) is composed only of finite single valued (in fact holomorphic) elements. Also for all \(\omega \in P\), \(C(\omega)\) is a holomorphic vector. Let the zeroes of \(|M(\omega)|\) be \(\{\omega_j\}\) for \(j = 1, 2, \ldots\). Clearly these are the poles of \(|[M(\omega)]^{-1}|\). Designate by \(\{\hat{\omega}_j\}\) those zeroes of \(|M(\omega)|\) which lie in the finite half plane of \(\omega\) s.t. Re\((\omega)\leq0\). Obviously these zeroes may lead to poles of the vector \(G(\omega)\).

Designate by \(\{\omega_j\}\) for \(j = 1, 2, \ldots\) those zeroes of \([M(\omega)]\) such that Re\((\omega)\geq0\). Clearly these cannot correspond with poles of \(G(\omega)\) since all of its elements are analytic for all finite \(\omega\) s.t. Re\((\omega)\geq0\), and finite if Re\((\omega)\geq0\). Hence the zeroes \(\{\hat{\omega}_j\}\) are zeroes of the vector \(G(\omega)\).

Consider now the point \(\omega = \lambda\). Clearly every element of
\[ M(\omega) \] and thus \( |M(\omega)| \) diverges here as does every element of \( C_\chi(\omega) \). However all the elements of \( G_\chi(\omega) \) exist at this point.

Hence the divergences of the elements of \( C_\chi(\omega) \) and the zero of \( [M(\omega)]^{-1} \) when \( \omega=\lambda \) must so compensate that \( G_\chi(\omega) \) is regular (and non-zero) at this point.

Although the general method of solution of the \( E_K|E_L|^2 \) queue for arbitrarily large \( L \), may be deduced from further consideration of these relationships it will be easier to consider some of the simpler cases which arise when \( L \) is small.
12. The Solution of $E^K|E_1(\Xi M)|2$

This case is of interest partly because it is simple and partly because if $K=1$ we seek A.K. Erlang's historic result for $M|M|2$. See Jensen [4]. GI$|M|C$ has also been investigated by Kendall [6] and others.

Consider the functional equation:

\[
K \varepsilon_d (\theta) \sum_{n=0}^{K} \phi^n = (\mu+\phi)^K (\mu+\theta) \chi^*_{+}(\phi, \theta)
\]

which is the appropriate form of (11.1). Change the notation to $v$ and $\omega$ as used in equations (11.2) and (11.3) and obtain;

\[
(\lambda \varepsilon_{\mu} - (\lambda - \omega)^K) (2\mu+\omega-v) \chi^*_{+}(v-\mu, \mu+\omega-v)
\]

\[
+ \lambda \varepsilon_{\mu} (2v-\omega-2\mu) \chi^*_{+}(-\mu, \mu+\omega)
\]

\[
= v\{f_0(\omega) + v\ f_1(\omega)\}
\]

analogous to (11.3).

Write;

\[
v = \mu \left(\frac{\lambda}{\lambda - \omega}\right)^K
\]

(for the single branch) and obtain;

\[
\chi^*_{+}(-\mu, \mu+\omega) = \frac{\mu \left(\frac{\lambda}{\lambda - \omega}\right)^K \{f_0(\omega) + \mu \left(\frac{\lambda}{\lambda - \omega}\right)^K f_1(\omega)\}}{\lambda \varepsilon_{\mu} (2\mu(\frac{\lambda}{\lambda - \omega})^K - 1) - \omega)}
\]

(12.4)
either directly from equation (12.2) or by following the
degenerate form of the process specified by equations (11.7)
to (11.9).

Clearly \( \chi^*_+(-\mu, \mu+\omega) \) is rational. Also it will be obvious
that since \( \chi^*_+(-\mu, \mu+\omega) < 0 \) if \( |\omega| < \infty \) and \( \Re(\omega) > 0 \) (see (12.12)
below) that,

\[
 f_1(\omega) = (\lambda - \omega)^K \tilde{f}_1(\omega)
\]

where \( \tilde{f}_1(\omega) \) is of lower order.

Thus;

\[
 \chi^*_+(-\mu, \mu+\omega) = \frac{f_0(\omega) + \mu \tilde{f}_1(\omega)}{2\mu \lambda^K - (\lambda - \omega)^K(2\mu + \omega)}
\]  

Consider the denominator of (12.7). If we write;

\[
 (\lambda - \omega)^K(2\mu + \omega) - 2\mu \lambda^K = 0
\]

and use Rouche's Theorem and the contours \( |\lambda - \omega| = \lambda \) and
\( |2\mu + \omega| = 2\mu \), we deduce that (12.8) has \( K \) roots in the finite
half-plane of \( \omega \) s.t. \( \Re(\omega) > 0 \) and one in the finite half plane
s.t. \( \Re(\omega) < 0 \).

We also note that generally \( \exists \) only one root at \( \omega = 0 \).

Thus;

\[
 \chi^*_+(-\mu, \mu+\omega) = \pi_0 \frac{\omega + \delta}{\omega + \gamma}
\]

where

\[
 \omega = -\gamma
\]

is the single root of (12.8) in the half-plane s.t. \( \Re(\omega) < 0 \)
and \( \delta \) is as yet an unknown constant and \( \pi_0 \) is the probability
of emptiness.
Proof: Since (12.7) exists when \( \omega = 0 \) and for all other roots of \( (12.8) \) such that \( \text{Re}(\omega) > 0 \), these are roots of the numerator of (12.7) and \( \chi^*_+(-\mu, \mu + \omega) \) must be of the form;

\[
\chi^*_+(-\mu, \mu + \omega) = \text{Const.} \frac{\omega + \beta}{\omega + \tau} \tag{12.10}
\]

since

\[
\chi^*_+(-\mu, \mu + \omega) < \infty \text{ if } \omega \rightarrow \infty
\]

whilst;

\[
\lim_{\text{Re}(\omega) \rightarrow \infty} \chi^*_+(-\mu, \mu + \omega) = \pi_0 \tag{12.11}
\]

both by Lemma (6.16); and the very simple form (12.9) is clearly correct.

Excursus:

\[
\chi^*_+(-\mu, \mu + \omega) < \infty \tag{12.12}
\]

if \( |\omega| < \infty \) and \( \text{Re}(\omega) > 0 \)

since;

\[
\chi^*_+(-\mu, \mu + \omega) = \int \int_{\infty} e^{-\mu(x + (\mu + \omega) y) d_\chi^+_+(x, y)}
\]

\[
\leq \int \int_{\infty} e^{-\text{Re}(\omega) y} d_\chi^+_+(x, y)
\]

because \( y - x > 0 \) and \( \mu > 0 \),

\[
\leq 1
\]

if \( \text{Re}(\omega) > 0 \) since \( y > 0 \).

The use of equation (12.9) in equation (12.2) and some obvious rearrangement produces;

\[
\chi^*_+(\nu + \mu, \mu + \omega - \nu) = \frac{v_0^2(\omega) + \nu f_1(\omega) - \lambda^\nu \mu (2v - \omega - 2\mu) \pi_0 \frac{\omega + \beta}{\omega + \tau}}{(2\mu + \omega - \nu)(\lambda^\nu \mu - (\lambda - \omega)^K)} \tag{12.13}
\]

which is clearly rational. Also since none of the zeroes of
\[ \lambda^K_{\mu} - (\lambda - \phi)^K_{\mu+\phi} \] may be singularities of \( \chi^*_+ (\phi, \theta) \) [See proof of Theorem (9.4)] it follows that none of the roots of

\[ \lambda^K_{\mu} - (\lambda - \omega)^K_{\mu} \] may be poles of \( \chi^*_+ (\nu - \mu, \mu + \omega - \nu) \). Hence this factor of the denominator of (12.13) also factors the numerator.

Thus,

\[ \chi^*_+ (\nu - \mu, \mu + \omega - \nu) = \frac{h_0 (\omega) + \nu h_1 (\omega)}{2 \mu + \omega - \nu} \]  

(12.14)

after division, where both \( h_0 (\omega) \) and \( h_1 (\omega) \) may be rational functions. Place \( \nu = 0 \) in equation (12.14) and obtain;

\[ \chi^*_+ (-\mu, \mu + \omega) = \frac{h_0 (\omega)}{2 \mu + \omega} = \pi_0 \left( \frac{\omega + \delta}{\omega + r} \right) \]

by use of (12.9).

Hence;

\[ h_0 (\omega) = \pi_0 \left( \frac{(2 \mu + \omega) (\omega + \delta)}{\omega + r} \right) \]  

(12.15)

Consider now;

\[ \lim_{\text{Re}(\nu) \to -\infty} \{ \chi^*_+ (\nu - \mu, \mu + \omega - \nu) \} = \pi_0 + \pi_+^*(\omega) \]  

(12.16)

for all finite \( \omega \) by Lemma (6.18)

where \( \pi_+^*(\omega) = (\text{Laplace Transformation of any concentration of}) \) 

(probablity on the line \( y = x > 0 \) w.r. to \( \omega \)).

and \[ \lim_{\text{Re}(\omega) \to \infty} \{ \pi_+^*(\omega) \} = 0 \]

since any such concentration must be on the strictly positive half line.

But by use of (12.14) we obtain;

\[ \lim_{\text{Re}(\nu) \to -\infty} \{ \chi^*_+ (\nu - \mu, \mu + \omega - \nu) \} = -h_1 (\omega) \]  

(12.17)

Hence,

\[ h_1 (\omega) = -\{ \pi_0 + \pi_+^*(\omega) \} \]  

(12.18)
by comparison of (12.16) and (12.17). However $m_+^*(\omega) = 0$ since if we consider the equation:

$$
(\lambda - \omega)^Kv(2\mu + \omega - v) \chi_+^*(v-\mu, \mu + \omega - v) = c(v-\mu, \mu + \omega - v)
$$

$$
= (\lambda^K\mu - (\lambda - \omega)^Kv)(2\mu + \omega - v) \chi_+^*(v-\mu, \mu + \omega - v)
$$

$$
+ \lambda^K\mu(2\mu - 2\mu) \chi_+^*(-\mu, \mu + \omega)
$$

and take:

$$
\{c(v-\mu, \mu + \omega - v)\}
$$

$$
v - (\lambda - \omega)^Kv^2 \{-m_+^*(\omega)\}
$$

(12.19)

by use of LHS of (12.19) and by the properties of $\chi_+^*(\sigma, \mu, \mu + \omega - \sigma)$

$$
v (\lambda - \omega)^Kv^2 \{-m_+^*(\omega)\}
$$

(12.20)

by use of RHS of (12.19) and as $|v| \to \infty$, in both cases.

Where:

$$
m_+^*(\omega) = (\text{Laplace Transformation of any probability \quad (concentration on the half line s.t. } -\infty < x < 0.)
$$

Hence by comparison of (12.20) and (12.21)

$$
m_+^*(\omega) = -m_-^*(\omega)
$$

(12.22)

for all finite $\omega$. However this is only possible if both are zero.

Thus both are zero as asserted and;

$$
h_1(\omega) = -m_0
$$

(12.23)

whence we may write;

$$
\chi_+^*(v-\mu, \mu + \omega - v) = m_0 \frac{(2\mu + \omega)(\omega + \delta) - \nu(\omega + \tau)}{(\omega + \tau)(2\mu + \omega - v)}
$$

(12.24)

We may check the correctness of this form as follows.
Consider the equation:

\[ vf_0(\omega) + v^2 f_1(\omega) = (\lambda K + (\lambda - \omega)) \pi \left\{ \frac{(2(\mu + \omega)(\omega + \phi) - \nu(\omega + r))}{\omega + r} \right\} + \lambda K(2v - \omega - 2\mu) \pi \left( \frac{\omega + \phi}{\omega + r} \right) \]

which may be obtained by use of (12.9) and (12.24) in (12.2) and equate the coefficients of the various powers of \( v \). Thus:

\[ v^0 \text{ gives,} \]

\[ 0 = \pi \left\{ \lambda K \left( \frac{2(\mu + \omega)(\omega + \phi)}{\omega + r} \right) - \lambda K \left( \frac{2(\mu + \omega)(\omega + \phi)}{\omega + r} \right) \right\} \]  

(12.26)

which is consistent.

\[ v^1 \text{ gives,} \]

\[ f_0(\omega) = \pi \left\{ \lambda K \left( \frac{2(\omega + \phi)}{\omega + r} \right) - (\lambda - \omega)K \left( \frac{2(\mu + \omega)(\omega + \phi)}{\omega + r} \right) \right\} \]  

(12.27)

\[ v^2 \text{ gives.} \]

\[ f_1(\omega) = \pi \left( \lambda - \omega \right)^K \]

whence,

\[ \tilde{f}_1(\omega) = \pi \]  

(12.28)

which is also consistent.

From (12.27) write:

\[ (\omega + r) f_0(\omega) = \pi \left\{ \lambda K \left( 2(\omega + \phi) - (\omega + r) \right) - (\lambda - \omega)K \left( 2(\mu + \omega)(\omega + \phi) \right) \right\} \]  

(12.29)

Place \( \omega = -r \) and obtain:

\[ ((\lambda + r) \left( K \left( 2\mu - r \right) - 2\lambda K \right) \phi - r) = 0 \]  

(12.30)

whence the requirement that \( -r \) be the root of

\[ (\lambda - \omega)K \left( 2(\mu + \omega) - 2\lambda K \right) = 0 \]

can be satisfied. (Compare (12.7) and (12.9))
Consider \( C(0, \theta) \). The left-hand side of equation (7.17) shows that since \( \chi^*(0, \theta) = \chi^*(0, \theta) \) by lemma (6.13),

\[ C(0, -\mu) = 0. \]

We may use equations (12.27) and (12.28) to write;

\[
C(\phi, \theta) = \pi_0 (\mu+\phi) \left[ (\mu+\phi) (\lambda-\phi) K+\lambda \mu \left( \frac{2(\theta+\phi+\theta)}{\theta+\phi+\tau} - 1 \right) \right. \\
\left. - (\lambda-\phi) K \left( \frac{(2\mu+\phi+\theta)}{\theta+\phi+\tau} \right) \right] 
\]

whence if \( \phi = 0 \) and \( \theta = -\mu \);

\[
C(0, -\mu) = 0 = \pi_0 \lambda \mu \left[ (\lambda+\mu) K \mu K \mu \left( \frac{2(\theta-\mu)}{\theta-\mu} - 1 \right) \right. \\
\left. - (\lambda+\mu) K \frac{(\theta-\mu)}{\theta-\mu} \right] 
\]

whence,

\[
\delta = \frac{(\lambda+\mu) K \mu - \lambda K \mu}{(\lambda+\mu) K - 2\lambda K} 
\]

and this equation will determine the constant \( \delta \) for at least all finite \( K \).

**Excursus. Check Solution of \( M|\delta|2 \)**

If \( K = 1 \) we require \( r \) to be the appropriate root of;

\[
(\lambda+r) (2\mu-\lambda) - 2\mu \lambda = 0
\]

whence;

\[
r = 2\mu - \lambda
\]

Use of (12.35) in (12.33) gives;

\[
\delta = 2\mu
\]

Therefore;

\[
\chi^*(\phi, \theta) = \frac{2\mu-\lambda}{2\mu+\lambda} \left( \frac{(2\mu+\phi+\theta)^2 - (\mu+\phi)(2\mu-\lambda+\theta+\phi)}{(\mu+\theta)(2\mu-\lambda+\theta+\phi)} \right)
\]
after using the obvious normalizing requirement, namely:

\[ \chi^*_+(0,0) = 1 \]  (12.38)

for the Laplace Transformation of any bivariate distribution function. Thus;

\[ \chi^*_+(\phi,0) = \chi^*_+(\phi) = \left(\frac{2\mu-\lambda}{2\mu+\lambda}\right) \left\{1+\frac{\lambda}{\mu} + \frac{\lambda^2}{(2\mu-\lambda+\phi)} \right\} \]  (12.39)

by rearrangement and by placing \( \theta = 0 \), where \( \chi^*_+(\phi) \) is the L.P.T. of the Delay Distribution of the \( M|M|2 \) queue. Direct comparison shows that this result contains identical probabilities and the same exponential distribution as A.K. Erlang's now classical result. Thus Erlang obtained the following for the ergodic \( M|M|2 \) queue;

\[ P(0) = \text{Probability of emptiness} \]  (12.40)

\[ P(1) = \text{Probability of one server busy} \]

\[ = \frac{\lambda}{\mu} P(0) \]  (12.41)

\[ P(2) = \text{Probability of two busy servers and from 0 to } \infty \text{ customers waiting,} \]

\[ = \frac{\lambda^2}{2\mu} \left(\frac{2\mu-\lambda}{2\mu+\lambda}\right) P(0) \]

\[ = \frac{\lambda^2}{\mu(2\mu-\lambda)} P(0) \]  (12.42)

with the normalizing relationship;

\[ P(0) = P(1) = P(2) = 1 \]  (12.43)

whence,

\[ P(0) = \frac{2\mu-\lambda}{2\mu+\lambda} \]  (12.44)

\[ = \pi_0 \] as defined in equation (12.37).
Erlang also deduced the delay distribution to be a single exponential and to have the form:

\[ P(T > t) = P(2)e^{-(2-\frac{\lambda}{\mu})t/h} \]  

(12.45)

where \( h = \text{mean service time} \)

\[ = P(2)e^{-(2\mu-\lambda)t} \]  

(12.46)

since

\[ h = \mu^{-1} \]  

(12.47)

for a negative exponential service time distribution with service rate \( \mu \). Thus from (12.45) one may obtain;

\[ \int_{\sigma^+} e^{-\phi t}dP(T < \tau) = (2\mu-\lambda)P(2)/(2\mu-\lambda+\phi) \]

\[ = \frac{\lambda^2}{\mu(2\mu-\lambda+\phi)} \left( \frac{2\mu-\lambda}{2\mu+\lambda} \right) \]  

(12.48)

and one sees that the results (12.44), (12.41), (12.42) and (12.48) may be assembled to give (12.39).

We have therefore roughly verified the correctness of our assertions in at least this the simplest of all possible cases.

\textbf{Remark:} We note that the root \( \omega = -r \) of the polynomial (12.8) is;

\[ r = 2\mu-\lambda \]  

(12.35)

whilst if \( \rho \) is the mean load per server, whence

\[ \rho = \frac{\bar{\rho}}{2\bar{\mu}} \]

where the bars indicate means, then;

\[ \rho = \frac{\lambda}{2\mu} \]
since
\[ S = \mu^{-1} \]
\[ \bar{S} = \lambda^{-1} \]

for M|M|2

Thus as \( \rho \to 1, \tau \to 0 \) and it appears that \( \chi^*_+(\phi, \theta) \) may cease to be analytic at the point \( \phi = \theta = 0 \). However an analytic (as opposed to the Kiefer and Wolfowitz probabilistic study [7]) investigation of the Necessary and Sufficient Conditions for Ergodicity was attempted and produced the result;

If \( \rho \to 1, \tau \to 0 \) and \( \chi^*_+(\phi, \theta) \to 0 \) for all finite \( \phi \) and \( \theta \). This is scarcely surprising.

It appears also that for \( E_K |E_L|2 \) class of queues (and perhaps others) there always exists a real \( r_1 \) or a set \( \{r_i\} \) for \( i = 1, \ldots, L \) for an \( E_L \) service time distribution such that as \( \rho \to 1, r_1 \to 0 \) Re(\( r_1 \)) \to 0 \) and \( \chi^*_+(\phi, \theta) \to 0 \) for all finite \( \phi \) and \( \theta \). This root may be readily identified since \( |r_1| < \min_{i>1} |r_i| \).

Hence we propose to call this first singularity the Kiefer Wolfowitz Singularity in view of its relevance to the existence of \( \chi^*_+(\phi, \theta) \) as the load per server \( (\rho) \) approaches unity.
13. The Solution of $E_K|E_2|^2$

$E_K|E_2|^2$ is the simplest situation wherein the matrix relationship (11.8) is not completely degenerate. Thus we use this as a further demonstration of the established principles and to show how rapidly the process increases in complexity as $L$ becomes larger. It also possesses the undeniable advantage that both $r_1$ and $r_2$ defined for equations (13.7.1) and (13.7.2) are real whence the simultaneous linear equations which arise possess only real coefficients.

The functional equation to be solved is:

$$\left(\lambda - \phi - \phi \right)^K (\mu+\phi)^2 (\mu+\theta)^2 \chi^*(\phi,\theta)$$

$$= C(\phi,\theta) \quad (13.1.1)$$

$$= (\mu+\phi)^2 \sum_{m=0}^{K} \phi^m d_m(\theta) \quad (13.1.2)$$

$$= (\lambda^K - (\lambda - \phi)^K (\mu+\phi)^2) (\mu+\theta)^2 \chi^*(\phi,\theta)$$

$$+ \lambda^2 \left[ (\mu+\phi)^2 \left[ \chi^*_+(-\mu,\mu+\phi)+(\mu+\theta)D_z \{ \chi^*_+(z-\mu,\mu+\phi+z) \}_{z=0} \right] \right]$$

$$- (\mu+\phi)^2 \left[ \chi^*_+(-\mu,\mu+\phi)+(\mu+\phi)D_z \{ \chi^*_+(z-\mu,\mu+\phi+z) \}_{z=0} \right] \quad (13.1.3)$$

where,

$$C(\phi,\theta) = 0(|\phi|^{K+2}) \quad (13.2.1)$$

if $|\phi| \to \infty$ and $|\theta| \to \infty$

$$= 0(|\theta|^{K+2}) \quad (13.2.2)$$

if $|\theta| \to \infty$ and $|\phi| \to \infty$

by the use of Theorems (9.3.2) and (9.7)
As before, write:

\[ \phi = v - \mu \]  \hspace{1cm} (13.3.1)

\[ \theta = \omega - \phi = \mu + \omega - v \]

and now obtain:

\[
(\lambda - \omega)^K (2\mu + \omega - v)^2 v^2 \chi^-_\text{top} (v-\mu, \mu + \omega - v) \\
= C(v-\mu, \mu + \omega - v) \hspace{1cm} (13.4.1)
\]

\[
= v^2 \sum_{m=0}^{K} (v-\mu)^m d_m (\mu + \omega - v) \hspace{1cm} (13.4.2)
\]

\[
= (\lambda \mu^2 - (\lambda - \omega)^2) (2\mu + \omega - v)^2 \chi^-_\text{top} (v-\mu, \mu + \omega - v) \hspace{1cm} (13.4.3)
\]

\[
+ \lambda^2 (2\mu + \omega - v)^2 \left[ \sum_{m=0}^{\infty} (\chi^-_\text{top} (z-\mu, \mu + \omega - z) D_z (\chi^-_\text{top} (z-\mu, \mu + \omega - z)) \right] \\
\]

We now show that:

\[
C(v-\mu, \mu + \omega - v) = O(|v|^4) \hspace{1cm} (13.5.1)
\]

as \( |v| \to \infty \) for all finite \( \omega \).

**Proof:** \( C(\phi, \theta) \) is a polynomial in both \( \phi \) and \( \theta \). Hence the substitutions (13.3.1) and (13.3.2) create a bivariate polynomial in \( v \) and \( \omega \) and it merely remains to determine the order.

By Theorem (10.1) \( \chi^-_\text{top} (\phi, \theta) \) is rational and \( O(1) \) as either \( |\phi| \) or \( |\theta| \to \infty \). Thus \( \chi^-_\text{top} (v-\mu, \mu + \omega - v) \) is also rational in \( v \) and \( \omega \) and as \( |v| \to \infty \) we find:

\[
(\lambda - \omega)^K (2\mu + \omega - v)^2 v^2 \chi^-_\text{top} (v-\mu, \mu + \omega - v) \\
\sim (\lambda - \omega)^K v^4 (-\pi_o + m^*_\text{top} (\omega)) \hspace{1cm} (13.5.2)
\]

at least for all finite \( \omega \) s.t. \( m^*_\text{top} (\omega) \) exists where;
\[ m_*(\omega) = \int_{-\infty}^{0} e^{-\omega x} d\chi_-(x,x) \]  

(13.5.3)

and is the Laplace Transformation of any probability concentrated on the line \( x=y, x (-\infty,0) \), since;

(i) \[ \lim_{\text{Re}(v) \to \infty} \{ \chi^*(v-\mu+\omega-v) \} \]

\[ = \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{0} \int_{x+\varepsilon}^{x+\varepsilon} e^{-\omega y} d\chi_-(x,y) \right\} \]  

(13.5.4)

\[ = m^*(\omega) - \pi_0 \]  

(13.5.5)

by the definition of \( m^*(\omega) \), \( \chi^*(x,y) \) and \( \pi_0 \), and

(ii) It is clear that if \( \chi^*(v-\mu+\omega-v) \) is rational and contains \( m^*(\omega) \), then this is also rational. Thus (13.5.2) is true except possibly where \( m^*(\omega) \) has its poles, if any.

We may also consider \( \lim_{|v| \to \infty} \{ C(v-\mu+\omega-v) \} \) by use of equation (13.4.3) and show that;

(i) \[ M(v-\mu+\omega-v) = O(|v|^3) \]  

(13.5.6)

as \( |v| \to \infty \) for all finite \( \omega \).

Note: \( M(\cdot,\cdot) \) is defined by equation (7.19),

and (ii) as \( |v| \to \infty \)

\[ (\lambda - (\lambda - \omega)^2)(2\mu + \omega - v)^2 \chi^*(v-\mu+\omega-v) \]

\[ - (\lambda - \omega)^4 (\pi_0 + m^*(\omega)) \]  

(13.5.7)

where \( m^*(\omega) \) is the Laplace Transformation of any concentration of probability which may exist on the line \( y=x, x \in (0,\infty) \). Hence (13.5.2) and (13.5.7) both assess

\[ \lim_{|v| \to \infty} \{ C(v-\mu+\omega-v) \} \]  

for the polynomial \( C(v-\mu+\omega-v) \) and

it is necessary that;
(i) \[ C(v-u,\mu+\omega-v) = 0(|v|^4) \]
as asserted since if it were a polynomial of any higher order
there would exist a \( \omega \) s.t. for all \(|v| > \omega \) and any \( \Lambda(\alpha) \in [0, 2\pi] \)
and any finite \( \omega \),
\[ C(v-u,\mu+\omega-v) = 0(|v|^4) \]
which would be a contradiction.

(ii) \[ (\lambda-\omega)^K_4 (m^*_-(\omega) - \pi_0) \] \[ = -(\lambda-\omega)^K_4 (\pi_0 + m^*_+(\omega)) \]
as \(|v| \to \infty \) and for all finite \( \omega \) s.t. \( m^*_-(\omega) \) and \( m^*_+(\omega) \) are both
analytic. Thus we have the useful corollary that these are
both entire and;
\[ m^*_+(\omega) = -m^*_-(\omega) = 0 \]
\( \forall \omega \) as the Laplace Transforms of probability measures on
the strictly positive and negative half lines may only be equal\(^{\text{mod} 1} \)
(and thus define an entire function by Schwartz continuation)
if both are zero.

We may therefore write;
\[ C(v-u,\mu+\omega-v) = v^2 \{ f_0(\omega) + v f_1(\omega) + v^2 f_2(\omega) \} \]
(13.5.10)
where \( f_0(\omega) \), \( f_1(\omega) \) and \( f_2(\omega) \) are polynomials in \( \omega \), by use of
(13.4.2) and (13.5.1).

Write;
\[ v = h(\omega) = \pm \{ \frac{\lambda}{\lambda-\omega} \}_{p}^{K/2} \]
(13.6.1)
where the notation \( \{ \}_{p} \) denotes the principal (i.e. positive
in this case) branch and note that;
\[ C(v-u,\mu+\omega-v) = M(v-u,\mu+\omega-v) \]
(13.6.2)
if \( v \) is defined by (13.6.1). [This is by Theorem (9.4).]

Use of the positive branch of (13.6.1) gives;

\[
C_1(\omega) = \frac{\lambda K^2}{(\lambda - \omega)^K} \left\{ f_0(\omega) + \mu \left( \frac{\lambda}{\lambda - \omega} \right)^{K/2} f_1(\omega) + \frac{\lambda K^2}{(\lambda - \omega)^K} f_2(\omega) \right\}
\]

\[
= \lambda K^2 \left[ \frac{\lambda K^2}{(\lambda - \omega)^K} \left[ G_0(\omega) + \left( 2\mu + \omega - \mu \left( \frac{\lambda}{\lambda - \omega} \right)^{K/2} \right) G_1(\omega) \right] \right]
\]

(13.6.3)

whilst the negative branch gives;

\[
C_2(\omega) = \frac{\lambda K^2}{(\lambda - \omega)^K} \left\{ f_0(\omega) - \mu \left( \frac{\lambda}{\lambda - \omega} \right)^{K/2} f_1(\omega) + \frac{\lambda K^2}{(\lambda - \omega)^K} f_2(\omega) \right\}
\]

\[
= \lambda K^2 \left[ \frac{\lambda K^2}{(\lambda - \omega)^K} \left[ G_0(\omega) + \left( 2\mu + \omega + \mu \left( \frac{\lambda}{\lambda - \omega} \right)^{K/2} \right) G_1(\omega) \right] \right]
\]

(13.6.4)

where as before;

\[
G_0(\omega) = \chi^*_+(-\mu, \mu + \omega)
\]

and,

\[
G_1(\omega) = D_2 \left\{ \chi^*_+(z - \mu, \mu + \omega - z) \right\}_{z=0}
\]

Remark: The Example matrix of Section 11 applies to this system.

Solution of (13.6.3) and (13.6.4) as a pair of simultaneous linear equations gives;
\[ G_0(\omega) = \frac{6\mu^2 (2\mu+\omega) (\frac{\lambda}{\lambda-\omega})^{K}[c_1(\omega) - c_2(\omega)]}{4\mu^3 (2\mu+\omega)^2 \{4\lambda \mu^2 -(2\mu+\omega)^2 (\lambda-\omega)K\}} \]  \hspace{1cm} (13.6.5) \\
\[ G_1(\omega) = \frac{3K}{(\frac{\lambda-\omega}{\lambda})^2} + \frac{[c_1(\omega) + c_2(\omega)] \{4\mu^3 (\frac{\lambda}{\lambda-\omega})^2 + 2\mu (2\mu+\omega)^2 (\frac{\lambda}{\lambda-\omega})K/2\}}{4\mu^3 (2\mu+\omega)^2 \{4\lambda \mu^2 -(2\mu+\omega)^2 (\lambda-\omega)K\}} \]  \hspace{1cm} (13.6.6) \\
and, \\
\[ G_1(\omega) = \frac{\lambda K/2 + 2\mu K/2}{2(\mu)^2 - (2\mu+\omega)(\lambda-\omega)K/2} c_1(\omega) \]  \\
\[ G_2(\omega) = \frac{2(\mu)^2 - (2\mu+\omega)(\lambda-\omega)K/2}{4\lambda \mu^2 -(2\mu+\omega)^2 (\lambda-\omega)K} \]

which are not necessarily rational. However they must both exist when \( \omega=\lambda \) and for at least all \( \omega \) s.t. \( \text{Re}(\omega)>0 \). (See previous discussion in Section 11 if this is not obvious)

It is therefore necessary that \( G_0(\omega) \) and \( G_1(\omega) \) be of the form;

\[ G_0(\omega) = \frac{H_0(\omega)}{(2\mu+\omega)^2 \prod_{i=1}^{2} (\omega + \omega_i)} \]  \hspace{1cm} (13.7.1) \\
and,

\[ G_1(\omega) = \frac{H_1(\omega)}{2 \prod_{i=1}^{2} (\omega + \omega_i)} \]  \hspace{1cm} (13.7.2) \\

where \( H_0(\omega) \) and \( H_1(\omega) \) are polynomials s.t.;

\[ H_0(\omega) = 0(|\omega|^4) \]  \hspace{1cm} (13.7.3) \\
and,

\[ H_1(\omega) = 0(|\omega|^2) \]  \hspace{1cm} (13.7.4) \\

that \( f_2(\omega) \) be of the form.
\[ f_2(\omega) = (\lambda - \omega)^K f_2(\omega) \quad (13.7.5) \]

and that \(-r_1\) and \(-r_2\) be the two roots of

\[ 4\lambda^2 - (2\mu+\omega)^2 (\lambda - \omega)^K = 0 \]

which lie in \(\text{Re}(\omega) < 0\).

We may prove each of the above points as follows:

(i) \(G_0(\omega)\) and \(G_1(\omega)\) would both be rational if the terms

in \((\lambda - \omega)^K/2\) or \((\lambda - \omega)^K/2\) did not exist in the numerators. However if these functions are not rational then the point \(\omega = \lambda\)

must be a branch point and thus a singular point of both, whence the branch point does not in fact exist since both \(G_0(\omega)\) and \(G_1(\omega)\) must be holomorphic if \(|\omega| < \infty\) and \(\text{Re}(\omega) > 0\). See Lemmas (6.14) and (6.21).

Hence both \(G_0(\omega)\) and \(G_1(\omega)\) are rational.

(ii) By (13.6.5) it is clear that the only possible poles of \(G_0(\omega)\) which will satisfy the requirement that it be analytic

if \(\text{Re}(\omega) > 0\) and exist when \(\omega = 0\) are those shown in the denominator

of (13.7.1)

(iii) By Lemma (6.16)

\[ \lim_{\text{Re}(\omega) \to \infty} \{ \chi_+(-\mu, \mu+\omega) \} \]

\[ = \lim_{\text{Re}(\omega) \to \infty} \{ G_0(\omega) \} \]

\[ = \pi_0 > 0 \quad (13.7.6) \]

Hence the rational quotient (13.7.1) exists as \(|\omega| \to \infty\) and

(13.7.3) is clearly necessary.
(iv) By similar argument to (i) the only possible poles of \( G_1(\omega) \) are those shown in (13.7.2).

(v) By Lemma (6.22),

\[
\lim_{\Re(\omega) \to \infty} \{ G_1(\omega) \} = 0
\]

whence the rational quotient (13.7.2) must be such that

\[
\lim_{|\omega| \to \infty} \{ G_1(\omega) \} = 0
\]

whence (13.7.4) is necessary.

(vi) If (13.7.5) were not true, then as \( \omega \to \lambda \) in equation (13.6.6) we should obtain;

\[
G_1(\omega) \propto \frac{f_2(\omega)}{(2\mu+\omega)(\lambda-\omega)^K}
\]

after substituting for \( C_1(\omega) \) and \( C_2(\omega) \) by use of the LH sides of (13.6.3) and (13.6.4). However \( G_1(\omega) \) is clearly analytic at \( \omega = \lambda \) and (13.7.9) cannot therefore be true. Hence (13.7.5) is true as asserted.

Remarks: 1. Nothing may however be said yet about \( f_1(\omega) \) as \( \omega \to \lambda \).

2. Obviously something more may be said about \( G_0(\omega) \) and \( G_1(\omega) \) and \( C_1(\omega) \) and \( C_2(\omega) \) since the necessity of (13.7.1) and (13.7.2) etc. implies a good deal more about the numerators of (13.6.5) and (13.6.6). However \( G_0(\omega) \) and \( G_1(\omega) \) are very simple forms and it is easier to proceed otherwise.
Write:

\[ G_0(\omega) = \frac{\pi_0 \left\{ \omega^4 + S_1 \omega^3 + S_2 \omega^2 + S_3 \omega + S_4 \right\}}{(2\mu + \omega)^2 \prod_{i=1}^2 (\omega + r_i)} \]  \hspace{1cm} (13.8.1)

by use of (13.7.1) and (13.7.3) where the \( S_i \) for \( i = 1, \ldots, 4 \) are constants analogous to the \( s \) defined in equation (12.33).

Write:

\[ H_1(\omega) = \pi_0 \left\{ k_{12} \omega^2 + k_{11} \omega + k_{10} \right\} \]  \hspace{1cm} (13.8.2)

where \( k_{12}, k_{11} \) and \( k_{10} \) are constants, from (13.7.4). Rewrite (13.4.3) in the form:

\[ v^2 \sum_{m=0}^2 v^m f_m(\omega) \]

\[ = (\lambda^2 - (\lambda-\omega)k^2)(2\mu + \omega - v)^2 \chi_+^*(v-\mu, \mu + \omega - v) \]  \hspace{1cm} (13.8.3)

\[ + \lambda^2 k^2 \left[ v^2 - (2\mu + \omega - v)^2 \right] \pi_0 \frac{(\omega + S_1 \omega^3 + S_2 \omega^2 + S_3 \omega + S_4)}{(2\mu + \omega)^2 \prod_{i=1}^2 (\omega + r_i)} \]

\[ + \lambda k^2 \frac{(2\mu + \omega - v)(2v - \omega - 2\mu)\pi_0 \left\{ k_{12} \omega^2 + k_{11} \omega + k_{10} \right\}}{(2\mu + \omega)^2 \prod_{i=1}^2 (\omega + r_i)} \]

Rearrange (13.8.3) as follows:

\[ \chi_+^*(v-\mu, \mu + \omega - v) \]
\[ \sum_{m=0}^{2} \nu^m f_m(\omega) = \left( \lambda \mu^2 \right)^{-2} \frac{2^2 (2\mu+\omega-v)^2}{(2\mu+\omega)^2 \prod_{i=1}^{2} (\omega+i)} \left[ v^2 - (2\mu+\omega-v)^2 \right] \left[ \omega^4 + s_1 \omega^3 + s_2 \omega^2 + s_3 \omega + s_4 \right] \]

\[ + v (2\mu+\omega-v)(2\nu-2\mu) \left( 2\mu+\omega \right) \left[ k_{11} \omega + k_{10} + k_{12} \omega^2 \right] \]

\[ (\lambda - \omega)^2 \nu^2 (2\mu+\omega-v)^2 \]

It was shown in the proof of Theorem (9.4) and again discussed in the proof of Theorem (10.4) that none of the zeroes of;

\[ \lambda \mu^2 - (\lambda - \omega)^2 \nu^2 \]

had loci which were such as to be the loci of acceptable singularities of \( \chi^*_+(v-\mu, \mu+\omega-v) \). Thus the constants \( s_i^0 \) for \( i=1, \ldots, 4 \) and \( k_{12} \), \( k_{11} \) and \( k_{10} \) must be such that (13.8.5) is a factor of the numerator of (13.8.4). Thus by consideration of the possible order of the function which may result in the numerator of the quotient one obtains the form;

\[ \chi^*_+(v-\mu, \mu+\omega-v) = \frac{j_0(\omega)+vj_1(\omega)+v^2j_2(\omega)}{(2\mu+\omega-v)^2} \]

where the necessarily rational functions \( j_0(\omega), j_1(\omega) \) and \( j_2(\omega) \) remain to be determined.

By Lemma (6.18);

\[ \lim_{\text{Re}(\nu) \to -\infty} \{ \chi^*_+(v-\mu, \mu+\omega-v) \} = \pi_0 + m^*(\omega) \]

at least for all finite \( \omega \) s.t. the Laplace Transformation \( m(\omega) \) exists. However;

\[ m^*(\omega) = 0 \]

in this situation since;
(i) By Remark 3 to Lemma (6.18); 
\[ \lim_{\omega \to \infty} m^*(\omega) = 0 \]

and (ii) \( m(\omega) \) is clearly rational in this case since \( \chi_+^{*(v-\mu, \mu+\omega-v)} \) is and contains it. Thus by use of (13.8.7) and (13.8.8) in \( \lim_{v \to \infty} \{13.8.6\} \) one obtains;

\[ j_2(\omega) = \pi_0 \]  

(13.8.9)

Also, since;

\[ G_0(\omega) = \{ \chi_+^{*(v-\mu, \mu+\omega-v)} \}_{v=0} \]

\[ = \frac{j_0(\omega)}{(2\mu+\omega)^2} \]

\[ = \frac{\pi_0 \{ \omega^4 + S_1 \omega^3 + S_2 \omega^2 + S_3 \omega + S_4 \}}{(2\mu+\omega)^2} \prod_{i=1}^{2} (\omega+\tau_i) \]  

(13.8.10)

one obtains;

\[ j_0(\omega) = \pi_0 \frac{\omega^4 + \sum_{i=1}^{4} S_i \omega^{4-i}}{\prod_{i=1}^{2} (\omega+\tau_i)} \]  

(13.8.11)

Moreover by,

\[ G_1(\omega) = \{ \chi_+^{*(v-\mu, \mu+\omega-v)} \}_{v=0} \]

\[ = \frac{j_1(\omega)}{(2\mu+\omega)^2} + \frac{2j_0(\omega)}{(2\mu+\omega)^3} \]

\[ = \frac{\pi_0 [k_{12} \omega^2 + k_{11} \omega + k_{10}]}{(2\mu+\omega)^2} \prod_{i=1}^{2} (\omega+\tau_i) \]  

(13.8.12)

one obtains;
\[ j_1(\omega) = \frac{\pi_0 \{(2\mu+\omega)^2 \left[ k_{12} \omega^2 + k_{11} \omega + k_{10} \right] - 2 \left[ \omega^4 + \sum_{i=1}^{4} S_i \omega^{4-i} \right]\}}{(2\mu+\omega)^{\sum_{i=1}^{2} (\omega+x_1_i)}} \]  

(13.8.13)

One may now rewrite (13.8.4) as follows:

\[ \chi_+(\nu-\mu, \mu+\omega-\nu) \]

\[ \pi_0 \left\{ \omega^4 + S_1 \omega^3 + S_2 \omega^2 + S_3 \omega + S_4 \right\} \]

\[ +(\nu) \left\{ (2\mu+\omega) \left[ k_{12} \omega^2 + k_{11} \omega + k_{10} \right] \right\} \]

\[ - \left( \frac{2}{2\mu+\omega} \right) \left\{ \omega^4 + S_1 \omega^3 + S_2 \omega^2 + S_3 \omega + S_4 \right\} \]

\[ + \nu^2 \prod_{i=1}^{2} (\omega+x_1_i) \]

(13.9)

Substitute:

\[ \omega = \theta + \phi \quad \text{and} \quad \nu = \mu + \phi \]  

(13.10.1)

(13.10.2)

to invert (13.3.1) and (13.3.2) and obtain the rational form:

\[ \chi_+^*(\phi, \theta) = \frac{(2\mu+\theta+\phi) \left\{ (\theta+\phi)^4 + \sum_{i=1}^{4} (\theta+\phi)^{4-i} S_i \right\}}{(2\mu+\theta+\phi)^{\sum_{i=1}^{2} (\theta+\phi+x_1_i)}} \]

(13.10.3)
Remark: Of course the fascinating thing is that poles (or perhaps in the more general case singularities with loci of the same form) such as \((\theta+\phi+r_1)\) were not intuitively obvious to somebody many years ago. They are obvious once one looks at the multi-server queueing process in the right way. See the remarks on the "Queuelet Queue" in Section 16.

By Lemma (6.3);

\[
\lim_{\text{Re}(\phi) \to \infty} \left\{ \pi_1^*(\phi, \theta) \right\} = \pi_0 + \pi_1^*(\theta) \tag{13.11.1}
\]

since \(\pi_1^*(\theta) = 0\) for queues in this class. (See Corollary to Theorem (9.3)) whilst;

\[
\lim_{\text{Re}(\theta) \to \infty} \{\pi_1^*(\theta)\} = 0 \tag{13.11.2}
\]

and

\[
\pi_1^*(\theta) \leq \pi_1^*(0) = \pi_1^* < 1 \tag{13.11.3}
\]

at least for all finite \(\theta\) s.t. \(\text{Re}(\theta) > 0\), by the obvious properties of the Laplace Transformation of measures on \((0, \infty)\). Hence the numerator of the rational fraction (13.10.3) must be \(O(|\phi|^3)\) for all finite \(\theta\), although the form appears to contain powers up to \(\phi^5\).

The coefficients of \(\phi^5\) all cancel iff;

\[
k_{12} = 0 \tag{13.11.4}
\]

although the coefficients of \(\phi^4\) in the numerator are;

\[
\pi_0 \left\{ 2\mu + \mu_1 + 4\phi + k_{11} - 2[\mu + \mu_1 + 4\phi] + 4\mu + r_1 + r_2 + 3\phi \right\} = 0 \tag{13.11.5}
\]

by virtue of the limited order. Thus we obtain the useful linear relationship;
\[ k_{11} = S_1 - 4\mu - R_1 \] \hspace{1cm} (13.11.6)

where;
\[ R_1 = r_1 + r_2 \] \hspace{1cm} (13.11.7)

It will also be obvious that since \( \chi_{+}(x,y) \) is a proper distribution on \( 0 \leq x < y < \infty \),
\[ \chi_{+}^*(0,0) = 1 \] \hspace{1cm} (13.11.8)

for normalization whence,
\[ \pi_o = \frac{r_1 r_2}{2k_{10} + r_1 r_2} \] \hspace{1cm} (13.11.9)

One may now rewrite (13.8.3) in the form given on the next page.
in order to equate the coefficients of the various powers of \( v \).

Thus one may easily verify that the coefficients of \( v^0 \) and \( v^1 \) are zero in the RHS of (13.13.1) whilst the coefficients of \( v^2 \) are;

\[
\begin{align*}
\pi_0 (\omega) &= \frac{4^0 \sum_{i=1}^{4} S_i \omega^{4-i}}{2 (\omega+r_1)} \\
\pi_0 (\omega) &= \frac{4^0 \sum_{i=1}^{2} S_i \omega^{4-i}}{2 (\omega+r_1)}
\end{align*}
\]

(13.12.2)

Also the coefficients of \( v^3 \) are;

\[
\begin{align*}
f_1 (\omega) &= -\pi_0 (\lambda-\omega) \left[ \frac{2 (\omega+r_1)^2 (k_{11} + k_{10}) - 2 (\omega^4 + \sum_{i=1}^{4} S_i \omega^{4-i})}{(2\mu+\omega) \sum_{i=1}^{2} (\omega+r_1)} \right] \\
f_1 (\omega) &= -\pi_0 (\lambda-\omega) \left[ \frac{2 (\omega+r_1)^2 (k_{11} + k_{10}) - 2 (\omega^4 + \sum_{i=1}^{4} S_i \omega^{4-i})}{(2\mu+\omega) \sum_{i=1}^{2} (\omega+r_1)} \right]
\end{align*}
\]

(13.12.3)

whilst the coefficients of \( v^4 \) are simply;

\[
f_2 (\omega) = -\pi_0 (\lambda-\omega)^2
\]

(13.12.4)

which confirms (12.7.5).

If one multiplies both LHS and RHS of (13.12.2) by \( \sum_{i=1}^{2} (\omega+r_1) \) and places \( \omega = -r_j \) for \( j=1,2 \), one obtains;

\[
3\lambda^2 (k_{10} - r_j k_{11}) = (\lambda+r_j)^2 \left[ (-r_j)^{4+i} + \sum_{i=1}^{4} S_i (-r_j)^{4-i} \right]
\]

(13.12.5)

for \( j = 1, 2 \)

which are two useful linear relationships between the 6 constants.
By similar operations on (13.12.3) one may obtain:
\[ (\lambda + r_j)^K((2\mu - r_j)^2[k_{10} - r_j k_{11}] - 2[(-r_j)^4 + \sum_{i=1}^{4} S_i (-r_j)^{4-i}]) \]
\[ + 2\lambda \mu^2 [k_{10} - r_j k_{11}] \]
\[ = 0 \quad \text{for } j = 1, 2. \]

Also by use of \( \omega = -2\mu \) in (13.12.3) one obtains;
\[ (\lambda + 2\mu)[(-2\mu)^4 + \sum_{i=1}^{4} S_i (-2\mu)^{4-i}] = \lambda \mu^2 (k_{10} - 2\mu k_{11}) \]  

However the use of (13.12.5) in (13.12.6) implies that;
\[ ((\lambda + r_j)^K(2\mu - r_j)^2 - 4\lambda \mu^2)(k_{10} - r_j k_{11}) = 0 \]  

whence if the \(-r_j\) are any suitable roots of the polynomial
\[ (\lambda - \omega)^K(2\mu + \omega)^2 - 4\lambda \mu^2 = 0 \]
as previously deduced, neither \( k_{11} \) nor \( k_{10} \) need be zero and a non-trivial solution is possible.

Remarks: Clearly the form (13.10.3) implies;
\[ \lim_{|\phi| \to \infty} \lim_{|\theta| \to \infty} \{X^*_+(\phi, \theta)\} \]
\[ = \lim_{|\theta| \to \infty} \lim_{|\phi| \to \infty} \{X^*_+(\phi, \theta)\} \]
\[ = \lim_{|\phi| = |\theta| \to \infty} \{X^*_+(\phi, \theta)\} \]
\[ = \pi_o \]

which is consistent with the existance of \( \pi_o \) as a concentration of probability at the point \( x = y = 0 \).
We now possess four linear equations for the six unknowns $k_{11}$, $k_{10}$, $S_1$, $S_2$, $S_3$, and $S_4$ and a fifth equation for $\pi_0$. The two remaining equations are obtained from the roots of $C(\phi, \theta)$. We therefore construct:

$$C(v-\mu, \mu+\omega-v)$$

$$= v^2 \sum_{m=0}^{\infty} \left[ \prod_{i=1}^{2} (\omega+r_{11})^{-1} \right] \left[ (\lambda^2 + 2 \mu \pi) \prod_{i=1}^{2} (\omega+r_{11})^{-1} \right]$$

$$+ 3\lambda \mu^2 (k_{11} + k_{10})^2$$

$$\pi_0 v^2 \left[ \prod_{i=1}^{2} (\omega+r_{11}) \right]$$

and firstly note that if:

$$\omega = \theta + \phi = 0$$

and,

$$v = \mu + \phi = \lambda$$

whence

$$\theta = \phi = 0$$

then

$$C(0,0) = 0$$

(13.13.3)
which is in obvious agreement with the result which arises if we place \( \theta = \phi = 0 \) in equation (13.1.3). However,

\[
C(0, \theta) = (\mu + \theta)^2 \tilde{C}(0, \theta)
\]

(13.13.4)

since generally,

\[
C(0, \theta) = (\mu + \theta)^L \tilde{C}(0, \theta)
\]

(13.13.5)

and we may use (13.13.4) to obtain the remaining equations.

The proof of (13.13.5) is as follows:

By Lemma (6.13):

\[
\chi_-(0, \theta) = \chi_-(0, \theta)
\]

for at least all finite \( \theta \) s.t. \( \chi_-(0, \theta) \) exists. However this is analytic if \( \text{Re}(\theta) < 0 \), whence;

\[
\chi_-(0, \theta) < \infty
\]

(13.13.6)

if \( \text{Re}(\theta) < 0 \). In particular, the product

\[
(\lambda - \theta)^K \mu (\mu + \theta)^L \chi_-(0, \theta)
\]

\[
= (\lambda - \theta)^K \mu (\mu + \theta)^L \chi_-(0, \theta)
\]

\[
= (\mu + \theta)^L \tilde{C}(0, \theta)
\]

where \( \tilde{C}(0, \theta) \) is an analytic function of \( \theta \) if \( |\theta| < \infty \) since \( C(\phi, \theta) \) is entire. Hence (13.13.5) is true as asserted.

Thus;
by (13.13.4). Rearrangement gives:

\[ R_2 = \mu R_1 + \mu^2 \]

\[ \left( \lambda K \mu^2 - (\lambda + \mu) K \mu^2 \right) \left[ R_2 - \mu R_1 + \mu^2 \right] \]

\[ \left\{ \lambda K \mu^2 - (\lambda + \mu) K \mu^2 \right\} \left[ \mu^2 (k_{10} - \mu k_{11}) - 2(-\mu)^2 + \sum_{i=1}^{4} S_i (-\mu)^{4-i} + 3\lambda K \mu^2 (k_{10} - k_{11}) \right] \]

\[ 0 \]

where

\[ R_2 = r_1 r_2 \]

and

\[ R_1 = r_1 + r_2 \]

Differentiation of (13.13.2) with respect to \( \omega \) followed by substitution of \( v = \mu \) and \( \omega = -\mu \) and the use of (13.13.9) and (13.13.10) gives;
\[
\begin{align*}
(\lambda \mu^2 - (\lambda + \mu) K \mu^2 + K(\lambda + \mu)^{K-1} \mu^3) R_2 & \\
-\lambda (\lambda + \mu) K^{K-1} \mu^4 R_1 & \\
-((\lambda + \mu) K + K(\lambda + \mu) K^{-1} \mu)^0 S_4 & \\
+(2(\lambda + \mu) K + K(\lambda + \mu) K^{-1} \mu^2)^0 S_3 & \\
-(3(\lambda + \mu) K^2 + K(\lambda + \mu) K^{-1} \mu^3)^0 S_2 & \\
+(4(\lambda + \mu) K^3 + K(\lambda + \mu) K^{-1} \mu^4)^0 S_1 & \\
+(3\lambda K \mu^2 + K(\lambda + \mu) K^{-1} \mu^3 - 2(\lambda + \mu) K^2) k_{10} & \\
-(2\lambda K^3 + K(\lambda + \mu) K^{-1} \mu^4 - (\lambda + \mu) K^3) k_{11} & \\
-4(\lambda + \mu) K \mu^4 + \lambda K \mu^4 & \\
\end{align*}
\]

which completes the tally of 6 simultaneous linear equations required to fully specify the solution of equation (13.1.3)
14. Summary of Equations for $E_K | E_L | 2$

The solution is of the form:

$$
\chi^*_+(\phi, \theta) = \pi_0 \begin{pmatrix}
(2\mu+\theta+\phi)(\theta+\phi)^4 + \sum_{i=1}^{4} S_i (\theta+\phi)^{4-i} \\
+(\mu+\phi) \left[ (2\mu+\theta+\phi)^2 [k_{11}(\theta+\phi)+k_{10}] \right] - 2\{ (\theta+\phi)^4 + \sum_{i=1}^{4} S_i (\theta+\phi)^{4-i} \} \\
+(\mu+\phi)^2 (2\mu+\theta+\phi) \left[ \frac{2}{\Pi} (\theta+\phi+\mu_i) \right] - 2\{ (\theta+\phi)^4 + \sum_{i=1}^{4} S_i (\theta+\phi)^{4-i} \}
\end{pmatrix}
$$

which is clearly rational.

$$
\pi_0 = \frac{r_1 r_2}{r_1 r_2 + 2k_{10}} = \frac{R_2}{R_2 + 2k_{10}}
$$

$$
k_{11} = S_1 - 4\mu - r_1 - r_2 = S_1 - 4\mu - R_1
$$

$$
(\lambda+2\mu)^K (-2\mu)^4 + \sum_{i=1}^{4} S_i (-2\mu)^{4-i} = \lambda^K \mu^2 (k_{10} - 2\mu k_{11})
$$

$$
(\lambda+r_1)^K (-r_1)^4 + \sum_{i=1}^{4} S_i (-r_1)^{4-i} = 3\lambda^K \mu^2 (k_{10} - r_1 k_{11})
$$

$$
(\lambda+r_2)^K (-r_2)^4 + \sum_{i=1}^{4} S_i (-r_2)^{4-i} = 3\lambda^K \mu^2 (k_{10} - r_2 k_{11})
$$

and the two (i.e. L) rather messy arisings of $C(0, -\mu)=0$, namely:

$$
\begin{pmatrix}
\mu^2 (\lambda^K - (\lambda+\mu)^K) \left[ \frac{2}{\Pi} (r_1 - \mu) \right] \\
+ (\lambda+\mu)^K (-\mu)^4 + \sum_{i=1}^{4} S_i (-\mu)^{4-i} \\
+ \mu^2 (\lambda^K - (\lambda+\mu)^K) (k_{10} - \mu k_{11})
\end{pmatrix} = 0
$$
and

\[
\begin{align*}
&\left(\lambda^{K} - (\lambda+\mu)^{K} + K(\lambda+\mu)^{K-1}\mu\right)\mu^{2}R_{2} \\
&-K(\lambda+\mu)^{K-1}\mu^{4}R_{1} \\
&-\left((\lambda+\mu)^{K} - K(\lambda+\mu)^{K-1}\mu\right)^{0}S_{4} \\
&+(2(\lambda+\mu)^{K} + K(\lambda+\mu)^{K-1}\mu^{2})^{0}S_{3} \\
&-(3(\lambda+\mu)^{K} + K(\lambda+\mu)^{K-1}\mu^{2})^{0}S_{2} \\
&+(4(\lambda+\mu)^{K} + K(\lambda+\mu)^{K-1}\mu^{3})^{0}S_{1} \\
&+(3\lambda^{K} + K(\lambda+\mu)^{K-1}\mu^{2} - 2(\lambda+\mu)^{K})\mu^{2}k_{10} \\
&-(2\lambda^{K} + K(\lambda+\mu)^{K-1}\mu^{3} - (\lambda+\mu)^{K})\mu^{3}k_{11} \\
&= (4(\lambda+\mu)^{K} + K)^{4}\mu^{4}
\end{align*}
\]  

where:

\[
R_{1} = r_{1} + r_{2}
\]

\[
R_{2} = r_{1}r_{2}
\]

and \(R_{1}\) and \(R_{2}\) are numbers such that;

\[
(\lambda+r_{j})^{K} (2\mu-r_{j})^{2} - 4\lambda^{K}\mu^{2} = 0
\]

and Re(\(r_{j}\))>0 for \(j = 1, 2\), and provided that the constants \(\lambda, \mu\) and \(K\) are such that two such numbers do in fact exist.

If two such numbers do not exist the queue is so loaded that an ergodic limiting distribution cannot exist whence all of the foregoing analysis is obviously irrelevant.
15. Sample Solution of $E_2/E_2^2/2$ and Simulation Check

15.1 Theoretical Values

The derivation and numerical solution of equations (14.1) to (14.7) is sufficiently complex to justify an independent numerical check. Consequently the equations were solved for $E_2/E_2^2/2$ with a mean load per server ($\rho$) of 0.75 and $\mu = 1$ to simplify the calculations.

The following are the parameters which were used:

\[
\begin{align*}
\mu &= 1.000\,000\,000 \\
\lambda &= 1.500\,000\,000 \\
r_1 &= 0.500\,000\,000 \\
r_2 &= 2.712\,214\,451 \\
R_1 &= 3.212\,214\,451 \\
R_2 &= 1.356\,107\,226
\end{align*}
\]

whence the following results were obtained:

\[
\begin{align*}
k_{11} &= 0.926\,744 \\
k_{10} &= 3.426\,673 \\
\pi_o &= 0.165\,188 \\
\bar{s}_4 &= 15.966\,039 \\
\bar{s}_3 &= 32.267\,360 \\
\bar{s}_2 &= 24.492\,325 \\
\bar{s}_1 &= 8.138\,958
\end{align*}
\]

all of which appear to be solutions to equations (14.1) to (14.7) correct to six decimal places.
Hence one may write:

\[
X_+^*(\phi, \theta) = \frac{0.165188^{(\phi+1)} \left( (\theta+\phi+2)(0.926744(\theta+\phi)+3.426673) \right) - 2 \left( (\theta+\phi)^4 + 8.138958(\theta+\phi)^3 \right) + 24.492325 (\theta+\phi)^2 + 32.267360 (\theta+\phi) + 15.966039}{(\theta+1)^2(\theta+\phi+2)(\theta+\phi+0.500000)(\theta+\phi+2.712214)}
\]

(15.1.3)
Whence for $\theta = 0$ the Laplace Transformation of the delay distribution may be written in the form:

$$X_+^{*}(\phi, 0) = 0.4584 + \frac{0.290334}{\phi+0.500000} + \frac{0.015311}{\phi+2.712214}$$

$$- \frac{0.08936}{\phi+2.000000}$$  \hspace{1cm} (15.1.4)

iff $\mu = 1$.

Thus the inversion of this simple form gives;

$$P(W_{<w}) = 1.0000 - 0.5807 -0.500\mu w$$

$$- 0.0056 -2.7122\mu w$$

$$+ 0.0447 -2.000\mu w$$  \hspace{1cm} (15.1.5)

for $E_2/E_2/2$ with $\rho = 0.75$

and where $\mu$ has been reintroduced to permit the time parameter to be rescaled.

Computation of 15.5 for $\mu = 2$ which corresponds with the time scale of the simulations if $L = 2$ gives the following values

<table>
<thead>
<tr>
<th>$w/\bar{w}$</th>
<th>$P(W_{&lt;w})$</th>
<th>$\sigma_{10,000}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>0.4584</td>
<td>0.0050</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.5012</td>
<td>0.0050</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.5427</td>
<td>0.0050</td>
</tr>
<tr>
<td>0.3000</td>
<td>0.5822</td>
<td>0.0049</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.6191</td>
<td>0.0049</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.6535</td>
<td>0.0048</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.6835</td>
<td>0.0047</td>
</tr>
<tr>
<td>0.7000</td>
<td>0.7143</td>
<td>0.0045</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.7409</td>
<td>0.0044</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.7651</td>
<td>0.0043</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.7872</td>
<td>0.0041</td>
</tr>
</tbody>
</table>

TABLE 15.1
1.2000  0.8256  0.0037
1.5000  0.8706  0.0034
2.0000  0.9214  0.0027
2.5000  0.9523  0.0021
3.0000  0.9711  0.0017

* σ = Standard Direction

where all figures are correct for four places and;

\[
\sigma_{10,000} = \left( \frac{10,000 \ P(W\leq w) (1\leq P(W\leq w))}{10,000} \right)^{1/2}
\]  

(15.1.6)

since the individual simulation runs each involved 10,000 arrivals.

15.2 The Simulations

The programme which was used to obtain the comparison results is given in Appendix I with some explanatory notes.

For the record, however, three variants on the same programme were tried, of which the first and third were, in fact, satisfactory, although the first appeared to give some trouble generating correct and independent values of \( s(n) \) and \( a(n) \). This apparent difficulty led to the development of Programme II which was obviously faulty, but which clarified the problems of Programme I. Consequently Programme III was written, which operated as follows;

Two independent sequences of pseudo random numbers were generated by the power residues method. (RANDU in the programme listing.) These were floating point numbers uniformly distributed on the interval from 0.000 000 000 to 0.999 999 999. The two sequences used different multipliers of the required 8t
±3 form (t is an integer) and one was used to generate the $a^{(n)}$ and the other the $s^{(n)}$. The required $E_2$ distributions were obtained by drawing two random numbers in sequence for each $a^{(n)}$ or $s^{(n)}$, multiplying these and then taking a suitably scaled natural logarithm of the product. Thus each $E_2$ distribution was obtained by the convolution of two exponentials.

Note: Programmes I and II were more general in that the distributions $A(a)$ and $B(s)$ were computed for 101 equally spaced values of both $a$ and $s$ such that any sequence of uniformly distributed random number $\varepsilon(0,1)$ could be used to obtain an appropriately distributed sequence of values of $a^{(n)}$ or $s^{(n)}$ by interpolation. Fourth order Gregory Newton interpolation was used but gave difficulties with minor inaccuracies for small $s^{(n)}$ and particularly $a^{(n)}$ values. Programme II endeavoured to correct this but managed to produce a partial "shaving off" of the lower tail of the $A(a)$ and $B(b)$ distributions with a consequent rather disturbing improvement in the $P(W < w)$ distribution. [Apparently by the deletion of most of the closely spaced "second arrivals" which nearly always have to wait.]

Programme III is believed to be free from any such bias sources and to thus be capable of producing as nearly a correct estimate of the $E_2/E_2/2$ Delay Distributions as may be obtained by any computer based simulation process. In any event, it is available for inspection and the card decks have been retained and may be rerun if this proves necessary.
15.3 Comparisons of Results

This is based on Table 15.2 and figure 15.1;

Table 15.2 follows on page 223

Table 15.2 contains the results of the 10 simulation runs for the points \( w/s = 0.0000 \) \((0.5000)\) 3.0000, the overall parameters of the individual runs and the theoretical values. It will be noticed that the largest individual discrepancy between any theoretical result and the mean result obtained from the 10 simulation runs is 0.0029 which corresponds with the event that 290 fewer arrivals (out of 100,000) experienced a delay of less than two service times than was predicted by the theory. Student's "T" Test was applied to this difference by using the sample standard deviation and gave \( t = 0.245 \) which must be compared with \( t = 1.835 \) for the 5% rejection level with 9 degrees of freedom. As may also be deduced from the graph (Figure 15.1) the standard deviations of the sample sets at the other points are comparable, whilst the differences between the theoretical and the simulation results are comparable or smaller.

Hence it is clear that there are no, in the least, significant statistical deviations between the theoretical and the simulation results for any of the points considered. This is remarkably good agreement in view of the variation of \( \rho \) for the individual simulation runs over the range \( 0.7448 \leq \rho \leq 0.7567 \)

for a mean of \( \bar{\rho} = 0.7541 \)
<table>
<thead>
<tr>
<th>( w/s )</th>
<th>Theoretical Values</th>
<th>Run No.</th>
<th>Simulation Results</th>
<th>Mean of ( p(w^2) )</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( P(\text{w} &lt; w) \times 10^4 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.0000</td>
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<td>.4444</td>
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<td>.4897</td>
<td>.4830</td>
</tr>
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<td>( s )</td>
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<td>71.155 70.635 68.966 71.498 71.237</td>
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<tr>
<td>( \rho )</td>
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<td>47.330 46.876 46.915 47.291 47.249</td>
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<table>
<thead>
<tr>
<th>( w/s )</th>
<th>Run No.</th>
<th>Simulation Results</th>
<th>Mean of ( p(w^2) )</th>
<th>Diff.</th>
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<td>100.151</td>
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<td>( \sigma_s )</td>
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<td>( \sigma_a )</td>
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<tr>
<td>( \rho )</td>
<td>46.812 46.175 46.539 46.460 47.207 46.685</td>
<td>46.455</td>
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<td></td>
</tr>
</tbody>
</table>

* Gives "t" = 0.245 as worst value for "t" Test. Not significant.
Figure 15.1 shows;

(i) A solid curve which is the theoretical result.

(ii) Two dotted curves which are the ±3 Standard Deviation limits for the binomial distribution applied to the theoretical probabilities.

(iii) The results of the individual simulation runs plotted as simple points based on $\bar{s} = 100.0000$ in each instance.

and (iv) The mean probabilities for the 10 runs plotted as larger ringed points.

It will be noticed that the probabilities produced by the individual runs are rather more widely dispersed than would be predicted by the binomial model (actually the Normal approximation thereto), but this is scarcely surprising in view of the variations in $\rho$ and $\bar{s}$ which arose between the individual simulation runs.

Thus it will be noticed that Run No.3, which persistently produced the largest $P(W<w)$ values and thus the delay distribution with the smallest moments, has the smallest $\bar{s}$, quite the smallest $\sigma_s$ and one of the smaller $\sigma_a$. See Table 15.2 for these details.

On the other hand Run No.4 which produces consistently the lowest $P(W<w)$ values, and thus the worst delay distribution, has the largest $\bar{s}$, the largest $\sigma_s$, the second highest $\sigma_a$ and the highest $\rho$.

One therefore concludes that much of the variation between
theoretical result and the results of the individual simulation runs is due to the variation in the exact parameters of the traffic generated by the simulator from one run to the next, rather than any abnormal variation in the responses of the queue being simulated to the applied loads. That is, it seems likely that if we could carry out 10 independent simulation runs each such that the estimated parameters $\rho$, $\bar{s}$, $\bar{a}$, $\gamma_s$ and $\gamma_a$ were all at least as close to the desired values as are the estimated parameters for the mean of the 10 runs, we should find that the binomial probability limits would hold quite well.

In these circumstances, it seems reasonable to assert that the agreement between the theoretical and simulation results is at least sufficiently good to not disprove the correctness of the preceding theoretical work. Furthermore the pattern of such differences as do exist appears to be quite normal for simulation verifications of this type.

Hence there is no evidence from this source which suggests that the theoretical solutions are in any wise defective.
16. **Interpretations and Conjectures**

The purpose of this section is to review some of the results obtained in the previous sections and to suggest explanations for some phenomena which may call for additional work.

16.1. **Systems of the E\(_k\)/E\(_L\)/2 Class**

16.1.1 **Forms of Solutions**

The results given for E\(_k\)/M/2 and E\(_k\)/E\(_2\)/2 suggest that the form of the Laplace Transformation which solves equation (7.17) depends only on \(L\) and thus the form of the service time distribution function alone. This sole dependence on \(L\) also appears to be implicit in the \(L\) by \(L\) dimensionality of the matrix \([M(\omega)]\) introduced in section 11, and the controlling role which this plays in the determination of the form of \(X_+^*(\nu-\mu,\mu+\omega-\nu)\) and thence trivially, \(X_+^*(\phi,\theta)\).

Some such result is scarcely surprising in view of W.L. Smith's equivalent observation for the queue GI/G/2. See [16].

The following explanation is offered as a reasonable conjecture which, if it is true, has quite wide implications.

"If a process is based on an embedded Markov chain which follows arrivals, and which therefore seeks the distribution of a single variable, or the joint distribution of several variables, where this is \(a\), or these are, residual service time or times, or the sums of service times, and one residual service time for each such variable, one might expect that the resulting distribution would;

1. Contain a maximum amount of information about the
service time distribution function and

(ii) Use a minimum amount of information about the form of the interarrival time distribution function."

However, this conjecture quite fails to explain why singularities with loci of the form \( \theta + \phi + r_1 = 0 \) arise, although it may lead one to suppose, perhaps quite illogically, that these should be poles if \( \phi^*(\theta) \) has only poles for its singularities. A better explanation for this point may be Pollaczek's contention that if \( a^*(\cdot) \) and \( \beta^*(\cdot) \) are rational fractions the Laplace Transformation of the delay distribution will also be rational.

What now needs to be done is to show whether the forms of solution which arise for queues of the \( E_k/E_0/2 \) class also hold for ergodic queues of the \( GI/E_2/2 \) class and thence whether these also hold for the more general class where the interarrival intervals have a correlational structure such as that of a geometrically smoothed moving average.

16.1.2 Uniqueness of Solutions

As it is well known, no probablistic problem is adequately solved unless the solution can be shown to be unique. In the two cases which we have investigated, this uniqueness is ensured by the fact that the process reduces to the determination of a set of constants which is just sufficient to use all the information which can be deduced from the analysis. Also, in the case of \( E_k/E_2/2 \) the problem reduces to simultaneous linear equations wherein the roots \( r_1 \) and \( r_2 \) of the polynomial;

\[
(\lambda-\omega)^k(2\mu+\omega)^2-4\lambda^2\mu^2 = 0
\]  

(16.1.2.1)
enter in the fully permutable forms;

\[ R_1 = r_1 + r_2 \]
\[ R_2 = r_1 r_2 \]  

(16.1.2.2)

Despite the foregoing, it seems quite likely that equation (7.17) would have multiple solutions and we presume that the apparent uniqueness of the solution form which we have constructed arises from our requirement that \( \chi^*(\phi, \theta) \) and \( \chi_+^*(\phi, \theta) \) be the Laplace Transformation of proper distribution functions whilst \( \chi_-^*(\phi, \theta) \) shall also have an appropriate domain of analyticity. Certainly it is these constraints which ensure that \( C(K, L) \) be a bivariate polynomial for all finite \( K \) and \( L \).

We now need to see if it is true that the employment of these constraints for queues of the GI/G/2 class ensures that:

(i) \( C(\phi, \theta) \) is always bivariate entire, and,

(ii) The solution process always reduces to a set of simultaneous linear equations which specify a necessarily unique solution.

Unfortunately my own, as yet incomplete explorations of such systems as \( E_k/E_2/2 \) and \( E_k/E_4/2 \) suggest that the number of constants required to specify the form of solution and the complexity of the simultaneous equations, both increase so rapidly that alternative methods may be necessary.

16.1.3. The Meaning of the Singularities

Consider the form for \( \chi_+^*(\phi, \theta) \) given in equation (14.1). Clearly a pole of the form \((\mu + \theta)^{-2}\) can be understood to arise
from the presence of \( \pi_1^*(\theta) \) in this transform and the contribution made by the up to at most two stages of the distribution \( \pi_1(y) \) when \( B(s) \) has two stages. See in particular the proof of Lemma (9.8). We should also expect this to generalize to the form \( (\mu+\theta)^{-L} \) for an \( E_L \) service time distribution function.

However, poles of the form \( (2\mu+\theta+\phi)^{-1} \) or \( (\theta+\phi+r_i)^{-1} \) for \( i = 1, 2, \ldots \) are more difficult to understand. A partial interpretation is possible in the following manner.

(i) Consider a bivariate Laplace Transformation of the form

\[
E(e^{-\phi x-\theta y}) = \frac{q}{q+\theta+\phi}
\]

where \( q \) is, say, real. One notes either by direct inversion, prior knowledge or an intuition based on the recurrences (2.16) and (2.17), that this is the Laplace Transformation of a degenerate bivariate distribution concentrated on the line \( y = x \) for all \( x \in [0, \infty) \). Clearly this is also an exponential density function with mean \( q^{-1} \).

(ii) We can now interpret poles of the form \( (\theta+\phi+r_i)^{-1} \) as evidence that the walk defined by the recurrences (2.16) and (2.17) is trying to concentrate at least some of its probability on the line \( y = x \) or at least in a band close to this line. Furthermore this notion admits of a direct probabilistic demonstration, namely;
(iii) Consider an ergodic GI/G/2 queue which obeys the first come first served discipline. Traditionally the queue mechanism has been thought of as a single waiting line in which all customers wait in order of arrival until they reach the head and a server becomes free and they are released to commence service. However it is more useful to consider the equivalent system wherein each server has a "queuelet" before him and each customer as he arrives determines which queuelet will have the shorter time to extinction, joins this and remains there until he completes his service. Clearly the two systems are strictly equivalent because any of the first n-1 customers who are still in the queuelets when the n-th arrives will commence service no later than he whilst he will commence service no later than the (n+1)-th customer to arrive unless he (the n-th) joins the wrong queuelet.

(iv) The significant property which is apparent from queuelet operation is that since each successive customer adds his service time to the lesser queuelet time, the arriving stream tries to keep a near parity between the queuelet times. That is, the system does try to place the probability on the line \( Y^+ = X^+ \). Furthermore the instantaneous delay state of the system \( x^+ \) will be very responsive to changes in the mean \( (x^+ + y^+)/2 \), which is also the variable which increases without limit.
if the queue becomes overloaded whence \( \{x\}^+ \) will also increase without limit since the mean of \( \{y\}^+ - \{x\}^+ \) is finite if the mean service time is finite. See the proof of Lemma (6.5) and Equation (6.5.3.5) in particular. See also [7*] where this notion appears as an essential principle.

(v) This argument may be repeated for any ergodic GI/G/C queue for \( C>2 \) and we may reasonably conjecture that the distributions which arise from these must be such that their Laplace Transformations will possess analogous singularity loci of the form;

\[
\sum_{j=1}^{C} \theta_j r_i = 0 \tag{16.13.2}
\]

for \( i = 1,2,\ldots, \geq L \)

where the \( \theta_j \) for \( j = 1,2,\ldots,C \) are the complex variables of the Laplace Transformation and the \( r_i \) are complex constants.

(vi) We will now see how to locate at least some, if not all, of these \( r_i \) without first solving the whole queue equation.

16.2 The Queue GIXC/G/1

The preceding discussion of the queuelet queue, and in particular paragraph (iv), suggests that one should consider a very approximately equivalent system in which the queuelets are always kept strictly equal since this will permit uni-
variative complex analysis to be applied to a simple single server system.

The queuelets can be kept strictly equal if each customer spends an equal time with each server. Thus either each customer divides himself and his service times into \( C \) equal parts and assigns one such pair of pieces to each server or he visits each of the servers in turn and takes his full service time with each. In either event only one of the \( C \) servers need be investigated, but in the first case the \( m^{th} \) moment of the resulting delay distribution will be scaled down by the factor \( C^m \), whilst in the second the moments of the interarrival time distribution will have to be scaled up by \( C^m \) to preserve the correct mean traffic load per server.

We adopt the second procedure for reasons which will become more obvious in sequel, and consider the equivalent single server ergodic queue equation. This is analogous to equation (4.18) and uses the same notation. It is:

\[
(1 - \beta^*(z) \alpha^*(-C,z)) \delta^*(z) = -\delta^*_-(z) \tag{16.2.1}
\]

Consider now equation (4.48) for the queue GI/G/2 which reads as follows:

\[
\chi^*(\phi, \theta) = \alpha^*(-\delta + \phi) \left\{ \beta^*(\theta) \chi^*_+(\theta, \phi) \right\}
+ \left\{ \int_0^\infty e^{-\theta y - \phi x} \beta^*(y-x, \phi) \right\} d\chi_+(x, y) \tag{16.2.2}
\]

and make the substitutions

\[
\theta = \phi = z \tag{16.2.3}
\]

\[
\chi^*_+(z, z) = \delta^*_+(z) \tag{16.2.4}
\]
\[ \chi^*(z,z) = \delta^*_+(z) + \delta^*_-(z) \]  \hspace{1cm} (16.2.5)

and obtain

\[ (1-\beta^*(z) \alpha^*(-2z) ) \delta^*_+(z) = -\delta^*_-(z) \]  \hspace{1cm} (16.2.6)

Clearly this is very similar to (16.2.1) for \( C = 2 \) and would be identical if we knew that;

\[ \delta^*_-(z) = \delta^*_+(z) \]  \hspace{1cm} (16.2.7)

However we know that the domain of analyticity of

\[ \chi^*_-(z,z) = \delta^*_-(z) \]  \hspace{1cm} (16.2.8)

is likely to be at best a strip such that;

\[ -\sigma_1 < \Re(z) < \sigma_2 \]  \hspace{1cm} (16.2.9)

for all \(-\infty < \Im(z) < \infty\)

and we infer that the two very similar looking equations (16.2.1) and (16.2.6) demand slightly different solutions.

The reason is this. If we solve (16.2.1) and use the result as an approximate solution for (16.2.6) we sweep up the probability from the region under the distribution \( \pi(x,y) \) such that \( x+y \leq 0 \) and associate all that probability with the condition that the customer does not wait, whereas the correct sweeping up which will satisfy (16.2.6) is that the customer does not wait if \( x < 0 \) for all \( y \in (-\infty, \infty) \).

Therefore the use of the approximation;

\[ \delta^*_-(z) = \delta^*_+(z) \]  \hspace{1cm} (16.2.10)

implies an underestimation of the probability of not waiting.

However the \( \text{E}_K/\text{E}_L/2 \) class of queues give rise to the following empirical observation. The factor

\[ 1-\beta^*(z) \alpha^*(-2z) \]  \hspace{1cm} (16.2.11)
generates the polynomial

\[ \lambda^L (2\mu)^L - (\lambda - z)^L (2\mu + z)^L \]  

L roots of which locate those singularities of \( \chi_+(\phi, \theta) \) which are of the form

\[ (\theta + \phi + r_i)^{-1} \]  

for \( i = 1, 2, \ldots, L \).

These are, of course, not necessarily all of the singularities with such loci as has been demonstrated.

We therefore conjecture that equation (16.2.1) is likely to be useful at least as a starting aid in the investigation of multi-server queues.

One may also apply the standard univariate generating function methods to analogous equations such as:

\[ (n+1) \delta^*_+(z) + (n+1) \delta^*_+(z) = \beta^*(z) \alpha^*(-Cz) (n) \delta^*_+(z) \]  

where \( \_ \) denotes either function and very quickly show the necessity of the main Kiefer-Wolfowitz ergodicity condition, namely that the mean traffic per server \( \rho < 1 \).

What needs now to be done is to systematically investigate (16.2.1) and (16.2.6) to see if the solution of (16.2.1) will provide either a strict lower bound for the true delay distribution or a good heavy traffic approximation, or both.
17. A Continuation and Completion Theorem

17.1 Introduction

The method which W.L. Smith used in [16] for the solution of the analytic problems of GI/G/1 is delightfully simple. By comparison the arguments of this Thesis are restricted, tortuous and distinctly unpleasing. In part this may be due to the difference in the problem, but the comparison does suggest that a good deal more simplification and generalization may be possible.

The purpose of this section is to show where, and perhaps how, such simplifications might be found.

Consider the form of the solution transform for the \( E_k/E_2/2 \) queue. We note that the form appears to depend only on the \( E_2 \) service time distribution, and that this non-dependence on \( K \) extends an equivalent observation of W.L. Smith's regarding queues of the GI/G/1 class. Furthermore the process for the determination of the solution transform for \( E_k/E_2/2 \) appears to be such that a similar process should be possible for much more general interarrival time distribution. Also, as is well known, the limit as \( K \to \infty \) of an \( E_k \) distribution can be made the deterministic distribution by sealing \( \lambda \) so that \( K/\lambda = \bar{\lambda} \) It therefore appears possible that the simultaneous equations derived in Section 13 may have a limiting form as \( K \to \infty \) with \( K/\lambda = \bar{\lambda} \) which defines the solution transform for the ergodic queue \( D/E_2/2 \) in the Kendall notation.
However any such limiting process cannot be mathematically justified unless the derivation of the formulae can be shown to be sound in the limiting situation. This we have not yet done although Remark 2 after the proof of Lemma 9.8 suggests that some such limiting process may be justified.

We now require to prove that a limiting process of this type will give a solution for D/E\(_K\)/2 and thence extend these processes to the solution of the functional equations for queues of the GI/E\(_K\)/2 class.

It appears that given some restrictions in the form of \(\alpha^*(\theta)\), and \(\beta^*(\theta)\), this may in fact be possible.

The main result of this section is to show that equation (4.48) can always be rearranged to define a bivariate entire Centre Function \(C(\phi,\theta)\) if \(\alpha^*(\theta)\) and \(\beta^*(\theta)\) are much more mildly restricted than hitherto. We shall show that provided 3 entire functions \(N_\alpha(\theta)\) and \(N_\beta(\theta)\) s.t. the products \(N_\alpha(\theta)\alpha^*(-\theta)\) and \(N_\beta(\theta)\beta^*(\theta)\) are both entire and that \(B(s)\) is any mixture of any number of distributions either with entire Laplace Transformations or composed of any finite numbers of exponential stages, we can always find such a Centre Function.

Note that these definitions include service time distribution functions \(B(s)\) which may contain component distributions which are discrete or even range limited continuous, but we do not claim to necessarily know how to completely solve the resulting equations, merely that these may always be arranged
to define a bivariate entire centre function.

Consider equation (4.48) for the ergodic queue, namely;

\[
\chi^*(\phi, \theta) = \alpha^*(-\phi) \left\{ \beta^*(\theta) \chi^*_\alpha(\theta, \phi) + \int_{-\infty}^{\infty} \int_{0-x}^{\infty} e^{-\theta y - \phi x} \chi^*_\beta(y-x, \phi) d\chi^*(x, y) \right\}
\]

(17.1.1)

and make these assumptions.

(i) The distribution \( A(a) \) for \( a \in [0, \infty) \) is such that its Laplace Transformation \( \alpha^*(\theta) \) is such that there exists an entire function \( N_\alpha(\theta) \) which can nullify the singularities of \( \alpha^*(-\theta) \), if any, so that the product \( N_\alpha(\theta)\alpha^*(-\theta) \) is also entire.

Note: Clearly if \( \alpha^*(\theta) \) is rational or a sum of rational and entire elements, \( N_\alpha(\theta) \) is an easily defined polynomial.

(ii) The distribution \( B(s) \) for \( s \in [0, \infty) \) is such that its Laplace Transformation \( \beta^*(\theta) \) is such that an entire \( N_\beta(\theta) \) exists which is such that the product \( \beta^*(\theta) N_\beta(\theta) \) is also entire.

(iii) \( B(s) \) is such that \( \beta^*(\theta) \) is analytic for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \).

and
(iv) \( B(s) \) is at most a mixture of distributions composed of exponential stages and other distributions with entire Laplace Transformations.

**Note:** Clearly (iv) implies (ii) and (iii) if the stage rates \( \mu_1, \mu_2, \ldots \) etc. are all non-zero, but we give (ii) and (iii) as separate assumptions since we will discuss the problems which arise when we attempt to prove our main result without (iv).

**17.2 Theorem** If assumption (iii) above is true, then the sum of integrals which defines \( \chi^*(\phi, \theta) \) exists for all \( \phi \) and \( \theta \) in the convex tube \( R_- \) where

\[
R_- = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta + \phi) \leq 0 \text{ and } \text{Re}(\theta) > -\sigma \} 
\]

(17.2.1)

Hence \( \chi^*(\phi, \theta) \) is bivariate analytic in \( D_- \) and continuous within \( R_- \) and in particular onto and on the hyperplane s.t. \( \text{Re}(\theta + \phi) = 0 \) when \( \text{Re}(\theta) \geq -\sigma \) where;

\[
D_- = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta + \phi) < 0 \text{ and } \text{Re}(\theta) > -\sigma \} 
\]

(17.2.2)

**Proof:** This result is really proven in the proof of Corollaries 4 and 5 to Lemma 6.11 where the defining integral for \( \chi^*(\phi, \theta) - \pi_1^*(\theta) \) was shown to exist in \( R_- \) whence \( \chi^*(\phi, \theta) \) is bivariate analytic in \( D_- \) since \( \pi_1^*(\theta) \) is analytic if \( \theta \) is finite and \( \text{Re}(\theta) > -\sigma \), and we may apply Theorem 5.3.6 to obtain the asserted result.
17.3 Theorem  If assumption (iii) above is true then

\( \chi^*(\phi, \theta) \) is analytic in the open convex tube \( D_+ \) and continuous within and onto and on all boundary hyperplanes of \( R_+ \) where;

\[
D_+ = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \Re(\theta + \phi) > 0 \quad \Re(\theta) > -\sigma \}
\]

(17.3.1)

\[
R_+ = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \Re(\theta + \phi) \geq 0 \quad \Re(\theta) > -\sigma \}
\]

(17.3.2)

Proof:  This follows the method used in the proof of Lemma 6.5 and uses results from that proof:

Define

\[
I_+ (\phi, \theta) = \int_{0+}^{\infty} \int_{x-}^{\infty} e^{-\phi x - \theta y} d\pi(x,y)
\]

(17.3.3)

\[
= \int_{0+}^{\infty} \int_{\hat{e}=0-}^{\infty} e^{-(\theta+\phi)x - \theta \hat{e}} d\pi(x,x+\hat{e})
\]

(17.3.4)

where \( \hat{e} = y-x \). See Lemma 6.5.

\[
= \int_{0+}^{\infty} \int_{\hat{e}=0-}^{\infty} e^{-(\theta+\phi)x - \theta \hat{e}} d_{\hat{e}} \{ \Pi(x)d_{\hat{e}} \hat{Q}(\hat{e}|x) \}
\]

(17.3.5)

where \( \hat{Q}(\hat{e}|x) \) is a conditional distribution function defined in the proof of Lemma 6.5

\[
\leq \int_{0+}^{\infty} d_{x} \Pi(x) \int_{\hat{e}=0-}^{\infty} e^{-\theta \hat{e}} d_{\hat{e}} \hat{Q}(\hat{e}|x)
\]

(17.3.6)

if \( \Re(\theta + \phi) \geq 0 \).

However in the proof of Lemma 6.5 we showed that

uniformly for any \( x \in (-\infty, \infty) \) and any \( n = 1, 2, \ldots \).
\[(n)Q(\omega|x) - (n)Q(\hat{e}|x) = o(\|e^{-\sigma\hat{e}}|) \quad (17.3.7)\]

as \( \hat{e} \to -\infty \)

where \( n \) was the number of arrivals since the queue commenced operation. This was extended to the ergodic queue by:

(i) Showing that as \( \hat{e} \to +\infty; \)

\[(m)Q(\omega|x) - (m)Q(\hat{e}|x) = o(\|e^{-\sigma\hat{e}}|) \quad (17.3.8)\]

again uniformly for any \( x \in (-\infty, \infty) \) and any \( m=1,2,\ldots \) where \( m \) became the number of arrivals between the moment just before an arbitrary arrival and just before the arrival when the queue was last empty and

(ii) Showing that if the queue were ergodic then existed a proper distribution \( P(m) \) for \( m=1,2,\ldots \) which defined the probabilities of the various possible numbers of arrivals since the queue was last empty.

We can use these notions to show that \( \tilde{I}(\phi,\theta) \) exists in \( \mathbb{R}_+ \) as follows. Consider:

\[
\sum_{m=1}^{\infty} P(m) \int_{0^+}^{\infty} d_x^{(m)} \Pi(x) \int_{\hat{e}=0}^{\infty} |e^{-\theta\hat{e}}| d_{\hat{e}}^{(m)} \hat{Q}(\hat{e}|x) \quad (17.3.9)
\]

Clearly by (17.3.8) the integral;

\[
\int_{\hat{e}=0}^{\infty} |e^{-\theta\hat{e}}| d_{\hat{e}}^{(m)} \hat{Q}(\hat{e}|x) \quad (17.3.10)
\]

will exist for all finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) and
uniformly for all \( x \in (-\infty, \infty) \) and \( m = 1, 2, \ldots \) therefore by Fubini's Theorem

\[
\sum_{m=1}^{\infty} P(m) \int_{0}^{\infty} e^{-\theta \hat{e}} d\hat{e} \int_{0+} d\xi (m) \Pi(x) (m) \hat{Q}(e|\xi) \tag{17.3.11}
\]

will exist whence;

\[
\int_{0}^{\infty} e^{-\theta \hat{e}} d\hat{e} \sum_{m=1}^{\infty} P(m) \int_{0+} d\xi (m) \Pi(x) (m) \hat{Q}(e|\xi) \tag{17.3.12}
\]

will exist.

However (17.3.12) and (17.3.6) are equivalent and we have shown that \( I_+(\phi, \theta) \) exists in the asserted region. Hence by Theorems (5.3.5) and (5.3.6) \( I_+^{*}(\phi, \theta) \) is analytic in \( \mathcal{D}_+ \) and continuous within \( R_+ \) whence \( \chi_+^{*}(\phi, \theta) \) also has these properties since

\[
\chi_+^{*}(\phi, \theta) = I_+(\phi, \theta) + \pi_1^{*}(\theta) + \hat{\pi}_1^{*}(\theta) + \pi_0 \quad \text{and both} \\
\pi_1^{*}(\theta) \text{ and } \hat{\pi}_1^{*}(\theta) \text{ if it is not zero, are both analytic} \\
\text{functions of } \theta \text{ in } \mathcal{D}_+ \text{ which suffices since } \pi_0 \text{ is a constant and thus entire.}
\]

17.4 Theorem

The integral

\[
I_+^{*}(\phi, \theta) = \int_{0}^{\infty} \int_{0+} e^{-\phi x - \theta y} d\xi (x,y) \beta^*(y-x, \phi) \text{ d}x_+ (x,y) \tag{17.4.1}
\]

exists in the region \( R^{(+)}_+ \) and defines a bivariate analytic function in \( D^{(+)}_+ \) where
\[ R^{(+)}_\phi = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta+\phi) > 0, \text{Re}(\phi), \text{Re}(\theta) > -\sigma \} \]

\[ D^{(+)}_\phi = \{ \text{All finite } \phi \text{ and } \theta \text{ s.t. } \text{Re}(\theta+\phi) > 0, \text{Re}(\phi), \text{Re}(\theta) > -\sigma \} \]

whence \( I^{(+)}_\phi (\phi, \theta) \) is continuous on the hyperplane s.t. \( \text{Re}(\theta+\phi)=0 \) \(|\text{Re}(\phi)| < \sigma \) and onto this from all adjacent points within \( D^{(+)}_\phi \)

**Proof:** By assumption (iii) \( \beta^*(\phi) \) is analytic if \( \phi \) is finite and \( \text{Re}(\phi) > -\sigma \). Also \( \beta^*(y-x,\phi) \) is entire for any finite \( y-x \). Thus we may find

\[
\hat{\beta} = \max_{\phi, y-x} \{|\beta^*(y-x,\phi)|\} \quad (17.4.4)
\]

\[
\text{if } \text{Re}(\phi) > -\sigma \quad |\phi| < \infty \quad \text{and } y-x \in [0,\infty)
\]

whence

\[
|I^{(+)}_\phi (\phi, \theta)| \leq \hat{\beta} \int_0^\infty \int e^{-\phi x - \theta y} |d\chi_+(x,y)\) (17.4.5)
\]

\[
\text{if } \text{Re}(\phi) > -\sigma
\]

However, by Theorem (17.3) the integral;

\[
\int_0^\infty \int e^{-\phi x - \theta y} |d\chi_+(x,y)\) (17.4.6)
\]

will exist in the tube \( R^+ \) defined by equation (17.3.2) whence (17.4.5) will be finite in \( R^{(+)}_\phi \)

The asserted results then follow by use of Theorems
Corollary: The equivalent integral $I_\theta^{(+)}(\theta, \phi)$ which is obtained by exchange of $\phi$ and $\theta$ in (17.4.1) also exists in $R^{(+)}_\phi$ and defines an analytic function in $D^{(+)}_\phi$.

Proof: Obvious exchange of variables.

Remark: Implicit in the above is the observation that if $B(s)$ is range limited i.e. $dB(s)=0$ for all $s > s^\wedge$ where $s^\wedge$ is a finite maximum or deterministic (which is the same thing) $I^{(+)}_\phi(\phi, \theta)$ and $I_\theta^{(+)}(\theta, \phi)$ both exist if $\phi$ and $\theta$ are finite and $\text{Re}(\theta + \phi) > 0$.

17.5 Definition of $C(\phi, \theta)$

We now rearrange equation (4.48) as follows:

$$N_\alpha(\theta + \phi) N_\beta(\theta) N_\beta(\phi) \chi^*_-(\phi, \theta) = C(\phi, \theta) \quad (17.5.1)$$

$$= N_\alpha(\theta + \phi) \alpha^*(-\theta + \phi) N_\beta(\phi) N_\beta(\theta) \left[ \beta^*(\theta) \chi^*_+(\theta, \phi) \right]$$

$$+ \left\{ \int_0^\infty \int_0^\infty e^{-\phi x - \theta y} \beta^*(y-x,\phi) \chi^*_+(x,y) \, dx \, dy \right\}$$

$$- N_\alpha(\theta + \phi) N_\beta(\theta) N_\beta(\phi) \chi^*_+(\phi, \theta)$$

$$= N_\alpha(\theta + \phi) \alpha^*(-\theta + \phi) N_\beta(\phi) N_\beta(\theta) \left[ \beta^*(\theta) \chi^*_+(\theta, \phi) \right]$$

$$+ I^{(+)}_\phi(\phi, \theta)$$

$$- I^{(+)}_\theta(\theta, \phi) \quad (17.5.2)$$
Theorem

\( C(\phi, \theta) \) as defined by equations (17.5.1) and (17.5.2) is analytic for all finite \( \phi \) and \( \theta \) s.t. \( \Re(\theta) > -\sigma \).

Proof:

The product

\[ N_\alpha(\theta+\phi)N_\beta(\theta)N_\beta(\phi) \chi^*_+(\phi, \theta) \]  \hspace{1cm} (17.6.1)

is entire by assumption

\[ \chi^*_+(\theta, \phi) \] is analytic for all finite \( \phi \) and \( \theta \) s.t. \( \Re(\theta+\phi) > 0 \) and \( \Re(\phi) > -\sigma \) and continuous onto the hyperplane where \( \Re(\theta+\phi) = 0 \) and \( \Re(\phi) > -\sigma \)

\[ I^{(+)}_\theta(\theta, \phi) \] and \( I^{(+)}_\phi(\phi, \theta) \) are both analytic in \( D^{(+)}_\phi \) and continuous in \( R^{(+)}_\phi \) respectively.

Thus the R.N.S. of (17.5.3) is analytic if \( \theta \) and \( \phi \) are finite \( \Re(\theta+\phi) > 0 \) and \( \Re(\theta), \Re(\phi) > -\sigma \).

This is also continuous from within this convex tube onto the hyperplane where \( \Re(\theta+\phi) = 0 \) and \( -\sigma < \Re(\phi) < \sigma \)

Similarly the product:

\[ N_\alpha(\theta+\phi)N_\beta(\theta)N_\beta(\phi) \chi^*_-(\phi, \theta) \]  \hspace{1cm} (17.6.2)

is analytic wherever \( \chi^*_-(\phi, \theta) \) is analytic
and continuous wherever this is continuous. These are the open tube $D_-$ and the partly closed tube $R_-$ defined by Theorem 17.2. We may now choose any constant $\phi_0$ s.t. $-\sigma < \text{Re}(\phi_0) < \sigma$ and use equation (17.5.2) to define a Schwartz continuation in $\theta$. This may be repeated for any other $\phi_0$ s.t. $-\sigma < \text{Re}(\phi_0) < \sigma$ and thus defines a continuation in $\theta$ for all constant finite $\phi$ s.t. $-\sigma < \text{Re}(\phi) < \sigma$ and any finite $\theta$ s.t. $\text{Re}(\theta) > -\sigma$.

Similarly we may continue in $\phi$ for any constant $\theta_0$ if $-\sigma < \text{Re}(\theta_0) < \sigma$ and thus define $C(\phi, \theta)$ analytic in $\phi$ for all finite $\phi$ in this domain.

Hence $C(\phi, \theta)$ is bivariate analytic in the domain such that $\theta$ and $\phi$ are finite and $-\sigma < \text{Re}(\theta), \text{Re}(\phi) < \sigma$, by use of Hartogs' Theorem. However ordinary continuation then gives $C(\phi, \theta)$ bivariate analytic in the tube such that $\theta$ and $\phi$ are finite and

\begin{align*}
(i) \quad & \text{Re}(\phi) > -\sigma \quad \text{Re}(\theta) > -\sigma \quad (17.6.3) \\
(ii) \quad & \text{Re}(\phi) < -\sigma \quad \text{Re}(\theta) > -\sigma \quad \text{Re}(\theta+\phi) < 0
\end{align*}

However this is a tube with a non-convex basis which admits of an immediate analytic completion to the convex tube such that $\theta$ and $\phi$ are both finite and $\text{Re}(\theta) > -\sigma$.

Hence the Theorem is true as asserted.

We now require to continue $C(\phi, \theta)$ into the region wherein $\text{Re}(\theta) \leq \sigma$. 

17.7 Theorem  If \( B(s) \) satisfies assumption (iv), \( \chi_\star(\phi, \theta) \) has no other singularities in the finite domain where \( \text{Re}(\phi) < 0 \) and \( \text{Re}(\theta) \leq -\sigma \) other than poles in the plane of \( \theta \) and these are all such that \( N_\beta(\theta) \) will nullify them.

Proof: This follows a method of argument introduced in Remark 2 after the proof of Lemma 9.8.

By definition (6.10)
\[
\chi_\star(\phi, \theta) = \int \int 1 e^{-\phi x - \theta y} d\pi(x, y) - \pi_1^\star(\theta) \quad (17.7.1)
\]
where;
\[
\pi_1^\star(\theta) = \int \int 1 e^{-\theta y} d\pi(x, y) \quad (17.7.2)
\]
and we require to show that both \( \pi_1^\star(\theta) \) and the integral have the desired properties.

The distribution \( \pi(x, y) \) holds at the moment just before an arbitrary arrival. We may therefore select from the sequence of all arrivals that subsequence who encounter one server busy (with the necessarily lone remaining customer in the system) and the other free for a time \( x \) s.t. \( x \in [-x^1, -x^1 - \varepsilon) \)

where \( \varepsilon > 0 \) but small.

If the service time distribution \( B(s) \) is a mixture of component distributions we may now define a probability distribution \( p(i/x) \) for the probability that a particular lone remaining customer in this subsequence will have the \( i \)-th component of \( B(s) \) as his original service time
distribution, given that the other server has been free for a time $x$. That is we may decompose the subsequence of events into subsequences of component events and associate probabilities with these.

Next, if the typical lone remaining customer was such that his original service time distribution was of the $i^{\text{th}}$ sort and one server has been free a time $x$, we may define $P_i(y|x)$ for the probability distribution of his residual service time. We can now say the following things about the $P_i(y|x)$:

(i) If the $i^{\text{th}}$ component distribution in $B(s)$ was range limited (which includes all discrete distributions with finite means) with range $R$ the $P_i(y|x)$ will be range limited with range $R - x$ and

(ii) If the $i^{\text{th}}$ original component distribution contained say $L_i$ exponential stages the $P_i(y|x)$ will also be composed of from 1 to $L_i$ unexpended stages and we may now write $P_i(j|x)$ for the probability that it contains $j$ such stages.

Naturally, being unexpended, these stages have the same rates as the stages originally part of the $i^{\text{th}}$ component of $B(s)$.

The integral:

$$\int_{-\infty}^{0} \int_{-\infty}^{\infty} e^{-\theta y} d\pi(x,y)$$
where \( \Pi(x) \) is the unconditional probability distribution for \( x \in (-\infty, \infty) \). Suppose now that for \( i=1,2,.. \), the \( P_i(y|x) \) are all finite range limited whilst for \( i=I+1, I+2,.. \) the \( P_i(y|x) \) are all composed of from \( j=1,.., L_i \) stages, each such stage having service rate \( \mu_{ij} \). Then we may rewrite (17.7.3) in the form;

\[
\sum_{i=1}^{I} \int_{-\infty}^{0} \exp(-\phi x) \int_{0}^{\infty} \exp(-\theta y) dy \sum_{j=1}^{L_i} P_i(j|x) \frac{(-\mu_i}{\mu_i+\theta})^j |dx| \Pi(x) \]  

\[
\text{Limit } 0^- + \sum_{i=I+1}^{\infty} \int_{-\infty}^{0} \exp(-\phi x) \sum_{j=1}^{L_i} P_i(j|x) \frac{(-\mu_i}{\mu_i+\theta})^j |dx| \Pi(x) \]  

where \( \mu_i \geq \sigma \) for all \( i=1,2,.. \) and the \( P_i^*(\theta|x) \) for \( i=1,2,.., I \) are a set of entire functions.

If we now choose any finite \( \theta \) s.t. \( \text{Re}(\theta) > -\sigma \) all functions of the form

\[
\frac{-\mu_i}{\mu_i+\theta}^j
\]

will be of finite modulus for all \( i=I+1, I+2,.. \) as will all of the \( P_i^*(\theta|x) \) for all \( x \in (-\infty, 0) \).

Therefore the integrals;

\[
\int_{-\infty}^{0} \exp(-\phi x) P_i^*(\theta|x) |d \Pi(x) \]  

(17.7.5)
will exist for any $i=1,2,...$, I if $\text{Re}(\theta) > -\sigma$ and
$\text{Re}(\phi) < 0$ as will any integral;

$$
\int_{-\infty}^{0} e^{-\phi x} \left| \sum_{j=1}^{L_i} \left( \frac{\mu_i}{\mu_i + \theta} \right)^j \right| \cdot \Pi_i^*(x) \, \Pi(x) \quad (17.7.6)
$$

for any $i=I+1,...$ and for all $x \in (-\infty,0)$ and $\text{Re}(\theta) > -\sigma$ since the $\Pi_i^*(j|x)$ are probabilities.

We now note that (17.7.5) will also exist for any finite $\theta$ and any finite $\phi$ s.t. $\text{Re}(\phi) \leq 0$. Whence this defines a bivariate analytic function if $\phi$ and $\theta$ are finite and $\text{Re}(\phi) < 0$.

Also it will be clear that (17.7.6) may be written in the form:

$$
\int_{-\infty}^{0} e^{-\phi x} \left| \sum_{j=1}^{L_i} \left( \frac{\mu_i}{\mu_i + \theta} \right)^j \right| \cdot \Pi_i^*(x) \, \Pi(x) \quad (17.7.7)
$$

Whence we may write the second sum in (17.7.4) in the form:

$$
\lim_{i=I+1} \sum_{j=1}^{L_i} \left( \frac{\mu_i}{\mu_i + \theta} \right)^j \int_{-\infty}^{0} e^{-\phi x} \Pi_i(j|x) \, \Pi(x) \quad (17.7.8)
$$

and we note that each integral in this sum will always exist provided that $\text{Re}(\phi) \leq 0$. Thus we may rewrite this sum as;

$$
\lim_{i=I+1} \sum_{j=1}^{L_i} \left( \frac{\mu_i}{\mu_i + \theta} \right)^j \Pi_i^*(\phi) \quad (17.7.9)
$$

where the $\Pi_i^*(\phi)$ are analytic functions for all $i=I+1,..$, if $\text{Re}(\phi) < 0$.

Thus we have shown that if $\text{Re}(\phi) < 0$, the singularities of (17.7.4) are only poles at the set of points $\{-\mu_i\}$ for
i = I+1, I+2, ... in the \( \Theta \) plane and that the maximum order of these poles is \( L_i \) for each \( i \).

However the poles of \( \beta^*(\Theta) \) are of the form:

\[
\frac{\mu_i^{-L_i}}{(\mu_i + \Theta)}
\]

(17.7.10)

for each \( i = I+1, I+2, ... \)

Whence \( N_\beta(\Theta) \) will nullify all of the poles of (17.7.9) since it must contain the product:

\[
\lim_{i=I+1} \pi_i^{-L_i} (\mu_i + \Theta)
\]

(17.7.11)

Similarly \( \pi_1^*(\Theta) \) shares this property since equation (17.7.4) defines \( \pi_1^*(\Theta) \) if \( \phi = 0 \).

We have therefore shown that if \( \Theta \) and \( \phi \) are finite and \( \text{Re}(\phi) < 0 \), equation (17.7.1) defines \( \chi_{-+}^*(\phi, \Theta) \) to be a function of which the singularities are at most only poles in the plane of \( \Theta \) and of such order as to be nullified by \( N_\beta(\Theta) \).

Hence the Theorem is true as asserted.

17.8 Theorem

The function \( C(\phi, \Theta) \) defined by equations (17.5.1) and (17.5.3) is bivariate entire.

Proof:

Use Theorem (17.7) and ordinary continuation in each
complex variable in turn when the other is a complex constant to continue $C(\phi, \theta)$ into the tube such that

(i) If $\text{Re}(\phi) < 0$ and $|\phi| < \infty$ then $C(\phi, \theta)$ is bivariate analytic for all finite $\theta$ and

(ii) If $\text{Re}(\phi) > 0$ and $|\phi| < \infty$ then $C(\phi, \theta)$ is bivariate analytic for all finite $\theta$ s.t. $\text{Re}(\theta) > -\infty$.

However this tube has a non-convex base and thus admits of an immediate completion to the entire finite basis plane. Thus $C(\phi, \theta)$ is entire.

17.9 More General Service Time Distributions

Although the restrictions of $B(s)$ by assumption (iv) may be of little practical significance, it is mathematically unpleasing. If this difficulty possesses a solution it appears that it may be obtained by so strengthening the proofs of Results 1 and 2 used to prove Lemma (6.5) that assumptions (ii) and (iii) will be sufficient to prove an equivalent result to that somewhat heuristically shown in Theorem (17.7).

This upgrading process is easy for Result 1 to Lemma (6.5), but the equivalent for Result 2 appears more difficult unless the variables are independent.

Such extensions and the actual solutions of the functional or integral equations such as may now be accessible are more appropriately dealt with elsewhere.
References

The following are all of the references actually referred to in the text of the Thesis.

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Additional references of some relevance are:

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Pergamon (1958)

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The following fortran listing was printed out by the A.N.U. computer immediately before the 10 sets of results used for comparison purposes in Section 15.

Explanatory notes are given in the right margin.

One may change the traffic load per server by altering the constants in either of the statements tagged 502 or 504 or both, but it will clearly be more useful to change only 502 and thence a.
APPENDIX I

DIMENSION ARL(110), ARK(110), W(1001)  

1 FORMAT (212, F7.5)  
2 FORMAT (4F10.6)  
3 FORMAT (31HA AND S DISTRIBUTION PARAMETERS)  
4 FORMAT (23HDISTRIBUTION PARAMETERS)  
5 FORMAT (21HNUMBERS OF POOR EXITS)  
6 FORMAT (4HLPF=14, 4HKPF=14)  
7 FORMAT (7HRESULTS)  
8 FORMAT (6HS BAR=F7.3, 6HA BAR=F7.3, 4HROE=F7.5)  
9 FORMAT (20HDENSITY DISTRIBUTION)  
10 FORMAT (20F6.4)  
11 FORMAT (27HDISTRIBUTION AT TENTH STEPS)  
100 FORMAT (18HABORIYING THIS RUN.)  

RESERVES SPACE

Read & write formats

and output headings

Read & write run details

These are the power residue multipliers

\[
|8t+3|
\]

Sees 10 runs of 10,000 arrivals

Clears array which holds \( P(T_t) \)

Clears array which holds \( P(T_{<t}) \)

Clears queuelets and statistical scores

Ensures 101 sets of 100 arrivals. The

first set of 100 is not used to produce

data. It merely initiates \( V_1 \) and \( V_2 \).

Generates \( \Lambda^{(n)} \) for \( E_1 \) distribution with

nominal mean \( \bar{a} = 66.6 \) clock units.

Generates \( S^{(n)} \) for \( E_2 \) distribution with

nominal mean 100.0 units

Recurrence as by equations 2.16 and 2.17
73 \textit{VI}=0. \\
74 \textit{IF(V2) 75,76,76} \\
75 \textit{V2}=0. \\
76 \textit{IF(12-1) 80,80,77} \\
77 \text{INX=V2} \\
78 \text{IF(1000-INX) 78,78,79} \\
79 \text{INX=INX+1} \\
\text{W(INX)=W(INX)+.0001} \\
\text{AS=AS+A} \\
\text{SS=SS+S} \\
\text{ST=ST+S*S} \\
\text{AT=AT+A*A} \\
80 \text{CONTINUE} \\
81 \text{WRITE (3,5)} \\
82 \text{WRITE (3,6) LPE,KPE} \\
83 \text{SS=SS/10000.} \\
84 \text{AS=AS/10000.} \\
85 \text{ST=ST/10000.} \\
86 \text{AT=AT/10000.} \\
87 \text{PROE=SS/(2.*AS)} \\
88 \text{WRITE (3,7)} \\
89 \text{WRITE (3,8) SS,AS,PROE} \\
90 \text{SS=(ST-SS*SS)**.5} \\
91 \text{AS=(AT-AS*AS)**.5} \\
92 \text{PROE=123,456} \\
93 \text{WRITE(3,7)} \\
94 \text{WRITE(3,8) SS,AS,PROE} \\
95 \text{WRITE(3,9)} \\
96 \text{WRITE(3,10)(W(J),J=1,1001,1) Writes out delay density distribution.} \\
97 \text{DO 81 I3=1,1000,1} \\
98 \text{W(I3+)=W(I3)+W(I3+1)} \\
99 \text{GO TO 81} \\
\text{82 WRITE(3,10)(W(J),J=1,1001,10) Writes out P(T=t) for 0 \leq t < 10 \leq 10 to \infty modulo 0.1s.} \\
\text{END} \\
\text{A photostat of the actual programme is given below to ensure that no errors in transcription need remain undetected.}
DIMENSION ARL(110), ARK(110), W(1001)

1 FORMAT(212, F7.5)
2 FORMAT(410.6)
3 FORMAT(31HA AND S DISTRIBUTION PARAMETERS)
4 FORMAT(23HDISTRIBUTION PARAMETERS)
5 FORMAT(21HNUMBERS OF POOR EXITS)
6 FORMAT(4HLPE=14, 4HKPE=I4)
7 FORMAT(7HRESULTS)
8 FORMAT(6HS BAR=F7.3, 6HA BAR=F7.3, 4HROE=F7.5)
9 FORMAT(2HDENSITY DISTRIBUTION)
10 FORMAT(2OF6.4)
11 FORMAT(2FDISTRIBUTION AT TENTH STEPS)
100 FORMAT(18HAaborting THIS RUN.)

READ(1,1) K, L, DROE
WRITE(3,1) K, L, DROE.
LSD=375323
LSE=123459
DO 82 KLOT=1,10,1
DO 81 12=1,1001,1
51 W(IX)=0.
V1=0.
V2=0.
SS=0.
AS=0.
ST=0.
AT=0.
DO 80 I2=1,101,1
DO 80 I4=1,100,1
CALL RANDU (LSE, ISTQ, X)
LSE=ISTQ
CALL RANDU (LSE, ISTQ, XY)
LSE=ISTQ
X=X*XY
IF (X-1.E-20) 501,502,502
X=1.E-20
A=-.3333*alog(X)
CALL RANDU (LSD, ISTP, X)
LSD=ISTP
CALL RANDU (LSD, ISTP, XY)
LSD=ISTP
X=X*XY
IF (X-1.E-20) 503,504,504
504 S=0.-50.*alog(X)
V2=V2+S
IF(V1-V2) 71,72,72
71 A=V1
V1=V2
V2=V1-A
IF(V1-V2) 73,74,74
73 V1=0.
74 IF(V2) 75,76,76
75 V2=0.
76 IF(I2-1) 80,80,77
77 I2=I2+1
IF(1000-IX) 78,78,79
78 INX=1000
79 INX=INX+1
W(INX)=W(INX)+.0001
AS=AS+A
SS=SS+S
ST=ST+S*S
AT=AT+A*A
80 CONTINUE
WRITE(3,5)
WRITE (3,6) LPE, KPE
SS=SS/10000.
AS=AS/10000.
ST=ST/10000.
AT=AT/10000.
PROE=SS/(2.*AS)
WRITE(3,7)
WRITE(3,8) SS, AS, PROE
SS=(ST-SS*SS) **.5
AS=(AT-AS*AS) **.5
PROE=123.456
WRITE(3,7)
WRITE(3,8) SS, AS, PROE
WRITE(3,9)
81 W(I3+1)=W(I3)+W(I3+1)
WRITE(3,11)
82 WRITE(3,10)(W(J), J=1,1001,1)
83 GO TO 84
84 STOP END
The Method of Dr. F. Pollaczek

This appendix does not review the Pollaczek method in detail. It merely gives a number of general observations mainly based on Pollaczek's publication [15] which arise as a result of my own experience with GI/G/2 and the comments of Kingman [9].

Although I cannot claim to be an expert on Pollaczek's methods it is clear to me that beneath the, at times, severe complexities of his formal operations there hides a very simple although incomplete approach to the problem of GI/G/C. The simplicity of his approach lies in the use of expressions symmetric in the queuelet lengths. The difficulty arises from the need to make this formulation for the Markov process iterate correctly. As we shall see, it is this weakness which leads Pollaczek to transform the problem of GI/G/C into that of solving C+1 simultaneous integral (or functional) equations whilst the earlier simplicity denies him much of the (essentially probabilistic) information required to specify that the solution is unique and of the appropriate type.

The following detailed comments follow the notation used in [15] and also refer to equation numbers in that publication.

(i) Pollaczek's Markov chain formulation is essentially the same as that of Kiefer & Wolfowitz or this Thesis, but he does not order the variables as to magnitude. See equations (1), (2), (8) and (9) of [15]. We may therefore compare
his "last ends" with arbitrarily numbered queuelet lengths.

(ii) The usage of the expression:

\[ s \min (a_1, \ldots, a_s) = \prod_{v=1}^{s} s(a_v) \]

in the form of a multiple inversion integral (See (10) et seq of [15]) is very ingenious, particularly where he extends it to express:

\[ s \min (a_1-t, \ldots, a_s-t) \]

(iii) The use of this method represents a very simple minimum step away from the method which he successfully used for GI/G/1. What are perhaps not quite obvious are the following:

(a) This operator is clearly intended to "Sweep up" probability from that part of \( R_s \) where \( \min\{a_1, \ldots, a_s\} \) is negative but the product \( \prod_{v=1}^{s} s(a_v) \) implies the creation of numerous probability concentrations in that point or those hyperplanes where any one or more (and thence up to all) of the \( a_v < 0 \) for \( v=1,2,\ldots,s \) which implies a failure to obtain potentially valuable information.

Clearly the \( \pi_0 \) and \( \pi_1(y) \) in the notation of this Thesis are simple examples of such concentrations, and

(b) This process will operate for any symmetric or
asymmetric distribution on a set of variables, and symmetry in the variables is not necessary.

(iv) Pollaczek appears to have applied this operator to a process which he has chosen to make symmetric in its variables possibly in the mistaken belief that symmetry is the best possible basis for the Markov chain iterations. This implies that if we seek the ergodic limiting distribution for a suitably loaded queue of the GI/G/S class this will have S fold symmetry in its variables and possess concentrations of probability in one point and $2^S - 2$ hyperplanes of orders from 1 to $S-1$ symmetrically disposed in the space such that

$$q_v(+) \in [0, \infty)$$

for all $v = 1, 2, \ldots, n$

where $q_v$ is the length of the $v$th queuelet.

Note: $q_v = \lim_{n \to \infty} \{t_{n,v}\}$

where $t_{n,v}$ is defined by [15] equation (3).

Clearly this type of symmetry in distribution will also exist in the distribution of Pollaczek's $t_{n,t}$ for all $n = 1, 2, \ldots$ if the queue starts from $t_{0,v} = 0$ for all $v = 1, 2, \ldots, S$ since the system is then symmetry preserving.

Nowhere as far as I am aware does Pollaczek mention this feature which might be of considerable help when one seeks to solve his simultaneous integral equations.

(v) Pollaczek leans very heavily on symmetry when developing
his recurrence relationships. Thus resymmetrization is the essential motivation of the operations which proceed from equation (12) of [15] to equation (22) and thence (24) and (25). However it is this need to resymmetrize the system in order to make the Markov chain iterate, which causes the problem made so apparent in paragraph 4 of [15].

Since the GI/G/s queue has s queuelets the delay experienced by an arbitrary arrival may be any one of s queuelet lengths and it is necessary for the process to somehow keep track of which. However these queuelet lengths are mutually distributed non-independent random variables with positive covariances, which have somehow to be kept in a state of "stochastic balance". Therefore any symmetrized process must either:

(a) Produce a set of simultaneous relationships which will maintain this balance and also normalize the system, or

(b) Express the normalized joint distribution of all the variables at the n\(^{th}\) step in terms of that at the (n-1)\(^{th}\) step.

Thus since Pollaczek makes no attempt whatsoever to do (b), it is clear that he runs direct to (a) and that this is why he obtains s+1 simultaneous relationships between s+1 generating functions.
On the other hand it is also clear that if one attempts (b) then irrespective of whether one is merely seeking the ergodic limiting distribution for the queue or a generating function which will define all the distributions, one must expect to obtain a single quite complex functional or integral equation. Also it appears to me that any such single equation will be more easily handled if one orders the variables of the problem as is done in this Thesis and then accepts the resulting asymmetric distributions, since the number of points and hyperplanes in which probability may be concentrated is reduced from $2^S - 1$ to $S$ and this is clearly important when $S$ is large.

(vii) A further problem with Pollaczek's method which we have already touched on is the shortage of information wherewith to adequately characterize the functions of the problem. That is, the functions which solve integral equations are not necessarily unique and clearly depend on where these are required to be analytic. Certainly it is clear that Pollaczek appears to provide very little, if any, insight in [15] as to how the necessarily unique sequence of Laplace Transformation of the desired sequence of distributions shall be determined. One has only to compare the amount of work required to define the single Integral Equation of this Thesis with that required to fully solve it for two very simple
special cases (which arise from the same functional
equation anyway), to wonder if Pollaczek's derivation
of his simultaneous Integral Equations is no more than
perhaps one tenth of the work to be done before he has
a complete method.

To summarize: The general problem is very difficult:
Pollaczek's manipulations are very ingenious and at times
almost bewildering, but there is a latent ambiguity, namely
which queuelet shall cause whom to wait and the potential
user lacks information about the types of functions which
can be expected to be correct solutions to the simultaneous
Integral Equations.

Thus I agree with Kingman's comment that a new approach
is needed. However, I cannot assert that the rather
pedestrian methods of this Thesis are in any wise "radically
new". To the contrary, they simply appear to me as simple
and I hope logical extensions of some very old ideas
spurred on by a few elementary notions from Multivariate
Complex Analysis.