Global Solutions to Fractional Programming Problem with Ratio of Nonconvex Functions

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Abstract
This paper presents a canonical dual approach for minimizing a sum of quadratic function and a ratio of nonconvex functions in $\mathbb{R}^n$. By introducing a parameter, the problem is first equivalently reformed as a nonconvex polynomial minimization with elliptic constraint. It is proved that under certain conditions, the canonical dual is a concave maximization problem in $\mathbb{R}^2$ that exhibits no duality gap. Therefore, the global optimal solution of the primal problem can be obtained by solving the canonical dual problem.

Keywords: Nonconvex fractional programming; Sum-of-ratios; Global optimization; Canonical duality theory

1. Introduction

We intend to solve the following nonconvex fractional programming problem:

\[
(P): \min \left\{ P_0(x) = f(x) + \frac{g(x)}{h(x)} : x \in \mathcal{X} \right\},
\]

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and

\[
f(x) = \frac{1}{2} x^T Q x - f^T x,
\]

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\[
g(x) = \frac{1}{2} \left( \frac{1}{2} |Bx|^2 - \lambda \right)^2 - c^T x,
\]
\[
h(x) = \frac{1}{2} x^T H x - b^T x,
\]
with \( B \in \mathbb{R}^{m \times n} \), \( Q \in \mathbb{R}^{n \times n} \) being symmetric, \( H \in \mathbb{R}^{n \times n} \) negative definite, \( \lambda \in \mathbb{R}^+ \), and \( f, c, b \in \mathbb{R}^n \) are given vectors. In this paper, the notation \(|v|\) denotes the Euclidean norm of \( v \). Assume that \( \mu_1^{-1} = h(H^{-1}b) > 0 \) and \( \delta \in (0, \mu_1^{-1}] \), the feasible domain \( \mathcal{X} \) is defined by
\[
\mathcal{X} = \{ x \in \mathbb{R}^n \mid h(x) \geq \delta > 0 \},
\]
which is a constraint of elliptic type.

Problem \((P)\) belongs to a class of “sum-of-ratios” problems that have been actively studied for several decades. The ratios often stand for efficiency measures representing performance-to-cost, profit-to-revenue or return-to-risk for numerous applications in economics, transportation science, finance, and engineering (see [1, 6, 11, 22, 24, 25, 29, 32]). The nonconvex function \( g(x) \) is the well-known double-well potential, which appears frequently in chaotic dynamics [15], phase transitions of solids [16], and large deformation mechanics [21]. Depending on the nature of each application, the functions \( f, g, h \) can be affine, convex, concave, or neither. However, even for the simplest case in which \( f, g, h \) are all affine functions, the problem \((P)\) is still a global optimization problem that may have multiple local optima [5, 28]. In particular, Freund and Jarre [13] showed that the sum-of-ratios problem \((P)\) is NP-complete when \( f, g \) are convex and \( h \) is concave.

Canonical duality theory is a powerful methodological theory which can be used for solving a large class of nonconvex/nonsmooth/discrete problems in nonlinear analysis and global optimization [14, 18, 19]. The main goal of this paper is to solve the problem \((P)\) by the canonical duality theory. In Section 2, the problem \((P)\) is first parameterized equivalently as a nonconvex polynomial minimization \((P_\mu)\) with an elliptic constraint. For each given parameter, the canonical dual problem is derived by the standard canonical dual transformation. The global optimality condition is proposed in Section 3. An example is illustrated in Section 4. Conclusion is provided in the last section.
2. Canonical Dual Problem

In order to solve the problem \((P)\), we consider the following parameterized subproblem:

\[
(P_\mu) : \min \left\{ P_\mu(x) = \frac{1}{2} x^T Q x - f^T x + \mu g(x) : x \in X_\mu \right\},
\]

where \(\mu \in [\mu_0, \delta^{-1}]\) and

\[X_\mu = \{ x \in \mathbb{R}^n \mid h(x) \geq \mu^{-1} \geq \delta > 0 \}\]

is a convex set. We immediately have the following result:

**Lemma 1.** Problem \((P)\) is equivalent to \((P_\mu)\) in the sense that

\[
\inf_{x \in \mathcal{X}} P_0(x) = \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{x \in X_\mu} P_\mu(x).
\]

**Proof.** It is easy to see that

\[
\inf_{x \in \mathcal{X}} P_0(x) = \inf_{x \in X_\mu} \left\{ \frac{1}{2} x^T Q x - f^T x + \frac{g(x)}{h(x)} \right\}
\]

\[
= \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{h(x) = \mu^{-1}} \left\{ \frac{1}{2} x^T Q x - f^T x + \frac{g(x)}{h(x)} \right\}
\]

\[
= \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{h(x) = \mu^{-1}} \left\{ \frac{1}{2} x^T Q x - f^T x + \mu g(x) \right\}
\]

\[
\geq \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{x \in X_\mu} \left\{ \frac{1}{2} x^T Q x - f^T x + \mu g(x) \right\}
\]

\[
= \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{x \in X_\mu} P_\mu(x).
\]

Conversely,

\[
\inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{x \in X_\mu} \left\{ \frac{1}{2} x^T Q x - f^T x + \mu g(x) \right\}
\]

\[
= \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{h(x) \geq \mu^{-1}} \left\{ \frac{1}{2} x^T Q x - f^T x + \mu g(x) \right\}
\]

\[
\geq \inf_{\mu \in [\mu_0, \delta^{-1}]} \inf_{h(x) \geq \mu^{-1}} \left\{ \frac{1}{2} x^T Q x - f^T x + \frac{g(x)}{h(x)} \right\} \quad \text{(since } g(x) > 0)\)
\]

\[
= \inf_{x \in \mathcal{X}} P_0(x).
\]
This completes the proof of the lemma.

Now, for any \( \mu \in [\mu_0, \delta^{-1}] \), we define

\[
G_\mu(\varsigma, \sigma) = Q + \mu \varsigma B^T B - \sigma H,
\]

where ‘\( \succ \)’ means positive definiteness of a matrix. Let \( \partial S_\mu^+ \) denote a singular hyper-surface defined by

\[
\partial S_\mu^+ = \{ (\varsigma, \sigma) \in \mathbb{R}^2 \mid \varsigma \geq -\lambda, \ \sigma \geq 0, \ G_\mu(\varsigma, \sigma) \succcurlyeq 0 \},
\]

Then, the canonical dual problem \( (P_d) \) can be proposed as the following:

\[
(P_d) : \sup \{ P_d(\varsigma, \sigma) \mid (\varsigma, \sigma) \in S_\mu^+ \}.
\]

**Theorem 1.** (Weak Duality) If there exists a global maximizer \( (\varsigma_\mu, \sigma_\mu) \) of \( P_d(\varsigma, \sigma) \) over \( S_\mu^+ \), then the vector

\[
x_\mu = G_\mu^{-1}(\varsigma_\mu, \sigma_\mu) (f + \mu c - \sigma b)
\]

is a global minimizer of \( (P_d) \) over \( X_\mu \) and

\[
P_\mu(x) \geq P_d(\varsigma, \sigma), \ \forall x \in X_\mu, \ (\varsigma, \sigma) \in S_\mu^+.
\]

**Proof.** Let \( \Lambda(\cdot) : \mathbb{R}^n \to \mathbb{R} \) be the geometrical transformation [14, 19, 20] defined by

\[
\xi = \Lambda(x) = \frac{1}{2} |Bx|^2 - \lambda
\]

and let

\[
U(\xi) = \frac{1}{2} \xi^2
\]

Then, Problem \( (P_d) \) in (2) can be written in the following form

\[
\min \left\{ P(x) = \frac{1}{2} x^T Q x - f^T x + \mu (U(\Lambda(x)) - \sigma(h(x) - \mu^{-1})) | x \in \mathbb{R}^n \right\}.
\]
Let $\varsigma$ be the dual variable of $\xi$, i.e., $\varsigma = \nabla U(\xi) = \xi$, the Legendre conjugate $U^*(\varsigma)$ can be uniquely defined by

$$U^*(\varsigma) = \text{sta}\{\xi \varsigma - U(\xi) \mid \xi \geq -\lambda\} = \frac{1}{2} \varsigma^2 \quad (14)$$

where $\varsigma \in V_0^* = \{\varsigma \in \mathbb{R} \mid \varsigma \geq -\lambda\}$.

By replacing $U(\Lambda(x))$ with $\Lambda(x)^T \varsigma - U^*(\varsigma)$ in (13), the Gao-Strang total complementary function for this fractional problem can be obtained as the following

$$\Xi(x, \varsigma, \sigma) = \frac{1}{2} x^T Q x - f^T x + \mu U(\Lambda(x)) - \sigma \left( \frac{1}{2} x^T H x - b^T x - \mu^{-1} \right)$$

$$= \frac{1}{2} x^T Q x - f^T x + \mu \left( \Lambda(x) \varsigma - U^*(\varsigma) - c^T x \right) - \sigma \left( \frac{1}{2} x^T H x - b^T x - \mu^{-1} \right)$$

$$= \frac{1}{2} x^T G_\mu(\varsigma, \sigma) x - (f + \mu c - \sigma b)^T x - \mu \varsigma - \frac{\mu}{2} \varsigma^2 + \frac{\sigma}{\mu},$$

where $G_\mu(\varsigma, \sigma)$ is defined in (4). Note that $\Xi(x, \varsigma, \sigma)$ is convex in $x \in \mathbb{R}^n$ for any given $(\varsigma, \sigma) \in S^+_\mu$ and concave in $(\varsigma, \sigma)$ for any given $x \in \mathbb{R}^n$. By the criticality condition

$$\frac{\partial \Xi}{\partial x} = G_\mu(\varsigma, \sigma)x - (f + \mu c - \sigma b) = 0 \quad (15)$$

we have $x(\varsigma, \sigma) = G_\mu^{-1}(\varsigma, \sigma)(f + \mu c - \sigma b)$, which is the global minimizer of $\Xi(x, \varsigma, \sigma)$. Moreover,

$$\min_{x \in \mathbb{R}^n} \Xi(x, \varsigma, \sigma) = \Xi(x(\varsigma, \sigma), \varsigma, \sigma)$$

$$= \frac{1}{2} x(\varsigma, \sigma)^T (G_\mu(\varsigma, \sigma)) x(\varsigma, \sigma) - (f + \mu c - \sigma b)^T x(\varsigma, \sigma) - \mu \lambda \varsigma - \frac{\mu}{2} \varsigma^2 + \frac{\sigma}{\mu}$$

$$= \frac{1}{2} x(\varsigma, \sigma)^T (f + \mu c - \sigma b) x(\varsigma, \sigma) - \mu \lambda \varsigma - \frac{\mu}{2} \varsigma^2 + \frac{\sigma}{\mu}$$

$$= -\frac{1}{2} (f + \mu c - \sigma b)^T x(\varsigma, \sigma) - \mu \lambda \varsigma - \frac{\mu}{2} \varsigma^2 + \frac{\sigma}{\mu}$$

$$= -\frac{1}{2} (f + \mu c - \sigma b)^T G_\mu^{-1}(\varsigma, \sigma)(f + \mu c - \sigma b) - \mu \lambda \varsigma - \frac{\mu}{2} \varsigma^2 + \frac{\sigma}{\mu}$$

$$= P^d(\mu, \varsigma).$$
By the assumption, \((\varsigma_\mu, \sigma_\mu)\) is a global maximizer of \(P^d_\mu(\varsigma, \sigma)\). If \((\varsigma, \sigma)\) belongs to an interior of \(S^+_\mu\), then \(\frac{\partial}{\partial \varsigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = 0\), and \(\frac{\partial}{\partial \sigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = 0\). Otherwise, we have \(\frac{\partial}{\partial \varsigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = 0\), \(\sigma_\mu = 0\), \(\frac{\partial}{\partial \sigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) \leq 0\). In either case, if we denote \(x_\mu = x(\varsigma_\mu, \sigma_\mu) = C^{-1}_\mu(\varsigma_\mu, \sigma_\mu)(f + \mu c - \sigma_\mu b)\), we have

\[
\frac{\partial}{\partial \varsigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = \mu \left( \frac{1}{2} x_\mu^T B^T B x_\mu - \lambda - \varsigma \right) = 0,
\]
\[
\frac{\partial}{\partial \sigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = \frac{1}{\mu} - x_\mu^T \left( \frac{1}{2} H x_\mu - b \right) \leq 0.
\]
That is,

\[
\varsigma = \frac{1}{2} |B x_\mu|^2 - \lambda,
\]
\[
\frac{1}{2} x_\mu^T H x_\mu - b^T x_\mu - \mu^{-1} \geq 0.
\]

Therefore, \(x_\mu \in X_\mu\), and for any \((\varsigma, \sigma) \in S^+_\mu\), we have

\[
P^d_\mu(\varsigma, \sigma) \leq P^d_\mu(\varsigma_\mu, \sigma_\mu)
= \min_{x \in X_\mu} \Xi(x, \varsigma_\mu, \sigma_\mu) = \Xi(x_\mu, \varsigma_\mu, \sigma_\mu)
= \frac{1}{2} x^T Q x - f^T x + \mu (\Lambda(x)^T \varsigma - U^*(\varsigma) - c^T x)
- \sigma \left( \frac{1}{2} x^T H x - b^T x - \mu^{-1} \right)
\leq \frac{1}{2} x^T Q x - f^T x + \mu (\Lambda(x)^T \varsigma - U^*(\varsigma) - c^T x)
= \frac{1}{2} x^T Q x - f^T x + \mu \left( \frac{1}{2} |B x|^2 - \lambda \right) - c^T x = P(\mu)(x).
\]

This completes the proof. \(\square\)

3. Sufficient Condition for Global Optimality

In this section we prove that canonical dual problem is a concave maximization problem. And we derive global optimality condition based on canonical duality theory.

Lemma 2. For any \(\mu \in [\mu_0, \delta^{-1}]\), the canonical dual function \(P^d_\mu(\varsigma, \varsigma)\) is a two-dimensional concave function over \(S^+_\mu\).
Proof. Notice that the Hessian Matrix of the dual objective function is

\[ \nabla^2 P_d(\sigma, \varsigma) = S = \begin{pmatrix} H_{\sigma^2} & H_{\sigma\varsigma} \\
H_{\varsigma\sigma} & H_{\varsigma^2} \end{pmatrix}, \]

where

\[
\begin{align*}
H_{\sigma^2} &= -(Hx_\mu - b)^T G_\mu^{-1}(\varsigma, \sigma)(Hx_\mu - b), \\
H_{\varsigma^2} &= -\mu^2 x_\mu^T (B^T B) G_\mu^{-1}(\varsigma, \sigma)(B^T B)x_\mu - \mu, \\
H_{\sigma\varsigma} &= \mu (Hx_\mu - b)^T G_\mu^{-1}(\varsigma, \sigma)(B^T B)x_\mu, \\
H_{\varsigma\sigma} &= \mu x_\mu^T (B^T B) G_\mu^{-1}(\varsigma, \sigma)(Hx_\mu - b). 
\end{align*}
\]

In order to show the dual function is a concave function, it is equivalent to show that

\[ S_0 = \begin{pmatrix} H_{\sigma^2} & H_{\sigma\varsigma} \\
H_{\varsigma\sigma} & H_{\varsigma^2} + \mu \end{pmatrix}, \]

is semi-negative definite. By Sylvester’s Criterion, it suffices to show that all the leading principal minors have a non-positive determinant. Obviously, the first \(1 \times 1\) leading principal minors have non-positive determinants, since

\[ -(Hx_\mu - b)G^{-1}(\varsigma, \sigma)(Hx_\mu - b) \leq 0. \quad (16) \]

It is left to show \(\det(S_0) \leq 0\). Note that

\[
S_0 = CD = \begin{pmatrix} -(Hx_\mu - b)^T G_\mu^{-1}(\varsigma, \sigma) & 0 \\
0 & \mu x_\mu^T B^T B G_\mu^{-1}(\varsigma, \sigma) \end{pmatrix}
\begin{pmatrix}
Hx_\mu - b & -\mu(B^T B)x_\mu \\
Hx_\mu - b & -\mu(B^T B)x_\mu 
\end{pmatrix}
\]

Apparently, \(\text{Rank}(CD) \leq \text{Rank}(D) \leq n\). We can make a conclusion that \(\det(S_0) = 0\). Thus, \(S\) is semi-negative definite, which implies that dual function is concave function. \(\square\)

Theorem 2. (Strong Duality) If \((\varsigma_\mu, \sigma_\mu)\) is a critical point of \(P_\mu^d(\varsigma, \sigma)\) over \(S^+_\mu\), then \((P_\mu^d)\) is perfectly dual to \((P_\mu)\) in the sense that the vector

\[ x_\mu = G_\mu^{-1}(\varsigma_\mu, \sigma_\mu)(f + \mu c - \sigma_\mu b) \quad (17) \]
is a global minimizer of \((\mathcal{P}_\mu)\) and \((\varsigma_\mu, \sigma_\mu)\) is a global maximizer of \((\mathcal{P}^d_\mu)\), and
\[
\min_{x \in \mathcal{X}_\mu} P_\mu(x) = P_\mu(x_\mu) = P^d_\mu(\varsigma_\mu, \sigma_\mu) = \max_{(\varsigma, \sigma) \in S^+_\mu} P^d_\mu(\varsigma, \sigma). \tag{18}
\]

**Proof.** The proof basically follows that of the former weak duality Theorem. The only difference lies in the assumption that \((\varsigma_\mu, \sigma_\mu)\) is a critical point of \(P^d_\mu(\varsigma, \sigma)\) over \(S^+\). In this case, \(\frac{\partial}{\partial \varsigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = 0\), and \(\frac{\partial}{\partial \sigma} P^d_\mu(\varsigma_\mu, \sigma_\mu) = 0\). So
\[
 x_\mu = x(\varsigma_\mu, \sigma_\mu) = G_\mu(\varsigma_\mu, \sigma_\mu)(f + \mu c - \sigma b) \text{ is on the boundary of } \mathcal{X}_\mu'.
\]
That is, \(\frac{1}{2} x^T H x - x^T f - \mu c - \sigma b = 0\). This further implies that
\[
P^d_\mu(\varsigma_\mu, \sigma_\mu) = \Xi(x_\mu, \varsigma_\mu, \sigma_\mu) = P_\mu(x_\mu) \tag{19}
\]
and the equation (18) follows naturally. \(\square\)

The above results immediately lead to the following sufficient condition for finding the global optimizer of problem \((\mathcal{P})\):

**Corollary 1.** If there exists a critical point \((\varsigma_\mu, \sigma_\mu) \in S^+_\mu\) for every \(\mu \in [\mu_0, \delta^{-1}]\), then
\[
\min_{x \in \mathcal{X}} P_0(x) = \min_{\mu \in [\mu_0, \delta^{-1}]} P^d_\mu(\varsigma_\mu, \sigma_\mu). \tag{20}
\]

4. **Numerical Example**

In order to demonstrate the application of the theoretical results presented in this paper, let us consider the following one-dimensional problem
\[
\min_{x} P_0(x) = -5x^2 - x + \frac{0.5(0.5x^2 - 2)^2 - 0.5x}{-0.5x^2 - x}
\]
over the feasible domain
\[
\mathcal{X} = \{x \in \mathbb{R} \mid -0.5x^2 - x \geq 0.01\} = \{-1.9899 \leq x \leq -0.01005\}.
\]

The target function \(P_0(x)\) has singularities at \(-2, 0\) and is nonconvex over \(\mathcal{X}\). (See Figure 2 for the graph of \(P_0(x)\).)

By Lemma 1, we have \(\min_{x \in \mathcal{X}} P_0(x) = \min_{\mu \in [2, 100]} \min_{x \in \mathcal{X}_\mu} P_\mu(x)\) with
\[
P_\mu(x) = \frac{1}{2} x^T Q x - x^T f + \mu g(x)
\]
\[
= -5x^2 - x + \mu(0.5(0.5x^2 - 2)^2 - 0.5x)
\]

\[ \]
and $X_\mu = \{-1 - \sqrt{1 - 2\mu^{-1}} \leq x \leq -1 + \sqrt{1 - 2\mu^{-1}}\}$. For each $\mu \in [2, 100]$, the canonical dual function can be obtained as

$$P_\mu^d(\varsigma, \sigma) = -\frac{1}{2}(f + \mu c - \sigma b)^T G_\mu^{-1}(\varsigma, \sigma)(f + \mu c - \sigma b) - \mu \lambda \varsigma - \frac{\mu^2}{2} \varsigma^2 + \frac{\sigma}{\mu}$$

over $S^+_\mu = \{\varsigma \in \mathbb{R}, \sigma \in \mathbb{R} \mid \varsigma \geq -2, \sigma \geq 0, \mu \varsigma + \sigma > 10\}$. The partial derivatives of $P_\mu^d(\varsigma, \sigma)$ become

$$\frac{\partial}{\partial \varsigma} P_\mu^d(\varsigma, \sigma) = \frac{\mu(1 + 0.5\mu - \sigma)^2}{2(-10 + \mu \varsigma + \sigma)^2} - 2\mu - \mu \varsigma,$$

and

$$\frac{\partial}{\partial \sigma} P_\mu^d(\varsigma, \sigma) = \frac{1}{2} \left( \frac{1 + 0.5\mu - \sigma}{-10 + \mu \varsigma + \sigma} \right)^2 + \frac{1 + 0.5\mu - \sigma}{-10 + \mu \varsigma + \sigma} + \frac{1}{\mu},$$

It is obvious that

$$\lim_{\varsigma \to \infty} \frac{\partial}{\partial \varsigma} P_\mu^d(\varsigma, \sigma) < 0 \quad \text{for} \quad \forall \mu \in [2, 100].$$
\[
\lim_{\sigma \to \infty} \frac{\partial}{\partial \sigma} P^d(\zeta, \sigma) = \frac{1}{2} - 1 + \frac{1}{\mu} = -\frac{1}{2} + \frac{1}{\mu} < 0 \text{ for } \mu > 2.
\]

Therefore, the maximizer \((\zeta_\mu, \sigma_\mu)\) of \(P^d_\mu(\zeta, \sigma)\) does not exist on \(\mu = 2\).

Suppose the maximizer of \(P^d_\mu(\zeta, \sigma)\) is \((\zeta, \sigma)\) for given \(\mu\), we define \(P^d_\mu(\mu) = P^d_\mu(\zeta, \sigma)\). To minimize \(P^d_\mu(\mu)\) over \(\mu \in [\mu_0, \delta^{-1}]\), we use the line search with the Armijo’s rule. Suppose the current iterate is at \(\mu_k \in [\mu_0, \delta^{-1}]\), we may approximate the derivative of \(P^d_\mu(\mu)\) at \(\mu = \mu_k\) by

\[
d_k = \frac{d}{d\mu} P^d(\mu)|_{\mu = \mu_k} = \frac{P^d(\mu_k + \epsilon) - P^d(\mu_k)}{\epsilon}
\]

where \(\epsilon\) is a selected parameter and the two terms in the numerator can be evaluated by the interior point method in the optimization Toolbox within the Matlab environment. If \(d_k > 0\), the full step size \(s\) can be taken as the distance from the left boundary to \(\mu_k\), i.e., \(s = \mu_0 - \mu_k\). Otherwise, we take \(s = \delta^{-1} - \mu_k\) from the other end. Then we select two parameters such that parameter \(\alpha\) is close to 0 for scaling the slope \(d_k\), and parameter \(\beta \in (0, 1)\) for scaling the full step size \(s\). Choose \(m\) to be the first non-negative integer such that

\[
m = \min\{n \geq 0 | P^d(\mu_k + (\beta^n s)(\alpha d_k)) < P^d(\mu_k) + (\beta^n s)(\alpha d_k)\}\]

Then, we can update

\[
\mu_{k+1} = \mu_k + (\beta^m s)(\alpha d_k)
\]

and repeat until \(d_k\) is nearly 0.

In our example, we use \(\beta = 2/3\), \(\alpha = 0.001\) and \(\epsilon = 0.01\). It reaches the global minimum of \(P^d_\mu(\zeta, \sigma)\) at \(\mu = 4.8863\) with a value of \(-9.0851\). The corresponding primal solution is \(x = -1.7686\), and \(P^d_\mu(x) = -9.0851\). See Fig. 2 for the graph of \(P^d_{4.8863}(\zeta, \sigma)\).

5. Conclusions

The paper presents a new method for solving a special fractional programming problem with sum of a quadratic function and the ratio of nonconvex function and quadratic function as its target function. By using the canonical duality theory, this challenging problem is reformulated as a concave maximization dual problem over a convex domain, which can be solved easily if
the canonical dual function $P^d_{4.8863}(\varsigma, \sigma)$ has a critical point in $S^+_n$. Otherwise, the primal problem could be really NP-hard, which is a conjecture proposed in [17]. Recently, we noted a new methodology has been proposed by Sergeryev [7, 30] for performing calculations with infinite and infinitesimal quantities, by introducing an infinite unit of measure expressed by the numeral grossone. We believe this new concept could be important for the future study of the fractional optimization.

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[3] H.P. Benson, Using concave envelopes to globally solve the nonlinear


