MARTINGALE LIMIT THEORY

by

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INTRODUCTION

A central limit theorem for martingales was considered as early as 1935, when Paul Lévy established several results. However, it is only since the 1960's that the field has received much attention, and some of the most important work was published by Brown [9] in 1971. In this dissertation we take a critical look at the present situation of martingale central limit theory. We change the form of the classical central limit problem and show that in this new form it has a solution under more general conditions. This leads us to analogous results for reverse martingales, and to a new form of the law of the iterated logarithm.

Chapter I serves both as a historical introduction and as a survey of recent developments in martingale central limit theory. It shows how closely linked the martingale and independence theories are, but points out that some important independence results do not appear to carry over to martingales. Brown's results are introduced here in section 1.6. They contain conditions under which the traditional form of the central limit theorem for sums of independent variables can be applied to sums of dependent variables. One of these conditions is that

\begin{equation}
(0.1) \quad s_n^{-2} \frac{V_n^2}{n} \overset{P}{\rightarrow} 1,
\end{equation}

where \( s_n^2 = \sum_{j=1}^{n} E(X_j^2) \) and \( V_n^2 = \sum_{j=1}^{n} E(X_j^2 | X_1, \ldots, X_{j-1}) \). This is trivially satisfied if \( \{X_n\} \) are independent but does not hold so generally when all we know is that \( \{X_n\} \) are martingale differences. Chapter II examines the behaviour of the sum \( \sum_{j=1}^{n} X_j \) when condition (0.1) no longer holds.

Chapters II, III and IV are devoted to original work, and Chapter II contains the most important results. Section 2.2 investigates why
Brown's Theorems 2 and 3 fail when condition (0.1) is weakened, and produces two new results on convergence to mixtures of normal distributions. Section 2.3 shows that if we consider the randomly normed sums
\[ V_n^{-1} \sum_{j=1}^{n} X_j \] rather than the sums with the traditional constant norming,
\[ S_n^{-1} \sum_{j=1}^{n} X_j, \] then an invariance principle is still valid.

Chapter III remarks that most of the results of Chapter II have analogues for reverse martingales. This parallels Scott's extension [36] of Brown's work, and shows the dual nature of reverse martingale theory. The proofs are very similar to those in Chapter II, and are dismissed after some short notes.

The last chapter in the thesis takes a look at the law of the iterated logarithm for (forward) martingales. Heyde and Scott [20] considered the law for martingales with a constant norming, and in the light of the results of Chapter II we consider it with two sorts of random norming. We speculate on an extension to reverse martingales.

Many of the results in Chapters II and III are obtained using an approximation technique. Basically, the reason for its introduction is this. Frequently during the proofs we desire to approximate to conditional expectations like \( E[X^{-1} Y I(A) | G] \) by \( X^{-1} E(Y | G) I(A) \), where \( X \) and \( Y \) are random variables, \( I(A) \) is the indicator function of the set \( A \) and \( G \) is a \( \sigma \)-field. Of course, it is trivial that if \( A \in G \) and \( X \) is \( G \)-measurable,

\[ E[X^{-1} Y I(A) | G] = X^{-1} E(Y | G) I(A). \]

If \( A \notin G \), we may be able to approximate \( A \) by a set \( B \in G \). Often \( A = X^{-1}(E) \) where \( E \) is a Borel set, and so we can tackle the problem by finding a \( G \)-measurable random variable \( Z \) which approximates \( X \) and setting \( B = Z^{-1}(E) \). Then (hopefully)
\[ E[X^{-1} Y I(A) | G] \sim E[Z^{-1} Y I(B) | G] \]
\[ = Z^{-1} E(Y | G) I(B) \]
\[ \sim X^{-1} E(Y | G) I(A). \]

This argument is extremely simple in concept, but to set it up rigorously is a little tedious. We often have to show that if \( Z \) is sufficiently close to \( X \) then

\[ P(|E[X^{-1} Y I(A) | G] - Z^{-1} E(Y | G) I(B)| > \epsilon) < \epsilon. \]

Probabilities like this are most easily estimated using Chebychev-type inequalities, so it may be necessary to prove that a term like that within modulus signs, converges to zero in the mean of order 2. A truncation argument is usually sufficient to do this, but truncations always introduce a rather depressing amount of algebra to the calculations.
The following basic notation will be adopted throughout this thesis. Further notation will be introduced from time to time.

Random variables (rv's) will be denoted by upper case letters \((X, X_n, S_n)\) and \(\sigma\)-fields by script letters \((F, F_n, G_n)\). Sets in \(\sigma\)-fields will be denoted by the letters \(E, F, G\) and \(H\). Most of the rv's we consider may be supposed defined on a single probability space \((\Omega, F, P)\). If \(X_1, \ldots, X_n\) are defined on this space, \(F\{X_1, \ldots, X_n\}\) will denote the \(\sigma\)-field generated by \(X_1, \ldots, X_n\). \(F_0\) will stand for the trivial \(\sigma\)-field \(\{\phi, \Omega\}\) and \(\omega\) will denote an element of \(\Omega\).

If \(s(X)\) is statement about the r.v. \(X\) (such as "\(X \geq 0\)"), we will write \(\{s(X)\}\) for \(\{\omega \in \Omega | s(X(\omega)) \text{ is true}\}\).

In an abuse of notation, "\(X \in G\)" will mean "\(X\) is \(G\)-measurable".

\(I(E)\) will denote the indicator function of the set \(E \in F\); that is,

\[
I(E)(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{otherwise.} \end{cases}
\]

\(I(E)\) need not be random; thus if \(s_1, s_n\) and \(u\) are real numbers,

\[
I(s_n \geq s_j < u) \equiv \begin{cases} 1 & \text{if } s_n \geq s_j < u \\ 0 & \text{otherwise.} \end{cases}
\]

If \(X\) and \(Y\) are rv's we will adopt the convention that \(X^{-1} Y = \infty\) whenever \(X = 0\), whatever the value of \(Y\).

All equalities and inequalities between rv's will be thought of as holding in the almost sure (a.s.) sense, with respect to the measure \(P\).

\(N(0, s^2)\) will denote a random variable having the normal distribution with mean zero and variance \(s^2\), while \(W(u) (u \geq 0)\) will stand for a standard Brownian motion.

Convergence properties for random variables will be indicated by \(\mathbb{P}, a.s., L^1\) and \(P\), meaning respectively convergence in probability,
almost sure convergence, convergence in the mean of order 1 and convergence in distribution. The rv's $X_n$ will be said to be "o(1) in probability" if $X_n \overset{p}{\to} 0$ as $n \to \infty$. Weak convergence of measures will be denoted by $\Rightarrow$.

$C[0,1]$ will stand for the metric space of continuous functions on the interval $[0,1]$, with the uniform metric

$$\rho(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|.$$  

$D[0,1]$ will represent the space of functions on $[0,1]$ which are right-continuous and have left-hand limits, and we will always give $D$ the uniform metric.

The end of the proof of a lemma, proposition, theorem or corollary will be indicated by the symbol **.

In Chapter II, $\{(S_n, F_n)\}_{n=1}^{\infty}$ will denote a zero-mean, square-integrable martingale on $(\Omega, F, P)$. That is, for each $n$,

1. $F_n \subseteq F_{n+1}$ and $S_n \in F_n$,
2. $E(S_n^2) < \infty$ and $E(S_n) = 0$,
3. $E(S_{n+1}|F_n) = S_n$.

We will define

$$X_n = S_n - S_{n-1} \quad (S_0 = 0),$$
$$s_n^2 = E(S_n^2) = \sum_{j=1}^{n} E(X_j^2),$$
$$v_n^2 = \sum_{j=1}^{n} E(X_j^2|F_{j-1}).$$

Neveu [29] introduces the rv $V_n^2$ via Doob's decomposition ([14], p. 297) of the submartingale $\{(S_n^2, F_n)\}_{n=1}^{\infty}$. We can write

$$S_n^2 = M_n + A_n$$

where $\{(M_n, F_n)\}_{n=1}^{\infty}$ is a martingale and $\{A_n\}_{n=1}^{\infty}$ is an increasing
sequence of non-negative rv's. This decomposition is uniquely determined a.s. by
\[ A_{n+1} - A_n = E(S_{n+1}^2 | F_n) - S_n^2 = E[(S_{n+1} - S_n)^2 | F_n] = E(X_{n+1}^2 | F_n) \]
and so \( A_n = V_n^2. \)

\( \{X_n\}_{n=1}^\infty \) will be known as a martingale difference sequence. It has the property
\[ E(X_n | F_{n-1}) = 0. \]

Let us state here an elementary result which will be used repeatedly in Chapter II:

If \( \{(S_j, F_j)\}_{j=1}^\infty \) is a martingale and
\[ X_j = S_j - S_{j-1}, \quad j = 1, 2, \ldots, n \quad (S_0 = 0), \]
then
\[ E(S_n^2) = E[E(X_j^2 | F_{j-1})]. \]

In Chapter III, \( \{(S_n, F_n)\}_{n=1}^\infty \) will denote a zero-mean, square-integrable reverse martingale on \((\Omega, F, P)\). That is, for each \( n, \)

(i) \( F \subseteq F_n \supseteq F_{n+1} \) and \( S_n \in F_n. \)
(ii) \( E(S_n^2) < \infty \) and \( E(S_n) = 0, \)
(iii) \( E(S_{n-1} | F_n) = S_n. \)

Theorem 4.2, Chapter VII of Doob [14] gives us
\[ S_n \overset{a.s.}{\longrightarrow} E(S_m | F_{\infty}) = X \quad \text{say, as } n \to \infty, \]
where \( F_\infty = \cap_{n=1}^\infty F_n \) and \( X \) does not depend on \( m. X \) is \( F_n \)-measurable for each \( n, \) and so \( \{(S_n - X, F_n)\}_{n=1}^\infty \) is a reverse MG. Hence there is no loss of generality in supposing that \( X = 0 \) a.s.. In this case, define
\[ X_n = S_n - S_{n+1}, \]
\[ s_n^2 = E(S_n^2) = \sum_{j=1}^n E(X_j^2), \quad \text{and} \]
\[ V_j^n = \sum_{n}^\infty E(x_j^n | F_{j+1}). \]

\( \{X_n\}^\infty \) is a reverse martingale difference sequence, and has the property

\[ E(x_n | F_{n+1}) = 0. \]

Note that if \( \{(S_n, F_n)\}^\infty \) is a reverse MG then \( \{(S_{-n}, F_{-n})\}^{-\infty} \) is an MG.
1.1 Introduction

In this chapter we will sketch the development of martingale central limit theory, starting with the work of Lévy in 1935. We will relate the various early and contemporary results and indicate some areas to which it has so far proved difficult to extend them. There is as yet no unifying theory as there is for sequences of sums of independent random variables. Those limit theorems which do exist often apply to martingales satisfying conditions which are either uncomfortably strong or are difficult to verify in practice.

It will be necessary to state a sequence of theorems without giving proofs. The theorems will be quoted in a form which relates them to one another and to the work in the following chapters. We will try to present a common notation throughout the discussion, even to the point where it conflicts with the notation of other writers.

1.2 Generalising the Independence Case

A reasonably complete answer to the question of asymptotic normality of sums of independent, square-integrable rv's was given by Lindeberg and Feller (see for example Chung [12], Theorem 7.2.1):

Theorem 1.1 [Lindeberg-Feller]. If \( \{X_j\}_{j=1}^{\infty} \) is a sequence of independent, square-integrable rv's with zero means and if \( S_n = \sum_{j=1}^{n} X_j \) and \( s_n^2 = E(S_n^2) \), then the following two conditions are equivalent:

\[
\begin{align*}
(1.1) & \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^{n} \mathbb{E}[X_j^2 I(|X_j| > \varepsilon s_n)] + 0, \\
(1.2) & \quad (i) \quad s_n^{-1} S_n \overset{D}{\rightarrow} N(0,1), \text{ and } \\
& \quad (ii) \quad s_n^{-2} \max_{j \leq n} \mathbb{E}(X_j^2) + 0
\end{align*}
\]
Their result was extended to a functional central limit theorem by Prokhorov [33].

One direction in which we might seek to extend Theorem 1.1 is to relax the condition of independence. There are many alternative conditions we may impose, such as exchangeability, various types of mixing, and $m$-dependence. Billingsley [7], Chapter 4, contains limit theorems for the first two kinds of dependence, while Theorem 7.3.1 of Chung [12] is a result for uniformly bounded $m$-dependent rv's. Bergström [3] has used a comparison method to prove several central limit theorems for various kinds of dependence. This method is outlined in Loève [26] (p. 375) and is based on comparing partial sums

$$S_{m,n} = \sum_{m+1}^{n} X_j \quad (m < n)$$

of dependent variables with partial sums

$$S'_{m,n} = \sum_{m+1}^{n} X'_j \quad (m < n)$$

of independent normal variables.

Perhaps the most attractive dependence condition is the martingale property, for it is basically just a first moment condition and hence is quite easy to work with and relatively simple to verify in practice. Moreover, results for other types of dependence can often be obtained from martingale results - see for example Scott [35].

1.3 The Work of Lévy

The word "martingale" was first introduced to mathematics by Jean Ville [41] in 1939, and he and Paul Lévy were the first to study such processes in general. In the mid 1930's Lévy [22], [23] and [24] provided the first martingale central limit theorems. He also introduced the "conditional variance" for martingales,

$$\gamma^2_n = \sum_{1}^{n} E(X_j^2 | F_{j-1})$$
which seems to play an important role in modern martingale limit theory.

His early results required the strong assumption that for each \( n \),
\( V_n^2 \) is a.s. constant, and this assumption recurs even in contemporary
work - see for example Csörgő [13]. Many of Lévy's results require some
sort of boundedness condition. The theorem we quote here is equivalent
to his Theorem 67.3, p. 246 of [24], and is not unlike a recent result
due to Drogin [15].

**Theorem 1.2** [Lévy]. Suppose \( \{X_n\}_0^\infty \) is a sequence of square-integrable
rv's for which

\[
E(X_n | X_1, \ldots, X_{n-1}) = 0, \quad n = 1, 2, \ldots \quad (X_0 = 0),
\]

\[
V_n^2 = \sum_{j=1}^{n} E(X_j^2 | X_1, \ldots, X_{n-1}) \quad \text{diverges a.s.},
\]

and for some increasing sequence of positive constants \( \{B_n\}_1^\infty \),

\[
|X_n| < B_n, \quad n = 1, 2, \ldots, \quad \text{and} \quad V_n^{-1} B_n \xrightarrow{a.s.} 0.
\]

For \( t > 0 \) define

\[
\zeta(t) = \inf\{j | V_j^2 \geq t\}, \quad \text{and}
\]

\[
S(t) = \sum_{j=1}^{\zeta(t)} X_j + c_t \zeta(t)
\]

where

\[
0 < c_t \leq 1 \quad \text{and} \quad c_t^2 \zeta(t) + (1-c_t^2) V_{\zeta(t)}^2 = t.
\]

Then

\[
t^{1/2} S(t) \xrightarrow{D} N(0,1) \quad \text{as} \quad t \to \infty.
\]

Lévy's proof is very straightforward, in that he uses probabilistic
techniques to estimate directly the error

\[
|P(t^{-1/2} S(t) \leq x) - (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-i x^2) dx|.
\]

Of course, if \( V_n^2 \) is a.s. constant (i.e. \( V_n^2 = s_n^2 \text{ a.s.} \)) we can
let \( t \to \infty \) along the sequence \( \{V_n^2\}_1^\infty \) and derive as a corollary to
Theorem 1.2,
\[ v_n^{-1} S_n \xrightarrow{\mathcal{D}} N(0,1); \]

compare this with our Theorem 2.3.

The next major advance in martingale theory subsequent to the work of Levy was the appearance in 1953 of Doob's classic "Stochastic Processes" [14]. He mentions Lévy's Theorem 67.1 of [24] and sketches a characteristic function proof, remarking that "the central limit theorem is applicable to martingales much as it is to sums of mutually independent random variables". However, little was published to enlarge upon this statement until the early 1960's.

1.4 Martingales with Stationary, Ergodic Differences

Billingsley [6] in 1961, and independently Ibragimov [21] in 1963, provided proofs that the central limit theorem applies to square-integrable martingales with stationary, ergodic differences. Each acknowledges a debt to the work of Lévy although both tackle the problem via the convergence of characteristic functions. We state their result as Theorem 1.3.

**Theorem 1.3** [Billingsley; Ibragimov]. If \( \{X_n\}_{n=1}^{\infty} \) is a stationary, ergodic sequence of square-integrable rv's satisfying

\[
E(X_n | X_1, \ldots, X_{n-1}) = 0, \quad n = 1, 2, \ldots \quad (X_0 = 0),
\]

and if \( S_n = \sum_{j=1}^{n} X_j \) and \( \sigma^2 = E(X_1^2) \), then

\[
(\sigma^2)^{1/2} S_n \xrightarrow{\mathcal{D}} N(0,1).
\]

Following Lévy in Theorem 1.2, Billingsley proves first that

\[
t^{1/2} S(t) \xrightarrow{\mathcal{D}} N(0,1).
\]

To do this he uses the comparison method. In the notation of Theorem 1.2, define
\[ X_j(t) = \begin{cases} X_j & \text{if } j < \ell(t) \\ c_t X_j & \text{if } j = \ell(t) \\ 0 & \text{if } j > \ell(t), \end{cases} \]

\[ \bar{\sigma}_j^2(t) = \begin{cases} \mathbb{E}(X_j^2 \mid F_{j-1}) & \text{if } j < \ell(t) \\ c_t^2 \mathbb{E}(X_j^2 \mid F_{j-1}) & \text{if } j = \ell(t) \\ 0 & \text{if } j > \ell(t), \end{cases} \]

\[ S_n(t) = \sum_{j=1}^n \tilde{X}_j(t) + \sum_{j=n+1}^\infty \bar{\sigma}_j Y_j, \quad n = 0, 1, 2, \ldots, \]

where \( \{Y_n\}_n^\infty \) is a sequence of independent \( \mathcal{N}(0,1) \) variables, independent of the sequence \( \{X_n\}_n^\infty \). \( S_0(t) \) has the normal distribution with mean zero and variance \( t \), and

\[ |E \exp(is \tilde{S}(t)) - \exp(-is^2)| = \]

\[ = \left| \sum_{j=1}^\infty E[\exp(is \tilde{S}_n(t)) - \exp(is \tilde{S}_{n-1}(t))] \right| \]

\[ \leq \sum_{j=1}^\infty E[|\exp(is \tilde{S}_n(t)) - \exp(is \tilde{S}_{n-1}(t))|]. \]

(The infinite sum in the expression preceding the inequality has, with probability 1, only a finite number of non-zero terms.)

Billingsley shows that the last-written sum converges to zero as \( n \to \infty \), and the proof of Theorem 1.3 is completed by showing that

\[ n^{-\frac{1}{2}} |S_n - S(n\sigma^2)| \overset{p}{\to} 0. \]

Ibragimov's proof does not appeal directly to the work of Lévy, but like Billingsley he uses the comparison method. If \( \{Y_n\}_n^\infty \) are independent \( \mathcal{N}(0,1) \) variables independent of \( \{X_n\}_n^\infty \) and if we define

\[ T_j(n) = S_j + \sum_{k=j+1}^n Y_k \quad \text{for } j \leq n \]

then in the case \( \sigma = 1 \)

\[ |E \exp(is \frac{1}{n} S_n) - \exp(-is^2)| = \]

\[ = \left| \sum_{j=1}^n E[\exp(is \frac{1}{n} T_j(n)) - \exp(is \frac{1}{n} T_{j-1}(n))] \right|. \]
Using techniques of Bernstein [4] and [5], Ibragimov proves that the telescoping sum on the RHS can be made small.

In 1967 Rosén ([34], Theorem 4) provided a central limit theorem for triangular arrays of dependent variables satisfying several asymptotic moment conditions, and specialised this to obtain Theorem 1.3. His work contains an interesting technique which is worth mentioning here. He first proves Theorem 1.3 in the case where the $X_n$ have a continuous distribution; this allows him to truncate the $X_n$ above and below zero, obtaining new variables $X_n^C$ which continue to satisfy $E(X_n^C) = 0$ and for which $E(X_n^C)^2 = (1-\varepsilon)e^2$. Such variables satisfy his moment conditions and the theorem is proved in the continuous case. He then remarks that if $\{Z_n\}_{n=1}^\infty$ is any sequence of independent $N(0,1)$ variables, independent of the sequence $\{X_n\}_{n=1}^\infty$, then $\{X_n + Z_n\}_{n=1}^\infty$ is a continuous stationary ergodic martingale difference sequence with respect to the sequence of $\sigma$-fields $\{X_1, \ldots, X_n, Z_1, \ldots, Z_n\}$, $n = 1, 2, \ldots$, and hence

$$n^{-\frac{1}{2}} \sum_{j=1}^{n} (X_j + Z_j) \Rightarrow N(0, e^2 + 1).$$

Theorem 1.3 follows immediately.

In his Theorems D and E Rosén provides sufficient conditions to extend his central limit theorem to a functional limit theorem in $C[0,1]$. He applies these to generalise Theorem 1.3 in the case where $|X_n|$ is uniformly bounded.

In 1968 Billingsley [7] gave a final answer in the case of a stationary ergodic difference sequence, extending Theorem 1.4 to a functional central limit theorem in $D[0,1]$.

1.5 Reverse Martingales

Theorem 1.3 inspired Loynes [27] to prove an analogous result for reverse martingales. In the case of a forward martingale with stationary, ergodic differences, the ergodic theorem gives us
On the other hand, if \( \{(S_n, F_n)\}^\infty_1 \) is a reverse martingale and
\[
F_\infty = \cap \, F_n,
\]
then
\[
S_n \overset{\mathcal{L}}{\to} E(S_1|F_\infty).
\]
These similarities indicate that we might seek a central limit theorem for reverse martingales. In Theorem 1.4 we quote a special case of Loynes' Theorem 1.

**Theorem 1.4 [Loynes].** Suppose \( \{(S_n, F_n)\}^\infty_1 \) is a square-integrable reverse martingale for which \( S_n \to 0 \) a.s.. Define
\[
X_n = S_n - S_{n+1}, \quad n = 1, 2, \ldots,
\]
\[
s_n^2 = E(S_n^2) = \sum_{j=1}^\infty E(X_j^2), \quad \text{and}
\]
\[
v_n^2 = \sum_{j=0}^\infty E(X_j^2|F_{j+1}).
\]
If
\[
(1.3) \quad s_n^{-2} v_n^2 \overset{\mathcal{L}}{\to} 1, \quad \text{and}
\]
\[
(1.4) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^\infty E[X_j^2 1(|X_j| > \varepsilon s_n)|F_{j+1}] \overset{\mathcal{L}}{\to} 0,
\]
then
\[
s_n^{-1} S_n \overset{D}{\to} N(0,1).
\]
Loynes' proof is modelled on that of Billingsley [6].

A later paper of Loynes [28] extended this result to a functional central limit theorem, showing that if \( \xi_n \) is the random function obtained by linearly interpolating between the points
\[
\ldots, (s_{n-1}^2 s_n^2 s_{n+2}, s_n^2 s_{n+1}^2 s_{n+3}), (s_n^2 s_{n+1}^2 s_{n+2}^2, s_n^2 s_{n+1}^2 s_{n+2}), (1, s_n^2 s_{n+1}^2), n = 1, 2, \ldots
\]
if \( P_n \) and \( P \) are the measures induced on \( C[0,1] \) by \( \xi_n \) and a Brownian motion respectively, and if (1.3) and (1.4) hold, then
\[
P_n \overset{w}{\to} P \quad \text{in } C[0,1].
In the same paper Loynes provided a useful tool in the study of martingale convergence. In general, if \( \{Q_n\}_1^\infty \) is a sequence of measures on \( C[0,1] \) or \( D[0,1] \) and \( P \) is the measure induced by Brownian motion, to prove that

\[ Q_n \xrightarrow{w} P \]

it is necessary to prove both that the finite distributions of \( Q_n \) converge to those of \( P \) and that \( \{Q_n\}_1^\infty \) is tight. However, if \( \{(S_n, F_n)\}_1^\infty \) is a forward or reverse MG and \( \xi_n \) is obtained by interpolating between the points \( (s_n^{-2}, s_n^2, s_n^{-1}, S_n) \) in the natural way, then the measures induced by the \( \xi_n \) converge to \( P \) if the finite dimensional distributions converge. Loynes' proofs are based on the sufficient condition for tightness given in Theorem 15.5 of Billingsley [7].

More recently Scott [36] has used this result to show that the conclusion of Loynes' functional central limit theorem remains true if the almost sure convergence in (1.3) and (1.4) is weakened to convergence in probability. To prove that the finite dimensional distributions converge, Scott adjusts the proof of Theorem 2 of Brown [9] to the case of reverse martingales.

In Chapter III we will show that Loynes' work can be extended even further.

1.6 Generalising Billingsley's Theorem

Theorem 1.4 suggests a generalisation of Theorem 1.3 to forward martingales satisfying a Lindeberg-type condition and \( s_n^{-2} V_n^2 \to 1 \) (the convergence being in a sense to be determined). Historically this is reminiscent of Lindeberg's extension [23] of the central limit theorem for independent, identically distributed random variables.

In 1971 Brown [9] published a proof that the central limit theorem applies to martingales satisfying \( s_n^{-2} V_n^2 P \to 1 \) and a Lindeberg condition.
His techniques are drawn from Billingsley's proof of Theorem 1.3, and we rely heavily on them to produce the results in Chapters II and III. He first proves the convergence of the finite distributions and then obtains tightness using properties of the normal distribution.

Brown's Theorems 2 and 3 are combined here as Theorem 1.5.

**Theorem 1.5** [Brown]. Suppose \( \{(S_n, F_n)\}_{n=1}^{\infty} \) is a zero-mean, square-integrable MG and define

\[
X_n = S_n - S_{n-1}, \quad n = 1, 2, \ldots \quad (S_0 = 0),
\]

\[
s_n^2 = E(S_n^2) = \sum_{j=1}^{n} S_j^2
\]

\[
y_n^2 = \sum_{j=1}^{n} E(X_j^2 | F_{j-1}).
\]

Let \( \xi_n \) be the random function obtained by linearly interpolating between the points

\((0,0), (s_n^{-2} s_1, s_n^{-2} s_1 S_1), (s_n^{-2} s_n^2, s_n^{-1} S_2), \ldots, (1, s_n^{-1} S_n)\), \( n = 1, 2, \ldots \),

and let \( P_n \) and \( P \) be the measures induced on \( C[0,1] \) by \( \xi_n \) and a Brownian motion, respectively.

If

\[
(1.5) \quad s_n^{-2} y_n^2 \underset{P_n}{\rightarrow} 1, \quad \text{and}
\]

\[
(1.6) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \varepsilon s_n)] F_{j-1} \underset{P}{\rightarrow} 0,
\]

then

\[
P_n \overset{y}{\rightarrow} P.
\]

The corollary that \( s_n^{-1} S_n \overset{P}{\rightarrow} N(0,1) \) strongly resembles a result given in Theorem 2.2 of Dvoretzky [16]. This is stated for variables satisfying a slightly weaker first moment condition than is required by the martingale definition.

In Chapter II we will study the behaviour of \( s_n^{-1} S_n \) (and \( V_n^{-1} S_n \)) when condition (1.5) is relaxed to
where $\eta^2$ is a random variable.

Scott [35] showed that Theorem 1.5 could be proved using the Skorokhod representation, and restated the result in $D[0,1]$ rather than $C[0,1]$. He also showed that conditions (1.5) and (1.6) are jointly equivalent to several other pairs of conditions. In fact they are equivalent to the apparently stronger conditions

$$s_n^{-2} \frac{\nu^2}{\eta^2} \overset{L_1}{\rightarrow} 1,$$

and

$$s_n^{-2} \frac{\nu^2}{\eta^2} \sum_{1}^{n} \mathbb{E}[X_j^2 \mathbb{I}(|X_j| > \epsilon s_n)] \rightarrow 0.$$ 

The importance of the Skorokhod representation for martingales (Theorem 4.3 of Strassen [40]) is that it allows us to "embed" the martingale in a Brownian motion. Instead of working with the sequence

$$\left(s_n^{-1} S_1, \ldots, s_n^{-1} S_n\right)$$

we consider the more pertinent sequence

$$\left(W(T_1(n)), \ldots, W(T_n(n))\right)$$

where $W(t)$ is a Brownian motion and $T_1(n), \ldots, T_n(n)$ are non-negative stopping times. The existence of the representation provides a direct tie between martingale theory and the theory for sums of independent random variables, for which Skorokhod first established his representation.

There is as yet no Skorokhod representation for reverse martingales. It has been postulated (Loynes [28]) that if $\{(S_n, F_n)\}_1^\infty$ is a reverse martingale with

$$F_n = F\{S_n, S_{n+1}, \ldots\},$$

then there exists a probability space on which are defined a Brownian motion $W$ and a sequence of non-negative stopping times $\{T_n\}_1^\infty$ such that for any sequence of integers

$$0 < n_1 < \cdots < n_k,$$
\((W(\sum_{j=1}^{n_k} T_j), \ldots, W(\sum_{j=1}^{n_k} T_j))\)

has the same distribution as

\((S_{n_1}, \ldots, S_{n_k})\).

Brown and Eagleson [11] have generalised the corollary to

Theorem 1.5,

\(S_{n}^{-1} S_n \xrightarrow{D} N(0,1),\)

to obtain a result on convergence to infinitely divisible distributions. Their result parallels the extension of Theorem 1.1 to a limit theorem for infinitely divisible laws.

1.7 Necessary and Sufficient Conditions

A disappointing aspect of all the martingale limit theorems described so far is that they contain only sufficient conditions for the convergence of normed martingales to a limit law. It is tempting to try to extend the necessity part of Theorem 1.1 to sequences of martingales, but unfortunately any attempt to generalise the classical proofs seems to founder at some stage. However, several "necessary and sufficient" results are known.

Brown has obtained two conditions which are equivalent to the Lindeberg condition, (1.6). One is directly comparable with (1.2) (they are equivalent under the assumption of independence) but the other has no predecessor in the independence case. We combine Theorem 1 of [9] and a version of Theorem 2 of [10] as our Theorem 1.6.

**Theorem 1.6** [Brown]. Adopt the notation and definitions of Theorem 1.5, and let

\[ f_n(t) = \prod_{j=1}^{n} E[\exp(it_{n}^{-1} X_j)|F_{j-1}]. \]
If \((r_{n1}, \ldots, r_{nn})\) is any randomly chosen permutation of the integers \((1, \ldots, n)\), define

\[ Y_{nj} = X_{r_{nj}}, \quad j = 1, \ldots, n, \]

and define the random function \(\eta_n\) by

\[ \eta_n(u) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nu]} Y_{nj} \quad \text{for each } u \in [0,1]. \]

Let \(Q_n\) and \(P\) be the measures induced on \(D[0,1]\) by \(\eta_n\) and a Brownian motion, respectively. If condition (1.5) holds, then (1.6) is equivalent to each of the following conditions:

\[ \begin{align*}
(1.7) & \quad (i) \text{ for all } t, \quad f_n(t) \overset{P}{\to} \exp(-2t^2) \quad \text{and} \\
(ii) & \quad \max_{j \leq n} E(X_j^2 | F_{j-1}) \overset{P}{\to} 0; \\
(1.8) & \quad Q_n \overset{\mathcal{F}}{\Rightarrow} P.
\end{align*} \]

Brown proves the equivalence of (1.6) and (1.8) as a corollary to a more general result which applies to triangular arrays of rv's which are exchangeable in each row. (A set of rv's \(\{X_1, \ldots, X_n\}\) is said to be exchangeable if for each (fixed) permutation \((r_1, \ldots, r_n)\) of \((1, \ldots, n)\), \((X_{r_1}, \ldots, X_{r_n})\) has the same distribution as \((X_1, \ldots, X_n)\).) The set of rv's \(\{Y_{n1}, \ldots, Y_{nn}\}\) defined in Theorem 1.6 is exchangeable, and Brown's theorem follows almost immediately from his earlier result.

Drogin [15] has also provided a "necessary and sufficient" result. His has the drawback that it applies only to a random subsequence of a normed martingale, and hence it is not easy to compare it with the previous results. Adler [2] has shown that if we alter the definition of the process then the subsequence can be made nonrandom, but unfortunately it appears that in doing this we lose the necessity part of the result. Basically what Drogin does is to choose a subsequence of \(\{S_n\}_{n=1}^{\infty}\) so that the corresponding subsequence of \(\{V_n^2\}_{n=1}^{\infty}\) behaves like
{n}_{1}^{\infty}: Adler chooses a subsequence of \( \{S_{n}\}_{1}^{\infty} \) so that the subsequence of \( \{s_{n}\}_{1}^{\infty} \) behaves like \( \{n\}_{1}^{\infty} \).

We quote a portion of Drogin's Theorem 1 as our Theorem 1.7:

**Theorem 1.7** [Drogin]. Adopt the notation and definitions of Theorem 1.5, and define

\[ T_{n} = \inf\{j|v_{j}^{2} > n\}. \]

Let \( \xi_{n} \) be the random function obtained by linearly interpolating between the points

\[ (0,0), (n^{-1}v_{1}^{2}, n^{-2}S_{1}), (n^{-1}v_{2}^{2}, n^{-2}S_{2}), \ldots, \]

and restrict \( \xi_{n} \) to the interval \([0,1]\). Let \( R_{n} \) and \( P \) be the measures induced on \( C[0,1] \) by \( \xi_{n} \) and a Brownian motion, respectively.

Conditions (1.9) and (1.10) are equivalent:

(1.9) for all \( \varepsilon > 0 \), \( n^{-1} \sum_{1}^{T_{n}} x_{j}^{2} I(x_{j}^{2} > n\varepsilon) \xrightarrow{L_{1}} 0 \) as \( n \to \infty \),

(1.10) \( R_{n} \xrightarrow{v} P \) and

\[ n^{-1} v_{n}^{2} \xrightarrow{L_{1}} 1. \]

Drogin's proof is based on an invariance principle in which the exemplary distribution is that of the number of heads in a fair coin-tossing experiment. It does not involve characteristic functions or the Skorokhod representation.

Unfortunately all of these results miss the essential character of the necessity half of Theorem 1.1. In each case the "convergence to normality part" of the necessary condition is a little stronger than just \( s_{n}^{-1} S_{n} \xrightarrow{D} N(0,1) \) or \( P_{n} \xrightarrow{P} P \). It is still not clear what form a more basic necessary and sufficient condition should take.
1.8 Removing Uniform Asymptotic Negligibility

Let us pause for a moment and return to the independence case and Theorem 1.1. By considering sequences of independent normal variables it is easy to construct examples for which the uniform asymptotic negligibility condition (1.2) (ii) is not satisfied but (1.2) (i) is true. Just how "close" is (1.1) to being necessary for (1.2) (i)?

Lévy [24] considered this sort of problem and suggested that a necessary and sufficient condition for (1.2) (i) might take the following form:

(i) Each non-negligible summand is almost normal, and

(ii) the largest of the individually negligible summands is negligible itself.

In 1967 Zolotarev [142] provided a precise formulation of these ideas:

**Theorem 1.8 [Zolotarev].** Adopt the notation and definitions of Theorem 1.1 and let \( \sigma_{nj}^2 = s_n^{-2} E(X_j^2) \). Let \( F_{nj}, F_n, \phi_{nj} \) and \( \phi \) denote the distribution functions of \( s_n^{-1} X_j, s_n^{-1} S_n, N(0, \sigma_{nj}^2) \) and \( N(0,1) \) respectively, and let \( L(F,G) \) represent the Lévy distance between the distribution functions \( F \) and \( G \).

Conditions (1.11) and (1.12) are equivalent:

\[
(1.11) \quad (i) \quad \alpha_n = \sup_{j \leq n} L(F_{nj}, \phi_{nj}) \to 0, \quad \text{and} \\
(ii) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{U_n} E[X_j^2 I(|X_j| > \varepsilon s_n)] \to 0,
\]

where \( U_n \) is the set of indices

\[
U_n = \{ j \leq n | \sigma_{nj}^2 < \sqrt{n} \};
\]

\[
(1.12) \quad L(F_n, \phi) \to 0.
\]

Once again we might try to extend Theorem 1.8 by relaxing the independence condition. Following on from some work of Adler [1], Scott...
[37] has provided the following result. We quote a special case of his main Theorem:

**Theorem 1.9** [Scott]. Adopt the notation and definitions of Theorem 1.5 and let $\sigma^2_{nj} = s_n^{-2} E(X_j^2)$ and $\phi_{nj}$ denote the distribution function of $N(0, \sigma^2_{nj})$. Consider the following two conditions:

1. There exists a sequence of positive numbers $\{\gamma_n\}$ converging to zero such that if
   
   $$U_n = \{j \leq n | \sigma^2_{nj} \leq \gamma_n\} \text{ and}$$
   
   $$a_n = \sup_{j \notin U_n} \sup_{x \in R} E|P(s_n^{-1} X_j \leq x | F_{j-1}) - \phi_{nj}(x)|$$

   then
   
   $$\gamma_n^{-1} a_n^{3/4} \to 0; \quad \text{and}$$

   $$(1.13) \quad \text{s}^{-2} \sum_{j \in U_n} [E(X_j^2 | F_{j-1}) - E(X_j^2)] \overset{P}{\to} 0 \quad \text{and}$$

   $$\sum_{j \in U_n} \text{I}(|X_j| > \varepsilon s_n) |F_{j-1}| \overset{P}{\to} 0.$$

If (1.13) and (1.14) hold then

$$s_n^{-1} S_n \overset{P}{\to} N(0,1).$$

Theorem 1.9 contains the one-dimensional central limit corollary to Theorem 1.5 (to show this set $\gamma_n = (1 + \frac{1}{n}) \sup_{j \leq n} \sigma^2_{nj}$) and the sufficiency part of Zolotarev's results. In the absence of asymptotic negligibility it will not necessarily be true that $s_n^{-2} s_{n+1}^2 \to 1$ and hence it will not always be possible to obtain a functional central limit theorem under conditions (1.13) and (1.14).
1.9 Rates of Convergence

If \( \{X_n\}_{n=1}^{\infty} \) are independent rv's, the rate of convergence of \( S_n^{-1} S_n \) to normality is neatly determined by the Berry-Esseen bound. When the \( X_n \) are independently and identically distributed with finite third moment and variance \( \sigma^2 \), the bound implies

\[
\sup_x |P(S_n - \sigma \sqrt{n} x) - \Phi(x)| = O(n^{-\frac{1}{2}}) \quad \text{as} \quad n \to \infty.
\]

Heyde and Brown [19] have provided an analogous result for martingales satisfying two asymptotic moment conditions. They have obtained bounds for

\[
\sup_x |P(S_n - s_n x) - \Phi(x)|
\]

based on universal constants and \( 2(1+\delta) \)-order moments \( (0 < \delta \leq 1) \). When the \( X_n \) are independent and identically distributed with 
\( 2(1+\delta) \)-order moments and variance \( \sigma^2 \), Heyde and Brown's Theorem gives us

\[
\sup_x |P(S_n - \sigma \sqrt{n} x) - \Phi(x)| = O(n^{-\delta/(2\delta+1)})
\]

which is not quite so precise as the Berry-Esseen bound.

In this brief survey we have emphasized the parallel development of martingale central limit theory and central limit theory for sums of independent random variables. In the past the independence theory has been used to indicate possible directions for future martingale research. However, several important independence results do not hold for martingales. The object of the remaining chapters of this thesis is to investigate the behaviour of martingales whose differences obey conditions which are not characteristic of independent variables.
CHAPTER II
SOME WEAK CONVERGENCE RESULTS FOR FORWARD MARTINGALES

2.1 Introduction and Summary

Let \( (S_n, F_n)_{1}^{\infty} \) be a zero-mean, square-integrable MG, and define

\[
X_n = S_n - S_{n-1} \quad (S_0 = 0),
\]

\[
s_n^2 = E(S_n^2)
\]

and

\[
V_n^2 = \sum_{j=1}^{n} E(X_j^2 | F_{j-1}).
\]

Most central limit theorems for MG's imitate limit theorems for sums of independent rv's, in the sense that they describe the asymptotic behaviour of \( s_n^{-1} S_n \). They require that \( s_n^{-2} V_n^2 \) converges to 1 in some sense; see for example condition (1) of Brown [9]. This often holds trivially for sums of independent rv's, for when \( \{X_n\}_{n=1}^{\infty} \) are independent and \( F_n = F(X_1, \ldots, X_n) \) we have \( V_n^2 = s_n^2 \) a.s.. However, it frequently does not hold for MG sequences.

The question arises as to how \( \{S_n\}_{n=1}^{\infty} \) behaves under milder conditions, such as \( s_n^{-2} V_n^2 \rightarrow q^2 \) where \( q^2 \) is an rv. There appears to be little published work on the subject of weak convergence under conditions like this. Heyde [18] has given a central limit theorem for randomly normed estimators of parameters in a branching process. Eagleson [17] has proved that if \( s_n^{-2} V_n^2 \rightarrow q^2 \) where \( q \) is an a.s. positive, \( F_1 \)-measurable rv and if the MG differences \( X_n \) satisfy Lindeberg and boundedness conditions, then \( s_n^{-1} S_n \overset{D}{\rightarrow} N(0, q^2) \) where \( q \) and \( N(0,1) \) are independent.
The work in this chapter begins with a reappraisal of Eagleson's result on convergence to mixtures of normal laws. Theorem 2.1 is a generalisation, and is given a classical characteristic function proof. The theorem could have been stated for triangular arrays of MG differences, but in its present form it leads more naturally to Theorems 2.2 and 2.3. It is interesting to note that convergence in probability in Theorem 2.1 cannot be replaced by convergence in distribution; a counter-example appears in Dvoretzky [16].

In Theorem 2.2 we extend Theorem 2.1 to a functional weak convergence result. Unfortunately to do this we have to strengthen $s_n^{-2} v_n^{-2} \to n^2$ to $s_n^{-2} v_n^{-2} \overset{a.s.}{\to} n^2$. It has not been possible to obtain either the convergence of finite distributions or tightness without the stronger condition.

Section 2.3 contains the most important work in this chapter. Proposition 2.1 extends some results in Theorem 1 of Scott [36], and Proposition 2.2 relates the Lindeberg condition of previous authors to our own, (2.23). Next we introduce a "measurability condition", (2.24), which describes how "close" the rv's $V^2_n$ are to being measurable in smaller $\sigma$-fields than $F_{n-1}$.

From Theorem 2.1 the question arises as to whether

$$S_n/V_n = (s_n^{-1} s_n)(s_n^{-1} V_n) \overset{D}{\to} N(0, n^2)/\eta = N(0, 1).$$

Theorem 2.3 answers this question in a slightly more general context and resembles Theorems 2 and 3 of Brown [9]. A specialised answer is given in the corollary, which also considers the case where $\{X_n\}_{n=1}^{\infty}$ is a stationary sequence.
2.2 Convergence to Mixtures of Normal Laws

For \( u \in [0,1] \) define

\[
k = k(n,u) = \max\{j < n | s_j^2 \leq u s_n^2 \} \quad \text{and} \quad \xi_n(u) = s_n^{-1}[s_k + (u s_n^2 - s_k^2)^{-1}s_{k+1}^2 - s_{k+1}^2]^{-1}X_{k+1}
\]

so that \( \xi_n \) is obtained by linearly interpolating between the points of the sequence

\[
(0,0), (s_n^{-2}s_1^2, s_n^{-1}s_1), (s_n^{-2}s_2^2, s_n^{-1}s_2), \ldots, (1, s_n^{-1}s_n^2).
\]

Let \( P_n \) be the probability measure induced on \( C[0,1] \) by \( \xi_n \). Let \( \eta^2 \) be a random variable on \( (\Omega, F, \mathbb{P}) \), and on some probability space define a standard Brownian motion \( W(u) \) \( (u > 0) \) and a copy \( \eta_{\perp}^2 \) of \( \eta^2 \), such that \( W \) and \( \eta_{\perp}^2 \) are independent. For \( u \in [0,1] \) let

\[
Z(u) = W(\eta_{\perp}^2 u)
\]

and let \( P(\eta_{\perp}^2) \) be the measure induced on \( C[0,1] \) by \( Z \).

Theorem 2.1 If

\[
\text{(2.1)} \quad s_n^{-2}\eta_{\perp}^2 \xrightarrow{P} \eta^2 < \infty \quad \text{a.s.}
\]

and

\[
\text{(2.2)} \quad \text{for all } \epsilon > 0, \quad s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \epsilon s_n)|F_{j-1}] \xrightarrow{P} 0,
\]

then

\[
s_n^{-1}s_n \xrightarrow{D} F
\]

where \( F \) is the distribution with characteristic function

\[
f(t) = E \exp(-\frac{1}{2} \eta^2 t^2).
\]
Theorem 2.2  If

\[ s_n^{-2} \sum_{i=1}^{n} \epsilon_i^2 \xrightarrow{a.s.} n < \infty \text{ a.s.} \]

and if (2.2) holds, then

\[ P_n \xrightarrow{w} P(\eta^2) . \]

Before proceeding to the proofs of Theorems 2.1 and 2.2, let us give some simple lemmas.

Lemma 2.1  If condition (2.1) holds, then for each \( \epsilon > 0 \) there exists a positive integer \( m \) and an \( F_m \)-measurable (simple) function \( \sigma^2 \) satisfying

\[ P(|\eta^2 - \sigma^2| > \epsilon) < \epsilon . \]

Lemma 2.2  [Kolmogorov-Brown]. If \( \{(S_j, F_j)\}_1^n \) is an MG and \( \epsilon > 0 \) then

\[ \max P(j \leq n | S_j | > \epsilon) \leq \epsilon^{-2} \mathbb{E}(S_n^2) \]

(Kolmogorov's inequality), and

\[ \max P(j \leq n | S_j | > 2\epsilon) \leq \epsilon^{-1} \int_{\{|S_n| > \epsilon\}} |S_n|^2 dP \]

(Brown's inequality).

PROOFS OF RESULTS IN §2.2

Proof of Lemma 2.1  Let \( F_\infty \) be the \( \sigma \)-field generated by the field

\[ \bigcup_{n=1}^{\infty} F_n . \]

Since \( \eta^2 \) is the a.s. limit of a subsequence of the sequence \( \{s_n^{-2} \sum_{i=1}^{n} \epsilon_i^2\}_1^{\infty} \) of \( F_\infty \)-measurable rv's, then \( \eta^2 \) is measurable in the completion of \( F_\infty \). Hence \( \eta^2 \) is a.s. equal to an \( F_\infty \)-measurable rv, and so we may
suppose without loss of generality that \( \eta^2 \) is itself \( F_\infty \)-measurable.

If \( \varepsilon > 0 \) we can choose a simple function

\[
\tau^2 = \sum_{i=1}^{r} a_i I(A_i),
\]

where each \( A_i \in F_\infty \) and \( a_i > 0 \), such that

\[
P(\left| \eta^2 - \tau^2 \right| > \varepsilon) < \varepsilon/2.
\]

For each \( j \leq n \) we can choose a positive integer \( m_j \) and a set \( B_j \in F_{m_j} \) such that

\[
P(A_j \Delta B_j) < \varepsilon/2r,
\]

where \( \Delta \) denotes the symmetric difference. (See for example Theorem 8.1.1 of Chung [12].) Let

\[
\sigma^2 = \sum_{i=1}^{r} a_i I(B_i)
\]

and

\[
m = \max_{j \leq r} m_j.
\]

Then \( \sigma^2 \) is \( F_m \)-measurable, and \( \sigma^2 = \tau^2 \) except possibly on the set

\[
\bigcup_{i=1}^{r} (A_i \Delta B_i)
\]

which has measure less than \( \varepsilon/2 \). Hence

\[
P(\left| \eta^2 - \sigma^2 \right| > \varepsilon) < \varepsilon.
\]

A proof of Kolmogorov's inequality can be found on p. 314 of Doob.
(Note that \( \{(S_j^2, F_j)\}_{1}^{n} \) is a sub MG.) A proof of Brown's inequality appears in [9].
Proof of Theorem 2.1  The method of proof is straightforward. We approximate to \( \eta^2 \) by an \( F_m \)-measurable rv \( \sigma^2 \), then approximate to \( S_n \) by the function

\[
s_{n}^{T_n} = \sum_{m=1}^{n} X_j I(s_{n}^{-2}V_{m}^2 < \sigma^2 + \delta), \quad n = m+1, m+2, \ldots,
\]

and finally we apply a technique of Brown [9] to the sequence \( \{ T_n \}_{m+1}^{\infty} \).

If \( \delta > 0 \) choose a positive integer \( m(\delta) \) and a simple function \( \sigma^2 \in F_m \) as in Lemma 2.1, s.t.

\[
P(|\eta^2 - \sigma^2| > \delta) < \delta.
\]

If \( N \) is chosen sufficiently large we have for all \( n > N \),

\[
P(| \eta^2 - \sigma^2 | < \delta; |\eta^2 - \sigma^2| < \delta; s_{n}^{-2}V_{m}^2 < \delta; s_{n}^{-1}|S_m| < \delta) > 1- \delta.
\]

For \( j, r, n > \max(m, N) \) define

\[
Y_j = Y_{j}(n, \delta, \sigma^2) = s_{n}^{-1}X_{j} I(s_{n}^{-2}V_{j}^2 < \sigma^2 + \delta),
\]

\[
T_r = T_{r}(n, \delta, \sigma^2) = \sum_{j=m+1}^{n} Y_{j}
\]

and

\[
U_{r}^2 = U_{r}^2(n, \delta, \sigma^2) = \sum_{j=m+1}^{n} E(Y_{j}^2 | F_{j-1}).
\]

We wish to show that

\[
|E \exp(it\eta_n^{-1}S_n) - E \exp(-\frac{1}{2}\eta^2t^2)| \to 0 \quad \text{as} \quad n \to \infty.
\]

Now,

(2.4)  \[
|E \exp(it\eta_n^{-1}S_n) - E \exp(-\frac{1}{2}\eta^2t^2)| \leq E|\exp(it\eta_n^{-1}S_n) - \exp(itT_n)| +
+ E|\exp(itT_n) - \exp(itT_n + \frac{1}{2}\sigma^2U_{n}^2 - \frac{1}{2}\sigma^2t^2)| +
+ |E[(\exp(itT_n + \frac{1}{2}\sigma^2U_{n}^2) - 1)\exp(-\frac{1}{2}\sigma^2t^2)]| + E|\exp(-\frac{1}{2}\sigma^2t^2) -
- \exp(-\frac{1}{2}\eta^2t^2)|.
\]
Since

\[ |s^{-1}_n S_n - T_n| \leq s^{-1}_n |S_m| + s^{-1}_n \sum_{m+1}^n X_j I(s_n^{-2}v_j^2 > \sigma^2 + 2\delta) \]

< \delta with probability > 1-\delta if n > N,

then the 1st term on the right hand side of (2.4) can be made arbitrarily small by choosing \( \delta \) sufficiently small and then \( n \) sufficiently large. The 2nd term on the right hand side of (2.4) equals

\[ E[1 - \exp \left( \frac{1}{2} t^2 (U_n^2 - \sigma^2) \right)] \]

which can be made small in a similar way, since

\[ U_n^2 = \sum_{n}^{\infty} E(X_j^2 | F_{j-1}) I(s_n^{-2}v_j^2 < \sigma^2 + 2\delta) < \sigma^2 + 2\delta \]

with probability 1, and

\[ |U_n^2 - \sigma^2| \leq \sum_{n}^{\infty} E(X_j^2 | F_{j-1}) I(s_n^{-2}v_j^2 < \sigma^2 + 2\delta) - n^2 |n^2 - \sigma^2| \]

< 3\delta with probability > 1-\delta if n > N.

The last term on the right hand side of (2.4) can also be made small by choosing \( \delta \) small and then \( n \) large, since \( f(x) = \exp(-\frac{1}{2} x^2) \) is bounded and uniformly continuous.

Hence it suffices to prove that the 3rd term of (2.4),

\[ A_n = |E[\exp(itT_n + \frac{1}{2} t^2 U_n^2) - 1] \exp(-\frac{1}{2} \sigma^2 t^2)| \]

converges to zero as \( n \to \infty \). To this end, define

\[ Z_j = \exp(itT_j + \frac{1}{2} t^2 U_j^2) - \exp(itT_{j-1} + \frac{1}{2} t^2 U_{j-1}^2) \]

\[ = [\exp(itY_j) - \exp(-\frac{1}{2} t^2 E(Y_j^2 | F_{j-1}) \}] \exp(itT_{j-1} + \frac{1}{2} t^2 U_{j-1}^2) \]

\[ = W_j \exp(itT_{j-1} + \frac{1}{2} t^2 U_{j-1}^2) \], say.
Then

\[ A_n = |E\{\exp\left(-\frac{1}{2} \sigma^2 t^2\right) \sum_{m+1}^{n} Z_j\}| \]  

(here it is convenient to define \( T_m = U_m^2 = 0 \))

\[ = |E\{\exp\left(-\frac{1}{2} \sigma^2 t^2\right) \sum_{m+1}^{n} E(Z_j|F_{j-1})\}| \]  

since \( \sigma^2 \) is \( F_m \)-measurable

\[ \leq E\{\exp\left(-\frac{1}{2} \sigma^2 t^2\right) \sum_{m+1}^{n} |E(Z_j|F_{j-1})|\} . \]

But

\[ E(Z_j|F_{j-1}) = E(W_j|F_{j-1}).\exp(itT_{j-1}+\frac{1}{2} t^2U_j^2) \]

since \( T_{j-1} \) and \( U_j^2 \) are \( F_{j-1} \)-measurable, and hence

\[ A_n \leq E\left( \sum_{m+1}^{n} \exp\left[\frac{1}{2} t^2(U_j^2-\sigma^2)\right]|E(W_j|F_{j-1})|\right) . \]

Since \( U_j^2 \leq \sigma^2 \leq \sigma^2+2\delta \) it suffices to prove that

\[ E\left( \sum_{m+1}^{n} |E(W_j|F_{j-1})|\right) \]

converges to zero as \( n \to \infty \).

Define \( Q(x) \) by \( e^{ix} = 1+ix-\frac{1}{2} x^2+\frac{1}{2} x^2 Q(x) \) and \( Z(x) \) by \( e^{-x} = 1-x-Z(x) \).

For real \( x \), \( |Q(x)| \leq \min\left(\frac{1}{3} |x|, 2\right) = M(x) \) say, and for positive \( x \), \( |Z(x)| \leq \frac{1}{2} x^2 . \)

Then we can write

\[ E(W_j|F_{j-1}) = 1+itE(Y_j|F_{j-1})-\frac{1}{2} t^2E(Y_j^2|F_{j-1})+\frac{1}{2} t^2E[Y_j^2Q(tY_j)|F_{j-1}] - \]

\[ - 1+\frac{1}{2} t^2E(Y_j^2|F_{j-1})+Z\left[\frac{1}{2} t^2E(Y_j^2|F_{j-1})\right] . \]
The 1st and 5th and the 3rd and 6th terms cancel. Since \( j > m \) and \( \sigma^2 \) is \( F_m \)-measurable, the 2nd term equals
\[
\lim_{n \to \infty} E(X_j | F_{j-1}) I(s_n^{-2}v_j^2 \leq \sigma^2 + 2\delta) = 0.
\]
Hence it suffices to show that the expectations of the sums of the moduli of the 4th and 7th terms in (2.5) converge to zero as \( n \to \infty \), i.e. that
\[
B_n = n^{-2} E\{ \sum_{m+1}^{n} E[X_j^2 I(s_n^{-2}v_j^2 \leq \sigma^2 + 2\delta) M(s_n^{-1}|tX_j| | F_{j-1})] \}
\]
and
\[
C_n = E\{ \sum_{m+1}^{n} Z_n^2 E(s_n^{-2}X_j I(s_n^{-2}v_j^2 \leq \sigma^2 + 2\delta) | F_{j-1})] \}
\]
converge to 0 as \( n \to \infty \).

If \( \epsilon > 0 \) then
\[
B_n \leq n^{-2} E\{ \sum_{m+1}^{n} E[X_j^2 I(|X_j| \leq \epsilon s_n) M(s_n^{-1}|tX_j| | F_{j-1})] + \]
\[
+ n^{-2} E\{ \sum_{m+1}^{n} E[X_j^2 I(|X_j| > \epsilon s_n) M(s_n^{-1}|tX_j| | F_{j-1})] I(s_n^{-2}v_j^2 \leq \sigma^2 + 2\delta) \}
\]
\[
\leq \frac{1}{3} \epsilon |t| n^{-2} E\{ \sum_{m+1}^{n} X_j^2 \} + 2E(s_n^{-2} \sum_{m+1}^{n} E[X_j I(|X_j| > \epsilon s_n) | F_{j-1}] I(s_n^{-2}v_j^2 \leq \sigma^2 + 2\delta) \}
\]
The 1st term after the last inequality doesn't exceed \( \frac{1}{3} \epsilon |t| \), and the 2nd term is \( o(1) \), since the integrand is \( o(1) \) in probability and is dominated by the integrable (simple) function \( (\sigma^2 + 2\delta) \). Hence \( B_n \to 0 \) as \( n \to \infty \).

Since \( |Z(x)| \leq \frac{1}{2} x^2 \), to prove that \( C_n \to 0 \) it suffices to prove that
\[ D_n = s^{-4} n \sum_{m+1}^n \left( E(X_m^2 | F_{j-1}) \right)^2 I(s_n^{-2} v_j^2 < \sigma^2 + \delta) \to 0. \]

Now, if \( \epsilon > 0 \) then

\[ D_n \leq 2 s^{-4} n \sum_{m+1}^n \left( E(X_m^2 I(|X_m| < \epsilon s_n) | F_{j-1}) \right)^2 + \]

\[ + 2 E(s_n^{-4} \sum_{m+1}^n \left( E[X_m^2 I(|X_m| > \epsilon s_n) | F_{j-1}) \right)^2 I(s_n^{-2} v_j^2 < \sigma^2 + \delta)) \]. \]

An application of Jensen's inequality bounds the 1st term on the right hand side by

\[ 2 s^{-4} n \sum_{m+1}^n E(X_m^4 I(|X_m| < \epsilon s_n) \leq 2 \epsilon^2 s^{-2} n \sum_{m+1}^n E(X_m^2) \leq 2 \epsilon^2. \]

The integrand of the 2nd term does not exceed

\[ \{s_n^{-2} \sum_{m+1}^n E[X_m^2 I(|X_m| > \epsilon s_n) | F_{j-1}] I(s_n^{-2} v_j^2 < \sigma^2 + \delta)) \}^2 \]

which \( \to 0 \), since it is \( o(1) \) in probability and is dominated by the integrable function \( (\sigma^2 + \delta)^2 \).

Hence \( C_n \to 0 \), and the proof is complete. **

Proof of Theorem 2.2 Theorem 2.2 will be proved in two parts. We will first show (Part I) that the finite distributions of \( P_n \) converge to those of \( P(\eta^2) \) and then (Part II) that the sequence of measures \( \{P_n\}_{m+1}^\infty \) is tight (see Billingsley [7], p.54).

To establish Part I we will set up a sequence of lemmas. The most important of these (Lemma 3) shows that \( \{\xi_n\}_{m+1}^\infty \) can be approximated by a sequence \( \{\eta_n\}_{m+1}^\infty \), from which it is more easy to obtain the conclusions
of the Theorem. The proof of Part I is then completed using techniques
due to Brown [9]. Basically, it is an extension of the proof of
Theorem 2.1.

The proof of Part II is modelled on that of Brown's Theorem 3.
The result is first obtained in the case $E(\eta^2) < \infty$, and then is
generalised using a truncation argument.

**Proof of Part I**: Convergence of finite dimensional distributions

Let $0 < \delta < 1$ and choose a positive integer $m$ and an $F_m$-measurable
simple function $\sigma^2$ s.t.

$$P(|\eta^2 - \sigma^2| > \delta) < \delta.$$ 

$m$ may be chosen so large that

$$P(\text{for all } n > m, |s_n^{-2} \gamma_n^2 - \eta^2| \leq \delta) > 1-\delta.$$ 

**Lemma 1** Under conditions (2.1) and (2.2), we can choose $N = N(\delta) > m$
so large that for all $n \geq N$,

(2.6)  $P(\max_{j \leq m} s_n^{-1} |S_j| > \delta) < \delta$,

(2.7)  $P(s_n^{-2} \gamma_n^2 > \delta) < \delta$,

(2.8)  $P(\max_{j \leq n} s_n^{-1} |X_j| > \delta) < \delta$, and

(2.9)  $P(s_n^{-2} \max_{j \leq n} E(\gamma_j^2|F_{j-1}) > \delta) < \delta$.

**Proof of Lemma 1** (2.6) and (2.7) are a consequence of the fact that
(2.2) implies $s_n \to \infty$. To prove (2.8), let $c > 0$ and write

$$p(c) = P(\eta^2 > c) \to 0 \text{ as } c \to \infty.$$ 

Then
\[ \Pr(s_n^{-1} \max_j j \leq n | X_j | > \delta) \leq \Pr(s_n^{-2} \max_j j \leq n X_j^2 I(s_n^{-2} \nu^2_j \leq 2c) > \delta^2) + p(c) + o(1) \]

\[ = \Pr(s_n^{-2} \max_j j \leq n X_j^2 I(s_n^{-2} \nu^2_j \leq 2c; |X_j| > \delta s_n) > \delta^2) + p(c) + o(1) \]

\[ \leq \delta^{-2} E[s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \delta s_n)|F_{j-1}] I(s_n^{-2} \nu^2_j \leq 2c)] + p(c) + o(1) \]

\[ = p(c) + o(1), \]

since the integrand of the expectation in the second-last line is \( o(1) \) in probability and is bounded by \( 2c \). This is true for all \( c > 0 \), and (2.8) follows.

To prove (2.9), write

\[ \Pr(s_n^{-2} \max_j j \leq n E(X_j^2|F_{j-1}) > \delta) \leq \]

\[ \leq \Pr(s_n^{-2} \max_j j \leq n E[X_j^2 I(|X_j| \leq \sqrt{\frac{1}{2}} \delta s_n)|F_{j-1}] + s_n^{-2} \max_j j \leq n E[X_j^2 I(|X_j| > \sqrt{\frac{1}{2}} \delta s_n)|F_{j-1}] > \delta) \]

\[ \leq \Pr(s_n^{-2} \max_j j \leq n E[X_j^2 I(|X_j| > \sqrt{\frac{1}{2}} \delta s_n)|F_{j-1}] > \frac{1}{2}) \]

\[ \leq \Pr(s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \sqrt{\frac{1}{2}} \delta s_n)|F_{j-1}] > \frac{1}{2}) \]

\[ = o(1). \]

We now introduce a random function which approximates \( \xi_n \). For \( u \in [0,1] \) and \( n \geq N \), define

\[ \eta_n(u) = s_n^{-1} \sum_{j=m+1}^{n} X_j I(s_n^{-2} \nu^2_j \leq u(\sigma^2 + 2\delta); s_n^{-2} E(X_j^2|F_{j-1}) \leq \delta). \]

**Lemma 2** Under conditions (2.2) and (2.3), for all \( n \geq N \) and for all \( u \in [0,1] \),
(2.10) \[ P(\sum_{m+1}^{n} X_j I(s_n^{-2}s_j^2 \leq u) - \sum_{m+1}^{n} X_j I(s_n^{-2}s_j^2 < u(\sigma^2 + 2\delta); s_n^{-2}E(X_j^2|F_{j-1}) \leq \delta) > \delta^3) < \delta^3. \]

Proof of Lemma 2 On the set

\[ E = \{ \text{for all } n > m, \ |s_n^{-2}s_n^2 - n^2| \leq \delta \} \cap \{|n^2 - \sigma^2| \leq \delta\} \]

we have for all \( j \) in \( m < j \leq n \),

\[ I(s_n^{-2}s_j^2 \leq u(\sigma^2 - 2\delta)) \leq I(s_n^{-2}s_j^2 \leq u) \leq I(s_n^{-2}s_j^2 \leq u(\sigma^2 + 2\delta)), \]

and on

\[ F = \{s_n^{-2} \max_{j < n} E(X_j^2|F_{j-1}) \leq \delta\} \]

we have for any event \( A \) and any \( j \leq n \),

\[ I(A) = I(A; s_n^{-2}E(X_j^2|F_{j-1}) \leq \delta). \]

Hence on \( E \cap F \), the term within modulus signs in (2.10) does not exceed

\[ \max_{m < r \leq s \leq n} \left| \sum_{r}^{s} X_j I(u(\sigma^2 - 2\delta) < s_n^{-2}s_j^2 \leq u(\sigma^2 + 2\delta); s_n^{-2}E(X_j^2|F_{j-1}) \leq \delta) \right| \leq 2 \max_{m < r \leq n} \left| \sum_{m+1}^{r} X_j I(u(\sigma^2 - 2\delta) < s_n^{-2}s_j^2 \leq u(\sigma^2 + 2\delta); s_n^{-2}E(X_j^2|F_{j-1}) \leq \delta) \right| \]

\[ = 2 \max_{m < r \leq n} |s_n M_r|, \text{ say.} \]

Applying Kolmogorov's inequality to the MG \( \{M_r\}_{m+1}^{n} \) we obtain

\[ P(\max_{m < r \leq n} |M_r| > \delta^3) \leq \frac{1}{\delta^3} \left[ \frac{2}{\delta^3} E(s_n^{-2} \sum_{m+1}^{n} E(X_j^2|F_{j-1}) I(u(\sigma^2 - 2\delta) < s_n^{-2}s_j^2 \leq u(\sigma^2 + 2\delta); s_n^{-2}E(X_j^2|F_{j-1}) \leq \delta) \} \right] \]
Since $P(E \cap F) > 1-3\delta$, the result follows. **

**Lemma 3** Under conditions (2.2) and (2.3), for all $n > N$ and for all $u \in [0,1],

$$P(|\xi_n(u) - \eta_n(u)| > \frac{1}{6^3}) < \frac{1}{106^3}.$$ 

**Proof of Lemma 3**

$$|\xi_n(u) - \eta_n(u)| =$$

$$= \left| s_n^{-1} \sum_{j=1}^{n} X_j I(s_n^{-2} \sum_{j=1}^{n} X_j^2 \leq u) - s_n^{-1} \sum_{j=1}^{m} X_j I(s_n^{-2} \sum_{j=1}^{m} X_j^2 \leq u(\sigma^2 + 2\delta)); s_n^{-2} E(X_j^2 | F_{j-1}) \leq \delta) + s_n^{-1} (\sum_{k=1}^{n} X_k - s_n^{-2} \sum_{k=1}^{n} X_k^2)^{-1} X_{k+1}\right|$$

$$\leq s_n^{-1} \max_{j \leq m} |S_j| + s_n^{-1} \max_{j \leq n} |X_j| +$$

$$+ s_n^{-1} \left| \sum_{m+1}^{n} X_j I(s_n^{-2} \sum_{j=1}^{m} X_j^2 \leq u) - s_n^{-1} \sum_{m+1}^{n} X_j I(s_n^{-2} \sum_{j=1}^{m} X_j^2 \leq u(\sigma^2 + 2\delta)); s_n^{-2} E(X_j^2 | F_{j-1}) \leq \delta)\right|$$

$$< \delta + \delta + 2\delta^3 \text{ with probability } > 1-\delta-\delta-8\delta^3 > 1-10\delta^3,$$

using Lemmas 1 and 2. **

Now we must introduce some more notation. Suppose

$$0 = u_0 < u_1 < \ldots < u_p \leq 1.$$ 

Let $t_1, \ldots, t_p$ be real numbers and define

$$\alpha^2 = \eta^2 \sum_{k=1}^{p} (u_k - u_{k-1})^2 t_k^2,$$

$$\beta^2 = \sigma^2 \sum_{k=1}^{p} (u_k - u_{k-1})^2 t_k^2.$$
\[ t = \max |t_k| , \]

\[ \theta_j = \sum_{k=1}^{P} t_k I(u_k - 1(\sigma^2 + 2\delta) < s_n^{-2} v_j^2 \leq u_k(\sigma^2 + 2\delta); s_n^{-2} E(X_j^2 | F_{j-1}) \leq \delta) , \]

and for \( j \geq N \) and \( n \geq r \geq N, \)

\[ Y_j = s_n^{-1} \theta_j X_j , \]

\[ T_r = \sum_{m=1}^{n} Y_j , \quad \text{and} \]

\[ U_r^2 = \sum_{m=1}^{n} E(Y_j^2 | F_{j-1}) . \]

\( \theta_j \) is an \( F_{j-1} \)-measurable rv and \( |\theta_j| \) does not exceed \( t I(s_n^{-2} v_j^2 < \sigma^2 + 2\delta) . \)

We note that for all \( k, \)

\[ \eta_n(u_k) - \eta_n(u_k - 1) = s_n^{-1} \sum_{j=m+1}^{n} X_j I(u_k - 1(\sigma^2 + 2\delta) < s_n^{-2} v_j^2 \leq u_k(\sigma^2 + 2\delta); s_n^{-2} E(X_j^2 | F_{j-1}) \leq \delta) \]

and

\[ \sum_{k=1}^{P} t_k (\eta_n(u_k) - \eta_n(u_k - 1)) = \]

\[ = s_n^{-1} \sum_{j=m+1}^{n} X_j \sum_{k=1}^{P} t_k I(u_k - 1(\sigma^2 + 2\delta) < s_n^{-2} v_j^2 \leq u_k(\sigma^2 + 2\delta); s_n^{-2} E(X_j^2 | F_{j-1}) \leq \delta) \]

\[ = s_n^{-1} \sum_{m=1}^{n} \theta_j X_j \]

\[ = T_n . \]

We must prove that

\[ |\text{Eexp} \left[ \sum_{k=1}^{P} t_k (\xi_n(u_k) - \xi_n(u_k - 1)) \right] - \text{Eexp} (-\frac{1}{2} \sigma^2) | \rightarrow 0 \quad \text{as} \ n \rightarrow \infty . \]
As in the proof of Theorem 2.1, we decompose the LHS into several pieces and show that each can be made small:

\[
\text{(2.11) } \quad |\mathbb{E}\exp\left[i \sum_{k=1}^{P} t_k(\xi_n(u_k) - \xi_n(u_{k-1}))\right] - \mathbb{E}\exp\left(-\frac{1}{2} \alpha^2 \right)| \leq \\
\leq \mathbb{E}\left|\exp\left[i \sum_{k=1}^{P} t_k(\xi_n(u_k) - \xi_n(u_{k-1}))\right] - \exp\left[i \sum_{k=1}^{P} t_k(\eta_n(u_k) - \eta_n(u_{k-1}))\right]\right| + \\
+ \mathbb{E}\left|\exp(it_n) - \exp(it_n + \frac{1}{2} \frac{U_n^2}{n} - \frac{1}{2} \beta^2)\right| + \left|\mathbb{E}\left[\exp(it_n + \frac{1}{2} \frac{U_n^2}{n}) - 1\right]\frac{1}{2} \beta^2\right| + \\
+ \mathbb{E}\left|\exp\left(-\frac{1}{2} \beta^2\right) - \exp\left(-\frac{1}{2} \alpha^2\right)\right| .
\]

Since

\[
\left|\sum_{k=1}^{P} t_k(\xi_n(u_k) - \xi_n(u_{k-1})) - \sum_{k=1}^{P} t_k(\eta_n(u_k) - \eta_n(u_{k-1}))\right| < 8\pi \delta^3
\]

with probability \( > 1 - 10\pi \delta^3 \) if \( n > N \) (Lemma 3) then by choosing \( \delta \) sufficiently small and then \( n \) sufficiently large, the 1st term on the RHS of (2.11) can be made arbitrarily small.

The second term on the RHS of (2.11) equals

\[
\mathbb{E}\left|1 - \exp\left(\frac{1}{2} \frac{U_n^2 - \beta^2}{n}\right)\right|
\]

which, in view of the following lemma, can be made small in a similar way.

**Lemma 4** For all \( n \), \( U_n^2 - \beta^2 < 3\delta pt^2 \) with probability 1, and for all \( n > N \),

\[
P(U_n^2 - \beta^2 \geq -2\delta pt^2) > 1 - 2\delta .
\]

**Proof of Lemma 4**

\[
U_n^2 = s_n^{-2} \sum_{j=m+1}^{n} \theta_j^2 E(X_j^2 | F_{j-1})
\]
which proves the 1st result. To obtain the 2nd result, note that on the set

\[ G = \{s_n^{-2} \max_{j \leq n} E(X_j^2 | F_{j-1}) \leq \delta \} \cap \{s_n^{-2} \leq \delta \} \]

we have

\[ U_n^2 = \sum_{k=1}^{p} t_k^2 \left[ s_n^{-2} \sum_{j=m+1}^{n} E(X_j^2 | F_{j-1}) \right] I(s_n^{-2} \leq u_k(\sigma^2 + 2\delta)) - \]

\[ - s_n^{-2} \sum_{j=m+1}^{n} E(X_j^2 | F_{j-1}) \right] I(s_n^{-2} \leq u_k(\sigma^2 + 2\delta)) \]

\[ \geq \sum_{k=1}^{p} t_k^2 \left[ s_n^{-2} \sum_{j=1}^{n} E(X_j^2 | F_{j-1}) \right] I(s_n^{-2} \leq u_k(\sigma^2 + 2\delta)) - \]

\[ - s_n^{-2} \sum_{j=1}^{n} E(X_j^2 | F_{j-1}) \right] I(s_n^{-2} \leq u_k(\sigma^2 + 2\delta)) \]
\[
\sum_{k}^{p} \frac{t_k^2}{u_k} = u_k (\sigma^2 + 2\delta) - \delta\cdot u_{k-1} (\sigma^2 + 2\delta) - \delta pt^2 \\
\geq \beta^2 - 2\delta pt^2.
\]

Since \( P(G) > 1 - 2\delta \), the 2nd result is proved.

The last term on the RHS of (2.11) can also be made small. Therefore it suffices to prove that the second last term,

\[
A_n = \left| \mathbb{E} \{ \exp \left( i T_n + \frac{1}{2} U_n^2 \right) \} \right|
\]

converges to zero as \( n \to \infty \). To this end, define

\[
Z_j = \exp \left( i T_{j} + \frac{1}{2} U_{j}^2 \right) - \exp \left( i T_{j-1} + \frac{1}{2} U_{j-1}^2 \right)
\]

\[
= \exp \left( i Y_j \right) - \exp \left( -\frac{1}{2} \mathbb{E} (Y_j^2 | F_{j-1}) \right) \exp \left( i T_{j-1} + \frac{1}{2} U_{j}^2 \right)
\]

\[
= W_j \exp \left( i T_{j-1} + \frac{1}{2} U_{j}^2 \right), \quad \text{say.}
\]

Then

\[
A_n = \left| \mathbb{E} \{ \exp \left( -\frac{1}{2} \beta^2 \right) \sum_{m=1}^{n} Z_j \} \right| \quad \text{(here it is convenient to define \( T_m = U_m^2 = 0 \))}
\]

\[
= \left| \mathbb{E} \{ \exp \left( -\frac{1}{2} \beta^2 \right) \sum_{m=1}^{n} \mathbb{E} (Z_j | F_{j-1}) \} \right|
\]

\[
\leq \mathbb{E} \{ \sum_{m=1}^{n} \mathbb{E} (Z_j | F_{j-1}) \}.
\]

But

\[
\mathbb{E} (Z_j | F_{j-1}) = \mathbb{E} (W_j | F_{j-1}) \exp \left( i T_{j-1} + \frac{1}{2} U_{j}^2 \right)
\]

and hence

\[
A_n \leq \mathbb{E} \left\{ \sum_{m=1}^{n} \exp \left[ \frac{1}{2} (U_j^2 - \beta^2) \right] \mathbb{E} (W_j | F_{j-1}) \right\}.
\]

Since

\[
U_j^2 \leq U_n^2 \leq \beta^2 + 3\delta pt^2,
\]
it suffices to prove that

\[ E_\infty \left( \sum_{m+1}^{n} \left| E(W_j | F_{j-1}) \right| \right) \to 0 \quad \text{as} \quad n \to \infty . \]

Define \( Q(x), Z(x) \) and \( M(x) \) as in the proof of Theorem 2.1, and decompose \( E(W_j | F_{j-1}) \) as in (2.5). The argument following (2.5) can be used to show it is sufficient to prove that

\[ B_n = s_n^{-2}E_\infty \left( \sum_{m+1}^{n} \theta_j^2 E(x_j^2 | Q(s_n^{-1} \theta_j x_j) | F_{j-1}) \right) \]

and \( C_n = E_\infty \left( \sum_{m+1}^{n} Z(s_n^{-2} \theta_j^2 X_j^2 | F_{j-1}) \right) \)

converge to zero as \( n \to \infty \).

If \( \epsilon > 0 \) then

\[ B_n \leq t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ M(s_n^{-1} | tX_j |) | F_{j-1}) \ I(s_n^{-2} v_j^2 \leq \sigma^2 + 2\delta) \right) \]

\[ \leq t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ M(s_n^{-1} | tX_j |) \ I(|X_j| \leq \epsilon s_n) | F_{j-1}) \right) + \]

\[ + t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ M(s_n^{-1} | tX_j |) \ I(|X_j| > \epsilon s_n) | F_{j-1}) \ I(s_n^{-2} v_j^2 < \sigma^2 + 2\delta) \right) \]

\[ \leq t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ \frac{1}{3} s_n^{-1} | tX_j | \ I(|X_j| \leq \epsilon s_n) | F_{j-1}) \right) + \]

\[ + t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ 2 \ I(|X_j| > \epsilon s_n) | F_{j-1}) \ I(s_n^{-2} v_j^2 < \sigma^2 + 2\delta) \right) \]

\[ \leq \frac{1}{3} \epsilon t^3 s_n^{-2}E_\infty (x_j^2) + 2t^2 s_n^{-2}E_\infty \left( \sum_{m+1}^{n} E(x_j^2 \ I(|X_j| > \epsilon s_n) | F_{j-1}) . I(s_n^{-2} v_j^2 < \sigma^2 + 2\delta) \right) . \]
The 1st term $\leq \frac{1}{3} \varepsilon t^3$ and the second is $o(1)$, since the integrand is $o(1)$ in probability and is dominated by the integrable (simple) function $(\sigma^2 + 2\delta)$. Hence $B_n \to 0$ as $n \to \infty$.

Since $|Z(x)| \leq \frac{1}{2} x^2$, to prove that $C_n \to 0$ it suffices to prove that

$$D_n = s_n^{-4} E\left\{ \sum_{m+1}^n \theta_j^4 \left( \sum_{j=1}^m E(X_j^2 | F_{j-1}) \right)^2 \right\} \to 0.$$

Now, if $\varepsilon > 0$ then

$$D_n \leq 2t^4 s_n^{-4} \sum_{m+1}^n \theta_j^4 \left( E(X_j^2 | I(|X_j| \leq \varepsilon s_n^2 | F_{j-1})) \right)^2 +$$

$$+ 2t^4 E\left\{ \sum_{m+1}^n \theta_j^4 \left( E(X_j^2 | I(|X_j| > \varepsilon s_n^2 | F_{j-1})) \right)^2 \right\}$$

$$\leq 2t^4 s_n^{-4} \sum_{m+1}^n \left( E(X_j^2 | I(|X_j| \leq \varepsilon s_n^2 | F_{j-1})) \right)^2 +$$

$$+ 2t^4 s_n^{-4} \sum_{m+1}^n \left( E(X_j^2 | I(|X_j| > \varepsilon s_n^2 | F_{j-1})) \right)^2 I(s_n^{-2} v_j^2 \leq \sigma^2 + 2\delta).$$

By applying Jensen's inequality we can bound the 1st term on the RHS by

$$2t^4 s_n^{-4} \sum_{m+1}^n X_j^4 I(|X_j| \leq \varepsilon s_n) \leq 2t^4.$$

The integrand of the 2nd term does not exceed

$$\{s_n^{-2} \sum_{m+1}^n E[X_j^2 | I(|X_j| > \varepsilon s_n | F_{j-1})] I(s_n^{-2} v_j^2 \leq \sigma^2 + 2\delta) \}^2$$

which, as in the proof of Theorem 2.1, $\to 0$.

Hence $C_n \to 0$, and the proof of Part I is complete. **
Proof of Part II: Tightness

We will continue to use the notation introduced in Part I.

Tightness prevails if

\[(2.12) \quad \text{for all } \varepsilon > 0, \lim_{n \to \infty} \limsup_{h \to 0} \sum \Pr( \sup_{kh < u \leq (k+1)h} |\xi_n(u) - \xi_n(kh)| > 4\varepsilon) = 0\]

(Parthasarathy [31], p. 222). Now,

\[
|\xi_n(u) - \xi_n(kh)| \leq 2s_n^{-1} \max_{j \leq m} |X_j| + 2s_n^{-1} \max_{j \leq m} |S_j| + s_n^{-1} \sum_{j=m+1}^{n} X_j I(s_n^{-2}S_j^2 \leq u) - \sum_{j=m+1}^{n} X_j I(s_n^{-2}S_j^2 \leq kh) \cdot I(s_n^{-2}S_j^2 \leq kh).
\]

On the set

\[E = \{\text{for all } n \geq m, \{s_n^{-2}V_n^2 - \eta^2 \leq \delta\} \cap \{\eta^2 - \sigma^2 \leq \delta\} \}
\]

we have for \(kh < u \leq (k+1)h\) and all \(j \geq m,\)

\[I(s_n^{-2}V_j^2 \leq kh(\sigma^2 + 2\delta)) \leq I(s_n^{-2}S_j^2 \leq u) \leq I(s_n^{-2}S_j^2 \leq u) \leq I(s_n^{-2}V_j^2 \leq (k+1)h(\sigma^2 + 2\delta)) \]

and hence on \(E,\)

\[
\sup_{kh < u \leq (k+1)h} s_n^{-1} \left| \sum_{j=m+1}^{n} X_j I(s_n^{-2}S_j^2 \leq u) - \sum_{j=m+1}^{n} X_j I(s_n^{-2}S_j^2 \leq kh) \right| \leq s_n^{-1} \max_{m \leq r \leq s_1} \left| \sum_{j=r}^{s} X_j I(kh(\sigma^2 - 2\delta) < s_n^{-2}V_j^2 \leq (k+1)h(\sigma^2 + 2\delta)) \right|
\]

\[
\leq 2s_n^{-1} \max_{m \leq r \leq s_1} \left| \sum_{j=r}^{s} X_j I(kh(\sigma^2 - 2\delta) < s_n^{-2}V_j^2 \leq (k+1)h(\sigma^2 + 2\delta)) \right|.
\]

Therefore for \(n \geq N,\)

\[
\Pr\left( \sup_{kh < u \leq (k+1)h} |\xi_n(u) - \xi_n(kh)| > 4\varepsilon \right) = P_{n,k} (\text{say})
\]
\[ \leq P(\frac{s_n^{-1}}{m-1} \max_{m \leq r \leq n} \sum_{j=1}^{r} I(kh(\sigma^2-2\delta) < s_n^{-2}E_{j-1}^2 < (k+1)h(\sigma^2+2\delta); \]
\[ s_n^{-2}E(X_{j}^2|F_{j-1}) \leq \delta) | > 2(\epsilon-\delta) + 5\delta \]

(using Lemma 1)

\[ \leq P(\frac{s_n^{-1}}{m-1} \max_{m \leq r \leq n} \sum_{j=1}^{r} I(kh(\sigma^2-2\delta) < s_n^{-2}E_{j-1}^2 < (k+1)h(\sigma^2+2\delta); \]
\[ s_n^{-2}E(X_{j}^2|F_{j-1}) \leq \delta) | > \frac{1}{\delta^3} + \frac{1}{3} \]

\[ + \frac{1}{\delta^3} E[kh(\sigma^2+2\delta)-kh(\sigma^2-2\delta)+6] \leq 5\delta^3 \cdot \frac{1}{3} \]

An application of Brown's inequality (see Lemma 2.2) to the MG

\[ M_r = s_n^{-1} \sum_{m+1}^{r} I(kh(\sigma^2+2\delta) < s_n^{-2}E_{j-1}^2 < (k+1)h(\sigma^2+2\delta); s_n^{-2}E(X_{j}^2|F_{j-1}) \leq \delta), \]
\[ r = m+1, \ldots, n, \]

bounds the 2nd term by

\[ \Delta^{-1} \int_{\{M_n \mid \Delta \}} |M_n| |dP| , \]

where \( \Delta = \epsilon - 2\delta^3 \). Hence

\[ p_{n,k} \leq \Delta^{-1} \int_{\{M_n \mid \Delta \}} |M_n| |dP| + 10\delta^3 \]
where $c$ is a large positive constant, $H$ is the set
\[
H = \{|\xi_n(kh)-\eta_n(kh)| < \frac{1}{4\delta^3}; |\xi_n((k+1)h)-\eta_n((k+1)h)| < \frac{1}{4\delta^3}\},
\]
and $\overline{H}$ is the complement of $H$. In view of our Lemma 3, $H$ has probability $> 1-20\delta^3$.

A Chebyshev-type inequality can be used to bound the 1st term in (2.13) by $(\Delta c)^{-1}E(\sigma^2+2\delta)$. The 2nd does not exceed \[\Delta^{-1}\sqrt{E(\sigma^2+2\delta) P(\overline{H})}\] (Cauchy-Schwartz inequality). Since
\[
|M_n| = |\eta_n((k+1)h)-\eta_n(kh)|
\]
\[
\leq |\xi_n((k+1)h)-\xi_n(kh)| + \frac{1}{4\delta^3} \quad \text{on } H
\]
\[
= |M'_n| + \frac{1}{4\delta^3} \quad \text{say},
\]
then the 3rd term on the RHS of (2.13) is bounded by
\[
\Delta^{-1}\left(\frac{1}{4\delta^3} + \int_{\{|c+8\delta^3 > |M'_n| > \Delta - 8\delta^3\}} |M'_n| dP \frac{1}{4\delta^3}\right).
\]
Consequently
\[
P_{n,k} \leq \Delta^{-1}[c^{-1}E(\sigma^2+2\delta) + \sqrt{E(\sigma^2+2\delta) 20\delta^3} + \frac{1}{8\delta^3} + \int_{\{|c+8\delta^3 > |M'_n| > \Delta - 8\delta^3\}} |M'_n| dP \frac{1}{8\delta^3}] .
\]

Since the finite dimensional distributions of $\xi_n$ converge to those of $Z$, and
\[
M'_n = \xi_n((k+1)h)-\xi_n(kh),
\]
then
\[
(2.14) \quad \limsup_{n \to \infty} p_{n,k} \leq \Delta^{-1} \left[ c^{-1} E(\sigma^2 + 2\delta) + \sqrt{E(\sigma^2 + 2\delta) + \frac{1}{3} + \frac{1}{3}} \right. \\
+ E\left\{ \int_{c+2\delta}^{c+8\delta} \frac{1}{\Delta-8\delta^3} x \, dP\left( |N(0,\eta^2 h)| \leq x \right) \right\}
\]

where in the innermost integral in the last term, \( \eta^2 \) is held constant.

At this stage in the proof we will suppose that \( E(\eta^2) < \infty \). Then for each \( \delta > 0 \) it is possible to choose \( \sigma^2 \) s.t. \( E(\sigma^2) < 2E(\eta^2) \). For such a choice of \( \sigma^2 \), the terms \( E(\sigma^2 + 2\delta) \) in (2.14) are always bounded by \( 2(E\eta^2 + 1) \).

The inequality (2.14) holds for all \( \delta \) and \( c > 0 \), and hence is true in the limit as \( \delta \to 0 \) and \( c \to \infty \). Taking these limits,

\[
\limsup_{n \to \infty} p_{n,k} \leq \epsilon^{-1} E\left\{ \int_\epsilon^\infty x \, dP\left( |N(0,\eta^2 h)| \leq x \right) \right\}
\]

\[
= \epsilon^{-1} E\left\{ \frac{\sqrt{2/\pi}}{\eta} \sqrt{h} \exp\left( -\frac{1}{2} \epsilon^2 h^{-1} \eta^{-2} \right) \right\}.
\]

Hence, returning to (2.12),

\[
\limsup_{n \to \infty} E P\left( \sup_{kh<1} \left| \xi_n(u) - \xi_n(kh) \right| > 4\epsilon \right) \leq \epsilon^{-1} E\left\{ \frac{\sqrt{2/\pi}}{\eta} \sqrt{h} \exp\left( -\frac{1}{2} \epsilon^2 h^{-1} \eta^{-2} \right) \right\}.
\]

Since

\[
\frac{1}{\eta h^{-1} \exp(-h^{-1} \eta^{-2})} \leq \eta^2
\]

and the LHS converges to zero as \( h \to 0 \), then the dominated convergence theorem provides that

\[
E\left( \frac{1}{\eta h^{-2} \exp(-h^{-1} \eta^{-2})} \right) \to 0 \quad \text{as} \quad h \to 0.
\]
It follows that

$$\lim \limsup \sum P(\sup_{h \to 0} \sup_{n \to \infty} |\xi_n(u) - \xi_n(kh)| > 4\varepsilon) = 0,$$

and the theorem is proved for $E(\eta^2) < \infty$.

Before considering the case of a more general $\eta^2$, let us remark that if $\{u_n\}_1^\infty$ is any increasing sequence of positive numbers s.t.

$$s_n^{-2}u_n^2 \to k^2, \quad 0 < k < \infty,$$

and if in the definition of $\xi_n$ we replace $s_j$ by $u_j$ wherever it occurs, obtaining a new process $\xi_n$, then the above proofs of Parts I and II can be reworked to show that if $E(\eta^2) < \infty$,

$$\xi_n \overset{D}{\to} W(k^{-2}\eta^2).$$

Now suppose that $\eta^2$ is any a.s. finite rv which is not a.s. zero. Let $c > 0$ be a continuity point of $\eta^2$ and consider the MG difference sequence

$$Y_j = X_j I(s_j^{-2}v_j^2 \leq c), \quad j = 1,2,... .$$

Define

$$T_n = \sum_{j=1}^n Y_j,$$

$$t_n^2 = ET_n^2,$$

$$w_n^2 = \sum_{j=1}^n E(Y_j^2|F_{j-1}) = \sum_{j=1}^n E(X_j^2|F_{j-1}) I(s_j^{-2}v_j^2 \leq c).$$

If

$$s_n^{-2}v_n^2(\omega) \to \eta^2(\omega) < c$$

then for all sufficiently large $j$,

$$s_j^{-2}v_j^2(\omega) < c.$$
Consequently, since \( s_n \to \infty \),

\[
\frac{s_n^{-2}}{\varphi_n^2}(\omega) = \frac{s_n^{-2}}{\varphi_n} \sum_{j=1}^{\infty} E(X_j^2 | F_{j-1})(\omega) I(s_j^{-2} \varphi_j^2 \leq c)(\omega) \to \eta^2(\omega).
\]

If

\[
\frac{s_n^{-2}}{\varphi_n^2}(\omega) + \eta^2(\omega) > c
\]

then a similar argument shows that

\[
\frac{s_n^{-2}}{\varphi_n^2}(\omega) \to 0.
\]

Since

\[
P(\eta^2 = c) = 0
\]

then

\[
\frac{s_n^{-2}}{\varphi_n^2} \overset{a.s.}{\to} (\eta_c)^2 = \eta^2 I(\eta^2 \leq c).
\]

Choose \( c \) sufficiently large for

\[
\lambda^2 = E(\eta_c)^2 > 0.
\]

\( \{s_n^{-2}\varphi_n^2\}_{n=1}^\infty \) is uniformly integrable (in fact, it is uniformly bounded) and hence

\[
\frac{s_n^{-2}}{\varphi_n^2} E(\eta_n^2) \to E(\eta_c)^2,
\]

that is,

\[
\frac{s_n^{-2}}{\varphi_n^2} \lambda_n^2 \to \lambda^2.
\]

Consequently

\[
\frac{s_n^{-2}}{\varphi_n^2} \overset{a.s.}{\to} \lambda^2(\eta_c)^2 \quad \text{as} \quad n \to \infty.
\]

Define the random function \( \xi_n \) for the MG \( \{(T_n, F_n)\}_{n=1}^\infty \) in the same way as we defined \( \xi_n \) for the MG \( \{(S_n, F_n)\}_{n=1}^\infty \). Since

\[
E(\lambda^{-2}(\eta_c)^2) < \infty
\]

then the results proved so far imply

\[
\xi_n \overset{\mathcal{D}}{\to} W(\lambda^{-2}(\eta_c)^2 t).
\]
Define the random function \( \tilde{\xi}_n \) by replacing \( t_j \) by \( s_j \) wherever it occurs in the definition of \( \tilde{\xi}_n \). That is, \( \tilde{\xi}_n \) is the function obtained by linearly interpolating between the points

\[(0,0), (s_{n-1}^2, s_{n-1}T_1), (s_{n-1}^2, s_{n-1}T_2), \ldots, (1, s_{n}^{-1}T_n) .\]

The remark following the proof of Part II allows us to say that

\[\tilde{\xi}_n \xrightarrow{D} W((\eta_c)^2t) ,\]

and hence the sequence \( \{\xi_n\}_{n=1}^\infty \) is tight.

Now let us return to the sequence \( \{\xi_n\}_{n=1}^\infty \). \( \{\xi_n\}_{n=1}^\infty \) is tight iff

for all \( \varepsilon > 0 \),

\[\lim_{h \to 0} \limsup_{n \to \infty} P(\sup_{\lvert u-v \rvert \leq h} \lvert \xi_n(u) - \xi_n(v) \rvert > \varepsilon) = 0\]

(Billingsley [7], p.55).

Since

\[\lvert \xi_n(u) - \xi_n(v) \rvert \leq \lvert \xi_n(u) - \tilde{\xi}_n(u) \rvert + \lvert \tilde{\xi}_n(u) - \tilde{\xi}_n(v) \rvert + \lvert \tilde{\xi}_n(v) - \xi_n(v) \rvert\]

and for all choices of \( c \), \( \{\tilde{\xi}_n\}_{n=1}^\infty \) is tight, it is sufficient to prove that

for all \( \varepsilon > 0 \),

\[\limsup_{n \to \infty} P(\sup_{u \in [0,1]} \lvert \xi_n(u) - \tilde{\xi}_n(u) \rvert > \varepsilon) = 0\]

can be made arbitrarily small by choosing \( c \) sufficiently large.

If \( P(\eta^2 > c) = \delta > 0 \), choose a positive integer \( M \) so large that

\[P(\text{for all } n > M, \, s_n^{-2}V_n^2 \leq c) > 1 - 2\delta .\]

If \( n > M \),

\[\lvert \xi_n(u) - \tilde{\xi}_n(u) \rvert \leq s_n^{-1} \max_{1 < r \leq n} \left\{ \sum_{j=1}^{r} X_j I(s_j^{-2}V_j^2 > c) \right\} \]

\[= s_n^{-1} \max_{1 < r \leq M} \left\{ \sum_{j=1}^{r} X_j I(s_j^{-2}V_j^2 > c) \right\} .\]
with probability $> 1 - 2\delta$.

Consequently, since $s_n \to \infty$,

$$P\left( \sup_{u \in [0,1]} |\xi_n(u) - \xi_n(u)| > \epsilon \right) < 2P(\eta^2 > c) + o(1)$$

and so

$$\limsup_{n \to \infty} P\left( \sup_{u \in [0,1]} |\xi_n(u) - \xi_n(u)| > \epsilon \right) < 2P(\eta^2 > c).$$

The proof of Theorem 2.2 is complete.

2.3 Convergence to Brownian Motion

Proposition 2.1 Consider the following three sets of conditions, where $\eta^2$ is an rv:

(A) (2.15) $s_n^{-2} \sum_{j=1}^{n} E(X_j^2 | F_{j-1}) \xrightarrow{p} \eta^2$, and

(2.16) for all $\epsilon > 0$, $s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \epsilon s_n)] \xrightarrow{p} 0$;

(B) (2.17) $s_n^{-2} \sum_{j=1}^{n} X_j^2 \xrightarrow{p} \eta^2$, and

(2.18) for all $\epsilon > 0$, $s_n^{-2} \sum_{j=1}^{n} X_j^2 I(|X_j| > \epsilon s_n) \xrightarrow{p} 0$;

(C) (2.19) $s_n^{-2} \sum_{j=1}^{n} X_j^2 \xrightarrow{p} \eta^2$, and

(2.20) $s_n^{-2} \sup_{j \leq n} X_j^2 \xrightarrow{p} 0$.

If $E(\eta^2) = 1$, they are equivalent. In fact, they are equivalent to the apparently stronger conditions $(A')$, $(B')$ and $(C')$ which are obtained from (A), (S) and (C) by replacing $"p"$ by "$\Rightarrow"$ wherever it occurs.
Proposition 2.2  The condition

(2.21) for all \( \varepsilon > 0 \), \( s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \varepsilon s_n)|F_{j-1}] \rightarrow 0 \)

is sufficient for the condition

(2.22) for all \( \varepsilon > 0 \), \( s_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \varepsilon v_j)|F_{j-1}] \rightarrow 0 \)

and if (2.1) holds it is also necessary. If (2.1) holds and \( \eta^2 > 0 \) a.s., (2.21) is equivalent to

(2.23) for all \( \varepsilon > 0 \), \( v_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \varepsilon v_j)|F_{j-1}] \rightarrow 0 \).

Note that convergence in probability in (2.23) is equivalent to convergence in the mean of order 1, since the term on the LHS is bounded by 1.

A Measurability Condition

In Theorem 2.3 it will be necessary to impose a measurability condition on the rv's \( v_n^2 \). Of course, \( v_n^2 \) is measurable in the \( \sigma \)-field \( F_{n-1} \); but just how "close" is it to being measurable in smaller \( \sigma \)-fields? The following condition is a requirement that all the \( v_n^2 \) be "uniformly close" to being measurable in earlier \( \sigma \)-fields than \( F_{n-1} \):

(2.24) For each \( \delta > 0 \), there exists a positive integer \( m \) and a sequence of \( F_m \)-measurable rv's \( \{v_n^2\}_{n=1}^{\infty} \) such that for all \( n \),

\[ P(|\sigma_n^{-2}v_n^2 - 1| > \delta) < \delta. \]

It is not difficult to see that (2.24) is equivalent to the more compactly expressed condition

(2.25) for all \( \delta > 0 \), \( \lim_{m \to \infty} \limsup_{n \to \infty} \inf_{\sigma^2 \in F_m} P(|\sigma_n^{-2}v_n^2 - 1| > \delta) = 0. \)

(2.25) is satisfied if
for all $\delta > 0$, \( \lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|E(X_j^2 \mid F_m)^{-1}X_j^2 - 1| > \delta) = 0. \)

In fact, our proof of Theorem 2.3 can be worked with a slightly weaker condition than (2.24). We need only suppose that $\sigma_n^2 \in F_m(n)$ where the sequence \( \{m(n)\}_{1}^{\infty} \) does not increase too fast, the rate of increase being sufficiently slow for quantities like

\[
\nu_n^{-1} \max_{j \leq m(n)} |S_j| \quad \text{and} \quad \nu_n^{-2} \nu_n^2
\]

to converge to zero in probability. However, we will not complicate our proofs by introducing this condition.

Now let us define a new random function. For $u \in [0,1]$ let

\[
\mathcal{X}(n,u) = \max\{j \leq n \mid \nu_j^2 \leq u \nu_n^2\}
\]

\[
\alpha_n(u) = \nu_n^{-1} [S_{\mathcal{X}}(u) + (u \nu_n^2 - \nu_n^2)(\nu_{\mathcal{X}+1}^2 - \nu_{\mathcal{X}}^2)^{-1}X_{\mathcal{X}+1}]
\]

so that $\alpha_n$ is obtained by linearly interpolating between the points of the sequence

\[
(0,0), (\nu_n^{-2} \nu_n^{-1} S_1), (\nu_n^{-2} \nu_n^{-1} S_2), \ldots, (1, \nu_n^{-1} S_n)
\]

Let $\mathbb{P}_n^*$ and $\mathbb{P}^*$ be the measures induced on $\mathbb{C}[0,1]$ by $\alpha_n$ and a standard Brownian motion, respectively.

**Theorem 2.3** If conditions (2.23) and (2.24) hold then

(2.26) $\mathbb{P}_n^* \overset{w}{\to} \mathbb{P}^*.$

**Proposition 2.3** If $\{c_n\}_{1}^{\infty}$ is a sequence of positive numbers and $\eta^2$ is an a.s. finite and non-zero rv such that

(2.27) $c_n^{-1} \nu_n^{-2} \overset{P}{\to} \eta^2$

then condition (2.24) holds. In particular, if (2.27) is true with $c_n = s_n^2$ then (2.24) holds.
Proposition 2.4 If \( \{X_n\}_{n=1}^{\infty} \) is a stationary square-integrable MG difference sequence with \( P(\text{for all } n, X_n = 0) = 0 \) then conditions (2.15) and (2.16) hold with \( \eta^2 > 0 \) and \( E(\eta^2) = 1 \).

Corollary to Theorem 2.3 If conditions (2.15) and (2.16) hold for some a.s. finite and non-zero rv \( \eta^2 \) then

\[
(2.28) \quad \frac{-1}{\eta^2} \sum_{n=1}^{\infty} X_n \overset{\mathcal{D}}{\rightarrow} N(0,1).
\]

If \( E(\eta^2) = 1 \) or if for some real number \( c > 0 \), \( |X_n| \leq c \) for all \( n \), then

\[
(2.29) \quad \left( \sum_{j=1}^{n} X_j \right)^{-\frac{1}{2}} \left( \sum_{j=1}^{n} X_j \right) \overset{\mathcal{D}}{\rightarrow} N(0,1).
\]

If \( \{X_n\}_{n=1}^{\infty} \) is stationary with \( P(\text{for all } n, X_n = 0) = 0 \), then (2.28) and (2.29) hold.

PROOFS OF RESULTS IN §2.3

Proof of Proposition 2.1 (After Scott [35]). That

\[
(s_n^{-2} \sum_{j=1}^{n} E(X_j^2|F_{j-1}) + \eta^2) \overset{\mathcal{D}}{\rightarrow} \left( s_n^{-2} \sum_{j=1}^{n} E(X_j^2|F_{j-1}) + \eta^2 \right)
\]

and

\[
(s_n^{-2} \sum_{j=1}^{n} X_j^2 \rightarrow \eta^2) \overset{\mathcal{D}}{\rightarrow} \left( s_n^{-2} \sum_{j=1}^{n} X_j^2 \rightarrow \eta^2 \right)
\]

follows immediately from Theorem 4.5.4, p. 90 of Chung [12], in view of the fact that \( E(\eta^2) = 1 \). The equivalence of (A) and (A'), of (B) and (B') and of (C) and (C') then follows from Theorem 1 of Pratt [32].

Since for any \( \delta, \varepsilon \) for which \( 0 < \delta < \varepsilon \) we have

\[
s_n^{-4} \sum_{j=1}^{n} X_j^4 \mathbb{I}(|X_j| \leq \varepsilon s_n) \leq \delta^2 + \varepsilon^2 s_n^{-2} \sum_{j=1}^{n} X_j^2 \mathbb{I}(|X_j| > \delta s_n),
\]

\[
s_n^{-4} \sum_{j=1}^{n} X_j^4 \mathbb{I}(|X_j| \leq \varepsilon s_n) \leq \delta^2 + \varepsilon^2 s_n^{-2} \sum_{j=1}^{n} X_j^2 \mathbb{I}(|X_j| > \delta s_n),
\]
then under either \((A')\) or \((B')\),
\[
\mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon s_n^2) \leq 0.
\]

A Chebyshev-type inequality then provides
\[
\mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon s_n^2) = \mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon s_n^2 | F_{j-1}) \leq \mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon s_n^2 | F_{j-1}) \leq 0
\]
from which we can deduce the equivalence of \((A)\) and \((B)\).

Since
\[
\mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon^2 + s_n^{-2} \sum_{j=1}^{n} X_j^2 I(|X_j| > \varepsilon s_n))
\]
then \((B)\) implies \((C)\). To prove the converse, observe that for any \(\varepsilon, \delta > 0\),
\[
\mathbb{P}(\sum_{j=1}^{n} X_j^2 \leq \varepsilon^2 + s_n^{-2} \sum_{j=1}^{n} X_j^2 I(|X_j| > \varepsilon s_n)) \leq \mathbb{P}(\sum_{j=1}^{n} X_j^2 > \varepsilon^2 + (\varepsilon^2 + 1)) + \mathbb{P}(\sum_{j=1}^{n} X_j^2 > \varepsilon^2 + (\varepsilon^2 + 1))
\]
where \(c\) is a positive constant. The 1st term converges to 0 as \(n \to \infty\) under \((2.20)\), the 2nd can be made arbitrarily small by choosing \(c\) sufficiently large, and the last term does not exceed
\[
\mathbb{P}(\sum_{j=1}^{n} X_j^2 > \varepsilon^2 + 1)
\]
which converges to 0 under \((2.19)\).

Hence \((C)\) implies \((B)\), and the proof is complete. **
Proof of Proposition 2.2

**Sufficiency** Suppose (2.21) holds. Let \( \delta > 0 \).

\[
\frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon V_j] \mid F_{j-1} = \\
= \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon V_j] \mid F_{j-1} \left( V_j \leq \delta s_n^{-1} \right) + \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon \delta s_n^{-1}] \mid F_{j-1} \\
\leq \delta \varepsilon + o(1) \text{ in probability.}
\]

This is true for all \( \delta > 0 \) and so implies (2.22).

**Necessity** Suppose (2.22) and (2.1) hold. To prove (2.21), it suffices to prove that for arbitrary large \( c \),

\[
I(n^2 < c) \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon s_n^{-1}] \mid F_{j-1} = 0.
\]

The LHS does not exceed

\[
\frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon s_n^{-1} ; V_j^2 < 2cs_n^2] \mid F_{j-1} + \\
I(n^2 < c) \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon s_n^{-1}] \mid F_{j-1} \left( V_j^2 > 2cs_n^2 \right)
\]

\[
\leq \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid |X_j| > \varepsilon (2c)^{-1/2} V_j] \mid F_{j-1} \left( V_j^2 > 2cs_n^2 \right) + I(n^2 < c) \frac{s_n^{-2}}{n} \sum_{j=1}^{n} \mathbb{E}[X_j^2 \mid F_{j-1}].
\]

\[
\leq o(1) + I(n^2 < c ; s_n^{-2} V_n^2 > 2c) \frac{s_n^{-2}}{n} V_n^2 \text{ in probability} \\
< o(1) + I(s_n^{-2} V_n^2 - n^2 > c) \frac{s_n^{-2}}{n} V_n^2 \\
= o(1) \text{ in probability,}
\]

and so (2.21) is proved.
The truth of the final statement of the proposition follows from an elementary convergence result:

\[ \text{If } Y_n \xrightarrow{P} Y < \infty \text{ a.s. and } Z_n \xrightarrow{P} 0 \text{ then } Y_n Z_n \xrightarrow{P} 0. \]

**

Proof of Theorem 2.3  The simplest proof of Theorem 2.3 is very similar to that of Theorem 2.2, in that we use condition (2.24) to approximate to all random variables which are not measurable in small enough σ-fields. However, there are several important distinctions, notably that the a.s. convergence in condition (2.3) of Theorem 2.2 can be relaxed here to the "approximation in probability" condition, (2.24). Consequently we will rework most of the proof of Theorem 2.2.

Once again the proof is divided into two parts, Part I showing the convergence of finite distributions and Part II establishing tightness.

To prove Part I we first show (Lemma 4) that \( \{a_n\}_{n=1}^{\infty} \) can be approximated by a sequence \( \{\beta_n\}_{n=1}^{\infty} \) where \( \beta_n \) is a sum of MG differences. The proof of Part I can then be completed in two ways - either using the approach of Theorem 2.2, or by applying the Skorokhod representation as in Scott [35]. (This approach requires the trivial restriction that \( F_n \) is the σ-field generated by \( X_1, X_2, \ldots, X_n \).) Unfortunately the Skorokhod representation cannot be handled here with the ease of [35]. There are measurability difficulties which we overcome using a rather complicated truncation due to Strassen [40]. In the interests of brevity, we will do little more than outline the Skorokhod representation proof.

Part II is obtained in the same way as before.

Proof of Part I: Convergence of finite dimensional distributions

Let \( 0 < \delta < \frac{1}{2} \) and choose a positive integer \( m \) and a sequence \( \{\sigma_n^2\}_{n=1}^{\infty} \) of \( F_m \)-measurable functions as in condition (2.24).
LEMA 1 Under conditions (2.23) and (2.24) we can choose \( N = N(\delta) > m \) so large that for all \( n > N \)

(2.30) \[ P(V_{\infty}^{-1} \max_{j \leq m} |S_j| > \delta) < \delta, \]

(2.31) \[ P(\sigma_n^{-2} \sigma_{m+1}^2 > 2\delta) < 2\delta, \]

(2.32) \[ P(V_{\infty}^{-1} \max_{j \leq n} |X_j| > \delta) < \delta \quad \text{and} \]

(2.33) \[ P(\sigma_n^{-2} \max_{j \leq n} E(X_j^2 | F_{j-1}) > 2\delta) < 2\delta. \]

Proof of Lemma 1 (2.30) and (2.31) are a consequence of the fact that (2.23) implies \( V_n \rightarrow \infty \) a.s.

To prove (2.32) we make use of (2.24). Let \( \varepsilon, 0 < \varepsilon < 1 \), replace \( \delta \) in (2.24), giving an integer \( p \) rather than \( m \) and a sequence \( \{\tau_n\}^\infty_1 \) rather than \( \{\sigma_n^2\}^\infty_1 \). If \( n > p \),

\[
P(V_{\infty}^{-1} \max_{j \leq n} |X_j| > \delta) = P(V_{\infty}^{-1} \max_{p < j \leq n} |X_j| > \delta) + o(1) 
\leq P(\tau_n^{-2} \max_{p < j \leq n} X_j^2 I(\tau_n^{-2} \tau_j < 1 + \varepsilon) > \delta^2/(1-\varepsilon)) + \varepsilon + o(1) 
\leq P(\tau_n^{-2} \max_{p < j \leq n} X_j^2 I(\tau_n^{-2} \tau_j < 1 + \varepsilon; X_j^2 > \frac{1}{2} \delta^2 \tau_n^2) > \frac{1}{2} \delta^2) + \varepsilon + o(1) 
\leq 2\delta^{-2} E(\tau_n^{-2} \sum_{p+1}^n E[X_j^2 I(|X_j| > \frac{1}{2} \delta \tau_j \left[F_{j-1}\right] \left[F_{j-1}\right] I(\tau_n^{-2} \tau_j < 1 + \varepsilon)) + \varepsilon + o(1) 
\leq 2\delta^{-2} \varepsilon (1 + \varepsilon) + \varepsilon + o(1) \]
since \( V_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > \frac{1}{2} \delta V_j) | F_{j-1}] \) converges to zero in the mean of order 1.

By choosing \( \varepsilon \) sufficiently small and then \( n \) sufficiently large, we can obtain (2.32).

To prove (2.33) we first prove

\[
(2.34) \quad P(V_n^{-2} \max_{j \leq n} E(X_j^2 | F_{j-1}) > \delta) < \delta \quad \text{for sufficiently large } n.
\]

This is a consequence of (2.23) alone:

\[
P(V_n^{-2} \max_{j \leq n} E(X_j^2 | F_{j-1}) > \delta) \leq
\]

\[
\leq P(V_n^{-2} \max_{j \leq n} E[X_j^2 I(|X_j| < \sqrt{2 \varepsilon} V_j) | F_{j-1}] + V_n^{-2} \max_{j \leq n} E[X_j^2 I(|X_j| > \sqrt{2 \varepsilon} V_j) | F_{j-1}) > \delta)
\]

\[
\leq P(V_n^{-2} \max_{j \leq n} E[X_j^2 I(|X_j| > \sqrt{2 \varepsilon} V_j) | F_{j-1}] > \frac{1}{2} \delta)
\]

\[
\leq P(V_n^{-2} \sum_{j=m+1}^{n} E[X_j^2 I(|X_j| > \sqrt{2 \varepsilon} V_j) | F_{j-1}] > \frac{1}{2} \delta)
\]

\[
= o(1).
\]

(2.33) is a consequence of (2.34) and the definition of \( \sigma_n^2 \). **

We now introduce a random function which approximates \( \alpha_n \). For \( u \in [0,1] \) and \( n \geq N \), define

\[
\beta_n(u) = \sigma_n^{-1} \sum_{j=m+1}^{n} X_j I(\sigma_n^{-2} V_j^2 \leq u(1+\delta); \sigma_n^{-2} E[X_j^2 | F_{j-1}] \leq 2\delta).
\]

**LEMMA 2** Under conditions (2.23) and (2.24) and for all \( u \in [0,1] \),

\[
(2.35) \quad P(\left| \sigma_n^{-1} \sum_{j=m+1}^{n} X_j I(V_n^{-2} V_j^2 \leq u; \sigma_n^{-2} V_j^2 \leq 1+\delta) \right| > \delta^{-2}) < 2\delta.
\]
Proof of Lemma 2  The term within modulus signs in (2.35) does not exceed
\[
\max_{m<r\leq n} \left| \sigma^{-1}_n \sum_{m+1}^r x_j I(\sigma^{-2}_n v_j^2 \leq 1+\delta) \right| = \max_{m<r\leq n} |M_r| \quad \text{say.}
\]

Applying Kolmogorov's inequality for martingales (Lemma 2.2) we obtain
\[
P\left( \max_{m<r\leq n} |M_r| > \frac{1}{\delta^2} \right) \leq \delta \mathbb{E}\left[ \sigma^{-2}_n \sum_{m+1}^n \mathbb{E}(x_j^2 | F_{j-1}) I(\sigma^{-2}_n v_j^2 \leq 1+\delta) \right]
\leq \delta (1+\delta)
< 2\delta .
\]

**

**LEMMA 3**  Under conditions (2.23) and (2.24), for all \( n \geq N \) and for all \( u \in [0,1] \),
\[
(2.36) \quad P\left( \left| \sigma^{-1}_n \sum_{m+1}^n x_j I(\sigma^{-2}_n v_j^2 \leq u; \sigma^{-2}_n v_j^2 \leq 1+\delta) \right| - \sigma^{-1}_n \sum_{m+1}^n x_j I(\sigma^{-2}_n v_j^2 \leq u(1+\delta); \sigma^{-2}_n v_j^2 \leq 2\delta) \right| > \frac{1}{\delta^3} \) < \frac{1}{7\delta^3}.
\]

Proof of Lemma 3  On the set
\[
E = \{ |\sigma^{-2}_n v_n^2 - 1| \leq \delta \}
\]
we have for all \( u \in [0,1] \) and for all \( j \),
\[
I(\sigma^{-2}_n v_j^2 \leq u(1-\delta)) \leq I(\sigma^{-2}_n v_j^2 \leq u; \sigma^{-2}_n v_j^2 \leq 1+\delta) \leq I(\sigma^{-2}_n v_j^2 \leq u(1+\delta))
\]
and on
\[
F = \{ \sigma^{-2}_n \max_{j \leq n} \mathbb{E}(x_j^2 | F_{j-1}) \leq 2\delta \}
\]
we have for any event \( A \) and any \( j \leq n \),
\[
I(A) = I(A; \sigma^{-2}_n \mathbb{E}(x_j^2 | F_{j-1}) \leq 2\delta) .
\]
Hence on $E \cap F$, the term in modulus signs in (2.36) does not exceed
\[
\sigma_n^{-1} \max_{m<r<s \leq n} \left| \sum_{r}^{s} X_j I(u(1-\delta) < \sigma_n^{-2} \sigma_j^2 \leq u(1+\delta); \sigma_n^{-2} E(X_j^2|F_{j-1}) \leq 2\delta) \right|
\leq 2\sigma_n^{-1} \max_{m<r \leq n} \left| \sum_{r}^{m} X_j I(u(1-\delta) < \sigma_n^{-2} \sigma_j^2 \leq u(1+\delta); \sigma_n^{-2} E(X_j^2|F_{j-1}) \leq 2\delta) \right|
\leq 2 \max_{m<r \leq n} |M'_r|, \quad \text{say.}
\]

Applying Kolmogorov's inequality to the MG $\{M'_r\}_{m+1}^n$ we obtain
\[
P(\max_{m<r \leq n} |M'_r| > \delta^2) \leq \frac{2}{\delta^3} E\left(\sigma_n^{-2} \sum_{m+1}^{n} E(X_j^2|F_{j-1}) I(u(1-\delta) < \sigma_j^{-2} \sigma_j^2 \leq u(1+\delta); \sigma_j^{-2} E(X_j^2|F_{j-1}) \leq 2\delta)\right)
\leq \frac{2}{\delta^3} E\{u(1+\delta) - u(1-\delta) + 2\delta\}
\leq \frac{1}{\delta^3}.
\]

Since $P(E \cap F) > 1 - 3\delta$, the result follows.

**

**LEMMA 4** Under conditions (2.23) and (2.24), for all $n \geq N$ and for all $u \in [0,1]$,
\[
P(|a_n(u) - \beta_n(u)| > 6\delta^3) < \frac{1}{126^3}.
\]

Proof of Lemma 4
\[
|a_n(u) - \beta_n(u)| =
\]
\[
= |\sum_{1}^{n} X_j I(V_{n,j}^2 - V_{j+1}^2) - \sum_{m+1}^{n} X_j I(\sigma_n^{-2} \sigma_j^2 \leq u(1+\delta); \sigma_n^{-2} E(X_j^2|F_{j-1}) \leq 2\delta) +
\]
\[
+ (uV_n^2 - V_{n+1}^2)(V_{n+1}^2 - V_n^2)(V_{n+1}^2 - V_n^2)^{-1} V_n^{-1} X_{n+1}|
\]
\[
\leq V_n^{-1} \max_{j \leq m} |S_j| + V_n^{-1} \max_{j \leq n} |X_j| +
\]
\[
+ |V_n^{-1} \sum_{m+1}^{n} X_j I(\sigma_n^{-2}v_j^2 \leq u) - \sigma_n^{-1} \sum_{m+1}^{n} X_j I(\sigma_n^{-2}v_j^2 \leq u(1+\delta))| +
\]
\[
+ \sigma_n^{-1} \sum_{m+1}^{n} X_j I(\sigma_n^{-2}v_j^2 \leq u; \sigma_n^{-2}v_j^2 \leq 1+\delta) - \sum_{m+1}^{n} X_j I(\sigma_n^{-2}v_j^2 \leq u(1+\delta)) | +
\]
\[
\sigma_n^{-2}E(X_j^2|F_{j-1}) \leq 2\delta)
\]

on the set
\[
E = \{|\sigma_n^{-2}v_n^{-1}| \leq \delta \}.
\]

With probability \( > 1-\delta-\delta-(\delta+2\delta)-7\delta > 1-12\delta^3 \), the RHS does not exceed
\[
\delta+\delta+\delta(1-\delta)^{-1} \cdot \frac{1}{\delta^2} + \frac{1}{2\delta^3} < 6\delta^3,
\]
using Lemmas 1, 2 and 3. We have included \( E \) in the exceptional set whose probability \(< 12\delta^3 \), and so the lemma is proved.

**

Proof of Part I using characteristic functions (This proof is similar to that of Part I of Theorem 2.2).

First we must introduce further notation. Suppose
\[
0 = u_0 < u_1 < ... < u_p \leq 1.
\]
Let \( t_1, ..., t_p \) be real numbers and define
\[
\alpha^2 = \sum_{k}^{p} (u_k-u_{k-1}) t_k^2,
\]
\( t = \max |t_k| \),

\[
\theta_j = \sum_{k=1}^{p} t_k I(u_{k-1}(1+\delta) < \sigma_n^{-2}v_j^2 < u_k(1+\delta); \sigma_n^{-2}E(X_j^2|F_{j-1}) \leq 2\delta)
\]

and for \( j \geq N \) and \( n \geq r \geq N, \)

\[
Y_j = \sigma_n^{-1} \theta_j X_j,
\]

\[
T_r = \sum_{m+1}^{r} Y_j,
\]

\[
U_r^2 = \sum_{m+1}^{r} E(Y_j^2|F_{j-1}).
\]

\( \theta_j \) is an \( F_{j-1} \)-measurable rv and \( |\theta_j| \) does not exceed \( t I(\sigma_n^{-2}v_j^2 \leq 1+\delta) \).

We note that for all \( k, \)

\[
\beta_n(u_k) - \beta_n(u_{k-1}) = \sigma_n^{-1} \sum_{j=m+1}^{n} X_j I(u_{k-1}(1+\delta) < \sigma_n^{-2}v_j^2 \leq u_k(1+\delta)); \sigma_n^{-2}E(X_j^2|F_{j-1}) \leq 2\delta)
\]

and

\[
\sum_{k=1}^{p} t_k (\beta_n(u_k) - \beta_n(u_{k-1})) = \sigma_n^{-1} \sum_{j=m+1}^{n} X_j \sum_{k=1}^{p} t_k I(u_{k-1}(1+\delta) < \sigma_n^{-2}v_j^2 \leq u_k(1+\delta); \sigma_n^{-2}E(X_j^2|F_{j-1}) \leq 2\delta)
\]

\[
= \sigma_n^{-1} \sum_{j=m+1}^{n} \theta_j X_j
\]

\[
= T_n.
\]

We must prove that

\[
|\mathbb{E}\exp \left[ \sum_{k=1}^{p} t_k (\alpha_n(u_k) - \alpha_n(u_{k-1})) \right] - \exp(-\frac{1}{2} \alpha^2)| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
We decompose the LHS into several pieces and show that each can be made small:

\[
(2.37) \quad |E \exp \left[ \sum_{k=1}^{P} t_k (\alpha_n(u_k) - \alpha_n(u_{k-1})) \right] - \exp(- \frac{1}{2} \alpha^2)| \leq 
\]

\[
\leq E |\exp \left[ \sum_{k=1}^{P} t_k (\alpha_n(u_k) - \alpha_n(u_{k-1})) \right] - \exp \left[ \sum_{k=1}^{P} t_k (\beta_n(u_k) - \beta_n(u_{k-1})) \right]| + 
\]

\[
+ E |\exp(it_n) - \exp(it_n + \frac{1}{2} U_n^2 - \frac{1}{2} \alpha^2)| + |E \exp(it_n + \frac{1}{2} U_n^2 - 1)| \exp(- \frac{1}{2} \alpha^2). 
\]

Since

\[
|\sum_{k=1}^{P} t_k (\alpha_n(u_k) - \alpha_n(u_{k-1})) - \sum_{k=1}^{P} t_k (\beta_n(u_k) - \beta_n(u_{k-1}))| < 6p \delta^3 
\]

with probability > 1-12p\delta^3 (Lemma 4) then by choosing \( \delta \) sufficiently small and then \( n \) sufficiently large, the 1st term on the RHS of (2.37) can be made arbitrarily small.

The second term on the RHS of (2.37) equals

\[
E |1 - \exp \left( \frac{1}{2} U_n^2 - \alpha^2 \right)| 
\]

which, in view of the following lemma, can be made small in a similar way.

**Lemma 5** For all \( n \), \( U_n^2 - \alpha^2 \leq 3\delta p^2 \) with probability 1, and for all \( n > N \),

\[
P(U_n^2 - \alpha^2 > -4\delta p^2) < 4\delta. 
\]

**Proof of Lemma 5**

\[
U_n^2 = \sum_{m=1}^{n} \theta_j^2 \mathbb{E}(X_j^2 | F_{j-1}) 
\]

\[
= \sum_{j=m+1}^{n} \mathbb{E}(X_j^2 | F_{j-1}) + \sum_{k=1}^{P} t_k \mathbb{I}(u_{k-1}(1+\delta) < \sigma_n v_j \leq u_k(1+\delta); 
\]

\[
\sigma_n \mathbb{E}(X_j^2 | F_{j-1}) \leq 2\delta
\]
\[ P \left( \frac{\sigma_n^{-2} \sum_{j=m+1}^{n} E(X_j^2 | F_{j-1}) I(\sigma_n^{-2} v_j^2 \leq u_k(1+\delta))}{\sum_{k=1}^{p} t_k^2} \right) \leq 2\delta \]

\[ \leq \sum_{k=1}^{p} t_k^2 (u_k(1+\delta) - u_{k-1}(1+\delta) + 2\delta) \]

\[ \leq (1+\delta) a^2 + 2\delta p t^2 \]

\[ \leq a^2 + 3\delta p t^2, \]

which proves the 1st result. To obtain the 2nd result, note that on the set

\[ G = \{ \sigma_n^{-2} \max_{j \leq n} E(X_j^2 | F_{j-1}) \leq 2\delta \} \cap \{ \sigma_n^{-2} v_j^2 \leq 2\delta \} \]

we have

\[ u_n^2 = p \sum_{k=1}^{p} t_k^2 \left( \sigma_n^{-2} \sum_{j=m+1}^{n} E(X_j^2 | F_{j-1}) I(\sigma_n^{-2} v_j^2 \leq u_k(1+\delta)) - \sigma_n^{-2} \sum_{j=m+1}^{n} E(X_j^2 | F_{j-1}) I(\sigma_n^{-2} v_j^2 \leq u_{k-1}(1+\delta)) \right) \]

\[ + \sum_{k=1}^{p} t_k^2 \left( \sigma_n^{-2} \sum_{j=1}^{n} E(X_j^2 | F_{j-1}) I(\sigma_n^{-2} v_j^2 \leq u_k(1+\delta)) - \sigma_n^{-2} \sum_{j=1}^{n} E(X_j^2 | F_{j-1}) I(\sigma_n^{-2} v_j^2 \leq u_{k-1}(1+\delta)) \right) - 2\delta p t^2 \]

\[ \geq p \sum_{k=1}^{p} t_k^2 [u_k(1+\delta) - 2\delta - u_{k-1}(1+\delta)] - 2\delta p t^2 \]

\[ \geq a^2 - 4\delta p t^2. \]

Since \( P(G) > 1-4\delta \), the 2nd result is proved.

**
Therefore it suffices to prove that
\[ A_n = |E\exp(it_{n+1}^2/2 - 1) | \]
can be made small. To this end, define
\[ Z_j = \exp(it_{j+1/2}^2 - \exp(it_{j-1/2}^2 + 1/2 u_{j-1}^2) \]
\[ = \exp(iY_j^2 - \exp(-\frac{1}{2} E(Y_j^2 | F_{j-1}))) \exp(it_{j-1/2}^2 + 1/2 u_{j-1}^2) \]
\[ = W_j \exp(it_{j-1/2}^2 + 1/2 u_{j-1}^2) , \text{ say.} \]

Then
\[ A_n = \left| E \left( \sum_{m=1}^{n} Z_j \right) \right| \quad (\text{here it is convenient to define } T_m = u_m^2 = 0) \]
\[ = \left| E \left( \sum_{m=1}^{n} E(Z_j | F_{j-1}) \right) \right| \]
\[ \leq E \left( \sum_{m=1}^{n} \left| E(Z_j | F_{j-1}) \right| \right) . \]

But
\[ E(Z_j | F_{j-1}) = E(W_j | F_{j-1}) \exp(it_{j-1/2}^2 + 1/2 u_{j-1}^2) \]
and hence
\[ A_n \leq E \left( \sum_{m=1}^{n} \exp(1/2 u_{j-1}^2) E(W_j | F_{j-1}) \right) . \]

Since
\[ u_{j-1}^2 < u_{n}^2 = a^2 + 36pt^2 . \]
it suffices to prove that
\[ E \left( \sum_{m=1}^{n} \left| E(W_j | F_{j-1}) \right| \right) \]
can be made small.
Define $Q(x)$, $Z(x)$ and $M(x)$ as in the proof of Theorem 2.1, and decompose $E(W_j|F_{j-1})$ as in (2.5). The argument following (2.5) can be used to show that it is sufficient to prove

$$B_n = E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2|Q^{-1}_j x_j)|F_{j-1} \right)$$

and

$$C_n = E\left( \sum_{m+1}^{\infty} |Z(n^{-2}) E[X_j^2|F_{j-1}]| \right)$$

can be made small.

If $\varepsilon > 0$ then

$$B_n \leq t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 M(\sigma_n^{-1}|tX_j|)|F_{j-1}] I(\sigma_n^{-2} v_j^2 \leq 1+\delta) \right)$$

$$= t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 M(\sigma_n^{-1}|tX_j|)|F_{j-1}] I(|X_j| \leq \varepsilon \sigma_n)|F_{j-1} \right) I(\sigma_n^{-2} v_j^2 \leq 1+\delta) +$$

$$+ t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 M(\sigma_n^{-1}|tX_j|)|F_{j-1}] I(|X_j| > \varepsilon \sigma_n)|F_{j-1} \right) I(\sigma_n^{-2} v_j^2 \leq 1+\delta)$$

$$\leq t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 \frac{1}{3} \sigma_n^{-1}|tX_j| I(|X_j| \leq \varepsilon \sigma_n)|F_{j-1} \right) I(\sigma_n^{-2} v_j^2 \leq 1+\delta) +$$

$$+ t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 I(|X_j| > \varepsilon \sigma_n)|F_{j-1}] I(\sigma_n^{-2} v_j^2 \leq 1+\delta) \right)$$

$$\leq \frac{1}{3} \varepsilon t^3 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 |F_{j-1}] I(\sigma_n^{-2} v_j^2 \leq 1+\delta) \right) +$$

$$+ 2t^2 E\left( \sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 I(|X_j| > \varepsilon \sigma_n; \sigma_n^{-2} v_j^2 \leq 1+\delta)|F_{j-1}] \right).$$

The 1st term $\leq \frac{1}{3} \varepsilon t^3 (1+\delta)$. The integrand of the 2nd is bounded by $(1+\delta)$ and does not exceed

$$\sigma_n^{-2} \sum_{m+1}^{\infty} E[X_j^2 I(|X_j| > \varepsilon(1+\delta) \sigma_n^{-2} v_j)|F_{j-1}]$$
which with probability $> 1-\delta$ does not exceed
\[
(1+\delta)^{-2} \sum_{m+1}^{n} \mathbb{E}[X_j^2 I(|X_j| > \epsilon(1+\delta) V_j^{-1/2}) | F_{j-1}]
\]
which converges to zero in the mean of order 1.

Hence the 2nd term $\leq \delta(1+\delta)+o(1)$, so that
\[
B_n \leq \frac{1}{3} \epsilon t^3(1+\delta) + \delta(1+\delta) + o(1)
\]
This is true for all $\epsilon > 0$. Therefore by choosing $\delta$ sufficiently small and then $n$ sufficiently large, $B_n$ can be made arbitrarily small.

Since $|Z(x)| \leq \frac{1}{2} x^2$, to prove that $C_n$ can be made small it suffices to prove that
\[
D_n = \mathbb{E}\{a_4 \sum_{j}^{} \mathbb{E} \{X_j^4 | F_{j-1}\}\}
\]
can be made small.

Now, if $\epsilon > 0$ then
\[
D_n \leq 2 t^4 \mathbb{E}\{\sigma_n^{-4} \sum_{m+1}^{n} (\mathbb{E}[X_j^2 I(|X_j| \leq \epsilon \sigma_n) | F_{j-1}]^2 I(\sigma_n^{-2} V_j^2 \leq 1+\delta)) + \\
+ 2 t^4 \mathbb{E}\{\sigma_n^{-4} \sum_{m+1}^{n} (\mathbb{E}[X_j^2 I(|X_j| > \epsilon \sigma_n) | F_{j-1}]^2 I(\sigma_n^{-2} V_j^2 \leq 1+\delta))
\}
\]
By applying Jensen's inequality we can bound the 1st term on the RHS by
\[
2 \epsilon^2 t^4 \mathbb{E}\{\sigma_n^{-4} \sum_{m+1}^{n} \mathbb{E}[X_j^4 I(|X_j| \leq \epsilon \sigma_n) | F_{j-1}] I(\sigma_n^{-2} V_j^2 \leq 1+\delta))
\]
\[
\leq 2 \epsilon^2 t^4 \mathbb{E}\{\sigma_n^{-2} \sum_{m+1}^{n} \mathbb{E}[X_j^2 | F_{j-1}] I(\sigma_n^{-2} V_j^2 \leq 1+\delta))
\]
\[
\leq 2 \epsilon^2 (1+\delta) t^4
\]
The integrand of the second term is not greater than

\[
\{\sigma_n^{-2} \sum_{m+1}^{n} E[X_j^2 I(|X_j| > \epsilon \sigma_n; \sigma_n^{-2}v_j^2 \leq 1+\delta)|F_{j-1}]}\}^2
\]

\[
\leq \{\sigma_n^{-2} \sum_{m+1}^{n} E[X_j^2 I(|X_j| > \epsilon(1+\delta)^{-2}v_j)|F_{j-1}]} I(\sigma_n^{-2}v_j^2 \leq 1+\delta)\}^2
\]

which is bounded by \((1+\delta)^2\) and with probability \(> 1-\delta\) does not exceed

\[
\{(1+\delta)^{-2} \sum_{m+1}^{n} E[X_j^2 I(|X_j| > \epsilon(1+\delta)^{-2}v_j)|F_{j-1}]} I(\sigma_n^{-2}v_j^2 \leq 1+\delta)\}^2
\]

which converges to zero in the mean of order 1, since it is \(o(1)\) in probability and is bounded by \((1+\delta)^2\).

Hence the 2nd term < 2\delta(1+\delta)^2t^4 + o(1), so that

\[
D_n \leq 2\epsilon^2(1+\delta)^2t^4 + 2\delta(1+\delta)^2t^4 + o(1).
\]

Once again, by choosing \(\delta\) small and then \(n\) large, \(D_n\) can be made arbitrarily small.

This completes the proof of Part I. **

Proof of Part I using the Skorokhod representation

Suppose \(\mathcal{F}_n\) is the \(\sigma\)-field generated by \(X_1, \ldots, X_n\). To avoid trivial complications, suppose that \(X_1^2 \geq 1\). Then each \(\sigma_n\) can be chosen \(\geq 1\). Define

\[
\tilde{X}_j = X_j I(2^{-j} < |X_j| \leq \delta v_j) + \frac{1}{2} X_j I(|X_j| \leq 2^{-j}) + \text{sgn}(X_j).
\]

\[.2^{-j-1}(1+\delta v_j |X_j|^{-1}) I(|X_j| > \delta v_j),\]

where in the case \(\delta v_j(\omega) \leq 2^{-j}\) we interpret \(I(2^{-j} < |X_j| \leq \delta v_j)(\omega) = 0\), and for \(n > m\) let
\[ X_j^* = \begin{cases} \frac{1}{(nV_j)^{-1}}[\tilde{X}_j - E(\tilde{X}_j | F_{j-1})], & 1 \leq j \leq m, \\ \sigma_n^{-1}[\tilde{X}_j - E(\tilde{X}_j | F_{j-1})], & m < j \leq n, \end{cases} \]

and

\[ S_r^* = \sum_{j=1}^{r} X_j. \]

(Of course, \( X_j^* \) depends on \( n \) and \( \delta \).

For \( u \in [0,1] \) define

\[ \gamma_n(u) = \sum_{j=1}^{n} X_j^* I(\sigma^{-2}v_j^2 \leq u(1+\delta)). \]

\( \{X_j^*\}_{j=1}^{n} \) is a sequence of MG differences and \( F_n \) is the \( \sigma \)-field generated by \( X_1^*, \ldots, X_n^* \), as will shortly be proved.

The peculiar truncations in the definition of \( \tilde{X}_j \) are designed to suppress the size of the tail of \( X_j \) without removing any of the "information" about \( X_j \) which is contained in its tail. That is to say, the truncations are designed to give us new rv's \( \tilde{X}_j \) which are like \( X_j I(|X_j| \leq \delta V_j) \) and are such that \( X_1, \ldots, X_n \) and \( \tilde{X}_1, \ldots, \tilde{X}_n \) generate the same \( \sigma \)-field for each \( n = 1, 2, \ldots \). Since the rv's \( X_j I(|X_j| \leq \delta V_j) \) contain (in general) less information than the variables \( X_j \), they will generate a smaller \( \sigma \)-field.

Strassen [40], has used a similar truncation to ours in the proof of his Theorem 4.4.

The first \( m \) \( X_j^* \)'s are normed differently to the rest in order to preserve their measurability in the respective \( \sigma \)-fields \( F_{j-1} \). The norming is designed to make them asymptotically negligible.
Lemma 6 For \( m \leq j \leq n \), \( \{X_1, \ldots, X_j\} \) and \( \{X_1^*, \ldots, X_j^*\} \) generate the same \( \sigma \)-field.

Proof of Lemma 6 First we will use an induction argument to prove that \( \{X_1, \ldots, X_j\} \) and \( \{
\tilde{X}_1, \ldots, \tilde{X}_j\} \) generate the same \( \sigma \)-field. Consider the graph of \( \tilde{X}_j \) against \( X_j \) in our Figure 1. This is a 1-1 transformation, and we can write

\[
X_j = f_j(\tilde{X}_j, V_j)
\]

where \( f_j \) is a Borel-measurable function of two variables. Suppose that \( X_1, \ldots, X_{j-1} \) and \( \tilde{X}_1, \ldots, \tilde{X}_{j-1} \) generate the same \( \sigma \)-field, \( F_{j-1} \). Since \( V_j \) is \( F_{j-1} \)-measurable, we can write

\[
V_j = g_j(\tilde{X}_1, \ldots, \tilde{X}_{j-1})
\]

where \( g_j \) is a Borel-measurable function of \( j-1 \) variables. Then

\[
X_j = f_j(\tilde{X}_j, g_j(\tilde{X}_1, \ldots, \tilde{X}_{j-1}))
\]

so that \( X_j \) is measurable in the \( \sigma \)-field generated by \( \tilde{X}_1, \ldots, \tilde{X}_j \). Therefore

\[
F\{X_1, \ldots, X_j\} \subseteq F\{\tilde{X}_1, \ldots, \tilde{X}_j\}
\]

and since clearly

\[
F\{\tilde{X}_1, \ldots, \tilde{X}_j\} \subseteq F\{X_1, \ldots, X_j\}
\]

then

\[
F\{X_1, \ldots, X_j\} = F\{\tilde{X}_1, \ldots, \tilde{X}_j\}.
\]

For \( j = m \) both \( \sigma \)-fields are equal to that generated by \( X_1, \ldots, X_m \), and the induction is complete.

A similar inductive argument shows that

\[
F\{\tilde{X}_1, \ldots, \tilde{X}_j\} = F\{X^*_1, \ldots, X^*_j\}.
\]
Figure 1. Transformation of $X_j$ to $\tilde{X}_j$. 
The first step in the Skorokhod representation proof is to prove a new approximation lemma:

**Lemma 7** If \( n \) is sufficiently large then
\[
P\left(\left| a_n(u) - \gamma_n(u) \right| > 20\delta^3 \right) < 30\delta^3.
\]

Next we introduce the Skorokhod representation (see Strassen [40], Theorem 4.3). Since \( \{X_j^*(n)\}_{j=1}^n \) is a sequence of MG differences relative to the sequence \( \{F_j\}_{j=1}^n \) of \( \sigma \)-fields, and since
\[
F_j = F\{X_1^*(n), \ldots, X_j^*(n)\},
\]
then there exists a probability space \( (\Omega', F', P') \) on which are defined a Brownian motion \( W \) and a triangular array \( \{T_j(n)\}_{j=1}^{\infty} \), \( \{T_j(n)\}_{j=1}^{\infty} \) of non-negative rv's s.t.
\[
(S_1^*(n), S_2^*(n), \ldots, S_n^*(n)) = (W(T_1(n)), W(T_j(n)), \ldots, W(T_j(n)))
\]
has the same distribution as
\[
(S_1^*(n), S_2^*(n), \ldots, S_n^*(n))
\]
for each \( n \geq 1 \).

Let \( \overline{X}_j(n) = S_j(n) - S_{j-1}(n) \). If \( \overline{F}_j(n) \) is the \( \sigma \)-field generated by \( \overline{X}_1(n), \ldots, \overline{X}_j(n) \) and \( \overline{G}_j(n) \) is the \( \sigma \)-field generated by \( \overline{X}_1(n), \ldots, \overline{X}_j(n) \) and \( W(t) \) for \( 0 \leq t \leq \sum_{i=1}^{j} T_i(n) \), \( 1 \leq j \leq n \), then
\[
E(T_j(n) | \overline{G}_{j-1}(n)) = E(\overline{X}_j^2(n) | \overline{G}_{j-1}(n)) = E(\overline{X}_j^2(n) | \overline{F}_{j-1}(n))
\]
and for \( r > 1 \),
\[
E(\tau_j(n)|\overline{G}_{j-1}(n)) \leq L_r E(\tau_{2r}(n)|\overline{G}_{j-1}(n)) = L_r E(\tau_{2r}(n)|\overline{F}_{j-1}(n))
\]

where \(L_r\) is a constant depending only on \(r\) and \(n\).

The proof of Part I is concluded using the approximation procedures introduced in our earlier proof and the techniques of Scott [35].

**

Proof of Part II: Tightness

We will continue to use the notation introduced in Part I.

Tightness prevails if

\[
(2.38) \quad \text{for all } \varepsilon > 0, \lim_{h \to 0} \limsup_{n \to \infty} \sum_{h < u \leq (k+1)h} P(\sup_{kh < u \leq (k+1)h} |a_n(u) - a_n(kh)| > 4\varepsilon) = 0
\]

(Parthaasarathy [31], p. 222). Now,

\[
|a_n(u) - a_n(kh)| \leq 2\sigma_n^{-1} \max_j X_j + 2\sigma_n^{-1} \sum_{j \leq m} I(V_n^{-2} V_j^2 < u) - \sum_{m+1}^n I(V_n^{-2} V_j^2 \leq kh).
\]

On the set

\[
E = \{ |\sigma_n^{-2} V_{j+1}^2 | \leq \delta \}
\]

we have for \(kh < u \leq (k+1)h\) and for all \(j\),

\[
I(\sigma_n^{-2} V_j^2 \leq kh(1-\delta)) \leq I(V_n^{-2} V_j^2 \leq u) \leq I(V_n^{-2} V_j^2 \leq kh) \leq I(\sigma_n^{-2} V_j^2 \leq (k+1)h(1+\delta))
\]

and hence on \(E\),

\[
\sup_{kh < u \leq (k+1)h} \sum_{m+1}^n \left| \sum_{j \leq m} I(V_n^{-2} V_j^2 \leq u) - \sum_{m+1}^n I(V_n^{-2} V_j^2 \leq kh) \right| \leq 2\sigma_n^{-1} \max_j \sum_{m+1}^n I(\sigma_n^{-2} V_j^2 \leq (k+1)h(1+\delta))
\]

\[
\leq 2\sigma_n^{-1} \max_{m<r<s} \sum_{r}^s I(\sigma_n^{-2} V_j^2 \leq (k+1)h(1+\delta))
\]
\[
\frac{1}{\sigma_n^2} \max_{m < r \leq n} \left| \sum_{j=1}^{r} X_j \right| \leq 2(1+\delta) \frac{\max_{m < r \leq m+1} \left| \sum_{j=1}^{r} \sigma_n^{-2} V_j^2 \right| \leq (k+1)(l+\delta)}{\sigma_n^2}.
\]

Therefore for \( n \geq N \),

\[
P(\sup_{kh<u<(k+1)h} |a_n(u) - a_n(kh)| > 4\epsilon) =
\]

\[= P_n,k \ (\text{say})
\]

\[
\leq P(\max_{m < r \leq n} \left| \sum_{j=1}^{r} X_j \right| I(kh(1+\delta) < \sigma_n^{-2} V_j^2 \leq (k+1)(l+\delta); \sigma_n^{-2} E(X_j^2 | F_{j-1}) \leq 2\delta) > 4\epsilon)
\]

\[> 2\sqrt{1-\delta}(\epsilon-\delta)+5\delta
\]

(\text{using Lemma 1})

\[
\leq P(\max_{m < r \leq n} \left| \sum_{j=1}^{r} X_j \right| I(kh(1+\delta) < \sigma_n^{-2} V_j^2 \leq (k+1)(l+\delta); \sigma_n^{-2} E(X_j^2 | F_{j-1}) \leq 2\delta) > \frac{1}{3}) + \frac{1}{\delta^3}
\]

\[
+ P(\max_{m < r \leq n} \left| \sum_{j=1}^{r} X_j \right| I(kh(1+\delta) < \sigma_n^{-2} V_j^2 \leq (k+1)(l+\delta); \sigma_n^{-2} E(X_j^2 | F_{j-1}) \leq 2\delta) > \frac{1}{\delta^3}) + 5\delta
\]

Kolmogorov's inequality bounds the 1st term on the RHS by

\[
\frac{2}{\delta^3} E[kh(1+\delta)-kh(1-\delta)+2\delta] \leq \frac{1}{\delta^3}.
\]

An application of Brown's inequality (see Lemma 2.2) to the MG

\[
M_r = \sigma_n^{-1} \sum_{j=1}^{r} X_j \ I(kh(1+\delta) < \sigma_n^{-2} V_j^2 \leq (k+1)(l+\delta); \sigma_n^{-2} E(X_j^2 | F_{j-1}) \leq 2\delta),
\]

\( r = m+1, \ldots, n \),

bounds the 2nd term by

\[
\Delta^{-1} \int_{\{M_n \geq \Delta\}} |M_n| dP,
\]

where \( \Delta = \sqrt{1-\delta}(\epsilon-2\delta^3) \). Hence
\[ p_{n,k} \leq \Delta^{-1} \int_{\{|M_n| > \Delta\}} \frac{1}{9 \delta^3} |M_n| dP \]

\[ (2.39) \leq \Delta^{-1} \int_{\{|M_n| > c\}} \frac{1}{9 \delta^3} |M_n| dP + \Delta^{-1} \int_{\Delta H} \frac{1}{9 \delta^3} |M_n| dP + \frac{1}{9 \delta^3} |M_n| dP \]

where \( c \) is a large positive constant, \( H \) is the set

\[ H = \{ |\alpha_n(kh) - \beta_n(kh)| \leq 6 \delta^3; |\alpha_n((k+1)h) - \beta_n((k+1)h)| \leq 6 \delta^3 \} , \]

and \( \overline{H} \) is the complement of \( H \). In view of our Lemma 4, \( H \) has probability \( \geq 1 - 24 \delta^3 \).

A Chebyshev-type inequality can be used to bound the 1st term in (2.39) by \((\Delta c)^{-1}(1+\delta)\). The 2nd does not exceed \( \Delta^{-1} \sqrt{(1+\delta) P(\overline{H})} \) (Cauchy-Schwartz inequality). Since

\[ |M_n| = |\beta_n(kh) - \beta_n(kh)| \leq |\alpha_n((k+1)h) - \alpha_n(kh)| + 12 \delta^3 \text{ on } H \]

\[ = |M'_n| + 12 \delta^3 \text{ say,} \]

then the 3rd term in (2.39) is bounded by

\[ \Delta^{-1} \left( \frac{1}{12 \delta^3} + \int_{\{|M'_n| > \Delta - 12 \delta^3\}} \frac{1}{9 \delta^3} |M'_n| dP \right) . \]

Consequently

\[ p_{n,k} \leq \Delta^{-1} \left( c^{-1}(1+\delta) + \sqrt{(1+\delta) 24 \delta^3} + 12 \delta^3 + \int_{\{|M'_n| > \Delta - 12 \delta^3\}} \frac{1}{9 \delta^3} |M'_n| dP \right) + 9 \delta^3 . \]
Since the finite dimensional distributions of \( \alpha_n \) converge to those of \( W \), and

\[
M'_n = \alpha_n((k+1)h) - \alpha_n(kh),
\]
then

\[
\limsup_{n \to \infty} p_{n,k} \leq \Delta^{-1} \left( c^{-1}(1+\delta) + \sqrt{1+\delta} + 2 \delta^3 + \frac{1}{3} \right)
\]

\[
+ \int_{-\Delta-2\delta}^{c+12\delta} \int_0^x dP(|N(0,h)| \leq x) + 9 \delta^3.
\]

This is true for all \( \delta, c > 0 \), and hence is true in the limit as \( \delta \to 0 \) and \( c \to \infty \):

\[
\limsup_{n \to \infty} p_{n,k} \leq c^{-1} \int_0^\infty x dP(|N(0,h)| \leq x)
\]

\[
= c^{-1} \sqrt{2h/\pi} \exp\left(-\frac{1}{2} c^2 h^{-1}\right).
\]

Hence, returning to (2.38),

\[
\limsup_{n \to \infty} \mathcal{E} \mathcal{P}(\sup_{kh<1} |\alpha_n(u) - \alpha_n(kh)| > 4 \epsilon) \leq
\]

\[
\leq h^{-1} c^{-1} \sqrt{2h/\pi} \exp\left(-\frac{1}{2} c^2 h^{-1}\right)
\]

\[
\to 0 \quad \text{as} \quad h \to 0.
\]

This completes the proof of Theorem 2.3.

**Proof of Proposition 2.3** It is sufficient to show that the approximation in (2.24) can be made for all sufficiently large \( n \), say for all \( n \geq N \).

For then we can replace \( m \) by \( m' = \max(m,N) \) and set \( \sigma_n^2 = v_n^2 \) for \( n < m' \), and the approximation will hold for all \( n \).

If \( \delta > 0 \) choose \( \lambda < 1 \) so small that

\[
P(\eta^2 \leq \lambda) < \delta^2 (1+\delta)^{-1}.
\]
Let $\varepsilon = \lambda \delta (1+\delta)^{-1}$ and choose $\sigma^2$ as in Lemma 2.1. Since

$$\sigma^{-2} |\eta^2 - \sigma^2| \leq (\lambda - \varepsilon)^{-1} \varepsilon$$

on the set

$$\{|\eta^2 - \sigma^2| \leq \varepsilon ; \eta^2 > \lambda\}$$

then

$$P(\sigma^{-2} |\eta^2 - \sigma^2| > (\lambda - \varepsilon)^{-1} \varepsilon) \leq P(|\eta^2 - \sigma^2| > \varepsilon) + P(\eta^2 \leq \lambda)$$

$$< \lambda \delta (1+\delta)^{-1} + \delta^2 (1+\delta)^{-1}$$

$$< \delta \,.$$ 

Since $(\lambda - \varepsilon)^{-1} \varepsilon = \delta$ then

$$P(\sigma^{-2} |\eta^2 - \sigma^2| > \delta) < \delta \,.$$ 

Let $\sigma_n^2 = c_n \sigma^2$ for large $n$. Since

$$\sigma_n^{-2} \sum_{j=1}^{n} X_j^2 - \sigma^{-2} \eta^2 = 1+\sigma^{-2} (\eta^2 - \sigma^2)$$

then

$$\limsup_{n \to \infty} P(\sigma_n^{-2} \sum_{j=1}^{n} X_j^2 - \sigma^{-2} \eta^2 > \delta) < \delta \,,$$

and so the approximation in (2.24) can be made for sufficiently large $n$.

**Proof of Proposition 2.4** Condition (2.18) holds trivially for a stationary square-integrable sequence. Thus, in view of the results of Proposition 2.1, it is sufficient to prove that for some a.s. non-zero rv $\eta^2$ with $E(\eta^2) = 1$,

$$\sigma_n^{-2} \sum_{j=1}^{n} X_j^2 - \sigma^{-2} \eta^2 \rightarrow \eta^2 \,.$$ 

In fact, if $T$ is the $\sigma$-field of events which are invariant with respect to $\{X_n\}_1^\infty$, then
\[ s_n^{-2} \sum_{i=1}^{n} \mathbb{E} \mathbb{X}^2_{\mathbb{J}} \xrightarrow{a.s.} \mathbb{E}(X^2_1)^{-1}\mathbb{E}(X^2_1 | T) = \eta^2 \]

(Breiman [8], p. 118). \( \mathbb{E}(\eta^2) = 1 \), and so it remains only to show that \( \mathbb{E}(X^2_1 | T) \) is a.s. non-zero.

If 
\( \mathbb{E} = \{ \mathbb{E}(X^2_1 | T) = 0 \} \)

then
\[
\int_{\mathbb{E}} X^2_n \, dP = \int_{\mathbb{E}} X^2_1 \, dP \quad \text{since } \mathbb{E} \text{ is invariant}
\]
\[
= \int_{\mathbb{E}} \mathbb{E}(X^2_1 | T) \, dP
\]
\[
= 0
\]

and so \( X_n = 0 \) a.s. on \( \mathbb{E} \). This is true for each \( n \), and \( P(\text{for all } n, \; X_n = 0) = 0 \). Hence
\[ P(\mathbb{E}) = 0. \]

**

Proof of Corollary to Theorem 2.3

The results follow directly from Theorem 2.3, Propositions 2.3 and 2.4, and the Remark following Corollary VII-2-6, p. 152 of Neveu [30].
CHAPTER III

SOME WEAK CONVERGENCE RESULTS FOR REVERSE MARTINGALES

3.1 Introduction and Summary

Let \( \{(S_n^-, F_n)\} \) be a square-integrable reverse MG satisfying a.s. \( S_n \to 0 \), and define

\[
X_n = S_n - S_{n+1} ,
\]

\[
s_n^2 = \mathbb{E}S_n^2 , \quad \text{and}
\]

\[
V_n^2 = \sum_{j=1}^{\infty} \mathbb{E}(X_j^2 | F_{j+1}) .
\]

Scott [36] has verified that the reverse MG analogues of Brown's Theorems 2 and 3 [9] are true. Since the work in our Chapter II is little more than an extension of Brown's results, it is perhaps not surprising to learn that it also has an analogue in the theory of reverse MG's. In sections 3.2 and 3.3 we present results which parallel those of sections 2.2 and 2.3, and in section 3.4 we give a short note on their proofs.

3.2 Convergence to Mixtures of Normal Laws

For \( u \in [0,1] \) define

\[
k = k(n, u) = \min\{ j \geq n | s_j^2 \leq u s_n^2 \}
\]

and

\[
\xi_n(u) = s_n^{-1}[s_k + (u s_n^2 - s_k^2)(s_{k-1}^2 - s_k^2)^{-1} x_{k-1}]
\]

so that \( \xi_n \) is obtained by linearly interpolating between the points of the sequence

\[
\ldots , (s_n^{-2}, s_{n+2}^{-2}, s_{n+4}^{-2}, (s_{n+2}^{-2}, s_{n+4}^{-2}, (s_{n+4}^{-2}, s_{n+6}^{-2}, (1, s_n^{-1} s_n) .
\]

Let \( P_n \) be the probability measure induced on \( C[0,1] \) by \( \xi_n \). Let \( \eta^2 \) be a random variable on \( (\Omega, F, P) \), and on some probability space define a standard Brownian motion \( W(u) \) \( (u > 0) \) and an independent copy \( \eta^2_1 \) of \( \eta^2 \). For \( u \in [0,1] \) let
\( Z(u) = W(n^2 u) \)

and let \( P(n^2) \) be the measure induced on \( C[0,1] \) by \( Z \).

**Theorem 3.1** If

\[ s_n^{-2} \sum_{i}^{n} \frac{P}{n} + n^2 < - \infty \text{ a.s.} \quad \text{and} \]

\[ s_n^{-2} \sum_{i}^{n} E[X_j^2 I(|X_j| > \epsilon s_n)] |F_{j+1}| \rightarrow 0 \]

then

\[ s_n^{-1} s_n^{-1} \rightarrow F \]

where \( F \) is the distribution with characteristic function

\[ f(t) = \text{E} \exp(-\frac{1}{2} \eta^2 t^2) \]

**Theorem 3.2** If

\[ s_n^{-2} \sum_{i}^{n} \frac{a.s.}{n} + n^2 < - \infty \text{ a.s.} \]

and if (3.2) holds, then

\[ P_n \rightarrow P(n^2) \]

### 3.3 Convergence to Brownian Motion

**Proposition 3.1** Consider the following three sets of conditions, where \( n^2 \) is an rv:

(A) \( s_n^{-2} \sum_{i}^{n} \frac{P}{n} E[X_j^2 |F_{j+1}| \rightarrow n^2 \), \quad \text{and} \]

\[ s_n^{-2} \sum_{i}^{n} E[X_j^2 I(|X_j| > \epsilon s_n)] |F_{j+1}| \rightarrow 0 \]

for all \( \epsilon > 0 \),
(B) \( (3.6) \quad s_n^{-2} \sum_{j=1}^{\infty} x_j^2 \xrightarrow{p} \eta^2, \quad \text{and} \)

\( (3.7) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^{\infty} x_j^2 \mathbb{I}(|x_j| > \varepsilon s_n) \xrightarrow{p} 0; \)

(C) \( (3.8) \quad s_n^{-2} \sum_{j=1}^{\infty} x_j^2 \xrightarrow{p} \eta^2, \quad \text{and} \)

\( (3.9) \quad s_n^{-2} \sup_{j \geq n} x_j^2 \xrightarrow{p} 0. \)

If \( E(\eta^2) = 1, \) they are equivalent. In fact, they are equivalent to the apparently stronger conditions (A'), (B') and (C') which are obtained from (A), (B) and (C) by replacing "\( \xrightarrow{p} \)" by "\( \xrightarrow{L_1} \)" wherever it occurs.

**Proposition 3.2** The condition

\( (3.10) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^{\infty} E[x_j^2 \mathbb{I}(|x_j| > \varepsilon s_n)|F_{j+1}] \xrightarrow{p} 0. \)

is sufficient for the condition

\( (3.11) \quad \text{for all } \varepsilon > 0, \quad s_n^{-2} \sum_{j=1}^{\infty} E[x_j^2 \mathbb{I}(|x_j| > \varepsilon v_j)|F_{j+1}] \xrightarrow{p} 0. \)

and if (3.1) holds it is also necessary. If (3.1) holds and \( \eta^2 > 0 \) a.s.,

(3.10) is equivalent to

\( (3.12) \quad \text{for all } \varepsilon > 0, \quad v_n^{-2} \sum_{j=1}^{\infty} E[x_j^2 \mathbb{I}(|x_j| > \varepsilon v_j)|F_{j+1}] \xrightarrow{p} 0. \)

Note that convergence in probability in (3.12) is equivalent to convergence in the mean of order 1, since the term on the LHS is bounded by 1.

**A Measurability Condition**

Arguing by analogy with section 2.3, we impose a measurability condition on the rv's \( v_n^2. \) Let \( F_\infty = \bigcap_{1}^{\infty} F_n. \) The following condition is a
requirement that all the \( v_{n}^{2} \) are "uniformly close" to being measurable in \( F_{\infty} \):

\[(3.13) \text{ For each } \delta > 0, \text{ there exists a sequence of } F_{\infty}-\text{measurable rv's } \{\sigma_{n}^{2}\}
\text{ such that for all sufficiently large } n,
\]
\[
P(|\sigma_{n}^{2}v_{n}^{2}-1| > \delta) < \delta.
\]

(3.13) is equivalent to

\[(3.14) \text{ for all } \delta > 0, \limsup_{n \to \infty} \inf \sigma_{n}^{2} \epsilon F_{\infty} \]

which is satisfied if

\[
\text{for all } \delta > 0, \limsup_{n \to \infty} P(|E(V_{n}^{2}|F_{\infty})^{-1}v_{n}^{2}-1| > \delta) = 0
\]

that is if

\[
v_{n}^{-2}E(V_{n}^{2}|F_{\infty}) \rightarrow 1.
\]

Now let us define a new random function. For \( u \in [0,1] \), let

\[
\ell = \ell(n,u) = \min\{j \geq n | v_{j}^{2} < u v_{n}^{2}\}
\]

and

\[
\alpha_{n}(u) = v_{n}^{-1}[X_{\ell} + (u v_{n}^{2} - v_{\ell}^{2})(v_{\ell+1}^{2} - v_{\ell}^{2})^{-1}X_{\ell+1}]
\]

so that \( \alpha_{n} \) is obtained by linearly interpolating between the points of the sequence

\[
..., (v_{n}^{-2}v_{n+2}, v_{n+2}), (v_{n}^{-2}v_{n+1}^{2}, v_{n+1}^{2}), (1, v_{n}^{-1}v_{n}^{2}).
\]

Let \( P^{*} \) and \( P_{n}^{*} \) be the measures induced on \( C[0,1] \) by \( \alpha_{n} \) and a standard Brownian motion, respectively.
Theorem 3.3 If conditions (3.12) and (3.13) hold then
\[ p_n^* \xrightarrow{w} p^* . \]

Proposition 3.3 If \( \{c_n\}_{n=1}^{\infty} \) is a sequence of positive numbers and \( \eta^2 \) is an a.s. finite and non-zero rv such that
\[ c_n^{-1}v_n^2 \xrightarrow{P} \eta^2 \]
then condition (3.14) holds. In particular, if (3.15) is true with \( c_n = s_n^2 \) then (3.14) holds.

Corollary to Theorem 3.3 If conditions (3.4) and (3.5) hold for some a.s. finite and non-zero rv \( \eta^2 \) then
\[ V_n^{-1}s_n \xrightarrow{D} \mathcal{N}(0,1) . \]
If \( E(\eta^2) = 1 \) then
\[ \left( \sum_{j} X_j^2 \right)^{-1/2} \left( \sum_{j} X_j \right) \xrightarrow{D} \mathcal{N}(0,1) . \]

3.4 Notes on the proofs

The proofs of most of the results in Chapter II can be reworked for reverse MG's, and in fact they are often a lot simpler than they are for forward MG's.

If \( \eta^2 \) is an rv such that
\[ s_n^{-2}v_n^2 \xrightarrow{P} \eta^2 \]
then \( \eta^2 \) is \( F_\infty \)-measurable, and so to prove Theorems 3.1 and 3.2 it is not necessary to approximate \( \eta^2 \) by a simple function \( \sigma^2 \). If \( \eta^2 \) is not square-integrable we must resort to a truncation argument to obtain both results, but the techniques involved are quite simple.
The proofs of Theorem 3.3 and Propositions 3.1 and 3.2 are very similar to those of Theorem 2.3 and Propositions 2.1 and 2.2, respectively. Proposition 3.3 is trivial, since \( \eta^2 \) must be \( F_\infty \)-measurable and so if \( \sigma_n^2 = c_n \eta^2 \in F_\infty \) we have
\[
\sigma_n^{-2} \mathbb{V}_n^2 \xrightarrow{p} 1.
\]
In fact, if \( s_n^{-2} \mathbb{V}_n^2 \xrightarrow{p} \eta^2 \) and (3.5) holds, (3.16) is quite easily obtained by showing that
\[
(s_n \eta)^{-1} s_n \xrightarrow{d} N(0,1).
\]

We will not persevere any further with the proofs except to give two lemmas.

Lemma 3.1 [Kolmogorov - Brown]. If \( \{(S_n^j, F_n^j)\}_{j=1}^{\infty} \) is a reverse MG and \( \epsilon > 0 \), then
\[
P(\max\{S_j^j\} > \epsilon) \leq \epsilon^{-2} \mathbb{E}(S_n^2)
\]
(Kolmogorov's inequality), and
\[
P(\max\{S_j^j\} > 2\epsilon) \leq \epsilon^{-1} \int \{S_n^j \mid dP \}
\]
(Brown's inequality).

Lemma 3.2 If \( S_n^j \rightarrow 0 \) then
\[
\mathbb{E}(S_n^2) = \mathbb{E}\{E(X_j^2 \mid F_{j+1})\}.
\]
Proof of Lemma 3.2 It is easy to show that for each \( m > n \),
\[
S_n^m = \sum_{j=n}^{m} X_j + S_{m+1}
\]
\[ E(S_n^2) = \sum_{j=1}^{m} E(X_j^2) + E(S_{m+1}^2) \]
\[ = E[\sum_{j=1}^{m} E(X_j^2|F_{j+1})] + E(S_{m+1}^2). \]

The result will follow from the monotone convergence theorem if we show

\[ E(S_n^2) \to 0. \]

\( a.s. \) Since \( S_n \to 0 \) this is equivalent to the uniform integrability of \( \{S_n^2\}_{n=1}^{\infty} \). The reverse MG property implies

\[ S_n = E(S_n^2|F_n) \]

and hence from Jensen's inequality

\[ S_n^2 \leq E(S_n^2|F_n). \]

Thus

\[ \int \{|S_n| > c\} S_n^2 \, dP \leq \int \{|S_n| > c\} E(S_n^2|F_n) \, dP = \int \{|S_n| > c\} S_n^2 \, dP \]

and so \( \{S_n^2\}_{n=1}^{\infty} \) is uniformly integrable if

\[ P(|S_n| > c) \to 0 \text{ uniformly in } n \text{ as } c \to \infty. \]

But

\[ P(|S_n| > c) \leq c^{-2} E(S_n^2) \leq c^{-2} E(S_1^2) \to 0 \text{ as } c \to \infty. \]

The lemma is proved.

\[ ** \]

A proof of Kolmogorov's inequality can be found in Doob [14] (p. 314) and a proof of Brown's inequality appears in Brown [9].
CHAPTER IV

SOME LAWS OF THE ITERATED LOGARITHM FOR MARTINGALES

4.1 Introduction and Summary

Let \( \{(S_n, F_n)\}_{n=1}^{\infty} \) be a (forward) MG and adopt the notation of Chapter II.

Since we have obtained a central limit theorem for martingales with a random norming it seems natural to investigate the law of the iterated logarithm (LIL) with the same, or similar, norming. Most LIL's for MG's are based on the independence theory analogues and describe the a.s. asymptotic behaviour of quantities like \( \frac{S_n}{\phi(n)} \), where \( \phi(.) \) is an increasing function. However, there are several prominent exceptions. Theorem 4.4 of Strassen [40] is an iterated logarithm-type result for MG differences whose tails obey a strong negligibility condition. Proposition VII-2-7 of Neveu [30] is a law of the iterated logarithm for sequences of uniformly bounded MG differences. Theorem 3 of Stout [38] generalises both these results and appears to be quite close to our Theorem 4.1. All of these results involve a norming by the random variables \( V_n^2 \) rather than by the constants \( s_n^2 \).

In section 4.2 we generalise the functional law of the iterated logarithm of Heyde and Scott [20]. Using their techniques and those of Strassen [39] we produce two very similar results in Theorems 4.1 and 4.2. The Corollary to Theorem 4.2 contains the results of Heyde and Scott and also considers the case of a stationary sequence.

Our proofs of Theorems 4.1 and 4.2 are via the Skorokhod representation. Since there is as yet no representation for reverse MG's, it is not possible to generalise these results as we did those of Chapter II. However, it is tempting to postulate that the reverse MG analogues of Theorems 4.1 and 4.2 are true.
4.2 Some Laws of the Iterated Logarithm

In the notation of Chapter II, let $F_n$ be the $\sigma$-field generated by $X_1, \ldots, X_n$ ($n \geq 1$) and define

$$U_n^2 = \sum_{j=1}^{n} X_j^2.$$  

To avoid trivial complications in the following work let us suppose that $X_1^2 > e$. Since the limit laws we establish are invariant under any transformation of a finite number of the $X_j$, this assumption can be made without loss of generality. Let $\phi$ be the real function on $(e, \infty)$ defined by

$$\phi(u) = \frac{1}{(2u \log \log u)^2}$$

and for $u \in [0,1]$ define

$$\ell = \ell(n,u) = \max\{j \leq n | V_j^2 \leq uV_n^2\},$$
$$m = m(n,u) = \max\{j \leq n | U_j^2 \leq uU_n^2\}$$

and

$$v_n(u) = [\phi(V_n^2)]^{-1}[S_{\ell} + (uV_n^2 - V_n^2)(V_{\ell+1}^2 - V_n^2) X_{\ell+1}],$$
$$u_n(u) = [\phi(U_n^2)]^{-1}[S_m + (uU_n^2 - U_n^2)(U_{m+1}^2 - U_m^2) X_{m+1}].$$

Let $K$ be the set of absolutely continuous functions $x \in C[0,1]$ such that

$$x(0) = 0 \quad \text{and} \quad \int_0^1 x^2 \, dt \leq 1.$$  

Theorem 4.1 Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of non-negative rv's such that $Z_n \in F_{n-1}$. If

$$\limsup_{n \to \infty} V_n^{-1} \sum_{j=1}^{n} I(|X_j| > Z_j) - E[X_j I(|X_j| > Z_j)|F_{j-1}] < \infty \text{ a.s.},$$

(4.1)
\[(4.2) \quad V_n^{-2} \sum_{j=1}^{n} E[X_j^2 I(|X_j| > Z_j)|F_{j-1}] \rightarrow 0 \text{ a.s.}\]

\[(4.3) \quad \sum_{j=1}^{n} E[|X_j|^4 I(|X_j| \leq Z_j)|F_{j-1}] < \infty \text{ a.s.}, \text{ and} \]

\[(4.4) \quad V_{n+1}^{-1} V_n \rightarrow 1 \text{ and } V_n \rightarrow \infty, \]

then \(\{u_n\}_n^\infty\) is relatively compact in \(C[0,1]\) and the set of its a.s. limit points coincides with \(K\).

**Theorem 4.2** Let \(\{Z_n\}_1^\infty\) be as in Theorem 4.1. If

\[(4.5) \quad \limsup_{n \rightarrow \infty} V_n^{-1} \sum_{j=1}^{n} E[X_j I(|X_j| > Z_j) - E[X_j I(|X_j| > Z_j)|F_{j-1}]| < \infty \text{ a.s.}, \]

\[(4.6) \quad V_n^{-1} \sum_{j=1}^{n} X_j^2 I(|X_j| > Z_j) \rightarrow 0 \text{ a.s.}, \]

\[(4.7) \quad \sum_{j=1}^{n} E[X_j^4 I(|X_j| \leq Z_j)|F_{j-1}] < \infty \text{ a.s.}, \text{ and} \]

\[(4.8) \quad V_{n+1}^{-1} V_n \rightarrow 1 \text{ and } V_n \rightarrow \infty, \]

then \(\{u_n\}_n^\infty\) is relatively compact in \(C[0,1]\) and the set of its a.s. limit points coincides with \(K\).

**Corollary to Theorem 4.2** If for some a.s. finite and non-zero rv \(n^2\) we have

\[(4.9) \quad s_n^{-2} u_n^{-2} \rightarrow n^2 \text{ a.s.,} \quad s_n \rightarrow \infty, \]

\[(4.10) \quad \text{for all } \epsilon > 0, \quad E[s_j^{-1} E[|X_j| I(|X_j| > \epsilon s_j)] < \infty \quad \text{and} \]

\[(4.11) \quad \text{for some } \delta > 0, \quad E[s_j^{-4} E[X_j^4 I(|X_j| \leq \delta s_j)] < \infty, \]
then the conclusions of Theorem 4.2 hold. In particular, they hold if \( \{X_n\}_1^\infty \) is stationary.

Before proceeding with the proofs of Theorems 4.1 and 4.2, let us state as a lemma a result contained in Proposition IV 6.2 of Neveu [19].

**Lemma 4.1** [Neveu]. Suppose \( \{(T_n, G_n)\}_1^\infty \) is a martingale and \( Y_n = T_n - T_{n-1}, \ n = 1, 2, \ldots \ (T_0 = 0) \). Let \( \{W_n\}_1^\infty \) be an increasing sequence of positive rv's such that for each \( n \), \( W_n \) is \( G_{n-1} \)-measurable a.s. \( (G_0 = \{\emptyset, \Omega\}) \) and \( W_n \to \infty \). If

\[
\sum_{j=1}^\infty E(Y_j^2|G_{j-1}) < \infty \text{ a.s.},
\]

then

\[
W_1^n \overset{a.s.}{\to} 0.
\]

**PROOFS OF RESULTS IN §4.2**

**Proof of Theorem 4.1** The proof is based on that of Heyde and Scott and Strassen and uses a truncation like that introduced in the Skorokhod representation proof of Theorem 2.3.

Define

\[
\tilde{x}_j = x_j I(2^{-j} < |x_j| \leq Z_j) + \frac{1}{2} x_j I(|x_j| \leq 2^{-j}) + \sgn(x_j) 2^{-j-1}(2+Z_j|x_j|^{-1}) I(|x_j| > Z_j)
\]

and 

\[
x_j^* = \tilde{x}_j - E(\tilde{x}_j|F_{j-1}).
\]

(If \( Z_j(\omega) < 2^{-j} \), let \( I(2^{-j} < |x_j| \leq Z_j)(\omega) = 0 \).)

Then \( x_1^*, \ldots, x_n^* \) generate the \( \sigma \)-algebra \( F_n \), as can be proved using the arguments in Lemma 5 of Theorem 2.3. Note that
\[ |X_j - X_j^*| I(|X_j| > Z_j) - E[X_j I(|X_j| > Z_j)|F_{j-1}]| \leq 3.2^{-j}. \]

Define
\[
S_n^* = \sum_{j=1}^{n} X_j^*,
\]
\[
V_n^{*2} = \sum_{j=1}^{n} E(X_j^{*2}|F_{j-1}),
\]
and if \( u \in [0,1] \) let
\[
\nu_n^*(u) = [\phi(V_n^{*2})^{-1} \left( S_n^* + uV_n^{*2} - V_n^{*2} \right) (V_n^{*2} + 1 - V_n^{*2})^{-1} X_n^*].
\]

Then
\[
\sup_{u \in [0,1]} |\nu_n(u) - \nu_n^*(u)| \leq [\phi(V_n^{*2})^{-1} \sup_{1 \leq r \leq n} |\Sigma(X_j - X_j^*)| \leq \]
\[
\leq [\phi(V_n^{*2})^{-1} \sup_{1 \leq r \leq n} |\Sigma X_j I(|X_j| > Z_j) - E[X_j I(|X_j| > Z_j)|F_{j-1}]| +
\]
\[
+ [\phi(V_n^{*2})^{-1} \sum_{1 \leq j \leq n} 3.2^{-j}, \text{ using (4.12),}
\]
\[ a.s. \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ due to (4.1) and (4.4).} \]

That is,
\[ (4.13) \sup_{u \in [0,1]} |\nu_n(u) - \nu_n^*(u)| \rightarrow 0. \]

Next we introduce the Skorokhod Representation (see Strassen [40], Theorem 4.3). By extending the original probability space if necessary we may suppose that there exists a Brownian motion \( W \) and a sequence \( \{T_n\}_{1}^{\infty} \) of non-negative rv's defined on our probability space, such that
\[
(c_1^*, s_2^*, \ldots, s_n^*) = (W(T_1), W(T_2), \ldots, W(T_n)).
\]
If \( G_n \) is the \( \sigma \)-field generated by \( X_1, \ldots, X_n \) and \( W(u) \) for \( u \leq \sum_{j=1}^{n} T_j \) then \( T_n \) is \( G_n \)-measurable,

\[
E(T_n | G_{n-1}) = E(X_n^{k2} | G_{n-1}) = E(X_n^{k2} | F_{n-1}) ,
\]

and for some constant \( L_r \) depending only on \( r \),

\[
E(T_n^r | G_{n-1}) \leq L_r E(X_n^{k2r} | G_{n-1}) = L_r E(X_n^{k2r} | F_{n-1}) .
\]

Define \( \mu(.) \) on \([0,\infty)\) by

\[
\mu(u) = S^*(u - V_{2}^{2}) (V_{2}^{2} - V_{2}^{2})^{-1} X_{p+1}^{*}
\]

where

\[
p = p(u) = \max\{j | V_{j}^{2} \leq u\} .
\]

Then

\[
\mu_n(u) = [\phi(V_{n}^{2})]^{-1} \mu(V_{n}^{2} u)
\]

so that in view of (4.13) and Corollary 1 of Strassen [39], it suffices to prove

\[
(4.14) \quad \lim_{t \to \infty} [\phi(t)]^{-1} \sup_{u \leq t} |\mu(u) - W(u)| = 0 \text{ a.s.}
\]

To this end we first wish to prove that

\[
(4.15) \quad \sum_{j=1}^{n} T_j - V_{2}^{2} = o(V_{2}^{2}) \text{ a.s. as } n \to \infty .
\]

Since

\[
E(X_j^{k4} | F_{j-1}) = E(\tilde{X}_j^{4} | F_{j-1}) - 4E(\tilde{X}_j^{3} | F_{j-1}) \cdot E(\tilde{X}_j | F_{j-1}) + 6E(\tilde{X}_j^{2} | F_{j-1}) \cdot E(\tilde{X}_j | F_{j-1}) .
\]

\[
[E(\tilde{X}_j | F_{j-1})]^2 - 3E[(\tilde{X}_j | F_{j-1})]^4
\]
\[ \leq 11E\left(\tilde{X}_j^4 | F_{j-1}\right) \]
\[ \leq 11E\left[X_j^4 I(|X_j| \leq Z_j) | F_{j-1}\right] + 11.2^{-4j} \]
then in view of (4.3) and (4.4),
\[ \sum_1^\infty V_j^{-4}E\left(X_j^4 | F_{j-1}\right) < \infty \quad \text{a.s.} \]
from which, using Lemma 4.1, we can derive
\[ \sum_1^n \{T_j - E(T_j | G_{j-1})\} = o\left(V_n^2\right) \quad \text{a.s.} \]
This is equivalent to (4.15).
Also,
\[ |X_j^2 - X_j^*|^2 \leq X_j^2 I(|X_j| > Z_j) + 2^{-2j} \]
so that
\[ |E(X_j^2 | F_{j-1}) - E(X_j^*^2 | F_{j-1})| \leq E\left[X_j^2 I(|X_j| > Z_j) | F_{j-1}\right| + 2^{-2j} \]
and hence after summing from \( j = 1 \) to \( n \), we deduce that
\[(4.16) \quad V_n^2 - V_n^*^2 = o\left(V_n^2\right) \quad \text{a.s. as } n \to \infty \]
under (4.2) and (4.4).
Combining (4.15) and (4.16) we have
\[(4.17) \quad V_n^2 \sum_1^n T_j \rightarrow 1 \quad \text{a.s.} \]
Under (4.4), \( p(u) \to \infty \) as \( u \to \infty \), and hence
\[ 1 \geq u^{-1} V_n^2 p(u) \geq V_n^{-2} p(u) + V_n^2 p(u) \rightarrow 1 \quad \text{as } u \to \infty \]
so that
\[ u^{-1} V_n^2 \quad \text{a.s.} \ \Rightarrow 1 \quad \text{as } u \to \infty . \]
Similarly
\[
\frac{u^{-1}p(u)}{p(u)+1} \rightarrow 1 \text{ as } u \to \infty
\]
and combining these and (4.17),
\[
\frac{u^{-1}p(u)}{1} \text{ a.s.} \quad \text{and} \quad \frac{u^{-1}p(u)+1}{1} \text{ a.s.} \quad \text{as } u \to \infty.
\]
Since
\[
|\mu(u)-W(u)| \leq \max\{|W(\sum T_j)-W(u)|,|W(\sum T_j)-W(u)|\}
\]
then (4.14) follows on working through Strassen's proof.

Proof of Theorem 4.2  This proof is the same as that of Theorem 4.1 except for some minor changes. Instead of \( v^*, \mu^* \) and \( \mu \) we define
\[
u_n^{*2} = \sum_{1}^{n} X_j^{*2},
\]
\[
u_n^{*}(u) = [\varphi(u_n^{*2})]^{-1}[S^{*+}(uU_n^{*2}-U_n^{*2})(U_n^{*2}-U_n^{*2})^{-1}X_n^{*}] \quad \text{and}
\]
\[
u(u) = S^{*+}(u-U_q^{*2})(U_q^{*2}-U_q^{*2})^{-1}X_q^{*+}
\]
where
\[
q = q(u) = \max\{j|U_j^{*2} \leq u\}.
\]
Instead of (4.15) we prove
\[
(4.18) \quad \sum_{1}^{n} T_j - \nu_n^{*2} = o(U_n^{*2}).
\]
Arguing as before, we can show that
\[
(4.19) \quad \sum_{1}^{\infty} U_j^{-4}E(X_j^{*4}|F_{j-1}) < \infty \text{ a.s.},
\]
which, in view of (4.8), is equivalent to
Lemma 4.1 applied to this gives us
\[ \sum_{j=0}^{n} [T_j - E(T_j | G_{j-1})] = o(U_{n-1}^2) \quad \text{a.s.,} \]
which is equivalent to (4.18).

Instead of (4.16) we prove
\[ (4.20) \quad U_n^2 - u_n^2 = o(U_n^2) \]
which follows from the inequality
\[ |X_j^2 - u_j^2| \leq X_j^2 \mathbb{I}(|X_j| > z_j) + 2^{-2j}. \]

We also require
\[ (4.21) \quad u_n^2 - v_n^2 = o(U_n^2), \]
that is,
\[ \sum_{j=0}^{n} [X_j^2 - E(X_j^2 | G_{j-1})] = o(U_n^2), \]
which follows from (4.19) and Lemma 4.1.

Combining (4.18), (4.20) and (4.21) we obtain
\[ (4.22) \quad U_n^{-2} \sum_{j=1}^{n} T_j \xrightarrow{a.s.} 1 \]
which can be compared with (4.17). The proof is completed as before. **

Proof of Corollary to Theorem 4.2 In view of (4.9), (4.5)-(4.8) will hold if the equivalent conditions with \( U_j \) replaced by \( s_j \) hold.

Let \( Z_j = \delta s_j. \) (4.5) and (4.7) follow immediately from (4.10) and (4.11), respectively. (4.10) implies, via the Kronecker Lemma, that
\[ \text{for all } \varepsilon > 0, \quad s_n^{-1} \sum_{j=1}^{n} |X_j| \mathbb{I}(|X_j| > \varepsilon s_j) \xrightarrow{a.s.} 0 \]
so that
def for all \( \varepsilon > 0 \), \( s_n^{-1} \sup_{j \leq n} \left| X_j \right| I(\left| X_j \right| > \varepsilon s_j) \rightarrow 0 \) a.s.

and hence
\[
s_n^{-1} \sup_{j \leq n} \left| X_j \right| \rightarrow 0.
\]

Now,
\[
s_n^{-2} \sum_{j=1}^{n} X_j^2 I(\left| X_j \right| > \delta s_j) \leq (s_n^{-1} \sup_{j \leq n} \left| X_j \right|) s_n^{-1} \sum_{j=1}^{n} \left| X_j \right| I(\left| X_j \right| > \delta s_j)
\]
and so (4.6) is true. Finally,
\[
s_n^{-2} \sup_{j \leq n} E(X_j^2) \leq E(s_n^{-2} \sup_{j \leq n} X_j^2) \rightarrow 0 \text{ as } n \rightarrow \infty
\]
using Theorem 1 of Pratt [32], and hence
\[
s_{n+1}^{-1}s_n + 1
\]
so that (4.8) is true.

If \( \{X_n\}_{n=1}^{\infty} \) is stationary with the distribution of \( X \),
\[
s_n^{-2} U_n^2 \rightarrow E(X^2)^{-1} E(X_1^2|T)
\]
where \( T \) is the invariant \( \sigma \)-field wrt the sequence \( \{X_n\}_{n=1}^{\infty} \). To establish (4.10) note that when \( E(X^2) = 1 \),
\[
\sum_{j=1}^{\infty} s_j^{-1} E[\left| X_j \right| I(\left| X_j \right| > \varepsilon s_j)] = E[\left| X \right| \sum_{j=1}^{\infty} s_j^{-2} I(\left| X \right| > \varepsilon n^2)]
\]
and
\[
\sum_{j=1}^{\infty} s_j^{-2} I(\left| X \right| > \varepsilon n^2) = \sum_{j=1}^{\infty} \left[ e^{-s_j^2} \right] \frac{1}{n^2}
\]
where \( [x] \) denotes the integer part of \( x \)
\[
\leq \int_{0}^{\varepsilon^{-2}x^2} \frac{1}{2} x^2 dx
\]
\[
= 2\varepsilon^{-1}\left| X \right|
\]
so that
\[
\sum_{j=1}^{\infty} \frac{1}{s_j} \left[ \mathbb{E}[X_j \mid |X_j| > \varepsilon s_j] \right] \leq 2\varepsilon^{-1} \mathbb{E}(X^2) < \infty.
\]

(4.11) follows from a similar argument:
\[
\sum_{j=1}^{\infty} \frac{1}{s_j} \mathbb{E}[X_j \mid |X_j| \leq \delta s_j)] = \mathbb{E}[X^4 \sum_{j=1}^{\infty} \frac{1}{n^2} \mathbb{I}(|X| \leq \delta n^2)]
\]
and if \( X \neq 0, \)
\[
\sum_{j=1}^{\infty} \frac{1}{n^2} \mathbb{I}(|X| \leq \delta n^2) = \sum_{j=1}^{\infty} \frac{1}{n^2} + \delta^4 X^{-4} \mathbb{I}(|X| = \delta n^2)
\]
\[
\leq \int_{\delta^{-2}X^2}^{\infty} x^{-2}dx + 2\delta^4 X^{-4}
\]
\[
= \delta^2 X^{-2} + 2\delta^4 X^{-4},
\]
so that
\[
\sum_{j=1}^{\infty} \frac{1}{s_j} \mathbb{E}[X_j \mid |X_j| \leq \delta s_j)] \leq 2\delta^2 \mathbb{E}(X^2) + 2\delta^4 < \infty. \quad \text{**}
\]
BIBLIOGRAPHY


