On solutions to the Yang-Baxter equation related to sl(n)

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I declare that the work contained in this thesis is my own original research, obtained in collaboration with my supervisor Dr. Vladimir Mangazeev at the Australian National University, Canberra. All material taken from other sources is referenced and acknowledged as such. I also declare that none of the material contained in this thesis has been submitted for another degree at this or another university.

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Abstract

In this thesis the problem of constructing solutions to the Yang-Baxter equation is considered. Such solutions are known as R-matrices and we study a certain class of these related to the quantum group $U_q(\widehat{sl_n})$. Using a variety of unrelated methods the matrix elements for different representations of the quantum group are constructed. In the process the structure of the solutions and their symmetries are detailed including a realisation of the R-matrix as a "composite object".

Among the new results obtained is a formula for the elements of the general $U_q(\widehat{sl_n})$ R-matrix for symmetric tensor representations with arbitrary weights in terms of multivariable q-hypergeometric series. This formula is shown to be factorised by more elementary R-matrices without the difference property. An explicit formula for the factors in terms of simple products is derived from the general formula by evaluating the R-matrix at special values of the spectral parameter. Using this factorisation a simple proof that the newly obtained R-matrix can be stochastic is given. Symmetries of the R-matrix generate identities of hypergeometric series which may be unknown.

This new factorised representation of the R-matrix is compared with other constructions developed in the literature. It is shown that there is agreement up to simple transforms between all the R-matrices considered, thereby linking different approaches to solving Yang-Baxter equation. In the process comparisons between different formulae for the matrix elements are made which reveal that the 3D approach based on a new solution to the tetrahedron equation is the most efficient construction for this class of R-matrices. In some cases comparisons can only be made in the rational limit $q \to 1$ and using the newly obtained trigonometric R-matrix a quantum deformation of their construction is given. These deformations are used to discover new structure of the trigonometric R-matrix, such as a new L-operator factorisation in the rank 1 case as well some new formulae for the generating function of the operator action.

Some progress is made towards a more general formula for matrix elements in the case of arbitrary highest weight representations of sl_n. Using a factorisation approach by Derkachov et al. explicit formulae for the elements of the factors in the case $n = 3$ is presented. These factors are shown to be related to the new trigonometric factorisation presented in this thesis.

Finally, the stochastic R-matrix is linked to recent developments in near-equilibrium stochastic systems of interacting particles of KPZ universality class. The factorisation of the matrix is shown to be equivalent to a "convolution" of the probability function describing these models. A generalisation of this probability function in the case of $sl(3)$ is proposed which contains an extra parameter and seems to satisfy the sum-to-unity rule.

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Chapter 1

Thesis Overview

In this thesis we consider the problem of constructing solutions to the parametrised quantum Yang-Baxter equation. That is, the linear operator equality

$$\mathcal{R}_{12}(\lambda_1,\lambda_2)\mathcal{R}_{13}(\lambda_1,\lambda_3)\mathcal{R}_{23}(\lambda_2,\lambda_3) = \mathcal{R}_{23}(\lambda_2,\lambda_3)\mathcal{R}_{13}(\lambda_1,\lambda_3)\mathcal{R}_{12}(\lambda_1,\lambda_2), \qquad (1.0.1)$$

acting on a tensor product of vector spaces $V_1 \otimes V_2 \otimes V_3$. A solution $\mathcal{R}(\lambda)$ is a linear 'R-operator/R-matrix' acting on $W \otimes U$ such that $\mathcal{R}_{ij}(\lambda_i,\lambda_j)$ acts as $W = V_i, U = V_j$ and trivially in the third space. It is also known as the triangle equation or 2-simplex equation. Ever since the connection between this equation and the solvability of quantum systems was realised, many approaches have been developed over the years as means of constructing solutions and thereby examples of quantum systems that are solvable. The list of all approaches, as they are currently known are

1. Direct evaluation of the universal R-matrix [1; 2; 3]
2. Fusion procedure [4; 5; 6]
3. Projection of a 3D Integrable Model [7; 8; 9; 10]
4. Matrix factorization of the L-operator [11; 12]
5. Factorization of the R-matrix by Q-operators [13; 14]
6. Coherent state action on the holomorphic basis [15]
7. Spectral decomposition [16; 17; 18]
8. Direct solution to recurrence relations

Development of some approaches, such as 3, 4 and 5 are still ongoing. Others, such as 1, 2, 6 and 7 have seen little progress in the last few decades and are perhaps (with the exception of 6) more "mature" in their development. In most cases, the R-matrix is presented in an abstract "black box" form in which the finer features of the solution are opaque. For example, up until now, matrix elements were only known explicitly for very few solutions whose structure is usually described by some low rank Lie algebra under a low dimensional fundamental or adjoint representation. We look to these methods in an attempt to construct matrix elements for higher rank Lie algebras for representations acting on finite and infinite dimensional spaces. Every method we consider, offers in theory a way of doing this for at least some classes of solutions. However in practice it is all too often an exhaustive computation and not feasible even with modern computer algebra. Furthermore some approaches are instrinsically limited or not yet developed enough to construct many solutions. Therefore there are many solutions which we know to exist but we cannot write down their elements.

One of the major results of this thesis featured in chapter 3 is the construction of a master formula (3.3.24) giving the elements of every R-matrix related to $U_q(\widehat{sl_n})$ for symmetric tensor representations with arbitrary weights. The formula is presented in terms of a multivariable basic hypergeometric series and is always a finite algebraic expression which is very efficient to compute. This includes the case of complex weight parameters where the R-matrix is irreducible and acts on a tensor product infinite-dimensional Verma modules. This work is a generalisation of the 3D projection approach used in [10] where the formula (3.8.3) was obtained in case $n = 2$. The formula is given in terms of a single variable basic hypergeometric series which matches the q-Racah polynomials [19] known in combinatorics. In chapter 4 we give another proof/derivation of this formula using the representation theory of quantum groups directly.

We spend a great deal of time studying the structure of this formula, including its symmetries, degenerations, and comparing it with other results obtained elsewhere in the literature. These are some minor results contained in this thesis. Actually, the most interesting thing to come out of this study is a new factorization for this class of R-matrices in terms of two simpler R-matrices with two independent spectral parameters. We say they are simpler because they are obtained from the general R-matrix by evaluating it at two special values of the spectral parameter. In these cases our formula (3.3.24) reduces to a simple binomial product with weight parameters entering the expression algebraically. Viewing these as spectral parameters it solves (1.0.1). Multiplying these matrices back together with special arguments we can reconstruct the original R-matrix thereby factorising it. This is encapsulated in (3.5.6) and we consider it another major result of this thesis.

We are very excited by this factorisation, and we believe there is still much more to the story regarding its reason for being and its applications. For example, the binomial product function determining matrix elements of the factors is actually the probability distribution function Φ (8.0.4) first introduced in [20] as a higher rank generalisation of a four parameter family of integrable stochastic zero range processes [21; 22; 23; 24]. Contained within this family are non-equilbrium systems of interacting particles belonging to the Kardar-Parisi-Zhang universality class. It is clear that the Yang-Baxter equation is behind the integrability of these models and our factorisation implies that the matrix elements we have constructed are some kind of convolution of these probability distribution functions. The explicit relation is given by (8.0.10) which we rewrite here in a more abstract form as

$$R(\lambda) = \Phi * \Phi, \tag{1.0.2}$$

where $R(\lambda)$ is a "twisted" version (8.0.1) of the R-matrix constructed in chapter 3. This elegant connection between the $U_q(\widehat{sl_n})$ R-matrix, typically known to describe a family of six-vertex-like models and 1D quantum spin-chain magnets, and these stochastic ZRP models is unknown and therefore is another major result of this thesis. In addition, we use this result to give a simple proof that our R-matrix can be stochastic by showing that its elements sum to unity down columns and are positive.

All of the results mentioned so far are obtained by using a solution (3.2.13) [9] to the more general tetrahedron/3-simplex equation and projecting it out in one direction. We are interested in other ways R-matrices can be constructed, with the goal of unifying these approaches and constructing the elements of even more solutions to the Yang-Baxter equation which we expect also has interesting structure such as factorisation and stochasticity. We also want to see if it is possible to find a "better" formula than the one we have obtained and therefore find some interesting identities for hypergeometric series. As we show in this thesis, each construction yields a very different presentation of the R-matrix. Even when two methods supposedly construct an identical matrix, the formula for the elements is often completely different. To illustrate, for the case of sl_2 the method of spectral decomposition in chapter 4 constructs a triple summation formula (4.2.9), while the factorisation methods of chapters 5 and 6 produce double summation and single summation (5.5.14) formulae respectively. Yet each formula produces (up to normalisation and simple transformations) the same R-matrices. In this thesis we spend much time showing how all these different presentations are actually the same, by using identities for hypergeometric series to transform and sum up all these different formulae to the same result. So far we have completed this unification for sl_2, where we have derived the single summation formula (3.8.3) and its rational limit from all constructions considered. From this exercise we conclude that this formula is simplest presentation of sl_2 related R-matrices. We do not believe it can be summed up further or transformed into an even simpler series. This unification is a main result of our work.

We believe this to be also true of our trigonometric R-matrix formula (3.3.24) and we have made some progress towards reconstructing it with other approaches. Before we elaborate, let us be clear that the 3D and spectral decomposition methods are the only ones considered in this thesis that construct trigonometric R-matrices. The methods in other chapter construct rational R-matrices which are a special case obtainable from the trigonometric ones in the limit $q \to 1$. We have successfully taken this limit in (3.3.24) to give another formula (3.7.4) for a family of sl_n related factorised rational R-matrices which we can directly compare to the constructions where the trigonometric version is unknown. Besides the aforementioned $n = 2$ case which we have unified in this thesis, we have succeeded in the case of $n = 3$ in section 5.6.4 for the rational R-matrix. One thing we have learnt is that the computational task involved in these alternative methods is far greater than the 3D model projection method of chapter 3, and the resultant formulae are much more complicated. Hence our assertion that the 3D construction is the most efficient for this particular class of R-matrices.

Given our unification and the unknown quantum deformation of the factorisation methods of chapters 6, 5 and Sklyanin's method in chapter 7 we use our trigonometric R-matrix to work out how they deform. We start from our $U_q(\widehat{sl_2})$ R-matrix (3.8.3) and reverse the arguments made in the rational case and replace them with their quantum analogues. We obtain the q-deformations of all the theory developed in what is essentially the $q \to 1$ limit and there are number of new results here. The main one is a new higher spin $U_q(\widehat{sl_2})$ L-operator/R-matrix factorisation (6.3.17) with explicit formulae for all the factors generalising the results in [12]. We also find that our factorisation (3.5.6) in the $n = 2$ case is actually the q-deformation of the Q-operator factorisation of chapter 5 derived using a very different Lie group oriented approach [25; 14]. Therefore our factorisation is also a higher rank trigonometric generalisation of this construction.

In these rational R-matrix constructions we often have to consider a generating function for the operator action to extract matrix elements. We also present a deformation of these functions for the action of the trigonometric R-matrix. For example, the R-matrix constructed in chapter 7 has a generating function presented as a terminating ${}_2F_1$ hypergeometric series. This presentation is particularly nice because its dependence on the holomorphic basis of the underlying space is a function of only a single variable.

Besides showing that this R-matrix is exactly the same as that obtained from the 3D approach, we also found its quantum deformation and its generating function which turns out to be a balanced and terminating ${}_4\phi_3$ basic hypergeometric series and it appears that it no longer depends on a single variable. This is unfortunate, but the result still may be useful in integration involving quantum groups although this is a question we did not get around to investigating.

We are particularly interested in the construction of chapter 5 for a number of reasons. The first is because it is a factorisation in terms of objects similar to ones we obtained in (3.5.6), (3.7.18) and so the construction can probably be generalised in this direction. Secondly because it applies for all highest weight representations of sl_n and hence we could obtain a more general formula for a larger family of R-matrices. We attempted this for the case of sl_3 and mostly succeeded, where we constructed explicit formula for the elements of the three factors (5.6.15), (5.6.19), and (5.6.21) composing the full R-matrix. The functions for their elements are already quite complicated and the resultant function obtained by composing them together is even more complicated - containing 12 summations - so we do not write it down until we can find a way of summing it. Nevertheless the factors are interesting objects in their own right because they are a kind of R-matrix, satisfying identity (5.3.4d). This identity is essentially the Yang-Baxter equation but with the extra complexity of the intertwining of representations by each factor. This complexity can be removed at least in the $n = 2$ case by making the right variable substitution as in (5.5.17) where we showed it is the same as our factorisation. It is probable that the same can be done for the $n = 3$ factors we construct with this method. We also mention as another application that they are building blocks for constructing Q-operators as explained in [26; 27; 14]. This is not explored any further in our thesis but it is something we would like to investigate in the future.

We also think there is an application of the chapter 5 construction to the aformentioned stochastic ZRP models. Given the similarity of that factorisation with ours (3.5.6) we ask if they are also stochastic R-matrices. If they are then they must be something more general because they are operators acting with more parameters. As we considered the sl_3 construction we saw that one of the factors is described by a function that seems to satisfy the sum-to-unity property. It has an extra parameter compared to the function (8.0.10) (for $n = 3$) describing stochastic R-matrices and it reduces to it when this extra parameter is set to 0. We cannot at this time give a proof of the sum to unity property or its positivity regimes, but propose this function as a possible generalisation which may be interpreted as a new collection of stochastic models - a conjecture.

The structure of this thesis is summarised as follows. Chapter 2 is a brief background of Yang-Baxter integrability. We present some history of the notion of integrability and the motivation for studying it as an enquiry into some of the most fundamental problems in physics. We use this as a foundation to formally introduce the Yang-Baxter equation; how it appears and some of the discoveries made in attempts to solve it. The goal of this chapter is to provide some context for our work, and to explain where it fits in the body of research on this topic. We also use it to introduce some of the notations, theory, and terminology used in this thesis and elsewhere, such as quantum groups and their representation theory. We will briefly mention some constructions that we did not consider in this thesis, such as the universal R-matrix construction.

The rest of the thesis is divided into chapters based on each particular R-matrix construction we have considered. Chapter 3 is dedicated to the 3D model projection approach where we use a solution of the tetrahedron equation to construct R-matrices. Chapter 4 is dedicated to the method of "spectral decomposition" where we use the representation theory of quantum groups to write down the R-matrix in terms of its eigenvalue decomposition on its subspaces which are described by Clebsch-Gordan coefficients. Chapter 5 is about a R-matrix factorisation in terms of more elementary intertwining operators which are building blocks of Q-operators. Chapter 6 is another factorisation approach that can be considered a continuation of chapter 5 when one tries to restrict that construction to the case of finite-dimensional R-matrices. Chapter 7 is a construction of a SU(2) invariant R-matrix by considering its action on the coherent state vector where it turns out to have nice transformation properties that can be exploited. Chapter 8 is an investigation of the R-matrices considered as a stochastic object where we prove its sum-to-unity property and link it back to the stochastic models. The appendices are dedicated to hypergeometric series where we list all the definitions and identities we use in writing down and transforming the formulae we derive in the main text.

Finally, we would like to mention that some of this work has already been published in journals. The research on the 3D model approach in chapter 3 for $U_q(\widehat{sl_n})$ and its stochastic interpretation in chapter 8 appears in [28]. The main results in chapters 6 and 7 will appear in [29]. The results of other chapters may appear in later papers.

Chapter 2

Yang-Baxter Integrability

2.1 Integrable systems

In the discipline of physics one attempts to model the physical universe by building a complete, consistent theory from which predictions can be formulated and tested against experiment. In modern physics, popular theories such as the *standard model* are almost completely mathematical, such that the core principles of the theory are written in the formal language of mathematics. This formalisation of physics is perhaps an ongoing process since the time of Newton. His description of planetary motion as the solution to some collection of mathematical equations birthed the fundamental physics commonly known today as *Classical Mechanics*.

Historically, it is also Newton that one can trace back the early notions of an *Integrable System*. Consider a system containing three 'bodies': the Earth, Sun and Moon, each with an intrinsic property m_i, position and vector $\mathbf{x}_i$ relative to some origin. If given the position $\mathbf{x}_i(0)$ of each body at some time $t = 0$, is it possible to construct a function $\mathbf{x}_i(t)$ whose output gives the position each body at some time t in the future? Assuming the bodies interact with each other only through the gravitational force as described by Newton, such a function must be a solution to the equations

$$\begin{aligned}
\frac{d^2\mathbf{x}_1}{dt^2} &= -\frac{Gm_2}{(x_1 - x_2)^3}(\mathbf{x}_1 - \mathbf{x}_2) - \frac{Gm_3}{(x_1 - x_3)^3}(\mathbf{x}_1 - \mathbf{x}_3), \\
\frac{d^2\mathbf{x}_2}{dt^2} &= -\frac{Gm_3}{(x_2 - x_3)^3}(\mathbf{x}_2 - \mathbf{x}_3) - \frac{Gm_1}{(x_2 - x_1)^3}(\mathbf{x}_2 - \mathbf{x}_1), \\
\frac{d^2\mathbf{x}_3}{dt^2} &= -\frac{Gm_1}{(x_3 - x_1)^3}(\mathbf{x}_3 - \mathbf{x}_1) - \frac{Gm_2}{(x_3 - x_2)^3}(\mathbf{x}_3 - \mathbf{x}_2).
\end{aligned} \tag{2.1.1}$$

In mathematics (2.1.1) may be identified as a system of coupled, second-order linear ordinary differential equations. One can imagine that a solution is of interest to mathematicians as a formal object unto itself. Yet it also practical applications,such as accurate prediction of future solar events.

In any case, it turns out a general solution satisfying any initial condition $\mathbf{x}_i(0)$ does not have a 'closed' form. That is, the function $\mathbf{x}_i(t)$ cannot be written down in terms of elementary functions such as trignometric, exponential, logarithmic and algebraic functions. This was proven by Bruns and Poincare. Indeed, Newton himself had attempted a solution but failed. Sundman's theorem gives an expression for $\mathbf{x}_i(t)$ as an infinite power series but this is not considered a closed-form solution. Furthermore, its convergence is so slow that computing a good approximation of the function is impractical. Perhaps it is worth noting that *closed-form* solutions have been found in some special cases of the problem [30; 31; 32; 33].

This particular example is often referred to as the classical *three-body problem* and (2.1.1) its equations of motion. We have presented it in order to illustrate a system that is considered to be NOT integrable. It also serves as a special case of a more general problem in physics - the *many-body* problem. Many observable phenomena can be modelled as a system of interacting bodies. For example, the motion of galaxies is composed of the motion of many stars and their satellites interacting through gravity. The motion of a gas is composed of the motion of molecules interacting through intermolecular forces. These are systems with many thousands of bodies. So it seems to be somewhat demoralising that we cannot give a proper expression for the motion of just three bodies in general. Fortunately, in many cases a partial solution can be given. With the advent of computers sophisticated numerical techniques have been developed to provide an **approximate** answer to a systems behaviour - but not the exact motion!

It is the many-body systems whose equations of motion DO have a closed-form solution that are of particular interest to physicists and mathematicians alike. The elegant nature of their solution makes them special. These are the systems that we loosely refer to as *Integrable*. Loosely, because there are actually a few different definitions of an integrable system; depending on the theory we use to model it. But they all express the same theme, and that is the solvability of a system. Let us consider a few examples.

Say we want to model the dynamics of an n-body system using classical mechanics, whereby one realises it as a *Hamiltonian system*. Integrability in this context refers to *Liouville integrability*. In this formalism the system is described by a 2n-dimensional state vector $(\mathbf{q}, \mathbf{p})$, with each component (q_i, p_i) of the coordinate referring to the position and momentum state of the ith body in the system. The vector space of all configurations is known as *phase space*. The motion of the system can then be thought of as a curve embedded in phase space. To determine this curve one considers the **Hamiltonian** $\mathcal{H}(\mathbf{q}, \mathbf{p}; t)$ constructed from the axioms of the particular system. The curve is determined by *Hamilton's equations*,

$$\begin{aligned} \frac{d\mathbf{q}}{dt} &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \\ \frac{d\mathbf{p}}{dt} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}. \end{aligned} \tag{2.1.2}$$

Naturally one may consider other scalar-valued functions defined on the phase space. Given two such functions f, g let us also consider the operation

$$\{f, g\} := \sum_{i=0}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \tag{2.1.3}$$

which (assuming it is well-defined) is also scalar-valued function on the phase space. We call $\{\cdot, \cdot\}$ the Poisson bracket. The system is considered to be Liouville integrable when the system admits a sufficient number of independent functions $\mathcal{X}_i(\mathbf{q}, \mathbf{p}; t)$ such that

$$\{\mathcal{X}_i, \mathcal{H}\} = 0, \quad \{\mathrm{X}_i, \mathrm{X}_j\} = 0, \quad \frac{d\mathcal{X}_i}{dt} = 0. \tag{2.1.4}$$

$\mathcal{X}_i$ are often called conserved quantities or integrals of motion. Their existence allows one to write down the solution to (2.1.2) in a closed-form.

The Hamiltonian formalism of a physical system and its integrability is a good place to start in introducing the background of this thesis. We will consider the integrability of a system under the theory quantum mechanics. It is true that so far we have only discussed classical mechanics. But it is the power of the Hamiltonian formalism that allows us to talk about the same system in both a classical setting AND a quantum setting. Roughly speaking, phase space in the classical system becomes a *Hilbert space* in the quantum system, and functions on phase space become operators acting on the Hilbert space. Expressions represented by Poisson brackets become expressions represented by commutator brackets under the rule

$$\{\cdot, \cdot\} \to \frac{1}{i\hbar}[\cdot, \cdot]. \tag{2.1.5}$$

This process is commonly known as *Canonical quantization* of a classical system. The details are rather technical and not the subject of this thesis so we will not go into them here. The main message is that the notion of integrability of a classical system (2.1.4) has a quantum analogue - sufficiently many conserved quantities commuting with the Hamiltonian operator and each other under the commutator bracket.

2.2 Statistical mechanics

In large systems, we may not be concerned with the exact motion of every single body. Indeed, in the last section we established that for anything more than 2 bodies this is not possible except under special circumstances. But perhaps we want to model the systems 'average' behaviour. We may be interested in macroscopic properties that we can measure. For example, the density and temperature of a gas or the magnetisation of a magnet. These are not mechanical 'motion' but they are measurable quantities familiar in the study of thermodynamics. It is perhaps true that the thermodynamics of a system arise from the microscopic mechanical motions and interactions of its smallest constituents. *Statistical Mechanics* aims to clarify this link between mechanics and thermodymanics; to model macroscopic behaviour starting from only its most fundamental microscopic interactions.

The approach taken is to consider all possible configurations C of the system. A configuration being a labelling of the state of each component. Assign to each configuration an energy $E(C)$ and let us assume that the probability $p(C)$ of observing the system in a particular configuration is given by

$$p(C) = \mathcal{Z}^{-1}\mathcal{W}(C), \quad \mathcal{W}(C) := \exp(-\frac{E(C)}{kT}). \tag{2.2.1}$$

$\mathcal{W}(C)$ is known as the *Boltzmann weight* of a configuration and this collection of weights defines the *Boltzmann probability distribution*. The object $\mathcal{Z}$ is known as the *partition function* and the sum-to-unity requirement of the probability distribution lets us realise it as

$$\mathcal{Z} = \sum_C \exp(-\frac{E(C)}{kT}). \tag{2.2.2}$$

We will not spend time justifying the assumption of a Boltzmann probability distribution of the states, other than to say that in many physical applications it leads to useful predictions. Another assumption that is implicit in this distribution is that the system is in equilibrium with its surroundings. By that we mean that the collection of configurations and their probabilities is static in time. This means there are no net particle or energy flow in or out of the system.

Given some observable quantity $\mathcal{O}$ of the system with value $\mathcal{O}(C)$ then its expected value is

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \sum_{C} \mathcal{O}(C) \exp(-\frac{E(C)}{kT}). \tag{2.2.3}$$

For example, consider the internal energy E of the system, then

$$\langle E \rangle = \mathcal{Z}^{-1} \sum_{C} E(C) \exp(-\frac{E(C)}{kT}) = kT^2 \frac{\partial}{\partial T} \log \mathcal{Z}. \tag{2.2.4}$$

It is easily calculated once of the partition function is known. This is also true of many other macroscopic quantities one may be interested in such as correlation functions. Furthermore, the partition function can exhibit the critical behaviour of a system. That is, for some parameters determining the Boltzmann weights, the system may drastically change its behaviour. Real world examples of this include the change of water into steam or ice dependent on the temperature and pressure. Phenomena such as these are known as a *phase transition*, and the collection of values for the parameters that cause it are the systems *critical points*. In statistical models they manifest in the partition function of a system as some kind of singularity. As we can see, the partition function contains most, if not all of the answers to the interesting questions we can ask about a systems behaviour.

Therefore the primary goal is to calculate the partition function of a given system. In computing a systems partition function one can say the system is effectively 'solved'. This is much easier said than done. Generally speaking, for real world many-body systems the number of possible configurations one could observe it in is exhaustively large. Computing a sum over all these configurations seems hopeless.

Instead, let us ask: For what kind of systems can we compute the partition function? For what kinds of systems might the expression for the partition function have an analytic or even closed form? What is the microscopic nature of such a system? Are such systems somehow related? These questions are central to much of the research and progress made in statistical mechanics throughout the 20^{th} century.

Many examples of solvable models, both classical and quantum, were found over the last century. Some examples that have guided much of the research include the Ising model [34], Eight-vertex model [35] and the XYZ Heisenberg spin chain [36]. It is only in recent decades though that an underlying theme has emerged that seems to explain their solvability. This theme has become to be known as the famous **Yang-Baxter equation** [37; 35]. The idea is that every model that can be solved exactly admits a solution to the Yang-Baxter equation. Conversely, every solution to the Yang-Baxter equation describes a solvable model.

Maybe that last paragraph is a bit bold and needs some qualifications. It is true when talking about a special class of systems known as the two-dimensional vertex models. These are systems such as the six and eight-vertex models where the interacting bodies are contrained to a lattice in two space dimensions. Each site in the lattice is a vertex with edges drawn between interacting sites. For two-dimensional solvable vertex models the construction of a solution to the Yang-Baxter equation is usually obvious. For vertex models in higher dimensions one talks about solutions to a generalised Yang-Baxter equation. For example, solvable three-dimensional vertex models admit solutions to what is known as **Zamolodchikov's tetrahedron equation** [38; 39] - a 3D Yang-Baxter equation. For solvable systems that are not posed as a vertex model, the Yang-Baxter link is less obvious. Progress has been made in figuring this out; certain models are found to be 'equivalent' to a vertex model. For example, a duality exists between vertex models and another class of models known as *interaction-round-a-face* models. An equivalence also exists between a n-dimensional classical system and a $(n-1)$-dimensional quantum system; the most famous example being the equivalence of the 2D eight-vertex model and the 1D XYZ Heisenberg spin chain.

Unfortunately for some solvable models the role of the Yang-Baxter equation (and its generalisations) is still not known. Regardless, at this point in time the consensus seems to be that it does play a role, and only the details need to worked out. Since it is known in a large array of cases already, the notion of *Yang-Baxter integrability* as another formulation of integrability has resulted in a hive of research activity over the past few decades. There is now a formal definition for integrable systems in this sense, and we will get to it. But first we find it more appropriate introduce a concrete example of a solvable model in statistical mechanics. Let us introduce the **six-vertex model** and use it to show how the Yang-Baxter equation appears as a necessary condition for solvability. We also like this example because it serves as a base special case for main results presented in this thesis.

2.2.1 The six-vertex model

Consider a two-dimensional $N \times M$ lattice where each site takes on a configuration from a set of 'allowed' local configurations. The local configuration expresses the idea of an interaction occuring between nearest neighbours, and therefore we can represent this graphically by drawing an edge between each neighbouring lattice site, with a labelling of the edge indicating the nature of the interaction. The six-vertex model [40; 41] is the lattice where the allowed configurations are given by Figure 2.1.

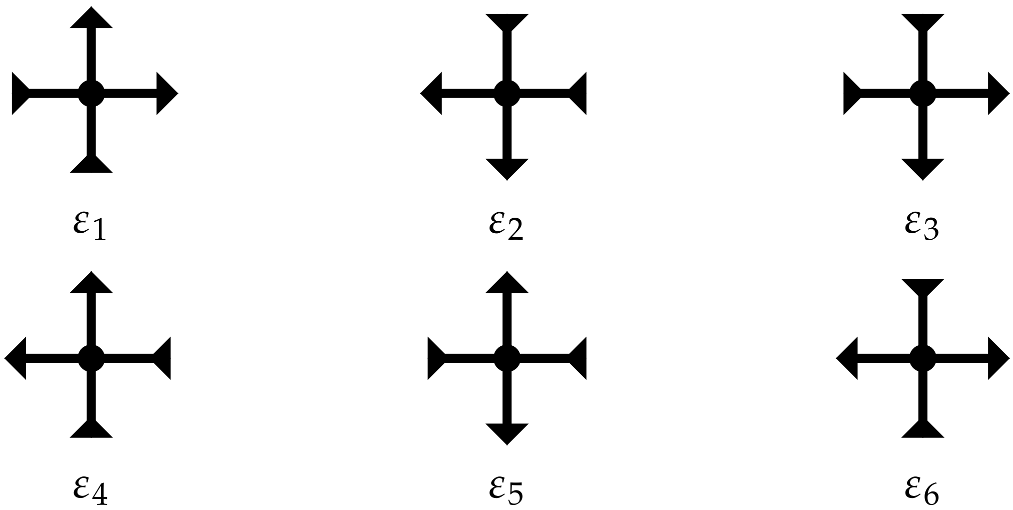

Figure 2.1: Local site configurations allowed in the six-vertex model.

One notices that each vertex has four edges, with each edge labelled by either an inward or outward facing arrow, such that every vertex always has 2 of each kind. This is perhaps a defining characteristic of the six-vertex model, and is sometimes referred to as the *ice-rule*. This is because it belongs to a well-known class of models called the *ice-type* models in which the allowed vertices have a similar kind of 'conservation law/ice-rule' on the edges pointing inwards and outwards.

Physically, edge configurations are supposed to model hydrogen bonding in ice and similar crystals, where each lattice site represents an oxygen atom and the edge a hydrogen bond; its edge direction indicating which atom the hydrogen ion is closer to. By specifying a local configuration at each site we determine the global configuration of the system. For example, a possible global configuration of a $N = M = 4$ system is given by Figure 2.2. The Boltzmann weight $\mathcal{W}(C)$ of each global configuration C is the

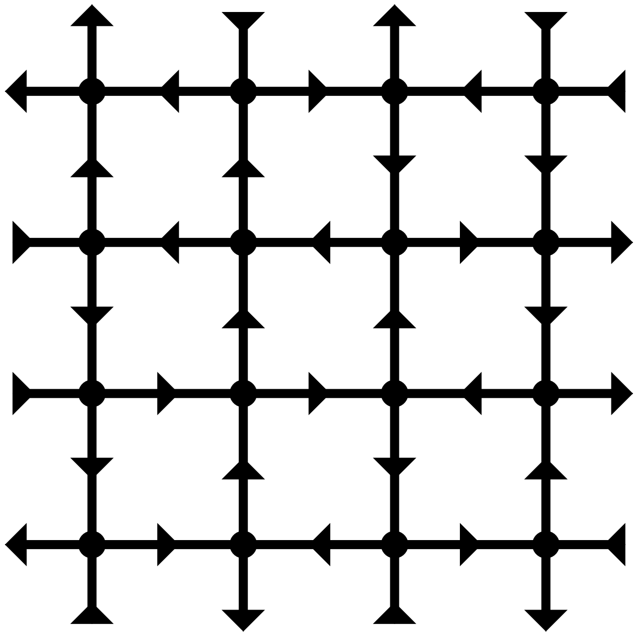

Figure 2.2: A possible configuration of a 4×4 six-vertex model with periodic boundary conditions.

product of the weights appearing locally. In the six-vertex model we assign the energies ε_i to vertices as in Figure 2.1, then the local weights are defined by

$$\mathcal{W}_i = \exp(-\frac{\varepsilon_i}{kT}). \tag{2.2.5}$$

Let us consider Figure 2 as an example. Let us label this configuration as C_1. Its Boltzmann weight can be expressed as

$$\mathcal{W}(C_1) = \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3^2\mathcal{W}_4^2\mathcal{W}_5^5\mathcal{W}_6^5. \tag{2.2.6}$$

As discussed earlier to solve this model we need to be able to sum over the weights of all possible configurations - the partition function. A popular way to express this complicated summation is with a mathematical object known as the **transfer matrix**.

2.2.2 The transfer matrix

The problem of computing the partition function can be posed as an eigenvalue problem of a matrix which is often referred to as a transfer matrix. In this section we will show how the transfer matrix is constructed, techniques used to find its eigenvalues

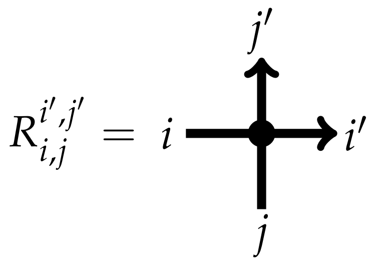

Figure 2.3: Matrix elements for an operator R given by vertex configurations.

and how it is related to the summation in (2.2.2).

First, consider a matrix R whose entries contain all local weights (2.2.5) of the six-vertex model. We use vertex edges to index the elements $R_{i,j}^{i',j'}$ as in Figure 2.3 where an edge with an upward/rightward arrow is labelled as zero, and an edge with a leftward/downward arrow is labelled as one. The arrowheads in Figure 2.3 refer to the edges which are 'upper/column indices' of the matrix, we stress they are **not** related to the arrows in Figure 2.1. Notice that the only indices that correspond to valid vertex configurations are those that satisfy

$$i + j = i' + j'. \tag{2.2.7}$$

We also make an extra assumption; the energy of a local configuration does not change under a reversal of its arrows. This is a symmetry of the system which means that

$$a = \mathcal{W}_1 = \mathcal{W}_2, \quad b = \mathcal{W}_3 = \mathcal{W}_4 \quad c = \mathcal{W}_5 = \mathcal{W}_6 \tag{2.2.8}$$

and the matrix R can be written down as

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}. \tag{2.2.9}$$

Now let us consider the matrix $\mathcal{T}(M)$ constructed by taking M sites(Figure 2.3) with entries indexed by the vertical edges $j_1^{(')}, j_2^{(')}, \ldots, j_m^{(')}$ and calculated by summing over all possible horizontal edge configurations such that $i'_s = i_{s+1}, i'_M = i_1$. That is, matrix

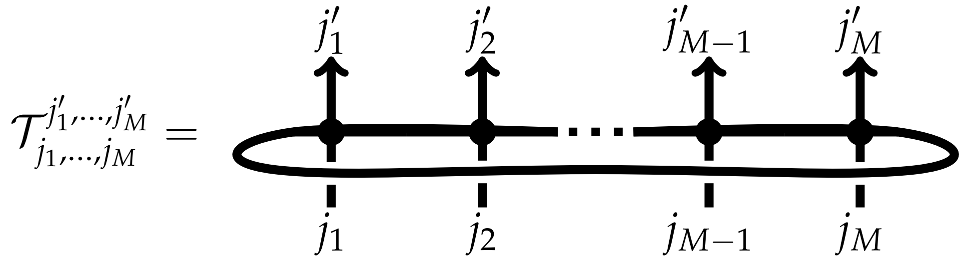

Figure 2.4: Graphical notation of the row-to-row transfer matrix $\mathcal{T}$ for M sites.

elements have the form

$$[\mathcal{T}(M)]^{j'_1,\dots,j'_M}_{j_1,\dots,j'_M} = \sum_{i_1,\dots,i_M} \prod_{s=1}^{M} R^{i'_s,j'_s}_{i_s,j_s}. \tag{2.2.10}$$

The expression (2.2.10) is somewhat awkward to use in discussions. Quite often we find it more convenient to use a graphical notation to express complicated objects such as $\mathcal{T}$ which we feel better illustrates its construction from more elementary objects R. We present Figure 2.4 as a graphical representation of (2.2.10). Here external edges connected to only one vertex are the matrix indices, and each possible configuration corresponds to a particular matrix element. Edges that are enclosed by two vertices are summed over all possible edge states. The loop through all horizontal edges in Figure 2.4 means we sum over possible horizontal configurations in determining an element of the matrix $\mathcal{T}$. Let us note that in imposing periodic boundary conditions, we are discussing a special case of the six-vertex model on a torus.

The graphical notation we have used in Figure 2.3 and 2.4 is essentially the *Penrose graphical notation* used in multilinear algebra. Of course, R and $\mathcal{T}$ are multidimensional arrays and can be thought of as tensor objects with an underlying basis. The *order* of a tensor is number of indices needed to specify an element of the array - graphically the number of edges passing through the vertex. So R is an order 4 tensor and $\mathcal{T}$ is an order 2M tensor. Because the number of lower indices (incoming edges) is equal to the number of upper indices (outgoing edges), R is a multilinear operator acting on a vector space $V_1 \otimes V_2$ and likewise for $\mathcal{T}$ (but with M factors).

With the machinery we have established so far, it is not hard to see that the partition function $\mathcal{Z}$ for the $N \times M$ six vertex model is given by

$$\mathcal{Z} = \text{Trace}[\mathcal{T}^N(M)]. \tag{2.2.11}$$

In (2.2.11) we have matrix multiplied N copies of $\mathcal{T}(M)$ which sums over all possible vertical edge configurations except at the lattice boundary. The trace imposes periodic boundary conditions for the boundary edges and sums over all their possible configurations. The result is the partition function.

It is well known that the trace of a linear operator is a sum of its eigenvalues. If we know the eigenvalues of $\mathcal{T}$ then we know it for any power of $\mathcal{T}$ and therefore we know $\mathcal{Z}$. Therefore the problem of solving a vertex model is equivalent to diagonalising $\mathcal{T}$ - thereby calculating its eigenspectrum.

2.3 Yang-Baxter integrability

There are two main approaches to calculating the spectrum of $\mathcal{T}$; the *Bethe Ansatz* and the method of *Commuting Transfer matrices*. These methods are actually equivalent in a way that was made precise through the *Baxter Q-Operators* but we will save that discussion for later. Now we will give an overview of the two techniques and introduce the Yang-Baxter equation.

2.3.1 Bethe ansatz

The Bethe ansatz was a method introduced by Hans Bethe [42] to solve the spin-1/2 1D Heisenberg spin by diagonalising its Hamiltonian. Lieb [40] discovered that this technique also works for the six-vertex model, and was able to solve it for a number of cases.

The Bethe ansatz aims to solve the eigenvalue equation

$$\mathcal{T}\,|\boldsymbol{X}\rangle = \Lambda\,|\boldsymbol{X}\rangle \tag{2.3.1}$$

directly by assuming the eigenvectors are given by a superposition of plane-waves, that is

$$|\boldsymbol{X}\rangle = \sum_{\sigma} A_{\sigma} \exp(i\sigma \boldsymbol{k} \cdot \boldsymbol{X}). \tag{2.3.2}$$

where the sum is over all permutations $\boldsymbol{\sigma}$ of the components in the wavevector $\boldsymbol{k}$. This special form of the eigenvectors is the 'ansatz' and substituting (2.3.2) into the left-hand side of (2.3.1) leads to the right-hand side plus a collection of extra terms. Requiring that these terms cancel leads to a set of equations for the coefficients A_σ and wavevector components k_s commonly known as the *Bethe Ansatz* equations.

The ice-rule mentioned in the last section plays an important role in applying (2.3.1) to the six-vertex model. It is easy to see that rule (2.2.7) for each site implies a global rule

$$j_1 + \cdots + j_M = j'_1 + \cdots + j'_M = n \tag{2.3.3}$$

for the M-site transfer matrix $\mathcal{T}$. Therefore $\mathcal{T}$ has a M+1 block-diagonal form and in diagonalising it one can restrict the problem to each block indexed by n =0,. . . ,M. Once a block n is fixed a state $|\boldsymbol{X}\rangle$ is specified by $\boldsymbol{X} = (x_1, \ldots, x_n)$ where x_i is the position of the ith down configuration in the lattice/one index in $\mathcal{T}$. The sum in (2.3.2) is over the $n!$ permutations of $\boldsymbol{k} = (k_1, \ldots, k_n)$.

Peforming the necessary computations one finds that the extra terms in (2.3.1) disappear provided that

$$\exp(iMk_j) = (-1)^{n-1} \prod_{l=1}^{n} \frac{1 - 2\boldsymbol{\Delta}\exp(iMk_j) + \exp(iM(k_j + k_l))}{1 - 2\boldsymbol{\Delta}\exp(iMk_l) + \exp(iM(k_j + k_l))}, \tag{2.3.4}$$

$$\boldsymbol{\Delta} := (a^2 + b^2 - c^2)/2ab. \tag{2.3.5}$$

(2.3.4) are the Bethe ansatz equations for the six-vertex model. They are a set of n equations determining $\boldsymbol{k}$. Once they are solved, A_σ and Λ_n in (2.3.1) can be computed by

$$A_\sigma = \epsilon_\sigma \prod_{1 \le i < j \le n} \left(1 - 2\Delta\exp(iM\sigma_i) + \exp(iM(\sigma_i + \sigma_j))\right), \tag{2.3.6}$$

$$\Lambda_n = a^M \prod_{s=1}^{n} L(k_s) + b^M \prod_{s=1}^{n} M(k_s), \tag{2.3.7}$$

$$L(k_s) = \frac{ab - (c^2 - b^2)\exp(iMk_s)}{a^2 - ab\exp(iMk_s)}, \tag{2.3.8}$$

$$M(k_s) = \frac{a^2 - c^2 - ab\exp(iMk_s)}{ab - b^2\exp(iMk_s)}. \tag{2.3.9}$$

Clearly (2.3.4) are transcendental equations and therefore no closed-form solution exists. In fact, these equations remain unsolved for general finite n,M. The only case where they have been solved exactly is in the (thermodynamic) limit $M \to \infty$, and only for the maximum eigenvalue. Perhaps this is fortunate, because for the $N \times M$ lattice in the limit $M, N \to \infty$

$$\mathcal{Z} \sim \Lambda_{MAX}^{N}, \tag{2.3.10}$$

so we can say the infinite six-vertex model on a torus can be solved by the Bethe ansatz. At this point we remind the reader of the many-body problem outlined in the last section. Certainly, the Bethe ansatz equations are in a sense 'equations of motion' for the six-vertex model - describing its behaviour on average. It seems counter intuitive that the equations for finitely many bodies are harder to solve than for infinitely many interacting bodies. Nevertheless, in principle the Bethe ansatz equations admit enough solutions $\boldsymbol{k}$ to determine all eigenvalues and eigenvectors of the transfer matrix.

An important point that we want to make is that this method relied on the ice rule to specify states and the terms in the eigenfunction (2.3.2) by co-ordinates $\boldsymbol{X}$. For this reason this technique has come to be known as the *coordinate Bethe ansatz*. However, it is not immediately clear how one could apply this technique if such a conservation rule does not exist. This is a problem encountered in solving the eight-vertex model. Regardless, the eight-vertex model was solved by Baxter [43] using what has come to be known as the commuting transfer matrix method.

2.3.2 Commuting transfer matrices

These two methods on the surface seem quite different but they actually imply one another. That is, the Bethe ansatz equations imply that transfer matrices with different Boltzmann weights commute. Conversely, the premise of commuting transfer matrices implies the Bethe ansatz equations. This fact allows a more algebraic formulation of the coordinate bethe ansatz approach outlined in the last section. The Yang-Baxter equation arises as a sufficiency condition for the commutativity of transfer matrices and therefore the existence of the Bethe ansatz equations.

The crucial observation to make is that the Boltzmann weights a,b,c of equation (2.2.8) only enter the Bethe ansatz equations (2.3.4) through $\boldsymbol{\Delta}$ (2.3.5) and therefore this is the only variable the eigenvectors depend on. That means that if we choose different weights a', b', c' with the same $\boldsymbol{\Delta}$ then the transfer matrix has the same set of eigenvectors. At this point it is appropriate to introduce a parametrisation of the weights. Let us define

$$a = \rho q(1-\lambda^2), \quad b = \rho(q^2-\lambda^2), \quad c = \rho(q^2-1)\lambda, \tag{2.3.11}$$

$$\mathbf{\Delta} = \frac{q+q^{-1}}{2} \tag{2.3.12}$$

so that a,b,c are entire functions of q, λ. For fixed q, the Boltzmann weights $a(\lambda), b(\lambda)$, $c(\lambda)$ lie on a curve parametrised by λ. For every point we can associate a transfer matrix $\mathcal{T}(\lambda)$. Every transfer matrix lying on this curve will have the same eigenvectors because $\mathbf{\Delta}$ depends only on q. Therefore they are simultaneously diagonalisable and hence commute.

$$[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0 \quad \forall \lambda, \mu \in \mathbb{C}. \tag{2.3.13}$$

Now supposing that (2.3.13) holds it is possible to recover the Bethe ansatz equations. The approach is to construct operators $\mathcal{Q}(\lambda)$ that satisfy

$$\mathcal{T}(\lambda)\mathcal{Q}(\lambda) = \left[\lambda q^{-1}\right]^M \mathcal{Q}(q\lambda) + [\lambda]^M \mathcal{Q}(q^{-1}\lambda), \tag{2.3.14}$$

$$[\mathcal{T}(\lambda), \mathcal{Q}(\mu)] = [\mathcal{Q}(\lambda), \mathcal{Q}(\mu)] = 0. \tag{2.3.15}$$

(2.3.14) is known as the Baxter TQ-relation. The commutativity (2.3.15) of these operators with the transfer matrix means that they same have the eigenvectors and so can we diagonalize them simultaneously. Define diagonal matrices $\mathcal{T}_d(\lambda)$ and $\mathcal{Q}_d(\lambda)$ by

$$\mathcal{T}(\lambda) = \mathcal{M}^{-1}\mathcal{T}_d(\lambda)\mathcal{M}, \quad \mathcal{Q}(\lambda) = \mathcal{M}^{-1}\mathcal{Q}_d(\lambda)\mathcal{M} \tag{2.3.16}$$

then the TQ-relation becomes 2^M equations of the form

$$\Lambda(\lambda) = \frac{\left[\lambda q^{-1}\right]^M \mathcal{A}(q\lambda) + [\lambda]^M \mathcal{A}(q^{-1}\lambda)}{\mathcal{A}(\lambda)} \tag{2.3.17}$$

for each entry on the main diagonal $\mathcal{T}_d(\lambda)\mathcal{Q}_d(\lambda)$ and eigenvalues$\Lambda(\lambda), \mathcal{A}(\lambda)$ respectively. $\Lambda(\lambda)$ is an entire function in λ by (2.3.16) since $\mathcal{M}$ does not depend on λ and $\mathcal{T}(\lambda)$ is entire. Therefore the right-hand side of (2.3.17) is entire. This means that given zeroes $\{\lambda_j\}$ of $\mathcal{A}(\lambda)$ the numerator must also vanish, therefore

$$\frac{[\lambda_j]^M}{\left[\lambda_j q^{-1}\right]^M} = -\frac{\mathcal{A}(q\lambda_j)}{\mathcal{A}(q^{-1}\lambda_j)} \tag{2.3.18}$$

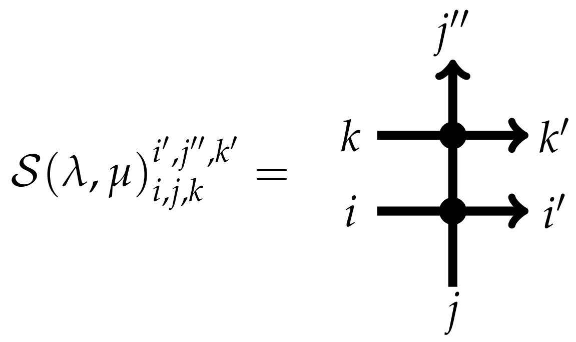

Figure 2.5: Matrix $[\mathcal{S}(\lambda,\mu)]$

which are equivalent to the Bethe ansatz equations (2.3.4). To make the connection we identify

$$\exp(ik_j) = \frac{[\lambda_j]}{[\lambda_j q^{-1}]}, \quad \mathcal{A}(\lambda) = \prod_{l=1}^{n} \left[\lambda\lambda_l^{-1}\right] \tag{2.3.19}$$

and so the solutions k_j of (2.3.4) correspond to solutions λ_j of (2.3.18).

Of course the argument rests upon the assumption that transfer matrices commute for all values of the spectral parameter - it may not be true. But even if it is true it is not clear from the arguments how one could construct $\mathcal{Q}(\lambda)$ if the Bethe-ansatz equations were not already known. The problem of constructing Q-operators has been considered in many texts, beginning with Baxter in [43] for eight-vertex model, known as the *propagation through the vertex* method. Another method due to Bazhanov-Lukyanov-Zamalodchikov is given in [44; 45] and yet another method by Chicherin-Derkachov-Karakhanyan in [46; 27; 47].

On the problem of commuting transfer matrices, we may ask, what is a sufficient condition for (2.3.13) to hold? Consider a matrix $\mathcal{S}(\lambda,\mu)$ with elements defined by

$$[\mathcal{S}(\lambda,\mu)]_{i,j,k}^{i',j'',k'} := \sum_{j'} R(\lambda)_{i,j}^{i',j'} R(\mu)_{k,j'}^{k',j''} \tag{2.3.20}$$

and represented graphically in Figure 2.5. We can write $\mathcal{T}(\mu)\mathcal{T}(\lambda)$ and $\mathcal{T}(\lambda)\mathcal{T}(\mu)$ in terms of $\mathcal{S}$. It is obvious that

$$[\mathcal{T}(\lambda)\mathcal{T}(\mu)]_{j_1,\ldots,j_M}^{j_1'',\ldots,j_M''} = \text{Trace} \prod_{k=1}^{M} [\mathcal{S}(\lambda,\mu)]_{j_k}^{j_k''} \tag{2.3.21}$$

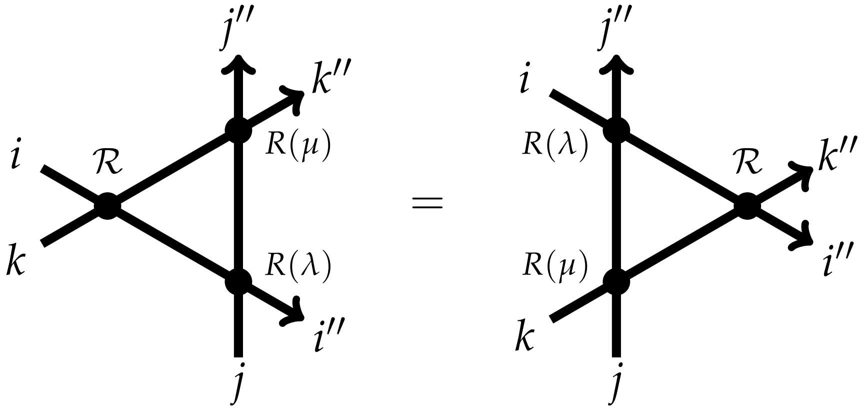

Figure 2.6: Sufficiency condition for $\mathcal{T}(\lambda)\mathcal{T}(\mu) = \mathcal{T}(\mu)\mathcal{T}(\lambda)$

where we have performed a matrix product of the factors $\mathcal{S}$ over $i_l^{(')}, k_l^{(')}$, and traced over the edge states. $j_l^{(')}$ indices in $\mathcal{S}$ are fixed because we want to consider individual elements of (2.3.13). For $\mathcal{T}(\mu)\mathcal{T}(\lambda) = \mathcal{T}(\lambda)\mathcal{T}(\mu)$ to hold it would be sufficient if there was a matrix $\mathcal{R}$ such that

$$[\mathcal{S}(\mu,\lambda)]_{j_k}^{j_k''} = \mathcal{R}\,[\mathcal{S}(\lambda,\mu)]_{j_k}^{j_k''}\,\mathcal{R}^{-1} \tag{2.3.22}$$

because the trace is not affected. Post-multiplying by $\mathcal{R}$, matrix elements of (2.3.22) satisfy

$$\sum_{i',k',j'} \mathcal{R}_{i,k}^{i',k'} R(\lambda)_{i',j}^{i'',j'} R(\mu)_{k',j'}^{k'',j''} = \sum_{i',k',j'} R(\mu)_{i,j}^{i',j'} R(\lambda)_{k,j'}^{k',j''} \mathcal{R}_{i',k'}^{i'',k''}. \tag{2.3.23}$$

The graphical represention given by Figure 2.6 is somewhat more informative. One can think of the equation as describing any of the three edges "moving through" the opposite vertex and vice versa. The internal summation represented by the inner triangle has very different structure on both sides but somehow yield the same output for fixed external edge indices.

For the six-vertex model the equations (2.3.23) are simple enough to solve directly, and find

$$\mathcal{R} = R(\lambda/\mu). \tag{2.3.24}$$

With this subtitution (2.3.24) for $\mathcal{R}$, the set of equations (2.3.23) with fixed $\{i, i'', j, j'', k, k''\}$ is a matrix formulation of the **parametrised quantum Yang-Baxter equation**. We can now introduce it formally.

Definition 2.3.0.1 (Quantum Yang-Baxter equation)**.** The linear operator equality

$$\mathcal{R}_{12}(\lambda)\mathcal{R}_{13}(\lambda\mu)\mathcal{R}_{23}(\mu) = \mathcal{R}_{23}(\mu)\mathcal{R}_{13}(\lambda\mu)\mathcal{R}_{12}(\lambda), \tag{2.3.25}$$

acting on the vector space $V_1 \otimes V_2 \otimes V_3$ is the quantum Yang-Baxter equation. A solution $\mathcal{R}(\lambda)$ is a linear 'R-operator' acting on $W \otimes U$ such that $\mathcal{R}_{ij}(\lambda)$ acts as $W = V_i, U = V_j$ and trivially in the third space.

The relation (2.3.25) is probably the most common presentation of the Yang-Baxter equation found in the literature. We refer to solutions $\mathcal{R}(\lambda)$ as R-operators when discussing them independent of a basis. Of course, if we choose an orthonormal basis $|i,j\rangle := |i\rangle \otimes |j\rangle \in W \otimes U$ then the operator can be realised as a **R-matrix** $R(\lambda)$ with elements

$$R(\lambda)_{i,j}^{i',j'} = \langle i,j| \mathcal{R}(\lambda) |i',j'\rangle \tag{2.3.26}$$

and (2.3.25) can be written as (2.3.23) with (2.3.24) and re-scaling $\lambda := \lambda\mu$.

Figure 2.6 illustrates clearly how $\mathcal{R}$ in (2.3.25) are composed with each other. The three edges spanned by $\{i,i''\},\{j,j''\},\{k,k''\}$ represent the action in each factor of $V_1 \otimes V_2 \otimes V_3$. With a basis (2.3.26) the Yang-Baxter equation expresses an equality of two 6-order tensors.

Using operator language, we can reformulate the transfer matrix (2.2.10)/Figure 2.4 as a 'global' operator

$$\mathcal{T}(\lambda) := \mathrm{Trace}_{V_0}\left[\mathcal{M}_0(\lambda)\right], \tag{2.3.27}$$

$$\mathcal{M}_0(\lambda) := \mathcal{R}_{01}(\lambda) \otimes \mathcal{R}_{02}(\lambda) \otimes \cdots \otimes \mathcal{R}_{0M}(\lambda) \tag{2.3.28}$$

acting in the space $V_1 \otimes \cdots \otimes V_M$. This is sometimes referred to as the *quantum space*. The operator $\mathcal{M}_0(\lambda)$ is called the *Monodromy operator* and the transfer matrix is formed by taking the trace over V_0 also known as the *auxiliary space*. Graphically, it is fairly easy to see that

$$\mathcal{R}_{12}(\lambda/\mu)\mathcal{M}_1(\lambda)\mathcal{M}_2(\mu) = \mathcal{M}_2(\mu)\mathcal{M}_1(\lambda)\mathcal{R}_{12}(\lambda/\mu) \tag{2.3.29}$$

by starting with the left hand side and moving $\mathcal{R}$ 'through' each lattice site swapping λ and μ as in Figure 2.6. If $\mathcal{R}$ is non-singular (2.3.29) implies (2.3.13) so this relation is somehow a more general statement. $\mathcal{R}$ can be thought of as 'intertwining' representations of $\mathcal{M}$, in the sense of a module homomorphism preserving the action of $\mathcal{M}$ on some representation space. We will elaborate on this later.

Some parallels appear to emerge between integrability in the Liouville sense and in the Yang-Baxter sense. Just as a Liouville integrable system in classical mechanics admit sufficiently many commuting conserved quantities, a Yang-Baxter integrable system in statistical mechanics admits infinitely many commuting transfer matrices. The Yang-Baxter equation is a sufficiency condition for commutativity and therefore each R-matrix solution encodes a model in statistical mechanics that is exactly solvable by the Bethe ansatz/$\mathcal{Q}$-operators. A straightforward decoding of the R-matrix is to interpet its elements as Boltzmann weights of a vertex model just like (2.2.9) and Figure 2.1. Obviously not all solvable models are posed as vertex models and the process of relating such to an R-matrix can be highly non-trivial. The details are not important in our work, we only remark that Yang-Baxter integrability is not always obvious and can be hidden.

The question we want to answer is; **How to solve the Yang-Baxter equation? How to construct R-matrices?**. In this thesis we have constructed R-matrix solutions to the Yang-Baxter equation using a variety of methods. It is an interesting walkthrough of different approaches to constructing matrix elements, which all produce the same results. We are also interested in the structure of the solutions, and we find that the solutions have an interesting form in terms of hypergeometric series, relating them to certain special classes of orthogonal polynomials. We also find some more elementary objects related to the R-matrix. All R-matrices in this thesis are related to a special class of algebras known as 'quantum groups', which we will introduce now.

2.4 Quantum groups

Quantum groups were first found by the Leningrad school under Ludwig Fadeev as a consequence of their approach known as the *quantum inverse scattering method* [48; 5; 49] to constructing and solving integrable systems. They were introduced formally by Drinfel'd [50; 51] and Jimbo [16] as *Hopf algebras* with extra structure that makes them *quasitriangular*. Roughly speaking, each algebra contains the symmetries of an entire class of R-matrices [52; 53]. Given such an algebra, it is usually possible to construct an element of the algebra that represents this class - the so-called *universal R-matrix*

[50; 54; 55; 56; 3; 57]. Applying a representation of the algebra to the universal R-matrix allows one to realise it as an actual matrix.

Many of these algebras are deformations of *Universal enveloping algebras* $U(g)$ of a Lie algebra g. This unexpected link between statistical mechanics and Lie algebras was perhaps the reason for so much research interest over the past few decades. To show how quantum groups arise in studying R-matrices we will give an overview of some of the steps involved in the quantum inverse scattering method where they were originally discovered.

2.4.1 Quantum inverse scattering method

In the last section we constructed a collection of transfer matrices from the six-vertex model and found that they commute. In general, the existence of an invertible matrix R which satisfies the Yang-Baxter equation (2.3.25) was seen to be a sufficient condition for commutativity. Now we ask the converse question: given an R-matrix, what kind of commuting transfer matrices can be constructed? That is, we want to solve equation (2.3.22) for fixed $\mathcal{R}$ and variable $\mathcal{S}$.

This equation is just a local form of (2.3.29) (or $M = 1$) for $\mathcal{M}_0$ (2.3.28), but this is too specific because it is built out of factors R. We use a more general form

$$\mathcal{M}_0^L(\lambda) := \mathcal{L}_{01}(\lambda) \otimes \mathcal{L}_{02}(\lambda) \otimes \cdots \otimes \mathcal{L}_{0M}(\lambda), \tag{2.4.1}$$

where the $\mathcal{L}(\lambda)$ are known as *Local operators* or L-operators. They act in the auxiliary space V_0 as a $n \times n$ matrix with each entry an operator acting in the quantum space $V_1 \otimes \cdots \otimes V_M$. We also make the assumption that its dependence on λ only enters as

$$\mathcal{L}(\lambda) = \lambda \mathcal{L}^+(\lambda) + \lambda^{-1}\mathcal{L}^-(\lambda) \tag{2.4.2}$$

where $\mathcal{L}^+(\lambda)$ is an upper triangular matrix and $\lambda^{-1}\mathcal{L}^-(\lambda)$ is a lower triangular matrix. For example, when $n = 2$ it can be written as

$$\mathcal{L}(\lambda) = \begin{pmatrix} L_{11}^-\lambda^{-1} + L_{11}^+\lambda & \lambda L_{12} \\ \lambda^{-1}L_{21} & L_{22}^-\lambda^{-1} + L_{22}^+\lambda \end{pmatrix}, \tag{2.4.3}$$

a 2×2 matrix with operator entries L_{ij}. The idea is that a *representation* of these operators will give commuting transfer matrices by construction and hence an integrable

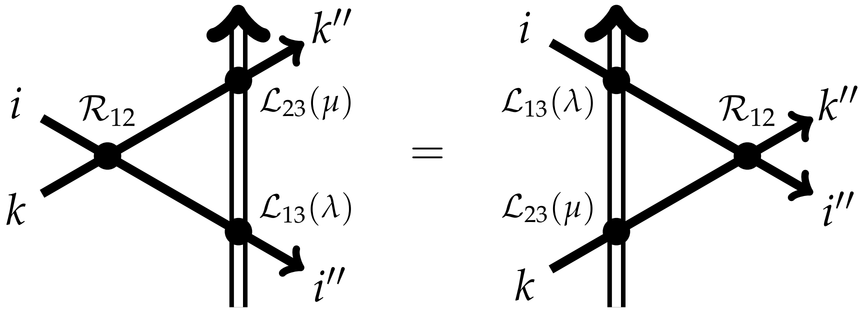

Figure 2.7: Yang-Baxter *RLL*-relation

model. Since $\mathcal{R}$ satisfies (2.3.22), the equation (2.3.25) can be written as

$$\mathcal{R}_{12}(\lambda/\mu)\mathcal{L}_{13}(\lambda)\mathcal{L}_{23}(\mu) = \mathcal{L}_{23}(\mu)\mathcal{L}_{13}(\lambda)\mathcal{R}_{12}(\lambda/\mu), \tag{2.4.4}$$

which is sometimes referred to as the *RLL-relation* or a weaker *RLL-form Yang-Baxter equation*. A graphical representation of this equation is given in Figure 2.7

We will also work with an $n \times n \times n \times n,\ n \geq 2$ R-matrix whose elements are given by the function

$$[\mathcal{R}(\lambda)]_{i,j}^{i',j'} = \delta_{i,i'}\delta_{j,j'}\delta_{i,j}(q-1)(\lambda + \lambda^{-1}q^{-1}) + \delta_{i,i'}\delta_{j,j'}(\lambda - \lambda^{-1}) + \delta_{i,j'}\delta_{j,i'}\sigma_{i,i'} \tag{2.4.5}$$

for $1 \leq i, j, i', j' \leq n$ and

$$\sigma_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ (q - q^{-1})\lambda & \text{if } i < j, \\ (q - q^{-1})\lambda^{-1} & \text{if } i > j. \end{cases} \tag{2.4.6}$$

For $n = 2$ this is equivalent to the six-vertex model R-matrix (2.3.24) and (2.2.8) but with a change of variables. Substituting this R-matrix and (2.4.3) in the RLL relation (2.4.4) defines relations for the operators $L^{\pm}_{\alpha\beta}$. In particular,

$$[L^{\pm}_{\alpha\alpha}, L^{\pm}_{\beta\beta}] = [L^{+}_{\alpha\alpha}, L^{-}_{\beta\beta}] = 0, \tag{2.4.7a}$$

$$L^{\pm}_{\alpha\alpha}L_{\beta\gamma} = q^{\mp\delta_{\alpha\beta}\pm\delta_{\alpha\gamma}}L_{\beta\gamma}L^{\pm}_{\alpha\alpha}, \tag{2.4.7b}$$

$$L_{\alpha\beta}L_{\beta\alpha} - L_{\beta\alpha}L_{\alpha\beta} = (q - q^{-1})(L^{+}_{\alpha\alpha}L^{-}_{\beta\beta} - L^{+}_{\beta\beta}L^{-}_{\alpha\alpha}), \tag{2.4.7c}$$

$$L_{\alpha\beta}L_{\alpha\gamma} = q^{-\epsilon_{\alpha\beta\gamma}}L_{\alpha\gamma}L_{\alpha\beta} \quad n > 2, \tag{2.4.7d}$$

$$L_{\alpha\gamma}L_{\beta\gamma} = q^{-\epsilon_{\alpha\beta\gamma}}L_{\beta\gamma}L_{\alpha\gamma} \quad n > 2, \tag{2.4.7e}$$

$$L_{\alpha\beta}L_{\beta\gamma} - L_{\beta\gamma}L_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}(q - q^{-1})L_{\beta\beta}^{\epsilon_{\alpha\beta\gamma}}L_{\alpha\gamma} \quad n > 2, \tag{2.4.7f}$$

$$L_{\alpha\beta}L_{\gamma\delta} - L_{\gamma\delta}L_{\alpha\beta} = (q^{\epsilon_{\alpha\beta\gamma}} - q^{\epsilon_{\alpha\beta\delta}})L_{\gamma\beta}L_{\alpha\delta} \quad n > 3 \tag{2.4.7g}$$

where $\alpha, \beta, \gamma, \delta \in \{1, \dots, n\}$ do not coincide for relations (2.4.7d) – (2.4.7g) and $\epsilon_{\alpha\beta\gamma}$ is an antisymmetric function such that

$$\epsilon_{\alpha\beta\gamma} = 1, \quad \text{if} \quad \alpha < \beta < \gamma. \tag{2.4.8}$$

The relations (2.4.7)-(2.4.7g) define the *L-operator algebra* $\mathcal{R}_L$ for the R-matrix (2.4.5). $\mathcal{R}_L$ also admits the comultiplication $\Delta : \mathcal{R}_L \to \mathcal{R}_L \otimes \mathcal{R}_L$ given by

$$\Delta(L_{\alpha\beta}) = \sum_{\gamma} L_{\alpha\gamma} \otimes L_{\gamma\beta} \tag{2.4.9}$$

which makes it a Hopf algebra. $\mathcal{R}_L$ is one presentation of the quantum group commonly known as $U_q(sl_n)$ in the literature. If we make the identification

$$L_{ii}^{\pm} = q^{\pm \sum_{s=1}^{n-1} \frac{(n-s)H_s}{n} \mp \sum_{s=1}^{i-1} H_s}, \tag{2.4.10}$$

$$L_{i,i+1} = (q - q^{-1})q^{\sum_{s=1}^{n-1} \frac{(n-s)H_s}{n} - \sum_{s=1}^{i} H_s} F_i, \tag{2.4.11}$$

$$L_{i+1,i} = (q^{-1} - q)q^{-\sum_{s=1}^{n-1} \frac{(n-s)H_s}{n} + \sum_{s=1}^{i} H_s} E_i \tag{2.4.12}$$

the defining relations (2.4.7)-(2.4.7g) of the L-operator algebra $\mathcal{R}_L$ are equivalent to the standard definition of $U_q(sl_n)$ introduced independently by Drinfeld and Jimbo. We will now formally introduce this quantum group [50; 51; 16] starting with the more general algebra $U_q(\widehat{sl_n})$, and explain its reductions.

2.4.2 The quantum group $U_q(\widehat{sl_n})$

$U_q(\widehat{sl_n})$ is the algebra generated by elements $\{q^{\pm H_i}, E_i, F_i\}_{i=0,\dots,n-1}$ over the field of rational functions $\mathbb{C}(q)$ subject to the relations

$$q^{H_i}q^{H_j} = q^{H_j}q^{H_i}, \quad q^{H_i}q^{-H_i} = q^{-H_i}q^{H_i} = 1, \tag{2.4.13a}$$

$$q^{H_i}E_j = q^{a_{ij}}E_j q^{H_i}, \quad q^{H_i}F_j = q^{-a_{ij}}F_j q^{H_i}, \tag{2.4.13b}$$

$$[E_i, F_j] = \delta_{i,j}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \tag{2.4.13c}$$

$$E_i^2 E_j - (q + q^{-1})E_i E_j E_i + E_j E_i^2 = 0 \quad |i - j| = 1 \text{ mod } n, \tag{2.4.13d}$$

$$F_i^2 F_j - (q + q^{-1})F_i F_j F_i + F_j F_i^2 = 0 \quad |i - j| = 1 \text{ mod } n \tag{2.4.13e}$$

where

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{ij} \tag{2.4.14}$$

is the $n \times n$ generalised Cartan matrix of the untwisted affine Lie algebra $\widehat{sl_n}$ and $(.,.)$ is the standard invariant bilinear form defined on the space $h^* = \bigoplus_{i=0}^{n-1} \mathbb{C}\alpha_i$ for simple roots α_i.

$U_q(\widehat{\mathfrak{sl}_n})$ has Hopf algebra structure with comultiplication

$$\Delta(q^{\pm H_i}) = q^{\pm H_i} \otimes q^{\pm H_i}, \tag{2.4.15}$$
$$\Delta(F_i) = F_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes F_i, \tag{2.4.16}$$
$$\Delta(E_i) = E_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i. \tag{2.4.17}$$

We distinguish between $U_q(\widehat{\mathfrak{sl}_n})$ and $U_q(\mathfrak{sl}_n)$. While both are examples of quantum groups, the former is additionally an *affine quantum group*. The latter does not contain the triple $\{E_0, F_0, q^{\pm H_0}\}$ in the list of generators and the first row and column (2.4.14) are hence also removed. We will refer to these Cartan matrices as $A_{\widehat{\mathfrak{sl}_n}}$ and $A_{\mathfrak{sl}_n}$ respectively.

This presentation was introduced independently by Drinfel'd and Jimbo and hence $U_q(\widehat{\mathfrak{sl}_n})$ is sometimes referred to as a *Drinfeld-Jimbo type quantum group*. It is also often referred to as a *quantised universal enveloping algebra* - that is an algebra with an extra *deformation parameter q* whereby

$$\lim_{q \to 1} U_q(\widehat{\mathfrak{sl}_n}) = U(\widehat{\mathfrak{sl}_n}), \tag{2.4.18}$$

and the 'classical' universal enveloping algebra is recovered as a limiting case. Roughly speaking, the universal enveloping algebra $U(\mathfrak{g})$ is defined as the smallest ring containing the Lie algebra $\mathfrak{g}$ where the Lie Bracket $[a, b]$ is replaced with the commutator bracket $[a, b] := ab - ba$. In some presentations of the affine quantum group an additional generator d called the *derivation* can be added satisfying

$$dq^{H_i} = q^{H_i} d, \quad [d, E_i] = \delta_{i0} E_i, \quad [d, F_i] = -\delta_{i0} F_i, \tag{2.4.19}$$

$$\Delta(d) = d \otimes 1 + 1 \otimes d. \tag{2.4.20}$$

This is because the Cartan matrix has corank 1 and it can be shown that the *Cartan subalgebra* of the underlying Lie algebra must have dimension $n + 1$. A further element c called the *central charge* is also present, but for our purposes we have set $c = 0$.

For the sake of completeness, we will give the definition of $\widehat{\mathfrak{sl}_n}$. It is the vector space over $\mathbb{C}$ generated by $\{E_i, F_i, H_i\}_{i=0}^{n-1}$ with Lie Bracket $[\cdot, \cdot]$ such that

$$[H_i, H_j] = 0, \quad [E_i, F_j] = \delta_{i,j} H_i, \tag{2.4.21a}$$

$$[H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j, \tag{2.4.21b}$$

$$\mathrm{ad}_{E_i}^{1-a_{ij}}(E_j) = 0, \quad \mathrm{ad}_{F_i}^{1-a_{ij}}(F_j) = 0 \tag{2.4.21c}$$

where a_{ij} is given in (2.4.14). $\widehat{\mathfrak{sl}_n}$ is an infinite dimensional affine lie algebra. It can be identified as $\widehat{\mathfrak{sl}_n} \cong \mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ for finite dimensional $\mathfrak{sl}_n$ and Laurent polynomials $\mathbb{C}[t, t^{-1}]$. Just like the quantum case, we can have extra elements c and d with relations analogous to (2.4.19). They generalise the Lie algebra to what is known as an *affine Kac-Moody algebra*. Our presentation corresponds to the case $c = 0$.

It is well known that finite semisimple Lie algebras $\mathfrak{g}$ admit the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{\Delta_-} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\Delta_+}, \tag{2.4.22}$$

$$\mathfrak{g}_{\Delta_\pm} = \bigoplus_{\lambda \in \Delta_\pm} \mathfrak{g}_\lambda, \tag{2.4.23}$$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H) X \ \forall H \in \mathfrak{h}\} \tag{2.4.24}$$

where $\mathfrak{h}$ is called the *Cartan subalgebra* and is the maximal commutative subalgebra of $\mathfrak{g}$. $\mathfrak{g}_{\Delta_\pm}$ are the positive and negative *root spaces* corresponding to roots $\lambda \in \mathfrak{h}^*$. Roots are characterised in terms of *simple roots* $\{\alpha_i\}_{i=1}^n$ where n is the *rank* of the algebra. Each root is an integral sum of simple roots $\lambda = \sum_i C_i \alpha_i, \quad C_i \in \mathbb{Z}$ and form the root system

$$\Delta = \Delta_+ \cup \Delta_-, \tag{2.4.25}$$

which splits into 'positive' roots Δ_+ or 'negative' roots Δ_- depending on the sign C_i which are either all positive or all negative. The entire root system can be constructed from the simple roots through the *Weyl group* although the details are not important to us here. The root vector E_λ corresponding to the root λ can be constructed from the generators in (2.4.21) by

$$
\begin{aligned}
E_{\alpha_i} &= E_i, \quad E_{-\alpha_i} = F_i, \quad H_{\alpha_i} = H_i, \quad i \geq 1 \\
[E_{\alpha_i}, E_{-\alpha_i}] &= H_{\alpha_i} \quad [E_\alpha, E_\beta] = \mu_{\alpha\beta} E_{\alpha+\beta} \text{ for } \alpha + \beta \neq 0, \; \mu_{\alpha\beta} \in \mathbb{Q}.
\end{aligned} \tag{2.4.26}
$$

The root vectors form a basis of the Lie algebra, sometimes called the *Cartan-Weyl basis* of the Lie algebra. With normalisation $\mu_{\alpha\beta} \in \mathbb{Z}$ it also sometimes called a *Chevalley basis*. For $\mathfrak{sl}_n$ the dimension of the Lie algebra is $n^2 - 1$.

The infinite dimensional affine Lie algebra $\widehat{\mathfrak{g}}$ case has a similar root space decomposition as (2.4.22) except the root system (2.4.25) is infinite because of the addition of an extra simple root α_0. The write the system down, we first locate the unique *maximal* root $\theta \in \Delta_+^{\mathfrak{g}}$ of the finite algebra $\mathfrak{g}$ and note that it satisfies

$$
a_{0i} = -2\frac{(\theta, \alpha_i)}{(\theta, \theta)}, \quad a_{i0} = -2\frac{(\alpha_i, \theta)}{(\alpha_i, \alpha_i)} \tag{2.4.27}
$$

where maximal is defined as being the root whereby $\theta - \gamma \notin \Delta_-^{\mathfrak{g}} \; \forall \gamma \in \Delta_+^{\mathfrak{g}}$. For $\mathfrak{sl}_n$ it is not too hard to see that $\theta = \sum_{i=1}^{n-1} \alpha_i$.

Next we define $\delta := \alpha_0 + \theta$ which is also the *null root* of the affine Lie algebra and satisfies

$$
(\delta, \delta) = 0, \quad (\delta, \theta) = (\theta, \delta) = 0 \tag{2.4.28}
$$

then the root system $\Delta_+^{\widehat{\mathfrak{g}}}$ of the affine Lie algebra $\widehat{\mathfrak{g}}$ can be written down as

$$
\begin{aligned}
\Delta_+^{\widehat{\mathfrak{g}}} &= \{\gamma + m\delta \mid \gamma \in \Delta_+^{\mathfrak{g}}, \; m \in \mathbb{Z}^+\} \cup \{m\delta \mid m \in \mathbb{Z}^+\} \\
&\cup \{(\delta - \gamma) + m\delta \mid \gamma \in \Delta_+^{\mathfrak{g}}, \; m \in \mathbb{Z}^+\}
\end{aligned} \tag{2.4.29}
$$

and similarly for the negative root system $\Delta_+^{\widehat{\mathfrak{g}}}$.

The corresponding root vectors can be constructed similar to (2.4.26) for the Lie algebra but we are interested in analogous elements of the quantum group which reduce to the Lie algebra root vectors in the limit $q \to 1$. Therefore we will only give the construction in the quantum case. To do this, we must first choose an ordering of the positive roots in (2.4.29). We choose the *normal* ordering

$$
\alpha \prec \alpha + \beta \prec \beta, \quad \forall \alpha, \beta \in \Delta_+^{\mathfrak{g}}, \tag{2.4.30}
$$

$$\gamma + m\delta \prec k\delta \prec (\delta - \gamma) + l\delta, \quad \forall \gamma \in \Delta_+^{\mathfrak{g}}. \tag{2.4.31}$$

For root vectors E_α, E_β in the quantum group we define the adjoint action and q-commutator in quantum group by

$$(\mathrm{ad}_q E_\alpha)(E_\beta) := [E_\alpha, E_\beta]_q := E_\alpha E_\beta - q^{(\alpha,\beta)} E_\beta E_\alpha. \tag{2.4.32}$$

The root vectors are constructed from the generators $\{E_i\}_{i=0}^{n-1}$ in (2.4.13) by first taking

$$E_{\delta-\theta} := E_0, \quad E_{\alpha_i} := E_i \text{ for } 1 \le i \le n-1, \tag{2.4.33}$$

$$E_\gamma := [E_\alpha, E_\beta]_q \quad \alpha + \beta = \gamma \in \Delta_+^{\mathfrak{g}}, \tag{2.4.34}$$

$$E_{\delta-\gamma} := [E_{\theta-\gamma}, E_{\delta-\theta}]_q \quad \gamma \in \Delta_+^{\mathfrak{g}} \tag{2.4.35}$$

where for E_γ we use α, β such that there are no closer roots α', β' such that $\alpha' + \beta' = \gamma$ in the ordering (2.4.30). The construction of root vectors involving $m\delta$ is more complicated. First we define $\bar{E}_{\delta,\gamma}$ by

$$\bar{E}_{\delta,\gamma} := [(\gamma,\gamma)]_q^{-1} [E_\gamma, E_{\delta-\gamma}]_q \tag{2.4.36}$$

then we have

$$E_{\gamma+m\delta} := (-1)^n \left(\mathrm{ad}_q \bar{E}_{\delta,\gamma}\right)^m (E_\gamma), \tag{2.4.37}$$

$$E_{(\delta-\gamma)+m\delta} := \left(\mathrm{ad}_q \bar{E}_{\delta,\gamma}\right)^m (E_{\delta-\gamma}), \tag{2.4.38}$$

$$\bar{E}_{m\delta,\gamma} := [(\gamma,\gamma)]_q^{-1} [E_{\gamma+(m-1)\delta}, E_{\delta-\gamma}]_q. \tag{2.4.39}$$

Finally, the root vectors $E_{m\delta,\gamma}$ are defined by the relation

$$\bar{E}_{m\delta,\gamma} = \sum_{p_1+2p_2\cdots+mp_m} \frac{\left(q^{(\gamma,\gamma)} - q^{-(\gamma,\gamma)}\right)^{\sum_i p_i - 1}}{p_1! p_2! \ldots p_m!} E_{\delta,\gamma}^{p_1} E_{2\delta,\gamma}^{p_2} \ldots E_{n\delta,\gamma}^{p_n}. \tag{2.4.40}$$

The construction for the negative root vectors follows in a similar fashion. It can be obtained as the image of the above construction under the well known *Cartan anti-involution* ω defined by

$$\omega(E_\gamma) = F_\gamma, \quad \omega(F_\gamma) = E_\gamma, \quad \omega(q^{H_\gamma}) = q^{-H_\gamma}, \quad \omega(q) = q^{-1}. \tag{2.4.41}$$

It is clear given $\mathcal{R}$ in (2.4.5) that many matrix solutions $\mathcal{L}$ in (2.4.4) and hence $\mathcal{S}$ in (2.3.22) must obey the relations of the $U_q(\mathfrak{sl}_n)$ algebra. In particular, these matrices must be built from representations of the algebra. In principle, if the representation theory of the underlying algebra is understood then matrix solutions to (2.4.4) can be constructed.

Of course, the RLL-relation (2.4.4) is a weaker form of the Yang-Baxter equation (2.3.25) which is what we are interested in solving. Therefore solutions $\mathcal{L}$ must also describe the R-Operator solutions $\mathcal{R}$ and therefore we expect the quantum group $U_q(\mathfrak{sl}_n)$ also describes the structure of the R-matrix. In deriving the L-operator algebra $\mathcal{R}_L$ (2.4.7)-(2.4.7g) each side of (2.4.4) acting in the third space V_3 of $V_1 \otimes V_2 \otimes V_3$ was left as an abstract operator. In trying to solve (2.3.25) we could do the same thing instead with spaces V_1 or V_2 and derive the same algebra relations due to the symmetry of the equation. Therefore an operator $\mathcal{R} \in \mathrm{End}(V \otimes U)$ solving (2.3.25) acts with same quantum group structure in both spaces, and is some representation of an element $\mathscr{R} \in U_q(\widehat{\mathfrak{sl}_n}) \otimes U_q(\widehat{\mathfrak{sl}_n})$ which often referred to as the **universal R-matrix**. That is,

$$\mathcal{R} = (\pi_V \otimes \pi_W)(\mathscr{R}) \tag{2.4.42}$$

where $\pi_V : U_q(\widehat{\mathfrak{sl}_n}) \to \mathrm{End}(V)$ is a representation associates each element of the quantum group with a linear transformation on a *representation space* V. The Cartan-Weyl basis construction is useful because it turns out they 'building blocks' of the universal R-matrix - it can be written down explicitly in terms of the root vectors, which we will do in the next section.

In this section we started with an R-matrix (2.4.5) and showed that it has underlying $U_q(\widehat{\mathfrak{sl}_n})$ structure. It turns out that this matrix is only one instance of a large family of matrices encapsulated by the $U_q(\widehat{\mathfrak{sl}_n})$ universal R-matrix $\mathscr{R}$. It corresponds to the *fundamental symmetric tensor representation* of $\mathscr{R}$ which we will elaborate upon further when we discuss the representation theory of $U_q(\widehat{\mathfrak{sl}_n})$.

It seems that we can construct many new matrix solutions to the Yang-Baxter equation by starting with a universal R-matrix and 'simply' evaluating it for different representations as in (2.4.42). In principle this is doable for $U_q(\widehat{\mathfrak{sl}_n})$ as the universal R-matrix is known and the representation theory is well understood. Unfortunately the expression for the universal R-matrix is quite complicated and direct evaluation has so far only been successful for small n in low dimensional representations [1; 2]. A major result of this thesis is the construction of matrix solutions for all n and for higher finite dimensional and infinite dimensional representations. We do this by exploring some different methods of constructing R-matrices and show they are more efficient than direct evaluation of the universal R-matrix. Regardless, we find it appropriate to give the universal R-matrix and a brief overview of its construction. To do that we introduce some extra structure enjoyed by $U_q(\widehat{\mathfrak{sl}_n})$ that makes it an example of a quasitriangular Hopf algebra.

Quasitriangular Hopf Algebras

Quasitriangular Hopf algebra structure was formally introduced by Drinfeld [51] to describe extra properties of Hopf algebras that admit a universal R-matrix. That is, a Hopf algebra $\mathcal{A}$ is quasitriangular if there exists an element $\mathscr{R} = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}$ such that

$$\mathscr{R}\Delta(g) = \Delta^{op}(g)\mathscr{R} \quad \forall g \in \mathcal{A}, \tag{2.4.43}$$

$$(\Delta \otimes id)(\mathcal{R}) = \mathscr{R}_{13}\mathscr{R}_{23}, \tag{2.4.44}$$

$$(id \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}. \tag{2.4.45}$$

(2.4.43)-(2.4.45) are necessary and sufficient conditions for $\mathscr{R}$ to satisfy the Yang-Baxter equation

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}. \tag{2.4.46}$$

Note that this is an equality of two expressions in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ which is technically different to the equality (2.3.25) of multilinear transformations on a vector space. Of course they are related via a representation map as in (2.4.42). It also does not have any dependence on spectral parameters λ, μ, although for some quantum groups it can be introduced with a little more effort. An R-matrix without a spectral parameter is said to satisfy the *non-parameterised Yang-Baxter equation*.

The problem of constructing the universal R-matrix for a given algebra has been considered by many researchers. The first success was due to Drinfeld [51] who gave an explicit formula in the $U_q(\mathfrak{sl}_2)$ case using the *quantum double construction* technique. This was generalised to $U_q(\mathfrak{sl}_n)$ by Rosso [54] and $U_q(\mathfrak{g})$, $\mathfrak{g}$ any finite-dimensional simple Lie algebra by Kirillov & Reshetikhin [55]. The universal R-matrix for untwisted affine quantum groups was first given by Tolstoy and Khoroshkin [3]. Their formula for the arbitrary $U_q(\widehat{\mathfrak{g}})$ case reads

$$\mathscr{R}^{U_q(\widehat{\mathfrak{g}})} = \left[\prod_{m \in \mathbb{Z}^+, \gamma \in \Delta_+^{\mathfrak{g}}} \exp_{q_\gamma} \left(C_{\gamma+m\delta} E_{\gamma+m\delta} \otimes F_{\gamma+m\delta} \right) \right]$$

$$\times \exp\left(\sum_{m\in\mathbb{Z}^+}\sum_{i,j=1}^{r} S_{m,ij} E_{m\delta,\alpha_i}\otimes F_{m\delta,\alpha_j}\right)$$
$$\times \left[\prod_{m\in\mathbb{Z}^+,\gamma\in\Delta_+^{\mathfrak{g}}} \exp_{q_\gamma}\left(C_{(\delta-\gamma)+m\delta}E_{(\delta-\gamma)+m\delta}\otimes F_{(\delta-\gamma)+m\delta}\right)\right]$$
$$\times \exp\left(\hbar\sum_{i,j=1}^{r} b_{ij} h_{\alpha_i}\otimes h_{\alpha_j}\right) \tag{2.4.47}$$

where r is the rank of the Cartan matrix $A_{\mathfrak{g}}$ and the factors $C_\gamma, S_{m,ij}, b_{ij}$ are determined by the relations

$$[E_\gamma, F_\gamma] = \frac{q^{H_\gamma}-q^{-H_\gamma}}{C_\gamma}, \quad b_{ij} = \left(A_{\mathfrak{g}}^{-1}\right)_{ij}, \quad q = \exp(\hbar), \tag{2.4.48}$$

$$S_{m,ij} = \left(T_m^{-1}\right)_{ij}, \quad T_{m,ij} = (-1)^{m\delta_{ij}} m^{-1}\frac{q^{m(\alpha_i,\alpha_j)}-q^{-m(\alpha_i,\alpha_j)}}{(q_{\alpha_i}-q_{\alpha_i}^{-1})(q_{\alpha_j}-q_{\alpha_j}^{-1})}. \tag{2.4.49}$$

The ordering of the products follows the normal ordering of the roots given in (2.4.31). For example, in the case of $U_q(\widehat{\mathfrak{sl}_2})$ the Khoroshkin-Tolstoy formula for the universal R-matrix reads

$$\mathscr{R}^{U_q(\widehat{\mathfrak{sl}_2})} = \left(\prod_{n\geq 0}\exp_{q_\alpha}((q-q^{-1})(E_{\alpha+n\delta}\otimes F_{\alpha+n\delta}))\right)$$
$$\times \exp\left(\sum_{n\geq 0} n[n]_{q_\alpha}^{-1}(q_\alpha - q_\alpha^{-1})(E_{n\delta}\otimes F_{n\delta})\right)$$
$$\times \left(\prod_{n\geq 0}\exp_{q_\alpha}((q-q^{-1})(E_{(\delta-\alpha)+n\delta}\otimes F_{(\delta-\alpha)+n\delta}))\right) q^{\frac{1}{2}H_\alpha\otimes H_\alpha}. \tag{2.4.50}$$

As mentioned earlier, the universal R-matrix does not typically have a spectral parameter. For $U_q(\widehat{\mathfrak{sl}_n})$ a spectral parameter can be introduced by the well known *evaluation homomorphism* map [58] $\mathrm{eval}_x : U_q(\widehat{\mathfrak{sl}_n}) \to U_q(\mathfrak{sl}_n)\otimes\mathbb{C}[x,x^{-1}]$ by

$$\mathscr{R}(x,y) = (\mathrm{eval}_x\otimes\mathrm{eval}_y)(\mathscr{R}), \tag{2.4.51}$$

$$\mathrm{eval}_x(E_0) = xGF_\theta, \quad \mathrm{eval}_x(F_0) = x^{-1}G^{-1}E_\theta, \quad \mathrm{eval}_x(q^{H_0}) = q^{-H_\theta},$$
$$\mathrm{eval}_x(E_i) = E_i, \quad \mathrm{eval}_x(F_i) = F_i, \quad \mathrm{eval}_x(q^{H_i}) = q^{H_i}, \quad 1\leq i\leq n-1 \tag{2.4.52}$$

where $G := q^{\frac{1}{n}\sum_{i=1}^{n-1}(n-2i)H_i}$. Applying (2.4.51) to (2.4.47) one can check that the universal R-matrix with spectral parameter depends only on the difference of the parameters such that $\mathscr{R}(x,y) = \mathscr{R}(x/y)$. Such an element, by construction, is a solution to the parameterised Yang-Baxter equation

$$\mathscr{R}_{12}(\lambda)\mathscr{R}_{13}(\lambda\mu)\mathscr{R}_{23}(\mu) = \mathscr{R}_{23}(\mu)\mathscr{R}_{13}(\lambda\mu)\mathscr{R}_{12}(\lambda). \tag{2.4.53}$$

Now that we have a formula (2.4.47), (2.4.51) for a universal R-matrix with spectral parameter we could evaluate it directly for a given representation as in (2.4.42). As mentioned earlier this is a problem that has not been solved in general. In this thesis we will show that there are many other ways to evaluate this quantum group formula that are not only more efficient but reveal interesting structure of the R-matrix that is not obvious by focusing solely on the Khoroskin-Tolstoy construction.

In the next section we will discuss the representation theory of $U_q(\mathfrak{sl}_n)$ in the generic q case which is well understood. The classification of all representations of the algebra gives a classification of all matrix realisations of $\mathscr{R}(\lambda)$ and hence solutions to the Yang-Baxter equation with $\mathfrak{sl}_n$ structure.

Representation Theory

Given the isomorphism $\widehat{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathbb{C}[x, x^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ well known in the study of non-twisted affine Kac-Moody Lie algebras, one asks if a similar isomorphism exists for quantum groups $U_q(\widehat{\mathfrak{g}})$. Unfortunately it does not, except only for $\widehat{\mathfrak{sl}_n}$ where the evaluation homomorphism map (2.4.52) shows that

$$U_q(\widehat{\mathfrak{sl}_n}) \cong U_q(\mathfrak{sl}_n) \otimes \mathbb{C}[x, x^{-1}] \tag{2.4.54}$$

where the left-hand side is as we have presented it in (2.4.13) ignoring the elements c and d. This means that the extra generators $\{E_0, F_0, H_0\}$ of $U_q(\widehat{\mathfrak{sl}_n})$ exist inside $U_q(\mathfrak{sl}_n) \otimes \mathbb{C}[x, x^{-1}]$ and that representations $\widehat{\pi_V} : U_q(\widehat{\mathfrak{sl}_n}) \to \mathrm{End}(V)$ can be constructed by considering representations $\pi_V : U_q(\mathfrak{sl}_n) \to \mathrm{End}(V)$ of the non-affine algebra and pulling back by eval_x. That is,

$$\widehat{\pi_V} = \pi_V \circ \mathrm{eval}_x, \tag{2.4.55}$$

and so in discussing the $U_q(\mathfrak{sl}_n)$ R-matrix with spectral parameter one is really taking about the $U_q(\widehat{\mathfrak{sl}_n})$ R-matrix. The only other parameters needed to specify an R-matrix are the parameters fixing the underlying quantum group representation, and in this case the non-affine quantum group. We will not say anything about the representation theory of $U_q(\widehat{\mathfrak{g}})$ in general other than that irreducible representations can be constructed by "combining" irreducible representations of the non-affine quantum group in a manner that is highly technical and non-trivial.

We will assume that q is not a root of unity. In this case the irreducible representations of $U_q(\mathfrak{sl}_n)$ are classified by *highest weight representations*. These are determined by the linear functional

$$\boldsymbol{I} = \sum_{s=1}^{n-1} I_s \omega_s, \quad I_s \in \mathbb{C} \tag{2.4.56}$$

on the Cartan subalgebra $\mathfrak{h}$ where ω_s have the the action

$$\omega_i(H_j) = \delta_{i,j}. \tag{2.4.57}$$

For now we choose to only focus on the theory for the case $\boldsymbol{I} = I\omega_1$ for $I \in \mathbb{Z}^+$ also known as *symmetric tensor representations* and from now on we will refer to a specific representation by its single parameter 'I' or map π^I, and its underlying representation space V_I. This is because much of the thesis is dedicated to R-matrices in these representations. The theory in this case is treated explicitly in Chapter 5 of [59] where the underlying space is characterised as the subspace of homogenous polynomials in the ring $\mathbb{C}[X_1, X_2, \dots, X_n]$ with basis

$$|m_1, m_2, \dots, m_n\rangle = \frac{X_1^{m_1}}{[m_1]_q!} \frac{X_2^{m_2}}{[m_2]_q!} \cdots \frac{X_n^{m_n}}{[m_n]_q!}, \tag{2.4.58}$$

such that $I \geq m_1, m_2, \dots, m_n \geq 0$ and $\sum_s m_s = I$. This basis generates a space of dimension $\binom{I+n-1}{n-1}$ which we denote by V_I. Quantum group generators $\bar{K}_i^{\pm}, \bar{E}_i, \bar{F}_i, i = 1, 2, \dots n$ presented in [59] act on this basis by

$$\bar{K}_i \, |m_1, m_2, \dots, m_n\rangle = q^{m_i - m_{i+1}} \, |m_1, m_2, \dots, m_n\rangle \,, \tag{2.4.59a}$$

$$\bar{E}_i \, |m_1, m_2, \dots, m_n\rangle = H(m_{i+1})[m_i + 1]_q \, |m_1, \dots, m_i + 1, m_{i+1} - 1, \dots, m_3\rangle \,, \tag{2.4.59b}$$

$$\bar{F}_i \, |m_1, m_2, \dots, m_n\rangle = H(m_i)[m_{i+1} + 1]_q \, |m_1, \dots, m_i - 1, m_{i+1} + 1, \dots, m_n\rangle \,, \tag{2.4.59c}$$

where $H(i) = 0$ for $i \leq 0$ and 1 otherwise. The highest weight vector is $|I, 0, \dots, 0\rangle$, annihilated by the raising operators $\bar{E}_i$.

We will also use the same notation as above in the case $I \in \mathbb{C}$ where the representation space is an infinite-dimensional *Verma module*. Generally speaking, all highest weight representations can be characterised as Verma modules containing a finite-dimensional subrepresentation only when the weight is integral and dominant.

Some results have also been achieved for arbitrary highest weight representations in chapter 5 and we will introduce the theory there as we need it. A complete explicit treatment of all irreducible representations of $U_q(\mathfrak{sl}_n)$ is given in [60].

Fixing a representation of the algebra and a basis for the underlying vector space, one can realise the universal R-matrix explicitly as a matrix. We denote the matrix realisation acting on $V_I \otimes V_J$ by

$$R_{I,J}(\lambda) = (\pi^I \otimes \pi^J)\mathscr{R}(\lambda). \tag{2.4.60}$$

All of representation theory in this case is a 'deformation' of the $\mathfrak{sl}_n$ representation theory over $\mathbb{C}$ which can all be obtained from the above by taking the limit $q \to 1$.

Chapter 3

A 3D Integrable Model

In this thesis we are primarily concerned with how to solve the Yang-Baxter equation. We showed in the introduction that this equation arises as a sufficiency condition for a 2-dimensional classical lattice (or 1-dimensional quantum chain)model in statistical mechanics to be solvable. Of course, from a physical point of view what we would really like to do is model the behaviour of lattice structures in three dimensions. Just like the 2D case we ask under what conditions we can compute the partition function of a 3D lattice model. We can follow the same logic of commuting transfer matrices as in the 2D case to derive a generalised 3D Yang-Baxter equation - the famous *Zamolodchikov Tetrahedron equation* [38; 39]. Understanding this equation is an active area of research and some solutions are known.

In addition to 3D integrability a solution to the tetrahedron equation also describes an infinite family of solvable 2D lattice models. That is, a 3D lattice can be interpreted as a 2D lattice whereby the third direction represents an increased set of states/degrees of freedom for each edge on the 2D lattice. The tetrahedron equation guarantees that this 2D *composite weight* satisfies the Yang-Baxter equation and hence describes some R-matrix. The size of the 3D lattice in this third direction is connected with the rank of the algebra behind the 2D lattice through *rank-size duality* as it is known in the literature. In this way the size parametrises an infinite family of solvable 2D lattice models.

In this chapter we will review the construction of a solution recently obtained by Bazhanov, Mangazeev and Sergeev in [9]. Using this solution we take an n-layer projection in a third direction and extract a formula for the elements of a R-Matrix, which we show is the $U_q(\widehat{sl_n})$ R-matrix for symmetric tensor representations. We will then study this formula and simplify it further to give what we believe is the simplest expression for the matrix elements in this case. We will analyse this formula by discussing

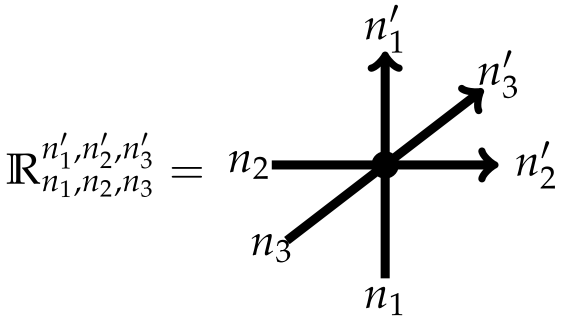

Figure 3.1: Matrix elements for an operator R given by vertex configurations.

its symmetries, degenerations and show that it factorises into a neat form which has interesting structure which we believe deserves further study.

3.1 The tetrahedron equation

Like the 2D models considered in this thesis, the 3D model considered in this chapter is formulated as an edge-spin model where the unit cell of the lattice is the vertex given by R in Figure 3.1. For this kind of model the tetrahedron equation is written as follows

Definition 3.1.0.1 (Tetrahedron Equation)**.** The linear operator equality

$$\mathbf{R}_{123}\mathbf{R}_{145}\mathbf{R}_{246}\mathbf{R}_{356} = \mathbf{R}_{356}\mathbf{R}_{246}\mathbf{R}_{145}\mathbf{R}_{123}, \tag{3.1.1}$$

acting on the vector space $V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5 \otimes V_6$ is the tetrahedron equation. A solution $\mathbf{R}$ is a linear 'R-operator' acting on $W \otimes U \otimes V$ such that $\mathbf{R}_{ijk}$ acts as $W = V_i, U = V_j, V = V_k$ and trivially in the other spaces.

Of course, as in the 2D case the operator form (3.1.1) can be expressed as a matrix equation once a basis for each space is fixed. Supposing there is an orthonormal basis $\{|n_1, n_2, n_3\rangle\}_{n_i \in \mathbb{Z}_{\geq 0}}$ for the space $V \otimes U \otimes W$ then $\mathbf{R}$ has matrix elements $\mathbb{R}^{n_1',n_2',n_3'}_{n_1,n_2,n_3}$ determined by the action

$$\langle n_1, n_2, n_3 |\, \mathbf{R} \,| n_1', n_2', n_3' \rangle = \mathbb{R}^{n_1',n_2',n_3'}_{n_1,n_2,n_3}, \tag{3.1.2}$$

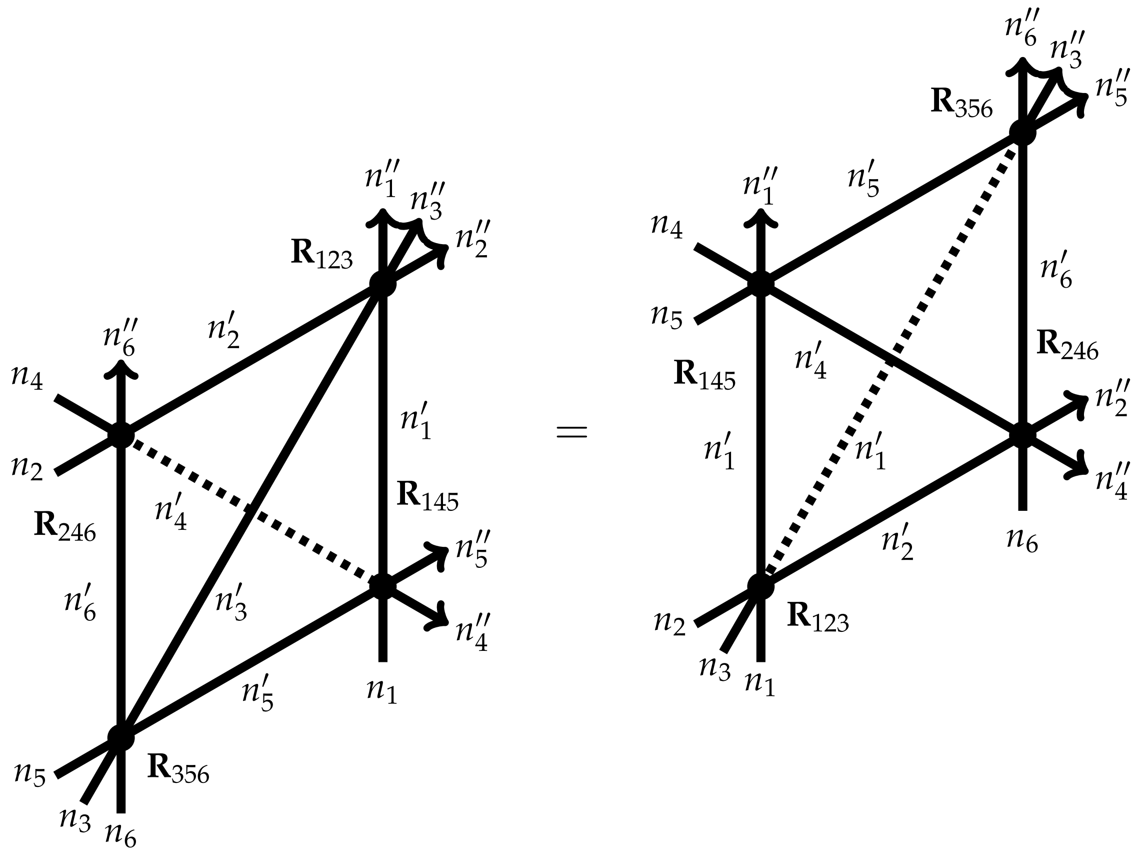

Figure 3.2: Tetrahedron equation in vertex form

and the tetrahedron equation (3.1.1) can be written as

$$\sum_{\substack{n_1',n_2',n_3' \\ n_4',n_5',n_6'}} \mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'} \mathbb{R}_{n_1',n_4,n_5}^{n_1'',n_4',n_5'} \mathbb{R}_{n_2',n_4',n_6}^{n_2'',n_4'',n_6'} \mathbb{R}_{n_3',n_5',n_6'}^{n_3'',n_5'',n_6''} = \sum_{\substack{n_1',n_2',n_3' \\ n_4',n_5',n_6'}} \mathbb{R}_{n_3,n_5,n_6}^{n_3',n_5',n_6'} \mathbb{R}_{n_2,n_4,n_6'}^{n_2',n_4',n_6''} \mathbb{R}_{n_1,n_4',n_5'}^{n_1',n_4'',n_5''} \mathbb{R}_{n_1',n_2',n_3'}^{n_1'',n_2'',n_3''}. \tag{3.1.3}$$

The equations (3.1.1) and (3.1.3) are represented graphically in Figure (3.2). One can see that the polyhedron contained by the four vertices is a tetrahedron. Starting from one side of the equation the other side can be obtained by selecting a vertex and 'moving through' the opposite face. In this process each face of the tetrahedron undergoes a Yang-Baxter transformation similar to Figure 2.6. This transformation of the faces allows us to interpret a 3D lattice as a 2D lattice with the third direction a decomposition of the states at each 2D lattice site. We will elaborate on this later.

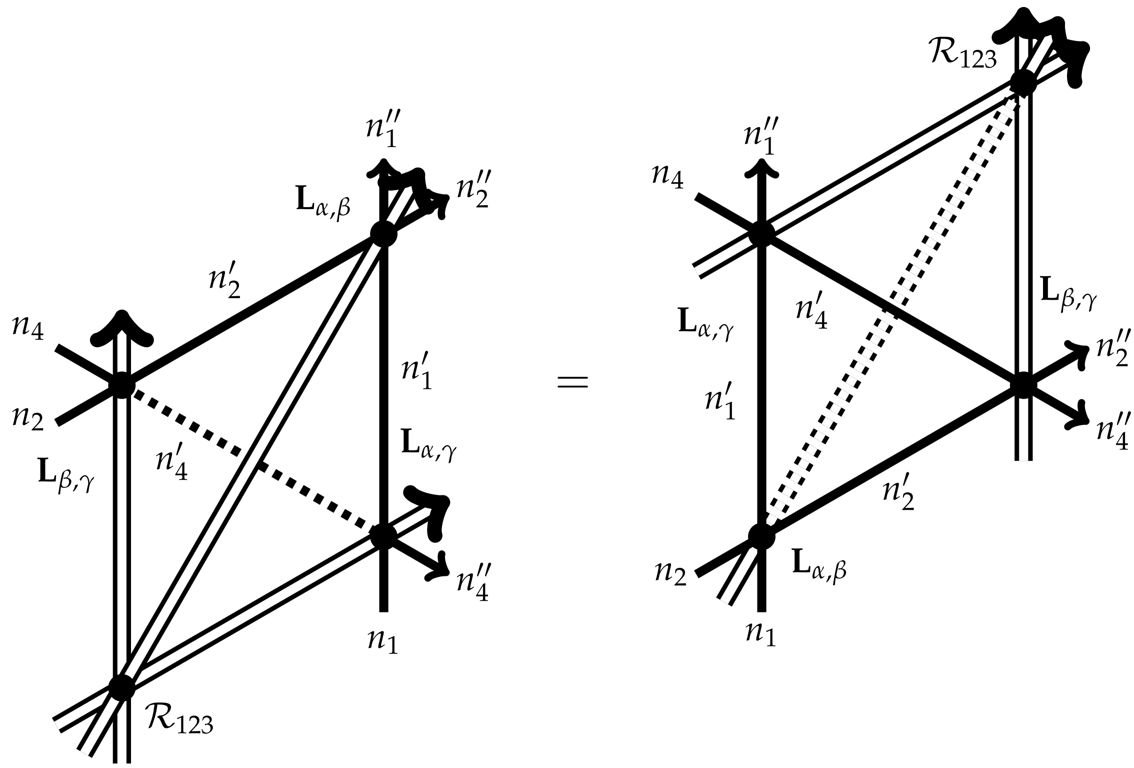

Figure 3.3: Tetrahedron RLLL relation

3.2 A solution with positive Boltzmann weights

In this section we review a solution to the tetrahedron equation originally found by Bazhanov, Mangazeev and Sergeev. The method involves the use of a functional form of the tetrahedron equation [7] whereby we consider the action of the map $\mathcal{R}_{123}$ on the q-oscillator algebra and under extra assumptions construct representations of it.

Just like in the 2D case where we considered a generalised RLL-Yang Baxter equation (2.4.4) represented graphically in Figure 2.7, we consider a generalised $RLLL$-tetrahedron equation of the form

$$\mathbf{L}_{\alpha,\beta} \boldsymbol{L}_{\alpha,\gamma} \boldsymbol{L}_{\beta,\gamma} \boldsymbol{R}_{123} = \mathbf{R}_{123} \boldsymbol{L}_{\beta,\gamma} \boldsymbol{L}_{\alpha,\gamma} \boldsymbol{L}_{\alpha,\beta} \tag{3.2.1}$$

illustrated in Figure 3.3. These L-operators are different to those used in (2.4.4). There R was already given explicitly and the remaining space was left as a representation space of L-operator algebra (2.4.7). In this case $\mathbf{R}$ is not known and we will consider the space it acts in(white lines in Figure 3.3) abstractly with (3.3) determining relations between the operators from each vertex. The tetrahedron equation (3.2) ensures that the L-operator algebra determined by (3.3) is associative.

For the solution constructed in this chapter we will consider an algebra constructed from a q-oscillator algebra placed at each vertex $\boldsymbol{L}$.

3.2.1 q-oscillator algebra and Fock space representations

Definition 3.2.0.1 (q-oscillator algebra Osc_q)**.** The q-oscillator algebra is generated by $\{\boldsymbol{k}, \boldsymbol{a}^+, \boldsymbol{a}^-\}$ over $\mathbb{C}(q, q^{-1})$ subject to the relations

$$\boldsymbol{k}\boldsymbol{a}^{\pm} = q^{\pm 1}\boldsymbol{a}^{\pm}\boldsymbol{k}, \quad q\boldsymbol{a}^+\boldsymbol{a}^- - q^{-1}\boldsymbol{a}^-\boldsymbol{a}^+ = q - q^{-1}, \tag{3.2.2a}$$

$$\boldsymbol{k}^2 = q(1 - \boldsymbol{a}^+\boldsymbol{a}^-) = q^{-1}(1 - \boldsymbol{a}^-\boldsymbol{a}^+). \tag{3.2.2b}$$

The q-oscillator algebra is a 'building block' of the solution to the tetrahedron equation we construct in this section. Therefore we also need to consider its representations. In particular, we will consider the infinite-dimensional irreducible anti-Fock space representation $\mathcal{F}_q^-$ spanned by $\{|n\rangle\}_{n\in\mathbb{Z}_{\geq 0}}$ with scalar product

$$\langle m|n\rangle = \delta_{n,m}, \quad \boldsymbol{N}\,|n\rangle = n\,|n\rangle\,, \quad \langle n|\,\boldsymbol{N} = \langle n|\,n \tag{3.2.3}$$

where $\boldsymbol{N}$ is the *occupation number* operator. The q-oscillator algebra action on this space is given by

$$\boldsymbol{k} = q^{-\boldsymbol{N}-1/2}, \tag{3.2.4a}$$

$$\boldsymbol{a}^+\,|0\rangle = 0, \quad \boldsymbol{a}^-\,|n\rangle = |n+1\rangle\,, \quad \boldsymbol{a}^+\,|n\rangle = (1 - q^{-2n})\,|n-1\rangle\,, \tag{3.2.4b}$$

$$\langle 0|\,\boldsymbol{a}^- = 0, \quad \langle n|\,\boldsymbol{a}^- = \langle n-1|\,, \quad \langle n|\,\boldsymbol{a}^- = \langle n+1|\,(1 - q^{-2-2n}). \tag{3.2.4c}$$

It is called the anti-Fock space representation because $\boldsymbol{a}^+$ annihilates the vacuum state $|0\rangle$. In the next section we use this representation to construct a solution to (3.1.1).

3.2.2 Functional Tetrahedron equation

The remaining spaces (denoted as black lines in Figure 3.3) are identified with $\mathbb{C}^2$. In particular, we choose **L** [9] as

$$\boldsymbol{L}(\boldsymbol{k},\boldsymbol{a}^{\pm}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \boldsymbol{k} & \boldsymbol{a}^{+} & 0 \\ 0 & \boldsymbol{a}^{-} & -\boldsymbol{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2.5}$$

acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$ with operator valued entries. Entries in (3.2.5) are indexed such that the indices for the second space index 2×2 blocks while indices for the first space index the entries in each block. In (3.3) we label these two-dimensional spaces by α, β and γ and label the $\boldsymbol{L}$ operators appropriately.

Now we define the map $\mathcal{R}_{123} : \mathrm{Osc}_q \otimes \mathrm{Osc}_q \otimes \mathrm{Osc}_q \to \mathrm{Osc}_q \otimes \mathrm{Osc}_q \otimes \mathrm{Osc}_q$ by the inner automorphism

$$\mathcal{R}_{123}(\boldsymbol{x}) = \mathbf{R}_{123} x \mathbf{R}_{123}^{-1} \quad \forall \boldsymbol{x} \in \mathrm{Osc}_q \otimes \mathrm{Osc}_q \otimes \mathrm{Osc}_q, \tag{3.2.6}$$

$$\mathcal{R}_{123}(\boldsymbol{a}_i^{\pm}) = \boldsymbol{a}_i'^{\pm}, \quad \mathcal{R}_{123}(\boldsymbol{k}_i) = \boldsymbol{k}_i', \quad i = 1, 2, 3. \tag{3.2.7}$$

Then we can rewrite (3.2.1) as

$$\begin{aligned} \boldsymbol{L}_{\alpha,\beta}(\boldsymbol{k}_1,\boldsymbol{a}_1^{\pm})\boldsymbol{L}_{\alpha,\gamma}(\boldsymbol{k}_2,\boldsymbol{a}_2^{\pm})\boldsymbol{L}_{\beta,\gamma}(\boldsymbol{k}_3,\boldsymbol{a}_3^{\pm}) &= \mathcal{R}_{123}\left(\boldsymbol{L}_{\beta,\gamma}(\boldsymbol{k}_3,\boldsymbol{a}_3^{\pm})\boldsymbol{L}_{\alpha,\gamma}(\boldsymbol{k}_2,\boldsymbol{a}_2^{\pm})\boldsymbol{L}_{\alpha,\beta}(\boldsymbol{k}_1,\boldsymbol{a}_1^{\pm})\right) \\ &= \boldsymbol{L}_{\beta,\gamma}(\boldsymbol{k}_3',\boldsymbol{a}_3'^{\pm})\boldsymbol{L}_{\alpha,\gamma}(\boldsymbol{k}_2',\boldsymbol{a}_2'^{\pm})\boldsymbol{L}_{\alpha,\beta}(\boldsymbol{k}_1',\boldsymbol{a}_1'^{\pm}). \end{aligned} \tag{3.2.8}$$

Substituting (3.2.5) into (3.2.8) we find that $\mathcal{R}_{123}$ is determined by the relations

$$\boldsymbol{k}_2'\boldsymbol{a}_1'^{\pm} = \boldsymbol{k}_3\boldsymbol{a}_1^{\pm} + \boldsymbol{k}_1\boldsymbol{a}_2^{\pm}\boldsymbol{a}_3^{\mp}, \tag{3.2.9a}$$

$$\boldsymbol{a}_2'^{\pm} = \boldsymbol{a}_1^{\pm}\boldsymbol{a}_3^{\pm} + \boldsymbol{k}_1\boldsymbol{k}_3\boldsymbol{a}_2^{\pm}, \tag{3.2.9b}$$

$$\boldsymbol{k}_2'\boldsymbol{a}_3'^{\pm} = \boldsymbol{k}_1\boldsymbol{a}_3^{\pm} + \boldsymbol{k}_3\boldsymbol{a}_1^{\mp}\boldsymbol{a}_2^{\mp}, \tag{3.2.9c}$$

$$\boldsymbol{k}_1'\boldsymbol{k}_2' = \boldsymbol{k}_1\boldsymbol{k}_2, \quad \boldsymbol{k}_2'\boldsymbol{k}_3' = \boldsymbol{k}_2\boldsymbol{k}_3, \tag{3.2.9d}$$

$$\begin{aligned} \left(\boldsymbol{k}_2'\right)^2 &= \boldsymbol{k}_1^2\boldsymbol{k}_2^2\boldsymbol{k}_3^2 + \boldsymbol{k}_1\boldsymbol{k}_3\left(q^{-1}\boldsymbol{a}_1^{+}\boldsymbol{a}_2^{-}\boldsymbol{a}_3^{+} + q\boldsymbol{a}_1^{-}\boldsymbol{a}_2^{+}\boldsymbol{a}_3^{-}\right) \\ &\quad + \boldsymbol{k}_1^2 + \boldsymbol{k}_3^2 - (q+q^{-1})\boldsymbol{k}_1^2\boldsymbol{k}_3^2. \end{aligned} \tag{3.2.9e}$$

Applying the representation (3.2.4) to the these relations one can determine recurrence relations for $\mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'}$ using (3.2.6), (3.2.7) and (3.1.2). Doing so we find

$$\mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'}\frac{q^{n_2(n_2+1)-(n_2-n_1')(n_2-n_3')}}{(q^2;q^2)_{n_2}}Q_{n_2}(x,y,z) \quad (3.2.10)$$

where $x = q^{-2n_1'}, y = q^{-2n_2'}, z = q^{-2n_3'}$ and

$$Q_{n+1}(x,y,z) = (x-1)(z-1)Q_n(xq^2,y,zq^2) + xz(y-1)q^{2n}Q_n(x,yq^2,z), \quad (3.2.11)$$

$$Q_0(x,y,z) = Q_0(xq^{-2},y,z) = Q_0(x,yq^{-2},z) = Q_0(x,y,zq^{-2}) \quad (3.2.12)$$

Choosing the normalisation $\mathbb{R}_{0,0,0}^{0,0,0} = 1$, equation (3.2.12) allows us to set $Q_0(x,y,z) = 1 \quad \forall x,y,z \in \{q^{-2n}\}_{n\in Z_{\geq 0}}$. Using this to solve the recurrence relation (3.2.11) we find $\mathbb{R}$ to be

$$\mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'}q^{-n_2(1+n_1+n_3)-n_1'n_3'} \times \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_{q^2} {}_2\phi_1\left(\begin{matrix} q^{-2n_2} & q^{-2n_1'} \\ & q^{-2n_1-2n_2} \end{matrix}\middle| q^2, q^{2(1+n_3')}\right), \quad (3.2.13)$$

solving (3.1.3) by construction. We shall refer to this matrix representation of the operator $\mathbf{R}$ as the 3D R-matrix. The formula contains q-binomials defined in (B.1.4) and a terminating basic hypergeometric series as defined in (B.1.9). We will use this notation regularly throughout the rest of the thesis.

We also notice that due to conservation laws we always have $n_2 \leq n_1+n_2$ and $n_1' \leq n_1+n_2$. As a result, for summands with a sufficiently large index, the numerator yields a double zero and the series terminates despite a simple zero in the denominator. In addition, the hypergeometric function in (3.2.13) does not require a regularisation and the range of the summation can be taken to be from 0 to $\min(n_2, n_1')$. All nonzero elements in (3.2.13) are positive for $0 < q < 1$ as explained in [9] where it was first introduced.

The R-Matrix (3.2.13) possesses a number of symmetries which are generated by two elementary ones

$$\mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \mathbb{R}_{n_3,n_2,n_1}^{n_3',n_2',n_1'}, \quad \mathbb{R}_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = q^{n_3-n_2+n_1^2-n_1'^2}\frac{(q^2;q^2)_{n_1'}}{(q^2;q^2)_{n_1}}\mathbb{R}_{n_1',n_3,n_2}^{n_1,n_3',n_2'}. \quad (3.2.14)$$

They can be proved by using Heine's transformations (B.2.6) for ${}_2\phi_1$ series . We list here two other useful symmetries which follow from (3.2.14):

$$\mathbb{R}_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} = q^{n_1-n_2+n_3^2-n_3'^2} \frac{(q^2;q^2)_{n'_3}}{(q^2;q^2)_{n_3}} \mathbb{R}_{n_2,n_1,n'_3}^{n'_2,n'_1,n_3} \tag{3.2.15}$$

and

$$\mathbb{R}_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} = q^{(n_3+n'_3+2n'_1-2n_2+1)(n_1-n'_1)} \prod_{i=1}^{3} \frac{(q^2;q^2)_{n'_i}}{(q^2;q^2)_{n_i}} \mathbb{R}_{n'_1,n'_2,n'_3}^{n_1,n_2,n_3}. \tag{3.2.16}$$

Let us notice that up to the factor $q^{-n_2(1+n_1+n_3)-n'_1n'_3}$ the expression (3.2.13) is a polynomial in $q^{2n'_3}$ and can be formally continued to negative values $n_3, n'_3 < 0$. So let us assume that $n_i, n'_i \geq 0$, $i = 1,2$ and $n_3, n'_3 \in \mathbb{Z}$ provided that all indices are still constrained by delta-functions entering (3.2.13). Then it is easy to find a transformation of matrix elements of the 3D R-matrix under the replacement $q \to q^{-1}$

$$\mathbb{R}_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3}\Big|_{q\to q^{-1}} = q^{(n_1-n'_2)(n_2-n'_2-1)} \mathbb{R}_{n_1,n_2,-n'_3-1}^{n'_1,n'_2,-n_3-1}. \tag{3.2.17}$$

Extra parameters can be added to (3.2.13) without affecting the equality of (3.1.3). In particular, the function

$$[\mathbb{R}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})]_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} = c_1^{n'_1-n_1} c_2^{n'_2-n_2} c_3^{n'_3-n_3} \left(\frac{\mu_3}{\lambda_1}\right)^{n_2} \left(\frac{\lambda_2}{\lambda_3}\right)^{n'_1} \left(\frac{\mu_1}{\mu_2}\right)^{n'_3} \mathbb{R}_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} \tag{3.2.18}$$

for triples $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}, \boldsymbol{\mu} = \{\mu_1, \mu_2, \mu_3\}$ and $\boldsymbol{c} = \{c_1, c_2, c_3\}$. The factors depending on $\boldsymbol{c}$ is just a diagonal similarity transform and it drops out of the tetrahedron trivially. The factors in $\boldsymbol{\mu}, \boldsymbol{\lambda}$ is also fairly easy to show; using the two delta functions at each vertex in (3.1.3) one can move all factors in $\boldsymbol{\mu}, \boldsymbol{\lambda}$ on the internal edges to the external edges. Comparing the external edges of both sides of the equation shows they are equal and hence the tetrahedron equation is not affected.

We will refer (3.2.18) as a 'dressed' 3D R-Matrix. These factors are important in the construction of certain objects of interest such as the layer-to-layer transfer matrix where they allow one to introduce two spectral parameters and thereby an infinite family of commuting operators for this 3D model. It also allows us to introduce a spectral parameter to a 2D R-matrix whose construction we will consider in the next section. This R-matrix related to $U_q(\widehat{sl_n})$ and using (3.2.18) we will give a new presentation for its matrix elements.

3.3 n-layer projection

We consider a 'composite weight' $\mathbb{S}$ constructed from the dressed R-Matrix (3.2.18) by multplying n copies of it together in the third space and taking the trace. That is

$$\mathbb{S}_{\mathbf{i},\mathbf{j}}^{\mathbf{i}',\mathbf{j}'} := \sum_{\boldsymbol{k}} \prod_{s=1}^{n} [\mathbb{R}(\boldsymbol{\lambda}_s, \boldsymbol{\mu}_s, \boldsymbol{c}_s)]_{j_s,i_s,k_s}^{j'_s,i'_s,k_{s+1}} \tag{3.3.1}$$

where $k_{n+1} = k_1, \boldsymbol{\lambda}_s = (\lambda_{1,s}, \lambda_{2,s}, \lambda_{3,s})$ and similarly for $\boldsymbol{\mu}_s, \boldsymbol{c}_s$. We note that the delta functions contained in each factor (3.2.13) imply the global conservation laws

$$I = I', \quad J = J', \tag{3.3.2}$$

$$I := \sum_{s=1}^{n} i_s, \quad I' := \sum_{s=1}^{n} i'_s, \quad J := \sum_{s=1}^{n} j_s, \quad J' := \sum_{s=1}^{n} j'_s. \tag{3.3.3}$$

We now consider the case where $\lambda_{3,s}$ and $\mu_{3,s}$ are equal for all s. Using these conservation laws for the indices we can rewrite $\mathbb{S}$ as

$$\mathbb{S}(w, \Phi, \Psi)_{\mathbf{i},\mathbf{j}}^{\mathbf{i}',\mathbf{j}'} = \Phi_h(\boldsymbol{i})\Phi_v(\boldsymbol{j}')\Psi_h(\boldsymbol{i}, \boldsymbol{i}')\Psi_v(\boldsymbol{i}, \boldsymbol{i}') \sum_{\boldsymbol{k}} w^{k_1} \prod_{s=1}^{n} \mathbb{R}_{j_s,i_s,k_s}^{j'_s,i'_s,k_{s+1}} \tag{3.3.4}$$

where

$$w = \prod_{s=1}^{n} \frac{\mu_{1,s}}{\mu_{2,s}} \tag{3.3.5a}$$

$$\Phi_h(\boldsymbol{i}) = \prod_{s=1}^{n-1} \phi_{h,k}^{i_k}, \quad \Phi_v(\boldsymbol{j}') = \prod_{s=1}^{n-1} \phi_{v,k}^{j'_k}, \tag{3.3.5b}$$

$$\Psi_h(\boldsymbol{i}, \boldsymbol{i}') = \prod_{s=1}^{n-1} \psi_{h,k}^{i'_k - i_k}, \quad \Psi_v(\boldsymbol{i}, \boldsymbol{i}') = \prod_{s=1}^{n-1} \psi_{v,k}^{j'_k - j_k}, \tag{3.3.5c}$$

$$\phi_{h,k} = \frac{\lambda_{1,n}}{\lambda_{1,k}}, \quad \phi_{v,k} = \frac{\lambda_{2,k}}{\lambda_{1,n}}, \tag{3.3.5d}$$

$$\psi_{h,k} = \frac{c_{2,k}}{c_{2,n}} \prod_{s=k}^{n-1} \frac{\mu_{1,s}}{\mu_{2,s}}, \quad \psi_{v,k} = \frac{c_{1,k}\lambda_{2,k}}{c_{1,n}\lambda_{1,n}}. \tag{3.3.5e}$$

Here we have also removed any factors in (3.3.1) depending solely on I, J because they are constants. The tetrahedron equation implies that this composite weight satisfies the Yang-Baxter equation

$$\sum_{\boldsymbol{i}',\boldsymbol{j}'\boldsymbol{k}'} \mathbb{S}_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'}(w, \Phi, \Psi)\mathbb{S}_{\boldsymbol{i}',\boldsymbol{k}}^{\boldsymbol{i}'',\boldsymbol{k}'}(w', \Phi', \Psi')\mathbb{S}_{\boldsymbol{j}',\boldsymbol{k}'}^{\boldsymbol{j}'',\boldsymbol{k}''}(w'/w, \Phi'', \Psi'') = \tag{3.3.6}$$

$$\sum_{i',j'k'} \mathbb{S}_{j,k}^{j',k'}(w'/w,\Phi'',\Psi'')\mathbb{S}_{i,k'}^{i',k''}(w',\Phi',\Psi')\mathbb{S}_{i',j'}^{i'',j''}(w,\Phi,\Psi)$$

provided that the functions $\{\Phi,\Psi,\Phi',\Psi',\Phi'',\Psi''\}$ are constrained by the relations

$$\phi_{v,k}=\phi'_{v,k},\quad \phi'_{h,k}=\phi''_{h,k},\quad \phi''_{v,k}=\phi^{-1}_{h,k},\quad \psi''_{v,k}=\frac{\psi_{h,k}\psi'_{v,k}\psi''_{h,k}}{\phi_{h,k}\psi_{v,k}\psi'_{h,k}}\quad \forall k=1,\dots n-1. \tag{3.3.7}$$

These constraints are again a consequence of the delta functions appearing in (3.2.13). It is clear from (3.3.6) that $\mathbb{S}(w)$ is a R-matrix with the parameter w in (3.3.4) playing the role of the spectral parameter. The other parameters Φ,Ψ and their constraints (3.3.7) give us some freedom in writing down the R-matrix. Their dependence on indices $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}',\boldsymbol{j}'$ are rather simple - the Ψ are gauge transformations of the R-matrix while the Φ are 'fields' in the physical sense which affect the Boltzmann weights of the underlying model.

The functions Φ,Ψ do play a nontrivial role when considering the transfer matrix built from $\mathbb{S}(w)$ as in (2.3.27) where they can affect the spectrum and commutativity if not chosen carefully. Since we are currently interested in the R-Matrix and not the global properties of the model, for now we can set these functions to 1 without loss of generality.

The R-Matrix $\mathbb{S}(w)$ is composite in the sense that it is a direct sum of "smaller" R-matrices. It is a fact which follows from considering the conservation laws (3.3.2) applied to each component in (3.3.6). By fixing the external indices in equation (3.3.6), it can be observed that the equation reduces to a tensor sum of an infinite number of the Yang-Baxter equations on subspaces indexed by global parameters $I,J,K=0,\dots,\infty$ defined in (3.3.3).

In particular, it was argued in [7; 9] that the subspace for each parameter I is in fact the underlying space of the rank I symmetric tensor representation of $U_q(\widehat{sl_n})$ and the action of $\mathbb{S}$ on this space is the corresponding R-matrix

$$\mathbb{S}_{i,j}^{i',j'}(w)=\bigoplus_{I,J=0}^{\infty} R_{I,J}^{(n)}(w). \tag{3.3.8}$$

The case $n=2$ in (3.3.4) was considered in [10] which resulted in a new formula for the matrix elements of the R-Matrix for $U_q(\widehat{sl_2})$ acting in the tensor product of representations of highest weight I and J. Setting $I=J=1$ the formula gives the R-matrix for

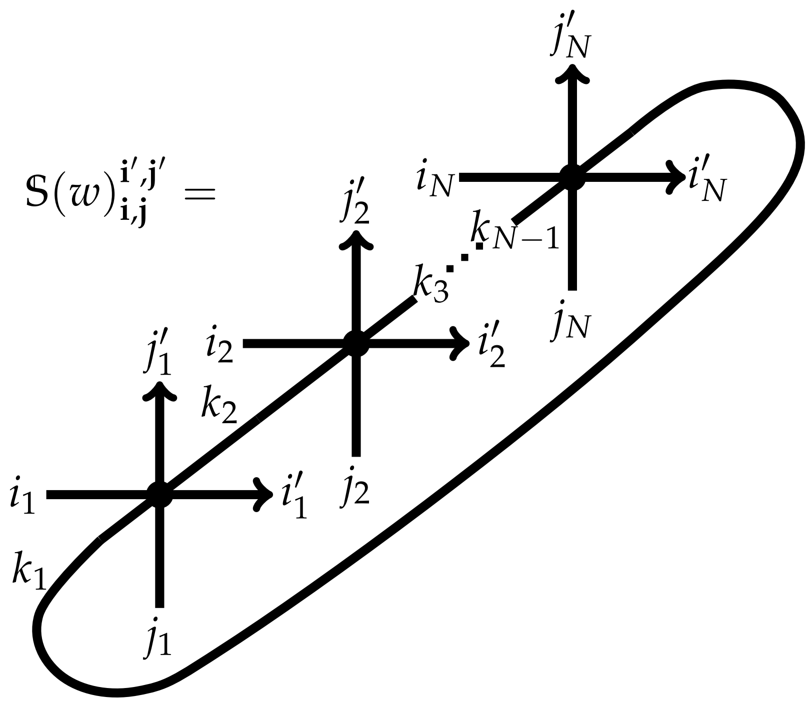

Figure 3.4: n-layer projection of the 3-dimensional model

the 6-vertex model.

The R-matrices $R^{(n)}_{I,J}(w)$ in (3.3.8) are irreducible and now we will give a new presentation for them by extracting a formula for them from the expression (3.3.1). First let us introduce some vector notations. We denote by $\boldsymbol{i} := \{i_1, \dots, i_r\}$ a vector with r components $i_k \in \mathbb{Z}_{\geq 0}$ and define

$$|\boldsymbol{i}| = \sum_{s=1}^{r} i_s, \quad (\boldsymbol{i}, \boldsymbol{j}) = \sum_{s=1}^{r} i_s j_s. \tag{3.3.9}$$

Addition is done component-wise and we introduce two permutations σ and τ acting on $\boldsymbol{k}$ as

$$\sigma\{k_1, \dots, k_r\} = \{k_2, \dots, k_r, k_1\}, \tag{3.3.10}$$

$$\tau\{k_1, \dots, k_r\} = \{k_r, k_{r-1}, \dots, k_1\} \tag{3.3.11}$$

of the vector coordinates. The dimension r can take values n and $n-1$ as explained below.

The Kronecker delta function of two vectors is zero unless all their components match, i.e.

$$\delta_{\boldsymbol{i},\boldsymbol{j}} = \prod_{s=1}^{r} \delta_{i_s,j_s}. \tag{3.3.12}$$

We also note that in discussing $\mathbb{S}(w)$ and $R^{(n)}_{I,J}(w)$ the vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}',\boldsymbol{j}'$ have different dimensions. When we use $\mathbb{S}(w)$, the n-layer composite weight, it is implied that the dimension $r = n$. When we derive the expression for the R-Matrix $R^{(n)}_{I,J}$, it is implied that the dimension $r = n-1$ because by fixing I, J the relation (3.3.3) implies that we can remove one of the indices. Typically we choose to remove last components i_n, j_n, i'_n, j'_n and replace them with $I - |\boldsymbol{i}|, J - |\boldsymbol{j}|$ etc. except in certain cases where it is more convenient to keep them. Of course, in evaluating final expressions the replacement has to be made regardless.

Combining (3.2.13) and (3.3.4) the composite weight $\mathbb{S}^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}(w)$ can be written as

$$\begin{aligned} \mathbb{S}^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}(w) = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} \sum_{\boldsymbol{k}\in\mathbb{Z}^n_{\geq 0}} \delta_{\boldsymbol{i}+\boldsymbol{k},\boldsymbol{i}'+\sigma\boldsymbol{k}}\, w^{k_1} q^{-|\boldsymbol{i}|-(\boldsymbol{i},\boldsymbol{j})-(\boldsymbol{k},\boldsymbol{i}+\sigma^{-1}\boldsymbol{j}')} \prod_{s=1}^{n} \begin{bmatrix} i_s + j_s \\ i_s \end{bmatrix}_{q^2} \\ \times \sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}} \prod_{s=1}^{n} \frac{(q^{-2i_s}, q^{-2j'_s}; q^2)_{m_s}}{(q^2, q^{-2i_s-2j_s}; q^2)_{m_s}} q^{2|\boldsymbol{m}|+2(\boldsymbol{m},\sigma\boldsymbol{k})}. \end{aligned} \tag{3.3.13}$$

The above formula contains $2n$ summations. The n summations in $\boldsymbol{k}$ are non-terminating. The n summations in $\boldsymbol{m}$ are restricted by $0 \leq m_s \leq \min(i_s, j'_s)$, $s = 1, \ldots, n$ due to the presence of Pochhammer symbols in the numerator. Let us also notice that all sums in m'_s terminates before the Pochhammer symbols in the denominator become zero. Therefore, there is no need for a regularisation.

This formula is quite easy to simplify. The presence of delta functions in (3.3.13) lead to the following global conservation laws for the spin indices $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}',\boldsymbol{j}'$,

$$i_1 + \cdots + i_n = i'_1 + \cdots + i'_n = I, \quad j_1 + \cdots + j_n = j'_1 + \cdots + j'_n = J \tag{3.3.14}$$

which allows us to remove one of the indices from $\boldsymbol{i},\boldsymbol{j},\boldsymbol{i}',\boldsymbol{j}'$ once we fix integers I, J. Furthermore, we can also express $k_2, \ldots, k_n$ in terms of k_1 by the relations

$$k_{s+1} = k_s + i_s - i'_s, \quad k_n = k_1 + \sum_{s=1}^{n-1} (i_s - i'_s), \tag{3.3.15}$$

which allows us to rewrite the sum in $\boldsymbol{k}$ as a single sum in k_1. However, some care must be taken in computing this sum. Note that when $i'_s > i_s$ for some s, the summation range of k_s implies contributions to the sum for negative values of k_{s+1} not included in the expression (3.3.13). These contributions turn out to be trivial. To see that we first notice that

$$\mathbb{R}^{n_1',n_2',n_3'}_{n_1,n_2,n_3} = 0, \quad n_3 < 0, \quad n_3' \geq 0 \tag{3.3.16}$$

This follows from (3.2.15) since the factor $1/(q^2;q^2)_{n_3}$ becomes zero and all other factors are nonzero. Now let us look at the product in (3.3.4) and assume that there are contributions from negative values for some k_s, $s = 1,\ldots,n$. All k_s cannot be negative, since $k_1 \geq 0$. Since the product is cyclic, we will always find at least one factor $\mathbb{R}^{j_s',i_s',k_{s+1}}_{j_s,i_s,k_s}$ such that $k_s < 0$ and $k_{s+1} \geq 0$. This factor will be equal to zero because of (3.3.16). Therefore, all factors which contain some negative k_s automatically disappear and we can safely sum over k_1 from 0 to ∞ in (3.3.13) with substitutions (3.3.15). As one can easily see the sum on k_1 becomes a geometric series which converges provided

$$wq^{-I-J} < 1, \quad 0 < q < 1. \tag{3.3.17}$$

Once this condition is satisfied for $w = \lambda^2 > 0$, the sum in (3.3.4) has only positive terms, since all matrix elements of the $3D$ R-matrix (3.2.13) are positive. Restricting the result to fixed positive values I, J we get the expression for matrix elements of the operator $R^{(n)}_{I,J}(w)$ in (3.3.8). The result reads

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'}\, q^{\Psi} \prod_{s=1}^{n} \begin{bmatrix} i_s + j_s \\ i_s \end{bmatrix}_{q^2} \sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}} \frac{q^{2|\boldsymbol{m}|+2\sum\limits_{k\geq l} m_k(i_l-i_l')}}{1-\lambda^2 q^{2|\boldsymbol{m}|-I-J}} \prod_{s=1}^{n} \frac{(q^{-2i_s}, q^{-2j_s'}; q^2)_{m_s}}{(q^2, q^{-2(i_s+j_s)}; q^2)_{m_s}} \tag{3.3.18}$$

where

$$\Psi = -2(\boldsymbol{i},\boldsymbol{j}) + (\boldsymbol{i}',\boldsymbol{j}') - (I-|\boldsymbol{i}|)(J-|\boldsymbol{j}|) + I(|\boldsymbol{i}'|-|\boldsymbol{i}|-1) + \sum_{1\leq k<l\leq n-1} (i_k' j_l' - i_k j_l). \tag{3.3.19}$$

Here in the left-hand side of (3.3.18) and in the expression for the phase factor (3.3.19) we used $(n-1)$-component indices, see (3.3.9) with $r = n-1$. However, in the right-hand side of (3.3.18) for compactness we kept n-component external indices assuming that we need to substitute i_n, j_n, i_n', j_n' from (3.3.14). The formula has n summation indices $m_1, m_2, \ldots, m_n$ which terminates after finitely many terms. Finally we notice that the sum $\sum_{k\geq l}$ in (3.3.18) taken over $n \geq k \geq l \geq 1$ can be restricted to the values $n-1 \geq k \geq l \geq 1$, since it is equal to zero for $k = n$ due to (3.3.14).

The case $n = 2$ of (3.3.18) was given in equation (75) of [9]. This formula generates elements of a $\binom{I+n-1}{n-1} \times \binom{J+n-1}{n-1}$-dimensional matrix determined by indices $0 \le |\boldsymbol{i}|, |\boldsymbol{i}'| \le I, 0 \le |\boldsymbol{j}|, |\boldsymbol{j}'| \le J$.

As the next step we shall evaluate one sum in (3.3.18) and reduce the total number of summations to $n-1$. We use the same method as in [10].

We start with the Lagrange interpolating formula

$$\sum_{l=0}^{k} \frac{x}{x-q^l} \frac{q^l (q^{-k};q)_l}{(q;q)_l} P_k(q^l) = \frac{P_k(x)(q;q)_k}{x^k (x^{-1};q)_{k+1}}, \tag{3.3.20}$$

which is valid for any polynomial $P_k(x)$ of degree at most k. First we define a new variable

$$l = m_1 + \cdots + m_n \tag{3.3.21}$$

which runs from 0 to I and use l instead of m_n. Then one can rewrite (3.3.18) as

$$\begin{aligned} \left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} &= \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} (-1)^{|\boldsymbol{i}|-|\boldsymbol{i}'|} q^{-2(\boldsymbol{i},\boldsymbol{j})+(\boldsymbol{i}',\boldsymbol{j}')-(I-|\boldsymbol{i}|)(J-|\boldsymbol{j}|)+I(|\boldsymbol{i}'|-|\boldsymbol{i}|-1)+\sum_{k<l}(i'_k j'_l - i_k j_l)} \\ \times \frac{q^{(|\boldsymbol{i}|-|\boldsymbol{i}'|)(|\boldsymbol{j}|+|\boldsymbol{j}'|-2J-1)}(q^{-2J};q^2)_{|\boldsymbol{j}|}}{(q^{-2J};q^2)_{|\boldsymbol{j}'|}(q^2;q^2)_I} & \prod_{s=1}^{n-1} \begin{bmatrix} i_s + j_s \\ i_s \end{bmatrix} \sum_{l=0}^{I} \frac{q^{2l}}{1-\lambda^2 q^{-I-J+2l}} \frac{(q^{-2I};q^2)_l}{(q^2;q^2)_l} P_I(q^{2l}). \end{aligned} \tag{3.3.22}$$

The summation in l matches (3.3.20) with $k = I$, $x = \lambda^{-2} q^{I+J}$ and

$$\begin{aligned} P_I(q^{2l}) &= q^{2l(I+|\boldsymbol{i}|-|\boldsymbol{i}'|)} \sum_{\boldsymbol{m}} q^{2(|\boldsymbol{m}|+\sum_{l>k} m_k (i'_l - i_l))} \prod_{s=1}^{n-1} \frac{(q^{-2i_s}, q^{-2j'_s}; q^2)_{m_s}}{(q^2, q^{-2(i_s+j_s)}; q^2)_{m_s}} \\ &\times (q^{-2l};q^2)_{|\boldsymbol{m}|} (q^{2(1-l+J-|\boldsymbol{j}'|+|\boldsymbol{m}|}; q^2)_{I-|\boldsymbol{i}'|} (q^{2(1-l+I-|\boldsymbol{i}|+|\boldsymbol{m}|)}; q^2)_{|\boldsymbol{i}|-|\boldsymbol{m}|}. \end{aligned} \tag{3.3.23}$$

The polynomial $P_I(x)$ in (3.3.23) has degree of at most I and therefore we can replace the sum in l in (3.3.22) with the right hand side of (3.3.20) to find the expression

Proposition 3.3.1.

$$\begin{aligned} \left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} &= \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} \left[A^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} B_{I,J}(\lambda)\, q^{(\boldsymbol{i}',\boldsymbol{j}')-(\boldsymbol{i},\boldsymbol{j})-J|\boldsymbol{i}|-I|\boldsymbol{j}'|+\sum\limits_{k>l}(i_k j_l + j'_k i'_l)} \\ \times \sum_{\boldsymbol{m}\in\mathbb{Z}^{n-1}_{\ge 0}} & \frac{(\lambda^2 q^{-I-J}, \lambda^2 q^{2+I+J-2|\boldsymbol{i}|-2|\boldsymbol{j}|}; q^2)_{|\boldsymbol{m}|}}{(\lambda^2 q^{2+I-J-2|\boldsymbol{i}|}, \lambda^2 q^{2+J-I-2|\boldsymbol{j}'|}; q^2)_{|\boldsymbol{m}|}} \prod_{s=1}^{n-1} \frac{(q^{-2i_s}, q^{-2j'_s}; q^2)_{m_s}}{(q^2, q^{-2(i_s+j_s)}; q^2)_{m_s}} q^{2(|\boldsymbol{m}|+\sum\limits_{k<l} m_k (i'_l - i_l))}. \end{aligned} \tag{3.3.24}$$

All external and summation indices in (3.3.24) have $n-1$ components and the coefficients $A^{(n)}_{I,J}(\lambda)^{\mathbf{i}',\mathbf{j}'}_{\mathbf{i},\mathbf{j}}$ and $B_{I,J}(\lambda)$ are given by

$$\left[A^{(n)}_{I,J}(\lambda)\right]^{\mathbf{i}',\mathbf{j}'}_{\mathbf{i},\mathbf{j}} = \frac{(\lambda^{-2}q^{I-J};q^2)_{|\mathbf{j}'|}(\lambda^{-2}q^{J-I};q^2)_{|\mathbf{i}|}(q^{-2J};q^2)_{|\mathbf{j}|}}{(\lambda^{-2}q^{-I-J};q^2)_{|\mathbf{i}+\mathbf{j}|}(q^{-2J};q^2)_{|\mathbf{j}'|}}\prod_{s=1}^{n-1}\begin{bmatrix} i_s+j_s \\ j_s \end{bmatrix}_{q^2}, \tag{3.3.25}$$

$$B_{I,J}(\lambda) = q^{-I-IJ}\frac{(\lambda^2 q^{-I-J};q^2)_{I+J+1}}{(\lambda^2q^{-I-J};q^2)_{I+1}(\lambda^2q^{-I-J};q^2)_{J+1}}. \tag{3.3.26}$$

The formula (3.3.24) provides an expression for the matrix elements of the $U_q(\widehat{sl_n})$ R-Matrix acting on the space $V_I \otimes V_J$ where

$$V_I \equiv \{|\mathbf{i}\rangle\}, \quad |\mathbf{i}| \leq I. \tag{3.3.27}$$

It follows from the tetrahedron equation for the 3D R-matrix (3.2.13) that (3.3.24) satisfies the Yang-Baxter equation

$$R^{(n)}_{I,J}(\lambda)R^{(n)}_{I,K}(\lambda\mu)R^{(n)}_{J,K}(\mu) = R^{(n)}_{J,K}(\mu)R^{(n)}_{I,K}(\lambda\mu)R^{(n)}_{I,J}(\lambda) \tag{3.3.28}$$

for any $I, J, K \in \mathbb{Z}_+$. However, one will notice that the coefficient $B_{I,J}(\lambda)$ is just a constant not depending on indices. We find it convenient to set this factor to 1. In what follows, we will use (3.3.24) with $B_{I,J}(\lambda) = 1$ unless stated otherwise. In this normalization we have

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\mathbf{0},\mathbf{0}}_{\mathbf{0},\mathbf{0}} = 1. \tag{3.3.29}$$

The main reason we do this is because (3.3.24) is now well defined even when $I, J \in \mathbb{C}$. Although the 3D model projection outlined in this paper satisfies the Yang-Baxter equation for integral weights by construction, the equation (3.3.28) remains valid even for complex weights $I, J, K \in \mathbb{C}$. The proof closely follows the arguments given in [20].

Consider a particular element of the Yang-Baxter equation $\langle \mathbf{i}, \mathbf{j}, \mathbf{k}|(3.3.28)|\mathbf{i}', \mathbf{j}', \mathbf{k}'\rangle$ with fixed external indices $\mathbf{i} = (i_1, \ldots, i_{n-1})$, etc. Due to the conservation law in (3.3.24) we have $|\mathbf{i}+\mathbf{j}+\mathbf{k}| = |\mathbf{i}'+\mathbf{j}'+\mathbf{k}'| \equiv m$ and all summation indices in (3.3.28) will also be limited by m. Choose an integer $N > m$ and assume that integer weights $I, J, K > N$. It is easy to see that all denominators in the R-matrices entering the Yang-Baxter equation are non-zero and (3.3.28) becomes the equality of two rational functions in variables $x = q^{-I}$, $y = q^{-J}$ and $z = q^{-K}$. After eliminating denominators we can rewrite (3.3.28) as equality of two polynomials in three variables x, y, z. The degree of these polynomials grows as a fixed polynomial in N. Now we know that the Yang-Baxter equation (3.3.28) is true for infinitely many integer variables $I, J, K > N$. It can only happen if

(3.3.28) reduces to a polynomial identity in $x, y, z \in \mathbb{C}$. Therefore, the Yang-Baxter equation with the R-matrix (3.3.24) and normalization (3.3.29) is satisfied for $I, J, K \in \mathbb{C}$. In this case it defines the infinite dimensional R-matrix corresponding to Verma module representations of $U_q(\widehat{sl_n})$.

To illustrate how formula (3.3.24) works, let us consider a special case $n = 2$. In this case matrix elements are indexed by indices i, j, i', j', and (3.3.24) becomes a single sum which is given by

$$\left[R^{(2)}_{I,J}(\lambda)\right]^{i',j'}_{i,j} = \delta_{i+j,i'+j'} q^{i'j'-ij-iJ-Ij'} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} \frac{(\lambda^{-2}q^{I-J};q^2)_{j'}(\lambda^{-2}q^{J-I};q^2)_i(q^{-2J};q^2)_j}{(\lambda^{-2}q^{-I-J};q^2)_{i+j}(q^{-2J};q^2)_{j'}}$$
$$\times\, {}_4\phi_3 \left(\begin{matrix} q^{-2i} & q^{-2j'} & \lambda^2 q^{-I-J} & \lambda^2 q^{2+I+J-2i-2j} \\ & q^{-2i-2j} & \lambda^2 q^{2+I-J-2i} & \lambda^2 q^{2+J-I-2j'} \end{matrix} \middle| q^2, q^2 \right). \qquad (3.3.30)$$

This is a balanced and terminating ${}_4\phi_3$ basic hypergeometric series for the elements of the $U_q(\widehat{sl_2})$ R-matrix. This case was already studied in [10] and the formula given there is of the same type as (3.3.30) but with different arguments. Most notably, the hypergeometric sum in [10] is a polynomial in the spectral parameter λ while (3.3.30) is a rational function.

Using Sears' transform (B.2.14) we can transform the sum in (3.3.30) to equation (5.8) in [10] by identifying

$$n = i, \quad a = q^{-2j'}, \quad b = \lambda^2 q^{-I-J}, \quad c = \lambda^2 q^{2+I+J-2i-2j},$$
$$d = \lambda^2 q^{2+I-J-2i}, \quad e = q^{-2i-2j}, \quad f = \lambda^2 q^{2+J-I-2j'}.$$

Let us note that (5.8) in [10] requires a regularisation but the expression (3.3.30) is free from any divergences.

One problem with (3.3.24) is that the hypergeometric sum is a rational function in λ. For integer I the whole expression (3.3.24) is a polynomial in λ up to an overall factor $(\lambda^2 q^{-I-J}; q^2)_{I+1}$. However, when both weights I, J are non-integral, this factor is no longer a polynomial in λ and no polynomial normalization exists. In this case we can adopt the normalization (3.3.29) where elements of the R-matrix are rational functions of λ.

As mentioned before, using Sears' transformation the hypergeometric sum in (3.3.24) for $n = 2$ can be transformed into a hypergeometric polynomial in λ up to simple q-

binomial factors. We are aware of the A_n multivariable generalizations of Sears' transformation in the literature, but they do not appear to be applicable to our expression for $n \geq 3$. Focusing on each summation index in (3.3.24) one can easily see that it is a ${}_4\phi_3$ basic hypergeometric series but it is not balanced and only one of the hypergeometric series has a q^2 argument so the A_1 Sears' transformations does not apply.

We expect that a formula with a hypergeometric sum being a polynomial in λ still exists but it probably requires a new yet to be discovered identity for multivariable hypergeometric series.

3.4 Symmetries and special cases

In this section we discuss symmetries of the R-matrix $R^{(n)}_{I,J}(\lambda)$ given by (3.3.24). They can be derived from the corresponding symmetries of the 3D R-Matrix generated by (3.2.14). It is actually more convenient to use (3.2.15-3.2.16) since we need to to keep a position of the third "hidden" direction where we take the trace. Applying these transformations to the factors in (3.3.4) we find two symmetries

$$\left[R^{(n)}_{I,J}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \lambda^{2(|\boldsymbol{i}'|-|\boldsymbol{i}|)}\left[R^{(n)}_{J,I}(\lambda)\right]_{\tau\boldsymbol{j},\tau\boldsymbol{i}}^{\tau\boldsymbol{j}',\tau\boldsymbol{i}'}, \tag{3.4.1}$$

$$\left[R^{(n)}_{I,J}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = q^{2[\boldsymbol{i}',\boldsymbol{j}']-2[\boldsymbol{i},\boldsymbol{j}]}\lambda^{2(|\boldsymbol{i}'|-|\boldsymbol{i}|)}\prod_{s=1}^{n}\frac{(q^2;q^2)_{i'_s}(q^2;q^2)_{j'_s}}{(q^2;q^2)_{i_s}(q^2;q^2)_{j_s}}\left[R^{(n)}_{I,J}(\lambda)\right]_{\tau\boldsymbol{i}',\tau\boldsymbol{j}'}^{\tau\boldsymbol{i},\tau\boldsymbol{j}}. \tag{3.4.2}$$

Let us explain some notations here. In the previous section we mentioned that for the R-matrix $R^{(n)}_{I,J}(\lambda)$ we are using $n-1$-component indices, i.e. $\boldsymbol{i} = \{i_1, \ldots, i_{n-1}\}$ with the last n-th component $i_n = I - |\boldsymbol{i}|$ removed and similar for j's. However, in (3.4.2) the product in the right-hand side is taken over $s = 1, \ldots, n$ where for $s = n$ we substitute the last component as above, i.e. $i_n = I - |\boldsymbol{i}|$, $j_n = J - |\boldsymbol{j}|$, etc. The transformation τ is defined in (3.3.10).

In addition, in (3.4.2) we used a notation $[\boldsymbol{i}, \boldsymbol{j}]$ for a convolution of n-component indices, i.e.

$$[\boldsymbol{i}, \boldsymbol{j}] = (\boldsymbol{i}, \boldsymbol{j}) + (I - |\boldsymbol{i}|)(J - |\boldsymbol{j}|). \tag{3.4.3}$$

There is also a symmetry of the R-matrix which corresponds the the cyclic permutation of the n $3D$ R-matrices in the "hidden" direction. Let us introduce the notation

$$\bar{\boldsymbol{\imath}} = \{I - |\boldsymbol{i}|, i_1, \ldots, i_{n-2}\}, \tag{3.4.4}$$

which is equivalent to the permutation $\sigma^{-1}\boldsymbol{i}$ for the n-tuple $\boldsymbol{i}$ but with the last component removed. Here we assume that $I, J \in \mathbb{Z}_+$. Performing a cyclic shift in (3.3.4) we easily obtain

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = \lambda^{2(|\boldsymbol{i}'|-|\boldsymbol{i}|)}\left[R^{(n)}_{I,J}(\lambda)\right]^{\bar{\boldsymbol{\imath}}',\bar{\boldsymbol{\jmath}}'}_{\bar{\boldsymbol{\imath}},\bar{\boldsymbol{\jmath}}}\,. \tag{3.4.5}$$

For example, when $n = 2$ this corresponds to $i \to I - i$ and similarly for other indices.

The last symmetry follows from the transformation of the $3D$ R-matrix (3.2.17). After simple calculations one can obtain the following result

$$\left[R^{(n)}_{I,J}(\lambda, q)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = q^{[\boldsymbol{i},\boldsymbol{j}]-[\boldsymbol{i}',\boldsymbol{j}']}\left[R^{(n)}_{J,I}(\lambda^{-1}, q^{-1})\right]^{\boldsymbol{j}',\boldsymbol{i}'}_{\boldsymbol{j},\boldsymbol{i}}\,. \tag{3.4.6}$$

Finally, when $I = J$ and $\lambda = 1$ the R-matrix reduces to permutation operator

$$R^{(n)}_{I,I}(1) = \mathbb{P}_{1,2} \tag{3.4.7}$$

which can be seen from (3.5.2) in the next section.

3.5 Reductions and factorization

There are two special points in the spectral parameter $\lambda = q^{\pm(I-J)/2}$ where the multiple sum in (3.3.24) reduces to one non-zero summand. These specializations produce the R-matrix without difference property with weights I, J playing the role of spectral parameters. With the normalization (3.3.29) we can choose $I, J \in \mathbb{C}$ and obtain the R-matrix acting in the tensor product of two Verma modules.

For the case of the $U_q(\widehat{sl_2})$ algebra the importance of such reductions was first noticed in [22]. Under the choice $\lambda = q^{(J-I)/2}$ the $U_q(\widehat{sl_2})$ R-matrix of [10] reduces to the R-matrix of the Povolotsky model [61] which satisfies a stochasticity condition and defines a family of zero-range chipping models. A generalization of the Povolotsky model to arbitrary rank $n-1$ was obtained in the recent paper [20].

So let us start with the case $\lambda = q^{(I-J)/2}$, $I - J \in \mathbb{Z}_+$. The expression for the R-matrix (3.3.24) contains the factor $(\lambda^{-2}q^{I-J}; q^2)_{|\boldsymbol{j}'|}$ outside the sum which has the argument 1 after the above substitution. This factor is always zero for $|\boldsymbol{j}'| > 0$ unless it is canceled by the factor $(\lambda^2 q^{2+J-I-|\boldsymbol{j}'|}; q^2)_{|\boldsymbol{m}|}$ inside the sum. It can only happen when $|\boldsymbol{m}| = |\boldsymbol{j}'|$ or $\boldsymbol{m} = \boldsymbol{j}'$, since $m_s \le j'_s$, $s = 1, \ldots, n-1$. Let us note that the argument fails when $J - I$ is a positive integer because the other factor in the denominator can cancel with the zero of $(\lambda^{-2}q^{I-J};q^2)_{|\boldsymbol{j}'|}$ and multiple summands survive.

After simple algebra one can derive from (3.3.24) the following result

$$\left[R_{I,J}^{(n)}(q^{\frac{I-J}{2}})\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} q^{(\boldsymbol{i}',\boldsymbol{j}')-(\boldsymbol{i},\boldsymbol{j})-J|\boldsymbol{i}|-I|\boldsymbol{j}'|+2J|\boldsymbol{j}'|+\sum\limits_{k>l}(i_k j_l+j'_k i'_l-2j'_k j_l)} \times \frac{(q^{-2J};q^2)_{|\boldsymbol{j}|}(q^{2J-2I};q^2)_{|\boldsymbol{i}'|-|\boldsymbol{j}|}}{(q^{-2I};q^2)_{|\boldsymbol{i}'|}} \prod_{s=1}^{n-1} \begin{bmatrix} i'_s \\ j_s \end{bmatrix}_{q^2}. \tag{3.5.1}$$

Similarly, we can make the substitution $\lambda = q^{\frac{J-I}{2}}$ for $J - I \in \mathbb{Z}_+$. In this case the argument is the same except with the factors $(\lambda^{-2}q^{J-I};q^2)_{|\boldsymbol{i}|}$ and $(\lambda^2 q^{2+I-J-|\boldsymbol{i}|};q^2)_{|\boldsymbol{m}|}$ and so the only summand that contributes is $\boldsymbol{m} = \boldsymbol{i}$. Then we obtain

$$\left[R_{I,J}^{(n)}(q^{\frac{J-I}{2}})\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} q^{(\boldsymbol{i}',\boldsymbol{j}')-(\boldsymbol{i},\boldsymbol{j})-J|\boldsymbol{i}|-I|\boldsymbol{j}'|+2I|\boldsymbol{i}|+\sum_{k>l}(i_k j_l+j'_k i'_l-2i_k i'_l)} \times \frac{(q^{-2I};q^2)_{|\boldsymbol{i}|}(q^{2I-2J};q^2)_{|\boldsymbol{j}'|-|\boldsymbol{i}|}}{(q^{-2J};q^2)_{|\boldsymbol{j}'|}} \prod_{s=1}^{n-1} \begin{bmatrix} j'_s \\ i_s \end{bmatrix}_{q^2}. \tag{3.5.2}$$

Obviously these reductions are substantially simpler than the original R-matrix. As mentioned above I, J play role of the spectral parameters for these R-matrices and can now take arbitrary complex values.

In fact, one can construct the full R-matrix as a matrix product of (3.5.1)-(3.5.2). To explain this it is convenient to apply a simple similarity transformation in the first space and introduce

$$\boldsymbol{R}_{I,J}^{(n)}(\lambda) = U \otimes \mathbf{1}\, R_{I,J}^{(n)}(\lambda)\, U^{-1} \otimes \mathbf{1} \tag{3.5.3}$$

with

$$U_{\boldsymbol{i},\boldsymbol{i}'} = \delta_{\boldsymbol{i},\boldsymbol{i}'} \left(\lambda q^{(I-J)/2}\right)^{|\boldsymbol{i}|}. \tag{3.5.4}$$

We now define two operators $\boldsymbol{M}$ and $\boldsymbol{N}$ acting in the tensor product of two Verma modules by

$$\boldsymbol{M}(q^I, q^J) = \check{\boldsymbol{R}}_{I,J}^{(n)}(q^{\frac{I-J}{2}}), \quad \boldsymbol{N}(q^I, q^J) = \check{\boldsymbol{R}}_{I,J}^{(n)}(q^{\frac{J-I}{2}}) \tag{3.5.5}$$

where as usual $\check{R}_{1,2}(\lambda) = \mathbb{P}_{1,2} R_{1,2}(\lambda)$, etc. with $\mathbb{P}_{1,2}$ being the permutation operator. Both operators $\boldsymbol{M}(q^I, q^J)$ and $\boldsymbol{N}(q^I, q^J)$ of complex arguments q^I, q^J are defined by its matrix elements via (3.5.1)-(3.5.2) and (3.5.5).

With these notations one can easily derive from (3.3.24) the following factorization

Proposition 3.5.1.

$$\check{\mathbf{R}}^{(n)}_{I,J}(\lambda) = \mathbf{M}(\lambda q^{\frac{I+J}{2}}, q^J)\mathbf{N}(\lambda^{-1} q^{\frac{I+J}{2}}, q^J). \tag{3.5.6}$$

A similar factorization of the R-matrix appeared in [25] for the sl_2 case of the XXX model.

We can also rewrite a factorization formula (3.5.6) for the matrix elements of the original R-matrix $R^{(n)}_{I,J}(\lambda)$ as follows

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\mathbf{i}',\mathbf{j}'}_{\mathbf{i},\mathbf{j}} = \sum_{\mathbf{k}+\mathbf{l}=\mathbf{i}+\mathbf{j}} \tilde{\mathbf{M}}^{\mathbf{k},\mathbf{l}}_{\mathbf{i},\mathbf{j}}\, \tilde{\mathbf{N}}^{\mathbf{i}',\mathbf{j}'}_{\mathbf{k},\mathbf{l}}, \tag{3.5.7}$$

with

$$\tilde{\mathbf{M}}^{\mathbf{i}',\mathbf{j}'}_{\mathbf{i},\mathbf{j}} = \delta_{\mathbf{i}+\mathbf{j},\mathbf{i}'+\mathbf{j}'}\, q^{-J|\mathbf{i}|-(\mathbf{i},\mathbf{j})+\sum\limits_{k>l}(i_k j_l + j'_k i'_l - 2 j'_k j_l)} \frac{(q^{-2J};q^2)_{|\mathbf{j}|}(\lambda^{-2}q^{J-I};q^2)_{|\mathbf{i}'-\mathbf{j}|}}{(\lambda^2 q^{-I-J})^{|\mathbf{j}'|}(\lambda^{-2}q^{-I-J};q^2)_{|\mathbf{i}'|}} \prod_{s=1}^{n-1} \begin{bmatrix} i'_s \\ j_s \end{bmatrix}_{q^2}, \tag{3.5.8a}$$

$$\tilde{\mathbf{N}}^{\mathbf{i}',\mathbf{j}'}_{\mathbf{i},\mathbf{j}} = \delta_{\mathbf{i}+\mathbf{j},\mathbf{i}'+\mathbf{j}'}\, q^{(\mathbf{i}',\mathbf{j}')-I|\mathbf{j}'|+\sum\limits_{k>l}(j_k i_l + j'_k i'_l - 2 j_k i'_l)} \frac{(\lambda^2 q^{-I-J};q^2)_{|\mathbf{j}|}(\lambda^{-2}q^{I-J};q^2)_{|\mathbf{j}'-\mathbf{j}|}}{(q^{-2J};q^2)_{|\mathbf{j}'|}} \prod_{s=1}^{n-1} \begin{bmatrix} j'_s \\ j_s \end{bmatrix}_{q^2} \tag{3.5.8b}$$

where we removed some gauge factors which cancel in the matrix product (3.5.7).

3.6 Comparison with other results

In this and next sections we will compare (3.3.24) with some other presentations of the $U_q(\widehat{sl_n})$ related R-matrix given in the literature. We will establish a connection with the standard the $U_q(\widehat{sl_n})$ L-operator presented in [20] and also compare our results with some higher-spin examples of the $U_q(\widehat{sl_3})$ R-Matrix.

We start with some remarks regarding the coefficient $A^{(n)}_{I,J}(\lambda)$ in (3.3.25). For specific elements of the R-matrix the q-Pochhammer symbols are finite products as their arguments are integers. If we want to derive the formula for the L-operator as an $n \times n$-matrix with operator entries acting in the Verma modules spanned by $|\mathbf{j}\rangle = |j_1, \ldots, j_{n-1}\rangle$ we need to rewrite (3.3.25) in the form suitable for abstract values of $\mathbf{j}$ indices.

This is achieved by a slight change of normalization of the R-matrix

$$\bar{R}^{(n)}_{I,J}(\lambda) = \sigma_{I,J}(\lambda) R^{(n)}_{I,J}(\lambda), \tag{3.6.1}$$

with

$$\sigma_{I,J}(\lambda) = -\lambda^{-I} q^{\frac{I+J}{2}} (\lambda^2 q^{-I-J}; q^2)_{I+1}. \tag{3.6.2}$$

We also restore a coefficient $B_{I,J}(\lambda)$ in (3.3.26) and define

$$\bar{A}^{(n)}_{I,J}(\lambda) = \sigma_{I,J}(\lambda) A^{(n)}_{I,J}(\lambda) B_{I,J}(\lambda). \tag{3.6.3}$$

After simple calculations we obtain

$$\bar{A}^{(n)}_{I,J}(\lambda) = \frac{(\lambda^{-2} q^{-I-J+2|\boldsymbol{i}|+2|\boldsymbol{j}|}; q^2)_{I-|\boldsymbol{i}'|} (\lambda^{-2} q^{J-I}; q^2)_{|\boldsymbol{i}|}}{(-1)^{I+1} \lambda^{-I} q^{-\frac{I+J}{2}} (q^{-2J+2|\boldsymbol{j}|}; q^2)_{|\boldsymbol{i}-\boldsymbol{i}'|}} \prod_{s=1}^{n-1} \frac{(q^{2+2j_s}; q^2)_{i_s}}{(q^2; q^2)_{i_s}} \tag{3.6.4}$$

and this expression is a finite product for integer $\boldsymbol{i}$, $\boldsymbol{i}'$ and abstract values of $\boldsymbol{j}$'s. In short, a change of normalization is equivalent to replacing the product $A^{(n)}_{I,J}(\lambda) B_{I,J}(\lambda)$ in (3.3.24) with (3.6.4). The sum in (3.3.24) is still finite because it terminates by integer values of $\boldsymbol{i}'$s.

It is easier to write down explicit formulas in original n-component notations. Introduce n-component vectors $\boldsymbol{e}_\alpha = (0, \dots, 0, 1, 0, \dots, 0)$ with 1's at the α-th position from the left, $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^n_+$ with $|\boldsymbol{j}| = |\boldsymbol{k}| = J$. Then using (3.6.4) in (3.3.24) we obtain for the renormalized R-matrix (3.6.1)

$$\left[\bar{R}^{(n)}_{1,J}(\lambda)\right]^{\boldsymbol{e}_\beta, \boldsymbol{k}}_{\boldsymbol{e}_\alpha, \boldsymbol{j}} = \begin{cases} [\lambda q^{\frac{1-J}{2} + k_\alpha}] & \text{if } \alpha = \beta, \\ \lambda q^{\frac{1-J}{2} + \sum_{s=\beta}^{\alpha-1} k_s} [q^{k_\alpha}] & \text{if } \alpha > \beta, \\ \lambda^{-1} q^{\frac{1+J}{2} - \sum_{s=\alpha}^{\beta-1} k_s} [q^{k_\alpha}] & \text{if } \alpha < \beta, \end{cases} \tag{3.6.5}$$

where

$$[x] = x - x^{-1}. \tag{3.6.6}$$

In the recent paper [20] matrix elements for the $U_q(A^{(1)}_{n-1})$ R-Matrix $R^K(z)$ acting in the space $V_1 \otimes V_m$ were given by

$$\left[R^K_{1,m}(z)\right]^{e_k, \boldsymbol{\delta}}_{e_j, \boldsymbol{\beta}} = \begin{cases} q^{\beta_k + 1} \frac{1 - q^{-2\beta_k + m - 1} z}{q^{m+1} - z} & \text{if } j = k, \\ -q^{\beta_{j+1} + \cdots + \beta_{k-1}} \frac{1 - q^{2\beta_k}}{q^{m+1} - z} & \text{if } j < k, \\ -q^{m - (\beta_k + \cdots + \beta_j)} \frac{z(1 - q^{2\beta_k})}{q^{m+1} - z} & \text{if } j > k, \end{cases} \tag{3.6.7}$$

and the elements of $R^K(z)$ acting on $V_l \otimes V_1$ were given by

$$\left[R^K_{l,1}(z)\right]^{\gamma,e_k}_{\alpha,e_j} = \begin{cases} q^{\gamma_k+1}\frac{1-q^{-2\gamma_k+l-1}z}{q^{l+1}-z} & \text{if } j=k \\ -q^{l-(\alpha_j+\cdots+\alpha_k)}\frac{z(1-q^{2\alpha_k})}{q^{l+1}-z} & \text{if } j<k, \\ -q^{\alpha_{k+1}+\cdots+\alpha_{j-1}}\frac{1-q^{2\alpha_k}}{q^{l+1}-z} & \text{if } j>k, \end{cases} \tag{3.6.8}$$

where we write them in the same notations as in (3.6.5).
A direct comparison of (3.6.7) and (3.6.5) gives

$$\left[\bar{R}^{(n)}_{1,J}(\lambda)\right]^{e_\beta,\boldsymbol{k}}_{e_\alpha,\boldsymbol{j}} = \left[\lambda q^{\frac{1+J}{2}}\right] q^{[e_\beta,\boldsymbol{k}]-[e_\alpha,\boldsymbol{j}]} \left[R^K_{1,J}(\lambda^{-2})\right]^{e_\alpha,\boldsymbol{j}}_{e_\beta,\boldsymbol{k}}. \tag{3.6.9}$$

To compare matrix elements of $R^K_{l,m}(z)$ with our formula (3.3.24) for other cases we must first identify their parameters. So we set $l=I$, $m=J$ and $z=\lambda^{-2}$. The normalization of the R-matrix $R^K_{l,m}(z)$ is the same as (3.3.29) for $R^{(n)}_{I,J}(\lambda)$. Therefore, we expect that for arbitrary I, J

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = q^{[\boldsymbol{i}',\boldsymbol{j}']-[\boldsymbol{i},\boldsymbol{j}]} \left[R^K_{I,J}(\lambda^{-2})\right]^{\boldsymbol{i},\boldsymbol{j}}_{\boldsymbol{i}',\boldsymbol{j}'}. \tag{3.6.10}$$

The difference between two R-matrices in (3.6.10) is easy to explain. The matrix elements of $R^{(n)}_{I,J}(\lambda)$ are defined similar to (3.1.2), i.e.

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = \langle \boldsymbol{i},\boldsymbol{j}|R^{(n)}_{I,J}(\lambda)|\boldsymbol{i}',\boldsymbol{j}'\rangle. \tag{3.6.11}$$

However, the matrix elements of $R^K_{l,m}(z)$ in [20] are defined by the transposed action

$$\left[R^K_{l,m}(z)\right]^{\gamma,\delta}_{\alpha,\beta} = \langle \gamma,\delta|R^K_{l,m}(z)|\alpha,\beta\rangle. \tag{3.6.12}$$

It is easy to check that the extra "twist" factor $q^{[\boldsymbol{i}',\boldsymbol{j}']-[\boldsymbol{i},\boldsymbol{j}]}$ in (3.6.10) drops out from the Yang-Baxter equation.

We have checked that the relation (3.6.10) holds for the $U_q(A^{(1)}_1)$ and $U_q(A^{(1)}_2)$ R-matrices for all cases given in Appendix A of [20].

It is also interesting to compare our reductions (3.5.1) and (3.5.2) with that obtained in [20]. In particular, we expect that the **Theorem 2** in [20]

$$\left[R^K_{l,m}(q^{l-m})\right]^{\gamma,\delta}_{\alpha,\beta} = \delta_{\alpha+\beta,\gamma+\delta} q^{\psi} \begin{bmatrix} m \\ l \end{bmatrix}^{-1}_{q^2} \prod_{s=1}^{n+1} \begin{bmatrix} \beta_s \\ \gamma_s \end{bmatrix}_{q^2}, \tag{3.6.13}$$

$$\psi = \sum_{1\le s,t\le n+1} \alpha_s(\beta_t - \gamma_t) + \sum_{1\le s,t\le n+1} (\beta_s - \gamma_s)\gamma_t \qquad (3.6.14)$$

should correspond to the substitution $\lambda = q^{\frac{J-I}{2}}$ given by (3.5.2). A direct calculation shows the relation (3.6.10) also holds in this case.

Now let us turn to the $U_q(sl_n)$ L-operator. When $J = 1$ the expression (3.6.5) further reduces to the trigonometric n-state R-Matrix [62; 63]. We shall also use a *twisted* version of this R-matrix [64] which we give in the form stated in [65]

$$R^{\beta,\delta}_{\alpha,\gamma}(\lambda) = \delta_{\alpha,\beta}\delta_{\gamma,\delta}\delta_{\alpha,\gamma}(q-1)(\lambda+\lambda^{-1}q^{-1}) + \delta_{\alpha,\beta}\delta_{\gamma,\delta}\rho_{\alpha,\gamma}(\lambda-\lambda^{-1}) + \delta_{\alpha,\delta}\delta_{\beta,\gamma}\sigma_{\alpha,\beta}, \quad (3.6.15)$$

where

$$\sigma_{\alpha,\beta} = \begin{cases} 0 & \text{if } \alpha=\beta, \\ (q-q^{-1})\lambda & \text{if } \alpha<\beta, \\ (q-q^{-1})\lambda^{-1} & \text{if } \alpha>\beta \end{cases} \qquad (3.6.16)$$

and $\rho_{\alpha,\beta}$ are nonzero complex parameters such that

$$\rho_{\alpha,\alpha} = \rho_{\alpha,\beta}\rho_{\beta,\alpha} = 1, \quad \alpha,\beta = 1,\dots,n. \qquad (3.6.17)$$

Setting all $\rho_{\alpha,\beta} = 1$ and taking convention that all indices $\alpha,\beta,\gamma,\delta = 1,\dots,n$ in (3.6.15) denote positions of 1's counted from the right, i.e. $\alpha \equiv \boldsymbol{e}_{n-\alpha+1}$ we obtain that (3.6.15) is equivalent to (3.6.5) with $J = 1$.

Setting $I = J = 1$ in the Yang-Baxter equation (3.3.28) we obtain the L-operator algebra

$$R_{1,2}(\lambda/\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{1,2}(\lambda/\mu), \qquad (3.6.18)$$

where the $R_{1,2}(\lambda)$-matrix corresponds to the standard $U_q(A^{(1)}_{n-1})$ trigonometric R-matrix (3.6.5) with $J = 1$. The L-operators $L(\lambda)$ are identified with $\bar{R}^{(n)}_{1,K}(\lambda)$ (3.6.5) acting in the "quantum" space with the weight K.

To rewrite the L-operator in algebraic notations let us introduce Weil operators X_k, Z_k, $i = 1,\dots,n$ acting in the space of n-component vectors $|\boldsymbol{j}\rangle$, $j_s \in \mathbb{Z}$, $s = 1,\dots,n$ and their conjugates such that

$$Z_k|\boldsymbol{j}\rangle = q^{j_k}|\boldsymbol{j}\rangle, \quad X_k|j_1,\dots,j_n\rangle = |j_1,\dots,j_k+1,\dots,j_n\rangle, \qquad (3.6.19)$$

$$\langle \boldsymbol{j}|Z_k = q^{j_k}\langle \boldsymbol{j}|, \quad \langle j_1,\ldots,j_n|X_k = \langle j_1,\ldots,j_k-1,\ldots,j_n|. \tag{3.6.20}$$

They satisfy the Weil algebra relations

$$Z_k X_l = q^{\delta_{k,l}} X_l, Z_k, \quad k,l = 1,\ldots,n. \tag{3.6.21}$$

We can now define the L-operator $L(\lambda)$ as an $n \times n$ matrix with operator entries such that

$$\langle \boldsymbol{j}|L_{\alpha,\beta}(\lambda)|\boldsymbol{k}\rangle = \left[\bar{R}^{(n)}_{1,J}(\lambda)\right]^{e_\beta,\boldsymbol{k}}_{e_\alpha,\boldsymbol{j}}. \tag{3.6.22}$$

Using (3.6.19)-(3.6.21) we obtain

$$L_{\alpha,\beta}(\mu) = \begin{cases} [\mu Z_\alpha] & \text{if } \alpha = \beta, \\ \mu\, X_\alpha^{-1} X_\beta\, [Z_\alpha] \prod\limits_{s=\beta}^{\alpha-1} Z_s & \text{if } \alpha > \beta, \\ \mu^{-1} q X_\alpha^{-1} X_\beta\, [Z_\alpha] \prod\limits_{s=\alpha}^{\beta-1} Z_s^{-1} & \text{if } \alpha < \beta, \end{cases} \tag{3.6.23}$$

where we defined a rescaled spectral parameter $\mu = \lambda q^{\frac{1-J}{2}}$ and for any vector $|\boldsymbol{j}\rangle$, $|\boldsymbol{j}| = J$

$$\mathcal{Z}|\boldsymbol{j}\rangle = q^{|\boldsymbol{j}|}|\boldsymbol{j}\rangle, \quad \mathcal{Z} = \prod_{s=1}^{n} Z_s. \tag{3.6.24}$$

In fact, we can consider (3.6.23) as an operator solution of the algebra (3.6.18) since a rescaling of the spectral parameter does not affect (3.6.18). The operator $\mathcal{Z}$ commutes with (3.6.23) and all representations are characterised by its complex eigenvalue q^J.

3.7 Rational limit

In the chapters that follow we will consider some other approaches to constructing R-matrices. We will compare our results obtained from the 3D approach with these other methods. Unfortunately, not all of these approaches have been developed in the quantum case and from them we can only get a result for rational R-matrices. So the only way we can compare the results from this chapter is to construct rational R-matrices from our formula (3.3.24). To begin the construction, let us consider

$$q = e^h, \quad \lambda = e^{\mu h}, \quad h \to 0 \tag{3.7.1}$$

also known as the *rational limit* because entries of the R-matrix go from being trigonometric functions in h, μ to rational functions in μ. The limit $q \to 1$ is equivalent to calculating the limit $h \to 0$. To calculate this, we note that the leading term in the asymptotics of the q-Pochhammer symbol is

$$(q^{2x}; q^2)_n \sim (-2h)^n\, (x)_n, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{3.7.2}$$

as $h \to 0$ for any finite real x and integer n.

Substituting (3.7.2) into (3.3.24) with normalization (3.3.29) it is easy to see that because the number of factors in the numerator and denominator are always equal that the leading term in h of any nonzero matrix element of the R-matrix is finite and given by

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{i',j'}_{i,j} = R^{(n),r}_{I,J}(\mu)^{i',j'}_{i,j}\,(1 + O(h)), \tag{3.7.3}$$

with

Proposition 3.7.1.

$$\begin{aligned}\left[R^{(n),r}_{I,J}(\mu)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} &= \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} \left[A^{(n),r}_{I,J}(\mu)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} \\ &\times \sum_{\boldsymbol{m}\in\mathbb{Z}^{n-1}_{+}} \frac{(\mu - \frac{I}{2} - \frac{J}{2}, \mu + 1 + \frac{I}{2} + \frac{J}{2} - |\boldsymbol{i}| - |\boldsymbol{j}|)_{|\boldsymbol{m}|}}{(\mu + 1 + \frac{I}{2} - \frac{J}{2} - |\boldsymbol{i}|, \mu + 1 + \frac{J}{2} - \frac{I}{2} - |\boldsymbol{j}'|)_{|\boldsymbol{m}|}} \prod_{s=1}^{n-1} \frac{(-i_s, -j'_s)_{m_s}}{m_s!(-i_s - j_s)_{m_s}}.\end{aligned} \tag{3.7.4}$$

Like the q-deformed case all external and summation indices in (3.7.4) have $n-1$ components and the coefficient $A^{(n),r}_{I,J}(\mu)^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}$ is given by

$$\left[A^{(n),r}_{I,J}(\mu)\right]^{i',j'}_{i,j} = \frac{(-\mu + \frac{I}{2} - \frac{J}{2})_{|\boldsymbol{j}'|}(-\mu + \frac{J}{2} - \frac{I}{2})_{|\boldsymbol{i}|}(-J)_{|\boldsymbol{j}|}}{(-\mu - \frac{I}{2} - \frac{J}{2})_{|\boldsymbol{i}+\boldsymbol{j}|}(-J)_{|\boldsymbol{j}'|}} \prod_{s=1}^{n-1} \binom{i_s + j_s}{j_s}. \tag{3.7.5}$$

The normalisation (3.3.29) is important to the asymptotics. If we chose instead to leave the factor $B_{I,J}(\lambda)$ in (3.3.26) then the leading term in h would be $1/h$ in (3.7.3) instead. This is because the product in the numerator of (3.3.26) has one less factor than the denominator. The other benefit of leaving out $B_{I,J}(\lambda)$, as mentioned earlier in the quantum case, is that the formula remains valid even for complex weights. From the change of variables (3.7.1) it is easy that the R-matrix (3.7.4) satisfies the Yang-Baxter equation with additive spectral parameters

$$R^{(n),r}_{I_1,I_2}(u)R^{(n),r}_{I_1,I_3}(u+v)R^{(n),r}_{I_2,I_3}(v) = R^{(n),r}_{I_2,I_3}(v)R^{(n),r}_{I_1,I_3}(u+v)R^{(n),r}_{I_1,I_2}(u) \tag{3.7.6}$$

for all $I_1, I_2, I_3 \in \mathbb{C}$ similar to (3.3.24). As in the trigonometric case all sums over internal indices in (3.7.6) contain a finite number of terms even in the case of complex weights.

Now let us consider (3.7.4) for n=2. In this case the formula becomes

$$R^{(2),r}_{I,J}(\mu)^{i',j'}_{i,j} = \delta_{i+j,i'+j'} \binom{i+j}{i} \frac{(-\mu+\frac{I}{2}-\frac{J}{2})_{j'}(-\mu+\frac{J}{2}-\frac{I}{2})_{i}(-J)_{j}}{(-\mu-\frac{I}{2}-\frac{J}{2})_{i+j}(-J)_{j'}} \qquad (3.7.7)$$
$$\times\, {}_4F_3 \left(\begin{matrix} -i & -j' & \mu-\frac{I}{2}-\frac{J}{2} & \mu+1+\frac{I}{2}+\frac{J}{2}-i-j \\ & -i-j & \mu+1+\frac{I}{2}-\frac{J}{2}-i & \mu+1+\frac{J}{2}-\frac{I}{2}-j' \end{matrix} \middle| 1 \right).$$

When the weights I, J are integers we obtain the standard higher-spin XXX R-matrix with spins $I/2$ and $J/2$.

As we see from (3.7.4) and (3.7.7) the R-matrix theory in the rational limit is in some sense similiar to the q-deformed case. Basic hypergeometric series degenerate to classical hypergeometric series and q-pochammer symbols degenerate to classical pochammer symbols. Also, the phase factors present in the summation and products of (3.3.24) reduce to unity in the rational limit.

3.7.1 Rational L-operator

We are also interested in the L-operator in the rational limit. In later chapters we will make a comparison between the L-operator derived from the 3D model construction and those constructed by other means. For the same reasons explained in section 3.6 we need to multiply the formula (3.7.4) by a constant factor which we will call

$$\sigma^{\mathrm{r}}(\mu) = -(\mu - \frac{I}{2} - \frac{J}{2})_{I+1}, \qquad (3.7.8)$$

analogous to σ in (3.6.2). That is, we will work with the formula

$$\left[\bar{R}^{(n),r}_{I,J}(\mu)\right] = \sigma^{\mathrm{r}}(\mu) \left[R^{(n),r}_{I,J}(\mu)\right]. \qquad (3.7.9)$$

The only difference to (3.7.4) is the coefficient $A^{(n),r}_{I,J}(\mu)$ is replaced by a new coefficient $\bar{A}^{(n),r}_{I,J}(\mu)$ given by

$$\left[\bar{A}^{(n),r}_{I,J}(\mu)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = \frac{(-\mu-\frac{I}{2}-\frac{J}{2}+|\boldsymbol{i}|+|\boldsymbol{j}|)_{I-|\boldsymbol{i}'|}(-\mu+\frac{J}{2}-\frac{I}{2})_{|\boldsymbol{i}|}}{(-1)(-J+|\boldsymbol{j}|)_{|\boldsymbol{i}-\boldsymbol{i}'|}} \prod_{s=1}^{n-1} \frac{(1+j_s)_{i_s}}{i_s!} \qquad (3.7.10)$$

similar to (3.6.4). Now the formula is valid for abstract values of J,$\boldsymbol{j}$ indices. To write down the L-operator we will use the same n-component vector notation as in the quantum case (3.6.5). We have

$$\left[\bar{R}^{(n),r}_{1,J}(\mu)\right]^{e_\beta,\boldsymbol{k}}_{e_\alpha,\boldsymbol{j}} = \delta_{\alpha,\beta}(\frac{1}{2}+\mu-\frac{J}{2})+k_\alpha. \tag{3.7.11}$$

We can also write the rational L-operator with operator valued entries analogous to (3.6.23). Let us consider the case $J \in \mathbb{Z}^+$. Recall from section 2.4.2 that symmetric tensor representations of integral weight $J\omega_1$ of $U_q(\widehat{sl_n})$ act on the space of homogenous polynomials of degree J in n variables. Let us identify the abstract basis of n-component vectors $|\boldsymbol{j}\rangle = |j_1,\ldots,j_n\rangle$ used in writing down (3.6.23) with homogenous polynomials by

$$z_1^{j_1}\ldots z_n^{j_n} \leftrightarrow |j_1,\ldots,j_n\rangle\,, \tag{3.7.12}$$

then the rational L-operator $L^r(\mu)$ can be written down as a $n\times n$ matrix with operator entries

$$L^{(n),r}_{\alpha,\beta}(\mu) = \delta_{\alpha,\beta}(1+\mu-\frac{J}{2})+z_\beta\partial_\alpha. \tag{3.7.13}$$

For general $J\in\mathbb{C}$ the the entries of the $L^{(n),r}(\mu)$ act on an infinite dimensional Verma module and the space cannot be realised as the space of homogeneous polynomials. Instead, it can be realised as polynomials in $n-1$ variables of arbitrary degree - $\mathbb{C}[z_1,\ldots,z_{n-1}]$. Writing down a closed formula for the L-operator for any rank n like (3.7.13) is more difficult in this realisation of the representation space. However, for particular algebras we can write down an $n\times n$ matrix of operators. For example,

$$L^{(2),r}(\mu) = \begin{pmatrix} \frac{1}{2}+\mu+\frac{J}{2}-z\partial & Jz-z^2\partial \\ \partial & \frac{1}{2}+\mu-\frac{J}{2}+z\partial \end{pmatrix}, \tag{3.7.14a}$$

$$L^{(3),r}(\mu) = \begin{pmatrix} \frac{1}{2}+\mu+\frac{J}{2}-z_1\partial_1-z_2\partial_2 & z_2(J-z_1\partial_1-z_2\partial_2) & z_1(J-z_1\partial_1-z_2\partial_2) \\ \partial_2 & \frac{1}{2}+\mu-\frac{J}{2}+z_2\partial_2 & z_1\partial_2 \\ \partial_1 & z_2\partial_1 & \frac{1}{2}+\mu-\frac{J}{2}+z_1\partial_1 \end{pmatrix}. \tag{3.7.14b}$$

Of course (3.7.14) is equivalent to (3.7.13) for $J\in\mathbb{Z}^+$. In this case the difference is an extra variable z_n that homogenises the polynomials in (3.7.14).

3.7.2 Factorization

Given the reductions (3.5.1), (3.5.2) and factorization (3.5.6) of the trigonometric R-matrix there obviously must exist similar results for the rational R-matrix (3.7.4). One can start from (3.7.4) and make the substitutions $\mu = I - J, \quad I - J \in \mathbb{Z}^+$ and $\mu = J - I, \quad J - I \in \mathbb{Z}^+$ and make the same arguments given in the beginning of section (3.5) but for classical pochammer symbols to find

$$\left[R_{I,J}^{(n),r}(\frac{I-J}{2})\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-J)_{|j|}(J-I)_{|i'-j|}}{(-I)_{|i'|}} \prod_{s=1}^{n-1} \binom{i'_s}{j_s}, \quad I - J \in \mathbb{Z}^+ \tag{3.7.15}$$

$$\left[R_{I,J}^{(n),r}(\frac{J-I}{2})\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-I)_{|i|}(I-J)_{|j'-i|}}{(-J)_{|j'|}} \prod_{s=1}^{n-1} \binom{j'_s}{i_s}, \quad J - I \in \mathbb{Z}^+. \tag{3.7.16}$$

Of course one could also just take the rational limit of (3.5.1) and (3.5.2). In any case, we can define

$$\boldsymbol{M}^r(I,J) := \check{R}_{I,J}^{(n),r}(\frac{I-J}{2}), \quad \boldsymbol{N}^r(I,J) := \check{R}_{I,J}^{(n),r}(\frac{J-I}{2}) \tag{3.7.17}$$

analogous to (3.5.5). Let us note that no similarity transform on (3.7.4) is required here unlike (3.5.3) and (3.5.4) in the trigonometric case. Functions $\boldsymbol{M}^r(I,J), \boldsymbol{N}^r(I,J)$ depend only on complex parameters I, J and the dependence is algebraic. Using these functions the rational R-matrix $\check{R}_{I,J}^{(n),r}(\mu)$ can be written in the factorised form

Proposition 3.7.2.

$$\check{R}_{I,J}^{(n),r}(\mu) = \boldsymbol{M}^r(\mu + \frac{I+J}{2}, J)\boldsymbol{N}^r(-\mu + \frac{I+J}{2}, J). \tag{3.7.18}$$

Removing the permutation operator in $\check{R}$ gives back the original rational R-matrix (3.7.4), which at the level of matrix elements can be written as

$$\left[R_{I,J}^{(n),r}(\lambda)\right]_{i,j}^{i',j'} = \sum_{k+l=i+j} \tilde{\boldsymbol{M}}^{r\,k,l}_{\;\;i,j}\; \tilde{\boldsymbol{N}}^{r\,i',j'}_{\;\;k,l} \tag{3.7.19}$$

where

$$\tilde{\boldsymbol{M}}^{r\,i',j'}_{\;\;i,j} = \delta_{i+j,i'+j'} \frac{(-J)_{|j|}(-\mu + \frac{J-I}{2})_{|i'-j|}}{(-\mu - \frac{I+J}{2})_{|i'|}} \prod_{s=1}^{n-1} \binom{i'_s}{j_s}, \tag{3.7.20a}$$

$$\tilde{\boldsymbol{N}}^{r\,i',j'}_{\;\;i,j} = \delta_{i+j,i'+j'} \frac{(\mu - \frac{I+J}{2})_{|j|}(-\mu + \frac{I-J}{2})_{|j'-j|}}{(-J)_{|j'|}} \prod_{s=1}^{n-1} \binom{j'_s}{j_s}. \tag{3.7.20b}$$

We will return to these results in later chapters when making comparisons with some other factorisation techniques. To finish this section, we will write down the symmetries of (3.7.4) which are easily deduced from (3.4.1) - (3.4.6). They are

$$\left[R^{(n),r}_{I,J}(\mu)\right]^{i',j'}_{i,j} = \left[R^{(n),r}_{J,I}(\mu)\right]^{j',i'}_{j,i}, \tag{3.7.21a}$$

$$= \left[R^{(n),r}_{J,I}(\mu)\right]^{\tau j',\tau i'}_{\tau j,\tau i}, \tag{3.7.21b}$$

$$= \prod_{s=1}^{n} \frac{i'_s! j'_s!}{i_s! j_s!} \left[R^{(n),r}_{I,J}(\mu)\right]^{\tau i,\tau j}_{\tau i',\tau j'}, \tag{3.7.21c}$$

$$= \left[R^{(n),r}_{I,J}(\mu)\right]^{\bar{i}',\bar{j}'}_{\bar{i},\bar{j}}. \tag{3.7.21d}$$

3.8 A polynomial representation

In the last section we noted that the formula (3.3.24) contains λ in the denominator terms of the hypergeometric sum and hence each summand is a rational function in this variable. It would be interesting to determine whether a formula exists where the summands only contain λ in the numerator and hence are polynomials. We pose this as a problem; can (3.3.24) be transformed so that the sum only contains λ in the numerator?

We can at least give a partial answer to this problem. For $n = 2$ it can be done using Sears transform for ${}_4\phi_3$ hypergeometric series. For $n > 2$ we can also do it but only at the cost of adding extra summations to the expression. As mentioned earlier, at this stage we are not sure if it is possible to transform without adding extra sums, but if it is then it probably requires some yet to be discovered identities for multivariable hypergeometric series.

In any case, let us take (3.3.30) and apply a similarity transform in the auxiliary space by defining

$$\left[\tilde{R}^{(2)}_{I,J}(\lambda)\right]^{i',j'}_{i,j} = \lambda^{i-i'} \left[R^{(2)}_{I,J}(\lambda)\right]^{i',j'}_{i,j}. \tag{3.8.1}$$

It is easy to check that the Yang-Baxter equation is not affected by this modification. Now we make a Sears' transform (B.2.14) by identifying

$$n = i, \quad a = q^{-2j'}, \quad b = \lambda^2 q^{-I-J}, \quad c = \lambda^2 q^{2+I+J-2i-2j},$$

$$d = \lambda^2 q^{2+I-J-2i}, \quad e = q^{-2i-2j}, \quad f = \lambda^2 q^{2+J-I-2j'}. \tag{3.8.2}$$

The result is

$$\begin{aligned}\left[\tilde{R}^{(2)}_{I,J}(\lambda)\right]^{i',j'}_{i,j} &= \delta_{i+j,i'+j'}\,(-1)^i \frac{q^{i(i+j-2J-1)-Ij+i'(I+j')}}{\lambda^{i+i'}(q^2;q^2)_i}\frac{(q^{-2J};q^2)_j(\lambda^{-2}q^{+I-J};q^2)_{j-i'}}{(q^{-2J};q^2)_{j'}(\lambda^{-2}q^{-I-J};q^2)_{i+j}}\\ &\times {}_4\overline{\phi}_3\left(\begin{matrix} q^{-2i}; & q^{-2i'} & \lambda^{-2}q^{J-I} & \lambda^2 q^{2+J-I} \\ & q^{-2I} & q^{2(1+j-i')} & q^{2(1+J-i-j)}\end{matrix}\middle|\, q^2, q^2\right). \end{aligned} \tag{3.8.3}$$

This formula for the $U_q(\widehat{sl_2})$ R-matrix first appeared (up to normalisation) in (5.8) of [10]. Let us note that in writing down this formula we have used the regularisation scheme (B.1.10) and a regularised version of Sears' transform (B.2.14). The reason we regularise in this way is because the q-pochammer terms in the denominator are of the form $(q^{-2n};q^2)_r$ which is zero when $r > n$. Over the summation range from 0 to $\mathrm{Min}(i,i')$ this inequality is sometimes satisfied and the summation is not well defined. However, in all such cases these poles are cancelled off by zeroes in the products outside the ${}_4\phi_3$ summation and so the poles are can be removed by our choice of regularisation.

In most cases a regularisation is not necessary. Of course, (3.3.30) does not need it, but there are polynomial representations that mostly do not require it. For example, transforming (3.8.3) using Sears' transform (B.2.13) with

$$\begin{aligned} n = i, \quad a = q^{-2i'}, \quad c = \lambda^{-2}q^{2+J-I}, \quad d = \lambda^{-2}q^{J-I},\\ d = q^{-2I}, \quad e = q^{2J-2j-2i+2}, \quad f = q^{2+2j-2i'}, \end{aligned}$$

and removing regularisation we get the formula

$$\begin{aligned}\left[\bar{R}^{(2)}_{I,J}(\lambda)\right]^{i',j'}_{i,j} &= \delta_{i+j,i'+j'}\,q^{i'j'-ij+I(i-j')}\begin{bmatrix} i+j \\ i\end{bmatrix}_{q^2}\frac{(q^{-2I};q^2)_i(q^{-2J};q^2)_j(\lambda^{-2}q^{I-J};q^2)_{j-i'}}{\lambda^{i+i'}(q^{-2J};q^2)_{j-i'}(\lambda^{-2}q^{-I-J};q^2)_{i+j}}\\ &\times {}_4\phi_3\left(\begin{matrix} q^{-2i} & q^{-2i'} & \lambda^2 q^{-I-J} & \lambda^{-2}q^{-2-I-J} \\ & q^{-2I} & q^{-2(i+j)} & q^{-2(J-j+i')}\end{matrix}\middle|\, q^2, q^2\right). \end{aligned} \tag{3.8.4}$$

Unlike (3.3.30) and (3.8.3) this series does not appear to always terminate naturally after finitely many terms. This is because of the denominator arguments of the form q^{-m}, $m \in \mathbb{Z}^+$ which can cancel numerator terms that would otherwise terminate the series. However, this series should still be considered to be finite and truncated after i terms because it was transformed using identity (B.2.13) which is for finite sums. Of course, there is also the danger of poles from the denominator terms but this is not a problem for finite-dimensional R-matrices or when I, J are non-integral. In the former case the sum is truncated before these terms becomes zero and in the latter case the sum terminates. The only case where it is a problem is for infinite-dimensional R-matrices where $I, J \in \mathbb{Z}^+$ and $i^{(\prime)} > I, j^{(\prime)} > J$. In this case the expression requires a regularisation such as (B.1.10).

In the following chapters we want to compare some of the results obtained with (3.8.3) as well as its rational limit $q \to 1$. Following (3.7.1) - (3.7.3) we get

$$\left[R^{(2),r}_{I,J}(\mu)\right]^{i',j'}_{i,j} = \delta_{i+j,i'+j'}(-1)^i \frac{(-J)_j(-\mu+\frac{I}{2}-\frac{J}{2})_{j-i'}}{i!(-J)_{j'}(-\mu-\frac{I}{2}-\frac{J}{2})_{i+j}} \times {}_4\overline{F}_3\left(\begin{matrix}-i; & -i' & -\mu+\frac{J}{2}-\frac{I}{2} & \mu+1+\frac{J}{2}-\frac{I}{2} \\ & -I & 1+j-i' & 1+J-i-j\end{matrix}\middle| 1\right). \tag{3.8.5}$$

Obviously $R^{(2)}$ and $\tilde{R}^{(2)}$ have the same limit as $q \to 1$ because the leading term in the diagonal similarity transform (3.8.1) is 1.

The $n > 2$ case is more complicated, and best possible result we have found so far pertains to the rational limit, where we have succeeded at the cost of adding an extra summation to the expression so that it is an n-fold summation rather than an $(n-1)$-fold summation like (3.3.24) and (3.7.4). It would be nice to find a $(n-1)$-fold summation polynomial formula but it is at this stage unclear to us if it exists. In any case, we have found that

$$\left[R^{(n),r}_{I,J}(\mu)\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'} \frac{(-I)_{|\boldsymbol{i}|}(-J)_{|\boldsymbol{j}|}(-\mu+\frac{I-J}{2})_{|\boldsymbol{j}-\boldsymbol{i}'|}}{(-J)_{|\boldsymbol{j}-\boldsymbol{i}'|}(-\mu-\frac{I+J}{2})_{|\boldsymbol{i}+\boldsymbol{j}|}} \prod_{s=1}^{n-1}\binom{i_s+j_s}{i_s}$$
$$\times \sum_{\boldsymbol{m},m_n} \frac{(\mu+\frac{I+J}{2}-|\boldsymbol{i}+\boldsymbol{j}|+1)_{|\boldsymbol{m}|}}{(-|\boldsymbol{i}|,-|\boldsymbol{i}'|)_{|\boldsymbol{m}|}} \frac{(-|\boldsymbol{i}|,-|\boldsymbol{i}'|,\mu-\frac{I+J}{2})_{|\boldsymbol{m}|+m_n}}{m_n!(-I,J-|\boldsymbol{j}|+|\boldsymbol{i}'|)_{|\boldsymbol{m}|+m_n}} \prod_{s=1}^{n-1} \frac{(-1)^{m_s}(-i_s,-i'_s)_{m_s}}{m_s!(-i_s-j_s)_{m_s}},$$

where $\boldsymbol{m} = \{m_1, \ldots m_{n-1}\}$ as in (3.7.4), $0 \le \boldsymbol{m} \le \boldsymbol{i}$ and $0 \le m_n \le |\boldsymbol{i}-\boldsymbol{m}|$. We note that for finite matrices and non-integral Verma modules $I, J \in \mathbb{C}$ the formula does not need regularisation.

Chapter 4

Spectral decomposition

In this chapter we will review another method for constructing R-matrices. This method we will refer to as *spectral decomposition*. This method is well known in the literature [16; 17; 18], and is perhaps the oldest of all the approaches considered in this thesis. It is quite different to the 3D model approach of the last chapter because it uses quantum groups and their representation theory directly. We are interested in constructing the R-matrix using this approach and comparing it with the other newer constructions.

We can show that at least for $U_q(\widehat{\mathfrak{sl}_2})$ the R-matrix is the same as that obtained from the 3D approach. To show this we construct an explicit formula for the matrix elements for finite-dimensional highest weight representations. The formula at first sight appears to be very different to (3.3.24) obtained in Chapter 3 with very different structure. It turns out both formulas actually produce the same output and to prove they are indeed the same we will give the transformation between the two.

First we will give an overview of the method including all of the relevant theory and then we will apply it to the case of $U_q(\widehat{\mathfrak{sl}_2})$.

4.1 The Jimbo equations

Perhaps the central idea behind this method is notion of the R-Matrix as an intertwiner of two quantum group representations. Let us recall (2.4.43)

$$\mathscr{R}\Delta(g) = \Delta^{op}(g)\mathscr{R} \quad \forall g \in \mathcal{A}, \tag{4.1.1}$$

satisfied by the universal R-matrix. Because Δ is an algebra homomorphism it is enough to consider the relation on the generators of $U_q(\widehat{\mathfrak{g}})$. Indeed, for $U_q(\widehat{sl_n})$ under the evaluation homomorphism (2.4.52) the relation (4.1.1) can be written as n separate relations

$$\mathscr{R}(x,y)\Delta(E_i) = \Delta^{op}(E_i)\mathscr{R}(x,y) \quad 1 \leq i \leq n, \tag{4.1.2a}$$

$$\begin{aligned}&\mathscr{R}(x,y)(xGF_\theta \otimes q^{H_\theta/2} + q^{-H_\theta/2} \otimes yGF_\theta)\\ &= (yGF_\theta \otimes q^{-H_\theta/2} + q^{H_\theta/2} \otimes xGF_\theta)\mathscr{R}(x,y)\end{aligned} \tag{4.1.2b}$$

for the universal R-matrix and coproduct. Taking two affinisable finite-dimensional representations π^I, π^J and applying $\pi^I \otimes \pi^J$ to these relations as in (2.4.60) we get a set of linear equations that can be solved for $R_{I,J}(\lambda)$, $\lambda = x/y$. Just like (2.4.60) we will denote the representation map $\pi^I \otimes \pi^J$ applied to Δ by

$$\Delta_{I,J}(a) := (\pi^I \otimes \pi^J)(\Delta(a)). \tag{4.1.3}$$

It is true that $\pi^I \otimes \pi^J$ is a representation of $U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})$ on the representation space $V_I \otimes V_J$. That is,

$$\pi^I \otimes \pi^J : U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}}) \to \mathrm{End}(V_I) \otimes \mathrm{End}(V_J), \tag{4.1.4}$$

$$(\pi^I \otimes \pi^J)(a \otimes b) \cdot (v \otimes u) = (\pi^I(a) \cdot v) \otimes (\pi^J(b) \cdot u) \quad \forall a, b \in \widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}} \tag{4.1.5}$$

and $\mathrm{End}(V_I) \otimes \mathrm{End}(V_J) \cong \mathrm{End}(V_I \otimes V_J)$ at least for the finite dimensional representation spaces we consider.

Due to Hopf algebra structure of the quantum group Δ is an algebra homomorphism and hence $\Delta_{I,J}$ can be viewed as a representation of $U_q(\widehat{\mathfrak{g}})$ on the space $V_{\boldsymbol{I}} \otimes V_J$ - the pullback of $\pi^I \otimes \pi^J$ by Δ. With this in mind let us define a modified R-Matrix $\check{R}_{I,J} : V_I \otimes V_J \to V_J \otimes V_I$ by

$$\check{R}_{I,J} = \mathbb{P}R_{I,J}, \quad \mathbb{P}(v \otimes u) = u \otimes v, \tag{4.1.6}$$

and rewrite the relations (4.1.2) in terms of $\check{R}_{I,J}(\lambda)$ for a given representation $\Delta_{I,J}$. We have

$$\check{R}_{I,J}(\lambda)\Delta_{I,J}(a) = \Delta_{J,I}(a)\check{R}_{I,J}(\lambda), \quad \forall a \in U_q(\mathfrak{g}), \tag{4.1.7a}$$

$$\begin{aligned}&\check{R}_{I,J}(\lambda)(\lambda\pi^I(GF_\theta) \otimes \pi^J(q^{H_\theta/2}) + \pi^I(q^{-H_\theta/2}) \otimes \pi^J(GF_\theta))\\ &= (\pi^J(GF_\theta) \otimes \pi^I(q^{H_\theta/2}) + \lambda\pi^J(q^{-H_\theta/2}) \otimes \pi^I(GF_\theta))\check{R}_{I,J}(\lambda).\end{aligned} \tag{4.1.7b}$$

In other words, $\check{R}_{I,J}(\lambda)$ intertwines the representations $\Delta_{I,J}$ and $\Delta_{J,I}$. Equations (4.1.7) are often referred to as the *Jimbo equations*, having first appeared in [16] and been solved in the case of $U_q(\widehat{\mathfrak{sl}_2})$ when $I = J$. The case of untwisted $U_q(\widehat{\mathfrak{g}})$ was developed for affinisable finite-dimensional representations where $\boldsymbol{I} = \boldsymbol{J}$ and their tensor product is multiplicity free in [17]. This was generalised to $\boldsymbol{I} \neq \boldsymbol{J}$ in [18].

The solution uses the representation theory of quantum groups directly. An important result in the theory is that a tensor product of irreducible finite highest weight representations is reducible. That is, the modules have the direct sum decomposition

$$V_I \otimes V_J \cong \bigoplus_{\nu} V_\nu. \tag{4.1.8}$$

We denote by $P^{\nu}_{I,J} : V_I \otimes V_J \to V_\nu$ the projection operators from the tensor product to one of its summands. The solution $\check{R}(\lambda)$ satisfying (4.1.7a) then takes the form

$$\check{R}_{I,J}(\lambda) = \sum_{\nu} \rho_\nu(\lambda) \check{\boldsymbol{P}}^{\nu}_{I,J} \tag{4.1.9}$$

where ν is a summation over the components of the direct-sum decomposition (4.1.8). The $\check{\boldsymbol{P}}^{\nu}_{I,J}$ are operators defined by

$$\check{\boldsymbol{P}}^{\nu}_{I,J} := P^{\nu}_{J,I} \check{R}_{I,J}(1) = \check{R}_{I,J}(1) P^{\nu}_{I,J} \tag{4.1.10}$$

and are themselves a projection operators, satisfying

$$\check{\boldsymbol{P}}^{\nu}_{J,I} \check{\boldsymbol{P}}^{\nu'}_{I,J} = P^{\nu}_{I,J} P^{\nu'}_{I,J} = \delta_{\nu,\nu'} P^{\nu}_{I,J}. \tag{4.1.11}$$

The functions $\rho_\nu(\lambda)$ are eigenvalues associated with each projection operator. Their form can be determined by (4.1.7b) which was found in [18] to be

$$\rho_\nu(\lambda) = \frac{1 - \lambda q^{C(\nu')/2 - C(\nu)/2}}{\lambda - q^{C(\nu')/2 - C(\nu)/2}} \rho_{\nu'}(\lambda), \quad \nu \neq \nu' \tag{4.1.12}$$

where $C(\nu)$ is the eigenvalue of the Casimir operator action on the irreducible highest weight module $V(\nu)$. Picking a specific algebra this relation determines an explicit formula for $\rho_\nu(\lambda)$ up to normalisation.

It was also shown in [18] that given an orthonormal basis $\{|v^{(\nu)}_\alpha\rangle_{I\otimes J}\}_\alpha$ of $V_\nu \subset V_I \otimes V_J$

$$\begin{bmatrix} I & J & I+J-2k \\ i & j & m \end{bmatrix} =$$

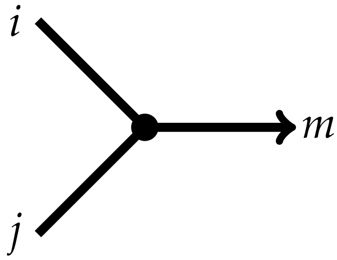

Figure 4.1: Tensor representation of Clebsch-Gordan coefficients

one can write down the projection operators $\check{P}^{\nu}_{I,J}$ as

$$\check{P}^{\nu}_{I,J} = \sum_{\alpha} |v^{(\nu)}_{\alpha}\rangle_{J\otimes I\ I\otimes J}\, \langle v^{(\nu)}_{\alpha}|\,. \tag{4.1.13}$$

This basis can be constructed from quantum Clebsch-Gordan coefficients, although this can be a difficult problem in itself. Nevertheless, combining (4.1.9), (4.1.12) and (4.1.13) we can construct the R-Matrix using the representation theory of the associated quantum group. In the section section we will apply this theory to the case of $U_q(\widehat{sl_2})$ for finite-dimensional highest weight representations.

4.2 The case $U_q(\widehat{sl_2})$

4.2.1 Quantum Clebsch-Gordan coefficients

For $U_q(\mathfrak{sl}_2)$ all finite-dimensional highest weight representation are specified by a positive integer I. The direct sum decomposition of the tensor product of two such representations is

$$V_I \otimes V_J \cong \bigoplus_{k=0}^{\min(I,J)} V_{I+J-2k}. \tag{4.2.1}$$

As mentioned in the last section, each space V_{I+J-2k} has a basis $\{|v^{(I+J-2k)}_m\rangle_{I\otimes J}\}_{m=0}^{I+J-2k}$ which we can write in terms of the natural basis $\{|i\rangle_I \otimes |j\rangle_J\}_{i,j=0}^{I,J}$ of $V_I \otimes V_J$ by

$$|v^{(I+J-2k)}_m\rangle_{I\otimes J} = \sum_{i,j=0}^{I,J} \begin{bmatrix} I & J & I+J-2k \\ i & j & m \end{bmatrix} |i\rangle_I \otimes |j\rangle_J\,. \tag{4.2.2}$$

The coefficients are often referred to as *Quantum Clebsch-Gordan coefficients* or *Quantum 3jm-symbols*. They are a third order tensor as illustrated in figure 4.1. Their explicit values depend on the choice of action of the algebra on the highest weight module V_I. In this section we will define the action of the generators $\{E, F, q^H, q^{-H}\}$ by

$$E\,|i\rangle_I = a(I,i)\,|i-1\rangle_I\,, \quad F\,|i\rangle_I = b(I,i)\,|i+1\rangle_I\,, \quad q^H\,|i\rangle_I = c(I,i)\,|i\rangle_I\,, \tag{4.2.3a}$$

$$e(I,i) = \sqrt{[i, I-i+1]_q}, \quad f(I,i) = \sqrt{[i+1, I-i]_q}, \quad h(I,i) = q^{I-2i}. \tag{4.2.3b}$$

This definition is different to our earlier definition in (2.4.59) although the modules are isomorphic. As discussed in the last section, this action can be extended to $V_I \otimes V_J$ using (4.1.3) and (2.4.15). The reason we use this action is that it enables us to construct coefficients such that $|v_m^{(I+J-2k)}\rangle_{I\otimes J}$ are self-dual with respect to the standard inner product. That is

$$\langle v_s^n | v_t^m \rangle = \delta_{n,m}\delta_{s,t}, \tag{4.2.4}$$

which makes the construction of matrix elements for $\check{\boldsymbol{P}}^{\nu}_{I,J}$ from the Clebsch-Gordan coefficients much simpler. In fact, defined like this the formula for matrix elements of the projection operators has the form

$$(\mathcal{P}^k_{I,J})^{i',j'}_{i,j} = \sum_{s=0}^{I+J-2k} \begin{bmatrix} J & I & I+J-2k \\ i & j & s \end{bmatrix} \begin{bmatrix} I & J & I+J-2k \\ i' & j' & s \end{bmatrix}. \tag{4.2.5}$$

We also need the eigenvalues $\rho_k(\lambda)$, which can be calculated explicitly from (4.1.12) as

$$\rho_k(\lambda) = \rho_0(\lambda) \prod_{s=1}^{k} \frac{1-\lambda q^{I+J-2s+2}}{\lambda - q^{I+J-2s+2}} \tag{4.2.6}$$

up to a normalisation ρ_0. Combining these objects together and using (4.1.9) and (4.1.6) we obtain the following formula for $R_{I,J}(\lambda)$ in terms of Clebsch-Gordan coefficients:

$$[R_{I,J}(\lambda)]^{i'j'}_{ij} = \\ = \rho_0(\lambda) \sum_{k=0}^{\min(I,J)} \sum_{s=0}^{I+J-2k} \prod_{j=1}^{k} \frac{1-\lambda q^{(I+J-2j+2)}}{\lambda - q^{(I+J-2j+2)}} \begin{bmatrix} J & I & I+J-2k \\ j & i & s \end{bmatrix} \begin{bmatrix} I & J & I+J-2k \\ i' & j' & s \end{bmatrix}. \tag{4.2.7}$$

This construction exhibits structure of the R-matrix that is unlike any of the other methods examined in this thesis. Its tensor composition from the base building block Clebsch-Gordan coefficients is represented in figure 4.2. All that is left is to write down

$$[R(\lambda)]_{i,j}^{i',j'} = \sum_{k} \rho_k(\lambda)$$

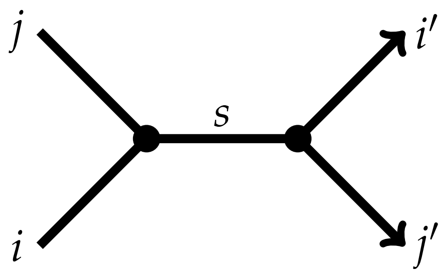

Figure 4.2: R-matrix representation as a sum of projection operators

the expression for these coefficients and we have a formula for the matrix elements of $R_{I,J}(\lambda)$. The coefficients we use are given by

$$\begin{bmatrix} I & J & I+J-2k \\ i & j & m \end{bmatrix} = \delta_{i+j,k+m}(-1)^{i+j} q^{-j^2+mj-2ij-2kj+(i-k)(J-k)}$$
$$\times \sqrt{\frac{(q^2;q^2)_i (q^2;q^2)_{J-j} (q^2;q^2)_{I-k} (q^2;q^2)_{I+J-2k-m} (q^4;q^2)_{I+J-2k}}{(q^2;q^2)_j (q^2;q^2)_{I-i} (q^2;q^2)_{J-k} (q^2;q^2)_k (q^2;q^2)_m (q^4;q^2)_{I+J-k}}}$$
$$\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j}; & q^{-2m} & q^{2+2J-2j} \\ & q^{2+2i-2m} & q^{2i-2m-2I} \end{matrix} \middle| q^2, q^{2i+2j+2k-2I-2J} \right). \tag{4.2.8}$$

We constructed this formula using the action (4.2.3). The highest weight vector $|v_0^\nu\rangle_{I\otimes J}$ in each submodule can be identified by solving

$$\Delta_{I,J}(E) \cdot |v_0^\nu\rangle_{I\otimes J} = 0.$$

The string basis of each submodule can be generated by repeated applications of the 'lowering operator' F to the highest weight vector. That is,

$$|v_m^\nu\rangle_{I\otimes J} = \Delta_{I,J}(F^m) \cdot |v_0^\nu\rangle_{I\otimes J}.$$

Some analysis of (4.2.8) is warranted. One can use the Karlsson-Minton summation formula (B.2.4) on the ${}_3\phi_2$ hypergeometric series with

$$a = q^{-2j}, \quad b_1 = q^{2i-2m-2I}, \quad m_1 = I - i,$$
$$b_2 = q^{2+2i-2m}, \quad m_2 = J + m - i - j,$$

to see that the quantum Clebsch-Gordon coefficients are non-trivially zero whenever $m > I + J - 2k$. In addition, it is easy to see that the expression is trivially zero when $m < 0$ due to the $(q^2;q^2)_m$ q-Pochhammer in the denominator.

This analysis is important when reducing the quadruple sum in (4.2.7) to a triple sum. The conservation law $\delta_{i+j,m+k}$ in (4.2.8) allows us to eliminate the sum over 's' in (4.2.7). In doing so the sum over 'k' may include terms containing Clebsch-Gordon coefficients with arguments outside its domain. In particular, it may happen that the inequality $0 \le m \le I + J - 2k$ is broken. Due to the discussion in the previous paragraph these terms will always be zero and not contribute to the sum.

4.2.2 An explicit formula

The conservation laws of both Clebsch-Gordon coefficients in (4.2.7) imply the conservation law $\delta_{i+j,i'+j'}$ for the R-matrix. After substituting (4.2.8) into (4.2.7), setting $\rho_0(\lambda) = 1$ and simplifying with q-Pochhammer identities we find the triple sum expression

$$\begin{aligned}
[R_{IJ}(\lambda)]_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} T_{i,j}^{i',j'} \frac{q^{i(3I+5J-2j+4)+j(3I+2J+2)} \left(q^2;q^2\right)_{i'} \left(q^2;q^2\right)_{I+J-i-j} \left(q^2;q^2\right)_{J-j'}}{q^{i'(I+2J+2)} \left(q^2;q^2\right)_i \left(q^2;q^2\right)_{i+j} \left(q^2;q^2\right)_{I+J} \left(q^2;q^2\right)_{J-j}} \\
&\times \sum_r \frac{(-1)^r \lambda^r \left(\lambda^{-1} q^{-I-J}, q^{-2i-2j}, q^{-2I-2J-2}; q^2\right)_r \left(q^{-2I-2J}; q^2\right)_{2r}}{q^{r(I+J+1+2j'-2j)-r^2} \left(q^2, \lambda q^{-I-J}, q^{2i-2I+2j-2J}; q^2\right)_r \left(q^{-2I-2J-2}; q^2\right)_{2r}} \\
&\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2i}; & q^{-2i+2I+2} & q^{-2i-2j+2r} \\ & q^{-2i+2r+2} & q^{-2i-2J+2r} \end{matrix} \middle| q^2, q^{2i-2I+2j-2J+2r} \right) \\
&\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-j'}; & q^{-2i'-2j'+2r} & q^{2J-2j'+2} \\ & q^{2r-2j'+2} & q^{-2I-2j'+2r} \end{matrix} \middle| q^2, q^{2i'-2I+2j'-2J+2r} \right)
\end{aligned} \tag{4.2.9}$$

for elements of the R-matrix where

$$T_{i,j}^{i',j'} = \sqrt{\frac{(q^2,q^2)_i \, (q^2,q^2)_j \, (q^{-2I},q^2)_{i'} \, (q^{-2J},q^2)_{j'}}{(q^2,q^2)_{i'} \, (q^2,q^2)_{j'} \, (q^{-2I},q^2)_i \, (q^{-2J},q^2)_j}}. \tag{4.2.10}$$

The sum over 'r' is finite and over the range $0 \le r \le \min(I, J, i+j, I+J-i-j)$. This is a refinement of the upper bound in (4.2.7) where it is $\min(I, J)$ because the sum can terminate sooner than this in a couple of ways. First, the q-Pochhammer $\left(q^{-2i-2j}; q^2\right)_r$ in the numerator will have a zero whenever $r > i + j$ which may be a smaller bound and therefore terminating the sum earlier. Second, it may happen that $r > I + J - i - j$ and therefore the q-Pochhamer $\left(q^{2i-2I+2j-2J}; q^2\right)_r$ in the denominator is zero. However, this pole is cancelled by the double zero (implied by (B.2.4)) of the two Clebsch-Gordon ${}_3\phi_2$ hypergeometric series and therefore no terms when $r > I + J - i - j$ contribute to the sum. Therefore summing over 'r' from 0 to $\min(I, J, i+j, I+J-i-j)$ includes all of the non-zero terms and the expression is well-defined.

Comparing (4.2.9) to the single summation formula of (3.8.3) we immediately notice the presence of square roots as a major difference between the formulae. It is obvious they come from the Clebsch-Gordan coefficients (4.2.8) which in turn comes our choice of action (4.2.3) in constructing them. Regardless, this is just a gauge transformation of the R-Matrix because these terms cancel off in the internal sum of the Yang-Baxter equation (2.3.23) and the terms left on the external edges are simply diagonal similarity transforms and hence it is unaffected. Actually, comparing the outputs of (4.2.9) and (3.8.3) we find they are related by this transformation as well a few other simple ones. Removing the term T from (4.2.9) the relation can be stated explicitly as

$$\begin{aligned}\tilde{R}^{(2)}_{I,J}(\lambda) &= (M(\lambda)U \otimes U)R_{I,J}(\lambda^{-2})(M^{-1}(\lambda)U^{-1} \otimes U^{-1}),\\ U^m_n &= \delta_{n,m} q^{n^2/2}, \quad M^m_n = \delta_{n,m}\lambda^n.\end{aligned} \tag{4.2.11}$$

These similarity transformations do not affect the Yang-Baxter equation. Applying the transformation (4.2.11) (with T removed) to (4.2.9) we can rewrite the matrix elements of the $U_q(\widehat{sl_2})$ R-matrix as

$$\begin{aligned}[R_{I,J}(\lambda)]^{i',j'}_{i,j} &= \delta_{i+j,i'+j'} \frac{q^{i(3I-3j+5J+i'+4)+j(3I+2J+i'+2)} \left(q^2;q^2\right)_{i'} \left(q^2;q^2\right)_{I+J-i-j} \left(q^2;q^2\right)_{J-j'}}{q^{i'(I+2J+2)+i'^2}\lambda^{i'-i} \left(q^2;q^2\right)_i \left(q^2;q^2\right)_{i+j} \left(q^2;q^2\right)_{I+J} \left(q^2;q^2\right)_{J-j}}\\ &\times \sum_r \frac{(-1)^r \lambda^{-2r} \left(\lambda^2 q^{-I-J}, q^{-2i-2j}, q^{-2I-2J-2}; q^2\right)_r \left(q^{-2I-2J};q^2\right)_{2r}}{q^{r(I+J+1+2j'-2j)-r^2} \left(q^2, \lambda^{-2}q^{-I-J}, q^{2i-2I+2j-2J}; q^2\right)_r \left(q^{-2I-2J-2};q^2\right)_{2r}}\\ &\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2i}; & q^{-2i+2I+2} & q^{-2i-2j+2r} \\ & q^{-2i+2r+2} & q^{-2i-2J+2r} \end{matrix} \middle| q^2, q^{2i-2I+2j-2J+2r} \right)\\ &\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j'}; & q^{-2i'-2j'+2r} & q^{2J-2j'+2} \\ & q^{2r-2j'+2} & q^{-2I-2j'+2r} \end{matrix} \middle| q^2, q^{2i'-2I+2j'-2J+2r} \right)\end{aligned} \tag{4.2.12}$$

The fact that this is consistent with the expression (3.8.3) will be proved in the next section.

4.2.3 Transformation to a single sum

Beginning with (4.2.12) we apply the identities (B.2.16) and (B.2.18) to obtain

$$[R_{I,J}(\lambda)]^{i',j'}_{i,j} = \delta_{i+j,i'+j'} \frac{(-1)^i \lambda^{i-i'} q^{i^2+2I^2+i(J-2I+1)+3i'j'-ij} \left(q^2;q^2\right)_{i'} \left(q^2;q^2\right)_{J-j'}}{q^{-j'(2J-I+2)} \left(q^2;q^2\right)_i \left(q^2;q^2\right)_{I-i} \left(q^2;q^2\right)_{i+j} \left(q^2;q^2\right)_{I+J}}$$

$$\times \sum_r \frac{q^{2r(i+j)+r(I+J)-2r^2}\lambda^{-2r}\left(\lambda^2 q^{-I-J}, q^{-2i-2j}, q^{-2I-2J-2}; q^2\right)_r \left(q^{-2I-2J}; q^2\right)_{2r}}{(\lambda^{-2}q^{-I-J}, q^2)_r \left(q^{-2I-2J-2}; q^2\right)_{2r}}$$
$$\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{2r-2I}; & q^{2i-2I} & q^{2J-2r+2} \\ & q^{-2I} & q^{2i-2I+2j+2} \end{matrix} \middle| q^2, q^2 \right) {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j'}; & q^{2r-2J} & q^{2I-2r+2} \\ & q^{-2J} & q^{-2i+2I-2j+2} \end{matrix} \middle| q^2, q^{-2i'} \right). \tag{4.2.13}$$

The summation range for r is the same as it was in equation (4.2.9). We rewrite (4.2.13) by bringing out the summations in the ${}_3\phi_2$ series outside the sum over 'r' - denoting their summation indices by 's' and 'l' respectively. In doing so we notice that the sum over 'r' is a very-well-poised ${}_6\phi_5$ series. More explicitly,

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \lambda^{i-i'}(-1)^{I+j+i'} \frac{q^{2i^2+i(-3I+j+J+i'+2)}(q^2;q^2)_I(q^2;q^2)_J}{q^{I(1-I)+(i'-j)(j+2i'+1-I)}(q^2;q^2)_{I+J}}$$
$$\times \frac{(q^2;q^2)_{I-i'}(q^2;q^2)_{i'}}{(q^2;q^2)_{I-i}(q^2;q^2)_i} \sum_{l,s=0}^{j',I-i} \frac{q^{2s-2li'}(q^{-2j'}, q^{2I+2}, q^2)_l \left(q^{2J+2}, q^{2i-2I}; q^2\right)_s}{(q^2;q^2)_l\,(q^2;q^2)_s\,(q^2;q^2)_{I+l-i-j}\,(q^2;q^2)_{i+j+s-I}}$$
$$\times {}_6\phi_5 \left(\begin{matrix} q^{-2I-2J-2} & -q^{-I-J+1} & q^{-I-J+1} & \lambda^2 q^{-I-J} & q^{2l-2J} & q^{2s-2I} \\ & -q^{-I-J-1} & q^{-I-J-1} & \frac{q^{-I-J}}{\lambda^2} & q^{-2I-2l} & q^{-2J-2s} \end{matrix} \middle| q^2, \frac{q^{I+J-2l-2s}}{\lambda^2} \right). \tag{4.2.14}$$

Interestingly, this series depends only on λ, I and J, a fact elaborated on in section 4.2.4. Even though the ${}_6\phi_5$ series has denominator terms of the form $q^{-m}, m \in \mathbb{Z}^+$ this sum does not have any poles because of the range of the summation over 'r' explained after equation (4.2.9). Therefore this series is truncated before the denominator terms can be zero. In fact, by noting the range of the sums over 's' and 'l' the range of the summation over 'r' can be re-expressed as $0 \le r \le \min(I-s, J-l)$. We use identity (B.2.5) with $a = q^{-2I-2J-2}$ and $n = I - s$ to write the ${}_6\phi_5$ series as a product. The triple sum therefore reduces to a double sum. Identifying the sum over 'l' as a ${}_3\phi_2$ series we have

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^{j'}\lambda^{3i-i'-2I}q^{i^2+I(I-J-2)+i(j+i'+3-2I)+(j-i')(j+2i'+1-I)}$$
$$\times \frac{(q^2;q^2)_{i'}(q^2;q^2)_{I-i'}(\lambda^{-2}q^{J-I};q^2)_i}{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_{I-i}(\lambda^{-2}q^{-I-J};q^2)_I} \sum_{l=0}^{j'} \frac{q^{-2li'}(q^{-2j'}, \lambda^2 q^{I-J+2}; q^2)_l}{(q^2;q^2)_{I+l-i-j}(\lambda^2 q^{-2i+I-J+2}; q^2)_l}$$
$$\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2I+2i}; & \lambda^2 q^{-I+J+2} & q^{2l+2} \\ & \lambda^2 q^{-I-J+2l+2} & q^{2i-2I+2j+2} \end{matrix} \middle| q^2, q^2 \right). \tag{4.2.15}$$

Transforming the ${}_3\phi_2$ using identity (B.2.19) we see that (4.2.15) becomes

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^{i+i'}\lambda^{i-2j-i'}q^{i^2+2j^2-2i'^2-jJ+i(j-J+i'-I+1)+i'(I+j-1)}$$

$$\times \frac{(q^2,q^2)_{i'}(q^2,q^2)_{I-i'}(\lambda^{-2}q^{J-I},q^2)_i}{(q^2,q^2)_i(q^2,q^2)_j(q^2,q^2)_{I-i}(\lambda^{-2}q^{-I-J},q^2)_{i+j}} \sum_{l=0}^{j'} \frac{q^{-2li'}(q^{-2j'},\lambda^2 q^{I-J+2},q^2)_l}{(q^2,\lambda^2 q^{I-J-2i+2},q^2)_l}$$
$$\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j}; & \lambda^2 q^{-I-J} & \lambda^2 q^{I+J-2i-2j+2} \\ & \lambda^2 q^{I-J-2i-2j+2l+2} & \lambda^2 q^{J-I-2j+2} \end{matrix} \middle| q^2, q^{2l+2} \right). \tag{4.2.16}$$

Now rewriting (4.2.16) by interchanging the order of the two sums, the sum over 'l' can be identified as a terminating ${}_2\phi_1$ series. That is,

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^{i+i'}\lambda^{j+j'} \frac{q^{i^2-2i'^2-i(I+J+j-i'-1)}(q^2;q^2)_{i'}(q^2;q^2)_{I-i'}}{q^{i'(1-I)+j(J-i'-2)}(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_{I-i}}$$
$$\times \frac{(\lambda^{-2}q^{I-J};q^2)_j(\lambda^{-2}q^{J-I};q^2)_{i+j}}{(\lambda^{-2}q^{-I-J};q^2)_{i+j}} \sum_{s=0}^{j} \frac{q^{2s}(q^{-2j},\lambda^2 q^{-I-J},\lambda^2 q^{I+J-2i-2j+2};q^2)_s}{(q^2,\lambda^2 q^{J-I-2j+2},\lambda^2 q^{I-J-2i-2j+2};q^2)_s}$$
$$\times {}_2\phi_1 \left(\begin{matrix} q^{-2j'} & \lambda^2 q^{I-J+2} \\ & \lambda^2 q^{I-J-2i-2j+2s+2} \end{matrix} \middle| q^2, q^{2s-2i'} \right). \tag{4.2.17}$$

The ${}_2\phi_1$ series can written as a product of binomials using the q-Chu-Vandermonde sum (B.2.2) with $n = j'$. Doing so yields the single summation

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{i'j'-ij-Ij-Ji'} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} \frac{(\lambda^{-2}q^{J-I};q^2)_{i'}(\lambda^{-2}q^{I-J};q^2)_j(q^{-2I};q^2)_i}{\lambda^{i-i'}(\lambda^{-2}q^{-I-J};q^2)_{i+j}(q^{-2I};q^2)_{i'}}$$
$$\times {}_4\phi_3 \left(\begin{matrix} q^{-2j} & q^{-2i'} & \lambda^2 q^{-I-J} & \lambda^2 q^{I+J-2i-2j+2} \\ & q^{-2i-2j} & \lambda^2 q^{J-I-2j+2} & \lambda^2 q^{I-J-2i'+2} \end{matrix} \middle| q^2, q^2 \right). \tag{4.2.18}$$

This is just (3.3.30) under the symmetry (3.4.1) and the similarity transform (3.8.1). We already showed this is equal to (3.8.3) using the Sears' transform (B.2.14) with identification (3.8.2). Therefore we have linked the two constructions and shown that the constructed R-matrix is the same.

4.2.4 Discussion

In transforming (4.2.13) to (4.2.14) we realised the summation over 'r' as a ${}_6\phi_5$ very-well-poised basic hypergeometric series that can be written as a product of binomials. This product depends only on λ, I, and J and not at all on indices i, j, i', j'. Moreover, the remaining summations do not depend on λ and are split into summations depending only on lower or upper indices. In other words, we can write (4.2.15) in the form

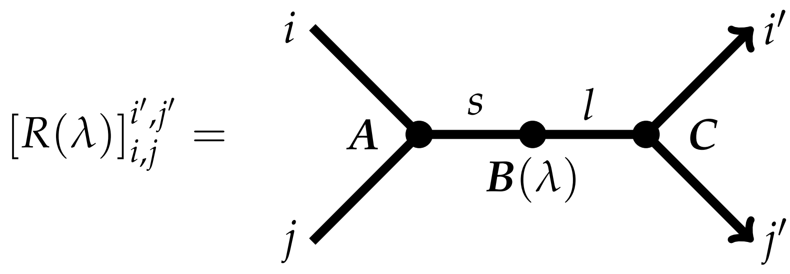

Figure 4.3: Graphical representation of the R-matrix (4.2.19)

Proposition 4.2.1.

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}\lambda^{i-i'}\kappa \sum_{s,l} \boldsymbol{A}_{i,j}^{s}\boldsymbol{B}_{s}^{l}(\lambda)\boldsymbol{C}_{l}^{i',j'} \tag{4.2.19}$$

where

$$\boldsymbol{A}_{i,j}^{s} = (-1)^{j}\frac{q^{2i^2+j^2+ij+j(1-I)+i(2-3I+J)+2s}(q^{2i-2I};q^2)_s}{(q^2;q^2)_s(q^2;q^2)_i(q^2;q^2)_{I-i}(q^2;q^2)_{s+i+j-I}}, \tag{4.2.20a}$$

$$\boldsymbol{B}_{s}^{l}(\lambda) = \frac{(q^2;q^2)_{s+l}(\lambda^2 q^{2+I-J};q^2)_l(\lambda^2 q^{2+J-I};q^2)_s}{(\lambda^2 q^{2-I-J};q^2)_{s+l}}, \tag{4.2.20b}$$

$$\boldsymbol{C}_{l}^{i',j'} = (-1)^{i'}\frac{q^{i'j'-i'^2+i'(I-1)-2li'}(q^{-2j'};q^2)_l(q^2;q^2)_{I-i'}(q^2;q^2)_{i'}}{(q^2;q^2)_l(q^2;q^2)_{l+I-i'-j'}}, \tag{4.2.20c}$$

$$\kappa = (-1)^{I}q^{I^2-IJ-I}\frac{(\lambda^{-2}q^{J-I};q^2)_I}{(\lambda^{-2}q^{-I-J};q^2)_I}. \tag{4.2.20d}$$

The summation indices s, l have the same range as they do in (4.2.13). κ is a multiplication of the R-matrix by a constant and the $\lambda^{i-i'}$ term is a gauge transform whose presence does not affect the Yang-Baxter equation - both terms can be removed if desired. A graphical representation of this factorisation is presented in figure 4.3. Although the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same valence as the Clebsch-Gordan coefficients, their elements are very different, given by binomial products rather than ${}_3\phi_2$ basic hypergeometric series.

Chapter 5

R-matrix factorisation by Q-operators

In this chapter we will review a factorisation of the R-matrix first introduced by Derkachov in [13] and developed by Derkachov-Manashhov in [66]. It is a factorisation of rational R-matrices with $SL(N, \mathbb{C})$ symmetry in terms of 'elementary' intertwining operators which are closely related to Baxter Q-operators [41]. We are particularly interested to compare the factorisation (3.5.6) obtained from the 3D model approach to the one here. Already we can note some important similarities and differences. Firstly, the factorisation (3.5.6) is for trigonometric R-matrices and so is a deformation of those constructed by Derkachov. A direct comparison is only possible after taking the rational limit $q \to 1$ as we did in (3.7.4). In this case, the rational limit of (3.5.6) - which we found to be (3.7.19) - is written as a product of two almost equal factors for symmetric tensor representations of $\widehat{sl_n}$ while Derkachov's factorisation for the same algebra contains n factors for any lowest weight representation. For the case $n = 2$ it turns out that the factorizations are exactly the same. For $n > 2$ however the situation is more complicated and it would be interesting to see how the n factors in Derkachov's construction reduce to the 2 stochastic R-matrices (3.5.8) in the restriction to symmetric tensor representations.

Derkachov's construction [66] uses the principal series representations of the Lie group $SL(n, \mathbb{C})$ to construct the R-operator that acts on this representation space and is invariant with respect to the Lie group action. The representation space is characterised as the space of functions in $n(n-1)/2$ variables and the R-operator is represented in a factorised form consisting of integral operators. The restriction of these operators to the space of polynomials [13] is equivalent to the R-matrix acting on Verma modules of $\widehat{sl_n}$.

The main advantage of this approach is that it constructs the R-matrix for any representation of $\widehat{sl_n}$. In the process we also construct elementary factors that are very closely related to local operators used to construct Baxter Q-operators. That means we can potentially generalise the formulae obtained in chapter 3 using the 3-dimensional approach. In this thesis we will consider the explicit construction of the R-matrix for the case $n = 2, 3$. For higher rank algebras the problem is no more difficult conceptually, but the computational challenge grows considerably with n, and at the time of writing this is still in the process of being solved.

In the first part of this chapter we will give an overview of the construction and then we will compute formulae for matrix elements of all the operators involved in the construction for $n = 2, 3$. We will then analyse and discuss the results.

5.1 Factorized ansatz

This factorisation approach can be seen as an alternative way of solving the Yang-Baxter RLL-relation, which we introduced in a particular case in (2.4.4). There it was presented as an equality of operators acting on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes V_3$ so that $\mathcal{R}_{12}$ acts on a finite dimensional vector space. In this chapter we will consider the same equation but acting on $V_1 \otimes V_2 \otimes \mathbb{C}^2$ (in the case of sl_2) so that $\mathcal{R}_{12}$ is an operator acting on $V_1 \otimes V_2 \cong V_I \otimes V_J$ with arbitrary weight parameters I and J. Both forms are a restriction of the general Yang-Baxter equation (2.3.25) to certain integral weights. Here we will rewrite the RLL-relation but with additive spectral parameters u, v as

$$\mathcal{R}_{12}(u-v)\mathcal{L}_1(u)\mathcal{L}_2(v) = \mathcal{L}_2(v)\mathcal{L}_1(u)\mathcal{R}_{12}(u-v). \tag{5.1.1}$$

For a fixed L-operator L_i acting non-trivially on $V_i \otimes \mathbb{C}^m$ this equation determines the R-matrix $\mathcal{R}_{12}(u-v)$ with arbitrary weights I, J. For example, in [13] the L-operator acting in $V_l \otimes \mathbb{C}^2$ is given by

$$L(u) = u + 2S \otimes \mathbf{s} + S_- \otimes \mathbf{s}_+ + S_+ \otimes \mathbf{s}_- = \begin{pmatrix} u+S & S_- \\ S_+ & u-S \end{pmatrix}, \tag{5.1.2}$$

where $\{S, S_\pm\}$ are generators of the sl_2 Lie algebra acting in the *lowest weight* representation space V_l with spin parameter l. These generators are related to $\{E_1, F_1, H_1\}$ presented in (2.4.21) by $2S = H_1, S_+ = E_1$, and $S_- = F_1$. The operators $\{\mathbf{s}, \mathbf{s}_\pm\}$ are the matrices

$$\mathbf{s} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{s}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5.1.3}$$

which is a representation of sl_2 acting in $V_{-\frac{1}{2}} \cong \mathbb{C}^2$. Realising the Verma module V_{l_i} acted upon by L_i as the space of the polynomials $\mathbb{C}[z_i]$ we can write the L-operator with differential operator entries as

$$L_i(u) = \begin{pmatrix} u + l_i + z_i\partial_i & -\partial_i \\ z_i^2\partial_i + 2l_iz_i & u - l_i - z_i\partial_i \end{pmatrix}. \tag{5.1.4}$$

This L-operator is equivalent to the one found in (3.7.14a) using the 3-dimensional approach, although the resulting presentation is different. There is the obvious difference that (5.1.2) is defined acting on $V_l \otimes \mathbb{C}^2$ with the operator entries acting on the first tensor space whereas they act on the second tensor space in (3.7.14). However, the L-operator is symmetric with respect to this swapping of the tensor factors which one can see from the symmetry (3.7.21). The difference in presentation is a change of basis and parameters. If we identify $\mu = u + \frac{1}{2}$ and $J = -2l$ and identify the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ in (5.1.2) and make the change $e_1 \to -e_2, e_2 \to e_1$ then we get (3.7.14a). In what follows we will work with the presentations (5.1.2) and (5.1.4).

The L-operator (5.1.4) is a function of variables u, l. After making a lightcone change of variables $u_1 := u + l$, $u_2 := u - l$ and by working with the permuted R-matrix $\check{\mathcal{R}}_{12} = \mathbb{P}_{12}\mathcal{R}_{12}$, the Yang-Baxter RLL relation (5.1.1) can be written as

$$\check{\mathcal{R}}_{12}L_1(u_1, u_2)L_2(v_1, v_2) = L_1(v_1, v_2)L_2(u_1, u_2)\check{\mathcal{R}}_{12}. \tag{5.1.5}$$

The R-matrix acts as an intertwiner of the two L-operators, interchanging the variables (u_1, u_2) with (v_1, v_2). Derkachov's approach [13] is to split this operation into two stages

$$\mathbb{R}_1 L_1(u_1, u_2)L_2(v_1, v_2) = L_1(v_1, u_2)L_2(u_1, v_2)\mathbb{R}_1, \tag{5.1.6a}$$

$$\mathbb{R}_2 L_1(u_1, u_2)L_1(v_1, v_2) = L_1(u_1, v_2)L_2(v_1, u_2)\mathbb{R}_2, \tag{5.1.6b}$$

and solve these simpler equations for $\mathcal{R}_{12} = \mathbb{P}_{12}\mathbb{R}_1\mathbb{R}_2$ instead. In other words, the R-matrix related to sl_2 factorises into more elementary intertwining operators $\mathcal{R}_1$ and $\mathcal{R}_2$. Solving these equations on the space $\mathbb{C}[z_1] \otimes \mathbb{C}[z_2]$ it was found in [13] that the factors are given (up to normalization) by

$$\mathcal{R}_1(u_1|v_1,v_2) = \frac{\Gamma(z_{21}\partial_2 + u_1 - v_2)}{\Gamma(z_{21}\partial_2 + v_1 - v_2)}, \tag{5.1.7a}$$

$$\mathcal{R}_2(u_1,u_2|v_2) = \frac{\Gamma(z_{12}\partial_1 + u_1 - v_2)}{\Gamma(z_{12}\partial_1 + u_1 - u_2)} \tag{5.1.7b}$$

where $z_{ij} = z_i - z_j$. The sl_2 R-matrix can therefore be represented as a ratio of gamma functions with operator arguments. The gamma functions could be expanded out as a formal series of operators, but a neater and more transparent way of representing this operator action is with the Euler beta integral given by equation (6.0.2) in Chapter 6 where it is considered in more detail. We stress that this is a factorisation of the R-matrix acting on infinite-dimensional Verma modules. While the overall R-matrix can be restricted to a finite-dimensional subspace for integral weights, the operator multiplication of factors is still over an infinite-dimensional space.

So far we have only considered the case of sl_2 to illustrate the main idea. In fact, this idea works for sl_n for arbitrary highest or lowest weight representations. The sl_n L-operator is a function of n variables $L(u_1, u_2, \ldots, u_n)$ and one can break the operation (5.1.5) into n stages so that the sl_n R-operator has the form

$$\mathcal{R}_{12} = \mathbb{P}_{12}\mathbb{R}^{(1)}\mathbb{R}^{(2)}\ldots\mathbb{R}^{(n)}, \tag{5.1.8}$$

$$\mathbb{R}^{(i)}L_1(u_1,\ldots,u_i,\ldots,u_n)L_2(v_1,\ldots,v_i,\ldots,v_n) = L_1(u_1,\ldots,v_i,\ldots,u_n)L_2(v_1,\ldots,u_i,\ldots,v_n)\mathbb{R}^{(i)}. \tag{5.1.9}$$

We would like to compute matrix elements of the factors R_i explicitly. For sl_2 we could do this using (5.1.7) but solving the RLL-relation directly for general n is difficult. Instead, we will construct the factors using a different approach [66] related to *principal series representations* of the Lie group $SL(n, \mathbb{C})$.

5.2 Principal series representations of $SL(n, \mathbb{C})$

The R-operator constructed in [66] acts in the tensor product of principal series representation spaces and is invariant under the action of the Lie group $SL(n, \mathbb{C})$. The representation space is the space of states for the non-compact spin magnet with $SL(n, \mathbb{C})$ symmetry. Of course, we are interested in the R-operator acting on sl_n Verma modules but it appears to be easier to first construct the factors as integral operators on the principal series representation space and then 'restrict' the action to the subspace of polynomials of arbitrary degree. This restriction was performed in [14] yielding a new presentation for the R-operator with sl_n symmetry acting on infinite dimensional Verma modules.

The Lie Group $SL(n,\mathbb{C})$ is defined as the group of $n \times n$ matrices with complex entries and determinant 1. The principal series representations we consider [14] are constructed as induced representations of the subgroup of upper triangular matrices, which we call H_+ (and H_- for the lower triangular subgroup). Also important to the construction are the subgroups of lower and upper unitriangular matrices, which we denote by Z_- and Z_+ respectively. More specifically, group elements have the form

$$z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ z_{21} & 1 & 0 & \cdots & 0 \\ z_{31} & z_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{n,n-1} & 1 \end{pmatrix} \in Z_-, \quad h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & 0 & h_{33} & \cdots & h_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{n,n} \end{pmatrix} \in H_+ \tag{5.2.1}$$

and similarly for Z_+, H_-. Representations of H_+ are classified by the character map $\boldsymbol{\alpha} : H_+ \to \mathbb{C}$ given by

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{n} h_{kk}^{\sigma_k - k} \bar{h}_{kk}^{\bar{\sigma}_k - k} \tag{5.2.2}$$

where $\bar{h}_{kk}$ is the complex conjugate of h_{kk} and the complex tuples $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\bar{\boldsymbol{\sigma}} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$ determine the character map. In general $\sigma_k, \bar{\sigma}_k$ are not complex conjugates because we require $\sigma_k - \bar{\sigma}_k \in \mathbb{Z}$ to ensure $\boldsymbol{\alpha}$ is a single-valued function. Because each element h has unit determinant it is easy to see that $\boldsymbol{\alpha}(h)$ only depends on the differences $\sigma_{k,k+1} := \sigma_k - \sigma_{k+1}$. Furthermore, elements in $\boldsymbol{\sigma}$ are constrained by the relation

$$\sigma_1 + \sigma_2 + \cdots + \sigma_n = n(n-1)/2. \tag{5.2.3}$$

The representation space is characterised as the space of functions on Z_-. The action of an element $g \in SL(n,\mathbb{C})$ on some function $\Phi(z) \equiv \Phi(z_{21}, z_{31}, \ldots, z_{n,n-1})$ is given by

$$\left[T^{\boldsymbol{\alpha}}(g)\Phi\right](z) = \boldsymbol{\alpha}(h)\Phi(z\bar{g}), \tag{5.2.4}$$

where g, z, h are related by the Gauss decomposition $g = zh$ and $z\bar{g}$ satisfies the relation $g^{-1} \cdot z = z\bar{g} \cdot h$.

We are interested in a special case of representations known as the *unitary principal series representations*. These are irreducible representations which parametrise $\sigma_{k,k+1}, \bar{\sigma}_{k,k+1}$ by

$$\sigma_{k,k+1} = -\frac{n_k}{2} + i\lambda_k, \quad \bar{\sigma}_{k,k+1} = \frac{n_k}{2} + i\lambda_k, \quad n_k \in \mathbb{Z}, \lambda_k \in \mathbb{R}. \tag{5.2.5}$$

For these representations $T^{\alpha}(g)$ is a unitary operator on the Hilbert space $L^2(Z_-)$ with inner product

$$\langle \Phi_1 | \Phi_2 \rangle = \int \prod_{1 \leq i \leq k \leq n} d^2 z_{ki} (\Phi_1(z))^* \Phi_2(z). \tag{5.2.6}$$

The representation theory of $SL(n, \mathbb{C})$ is closely related to that of the complex Lie algebra sl_n. Taking $g \in SL(n, \mathbb{C})$ close to the identity $\mathbb{1}$ we write it as $g = \mathbb{1} + \epsilon\mathcal{E} = \mathbb{1} + \sum_{1 \leq i,k \leq n} \epsilon^{ik} \mathcal{E}_{ki}$ where

$$(\mathcal{E}_{ik})_{nm} = \delta_{in}\delta_{km} - \frac{1}{n}\delta_{ik}\delta_{nm} \tag{5.2.7}$$

are the generators of $SL(n, \mathbb{C})$ in the fundamental representation. The operator $T^{\alpha}(g)$ then acts as

$$\left[T^{\alpha}(\mathbb{1} + \sum_{1 \leq i,k \leq n} \epsilon^{ik} \mathcal{E}_{ki}) \Phi \right](z) = \Phi(z) + \sum_{1 \leq i,k \leq n} \left(\epsilon^{ik} E_{ki} + \bar{\epsilon}^{ik} \bar{E}_{ki} \right) \Phi(z) + \mathcal{O}(\epsilon^2). \tag{5.2.8}$$

The linear operators $E_{ik}(\bar{E}_{ik})$ acting on variables $z_{mn}(\bar{z}_{mn})$ are generators of the Lie algebra sl_n. They satisfy the commutation relations

$$[E_{ik}, E_{nm}] = \delta_{kn} E_{im} - \delta_{im} E_{nk}. \tag{5.2.9}$$

They can be considered as the standard generators of the Lie algebra gl_n with the extra relation $\sum_{i=1}^{n} E_{ii} = 0$. This can be seen from their operator form given in [14] by

$$E_{ki} = -\sum_{mn} z_{im} \left(D_{nm} + \delta_{nm}\sigma_m \right) \left(z^{-1} \right)_{nk}, \tag{5.2.10}$$

$$D_{ki} = \sum_{m=k}^{n} z_{mk} \frac{\partial}{\partial z_{mi}} = -\sum_{m=1}^{i} \tilde{z}_{im} \frac{\partial}{\partial \tilde{z}_{km}} \quad k > i \tag{5.2.11}$$

where $\tilde{z}_{ki} = \left(z^{-1} \right)_{ki}, z_{ii} = 1$ and operators D_{nm} are non-zero only for $n > m$. The operators (5.2.10) define a highest weight representation of sl_n acting on a Verma module realised as the space of polynomials

$$\mathbb{V} = \{P(z_{21}, z_{31}, \ldots, z_{n,n-1}), \quad \deg(P) < \infty\}. \tag{5.2.12}$$

It's clear from (5.2.10) that the representation is determined by $\boldsymbol{\sigma}$, which relates the representation $T^{\boldsymbol{\alpha}}$ (5.2.2), (5.2.4) of $SL(n, \mathbb{C})$ to the Lie algebra. Similarly, the antiholomorphic $\bar{E}_{ik}$ from (5.2.8) are a representation of sl_n of the same form as (5.2.10) but with $\sigma_k \to \bar{\sigma}_k$, $z_{ij} \to \bar{z}_{ij}$ and acting on the space we call $\overline{\mathbb{V}}$, similar to $\mathbb{V}$ (but dual) consisting of polynomials in antiholomorphic variables $\bar{z}_{ij}$. Obviously this is determined by $\bar{\boldsymbol{\sigma}}$, which together with $\boldsymbol{\sigma}$ completely determines the character $\boldsymbol{\alpha}(h)$ and the representation $T^{\boldsymbol{\alpha}}$. Therefore a principal series representation of the Lie group gives rise to two representations of the corresponding Lie algebra which are dual to each other.

We want to make a comparison between the representation parameters $\boldsymbol{\sigma}$ and the language we have used in other chapters to describe sl_n representations (for example (2.4.56) and section 2.4.2. Firstly, let us note that in some instances it is more convenient to use the character notation (5.2.2) (especially when invoking similarities between $SL(n, \mathbb{C})$ and sl_n representations). When talking about sl_n representations we can say it is specified by $\boldsymbol{\sigma}$ or $\boldsymbol{\alpha}(h) = \prod_{k=1}^{n} h_{kk}^{\sigma_k - k}$ in the case of the generators E_{ki} acting on $\mathbb{V}$. We stress $\boldsymbol{\alpha}(h)$ in this context is not the character map (5.2.2) - it contains only the $SL(n, \mathbb{C})$ action (5.2.4) on the holomorphic variables of $\Phi(z, \bar{z})$. For the antiholomorphic variables, the conjugate factors $\bar{h}_{kk}$ and $\bar{\boldsymbol{\sigma}}$ detail the $SL(n, \mathbb{C})$ action and dual sl_n representation.

One can see from (5.2.10) that the element '1' is the lowest weight vector of $\mathbb{V}$ - it is annihilated by all $E_{ik}, \quad i > k$. Let us identify these generators with the generators $\{E_i, F_i, H_i\}_{i=1,\ldots,n-1}$ in (2.4.21) by

$$H_i = E_{ii} - E_{i+1,i+1}, \quad F_i = E_{i,i+1}, \quad E_i = E_{i+1,i}. \tag{5.2.13}$$

In the language of lowest weight representations the lowest weight vector 1 has lowest weight $(E_{ii} - E_{i+1,i+1}) \cdot 1 = (1 - \sigma_i + \sigma_{i+1}) \cdot 1$. If we denote $\lambda_i := 1 - \sigma_i + \sigma_{i+1}$ then the representation is determined by the tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-1})$ and has finite-dimensional subrepresentation if all the λ_i are negative integers. This is equivalent to the highest weight representation theory outlined in section 2.4.2 - the parameters are related by $\boldsymbol{\lambda} = -\boldsymbol{I}$ and thus the relation between components of $\boldsymbol{\sigma}$ and $\boldsymbol{I}$ is

$$I_k = \sigma_k - \sigma_{k+1} - 1 \iff \sigma_k = n - k + \sum_{s=k}^{n-1} I_s - \frac{1}{n} \sum_{s=1}^{n-1} -sI_s. \tag{5.2.14}$$

In what follows we will mostly use the $\boldsymbol{I}$ parameters in the context of sl_n Verma modules rather than $\boldsymbol{\sigma}$ (which we still use for $SL(n,\mathbb{C})$) in order to be consistent with other chapters and to make comparisons easier.

5.3 Factorised form and properties

It was proven in [66] that the $SL(n,\mathbb{C})$ invariant R-matrix acting on the tensor product of two unitary principal series representations $T^{\boldsymbol{\alpha}} \otimes T^{\boldsymbol{\beta}}$ has the factorised form

$$\mathcal{R}_{12}(u-v) = \mathbb{P}_{12}\mathbb{R}^{(1)}_{12}(u_1-v_1)\dots\mathbb{R}^{(n)}_{12}(u_n-v_n), \tag{5.3.1}$$

where u,v are spectral parameters and $u_i := u-\sigma_i,\ v_i := v-\rho_i$. The factors $\mathbb{R}^{(i)}(\lambda)$ solve (5.1.9) and are intertwiners of representations $T^{\boldsymbol{\alpha}} \otimes T^{\boldsymbol{\beta}}$ with $T^{\boldsymbol{\alpha}_{i,\lambda}} \otimes T^{\boldsymbol{\beta}_{i,\lambda}}$. That is,

$$\mathbb{R}^{(i)}(\lambda)\left(T^{\boldsymbol{\alpha}}(g)\otimes T^{\boldsymbol{\beta}}(g)\right) = \left(T^{\boldsymbol{\alpha}_{i,\lambda}}(g)\otimes T^{\boldsymbol{\beta}_{i,\lambda}}(g)\right)\mathbb{R}^{(i)}(\lambda) \quad \forall g \in SL(n,\mathbb{C}), \tag{5.3.2}$$

$$\boldsymbol{\alpha}_{i,\lambda}(h) = h_{ii}^{-\lambda}\bar{h}_{ii}^{-\bar{\lambda}}\boldsymbol{\alpha}(h), \quad \boldsymbol{\beta}_{i,\lambda}(h) = h_{ii}^{\lambda}\bar{h}_{ii}^{\bar{\lambda}}\boldsymbol{\beta}(h). \tag{5.3.3}$$

As we will see, the intertwining action of the $\mathbb{R}^{(i)}$ has to be carefully taken into account when multiplying factors such as in (5.3.1) because the representation parameters of the operator depend on its position in the product. We will sometimes write the operator as $\mathbb{R}^{(i)}(\lambda|\boldsymbol{\sigma}',\boldsymbol{\rho}')$ to indicate the transformed representation it acts as. Computing the transforms (5.3.3) over all $\mathbb{R}^{(i)}$ on the right hand side of (5.3.1) we see that the overall change to the representation parameters is nil, as we would expect from the left hand side.

The intertwiner factors are very interesting objects in their own right. Some non-trivial relations they satisfy include

$$\mathbb{R}^{(i)}_{12}(0) = id, \tag{5.3.4a}$$

$$\mathbb{R}^{(i)}_{12}(\lambda)\mathbb{R}^{(i)}_{12}(\mu) = \mathbb{R}^{(i)}_{12}(\lambda+\mu), \tag{5.3.4b}$$

$$\mathbb{R}^{(i)}_{12}(\lambda)\mathbb{R}^{(j)}_{23}(\mu) = \mathbb{R}^{(j)}_{23}(\mu)\mathbb{R}^{(i)}_{12}(\lambda), \tag{5.3.4c}$$

$$\mathbb{R}^{(i)}_{12}(\lambda)\mathbb{R}^{(i)}_{23}(\lambda+\mu)\mathbb{R}^{(i)}_{12}(\mu) = \mathbb{R}^{(i)}_{23}(\mu)\mathbb{R}^{(i)}_{12}(\lambda+\mu)\mathbb{R}^{(i)}_{23}(\lambda), \tag{5.3.4d}$$

$$\mathbb{R}^{(i)}_{12}(\lambda-\sigma_i+\rho_i)\mathbb{R}^{(j)}_{12}(\lambda-\sigma_j+\rho_j) = \mathbb{R}^{(j)}_{12}(\lambda-\sigma_j+\rho_j)\mathbb{R}^{(i)}_{12}(\lambda-\sigma_i+\rho_i). \tag{5.3.4e}$$

Identity (5.3.4d) states that the factors satisfy a Yang-Baxter type equation, although the operator composition à la (5.3.2) is more complicated. We already know from chapter 3 that at least for symmetric tensor representations the R-matrix factorises into 'smaller' R-matrices without the difference property. We will aim to clarify the connection between these factorisations. Equation (5.3.4e) states that the factors $\mathbb{R}^{(i)}(u_i - v_i)$ commute with each other.

The operators $\mathbb{R}^{(i)}$ acting on the principal series representation space of $SL(n, \mathbb{C})$ are represented in [66] as a product of even more elementary intertwining operators. The action of these elementary operators takes the form of an integral operator. We may view the R-operators as being composed of even smaller building blocks. However, unlike $\mathbb{R}^{(i)}$, these operators cannot be restricted to sl_n Verma modules. Therefore we should be able to represent the operator $\mathbb{R}^{(i)}$ restricted to Verma modules as a matrix, and calculate a formula for its entries. In principle, this could be done by evaluating the integral operator directly for monomial test functions but it appears to be simpler to write the action down in terms of the operator kernel with respect to a hermitian form on the Verma module.

5.4 Operator kernel and hermitian form

In this section we will present a construction of the factors $\mathbb{R}^{(i)}$ operating on sl_n Verma modules. It was first presented in [14], and we will follow it explicitly for the case of sl_2, sl_3 to construct a matrix representation of the operators and a formula for their elements.

Let us first give a brief overview of some concepts that we will use. A hermitian form $\Omega : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ can be specified in terms of a reproducing kernel. A reproducing kernel $\mathcal{E}(z, w)$ evaluates the point $P(z) \in \mathbb{V}$ through the hermitian form by

$$P(z) = \Omega\left(\mathcal{E}(z, \alpha), P(\alpha)\right). \tag{5.4.1}$$

Fixing a basis $\{e_n(z)\}$ of $\mathbb{V}$ we can represent the hermitian form as a matrix with elements $\Omega_{nm}(\overline{e_n}, e_m)$. Expanding $\mathcal{E}(z, \alpha)$ in this basis it is easy to see Ω^{-1} forms the coefficients of this expansion. That is,

$$\mathcal{E}(z, \alpha) = \sum_{m,n} e_m(z) \Omega^{-1}_{mn} \overline{e_n(\alpha)} \tag{5.4.2}$$

Generally speaking, we associate an operator with a kernel and hermitian form (and vice versa) - the reproducing kernel being associated with the identity operator. Given an arbitrary operator $\mathbb{A}$ acting on $\mathbb{V}$ we denote its kernel by $\mathcal{A}(z, w)$. The relation between an operator, its kernel and hermitian form is given by

$$[\mathbb{A}P](z) = \Omega(\mathcal{A}(z,\alpha), P(\alpha)), \tag{5.4.3}$$

$$\mathcal{A}(z,\alpha) = \sum_{nm} e_n(z) \left(A\Omega^{-1}\right)_{nm} \overline{e_m(\alpha)}, \tag{5.4.4}$$

of which (5.4.1) and (5.4.2) is a particular case.

We will construct the factors $\mathbb{R}^{(i)}$ by first constructing their kernels and the associated hermitian form, and then evaluating (5.4.3). We start from a special form for the reproducing kernel and kernels of the operators $\mathbb{R}^{(i)}$. For a sl_n Verma module of highest weight $\boldsymbol{I}$ the reproducing kernel has the form [14]

$$\mathcal{E}^{\boldsymbol{I}}(z,\alpha) = \prod_{k=1}^{n-1} (\Delta_k(\alpha^\dagger z))^{I_k}, \tag{5.4.5}$$

where $\Delta_k(M) = \mathrm{Det}(M_k)$ and M_k is the k-th main minor of the matrix M. The corresponding hermitian form will be denoted by $\Omega_{\boldsymbol{I}}$. In this Verma module we will work with the basis

$$e_{\boldsymbol{i}}(z) = \prod_{s>l} z_{sl}^{i_{sl}}, \tag{5.4.6}$$

indexed by the tuple $\boldsymbol{i} = (i_{21}, \ldots, i_{n,n-1})$. We use the same notation for the second tensor factor acted on by $\mathbb{R}^{(i)}$ and denote its reproducing kernel by $\mathcal{E}^{\boldsymbol{J}}(w, \beta)$. The kernel of $\mathbb{R}^{(i)}(\lambda)$ is denoted by $\mathcal{R}_{\lambda,\boldsymbol{IJ}}(z, w|\alpha, \beta)$. A presentation of this kernel was constructed in [14] by considering the operator action on the coherent state basis of principal series representations. We begin with this presentation which, in the notation of [14], is given by

$$\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}(z, w|\alpha, \beta) = A_i \left(\beta_w w^{-1} z \alpha_z^{-1}\right)^{\lambda}_{ii} \mathcal{E}^{\boldsymbol{I}}(z,\alpha)\mathcal{E}^{\boldsymbol{J}}(w,\beta). \tag{5.4.7}$$

Here, given matrices $z, w \in Z_-$, $\alpha^\dagger, \beta^\dagger \in Z_+$ as in (5.2.1) we form the decomposition

$$\alpha^\dagger z = z_\alpha d_{z,\alpha} \alpha_z, \quad \beta^\dagger w = w_\beta d_{w,\beta} \beta_w \tag{5.4.8}$$

where $z_\alpha, w_\beta \in Z_-$, $\alpha_z, \beta_w \in Z_+$ and $d_{z,\alpha}, d_{w,\beta}$ are diagonal matrices. A_i is a constant factor which for now we choose to be 1.

The action of the factor $\mathbb{R}^{(i)}(\lambda)$ on an element $\psi(z,w) \in \mathbb{V} \otimes \mathbb{V}$ can be written as

$$\left[\mathbb{R}^{(i)}(\lambda)\psi\right](z,w) = \Omega_{IJ}\left(\mathcal{R}^{(i)}_{\lambda,IJ}(z,w|\alpha,\beta), \psi(\alpha,\beta)\right). \tag{5.4.9}$$

We recall that the variables α and β in $\mathcal{R}$ are conjugated with respect to the analogous variables in ψ - see (5.4.7), (5.4.8). The hermitian form Ω_{IJ} is constructed as the kronecker product of the forms Ω_I, Ω_J.

5.5 Construction for $\mathfrak{sl}_2$

In this section we will explicitly carry out the construction outlined in the last section for the case of the Lie algebra sl_2. That is, we want to construct the sl_2 invariant R-matrix of the form (5.3.1). Here we will rewrite this equation for the case $n = 2$ with explicit dependence on the spectral parameter λ and weight parameters I, J. That is, we want to calculate

$$\mathcal{R}_{12}(\lambda) = \mathbb{P}_{12}\mathbb{R}^{(1)}_{12}(\lambda + \frac{J-I}{2}|\frac{I+J}{2} - \lambda, \frac{I+J}{2} + \lambda)\mathbb{R}^{(2)}_{12}(\lambda + \frac{I-J}{2}|I,J). \tag{5.5.1}$$

We will do this by first calculating the factors $\mathbb{R}^{(i)}_{12}(\lambda|I,J)$ and then composing them with the substituted variables. To do this, we must first calculate the kernels of the operators involved.

Throughout we work with 2×2 matrices belonging to $SL(2,\mathbb{C})$ and its subgroups (5.2.1). Note that the matrices in Z_- are functions of only a single variable z_{21}. Therefore we can drop the index and work with the variables z, w, α, β etc. Also the basis of monomials (5.4.6) is just the set $\{z^i\}_{i\geq 0}$.

Using (5.4.5) the reproducing kernel may be written in the form

$$\begin{aligned}\mathcal{E}^I(z,\alpha) &= \Delta_1(\alpha^\dagger z)^I \\ &= (1 + z\overline{\alpha})^I\end{aligned} \tag{5.5.2}$$

Using the binomial theorem to expand this in the monomial basis gives

$$\mathcal{E}^I(z,\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k(-I)_k}{k!}\overline{\alpha}^k z^k. \tag{5.5.3}$$

This terminates if and only if $I \in \mathbb{Z}^+$. Since $\overline{\alpha}^k$ and z^k are our basis elements and we know the reproducing kernel has the form (5.4.2), it is easy to see that Ω_I^{-1} is given by a diagonal matrix. Therefore we deduce that the elements of Ω_I are given by

$$[\Omega_I]_n^m = \delta_{n,m}(-1)^n \frac{n!}{(-I)_n}. \tag{5.5.4}$$

Next we want to calculate the kernels $\mathcal{R}^{(1)}_{\lambda,IJ}, \mathcal{R}^{(2)}_{\lambda,IJ}$. First we need the decomposition (5.4.8) for 2×2 matrices. It is

$$\begin{aligned} \alpha^\dagger z &= \begin{pmatrix} 1+\overline{\alpha}z & \overline{\alpha} \\ z & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{z}{1+\alpha z} & 1 \end{pmatrix} \begin{pmatrix} 1+\overline{\alpha}z & 0 \\ 0 & \frac{1}{1+\overline{\alpha}z} \end{pmatrix} \begin{pmatrix} 1 & \frac{\alpha}{1+\overline{\alpha}z} \\ 0 & 1 \end{pmatrix} \\ &= z_\alpha d_{z,\alpha} \alpha_z \end{aligned} \tag{5.5.5}$$

and similarly for $\beta^\dagger w$. We then calculate the matrix $\beta_w w^{-1} z \alpha_z^{-1}$ in (5.4.7) to be

$$\begin{aligned} \beta_w w^{-1} z \alpha_z^{-1} &= \begin{pmatrix} 1 & \frac{\beta}{1+\beta w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-\alpha}{1+\alpha z} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\beta z}{1+\beta w} & \frac{\beta-\alpha}{(1+\alpha z)(1+\beta w)} \\ z-w & \frac{1+\alpha w}{1+\alpha z} \end{pmatrix}. \end{aligned} \tag{5.5.6}$$

Combining (5.5.2) and (5.5.6) we see that the kernels (5.4.7) in the case of sl_2 can be represented by the functions

$$\mathcal{R}^{(1)}_{\lambda,\boldsymbol{IJ}}(z,w|\alpha,\beta) = (1+\overline{\alpha}z)^I(1+\overline{\beta}w)^{J-\lambda}(1+\overline{\beta}z)^\lambda, \tag{5.5.7a}$$

$$\mathcal{R}^{(2)}_{\lambda,\boldsymbol{IJ}}(z,w|\alpha,\beta) = (1+\overline{\alpha}z)^{I-\lambda}(1+\overline{\beta}w)^J(1+\overline{\alpha}w)^\lambda. \tag{5.5.7b}$$

We will calculate $\mathbb{R}^{(i)}(\lambda)$ by comparing the coefficients in the expansions of (5.5.7) and (5.4.4). Extracting the elements of $\mathbb{R}^{(i)}(\lambda)$ is easy because Ω is a diagonal matrix. The series expansion of the $\mathcal{R}^{(i)}_{\lambda,IJ}$ are given by

$$\mathcal{R}^{(1)}_{\lambda,IJ}(z,w|\alpha,\beta) = \sum_{s_1,s_2,s_3=0}^{\infty} (-1)^{s_1+s_2+s_3} \frac{(-I)_{s_1}(\lambda-J)_{s_2}(-\lambda)_{s_3}}{s_1!s_2!s_3!} z^{s_1+s_3} w^{s_2} \overline{\alpha}^{s_1} \overline{\beta}^{s_2+s_3}, \tag{5.5.8a}$$

$$\mathcal{R}^{(2)}_{\lambda,IJ}(z,w|\alpha,\beta) = \sum_{s_1,s_2,s_3=0}^{\infty} (-1)^{s_1+s_2+s_3} \frac{(\lambda-I)_{s_1}(-J)_{s_2}(-\lambda)_{s_3}}{s_1!s_2!s_3!} z^{s_1} w^{s_2+s_3} \overline{\alpha}^{s_1+s_3} \overline{\beta}^{s_2}. \tag{5.5.8b}$$

Now comparing to (5.4.4),

$$\mathcal{R}^{(k)}_{\lambda,IJ}(z,w|\alpha,\beta) = \sum_{i,j,i',j'=0}^{\infty} (-1)^{i'+j'} \left[\mathbb{R}^{(k)}(\lambda|I,J)\right]_{i,j}^{i',j'} \frac{(-I)_{i'}(-J)_{j'}}{i'!j'!} z^i w^j \overline{\alpha}^{i'} \overline{\beta}^{j'} \tag{5.5.9}$$

and one can see that (by equating coefficients) we must have

Proposition 5.5.1.

$$\left[\mathbb{R}^{(1)}(\lambda|I,J)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{j'!(-\lambda)_{j'-j}(\lambda-J)_j}{j!(-J)_{j'}(j'-j)!}, \tag{5.5.10a}$$

$$\left[\mathbb{R}^{(2)}(\lambda|I,J)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{i'!(-\lambda)_{i'-i}(\lambda-I)_i}{i!(-I)_{i'}(i'-i)!}. \tag{5.5.10b}$$

The ice rule $i+j=i'+j'$ enters by equating the exponents of (5.5.8) and (5.5.9). Immediately we see that both operators are defined by essentially the same function. They are related by a swapping of indices, that is

$$\left[\mathbb{R}^{(1)}(\lambda|I,J)\right]_{i,j}^{i',j'} = \left[\mathbb{R}^{(2)}(\lambda|J,I)\right]_{j,i}^{j',i'}. \tag{5.5.11}$$

We can now calculate the R-matrix (5.5.1). Let us first explain the variable substitutions we need to make in more detail. As mentioned in the last section, the operators $\mathbb{R}^{(i)}(\lambda)$ (factorising the $SL(2,\mathbb{C})$-invariant R-matrix) are intertwiners of principal series representations - (5.3.2) and (5.3.3). This is still true in the restriction to Verma modules. Using (5.3.3) it is easily to calculate how each factor transforms the representation parameters $\boldsymbol{I},\boldsymbol{J}$. For convenience we write it down explicitly:

$$\mathbb{R}^{(1)}(\lambda)\left(\pi^I\otimes\pi^J\right) = \left(\pi^{I+\lambda}\otimes\pi^{J-\lambda}\right)\mathbb{R}^{(1)}(\lambda), \tag{5.5.12a}$$

$$\mathbb{R}^{(2)}(\lambda)\left(\pi^I\otimes\pi^J\right) = \left(\pi^{I-\lambda}\otimes\pi^{J+\lambda}\right)\mathbb{R}^{(2)}(\lambda). \tag{5.5.12b}$$

Consequently, the representation parameters of $\mathbb{R}^{(i)}(\lambda|I,J)$ depend not only on the representation parameters of the operators preceding it but also the spectral parameters - a subtle detail. Given λ is generic, this means that $\mathbb{R}^{(i)}(\lambda)$ generally maps an integral highest weight representation to a non-integral one - it does not preserve finite dimensional subspaces in the Verma module acted upon by the R-matrix when I,J are integers. So even though the R-operator can be restricted to finite representations for integral weights, the multiplication of the operators comprising it is over an infinite dimensional space.

We can now write down a formula for elements of the R-matrix $R_{IJ}(\lambda)$. Using the constructed factors (5.5.10) we get

$$[R(\lambda)]_{i,j}^{i',j'} = \sum_{r,s}\left[\mathbb{R}^{(1)}(\lambda+\frac{J-I}{2}|\frac{I+J}{2}-\lambda,\frac{I+J}{2}+\lambda)\right]_{j,i}^{r,s}\left[\mathbb{R}^{(2)}(\lambda+\frac{I-J}{2}|I,J)\right]_{r,s}^{i',j'}$$
$$= \frac{(-I)_i i'!}{(-I)_{i'} i!}\sum_{r,s=0}^{i+j}\delta_{i+j,r+s,i'+j'}\frac{s!(-\lambda+\frac{I}{2}-\frac{J}{2})_{j-r}(-\lambda-\frac{I}{2}+\frac{J}{2})_{i'-r}(\lambda-\frac{I}{2}-\frac{J}{2})_r}{r!(j-r)!(i'-r)!(-\lambda-\frac{I}{2}-\frac{J}{2})_s} \tag{5.5.13}$$

Even though there are two summation indices this really only a single sum because of the kronecker delta function. We can use this relation to eliminate the sum over r or s; we choose s. We can see from the sum (5.5.13) that r runs from 0 to $\mathrm{Min}(i',j)$ and so can safely make the substitution $s = i+j-r$ without summing negative s values. As a result we obtain the expression

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}\frac{(i+j)!(-I)_i(-\lambda+\frac{I}{2}-\frac{J}{2})_j(-\lambda+\frac{J}{2}-\frac{I}{2})_{i'}}{i!j!(-I)_{i'}(-\lambda-\frac{I}{2}-\frac{J}{2})_{i+j}}$$
$$\times\,{}_4F_3\left(\begin{matrix}-i' & -j & \lambda-\frac{I}{2}-\frac{J}{2} & \lambda+\frac{I}{2}+\frac{J}{2}-i-j+1\\ & -i-j & \lambda+\frac{J}{2}-\frac{I}{2}-j+1 & \lambda+\frac{I}{2}-\frac{J}{2}-i'+1\end{matrix}\middle|\,1\right), \tag{5.5.14}$$

for elements of the sl_2-invariant R-matrix acting on Verma modules $\mathbb{V}\otimes\mathbb{V}$. They are ${}_4F_3$ balanced and terminating hypergeometric series. It is very easy to compare this formula with (3.7.7) obtained from the 3D model approach. They are the same up to a swapping of indices, that is

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \left[R_{J,I}^{(2),r}(\lambda)\right]_{j,i}^{j',i'}, \tag{5.5.15}$$

but this is just the symmetry (3.7.21a) found using the 3-dimensional approach. We conclude that we are dealing with the same R-matrix.

Actually, we can get (3.7.7) exactly from this chapter's construction without having to appeal to symmetry relations. It turns out the symmetry in this constructions manifests itself in the reordering of the factors (5.3.4e). Using this relation the R-matrix also factorises as

$$[R(\lambda)]_{i,j}^{i',j'} = \sum_{r,s}\left[\mathbb{R}^{(2)}(\lambda+\frac{I-J}{2}|\frac{I+J}{2}+\lambda,\frac{I+J}{2}-\lambda)\right]_{j,i}^{r,s}\left[\mathbb{R}^{(1)}(\lambda+\frac{J-I}{2}|I,J)\right]_{r,s}^{i',j'}$$
$$= \frac{j'!(-J)_j}{j!(-J)_{j'}}\sum_{r,s=0}^{i+j}\delta_{i+j,r+s,i'+j'}\frac{r!(-\lambda-\frac{I}{2}+\frac{J}{2})_{r-j}(-\lambda+\frac{I}{2}-\frac{J}{2})_{r-i'}(\lambda-\frac{I}{2}-\frac{J}{2})_s}{s!(r-i')!(r-j)!(-\lambda-\frac{I}{2}-\frac{J}{2})_r}$$

Eliminating the sum over r, we get

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(i+j)!(-J)_j(-\lambda+\frac{I}{2}-\frac{J}{2})_{j'}(-\lambda-\frac{I}{2}+\frac{J}{2})_i}{i!j!(-J)_{j'}(-\lambda-\frac{I}{2}-\frac{J}{2})_{i+j}}$$
$$\times {}_4F_3\left(\begin{matrix}-i & -j' & \lambda-\frac{I}{2}-\frac{J}{2} & \lambda+\frac{I}{2}+\frac{J}{2}-i-j+1 \\ & -i-j & \lambda+\frac{J}{2}-\frac{I}{2}-i+1 & \lambda+\frac{I}{2}-\frac{J}{2}-j'+1\end{matrix}\middle| 1\right), \quad (5.5.16)$$

which is exactly (3.7.7). We can also make a direct comparison with the factorisation (3.7.19) in the sl_2 case. Then

$$\boldsymbol{M}^r(\lambda+\frac{I+J}{2},J) = \mathbb{P}\mathbb{R}^{(2)}(\lambda+\frac{I-J}{2}|\frac{I+J}{2}+\lambda,\frac{I+J}{2}-\lambda), \quad (5.5.17a)$$
$$\boldsymbol{N}^r(-\lambda+\frac{I+J}{2},J) = \mathbb{R}^{(1)}(\lambda+\frac{J-I}{2}|I,J). \quad (5.5.17b)$$

Although we write the $\mathbb{R}^{(i)}(\lambda|I,J)$ as depending on three variables, it is obvious from (5.5.10) and (5.5.17) that they are functions of only two variables. We were able to derive the factors $\boldsymbol{M}^r, \boldsymbol{N}^r$ from the general R-matrix by making an appropriate substitution for the spectral parameter. Indeed a similar degeneration also exists for the factors $\mathbb{R}^{(i)}(\lambda)$ as can be seen from (5.3.4).

We have found a function determining matrix elements of the R-matrix and its factors. The identities satisfied by these operators should yield related identities for the matrix elements. We have already considered identity (5.3.4e) which turns out to be an identity for ${}_4F_3$ hypergeometric series and a symmetry of the R-matrix. For the sake of interest we will consider the other identities in (5.3.4) and write down their form at the level of matrix elements.

5.5.1 Factor Identities

For all of the identities in (5.3.4) we can consider only $\mathbb{R}^{(1)}$ without loss of generality because of the relation (5.5.15). In using the Kronecker delta relations for each factor to eliminate summation variables we have checked that the remaining sums start from non-negative values.

Zero spectral parameter

The identity (5.3.4a) is obvious by inspecting (5.5.10). When $\lambda = 0$ the functions in (5.5.10) are zero unless $j = j'$, $i = i'$, in which case all terms in the numerator and denominator cancel.

Yang-Baxter identity

The identity (5.3.4d) is a Yang-Baxter relation for the factors $\mathbb{R}^{(i)}(\lambda)$. This relation is not surprising given the link established by (5.5.17) between the factors $\mathbb{R}^{(i)}(\lambda)$ and those in (3.7.17). At the level of matrix elements, the Yang-Baxter equation for the factors is an identity for ${}_4F_3$ hypergeometric series. In constructing the identity we will denote the left hand side of (5.3.4d) by $\mathbb{L}$ and the right hand side by $\mathbb{R}$. The identity can be expressed as

$$[\mathbb{L}(\lambda,\mu)]_{i,j,k}^{i',j',k'} = \sum_{r,l,s=0} \left[\mathbb{R}_{12}^{(1)}(\lambda|I+\mu,J+\lambda)\right]_{i,j}^{r,l} \left[\mathbb{R}_{23}^{(1)}(\lambda+\mu|J-\mu,K)\right]_{l,k}^{s,k'} \left[\mathbb{R}_{12}^{(1)}(\mu|I,J)\right]_{r,s}^{i',j'},$$

$$[\mathbb{R}(\lambda,\mu)]_{i,j,k}^{i',j',k'} = \sum_{r,l,s=0} \left[\mathbb{R}_{23}^{(1)}(\mu|J-\mu,K-\lambda)\right]_{j,k}^{r,l} \left[\mathbb{R}_{12}^{(1)}(\lambda+\mu|I,J+\lambda)\right]_{i,r}^{i',s} \left[\mathbb{R}_{23}^{(1)}(\lambda|J,K)\right]_{s,l}^{j',k'}.$$

There are 3 delta functions on each side implying the global conservation law $i+j+k = i'+j'+k'$ as well as the bounds $0 \le s, l \le j'+k'$ and $0 \le r \le j+k$ on the summation indices. These bounds can be further refined by looking at the explicit expression. We also use the conservation laws to eliminate two summation variables on each side. On the left hand side we subsitute $r = i'+j'-s$, $l = k'-k+s$ and on the right hand side $l = j+k-r$, $s = r+i-i'$. Simplifying some of the Pochhammer symbols in the resulting single sum, we obtain

$$\begin{aligned}[\mathbb{L}(\lambda,\mu)]_{i,j,k}^{i',j',k'} &= \delta_{i+j+k,i'+j'+k'} \, [A(\lambda,\mu)]_{i,j,k}^{i',j',k'} \frac{(-J)_j(-\lambda-\mu)_{k'-k}(-\mu)_{j'}(-\lambda)_{k'-k-j}}{(-J)_{j'}(-J-\lambda)_{k'-k}(i-i')!} \\ &\times \sum_{s=0}^{j'} \frac{(-j',k'-k+1,\mu-J,i-i'-j'-\lambda)_s(1+i-i'-j'+s)_{j'-s}}{s!(1+\mu-j',-\lambda-J+k'-k)_s},\end{aligned} \tag{5.5.18a}$$

$$[\mathbb{R}(\lambda,\mu)]_{i,j,k}^{i',j',k'} = \delta_{i+j+k,i'+j'+k'} \, [A(\lambda,\mu)]_{i,j,k}^{i',j',k'} \frac{(-\lambda-\mu)_{i-i'}(-\mu)_j(-\lambda)_{i-i'-j'}}{(-J-\lambda)_{i-i'}(k'-k)!}$$

$$\times \sum_{r=0}^{j} \frac{(-j, 1+i-i', \mu-J, k'-k-j-\lambda)_r (1+k'-k-j+r)_{j-r}}{r!(1+\mu-j, -\lambda-J+i-i')_r}, \tag{5.5.18b}$$

$$[A(\lambda,\mu)]_{i,j,k}^{i',j',k'} = \frac{k'!(\lambda+\mu-K)_k}{j!k!(-K)_{k'}}. \tag{5.5.18c}$$

Both sides are given by balanced and terminating ${}_4F_3$ hypergeometric series. In fact, we can rewrite this identity so that is expresses a symmetry of the function for the elements on both sides. If we define $\overline{\mathbb{L}}(\lambda,\mu)$ and $\overline{\mathbb{R}}(\lambda,\mu)$ by multiplying the formulas for $\mathbb{L}(\lambda,\mu)$ and $\mathbb{R}(\lambda,\mu)$ by $(-J)_{j'}$ and then removing the $A(\lambda,\mu)$ term we obtain the symmetry

$$\left[\overline{\mathbb{R}}(\lambda,\mu)\right]_{i,j,k}^{i',j',k'} = \left[\overline{\mathbb{R}}(\lambda,\mu)\right]_{k',j',i'}^{k,j,i}. \tag{5.5.19}$$

This identity is a combination of Whipple transformations (A.2.7) and the numerator term terminating the series is changed from j' to j.

Additive spectral parameter identity

On the level of matrix elements we expect a summation formula, given the left hand side contains a sum over matrix elements and the right hand side is just a product. Indeed, the identity reads

$$[\mathbb{LHS}(\lambda,\mu)]_{i,j}^{i',j'} = \sum_{r,s=0}^{i+j} \left[\mathbb{R}^{(1)}(\lambda; I+\mu, J-\mu)\right]_{i,j}^{r,s} \left[\mathbb{R}^{(1)}(\mu; I, J)\right]_{r,s}^{i',j'}, \tag{5.5.20}$$

$$[\mathbb{RHS}(\lambda,\mu)]_{i,j}^{i',j'} = \left[\mathbb{R}^{(1)}(\lambda+\mu)\right]_{i,j}^{i',j'}. \tag{5.5.21}$$

The delta functions present on the left hand side imply the conservation law $i+j = i'+j'$ consistent with the right hand side. Eliminating the sum over 'r' we get

$$[\mathbb{LHS}(\lambda,\mu)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^j \frac{(-\mu)_{j'}(\lambda+\mu-J)_j}{j!(-J)_{j'}(1+\lambda)_j} \sum_{s=0} \frac{(-j', -\lambda-j)_s}{(s-j)!(\mu-j'+1)_s}, \tag{5.5.22a}$$

$$[\mathbb{RHS}(\lambda,\mu)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{j'!(-\lambda-\mu)_{j'-j}(\lambda+\mu-J)_{j'}}{(-J)_{j'}(j'-j)!j!}. \tag{5.5.22b}$$

Both sides are zero whenever $j' < j$. The sum on the left appears to be a terminating ${}_3F_2$ series with $j'-j$ terms but actually it is a ${}_2F_1$ series. To see this we reverse the order of the summation by substituting $s = j' - s$ which gives

$$[\mathbb{LHS}(\lambda,\mu)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^{j+j'}\frac{(-\mu)_{j'}(\lambda+\mu-J)_j(-\lambda-j)_{j'}j'!}{j!(-J)_{j'}(1+\lambda)_j(1+\mu-j')_{j'}(j'-j)!} \times {}_2F_1\left(\begin{matrix} j-j' & -\mu \\ & 1+\lambda+j-j' \end{matrix}\middle| 1\right). \tag{5.5.23}$$

This is essentially the Gauss summation formula (A.2.2) up to a change of variables.

Commutativity identity

Identity (5.3.4c) does not give a non-trivial identity at the level of matrix elements because both sides are the same product of binomials. Regardless, we will still write down the elements for both sides of (5.3.4c). There are two orderings for the factors in this identity - $\mathbb{R}^{(1)}\mathbb{R}^{(2)}$ and vice versa. For the former we have

$$\begin{aligned}
[\mathbb{LHS}(\lambda,\mu)]_{i,j,k}^{i',j',k'} &= \sum_{s=0}\left[\mathbb{R}_{12}^{(1)}(\lambda|I,J-\mu)\right]_{i,j}^{i',s}\left[\mathbb{R}_{23}^{(2)}(\lambda|J,K)\right]_{s,k}^{j',k'} \\
= [\mathbb{RHS}(\lambda,\mu)]_{i,j,k}^{i',j',k'} &= \sum_{s=0}\left[\mathbb{R}_{23}^{(2)}(\mu|J-\lambda,K)\right]_{j,k}^{s,k'}\left[\mathbb{R}_{12}^{(1)}(\lambda|I,J)\right]_{i,s}^{i',j'} \\
&= \delta_{i+j+k,i'+j'+k'}\frac{j'!(-\lambda)_{i-i'}(-\mu)_{k-k'}(\lambda+\mu-J)_j}{j!(i-i')!(k-k')!(-J)_{j'}}.
\end{aligned} \tag{5.5.24}$$

On the left hand side we have made the substitution $s = j' + k' - k$ and on the right hand side $s = j + k - k'$. The two delta functions in each factors also imply the delta function $i + j + k = i' + j' + k'$. For the other ordering we get

$$\begin{aligned}
[\mathbb{LHS}(\lambda,\mu)]_{i,j,k}^{i',j',k'} &= \sum_{s=0}\left[\mathbb{R}_{12}^{(2)}(\lambda|I,J+\mu)\right]_{i,j}^{i',s}\left[\mathbb{R}_{23}^{(1)}(\lambda|J,K)\right]_{s,k}^{j',k'} \\
= [\mathbb{RHS}(\lambda,\mu)]_{i,j,k}^{i',j',k'} &= \sum_{s=0}\left[\mathbb{R}_{23}^{(1)}(\mu|J+\lambda,K)\right]_{j,k}^{s,k'}\left[\mathbb{R}_{12}^{(2)}(\lambda|I,J)\right]_{i,s}^{i',j'} \\
&= \delta_{i+j+k,i'+j'+k'}\frac{i'!k'!(-\lambda)_{i'-i}(-\mu)_{k'-k}(\lambda-I)_i(\mu-K)_k}{i!k!(i'-i)!(k'-k)!(-I)_{i'}(-K)_{k'}}
\end{aligned} \tag{5.5.25}$$

Even though we have written the identity as a sum over s there is actually no summation involved. This is easy to see from (5.5.10) because $\mathbb{R}^{(2)}$ only depends on i indices and $\mathbb{R}^{(1)}$ only depends on k indices. So both sides of the identity are just pure Kronecker products of tensors.

5.5.2 Discussion

In this section we have constructed the rational sl_2 R-matrix at the level of matrix elements using methods developed in [66; 14]. We have derived formulae for the R-matrix itself as well as its factors and related objects such as kernels and a hermitian form. We have also considered some identities satisfied by the factors and showed that they are cases of well known identities in theory of hypergeometric series.

What is interesting is that out of all the methods for constructing the sl_2 R-matrix, this one seems to be the most efficient. All of the other methods considered give a double or triple summation formulae for the matrix elements and much work has to be done to simplify it to a single variable hypergeometric series. Of course, in this construction we only get a rational R-matrix and not the more general trigonometric R-matrix like in the methods of spectral decomposition and the 3D model projection. However, it seems like this construction should somehow generalise to trigonometric R-matrices, because we were able to construct a similar factorisation (3.5.6) from factors that are very similar to the ones constructed in this section.

We notice that there is a factorisation of the sl_2 R-matrix and also its trigonometric and elliptic deformations in [67] in terms of 'parameter permutation' operators. These operators can be combined to construct the factors $\mathbb{R}^{(i)}(\lambda)$ and suggests that the finer structure of the R-matrix discussed in this chapter splits even further. It would interesting to investigate this factorisation, which may offer another means of constructing (5.5.10), (3.5.6) and their elliptic generalisation. Another factorised form of the R-matrix related to the quantum modular double was constructed in [12] which is a combination of two $U_q(\widehat{sl_2})$ algebras but it is not clear how it can be restricted to $U_q(\widehat{sl_2})$ or even if the factorisation is similar to what we have found. It would be an interesting exercise to reverse the arguments starting from the trigonometric R-matrix to derive the q-deformation of all the objects in this section. This is something we may attempt in the future.

Another appealing feature of the construction in this section is that we can construct the factors from the 'ground up'. The other means of deriving them, as in section 3.5, is to start with the general higher spin R-matrix and make the appropriate subsitution to see it 'degenerate' into the desired factor. The general R-matrix is a far more complicated object than its factors and so this is not always feasible. We are very interested in these factors as standalone objects. The function describing their elements appears to be the same function appearing in other related studies. Firstly, they describe more elementary R-matrices without difference property which as shown can be combined to construct the general R-matrix. Secondly, with some small modifications [14] they describe a local form of the Q-operator. That is why we have named this chapter 'Factorisation by Q-operators'. The Q-operators are known to factorise the transfer matrix through what is known as the *fundamental fusion relation*, but this is a factorisation of global operators. The factorisation in this chapter seems to be a local form of this well-known relation and hence details another connection between an integrable model and its Q-operators.

Thirdly, and unexpectedly, the factors in this section are essentially the stochastic matrices that appear in the study of TASEP models [61; 22]. We study this aspect of the factors in greater detail in chapter 8 but in summary the function determining elements of $\mathbb{R}^{(1)}$ and $\mathbb{R}^{(2)}$ is a special case of the more general Φ function (8.0.4) which, as it turns out, factorises the general $U_q(\widehat{sl_n})$ R-matrix for symmetric tensor representations (8.0.10). However, it seems that this might be just a happy coincidence for sl_2, and for general n the correspondence is more complicated and maybe non-existent. In the next section we will see how these factors generalise in the case of sl_3.

5.6 Construction for sl_3

In this section we will repeat the steps of the last section but this time for the sl_3 invariant R-matrix acting on $\mathbb{V} \otimes \mathbb{V}$ under arbitrary highest weight representations $\pi^{\boldsymbol{I}} \otimes \pi^{\boldsymbol{J}} \cong \pi^{(I_1,I_2)} \otimes \pi^{(J_1,J_2)}$. In one way this construction gives a more general R-matrix than (3.3.24) derived using the 3D approach which is only for symmetric tensor representations $\pi^{(I,0)} \otimes \pi^{(J,0)}$ but as mentioned multiple times in this chapter in another way it is less general because it is a construction for the Lie algebra/rational R-matrices and not the quantum group/trigonometric R-matrices.

In this case the R-matrix factorises into three factors of the form (5.3.1). The factors with explicit dependence on spectral and weight parameters can be written as $\mathbb{R}^{(i)}(\lambda|I_1, I_2; J_1, J_2)$ like for the sl_2 case but expressions quickly become cumbersome so we will refrain unless necessary. Instead we write it as

$$R_{\boldsymbol{I},\boldsymbol{J}}(\lambda) = \mathbb{P}\mathbb{R}^{(1)}(\lambda_1|\boldsymbol{I}^{''};\boldsymbol{J}^{''})\mathbb{R}^{(2)}(\lambda_2|\boldsymbol{I}^{'};\boldsymbol{J}^{'})\mathbb{R}^{(3)}(\lambda_3|\boldsymbol{I};\boldsymbol{J}) \tag{5.6.1}$$

where

$$\lambda_1 = \frac{2J_1 + J_2 - 2I_1 - I_2}{3}, \quad \lambda_2 = \frac{I_1 - I_2 + J_2 - J_1}{3}, \quad \lambda_3 = \frac{2I_2 + I_1 - J_2 - J_1}{3},$$
$$\boldsymbol{I}' = (I_1, -\lambda + \frac{I_2 - I_1 + J_1 + 2J_2}{3}), \quad \boldsymbol{J}' = (J_1, \lambda + \frac{I_1 + 2I_2 - J_1 + J_2}{3}),$$
$$\boldsymbol{I}'' = (-\lambda + \frac{2I_1 + I_2 + J_1 - J_2}{3}, J_2), \quad \boldsymbol{J}'' = (\lambda + \frac{I_1 - I_2 + 2J_1 + J_2}{3}, I_2). \tag{5.6.2}$$

We now work with the group of 3×3 matrices $SL(3, \mathbb{C})$ and its subgroups (5.2.1). The subgroup Z_- is a function of 3 variables z_{21}, z_{31}, z_{32}, which for convenience we will relabel as z_1, z_2 and z_3 respectively. We relabel the variables w, α and β in the same way. The basis of monomials (5.4.6) is the set $\{\boldsymbol{z^i}\} := \{z_1^{i_1} z_2^{i_2} z_3^{i_3}\}_{i_1,i_2,i_3=0,\ldots}$ and similarly for the other variables. We also define $z_X := z_1 z_3 - z_2$ to simplify some of our expressions.

The reproducing kernel (5.4.5) now has the form

$$\begin{aligned}\mathcal{E}^I(z, \alpha) &= \Delta_1(\alpha^\dagger z)^{I_1} \Delta_2(\alpha^\dagger z)^{I_2} \\ &= (1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2})^{I_1}(1 + z_3\overline{\alpha_3} + (z_1 z_3 - z_2)(\overline{\alpha_1\alpha_3} - \overline{\alpha_2}))^{I_2}.\end{aligned} \tag{5.6.3}$$

We expand this out using the multinomial theorem

$$(1 + a_2 + \cdots + a_n)^m = \sum_{s_2,\ldots,s_n} D^m_{s_2,\ldots,s_m} \prod_{i=2}^n a_i^{s_i}, \tag{5.6.4a}$$

$$D^m_{s_2,\ldots,s_m} = (-1)^{s_2+\cdots+s_m} \frac{(-m)_{s_2+\cdots+s_n}}{\prod_{i=2}^n s_i!}, \tag{5.6.4b}$$

to get

$$\begin{aligned}\mathcal{E}^{\boldsymbol{I}}(z, \alpha) = \sum_{\boldsymbol{s},\boldsymbol{l}=0}^{\infty} &\frac{(-I_2)_{l_1+l_2+l_3+l_4+l_5}}{l_1! l_2! l_3! l_4! l_5!} (-1)^{s_1+s_2+l_1+l_3+l_5} \\ &\times z_1^{s_1+l_4+l_5} z_2^{s_2+l_1+l_2} z_3^{l_3+l_4+l_5} \overline{\alpha_1}^{s_1+l_2+l_5} \overline{\alpha_2}^{s_2+l_1+l_4} \overline{\alpha_3}^{l_2+l_3+l_5}.\end{aligned} \tag{5.6.5}$$

To extract matrix elements out of this we must set

$$i_1 = s_1 + l_4 + l_5, \quad i_2 = s_2 + l_1 + l_2, \quad i_3 = l_3 + l_4 + l_5,$$
$$i_1' = s_1 + l_2 + l_5, \quad i_2' = s_2 + l_1 + l_4, \quad i_3' = l_2 + l_3 + l_5.$$

In doing so we notice the conservation laws

$$i_2 + i_3 = i_2' + i_3', \quad i_1 + i_3' = i_1' + i_3, \quad i_1 + i_2 = i_1' + i_2'$$

although the second relation is dependent on the other two. We find that the formula for the matrix elements of the inverse hermitian form $\Omega^{-1}_{I_1,I_2}$ is given by

$$\left[\Omega^{-1}_{I_1,I_2}\right]^{i'_1,i'_2,i'_3}_{i_1,i_2,i_3} = \delta_{i_1+i_2,i'_1+i'_2}\delta_{i_2+i_3,i'_2+i'_3}(-1)^{i_1+i_2+i_3-i'_1}\sum_{s_1,s_2,l_1=0}\frac{(-I_1)_{s_1+s_2}(-I_2)_{i_2+i_3-s_2}}{s_1!s_2!l_1!}$$
$$\times\frac{(-1)^{s_1}}{(i_2-s_2-l_1)!(s_1+i_3-i_1)!(i_1+i_2-s_2-l_1-i'_1)!(l_1+i'_1+s_2-s_1-i_2)!}$$

This is a triple sum expression for the matrix elements and is somewhat complicated. However, it can be summed up twice to obtain a single sum ${}_4F_3$ hypergeometric series. That is, it can be simplified to

$$\left[\Omega^{-1}_{I_1,I_2}\right]^{i'_1,i'_2,i'_3}_{i_1,i_2,i_3} = \delta_{i_1+i_2,i'_1+i'_2}\delta_{i_2+i_3,i'_2+i'_3}\frac{(-1)^{i_1+i'_1}I_2!(I_1+I_2-i'_1-i_3)!(i_3+i'_1)!}{i_1!i_3!i'_1!i'_3!(I_1+I_2-i_2-i_3)!(i_2-i'_1)!(I_2-i_3-i'_1)!}$$
$$\times\sum_{s=0}^{i_1}\frac{(-i_1,-i'_1,i_1+i_2-i_3-i'_1-I_1,I_1+I_2-i_3-i'_1+1)_s}{s!(1+i_2-i'_1,-i_3-i'_1,I_2-i_3-i'_1+1)_s}. \tag{5.6.6}$$

The hypergeometric series is balanced and terminating and the output contains not only binomial terms but also polynomial factors in I_1, I_2. It is a far more complicated expression than what was found in the sl_2 case (5.5.3) in not just the number of terms but also because it is not a diagonal matrix. Calculating a formula for the inverse therefore is non-trivial. However we were able to do it by making a few key observations. First, (5.6.6) can be evaluated for abstract I_1, I_2 because they do not enter the product argument of pochhammer sumber. Second, the two conservation laws of Ω^{-1} imply that the matrix has a block diagonal form, indexed by the value of each conserved quantity. Each block is finite and restricting to each one we can calculate the inverse of the block - giving elements of Ω. Third, by inspection we notice that the polynomial factors in these elements are almost exactly the same as Ω^{-1} except shifted by $I_1 \to I_1 + 1$. Making this shift we see the difference between Ω and Ω^{-1} is just a product of binomial terms which are not difficult to work out. We find matrix elements of Ω are given by the formula

$$[\Omega_{I_1,I_2}]^{i'_1,i'_2,i'_3}_{i_1,i_2,i_3} = (-1)^{i'_2+i_3}\delta_{i_1+i_2,i'_1+i'_2}\delta_{i_2+i_3,i'_2+i'_3}$$
$$\times\frac{i_2!(i_1+i_2-i'_1)!(i_3+i'_1)!(-I_1+i_1-i_3-1)_{i_2-i'_1}(i_2+i_3-I_2-i'_2)_{i_1}}{(i_2-i'_1)!(-I_1)_{i_1+i_2}(-I_1-I_2-1)_{i_1+i_2}(-I_2)_{i_3}}$$
$$\times\sum_{s=0}^{i_1}\frac{(-i_1,-i'_1,i_1+i_2-i_3-j_1-I_1-1,I_1+I_2-j_1-i_3+2)_s}{s!(1+i_2-j_1,-i_3-j_1,1+I_2-i_3-j_1)_s}. \tag{5.6.7}$$

The fact that Ω^{-1} is not diagonal also means we cannot use a shortcut that we used in calculating the factors for sl_2, where instead of finding matrix elements by considering the action (5.4.9) we were able to separate them from the hermitian form in (5.4.4).

To calculate the kernels $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{(3)}$ we first need to calculate the decomposition (5.4.8) for 3×3 matrices. It is given by

$$\begin{aligned}
\alpha^\dagger z &= \begin{pmatrix} 1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2} & z_3\overline{\alpha_2} + \overline{\alpha_1} & \overline{\alpha_2} \\ z_2\overline{\alpha_3} + z_1 & 1 + z_3\overline{\alpha_3} & \overline{\alpha_3} \\ z_2 & z_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{z_1 + z_2\overline{\alpha_3}}{1+z_1\overline{\alpha_1}+z_2\overline{\alpha_2}} & 1 & 0 \\ \frac{z_2}{1+z_1\overline{\alpha_1}+z_2\overline{\alpha_2}} & \frac{z_3+z_X\overline{\alpha_1}}{1+z_3\overline{\alpha_3}+z_X\overline{\alpha_X}} & 1 \end{pmatrix} \\
&\times \begin{pmatrix} 1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2} & 0 & 0 \\ 0 & \frac{1+z_3\overline{\alpha_3}+z_X\overline{\alpha_X}}{1+z_1\overline{\alpha_1}+z_2\overline{\alpha_2}} & 0 \\ 0 & 0 & \frac{1}{1+z_3\overline{\alpha_3}+z_X\overline{\alpha_X}} \end{pmatrix} \begin{pmatrix} 1 & \frac{\overline{\alpha_1}+z_3\overline{\alpha_2}}{1+z_1\overline{\alpha_1}+z_2\overline{\alpha_2}} & \frac{\overline{\alpha_2}}{1+z_1\overline{\alpha_1}+z_2\overline{\alpha_2}} \\ 0 & 1 & \frac{\overline{\alpha_3}+z_1\overline{\alpha_X}}{1+z_3\overline{\alpha_3}+z_X\overline{\alpha_X}} \\ 0 & 0 & 1 \end{pmatrix} \\
&= z_\alpha d_{z,\alpha} \alpha_z
\end{aligned} \tag{5.6.8}$$

and similarly for $\beta^\dagger w$. The matrix $\beta_w w^{-1} z \alpha_z^{-1}$ is easy to calculate but the elements are large expressions that are messy to write down. Since we are only interested in the diagonal elements for calculating the kernels we will just list those. They are

$$\begin{aligned}
\left(\beta_w w^{-1} z \alpha_z^{-1}\right)_{11} &= \frac{1 + z_1\overline{\beta_1} + z_2\overline{\beta_2}}{1 + w_1\overline{\beta_1} + w_2\overline{\beta_2}}, \\
\left(\beta_w w^{-1} z \alpha_z^{-1}\right)_{22} &= \big[1 + z_3\overline{\beta_3} + (w_1 + w_2\overline{\beta_3})(\overline{\alpha_1} + z_3\overline{\alpha_2}) + z_X\overline{\beta_X}(w_1\overline{\alpha_1} + w_2\overline{\alpha_2}) \\
&\quad + z_X\overline{\alpha_X} + w_X\overline{\beta_X} + z_X\overline{\alpha_1}(\overline{\beta_3} - \overline{\alpha_3}) + w_1\overline{\beta_X}(z_3 - w_3)\big] \\
&\quad \times (1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2})^{-1}(1 + w_3\overline{\beta_3} + w_X\overline{\beta_X})^{-1}, \\
\left(\beta_w w^{-1} z \alpha_z^{-1}\right)_{33} &= \frac{1 + w_3\alpha_3 + w_X\alpha_X}{1 + z_3\alpha_3 + z_X\alpha_X}.
\end{aligned}$$

We use these to calculate the kernel functions $\mathcal{R}^{(i)}$ from (5.4.7). They are

$$\begin{aligned}
\mathcal{R}^{(1)}_{\lambda,\boldsymbol{IJ}}(z, w|\alpha, \beta) &= (1 + z_1\overline{\beta_1} + z_2\overline{\beta_2})^\lambda (1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2})^{I_1} (1 + w_1\overline{\beta_1} + w_2\overline{\beta_2})^{J_1 - \lambda} \\
&\quad \times (1 + z_3\overline{\alpha_3} + z_X\overline{\alpha_X})^{I_2} (1 + w_3\overline{\beta_3} + w_X\overline{\beta_X})^{J_2}
\end{aligned} \tag{5.6.9a}$$

$$\begin{aligned}
\mathcal{R}^{(2)}_{\lambda,\boldsymbol{IJ}}(z, w|\alpha, \beta) &= \big[1 + z_3\overline{\beta_3} + (w_1 + w_2\overline{\beta_3})(\overline{\alpha_1} + z_3\overline{\alpha_2}) + z_X\overline{\alpha_X} + z_X\overline{\beta_X}(w_1\overline{\alpha_1} + w_2\overline{\alpha_2}) \\
&\quad + w_X\overline{\beta_X} + z_X\overline{\alpha_1}(\overline{\beta_3} - \overline{\alpha_3}) + w_1\overline{\beta_X}(z_3 - w_3)\big]^\lambda \, (1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2})^{I_1 - \lambda} \\
&\quad \times (1 + z_3\overline{\alpha}_3 + z_X\overline{\alpha}_X)^{I_2} (1 + w_{21}\overline{\beta_1} + w_2\overline{\beta_2})^{J_1} (1 + w_3\overline{\beta_3} + w_X\overline{\beta_X})^{J_2 - \lambda}
\end{aligned} \tag{5.6.9b}$$

$$\mathcal{R}^{(3)}_{\lambda,\boldsymbol{IJ}}(z, w|\alpha, \beta) = (1 + w_3\overline{\alpha_3} + w_X\overline{\alpha}_X)^\lambda (1 + z_1\overline{\alpha_1} + z_2\overline{\alpha_2})^{I_1} (1 + w_1\overline{\beta_1} + w_2\overline{\beta_2})^{J_1}$$

$$\times\,(1+z_3\overline{\alpha_3}+z_X\overline{\alpha_X})^{I_2-\lambda}(1+w_3\overline{\beta_3}+w_X\overline{\beta_X})^{J_2} \tag{5.6.9c}$$

We want to expand these functions in the basis of monomials and calculate the coefficients $\left[\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}$. That is,

$$\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}(z,w|\alpha,\beta)=\sum_{\substack{\boldsymbol{i},\boldsymbol{j}\\ \boldsymbol{i}',\boldsymbol{j}'}}\left[\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}\,z^{\boldsymbol{i}}w^{\boldsymbol{j}}\overline{\alpha}^{\boldsymbol{i}'}\overline{\beta}^{\boldsymbol{j}'}. \tag{5.6.10}$$

The kernel functions (5.6.9) are quite complicated expressions which we can expand using the multinomial theorem (5.6.4) but it is immediately obvious the resulting expression will be hugely complicated with many terms and summation variables.For example, performing the expansion, in the same style as (5.6.5), for $\left[\mathcal{R}^{(1)}_{\lambda,\boldsymbol{IJ}}\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}$ one will see that the formula contains six summations. The expressions for $\mathcal{R}^{(2)}_{\lambda,\boldsymbol{IJ}}$ and $\mathcal{R}^{(3)}_{\lambda,\boldsymbol{IJ}}$ are even more complicated, containing twenty-five(!) and nine summations respectively. However, we succeeded in simplifying them quite considerably; getting the expressions down to a product of two ${}_4F_3$ hypergeometric series. The simplification is an enormous calculation (even compared to other simplifications in this thesis) that involves many identities and would take pages to write down. For now we will just give the final result for each coefficient, which all have the form

$$\left[\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}\right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}}=\overline{\delta}^{\boldsymbol{j}}_{\boldsymbol{i}}\left[A^{(i)}(\lambda)\right]^{\boldsymbol{i}'}_{\boldsymbol{i}}\left[B^{(i)}(\lambda)\right]^{\boldsymbol{j}'}_{\boldsymbol{j}},$$
$$\overline{\delta}^{\boldsymbol{j}}_{\boldsymbol{i}}:=\delta_{i_2+j_2+i_3+j_3,i'_2+i'_3+j'_2+j'_3}\delta_{i_1+j_1+i'_3+j'_3,i_3+j_3+i'_1+j'_1}.$$

The $\boldsymbol{i}$ and $\boldsymbol{j}$ indices can be separated and $\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}$, up to the delta function, is given by a Kronecker product. We have written both A and B as having a dependence on λ but actually depending on i only one of these functions may have a dependence.

For $\mathcal{R}^{(1)}$ we get

$$\left[A^{(1)}\right]^{\boldsymbol{i}'}_{\boldsymbol{i}}=\frac{(-1)^{i_2+i_3+i'_1+i'_2}(-I_2)_{i_3}(-I_1+i'_2-i_3)_{i_3+i'_1-i'_3-i'_2}}{i'_2!i_3!(i'_2+i'_3-i_3)!(i'_1+i_3-i'_3)!(i_2+i_3-i'_2-i'_3)!(i_1+i'_3-i'_1-i'_3)!}$$
$$\times\,{}_4\overline{F}_3\left(\begin{matrix}-i'_2; & i_3-i'_2-i'_3 & I_1+1+i_3-i'_2 & -I_1-I_2+i'_1+i_3\\ & -I_2+i_3-i'_2 & i_3-i'_2+1 & i_3+i'_1-i'_2-i'_3+1\end{matrix}\middle|\,1\right), \tag{5.6.11a}$$

$$\left[B^{(1)}(\lambda)\right]^{\boldsymbol{j}'}_{\boldsymbol{j}}=\frac{(-1)^{j'_1+j'_3}(-\lambda)_{j'_1+j'_2-j_1-j_2}(\lambda-J_1)_{j_1-j_3}(-J_2)_{j_3}}{j_2!j'_3!(j_2+j_3-j'_3)!(j_1+j'_3-j_3)!(\lambda-J_1)_{j_2-j'_3}}$$

$$\times {}_4\overline{F}_3 \left(\begin{matrix} -j_2; & j_3' - j_3 - j_2 & \lambda - J_1 - J_2 + j_1 + j_3' & -\lambda + J_1 + 1 + j_3' - j_2 \\ & 1 + j_3' - j_2 & 1 + j_1 + j_3' - j_2 - j_3 & -J_2 + j_3' - j_2 \end{matrix} \middle| 1 \right). \tag{5.6.11b}$$

Note that we have written them in terms of regularised hypergeometric functions according to (A.1.4). Both formulae are balanced and terminating hypergeometric series and can be transformed using (the regularised version of) the Whipple identity (A.2.7). Now for $\mathcal{R}^{(2)}$ we get

$$\left[A^{(2)}(\lambda)\right]_i^{i'} = \frac{(-1)^{i_1+i_2+i_1'+i_3'}(-\lambda)_{i_3+i_1'-i_3'-i_1}(-I_2)_{i_3'}(-I_1-I_2+i_3+i_1')_{i_2-i_1'}(-\lambda-I_2)_{i_3}}{i_1!i_2!i_1'!i_3'!(i_1'+i_2'-i_1-i_2)!(i_2+i_3-i_2'-i_3')!(-\lambda-I_2)_{i_3+i_1'-i_1}}$$
$$\times {}_4\overline{F}_3 \left(\begin{matrix} -i_1'; & I_1+I_2+1-i_1'-i_2' & I_1+I_2+1-\tilde{i}_{31} & \lambda+I_2+1+i_1-\tilde{i}_{31} \\ & I_1+I_2+2-i_1' & I_1+I_2+1-i_2-\tilde{i}_{31} & \lambda+I_2+1-\tilde{i}_{31} \end{matrix} \middle| 1 \right), \tag{5.6.12a}$$

$$\left[B^{(2)}(\lambda)\right]_j^{j'} = \frac{(-1)^{j_3+j_2'}(\lambda-J_2)_{j_3}(-J_2)_{j_3'}(-J_1-J_2)_{j_2'+j_3'}}{j_1!j_3!j_1'!j_2'!(-J_2)_{j_3'+j_1-j_1'}(-J_1-J_2)_{j_1+j_3'}}$$
$$\times {}_4\overline{F}_3 \left(\begin{matrix} -j_1; & J_1+J_2+1-j_1-j_2 & J_1+J_2+1-j_1-j_3' & J_2+1+j_1'-j_1-j_3' \\ & J_1+J_2+2-j_1 & J_1+J_2+1-j_1-j_2'-j_3' & J_2+1-j_1-j_3' \end{matrix} \middle| 1 \right) \tag{5.6.12b}$$

where we have defined $\tilde{i}_{31} := i_3 + i_1'$ for compactness. Both formulae are terminating hypergeometric series but in general they are not balanced - $A^{(2)}(\lambda)$ is $(1+i_1'+i_2'-i_1-i_2)$-balanced while $B^{(2)}(\lambda)$ is $(1+j_1+j_2-j_1'-j_2')$-balanced. Finally for $\mathcal{R}^{(3)}$ we get

$$\left[A^{(3)}(\lambda)\right]_i^{i'} = \frac{(-1)^{i_1+i_2+i_3+i_1'}(-\lambda)_{i_2'+i_3'-i_2-i_3}(\lambda-I_2)_{i_3}(\lambda-I_1-I_2+i_3-i_1')_{i_2-i_1'}}{i_1!i_2!i_1'!(i_3+i_1'-i_1)!(i_1'+i_2'-i_1-i_2)!(i_1+i_3'-i_1'-i_3)!}$$
$$\times {}_4\overline{F}_3 \left(\begin{matrix} -i_1'; & i_1-i_1'-i_3 & -\lambda+I_1+I_2+1-i_1'-i_3 & I_1-i_1'-i_2' \\ & -i_1'-i_3 & -\lambda+I_1+I_2+1-i_2-i_3-i_1' & I_1+1-i_1' \end{matrix} \middle| 1 \right), \tag{5.6.13a}$$

$$\left[B^{(3)}\right]_j^{j'} = \frac{(-1)^{j_1+j_2+j_3+j_1'}(-J_2)_{j_3}(-J_1-J_2)_{j_2'+j_3'}}{j_1!j_1'!j_2'!(j_3'+j_1-j_1')!(-J_1-J_2)_{j_1+j_3'}}$$
$$\times {}_4\overline{F}_3 \left(\begin{matrix} -j_1; & j_1'-j_1-j_3' & J_1+J_2+1-j_1-j_3' & J_1-j_1-j_2 \\ & -j_1-j_3' & J_1+J_2+1-j_1-j_2'-j_3' & J_1+1-j_1 \end{matrix} \middle| 1 \right). \tag{5.6.13b}$$

Just like (5.6.12) these formulae contain terminating hypergeometric series that are not balanced - $A^{(3)}(\lambda)$ is $(1+i_1'+i_2'-i_1-i_2)$-balanced while $B^{(3)}(\lambda)$ is $(1+j_1+j_2-j_1'-j_2')$-balanced.

Now we can calculate the factors $\mathbb{R}^{(i)}(\lambda)$ by combining (5.6.11)-(5.6.13) and (5.6.7) in (5.4.9). We will calculate a formula for the matrix elements by considering the action on the monomial basis $\{z^{\boldsymbol{i}'} w^{\boldsymbol{j}'}\}$, that is,

$$\begin{aligned}
\mathbb{R}^{(i)}(\lambda) \cdot z^{\boldsymbol{i}'} w^{\boldsymbol{j}'} &= \Omega_{\boldsymbol{IJ}} \left(\mathcal{R}^{(i)}_{\lambda,\boldsymbol{IJ}}(z,w|\alpha,\beta), \alpha^{\boldsymbol{i}'} \beta^{\boldsymbol{j}'} \right), \\
&= \sum_{\substack{\boldsymbol{ij} \\ \boldsymbol{sr}}} [\mathcal{R}_{\lambda,\boldsymbol{IJ}}]^{\boldsymbol{s},\boldsymbol{r}}_{\boldsymbol{i},\boldsymbol{j}} \, [\Omega_{\boldsymbol{IJ}}]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{s},\boldsymbol{r}} \, z^{\boldsymbol{i}} w^{\boldsymbol{j}}, && \text{(5.6.14a)} \\
&= \sum_{\boldsymbol{ij}} \left[\mathbb{R}^{(i)}(\lambda) \right]^{\boldsymbol{i}',\boldsymbol{j}'}_{\boldsymbol{i},\boldsymbol{j}} z^{\boldsymbol{i}} w^{\boldsymbol{j}}. && \text{(5.6.14b)}
\end{aligned}$$

All that is left is to calculate this expression. This is also an enormous calculation similar to what is required in obtaining (5.6.11)-(5.6.13). The steps are exhaustively long involving summing up the expression many times. There is a summation over 6 indices (5.6.14a) with each element of $\mathcal{R}$ and $\Omega_{\boldsymbol{IJ}}$ given by double sums. However, we utilise one observation that simplifies the calculations a great deal; the $\boldsymbol{i}$ and $\boldsymbol{j}$ indices in formulae for $\mathcal{R}$ and $\Omega_{\boldsymbol{IJ}}$ can be separated. This means we can split to simplification into two smaller computations. Perhaps a more efficient means of summing up and obtaining our final result exists but is unknown to us at this point.

5.6.1 The factor $\mathbb{R}^{(1)}(\lambda)$

The final result of the calculation (5.6.14a) is the formula

Proposition 5.6.1.

$$\begin{aligned}\left[\mathbb{R}^{(1)}(\lambda)\right]_{i,j}^{i',j'} &= \frac{\overline{\delta}_i^j \delta_{i_3,i'_3} j'_1!(j'_2+j'_3)!(-J_1-1)_{j'_2}(-\lambda+J_1+1)_{j_3}(\lambda-J_1)_{j_1-j_3}(-\lambda)_{j'_1+j'_2-j_1+j_3}}{j_1!j_2!j_3!(j'_2+j'_3-j_2-j_3)!(j'_1-j_1-j'_3+j_3)!(-J_1)_{j'_1+j'_2}(-J_1-J_2-1)_{j'_2+j'_3}} \\ &\times \sum_{m,l=0}^{j_2+j_3} \frac{(-j_3,\lambda-J_1+j_1-j_3,\lambda-J_1-J_2-1)_m(-j'_2,J_1+1-j'_1-j'_2,J_1+J_2+2-j'_2-j'_3)_l}{m!(-j_2-j_3,\lambda-J_1-j_3)_m l!(J_1+2-j'_2,-j'_2-j'_3)_l} \\ &\times \frac{(-1)^{j_2+j'_3}(-j_2-j_3)_{l+m}}{(\lambda+1+j_1-j_3-j'_1-j'_2)_{l+m}} \qquad (5.6.15)\end{aligned}$$

for the matrix elements of $\mathbb{R}^{(1)}(\lambda)$ in the case of sl_3. This expression does not depend on the i indices (except for the delta functions) just like the sl_2 case. It is a somewhat more complicated expression than the sl_2 given it is a double sum compared to a simple product. In each summation variable the summation is a terminating ${}_4F_3$ series that is not balanced. It is interesting to analyse this formula by finding certain arguments where the expression simplifies.

One immediately sees from the denominator terms that the expression is trivially zero whenever

$$j_1 - j'_1 > j_3 - j'_3 > j'_2 - j_2.$$

It reduces to a simpler expression whenever $j_3 = 0$. In this case the summation in 'm' disappears and we can rewrite the expression as the single sum

$$\begin{aligned}\left[\mathbb{R}^{(1)}(\lambda)\right]_{i,j}^{i',j'}\Bigg|_{j_3=0} &= \frac{\overline{\delta}_i^j \delta_{i_3,i'_3}(-1)^{j_2+j'_3} j'_1!(j'_2+j'_3)!(-J_1-1)_{j'_2}(\lambda-J_1)_{j_1}(-\lambda)_{j'_1+j'_2-j_1}}{j_1!j_2!(j'_2-j_2+j'_3)!(j'_1-j_1-j'_3)!(-J_1)_{j'_1+j'_2}(-J_1-J_2-1)_{j'_2+j'_3}} \\ &\times {}_4F_3\left(\begin{matrix}-j'_2 & -j_2 & J_1+1-j'_1-j'_2 & J_1+J_2+2-j'_2-j'_3 \\ & -j'_2-j'_3 & J_1+2-j'_2 & \lambda+1+j_1-j'_1-j'_2\end{matrix}\middle|\, 1\right).\end{aligned}$$

Actually, if we instead set $j'_3, J_2 = 0$ we get a reduction to a product expression. To see this, notice that the sum in 'l' becomes a ${}_2F_1$ series and can be summed using the Gauss identity (A.2.2).

The resultant formula still contains the summation in 'm' but this is now reduced to a ${}_2F_1$ series which can be summed up as well. The result is the product

$$\left[\mathbb{R}^{(1)}(\lambda)\right]_{i,j}^{i',j'}\Bigg|_{J_2,j_3'=0} = \frac{\bar{\delta}_i^j \delta_{i_3,i_3'} j_1'! j_2'! (\lambda - J_1)_{j_1+j_2} (-\lambda)_{j_1'+j_2'-j_1-j_2}}{j_1! j_2! (-j_3')! j_3'! (j_3 + j_1' - j_1)! (j_2' - j_2 - j_3)! (-J_1)_{j_1'+j_2'}}. \qquad (5.6.16)$$

Immediately we see that we must also have $j_3 = 0$. So really, the resulting expression is

$$\left[\mathbb{R}^{(1)}(\lambda)\right]_{i,j}^{i',j'}\Bigg|_{J_2,j_3,j_3'=0} = \bar{\delta}_i^j \delta_{i_3,i_3'} \frac{j_1'! j_2'! (\lambda - J_1)_{j_1+j_2} (-\lambda)_{j_1'+j_2'-j_1-j_2}}{j_1! j_2! (j_1' - j_1)! (j_2' - j_2)! (-J_1)_{j_1'+j_2'}}. \qquad (5.6.17)$$

This formula is almost exactly the same as that which appears in the factorisation (3.7.18). Indeed, we already made the connection to $\boldsymbol{N}^r(I,J)$ in the case of sl_2 by the relation (5.5.17b) and it seems that the same relation holds for sl_3 when the second representation parameter J_2 and indices j_3, j_3' are 'switched off'. Indeed,

$$\boldsymbol{N}^r(-\mu + \frac{I+J}{2}, J) = \mathbb{R}^{(1)}(\mu + \frac{J-I}{2}|\boldsymbol{I}, J_1, 0)\Bigg|_{j_3,j_3'=0}. \qquad (5.6.18)$$

This makes sense, the factor $\boldsymbol{N}^r(I,J)$ is a factor of sl_3 R-matrix only for symmetric tensor representations and $\mathbb{R}^{(1)}(\lambda)$ as we have constructed it (5.6.15) is a more general object - holding for all highest weight representations. The questions still remains: how does $\boldsymbol{M}^r(I,J)$ appear? For sl_2 we could identify it with $\mathbb{R}^{(2)}(\lambda)$ but now there are two factors which should somehow reduce to it in the restriction to symmetric tensor representations.

5.6.2 The factor $\mathbb{R}^{(2)}(\lambda)$

The final result of the calculation (5.6.14a) is the formula

Proposition 5.6.2.

$$\left[\mathbb{R}^{(2)}(\lambda)\right]_{i,j}^{i',j'} = \frac{\bar{\delta}_i^j i_2'! j_3'! (-I_1 + i_3 - i_3')_{i_2} (-\lambda + i_3 - i_3' - i_1)_{i_1} (-\lambda)_{i_3 - i_3' + i_1' - i_1} (\lambda - J_2)_{j_3}}{(-1)^{j_1 + i_1' + j_1' + i_2 + i_2'} (i_1, i_2, j_3, i_2 + i_3 - i_2' - i_3', j_1 - j_1', j_2 - j_2')! (-I_1)_{i_2'} (-J_2)_{j_3'}}$$
$$\times {}_4F_3\left(\begin{matrix} -i_1' & -I_1 - 1 & -I_1 + i_2 + i_3 - i_3' & -\lambda + i_3 - i_3' \\ & -I_1 + i_2' & -I_1 + i_3 - i_3' & -\lambda + i_3 - i_3' - i_1 \end{matrix}\middle| 1\right). \qquad (5.6.19)$$

This is a terminating $(1 + i_1' + i_2' - i_1 - i_2)$-balanced ${}_4F_3$ hypergeometric series formula for the matrix elements. This is a simpler expression than the one obtained for $\mathbb{R}^{(1)}(\lambda)$ and so it appears the relation (5.5.15) is broken for rank greater than 2. Another interesting difference is that the function depends on both $\boldsymbol{i}$ and $\boldsymbol{j}$ indices. The sum only depends on $\boldsymbol{i}$ but there are binomial factors outside the sum which depend strictly on $\boldsymbol{j}$ indices.

This formula also has a number of interesting reductions. In particular, it is immediately obvious the expression is trivially zero whenever

$$j_1' > j_1, \quad j_2' > j_2,$$
$$i_3' - i_3 > i_2 - i_2' \iff j_3 - j_3' > j_2' - j_2.$$

There is also a trivial zero whenever $j_3' - j_3 < 0$. Consider the delta relation

$$j_3' - j_3 = i_2 - i_2' + j_2 - j_2' + i_3 - i_3',$$

if $j_3' - j_3 < 0$ then either $i_2 - i_2' + i_3 - i_3' < 0$ or $j_2 - j_2' < 0$. These are two regimes that give trivial zeroes so $j_3 > j_3'$ is another regime that gives a trivial zero.

We note that the indices only enter the expression as their difference $i_3' - i_3$. There are two reductions in the cases $i_3' = i_3$ and $i_3' > i_3$. Let us consider the former case first. All indices i_3', i_3 disappear and we can apply the Karlsson-Minton summation formula (A.2.10) by making the identification

$$a = -i_1', \quad b = -I_1 - 1, \quad b_1 = -I_1 + i_2',$$
$$m_1 = i_2 - i_2', \quad b_2 = -i_1 - \lambda, \quad m_2 = i_1.$$

The only potential issue is that m_1 is negative when $i_2' > i_2$, but in this case the factorial $(i_2 + i_3 - i_2' - i_3')!$ has negative argument and the entire expression is zero. Therefore we can assume it is positive in applying the summation formula. Doing so we see that the expression is given by the product

$$\left[\mathbb{R}^{(2)}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'}\Bigg|_{i_3=i_3'} = \overline{\delta}_{\boldsymbol{i}}^{\boldsymbol{j}} \frac{(-1)^{i_1+i_2+j_1+i_1'+i_2'+j_1'} i_1'! j_3'! (-I_1)_{i_2} (-\lambda)_{i_1'-i_1} (\lambda - I_1)_{i_1} (\lambda - J_2)_{j_3}}{i_1! j_3! (i_2 - i_2')! (j_1 - j_1')! (j_2 - j_2')! (-I_1)_{i_2} (-I_1)_{i_1'} (-J_2)_{j_3'}}. \tag{5.6.20}$$

This reduced product formula has all the same trivial zeroes as the full form. In the case $i_3' > i_3$ the formula has a non-trivial zero. This can only be seen by applying the second Karlson-Minton summation (A.2.11), by identifying

$$a = -i_1', \quad b_1 = -I_1 + i_3 - i_3', \quad b_2 = -I_1 + i_2', \quad b_3 = i_3 - i_3' - i_1 - \lambda,$$

$$m_1 = i_3' - i_3 - 1, \quad m_2 = i_2 - i_2' + i_3 - i_3', \quad m_3 = i_1,$$

in (5.6.19). Then

$$\begin{aligned}\mathrm{Re}(-a) - m_1 - m_2 - m_3 &= i_1' + i_2' - i_1 - i_2 + 1 \\ &= j_1 - j_1' + j_2 - j_2' + 1.\end{aligned}$$

This can only be less than or equal to zero when either $j_1' > j_1$ or $j_2' > j_2$. But in these cases the factorials $(j_2 - j_2')!$ and $(j_1 - j_1')!$ in the denominator outside the sum diverge and the expression is trivially zero. We also require $m_2 = i_2 - i_2' + i_3 - i_3' \geq 0$ which can be assumed because of the denominator factorial $(i_2 - i_2' + i_3 - i_3')!$ diverging otherwise. Under these conditions the sum is nontrivially zero by (A.2.11). In either case, the overall expression for $i_3' > i_3$ is zero.

5.6.3 The factor $\mathbb{R}^{(3)}(\lambda)$

The final result of the calculation (5.6.14a) is the formula

Proposition 5.6.3.

$$\begin{aligned}\left[\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} &= \frac{\overline{\delta}_i^j (-1)^{j_1+j_1'} i_2'! i_3'! (i_1 + i_3' - i_3)!}{i_1! i_2! i_3! (i_3' - i_3)! (i_1 - i_1' + i_3' - i_3)! (j_1 - j_1')! (j_2 - j_2')!} \\ &\times \frac{(-\lambda)_{i_2'+i_3'-i_2-i_3} (\lambda - I_2)_{i_3} (\lambda - I_1 - I_2 - 1)_{i_2+i_3-i_3'}}{(-I_1 - I_2 - 1)_{i_2'} (-I_2)_{i_3'} (\lambda - I_1 - I_2 - 1)_{i_3 - i_3'}} \\ &\times {}_4F_3\left(\begin{matrix} -i_1' & -I_1 - I_2 - 2 & i_3 - i_3' & \lambda - I_1 - I_2 - 1 + i_3 - i_3' + i_2 \\ & -I_1 - I_2 - 1 + i_2' & i_3 - i_3' - i_1 & \lambda - I_1 - I_2 - 1 + i_3 - i_3' \end{matrix} \middle| 1 \right). \end{aligned} \quad (5.6.21)$$

The matrix elements, like $\mathbb{R}^{(2)}(\lambda)$ are given by a terminating $(1 + i_1' + i_2' - i_1 - i_2)$-balanced ${}_4F_3$ hypergeometric series. In fact the formulae are very similar. It almost depends completely on $\boldsymbol{i}$ indices except for two factorial terms in the denominator, therefore we cannot compare it to the sl_2 factors or (3.7.18).

By observing the factorial terms in the denominator we see that that the expression has trivial zeroes when

$$\begin{aligned}&j_1' > j_1, \quad j_2' > j_2, \quad j_3' > j_3, \quad i_3 > i_3', \\ &i_3 - i_3' > i_1 - i_1', \quad i_3 - i_3' > i_2' - i_2.\end{aligned} \quad (5.6.22)$$

There do not appear to be any non-trivial zeroes as the Karlsson-Minton sum does not fit. There is however an interesting reduction in the case $i_3 = i'_3$. In this case the expression becomes a product. Note that one of the terms in the numerator of the hypergeometric series is now zero. It is possible that this zero is cancelled off by the resulting $-i_1$ term in the denominator sum for summands with index $> i_1$ and the sum remains nontrivial. However, this is always cancelled by the other numerator sum term '$-i'_1$' when $i'_1 \leq i_1$ and by the denominator factorial $(i_1 - i'_1)!$ when $i'_1 > i_1$. This means in either case the sum never contributes to the overall expression and can be discarded. The resulting formula can then be written as

$$\left[\mathbb{R}^{(3)}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'}\Bigg|_{i_3=i'_3} = \frac{\overline{\delta}_{\boldsymbol{i}}^{\boldsymbol{j}}(-1)^{j_1+j'_1} i'_2!(-\lambda)_{i'_2-i_2}(\lambda-I_2)_{i_3}(\lambda-I_1-I_2-1)_{i_2}}{i_2!(i_1-i'_1)!(j_1-j'_1)!(j_2-j'_2)!(-I_1-I_2-1)_{i'_2}(-I_2)_{i_3}}. \tag{5.6.23}$$

5.6.4 Symmetric tensor representations

We want to take the matrix elements for factors that we constructed (5.6.15), (5.6.19), and (5.6.21) in the last section and put them together to find the R-matrix (5.6.1). This is easy in principle - it's just matrix multiplication - but the resulting formula as we mentioned in the last section is incredibly complicated and so far we have not simplified it. For now, we will consider the restriction to symmetric tensor representations where $\boldsymbol{I} = (I, 0), \boldsymbol{J} = (J, 0),\ I, J \in \mathbb{C}$ to compare with our results using the 3D approach. We will show, just like the sl_2 case, that the R-matrices are the same but the main objective is to clarify the link between the two factorisations, where (3.7.18) contains two factors while (5.6.1) contains three.

First, we write down how the intertwining relations (5.3.2) work for sl_3. They are

$$\begin{aligned}\mathbb{R}^{(1)}(\lambda)(\pi^{(I_1,I_2)}\otimes\pi^{(J_1,J_2)}) &= (\pi^{(I_1+\lambda,I_2)}\otimes\pi^{(J_1-\lambda,J_2)})\mathbb{R}^{(1)}(\lambda),\\ \mathbb{R}^{(2)}(\lambda)(\pi^{(I_1,I_2)}\otimes\pi^{(J_1,J_2)}) &= (\pi^{(I_1-\lambda,I_2+\lambda)}\otimes\pi^{(J_1+\lambda,J_2-\lambda)})\mathbb{R}^{(2)}(\lambda),\\ \mathbb{R}^{(3)}(\lambda)(\pi^{(I_1,I_2)}\otimes\pi^{(J_1,J_2)}) &= (\pi^{(I_1,I_2-\lambda)}\otimes\pi^{(J_1,J_2+\lambda)})\mathbb{R}^{(3)}(\lambda)\end{aligned} \tag{5.6.24}$$

which is where the values (5.6.2) come from. In restricting to symmetric tensor representations we need to set $I_2 = J_2 = 0$. We also know that the representation space in this case is realised as the space of polynomials in two variables while the general case is in three variables. It is easy to see from the action (5.2.10) that the variables z_3, w_3 are not present in this restriction - their coefficient is always zero. Therefore to restrict we must also set $i'_3 = j'_3 = 0$. We also need to make sure that this sets $i_3 = j_3 = 0$ so that our operator does not map out of the subspace $\mathbb{C}[z_1, z_2]$. We will soon show that is indeed the case. In this restriction the R-matrix (5.6.1) simplifies considerably, and the elements can be written as

$$\begin{aligned}[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \mathbb{P} \sum_{\substack{s,l \\ r,t}} & \left[\mathbb{R}^1(\lambda + \frac{2J-2I}{3} \middle| -\lambda + \frac{2I+J}{3}, 0; \lambda + \frac{2J+I}{3}, 0)\right]_{i_1,i_2,i_3,j_1,j_2,j_3}^{s_1,s_2,s_3,l_1,l_2,l_3} \\ \times & \left[\mathbb{R}^2(\lambda + \frac{I-J}{3} \middle| I, -\lambda + \frac{J-I}{3}; J, \lambda + \frac{I-J}{3})\right]_{s_1,s_2,s_3,l_1,l_2,l_3}^{r_1,r_2,r_3,t_1,t_2,t_3} \\ \times & \left[\mathbb{R}^3(\lambda + \frac{I-J}{3} \middle| I, 0; J, 0)\right]_{r_1,r_2,r_3,t_1,t_2,t_3}^{i'_1,i'_2,0,j'_1,j'_2,0} \end{aligned} \tag{5.6.25}$$

It appears that there are twelve summations but actually this can be reduced quite drastically. In fact, we claim that

$$r_3 = s_3 = l_3 = i_3 = j_3 = 0.$$

This is justified by our consideration of the reductions of each factor $\mathbb{R}^{(i)}$ in the previous section where we found regimes where the elements of each factor can be zero or a simple product. Let us expain, starting from the upper indices and working downward. The summation index r_3 must be zero because $i'_3 = 0$ and (5.6.22) implies that $r_3 = 0$ else the expression is trivial. $l_3 = 0$ by examining $\mathbb{R}^{(2)}$ (5.6.19) is trivially zero when $\lambda = J_2$ and $l_3 > 0$. In (5.6.25) this equality holds. $j_3 = 0$ in $\mathbb{R}^{(1)}$ because $J_2 = 0, l_3 = 0$ and the reduction (5.6.17). If $s_3 = 0$, then clearly by the delta function in (5.6.15) we must have $i_3 = 0$ also. The justification that $s_3 = 0$ is more complicated. While $s_3 > r_3 = 0$ in $\mathbb{R}^{(2)}$ (5.6.19) is in general non-zero, $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ in (5.6.25) it seems it is always zero. This we discovered by evaluating the resulting expression for this range of values, although it seems to be a highly non-trivial zero that results from some complicated interplay of a quadruple summation. We expect it to be true because of the way $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ in (5.6.25) intertwines the representation, i.e.

$$\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)(\pi^{(I,0)} \otimes \pi^{(J,0)}) = (\pi^{(-\lambda+\frac{2I+J}{3},0)} \otimes \pi^{(\lambda+\frac{2J+I}{3},0)})\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)$$

preserving the (non-)zero nature of each component. Therefore $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ should preserve the subspace $\mathbb{C}[z_1, z_2]$. We also remark that $\mathbb{R}^{(1)}$ in (5.6.25) preserves this subspace, as justified in the last paragraph, and indeed by (5.6.24) it preserves the weights. So we set $s_3 = 0$. Another interesting feature of (5.6.25) is that t_3 is a non-trivial summation and so it can be said that although the operator $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ together preserves the subspace the individual factors map out of it even when $I_2 = J_2 = 0$. Indeed, these weight arguments for $\mathbb{R}^{(2)}$ in (5.6.25) are non-zero and actually depend on I, J.

Let us now consider the implications of our discussion so far for the expressions for $\mathbb{R}^{(i)}$ we use in constructing the R-matrix (5.6.25). For $\mathbb{R}^{(1)}$ we use the product formula (5.6.17). For $\mathbb{R}^{(2)}$ we use the product formula (5.6.20) and for $\mathbb{R}^{(3)}$ we use the product formula (5.6.23). All the regimes where these reductions hold are satisfied. Following from (5.6.18) therefore we have already identified one of our factors (3.7.18) in (5.6.25) and now we will prove that the other factor comes from $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ in a non-trivial way. We will do this by showing the expression for its elements are given by the same product function as $\tilde{\boldsymbol{M}}^r$.

Putting together $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ we see that there are five summations r_3, r_2, t_1, t_2, t_3. There are also four kronecker delta functions which allow us to remove two of these variables. We choose to eliminate t_1, t_2 by making the substitution $t_1 := i_1' + j_1' + t_3 - r_1, t_2 := i_2' + j_2' - r_2 - t_3$. In what follows, we rename the remaining summation indices $s := r_1, t := r_2, l := t_3$, and also $s_1 := i_1, s_2 := i_2, l_1 := j_1, l_2 := j_2$ because these are now lower indices for the purposes of our computation. After all, we find the expression for the matrix elements is given by

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i_1+i_2+i_1'+i_2'}(\lambda - \frac{2I+J}{3})_{i_1}}{i_1! i_2! (i_2' - i_2)! (-1-I)_{i_2'} (\lambda + 1 + \frac{I-J}{3})_{i_1}} \tag{5.6.26}$$
$$\times \sum_{s,l,t=0} \frac{(-1)^s s! l! (-\lambda + \frac{J-I}{3} - i_1)_s (-i_2')_l}{(s-l-i_1, s-i_1', l+i_1'-s)!(-I)_s} {}_3\overline{F}_2 \left(\begin{matrix} l - i_2'; & \lambda - \frac{2I+J}{3} - 1 & -i_2 \\ & \lambda + \frac{I-J}{3} + 1 - i_2' & 1 + l - i_2 \end{matrix} \middle| 1 \right).$$

The formula is a triple summation in variables $0 \le l \le i_2'$, $i_1' \le s \le i_1' + l$ and the terminating ${}_3F_2$ hypergeometric series (whose summation variables we call 't') which terminates after $i_2' - l$ terms. These bounds ensure that the variable substitution made in the last paragraph do not sum up negative values of summation indices. For the Kronecker delta function we have used the notation (3.3.12). Note that the expression is trivially zero unless $i_2' \ge i_2$ because of the factorial term outside the sum which can be negative. The first step is to transform this identity with the second Thomae theorem (A.2.5) (with the given ordering) to obtain

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i_1+i_2+i_1'}(\lambda - \frac{2I+J}{3})_{i_1}}{i_1! i_2! (i_2' - i_2)! (-1-I)_{i_2'} (\lambda + 1 + \frac{I-J}{3})_{i_1}} \tag{5.6.27}$$
$$\times \sum_{s,l,t=0} \frac{(-1)^{l+s} s! l! (-\lambda + \frac{J-I}{3} - i_1)_s (-i_2')_l}{(s-l-i_1)!(s-i_1')!(l+i_1'-s)!(-I)_s} {}_3\overline{F}_2 \left(\begin{matrix} l - i_2'; & -i_2 & 2 + I - i_2' \\ & -i_2' & \lambda + \frac{I-J}{3} + 1 - i_2' \end{matrix} \middle| 1 \right),$$

as an intermediate step. Making this transform affects the expression for the other summations so that they can be transformed into a product. In particular, we rewrite (5.6.27) by bringing the summation in 'l' forward and identifying it as a ${}_3F_2$ hypergeometric series. That is

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i_2+i'_1} i'_2!(\lambda - \frac{2I+J}{3})_{i_1}(-\lambda + \frac{J-I}{3})_{i'_2}}{i_1!i_2!(i'_2-i_2)!(i'_1-i_1)!(-1-I)_{i'_2}} \tag{5.6.28}$$
$$\times \sum_{s,l,t=0} \frac{(-1)^s s!(-i_2, 2+I-i'_2)_t}{t!(s-i_1, s-i'_1)!(-I)_s(\lambda + \frac{I-J}{3} - i'_2 + 1)_t} {}_3\overline{F}_2 \left(\begin{matrix} i_1 - s; & t - i'_2 & 1 \\ & 1 + i'_1 - s & -\lambda + \frac{J-I}{3} \end{matrix} \middle| 1 \right).$$

This series can be summed to a product using the identity (A.2.12) but we must do this carefully. First let us note that we can assume $i'_1 \geq i_1$ and $i'_2 \geq i_2$ because the expression is trivially zero otherwise because of the denominator factorials outside the sum. Secondly, it is easy to see that the sum in $t \leq i_2$ and $s \geq i'_1$, therefore we can assume that $i_1 - s \leq 0$, $t - i'_2 \leq 0$ and $s - i_1 \geq s - i'_1$. Therefore we can safely apply (A.2.12) with $n_1 = s - i_1$, $n_2 = s - i'_1$ and $a = t - i'_2$ to obtain

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i_2+i'_1} i'_2!(-\lambda + \frac{J-I}{3})_{i'_1+i'_2-i_1}(\lambda - \frac{2I+J}{3})_{i_1}}{i_1!i_2!(i'_2-i_2)!(i'_1-i_1)!(i'_1+i'_2)!(-1-I)_{i'_2}(-I)_{i'_1+i'_2}} \tag{5.6.29}$$
$$\times \sum_{s,t=0} \frac{(-i_2, -i'_1-i'_2, 2+I-i'_2, I+1-i'_1-i'_2)_t}{t!(\lambda + \frac{I-J}{3} + 1 + i_1 - i'_1 - i'_2)_t} {}_3\overline{F}_2 \left(\begin{matrix} t - i'_1 - i'_2; & 1 & 1 \\ & 1 - i'_1 & -I \end{matrix} \middle| 1 \right),$$

where we have written the resulting double summation by bringing forward the series in 's' and identifying it as a ${}_3F_2$ hypergeometric series. It is easy to see that $t \leq i_2$ and hence $i_1 + i'_2 - t \geq i'_1 \geq 0$. Therefore it can be summed with identity (A.2.12) and doing so we obtain

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i_2} i'_1!i'_2!(-\lambda + \frac{J-I}{3})_{i'_1+i'_2-i_1}(\lambda - \frac{2I+J}{3})_{i_1}}{i_1!i_2!(i'_2-i_2)!(i'_1-i_1)!(-I)_{i'_1+i'_2}} \tag{5.6.30}$$
$$\times {}_2F_1 \left(\begin{matrix} -i_2 & I + 1 - i'_1 - i'_2 \\ & \lambda + \frac{I-J}{3} + 1 + i_1 - i'_1 - i'_2 \end{matrix} \middle| 1 \right).$$

The ${}_2F_1$ is easily transformed using Gauss' sum (A.2.2). Finally we find matrix elements are given by the product

$$\left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{i'_1!i'_2!(\lambda - \frac{2I+J}{3})_{i_1+i_2}(-\lambda + \frac{J-I}{3})_{i'_1+i'_2-i_1-i_2}}{i_1!i_2!(i'_2-i_2)!(i'_1-i_1)!(-I)_{i'_1+i'_2}}. \tag{5.6.31}$$

Immediately we see that this function is essentially the same as $\mathbb{R}^{(1)}(\lambda)$ in the reduction (5.6.17) but with $\boldsymbol{i}^{(')}$ and $\boldsymbol{j}^{(')}$ insides swapped. Specifically, when $i_3^{(')}, j_3^{(')} = 0$

$$\left[\mathbb{R}^{(1)}(\lambda + \frac{I-J}{3}|J,0;I,0)\right]_{j,i}^{j',i'} = \left[\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\lambda|I,J)\right]_{i,j}^{i',j'}. \tag{5.6.32}$$

Now we can make the connection with our factorisation in the rational limit (3.7.18). Since we already identified $\mathbb{R}^{(1)}$ with $\tilde{\boldsymbol{N}}^r$ in (5.6.18) we compare $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ to $\tilde{\boldsymbol{M}}^r$ and see that

$$\mathbb{R}^{(2)}\mathbb{R}^{(3)}(\mu|\mu + \frac{I+J}{2}, \mu + 2J - I) = \mathbb{P}\tilde{\boldsymbol{M}}^r, \tag{5.6.33}$$

so they are the same up to some change of variables for $\boldsymbol{I}, \boldsymbol{J}$. We remind the reader that though we have written the function with a dependence on three variables, there are only two independent ones. That is the substitution made in (5.6.33) is not unique.

Now we can evaluate (5.6.25) using (5.6.17) and (5.6.31). We first need to make the required substitutions in $\mathbb{R}^{(1)}(\lambda|I,J)$ for the variables λ, I, J. It is easy to see from (5.6.25) that the expression will have four summations in variables $s_{11}, s_{12}, l_{11}, l_{12}$ but also four Kronecker delta functions, which allow us to remove two of these summations. We eliminate variables l_{11}, l_{12} by making the subsitution $l_{11} := i'_1 + j'_1 - s_{11}$, $l_{12} := i'_2 + j'_2 - s_{12}$ to get a double sum formula for the elements of the sl_3 R-matrix acting on the tensor product of Verma modules $V_I \otimes V_J$. As expected, this double sum formula is essentially the same as (3.7.4) in the case $n = 3$. In fact, they are identical up to a symmetry and change of variables, the explicit relationship is

$$\left[R_{J,I}^{(3),r}(\lambda)\right]_{j,i}^{j',i'} = \left(\mathbb{P}\mathbb{R}^{(1)}\mathbb{R}^{(2)}\mathbb{R}^{(3)}\right)(\lambda + \frac{I-J}{6}), \tag{5.6.34}$$

where the argument on the right hand side is the overall dependence of the expression on the spectral parameter, not the individual factors. The left hand side has the indices and weight parameters swapped but we established in symmetry (3.7.21a) that this does not affect the function. Therefore they are equal. This swapping of indices is the same as the sl_2 case where to get an identical expression (5.5.16) we had to reverse the ordering of the factors. This is the same for the sl_3 case where we can just reverse the ordering of $\mathbb{R}^{(1)}$ and $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$, where we find

$$\left[R_{I,J}^{(3),r}(\lambda)\right]_{i,j}^{i',j'} = \left(\mathbb{P}\mathbb{R}^{(2)}\mathbb{R}^{(3)}\mathbb{R}^{(1)}\right)(\lambda + \frac{I-J}{6}). \tag{5.6.35}$$

Some care needs to be taken in computing this; the operators $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$ and $\mathbb{R}^{(1)}$ take different arguments for weight parameters I, J in (5.6.35) compared to (5.6.34) and (5.6.25) because of their new relative positions in the product and their nature as intertwiners of the representation.

5.6.5 Discussion

For the sl_3 case we have established the relation between our factorisation (3.7.18) and that of this chapter and shown that they are the same R-matrix. It is interesting to compare with what happens in the sl_2 case, whose $\mathbb{R}^{(1)}$ factor is clearly contained within the same factor for sl_3 but the second factor is actually contained in $\mathbb{R}^{(2)}\mathbb{R}^{(3)}$. It is interesting that although this product preserves the subspace $\mathbb{C}[z_1, z_2]$ there is a contribution (the variable t_3) to its overall form coming from outside the space.

We expect that our sl_n R-matrix (3.7.4) can be constructed by considering the factorisation (5.1.8) but this could be quite difficult because we have to construct each factor in its entire form for all highest weight representations and then restrict afterwards. From our experience of considering sl_4, the complexity of these objects grow incredibly fast with n and it appears hopeless unless a pattern for its growth can be found. For example, the complexity of the formula (3.3.18) using the 3D approach grows in a simple pattern with n and the Lagrange interpolation formula (3.3.20) can be used to sum them up all at the same time. However, even with each factor in its simplest form the restriction to symmetric tensor representation is quite a large computation. In deriving the reductions of the last section and then transforming (5.6.26) to (5.6.31) we see it is a non-trivial task using many powerful identities. We are confident then, that for this class of representations the 3D model approach is the most efficient at constructing the R-matrix.

Our primary interest for considering the restriction was to see how the n factors of (5.1.8) go to only two (essentially equal) factors in (3.7.18). We have shown how this occurs for $n = 2, 3$ and conjecture for general n that

$$\begin{aligned} \tilde{\boldsymbol{N}}^r &\sim \mathbb{R}^{(1)}, \\ \tilde{\boldsymbol{M}}^r &\sim \mathbb{P}\mathbb{R}^{(2)}\mathbb{R}^{(3)} \ldots \mathbb{R}^{(n)}. \end{aligned}$$

This identification is based on the way each factor intertwines the representations, that is

$$\mathbb{R}^{(k)}(\lambda)(\pi^{\boldsymbol{I}} \otimes \pi^{\boldsymbol{J}}) = (\pi^{\boldsymbol{I}+\boldsymbol{\lambda}_k} \otimes \pi^{\boldsymbol{J}-\boldsymbol{\lambda}_k})\mathbb{R}^{(k)}(\lambda),$$
$$\boldsymbol{I} \pm \boldsymbol{\lambda}_k = (I_1, \ldots, I_{k-1} \mp \lambda, I_k \pm \lambda, \ldots, I_{n-1})$$

and hence for sl_n

$$\begin{aligned}\mathbb{R}^{(1)}(\lambda)(\pi^{(I_1,0,\ldots,0)} \otimes \pi^{(J_1,0,\ldots,0)}) &= (\pi^{(I_1+\lambda,0,\ldots,0)} \otimes \pi^{(J_1-\lambda,0,\ldots,0)})\mathbb{R}^{(1)}(\lambda),\\ \mathbb{R}^{(2)}\mathbb{R}^{(3)}\ldots\mathbb{R}^{(n)}(\lambda)(\pi^{(I_1,0,\ldots,0)} \otimes \pi^{(J_1,0,\ldots,0)}) &= \\ &(\pi^{(I_1-\lambda,0,\ldots,0)} \otimes \pi^{(J_1+\lambda,0,\ldots,0)})\mathbb{R}^{(2)}\mathbb{R}^{(3)}\ldots\mathbb{R}^{(n)}(\lambda).\end{aligned}$$

So these operators preserve the (non-)zero nature of each weight component and should preserve the symmetric tensor representation subspace $\mathbb{C}[z_{21}, \ldots, z_{n1}]$.

Chapter 6

L-operator factorisation

In this chapter we will study a factorisation of the sl_2 L-operator and generalise it to $U_q(\widehat{sl_2})$. The construction of this factorisation gives another way of constructing a formula for the elements of the rational sl_2 R-matrix, which is different again to the formulae obtained by the methods in other chapters. The formula is a finite double sum expression but can to summed up to the ${}_4F_3$ (3.8.5) first obtained using the 3-dimensional projection approach. Using the quantum deformation (3.8.3) we can reverse the arguments to obtain the analogous double sum for the $U_q(\widehat{sl_2})$ R-matrix. Upon inspection of this formula, we see it can be split into factors of the same form as the rational case. The factorisation holds for the R-matrix acting on $\mathbb{C}^{I+1} \otimes V_J$ and therefore is a factorisation of the higher spin $U_q(\widehat{sl_2})$ L-operator.

The factorisation in the rational case is related to the construction in chapter 5 but is different. The factorisation in that chapter is for the R-matrix acting on Verma modules and although the action can be restricted to a finite dimensional subspace when the weights are integral, the individual factors cannot as they map out of the subspace - it is a 'rectangular' factorisation. In contrast, the factorisation we study in this chapter holds for finite dimensional representations and the factors can be written down as finite square matrices. In the case of the L-operator we get a square factorisation with operator entries.

The overview the method, developed in [11; 12], we begin by writing down the sl_2 R-matrix in the factorised differential operator form (5.1.7) and (5.1.8) first introduced in [13]:

$$\mathbb{R}_{IJ}(\mu) = \mathbb{P}_{12} \frac{\Gamma(z_{21}\partial_2 - I)}{\Gamma(z_{21} - \mu - \frac{I+J}{2})} \frac{\Gamma(z_{12}\partial_1 + \mu - \frac{I+J}{2})}{\Gamma(z_{12}\partial_1 - I)}. \tag{6.0.1}$$

The R-matrix in this form acts on a tensor product of Verma modules realised as the space of polynomials $\mathbb{C}[z_1] \otimes \mathbb{C}[z_2]$. In the presentation here we have combined the two factors and removed the lightcone coordinates in chapter 5. I, J are the usual weight parameters and are considered to be complex in (6.0.1). The content of this chapter arises out of the interest in restricting this operator to finite dimensional subspaces, which is possible when $I, J \in \mathbb{Z}^+$ - thereby finding finite solutions to the Yang-Baxter equation. One way to evaluate the operator on a test function $\Phi(z_1, z_2)$ is to rewrite it as an integral operator by means of the Euler beta integral

$$\frac{\Gamma(z_{12}\partial_1 + a)}{\Gamma(z_{12}\partial_1 + b)}\Phi(z_1, z_2) = \frac{1}{\Gamma(b-a)}\int_0^1 d\alpha \alpha^{a-1}(1-\alpha)^{b-a-1}\Phi(\alpha z_1 + (1-\alpha)z_2, z_2). \quad (6.0.2)$$

If we consider $I \in \mathbb{Z}^+$, $J \in \mathbb{C}$ then generally speaking only one of the tensor factors has a finite subspace, and the restricted R-operator should be presentable as a product of finite matrices acting on this space with operator entries acting on the second space. Indeed, in [11] using (6.0.2) to evaluate the action of (6.0.1) on the test function $(z_1 - x)^I \Phi(z)$ the operator action was found to have the form

$$\begin{gathered} \mathbb{R}_{IJ}(\mu) \cdot (z_1 - x)^I \Phi(z) = \\ (z-x)^{-\mu+I+J/2}(z_1 - z)^{\mu+1+I/2+J/2}\partial_z^I (z_1 - z)^{-\mu-1+I/2-J/2}(z-x)^{\mu+I/2-J/2}\Phi(z). \end{gathered} \quad (6.0.3)$$

The term $(z_1 - x)^I$ is a generating function for the monomial basis $\{1, z_1, \ldots, z^I\}$ of the $(I+1)$-dimensional subspace with x an auxiliary parameter. Also note that for the second space we have changed the labelling of the variable $z_2 := z$. Expanding both sides in x and equating coefficients we can extract matrix elements of $\mathbb{R}_{IJ}(\mu)$. For example, when $I = 1$ the action (6.0.3) as found in [12] is given by

$$\begin{aligned} \mathbb{R}_{1J}(\mu - 1/2) \cdot 1 &= z_1\partial + (u + \frac{J}{2} - z\partial), \\ \mathbb{R}_{1J}(\mu - 1/2) \cdot z_1 &= z_1(z\partial + \mu - \frac{J}{2}) + 1 \cdot (-z^2\partial + Jz) \end{aligned} \quad (6.0.4)$$

which is the same as the action in (3.7.14a). However, the matrix form is presented differently; in (3.7.14a) the basis is ordered by $e_{i+1} = z_1^i$, $i = 0, \ldots, I$ whereas [12] use the reverse order $e_{i+1} = z_1^{I-i}$. It is convenient to adopt their ordering for this section, with which (6.0.4) was observed to factorise as

$$\begin{aligned} \mathbb{R}_{1J}(\mu - 1/2) &= \begin{pmatrix} z\partial_z + \mu - \frac{J}{2} & \partial_z \\ -z^2\partial_z + Jz & \mu + \frac{J}{2} - z\partial_z \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & \partial \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \end{aligned} \quad (6.0.5)$$

where $u_1 = \mu - J/2 - 1$ and $u_2 = \mu + J/2$. It turns out that a factorisation of this form exists for all $I \in \mathbb{Z}^+, J \in \mathbb{C}$. That is,

$$\mathbb{R}_{IJ}(\mu - I/2) = Z^{-1}U^+(u_2)DU^-(u_1)Z. \tag{6.0.6}$$

The factors $Z, U^{\pm}(u)$ and D are $(I+1)$-dimensional square matrices in general. It can be shown they have the operator form

$$Z = \exp\left(z\partial_{z_1}\right), \quad D = \check{C}\exp(\partial_z\partial_{z_1})\check{C}, \quad U^-(u) = \frac{\Gamma(z_1\partial_{z_1} + u + 1 - I)}{\Gamma(u+1-I)},$$
$$U^+(u) = \check{C}U^-(u)\check{C}, \quad \check{C}\cdot z_1^i = z_1^{I-i} \tag{6.0.7}$$

acting on $\mathbb{C}[z_1] \otimes \mathbb{C}[z]$. By considering this operator action on $(z_1 - x)^I\Phi(z)$ Chicherin-Derkachov [12] derive the following formula:

$$\left[\mathbb{R}_{IJ}(\mu - \frac{I}{2})\right] \cdot \left[(z_1 - x)^I\Phi(z)\right] = \sum_{k=0}^{I} \frac{I!}{k!(I-k)!}\frac{\Gamma(u_1 + 1 - I + k)}{\Gamma(u_1 + 1 - I)}$$
$$\times \sum_{p=0}^{I-k} \frac{(I-k)!}{p!(I-k-p)!}\frac{\Gamma(u_2 + 1 - k - p)}{\Gamma(u_2 + 1 - I)}(z_1 - z)^{k+p}\partial_z^p \left[(z - x)^{I-k}\Phi(z)\right]. \tag{6.0.8}$$

It was proven in [12] that this action is the same as (6.0.3) thereby proving the factorisation (6.0.6-6.0.7). It is from (6.0.8) we begin our construction of a formula for the matrix elements of $\mathbb{R}_{IJ}(\mu)$ and show that it is the same formula as (3.8.5) and hence showing this method is equivalent to all of the others.

6.1 Factorization formula for the rational R-matrix

Let us substitute into (6.0.8) $\Phi(z) = z^{j'}$ and expand in powers of x. As an intermediate step we have

$$\sum_{i'=0}^{I} \frac{I!(-x)^{I-i'}}{(I-i')!i'!}\left[\mathbb{R}_{IJ}(u - I/2)\cdot z_1^{i'}z^{j'}\right] = \sum_{k=0}^{I}\frac{(-I)_k(-1)^k}{(1)_k}(u_1 - I + 1)_k$$
$$\times \sum_{p=0}^{I-k}\frac{(-1)^p(k-I)_p}{(1)_p}\frac{\Gamma(u_2 + 1 - k - p)}{\Gamma(u_2 + 1 - I)}\sum_{i=0}^{k+p}\frac{(-k-p)_i(-1)^{k+p}}{(1)_i}z_1^i z^{k+p-i}$$
$$\times \sum_{s=0}^{I-k}\frac{(I-k)!(I-k-s+j')!(-1)^s}{(I-k-s)!s!(I-k-s+j'-p)!}z^{I-k-s+j'-p}x^s, \tag{6.1.1}$$

where all we have done is expand out binomial terms with the binomial theorem and apply the differential operator action on the basis. In equating powers of x we set $s = I - i'$. We also fix i' on the left hand side and collect powers of z, z_1. After some easy simplifications of Gamma functions and pochammer symbols using (A.1.1) we get the following result for the action of the operator $\mathbb{R}_{IJ}(\mu - \frac{I}{2})$:

$$\mathbb{R}_{IJ}(\mu - \frac{I}{2}) \cdot z_1^{i'} z^{j'} = \sum_{i=0}^{I} z_1^i z^{i'+j'-i} \left[\mathbb{R}_{IJ}(\mu - \frac{I}{2})\right]_{i,i'+j'-i}^{i',j'}, \tag{6.1.2}$$

where

$$\left[\mathbb{R}_{IJ}(\mu - \frac{I}{2})\right]_{i,i'+j'-i}^{i',j'} = \frac{i'!}{I!i!} \sum_{k=0}^{I} \sum_{p=0}^{I-k} \frac{(-I, \mu - I - J/2)_k (I-k)!(k-I)_p}{k!p!} \times \frac{(i'-k+1)_{j'}(\mu - I + J/2 + 1)_{I-k-p}(-k-p)_i}{(i'+j'-k-p)!}. \tag{6.1.3}$$

The summation of i seems to imply that index $i' + j' - i$ can be negative. Indeed it can, however, the function is always zero in this case. This is a consequence of the numerator pochammer $(-k-p)_i$ eliminating summands where $k + p < i$ and the denominator term $(i' + j' - k - p)!$ eliminating summands where $i' + j' < k + p$. To compare (6.1.3) with (3.8.5) we first reverse the spectral parameter shift and divide by the factor

$$C(\mu; I, J) = (\mu + \frac{J-I}{2} + 1)_I, \tag{6.1.4}$$

in effect defining the function $\mathbb{R}^r_{IJ}(\mu)$ for the matrix elements by

$$[\mathbb{R}_{IJ}(\mu)]_{i,j}^{i',j'} = C(\mu; I, J) \left[\mathbb{R}^r_{IJ}(\mu)\right]_{i,j}^{i',j'}. \tag{6.1.5}$$

Making a few further simplifications we finally get the formula

$$\left[\mathbb{R}^r_{IJ}(\mu)\right]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \sum_{k=0}^{i'} \sum_{p=0}^{I-k} (-1)^k \frac{(-i', \mu - \frac{I+J}{2})_k (k-I, -i-j+k)_p (-k-p)_i}{i!\,k!\,p!(-\mu - \frac{I+J}{2})_{k+p}}. \tag{6.1.6}$$

Let us notice that the operator $\mathbb{R}^r_{IJ}(\mu)$ is normalised similar to (3.3.29), i.e.

$$\left[\mathbb{R}^r_{IJ}(\mu)\right]_{0,0}^{0,0} = 1. \tag{6.1.7}$$

It is easy to see that the operator $\mathbb{R}^r_{12}(\mu)$ can be now defined for any $I \in \mathbb{C}$. Initially the action of the operator $\mathbb{R}_{IJ}(\mu)$ was defined only for $I \in \mathbb{Z}^+$. However, after extracting the factor $C(\mu; I, J)$ the resulting expression can be transformed to the form (6.1.6) where the restriction $k \leq I$ can be lifted and replaced by $k \leq i'$. One can expect that the renormalized operator $\mathbb{R}^r_{IJ}$ is well defined for all $I, J \in \mathbb{C}$.

This is indeed the case. In the rest of this section we will show that the action of the operator $\mathbb{R}^r_{IJ}(\mu)$ exactly coincides with the action (3.8.5). Therefore, $\mathbb{R}^r_{IJ}(\mu)$ and the operator $R^{(2),r}_{I,J}(\mu)$ from chapter 3 can be identified.

A disadvantage of the action (6.1.6) is that it is a double sum. Besides being less efficient to evaluate, it is very hard to prove symmetries similar to (3.7.21) from this representation. Therefore, we try to convert it to a single sum.

The first step is to notice that the sum over p in (6.1.6) can be represented in terms of a regularised ${}_3\widetilde{F}_2$ series defined in (A.1.5), in particular

$$\begin{aligned}\left[\mathbb{R}^r_{IJ}(\mu)\right]^{i',j'}_{i,j} &= \delta_{i+j,i'+j'} \sum_{k=0}^{i'} (-1)^{i+k} \frac{(-i', \mu - \frac{I+J}{2})_k \Gamma(-\mu - \frac{I+J}{2})}{i!} \\ &\times {}_3\tilde{F}_2 \left(\begin{matrix} 1+k & k-I & k-i-j \\ 1-i+k & k-\mu-\frac{I+J}{2} & \end{matrix} \middle| 1 \right). \end{aligned} \tag{6.1.8}$$

Now we can apply to (6.1.8) the Thomae' theorem (A.2.4). When possible we always write the arguments of hypergeometric functions in the same order as in the matching identity.

It is easy to see that after application of (A.2.4) only one argument of the hypergeometric function ${}_3\widetilde{F}_2$ contains the index k. Expanding it back into a sum over p one can rewrite the sum over k as a ${}_2F_1$ series

$$\begin{aligned}\left[\mathbb{R}^r_{IJ}(\mu)\right]^{i',j'}_{i,j} &= \delta_{i+j,i'+j'} \frac{(-\mu + \frac{I-J}{2})_j (1 + \mu + \frac{J-I}{2})_i}{i!\,(-\mu - \frac{I+J}{2})_j} \\ \times \sum_{p=0}^{i} &\frac{(-i, -\mu - \frac{I+J}{2} - 1, -\mu + j + \frac{I-J}{2})_p}{p!\,(-\mu + j - \frac{I+J}{2}, -\mu - i + \frac{I-J}{2})_p} \, {}_2F_1 \left(\begin{matrix} -i' & \mu - \frac{I+J}{2} \\ \mu + \frac{J-I}{2} - j - p + 1 & \end{matrix} \middle| 1 \right). \end{aligned} \tag{6.1.9}$$

Now using a Gauss summation formula (A.2.2) we can sum up the ${}_2F_1$ series and rewrite (6.1.9) as a single sum which can be easily transformed into a regularised ${}_4\overline{F}_3$

series. We obtain

$$\left[\mathbb{R}^r_{IJ}(\mu)\right]^{i',j'}_{i,j} = \delta_{i+j,i'+j'} \frac{(-1)^i}{i!} \frac{(-J)_j(-\mu+\frac{I-J}{2})_{j-i'}}{(-J)_{j'}(-\mu-\frac{I+J}{2})_{i+j}} \times {}_4\overline{F}_3\left(\begin{matrix} -i; & j-J & -\mu-\frac{I+J}{2}-1 & -\mu+\frac{I-J}{2}+j-i' \\ & -\mu-\frac{I+J}{2}+j & -\mu+\frac{I-J}{2}-i & j-i'-J \end{matrix}\middle| 1\right). \tag{6.1.10}$$

As the last step we apply the identity (A.2.13) to (6.1.10) and obtain exactly the expression on the right-hand side of (3.8.5). Let us note that all transformations between (3.8.5) and (6.1.6) are valid for any complex I, J. Therefore it justifies our previous statement that the renormalized operator $\mathbb{R}^r_{12}$ is defined by its action (6.1.6) for $I, J \in \mathbb{C}$.

6.2 The trigonometric R-matrix

In this section we construct a trigonometric factorization of the general R-matrix acting in $V_I \otimes V_J^+$, $I \in \mathbb{Z}^+$, $J \in \mathbb{C}$ similar to [12]. However, there is a noticeable difference between our result and their approach. In [12] the authors considered a more general case of the so called modular double. As well known in the standard q-deformed case the R-matrix can not be calculated uniquely from the L-operator intertwining relations due to the presence of a large center in the $U_q(sl_2)$ algebra on the space of continuous functions. This problem can be naturally solved using the modular double of the quantum group [68].

In principle one can derive a factorization of the standard R-matrix restricting a construction of [12] to a "half" of the representation of the modular double. However, we prefer to reverse the arguments of the previous section and first obtain the q-analog of the formula (6.0.8) starting from the hypergeometric representation of the XXZ R-matrix (3.8.3). Then we show how this q-analog allows a natural factorization.

First we start from (3.8.3) with $I, J \in \mathbb{C}$ and apply the identity (B.2.14) with $n = i$ and

$$a = \lambda^2 q^{2-I+J}, \quad b = q^{-2i'}, \quad c = \lambda^{-2} q^{J-I},$$
$$d = q^{2(1-i-j+J)}, \quad e = q^{-2I}, \quad f = q^{2(1+j-i')}$$

to get

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^i\lambda^{i-i'}\frac{q^{i(i+J-j+1)+(i'-I)j'}(q^{-2J};q^2)_j(\lambda^{-2}q^{I-J};q^2)_{j-i'}}{(q^{-2J};q^2)_{j'}(q^2;q^2)_i(\lambda^{-2}q^{-I-J};q^2)_{i+j}}$$
$$\times\, {}_4\overline{\phi}_3\left(\begin{matrix} q^{-2i}; & q^{2j-2J} & \lambda^{-2}q^{-I-J-2} & \lambda^{-2}q^{I-J+2j-2i'} \\ & q^{2j-2i'-2J} & \lambda^{-2}q^{-I-J+2j} & \lambda^{-2}q^{I-J-2i}\end{matrix}\middle| q^2,q^2\right). \qquad (6.2.1)$$

Now we extend ${}_3\overline{\phi}_2$ in (6.2.1) in a sum over p and notice that

$$\frac{(q^{2j-2J},\lambda^{-2}q^{I-J+2j-2i'};q^2)_p}{(q^{2j-2J-2i'},\lambda^{-2}q^{I-J+2j};q^2)_p} = \frac{(\lambda^2q^{J-I+2-2j};q^2)_{i'}}{(q^{2-2j+2J};q^2)_{i'}}$$
$$\times\, {}_2\phi_1\left(\begin{matrix} q^{-2i'} & \lambda^2q^{-I-J} \\ & \lambda^2q^{J-I+2-2j-2p}\end{matrix}\middle| q^2,q^{2(J+i'+1-p-j)}\right) \qquad (6.2.2)$$

as a consequence of the q-Chu-Vandermonde sum (B.2.2) with $n=i'$. Let us substitute (6.2.2) into the expanded version of (6.2.1) and expand ${}_2\phi_1$ in the sum over k. Interchanging summations we can represent the sum over p as a sum of ${}_3\phi_2$ basic hypergeometric series. After some straightforward calculations we obtain

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^i\lambda^{i+i'}q^{i(i-j+J-I+1)}\frac{(\lambda^{-2}q^{I-J-2i};q^2)_i(\lambda^{-2}q^{-I-J+2j};q^2)_I}{q^{Ij+i'(J-j')}(q^2;q^2)_i(\lambda^{-2}q^{-I-J};q^2)_I}$$
$$\times\sum_{k=0}^{i'}\frac{q^{2k(J+1)}(q^{-2i'},\lambda^2q^{-I-J};q^2)_k}{q^{2k(j-i')}(q^2,\lambda^2q^{J-I+2-2j};q^2)_k}\,{}_3\phi_2\left(\begin{matrix} q^{-2i} & \lambda^{-2}q^{-2-I-J} & \lambda^{-2}q^{I-J+2j-2k} \\ & \lambda^{-2}q^{I-J-2i} & \lambda^{-2}q^{-J-I+2j}\end{matrix}\middle| q^2,q^2\right). \qquad (6.2.3)$$

Until now we did not make any assumptions on the values of the weights $I,J\in\mathbb{C}$. However, to perform the next step we need to assume that I is a positive integer. The reason for this is that in the rational case we have a non-terminating Thomae's theorem (A.2.4). However, in the q-deformed case we were able to construct only its terminating version (B.2.20) with the argument $z=q^2$ on both sides. Since we are aiming to construct a factorization of the R-matrix which acts in a finite-dimensional first space, it does not cause any problems.

So let us now choose $I\in\mathbb{Z}^+$ and set

$$n=I-k,\quad m=i,\quad b=q^{2+2k},\quad c=q^{-2i-2j+2k},\quad e=\lambda^{-2}q^{-I-J+2k} \qquad (6.2.4)$$

in the regularized identity (B.2.20). After straightforward transformation we get the following answer:

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^i \lambda^{i'-i} \frac{q^{i(i-j-1)+Ij+i'(j'-J)}}{(q^2;q^2)_i}$$
$$\times \sum_{k,p} \frac{(-1)^k}{\lambda^{2k}} \frac{(q^{-2i'}, \lambda^2 q^{-I-J}; q^2)_k (q^{2k-2i'-2j'}; q^2)_p (q^2, q^{-2I}; q^2)_{k+p}}{(q^2, q^{-2I}; q^2)_k (q^2;q^2)_p (q^2;q^2)_{k+p-i} (\lambda^{-2} q^{-I-J}; q^2)_{k+p}} q^{k^2+k(J-I-2j'+1)+2p}. \tag{6.2.5}$$

The summation variables k and p in (6.2.5) are restricted as

$$0 \leq k \leq i', \quad 0 \leq p \leq (i'-k)+j', \quad i \leq k+p. \tag{6.2.6}$$

The formula (6.2.5) gives a q-deformation of (6.1.6) as can be easily seen by taking the limit $q \to 1$.

6.3 A trigonometric factorization

A factorised form of the R-matrix acting on the tensor product of an arbitrary finite dimensional and arbitrary infinite dimensional representations of the modular double was derived in (44) of [12]. More explicitly, the authors of [12] considered only special restricted representations corresponding to a "half" of the modular double. It is natural to expect that the R-matrix (44) from [12] should be related to our higher-spin R-matrix (6.2.5). In this section we shall derive this type of factorization directly from (6.2.5). As we shall see there is a slight difference between our results and results of [12].

First, let us remind that the operator $\mathbb{R}_{I,J}(\lambda)$ is realized on the space of polynomials in variables z_1 and z by its action on the basis

$$\mathbb{R}_{I,J}(\lambda) \cdot z_1^{i'} z^{j'} = \sum_{i,j} [R_{I,J}(\lambda)]_{i,j}^{i',j'} z_1^i z^j. \tag{6.3.1}$$

When I is a positive integer we can represent $\mathbb{R}_{I,J}(\lambda)$ by the $(I+1) \times (I+1)$-dimensional matrix with operator entries acting in the space $\mathbb{C}[z]$. It is obvious that this matrix coincides with a particular realization of the $(I+1)$-dimensional L-operator acting in the Verma module V_J.

To write down the explicit formulas for the L-operator we make two simple similarity transformations

$$R_{I,J}(\lambda) = (U \otimes U)\overline{R}_{I,J}(\lambda)(U^{-1} \otimes U^{-1}), \tag{6.3.2}$$

$$U_n^m = \delta_{n,m} q^{n^2/2}. \tag{6.3.3}$$

The purpose of this transformation is to remove some q-factors which mix indices in the auxiliary and quantum spaces. It is easy to see that (6.3.2) does not affect the Yang-Baxter equation.

Now let us define the L-operator $\overline{L}_I(\lambda)$ acting in V_J by its matrix elements in V_I:

$$[\overline{L}_I(\lambda)]_i^{i'} \cdot z^{j'} = \sum_j [\bar{R}_{I,J}(\lambda)]_{i,j}^{i',j'} z^j \tag{6.3.4}$$

and its slightly transformed version

$$[L_I(\lambda)]_i^{i'} = q^{(i'-i)J/2}[\overline{L}_I(\lambda)]_i^{i'}. \tag{6.3.5}$$

The second equivalence transformation (6.3.5) is needed to eliminate a dependence on J in the L-operator in favor of two new variables

$$\lambda_1 = \lambda q^{1+\frac{J}{2}}, \quad \lambda_2 = \lambda q^{-\frac{J}{2}} \tag{6.3.6}$$

analogous to the variables u_1, u_2 used earlier in this chapter for the rational case. It immediately follows from the Yang-Baxter equation that the L-operator (6.3.5) satisfies the following algebra

$$\overline{R}_{I,J}(\lambda_1/\lambda_2)L_I(\lambda_1)L_J(\lambda_2) = L_J(\lambda_2)L_I(\lambda_1)\overline{R}_{I,J}(\lambda_1/\lambda_2) \tag{6.3.7}$$

for any $I, J \in \mathbb{Z}_+$. To find $L_I(\lambda)$ we need to describe the explicit action of its entries on the basis $z^{j'}$. After the transformations (6.3.2), (6.3.5) and substitution $j = i' + j' - i$ a dependence on j' in (6.2.5) becomes quite simple. The only nontrivial factor containing j' is $(q^{-2i'-2j'+2k}; q^2)_p$ which can be expanded as

$$(q^{-2i'-2j'+2k}; q^2)_p = \sum_{r=0}^{p} \frac{(q^{-2p}; q^2)_r}{(q^2; q^2)_r} q^{2r(k+p-i'-j')}. \tag{6.3.8}$$

Substituting (6.3.8) into (6.2.5) we can bring it to the form

$$[\bar{R}_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \rho_{I,J}(\lambda) \sum_s [N_1(\lambda_1)]_i^s [N_2(\lambda_2)]_s^{i'} q^{(I-2s)(i'+j')+(i-i')J/2}, \tag{6.3.9}$$

where $N_{1,2}(\lambda_{1,2})$ are two numerical $(I+1)\times(I+1)$-dimensional matrices, $\rho_{I,J}(\lambda)$ is a constant multiplier and $\lambda_{1,2}$ are defined in (6.3.6).

The explicit form of matrices $N_{1,2}(\lambda)$ can be derived from (6.2.5). To do that we need to change variables from k,p,r in (6.2.5), (6.3.9) to k,s,t with

$$s=k+r,\quad t=I-k-p. \tag{6.3.10}$$

Using (6.3.8) we obtain after straightforward calculations

$$\begin{aligned}
[N_1(\lambda_1)]_n^m &= (-1)^n\lambda_1^{I-n}q^{-I(I+n)+n^2}\frac{(q^2;q^2)_m(q^2;q^2)_I}{(q^2;q^2)_n(q^2;q^2)_{I-m}(q^2;q^2)_{I-n}} &(6.3.11)\\
&\times\sum_{t=0}^{I-n}(-1)^t\lambda_1^{-2t}q^{(3I-2m-2n-t+1)t}\frac{(q^{-2(I-m)};q^2)_t(q^{-2(I-n)};q^2)_t(\lambda_1^2q^{-I};q^2)_t}{(q^2;q^2)_t(q^{-2I};q^2)_t},\\
[N_2(\lambda_2)]_n^m &= (-1)^n\lambda_2^m q^{n(n-1)}\frac{(q^2;q^2)_I}{(q^2;q^2)_n^2} &(6.3.12)\\
&\times\sum_{k=0}^{n}(-1)^k\lambda_2^{-2k}q^{(2m+2n-I-k+1)k}\frac{(q^{-2n};q^2)_k(q^{-2m};q^2)_k(\lambda_2^2q^{-I};q^2)_k}{(q^2;q^2)_k(q^{-2I};q^2)_k},
\end{aligned}$$

where $0\le n,m\le I$ and we fixed the normalization factor in (6.3.9) as

$$\rho_{I,J}(\lambda)=\frac{(-1)^I q^{-IJ/2}}{\lambda^I(\lambda^{-2}q^{-I-J};q^2)_I}. \tag{6.3.13}$$

Formulas (6.3.11-6.3.12) can be significantly simplified. Both expressions (6.3.11-6.3.12) are ${}_3\phi_1$ hypergeometric series which can be transformed using the identity (B.2.15) in Appendix B. For $N_1(\lambda)$ we set

$$a=q^{-2I+2n},\quad b=q^{-2I+2m},\quad c=q^{-2I+2m+2n+2},\quad z=\lambda^{-2}q^{I+2}$$

and for $N_2(\lambda_2)$ we set

$$a=q^{-2n},\quad b=q^{-2m},\quad c=q^{2I-2m-2n+2},\quad z=\lambda_2^{-2}q^{I+2}.$$

Changing a summation variable in $M_1(\lambda)$ we can rewrite (6.3.11-6.3.12) as

$$[N_1(\lambda)]_n^m=V_m[M_1(\lambda)]_n^m,\quad [N_2(\lambda)]_n^m=V_n^{-1}[M_2(\lambda)]_n^m,\quad V_m=q^{-Im}\frac{(q^2;q^2)_m}{(q^2;q^2)_{I-m}}, \tag{6.3.14}$$

where

$$[M_2(\lambda)]_n^m = (-1)^n \lambda^m \frac{q^{n(n-1)+2nm-In}}{(q^2;q^2)_n} \, {}_2\overline{\Phi}_1 \left(\begin{matrix} q^{-2n}; & q^{-2m} \\ & q^{2(I-n-m+1)} \end{matrix} \middle| q^2, \frac{q^{I+2}}{\lambda^2} \right) \tag{6.3.15}$$

and

$$[M_1(\lambda)]_n^m = \lambda^{n+m-I} q^{n(1-I)} \, [M_2(\lambda)]_n^m. \tag{6.3.16}$$

Taking into account definitions (6.3.4), (6.3.9) and canceling the factors V_m from (6.3.14) we can obtain the matrix factorization of the L-operator as

Proposition 6.3.1.

$$L_I(\lambda) = Z^{-1} M_1(\lambda_1) D_I M_2(\lambda_2) Z, \tag{6.3.17}$$

where $\mathcal{Z}$ and $\mathcal{D}$ are the diagonal shift and multiplication operators acting in $\mathbb{C}[z]$

$$[D_I]_n^m = \delta_{n,m} \mathcal{D}^{I-2n}, \quad \mathcal{D} \cdot z^{j'} = q^{j'} z^{j'}, \tag{6.3.18}$$

$$Z_n^m = \delta_{n,m} \mathcal{Z}^n, \quad \mathcal{Z} \cdot z^{j'} = z^{j'+1} \tag{6.3.19}$$

In particular, for $I = 1$ we obtain

$$M_1^{(1)}(\lambda) = \begin{pmatrix} \lambda^{-1} & \lambda \\ -q^{-1} & -1 \end{pmatrix}, \quad M_2^{(1)}(\lambda) = \begin{pmatrix} 1 & \lambda \\ -q^{-1} & -\lambda^{-1} \end{pmatrix}. \tag{6.3.20}$$

Then for the L-operator (6.3.17) we obtain

$$L_1(\lambda) = \begin{pmatrix} \lambda^{-1} q^{-1-J/2} \mathcal{D} - \lambda q^{J/2} \mathcal{D}^{-1} & \mathcal{Z}(q^{-J}\mathcal{D} - q^J \mathcal{D}^{-1}) \\ q^{-1} \mathcal{Z}^{-1} (\mathcal{D}^{-1} - \mathcal{D}) & \lambda^{-1} q^{-1+J/2} \mathcal{D}^{-1} - \lambda q^{J/2} \mathcal{D} \end{pmatrix} \tag{6.3.21}$$

After a simple equivalence transformation $D = \mathrm{diag}(1, \lambda^{-1})$ in $\mathbb{C}^2$, a change of variable $\lambda \to \lambda q^{-1/2}$ and discarding a constant $-q^{1/2}$ we obtain the standard $U_q(\widehat{sl_2})$ L-operator

$$L(\lambda) = \begin{pmatrix} \lambda q^{H/2} - \lambda^{-1} q^{-H/2} & \lambda (q - q^{-1}) \, F \\ & \\ \lambda^{-1} (q - q^{-1}) E & \lambda q^{-H/2} - \lambda^{-1} q^{H/2} \end{pmatrix} \tag{6.3.22}$$

where E, F and H are the generators of the quantum algebra $U_q(sl_2)$ with the action similar to (2.4.59) for $n = 2$ but related by the well known automorphism [59] $\omega(E) = F$, $\omega(F) = E$, $\omega(H) = -H$.

Let us also notice that the matrices $M_{1,2}^{(1)}(\lambda)$ (6.3.20) are inverses of each other up to a simple constant. Therefore, we can expect a simple relation between such matrices for general I. This is indeed the case and the result reads

$$[M_2^{-1}(\lambda)]_{n,m} = (-1)^I \lambda^{n+m-2I} \frac{q^{m-2mn+I}}{(q^{2-I}/\lambda^2;q^2)_I} [M_2(\lambda/q)]_{n,m} . \tag{6.3.23}$$

This can be proved by taking the matrix product of (6.3.23) with $M_2(\lambda)$ and expanding two ${}_2\overline{\phi}_1$ series.

The representation of the L-operator (6.3.17) is well known for $I = 1$ case and corresponds to a factorization into the product of two simple L-operators [69]. To see that let us remind that the L-operator of a massless lattice sine-Gordon model can be written in the form

$$L_{SG} = \begin{pmatrix} aX & b\mathcal{D} \\ c\mathcal{D}^{-1} & dX^{-1} \end{pmatrix}, \tag{6.3.24}$$

where a, b, c, d are parameters and operators $\mathcal{D}$ and X satisfy commutation relations of the Weil algebra $\mathcal{W}$

$$X\mathcal{D} = q\mathcal{D}X. \tag{6.3.25}$$

Let us take two copies of the L-operators acting in $\mathcal{W}_1$ and $\mathcal{W}_2$ with a particular choice of parameters

$$L^{(1)}_{SG}(\lambda_1) = \begin{pmatrix} \lambda_1^{-1}X_1\mathcal{D}_1^{-1} & \lambda_1\mathcal{D}_1 \\ -q^{-1}\mathcal{D}_1^{-1} & -X_1^{-1}\mathcal{D}_1 \end{pmatrix}, \tag{6.3.26}$$

$$L^{(2)}_{SG}(\lambda_2) = \begin{pmatrix} X_2\mathcal{D}_2 & q\lambda_2\mathcal{D}_2 \\ -q^{-1}\mathcal{D}_2^{-1} & -q^{-1}\lambda_2^{-1}X_2^{-1}\mathcal{D}_2^{-1} \end{pmatrix}. \tag{6.3.27}$$

Note that we made two transformations in (6.3.24) $X_1 \to X_1\mathcal{D}_1^{-1}$ and $X_2 \to X_2\mathcal{D}_2$ which obviously do not affect commutation relation (6.3.25).

Comparing this to (6.3.20) we find that

$$L^{(1)}_{SG}(\lambda_1) = \begin{pmatrix} 1 & 0 \\ 0 & X_1^{-1} \end{pmatrix} M_1^{(1)}(\lambda_1) \begin{pmatrix} X_1\mathcal{D}_1^{-1} & 0 \\ 0 & \mathcal{D}_1 \end{pmatrix} \tag{6.3.28}$$

and

$$L^{(2)}_{SG}(\lambda_2) = \begin{pmatrix} X_2\mathcal{D}_2 & 0 \\ 0 & \mathcal{D}_2^{-1} \end{pmatrix} M_2^{(1)}(\lambda_2) \begin{pmatrix} 1 & 0 \\ 0 & X_2^{-1} \end{pmatrix}. \tag{6.3.29}$$

For the product of two L-operators (6.3.28-6.3.29) acting in $\mathcal{W}_1 \otimes \mathcal{W}_2$ we obtain

$$L(\lambda) = L_{SG}^{(1)}(\lambda_1) L_{SG}^{(2)}(\lambda_2) =$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & X_1^{-1} \end{pmatrix} M_1^{(1)}(\lambda_1) \begin{pmatrix} X_1 X_2 \mathcal{D}_1^{-1} \mathcal{D}_2 & 0 \\ 0 & \mathcal{D}_1 \mathcal{D}_2^{-1} \end{pmatrix} M_2^{(1)}(\lambda_1) \begin{pmatrix} 1 & 0 \\ 0 & X_2^{-1} \end{pmatrix}. \quad (6.3.30)$$

It is easy to check that the L-operator (6.3.30) commutes with the product $X = X_1 X_2$ and depends only on the combination $\mathcal{D} = \mathcal{D}_1^{-1}\mathcal{D}_2$ of the operators $\mathcal{D}_{1,2}$. Therefore, (6.3.30) nontrivially acts in the factor algebra $\mathcal{W} = \mathcal{W}_1 \otimes \mathcal{W}_2 / I_X$ where I_X is the two-side ideal spanned by $X = X_1 X_2$ (X obviously commutes with $\mathcal{D}$). Defining the operators $\mathcal{Z} = X_1 \equiv X_2^{-1}$ and $\mathcal{D} = \mathcal{D}_1^{-1}\mathcal{D}_2$ on $\mathcal{W}$ we obtain

$$\mathcal{D}\mathcal{Z} = q\mathcal{Z}\mathcal{D} \quad (6.3.31)$$

and (6.3.30) reproduces the factorization formula (6.3.17) for $I = 1$.

The factorization (6.3.30) of the L-operator into two simpler L-operators depending on λ_1 and λ_2 has a similar form to the BLZ factorization of the transfer-matrix acting in the Verma module V_J into the product of Q-operators [44; 45]. However, their structure is completely different, and we believe it is more closely related to the factorisation of chapter 5. Although both factorizations produce the same L-operator, the L-operators $L_{SG}^{(1,2)}(\lambda_{1,2})$ in (6.3.28-6.3.29) do not act invariantly in the Verma module V_J^+. Due to the presence of the operators $X_{1,2}^{\pm 1}$ they will act in the module generated by z^j with j running from $-\infty$ to $+\infty$. This causes trouble with a proper definition of traces. The only possible resolution of this problem is a restriction to the cyclic case $q^N = 1$. Then all representations become finite-dimensional and well defined [69].

The BLZ approach is very different in this respect. It allows to derive the $U_q(sl_2)$ L-operator in terms of the q-oscillator L-operators acting in the space spanned by z^j, $j = 0, \ldots, \infty$. A factorization of the L-operator is more involved in this case [70; 71].

Finally we would like to compare our trigonometric factorization formula (6.3.17) with a similar result obtained in [12] for the case of modular double. Unfortunately, no direct comparison is possible since the authors of [12] calculated the L-operator for a special case of modular representations. However, there is an explicit transformation between our function $M_2(\lambda)$ and their analog $M^{(I)}$ (defined after the formula (37) in [12]) which we denote as $M_{CD}^{(I)}(U)$. Namely, we checked that

$$M_{CD}^{(I)}(U)_{n,m} = (-1)^n U^{I-2n} q^{n+Im/2-2mn} [M_2(U^{-2} q^{-I/2})]_{n,m}. \quad (6.3.32)$$

So up to some simple transformations they define essentially the same function.

6.4 An alternative derivation

In the previous sections we used the R-operator action (6.0.8) but we could have alternatively started from (6.0.1). As mentioned earlier these actions are the same but we are curious to see what the equality looks like at the level of matrix elements (6.1.3). It turns out we get a similar but not identical formula that is arguably "nicer" because the symmetries of the R-matrix are more apparent in its presentation. We transform this formula to (6.0.8) and thereby giving another proof that the two actions are equal.

We proceed in the same way as before, expanding out (6.0.1) acting on $(z_1 - x)\Phi(z)$ with $\Phi(z) = z^{j'}$ in powers of x. As an intermediate step we have

$$\sum_{i'=0}^{I} \frac{(-I)_{i'} x^{I-i'}}{i'!} \mathbb{R}_{IJ}(\mu - \frac{I}{2}) \cdot z_1^{i'} z^{j'} = \sum_{k=0}^{\infty} \frac{(\mu - J/2 - I)_k}{k!} z^k (z_1 - z)^{\mu + J/2 + 1}$$
$$\times \partial_z^I (z_1 - z)^{-\mu + I - J/2 - 1} \sum_{r=0}^{\infty} \frac{(J/2 - \mu)_r}{r!} z^{r + j'} x^{I - k - r}.$$

Equating coefficients of x we set $i' = k + r$ and eliminate the summation variable r. Now expanding out the remaining binomials we get

$$\sum_{i'} \frac{(-I)_{i'}}{i'!} \mathbb{R}_{IJ}(\mu - I/2) \cdot z_1^{i'} z^{j'} = (-1)^I z^{\mu + J/2 + 1} \sum_{k=0}^{i'} \frac{(\mu - J/2 - I)_k (J/2 - \mu)_{i'-k}}{k!(i'-k)!}$$
$$\times \sum_{t=0}^{\infty} \frac{(-\mu - J/2 - 1)_t}{t!} z^{k-t} \partial_z^I \sum_{q=0}^{\infty} \frac{(\mu - I + J/2 + 1)_q}{q!} z_1^{q+t} z^{-\mu + I - J/2 - 1 + i' + j' - k - q}.$$

Because we want an equality of the form (6.1.2) we must set $i = q + t$. Now eliminating the summation variable q and fixing i' we get

$$\frac{(-I)_{i'}}{i'!} \mathbb{R}_{IJ}(\mu - I/2) \cdot z_1^{i'} z^{j'} = (-1)^I z^{\mu + J/2 + 1} z_1^i \sum_{k=0}^{i'} \frac{(\mu - J/2 - I)_k (J/2 - \mu)_{i'-k}}{k!(i'-k)!}$$
$$\times \sum_{t=0}^{i} \frac{(-\mu - J/2 - 1)_t (\mu - I + J/2 + 1)_{i-t}}{t!(i-t)!} z^{k-t} \partial_z^I z^{-\mu + I - J/2 - 1 + j' + i' - i + t - k}.$$

Finally applying the differential operator, simplifying pochammer expressions and shifting back the spectral parameter we obtain the formula

$$[\mathbb{R}_{IJ}(\mu)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i+i'}}{i!(-I)_{i'}} \times \sum_{k,t=0} \frac{(-i,-\mu-\frac{I+J}{2}-1)_t(-i',\mu-\frac{I+J}{2})_k(\mu+\frac{J-I}{2}+1-j+k-t)_I}{t!k!(-\mu+\frac{I-J}{2})_{t-i}(\mu+\frac{I-J}{2}+1)_{k-i'}}. \quad (6.4.1)$$

This double sum expression terminates after finitely many terms by the indices i and i'. Though the arguments are different to (6.1.3) it has a similar form. We expect that since the actions (6.0.1) and (6.0.8) are equal that the two formulae (6.4.1) and (6.1.3) are equal. Indeed they are, and this is pretty easy to show.

First we note an interesting feature of the summation in (6.4.1); each summation variable k,t yields a balanced and terminating ${}_3F_2$ hypergeometric series. These series are summable using the Pfaff-Saalschütz's theorem (A.2.3) and we can apply it to the sum in t with $n=i$ or in k with $n=i'$. It turns out either choice gives ${}_4F_3$ functions that appear elsewhere in this thesis. If we choose the latter, and simplify pochammer symbols we get exactly the formula (6.1.10) multiplied by the factor $C(\mu;I,J)$ (6.1.5) which was an intermediate step in proving (6.1.6) is the same as (3.8.5). This proves (6.4.1) is equal to (6.1.3) and equal to (3.8.5) up to normalisation (6.1.5).

If we choose to sum up the series in t instead, we get exactly the formula (3.7.7) with all i,j and I,J swapped and again multiplied by (6.1.5). The swapping of indices is just the symmetry (3.7.21a) so they are equal.

6.4.1 q-deformation

We may find the q-deformation of (6.4.1) in the same way as we found the deformation (6.2.5) of (6.1.6). We start from (6.2.1) and rather than introducing another summation by the reverse q-Chu-Vandemonde sum (6.2.2) we instead reverse using the q-Pfaff-Saalschütz's theorem (B.2.3). This reversal is encapsulated in the identity

$$\frac{(q^{2j-2J},\lambda^{-2}q^{I-J+2j-2i'};q^2)_t(q^{2j-2i'-2J},\lambda^{-2}q^{-I-J+2j},\lambda^{-2}q^{I-J-2i};q^2)_i(q^{-2J};q^2)_j}{(q^{-2J+2j-2i'};q^2)_t}$$
$$= \frac{(\lambda^2q^{2+J-I},\lambda^{-2}q^{-I-J+2j};q^2)_i(\lambda^{-2}q^{I-J};q^2)_{j+t}(\lambda^{-2}q^{J-I};q^2)_{i'}(q^{-2J};q^2)_{i+j-i'}}{(-1)^i\lambda^{2(i-i')}q^{i^2+i'(I+J)+i(J-I+1)}(q^{-2I};q^2)_{i'}(\lambda^{-2}q^{I-J};q^2)_{j-i'}}$$

$$\times\,{}_3\phi_2\left(\begin{matrix} q^{-2i'} & \lambda^2 q^{-I-J} & \lambda^2 q^{2+I+J-2j-2t} \\ & \lambda^2 q^{2+I-J-2i'} & \lambda^2 q^{2+J-I-2j-2t}\end{matrix}\middle|\, q^2, q^2\right). \tag{6.4.2}$$

We have written it so that all terms on the left hand side of this identity appear in the expression (6.2.1); the variable t is the summation variable of the ${}_4\phi_3$ series. Therefore the right hand side can be directly substituted, and after some simple q-pochammer transforms we obtain

$$[\mathbb{R}_{IJ}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}\frac{(-1)^{i+i'}\lambda^{i-i'}q^{i-i'+I(j-i-i')+Ji+i(i-j)+i'(i+j)}}{(q^2;q^2)_i(q^{-2I};q^2)_{i'}(\lambda^2q^{2+J-I};q^2)_I}$$
$$\times\sum_{k,t=0}\frac{(q^{-2i},\lambda^{-2}q^{-2-I-J};q^2)_t(q^{-2i'},\lambda^2q^{-I-J};q^2)_k(\lambda^2q^{2+J-I-2j+2k-2t};q^2)_I}{(q^2;q^2)_t(q^2;q^2)_k(\lambda^{-2}q^{I-J};q^2)_{t-i}(\lambda^2q^{2+I-J};q^2)_{k-i'}}q^{2k+2t(I+1)}. \tag{6.4.3}$$

Since we started from (6.2.1) - the q-deformation of (6.1.10) - which is equal to (6.4.1) up to normalisation (6.1.5), the formula (6.4.3) is the q-deformation of (6.4.1) up to normalisation

$$C_q(\lambda;I,J) = (\lambda^2q^{2+J-I};q^2)_I$$

which appears in the denominator outside the sum of (6.4.3) and can be removed so that (6.4.1) will be recovered in the limit $q\to 1$.

It is possible to reverse the arguments in going from (6.0.3) to (6.4.1) to obtain an operator action that yields (6.4.3). The reverse direction is not much more complicated than the forward direction, with binomial factors and theorem replaced with their q versions. The only added extra complexity are the extra q phase factors inside the summation and the product outside which mix indices in the quantum and auxiliary spaces. Fortunately, any problem terms can be removed without affecting the Yang-Baxter equation - they are essentially similarity and gauge transforms of the operator which drop out of the equation. There are also extra phase terms appearing depending on how we define the q-derivative. All of these considerations allow for some degree of freedom for the formula we start from and the generating function for the action that we obtain. We start from (6.4.3) and make the necessary transformations a posteriori by defining $\bar{R}_{IJ}(\lambda)$ by

$$[\bar{R}_{IJ}(\lambda)]_{i,j}^{i',j'} = \frac{(-1)^I\lambda^{i-i'-I}q^{2(i-i')(I-J)}(\lambda^2q^{2+J-I};q^2)_I}{q^{(i-i')(i'-j)+IJ/2-I/2}(1-q^2)^I}\,[R_{IJ}(\lambda)]_{i,j}^{i',j'}\,. \tag{6.4.4}$$

By reversing the arguments made from (6.0.3) to (6.4.1) we obtain the following form for the action:

$$\bar{R}_{IJ}(\lambda) \cdot (z_1 x^{-1}; q^2)_I \Phi(z) = z^{\mu + \frac{I+J}{2} + 1} (z x^{-1} q^{J-I}; q^2)_{-\mu + \frac{I+J}{2}} (z_1 z^{-1} q^{I-2J}; q^2)_{\mu + \frac{I+J}{2} + 1}$$
$$\times E_q^I \left[z^{-\mu + \frac{I-J}{2} - 1} (z_1 z^{-1} \lambda^2 q^{-J+I+2}; q^2)_{-\mu + \frac{I-J}{2} - 1} (z x^{-1} \lambda^{-2} q^{-I+2J}; q^2)_{\mu + \frac{I-J}{2}} \Phi(z) \right], \quad (6.4.5)$$

where $\lambda = q^\mu$ and E_q is the q-derivative acting on z^i by

$$E_q \cdot z^i = \frac{q^i - q^{-i}}{q - q^{-1}} z^{i-1}. \quad (6.4.6)$$

This is a q-deformation of (6.0.3) whereby expanding in powers of z, z_1 one can obtain the expressions (6.4.4) and (6.4.3) for the coefficients of the expansion.

Chapter 7

SU(2)-invariance and coherent state action

In this chapter we consider the SU(2)-invariant R-matrix derived using Sklyanin's work [15]. His construction is a solution to the additive Yang-Baxter equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \tag{7.0.1}$$

acting on a tensor product $V_1 \otimes V_2$ of finite-dimensional irreducible unitary representation spaces of the group $SU(2)$. Each V_i is realised as the space of polynomials in z_i of degree at most $2l_i \in \mathbb{Z}^+$. It turns out this R-matrix is same as (3.7.7) constructed by projecting out the solution (3.2.13) of the tetrahedron equation (and therefore the same as the other constructions in this thesis). We will prove this by constructing an expression for the matrix elements of Sklyanin's solution for the operator action and give the transformation of this expression to (3.7.7). In the same way as in section 6.2 of chapter 6 we will also find a q-deformation of the solution in terms of its matrix elements and a generating function for its action.

First we begin with the standard definition of the Lie group $SU(2)$ as

$$SU(2) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C},\ \det(g) = 1,\ g^\dagger g = \mathbb{1}\}. \tag{7.0.2}$$

The representations of this group we denote by T^{l_i}, act on elements $P(z_i) \in V_i$ by

$$T^{l_i}(g) \cdot P(z_i) = (bz_i + d)^{2l_j} P(g \cdot z_i), \quad g \cdot z_i = \frac{az_i + c}{bz_i + d} \tag{7.0.3}$$

The R-operator $\mathcal{R}(\lambda)$ is constructed by considering the element

$$e_{l_i,\overline{z_i}}(z_i) := (1 + z_i\overline{z_i})^{2l_i} \in V_i, \tag{7.0.4}$$

where $\overline{z_i}$ is an auxiliary parameter. This is the 'coherent state' with respect to the inner product

$$\langle P|Q\rangle = \frac{1}{2i}\int_{\mathbb{C}} P(\overline{z_i})Q(z_i)\mu(z_i,\overline{z_i})d\overline{z_i}\wedge dz_i \tag{7.0.5a}$$

$$\mu(z_i,\overline{z_i}) = \frac{2l_j+1}{\pi(1+z_i\overline{z_i})^{-2(l_i+1)}} \tag{7.0.5b}$$

on V_i and by which $T^{l_i}(g)$ act as unitary operators. The inner product also has a reproducing kernel $\mathcal{E}_{l_i}(z_i,\overline{z_i})$ given by

$$P(z_i) = \frac{1}{2i}\int_{\mathbb{C}} \mathcal{E}_{l_i}(z_i,\alpha)P(\alpha)\mu(\alpha,\overline{\alpha})d\overline{\alpha_i}\wedge d\alpha_i, \tag{7.0.6}$$

$$\mathcal{E}_{l_i}(z_i,\overline{z_i}) = e_{l_i,\overline{z_i}}(z_i), \tag{7.0.7}$$

which is essentially a Bergman kernel as it is known in the literature [72].

The action of $\mathcal{R}(\lambda)$ on the coherent state $e_{l_1,\overline{z_1}}(z_1)e_{l_2,\overline{z_2}}(z_2) \in V_1\otimes V_2$ we write as

$$\frac{(\mathcal{R}(\lambda)\cdot e_{\overline{z_1}}e_{\overline{z_2}})(z_1,z_2)}{E_1(z_1,\overline{z_1})E_2(z_2,\overline{z_2})} = \hat{\mathbb{R}}(z_1,z_2;\overline{z_1},\overline{z_2}) \tag{7.0.8}$$

with $\hat{\mathbb{R}}$ the generating function of the action. The key insight into this construction by Sklyanin was to notice that this generating function is a function of only one variable. This follows from $SU(2)$ invariance of the R-operator

$$\hat{\mathbb{R}}_{12}(z_1,z_2;\overline{z_1},\overline{z_2}) = \hat{\mathbb{R}}_{12}(g\cdot z_1, g\cdot z_2;(g^{-1})^t\cdot\overline{z_1},(g^{-1})^t\cdot\overline{z_2})$$

because of the existence of only one unique non-trivial $SU(2)$-invariant on the space $V_1\otimes V_2$ given by

$$\xi(z_1,z_2;\overline{z_1},\overline{z_2}) = \frac{(1+z_1\overline{z_2})(1+z_2\overline{z_1})}{(1+z_1\overline{z_1})(1+z_2\overline{z_2})}, \tag{7.0.9}$$

and therefore $\hat{\mathbb{R}}$ must be a polynomial of ξ. By solving a RLL-relation for $\mathcal{R}(\lambda)$ of the same form as (5.1.1) (with $m = 2$) it can be computed that $\hat{\mathbb{R}}(\xi)$ satisfies the hypergeometric equation

$$\xi(1-\xi)\hat{\mathbb{R}}'' + [c-(1+a+b)\xi]\,\hat{\mathbb{R}}' - ab\hat{\mathbb{R}} = 0$$

with

$$a = -2l_1, \quad b = -2l_2, \quad c = \frac{\lambda}{\eta} + (1-l_1-l_2)$$

whose solution is the Jacobi polynomial

$$\hat{\mathbb{R}}(\lambda,\xi) = {}_2F_1\left(\begin{matrix} -2l_1, & -2l_2 \\ \frac{\lambda}{\eta} - l_1 - l_2 + 1 & \end{matrix} \middle| \xi \right). \tag{7.0.10}$$

This equation along with (7.0.8) allows us to calculate the action of R on basis elements $z_1^{i'} z_2^{j'}$ and hence deduce explicit matrix elements $[R(\lambda)]_{i,j}^{i',j'}$ by noting the relation

$$\mathcal{R}(\lambda) \cdot z_1^{i'} z_2^{j'} = \sum_{i,j} [R(\lambda)]_{i,j}^{i',j'}\, z_1^i z_2^j. \tag{7.0.11}$$

Putting together (7.0.8) and (7.0.10) and making frequent use of the binomial theorem (A.2.1) we find (as an intermediate step)

$$\sum_{i',j'=0} \binom{2l_1}{i'} \binom{2l_2}{j'} (\mathcal{R}(\lambda) \cdot z_1^{i'} z_2^{j'}) \overline{z_1}^{i'} \overline{z_2}^{j'} = \sum_{n,m_i=0} \binom{2l_1-n}{m_1} \binom{2l_2-n}{m_2} \binom{n}{m_3} \binom{n}{m_4}$$
$$\times \frac{(2l_1)!(2l_2)!}{n!(2l_1-n)!(2l_2-n)!} \prod_{s=0}^{n-1} \frac{(z_1\overline{z_1})^{m_1}(z_2\overline{z_2})^{m_2}}{\lambda-(l_1+l_2-s-1)\eta} \tag{7.0.12}$$

with binomial coefficients terminating each summation. Observing that the action of $\mathcal{R}(\lambda)$ is a function of z_1 and z_2 only we can consider the above equation as a generating function for the action in the auxiliary variables $\overline{z_1}$ and $\overline{z_2}$. In particular, the coefficient of $\overline{z_1}^{i'} \overline{z_2}^{j'}$ gives the action of $\mathcal{R}(\lambda)$ on the basis element $z_1^{i'} z_2^{j'}$ in $V_1 \otimes V_2$. This coefficient can easily be found by placing the constraints

$$i' = m_1 + m_4, \quad j' = m_2 + m_3 \tag{7.0.13}$$

which removes two summations on each side in (7.0.12) yielding

$$\mathcal{R}(\lambda) \cdot z_1^{i'} z_2^{j'} = \binom{2l_1}{i'}^{-1} \binom{2l_2}{j'}^{-1} \sum_{n,m_1,m_2=0} \binom{2l_1-n}{m_1} \binom{2l_2-n}{m_2} \binom{n}{j'-m_2} \binom{n}{i'-m_1}$$
$$\times \frac{(2l_1)!(2l_2)!}{n!(2l_1-n)!(2l_2-n)!} \prod_{s=0}^{n-1} \frac{z_1^{m_1+j'-m_2} z_2^{m_2+i'-m_1}}{\lambda-(l_1+l_2-s-1)\eta} \tag{7.0.14}$$

as the action on the R-matrix on the basis elements. It can be seen that (7.0.14) is of the same form as (7.0.11) by defining

$$i := m_1 + j' - m_2, \quad j := m_2 + i' - m_1. \tag{7.0.15}$$

We also observe the conservation law

$$i + j = i' + j'$$

and the following double sum formula for $R(\lambda)$:

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \binom{2l_1}{i'}^{-1} \binom{2l_2}{j'}^{-1} \sum_{n,m_2=0} \binom{2l_1 - n}{m_2 + i - j'} \binom{2l_2 - n}{m_2} \binom{n}{j' - m_2} \binom{n}{j - m_2}$$
$$\times \frac{(2l_1)!(2l_2)!}{n!(2l_1 - n)!(2l_2 - n)!} \prod_{s=0}^{n-1} \frac{1}{\lambda - (l_1 + l_2 - s - 1)\eta}. \tag{7.0.16}$$

Finally, after some simplification using Pochhammer identities and making the identification

$$I = 2l_1, \quad J = 2l_2,$$

between Sklyanin's spin notation $\{l_1, l_2\}$ and our weight notation $\{I, J\}$ one can easily rewrite this formula as

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i-i'} i'!(-I)_{i-j'}}{j!(-I)_{i'}(-J)_{j'}}$$
$$\times \sum_{s,l} \frac{s!(-j)_l(-j')_l(-J)_{s+l}(i - j' - I)_{s+l}}{(\frac{\lambda}{\eta} - \frac{I+J}{2} + 1)_s t!(l + i - j')!(l + s - j)!(l + s - j')!} \tag{7.0.17}$$

where $0 \le l \le \mathrm{Min}(j, j')$ and $0 \le s \le J - l$. For $\eta = 1$ this expression produces the same output as (3.7.7) and (3.8.5) up to a factor $B(\lambda; I, J)$ such that

$$\left[R_{I,J}^{(2),r}(\lambda)\right]_{i,j}^{i',j'} = B(\lambda; I, J)\, [R(\lambda)]_{i,j}^{i',j'}, \tag{7.0.18a}$$

$$B(\lambda; I, J) = \frac{(\lambda - \frac{J}{2} - \frac{I}{2} + 1)_I}{(\lambda + \frac{J}{2} - \frac{I}{2} + 1)_I}. \tag{7.0.18b}$$

The expression (7.0.17) is yet another formula for a rational R-matrix, and has quite different structure to other double sum expressions (6.0.8), (6.4.1) and (3.3.18) found using other methods. Regardless, in the next section we will show they are equal.

7.0.1 Transformation to a single sum

In applying the identities of Appendix A to the expressions in this section the arguments of the hypergeometric functions are already written in the correct ordering.

At first glance it appears that (7.0.17) is an unbalanced ${}_4F_3$ hypergeometric series in each summation index and therefore difficult to transform with single sum identities. However, the expression simplifies to two ${}_3F_2$ series with the change of variables $r := s + l$ and eliminating s. Bringing the series in l forward the expression can be written as

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^j i'!(-I)_{i-j'}}{i!j!(-I)_{i'}(-J)_{j'}} \times \sum_{r=0}^{J} \frac{(-J)_r(i-j'-I)_r}{(r-j)!(\lambda-\frac{I}{2}-\frac{J}{2}+1)_r} {}_3\overline{F}_2 \left(\begin{matrix} -j'; & -j & -\lambda+\frac{I}{2}+\frac{J}{2}-r \\ & -r & 1+i-j' \end{matrix} \middle| 1 \right) \tag{7.0.19}$$

Applying the identity (A.2.14) with $n = j', m = i$ the ${}_3F_2$ the expression becomes

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i'}(-I)_{i-j'}}{i!(-I)_{i'}(-J)_{j'}(\lambda-\frac{I}{2}-\frac{J}{2}+1)_{i-j'}} \times \sum_{r=0}^{J} \frac{(-J,i-j'-I)_r}{(r-j)!(\lambda-\frac{I}{2}-\frac{J}{2}+1+i-j')_r} {}_3\overline{F}_2 \left(\begin{matrix} -i; & -i' & -\lambda+\frac{I}{2}+\frac{J}{2}-i-j \\ & -i-j & 1-i-j+r \end{matrix} \middle| 1 \right). \tag{7.0.20}$$

Note the change in regularisation with respect to j' in (7.0.19) to i in (7.0.20). Observing the summation in r we notice that it is now also a ${}_3F_2$ hypergeometric series. To see this more clearly, we rewrite (7.0.20) as

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{(-1)^{i-i'}(-I)_{i-j'}\Gamma(\lambda-\frac{I}{2}-\frac{J}{2}+1)(1+j)_i}{i!(-I)_{i'}(-J)_{j'}} \times \sum_{l=0}^{i} \frac{(-i,-i',-\lambda+\frac{I}{2}+\frac{J}{2}-i-j)_l}{l!(-i-j)_l} {}_3\tilde{F}_2 \left(\begin{matrix} 1 & -J & i-I-j' \\ & 1-i-j+l & \lambda-\frac{I}{2}-\frac{J}{2}+1+i-j' \end{matrix} \middle| 1 \right). \tag{7.0.21}$$

Note the change in regularisation used. This ${}_3F_2$ hypergeometric series can be summed up using (A.2.15) with $n = i + j - l$, which is always positive because l runs from 0 to $\mathrm{Min}(i, i')$. Summing up we obtain

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'}(-1)^i \frac{(-J)_j(-\lambda+\frac{I}{2}-\frac{J}{2})_{j-i'}(\lambda-\frac{I}{2}+\frac{J}{2}+1)_I}{i!(-J)_{j'}(-\lambda-\frac{I}{2}-\frac{J}{2})_{i+j}(\lambda-\frac{I}{2}-\frac{J}{2}+1)_I}$$

$$\times {}_4\overline{F}_3 \left(\begin{matrix} -i; & -i' & \lambda + \frac{I}{2} + \frac{J}{2} + 1 - i - j & -\lambda + \frac{I}{2} + \frac{J}{2} - i - j \\ & 1 - i - j + J & 1 - i - i' + I & -i - j \end{matrix} \middle| 1 \right). \tag{7.0.22}$$

The ${}_4F_3$ hypergeometric series is balanced and terminating and therefore can be transformed to (3.8.3) (up to the factor $B(\lambda; I, J)$) using Whipple's identity (A.2.7) with $n = i$.

We have shown that matrix elements of the $SU(2)$-invariant R-matrix for finite-dimensional representations is given by the same single sum ${}_4F_3$ formula as that obtained using the 3-dimensional, factorisation and spectral decomposition methods of earlier chapters.

7.0.2 q-deformation

Since we have linked the R-matrix of this section to those whom we know their q-deformation, we can reverse the arguments of the previous section starting from (3.8.3) and reverse the arguments of the last section to find a quantum deformation of (7.0.17). Taking (3.8.3) we apply Sears' transform (B.2.13) with $n = i$ and

$$a = q^{-2i'}, \quad b = \lambda^{-2} q^{J-I}, \quad c = \lambda^2 q^{2-I+J},$$
$$d = q^{2-2i-2j+2J}, \quad e = q^{-2I}, \quad f = q^{2+2j-2i'}$$

to find

$$\begin{aligned} [R_{I,J}(\lambda)]_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} B_q(\lambda; I, J)(-1)^i \frac{q^{i(i-J-1)+(I-i)(J-j)+i'(I+j')+2i(i+j-I)-IJ}}{\lambda^{i+i'}} \\ &\times \frac{(q^{-2J}; q^2)_j (\lambda^{-2} q^{I-J}; q^2)_{j-i'} (\lambda^2 q^{2-I+J}; q^2)_I}{(q^{-2J}; q^2)_{j'} (q^2; q^2)_i (\lambda^{-2} q^{-I-J}; q^2)_{i+j} (\lambda^2 q^{2-I-J}; q^2)_I} \\ &\times {}_4\overline{\phi}_3 \left(\begin{matrix} q^{-2i}; & q^{-2i'} & \lambda^2 q^{I+J+2-2i-2j} & \lambda^{-2} q^{I+J-2i-2j} \\ & q^{-2i-2j} & q^{2-2i-2i'+2I} & q^{2-2i-2j+2J} \end{matrix} \middle| q^2, q^2 \right) \end{aligned} \tag{7.0.23}$$

where

$$B_q(\lambda; I, J) = \frac{(\lambda^2 q^{2-I-J}; q^2)_I}{(\lambda^2 q^{2-I+J}; q^2)_I}. \tag{7.0.24}$$

The constant (7.0.24) is the deformation of the constant factor (7.0.18) which appears because we started from (3.8.3). We keep it in the steps that follow for consistency and correctness but in the context of deforming (7.0.17) in the previous section we may ignore it. That is, (7.0.23) with $B_q(\lambda; I, J) := 1$ is a quantum deformation of (7.0.22) and so forth in the steps that follow.

Next we reintroduce the second summation by expanding out factors in (7.0.23) using identity (B.2.21) with

$$n = I + j - i', \quad m = i + j - l,$$
$$b = q^{-2J}, \quad e = \lambda^2 q^{2-I-J-2j+2i'}$$

where we denote by l the summation index of the ${}_4\overline{\phi}_3$ series. l runs from 0 to $\mathrm{Min}(i, i')$ so $n \geq m$ and so the identity can safely be used. In particular,

$$\frac{(\lambda^2 q^{J+I+2-2i-2j}; q^2)_l (\lambda^2 q^{J-I+2}; q^2)_I (\lambda^{-2} q^{I-J}; q^2)_{j-i'} (q^{-2J}; q^2)_j}{(q^{2J-2j-2i+2}; q^2)_l (\lambda^{-2} q^{-I-J}; q^2)_{i+j}} = \frac{q^{i'-j+j^2+i'^2-2ji'}}{(q^2; q^2)_{I+j-i'}}$$
$$\times \frac{q^{-J(i+i')+I(i+2j-i')+2IJ}}{(q^{-2J+2j}; q^2)_i} {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2I-2j+2i'}; & q^{-2J} & q^2 \\ & q^{2-2i-2j+2l} & \lambda^2 q^{2-I-J-2j+2i'} \end{matrix} \middle| q^2, q^2 \right). \quad (7.0.25)$$

We have written the identity so that all terms on the left hand side appear in (7.0.23) and so we can substitute them for the right hand side directly. Combining (7.0.23) and (7.0.25) we have the double sum expression

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} B_q(\lambda; I, J) \frac{(-1)^{i-i'} \lambda^{i+i'} q^{-j-j'+i^2-ij+ii'+ji'-I(i+j)-J(i+i')+2IJ}}{(\lambda^2 q^{2-I-J}; q^2)_I}$$
$$\times \frac{(q^{-2I}; q^2)_{i-j'}}{(q^{-2I}; q^2)_{i'} (q^{-2J}; q^2)_{j'}} \begin{bmatrix} i+j \\ i \end{bmatrix}_{q^2} \sum_{l=0}^{i} q^{2l} \frac{(q^{-2i}, q^{-2i'}, \lambda^{-2} q^{I+J-2i-2j}; q^2)_l}{(q^2, q^{-2i-2j}; q^2)_l (q^2; q^2)_{l+I-i-i'}}$$
$$\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2I-2j+2i'}; & q^{-2J} & q^2 \\ & q^{2-2i-2j+2l} & \lambda^2 q^{2-I-J-2j+2i'} \end{matrix} \middle| q^2, q^2 \right). \quad (7.0.26)$$

Now rewriting (7.0.26) by identifying the ${}_3\phi_2$ hypergeometric series indexed by 'l' we have

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} B_q(\lambda; I, J) \frac{(-1)^j \lambda^{j+j'} q^{-j+i^2+j'^2+ij+ii'+ji'} (q^{-2I}; q^2)_{i-j'} (\lambda^{-2} q^{I+J}; q^2)_{j-i'}}{q^{2jj'-2i'j'+I(j+j')+J(i+j)-2IJ} (q^{-2I}; q^2)_{i'} (q^{-2J}; q^2)_{j'} (q^2; q^2)_i}$$
$$\times \sum_{r=0}^{J} \frac{q^{2r} (q^{-2I-2j+2i'}, q^{-2J}; q^2)_r}{(\lambda^2 q^{2-I-J-2j+2i'}; q^2)_r (q^2; q^2)_{r-j}} {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2i}; & q^{-2i'} & \lambda^{-2} q^{I+J-2i-2j} \\ & q^{-2i-2j} & q^{2-2i-2j+2r} \end{matrix} \middle| q^2, q^2 \right),$$
$$(7.0.27)$$

analogous to (7.0.20). Now reversing the transformation (7.0.19) to (7.0.20) using identity (B.2.23) on the ${}_3\overline{\phi}_2$ function with $n = i, m = j'$ we have

$$[R_{I,J}(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} B_q(\lambda; I, J) \frac{q^{i'j'-ij-J(i+j)-I(j+j')+2IJ+j(j-1)}(q^2;q^2)_{i'}(q^{-2I};q^2)_{i-j'}}{(-1)^j \lambda^{-j-j'}(q^{-2I};q^2)_{i'}(q^{-2J};q^2)_{j'}(q^2;q^2)_i(q^2;q^2)_j}$$
$$\times \sum_{r=0}^{J} \frac{q^{2r+2rj'}(q^{-2I-2j+2i'};q^2)_r(q^{-2J};q^2)_r}{(\lambda^2 q^{2-I-J};q^2)_r(q^2;q^2)_{r-j}} {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j'}; & q^{-2j} & \lambda^{-2}q^{I+J-2r} \\ & q^{-2r} & q^{2-2j+2i'} \end{matrix} \middle| q^2, q^2 \right) \quad (7.0.28)$$

which deforms (7.0.19). Finally we reverse the change of summation index introduced in transforming (7.0.17) to (7.0.19) by defining $r := s + l$ and eliminating r. We then rewrite the double sum by removing the nested summation implicit in the ${}_3\phi_2$ series (and whose summation variable is l). We also remove the inessential constant factor $B_q(\lambda; I, J)$ that only appears because we started from (3.8.3). Finally we see that (7.0.28) can be rewritten as

$$[R(\lambda)]_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \frac{q^{i^2-i(I-j+J+i'+1)-2Ij+2IJ+Ii'+2j^2-jJ-ji'-2j+i'}(q^2;q^2)_{i'}(q^{-2I};q^2)_{i-j'}}{(-1)^{i'-i}\lambda^{i'-i-2j}(q^2;q^2)_j(q^{-2I};q^2)_{i'}(q^{-2J};q^2)_{j'}}$$
$$\times \sum_{s,l} \frac{\lambda^{-2l}q^{l(I+J+2)+2s+2l}(q^2;q^2)_s(q^{-2j},q^{-2j'};q^2)_l(q^{-2J},q^{2i-2I-2j'};q^2)_{s+l}}{(q^2;q^2)_l(q^2;q^2)_{i+l-j'}(q^2;q^2)_{s+l-j}(q^2;q^2)_{s+l-j'}(\lambda^2 q^{-I-J+2};q^2)_s} \quad (7.0.29)$$

with summation variables running over the same values as its rational limit (7.0.17) whereby taking the limit of (7.0.29) as $q \to 1$ we obtain (7.0.17).

7.1 Generating function for q-action

In this section we will continue reversing the arguments of the last section to find a q-deformation of $\hat{\mathbb{R}}(\xi)$. It is interesting to see whether the generating function still remains a function of a single variable. Unfortunately, this does not appear to be the case. Instead, the ${}_2F_1$ expression in (7.0.10) becomes a balanced and terminating ${}_4\phi_3$ function.

The main problem we have is defining the q-analogue of the functions $e_{l_i,\overline{z_i}}(z_i)$ and $\mathcal{E}_{l_i}(z_i, \overline{z_i})$. These functions have special properties as coherent states and reproducing kernels with respect to the inner product (7.0.5) on the $SU(2)$ representation spaces considered in this chapter. As far as we are aware, this theory in the case of quantum groups is not completely known. Some work has been done in this area, and the notion of a q-coherent state ${}_qe_{\overline{z_i}}(z_i)$ was proposed in [73] which we will use in our calculations.

Let us define

$$ {}_q e_{\overline{z_i}}(z_i) := (-z_i\overline{z_i}q^{1-2l_i};q^2)_{2l_i} \quad (7.1.1a)$$
$$ {}_q\mathcal{E}(z_i,\overline{z_i}) := (-z_i\overline{z_i}q^{1-2l_i};q^2)_{2l_i} \quad (7.1.1b)$$
$$ 2l_1 := I, \quad 2l_2 := J \quad (7.1.1c)$$

In our calculations we make frequent use of the q-binomial theorem (B.2.1b) in the same way we labelled expressions C_n^k in (7.0.12) the same pochammer symbols appear in (7.0.29) as q-pochammer symbols. That is, starting from (7.0.29) (with $B_q(\lambda;I,J) := 1$) and reversing arguments we get the intermediate step

$$\sum_{i',j'} \frac{(q^{-2I};q^2)_{i'}(q^{-2J};q^2)_{j'}q^{Ii'+i'+Jj'+j'}}{(-1)^{i'+j'}(q^2;q^2)_{i'}(q^2;q^2)_{j'}}R(\lambda)\cdot z_1^{i'}z_2^{j'}\overline{z_1}^{i'}\overline{z_2}^{j'} =$$
$$= \sum_{s,m_1}\frac{(q^2;q^2)_I(q^2;q^2)_J q^{2s^2-2Is-2Js+2IJ}}{(q^2;q^2)_{I-s}(q^2;q^2)_{J-s}(q^2,\lambda^2q^{2-I-J};q^2)_s}{}_{q^2}D_{I-s}^{m_1}z_1^{m_1}\overline{z_1}^{m_1}(-1)^{m_1}q^{m_1-Im_1-Jm_1+2m_1s}$$
$$\times\sum_{m_2}D_{J-s}^{m_2}z_2^{m_2}\overline{z_2}^{m_2}(-1)^{m_2}q^{m_2-Im_2-Jm_2+2m_2s}\sum_{m_3}{}_{q^2}D_s^{m_3}\lambda^{m_3}z_1^{m_3}\overline{z_2}^{m_3}(-1)^{m_3}q^{m_3-Im_3}$$
$$\times\sum_{m_4}{}_{q^2}D_s^{m_4}\lambda^{m_4}z_2^{m_4}\overline{z_1}^{m_4}(-1)^{m_4}q^{m_4-Jm_4}q^{m_1m_3-m_2m_3-m_1m_4+m_2m_4},$$

where on the right hand side we have reintroduced summation indices m_i that we used to expand out ξ (7.0.9) in the classical case and run over the same values. The relation between the m_i and our tensor indices are exactly the same as in the classical case (7.0.13), (7.0.15).

The phase term $q^{m_1m_3-m_2m_3-m_1m_4+m_2m_4}$ is problematic because it couples the summation variables. However, this term can be removed, because

$$m_1m_3 - m_1m_4 - m_2m_3 + m_2m_4 = \frac{1}{2}\left(i^2+j^2-i'^2-j'^2\right),$$

which is just the similarity transform (6.3.2)

$$R_{I,J}(\lambda) = (U\otimes U)\overline{R}_{I,J}(\lambda)(U^{-1}\otimes U^{-1}),$$
$$U_n^m = \delta_{n,m}q^{n^2/2}$$

that we made in constructing the L-operator factorisation (6.3.17). As explained in chapter 6, although this gives a slightly different matrix element it does not affect the Yang-Baxter equation. Therefore to uncouple the sums over m_i, we from now on work with $\overline{R}(\lambda)$.

Using the binomial theorem (B.2.1a) to sum the m_i we are left with only a sum in s. After making some simplifications we obtain

Proposition 7.1.1.

$$\frac{\left(\overline{R}(\lambda)\cdot {}_q e_{\overline{z_1}}\, {}_q e_{\overline{z_2}}\right)(z_1,z_2)}{{}_qE(z_1,\overline{z_1})\,{}_qE(z_2,\overline{z_2})} = q^{2IJ}\frac{(-z_1\overline{z_1}q^{1-I-J};q^2)_I(-z_2\overline{z_2}q^{1-I-J};q^2)_J}{(-z_1\overline{z_1}q^{1-I};q^2)_I(-z_2\overline{z_2}q^{1-J};q^2)_J} \times {}_4\phi_3\left(\begin{matrix} q^{-2I} & q^{-2J} & -z_1\overline{z_2}\lambda q^{1-I} & -z_2\overline{z_1}\lambda q^{1-J} \\ & \lambda^2 q^{2-I-J} & -z_1\overline{z_1}q^{1-I-J} & -z_2\overline{z_2}q^{1-I-J}\end{matrix}\middle| q^2,q^2\right) \tag{7.1.2}$$

which is a q-deformation of (7.0.8) which we call $\hat{\mathbb{R}}_q(z_1,z_2;\overline{z_1},\overline{z_2})$.

Because the right hand side is a balanced and terminating ${}_4\phi_3$ hypergeometric series and $I,J\in\mathbb{Z}^+$ the expression has many equivalent forms generated by the Sears' transforms (B.2.13) and (B.2.14). Some of these are symmetries of $\hat{\mathbb{R}}_q$, and they can be generated from three elementary ones:

$$\hat{\mathbb{R}}_q(z_1,z_2;\overline{z_1},\overline{z_2}) = \hat{\mathbb{R}}_q(-z_1,-z_2;-\overline{z_1},-\overline{z_2}), \tag{7.1.3a}$$

$$\hat{\mathbb{R}}_q(z_1,z_2;\overline{z_1},\overline{z_2}) = \hat{\mathbb{R}}_q(\overline{z_1}^{-1},\overline{z_2}^{-1},z_1^{-1},z_2^{-1}), \tag{7.1.3b}$$

$$\hat{\mathbb{R}}_q(\lambda|z_1,z_2;\overline{z_1},\overline{z_2}) = \frac{q^{-IJ}}{B_q(\lambda;I,J)}\hat{\mathbb{R}}_q(\lambda^{-1}q^{-1}|z_1,-\overline{z_2}^{-1};\overline{z_1},-z_2^{-1}) \tag{7.1.3c}$$

where $B_q(\lambda;I,J)$ is given by (7.0.24).

Chapter 8

A stochastic R-matrix

In this chapter we consider the stochastic nature of the R-matrices constructed in this thesis. It turns out that with a small modification to the phase terms in (3.3.24) the columns of the R-matrix related to symmetric tensor representations of $U_q(\widehat{sl_n})$ sum to unity. The positivity of each element follows from the positivity of the 3-dimensional Boltzmann weights in the construction (3.3.4) and condition (3.3.17).

In the recent work of [21; 23] the $U_q(\widehat{sl_2})$ R-matrix at a special value of λ was considered and shown to describe the probability distribution function of a four parameter family of integrable stochastic zero range processes involving interacting particles on a line. For certain values of these parameters it is possible to recover many known integrable models in the Kardar-Parisi-Zhang universality class. A $U_q(\widehat{sl_n})$ generalisation $\Phi_q(\gamma|\beta;\lambda,\mu)$ was constructed in [20] and is essentially the reductions (3.5.1), (3.5.2) of our R-matrix. This is an important link in light of the factorisation (3.5.7), where we have reconstructed the full R-matrix from its values at two special points. It appears that the R-matrix constructed with the 3D approach is somehow a combination of two integrable stochastic models. In this chapter we will write the combination down explicitly and use it to prove the stochasticity of the general R-matrix.

In the second part of this chapter we will propose a possible generalisation for all highest weight representations in the case of sl_3. It comes from the factor $\mathbb{R}^{(1)}(\lambda)$ (5.6.15) constructed from Derkachov's factorisation method. It is true that this factor in the case of sl_2 is stochastic and it appears that it may also be the case for sl_3, where we have observed that it satisfies the sum-to-unity property. We cannot give a proof at this time, and we only present it as a possible generalisation. We wonder if it continues to hold for sl_n, and if so, what kind of stochastic processes can be modeled by it.

8.0.1 Symmetric tensor representations of $U_q(\widehat{sl_n})$

Let us define another R-matrix $S_{I,J}(\lambda)$ by

$$[S_{I,J}(\lambda)]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \rho_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} \left[R_{I,J}^{(n)}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'}, \quad \boldsymbol{i},\boldsymbol{j},\boldsymbol{i}',\boldsymbol{j}' \in \mathbb{Z}_+^{n-1}, \tag{8.0.1}$$

with

$$\rho_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = q^{[\boldsymbol{i},\boldsymbol{j}]-[\boldsymbol{i}',\boldsymbol{j}']+\sum\limits_{1\le k<l\le n}(j_k i_l - i'_k j'_l)} = q^{(\boldsymbol{i},\boldsymbol{j})-(\boldsymbol{i}',\boldsymbol{j}')-J|\boldsymbol{i}|+I|\boldsymbol{j}'|+\sum\limits_{1\le k<l<n}(j_k i_l - i'_k j'_l)}. \tag{8.0.2}$$

In [20] $S_{I,J}(z)$ was given in terms of $R_{I,J}^K(z)$ with $z=\lambda^{-2}$. Here we defined $S_{I,J}(\lambda)$ in terms of $R_{I,J}^{(n)}(\lambda)$ using the relation (3.6.10). Using quantum group arguments it was shown in [20] that (8.0.1) solves the Yang-Baxter equation and satisfies the stochasticity condition

$$\sum_{\boldsymbol{i},\boldsymbol{j}} [S_{I,J}(\lambda)]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = 1. \tag{8.0.3}$$

We can now give the direct proof of (8.0.3) using the explicit formula (3.3.24) for the R-matrix.

To do that we find it convenient to follow notations of [20]. Introduce the function

$$\Phi_q(\gamma|\beta;\lambda,\mu) = q^{\xi}\left(\frac{\mu}{\lambda}\right)^{|\gamma|} \frac{(\lambda;q)_{|\gamma|}(\frac{\mu}{\lambda};q)_{|\beta|-|\gamma|}}{(\mu;q)_{|\beta|}} \prod_{s=1}^{n-1} \begin{bmatrix}\beta_s\\ \gamma_s\end{bmatrix}_q, \tag{8.0.4}$$

$$\xi = \sum_{1\le l<k<n} (\beta_l-\gamma_l)\gamma_k, \tag{8.0.5}$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{Z}_+^{n-1}$, and $\lambda,\mu\in\mathbb{C}$. This function satisfies the sum-to-unity rule

$$\sum_{\boldsymbol{i}} \Phi_q(\boldsymbol{i}|\boldsymbol{j};\lambda,\mu) = 1. \tag{8.0.6}$$

Note that the sum in (8.0.6) is always finite since the summand is equal to zero unless $\mathbf{0}\le\boldsymbol{i}\le\boldsymbol{j}$, i.e. $0\le i_s\le j_s$ for all $s=1,\dots,n-1$. The relation (8.0.6) can be easily proved by induction in n, see [20] for details.

Using these definitions and the expansion (3.5.7) $R^{(n)}_{I,J}(\lambda)$ can be expressed as

$$\left[R^{(n)}_{I,J}(\lambda)\right]^{\boldsymbol{i'},\boldsymbol{j'}}_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i'}+\boldsymbol{j'}}\, q^{(\boldsymbol{i'},\boldsymbol{j'})-(\boldsymbol{i},\boldsymbol{j})-J(|\boldsymbol{i}|+|\boldsymbol{j}|)+I(|\boldsymbol{j}|-|\boldsymbol{j'}|)+\sum_{k>l}(i_k j_l+j'_k i'_l-2i_l j_k)} \times$$
$$\sum_{\boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i}+\boldsymbol{j}} \Phi_{q^2}(\boldsymbol{j}|\boldsymbol{m}; q^{-2J}, \lambda^{-2}q^{-I-J})\Phi_{q^2}(\boldsymbol{n}|\boldsymbol{j'}; \lambda^2 q^{-I-J}, q^{-2J}) q^{2|\boldsymbol{n}|J+\sum_{k>l}2(j_k n_l - j_l n_k)}. \quad (8.0.7)$$

where we imply that the sum is taken over $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^{n-1}_{+}$ with the sum $\boldsymbol{m}+\boldsymbol{n} = \boldsymbol{i}+\boldsymbol{j}$ fixed.

Using this presentation of $R^{(n)}_{I,J}(\lambda)$ in terms of Φ we can rewrite the expression for matrix elements of $S_{I,J}(\lambda)$ as

$$[S_{I,J}(\lambda)]^{\boldsymbol{i'},\boldsymbol{j'}}_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i'}+\boldsymbol{j'}} \left(\lambda^2 q^{I+J}\right)^{|\boldsymbol{j}|} \times$$
$$\sum_{\boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i}+\boldsymbol{j}} \Phi_{q^2}(\boldsymbol{j}|\boldsymbol{m}; q^{-2J}, \lambda^{-2}q^{-I-J})\Phi_{q^2}(\boldsymbol{n}|\boldsymbol{j'}; \lambda^2 q^{-I-J}, q^{-2J}) q^{-2|\boldsymbol{m}|J+\sum_{k>l}2(j_l m_k - j_k m_l)}. \quad (8.0.8)$$

This expression can be simplified using symmetries of the function Φ. Substituting the explicit form of Φ (8.0.4) one can easily check that

$$\Phi_q(\boldsymbol{m}-\boldsymbol{j}|\boldsymbol{m}, \mu/\lambda, \mu) = \Phi_q(\boldsymbol{j}|\boldsymbol{m}, \lambda, \mu)\, q^{\sum\limits_{k<l}(j_k m_l - m_k j_l)} \mu^{-|\boldsymbol{j}|}\lambda^{|\boldsymbol{m}|}. \quad (8.0.9)$$

Then we can rewrite (8.0.8) in a factorized form

Proposition 8.0.1.

$$[S_{I,J}(\lambda)]^{\boldsymbol{i'},\boldsymbol{j'}}_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i'}+\boldsymbol{j'}} \sum_{\boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i}+\boldsymbol{j}} \Phi_{q^2}\left(\boldsymbol{m}-\boldsymbol{j}|\boldsymbol{m}; \frac{q^{J-I}}{\lambda^2}, \frac{q^{-I-J}}{\lambda^2}\right) \Phi_{q^2}\left(\boldsymbol{n}|\boldsymbol{j'}; \frac{\lambda^2}{q^{I+J}}, q^{-2J}\right). \quad (8.0.10)$$

Now the relation (8.0.3) becomes trivial. Indeed, for fixed $\boldsymbol{i'}, \boldsymbol{j'}$ we have

$$\sum_{\boldsymbol{i},\boldsymbol{j}} [S_{I,J}(\lambda)]^{\boldsymbol{i'},\boldsymbol{j'}}_{\boldsymbol{i},\boldsymbol{j}} = \sum_{\substack{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{i'}+\boldsymbol{j'} \\ \boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i'}+\boldsymbol{j'}}} \Phi_{q^2}\left(\boldsymbol{m}-\boldsymbol{j}|\boldsymbol{m}; \frac{q^{J-I}}{\lambda^2}, \frac{q^{-I-J}}{\lambda^2}\right) \Phi_{q^2}\left(\boldsymbol{n}|\boldsymbol{j'}; \frac{\lambda^2}{q^{I+J}}, q^{-2J}\right) =$$
$$= \sum_{\boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i'}+\boldsymbol{j'}} \Phi_{q^2}\left(\boldsymbol{n}|\boldsymbol{j'}; \frac{\lambda^2}{q^{I+J}}, q^{-2J}\right) \sum_{\boldsymbol{i}+\boldsymbol{j}=\boldsymbol{m}+\boldsymbol{n}} \Phi_{q^2}\left(\boldsymbol{m}-\boldsymbol{j}|\boldsymbol{m}; \frac{q^{J-I}}{\lambda^2}, \frac{q^{-I-J}}{\lambda^2}\right)$$
$$= \sum_{\boldsymbol{m}+\boldsymbol{n}=\boldsymbol{i'}+\boldsymbol{j'}} \Phi_{q^2}\left(\boldsymbol{n}|\boldsymbol{j'}; \frac{\lambda^2}{q^{I+J}}, q^{-2J}\right) = 1, \quad (8.0.11)$$

where we used twice the relation (8.0.6).

Setting $\lambda = q^{\pm(J-I)/2}$ in (8.0.8) and using relations

$$\Phi_q(\boldsymbol{i}|\boldsymbol{j};1,\mu) = \delta_{\boldsymbol{i},0}, \quad \Phi_q(\boldsymbol{i}|\boldsymbol{j};\mu,\mu) = \delta_{\boldsymbol{i},\boldsymbol{j}}, \tag{8.0.12}$$

we obtain two nontrivial degenerations of the R-matrix $S_{I,J}(\lambda)$

$$\left[S^{(1)}(\mu,\nu)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} \equiv \left[S_{I,J}(q^{(J-I)/2})\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'}\Phi_{q^2}\left(\boldsymbol{i}|\boldsymbol{j}';\mu,\nu\right) \tag{8.0.13}$$

and

$$\left[S^{(2)}(\mu,\nu)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} \equiv \left[S_{I,J}(q^{(I-J)/2})\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \delta_{\boldsymbol{i}+\boldsymbol{j},\boldsymbol{i}'+\boldsymbol{j}'}\Phi_{q^2}\left(\boldsymbol{j}|\boldsymbol{i}';\nu,\mu\right)\mu^{-|\boldsymbol{j}|}\nu^{|\boldsymbol{i}'|}q^{2\sum\limits_{k<l}(j_k i'_l - i'_k j_l)}, \tag{8.0.14}$$

where $\mu = q^{-2I}, \nu = q^{-2J}$ play the role of (complex) spectral parameters. Similar formulas for the R-matrix $R_{I,J}^{(n)}(\lambda)$ have been already obtained in (3.5.1)-(3.5.2).

We can now derive the formula for the L-operator corresponding to the stochastic R-matrix (8.0.1). First, we choose $I = 1$, $J \in \mathbb{Z}_+$ and $\boldsymbol{i} = \boldsymbol{e}_\alpha$, $\boldsymbol{i}' = \boldsymbol{e}_\beta$. Let us notice that the exponent of the q-factor in (8.0.1) can be compactly written in n-component notations as follows:

$$[\boldsymbol{i},\boldsymbol{j}] - [\boldsymbol{i}',\boldsymbol{j}'] + \sum_{1\le k<l\le n}(j_k i_l - i'_k j'_l) = \sum_{k=1}^{\alpha} j_k - \sum_{k=\beta}^{n} j'_k. \tag{8.0.15}$$

In particular, for $J = 1$ it simplifies to

$$\rho_{\boldsymbol{e}_\alpha,\boldsymbol{e}_\gamma}^{\boldsymbol{e}_\beta,\boldsymbol{e}_\delta} = q^{\delta_{\alpha,\beta}\,\epsilon_{\alpha,\gamma}}, \tag{8.0.16}$$

for $\boldsymbol{e}_\alpha + \boldsymbol{e}_\gamma = \boldsymbol{e}_\beta + \boldsymbol{e}_\delta$ with

$$\epsilon_{\alpha,\gamma} = \begin{cases} 1, & \alpha > \gamma, \\ 0, & \alpha = \gamma, \\ -1, & \alpha < \gamma. \end{cases} \tag{8.0.17}$$

Let us comment that (8.0.16) corresponds to the case

$$\rho_{\alpha,\gamma} = q^{\epsilon_{\alpha,\gamma}} \tag{8.0.18}$$

in (3.6.15). It was shown in [65] that (8.0.18) leads to a factorization of the L-operators at roots of unity. It would be interesting to understand further the relation between stochasticity and factorization of L-operators.

We can now derive the formula for the L-operators corresponding to the stochastic R-matrix $S_{I,J}(\lambda)$. Using (8.0.15) for general J and (3.6.5) one can write it in terms of Weil generators (3.6.21) similar to (3.6.23) in a compact form

$$L^S_{\alpha,\beta}(\mu) = \mu^{\epsilon_{\alpha,\beta}}\, X_\alpha^{-1} X_\beta\, \left[\mu^{\delta_{\alpha,\beta}} Z_\alpha\right] \prod_{\gamma=1}^{n} Z_\gamma^{\epsilon_{\alpha,\gamma}}. \tag{8.0.19}$$

It satisfies the algebra

$$S_{1,2}(\mu/\nu) L_1^S(\mu) \otimes L_2^S(\nu) = L_2^S(\nu) \otimes L_1^S(\mu) S_{1,2}(\mu/\nu), \tag{8.0.20}$$

where $S_{1,2}(\lambda)$ is given by (8.0.1) with $I = J = 1$. This L-operator was first obtained in [65] in a slightly different form. The root of unity condition $q^N = 1$ used there does not affect the local structure of the algebra (8.0.20).

Choosing the eigenvalue of the operator $\mathcal{Z}$ in (3.6.24) as C one can rewrite (8.0.19) as

$$L_{\alpha,\beta}(\mu) = \mu^{\epsilon_{\alpha,\beta}+\delta_{\alpha,\beta}} C\, X_\alpha^{-1} X_\beta\, (1 - \mu^{-2\delta_{\alpha,\beta}} Z_\alpha^{-2}) \prod_{s=\alpha+1}^{n} Z_s^{-2}. \tag{8.0.21}$$

This L-operator contains two complex parameters μ and $C = q^J$, where J can be identified with the weight of representation. As well known one can multiply the L-operator (8.0.21) by arbitrary complex parameters a_i ("horizontal" fields) from the left without affecting the Yang-Baxter relation. It immediately follows from the property

$$[\boldsymbol{A}_1 \otimes \boldsymbol{A}_2, S_{1,2}(\mu)] = 0, \tag{8.0.22}$$

where $\boldsymbol{A} = \{a_1, \ldots, a_n\}$.

We can also remove one pair of Weyl operators Z_1, X_1 by setting

$$Z_1 = C \prod_{i=2}^{n} Z_i^{-1}, \quad X_1 \equiv 1. \tag{8.0.23}$$

Let us introduce another set of operators

$$k_i = q^{-2} Z_{i+1}^{-2}, \quad \phi_i^+ = X_{i+1}^{-1}(1 - Z_{i+1}^{-2}), \quad \phi_i = X_{i+1}, \quad i = 1, \ldots, n-1 \tag{8.0.24}$$

instead of Z_i, X_i, $i = 2, \ldots, n$. Each set k_i, ϕ_i, ϕ_i^+ forms a q-oscillator algebra

$$\phi k = q^2 k \phi, \quad \phi^+ k = q^{-2} k \phi^+, \quad \phi\phi^+ - q^2 \phi^+ \phi = 1 - q^2. \tag{8.0.25}$$

If we now choose

$$a_1 = -\mu C, \quad a_i = \frac{\mu v}{C} q^{2(i-1-n)}, \quad i = 2, \dots, n \tag{8.0.26}$$

and make a change of variables

$$C = \frac{\sqrt{uv}}{q^n}, \quad \mu = q\sqrt{\frac{x}{v}}, \tag{8.0.27}$$

then we get exactly the L-operator from the recent paper by Garbali, De Gier and Wheeler [74]

$$L^{GGW}_{i,j}(x) = a_i L_{i,j}(\mu), \tag{8.0.28}$$

with $L_{ij}(\mu)$ given by (8.0.21). Therefore, the L-operator $L^{GGW}(x)$ corresponds to the standard $U_q(\widehat{sl_n})$ L-operator for symmetric representations in the presence of twist and "horizontal" fields.

8.0.2 Towards an arbitrary highest weight generalisation

Let us introduce a function Ψ by

$$\begin{aligned}
&\Psi(\boldsymbol{j}, \boldsymbol{j}'|\lambda_1, \lambda_2, \lambda_3) = \\
&= \frac{(-1)^{j_2+j_3'} j_1'!(j_2'+j_3')!(\lambda_2-1)_{j_2'}(1-\lambda_1)_{j_3}(\lambda_1)_{j_1-j_3}(\lambda_2-\lambda_1)_{j_1'+j_2'-j_1+j_3}}{j_1!j_2!j_3!(j_2'+j_3'-j_2-j_3)!(j_1'-j_1-j_3'+j_3)!(\lambda_2)_{j_1'+j_2'}(\lambda_2-\lambda_3-1)_{j_2'+j_3'}} \\
&\times \sum_{m,l} \frac{(-j_3, \lambda_1+j_1-j_3, \lambda_1-\lambda_3-1)_m(-j_2', 1-\lambda_2-j_1'-j_2', 2+\lambda_3-\lambda_2-j_2'-j_3')_l}{m!l!(-j_2-j_3, \lambda_1-j_3)_m(2-\lambda_2-j_2', -j_2'-j_3')_l} \\
&\times \frac{(-j_2-j_3)_{l+m}}{(1+\lambda_1-\lambda_2+j_1-j_3-j_1'-j_2')_{l+m}}
\end{aligned} \tag{8.0.29}$$

with summation indices $0 \le m, l \le j_2 + j_3$. It appears to satisfy the sum-to-unity rule

Proposition 8.0.2.

$$\sum_{\boldsymbol{j}} \Psi(\boldsymbol{j}, \boldsymbol{j}'|\lambda_1, \lambda_2, \lambda_3) = 1. \tag{8.0.30}$$

The sum in $\boldsymbol{j} = \{j_1, j_2, j_3\}$ is finite and it is easy to see that it terminates due to denominator factorials such that

$$0 \le j_1 \le j_1' + j_2' \quad 0 \le j_2 \le j_2' + j_3', \quad 0 \le j_3 \le j_2' + j_3'.$$

As mentioned in the introduction, Ψ is just the function for elements of $\mathbb{R}^{(1)}(\lambda)$ (5.6.15) with relabeled variables. The relation is

$$\left[\mathbb{R}^{(1)}(\lambda)\right]_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{i}',\boldsymbol{j}'} = \overline{\delta}_{\boldsymbol{i}}^{\boldsymbol{j}} \delta_{i_3,i_3'} \Psi(\boldsymbol{j},\boldsymbol{j}'|\lambda - J_1, -J_1, J_2). \tag{8.0.31}$$

For special values of Ψ we recover the rational limit of Φ_q for $n = 3$. In particular

$$\Psi(\boldsymbol{j},\boldsymbol{j}'|\lambda,\mu,0) = \lim_{q\to 1} \Phi_q(\gamma|\beta;\lambda,\mu) \tag{8.0.32}$$

with $\boldsymbol{j} = \{j_1, j_2, 0\} = \gamma$ and $\boldsymbol{j}' = \{j_1', j_2', 0\} = \beta$. Therefore the rational limit of Φ_q is, up to $n = 3$, contained within a larger function related to arbitrary highest weight representations of sl_3. We expect this to be true for all n, and perhaps even in the trigonometric case, but as of yet we have not constructed such a function.

Chapter 9

Conclusion

In this thesis we have considered solutions to the Yang-Baxter equation related to sl_n. Such solutions are known as R-matrices, and we have considered five different approaches to constructing them on the level of matrix elements, thereby indirectly evaluating the universal R-matrix (2.4.50) for the symmetric tensor representations. In the process we have uncovered and clarified in explicit detail some of the structure of these solutions, and structure which is not at all obvious from the q-exponential form of the universal solution. It is clear that the R-matrix is a composite object, summarised in one way by the formula

$$R(\lambda) = \Phi * \Phi \tag{9.0.1}$$

detailed in (8.0.10), (8.0.4), and (3.5.6)-(3.5.8). We think this is a remarkable relation, particularly for the matrix elements. Elements of the left hand side are expressible by the $(n-1)$-fold sum multivariable basic hypergeometric series (3.3.24), but the elements of Φ functions on the right hand side are simple binomial products. Moreover, the functions Φ are objects of great interest at the time of writing, with a recent explosion of research in their application to integrable near-equilibrium stochastic models. Equation (9.0.1) is another step towards clarifying the relation these models have with the lattice models of equilibrium statistical mechanics that we gave a brief overview of in chapter 2, and where the notion of R-matrix and Yang-Baxter integrability was first formulated. The function Φ by itself defines an R-matrix (8.0.13), (8.0.14), and therefore the factorisation of $R(\lambda)$ is also a composition of 'smaller' R-matrices.

In chapter 3 we studied the 3-dimensional structure of the $U_q(\widehat{sl_n})$ R-matrix in the case of symmetric tensor representations. The formula (3.3.4) can be considered as a trace of an operator product, graphically represented in Figure 3.4. Though the notion of a projecting a 3-dimensional model is not new, the link between quantum groups and solutions to the tetrahedron equation is a relatively recent development. It is perhaps with this approach we can claim the most success so far. We are confident that the formula (3.3.24) obtained from this approach is the simplest, and neatest formula one could find for the R-matrix elements for these representations. We come to this conclusion after our investigation of other methods, where even in the case of sl_2 much more work was required in obtaining the single summation formula (3.8.3). We also mention that (3.3.24) makes it very easy to derive the reductions (3.5.1), (3.5.2) from which we derived (9.0.1). As argued in section 3.5, they exist within (3.3.24) as a single summand which is easy to extract. In other presentations, they exist in a more 'dispersed form', and it takes more technical arguments with summation formulae to extract their formula.

We also note that (3.3.24) is the most general formula for the R-matrix elements currently available in the literature. In fact, for the R-matrix with arbitrary weights for both representations, it may be the only known formula. Most presentations usually take a fundamental representation in one or both spaces, but in our formula the weights are arbitrary complex parameters. We matched our formula to these special cases in section 3.6, and we showed that with the right choice of normalisation a formula for the standard $U_q(\widehat{sl_n})$ L-operator (3.6.5) can be given. Actually, (3.6.1) gives us an entire family of higher spin L-operators for $I \in \mathbb{Z}^+, J \in \mathbb{C}$, without the need for fusion procedures.

We also remark that the symmetries of the R-matrix discussed in section 3.4 are much easier to see from the 3-dimensional approach compared to other methods. They manifest as higher dimensional symmetries of the 3-dimensional Boltzmann weights projected out in one direction. It is interesting to see how transformations of the internal degrees of freedom inside the R-matrix yield invariant transformations of the overall operator. A similar idea also happens in chapter 5 for the factorisation (5.1.8) where the identity (5.3.4e) allows us to reorder the factors without affecting the R-matrix. The symmetries usually give different formulae but with the same output, and so they describe a group of transformations of the hypergeometric series we have derived. For the special case of $U_q(\widehat{sl_2})$ the formula (3.3.30) these symmetries are contained within the transformation group of Sears' transforms (B.2.13), (B.2.14) and probably the symmetries of higher rank R-matrices are special cases of a more general yet to be discovered identity.

In chapter 4 we used the representation theory of quantum groups directly to construct the R-matrix elements in the case of $U_q(\widehat{sl_2})$. We exploit the fact that a tensor product of irreducible finite-dimensional representations of the quantum group is semisimple. The intertwining relation (4.1.7a) allows us to write down the R-matrix as a sum of projection operators onto these components, acting on each subspace by an eigenvalue (4.1.12) determined by the Jimbo equations. This approach, besides the universal R-matrix, perhaps most clearly demonstrates the link between quantum groups (in their modern form) and the Yang-Baxter equation, because the solution is essentially a special combination of Clebsch-Gordan coefficients - pure representation theoretical objects. One can therefore say that so is the R-matrix, and it is obvious that it is another step in complexity in the heirarchy of special functions that appear in representation theory. This fact is pretty well known, in particular that the R-matrix is closely related to $6j$-symbols which arise when considering the decomposition of the tensor product of three irreducible representations.

The main challenge therefore is constructing the Clebsch-Gordan coefficients. In the case of $U_q(\widehat{sl_2})$ which we have considered the formula is quite well known, but it appears to be a highly non-trivial and unsolved problem for quantum groups in general. Even when it is known, there are some serious limitations of the method, because it does not work when the representation space contains multiplicities. This rules out many possible solutions we could construct. In the simplest case, the solution (4.2.9) is the least elegant of all the solutions constructed in this thesis. It is a triple summation involving somewhat complicated hypergeometric series and it is only valid for integral weights. The tools we used to construct this formula have been known for quite some time, yet (at least to our knowledge) it does not appear anywhere in the literature. Perhaps until (3.8.3) appeared in [10] the general higher spin R-matrix was considered too complicated to write down. The main result of this chapter was showing that the formula is not that complicated at all, and is actually equivalent to the q-Racah single sum polynomial representation (3.8.3) obtained from the 3-dimensional approach, thereby proving these are equivalent constructions.

One surprise was the intermediate step (4.2.14) in our calculations, which allows us to write the R-matrix as (4.2.20). It is not of the same form as (4.1.9) and each factor is given by a binomial product expression and the overall expression a double summation. We think the expression has a 'nice' form, but currently we do not know if it has any significance.

In chapter 5 we considered the factorisation approach due to Derkachov and Manashov. The techniques have so far only been developed for the rational sl_n R-matrices but a major appeal is the construction of the general R-matrix for any highest weight representation. The components in (5.3.1) we have constructed in the case of sl_2 and sl_3

and we have shown how they relate to our factorisation (3.5.6) and (3.7.18). It appears that they are essentially the same idea, although we have not yet checked the reduced RLL-relations (5.1.9) for the q-deformation. Our factorisation is more general in the sense that it holds for trigonometric R-matrices, but (5.3.1) is more general in the sense that it holds for more representations. We have also shown that in the restriction to symmetric tensor representations, the equation (5.3.1) simplifies significantly into just two symmetrical factors and this occurs not by some factors reducing trivially but by a non-trivial operator composition.

We also considered and verified some interesting identities (5.3.4) satisfied by the 'building block' operators $\mathbb{R}^{(i)}(\lambda)$. For sl_2 they correspond to some well known identities in the theory of hypergeometric series. We were able to construct these expressions for sl_3 as well but the identities are far more complicated multivariable hypergeometric series and do not seem to be like anything we have encountered in the literature. This might indicate some new identities but we have not yet paid serious attention to refining and making sense of them so we did not include them in this thesis. All we can say is that they exist and we used them to check that our expressions for the sl_3 factors $\mathbb{R}^{(i)}(\lambda)$ are correct.

Our initial interest in the method of chapter 5 was to find a better formula than (3.3.24) in the case of sl_3. It looked promising after considering the sl_2 case where the single sum formula (3.7.7) emerged immediately, but for higher rank algebras the process seems to be far less efficient for the same representations. For example, if we want to calculate the sl_3 case for symmetric tensor representations (like we did in section 5.6.4) we need to construct the factors for arbitrary representations and then restrict afterwards. This is because of the way the factors intertwine the representations (as given in (5.3.2)), so there are always contributions to the operator composition from the entire space. The initial formula obtained by putting these factors together is a quadruple sum but it can be shown that this is equivalent to the double sum in (3.7.4).

In chapter 6 we considered another factorisation (6.0.6) of the R-matrix by Chicherin and Derkachov. At the time of writing, this factorisation is known for sl_2 as well as the quantum and elliptic doubles in the literature. In this thesis we have generalised the result to $U_q(\widehat{sl_2})$ (6.3.17). We did this by first constructing matrix elements for their R-matrix via its action on the monomial basis of Verma modules. We then proved it is the same as the R-matrix 3.8.5 and used its q-deformation (3.8.3) to deform their R-matrix and also its factorisation. It differs in some fundamental ways to that of chapter 5 (and section 3.5) even though they are derived from similar means. First, the factorisation in this chapter is for the action of the R-matrix on the first/auxiliary space in $V_I \otimes V_J$, and more specifically is a restriction of this space to the finite dimensional subspace valid for the weight $I \in \mathbb{Z}^+$. The action on the second space, parameterised by $J \in \mathbb{C}$ takes the form of differential operator-valued matrix entries. Notably, it is a square factorisation and also factorises the higher spin L-operator. In contrast, the factorisation of chapter 5 is a factorisation of the action on Verma modules, and in trying to restrict to finite-dimensional representations we get a rectangular factorisation.

We also considered two different actions (6.0.3) and (6.0.8) for their R-matrix and gave another proof that they are the same. The first one was derived from results obtained by considering the action on the representation space of principal series representations. The second is based off an ansatz by observing the L-operator factorisation for a few special cases. We gave another proof that these actions are equal. We found a q-deformation of (6.0.3) whose expansion gives the q-deformed R-matrix (6.4.4) of (6.4.1).

In chapter 7 we considered the $SU(2)$-invariant R-matrix for finite-dimensional irreducible unitary representations. Using Sklyanin's observation that the generating function of this operator on the coherent state is particularly simple (a function of a single variable), we constructed the matrix elements explicitly and showed that it precisely the R-matrix of (3.7.7). Using our quantum deformation we deformed Sklyanin's R-matrix. We were curious to see if the generating function of this q-lift is also a function of a single variable. Unfortunately, that does not appear to be the case, and the polynomial dependence on ξ (7.0.9) as a ${}_2F_1$ function becomes a more complicated ${}_4\phi_3$ function.

Finally in chapter 8 we linked our R-matrix to recent developments in integrable stochastic zero-range processes. We rewrote our factorisation (3.5.6) as (9.0.1) and used it to prove that the general R-matrix can be stochastic. We also proposed a possible generalisation in the case of sl_3, but we cannot yet give a formal proof of this. In any case it seems that there is a close connection between the six-vertex like models and stochastic systems of interacting particles and the development of this is ongoing.

9.1 Future Work

We consider the work done so far to be merely the tip of an iceberg in a large ocean of unknowns, and there many open questions that we are interested in investigating all related to R-matrices and their applications.

With regards to the 3-dimensional approach of chapter 3, it is possible to construct R-matrices related to other algebras. This is achievable with the same construction but with a modification to the boundary conditions [7]. It would be interesting to see what representations we get, and if the R-matrix can be factorised in these cases. There is also a stochastic interpretation to consider, and there are probably new models we could construct.

An obvious generalisation of (3.3.24) would be a formula that holds for all highest weight representations. Currently, there are not any methods to construct this besides solving recurrence relations directly or evaluating the universal R-matrix. The factorisation approach of chapter 5 can in principle construct the matrix elements in the rational limit and maybe this would be the best approach to take. What we know is that computational challenge is enormous and it is not obvious that it would yield a useful formula. For example, we investigated a formula for the full sl_3 R-matrix, built from the factors $\mathbb{R}^{(i)}(\lambda)$ of 5.6. The formula contains 12 summations and it is not clear if any of these can be summed up. It is much more complicated than the double sum (3.3.24) even though it contains only two extra parameters. With more research we could probably write it in a more appealing form and generalise further to sl_n. A general formula would be useful for applications, especially to the recent developments in stochastic models. If not the general R-matrix, then the factors $\mathbb{R}^{(i)}$ which as we have seen for rank 2,3 can be stochastic and more general than the currently known sl_n related models.

Another direction of research are the Q-operators. Examples of Q-operators are fairly easy to construct from our results, and we would like to go through the details and compare them to what is known in the literature. $U_q(\widehat{sl_2})$ Q-operators have already been constructed using the formula (3.8.3) in [75] and we could probably repeat this construction using (3.3.24) to get at least some higher rank Q-operators. There is an alternative construction that follows from [66; 14; 27; 47] where the factors $\mathbb{R}^{(i)}$ are essentially local operators from which Q-operators can be constructed. It would be interesting to compare these Q-operators with those obtained by Mangazeev, whose local form is given by a ${}_3\phi_2$ basic hypergeometric series. These are obviously different objects to the product formulae 5.5.10 but maybe a (local) relation exists. Certainly a global relation must exist and in the rational case has been given in [76], but we do not know how this lifts to the trigonometric case. We also mention that the factorisation (3.5.6) bears a resemblance to the fundamental fusion relation [44; 45] between transfer matrices and Q-operators, though the former is local and the latter is global. It is probable that the factors (3.5.5) can be interpreted as a local Q-operator building block since they are a q-lift of 5.5.10.

We suspect that the other factorisation considered in chapter 6 can be generalised to higher rank algebras. It is currently only known for $U_q(\widehat{sl_2})$ but we do not see why it should not hold for symmetric tensor representations of $U_q(\widehat{sl_n})$. We already found a formula for the higher spin L-operator in this case and gave a formula when $I = 1$ in (3.6.5). Given the similarities of the higher rank L-operators it would not be surprising if they have a similar factorisation.

Finally we would like to comment on elliptic R-matrices. These are a further deformation of the trigonometric R-matrices considered in this thesis. Their algebras, elliptic quantum groups/Sklyanin algebras [77; 78], are two parameter deformations of classical Lie algebras. We have given some consideration to these already. It would be (very) nice to find the elliptic deformation of all the results in this thesis, supposing they exist. Hopf algebra structure is not known in this case and the representation theory is not very well understood. Furthermore the simplest R-matrix in this category, of the eight-vertex model, does not have the usual ice-rule. The representation spaces of Sklyanin algebras are typically characterised as polynomials in theta functions and this alone presents many technical hurdles. However, Q-operators are well known and perhaps a deformation could be obtained by considering these in light of the factorisations in chapters 3 and 5. An elliptic deformation of the factorisation in chapter 5 has been found in [67] for sl_2 in terms of more elementary 'parameter permutation' operators. We also mention that the factorisation [12] in chapter 6 has a generalisation [79] to the Sklyanin algebra in terms of elliptic gamma functions and finite difference operators acting on spaces of even theta functions. It may be possible to extract matrix elements from these constructions.

Appendix A

Hypergeometric Series

A.1 Definitions

The classical Pochhammer symbol $(a)_n$ is defined for any integer n by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\dots(a+n-1)\ , & n \geq 0 \\ \dfrac{1}{(a-1)(a-2)\dots(a+n)}, & n < 0. \end{cases} \tag{A.1.1}$$

Given the standard definition of a classical hypergeometric series

$${}_rF_s\left(\begin{matrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right) = \sum_{m=0}^{n} \frac{z^m}{m!} \frac{(a_1, \dots, a_r)_m}{(b_1, \dots, b_s)_m}, \tag{A.1.2}$$

with

$$(a_1, \dots, a_r)_m = (a_1)_m \dots (a_r)_m, \tag{A.1.3}$$

we define for any positive integer n a regularised terminating version ${}_r\overline{F}_s$ of (A.1.2) by

$$\begin{aligned} {}_r\overline{F}_s\left(\begin{matrix} -n; & a_1 & a_2 & \dots & a_{r-1} \\ & b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right) &= (b_1, \dots, b_s)_n \cdot {}_rF_s\left(\begin{matrix} -n & a_1 & \dots & a_{r-1} \\ b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right) \\ = \sum_{m=0}^{n} z^m \frac{(-n)_m}{m!} (a_1, \dots, a_{r-1})_m (b_1+m, \dots, b_s+m)_{n-m}. \end{aligned} \tag{A.1.4}$$

Our definition (A.1.4) of the regularised series slightly differs from the standard one, which we denote by ${}_r\widetilde{F}_s$

$${}_r\widetilde{F}_s\left(\begin{matrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right) = \frac{1}{\Gamma(b_1)\dots\Gamma(b_s)}\, {}_rF_s\left(\begin{matrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right)$$

$$= \sum_{m=0}^{n} \frac{z^m}{m!} \frac{(a_1, \ldots, a_r)_m}{\Gamma(b_1 + m) \ldots \Gamma(b_s + m)}. \tag{A.1.5}$$

The main reason for introducing ${}_r\overline{F}_s$ is that the RHS of (A.1.4) is a polynomial in all a's and b's.

For compactness of our expressions, we sometimes use the notation

$$(a_1, a_2, \ldots, a_n)! = a_1! a_2! \ldots a_n! \tag{A.1.6}$$

A.2 Identities

Now we describe transformations and summation formulas used in the main text. Where applicable we use a regularised version of the identity.

We start with the simplest identity, the well-known binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \binom{n}{k} = \frac{(-n)_k (-1)^k}{k!}. \tag{A.2.1}$$

The next level in the chain of complexity is the Gauss summation formula (eq. (46), Sec. 2.8 in [80])

$${}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0 \tag{A.2.2}$$

and Pfaff-Saalschütz's Theorem (eq. (3), Sec. 4.4 in [80]) for balanced and terminating ${}_3F_2$ series

$${}_3F_2\left(\begin{matrix} -n & a & b \\ c & d & \end{matrix} \middle| 1 \right) = \frac{(c-a, c-b)_n}{(c, c-a-b)_n}, \tag{A.2.3}$$

where $n \in \mathbb{Z}_{\geq 0}$ and $c + d - a - b + n = 1$.

For general ${}_3F_2$ series there is the so called Thomae's theorem (eq. (11) in [81]). In modern notations it is given in Sec. 3.2 of [82], but we rewrite it in a regularised form

$${}_3\tilde{F}_2\left(\begin{matrix} a & b & c \\ e & f & \end{matrix} \middle| 1 \right) = \frac{\Gamma(s)}{\Gamma(a)} \, {}_3\tilde{F}_2\left(\begin{matrix} e-a & f-a & s \\ s+b & s+c & \end{matrix} \middle| 1 \right), \tag{A.2.4}$$

where $s = e + f - a - b - c$ and $\text{Re}(a),\ \text{Re}(s) > 0$.

There is also a second Thomae's theorem, (eq. (3.1.1) in [83]). We write it using our regularisation (A.1.4) as

$$ {}_3\overline{F}_2\left(\begin{matrix} -n; & a & b \\ & c & d \end{matrix}\middle| 1\right) = (-1)^n {}_3\overline{F}_2\left(\begin{matrix} -n; & c-a & b \\ & c & 1+b-d-n \end{matrix}\middle| 1\right). \tag{A.2.5} $$

When the ${}_3F_2$ is also terminating there are identities which are special cases of (A.2.7). A particular example which we use can be found in [84], eq. (2.5.11)

$$ {}_3F_2\left(\begin{matrix} -n & a & b \\ & e & f \end{matrix}\middle| 1\right) = \frac{(e-a, f-a)_n}{(e,f)_n} {}_3F_2\left(\begin{matrix} -n & 1-s & a \\ & 1+a-e-n & 1+a-f-n \end{matrix}\middle| 1\right), \tag{A.2.6} $$

where $s = e + f - a - b + n$.

As mentioned, this is a special case of Whipple's formula (see (2.10.5) in [83]) for balanced and terminating ${}_4F_3$ series. It is given by

$$ {}_4F_3\left(\begin{matrix} -n & a & b & c \\ d & e & f & \end{matrix}\middle| 1\right) = \frac{(e-a, f-a)_n}{(e,f)_n} {}_4F_3\left(\begin{matrix} -n & a & d-b & d-c \\ & d & 1+a-e-n & 1+a-f-n \end{matrix}\middle| 1\right) \tag{A.2.7} $$

where n is a non-negative integer and

$$ d + e + f - a - b - c + n = 1. \tag{A.2.8} $$

For general hypergeometric series there exists the identity which appears as (18) in [85]. It can also be derived from (III.30) of [83] by taking the limit $q \to 1$. It is given by

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a & b & b_1+m_1 & \dots & b_r+m_r \\ & b+c+1 & b_1 & \dots & b_r \end{matrix}\middle| 1\right) = \frac{\Gamma(b+c+1)\Gamma(1-a)}{\Gamma(b+1-a)\Gamma(c+1)} $$
$$ \times \prod_{i=1}^{r} \frac{(b_i-b)_{m_i}}{(b_i)_{m_i}} {}_{r+2}F_{r+1}\left(\begin{matrix} -c & b & b+1-b_1 & \dots & b+1-b_r \\ & b+1-a & 1+b-m_1-b_1 & \dots & 1+b-m_r-b_r \end{matrix}\middle| 1\right), \tag{A.2.9} $$

where $m, m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Re}(c-a) \geq m_1 + \cdots + m_r - 1$.

The identity (A.2.9) was derived as a generalisation of an important class of ${}_{r+2}F_{r+1}$ summation formulae called the Karlsson-Minton sums. They are useful in analysis of series because they give regimes where a complicated expression may be summable or non-trivially zero. It is given by

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a & b & b_1+m_1 & \dots & b_r+m_r \\ & b+1 & b_1 & \dots & b_r \end{matrix}\middle| 1\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \prod_{i=1}^{r} \frac{(b_i-b)_{m_i}}{(b_i)_{m_i}} \tag{A.2.10} $$

where $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$ and a is a negative integer or non-negative provided the series converges. An important corollary is

$$ {}_{r+1}F_{r}\left(\begin{matrix} a & b_1+m_1 & \dots & b_r+m_r \\ & b_1 & \dots & b_r \end{matrix}\middle| 1\right) = 0 \quad \text{if} \quad \mathrm{Re}(-a) > m_1 + \cdots + m_r. \tag{A.2.11} $$

All the identities presented so far are well known and can be found in most textbooks on the subject. In this thesis we frequently have to combine these identities to get the R-matrix elements into the desired form. The following subsections are dedicated to the specific identities we use and derive by composing the above classic identities together.

A.2.1 Chapter 5 identities

Here we give the summation identity used twice in transforming (5.6.26) to (5.6.31). It is given by

$$ {}_3\overline{F}_2\left(\begin{matrix} -n_1; & a & 1 \\ & 1-n_2 & d \end{matrix}\middle| 1\right) = (-1)^{n_2} n_1! (d-a)_{n_1-n_2} (a)_{n_2}, \tag{A.2.12} $$

and holds only when n_1, n_2 are non-negative integers and $n_1 \geq n_2$ to ensure the left hand side terminates. The identity is derived by starting from the left hand side and applying Thomae's theorem (A.2.4), after which the series is balanced and terminating and can be summed up using Pfaff-Saalschütz theorem (A.2.3) to obtain the right hand side.

A.2.2 Chapter 6 identities

Applying (A.2.7) twice and rewriting the result in terms of regularised series (A.1.4) we obtain the neat result

$$ {}_4\overline{F}_3\left(\begin{matrix} -n; & a & b & c \\ & d & e & f \end{matrix}\middle| 1\right) = {}_4\overline{F}_3\left(\begin{matrix} -n; & 1-e-n & d-a & f-a \\ & 1-a-n & 1+b-e-n & 1+c-e-n \end{matrix}\middle| 1\right) \tag{A.2.13} $$

provided (A.2.8) is satisfied. As follows from (A.1.4) the formula (A.2.13) is a polynomial identity in 5 independent variables a, b, c, d, e for any non-negative integer n.

A.2.3 Chapter 7 identities

Applying a regularised version of (A.2.6) with $f = 1 + m - n$, $m, n \in \mathbb{Z}_{\geq 0}$ and using Thomae's theorem (A.2.4) twice one can derive the identity

$$ \begin{aligned} {}_3\overline{F}_2\left(\begin{matrix} -n; & a & b \\ & e & 1+m-n \end{matrix}\middle| 1\right) &= (-1)^{n+m}(a,b)_{n-m} \\ &\times {}_3\overline{F}_2\left(\begin{matrix} -m; & a-m+n & a+b-e-m \\ & a-m & 1+a-e-m \end{matrix}\middle| 1\right). \end{aligned} \tag{A.2.14} $$

Let us set $a = 1$ and $e = 1 - n$, $n \in \mathbb{Z}_{\geq 0}$ in (A.2.4). Then the series in the right-hand side terminates and can be summed up using the Pfaff-Saalschütz Theorem (A.2.3). After simple transformations the result reads

$$ {}_3\widetilde{F}_2\left(\begin{matrix} 1 & a & b \\ 1-n & c & \end{matrix}\middle| 1\right) = \frac{(a,b)_n\Gamma(c-a-b-n)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.2.15} $$

Appendix B

Basic Hypergeometric Series

B.1 Definitions

Here we list standard definitions in q-series which we need in the main text

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i), \tag{B.1.1}$$

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{B.1.2}$$

$$(a_1,\dots,a_m;q)_n := \prod_{i=1}^{m}(a_i;q)_n, \tag{B.1.3}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}, \tag{B.1.4}$$

$$[x] := x - x^{-1}, \tag{B.1.5}$$

$$[x]_q := \frac{q^x - q^{-x}}{q-q^{-1}}, \tag{B.1.6}$$

$$[a_1,\dots,a_n]_q := \prod_{s=1}^{n}[a_i]_q, \tag{B.1.7}$$

$$[n]_q! = \prod_{s=1}^{n}[n-i+1]_q, \quad n \in \mathbb{Z}_{\geq 0}. \tag{B.1.8}$$

In the q-deformed case we will only use terminating analogs of (A.1.2-A.1.4) with $s = r-1$. We define terminating basic hypergeometric series by ${}_{r+1}\phi_r$

$${}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}\,\middle|\, q, z\right) = \sum_{k=0}^{n} z^k \frac{(q^{-n};q)_k}{(q;q)_k} \prod_{s=1}^{r} \frac{(a_s;q)_k}{(b_s;q)_k}. \tag{B.1.9}$$

We also introduce a regularised version of (B.1.9) as the analog of (A.1.4)

$$
\begin{aligned}
{}_{r+1}\overline{\phi}_r \left(\begin{matrix} q^{-n}; a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \middle| q, z \right) &= \prod_{s=1}^{r} (b_s; q)_n \cdot {}_{r+1}\phi_r \left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \middle| q, z \right) \\
&= \sum_{k=0}^{n} z^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{s=1}^{r} (a_s; q)_k (b_s q^k; q)_{n-k} \,. \qquad \text{(B.1.10)}
\end{aligned}
$$

B.2 Identities

In this section we give transformation and summation formulas for basic hypergeometric series that we use in the main text. Some of these identities, as they are written, are non-terminating and converge under the assumption that $|q| < 1$ and $|z| < 1$ where z is the phase argument of the series. However, in this thesis we only use these identities on terminating series and therefore convergence is never an issue.

Like the classical case, we start with the simplest identity, the q-binomial theorem

$$
{}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix} \middle| q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty}, \qquad \text{(B.2.1a)}
$$

$$
(y; q)_n = \sum_{k=0}^{n} {}_q D_n^k y^k, \quad {}_q D_n^k = \frac{q^{nk} (q^{-n}; q)_k}{(q; q)_k} = \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k(k-1)}. \qquad \text{(B.2.1b)}
$$

We also make frequent use of the q-Chu-Vandermonde sum (equation (1.5.2) in [83])

$$
{}_2\phi_1 \left(\begin{matrix} q^{-n} & b \\ & c \end{matrix} \middle| q, \frac{cq^n}{b} \right) = \frac{(c/b; q)_n}{(c; q)_n} \qquad \text{(B.2.2)}
$$

and the q-Pfaff-Saalschütz sum ((1.7.2) in [83])

$$
{}_3\phi_2 \left(\begin{matrix} q^{-n} & b & c \\ & d & e \end{matrix} \middle| q, q \right) = \frac{(d/b, d/c; q)_n}{(d, d/bc; q)_n} \qquad \text{(B.2.3)}
$$

where $de = q^{1-n} bc$.

Also important in analysing the Clebsch-Gordan coefficients is the q-deformation of the Karlsson-Minton sum (A.2.11)

$$
{}_3\phi_2 \left(\begin{matrix} a & b_1 q^{m_1} & b_2 q^{m_2} \\ & b_1 & b_2 \end{matrix} \middle| q, a^{-1} q^{-m_1 - m_2} \right) = 0 \qquad \text{(B.2.4)}
$$

for m_1, m_2 arbitrary non-negative integers. When $a = q^{-n},\ n \in \mathbb{Z}_{\geq 0}$ we also require that $n - m_1 - m_2 > 0$.

The final summation formula we need is for a very-well-poised ${}_6\phi_5$ series;

$$
{}_6\phi_5 \left(\begin{matrix} a & qa^{1/2} & -qa^{1/2} & b & c & q^{-n} \\ & a^{1/2} & -a^{1/2} & aq/b & aq/c & aq^{n+1} \end{matrix} \middle| q, \tfrac{aq^{n+1}}{bc} \right) = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \tag{B.2.5}
$$

We also make frequent use of transformation formulas for basic hypergeometric series. In generating the 3-dimensional model symmetries (3.2.14) we use the Heine's transformations of ${}_2\phi_1$ series ((III.1)-(III-3) in [83])

$$
{}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| q, z \right) = \frac{(az, b; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix} \middle| q, b \right) =
$$

$$
= \frac{(c/b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} abz/c, b \\ bz \end{matrix} \middle| q, c/b \right) = \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} c/a, c/b \\ c \end{matrix} \middle| q, abz/c \right). \tag{B.2.6}
$$

We also use a collection of transformations of ${}_3\phi_2$ series ((III.9)-(III.13) of [83])

$$
{}_3\phi_2 \left(\begin{matrix} a & b & c \\ & d & e \end{matrix} \middle| q, de/abc \right) = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a & d/b & d/c \\ & d & de/bc \end{matrix} \middle| q, e/a \right), \tag{B.2.7}
$$

$$
= \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty} {}_3\phi_2 \left(\begin{matrix} d/b & e/b & de/abc \\ & de/ab & de/bc \end{matrix} \middle| q, b \right), \tag{B.2.8}
$$

$$
{}_3\phi_2 \left(\begin{matrix} q^{-n} & b & c \\ & d & e \end{matrix} \middle| q, q \right) = \frac{(de/bc; q)_n}{(e; q)_n} \left(\frac{bc}{d} \right)^n {}_3\phi_2 \left(\begin{matrix} q^{-n} & d/b & d/c \\ & d & de/bc \end{matrix} \middle| q, q \right), \tag{B.2.9}
$$

$$
= \frac{(e/c; q)_n}{(e; q)_n} c^n {}_3\phi_2 \left(\begin{matrix} q^{-n} & c & d/b \\ & d & q^{1-n}/e \end{matrix} \middle| q, bq/e \right), \tag{B.2.10}
$$

$$
{}_3\phi_2 \left(\begin{matrix} q^{-n} & b & c \\ & d & e \end{matrix} \middle| q, \frac{deq^n}{bc} \right) = \frac{(e/c; q)_n}{(e; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n} & c & d/b \\ & d & q^{1-n}/e \end{matrix} \middle| q, q \right), \tag{B.2.11}
$$

where $n \in \mathbb{Z}_{\geq 0}$. Given non-negative integers m, m_1 we also have ((III.29) of [83])

$$
{}_3\phi_2 \left(\begin{matrix} a & b & b_1 q^{m_1} \\ & bq^{1+m} & b_1 \end{matrix} \middle| q, a^{-1} q^{m+1-m_1} \right) =
$$

$$
= \frac{(q, bq/a; q)_\infty (bq; q)_m (b_1/b; q)_{m_1}}{(bq, q/a; q)_\infty (q; q)_m (b_1; q)_{m_1}} {}_3\phi_2 \left(\begin{matrix} q^{-m} & b & bq/b_1 \\ & bq/a & bq^{1-m_1}/b_1 \end{matrix} \middle| q, q \right). \tag{B.2.12}
$$

For ${}_4\phi_3$ series the q-deformation of the Whipple transform (A.2.7) is the well-known Sears' transform. They are two transforms ((III.15)-(III.16) from [83]) that we write in terms of regularised functions ${}_4\overline{\phi}_3$ as

$$ {}_4\overline{\phi}_3\left(\begin{matrix} q^{-n}; & a & b & c \\ & d & e & f \end{matrix}\middle| q,q\right) = (ef/a)^n q^{n(n-1)}\,{}_4\overline{\phi}_3\left(\begin{matrix} q^{-n}; & a & d/b & d/c \\ & d & aq^{1-n}/e & aq^{1-n}/f \end{matrix}\middle| q,q\right) \tag{B.2.13}$$

$$ = (ad)^n q^{n(n-1)}\,{}_4\overline{\phi}_3\left(\begin{matrix} q^{-n}; & e/a & f/a & ef/abc \\ & ef/ab & ef/ac & q^{1-n}/a \end{matrix}\middle| q,q\right), \tag{B.2.14}$$

where $n \in \mathbb{Z}_{\geq 0}$ and $def = abcq^{1-n}$.

Finally, we give Jackson's transformation ((III.8) from [83])

$$ {}_2\overline{\phi}_1\left(\begin{matrix} q^{-n} & b \\ & c \end{matrix}\middle| q,z\right) = (c/b;q)_n b^n\,{}_3\phi_1\left(\begin{matrix} q^{-n} & b & q/z \\ & bq^{1-n}/c & \end{matrix}\middle| q,z/c\right), \tag{B.2.15}$$

which is important in the derivation of the R-operator factorisation (6.3.17). In the following sections we will give particular identities that we have derived by combining the standard ones in the literature presented in this section.

B.2.1 Chapter 4 identities

In transforming (4.2.12) to (4.2.13) we are required to transform two ${}_3\phi_2$ hypergeometric series. Here we will show the steps for deriving the identities we use in making the transformation. The hypergeometric series we are required to transform are special because all of their arguments are of the form $q^m,\ m \in \mathbb{Z}$. This can introduce poles to the expression and make the identities we use undefined for certain arguments. To avoid this, we use regularised versions of the identities which we obtain by multiplying each side of the identity by the appropriate factors which cancel the poles. We also remind the reader that in what follows $0 \leq i, i' \leq I,\ 0 \leq j, j' \leq J$ with $I, J \in \mathbb{Z}_{\geq 0}$ and $0 \leq r \leq \min(I, J, i+j, I+J-i-j)$.

For the transformation of the first series the first is to apply the identity (B.2.7) with

$$ a = q^{-2i-2j+2r}, \quad b = q^{-2i}, \quad c = q^{2-2i+2I}, $$

$$d = q^{2-2i+2r}, \quad e = q^{-2i-2J+2r},$$

to get

$$
\begin{aligned}
&{}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2i}; & q^{-2i+2I+2} & q^{-2i-2j+2r} \\ & q^{-2i+2r+2} & q^{-2i-2J+2r} \end{matrix} \middle| q^2, q^{2i-2I+2j-2J+2r} \right) = \\
&= \frac{q^{(i-I+r)(i-I+2j-2J+r-1)}(q^2;q^2)_r(q^2;q^2)_{J-j}}{(-1)^{i+r+I}(q^2;q^2)_{I-i}(q^2;q^2)_{I+J-i-j-r}} \\
&\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{2r-2I}; & q^{2i-2j+2r} & q^{2+2r} \\ & q^{2-2i+2r} & q^{-2I-2J+4r} \end{matrix} \middle| q^2, q^{2j-2J} \right).
\end{aligned}
$$

Next, we apply (B.2.12) to the hypergeometric series on the right hand side with

$$
\begin{aligned}
&m = I - i, \quad m_1 = I + J - r + 1, \quad a = q^{-2i-2j+2r}, \\
&b = q^{-2I+2r}, \quad b_1 = q^{-2I-2J+4r},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&{}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2i}; & q^{-2i+2I+2} & q^{-2i-2j+2r} \\ & q^{-2i+2r+2} & q^{-2i-2J+2r} \end{matrix} \middle| q^2, q^{2i-2I+2j-2J+2r} \right) = \\
&= \frac{q^{i^2+2(I-r)(J-j+r)+i(2j+4r-2J-4I-1)}(q^2;q^2)_r(q^2;q^2)_{J-j}}{(-1)^i(q^2;q^2)_{I-i}(q^2;q^2)_{I+J-i-j-r}} \\
&\times {}_3\overline{\phi}_2 \left(\begin{matrix} q^{2r-2I}; & q^{2i-2I} & q^{2+2J-2r} \\ & q^{-2I} & q^{2+2i+2j-2I} \end{matrix} \middle| q^2, q^2 \right).
\end{aligned}
\tag{B.2.16}
$$

We note the change of regularisation with respect to different integers.

The transformation of the second series is performed by first applying the identity (B.2.11) with

$$
\begin{aligned}
&n = j'', \quad b = q^{2+2J-2j'}, \quad c = q^{-2i'-2j'+2r}, \\
&d = q^{2-2j'+2r}, \quad e = q^{-2j'+2r-2I},
\end{aligned}
$$

to get

$$
\begin{aligned}
&{}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j'}; & q^{-2i'-2j'+2r} & q^{2J-2j'+2} \\ & q^{2r-2j'+2} & q^{2I-2j'+2r} \end{matrix} \middle| q^2, q^{2i'-2I+2j'-2J+2r} \right) = \\
&= (-1)^{j'} q^{2j'(j'-2I-1)} {}_3\overline{\phi}_2 \left(\begin{matrix} q^{-2j'}; & q^{-2i'-2j'+2r} & q^{2r-2J} \\ & q^{2-2j'+2r} & q^{2-2i'-2j'+2I} \end{matrix} \middle| q^2, q^2 \right).
\end{aligned}
$$

(B.2.17)

Next we apply the identity (B.2.10) to the right hand side with

$$n = j', \quad b = q^{-2i'-2j'+2r}, \quad c = q^{-2J+2r},$$
$$d = q^{2-2i'-2j'+2I}, \quad e = q^{2-2j'+2r},$$

to get the identity

$$
\begin{aligned}
&{}_3\overline{\phi}_2\left(\begin{matrix} q^{-2j'}; & q^{-2i'-2j'+2r} & q^{2J-2j'+2} \\ & q^{2r-2j'+2} & q^{-2I-2j'+2r}\end{matrix}\,\middle|\, q^2, q^{2i'-2I+2j'-2J+2r}\right) \\
&= q^{2j'(r+i'-I)}\,{}_3\overline{\phi}_2\left(\begin{matrix} q^{-2j'}; & q^{2+2I-2r} & q^{2r-2J} \\ & q^{-2J} & q^{2-2i'-2j'+2I}\end{matrix}\,\middle|\, q^2, q^{-2i'}\right). \qquad \text{(B.2.18)}
\end{aligned}
$$

In transforming (4.2.15) to (4.2.16) we use an identity that is constructed from identities (in order) (B.2.10) and (B.2.8). In particular,

$$
\begin{aligned}
{}_3\overline{\phi}_2\left(\begin{matrix} q^{-2n}; & b & c \\ & d & q^{2-2n+2m}\end{matrix}\,\middle|\, q^2, q^2\right) &= (-1)^{n-m}\frac{(b;q^2)_{n-m}(c;q^2)_{n-m}(d;q^2)_n(q^2/b;q^2)_m}{b^{-m}q^{(n-m)(n-m-1)}(d;q^2)_{n-m}} \\
&\times {}_3\phi_2\left(\begin{matrix} q^{-2m} & d/c & bq^{2n-2m} \\ & dq^{2n-2m} & bq^{-2m}\end{matrix}\,\middle|\, q^2, c\right). \qquad \text{(B.2.19)}
\end{aligned}
$$

Due to the regularisation (B.1.10) used the hypergeometric series in the identities above have no poles. The q-pochhammer expression $(q^2;q^2)_n$ is not defined for integer $n < 0$ and so product terms outside the summation may not be defined. For example, the identity (B.2.16) is not defined when $j > J$ but because we work with finite-dimensional representations this inequality is never satisfied.

B.2.2 Chapter 6 identities

Applying (B.2.9) twice we derive a regularised terminating q-analog of Thomae's theorem (A.2.4)

$$
\begin{aligned}
{}_3\overline{\phi}_2\left(\begin{matrix} q^{-2n}; & b & c \\ & bq^{-2m} & e\end{matrix}\,\middle|\, q^2, q^2\right) &= (b/e)^m c^n (b;q^2)_{n-m}(e/c;q^2)_{n-m} \qquad \text{(B.2.20)} \\
&\times {}_3\overline{\phi}_2\left(\begin{matrix} q^{-2m}; & e/b & eq^{2n-2m}/c \\ & q^{-2m}e/c & eq^{2n-2m}\end{matrix}\,\middle|\, q^2, q^2\right),
\end{aligned}
$$

where $n, m \in \mathbb{Z}_{\geq 0}$. Let us note there are other non-terminating q-analogs of (A.2.4) like (III.10) in [83]. However, in all such identities the argument z is a rational function of parameters. For our purposes we need $z = q^2$ on both sides. To our knowledge (B.2.20) is the only such identity which produces a terminating version of (A.2.4) when $q \to 1$.

B.2.3 Chapter 7 identities

Setting $n \geq m$ and $b = q^2$ in (B.2.20) we can calculate the right-hand side using (B.2.3) because it is now a balanced sum. As a result we get a summation formula

$$ {}_3\overline{\Phi}_2\left(\begin{matrix} q^{-2n}; & q^2 & c \\ & q^{2-2m} & e \end{matrix}\,\middle|\, q^2, q^2\right) = c^n (e/c; q^2)_{n-m} (q^2; q^2)_n (q^{2-2m}/c; q^2)_m, \quad n \geq m. \tag{B.2.21} $$

Applying (B.2.9-B.2.11) one after another with $d = bq^{-2m}$, $m \in \mathbb{Z}_{\geq 0}$ and using regularised functions we get the formula

$$ {}_3\overline{\Phi}_2\left(\begin{matrix} q^{-2n}; & b & c \\ & bq^{-2m} & e \end{matrix}\,\middle|\, q^2, q^2\right) = e^m c^{n-m} q^{-2m} (b; q^2)_{n-m} (e/c; q^2)_{n-m} \tag{B.2.22} $$

$$ \times\, {}_3\overline{\Phi}_2\left(\begin{matrix} q^{-2m}; & bq^{2n-2m} & q^2 c/e \\ & q^{2+2n-2m} & q^{2-2m} b/e \end{matrix}\,\middle|\, q^2, q^2\right), \tag{B.2.23} $$

which is the q-analogue of (A.2.14) with the left-hand side and right-hand side interchanged.

Bibliography

[1] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "Exercises with the universal R-matrix," *J.Phys. A* **43** (2010) 415208, arXiv:1004.5342 [math-ph]. (cited on pages 1 and 32)

[2] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "On the universal R-matrix for the Izergin-Korepin model," *J. Phys. A* **44** no. 35, (2011) 355202, 25. (cited on pages 1 and 32)

[3] V. N. Tolstoy and S. M. Khoroshkin, "Universal R-matrix for quantized nontwisted affine Lie algebras," *Funktsional. Anal. i Prilozhen.* **26** no. 1, (1992) 69–71. (cited on pages 1, 25, and 33)

[4] Y. G. Stroganov, "A new calculation method for partition functions in some lattice models," *Phys. Lett. A* **74** no. 1-2, (1979) 116–118. (cited on page 1)

[5] P. P. Kulish, N. Y. Reshetikhin, and E. K. Sklyanin, "Yang-Baxter equations and representation theory. I," *Lett. Math. Phys.* **5** no. 5, (1981) 393–403. (cited on pages 1 and 24)

[6] A. N. Kirillov and N. Y. Reshetikhin, "Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum," *J. Phys. A* **20** no. 6, (1987) 1565–1585. (cited on page 1)

[7] V. V. Bazhanov and S. M. Sergeev, "Zamolodchikov's tetrahedron equation and hidden structure of quantum groups," *J.Phys.* **A39** (2006) 3295–3310, arXiv:hep-th/0509181 [hep-th]. (cited on pages 1, 42, 48, and 162)

[8] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Quantum geometry of 3-dimensional lattices," *J.Stat.Mech.* **0807** (2008) P07004, arXiv:0801.0129 [hep-th]. (cited on page 1)

[9] V. V. Mangazeev, V. V. Bazhanov, and S. M. Sergeev, "An integrable 3D lattice model with positive Boltzmann weights," *J.Phys.* **A46** (2013) 465206, arXiv:1308.4773 [math-ph]. (cited on pages 1, 3, 39, 44, 45, 48, and 52)

[10] V. V. Mangazeev, "On the Yang-Baxter equation for the six-vertex model," *Nuclear Phys. B* **882** no. 1, (2014) 70–96, `arXiv:1401.6494 [math-ph]`. (cited on pages 1, 2, 48, 52, 54, 56, 68, and 159)

[11] D. Chicherin, S. E. Derkachov, and V. P. Spiridonov, "From Principal Series to Finite-Dimensional Solutions of the Yang-Baxter Equation," *SIGMA* **12** (2016) 028, `arXiv:1411.7595 [math-ph]`. (cited on pages 1, 121, and 122)

[12] D. Chicherin and S. Derkachov, "Matrix factorization for solutions of the Yang-Baxter equation," `arXiv:1502.07923 [math-ph]`. (cited on pages 1, 4, 101, 121, 122, 123, 126, 128, 133, and 163)

[13] S. E. Derkachov, "Factorization of the R-matrix I.," *Journal of Mathematical Sciences* **143** (2007) 2773–2790, `arXiv:math/0503396v1 [math.QA]`. (cited on pages 1, 83, 84, 85, and 121)

[14] S. E. Derkachov and A. N. Manashov, "Factorization of R-matrix and Baxter Q-operators for generic sl(n) spin chains," *J. Phys. A: Gen. Physics* **42** (2009) 35, `arXiv:0809.2050v2 [nlin.SI]`. (cited on pages 1, 4, 5, 87, 88, 91, 92, 101, 102, and 162)

[15] E. K. Sklyanin, "Classical limits of SU(2)-invariant solutions of the Yang-Baxter equation," *J. Soviet Math.* **40** (1988) 93–107. (cited on pages 1 and 139)

[16] M. Jimbo, "A q-difference analogue of $U(G)$ and the Yang-Baxter equation," *Lett. Math. Phys.* **10** no. 1, (1985) 63–69. (cited on pages 1, 24, 27, 71, and 73)

[17] R. Zhang, M. Gould, and A. Bracken, "From representations of the braid group to solutions of the Yang-Baxter equation," *Nucl.Phys. B* **354** (1991) 625–652. (cited on pages 1, 71, and 73)

[18] G. W. Delius, M. D. Gould, and Y.-Z. Zhang, "On the construction of trigonometric solutions of the Yang-Baxter equation," *Nucl.Phys.* **B432** (1994) 377–403, `arXiv:hep-th/9405030 [hep-th]`. (cited on pages 1, 71, and 73)

[19] R. Askey and J. Wilson, "A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols," *SIAM J. Math. Anal.* **10** no. 5, (1979) 1008–1016. (cited on page 2)

[20] A. Kuniba, V. V. Mangazeev, S. Maruyama, and M. Okado, "Stochastic R matrix for $U_q(A_n^{(1)})$," `arXiv:1604.08304 [math.QA]`. (cited on pages 3, 53, 56, 58, 59, 60, 149, and 150)

[21] A. Borodin, I. Corwin, and V. Gorin, "Stochastic six-vertex model," *Duke Math. J.* **165** no. 3, (2016) 563–624. (cited on pages 3 and 149)

[22] A. Borodin, "On a family of symmetric rational functions," `arXiv:1410.0976 [math.CO]`. (cited on pages 3, 56, and 102)

[23] I. Corwin and L. Petrov, "Stochastic higher spin vertex models on the line," *Comm. Math. Phys.* **343** no. 2, (2016) 651–700. (cited on pages 3 and 149)

[24] A. Borodin and L. Petrov, "Higher spin six vertex model and symmetric rational functions," `arXiv:1601.05770 [math.PR]`. (cited on page 3)

[25] S. E. Derkachov and A. N. Manashov, "A general solution of the Yang-Baxter equation with the symmetry group $\mathrm{SL}(n,\mathbb{C})$," *Algebra i Analiz* **21** no. 4, (2009) 1–94. (cited on pages 4 and 58)

[26] S. E. Derkachov, D. Karakhanyan, and R. Kirschner, "Baxter Q-operators of the XXZ chain and R-matrix factorization," *Nucl.Phys.* **B738** (2006) 368–390, `arXiv:hep-th/0511024 [hep-th]`. (cited on page 5)

[27] D. Chicherin, S. Derkachov, D. Karakhanyan, and R. Kirschner, "Baxter operators for arbitrary spin," *Nucl. Phys.* **B854** (2012) 393–432, `arXiv:1106.4991 [hep-th]`. (cited on pages 5, 21, and 162)

[28] G. Bosnjak and V. Mangazeev, "Construction of R-matrices for symmetric tensor representations related to $U_q(\widehat{sl_n})$," *J. Phys. A: Math. Theor.* **49** (2016) , `arXiv:1607.07968 [math-ph]`. (cited on page 6)

[29] G. Bosnjak and V. Mangazeev *In preparation* (2017) . (cited on page 6)

[30] L. Euler, "De moto rectilineo trium corporum se mutuo attrahentium," *Novo Comm. Acad. Sci. Imp. Petrop* **11** (1767) 144–151. (cited on page 8)

[31] L. Lagrange, "Dessai sur le probléme des trois corps.," *Euvres* **6** (1772) 229–331. (cited on page 8)

[32] G. Jacobi, "Sur le mouvement d'un point et sur un cas particulier du probléme des trois corps," *Comptes Rendus,* **3** (1836) 59–61. (cited on page 8)

[33] H. Poincare, *New Methods of Celestial Mechanics (English translation).* History of Modern Physics and Astronomy. AIP-Press, 1992. (cited on page 8)

[34] L. Onsager, "Crystal statistics. I. A two-dimensional model with an order-disorder transition," *Phys. Rev.* **65** (1944) 117–149. (cited on page 12)

[35] R. J. Baxter, "Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors. II. Equivalence to a generalized ice-type lattice model. III. Eigenvectors of the transfer matrix and Hamiltonian.," *Ann. Phys.* **76** (1973) 1–24, 25–47, 48–71. (cited on page 12)

[36] L. A. Takhtajan and L. D. Faddeev, "The quantum method for the inverse problem and the XYZ Heisenberg model," *Uspekhi Mat. Nauk* **34(5)** no. 5, (1979) 13–63. (English translation: Russian Math. Surveys **34(5)** (1979) 11–68). (cited on page 12)

[37] C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction," *Phys. Rev. Lett.* **19** (1967) 1312–1315. (cited on page 12)

[38] A. B. Zamolodchikov, "Tetrahedra equations and integrable systems in three-dimensional space," *Zh. Eksp. Teor. Fiz.* **79** (1980) 641–664. [English translation: Soviet Phys. JETP **52** (1980) 325-326]. (cited on pages 12 and 39)

[39] A. B. Zamolodchikov, "Tetrahedron equations and the relativistic S-matrix of straight strings in (2+1)-dimensions," *Commun. Math. Phys.* **79** (1981) 489–505. (cited on pages 12 and 39)

[40] E. H. Lieb, "Exact solution of the problem of the entropy of two-dimensional ice," *Phys. Rev. Lett.* **18** no. 17, (Apr, 1967) 692–694. (cited on pages 13 and 17)

[41] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*. Academic, London, 1982. (cited on pages 13 and 83)

[42] H. Bethe, "On The theory Of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain," *Z. Phys.* **71** (1931) 205. (cited on page 17)

[43] R. J. Baxter, "Partition function of the eight-vertex lattice model," *Ann. Physics* **70** (1972) 193–228. (cited on pages 19 and 21)

[44] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, "Integrable structure of conformal field theory. 2. Q operator and DDV equation," *Commun.Math.Phys.* **190** (1997) 247–278, `arXiv:hep-th/9604044 [hep-th]`. (cited on pages 21, 133, and 162)

[45] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, "Integrable structure of conformal field theory. 3. The Yang-Baxter relation," *Commun.Math.Phys.* **200** (1999) 297–324, `arXiv:hep-th/9805008 [hep-th]`. (cited on pages 21, 133, and 162)

[46] S. E. Derkachov, "Baxter's Q-operator for the homogeneous XXX spin chain," *J. Phys. A* **32** no. 28, (1999) 5299–5316. (cited on page 21)

[47] D. Chicherin, S. Derkachov, D. Karakhanyan, and R. Kirschner, "Baxter operators with deformed symmetry," *Nucl.Phys.* **B868** (2013) 652–683, `arXiv:1211.2965 [math-ph]`. (cited on pages 21 and 162)

[48] L. Faddeev, E. Sklyanin, and L. Takhtajan, "The Quantum Inverse Problem Method. 1," *Theor.Math.Phys.* **40** (1980) 688–706. (cited on page 24)

[49] P. P. Kulish and N. J. Reshetikhin, "Quantum linear problem for the sine-Gordon equation and higher representations," *Zap. Nauchn. Sem. LOMI,* **101** (1981) 101–110. English translation: J Math Sci (1983) **23**:4, 2435-2441. (cited on page 24)

[50] V. Drinfeld, "Hopf algebras and the quantum Yang-Baxter equation," *Sov.Math.Dokl.* **32** (1985) 254–258. (cited on pages 24, 25, and 27)

[51] V. G. Drinfeld, "Quantum groups," in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pp. 798–820. Amer. Math. Soc., Providence, RI, 1987. (cited on pages 24, 27, and 33)

[52] V. V. Bazhanov, "Integrable quantum systems and classical lie algebras.," *Comm. Math. Phys.* **113** no. 3, (1987) 471–503. (cited on page 24)

[53] M. Jimbo, "Quantum R matrix for the generalized Toda system," *Comm. Math. Phys.* **102** no. 4, (1986) 537–547. (cited on page 24)

[54] M. Rosso, "An analogue of P.B.W. theorem and the universal R-matrix for $U_h(sl_{n+1})$," *Comm. Math. Phys* **124** (1990) 307–318. (cited on pages 25 and 33)

[55] A. Kirillov and N. Reshetikhin, "q-Weyl group and a multiplicative formula for universal R-matrices," *Comm. Math. Phys* **134** (1990) 421–431. (cited on pages 25 and 33)

[56] S. M. Khoroshkin and V. N. Tolstoy, "Universal R-matrix for quantized (super)algebras," *Comm. Math. Phys.* **141** (1991) 599–617. (cited on page 25)

[57] S. M. Khoroshkin and V. N. Tolstoy, "The uniqueness theorem for the universal R-matrix," *Lett. Math. Phys.* **24** no. 3, (1992) 231–244. (cited on page 25)

[58] M. Jimbo, "A q-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation," *Lett. Math. Phys.* **11** no. 3, (1986) 247–252. (cited on page 34)

[59] J. Jantzen, *Lectures on Quantum Groups*. Dimacs Series in Discrete Mathematics and Theoretical Comput. American Mathematical Soc., 1996. (cited on pages 36 and 131)

[60] C. Burdik, R. King, and T. Welsh, "The explicit construction of irreducible representations of the quantum algebras $U_q(sl_n)$," *AIP conference proceedings* **589** (2001) 158–169. (cited on page 37)

[61] A. M. Povolotsky, "On the integrability of zero-range chipping models with factorized steady states," *J. Phys. A* **46** no. 46, (2013) 465205, 25. (cited on pages 56 and 102)

[62] I. V. Cherednik, "On a method of constructing factorized S-matrices in terms of elementary functions," *Teoret. Mat. Fiz.* **43** no. 1, (1980) 117–119. (cited on page 61)

[63] P. P. Kulish and E. K. Sklyanin, "Solutions of the Yang-Baxter equation," *J. Sov. Math.* **19** (1982) 1596–1620. (cited on page 61)

[64] J. H. H. Perk and C. L. Schultz, "New families of commuting transfer matrices in q-state vertex models," *Phys. Lett.* **A84** (1981) 407–410. (cited on page 61)

[65] V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev, and Y. G. Stroganov, "$(Z_N\times)^{n-1}$ generalization of the chiral Potts model," *Commun. Math. Phys.* **138** (1991) 393–408. (cited on pages 61, 152, and 153)

[66] S. E. Derkachov and A. N. Manashov, "R-matrix and Baxter Q-operators for the the Noncompact SL(N,ℂ) Invariant Spin Chain," *SIGMA* **2** (2006) 20, `arXiv:nlin/0612003`. (cited on pages 83, 86, 90, 91, 101, and 162)

[67] S. Derkachov, D. Karakhanyan, and R. Kirschner, "Yang-Baxter R-operators and parameter permutations," *Nuclear Physics B* **785** (2007) 263–285. (cited on pages 101 and 163)

[68] L. Faddeev, "Modular double of a quantum group," in *Conférence Moshé Flato 1999, Vol. I (Dijon)*, vol. 21 of *Math. Phys. Stud*, pp. 149–156. Kluwer Acad. Publ., Dordrecht, 2000. math/9912078. (cited on page 126)

[69] V. V. Bazhanov and Y. G. Stroganov, "Chiral Potts model as a descendant of the six-vertex model," *J. Statist. Phys.* **59** no. 3-4, (1990) 799–817. (cited on pages 132 and 133)

[70] V. V. Bazhanov, T. Lukowski, C. Meneghelli, and M. Staudacher, "A Shortcut to the Q-Operator," *J.Stat.Mech.* **1011** (2010) P11002, `arXiv:1005.3261 [hep-th]`. (cited on page 133)

[71] S. Khoroshkin and Z. Tsuboi, "The universal R-matrix and factorization of the L-operators related to the Baxter Q-operators," *J. Phys. A* **47** no. 19, (2014) 192003, 11. (cited on page 133)

[72] S. Bergman, *The kernel function and conformal mapping*. Mathematical Surveys V. American Mathematical Society, 531 West 116th St, New York, 1950. (cited on page 140)

[73] V. Mangazeev, "Q-operators for the XXZ model," *unpublished* . (cited on page 146)

[74] A. Garbali, J. De Gier, and M. Wheeler, "A new generalisation of Macdonald polynomials," *M. Commun. Mat* (2017) 1–32, `arXiv:1605.07200 [math-ph]`. (cited on page 154)

[75] V. V. Mangazeev, "Q-operators in the six-vertex model," *Nucl. Phys.* **B886** (2014) 166–184, `arXiv:1406.0662 [math-ph]`. (cited on page 162)

[76] D. Chicherin, S. Derkachov, D. Karakhanyan, and R. Kirschner, "Baxter operators for arbitrary spin II," *Nucl. Phys.* **B854** (2012) 433–465, `arXiv:1107.0643 [hep-th]`. (cited on page 162)

[77] E. K. Sklyanin, "Some algebraic structures connected with the Yang-Baxter equation," *Func. Anal. Appl.* **16** no. 4, (1982) 263–270. (cited on page 163)

[78] G. Felder and A. Varchenko, "Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(\mathrm{sl}_2)$," *Nuclear Phys. B* **480** no. 1-2, (1996) 485–503. (cited on page 163)

[79] D. Chicherin, S. Derkachov, and S. V.P., "New elliptic solutions of the Yang-Baxter equation," *V.P. Commun. Math. Phys.* **345** (2016) 507–543, `arXiv:1412.3383 [math-ph]`. (cited on page 163)

[80] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vol I*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. (cited on page 166)

[81] J. Thomae, "Ueber die Functionen, welche durch Reihen von der Form dargestellt werden," *J. Reine Angew. Math.* **87** (1879) 26–73. (cited on page 166)

[82] W. N. Bailey, *Generalized hypergeometric series*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 32. Stechert-Hafner, Inc., New York, 1964. (cited on page 166)

[83] G. Gasper and M. Rahman, *Basic hypergeometric series*, vol. 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second ed., 2004. (cited on pages 167, 172, 173, 174, and 177)

[84] L. J. Slater, *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966. (cited on page 167)

[85] G. Gasper, “Summation formulas for basic hypergeometic series,” *SIAM J. MATH* **12** (1981) 196–200. (cited on page 167)