THE ECONOMICS OF EXHAUSTIBLE RESOURCES:

A THEORETICAL CONTRIBUTION

by

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In compliance with the requirements relating to Examination, and Submission of Theses, for the Degree of Doctor of Philosophy of the Australian National University, it is affirmed that, unless otherwise stated, the work that follows is my own.

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C O N T E N T S

ABSTRACT

This is a theoretical study of some of the basic problems associated with the intertemporal conservation and use of exhaustible resources.

The emphasis throughout the thesis is on issues of social welfare — in particular we are concerned to find out how an economy should manage its resources in order to maximize the stream of returns which it receives from exploiting them over some time period. To this end a number of optimal control models are applied to the following questions:

(i) When will a community find it optimal to use up all of a scarce depletable asset?

(ii) For what duration of time should exploitation of such an asset continue?

(iii) What is the optimal intertemporal pattern of exploitation of the resource?

(iv) How are the decisions implicit (i) - (iii) affected by the community's set of preferences and the physical constraints to which the economy is subject?

(v) How is the optimal use pattern of the resource affected by:
   (a) the length of the economy's planning;
   (b) the size of the initial stock of the resource;
   (c) the economy's choice of discount rate?

(vi) What structural changes in the economy are required as the resource is depleted?

(vii) How and when should the economy respond to the prospect of exhaustion of a key resource? When will the economy find it optimal to develop substitutes?

(viii) What are the implications for a two-sector economy's specialization patterns over time when one of its two commodities is produced using an exhaustible resource?

CHAPTER 1

INTRODUCTION

Although economics has frequently been called the science of scarcity, through some strange oversight most economists have, until recently, ignored the most basic form of scarcity which economics can experience — that which arises from the ultimately finite supply of those natural resources which have become the foundation of industrial civilization. Indeed, it is only in the last decade, as the ultimate scarcity of such resources as petroleum, iron ore, gold and mercury has become evident that economists have felt a pressing need to find out more about the nature of such scarcity and how it should affect our present economic decisions. The reasons for the previous indifference to the problem are not hard to identify. In particular, ultimate scarcity had not (at least in the early 60's) made itself felt through significant price rises. Indeed, as though to allay all fears, a series of empirical studies[1] appearing in the late 50's and early 60's seemed to confirm that most key resources were becoming "cheaper" over time and that therefore scarcity was receding into the future. Herfindahl [17] concluded that in the case of the non-ferrous metals, "the deterioration in the underlying natural resource conditions has not been great enough to counterbalance the cost reduction that has taken place over the years." Barnett and Morse [6], using the same data source as Herfindahl went further and found the hypothesis "that the cost of extractive output

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1 Herfindahl [17], Barnett and Morse [6], Fisher and Potter [12].

would have increased had it not been for sociotechnical progress in the economy as a whole" to be generally invalid. Hence it was concluded that, in the long run, in the absence of general technological change (which had previously been thought the main reason for declining prices) the cost of most extractive outputs would still have fallen, thus indicating that scarcity was not making its presence felt. Fisher and Potter [12] concluded that for metallic minerals in particular "there are fair degrees of assurance for supplies at least as far as the year 2000." Beyond 2000 they were fairly confident that rising costs would induce substitution and technological progress at an accelerating rate for some time. Whenever the question of ultimate exhaustion of certain resources was raised the usual reaction was that the phasing in of substitutes would be induced by market forces. In a mood of such prevailing optimism, economists could hardly be blamed for shelving the problem for a few more years. Nevertheless, *within a few more years* the mood was to change from one of calm optimism to one of pessimism and panic. Works such as the Club of Rome's "Limits to Growth" (Meadows et al. [23]) were to appear and be widely read by an increasingly aware public. Alarm about a so-called impending "energy crisis" was also to become widespread. A pre-echo of this alarm was to be found in several of the papers published in 1969 by the National Academy of Sciences Committee on Resources and Man [24]. Lovering, in particular, in his paper, "Mineral Resources from the Land" ([24] Ch. 6), expressed the view that:

(i) Already known commercial deposits would become ore through technical innovations, future availability of cheaper transportation, or rise in price.

(ii) Deposits not yet discovered would be discovered in the relatively near future because of rapidly developing discovery techniques.

(iii) Such discoveries would take place at a diminishing rate as scarcity becomes imminent, and when scarcity appears the resultant rise in price might be sudden rather than gradual.

(iv) Ample lead time would be needed for technology to mitigate such scarcity so that research into obtaining more reliable estimates of reserves and more complete information on substitution possibilities could be initiated.

(v) More recent statistics show that, contrary to the findings of Barnett and Morse, technology is barely keeping pace with increasing costs in extractive industries.[2]

Cloud, in "Mineral Resources from the Sea" ([24] Ch. 7), cautioned against being misled into anticipating an abundant variety of resources to be extracted from the sea when land reserves are exhausted and stressed the technical difficulties and uncertainty associated with mineral extraction from the sea.

Hubbert, in "Energy Resources" ([24] Ch. 8) predicted that 90% of estimated crude oil reserves would have been extracted by 2032 at the latest. Hubbert's predictions are based on a bell-shaped time profile of extraction of resources which assumes that (i) the steady rates of growth sustained while a resource is plentiful cannot be maintained for longer periods of time, (ii) in the initial stages of resource use a positive exponential rate of increase is fairly inevitable and (iii) in the final stages an exponential rate of decline of production is

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2 [24], pp.124-5.

indicated. Hubbert's projections are therefore bound to be more optimistic than those obtained using the Club of Rome's exponentially increasing extraction curve.

In the light of the prevailing concern over the future of economies which are operating subject to almost inescapable resource constraints, it is the aim of this thesis to develop further what has become known as the economic theory of exhaustible resources.

Because any work of finite length must necessarily be limited in scope it is as well at this stage to define the scope of this work.

To begin with it is important that the reader should realize that the thesis will confine itself to the study of *exhaustible* or *non-renewable* resources (such as petroleum, metals, etc., in contrast to renewable resources such as fisheries, forests). Ciriacy-Wantrup [9] defines this type of resource as one whose "total physical quantity does not increase significantly with time" ([9], p.35) subject to a spatial constraint. In some ways the terminology is bad: renewable resources may be exhausted, as evidenced in the extinction of various animal species. However, the term "exhaustible" has historically come to be identified almost exclusively with minerals, so we shall adhere to tradition and use Ciriacy-Wantrup's definition here.

Secondly, the thesis will be entirely theoretical. It is felt that the gaps in the existing economic theory of exhaustible resource use are so large as to provide material for a library of theses. It is important (and urgent) that any empirical studies which are embarked on in the future should have a good body of theoretical literature to draw on. It is hoped that the present work will go some of the way towards filling these needs.

Thirdly, the range of problems to be analysed will also be limited. Most of the problems which are discussed will be concerned with the general problem of "conservation" as it was defined by an Australian geologist:

> "Conservation is the effort to ensure to society the maximum present and future benefit from the use of natural resources. It involves the inventory and evaluation of natural resources and requires the substitution where possible of renewable or inexhaustible resources for those which are non-renewable, and of the more abundant non-renewable resources for the less abundant ones. It thus appears that conservation involves the balancing of natural resources against human resources and the rights of the present generation against the rights of future generations."[3]

Because the author regards the conservation problem as the important underlying problem in all policy problems involving the use of exhaustible resources, all models presented in the thesis will be intertemporal models. Furthermore, because the issue of conservation carries with it a connotation of *social* welfare the emphasis throughout will be on models which describe social optimizing and social (or centralized) planning.

The thesis is divided into seven chapters. Of these, chapter 1 is this introduction and chapter 7 is a summary of the main conclusions of the thesis. The remaining chapters may be outlined as follows:

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[3] See [11], pp.15-16.

Chapter 2 surveys the theoretical literature which predates the recent spate of papers. Noting that the theory has its roots in early work by Malthus and Ricardo, it traces through some extensions of their theories. It also contains a survey of some models of production in natural resource industries and outlines the early work on intertemporal planning and resource depletion (Hotelling, Gray) and some more recent extensions (Scott, Gordon, Herfindahl).

Chapter 3 presents a basic single resource model which may be used to synthesize and extend some of the earlier intertemporal models. It attempts to answer at a basic level the questions which are fundamental to the whole thesis:

(i) How should a community distribute its use of an exhaustible resource over time so as to maximize the present value of the stream of returns from its extraction?

(ii) When will it be optimal to exhaust the resource?

(iii) When will it be optimal to cease extraction of the resource?

These questions are examined using various assumptions about the economy's production relationships and the community's set of preferences.

Chapter 4 is concerned with disaggregation of the basic model of chapter 3. The first section of chapter 4 disaggregates the economy into two, and then three, sectors to study the structural changes necessitated by sound resource management. In the second section, the resource is no longer assumed to be homogeneous in quality and a model containing two different grades of the same resource is developed.

Chapter 5 extends the "two-resource" case looked at in the second sector of chapter 4, and emphasizes the mechanism whereby one resource may be replaced by another substitute resource. Section II of this chapter allows for the "development" of a substitute by investment in either physical-capital or technical know-how. Section III presents a brief treatment of uncertainty.

Chapter 6 allows for importing and exporting the resource in a world where the terms of trade are given. The problem is solved firstly assuming that international payments are balanced at every point in time and secondly assuming that they are merely balanced over the whole planning period. In both cases the changes in the economy's structure and specialization are noted.

CHAPTER 2

THE EARLY LITERATURE*

Although the current high level of interest in the economic theory of scarcity is of recent origin (the last two or three years) the study of exhaustible resource problems dates back a long way and has produced a substantial body of literature which considerably pre-dates the current spate of papers. It is the aim of this chapter to survey these important contributions and to use them as a means of illuminating the modern approach to the problems.

The chapter attempts to present the development of the subject by adhering as closely as possible both to the chronological and logical sequence in which the theoretical framework has been developed. Accordingly, the chapter is divided into three sections reflecting the three main stages in which economic thinking on resource problems was developed: the classical Malthusian and Ricardian approach (framed in terms of land rather than depletable resources), depletion and production in resource industries, and intertemporal analysis of resource use.

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* During the period in which the author has been working on this thesis there has been a boom in research on the economic theory of exhaustible resources. Because some of the work contained in this thesis pre-dates most of the recent work (Chapters 3, 4 and 6), while the remaining parts of the thesis have proceeded concurrently with the recent research by others, it therefore seems convenient to isolate the literature available prior to commencement of the thesis in this chapter, and discuss more recent contributions in footnotes elsewhere.

1. THE CLASSICAL ECONOMISTS

The first important economic analysis of scarcity was due to Malthus [22] and while it did not have anything to say on exhaustible resources other than land, his ideas are worth brief mention for the light they throw on later developments. The basic thesis is familiar enough: Economic scarcity derives from the incompatibility of a finite amount of agricultural land with provision of subsistence to an ever growing population, which leads to an ultimate fall in output per head and a cessation of growth. For Malthus, scarcity is inherent in the finitude of man's stock of resources. It is notable however, that despite this resource (= land) limitation, Malthusian scarcity does not become apparent until all of the available land is in use. Then scarcity will be reflected in an increasing incremental labour-capital cost per unit of output.

This basic approach to the problem was extended by David Ricardo [27], whose model differs from the Malthusian one in two important respects:

(a) it drops the assumption that a given resource is homogeneous, and postulates that use of a resource will proceed according to grade, better grades being used first, poor grades later;[1]

(b) at no stage does Ricardo stipulate a finite limit to resource supplies (although each grade is in limited supply, recourse can always be had to a lower grade). Thus, as more of the resource is brought into use and the highest grade becomes fully utilized the incremental capital-labour cost per unit of

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[1] See the discussion of the Herfindahl model for the case of two grades of resource in Section III.

output will begin to rise. When it rises to the level associated with the next lowest grade that will be the signal to switch to the lower grade.

The Ricardian model may be formulated algebraically as follows:[2]

It has an aggregate production function of the form:

$$Y = F(K_1, \phi(K_2)) \, , \tag{2.1}$$

where K_1 is the amount of a variable input (capital or labour or some amalgam of the two) used directly in the production of final output and K_2 is the amount of that input used in the process of "resource conversion"[3] whereby "unhomogeneous" resources are converted into "homogeneous" resources (E)[4] according to:

$$E = \phi(K_2) \; .$$

The Ricardian assumption that resources (land) are used in order of declining economic quantity implies that ϕ is a strictly concave function of K_2.

The expansion path followed by the economy as the total endowment of K increases is the solution to the problem:

$$\max_{K_1, K_2} F(K_1, \phi(K_2))$$

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2 See Barnett and Morse [6], Ch. 5.

3 See Barnett and Morse [6], p.110.

4 This may simply mean that a certain amount of the input is needed to make the unhomogeneous resource as effective in production as some predetermined "standard" resource.

s.t. $$K \geqslant K_1 + K_2 \ ,$$

where K is the total endowment of the main input.

The locus of the expansion path is:

(2.2) $$F_E \, \phi_{K_2} = F_{K_1} \ .$$

Along it, it may be shown that:

$$\frac{dY}{dK} > 0$$

and $$\frac{d^2Y}{dK^2} > 0 \quad \text{if} \quad \phi_{K_2K_2} < 0$$

(declining resource quality), so the Ricardian claim of diminishing returns in production is verified. It is also easily shown that the production function, F, must exhibit decreasing returns to scale when it is regarded as a function of E and K, with the consequence that it becomes optimal to substitute a growing stock of K for "standard" or "homogeneous" resources.

The Ricardian model proves to be an extremely useful one for giving us ideas about the formulation of scarcity models and the sort of problems we might wish to encompass. The resource conversion function as developed by Barnett and Morse has many possibilities, not least of which is its possible use as a device for incorporating recycling into a theory of resource use (regarding scrap as a non-standard resource which may be effectively converted into a standard resource by the application of labour-capital, K). On the whole however, both the Ricardian and Malthusian models are rather unsatisfactory depictions of the scarcity issue in the modern world and tell us as much about the problems by what they omit as by what they include.

In the first place they are both framed in terms of a single resource — agricultural land, which differs from the main classes of "exhaustible" resources in that it cannot, in the strict sense of the word, be termed depletable. Its continued use does not entail a significant fall in the total stock available. Secondly, both models take no account of economic foresight. In the extreme Malthusian case scarcity does not affect the economy's calculations until the point of complete utilization is reached. It would seem more reasonable for firms to foresee exhaustion of an asset and bid up its price.

Thirdly, the Ricardian model is based on the unrealistic assumption that producers have the information, opportunity and inclination necessary to exploit resources in order of declining economic quality. There is of course no guarantee that the best resources will be discovered first and in fact there does not seem to be any reason why the reverse should not be true.

II. DEPLETION AND PRODUCTION

A useful synthesis and extension of the classical models was presented by Barnett and Morse [6]. Their main contribution is an extension of Ricardo's theory to incorporate depletion.

This entails rewriting the resource conversion function as:

$$E_t = \phi(K_{2_t}, X_t) , \quad \phi_X > 0 \tag{2.3}$$ [5]

where $\Delta X_{t-1} \equiv X_t - X_{t-1} = -g(E_t)$

for some increasing function g.

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[5] See [6], pp.118-120.

$X_t \equiv$ stock of resources available at time t

$E_t \equiv$ current input of resources (homogeneous)

and $E_t \leqslant X_t$.

If we have a social production function of the form (2.1) with the new resources function (2.3) substituted in for E, then we can easily check the effects of resource depletion on optimum output when K is constant:

$$\frac{dY}{dX} = F_E \, \phi_X > 0 \quad \text{using (2.2)}$$

so that as the resource is depleted we are forced to produce a smaller output with the same endowment of K.

If the aggregate supply of K is assumed to adjust so as to maintain constant production for the economy we find that

$$\frac{dK_i}{dX} < 0 \text{ [6]} \quad i = 1, 2 .$$

Hence for output to be held at an optimally (myopic) constant level in the face of depletion it is necessary for the labour/capital input in both sectors to be increased — up to the point where the marginal product of K, is zero. As that point is approached the economy will be forced to put more and more of its additional labour/capital into resource conversion and negligible amounts into producing final product. It is clear that an economy suffering depletion of its resources must eventually be prepared to substitute large amounts of labour/capital for the resource if it is to adhere to a myopic decision-making rule. If depletion is too rapid (or more precisely if production is particularly

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6 $$\frac{dK_1}{dX} = \frac{(F_E)^2 \, [\phi_X \, \phi_{K_2K_2} - \phi_{K_2} \, \phi_{K_2X}]}{F_{K_1}[F_{K_2}F_{EK_1} - F_{K_1K_1} + (\phi_{K_2})^2 \, F_{EE} + F_E\phi_{K_2K_2} - F_{K_1E}\phi_{K_2}]}$$

sensitive to the depletion effect) then the economy may be unable to induce a sufficiently high rate of growth of K and so will be forced to reduce output.

This model continues to adhere to the Ricardian assumption that resources will be used in order of declining economic quality. It also assumes that the stock of capital is exogenously given. If one considers the problem in a broader macroeconomic framework and supposes that production of capital goods is likely to impose a further drain on (possibly different) depletable resources, then it must be conceded that reliance on the Barnett and Morse model is likely to lead to overly optimistic conclusions. The model is, of course, based entirely on a myopic view of resource-use decisions and as such effectively ignores the finite nature of exhaustible resource stocks. Nevertheless, we are left with a useful synthesis and extension of the classical model which gives us a sound basis for the production side of otherwise more realistic models. The theory of production from natural resources is further developed by Smith [29] who develops a general model of production to be applied to resources as superficially dissimilar as fisheries and petroleum. His model seems to represent the first attempt to incorporate externalities (in particular, crowding and common ownership in the fishing industry) into the relevant production functions. However, the main emphasis in his paper is on renewable resources (particularly fisheries, see also Smith [30]) and externalities in petroleum production (for example) are not discussed at great length.[7] Smith allows for movement in and out of the relevant

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[7] An interesting discussion of petroleum production and its associated externalities in terms of the theory of user costs may be found in Khoury [20].

industry by having the capital stock in the industry change in proportion to current profits, a behavioural assumption which amounts to assuming myopic decision-making. This assumption is, of course, fairly reasonable for the case with which Smith is primarily concerned (fisheries), but for industries such as mining and petroleum most interest attaches to the intertemporal aspects of their use. It is these intertemporal issues with which this thesis is primarily concerned and we now turn to a consideration of their treatment by earlier authors.

III. INTERTEMPORAL MODELS

As noted already we are taking exhaustible resources to mean resources whose "total physical quantity does not increase significantly with time".[8] Associated with their current use is an opportunity cost of future consumption foregone. This opportunity cost, which may also be defined as a user cost,[9] is clearly a cost over and above the costs of processing and extraction and its existence constitutes the main difference between industries such as mining and ordinary manufacturing industries. The first serious attempts to acknowledge this difference and embody it in a coherent theory are to be found in two important papers by Gray [15] and Hotelling [19].

Gray's paper provides an illuminating introduction to the problem, framed, as it is, in terms of an extension of the traditional Ricardian notion of rent to account for depletion. Gray observes that while the Ricardian idea of rent as "a payment for the original and

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8 See Ciriacy-Wantrup [9], p.35.

9 See Scott [28], p.35.

indestructible qualities of the soil"[10] is appropriate when applied to, say, urban land, it is inadequate in the case where exhaustible resources are mined from the soil. In such a case, it is generally impossible to separate the value of the exhausted properties from the value of the inexhaustible properties, and Gray concludes that the real economic rent of such resources comprises the "entire net return from the rent-bearer, including the so-called royalty".[11]

Hotelling's paper outlines the basis of what has become the modern approach to resource questions. Using the calculus of variations he analyses a series of models of resource use under different economic regimes (competition, monopoly, social control). Most of the time he is dealing with specific functions. However, his conclusions embody some measure of generality and are worth noting. The four main results are:

1. The resource will be exhausted in a finite time if and only if zero demand for the resource good occurs at a finite price;
2. a socially efficient path for resource prices will be identical to the efficient path under perfect competition (as long as no externalities or common property phenomena are present);[12]
3. in the case of common property resources exploitation may occur too rapidly;
4. monopolistic control of the industry will in general produce a longer period of exploitation than is socially optimal.

In particular, Hotelling introduces the "rule" that along an efficient path for a competitive industry (net) prices should rise at the rate of

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10 Gray [15], p.246.

11 Gray [15], p.446.

12 This result has been proven as a general proposition in a recent paper by Sweeney [32].

discount (δ). As will be seen in later chapters of this thesis this rule is only applicable to the most basic depletion models. On the other hand it does form a basis for constructing the intertemporal price rules for more general models. It has also been given some emphasis in the more recent literature.[13] It would therefore seem worthwhile to spend some time understanding the intuitive basis of the Hotelling rule.

At its simplest, as a rule for the individual competitive firm, it is merely an illustration of Jevons' formula for interest ("the rate of increase of the produce divided by the whole produce") discussed in some detail by Wicksell [35].[14] In terms of the problem facing an individual mining firm it may be explained as follows: Suppose such a firm wishes to maximize the present value of its stream of profits over some optimal time horizon at the end of which it will have just exhausted its initial endowment ($\bar{X}$) of the resource. Clearly the firm will find it optimal (if feasible) to adjust its output so that the present value of profits earned from the extraction of an additional unit of the resource will be the same for all periods. Marginal profits will be a measure of both the net price of the resource and the user cost associated with extraction of an additional unit of the resource at a particular point in time. One case dealt with in some detail by Gordon [14] and Herfindahl [18] is that of the constant cost perfectly competitive firm. Such a firm has no control over the size of its own marginal profits and is entirely dependent on the movement of industry price. Its marginal profits can only grow at the rate of discount if industry price rises sufficiently rapidly. On the other hand, it is an

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13 In particular Solow [31] and Nordhaus [25].

14 See Wicksell [35], pp.172-184.

easy matter to show (see Gordon [14], p.279) that a firm with variable costs should, in general, produce the required rate of marginal profits growth by lowering its output over time.[15]

Suppose now that we have a competitive industry containing n firms, each possessing a given, finite endowment ($\bar{X}_i$, i = 1, ..., n) of the resource. If all of these firms are faced with variable costs then output for each firm will be determinate. Price will rise in such a way that demand for the resource (given by the industry demand function) will fall to zero at the same time as the resource is exhausted. Individual firms will adjust output in such a way as to ensure the correct rate of growth (δ) of net price. On the other hand, when firms are faced with constant costs it is not clear that net price should still rise at the rate of discount. Suppose, however, that industry price is growing at a rate faster than that which would produce a rate of growth, δ, in net price. Then all firms in the industry would defer production until the "excessive" growth in profits ceases. This would produce a market disequilibrium which would cause the market price in the earlier parts of the plan to rise. A series of instantaneous adjustments can be envisaged which would continue until the correct path (net price growing at rate δ) is attained. A similar argument rules out the case where net price is growing at a lower rate than δ. Thus Hotelling's "rule" is established.[16]

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[15] This question is also discussed in a verbal-diagrammatic context by Scott [28].

[16] This rule may also be verified by noting that, in the case where net price is growing faster than δ, the present value of profits will be increased by waiting as p increases towards $\bar{p}$ (= $\bar{P} - \bar{C}$) = g(0) - $\bar{C}$) at which point the present value will be zero. If net price were to grow at a lower rate than δ, all production would be in the present and price would be driven to zero, thus again yielding zero profits. In either case, the solution is non-optimal.

Herfindahl carries the analysis further by examining the effects of varying some of the parameters in the constant cost competitive model.

His model may be formulated algebraically as follows.

Let the industry demand curve be denoted by

$$D(t) = f(P(t))$$

with an inverse $P(t) = f^{-1}(D(t) \equiv g(D(t))$ so that at the end of the plan,

$$P(T) = \bar{P} \equiv g(0) \ . \tag{2.4}$$

Initial price is defined as the solution P_0 to

$$g(0) = (P_0 - \bar{C})e^{\delta T} + \bar{C} \ , \tag{2.5}$$

where $\bar{C}$ is the level of (constant) per unit costs.

Also, since the resource is just exhausted at time, T, we have:

$$\int_0^T D(t)dt = \bar{X} \equiv X(0) \ . \tag{2.6}$$

For market equilibrium D(t) = total extraction at time t. (2.6) may be written as

$$\int_0^T f\{(P_0 - \bar{C})e^{\delta t} + \bar{C}\}dt = \bar{X} \ . \tag{2.7}$$

Thus (2.5) and (2.7) will jointly determine P_0 and T, while P(T) is determined directly by (2.4). We may rewrite (2.7) in a modified form as:

$$\int_0^T f(t, P_0, \bar{C}, \delta, \varepsilon)dt = \bar{X} \ , \tag{2.8}$$

where ε is a shift parameter for the demand function. Differentiating

(2.8) totally gives the relationship between changes in the variables and parameters:

$$(2.9) \qquad dP_0 \int_0^T f_{P_0}(t)dt + d\bar{C} \int_0^T f_{\bar{C}}(t)dt + d\delta \int_0^T f_{\delta}(t)dt$$

$$+ d\varepsilon \int_0^T f_{\varepsilon}(t)dt = d\bar{X} \quad (\text{since } f(T) = 0) \ .$$

Using (2.9) it is a straightforward matter to determine the effect of changes in the parameters on initial price. For example, setting all variables except P_0 and δ constant in (2.9), we find that:

$$\frac{dP_0}{d\delta} = - \frac{\int_0^T f_{\delta}(t)dt}{\int_0^T f_{P_0}(t)dt} < 0 \quad \text{since} \quad f_{\delta}(t) < 0 \ , \quad f_{P_0}(t) < 0 \ .$$

Differentiating $f((P_0 - \bar{C})e^{\delta T} + \bar{C}) = 0$ we obtain

$$\frac{dT}{d\delta} = - \text{sgn} \left[\frac{dP_0}{d\delta} + (P_0 - \bar{C})T\right]$$

and

$$\frac{dP_0}{d\delta} + (P_0 - \bar{C})T > (P_0 - \bar{C})T - (P_0 - \bar{C})T \frac{\int_0^T f'(P)e^{\delta t}dt}{\int_0^T f'(P)e^{\delta t}dt} = 0 \ .$$

Thus $$\frac{dT}{d\delta} < 0 \ .$$

Thus a fall in the rate of discount implies a rise in the initial price level and a lengthening of the period of exploitation.

Similarly it may be shown that:

(a) A rise in the quantity of available deposits will make price lower at each point in time and so lengthen the period of exploitation;

(b) A fall in unit costs implies a fall in initial price and a shortening of the period of exploitation;

(c) An increase in demand (change in ε) will always shorten the period of exploitation provided the new demand curve does not cross the old one at any feasible point.

In addition, Herfindahl examines the Ricardian situation where different grades of a resource are available in limited quantities and concludes that if all such deposits are known beforehand they will be exploited separately (this conclusion depends in some measure on Herfindahl's formulation of the problem — see [18], p.72) in order of declining economic quality. The Scott-Herfindahl-Gordon model which we have been discussing in this section has been extended by Cummings [10]. Cummings contributes two important additions to the theory:

(a) the incorporation of a depletion effect in the cost function;

and (b) an analysis of common property phenomena in intertemporal planning.

In particular he shows that if the present value of marginal profits at t is less than marginal profits at time zero minus the present value of the total change in costs that results from the fall in resource stocks between t = 0 and t (the cumulative influence of the depletion effect over all periods up to t), then the firm will extract none of the resource at t. If the present value of marginal profits at t is greater than marginal profits at time zero minus the cumulative depletion effect then it will pay the firm to produce at its maximal rate at time t.

In his analysis of common property resources, Cummings assumes an industry with n firms extracting from a common pool, each firm making its decisions on the assumption that all other firms will use the resource at each point in time at rates which minimize the present value of the profits of the firm in question. Under this behavioural assumption he deduces the following optimal rule for firm j:

> "At any t, produce if possible at a rate such that the present value of marginal profits at t equals the difference between the present value of the change in firm j's costs at t resulting from resource use (at t) by the other (n-1) firms and the opportunity cost associated with an increment of the resource left in stock at t."[17]

If the relationship specified in this rule cannot hold with equality over the feasible range of output, then output will be either zero or the maximal rate of production.

When firms in the common property situation submit to the direction of a central authority the authority would regulate extraction by each firm so that marginal costs of all firms are equal. Cummings shows that there is a divergence between the optimal decentralized and centralized solutions.[18]

An attempt to equip the basic Hotelling model with a more elaborate system of production was made by Burt and Cummings [8] in a discrete-time framework using dynamic programming. Their paper attempts to incorporate such additional features of the problem as investment in resource industries, depletion effects in production of the resource

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[17] Cummings [10], p.28.

[18] This question has been given more detailed scrutiny in a paper by Quirk and Smith [26] in the context of renewable resources.

good, renewability of the resource and capital depreciation. The underlying production relationships owe much to the paper by Smith (discussed in section II of this chapter), however, the model is framed in terms of a social objective function and reproduces the Scott-Herfindahl decision in a social context: viz. user cost equals marginal social benefits. In this case user cost consists of two components, one "the discounted marginal value of a unit of resource retained in stocks instead of being used in current production" (as in the Scott and Herfindahl papers) and the other "the discounted marginal value of capital stocks consumed by increment to current production". However, beyond this generalization of the basic model, Burt and Cummings' paper breaks little new theoretical ground. Its main use lies in its possible application to empirical studies. It is, however, too general to yield any specific a priori theoretical conclusions.

An alternative approach to that of the abovementioned authors is adopted by Anderson [1] who adds exhaustible resources as a third factor input to the Ramsey-Shell optimal growth model. Anderson assumes that the stock of the resource is depleted at a rate proportional to output ($X(t) = -e^{-\alpha t}F(K(t), L(t)$, where α is the rate of exogeneous technical change). Although one would expect the addition of an extra differential equation to the Shell model to lead to insurmountable difficulties, it turns out that the state equation for the capital-labour ratio (k) and its associated co-state (ψ) equation separate out nicely from the rest of the system so that Anderson is able to examine the optimal paths in the k-ψ plane. When a terminal constraint on the resource is imposed ($X(T) \geqslant X_T$ for some X_T), Anderson finds that Shell's solution to the problem is no longer optimal (because it will require more resources than the above constraint will allow). When the rate of technical change exceeds the population growth rate plus the social discount rate, the optimal solution with a resource constraint will approach the solution for the unconstrained case (the technical change offsets the resource scarcity). On the other hand the general effect of the resource constraint is to make it optimal to postpone capital accumulation in order simultaneously to keep the growing workforce employed and meet the resource constraint. If the rate of population growth plus the social discount rate is sufficiently in excess of the rate of technical progress the economy will proceed along the path of minimum resource use and postpone all capital accumulation until the end of the planning period.

CONCLUSION

The models surveyed above provide what is essentially the background to the present study. While, taken collectively, they all touch on a large number of the important aspects of the resource problem, their approach is piecemeal and none provides a very unified view of the problem. It would, for example, be interesting to answer questions such as:

(a) What is the optimal intertemporal allocation of economic effort *between sectors* in a vertically integrated, resource-based economy?

(b) How is the optimal intertemporal resource use path affected by the range of possibilities available after the resource is exhausted (e.g. existence of a "backstop technology")?

(c) At what point, and on what scale should an economy prepare (if at all) for the exhaustion of the resource?

(d) How does a depletion effect influence the amount of the resource left ultimately unexploited?

It is the intention of this thesis to develop the theory to encompass these and other interesting questions. The next chapter is devoted to synthesizing and extending the basic Hotelling model.

CHAPTER 3

THE BASIC SINGLE RESOURCE MODEL*

It is the aim of this chapter to provide a framework for synthesizing and extending some of the models discussed in the previous chapter. The relatively simple model presented here will serve as a basis for subsequent analysis and also as a means of directly answering some fundamental questions associated with resource scarcity. Some of the more interesting such questions which will be examined here are:

(i) when will a community find it optimal to use up all of a scarce depletable asset?

(ii) for what duration of time should exploitation of such an asset continue?

(iii) what is the optimal intertemporal pattern of exploitation of the resource?

(iv) how are the decisions implicit in (i) - (iii) affected by the community's set of preferences and the physical constraints to which the economy is subject?

(v) how is the optimal use pattern of the resource affected by

(a) the length of the economy's planning horizon,

(b) the size of the initial stock of the resource?

The chapter is divided into three sections. In Section I the basic model will be presented and the forms of the functions of the

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* This chapter is a generalized version of another paper by the author [33]. The author wishes to thank John Pitchford and Ngo Van Long for comments.

model discussed. Section II will examine the problem of exploiting a non-renewable resource so as to maximize the present value of utility derived from the future stream of consumption produced by using the resource. In Section III we will extend the analysis of Section II to incorporate an explicit conservation motive on the part of consumers and/or a depletion effect in production.

I. THE BASIC MODEL

In this section we shall formulate the problem of how to distribute exploitation of a fixed stock of an exhaustible asset over time in an optimal fashion. In establishing the foundations of the model it will be necessary to examine closely the properties of the main functions involved.

For the time being we will only discuss the functions relevant in Section II; those relevant to Section III will be discussed there.

To begin let us simply note that social welfare (W) is a strictly concave function of total consumption (C):

$$(3.1) \qquad W = v(C); \quad v'(C) > 0 \quad v''(C) < 0$$
$$v \in \mathcal{C}^2 .$$

So that total consumption for the economy will never be zero (even when all of the resource has been used up) we assume that there is an alternative source of consumption in the economy which is not produced from the resource and which is permanently at a constant level, $\bar{C}$. This allows us to abstract from the question of what the community might and should do after the resource has been exhausted, a question which itself raises a whole range of other issues and will in any case be discussed in Chapter 5. By assuming this alternative source of consumption we also provide ourselves with a convenient simplification of the relevance of the rest of the economy to the resource-use decision.[1]

Production of consumption goods from resources (C_X) is assumed to be a function of the amount of the resource extracted (E), so that:

$$C_X = \phi(E) \ .$$

Thus total consumption is given by:

$$C = C_X + \bar{C} = \phi(E) + \bar{C} \ ,$$

and it is possible to regard the utility function simply as a function of E, which we shall denote u(E). Before we proceed we would like to know how the form of u is affected by the form of ϕ. We see that

$$u' = v'\phi'$$

$$u'' = v''(\phi')^2 + v'\phi'' \ .$$

For all the functions we shall consider, it is assumed that $\phi'(E) \geqslant 0$ so that $u'(E) \geqslant 0$. Also for the standard concave production function with $\phi'' < 0$ we have $u'' < 0$ in which case u has the same curvature properties as v. When ϕ is smoothly convex for low values of E and concave for higher values (Figure 3.1(a)) then for all but low values of E u will be concave, but it may be convex for low E. While there is the slight possibility that u may in this case have more than one point of inflexion we shall ignore this contingency here.

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1 The questions of (a) the economy's behaviour after a resource ceases to be economical to extract and (b) disaggregation of the economy into resource production and resource-use sectors, are to be discussed in Chapters 5 and 4 respectively.

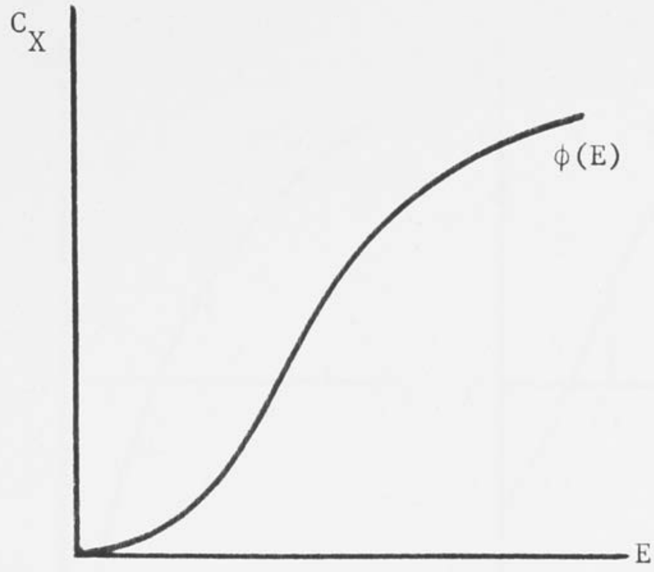

Figure 3.1(a)

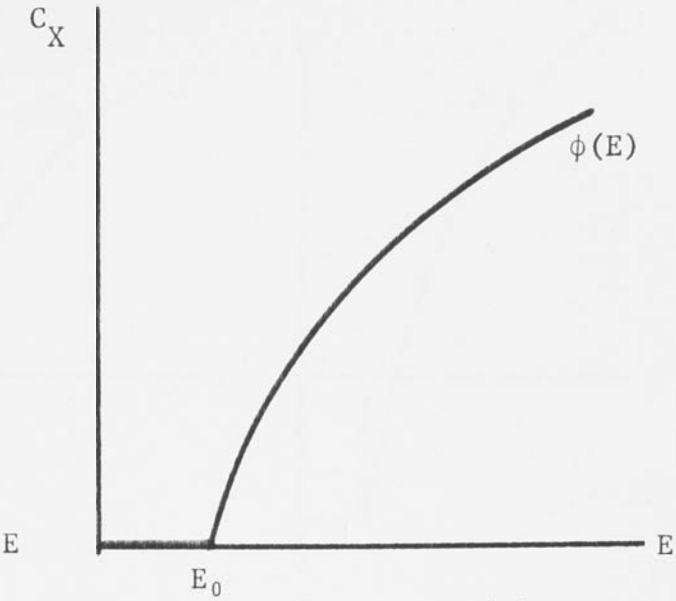

Figure 3.1(b)

It is also possible to envisage a production function of the form illustrated in Figure 3.1(b) and defined formally by:

$$(3.2)\quad \begin{cases} E > E_0 \Rightarrow \phi(E) > 0; \quad \phi'(E) > 0; \quad \phi''(E) < 0; \\ 0 \leqslant E < E_0 \Rightarrow \phi(E) = 0 \\ \text{for some } E_0 > 0 \,. \end{cases}$$

This form of production function would be relevant in a situation where a certain minimum level of extraction (E_0) is needed before any consumption goods can be produced. In this case the u function will have the same general form as ϕ (although the horizontal segment may correspond to negative values of u). For our purposes it will be convenient to distinguish three classes of u-function. They are illustrated in Figure 3.2 below.

(a) A class I u-function is defined by:

$$u''(E) > 0 \quad u''(E) < 0 \quad \forall_E \geqslant 0 \qquad \text{(Figure 3.2(a))}$$

(b) A class II u-function is defined by:

$$\left.\begin{array}{lll} u'(E) > 0 & \forall_E \geqslant 0 \\ u''(E) > 0 & 0 \leqslant E < E_0 \\ < 0 & E > E_0 \end{array}\right\} \qquad \text{(Figure 3.2(b))}$$

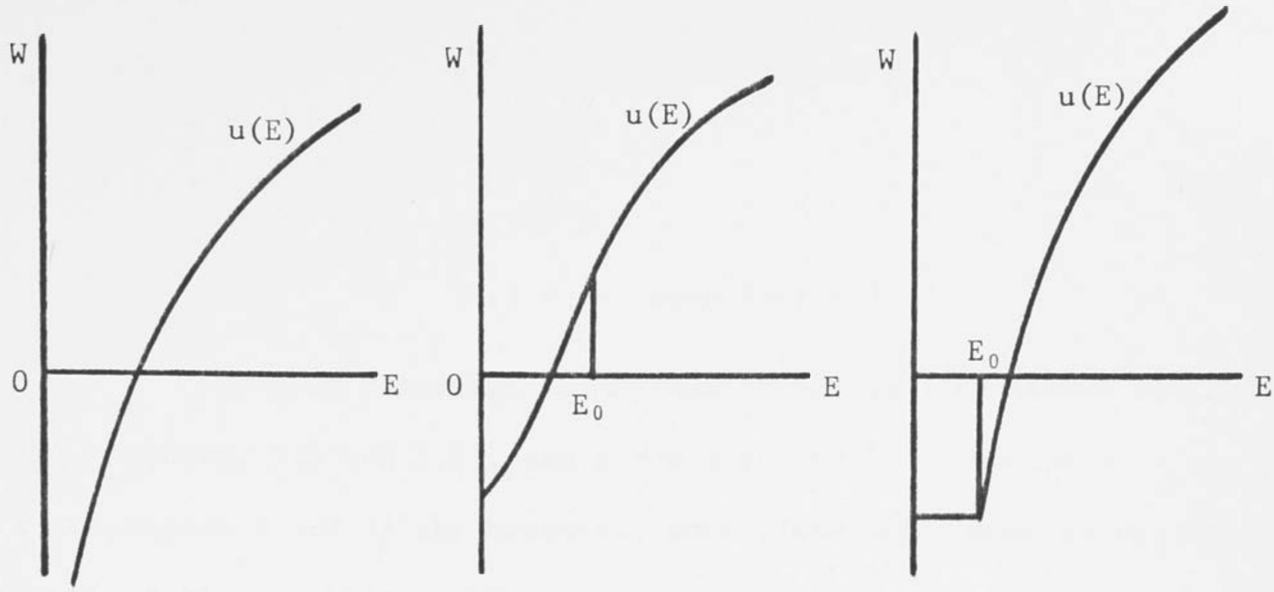

Figure 3.2(a) Figure 3.2(b) Figure 3.2(c)

(c) A class III u-function is defined by:

$$\left.\begin{array}{lll} u'(E) \equiv 0 & 0 \leqslant E < E_0 & \\ u'(E) > 0; & u''(E) < 0; & \text{for } E > E_0 \end{array}\right\} \quad \text{(Figure 3.2(c))}$$

This completes our specification of the functions involved. Before examining the workings of the model it remains to specify the dynamic equation for the system and state the optimization problem.

If the stock of the resource in the ground at time t is X(t) and extraction at time t is E(t), then the depletion of X proceeds according to

$$(3.3) \qquad \dot{X}(t) = -E(t) \qquad X(0) = X_0 \; .$$

We are concerned with the problem of selecting a time path for E(t) and a value for X(T) which will maximize the present value of the stream of consumption from time 0 to time T. Formulated mathematically the problem is to find E(t) and X(T) to

$$(3.4) \qquad \underset{E(t),X(T)}{\text{Max}} \int_0^T u(E)e^{-\delta t}dt \qquad \delta \geqslant 0, \text{ constant}$$

s.t. $\dot{X}(t) = -E(t) \qquad X(0) = X_0$

$$X(t) \geqslant 0$$

$$E(t) \geqslant 0$$

$$\dot{X}(t) = 0 \quad \text{when } X(t) = 0 \text{ .}$$ [2]

Applying Pontryagin's Maximum Principle (see Athans and Falb [4], Theorems 5.9 and 5.11, and Arrow and Kurz [3], Chapter 2, Propositions 4 and 7) the necessary conditions which must be satisfied by a solution to this problem are:

∃ a continuous function, ψ, such that (omitting time where it appears as an argument):

$$\mathcal{L} = u(E) - \psi E + \lambda_1 \dot{X} + \lambda_2 E \text{ ,}$$

(3.5) $$\psi\delta - \frac{\partial \mathcal{L}}{\partial X} = \dot{\psi} = \psi\delta \text{ ,}$$

(3.6) $$\frac{\partial \mathcal{L}}{\partial E} = 0 \Leftrightarrow \psi = u'(E) + \lambda_2 - \lambda_1 \text{ ,}$$

(3.7) $$\psi(T)\, X(T) = 0 \text{ ,}$$ [3]

(3.8) $$\begin{cases} \lambda_1 \geqslant 0, & \lambda_1 \dot{X} = \lambda_1 X = 0, \quad X \geqslant 0 \\ \lambda_2 \geqslant 0, & \lambda_2 E = 0, \quad\quad\quad\quad E \geqslant 0 \text{ .} \end{cases}$$

II. ANALYSIS OF THE MODEL

As we proceed it will become clear that there is a dichotomy of results between the model with class I u-functions on the one hand and the model with class II and III u-functions on the other. We will begin by looking at the most basic case, that of a zero discount rate.

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[2] See Arrow and Kurz [3], Proposition 5, part (g) and preceding discussion.

[3] Condition (3.7) is only a necessary condition when optimal T turns out to be finite. If T is infinite then we cannot use (3.8) as a necessary condition (see Arrow and Kurz [3], p.46).

Case 1: $\delta = 0$:

(a) Class I u-function:

It follows from (3.5) and (3.6) that the rate of exploitation is constant throughout the programme. Because of the necessary continuity of ψ we can easily deduce from (3.6) that it is never optimal to jump from a strictly positive E to E = 0 for X > 0 (it would lead to an upward jump in ψ). For X = 0 however such a jump in the E variable is permissible since the multiplier λ_1 can become positive and preserve the continuity of ψ (see Arrow [2] p.9, discussion prior to statement of Proposition 4). (3.7) tells us that when T is finite X(T) = 0 (since $\psi(t) > 0 \ \forall_t$). All of this gives us a clear picture of the optimal plan when T is any finite number. E will be set at some constant positive level until the resource is exhausted when it will fall to zero. What will be the optimal level of E? This may be determined by inspecting the present value integral:

$$P \equiv \int_0^T u(E)dt$$

$$= \int_0^{T'} u(E)dt + \int_{T'}^{T} u(0)dt \ ,$$

where T' is the time at which X is exhausted and therefore equals $\frac{X(0)}{E}$.

$$Tu(0) + X(0)\left(\frac{u(E) - u(0)}{E}\right) ,$$

and so maximizing P entails choosing E to maximize $\frac{u(E) - u(0)}{E}$. But for a class I function the relevant value of E is zero. Accordingly P will be maximized by choosing E to have as small a value as is consistent with resource exhaustion within the time available — this will be the value of E which spreads exploitation of the resource over the whole planning period from time 0 to time T, just exhausting the resource at time T. As the planning period, T, is lengthened, the optimal value of E will fall (see Figure 3.3) and as $T \to \infty$ optimal $E \to 0$. So that in the limiting case, $T = \infty$, there is no optimal solution. Any positive extraction path is dominated by a path with lower extraction while a zero extraction path (the limiting case) is dominated by *any* positive extraction path. This is of course a result which emerged from the Gale "cake-eating" example in [13].

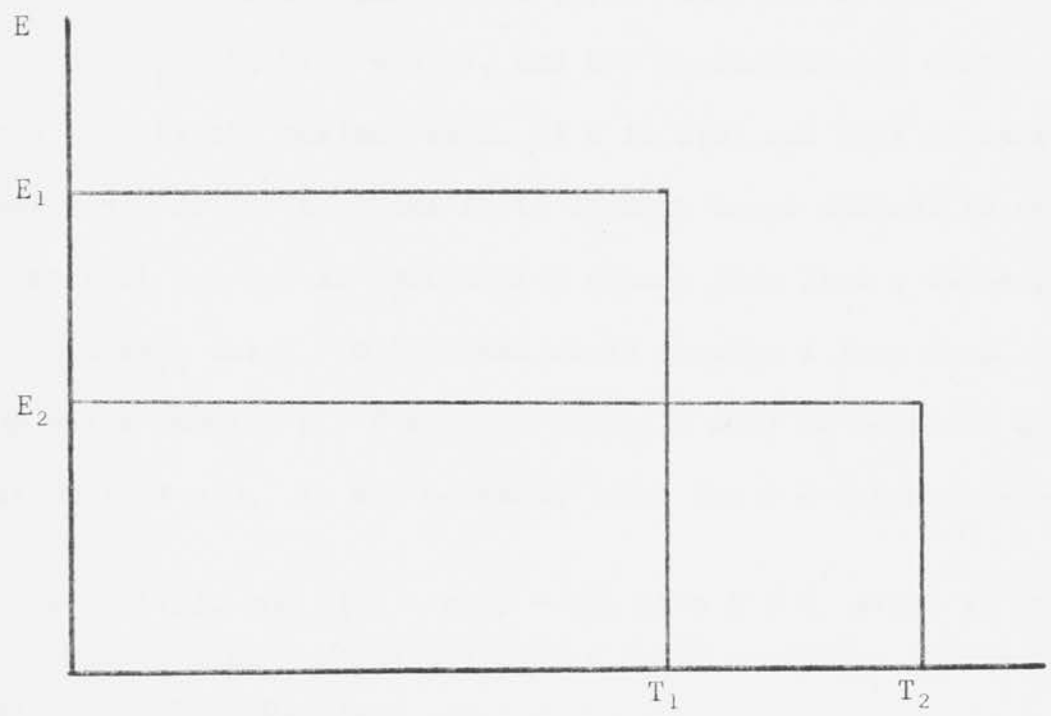

Figure 3.3

(b) Class II and III u-functions:

As before, when T is finite $X(T) = 0$. Now, we define $\hat{E}$ to be the value of E which maximizes

$$\frac{u(E) - u(0)}{E}$$

(in particular $\hat{E}$ will satisfy $u'(E) = \frac{u(E) - u(0)}{E}$). Because the Hamiltonian is non-concave for some values of E in this case the

Pontryagin necessary condition that the constrained Hamiltonian be maximized at each point in time must be approached from first principles. Simply, we wish to

$$\max_{E} H(E) = u(E) - \psi E$$

$$\text{s.t.} \quad E \geqslant 0 \ .$$

It is easily shown[4] that when $\psi < u'(\hat{E})$, optimal E will be the solution to $\psi = u'(E)$ with $E > \hat{E}$ (it can be seen diagrammatically with the aid of Figure 3.4 drawn for a class II u-function: in the diagram $\psi^1 > u'(\hat{E})$, $\psi^2 = u'(\hat{E})$ and $\psi^3 < u'(\hat{E})$). When $\psi > u'(\hat{E})$, $H(E) < u(0) \forall_E > 0$, $H(0) = u(0)$, and $E = 0$ consequently maximizes H. When $\psi = u'(\hat{E})$ the maximum value of H is u(0) and this is attained when either $E = 0$ or $E = \hat{E}$. Thus it is clearly never optimal to operate in the interval $0 < E < \hat{E}$. Moreover E cannot jump from a value greater than E to zero for $X > 0$ for this would require a jump in ψ, so for paths which involve $E > \hat{E}$ at some stage, E will be constant at that level until $X = 0$. It may be shown[5] that for $E \geqslant \hat{E}$ paths associated

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4 If $\psi < u'(\hat{E})$, and $H(E) - u(E) - \psi E$, then $E \neq 0$, since if it were:

$$H(E) > u(\hat{E}) - u'(\hat{E}).\hat{E} = u(\hat{E}) - (u(\hat{E}) - u(0)) = u(0) = H(0).$$

Also, for $E \in (0, \hat{E})$,

$$\begin{aligned} H(\hat{E}) - H(E) &= (u(\hat{E}) - u(0)) - (u(E) - u(0)) - \psi(\hat{E} - E) \\ &> (u(\hat{E}) - u(0)) - (u(E) - u(0)) - u'(\hat{E})(\hat{E} - E) \\ &> (u(\hat{E}) - u(0)) - \frac{E}{\hat{E}} (u(\hat{E}) - u(0)) - u'(\hat{E})(\hat{E} - E) \\ &= \frac{\hat{E} - E}{\hat{E}} [(u(\hat{E}) - u(0)) - \hat{E}.u'(\hat{E})] = 0. \end{aligned}$$

So we can assume an interior solution and differentiate H to obtain $u'(E) = \psi < u'(\hat{E})$ and $E > \hat{E}$ (since $H'(E) = u'(E) - \psi > 0$ in some neighbourhood of $\hat{E}$).

5 For $E > \hat{E}$, $[u(E) - u(0)]/E$ is a decreasing function of E. Since the present value integral equals

$$Tu(0) + X(0) \left(\frac{u(E) - u(0)}{E}\right)$$

this will be greater the lower is E, so long as $E > \hat{E}$.

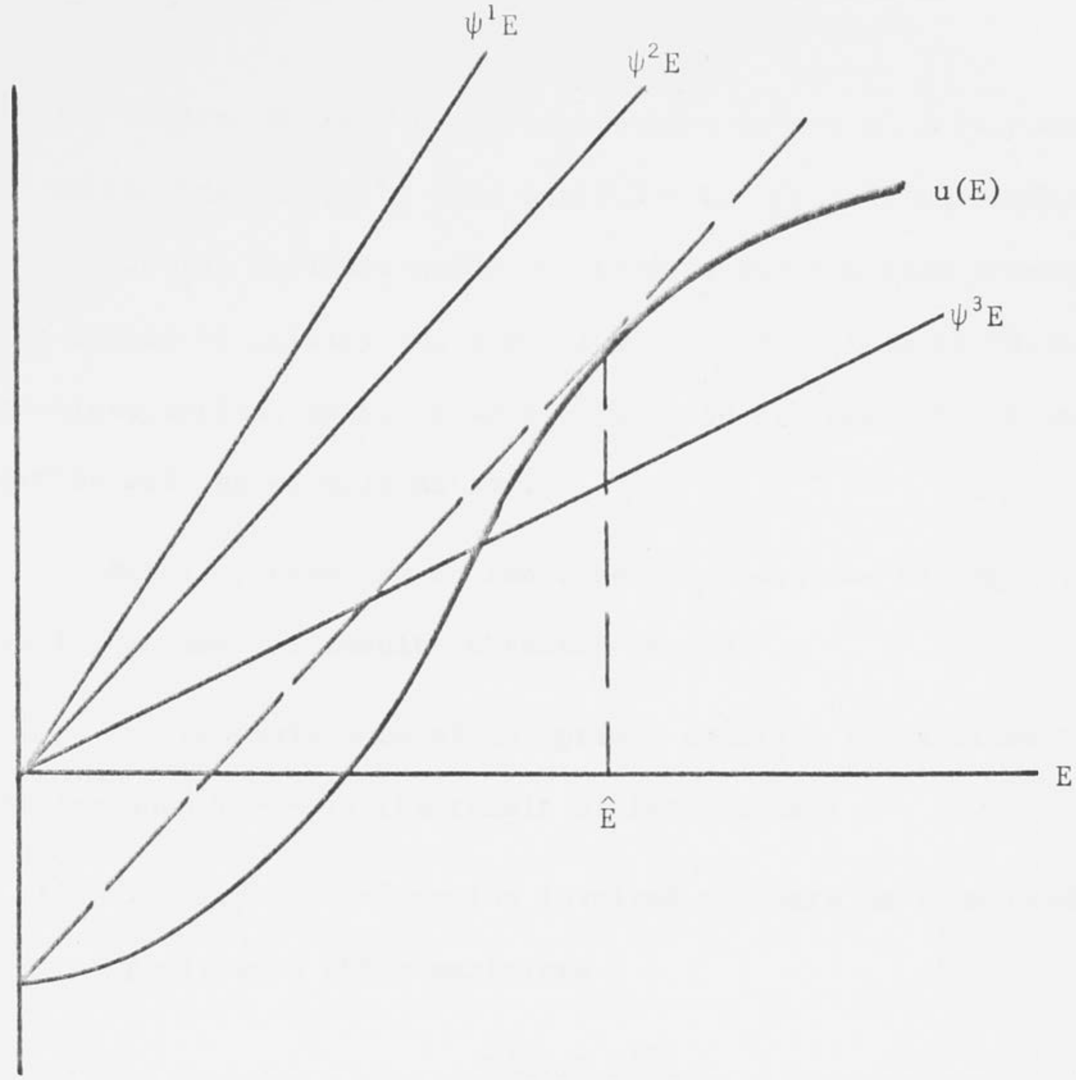

Figure 3.4

with lower values of E have a higher present value integral than paths for which E is high. Since X(T) is necessarily zero, for low values of T ($T < X(0)/\hat{E}$) the optimal path will entail E being set at that constant level consistent with exhausting the resource at time T. When $T = X(0)/\hat{E}$, then E will be set at $\hat{E}$ for the whole programme just running X to zero at time T. For longer planning horizons ($T > X(0)/\hat{E}$) clearly the appropriate positive level of extraction is $\hat{E}$ (since $\hat{E}$ dominates all higher values of E when it is feasible, and lower values of E have already been shown to be non-optimal). This will mean however, that for some of the programme E should be zero. Moreover, since when $\psi = u'(\hat{E})$, $H(0) = H(\hat{E}) = u(0)$ both $E = 0$ and $E = \hat{E}$ will maximize H and there is therefore, nothing to stop E jumping to zero before X is exhausted, later jumping back up to $\hat{E}$, etc. until X = 0. Clearly all such paths (and there are an infinite number of them) yield the same present value for the stream of utility and are all optimal solutions to (3.4). There is no unique optimal path. However, the path for which $E = \hat{E}$ until exhaustion will do as well as any.

Before proceeding to the case of a positive discount rate let us briefly assess the results obtained above:

The non-existence of an optimal solution for a class I u-function when $T = \infty$ is the result of two factors:

(i) The type of u-function involved prevents us from finding a positive E which maximizes

$$\frac{u(E) - u(0)}{E} .$$

Because this expression, which we will term "average excess utility" (AEU),[6] is always greater than marginal utility, it is a decreasing function of E so that the same number of total units of extraction would yield more "total excess utility" (TEU)

$$TEU \equiv \int_0^T (u(E) - u(0))dt$$

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[6] This concept of average excess utility has significance here for two reasons:

(i) We are interested in the excess of utility from extraction over the utility derived when all consumption is derived from the alternative source ($\bar{C}$);

(ii) Because we are trying to find the intertemporal allocation of resource extraction which gives us the highest total utility summed over all periods and because all periods are valued equally, we would like to know whether *each unit* of extraction is yielding as much excess utility as possible — i.e. is $\frac{u(E) - u(0)}{E}$ being maximized?

if spread as thinly as possible over the given time than if concentrated in a shorter interval of time (the AEU being higher in the former case). The TEU is therefore seen to rise as E moves closer to the E which maximizes AEU ($E \to 0$). When a positive E can be found to maximize AEU (as in the case of class II and III u-functions) it constitutes the optimal extraction level for long time horizons.

(ii) The absence of a discount rate with the present generation wanting the same benefits for all future generations as for itself means that when a class I u-function is involved the TEU over all periods will be greater the lower the level of extraction at any point in time to the point where extraction is zero in all periods. Given the scarcity of the resource, a positive discount rate may well be needed to help ration the limited stock of the resource among generations.

The absence of discounting is also responsible for the indeterminacy arising in the case of class II and III functions. Quite simply, without a discount rate we have no criterion for choosing between points in time and therefore no criterion for establishing one policy as uniquely optimal. We now turn to the case of a positive discount rate.

Case 2: $\delta > 0$:

As we would expect we again find that the class I and class II-III functions yield results which are superficially different but which are nevertheless related. In case 1 above, because a community faced with a class I function was prevented from attaining an interior maximum for average excess utility, in the infinite horizon case it attempted to approach that ideal situation by running the level of extraction to zero (a boundary maximum for AEU) and there was no optimal solution. For a different reason there were an infinite number of optimal solutions for class II-III functions. Naturally the introduction of a positive discount rate changes this situation.

(a) Class I u-function:

The interior solution to (3.5), (3.6) and (3.8) is characterized by the following relationships:

$$E > 0 \Rightarrow \lambda_1 = \lambda_2 = 0$$

$$\dot{\psi} = \psi\delta$$

$$\text{and} \qquad \psi = u'(E) \; .$$

Hence

$$\dot{E} = \frac{u'(E)\delta}{u''(E)} < 0 \quad \text{for} \quad \delta > 0 \; .$$

Thus E falls over time as the resource is depleted. For all finite T $X(T) = 0$. It turns out to be optimal for large T to run $E(t)$ continuously to zero so that it reaches zero at the time at which X is exhausted. In the ψ-X phase-plane (Figure 3.5(a) shows some of the paths satisfying the necessary conditions in ψ-X space) the optimal trajectory is the highest feasible path on or below path α in Figure 3.5(a). Optimality is established using the following comparison of integrals proof:

Let asterisk superscripts denote the path claimed to be optimal (viz. the longest exhaustion path feasible in the time available). Then we compare the present value integral along this path

(P*) with the present value along any other feasible Pontryagin path (i.e. some lower path in Figure 3.5(a)).

$$P^* - P \equiv \int_0^T [u(E^*) - u(E)]e^{-\delta t}dt$$

$$> \int_0^T u'(E^*)(E^* - E)e^{-\delta t}dt$$

(since u is strictly concave in E)

$$= \int_0^{T^*} u'(E^*)(E^* - E)e^{-\delta t}dt$$

(where T* is the time at which X is exhausted along the path claimed to be optimal; since this is the longest feasible exhaustion path

$E^*(t) = E(t) = 0 \ \forall_t \in (T^*, T])$

$$= \int_0^T \psi^*(t)e^{-\delta t}(E^* - E)dt$$

$$= \left[\psi_0^*(X - X^*)\right]_0^{T^*} = 0 \quad \text{since } X(T) = X^*(T) = 0$$

$$\text{and} \quad X(0) = X^*(0) = 0 \ .$$

Thus, in terms of Figure 3.5(a) the optimal trajectory will be the highest feasible path on or below path α. For a time horizon of T_1 say, (Figure 3.5(b)) path γ will be optimal. For a longer time horizon, say T_2 it will be feasible to spread extraction more over time so that path β is optimal. For time horizons greater than or equal to T_3, path α is optimal. In particular as $T \to \infty$ path α remains optimal. Provided $u'(0) < \infty$ (which is the likely situation when $\bar{C} > 0$) the resource will be exhausted in a finite time. This follows because

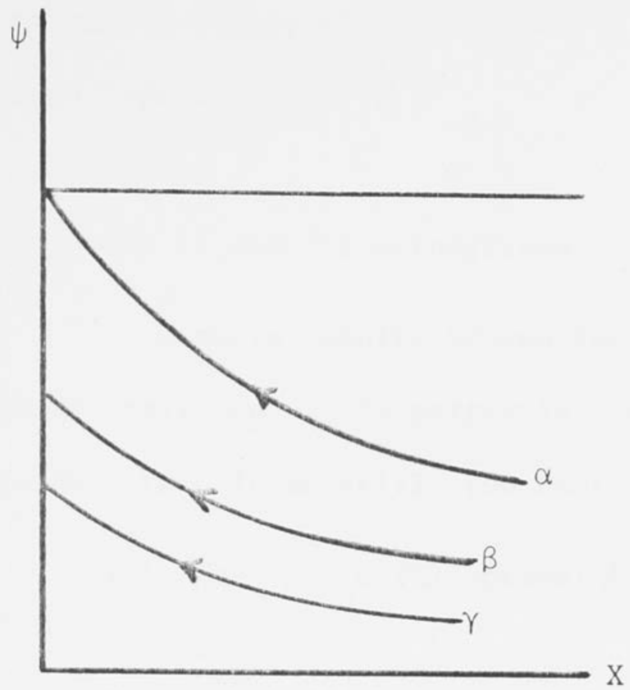

Figure 3.5(a)

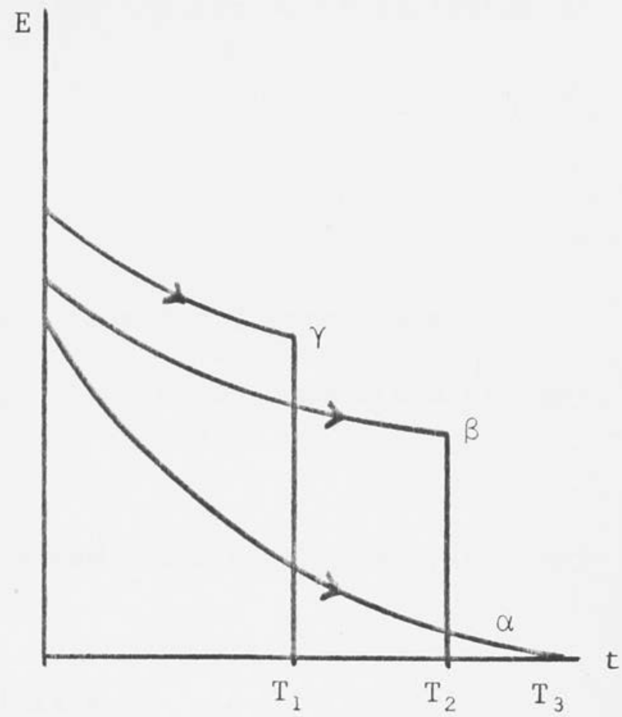

Figure 3.5(b)

$$\psi(t) = \psi_0 e^{\delta t} = u'(E) + \lambda_2 - \lambda_1$$

and when $\quad E > 0 \quad \psi_0 e^{\delta t} = u'(E)$;

therefore E cannot be positive for an infinite period of time since

$$\lim_{t \to \infty} \psi(t) = \infty \ .$$ [7]

Finally it is noted again that for long time horizons optimal E will fall continuously to zero reaching zero at the time (finite) at which X reaches zero. For the remainder of the programme the economy will gain all its consumption from $\bar{C}$, the alternative source. In the case where $\bar{C} = 0$ and $u'(0) = \infty$ the exploitation of the resource will be spread over an infinite time period with E going asymptotically to zero. As an intuitively appealing conjecture it would seem that as the

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7 After X = 0, if any of the programme remains, the multipliers λ_1 and λ_2 can change in such a way that ψ continues to grow exponentially over time.

alternative source of consumption, $\bar{C}$, falls the period of exploitation should increase.

(b) Class II and III u-functions:

Certain results proven for the case $\delta = 0$ carry over automatically here. In particular regardless of the magnitude of the discount rate it is still true that:

(i) When $\psi < u'(\hat{E})$ optimal E is the solution to $\psi = u'(E)$ with $E > \hat{E}$;

(ii) when $\psi > u'(\hat{E})$, $E = 0$ maximizes H;

(iii) when $\psi = u'(\hat{E})$ the maximum value of H is u(0), attained when $E = 0$ or $E = \hat{E}$;

(iv) it is never optimal to operate in the interval $(0, \hat{E})$.

When $\delta > 0$ for long time horizons the optimal path is the one for which E declines steadily to $\hat{E}$ reaching $\hat{E}$ at the time at which the resource is exhausted (path α in Figure 3.6). E then jumps to zero and consumption is $\bar{C}$ until the end of the plan. For shorter time horizons (when there is insufficient time for exhaustion of the resource along path α) the longest feasible exhaustion path will be optimal (This will be the highest path – say path β – below path α in Figure 3.6(a)).

Optimality is established as follows. Let asterisks denote the path claimed to be optimal (the longest feasible exhaustion path on or below α in Figure 3.6(a)) and let T* denote the exhaustion time along the said optimal path and T° the exhaustion time along some other arbitrarily chosen path. Then comparing present values:

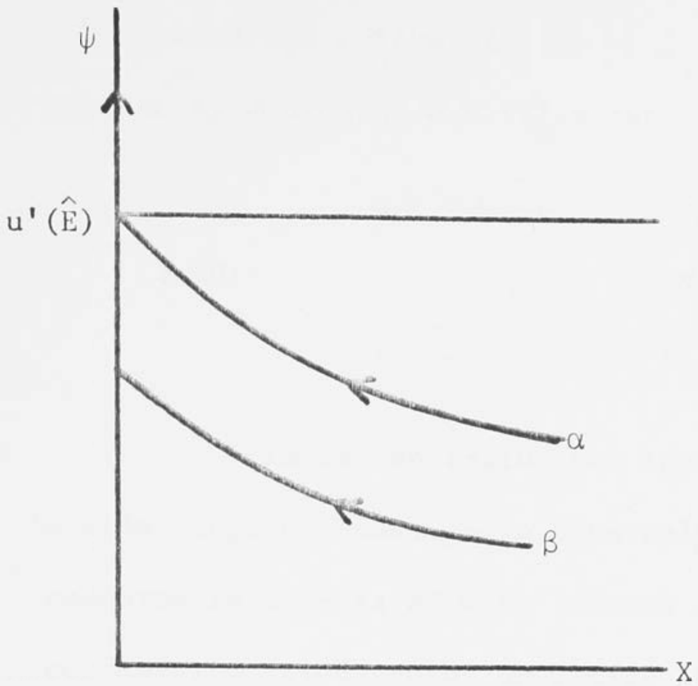

Figure 3.6(a)

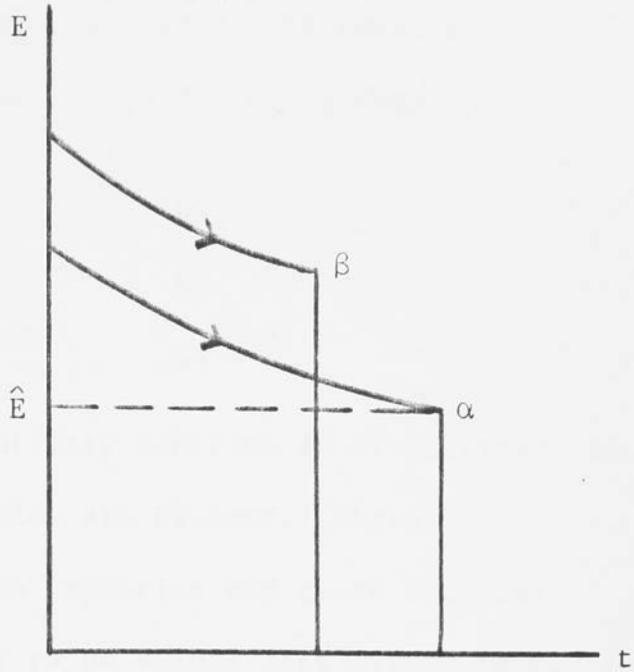

Figure 3.6(b)

$$P^* - P \equiv \int_0^T [u(E^*) - u(E)]e^{-\delta t}dt$$

$$> \int_0^T u'(E^*)(E^* - E)e^{-\delta t}dt$$

$$= \int_0^{T^\circ} \psi_0^*(E^* - E)dt + \int_{T^\circ}^{T^*} \psi_0^* E^* dt$$

$$= 0 \qquad\qquad \text{Q.E.D.}$$

III. DEPLETION OF A NON-RENEWABLE ASSET: UTILITY FUNCTION INCORPORATING A CONSERVATION MOTIVE AND/OR DEPLETION EFFECT

The simple form of utility function assumed, in section II, ignores an important phenomenon which is often associated with the depletion of a resource – a tendency to value the resource for its own sake independently of its value as a source of future consumption. Such a "conservation motive" (as we will refer to it) may be taken into account by employing a utility function of the following form:

$$
(3.10) \qquad \begin{array}{lll}
u = v(C, X) & v_C > 0 & v_{CC} < 0 \\
 & v_X > 0 & v_{XX} < 0 \\
v \in C^2 & v_{XC} \geqslant 0 & v_{CX} \geqslant 0 \, .
\end{array}
$$

Formulation (3.10) for the utility function is of relevance to a wide range of cases where externalities are present. Where the resource is associated with leisure its depletion may cause existing consumption (produced by depleting it) to be valued less highly (e.g. rutile mining and the defacing of beaches). Alternatively, the independent X in the utility function may be a dummy variable reflecting the uncertainty which the community feels about the feasibility of finding a substitute resource for producing its consumption goods when the present resource has been exhausted.

In addition there is the possibility of incorporating a *depletion effect* into the production function to reflect the increasing difficulty of extraction and/or the recourse to lower grades of the resource as it is depleted. This may be achieved by defining the production from resources as:

$$
\begin{array}{lll}
C_X = \phi(E, X) & \phi_X > 0 \, , \quad \phi_{XX} < 0 & {}_E > 0 \\
\phi \in C^2 & \phi_{EE} < 0 \, , & \\
 & \phi_{EX} = \phi_{XE} > 0 \, ,^{8} & \\
 & \phi_X(0, X) = \phi_{XX}(0, X) = 0 \, . &
\end{array}
$$

---

8 Note that, unlike the utility function, the production function is not allowed to be additively separable. Additive separability may be justified for a utility function but does not seem reasonable for a production function. Hence the assumption $\phi_{EX} = \phi_{XE} > 0$.

It is possible to subsume this depletion effect in the utility function and treat it in much the same way as the conservation motive. This is done as follows:

Let $u = v(\phi(E, X) + \bar{C}, X) \equiv u(E, X)$.

It is easily checked that $\text{sgn } u_E = \text{sgn } \phi_E$, $u_X > 0$, $u_{EE} < 0$ and $u_{XX} < 0$.

However because:

$$u_{EX} = u_{XE} = v_{CC}\phi_E\phi_X + v_{CX}\phi_E + v_C\phi_{EX}$$

the cross partials may be either positive or negative. Certainly, in the absence of a depletion effect we can say that $\text{sgn } u_{EX} = \text{sgn } \phi_E$, and, if we follow the not unreasonable assumption of section II that $\phi_E > 0$, then $u_{EX} = u_{XE} \geqslant 0$. However, in such a case $u_{EX} = u_{XE} = 0$ if and only $v_{CX} = v_{XC} = 0$, so that if there is no depletion effect then the u-function is additively separable if and only if the v-function is additively separable. We will initially confine our attention to this case in order to understand the basic structure of the problem. The optimization problem here is simply stated as that of selecting E(t) and X(T) to:

$$(3.11) \qquad \underset{E(t),X(T)}{\text{Max}} \int_0^T u(E, X)e^{-\delta t}dt \qquad \delta > 0 \text{ constant}$$

$$\text{s.t.} \quad \dot{X}(t) = -E(t) \qquad X(0) = X_C$$

$$X(t) \geqslant 0$$

$$E(t) \geqslant 0$$

$$\dot{X}(t) = 0 \quad \text{when } X(t) = 0 \,.$$

The necessary conditions to be satisfied by a solution to this problem are:

$\exists$ a continuous function, ψ, such that:

$$\dot{\psi} = \psi\delta - u_X \tag{3.12}$$

$$\psi = u_E(E, X) + \lambda_2 - \lambda_1 \, , \tag{3.13}$$

$$\psi(T)X(T) = 0 \qquad \text{for } T < \infty \, , \tag{3.14}$$

$$\begin{cases} \lambda_1 \geqslant 0 & \lambda_1\dot{X} = \lambda_1 X = 0 \, , \quad X \geqslant 0 \, , \\ \lambda_2 \geqslant 0 & \lambda_2 E = 0 \, , \qquad\qquad E \geqslant 0 \, . \end{cases} \tag{3.15}$$

(3.12) and (3.13) imply that when $E > 0$:

$$\dot{E} = \frac{u_E}{u_{EE}} \left(\delta - \frac{u_X}{u_E} + \frac{u_{EX}E}{u_E} \right) \tag{3.16}$$

and when u is additively separable in E and X:

$$\dot{E} = \frac{-u_E}{u_{EE}} \left(\frac{u_X}{u_E} - \delta \right) \; ; \quad \text{and} \tag{3.17}$$

$$\operatorname{sgn} \dot{E} = \operatorname{sgn} \left(\frac{u_X}{u_E} - \delta \right) \, . \tag{3.18}$$

Equation (3.18) simply states that E will tend to rise over time if the conservation motive is stronger than the preference for current consumption over future consumption (the discount motive) [$u_X > \delta u_E$] and E will tend to fall if the discount motive predominates.

We consider two cases:

Case 1: $u_X(0) \leqq \delta u_E(0)$:

This case may be thought of as that of a relatively weak conservation motive (weak relative to the discount motive). The graph of $\dot{E} = 0$[9] is shown in Figure 3.7(a) for $u_X(0) < \delta u_E(0)$ in the E-X plane together with the paths which satisfy conditions (3.12) - (3.15). Figure 3.7(b) shows these paths in the ψ-X phase plane. As in Section II, it is also optimal here to exhaust the resource when the planning horizon is finite (via (3.14) — along non-exhaustion paths we know that, in particular, $\psi(T) > 0$ and so (3.14) would be violated if X(T) were not zero) and it may be shown that in Figure 3.7(b) the highest feasible path on or below path α (in terms of Figure 3.7(a) the lowest path on or above α) yields a higher present value of the stream of utility than all other feasible[10] paths satisfying the necessary conditions and is therefore optimal (see Appendix for proof). For sufficiently large T, path α is optimal. Thus for long planning periods the optimal course is to exhaust the resource, running the level of extraction continuously to zero, E reaching zero at the same time as X. This solution will be less likely to be feasible the smaller is $\delta u_E(0) - u_X(0)$. In the limiting case where $\delta u_E(0) = u_X(0)$, along path α the relative strength of the conservation motive postpones exhaustion indefinitely meaning that exhaustion would take an infinite time if this path were followed, a

---

9
Let $L(E,X) = \dfrac{u_X}{u_E} - \delta$. Then $L_E = \dfrac{-u_X u_{EE}}{(u_E)^2} > 0$, and $L_X = \dfrac{u_E u_{XX}}{(u_E)^2} < 0$ so that $\left.\dfrac{dE}{dX}\right|_{\dot{E}=0} = \dfrac{-L_X}{L_E} > 0$.

10
As before some of the paths satisfying the necessary conditions may not exhaust the resource in the time.

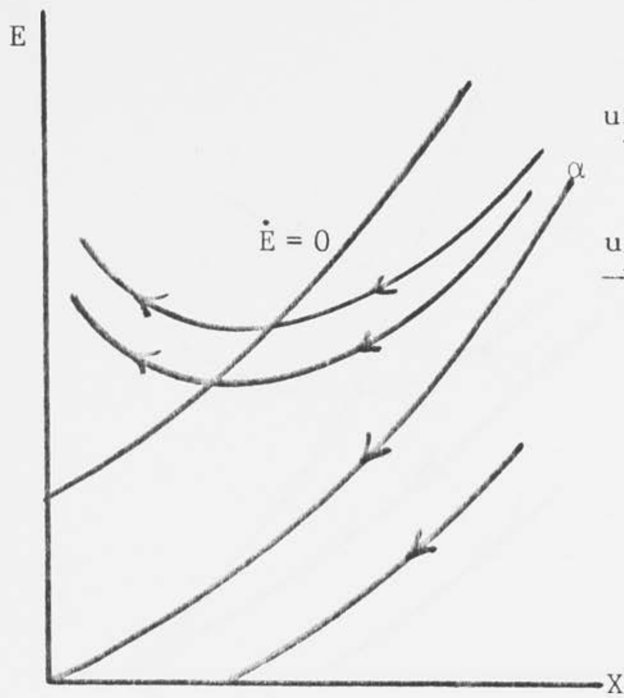

Figure 3.7(a)

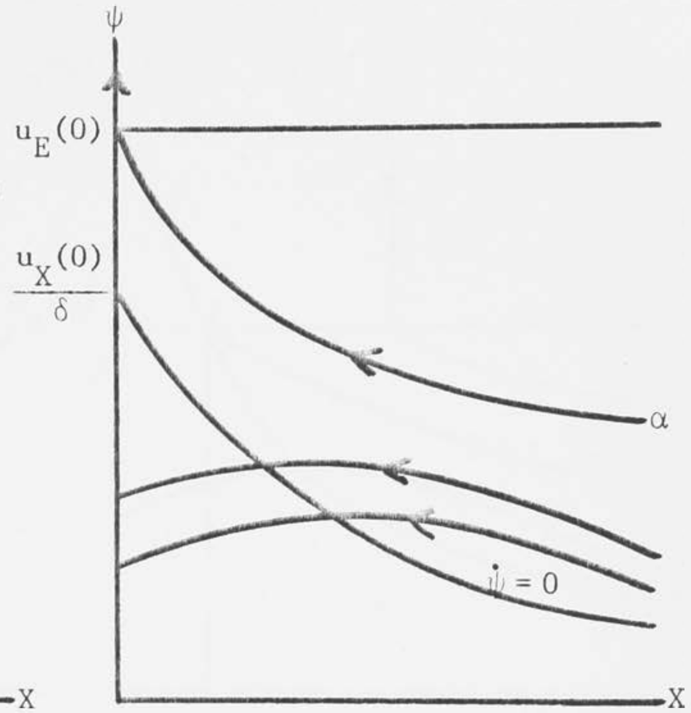

Figure 3.7(b)

fact which may be verified by noting that in this case the endpoint is the stationary solution for ψ and X. In such a case a more rapid exhaustion path will have to be followed for a finite horizon plan, although α would still be optimal if T were infinite. Clearly the stronger the conservation motive, the longer will be the optimal period of exploitation (assuming T is large enough to permit such flexibility of choice).

Case 2: $u_X(0) > \delta u_E(0)$:

With a strong conservation motive the $\dot{E} = 0$ locus will cut the X-axis to the right of the origin (Figure 3.8(a)). The path here which is comparable to path α for Case 1 will be β (Figures (3.8(a) and 3.8(b)) which leads to the equilibrium point P.

Obviously path β cannot be optimal for a finite horizon since it will take an infinite time even to reach P along it and for T finite

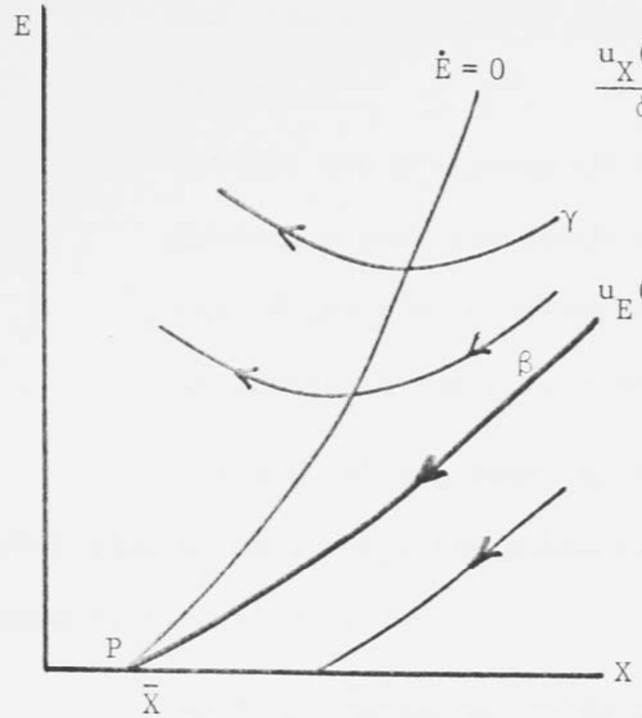

Figure 3.8(a)

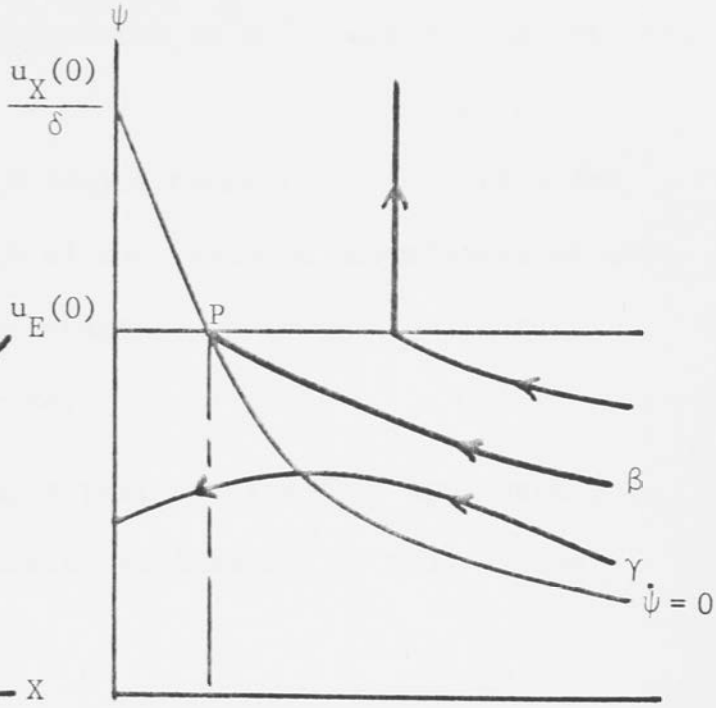

Figure 3.8(b)

we have already observed that X(T) = 0. It may be shown (see Appendix) that the lowest path in Figure 3.8(a) above β will be optimal (i.e. the highest path below β in Figure 3.8(b)) for T finite and path β will be optimal for T infinite. This means that although for all finite time horizons it is optimal to exhaust the resource, when the planning period is infinite it is optimal to leave a stock, $\bar{X}$, of the resource unexploited. The apparent paradox embodied in this result is resolved when the nature of the optimal path for a finite horizon is examined more closely. For a particular finite horizon the optimal path may be path γ (Figure 3.8). Along this trajectory extraction will initially decline (in the earlier stages of the programme the discount motive is naturally stronger than the conservation motive) but will eventually rise. The ultimate rise in E is attributable to a combination of:

(a) as time goes on the conservation motive increases in importance ($u_X(X)$ rises) relative to the discount motive —

this tends to make it attractive to defer extraction into the future;

(b) because the programme is finite there is no incentive for leaving a positive stock of the resource unexploited at the end of the plan — on the contrary there is a definite incentive to run X to zero.

(b) and (a) together imply that as time T is approached, E will rise so as to meet the exhaustion requirement as well as the conservation motive.

As T is increased the date of exhaustion is naturally moved on into the future so that the time at which E will have to start increasing in order to exhaust X is deferred (this is seen by noticing that for longer time horizons E will necessarily have to be spread more thinly over time; this will increase the valuation placed on an extra unit of current consumption relative to future consumption, $\delta u_E(E)$, and so X will be smaller by the time $u_X(X)$ becomes sufficiently large to cause E to rise over time). As $T \to \infty$ the exhaustion date for X is deferred indefinitely — there is thus no constraint on E to increase in the "final" stages of the programme. The thin intertemporal spread of E combined with the conservation motive, slows extraction down to the extent that E asymptotes to 0 over an infinite period of time and leaves an unexploited stock $\bar{X} > 0$ at $T = \infty$.

It is now time for us to see whether the above results still hold when we allow for u to be non-separable either because v is non-separable or because there is a depletion effect in production. We shall continue our taxonomic approach here and look initially at the cae where there is no depletion effect but v is non-separable and then we shall discuss the case where there is a depletion effect in production.

(a) v non-separable ($v_{CX} = v_{CX} > 0$), no depletion effect ($\phi_X \equiv 0$):

Now that the cross derivations u_{EX} enter the picture the behaviour of paths in the equivalent diagrams to Figures 3.7(a),3.8(a) in the E-X plane is difficult to determine. However we can be fairly explicit about the behaviour of the system in the ψ-X plane. The two main loci are $\dot{\psi} = 0$ and $E = 0$. The $E = 0$ locus is $\psi = u_E(0, X)$ and clearly has a positive slope. For the $\dot{\psi} = 0$ locus:

$$\frac{\partial \dot{\psi}}{\partial \psi} = \delta - \frac{u_{XE}}{u_{EE}} > 0$$

$$\frac{\partial \dot{\psi}}{\partial X} = \frac{u_{EE}u_{XX} - u_{XE}u_{EC}}{-u_{EE}} > 0$$

(if u is concave in E and X).

Thus $$\left.\frac{d\psi}{dX}\right|_{\dot{\psi}=0} < 0 .$$

As for the separable cases we consider two alternatives:

Case 1: $u_X(0, 0) \leqslant \delta u_E(0, 0)$:

In this case (see Figure 3.9), $\dot{\psi} = 0$ lies completely below $E = 0$[11], the only possible point of intersection in the two loci being

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11 Suppose this is not the case and $\dot{\psi} = 0$ intersects $E = 0$. Then above the $E = 0$ locus by definition $E = 0$ and $\dot{\psi} = \psi\delta - u_X(0, X)$. Then $\dot{\psi} = 0$ cuts the vertical axis at $\frac{u_X(0, 0)}{\delta}$ and so $u_X(0, 0) > \delta u_E(0, 0)$ which is a contradiction.

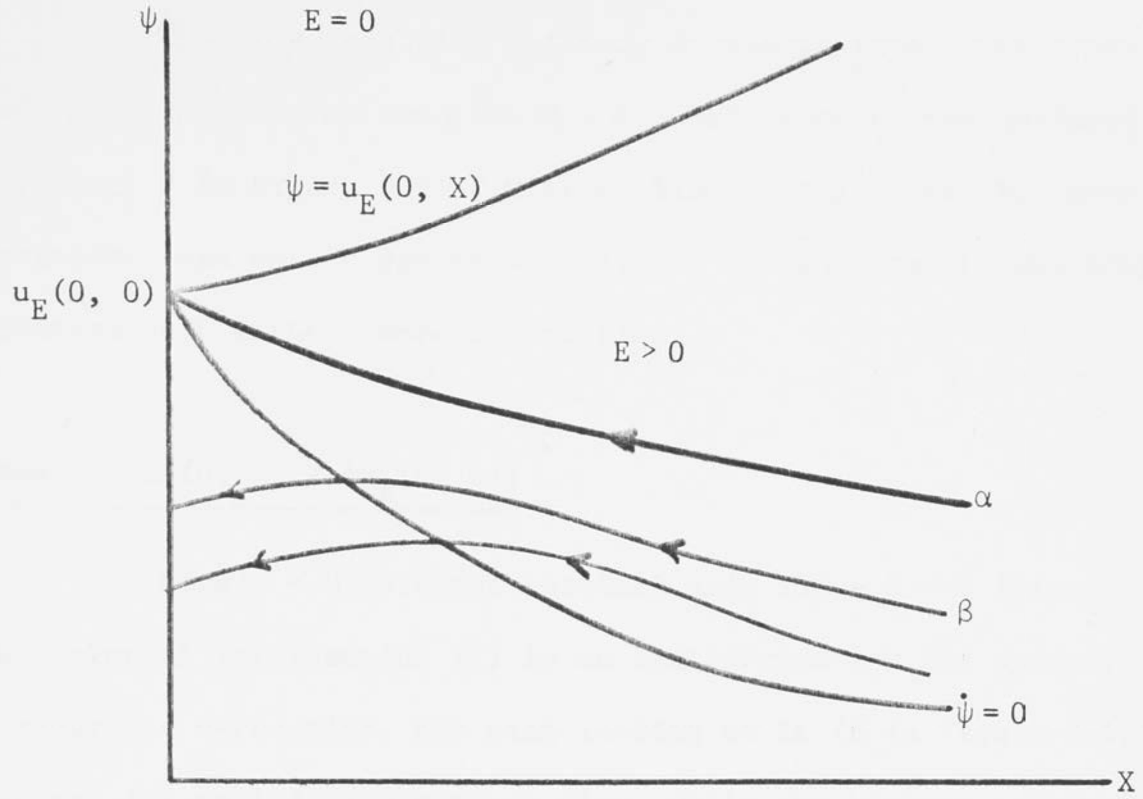


Figure 3.9

on the vertical axis (and this is only possible in the limiting case, $u_X(0, 0) = \delta u_E(0, 0)$).

It is proven in the Appendix that the optimal path here is the highest path on or below path α in Figure 3.9.

For long time horizons it will be optimal to follow trajectory α exhausting the resource in a finite time (except when $u_X(0, 0) = \delta u_E(0, 0)$ when an infinite time is required) and consuming $\bar{C}$ for the last stage of the programme when the path moves up the vertical axis. Along path α

$$\operatorname{sgn} \dot{E} = \operatorname{sgn}\left(\frac{u_X}{u_E} - \delta - \frac{u_{EX}E}{u_E}\right) .$$

But along α, $\dot{\psi} > 0$ so that using (3.12) and (3.13) $\frac{u_X}{u_E} - \delta < 0$ and so $\dot{E} < 0$.

Thus for long time horizons E will decline continuously to zero reaching zero at the time X = 0. For shorter time horizons (along, say, path β in Figure 3.9) E will decline initially and for more of the programme than when u was separable, but its subsequent behaviour is indeterminate without more information.

Case 2: $u_X(0, 0) > \delta u_E(0, 0)$:

Here $\dot{\psi} = 0$ cuts the vertical axis above E = 0 (Figure 3.10).[12] The point of intersection (P) is an equilibrium for the system. As for a separable u-function, the path leading to it (β in Figure 3.10) is optimal for an infinite horizon plan, while the highest path below it which is feasible for a given finite horizon is optimal for that horizon. As in Case 1, E declines continuously to zero along β and a positive stock $\bar{X}$ remains at $T = \infty$. Along the optimal path for a finite period plan, X(T) = 0. Thus the results for a separable v-function carry over when v is non-separable and there is no depletion effect in production.

(b) No conservation motive, depletion effect in production:

If we assume that $u_E(0, X^*) = 0$ for some $X^* > 0$, (it is likely that as X is severely depleted costs will rise to the point where the marginal cost of production exceeds the marginal product of the resource) then the boundary of the region E = 0 (namely $\psi = u_E(0, X)$ will cut the X-axis at $X^* > 0$ (Figure 3.11). The exact form of $\dot{\psi} = 0$ is not

---

12 Suppose $\dot{\psi} = 0$ does not pass above E = 0 then $\exists$ E > 0: $\frac{u_X(E, 0)}{\delta} \leqslant u_E(0, 0)$. But $u_X(E, 0) > u_X(0, 0)$. Thus $\delta u_E(0, 0) > \delta u_X(0, 0)$ which is a contradiction.

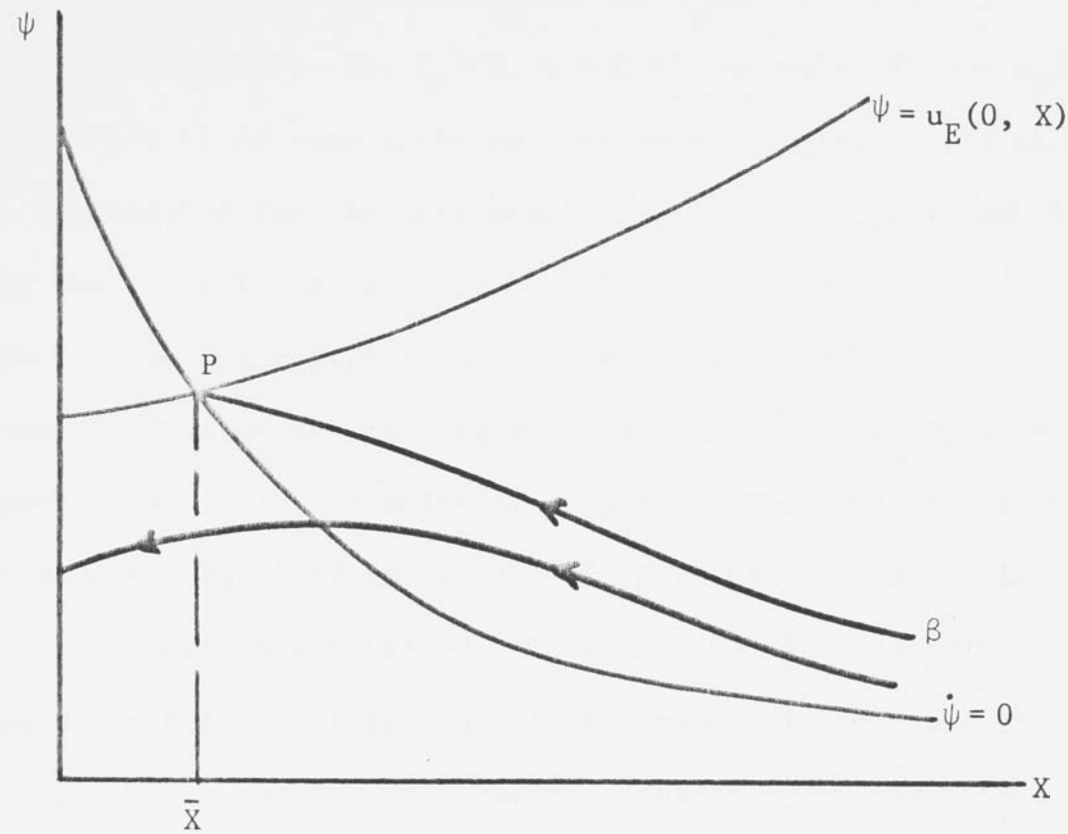
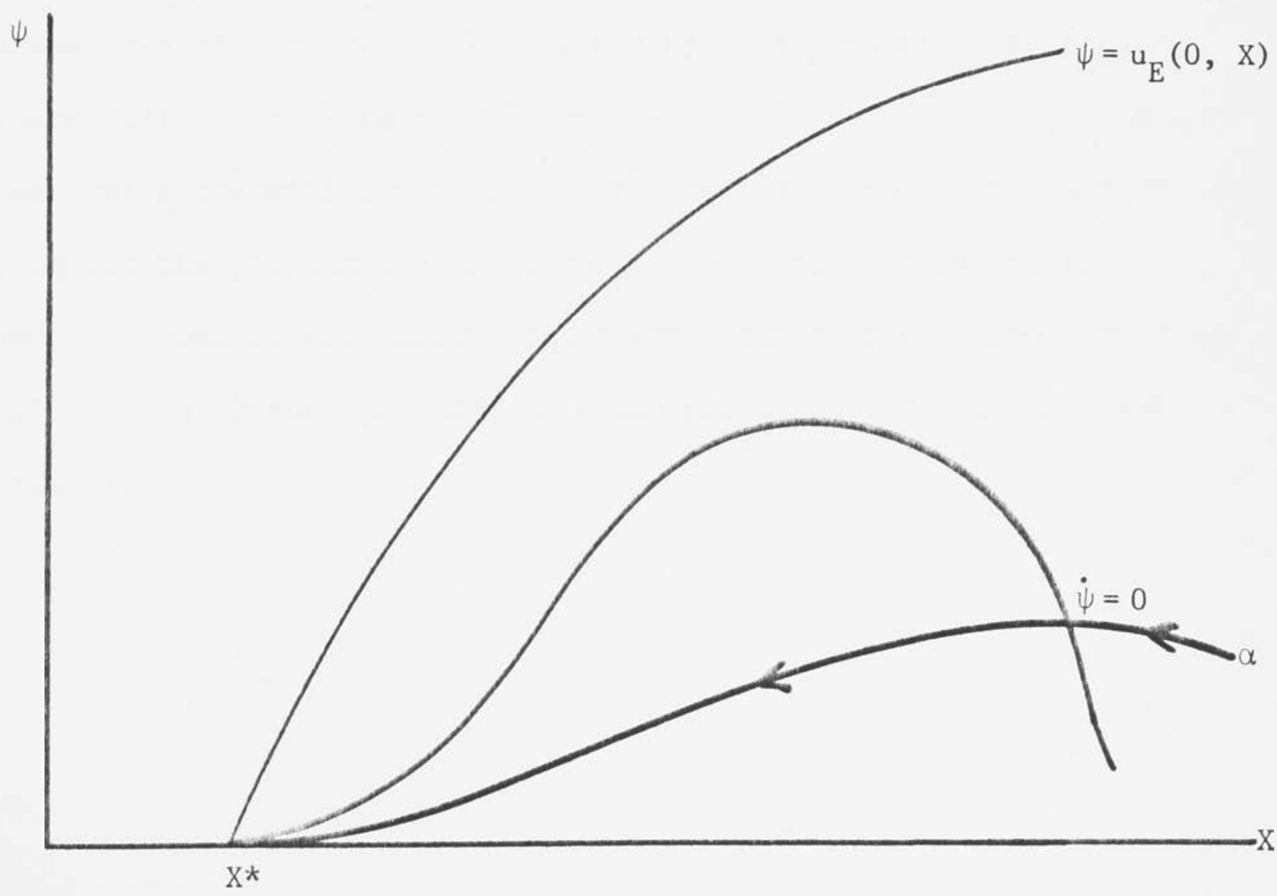


Figure 3.10



immediately clear. However, As $E \to 0$, $u_X(E, X) \to 0$ and $\dot{\psi} \to \psi\delta$, so that $\dot{\psi} = 0 \to$ the X-axis. For $E > 0$, $\dot{\psi} = 0$ if and only if $\psi\delta = u_X(E, X) > 0$. In addition it is reasonable to suppose that $\psi_X(E, X) \to 0$ as $X \to \infty$ (i.e. the depletion effect is irrelevant when the resource is not scarce). Thus the $\dot{\psi} = 0$ locus will have the form shown in Figure 3.11. There is clearly a unique equilibrium for the system of differential equations in ψ and X. This is not especially surprising since $u_X(0, 0) = 0$ and is greater than $\delta u_E(0, 0)$ which is negative. Thus the sufficient condition for a unique equilibrium established for (a) above is satisfied here and similar results are obtained. In this case there is no possibility of exhaustion being optimal since it is impossible to cross the locus $\psi = u_E(0, X)$. The resource becomes increasingly costly to extract until a point is reached where it is not worth continuing to extract it. Thus for a finite time horizon, the transversality condition (3.14) implies that $\psi(T) = 0$, and the optimal path is therefore that for which ψ reaches zero at time T. As $T \to \infty$, the optimal path tends to path α (Figure 3.11). The larger is T the lower will be the terminal stock of X, however there will always remain a terminal stock of at least X^*. Throughout the programme a positive shadow price is attached to the resource — however this social valuation eventually declines over time until the end of the programme at which point the resource is deemed worthless.

APPENDIX

OPTIMALITY IN SECTION III

(a) Conservation motive, no depletion effect:

$$P^* - P = \int_0^T [u(E^*, X^*) - u(E, X)]e^{-\delta t}dt$$

$$= \int_0^T [u(E^*, X^*) - u(E, X^*) + u(E, X^*) - u(E, X)]e^{-\delta t}dt$$

$$> \int_0^T [u_E(E^*, X^*).(E^* - E) + u_X(E, X^*).(X^* - X)]e^{-\delta t}dt$$

$$\geqslant \int_0^T [u_E(E^*, X^*).(E^* - E) + u_X(E^*, X^*).(X^* - X)]e^{-\delta t}dt$$

(since $X^*(t) \geqslant X(t)$, $E^*(t) \leqslant E(t)$ $\forall_t$ and $u_{EX} = u_{XE} \geqslant 0$, without a depletion effect).

$$= \int_0^T \{[\psi^* - \lambda_2^* + \lambda_1^*](E^* - E) + [\psi^*\delta - \dot{\psi}^*](X^* - X)e^{-\delta t}dt$$

$$\geqslant \int_0^T \psi^*(E^* - E)e^{-\delta t}dt - [\psi^* e^{-\delta t}(X^* - X)]_0^T$$

$$+ \int_0^T \psi^*(E - E^*)e^{-\delta t}dt$$

$$- \int_0^T \lambda_1^* \, Ee^{-\delta t}dt$$

$$= \psi^*(T)e^{-\delta T}[X^*(T) - X(T)] - \int_0^T \lambda_1^* \, Ee^{-\delta t}dt$$

$$= \psi^*(0)[X^*(T) - X(T)] - \int_0^T \lambda_1^* \, Ee^{-\delta t}dt \; .$$

(i) When T is finite $X^*(T) = X(T) = 0$ and $\lambda_1^* = 0$ along the longest feasible exhaustion path. Hence $P^* - P > 0$.

(ii) When T is infinite, either $X^*(T) = X(T) = 0$ and $\lambda_1^* = 0$ $(\delta u_E(0, 0) > u_X(0, 0))$, or $\delta u_E(0, 0) < u_X(0, 0)$, in which case there is an equilibrium in ψ-X space and $\psi^*(T)e^{-\delta T} \to 0$ as $T \to \infty$. Also, in such a case $\lambda_1^* = 0$ for the entire programme, so $P^* - P > 0$.

(b) <u>No conservation motive, depletion effect in production</u>:

Because only one path satisfies all of the necessary conditions, Proposition 5 in [2] may be invoked to establish optimality.

CHAPTER 4

DISAGGREGATION OF THE BASIC MODEL

In the previous chapter simplifying assumptions were made concerning: (a) the nature of the economy's resource stock; (b) the structure of the economy itself; and (c) the nature of the process whereby natural resources are used in production. The particular assumptions which interest us here are:

(i) the resource stock is essentially homogeneous in quality;

(ii) the economy consists of a single sector;

(iii) there are no possibilities explicitly acknowledged for substitution between the resource and another factor.

In this chapter these assumptions will be relaxed and a more complex picture of the resource depletion problem will be given.

In Section I, two two-sector models of resource extraction will be analysed. In the first of these one of the two sectors will be a resources sector and the other will be a manufacturing sector. The resource sector sells its commodity to the other sector which uses it as an input for producing the economy's only consumption good. In the other model, there will be two consumption goods sectors, only one of which uses the resource. The two sectors are assumed to employ labour as the other productive input.

In Section II, a model containing two resources will be presented. Each resource may be used to produce the same consumption good.

I

TWO-SECTOR ECONOMIES

In this section we retain the assumption made in Chapter 3 that the resource is homogeneous in quality and examine the implications of disaggregating such a single resource economy into two-sectors. As already suggested we shall develop our model in two stages. In the first instance (so as to understand the optimal workings of a vertically integrated economy using a resource) we shall consider the case where one sector extracts the exhaustible resource and sells it all as an intermediate good to the other (manufacturing) sector. In this case the outcome of industrial activity is a single consumption good. The two sectors involved in its production are assumed to draw from a labour supply which is fully mobile between them so that an allocation of labour to processing the resource into a finished good is viewed as a *substitute* for an allocation of labour into producing the resource input.

In the second model presented the sector which extracts the resource also produces a final consumption good from it; within this sector labour and the resource input are complementary. The other sector in the economy produces another consumption good without using the resource. Thus, in this case the trade-off is between labour being used to produce one or the other of the two consumption goods.

1. THE RESOURCE AS AN INTERMEDIATE GOOD

We assume two productive inputs: labour and an exhaustible resource. Both inputs are used to produce the economy's single consumption good, however only labour is used in the extractive sector.

We also allow for a depletion effect in the extractive sector, reflecting the additional amount of the variable factor needed to produce a given extractive output as the resource becomes more scarce. This depletion effect is allowed for as in Chapter 3 by including the remaining unexploited stock of the resource at any time t ($X(t)$) as an independent argument in the extractive sector's production function. The extraction of the resource in period t ($-\dot{X}(t)$) is used up immediately as an input in the production of consumption goods. Labour is assumed to be indispensable in both sectors. The production relationships for the two sectors are given by:

$$C = F(L_1, -\dot{X}) \tag{4.1}$$

$$\dot{X} = -G(L_2, X) \ , \tag{4.2}$$

where $C \equiv$ consumption

$L_i \equiv$ labour input into the ith sector.

In the light of the assumptions specified above and the assumptions which it is usual to make, the functions F and G have the following properties:

$$(4.3) \begin{cases} F_i > 0 \ , \quad F_{ii} < 0 \ , \quad F_{ij} > 0 \ , \quad i \neq j \quad i,j = 1,2 \\ F(0, \dot{X}) = F_2(0, \dot{X}) \equiv 0 \end{cases}$$

$$(4.4) \begin{cases} G_i > 0 \ , \quad G_{ii} < 0 \ , \quad G_{ij} > 0 \ , \quad i \neq j \quad i,j = 1,2 \\ G(0, X) = G_2(0, X) \equiv 0 \qquad \forall_X \geqslant 0 \\ G(L_2, 0) = G_1(L_2, 0) \equiv 0 \qquad \forall_{L_2} \in [0, L] \ . \end{cases}$$

The total labour supply is assumed to be fixed and equal to L:

$$L_1 + L_2 \leqslant L \ . \tag{4.5}$$

The conventional static allocation problem to be solved here would involve selecting non-negative L_1 and L_2 (subject to (4.5)) for given X, so as to maximize:

$$u(F(L_1, G(L_2, X))) ,$$

where $$u'(C) > 0 , \quad u''(C) < 0 \quad \forall_C \geqslant 0 .$$

The necessary conditions to be satisfied by a solution to this problem are:

$$\begin{cases} u'F_1 + \lambda_1 = u'F_2G_1 + \lambda_2 = \lambda_3 & & \\ \lambda_1 \geqslant 0 & L_1 \geqslant 0 & \lambda_1 L_1 = 0 \\ \lambda_2 \geqslant 0 & L_2 \geqslant 0 & \lambda_2 L_2 = 0 \\ \lambda_3 \geqslant 0 & L - L_1 - L_2 \geqslant 0 & \lambda_3(L - L_1 - L_2) = 0 . \end{cases}$$

As is usual in these problems $\lambda_3 > 0$ (for $X > 0$) and $L_1 + L_2 = L$. Thus:

$L_1 = L$ and $L_2 = 0$ when $F_1(L, 0) \geqslant F_2(L, 0).G_1(0, X)$,

$L_1 = 0$ and $L_2 = L$ when $F_1(0, G(L, X)) \leqslant F_2(0, G(L, X)).G_1(L, X)$

and L_1, L_2 each lies between 0 and L when:

$$(4.6) \quad F_1(L_1, G(L_2, X)) = F_2(L_1, G(L_2, X)).G_1(L_2, X).$$

In other words, when an interior solution is possible labour will be allocated between the sectors so as to equalize the *direct* marginal product of labour in the manufacturing sector (F_1) to the *indirect* marginal product of labour in that sector when the labour is channelled through the extractive sector (F_2G_1). This static optimum will bear some relevance to the solution of the dynamic problem which we are to examine.

The intertemporal problem to be solved involves finding the allocation $(L_1(t), L_2(t))$ of labour between the sectors over time which will maximize the present value of the stream of utility from consumption from time 0 until time T, where T is parametrically fixed. The terminal stock, X(T), of the resource is endogenous. We thus wish to:

$$(4.7) \qquad \underset{L_1(t),\ L_2(t),\ X(T)}{\text{Max}} \int_0^T u(C)\, e^{-\delta t} dt$$

$$\text{s.t.} \qquad \dot{X} = -G(L_2, X)$$

$$L_1 \geqslant 0$$

$$L_2 \geqslant 0$$

$$L - L_1 - L_2 \geqslant 0$$

$$X \geqslant 0$$

$$C = F(L_1, G(L_2, X)) \ .$$

The necessary conditions for the existence of such an optimal programme are that $\exists$ a continuous $\psi(t)$, such that:

$$\mathcal{L} = u(F(L_1, G(L_2, X))) - \psi G(L_2, X) + \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3(L - L_1 - L_2) - \lambda_4 G(L_2, X)$$

$$(4.8) \qquad \dot{\psi} = \psi(\delta + G_2) - u'F_2G_2 + \lambda_4 G_2$$

$$(4.9) \qquad u'F_1 + \lambda_1 = \lambda_3 = u'F_2G_1 - \psi G_1 + \lambda_2 - \lambda_4 G_1$$

$$(4.10) \qquad \psi(T)\, X(T) = 0 \quad \text{for} \quad T < \infty$$

$$(4.11) \qquad \begin{cases} \lambda_1 \geqslant 0 & L_1 \geqslant 0 & \lambda_1 L_1 = 0 \\ \lambda_2 \geqslant 0 & L_2 \geqslant 0 & \lambda_2 L_2 = 0 \\ \lambda_3 \geqslant 0 & L - L_1 - L_2 \geqslant 0 & \lambda_3(L - L_1 - L_2) = 0 \\ \lambda_4 \geqslant 0 & X \geqslant 0 & \lambda_4 X = \lambda_4 \dot{X} = 0 \ . \end{cases}$$

There are three alternative policies open to the economy:

$$\text{Policy A:} \quad L_1 = L \; ; \quad L_2 = 0$$

$$\text{Policy B:} \quad L_1 = 0 \; ; \quad L_2 = L$$

$$\text{Policy C:} \quad L_1 > 0 \; ; \quad L_2 > 0 \; .$$

A phase diagram showing the paths satisfying conditions (4.8) - (4.11) is given in Figure 4.1. The equation of the AC/CA switching surface is given by

$$\psi = \frac{u'(F(L,\ 0))[F_2(L,\ 0).G_1(0,\ X) - F_1(L,\ 0)]}{G_1(0,\ X)}$$

and is positively sloped in ψ-X space.

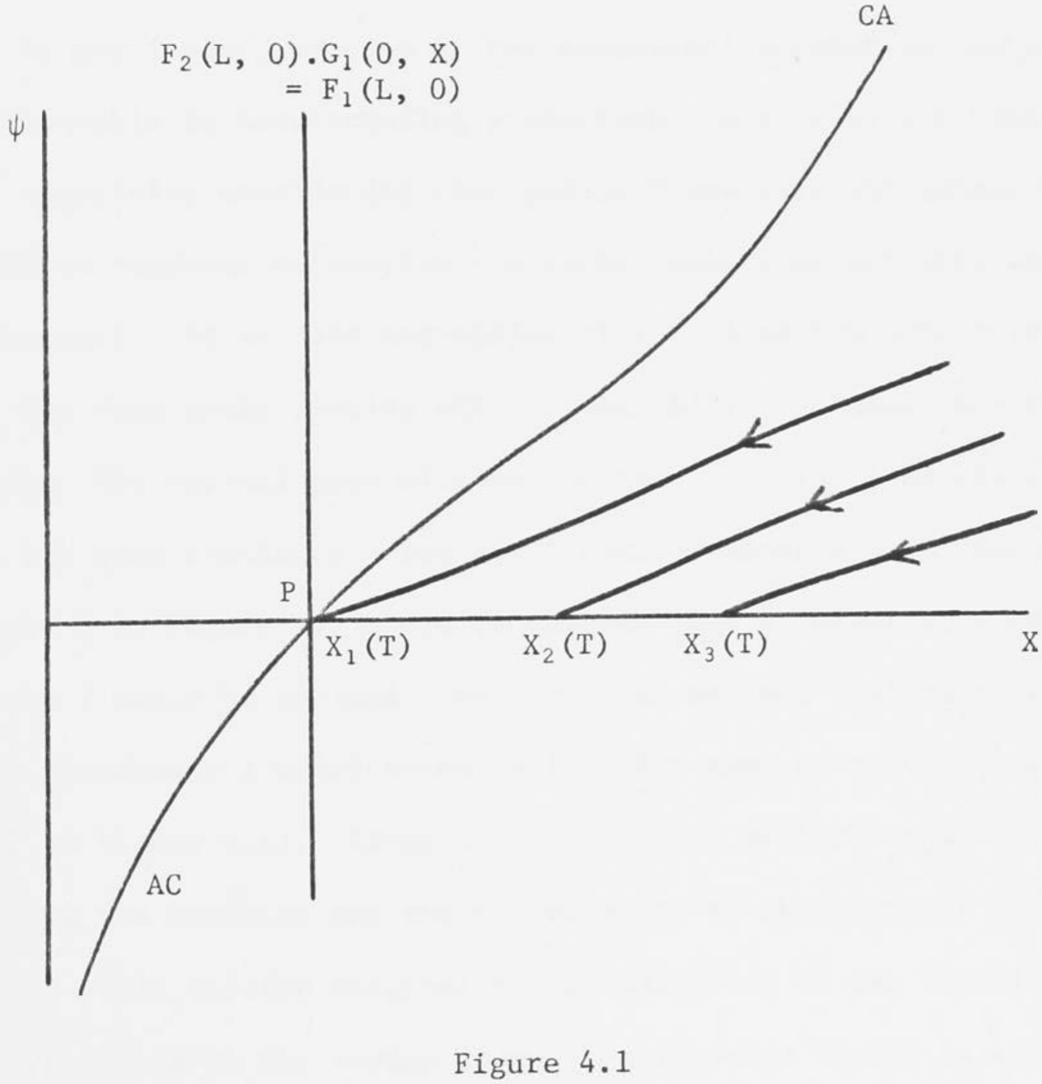


Figure 4.1

For a switch from policy A to policy C it is necessary that $\psi < 0$ at the switching surface and for a switch from C to A it is necessary that $\psi \geqslant 0$ at the surface. Switches between A and B are non-optimal because they involve a jump in ψ (since $F_1 - F_2G_1$ is a continuous decreasing function of L_1).

Equation (4.10) tells us that for all finite time horizons either the resource is exhausted or $\psi(T) = 0$. However the condition, $G(L_2, 0) = 0$, means that exhaustion of the resource along any Pontryagin path will take an infinite time. Thus, for any finite T, $\psi(T) = 0$. Because $\dot{\psi} < 0$ whenever $\psi < 0$ (and for some $\psi > 0$) this endpoint requirement immediately rules out all paths for which $\psi(t) < 0$ for any $t \leqslant T$. In particular, because of the additional assumption that labour is indispensable in manufacturing production, this rules out policy B (hardly surprising considering that policy B involves all labour being allocated to resource extraction — a rather pointless activity under the circumstances). It is also non-optimal for C to switch into A above the X-axis, for that would involve $\psi(T) > 0$ *and* $X(T) > 0$ (since $\dot{\psi} = \psi\delta$ for policy A). The optimal path will be the one for which ψ declines to zero in the time available. For short time horizons a path such as trajectory 3 in Figure 4.1 would be chosen. For a longer time horizon trajectory 2 would be optimal. As $T \to \infty$ the optimal trajectory will approach trajectory 1 which takes an infinite time to reach its endpoint (point P in Figure 4.1). Along all these paths policy C will be followed to the endpoint and the shadow price of the resource will fall over time. This falling marginal social valuation of the resource as an input corresponds to the rising labour requirements needed to extract a given number of units of the resource good as the resource is depleted. On the other hand it is impossible to say definitely in which direction the labour allocation will be moving although later we will attempt to identify the factors influencing the way in which labour moves from one sector to another over time.

It is interesting to note that for all time horizons the endpoint is the static optimum ($F_1 = F_2G_1$), and as the limiting case, $T = 0$, is approached the economy's optimal starting point would approach this static optimum ($\psi(0) \to 0$). There is thus a continuity of results between the myopic ($T = 0$) and intertemporal ($T > 0$) planning cases. We can also note that for the intertemporal problem the labour input into the extractive sector is less than the static optimum level at each point in time before the endpoint for the level of the resource stock (X) prevailing at the time (this follows from the fact that along policy C, sgn ψ = sgn$(F_2G_1 - F_1)$ and also from the fact that $F_2G_1 - F_1$ is a decreasing function of L_2).

It is also worth noting that in this model the conserving of a positive stock of the resource at the end of the programme is a consequence of the limited labour supply together with the assumption that the resource is dispensible in the production of manufacturing goods. In Figure 4.1, the vertical line, $F_2(L, 0).G_1(0, X) = F_1(L, 0)$, lies to the right of the ψ-axis, reflecting the dispensibility of resources in the consumption goods sector. As the resource becomes less dispensible this locus moves to the left and in the limiting case where it coincides with the vertical axis the resource is completely indispensible ($F(L_2, 0) = F_1(L_2, 0) = 0 \; \forall_{L_2 \in [0, L]}$). This case is illustrated in Figure 4.2. It is seen to be qualitatively the same as the case already examined, the only difference being the optimality of exhaustion over an infinite time.

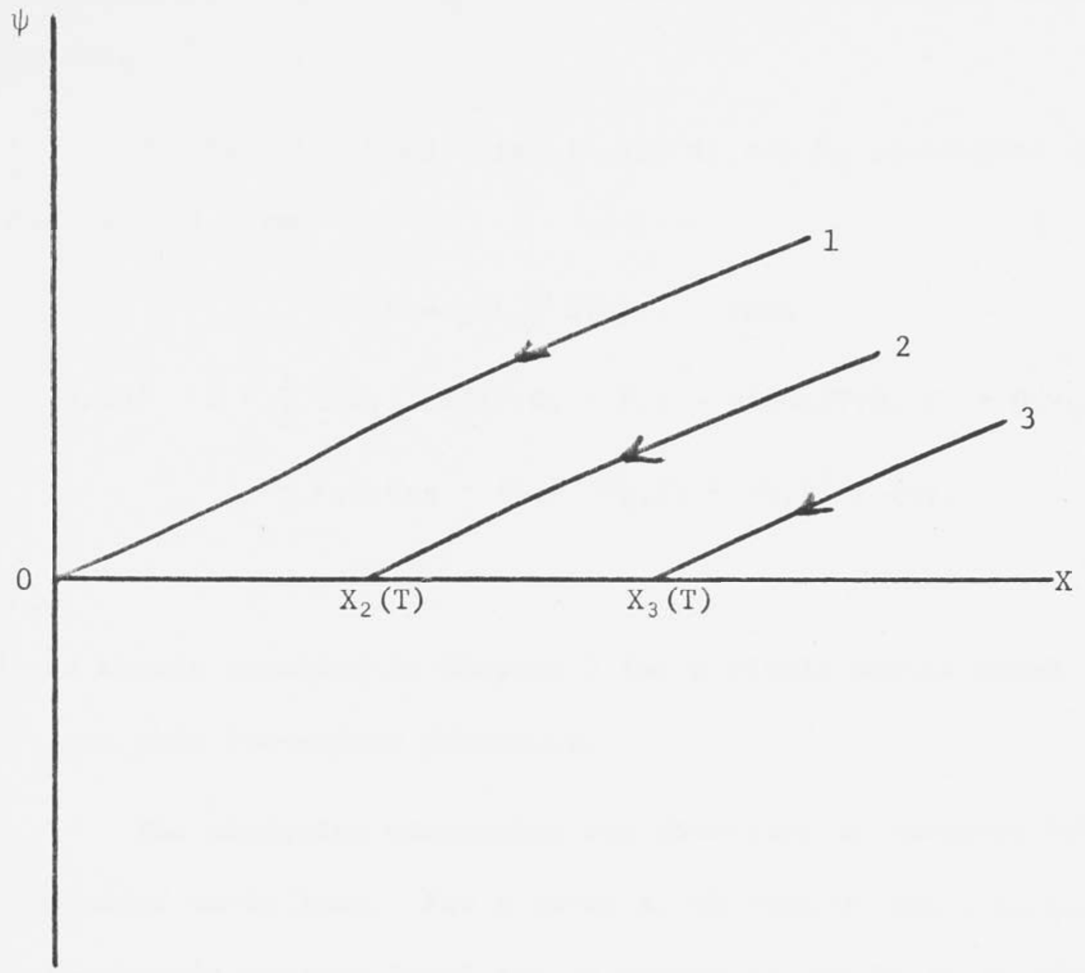


Figure 4.2

One question which remains unanswered concerns the time paths of L_1 and L_2 and the factors determining their behaviour.

It is a routine matter to check that

$$(4.12)\quad \dot{L}_2 = \frac{1}{\Delta}\{(G_1)^2\dot{\psi} - G_{12}GG_1\psi - Gu'G_1G_2(F_{12} - F_{22}G_1) + G_1Gu''F_2G_2(F_2G_1 - F_1)\} + \frac{1}{\Delta}Gu'G_1F_2G_{12}$$

where $\Delta = u''G_1(F_2G_1 - F_1)^2 + u'G_1(F_{22}(G_1)^2 + F_{11} - 2F_{12}G_1) + u'F_1G_{11} < 0.$

Thus, the first of the two terms on the R.H.S. of (4.12) is positive and the second term is negative, so that the sign of $\dot{L}_2$ is ambiguous.

On the other hand using (4.12) it may be shown that when G is of the special form:

$$G = g(L_2)\ \phi(X)\ , \quad \text{then}$$

$$(4.13) \quad \dot{C} = \frac{1}{\Delta}\{(G_1)^2\ \psi\delta(F_2G_1 - F_1) - u'GG_2(F_1G_{11}F_2 + G_1F_{11}F_2 - F_1G_1F_{12} - (G_1)^2\ F_{21}F_2 - (G_1)^2\ F_1F_{22})\}$$

$$< 0$$

and the result obtained in Chapter 3 for a single sector model carries over into this two-sector framework.

The ambiguity concerning the direction of movement of L_2 may be explained as follows. For a given X, throughout the plan L_2 is less than the static optimum level and is moving closer to it as time goes on. If there were no depletion effect then the only way in which the static optimum would be approached as $t \to T$ would involve L_2 rising over time (as is easily verified using (4.12), in such a case $\dot{L}_2 = (G_1)^2\dot{\psi}/\Delta > 0$; this, of course, is not to say that this represents the optimal behaviour of the system when there *is no* depletion effect — a separate problem which will be examined shortly). However because of the depletion effect, as X declines there may be a tendency for $F_2G_1 - F_1$ to fall independently of L_2 (if this happens the implication of the depletion effect is obviously that the amount of labour which it is optimal to use in resource extraction is lower because costs in that sector are relatively higher) and so in order for the static optimum to be reached at time T, L_2 may have to decline for part or all of the programme. Certainly as $T \to \infty$, $L_2(t) \to 0$ as $t \to T$. Increasing extraction costs make it optimal to make production of the consumption good increasingly labour intensive.

When there is no depletion effect in the model the whole outcome is somewhat different and has much in common with the model in section II, Chapter 3. It is proven in Appendix 4.1 that the optimal path is the longest feasible path of exploitation for which $\psi \geqslant 0$. There are two cases to distinguish. They are:

(i) $F_2(L, 0)\ G_1(0) > F_1(L, 0)$, and

(ii) $F_2(L, 0)\ G_1(0) \leqslant F_1(L, 0)$.

In the second of these two cases there is a net loss in output incurred from the first unit of labour allocated to extraction. Not surprisingly in such a case it is never optimal to extract the resource. The optimal course is to follow policy A and set $\psi = 0$ for the whole plan. X is never run below its initial level. In the case (i) however, output may be increased by allocating some labour to extraction. For very short time horizons it will not be possible to exhaust the resource in the time available with $\psi \geqslant 0$. The transversality condition (4.10) will be satisfied by setting $\psi = 0$ for the whole programme. Thus when the planning period is short the resource is relatively plentiful and may be viewed in the same way as any other good. Accordingly the dynamic and static optima will coincide. For somewhat longer time horizons setting $\psi = 0$ for the whole plan will exhaust the resource earlier than is necessary. The optimal path will be the highest path on or below path α in Figure 4.3 which exhausts the resource in the time available. ψ will be positive and rising over time while the labour allocation to the extractive sector will fall over time. Policy C is operative until X = 0.

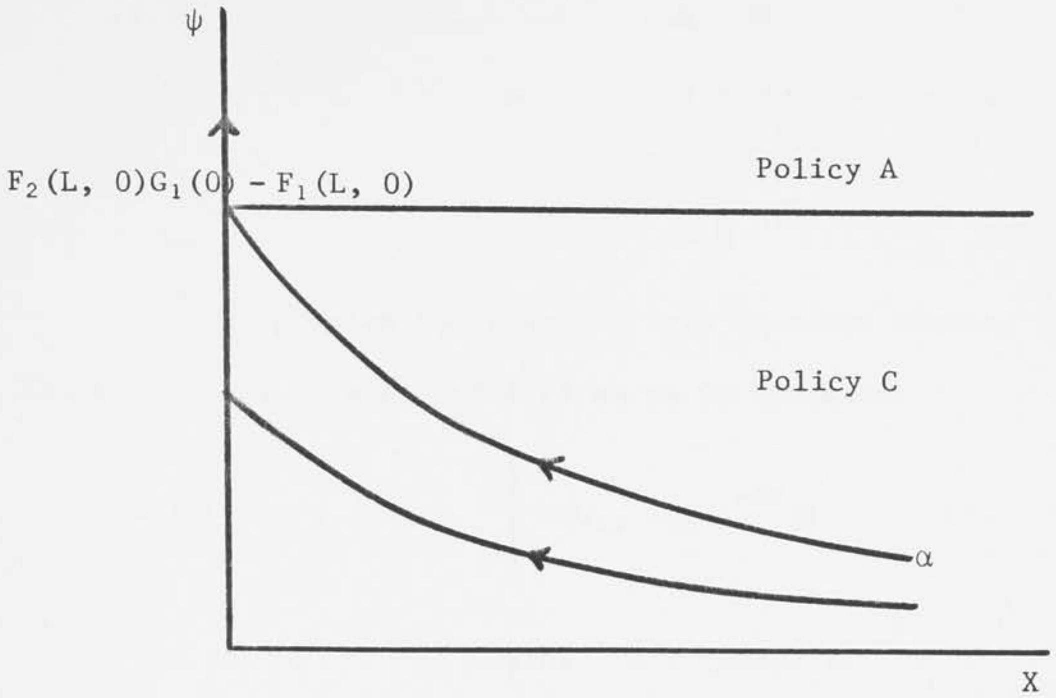


Figure 4.3

As for the model with depletion effect policy B is automatically ruled out once it is established that $\psi \geqslant 0$ along the optimal path. However here, with the possibility of exhaustion, the economy moves away from the static optimum over time rather than towards it.

2. A SECOND CONSUMPTION GOOD

Let us now extend our analysis to allow for a second consumption good in the economy. Consumption good no. 1 is produced using the resource according to:

$$C_1 = F(L_1, G(L_2, X)) \tag{4.14}$$

$$\dot{X} = -G(L_2, X) \ . \tag{4.15}$$

Consumption good no. 2 is produced without the resource, according to:

$$C_2 = H(L_3) \tag{4.16}$$

$$L \geqslant L_1 + L_2 + L_3 \ . \tag{4.17}$$

We will assume the standard concave utility function for two goods:

$$(4.18) \qquad u = u(C_1,\ C_2) \qquad u_1 > 0\ , \quad u_2 > 0\ ,$$
$$u_{11} \leqslant 0\ , \quad u_{22} \leqslant 0\ ,$$
$$u_{12} = u_{21} \geqslant 0$$
$$u \in C^{(2)}\ .$$

The problem to be solved here involves finding time paths for L_1, L_2, L_3 and a value for $X(T)$ so as to maximize:

$$(4.19) \qquad \int_0^T u(C_1,\ C_2)\ e^{-\delta t} dt$$

$$\text{s.t.} \quad \dot{X} = -G(L_2,\ X)$$
$$X \geqslant 0$$
$$L_1,\ L_2,\ L_3 \geqslant 0$$

and C_1 and C_2 are defined as above.

The necessary conditions to be satisfied by a solution to this problem are that $\exists$ continuous $\psi(t)$ for which

$$(4.20) \qquad \dot{\psi} = \psi(\delta + G_2) - u_1 F_2 G_2 + \lambda_5 G_2$$

$$(4.21) \qquad \psi = \frac{u_1(F_2 G_1 - F_1) + \lambda_2 - \lambda_1 - \lambda_5 G_1}{G_1}$$

$$(4.22) \qquad \psi = \frac{u_1 F_2 G_1 - u_2 H' + \lambda_2 - \lambda_3 - \lambda_5 G_1}{G_1}$$

$$(4.23) \qquad u_1 F_1 = u_2 H' + \lambda_3 - \lambda_1$$

$$(4.24) \qquad \psi(T)\ X(T) = 0 \qquad (T < \infty)$$

$$(4.25) \qquad \begin{cases} \lambda_1 \geqslant 0 & L_1 \geqslant 0 & \lambda_1 L_1 = 0 \\ \lambda_2 \geqslant 0 & L_2 \geqslant 0 & \lambda_2 L_2 = 0 \\ \lambda_3 \geqslant 0 & L_3 \geqslant 0 & \lambda_3 L_3 = 0 \\ \lambda_4 \geqslant 0 & L - L_1 - L_2 - L_3 \geqslant 0 & \lambda_4(L - L_1 - L_2 - L_3) = 0 \\ \lambda_5 \geqslant 0 & X \geqslant 0 & \lambda_5 X = \lambda_5 \dot{X} = 0\ . \end{cases}$$

We shall assume that both C_1 and C_2 are indispensible for consumers ($u_1(0, \cdot) = \infty = u_2(\cdot, 0)$). We also assume (i) it is possible to produce C_1 with labour only, and (ii) labour is indispensible in the production of both the resource good ($G(L_2, X)$) and C_2. This means that the only policies feasible for the economy are:

Policy A: $L_1 > 0$, $L_2 = 0$, $L_3 > 0$; and

Policy B: $L_1 > 0$, $L_2 > 0$, $L_3 > 0$.

Narrowing the problem down in this way enables us to focus on the interesting question of how the labour allocation between a resource-dependent sector ($L_1 + L_2$) and a non-resource-dependent sector (L_3) should change over time in order to get the "most" out of the exhaustible asset. In order to "dissect" the problem even further we shall examine firstly the case where there is no depletion effect ($C_1 = F(L_1, G(L_2))$).

The switching surface for a switch from policy B to policy A is in this case:

$$(4.26)\quad \bar{\psi} = \frac{u_1(F(L_1, 0), H(L_3))(F_2(L_1, 0).G'(0) - F_1(L_1, 0))}{G'(0)}$$

$$(4.27)\quad u_1(F(L_1, 0), H(L_3))F_1(L_1, 0) = u_2(F(L_1, 0), H(L_3)).H'(L_3)$$

$$(4.28)\quad L_1 + L_3 = L.$$

It may be shown that along policy B:

$$(4.29)\quad \dot{L}_3 = \frac{(u_1F_{11} - u_1F_{12}G_1) - u_{11}F_1(F_2G_1 - F_1) + u_{21}H'(F_2G_1 - F_1)}{A}\,\psi\delta G_1$$

$$(4.30)\quad \dot{L}_1 = \frac{\{u_{11}F_1F_2G_1 - u_{12}H'(F_2G_1 + F_1) + u_1F_{12}G_1 + u_{22}(H')^2 + u_2H''\}}{-u_{11}F_1(F_2G_1 - F_1) + u_2H'(F_2G_1 - F_1) + u_1F_{11} - u_1F_{12}G_1}\,\dot{L}_3$$

$$(4.31)\quad \dot{L}_2 = \frac{\{u_2H'' - 2u_{21}F_1H' + u_1F_{11} + u_{11}(F_1)^2\}}{u_{11}F_1(F_2G_1 - F_1) - u_{21}H'(F_2G_1 - F_1) - u_1F_{11} + u_1F_{12}G_1}\,\dot{L}_3$$

$$= \left\{\frac{u_2H'' - 2u_{21}F_1H' + u_1F_{11} + u_{11}(F_1)^2}{-A}\right\}\psi\delta G_1\,,$$

where A is a negative expression.

Thus L_2 is falling over time for policy B. The optimal path will be the longest exhaustion path feasible. For long time horizons, the economy will find it optimal to run L_2 continuously to zero, and, when L_2 reaches zero, to switch into policy A, producing both consumption goods with the appropriate quantities of labour (these will be the solutions, L_1 and L_3, to eqns. (4.27) and (4.28)).

While X is being exploited (policy B) the direction of movement of L_1 and L_3 is not clear. In the single consumption good model there was no difficulty in signing $\dot{L}_1$ because the only important considerations were technological. However when another consumption good is introduced into the model, the consumer's relative preferences for the two consumption goods also become important. So far as L_3 is concerned, if production relationships alone were important (in particular if $u_{11} = u_{21} = 0$[1]), then it would be optimal to increase L_3 as the resource is depleted. In other words, there should be a movement of the variable factor away from the resource-based sector (1 + 2) as time goes on and the resource is depleted. However the relative preferences for the resource good (which will become stronger as the resource is depleted and L_2 falls) will tend to offset the increase in L_3 warranted by production considerations. The relative strength of these two influences (($u_1F_{11} - u_1F_{12}G_1$) represents the influence of production and the remaining two terms in the numerator of the R.H.S. of (4.29) represent the influence of preferences) will determine whether L_3 should rise or fall as exploitation of X proceeds.

Similarly, if we were simply to consider the production aspects of the problem L_1 should be falling over time, reflecting the

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1 This would, for example, be the case if $u(C_1, C_2) = C_1 + pC_2$ for some $p > 0$.

general scaling down of activity in the resource based sector (this influence is represented by $u_1F_{12}G_1$ in the numerator on the R.H.S. of (4.30)), however innate preferences for the resource based good together with a diminishing marginal product of labour in the production of C_2 (represented by the remaining terms in the numerator on R.H.S. of (4.30)) will be working to prevent L_1 from falling. Again, the net result is indeterminate without more precise information about the functions involved.

When a depletion effect is introduced the model becomes even more complicated. We are not even able to say that the depletion effect will definitely cause the variables to move one way or the other. Now we must add to the expression in (4.29) for $\dot{L}_3$ two terms, a long expression ξ/A which is negative and a second

$$(4.32) \qquad \frac{F_1(u_1)^2G}{G_1A}\{G_{12}F_{11} - G_{12}F_{12} + G_{11}F_{12}G_2\} > 0 \ .$$

Again, in terms of pure production considerations one would expect the depletion effect to make it optimal to shift labour out of the resource-based industry in order to escape increasing costs of extraction as the resource is depleted (the expression in (4.32)). On the other hand there will again be an offsetting term (ξ/A, in this case) representing the community's desire to maintain production of C_1. Clearly the optimal movement of labour between the two main sectors of the economy (3 and combined 1-2) is more indeterminate than ever. However if we assume, for example, that utility is measured as some weighted sum of C_1 and C_2, say $C_1 + pC_2$ for some constant $p > 0$, and that the production function, F, for C_1 exhibits constant returns to scale then ξ will be zero and the sole influence of the depletion effect will be to shift some labour out of the resource-based industry as time goes on.

Nevertheless the final outcome is a far cry from the easily identified factor movements envisaged by Barnett and Morse [6] and which were discussed in Chapter 2.

As far as the general structure of the solution is concerned, the surface for switches between policies A and B is now given by:

$$\psi = \frac{u_1(F(L_1, 0), H(L_3))(F_2(L_1, 0).G_1(0, X) - F_1(L_1, 0))}{G_1(0, X)}$$

$$u_1(F(L_1, 0), H(L_3)).F_1(L_1, 0) = u_2(F(L_1, 0), H(L_3)).H'(L_3)$$

$$L_1 + L_3 = L .$$

This surface has slope given by:

$$\frac{d\psi}{dX} = \frac{u_1 F_1 G_{12}}{(G_1)^2} > 0$$

and is illustrated in Figure 4.4.

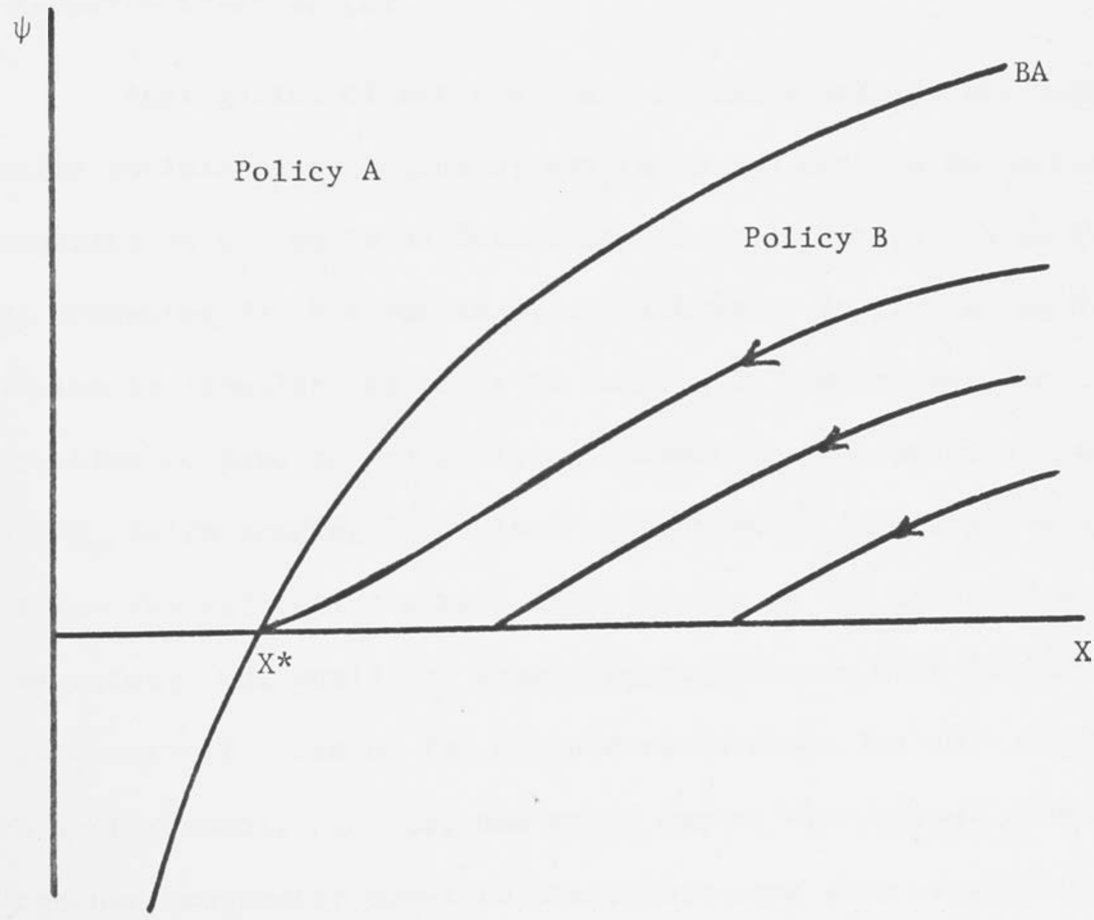

Figure 4.4

$X(T)$ is determined as the intersection of the optimal trajectory and $\psi = 0$. The optimal trajectory will be the one along which $\psi(t)$ reaches zero at $t = T$. As $T \to \infty$, $X(T) \to X^*$, which is defined as the X-solution to the system:

$$F_2(L_1, 0).G_1(0, X) = F_1(L_1, 0)$$

$$u_1(F(L_1, 0), H(L_3)).F(L_1, 0) = u_2(F(L_1, 0), H(L_3)).H'(L_3)$$

$$L_1 + L_2 = L \,.$$

Because of the difficulties which arise when working with three sectors it is natural to ask whether there is a valid, convenient way of aggregating the two sectors involved in the production of C_1 into a single sector and thereby reduce the above model to a two-sector model. This would have the advantage of both simplifying the exposition and making more precise the dichotomy between a resource-based sector and a non-resource-based sector.

Aggregation of sectors 1 and 2 should necessarily assume some specific optimizing behaviour within those two sectors and unfortunately, aggregation on the basis of *intertemporal* optimization (using the first model presented in this section), at each point in time is as difficult a problem to formulate as it is to solve. It may however, be interesting to make the simplifying assumption that sectors 1 and 2 combined, in responding to a given allocation of labour $(L_1 + L_2)$ determine the allocation within their sector on the basis of myopic maximization. One would not expect aggregation on this basis to lead to the same optimal rules as for those derived above for the underlying three sector model. Indeed, one would expect the optimal path derived for the new two-sector model to constitute some sort of second best solution. Nevertheless let us proceed with our new assumption and see where it leads us.

To begin with we shall redefine some variables. Writing

$$C_1 = F(L_1^*, G(L_2^*, X)) \quad \text{and} \quad C_2 = H(L_3^*) ,$$

we denote

$$L_1 = L_1^* + L_2^* \quad \text{and} \quad L_2 = L_3^* .$$

The next step is to derive a production function for the new vertically integrated sector 1 relating its output (C_1) to its total labour supply (L_1). If the sector wishes to allocate L_1^*, L_2^* so as to maximize C_1 subject to given L_1 at any point in time, then either $L_1^* = L_1$ or $L_2^* = L_1$ or L_1^* and L_2^* are both between 0 and L_1. Our assumption that $F_2(0, \dot{X}) = 0$ rules out the possibility that $L_1^* = 0$ and $L_2^* = L_1$. When $L_1^* = L_1$, $C_1 = F(L_1, 0)$ and the production relationship between C_1 and L_1 is immediately established. It remains for us to check the case where $L_1^*, L_2^* \in (0, L_1)$. In this case, condition (4.6) will hold and we find that:

$$\frac{\partial L_1^*}{\partial L_1} = \frac{F_{22}(G_1)^2 + F_2G_{11} - F_{12}G_1}{F_{11} - 2F_{12}G_1 + F_{22}(G_1)^2 + F_2G_{11}} , \quad \text{and}$$

$$\frac{\partial L_1^*}{\partial X} = \frac{F_{22}G_1G_2 + F_2G_{12} - F_{12}G_2}{F_{11} - 2F_{12}G_1 + F_{22}(G_1)^2 + F_2G_{11}} .$$

It may then be shown that if F and G are each homogeneous of degree 1:

$$\frac{\partial C}{\partial L_1} > 0 , \quad \frac{\partial C}{\partial X} > 0 , \quad \frac{\partial^2 C}{\partial L_1^2} < 0 , \quad \frac{\partial^2 C}{\partial X^2} < 0 , \quad \frac{\partial^2 C}{\partial X \partial L_1} > 0 .$$ [2]

Thus, in general we can write:

$$C_1 = f(L_1, X) , \tag{4.33}$$

where

---

2 The assumption that F and G are homogeneous of degree 1 is only required to prove $\frac{\partial^2 C}{\partial X \partial L_1} > 0$; the other results are true in general.

$$f_1 > 0 , \quad f_2 > 0 ,$$
$$f_{11} < 0 , \quad f_{22} < 0 ,$$
$$f_{12} = f_{21} > 0 , \quad f \in C^{(2)} ,$$
$$f(L_1, 0) = 0 ,$$

and we have a production function for the single vertically integrated sector relating that sector's output to its labour input and the stock of the resource. We now have a two sector economy with production relationships described by (4.33) and

$$C_2 = H(L_2) . \tag{4.34}$$

Now, for convenience, we define the stock of the resource (X) to be measured in terms of the amount of the consumption good which it will produce.[3] Then:

$$\dot{X} = -C_1 . \tag{4.35}$$

Noting that $L = L_1 + L_2$, it is possible to write $L_2 = k(C_1, X)$ and using (4.34):

$$C_2 = \phi(C_1, X) \quad \begin{cases} \phi_1 < 0 , & \phi_2 > 0 , \\ \phi_{11} < 0 , & \phi_{22} < 0 , \\ \phi_{12} > 0 , & \phi \in C^{(2)} , \\ \phi_2(0, X) = 0 & \\ \lim_{X \to 0} \phi_1(0, X) = -\infty . & \end{cases} \tag{4.36}$$

(4.36) is simply the usual sort of equation for a concave production frontier in a two sector economy. The frontier, relating output of one

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3 This is essentially the same assumption as we were making implicitly in the first model presented in this chapter, where X represented the total amount of E which it would produce according to $E = G(L_2, X)$.

sector to output of the other contracts inwards as the resource is depleted (Figure 4.5).

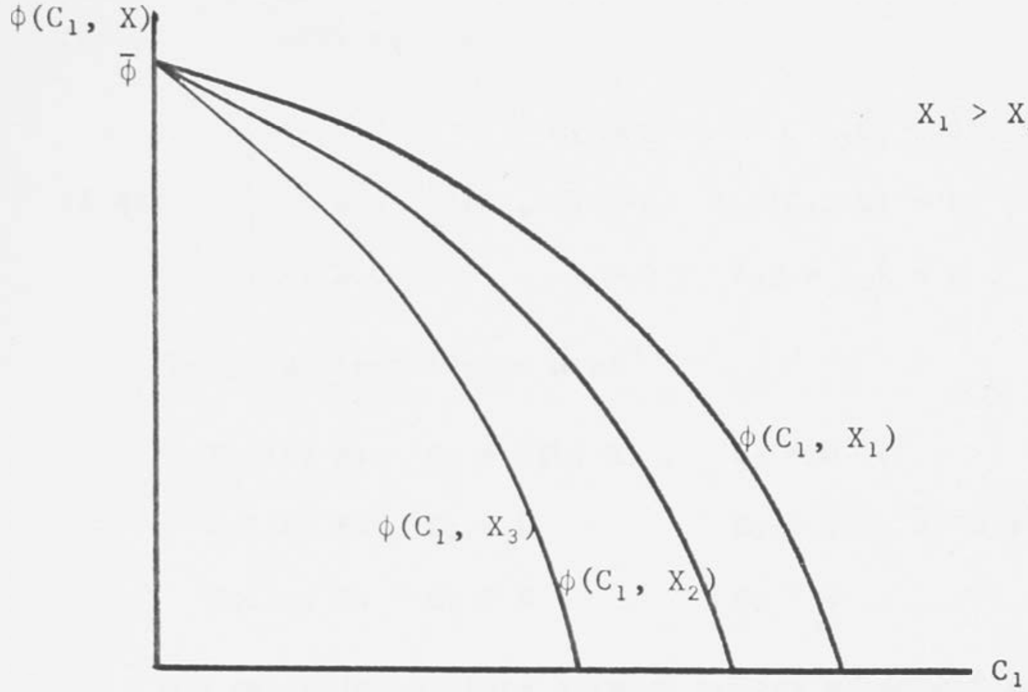

Figure 4.5

The problem to be solved here involves finding a time path for $C_1(t)$ and a value for $X(T)$ so as to maximize

$$(4.37) \qquad \int_0^T u(C_1, C_2)\, e^{-\delta t} dt$$

$$\text{s.t.} \quad \dot{X} = -C_1$$

$$C_2 = \phi(C_1, X)$$

$$C_1, C_2 \geqslant 0$$

$$X \geqslant 0 ,$$

where T is parametrically fixed and u has the properties listed in (4.18).

The set of necessary conditions are:

(4.38) $\dot{\psi} = \psi\delta - u_1\phi_2 - \lambda_2\phi_2$

(4.39) $\psi = u_1\phi_1 + u_2 + \lambda_2\phi_1 + \lambda_1 - \lambda_3$

(4.40) $\psi(T)\ X(T) = 0$

(4.41) $$\begin{cases} \lambda_1 \geqslant 0 & C_1 \geqslant 0 & \lambda_1 C_1 = 0 \\ \lambda_2 \geqslant 0 & \phi(C_1, X) \geqslant 0 & \lambda_1\phi(C_1, X) = 0 \\ \lambda_3 \geqslant 0 & X \geqslant 0 & \lambda_3 X = \lambda_3\dot{X} = 0\ . \end{cases}$$

The possible policies are:[4]

Policy A: $C_1 = f(L, X)$, $C_2 = 0$

Policy B: $C_1 = 0$ $C_2 = \phi(0, X) \equiv \bar{\phi}$

Policy C: $C_1 > 0$ $C_2 > 0$.

The switching surface between policy B and policy C is given by:

$$\psi = u_1(\bar{\phi}, 0).\phi_1(0, X) + u_2(\bar{\phi}, 0) \equiv \eta(X)\ .$$

If we assume (not unreasonably) that $\exists\ X > 0$ for which $\eta(X) > 0$, the assumption that $\lim_{X\to 0} \phi_1(0, X) = -\infty$ ensures that $\exists\ X^* > 0$, s.t. $\eta(X^*) = 0$, and the surface is as in Figure 4.6.

Policies A and C are both confined to regions below the $\eta(X)$ locus. Paths for which $\psi < 0$ are exhaustion paths, however, they take an infinite time to run X to zero and thus cannot satisfy the transversality conditions (4.40). Thus, as we found for the previous model (and by the same reasoning), the optimal path will be the one for which $\psi(t) > 0\ \forall_t < T$ and $\psi(T) = 0$ (e.g. path 1 in Figure 4.6 for some finite T; the optimal path $\to$ path 2 in Figure 4.6 as $T \to \infty$). In particular we again find that it is always optimal to leave a positive

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4 In this model we are not assuming (as we did for the three sector model) that both C_1 and C_2 are indispensible consumption goods (i.e. we are not specifically assuming $u_1(0, \cdot) = u_2(\cdot, 0) = \infty$).

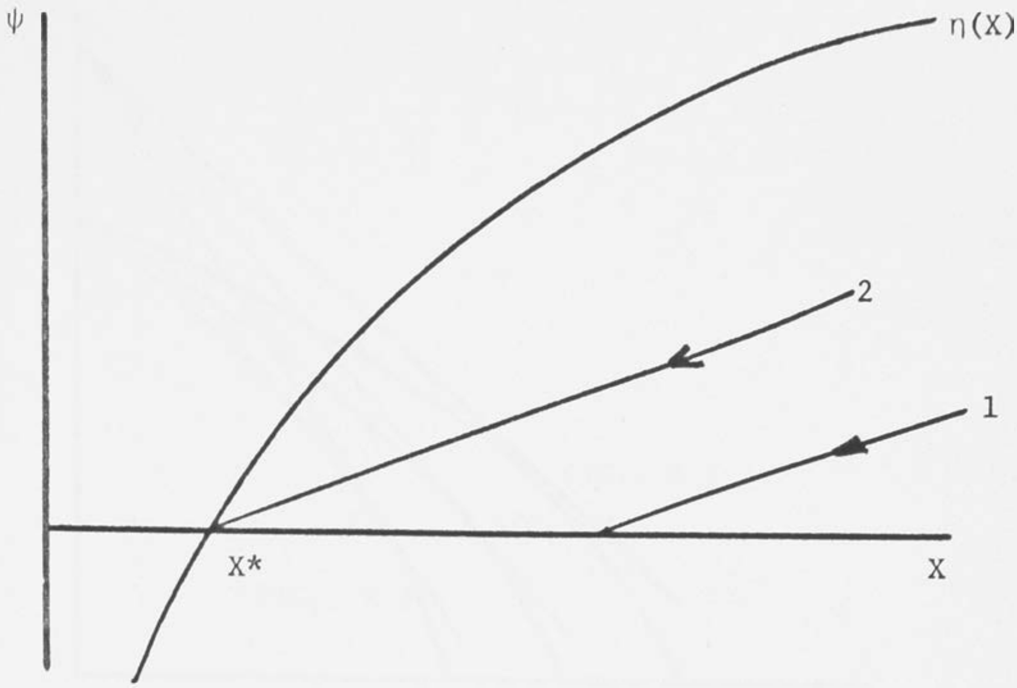

Figure 4.6

stock of X unexploited. This arises naturally because the increasing labour allocation to the resource-based sector required to offset the depletion effect in that sector squeezes consumption of C_2. This cannot proceed beyond the point at which the marginal social valuation of the resource (ψ) equals zero. Now, $\psi = 0$ if $\frac{u_2}{u_1} = -\phi_1$, which in a competitive system would represent a static optimum. Thus "extraction" (C_2) is initially less than the static optimum level. But as time passes and resource depletion shifts the production frontier inwards the origin (Figure 4.7), the economy moves towards the static optimum, reaching it at the end of the programme. Figure 4.7 shows a possible path (PQR) in terms of the shifting production frontier (as X falls from X_3 to X_2 to X_1). C_1 may be rising or falling over time depending on the properties of the functions involved. Policy C is followed for the entire programme.

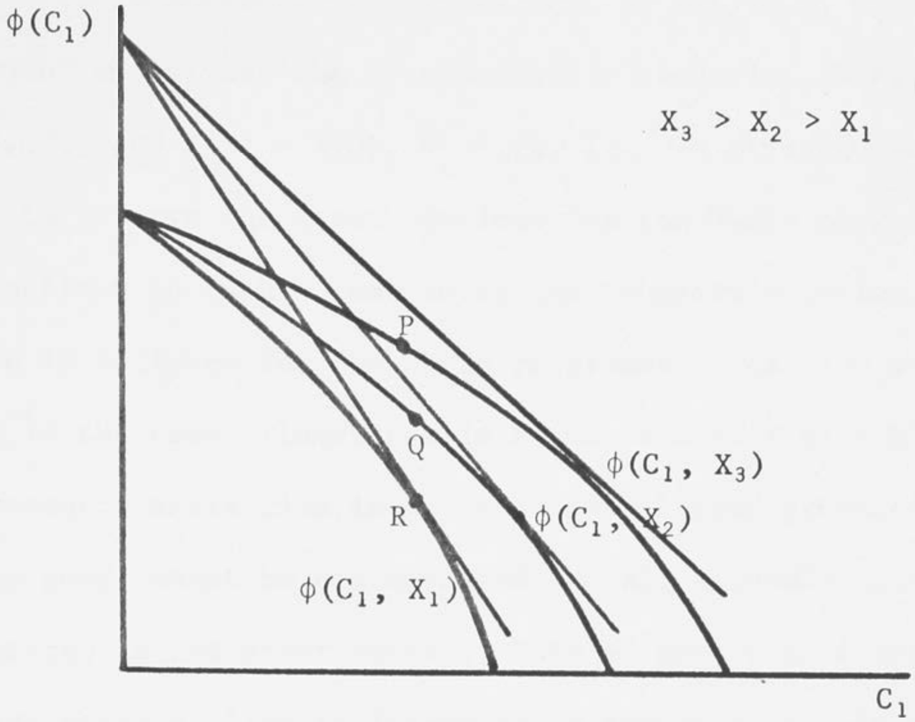

Figure 4.7

In the event that $u_2(\phi, 0) = \infty$, the locus, $\eta(X)$, will lie above the X-axis $\forall_X > 0$, and for finite T, the optimal path will be like path 1 in Figure 4.8 and will tend to path 2 (for which $X(T) = 0$) as $T \to \infty$.

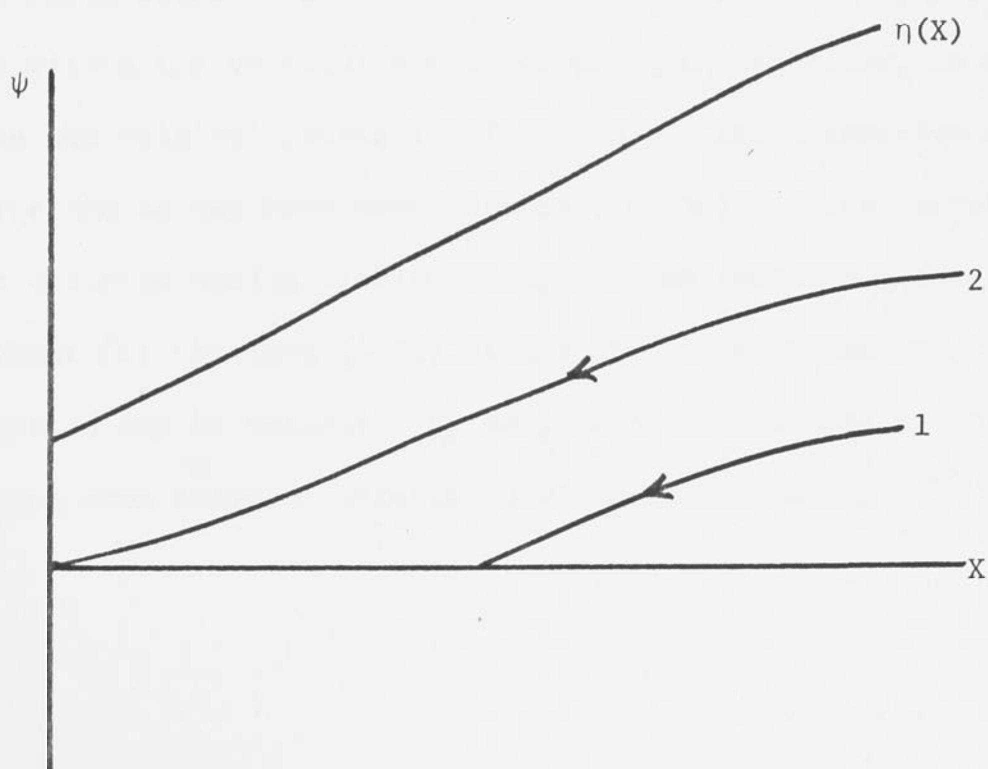

Figure 4.8

It is interesting to note that if $\eta(X) < 0$, $\forall_X \geqslant 0$, then the only "path" satisfying the transversality condition (4.40) is $\psi(t) = 0$, $\forall_t \in [0, T]$, and $X(t) = X(0)$, $\forall_t \in [0, T]$. In other words, it will be optimal to observe the static optimum for the whole programme, the static optimum in such a case being the boundary solution, $C_1 = 0$. Policy B is followed for the whole programme. The same course would be optimal in the case illustrated in Figure 4.6 if $X(0) < X^*$. In such cases resource extraction is so uneconomical that production of the resource good cannot be contemplated and all economic activity is concentrated in the other sector. This of course also depends upon consumers being willing to forego the resource good. In the extreme case where $u_2(\phi, 0) = \infty$, (and $\eta(X) > 0$ $\forall_X$) they will not, and both goods will be produced for the entire programme regardless of depletion effects and the initial endowment of the resource.

In concluding this section of the chapter it is worth noting that the production function (f) obtained by aggregating sectors 1 and 2 of the three sector model, while obtained by assuming myopic decision making within the vertically integrated resource sector, is of the same form as the original production function (G) for extraction of the resource and so may have some intuitive appeal. We can merely say that myopic decision making within the aggregated resource sector is sufficient for the form (4.33) of the production function. Such an aggregation may be validated by many other behavioural assumptions including some forms of intertemporal decision making.

II

NON-HOMOGENEOUS RESOURCES

At this stage it is probably as well to stop for a moment and reassess things. It should by now be clear that the scarcity inherent in exhaustible resources arises not from one cause, but from many. Firstly, there is the elementary scarcity implied by the finitude of the stock of the resource. Secondly, there is the intensifying of scarcity via depletion effects which reflect the increasing costs encountered as less accessible deposits are mined. Thirdly, there are the extra limitations imposed on the supply of the resource to future generations by planning with a positive discount rate. Finally, there is the usual type of scarcity associated with the production of any economic good derived from the limited supply of other factors of production. This type of scarcity will imply, in the case of resources, that some resource deposits will exhibit lower unit costs of extraction than others (possibly because of different transport costs, or different geological formations, etc.). It is this innate difference in costs together with the relative strengths of depletion effects at different stages of exploitation which constitutes the main economic distinction between "grades" of a resource. Such differences will of course also tell us something about the relative valuations of *different resources*. In this section we intend to pursue these considerations in the context of a one-sector, "two-resource" planning model. The two resources are assumed to be perfect substitutes in the production of a single consumption good. It is also assumed that each resource is itself homogeneous in quality.

Let $X(t)$ be the unexploited stock of one resource and $R(t)$ the corresponding stock of the other. Let $E_1(T)$ denote extraction of resource at time t and $E_2(t)$ the corresponding extraction of resource R. Y(t) denotes gross output of the consumption good at time t, produced using E_1 and E_2 according to:

$$Y = F(E_1 + E_2) \;, \qquad F' > 0 \,, \quad F'' < 0 \,.$$

The cost functions, V_1 and V_2, for the extraction of X and R respectively are given by:

$$(4.42) \quad V_1 = aE_1\,\phi(X) \qquad \phi' < 0 \,, \quad \phi'' > 0 \,, \quad \phi(0) = \infty \,,$$

$$(4.43) \quad V_2 = bE_2\,\eta(R) \qquad \eta' < 0 \,, \quad \eta'' > 0 \,, \quad \eta(0) = \infty \,.$$

The functions ϕ and η represent the depletion effects associated with the extraction of X and R respectively. Total and marginal costs of extraction become infinite as the resource approaches exhaustion.

Net output of the consumption good is clearly given by:

$$(4.44) \qquad C = F(E_1 + E_2) - aE_1\,\phi(X) - bE_2\,\eta(R) \,.$$ [5]

Now, suppose the economy is planning over a parametrically fixed time horizon and wishes to select time paths for $E_1(t)$ and $E_2(t)$ and terminal resource stocks, X(T) and R(T), so as to maximize the present value of the stream of consumption, i.e.

$$\max_{E_1, E_2, X(T), R(T)} \int_0^T Ce^{-\delta t}\, dt$$

$$\text{s.t.} \qquad C = F(E_1 + E_2) - aE_1\,\phi(X) - bE_2\,\eta(R)$$

---

5 In Chapter 3, it was possible to simply write net output as a function F(E) because a single *homogeneous* resource was involved. The non-homogeneity considered in this section makes it important to specify the components of net output (viz. gross output and the respective costs).

$$\dot{X} = -E_1$$

$$\dot{R} = -E_2$$

$$E_1, E_2, X, R \geqslant 0 \ .$$

The necessary conditions to be satisfied by a solution to this problem are that ∃ continuous functions of time, ψ_1 and ψ_2, and a function, $\mathcal{L}$, such that:

$$\mathcal{L} = F(E_1 + E_2) - aE_1\ \phi(X) - bE_2\ \eta(R) - \psi_1 E_1 - \psi_2 E_2$$
$$+ \psi_1 E_1 + \lambda_2 E_2 - \lambda_3 E_1 - \lambda_4 E_2 \ .$$

(4.45) $$\dot{\psi}_1 = \psi_1 \delta + aE_1\ \phi'(X)$$

(4.46) $$\dot{\psi}_2 = \psi_2 \delta + bE_2\ \eta'(R)$$

(4.47) $$\psi_1 = F'(E_1 + E_2) - a\ \phi(X) + \lambda_1 - \lambda_3$$

(4.48) $$\psi_2 = F'(E_1 + E_2) - b\ \eta(R) + \lambda_2 - \lambda_4$$

(4.49) $$\psi_1(T)\ X(T) = \psi_2(T)\ R(T) = 0 \qquad (T < \infty)$$

(4.50) $$\begin{cases} \lambda_1 \geqslant 0 & E_1 \geqslant 0 & \lambda_1 E_1 = 0 \\ \lambda_2 \geqslant 0 & E_2 \geqslant 0 & \lambda_2 E_2 = 0 \\ \lambda_3 \geqslant 0 & X \geqslant 0 & \lambda_3 X = \lambda_3 \dot{X} = 0 \\ \lambda_4 \geqslant 0 & R \geqslant 0 & \lambda_4 R = \lambda_4 \dot{R} = 0 \ . \end{cases}$$

There are four policies open to the economy. They are set out in Table 4.1.

Table 4.1

Policy	Control	
	E_1	E_2
A	> 0	0
B	0	> 0
C	> 0	> 0
D	0	0

The necessary conditions for the various policy switches are set out in Table 4.2. The derivation of these conditions is presented in Appendix 4.2. The information contained in Table 4.2 assumes that $\lambda_3 = \lambda_4 = 0$. In the event that either R = 0 or X = 0, or both, then some switches may take place under more general conditions. These switches will be taken acount of in the sufficiency proof in Appendix 4.3.

Table 4.2

Necessary conditions for policy switches

		Switches into			
		A	B	C	D
Switches out of	A	•	$a\phi(X) < b\eta(R)$	$a\phi(X) < b\eta(R)^{+}$	$F'(0) > a\phi(X)$
	B	$a\phi(X) > b\eta(R)$	•	$a\phi(X) > b\eta(R)^{+}$	$F'(0) > b\eta(R)$
	C	$a\phi(X) > b\eta(R)^{*}$	$a\phi(X) < b\eta(R)^{*}$	•	$F'(0) > a\phi(X)$ $F'(0) > b\eta(R)$
	D	$F'(0) < a\phi(X)$	$F'(0) < b\eta(R)$	$F'(0) < a\phi(X)$ $F'(0) < b\eta(R)$	•

* A condition which must hold immediately after a switch takes place.

\+ A condition which must hold immediately before a switch takes place.

Policy C is a single trajectory in the (R, X) plane with equation $a\phi'(X) = b\eta'(R)$ and turns out to be part of the optimal programme for most sets of initial conditions (X(0), R(0)). With this in mind we define an *arterial path* to be a Pontryagin path which ultimately meets and moves along path γ_1 (Figure 4.9(a)) if $a\phi(0) > b\eta(0)$ or path γ_2 (Figure 4.9(b)) if $a\phi(0) < b\eta(0)$.

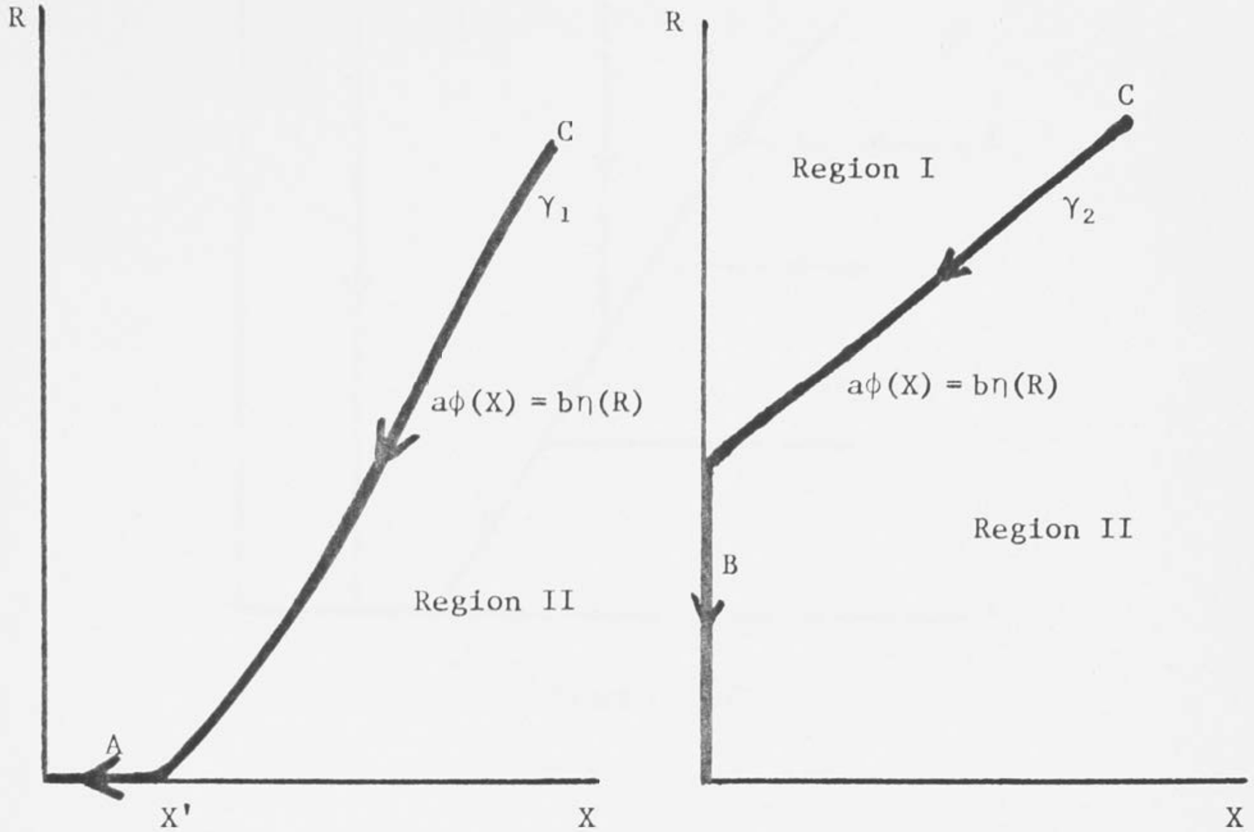

Figure 4.9(a) $a\phi(0) > b\eta(0)$

Figure 4.9(b) $a\phi(0) < b\eta(0)$

Paths γ_1 and γ_2 are switched into from above via policy B and from below via policy A. An example of a set of arterial paths is illustrated in Figure 4.10 for the case $a\phi(0) > b\eta(0)$. There is a unique arterial path for each set of initial conditions $(X(0), R(0))$.

Assuming that for X sufficiently large $F'(0) > a\phi(X)$ and for R sufficiently large $F'(0) > b\eta(R)$, then $\exists\ X'$: $F'(0) = a\phi(X')$ and R': $F'(0) = b\eta(R')$. In addition, the assumptions that $\phi(0) = \infty = \eta(0)$ imply that exhaustion of either resource would take an infinite time. Hence for all finite T, $\psi_1(T) = \psi_2(T) = 0$. The optimal path will be the appropriate arterial path (for the relevant $(X(0), R(0))$), and the time profiles of extraction of X and R will be determined by the requirement that ψ_1 and ψ_2 are run to zero at time T. In particular as $T \to \infty$, $E_1(T)$

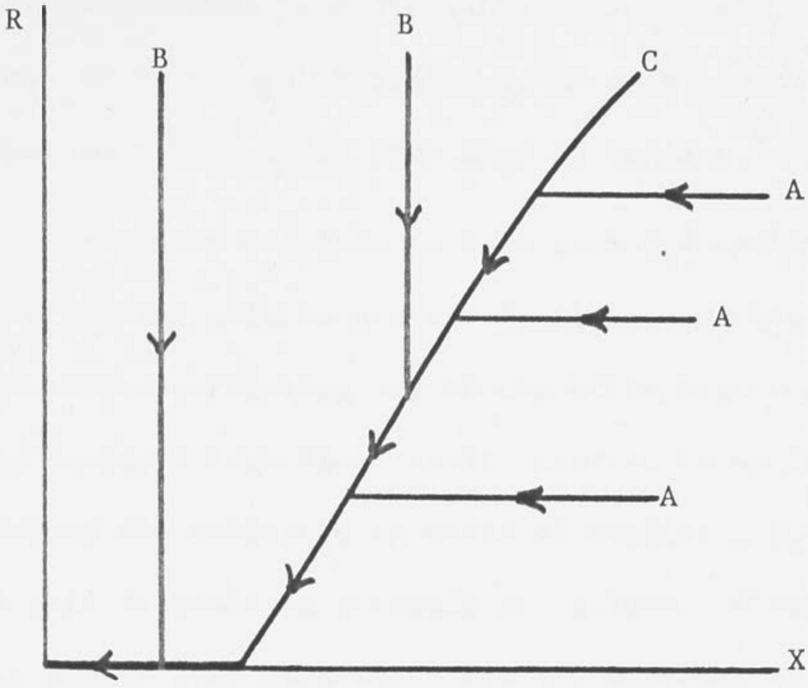

Figure 4.10

and $E_2(T)$ will $\rightarrow$ 0 and the endpoint $(X(T), R(T), \psi_1(T), \psi_2(T))$ will approach (X', R', 0, 0) where X' and R' are defined as above. The lower the cost coefficient (a) for resource X, say, the lower will be the ultimate terminal stock of the resource (X').

It is clear that above policy C (in region I — Figure 4.9) is a region of relative surplus of R while to the right of it (region II) is a region of relative surplus of X. We see that the optimal course for the economy if it is initially situated in region I is to produce all output using R (policy B) until a situation of balance is reached in which marginal costs of extraction are equated ($a\phi(X) = b\eta(R)$). When such a point is reached, policy C should become operative (exploiting both resources so that their marginal costs of extraction are kept equal).

When there are no depletion effects involved in the extraction of the two resources the marginal costs of extraction (a and b) are given constants and cannot be manipulated by controlling the time paths of extraction. It is not surprising that in such a model it is never optimal to operate policy C for more than an instant.[6]

It is actually possible to show that the optimal course when there are no depletion effects present is to exhaust the cheaper of the two resources before extracting any of the other resource. This of course is the standard Ricardian result referred to by Solow in [31] (p.4). Resources are exploited in order of declining economic quality. The proof of this proposition proceeds as follows. Firstly, it is reasonable to assume that both resources are economic (i.e. $F'(0) - a$ and $F'(0) - b$ are both positive), otherwise we have a trivial problem. Then ψ_1 and ψ_2 are both positive for the entire programme. It follows from the transversality condition that $X(T) = R(T) = 0$. Since C can only hold for a moment we have to decide when the remaining policies should operate. We shall consider the case where $a < b$ (i.e. X is the cheaper grade of resource). It is not optimal to switch out of A before X is exhausted (since it is impossible to switch back from D or B into A — a fact which is easily verified by inspecting the movement of ψ_1 — if it is not possible to switch back X cannot be exhausted). In addition, the only time when it is possible to switch out of B is when R is exhausted. (For policy B, $\psi_1 + a > \psi_2 + b$ and $\frac{d}{dt}(\psi_1 + a) > \frac{d}{dt}(\psi_1 + b)$ — a switch into A is only possible when λ_4 can become positive and preserve the continuity of the co-states; also if B switches into D before R is exhausted it cannot later switch back, so that the exhaustion requirement is violated.) Thus the optimal policy sequence

---

6 Policy C involves having $\psi_1 + a = \psi_2 + b$. For this equation to continue to hold $\dot{\psi}_1 = \dot{\psi}_2 \Leftrightarrow \psi_1\delta = \psi_2\delta \Leftrightarrow \psi_1 = \psi_2 \Leftrightarrow a = b$, which will only be true for a homogeneous resource, a case we are not interested in here.

must be ABD or BAD, with each resource being exhausted before exploitation of the other one begins. Our contention here is that since X is the cheaper resource, the sequence will be ABD. For an AB switch it is possible for λ_2 to decline to zero while policy A is in operation and λ_1 to increase from zero during policy B. That such a case is optimal may be verified by comparing integrals (letting asterisk superscripts denote the path claimed to optimal). We have:

$$\int_0^T \{F(E_1^* + E_2^*) - aE_1^* - bE_2^*) - (F(E_1 + E_2) - aE_1 - bE_2)\}e^{-\delta t}dt$$

$$> \int_0^T \psi_1^*(0)(E_1^* - E_1)dt + \int_0^T \psi_2^*(0)(E_2^* - E_2)dt$$

$$+ \int_0^T \lambda_1^* \, E_1 e^{-\delta t}dt + \int_0^T \lambda_2^* \, E_2 e^{-\delta t}dt$$

$$> 0 \; .$$

It is now clear that when one resource is cheaper to extract than another it will be optimal to exploit that resource exclusively. When there are no depletion effects this will simply involve exhausting one resource deposit and then commencing on the next most costly one. However, when extraction is characterized by depletion effects, it is possible to control extraction costs to some extent. If resource X is initially cheaper than R (i.e. $a\phi(X_0) < b\eta(R_0)$) it will initially be exploited exclusively. However as depletion effects set in, the cost advantage of X will be eroded. When the two marginal costs are the same net returns over time are increased if both resources are exploited simultaneously, keeping their marginal costs equal and postponing the full force of the depletion effect for each of them.

Having concentrated up to this point on the production aspects of the two-resource problem it would be interesting now to turn our attention to the social welfare problem where the community has its own innate valuation of each resource. This problem was examined in Chapter 3 for the case of a single resource by incorporating a "conservation motive" in the utility function. Here we generalize this procedure by defining a social welfare function of the form:

$$
\begin{aligned}
u = u(C, X, R) \qquad & u_C > 0 , \quad u_{CC} < 0 , \\
& u_X > 0 , \quad u_{XX} < 0 , \\
& u_R > 0 , \quad u_{RR} < 0 , \\
& u_{XC} = u_{CR} = u_{RC} = u_{CR} \equiv 0 , \\
& u_{XR} = u_{RX} \geqslant 0 .
\end{aligned}
$$

This utility function is assumed to be strongly separable between consumption and the two resources. This of course means that we can write:

$$u(C, X, R) = f(C) + g(X, R) .$$

We shall assume that $f'(0)$ is finite and that g is concave in X and R. To keep the analysis manageable we shall have to ignore depletion effects and also assume that both resources are characterized by identical (constant) unit costs (a). This means that C will now be written as:

$$C = F(E_1 + E_2) - a(E_1 + E_2) .$$

It is obviously convenient to write u in the form $u(E, X, R)$, where $E = E_1 + E_2$. The properties of u with respect to E are the same as its properties with respect to C.

We now wish to solve problem (4.44) with C replaced inside the integral sign by $u(E, X, R)$. The necessary conditions are:

$$\dot{\psi}_1 = \psi_1\delta - u_X(X, R) \tag{4.51}$$

$$\dot{\psi}_2 = \psi_2\delta - u_R(X, R) \tag{4.52}$$

$$\psi_1 = u_E(E) + \lambda_1 - \lambda_3 \tag{4.53}$$

$$\psi_2 = u_E(E) + \lambda_2 - \lambda_4 \tag{4.54}$$

$$\psi_1(T)\, X(T) = \psi_2(T)\, R(T) = 0 \tag{4.55}$$

$$\begin{cases} \lambda_1 \geqslant 0 & E_1 \geqslant 0 & \lambda_1 E_1 = 0 \\ \lambda_2 \geqslant 0 & E_2 \geqslant 0 & \lambda_2 E_2 = 0 \\ \lambda_3 \geqslant 0 & X \geqslant 0 & \lambda_3 X = \lambda_3 \dot{X} = 0 \\ \lambda_4 \geqslant 0 & R \geqslant 0 & \lambda_4 R = \lambda_4 \dot{R} = 0\,. \end{cases} \tag{4.56}$$

The policies available in the economy are as set out in Table 4.1. The necessary conditions for the policy switches (for $X > 0$, $R > 0$) are set out in Table 4.3.

Table 4.3

		Switches into			
		A	B	C	D
Switches out of	A	•	$u_X < u_R$	$u_X < u_R^{+}$	$\delta u_E(0) > u_X$
	B	$u_X > u_R$	•	$u_X > u_R^{+}$	$\delta u_E(0) > u_R$
	C	$u_X > u_R^{*}$	$u_X < u_R^{*}$	•	$\delta u_E(0) > u_X$ $\delta u_E(0) > u_R$
	D	$\delta u_E(0) < u_X$	$\delta u_E(0) < u_R$	$\delta u_E(0) < u_X$ $\delta u_E(0) < u_R$	•

$^{+}$ condition immediately before a switch.

* condition immediately after a switch.

As in the previous model, the optimal path will lie along one of the arterial paths defined in terms of Figure 4.9, with the difference that policy C now has as its equation $u_X(X, R) = (X, R)$. It will however still be positively sloped. Assuming again that $F'(0) > a$, we can deduce from (4.55) that for *finite* time horizons $X(T) = R(T) = 0$ (since either $\psi_i(T) > 0$, $i = 1,2$, or $\psi_i(T) \leqslant 0$ in which case the appropriate one of λ_3, λ_4 would have to be positive). On the other hand when $T = \infty$, there would appear to be the possibility (as there was for the single resource model of Chapter 3) that it may not be optimal to exhaust the resource. This would be the case if it were optimal to switch into policy D permanently before both resources are exhausted.

Because of its relative simplicity we will dispose of the case where T is finite first of all. It is proven in Appendix 4.3 that the optimal plan is the one which takes the longest time feasible to exhaust both resources moving along the appropriate arterial path (i.e. the one which satisfies the initial conditions). Thus when $u_X(0, 0) > u_R(0, 0)$ the optimal policy sequence will be:

(i) BCA if $X(0) > X'$ (Figure 4.9(a)
BA if $X'' < X(0) < X'$ for some $X'' > 0$
BDA if $X(0) < X''$
} when $(X(0), R(0))$ is in region I (Figure 4.9(a))

(ii) ACA when $(X(0), R(0)) \in$ region II .

Similarly, when $u_X(0, 0) < u_R(0, 0)$, the optimal policy sequence will be:

(i) ACB if $R(0) > R'$ (Figure 4.9(b)
AB if $R'' < R(0) < R'$, for some $R'' > 0$
ADB if $R(0) < R''$
} when $(X(0), R(0))$ is in region II (Figure 4.9(b))

(ii) BCB when $(X(0), R(0) \in$ region I.

X' and R' are here defined as the respective solutions to $u_X(X, 0) = u_R(X, 0)$ and $u_X(0, R) = u_R(0, R)$. It is clearly optimal for the economy, wherever possible, to move towards a situation where its intrinsic valuation of an additional unit of resource stock is the same for both resources ($u_X = u_R$). It will exclusively exploit the less valued resource until this state of balance is achieved. We may also note that resource R will be exhausted first if and only if $u_X(0, 0) > u_R(0, 0)$ (i.e. the intrinsic valuation of the last unit of X, when there is no R left is greater than the intrinsic valuation of the last unit of R when X is exhausted).

In analysing the case $T = \infty$, there are several cases to be distinguished and to facilitate the exposition we define:

$$\alpha(X, R) \equiv \delta u_E(0) - u_X(X, R)$$

$$\beta(X, R) \equiv \delta u_E(0) - u_R(X, R) .$$

There are several cases to be considered:

<u>Case 1: $\alpha = 0$ lies below $\beta = 0$ for some $X \geqslant 0$</u>:

We shall define $(\hat{X}, \hat{R})$ to be the solution of the equation system $\alpha(X, R) = 0 = \beta(X, R)$. The existence and uniqueness of $(\hat{X}, \hat{R})$ is guaranteed if we make the not unreasonable assumptions that $\lim_{X \to \infty} u_X(X, R) = 0 \ \forall_R$ and $\lim_{R \to \infty} u_R(X, R) = 0 \ \forall_X$. $(\hat{X}, \hat{R}, \hat{\psi}_1, \hat{\psi}_2)$ constitutes an equilibrium of the system of differential equations:

$$\dot{X} = -E_1$$

$$\dot{R} = -E_2$$

$$\dot{\psi}_1 = \psi_1 \delta - u_X(X, R)$$

$$\dot{\psi}_1 = \psi_2 \delta - u_R(X, R) ,$$

where $\hat{\psi}_1 = \hat{\psi}_2 = u_E(0)$.

In Appendix 4.3 it is proven that the optimal course for the economy is as follows:

(a) When $(X(0), R(0)) \in$ region I and $X(0) > \hat{X}$ (see Figure 4.11) a policy sequence BCD should be followed (say path P_1 in Figure 4.11); exploitation of both resources continues for an infinite time and positive stocks ($\hat{X}$ and $\hat{R}$ respectively) of the two resources remain; for this to happen, $E_i(t) \downarrow 0$ $(i = 1,2)$ as $t \to \infty$. $(X(t), R(t), \psi_1(t), \psi_2(t)) \to (\hat{X}, \hat{R}, \hat{\psi}_1, \hat{\psi}_2)$ as $t \to \infty$.

(b) When $(X(0), R(0)) \in$ region II and $R(0) > \hat{R}$, a policy sequence ACD is optimal (say Path P_2 in Figure 4.11); as in case (a) above, it takes an infinite time for the economy to reach the equilibrium $(\hat{X}, \hat{R}, \hat{\psi}_1, \hat{\psi}_2)$ and $E_i(t) \downarrow 0$ (i.e. 1,2) as $t \to \infty$.

(c) If $(X(0), R(0)) \in$ region I and $X(0) = \hat{X}$, then the economy follows BD (Path P_3) and if $(X(0), R(0)) \in$ region II and $R(0) = \hat{R}$, AD is optimal (Path P_4).

(d) If $(X(0), R(0)) \in$ region I above $\beta = 0$ and $X(0) < \hat{X}$, then the economy should follow B down to $\beta = 0$, letting $E_2(t) \downarrow 0$ and switch into D when the path (P_5, say) hits $\beta = 0$, at which point $\dot{\psi}_2 = 0$ and the subsystem of differential equations for $\dot{R}$ and $\dot{\psi}_2$ is in equilibrium; the process will again take an infinite time. Because $X(0)$ is less than the level of X which a more abundantly endowed economy would choose to conserve $(\hat{X})$ more R has to be used up in order to compensate; similarly when $(X(0), R(0)) \in$ region II to the right of $\alpha = 0$ with $R(0) < \hat{R}$, A should be followed to $\alpha = 0$ (P_6), with $E_1(t) \downarrow 0$ and D becoming "operative" at $t = \infty$ when the path meets $\alpha = 0$.

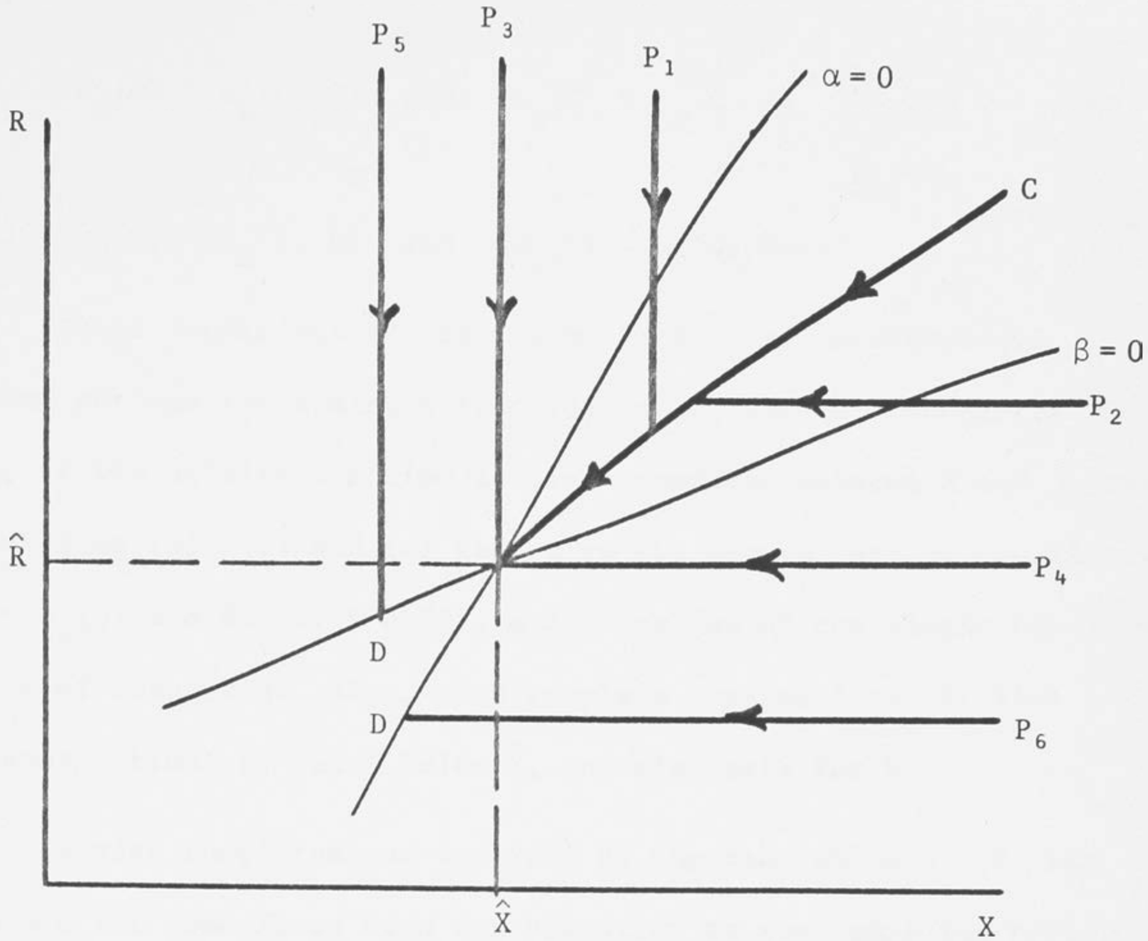

Figure 4.11

(e) Finally we consider the case where $X(0) < \hat{X}$, $R(0) < \hat{R}$ and $(X(0), R(0))$ lies between $\alpha = 0$ and $\beta = 0$. In this case the economy's resource endowments are below any level which the community regards as acceptable and so the optimal course is to do nothing — i.e. policy D is operative for the whole plan and $X(t) = X(0)$, $R(t) = R(0)$ $\forall_t$.

For all initial conditions considered here it is optimal to leave positive stocks of both resources unexploited at the endpoint. This result is a consequence of the relatively strong conservation motives assumed for both X and R. We are assuming in particular that one of the following three conditions holds:

(a) $\delta u_E(0) < u_X(0, 0)$ and $\delta u_E(0) < u_R(X, 0)$ $\forall_X \geqslant \underline{X}$ for some $\underline{X} > 0$,

(b) $\delta u_E(0) < u_R(0, 0)$ and $\delta u_E(0) < u_X(0, R)$ $\quad V_R \geqslant \underline{R}$ for some $\underline{R} > 0$,

(c) $\delta u_E(0) < u_R(0, 0)$ and $\delta u_E(0) < u_X(0, 0)$.

These conditions are very similar to the non-exhaustion conditions derived for a single resource in section III of Chapter 3. In fact, if the utility function is also separable between X and R, then the conditions (a), (b) and (c) reduce to the single part of conditions $\delta u_E(0) < u_X(0)$ and $\delta u_E(0) < u_R(0)$, a duplication of the single resource condition of Chapter 3. Also, with complete separability, if $X(0) > \hat{X}$, it is never optimal to run X below $\hat{X}$, and similarly for R.

Having completed our analysis of the case where $\alpha = 0$ lies below $\beta = 0$ for some X, we turn our attention to the remaining cases, where $\alpha = 0$ lies everywhere above $\beta = 0$ in the (X, R) plane.

Case 2: $\delta u_E(0) < u_X(0, 0)$:

This assumption implies in particular that $\delta u_E(0) > u_R(0, 0)$, so one would expect that the optimal path would involve exhaustion of R and non-exhaustion of X. This is established using the optimality proof of Appendix 4.3. The optimal path is the appropriate arterial path leading to X^* (Figure 4.12) provided $X(0) > X^*$. This entails a policy sequence BCAD (BAD if $X^* < X(0) < X'$) or ACAD depending on whether the economy is initially in region I or II. R will be exhausted in a finite time (the longest such time feasible) after which the remaining stock of X will be exploited exclusively until $t = \infty$ when a positive stock X^* of X remains. X^* is the solution to $\delta u_E(0) = u_X(X^*, 0)$. Once $R = 0$ and the economy is following policy A, the system of equations for the economy is:

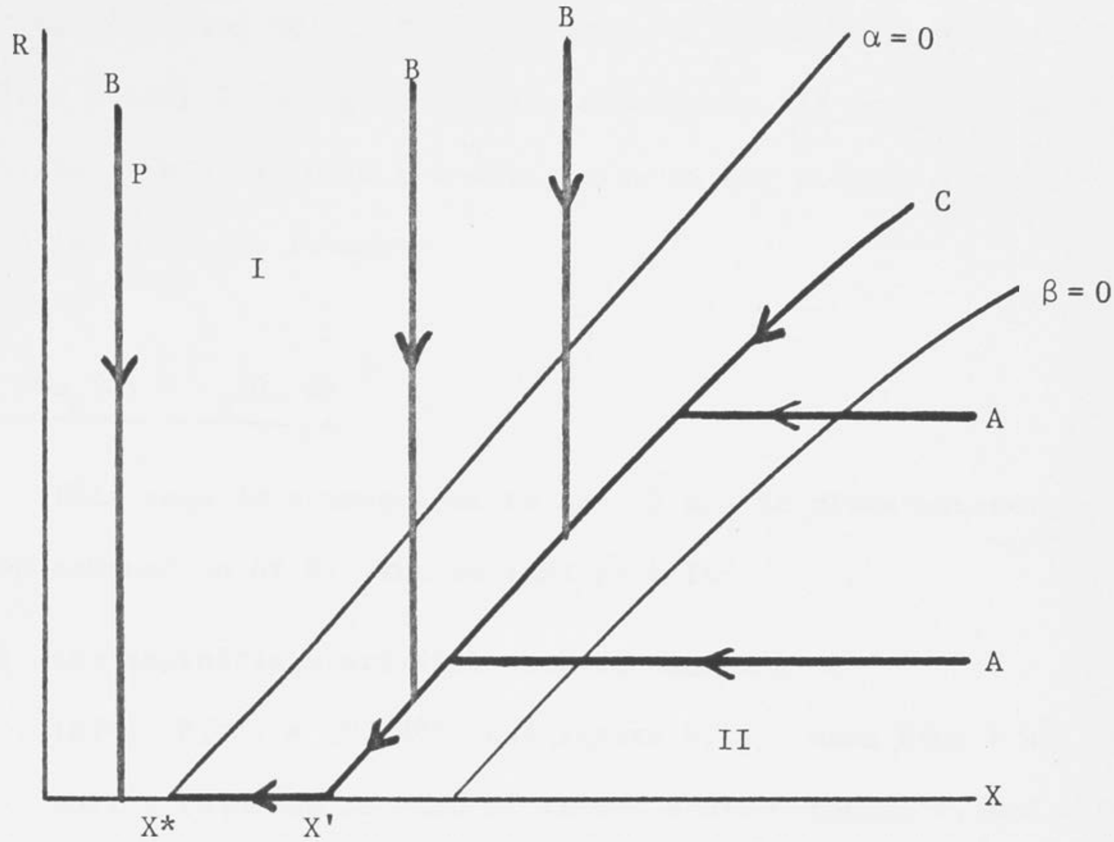

Figure 4.12

$$\dot{X} = -E_1$$
$$\dot{\psi}_1 = \psi_1\delta - u_X(X, 0)$$
$$\psi_1 = u_E(E_1)$$

and this is an entirely self-contained system in the (X, ψ) plane with $(\hat{X}, \hat{\psi}_1) = (X^*, u_E(0))$ as its equilibrium. It will take an infinite time to attain this equilibrium and the path leading to it (with $E_1(t) \downarrow 0$ as $t \to \infty$) will be the optimal path. Summing up, when $X(0) > X^*$, the economy will find it optimal to:

(a) exhaust R at the same time as $X = X'$ (i.e. as late in the plan as is feasible); and

(b) continue to exploit X forever but leave an amount, X^*, unexploited at the end of the programme.

When $X(0) < X^*$, the economy will adopt the longest possible exhaustion path for R, employing policy B until R = 0, at which point it will switch into policy D for the rest of the programme (a path such as P in Figure 4.12). We again note the similarity to the results obtained in Chapter 3 for a single resource.

Case 3: $\delta u_E(0) < u_R(0, 0)$:[7]

This case is symmetrical to case 2 and involves exhaustion of X and non-exhaustion of R. The optimal path is:

(a) the appropriate arterial path terminating at $(X(\infty), R(\infty)) = (0, R^*)$ (see Figure 4.13), when $R(0) > R^*$; the policy sequence is BCBD if $(X(0), R(0)) \in$ region I, and ACBD if $(X(0), R(0)) \in$ region II and $R(0) > R'$ or ABD if $R^* < R(0) < R'$.

(b) the appropriate path, AD (say path ξ in Figure 4.13) if $R(0) < R^*$.

Case 4: $\delta u_E(0) > u_X(0, 0)$; $\delta u_E(0) > u_R(0, 0)$:

Because the conservation motives for both resources are weak it turns out to be optimal to exhaust them both in a finite time (because of the assumption that $u_E(0) < \infty$). In Figure 4.14 (drawn on the assumption that $u_X(0, 0) > u_R(0, 0)$), the optimal plan is the longest exhaustion path for both resources along the appropriate arterial path. As such, it coincides with the optimal exhaustion path for a long finite time horizon. When both X and R have been exhausted

7 This condition, together with the fact that $\alpha = 0$ is assumed to lie above $\beta = 0$, implies that $\delta u_C(0) > u_X(0, 0)$.

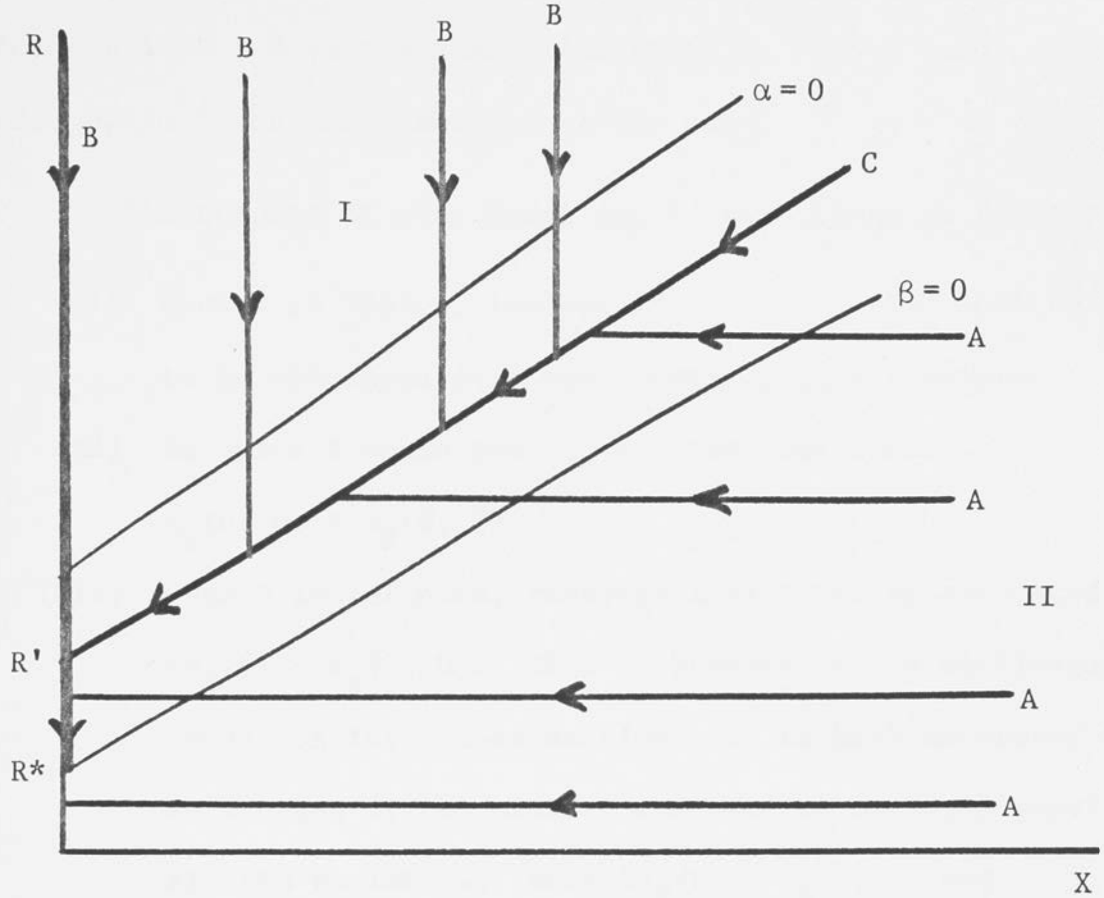

Figure 4.13

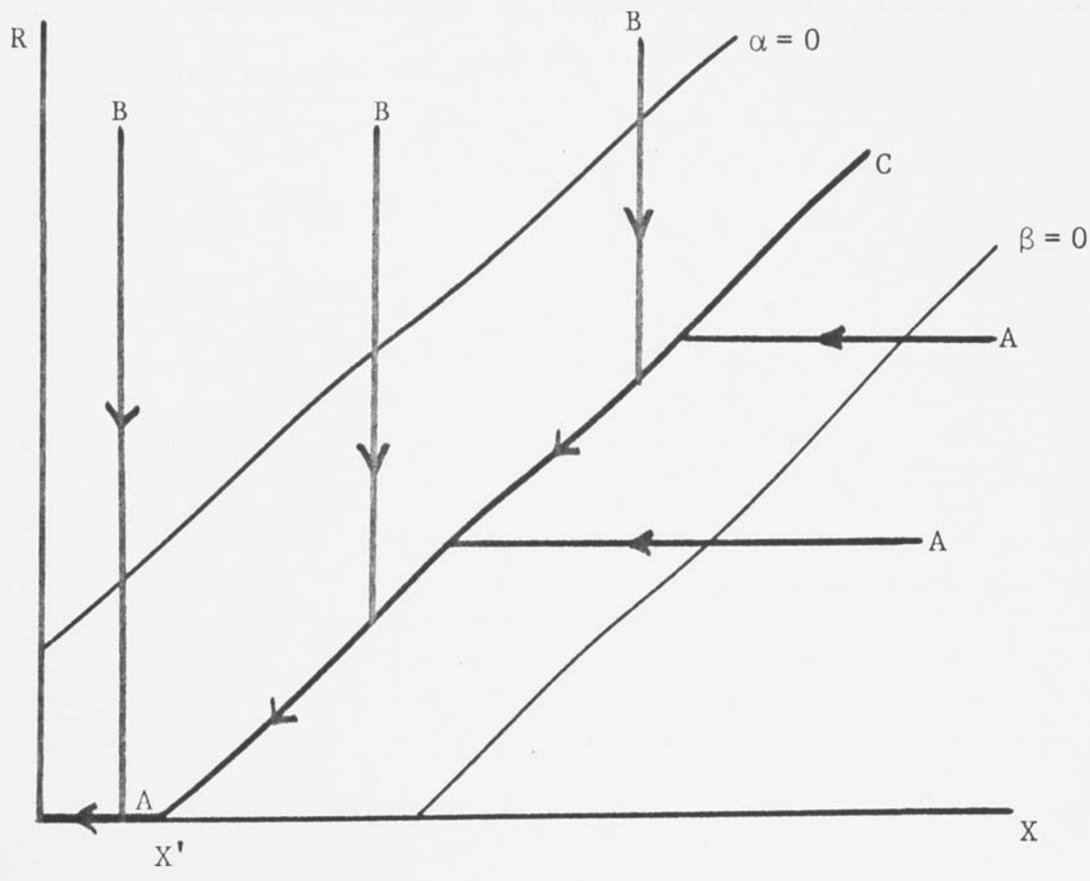

Figure 4.14

($E_1(t) \downarrow 0$ as $X \downarrow 0$ in the case illustrated in Figure 4.14), policy D becomes optimal for the remainder of the plan.

The results of this model may be summarized as follows:

(i) When T is finite, the optimal course is for both resources to be exhausted over the longest period feasible.

(ii) Resource R would tend to be exhausted first if $u_X(0, 0) > u_R(0, 0)$.

(iii) When T is infinite, resource X will not be exhausted if $\delta u_E(0) < u_X(0, 0)$. This is however only a sufficient condition for non-exhaustion. It is both necessary and sufficient if the utility function is strongly separable in all the variables. When $\delta u_E(0) > u_X(0, 0)$ *and* $\delta u_E(0) > u_R(0, 0)$, it is optimal to exhaust *both* resources. The exhaustion and non-exhaustion conditions are thus seen to be very similar to those derived for a single resource in Chapter 3.

APPENDIX 4.1

PROOF OF OPTIMALITY IN SECTION I

As in other proofs of this type in the thesis we let * superscripts denote the path claimed to be optimal. The difference between the value of the present value integral along the optimal path (P*) and along any other Pontryagin path (P) is:

$$P^* - P = \int_0^T [u(C^*) - u(C)]e^{-\delta t}dt$$

$$> \int_0^T u_C^* F_1^*(L_1^* - L_1)e^{-\delta t}dt + \int_0^T u_C^* F_2^*(G^* - G)e^{-\delta t}dt \ .$$

By the concavity of G:

$$G - G^* < G_1^*(L_2 - L_2^*) = G_1^*(L_1^* - L_1) \ .$$

Hence,

$$P^* - P > -\int_0^T u_C^* \frac{F_1^*}{G^*}(G^* - G)e^{-\delta t}dt + \int_0^T u_C^* F_2^*(G^* - G)e^{-\delta t}dt$$

$$= \int_0^T \left(\psi^* - \frac{\lambda_2^*}{G_1^*} + \frac{\lambda_1^*}{G_1^*} + \lambda_4^*\right)(G^* - G)e^{-\delta t}dt$$

$$> \int_0^T \psi^* e^{-\delta t}(G^* - G)dt + \int_0^T \frac{\lambda_2^*}{G_1^*} G_1(L_2 - L_2^*)e^{-\delta t}dt$$

$$+ \int_0^T \frac{\lambda_1^*}{G_1^*} G_1^*(L_1 - L_1^*)e^{-\delta t}dt + \int_0^T \lambda_4^*(G^* - G)e^{-\delta t}dt$$

$$\geqslant \psi_0^* \int_0^T (G^* - G)dt - \int_0^T \lambda_4^* G\, e^{-\delta t}dt$$

$$= \psi_0^*[X(T) - X^*(T)]$$

$$\geqslant -\psi_0^* X^*(T)$$

$= 0$ along all the paths claimed to be optimal.

QED

APPENDIX 4.2

DERIVATION OF SWITCHING CONDITIONS IN SECTION II

AB: For policy A: $\psi_1 = F'(E_1) - a\phi(X)$

$$\psi_2 = F'(E_1) - b\eta(R) + \lambda_2$$

For policy B: $\psi_1 = F'(E_2) - a\phi(X) + \lambda_1 - \lambda_3$

$$\psi_2 = F'(E_2) - b\eta(R) \ .$$

When $X > 0$ and $\lambda_3 = 0$, for policy A, $\psi_1 \leqslant \psi_2 + b\eta(R) - a\phi(X)$, while for policy B, $\psi_1 \geqslant \psi_2 + b\eta(R) - a\phi(X)$. In this case a switch can only take place if:

$$\dot{\psi}_1 > \dot{\psi}_2 - a\phi'(X).\dot{X} \quad \text{immediately before a switch}$$

$$\Leftrightarrow \psi_1\delta + aE_1\phi'(X) > \psi_2\delta + a\phi'(X)E_1$$

$$\Leftrightarrow \psi_1 > \psi_2$$

$$\Rightarrow a\phi(X) < b\eta(R) \ .$$

It may be similarly verified that this condition must also hold immediately after a switch. In addition, when $X = 0$, the multiplier, λ_3, can always jump to make a switch possible even if the above condition is not satisfied.

When $R > 0$, the condition for a BA switch is $a\phi(X) < b\eta(R)$, but when $R = 0$, λ_4 can always jump to make a switch possible regardless of this condition. These possibilities will be commented on in Appendix 4.3.

AC: For A: $\psi_1 \leqslant \psi_2 + b\eta(R) - a\phi(X)$, and for C: $\psi_1 = \psi_2 + b\eta(R) - a\phi(X)$. An AC switch therefore requires that $\dot{\psi}_1 > \dot{\psi}_2 - a\phi'(X)\dot{X}$ immediately before a switch occurs, so that we must have $a\phi(X) < b\eta(R)$ immediately before the switch. Similarly, a CA switch requires $a\phi(X) > b\eta(R)$ immediately after the switch occurs.

AD: When $R > 0$,

$$\psi_1 < F'(0) - a\phi(X) \quad \text{for A, and}$$

$$\psi_1 \geqslant F'(0) - a\phi(X) \quad \text{for D .}$$

A switch requires $\dot{\psi}_1 > 0$ at the switching surface.

$$\Leftrightarrow \quad \psi_1 > 0 \quad \text{at the surface}$$

$$\Leftrightarrow \quad F'(0) > a\phi(X) \ .$$

As for AB, when $R = 0$, λ_4 may jump to preserve the continuity of ψ_1 and thus make the switch possible without the above restriction.

The switching conditions for BC, BD and CD, etc., are derived similarly to the above.

APPENDIX 4.3

IDENTIFICATION OF THE OPTIMAL PATH IN SECTION II

$$P^* - P = \int_0^T [u(E^*, X^*, R^*) - u(E, X, R)]e^{-\delta t}dt$$

$$> \int_0^T u_E(E^*)(E^* - E)e^{-\delta t}dt + \int_0^T u_X(X^*, R^*)(X^* - X)e^{-\delta t}dt$$

$$+ \int_0^T u_R(X^*, R^*)(R^* - R)e^{-\delta t}dt$$

$$= \int_0^T (\psi_1^* - \lambda_1^* + \lambda_3^*)(E_1^* - E_1)e^{-\delta t}dt + \int_0^T (\psi_2^* - \lambda_2^* + \lambda_4^*)(E_2^* - E_2)e^{-\delta t}dt$$

$$+ \int_0^T (\psi_1^*\delta - \dot{\psi}_1^*)e^{-\delta t}(X^* - X)dt + \int_0^T (\psi_2^*\delta - \dot{\psi}_2^*)e^{-\delta t}(R^* - R)dt$$

$$\geqslant \int_0^T (\psi_1^* + \lambda_3^*)(E_1^* - E_1)e^{-\delta t}dt + \int_0^T (\psi_2^* + \lambda_4^*)(E_2^* - E_2)e^{-\delta t}dt$$

$$- [\psi_1^* e^{-\delta t}(X^* - X)]_0^T - [\psi_2^* e^{-\delta t}(R^* - R)]_0^T$$

$$+ \int_0^T \psi_1^* e^{-\delta t}(\dot{X}^* - \dot{X})dt + \int_0^T \psi_2^* e^{-\delta t}(\dot{R}^* - \dot{R})dt .$$

Hence we have:

$$(4.57)\quad P^* - P > \psi_1^*(T)e^{-\delta T}(X(T) - X^*(T)) + \psi_2^*(T)e^{-\delta T}(R(T) - R^*(T))$$

$$- \int_0^T e^{-\delta t}\lambda_3^* E_1 dt - \int_0^T e^{-\delta t}\lambda_4^* E_2 dt .$$

Using this expression, we can identify the optimal path in each of the cases discussed in the text of the chapter.

1. <u>T finite</u>: When T is finite, along all Pontryagin paths $X(T) = R(T) = 0$, so that $P^* - P$ reduces to:

$$P^* - P > - \int_0^T \lambda_3^* E_1 e^{-\delta t} dt - \int_0^T \lambda_4^* E_2 e^{-\delta t} dt$$

and the optimal path is the longest feasible arterial exhaustion path for *both* resources. Along such a path exploitation of X (R) continues until time T in which case $\lambda_3^* = 0$ ($\lambda_4^* = 0$) for the whole programme, or exploitation ceases before the endpoint, in which case the longest arterial exhaustion path for X (R) is again chosen because along it λ_3^* (λ_4^*) will be zero for the whole programme. The sufficiency proof effectively rules out sequences such as AB in region I (path β in Figure 4.15 below). For such a path the AB switch requires that λ_3^* become positive and grow over time and this would prevent us from establishing the superiority of path β.

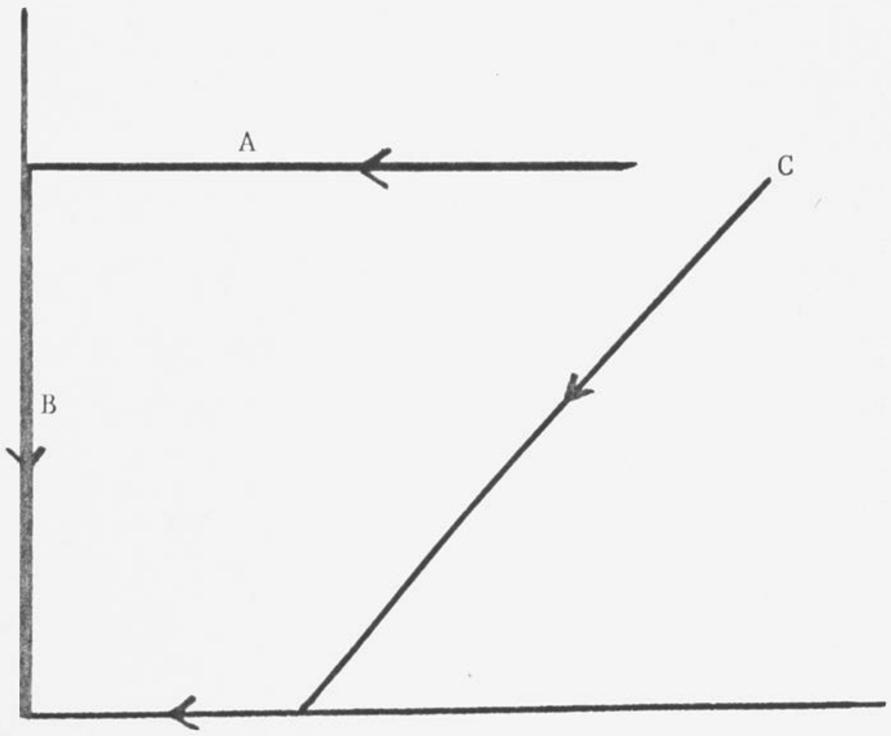

Figure 4.15

2. <u>$T = \infty$</u>: Each of the paths claimed to be optimal for cases 1 - 4 in the text has the property that $\lambda_3^* = \lambda_4^* = 0$ for the whole programme. This implies that $\psi_i^* \geqslant u_E^* > 0$, $i = 1,2$, and so

$$(4.58) \qquad P^* - P > -\psi_1^*(T)\ X^*(T)\ e^{-\delta T} - \psi_2^*(T)\ R^*(T)\ e^{-\delta T}$$

and for each of the paths claimed to be optimal one of the following will hold:

(i) $\psi_1^*(t) \to \hat{\psi}_1$, $\psi_2^*(t) \to \hat{\psi}_2$ (Case 1)

(ii) $\psi_1^*(t) \to \hat{\psi}_1$, $R^*(\infty) = 0$ (Case 2)

(iii) $\psi_2^*(t) \to \hat{\psi}_2$, $X^*(\infty) = 0$ (Case 3)

(iv) $X^*(\infty) = 0 = R^*(\infty)$ (Case 4) .

In each of these cases the R.H.S. of (4.58) is zero and $P^* - P > 0$.

CHAPTER 5
THE AVAILABILITY AND USE OF SUBSTITUTES

Up to this point we have carefully ignored the nature of the process whereby one resource is replaced by another in the process of production. We have also not paid any attention to the way in which a particular resource may become an economic alternative to an exhaustible resource whose cost of extraction is rising. In this chapter allowance will be made for a substitute resource to be phased in at some stage of the programme. It will be incorporated into the basic model (Chapter 3) in a way which acknowledges some of the more crucial aspects of the "substitutes" problem.

I
THE BACKGROUND

The issue of the development and use of substitute natural resources as a means of mitigating scarcity of existing resources has had a curious role to play in most discussions of resource policy. It is usually present in the background when the question of imminent exhaustion of a key resource is discussed and in such cases it serves as a useful foil for the inherently optimistic economist who is apt to point out that while one resource may be approaching exhaustion the pattern of adaptive behaviour induced by its increasing scarcity (rising costs of extraction and a resulting fall in profits) will bring forth a substitute resource to fill the breach. However, despite the importance placed on the development of substitutes by such economists, they have been reluctant to subject the process by which substitutes come into an economic existence to detailed analysis. A typical statement of this generally optimistic point of view is:

> "the heritage of knowledge, equipment, and economic institutions that the industrial nations are able to transmit to future generations is sufficient to overcome the potentially adverse effects of continual and unavoidable shift to natural resources with properties which on the basis of past technologies and products would have been economically inferior".[1]

Those economists who have been more precise about the role of substitution in the mitigation of scarcity have generally kept their references to it brief and have not ventured beyond the realm of possibilities:

> "Exhaustion is not necessarily desirable. Just as machines can become obsolete before they wear out, extraction of minerals can become unnecessary before the supply is depleted. Scrap availability might make mining undesirable; solar energy might displace mineral fuels".[2]

Although such statements as the above vastly oversimplify the situation, in the face of a shortage of more detailed appraisals, they have inevitably provided us with many of our impressions of the relationship between substitution and scarcity.

As already noted (see Chapter 1) a recent work by William Nordhaus [25] is mainly devoted to an empirical analysis of the phasing

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1 Barnett and Morse [6].

2 Gordon [14].

in of what he terms a "backstop technology".[3] There is also an interesting paper by Heal and Dasgupta [16] in which the date of availability of the substitute (which may well be a backstop technology) is subject to uncertainty. Heal and Dasgupta use their model to argue against the use of an a priori certainty equivalent discount rate in resource planning because of the bias involved. However neither the Heal and Dasgupta nor Nordhaus papers are particularly concerned with the economic process whereby one resource is replaced by another in the production of a consumption good. This chapter represents an attempt to model this process.

There seem to be three things which characterize the problem:

(i) A depletion effect in the production of the scarce resource makes that resource increasingly costly and provides one incentive for its replacement by something else;

(ii) The economic production of the substitute will require investment and the consequent building up of a stock of knowledge or physical capital; as this stock rises, the substitute should become cheaper to produce;

(iii) There will be uncertainty about the date at which the substitute will become available.

It is obviously impossible to incorporate all of these various aspects of the problem in a single model and so a series of models will be used. In Section II a simple certainty model is constructed in which two resources are perfect substitutes as an input into a consumption sector. One of the resources is in relatively short supply and its

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3 See also Solow [31], pp.4-5.

extraction is subject to a depletion effect. The other resource is plentiful but possibly costly to extract (a backstop technology). It is however, available to be extracted from the beginning of the programme. In Section III a two state variable model involving capital accumulation is used. Capital accumulation lowers the cost of the backstop technology. Section IV examines some other aspects of the Heal-Dasgupta formulation of the problem.

II

THE BASIC MODEL

1. THE OPTIMAL PATH

There are two resources in the model; the extraction of the first resource is E_1. This resource is in finite supply and its stock (X) changes according to:

$$(5.1) \qquad \dot{X} = -E_1 \ .$$

The other resource is not subject to any stock constraint and is thus regarded as being in abundant supply. It is accordingly unnecessary to specify an equation of the form (5.1) for the second resource. Its extraction is denoted as E_2.

We shall assume for simplicity that the two resources are perfect substitutes and that the production of consumption goods may accordingly be written:

$$(5.2) \quad C = F(E_1 + E_2) \quad F' > 0, \quad F'' < 0, \quad F'(\infty) = 0, \quad F(0) = 0 \ .$$

The scarce resource is assumed to have its scarcity reflected in the fact that its cost of extraction rises at an increasing rate as it is depleted according to the following variable cost function:

$$(5.3)\quad V = V(E_1, X) \qquad \left.\begin{array}{l} V_1 > 0, \quad V_2 < 0, \\ V_{11} > 0, \quad V_{22} > 0, \\ V_{12} = V_{21} < 0, \end{array}\right\} \text{for } E_1, X > 0$$

$$V_{22}(0, X) = V_2(0, X) = 0 = V(0, X), \quad X \geqslant 0$$

$$V_1(E_1, 0) = \infty, \quad E_1 \geqslant 0,$$

$$V_2(E_1, X) \to 0 \quad \text{as} \quad X \to \infty .$$

The variable cost function for the other resource is simply:

$$(5.4)\qquad S = S(E_2) , \quad S' > 0, \quad S'' > 0, \quad S(0) = 0 .$$

We assume that the economy wishes to select $E_1(t)$, $E_2(t)$ and $X(T)$ so as to maximize the present value of its stream of consumption up to a fixed time T. The problem is therefore to:

$$(5.5)\qquad \underset{E_1(t), E_2(t), X(T)}{\text{Max}} \int_0^T Ce^{-\delta t} dt$$

$$\text{s.t.} \quad C = F(E_1 + E_2) - V(E_1, X) - S(E_2)$$

$$\dot{X} = -E_2$$

$$E_1 \geqslant 0$$

$$E_2 \geqslant 0$$

$$X \geqslant 0 .$$

The solution to the problem (5.5) must satisfy the following necessary conditions:

$\exists$ a continuous function ψ, s.t.

$$(5.6)\qquad \dot{\psi} = \psi\delta + V_2(E_1, X)$$

$$(5.7)\qquad \psi = F'(E_1 + E_2) - V_1(E_1, X) + \lambda_1 - \lambda_3$$

$$(5.8)\qquad F'(E_1 + E_2) - S'(E_2) + \lambda_2 = 0$$

$$(5.9)\qquad \psi(T)X(T) = 0 , \quad T < \infty$$

$$(5.10) \quad \begin{cases} \lambda_1 \geqslant 0 & E_1 \geqslant 0 & \lambda_1 E_1 = 0 \\ \lambda_2 \geqslant 0 & E_2 \geqslant 0 & \lambda_2 E_2 = 0 \\ \lambda_3 \geqslant 0 & X \geqslant 0 & \lambda_3 X = \lambda_3 \dot{X} = 0 \ . \end{cases}$$

The four policies open to the economy are:

Policy A:	$E_1 > 0$	$E_2 = 0$
Policy B:	$E_1 = 0$	$E_2 > 0$
Policy C:	$E_1 > 0$	$E_2 > 0$
Policy D:	$E_1 = 0$	$E_2 = 0$.

The question may arise of whether consumption may become negative or zero (or in fact be always negative) along the optimal path. If we simply assume that $F'(0) > S'(0)$ (i.e. the backstop technology is always profitable by itself, even though it may be unprofitable *relative* to the exhaustible resource in the early stages when that resource is abundant and cheap to extract), then it follows that C will always be positive (since the backstop technology is always available as a source of positive C).

The switching surface for a switch from policy A to policy C in the ψ-X plane is given by the equations:

$$(5.11) \quad \begin{cases} \psi = S'(0) - V_1(E_1, X) \\ F'(E_1) = S'(0) \ . \end{cases}$$

The switching surface between B and C has a similar form. It is given by the equations:

$$(5.12) \quad \begin{cases} \psi = S'(E_2) - V_1(0, X) \\ F'(E_2) = S'(E_2) \ . \end{cases}$$

A BC switch requires $\psi < 0$ and a CB switch $\psi > 0$. It is an easy matter to show that the AC switching surface lies below the CB surface. We will assume that $\exists\ E_2 > 0$ s.t. $F'(E_2) = S'(E_2)$. Then policy D for which $F'(0) \leqslant S'(0)$ is ruled out. This leaves two main cases to be considered. They are:

(i) $F'(0) > V_1(0, X)$

(ii) $F'(0) \leqslant V_1(0, X)$,

The form of $\dot{\psi} = 0$ presents some minor difficulties. However we can say that:

(i) for case (ii) above $\dot{\psi} \gtreqless 0 \Leftrightarrow \psi \gtreqless 0$;

(ii) for case (i) as $X \to X^*$, $\dot{\psi} \to \psi\delta$ and for $X \leqslant X^*$, $\dot{\psi} = 0 \Leftrightarrow \psi = 0$, where X^* is the solution to $S'(E_2) = F'(E_2) = V_1(0, X)$ (see Figure 5.1);

(iii) for case (i) as $X \to \infty$, $\psi \to 0$ along $\dot{\psi} = 0$; and

(iv) $\dot{\psi} = 0$ lies below the CB switching surface.

This leads to a locus like that in Figure 5.1. For finite time horizons, $\psi(T)X(T) = 0$ which implies that $\psi(T)$ (as in Chapters 3 and 4) because of the assumption that $\lim_{X \to 0} V_1(E_1, X) = \infty \quad V_{E_1} \geqslant 0$. The optimal path is identified as follows:

(a) When $F'(0) > V_1(0, X)$, the optimal path is that path for which $\psi(T) = 0$. When $X(0) > X^*$ it is the path along which ψ declines to zero, reaching zero at time T (e.g. path 1 in Figure 5.1). The policy sequence will be AC or possibly just A if initial X is relatively large and T is relatively small. The latter possibility arises because if X is large and T short, the scarcity of X is not very apparent. It will be exploited at a level close to the static optimum tending to the static optimum as $t \to T$ and there will be no need to

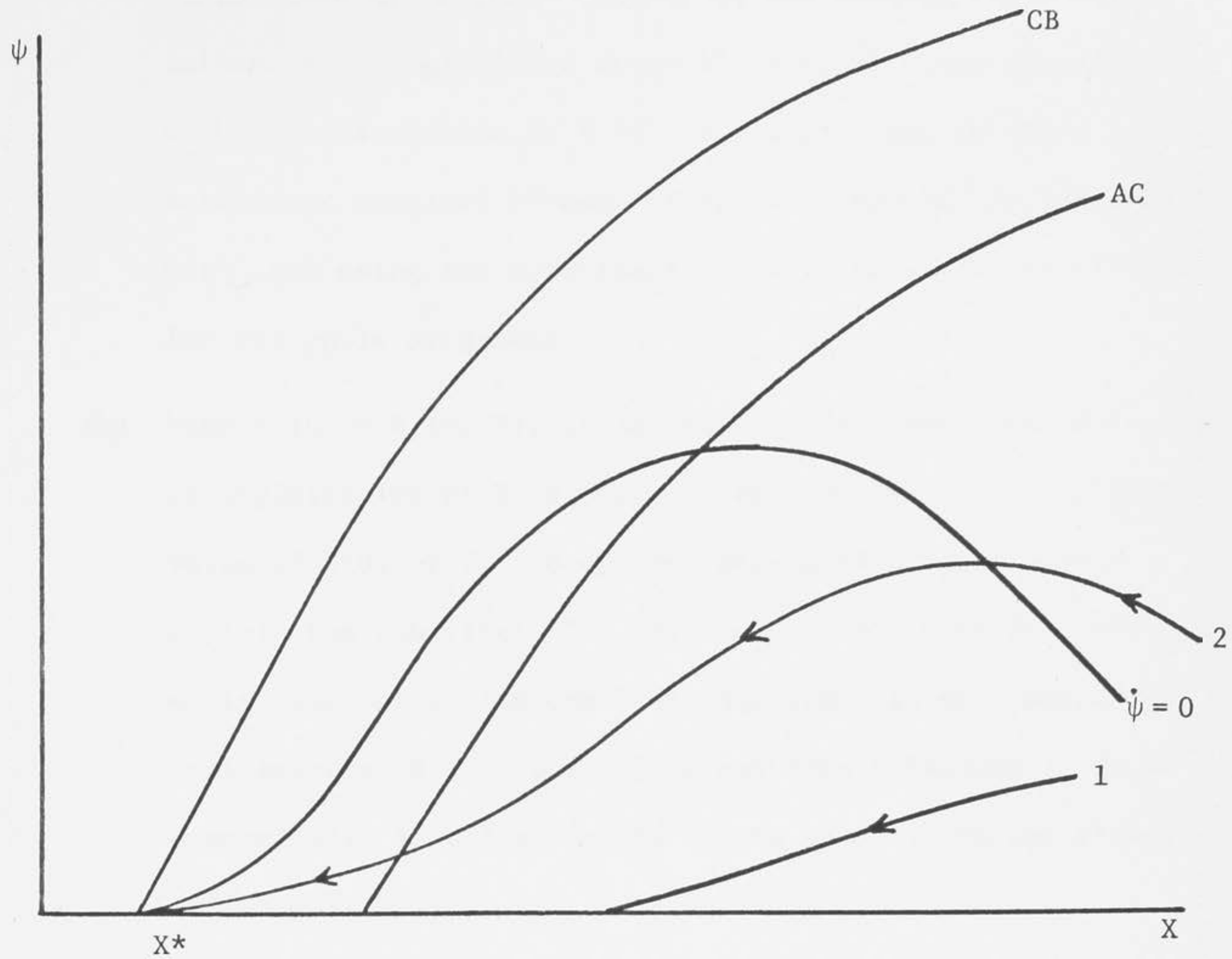

Figure 5.1

contemplate using the substitute. However when X is more scarce and T is large, the rationing of the exploitation of X over time becomes important and as the cost of extraction of X becomes too great the substitute will be needed to alleviate the pressure on the cost of X. This point is reached when the user cost of the scarce resource plus the marginal cost of extraction of X equals the marginal cost of zero production of the substitute ($\psi = S'(0) - V_1(E_1, X)$). Then the economy switches from A into C and exploits both resources until time T. As $T \to \infty$, the optimal path tends to path 2 in Figure 5.1. When $X(0) \leqslant X^*$, $\psi(t) \equiv 0 \equiv E_1(t)$ for the whole programme.

Policy B is operative. Clearly if the economy finds itself initially in a situation where the marginal cost associated with zero extraction of X ($\psi + V_1(0, X)$) exceeds the associated marginal product ($F'(E_2)$) it should content itself with just using the substitute. X will remain equal to X(0) for the whole programme.

(b) When $F'(0) \leqslant V_1(0, X)$, it is immediately clear that any level of exploitation of X is uneconomical and regardless of the value of X(0) or T, the optimal course will be to simply exploit the substitute for the whole plan. ψ is zero for the whole time reflecting the fact that there is no opportunity cost associated with using X at any time t because it is as economically valueless in the future as it is in the present ($F'(0) \leqslant V_1(0, X)$).

As noted in Chapter 4, when T = 0, the myopic and intertemporal plans coincide. Furthermore, for the case $F'(0) > V_1(0, X)$ as T increases the terminal stock of X falls. This is simply because with a longer planning horizon with discounting the higher costs of exploitation of X as it is depleted carry less weight in the present if they occur near the end of the longer programme.

2. COMPARATIVE DYNAMICS

Let us now turn to the question of how different cost functions for E_2 will affect the optimal path for an infinite horizon plan. In the next section a more "controlled" view of this problem will be taken with the cost of the substitute being deliberately reduced over time by means of investment. However here we are concerned with a problem of comparative dynamics, with the optimal path being shifted in some way by a change in a cost parameter. Because it is difficult to obtain definite results for the variable costs case using general functions, for the remainder of this section we shall assume "constant" costs in production of both resources. Thus

(5.13) $V = aE_1\phi(X)$ $\quad a > 0$, constant

$\phi(0) = \phi'(0) = \infty, \quad \phi'(X) < 0, \quad \phi''(X) > 0$.

(5.14) $S = bE_2$ $\quad b > 0$, constant .

Before looking at the comparative dynamics of the model, we shall check the nature of the optimal paths for the constant costs model. We will do this for the case without a depletion effect as well as for the case where a depletion effect is present.

Without a depletion effect the optimal path looks, predictably, like that shown in Figure 5.2.

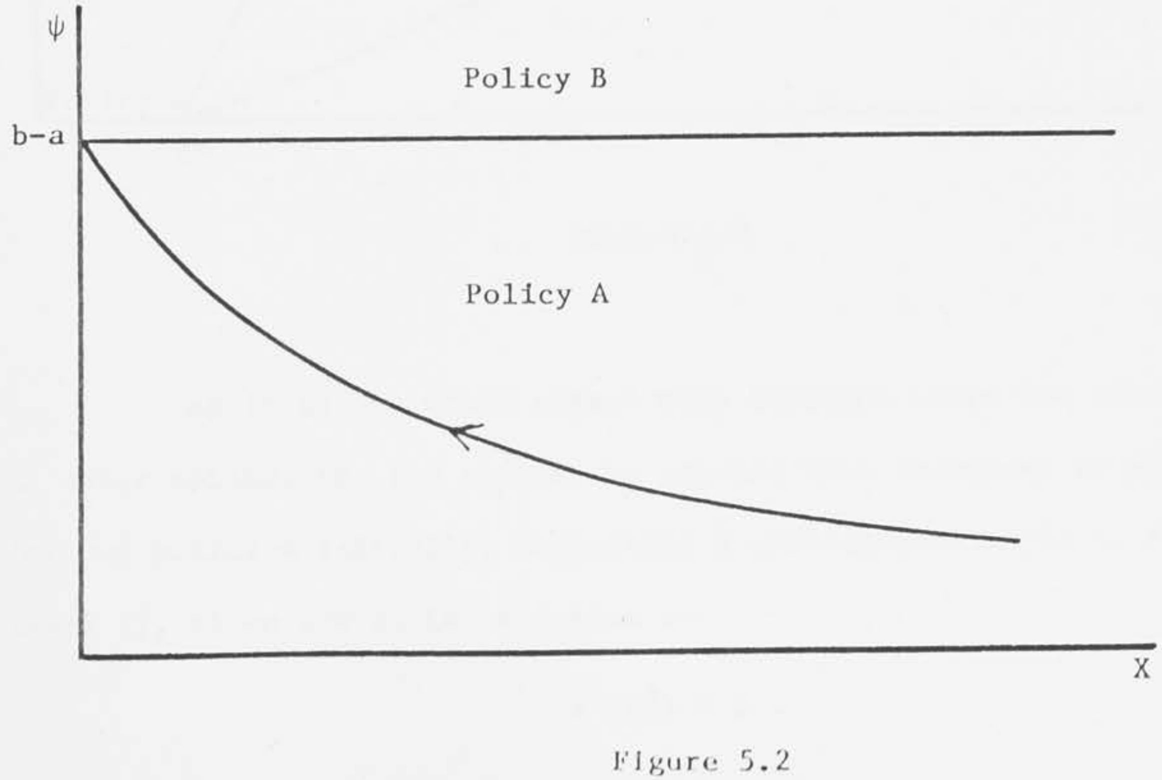

Figure 5.2

The resource is exploited exclusively, exhausted in a finite time, and then replaced by the substitute. The rate of exploitation of X does not run continuously to zero as in Chapter 3; instead E_1 falls to a positive level E_1^* which is the solution to $F'(E_1^*) = b$. At that point the substitute is introduced into the productive process and exploited at a rate equal to E_1^* until $t = \infty$.

When a depletion effect is present, the situation is as shown in Figure 5.3.

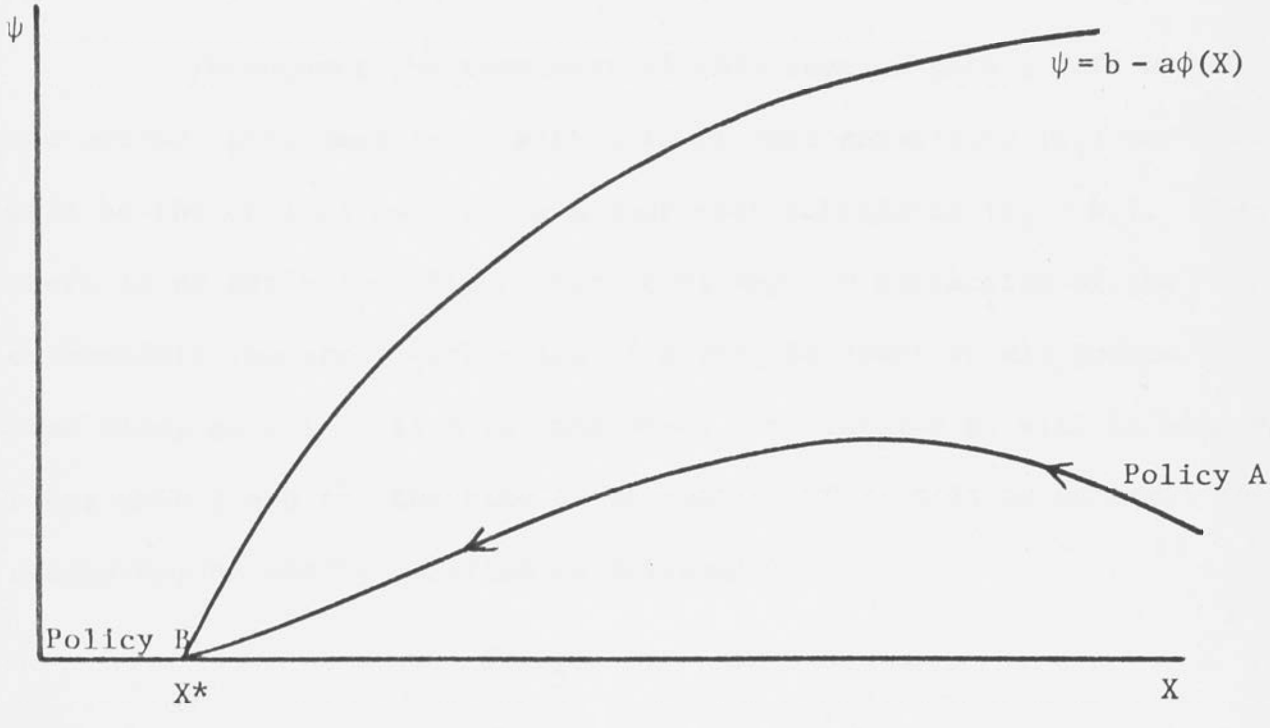

Figure 5.3

Again as one would expect when constant costs are involved, it is never optimal for the economy to produce both resources at once. It follows policy A initially, exploiting X exclusively, until E_1 falls to level E_1^*, given now as the solution to

$$F'(E_1^*) = b \ .$$

At that point X will equal X* (in this case the solution to $b = a\phi(X^*)$) and the economy should switch into policy B, using the substitute at a rate $E_2 = E_1^*$ for the remainder of the plan. In contrast to the model with variable costs, X will be run down to its terminal level (X*) in a finite time.

Our main concern now is to analyse the effects of a change in b on the optimal path. There are two cases to be considered separately:

(i) No depletion effect:

Throughout the remainder of this section path 1 will denote the optimal path associated with a lower cost substitute (b_1) and path 2 will be the optimal path for a higher cost substitute ($b_2 > b_1$). When there is no depletion effect characterizing the extraction of the exhaustible resource, extraction of X will be lower at all points of time along path 1 until X is exhausted. In addition E_2 will be higher along path 1 and t^*, the time of exhaustion of X, will be sooner. These claims may be easily verified as follows:

$$\psi = F'(E_1) - a \quad \text{and} \quad \dot{\psi} = \psi\delta ,$$

together imply that

$$\text{sgn}\,\frac{dE_1(t)}{db} = -\text{sgn}\,\frac{d\psi_0}{db} = -\text{sgn}\,\frac{d\psi(t)}{db} .$$

From $F'(E_1(t^*)) = b$ we find that

$$\frac{dt^*}{db} = \lim_{t \to t^*-} \frac{1}{F''(E_1(t)).\dot{E}_1(t)} > 0 .$$

Now, if we compare optimal plans for two values of b, and if we let t_1^* represent the exhaustion time of X on path 1 (see Figure 5.4), then $X(t_1^*)$ will be lower on path 1 than on path 2. Consequently path 1 must

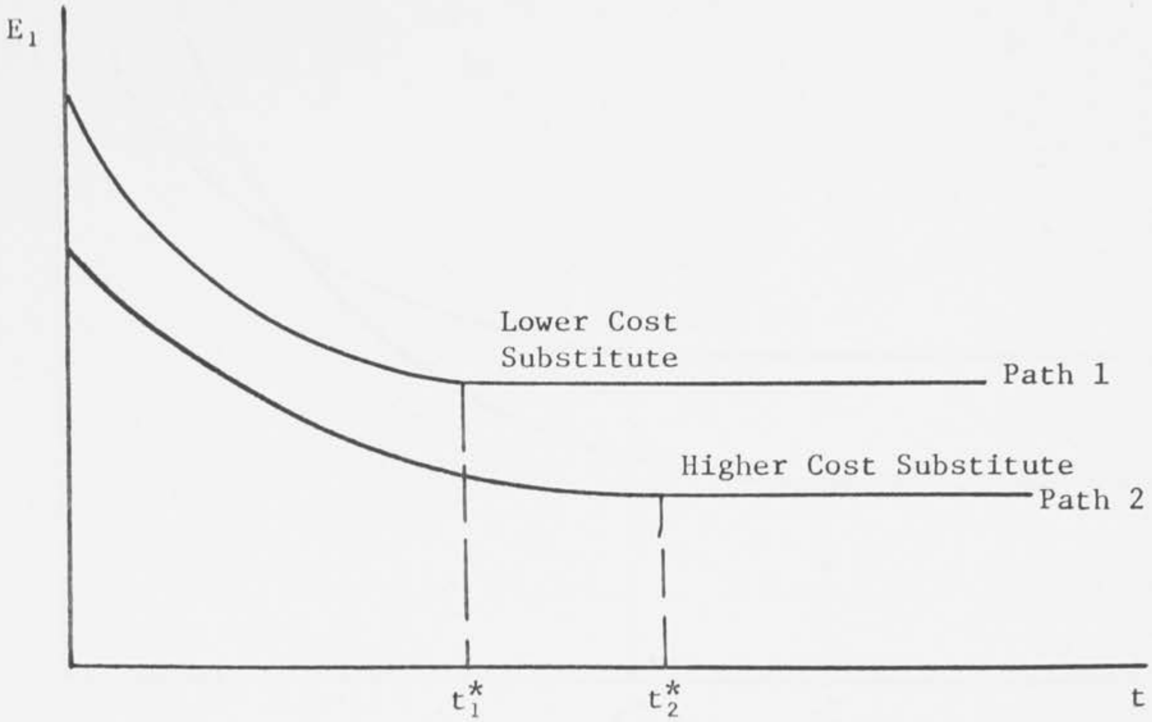

Figure 5.4

lie above path 2 for some t. Therefore it must lie above it for all t (since $\operatorname{sgn} \frac{dE_1(t)}{db} = -\operatorname{sgn} \frac{d\psi_0}{db}$ which is the same $\forall_t$).

(ii) Depletion effect:

In this case things are a little more difficult. However it is still possible to gain a fairly clear picture of the change in the time profile of E_1. As in the previous case the date at which the economy should switch into the backstop technology is brought forward when b is lowered (path 1 in Figure 5.5).[4] In addition the terminal stock (X^*) of the exhaustible resource is increased when b is lowered.[5] On the other hand in contrast to case (i) above, E_1 is not higher for

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4 t^* is defined by $b = a\phi(X(t^*))$ so that

$$\frac{dt^*}{db} = \lim_{t \to t^*-} \frac{-1}{a\phi'(X(t))E_1(t)} > 0 .$$

5 $b = a\phi(X^*) \Rightarrow \frac{dX^*}{db} = \frac{1}{a\phi'(X^*)} < 0.$

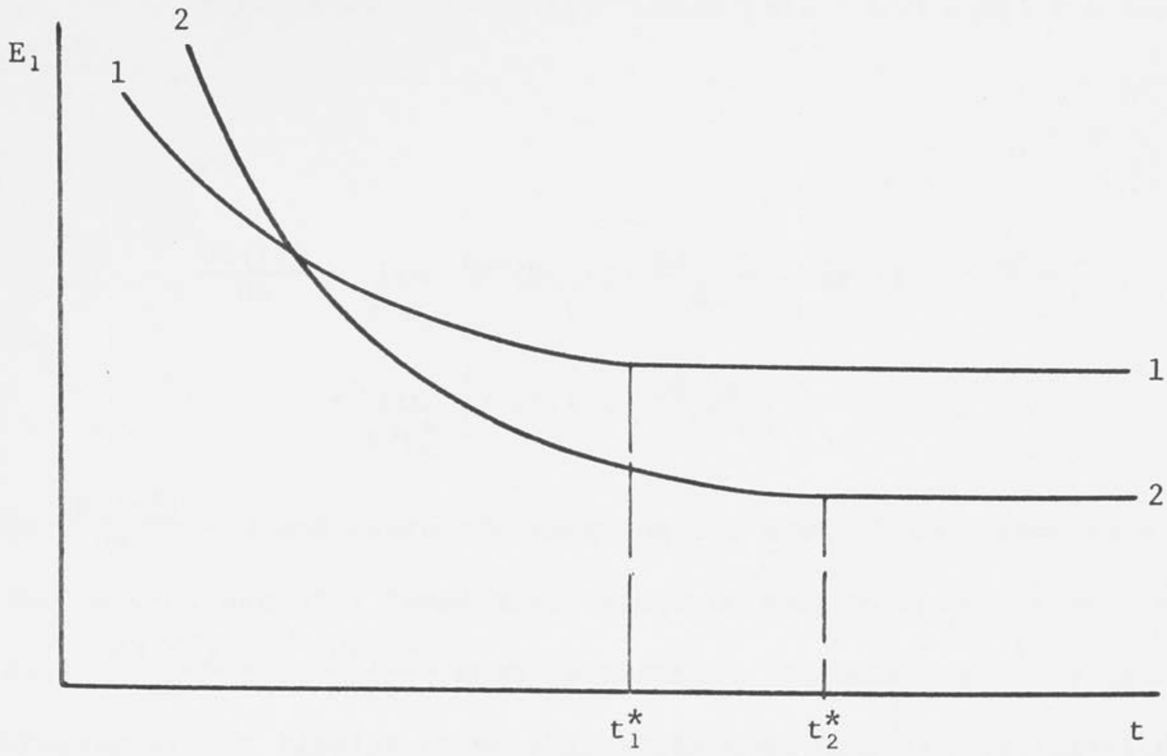

Figure 5.5

all values of t along path 1. In fact in the earlier stages of the programme we can expect it to be *lower*. This may be verified as follows:

At the switching time between policies A and B,

$$X_0 - X^* = \int_0^{t^*} E_1(t)dt$$

$$\Rightarrow \qquad \frac{-dX^*}{db} = \frac{dt^*}{db} E_1(t^*) + \int_0^{t^*} \frac{dE_1(t)}{db} dt$$

$$\Rightarrow \int_0^{t^*} \frac{dE_1(t)}{db} dt = 0 \ .$$

This last result implies that if we are comparing paths for two values of b, when $X(t_1^*) = X_1^*$ along path 1, $X(t_1^*)$ will also equal X_1^* along path 2, i.e. $\frac{dX(t_1^*)}{db} = 0$.

In addition since $\psi(t_1^*) = 0$ on path 1 and $\psi(t_1^*) > 0$ on path 2. $\frac{d\psi(t_1^*)}{db} > 0$.

Now,

$$\frac{d\psi(t_1^*)}{db} = \lim_{t\to t_1^*} \left(F''(E_1(t)) \frac{dE_1(t)}{db} - a\phi'(X(t)) \frac{dX(t)}{db} \right)$$

$$= \lim_{t\to t_1^*} \left(F''(E_1(t)) \frac{dE_1(t)}{db} \right) .$$

So $\frac{dE_1(t_1^*)}{db} < 0$ and therefore lowering the cost of the substitute raises the rate of use of X immediately prior to switching out of it. But since $\frac{dX(t_1^*)}{db} = 0$, $\frac{dE_1(t)}{db}$ must be positive for some t earlier in the programme: In simpler terms this would mean that the availability of cheaper substitutes makes it optimal to use *less* of an exhaustible resource in the earlier stages of a plan. This would seem to run counter to the prevailing intuition which says: "cheap substitutes will be available in the future, so why conserve our resources now". Obviously such arguments ignore depletion effects. In the case where there are no depletion effects, the popular view is supported — extraction of the exhaustible resource should be higher at all points of time until it is exhausted. The availability of a relatively cheap substitute renders the exhaustible resource less of a scarce commodity and results in it being used up more quickly. When a depletion effect is introduced, the popular view is supported in so far as, the "scarce" resource being more dispensible, less of it is ultimately used. Moreover it is optimal both to cease exploiting it sooner and to use more of it at some points of time. However precisely because it is optimal to use more of it at some points of time it is also optimal to delay the onset of depletion effects as long as possible. This involves *lowering* extraction in the earlier stages of the plan so as not to drive up costs too rapidly. Naturally there may be other constraints in the economy which will prevent E_1 from being lowered, but we are concerned here with the basic mechanism of the resource planning process. Having exhausted the implications of this simple model, we shall now proceed to analyse the more complicated problem of making a substitute an economic alternative by means of investment.

III

DEVELOPMENT OF A SUBSTITUTE

The previous section presented a very simplified view of the substitution problem. In particular it assumed that the cost function for production of the substitute is unchanging over time. This clearly ignores one important aspect of the economic "birth" of substitutes — namely the role of investment in making the substitute a better economic proposition by lowering its cost function over time. In this section we are concerned with incorporating the *development* of the substitute resource into the model of Section II.

The basic model to be used is that of Section II. Once again the substitute resource is assumed to be in abundant supply in the sense that sufficiently large stocks of it are in the ground for depletion to make no difference to its average or marginal costs of production. It is of course highly unlikely that an economic agent would even consider developing a replacement for a scarce resource which is characterized by similar depletion effects in the medium to short run. Nevertheless, especially in the early stages of the resource's development, certain inelasticities are bound to affect the costs of producing the new resource. In particular, costs of production may remain prohibitively high until the economy has a sufficiently high stock of the type of capital equipment used in extracting and processing the substitute. In addition low cost production of the substitute may depend on the acquisition of a certain minimum level of technical knowledge. Denoting the stock of capital associated with extraction of resource 2 as K (it may refer either to physical equipment or a form of technical progress) we may write the variable cost function for this resource as

$$(5.15)\quad S = S(E_2, K) \qquad S_1 > 0 \qquad S_{11} > 0$$
$$S \in C^{(2)} \qquad S_2 < 0 \qquad S_{22} > 0$$
$$S_{12} = S_{21} < 0 \quad \text{for} \quad E_2, \quad K > 0$$
$$S(E_2, 0) = S_1(E_2, 0) = \infty$$
$$S(0, K) = S_2(0, K) = S_{22}(0, K) = 0, \quad K \geqslant 0.$$

The stock of capital, K, is assumed to change according to:

$$(5.16)\qquad \dot{K} = I ,$$ [6]

where I denotes gross investment which is subject to an adjustment cost function:

$$(5.17)\qquad A = g(I) \qquad g' > 0, \quad g'' > 0 .$$ [7]

Because we are largely concerned with a situation where capital will have to be *accumulated*, we are ignoring the form of the

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[6] Since the firm is likely to be in the position of holding a stock of capital idle for some time, the conventional exponential depreciation assumption is inappropriate here. Depreciation would need to be some function of output of E_2. As introducing such an hypothesis would make the analysis rather more complicated than it is already, it has been omitted from the model.

[7] The marginal cost of investment is increasing, reflecting the fact that large bursts of investment spending at a point in time will be unprofitable. The firm is forced to spread its investment spending over time. This captures part of the essence of the problem of development of a substitute: the need for investment substantially in advance of production of the substitute.

adjustment cost function for $I < 0$, and we impose a non-negativity constraint on I. The optimization problem to be solved by the economy is

$$(5.18) \qquad \underset{\substack{E_1(t), E_2(t), I(t), \\ X(T)\ K(T)}}{\text{Max}} \int_0^T Ce^{-\delta t} dt \quad \text{T parametrically fixed.}$$

$$\text{s.t.} \quad C = F(E_1 + E_2) - V(E_1, X) - S(E_2, K) - g(I)$$

$$\dot{X} = -E_1 \qquad X(0) = X_0$$

$$\dot{K} = I \qquad K(0) = K_0$$

$$E_1 \geqslant 0$$

$$E_2 \geqslant 0$$

$$I \geqslant 0$$

$$X \geqslant 0 \ .$$

The necessary conditions are:

$\exists$ a continuous $\psi(t)$ such that:

$$(5.19) \qquad \dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$$

$$(5.20) \qquad \dot{\psi}_2 = \psi_2\delta + S_2(F_2, K)$$

$$(5.21) \qquad \psi_1 = F'(E_1 + E_2) - V_1(E_1, X) + \lambda_1 - \lambda_4$$

$$(5.22) \qquad \psi_2 = g'(I) - \lambda_3$$

$$(5.23) \qquad F'(E_1 + E_2) = S_1(E_2, K) + \lambda_2 = 0$$

$$(5.24) \qquad \psi_1(T)\, X(T) = \psi_2(T)\, K(T) = 0 \ , \qquad T < \infty$$

$$(5.25) \qquad \begin{cases} \lambda_1 \geqslant 0 & E_1 \geqslant 0 & \lambda_1 E_1 = 0 \\ \lambda_2 \geqslant 0 & E_2 \geqslant 0 & \lambda_2 E_2 = 0 \\ \lambda_3 \geqslant 0 & I \geqslant 0 & \lambda_3 I = 0 \\ \lambda_4 \geqslant 0 & X \geqslant 0 & \lambda_4 X = \lambda_4 \dot{X} = 0 \ . \end{cases}$$

There are eight possible policies open to the economy. They are summarized in Table 5.1. These policies are briefly analysed in Appendix 5.1. Table 5.2 summarizes the necessary conditions which must be satisfied in order for one specified policy to switch into another specified policy. These conditions are derived in Appendix 5.2.

Table 5.1

Policies	Control Variable		
	E_2	E_1	I
A	> 0	0	0
B	0	> 0	0
C	0	0	> 0
D	> 0	> 0	0
E	> 0	0	> 0
F	0	> 0	> 0
G	> 0	> 0	> 0
H	0	0	0

Since $X(T) \neq 0$ for all finite T (since exhaustion of X would take an infinite time in view of our assumption that $V_1(E_1, 0) = \infty$, $E_1 \geqslant 0$), $\psi_1(T) = 0$ for $T < \infty$. Also whenever $K(T) \neq 0$. $\psi_2(T) = 0$. If $K(T) = 0$, $\psi_2(T)$ may be $\gtreqless 0$. If $K(T) > 0$, then the condition, $\psi_2(T) = 0$, requires that at the endpoint (and indeed for a time before the endpoint, since $g'(0) > 0$) the economy must be operating under either policy B or policy D. In other words if the economy finds it optimal to accumulate capital it will attain its final target capital stock and cease investment *before the end* of the programme and will be producing the substitute *at the end* of the programme. Clearly for capital accumulation to be worthwhile there must be a period (the final stage of the programme) during which the accumulated capital "pays for itself". This is a consequence of assuming:

(a) a fixed finite time horizon;

(b) $g'(0) > 0$; and

(c) the capital stock has zero scrap value.

Assumption (c) is not unreasonable if the development expenditure (I) is devoted towards building up a stock of specialized technical know-how. It remains for us to decide between policies B and D as the endpoint policy when $K(T) > 0$. As is seen from Table 5.2, B can only be switched into from one of policies D, F and G. It cannot however, be switched into *as an endpoint policy* via D or G when T is finite since as the surface for such a switch is approached from either D or G, we have $\dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$ and $\psi_1 \to 0$, $V_2(E_1, X) \to 0$. This process must necessarily take an infinite period of time. It remains for us to examine a switch into policy B via F. Along policy F, E_2 is growing so that $F'(E_2)$ is falling. In order to have $\psi_1(T) = 0$, $F'(E_2)$ must be less than $V_1(0, X)$ for policy F (E_2 being the solution to $F'(E_2) = S_1(E_2, K)$). This means that when $F'(E_2) \leqslant V_1(0, X_0)$, $(F'(E_2) = S_1(E_2, K_0))$ (i.e. to the left of the unbroken curve $\hat{X}MPQ$ in Figure 5.6) policy B is operative in the final stage of the programme (including the endpoint). On the other hand, when $F'(E_2) > V_1(0, X_0)$ $(F'(E_2) = S_1(E_2, K_0)$ (i.e. to the right of $\hat{X}MPQ$), policy D is optimal for the final stage of the programme. This all, of course, assumes that $K(T) > 0$. When $K(T) = 0$ the optimal policy at T is either policy A if $F'(0) > V_1(0, X(T))$ or policy H if $F'(0) \leqslant V_1(0, X(T))$. The question of when optimal K(T) should be positive and when it should be zero will be discussed as we proceed. To assist the exposition Figure 5.6 illustrates possible optimal paths when $K(0) = 0$ and T is finite. In analysing the optimal paths we distinguish two cases, which depend on the initial value of the resource stock.

Table 5.2

Into / Out of	A	B	C	D	E	F	G	H
A	•	0	$F'(0)>V_1(0,X)$	$S_1(0,K)>V_1(E_1,K)$ $F'(E_1)=S_1(0,K)$	$\psi_2=g'(0)$	0	$S_1(0,K)>V_1(E_1,X)$ $F'(E_1)=S_1(0,K)$	$F'(0)>V_1(0,X)$
B	0	•	0	$F'(E_2)<V_1(0,X)$ $F'(E_2)=S_1(E_2,K)$	0	$g'(0)\delta>-S_2(E_2,K)$ $F'(E_2)=S_1(E_2,K)$	BF+BD conditions	0
C	0	0	•	0	$F'(0)<V_1(0,X)$	At $F'(0)=S_1(0,K)$	At $F'(0)=S_1(0,K)$ $F'(0)<V_1(0,K)$	0
D	$\psi_1=S_1(0,K)-V_1(E_1,X)$ $F'(E_1)=S_1(0,K)$	$F'(E_2)\geqslant V_1(0,X)$ $F'(E_2)=S_1(E_2,K)$	0	•	$\psi_1=S_1(0,K)-V_1(E_1,X)$ $F'(E_1)=S_1(0,K)$	$F'(E_2)>V_1(0,X)$ $F'(E_2)=S_1(E_2,K)$ $g'(0)\delta>-S_2(E_2,K)$	$g'(0)\delta>-S_2(E_2,K)$ $S_1(E_2,K)=F'(E_1+E_2)$	0
E	0	0	$F'(0)>V_1(0,X)$	0	•	0	$S_1(0,K)>V_1(E_1,X)$ $F'(E_1)=S_1(0,K)$	0
F	0	$g'(0)\delta<-S_2(E_2,K)$ $F'(E_2)=S_1(E_2,K)$	0	$F'(E_2)<V_1(0,X)$ $F'(E_2)=S_1(E_2,K)$ $g'(0)\delta<-S_2(E_2,K)$	0	•	$F'(E_2)<V_1(0,K)$ $F'(E_2)=S_1(E_2,K)$	0
G	0	DB+FB conditions	0	$g'(0)\delta<-S_2(E_2,K)$ $S_1(E_2,K)=F'(E_1+E_2)$	$\psi_1=S_1(0,K)-V_1(E_1,X)$ $F'(E_1)=S_1(0,K)$	$F'(E_2)>V_1(0,X)$ $F'(E_2)=S_1(E_2,K)$	•	0
H	$F'(0)<V_1(0,X)$	0	$\psi_2=g'(0)$	0	$F'(0)<V_1(0,X)$	0	0	•

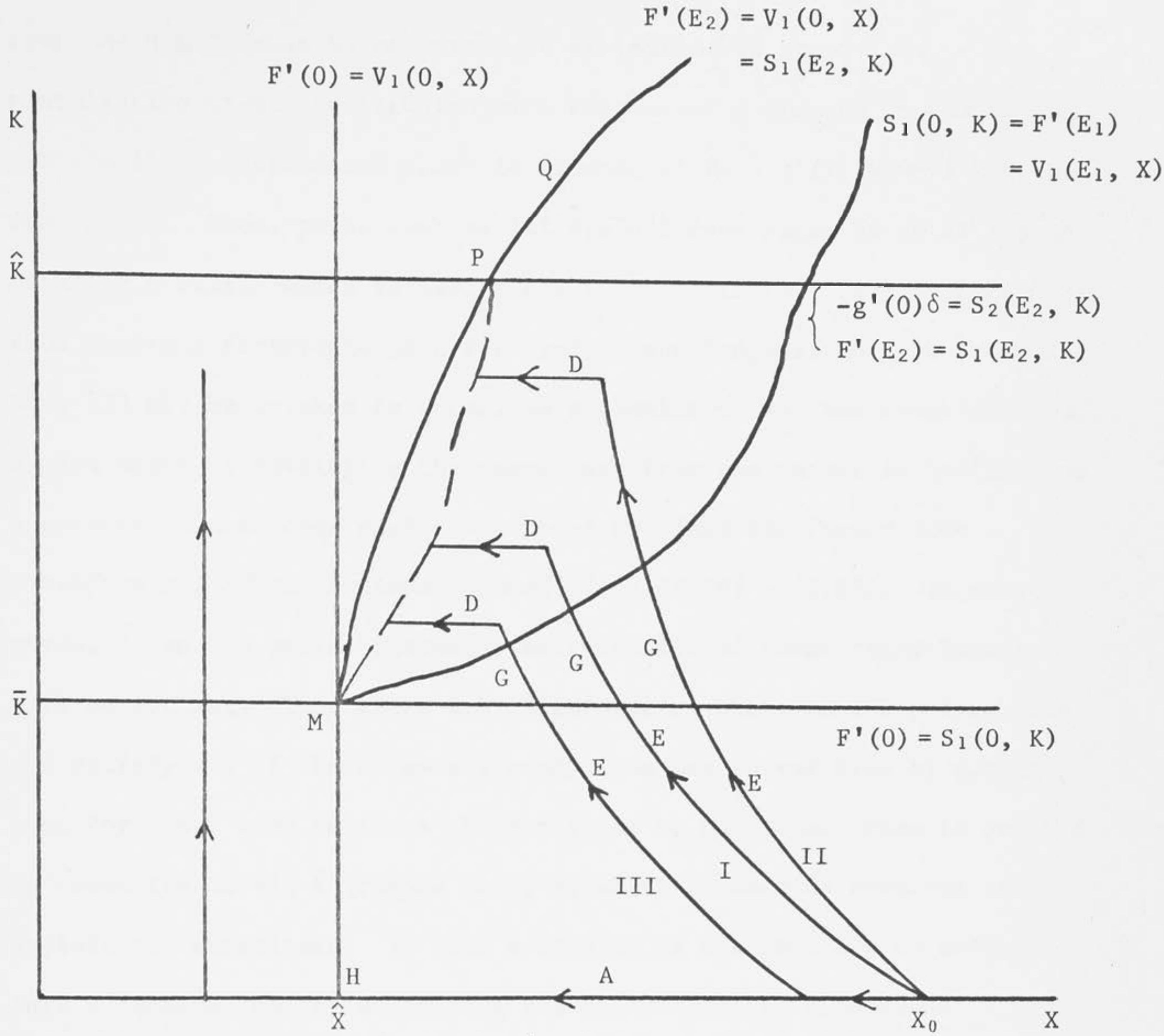

Figure 5.6

Case 1: $F'(0) > V_1(0, X)$:

In this case extraction of X is still economical and all policies used involve a possible E_1. There initially appears to be some difficulty in identifying the optimal path, because for any T, there are two paths satisfying (5.19) - (5.25). In terms of Figure 5.6, there will be a path like path III which takes the same time to reach its endpoint as a path like II. Both paths satisfy conditions (5.19) - (5.25). Nevertheless, on closer inspection we find that path III does not yield a maximum, for it does not satisfy the Pontryagin condition that the Hamiltonian be maximized at all points in time: the maximization of the Hamiltonian with respect to I amounts to maximizing $\psi_2 I - g(I)$, an expression which is greater if $\psi_2 = g'(I)$ when $I > 0$ than when $I = 0$. Thus, paths such as III are not Pontryagin paths if a path with $I > 0$ exists which satisfies conditions (5.19) - (5.25). There is thus a unique Pontryagin path for every T and Proposition 6 in Arrow and Kurz [3] may be invoked to establish optimality. We thus establish that a path which is developing the substitute from the outset is better than a path which postpones such development *provided the former path satisfies all of the Pontryagin conditions (5.19) - (5.25).* In other words, if an EGD policy sequence satisfies all of these conditions it will be optimal. That there are circumstances where an EGD policy does not satisfy all of the necessary conditions may be verified by noting that for short time horizons there may not be sufficient time to build K up above the level, $\bar{K}$ (Figure 5.6), at which it becomes economic to exploit the substitute. In such a case it is not possible to switch into a terminal policy satisfying the transversality conditions (5.24) for $0 < K \leqslant \bar{K}$.[8] Thus for such short programmes the only possibility is to leave K zero for the whole programme and concentrate on extracting X for as long as possible. This would mean following policy A with $\psi_1 > 0$ and falling to zero at time T and $\psi_2 = 0$. The preceding analysis may be summarized in the following conclusion: *Unless the planning period is long enough to permit optimal accumulation of an economically worthwhile stock of capital ($K > \bar{K}$) it is better not to accumulate any capital and just exploit the existing resource deposits for the whole programme.*

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8 The only possible such policies are C and E both of which involve $\psi_2 > 0$ and growing — it is not possible to switch back into H or A via C or E once K has become positive.

This suggests that short term planning will neglect the development of substitutes where long term planning would provide for it. This result is naturally dependent on the assumption that the capital in the system has zero scrap value, an assumption discussed briefly earlier in this section. Clearly for longer time horizons there is time for K to exceed $\bar{K}$, but also for the accumulated capital to be used in the production of E_2 and thus to "pay for itself". The longer the time horizon the higher will be the optimal level of the terminal capital stock (path II in Figure 5.6 is for a longer time horizon than is path I). When it is feasible to have K(T) satisfying $F'(0) > S_1(0, K(T))$ the optimal path will be the unique one satisfying all the necessary conditions and employing a policy sequence EGD. Logically enough in the early stages (until $F'(0) > S_1(0, K)$) capital for producing the substitute will be accumulated at the same time as X is extracted, then when the substitute is an "economic" proposition the resource good will be supplied from both sources. Ultimately there will be enough capital and because the "end of the world" (T) is in sight there is nothing to be gained from further accumulation so development expenditure will cease before the endpoint and policy D will become optimal. As was the case in the model of section II the assumption that the depletion effect for X can be made infinitesimally small by making E_1 arbitrarily small makes it optimal to spread extraction of X indefinitely. As $T \to \infty$ the terminal point for the programme will tend to point P in Figure 5.6 and the terminal policy will be policy B. Point P represents an equilibrium for the autonomous system of differential equations in $(\psi_1(t), \psi_2(t), X(t), K(t))$ and at P,

$$\begin{cases} g'(0)\delta = -S_2(E_2, K), \quad F'(E_2) = S_1(E_2, K), \ \psi_2 = g'(0), \ (\dot{\psi}_2 = 0, \ \dot{K} = 0) \\ \qquad\qquad F'(E_2) = V_1(0, X) = S_1(E_2, K) \ , \ (\dot{\psi}_1 = 0, \ \dot{X} = 0). \end{cases}$$

Thus as $t \to \infty$

$$X(t) \to \hat{X} \qquad \psi_1(t) \to 0$$

$$K(t) \to \hat{K} \qquad \psi_2(t) \to g'(0) \ .$$

It is easy to check for this limiting endpoint that

$$\frac{d\hat{K}}{d\delta} = \frac{g'(0)(F'' - S_{11})}{S_{11}S_{22} - S_{12}S_{21} - F''S_{22}}$$

which is negative if S is convex at the point $(\hat{X}, \hat{K})$, and

$$\frac{d\hat{X}}{d\delta} = \frac{F''S_{12}}{V_{12}(F'' - S_{11})} \frac{dK}{d\delta}$$

is also negative. Thus, as we might expect, if the future is discounted more heavily the optimal terminal capital stock as $T \to \infty$ will fall and more of the exhaustible resource should be used. Finally, we should note the important conclusion that if the substitute is worth developing, as it will be if the economy is committed to sufficiently long-term planning, then *it should be developed from the outset regardless of the initial stock of the exhaustible resource.*

Case 2: $F'(0) \leqslant V_1(0, X_0)$:

As for case 1, if T is not large enough to permit K to rise above $\bar{K}$, no investment at all should be undertaken. However here policy A would involve $\psi_1(t) < 0 \ V_t$ and so is non-optimal. So for small T the economy will find it optimal to do nothing at all — i.e. follow policy H for the whole programme with $\psi_1 = 0$. When T is large enough to permit K to rise above $\bar{K}$ along a Pontryagin path, the economy will build up its stock of K_1 initially via policy C ($\psi_1 = 0$), switching into policy F when $K = \bar{K}$ and producing the substitute until the endpoint. Policy B is switched into at some time before the endpoint, thus allowing ψ_2 to fall to zero. Production of E_1 is inefficient so that no X is exploited.

IV

UNCERTAINTY AND RESOURCE SUBSTITUTION

In this section we analyse the question of substitution of one resource input for another when there is uncertainty concerning the date at which the second, more plentiful resource is to become available. For the sake of mathematical viability the capital accumulation incorporated into the model of Section III is ignored here and a probability distribution is added to the certainty formulation of Section II. It is assumed that planning takes place over an infinite time horizon and that conditional upon a substitute ever becoming available, one "becomes available" at time t with probability, w(t). The essence of this formulation derives from a paper by Heal and Dasgupta [16]. In addition we define the variable

$$\Omega(t) \equiv \int_t^{\infty} w(\tau)d\tau \ ,$$

and denote $\eta \leqslant 1$ as the probability that the substitute will ever become available. η is assumed parametrically fixed.

Then if we denote

$$C_1 \equiv F(E_1) - V(E_1, X)$$

$$C_2 \equiv F(E_1 + E_2) - V(E_1, X) - S(E_2) \ ,$$

the optimization problem to be solved is to select time paths for E_1 and E_2 and a terminal value for X, so as to maximize the expected value of the discounted stream of consumption from both resources over all time periods, viz:

$$(5.26) \qquad \underset{E_1(t),E_2(t),X(\infty)}{\text{Max}} \int_0^{\infty} \{(1-\eta)C_1 + \eta[\Omega(t)C_1 + (1-\Omega(t))C_2]\}e^{-\delta t}dt$$

$$\text{s.t.} \qquad \dot{X} = -E_1 \qquad E_1 \geqslant 0$$

$$E_2 \geqslant 0 \qquad X \geqslant 0 \ .$$

The Pontryagin conditions which must be satisfied by a solution to this problem are that ∃ a continuous $\psi(t)$, satisfying

$$(5.27) \qquad \dot{\psi} = \psi\delta + V_2(E_1, X)$$

$$(5.28) \qquad \psi = (1 - \eta + \eta\Omega(t))F'(E_1) + \eta(1 - \Omega(t))F'(E_1 + E_2) - V_1(E_1, X) + \lambda_1 - \lambda_1$$

$$(5.29) \qquad \eta(1 - \Omega(t))[F'(E_1 + E_2) - S'(E_2)] + \lambda_2 = 0$$

$$(5.30) \qquad \begin{cases} \lambda_1 \geqslant 0 & E_1 \geqslant 0 & \lambda_1 E_1 = 0 \\ \lambda_2 \geqslant 0 & E_2 \geqslant 0 & \lambda_2 E_2 = 0 \\ \lambda_3 \geqslant 0 & X \geqslant 0 & \lambda_3 X = \lambda_3 \dot{X} = 0 \ . \end{cases}$$

As in Section II, there remain four policies open to the economy. We list them again here:

Table 5.3

Policy	Control Variable	
	E_1	E_2
A	> 0	0
B	0	> 0
C	> 0	> 0
D	0	0

The introduction of a time-dependent probability function into the model gives us a non-autonomous system of differential equations in ψ and X and is therefore difficult to describe in ψ-X space. On the other hand it is possible to say that:

(i) the switching surface between A and C lies below and to the right of the switching surface between C and B; however, the AC surface is not time dependent like the CB surface;

(ii) the equilibrium solution for the system constitutes the optimal endpoint and is here given by $\hat{\psi} = 0$, $E_1 = 0$, and $\hat{X}$ defined as the solution to

$$(1 - \eta)F'(0) + \eta F'(E_2) = V_1(0, \hat{X})$$

where

$$F'(E_2) = S'(E_2) ;$$

it is easily verified that $\frac{d\hat{X}}{d\eta} > 0$, reflecting the intuitively appealing conclusion that the more certain the *ultimate* availability of the substitute (the higher is η) the higher will be the optimal terminal stock of X. Clearly as it becomes more likely that a substitute will never be found (η becomes small) the economy will feel the need to push production of X into less efficient stages and extract deposits which are either of a lower grade or are more inaccessible and costly to extract so as to compensate for the unlikely availability of the substitute. In the limiting case as $\eta \to 0$, we are back to the last single-resource model presented in Chapter 3, Section III. When $\eta = 1$ we have the other extreme case in which the substitute is certain to become available ultimately. The terminal stock of the resource is the same as for the case examined in Section II, where the second resource was available with certainty from the beginning of the programme. This result is hardly surprising for once the resource is available with certainty before $t = \infty$ the optimal terminal stock of X is determined by technological considerations only, independent of the probability distribution $w(t)$.

APPENDIX 5.1

ANALYSIS OF POLICIES IN MODEL OF SECTION III

Policy A:

$$\dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$$

$$\dot{\psi}_2 = \psi_2\delta$$

$$\psi_1 = F'(E_1) - V_1(E_1, X)$$

$$\psi_2 = g'(0) - \lambda_3$$

$$F'(E_1) - S_1(0, K) + \lambda_2 = 0$$

Policy B:

$$\dot{\psi}_1 = \psi_1\delta$$

$$\dot{\psi}_2 = \psi_2\delta + S_2(E_2, K)$$

$$\psi_1 = F'(E_2) - V_1(0, X) + \lambda_1$$

$$\psi_2 = g'(0) - \lambda_3$$

$$F'(E_2) = S_1(E_2, K)$$

This last equation implies that $F'(0) > S_1(0, K)$.

Policy C:

$$\dot{\psi}_1 = \psi_1\delta$$

$$\dot{\psi}_2 = \psi_2\delta$$

$$\psi_1 = F'(0) - V_1(0, X) + \lambda_1$$

$$\psi_2 = g'(I)$$

$$F'(0) - S_1(0, K) + \lambda_2 = 0$$

$$\Rightarrow F'(0) \leqslant S_1(0, K)$$

Policy D:

$$\dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$$

$$\dot{\psi}_2 = \psi_2\delta + S_2(E_2, K)$$

$$\psi_1 = F'(E_1 + E_2) - V_1(E_1, X)$$

$$\psi_2 = g'(0) - \lambda_3$$

$$F'(E_1 + E_2) = S_1(E_2, K)$$

$$\Rightarrow F'(0) > S_1(0, K).$$

Also the solution E_2^* of $F'(E_1 + E_2^*) = S_1(E_2^*, K)$ is less than the solution E_2 of $F'(E_2) = S_1(E_2, K)$.

Policy E:

$$\dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$$
$$\dot{\psi}_2 = \psi_2\delta$$
$$\psi_1 = F'(E_1) - V_1(E_1, X)$$
$$\psi_2 = g'(I)$$
$$F'(E_1) - S_1(0, K) + \lambda_2 = 0$$

Policy F:

$$\dot{\psi}_1 = \psi_1\delta$$
$$\dot{\psi}_2 = \psi_2\delta + S_2(E_2, K)$$
$$\psi_1 = F'(E_2) - V_1(0, X) + \lambda_1$$
$$\psi_2 = g'(I)$$
$$F'(E_2) = S_1(E_2, K)$$
$$\Rightarrow F'(0) > S_1(0, K)$$

Policy G:

$$\dot{\psi}_1 = \psi_1\delta + V_2(E_1, X)$$
$$\dot{\psi}_2 = \psi_2\delta + S_2(E_2, K)$$
$$\psi_1 = F'(E_1 + E_2) - V_1(E_1, X)$$
$$\psi_2 = g'(I)$$
$$F'(E_1 + E_2) = S_1(E_2, K)$$
$$\Rightarrow F'(0) > S_1(0, K)$$

Policy H:

$$\dot{\psi}_1 = \psi_1\delta$$
$$\dot{\psi}_2 = \psi_2\delta$$
$$\psi_1 = F'(0) - V_1(0, X) + \lambda_1$$
$$\psi_2 = g'(0) - \lambda_3$$
$$F'(0) = S_1(0, K) - \lambda_2$$
$$\Rightarrow F'(0) \leqslant S_1(0, K) .$$

APPENDIX 5.2

DERIVATION OF SWITCHING CONDITIONS: MODEL OF SECTION III

AB: For policy A $\psi_1 < S_1(0, K) - V_1(0, X)$, and for policy B

$$\psi_1 > S_1(0, K) - V_1(0, X) .$$

Thus a switch between A and B involves a jump in ψ_1 and so is non-optimal.

AC: For A: $\psi_1 < F'(0) - V_1(0, X)$; $\psi_2 \leqslant g'(0)$; and for C:

$$\psi_1 \geqslant F'(0) - V_1(0, X); \quad \psi_2 > g'(0).$$

For a switch to take place it is necessary that:

(i) $\dot{\psi}_1 > 0$ at surface

i.e. $\psi_1 > 0$ at surface

i.e. $F'(0) > V_1(0, X)$ at surface

(ii) $\psi_2 > 0$ at surface which is always true.

Because $\psi_2 > 0$ a CA switch is impossible.

AD: For A: $\psi_1 \leqslant S_1(0, K) - V_1(E_1, X)$;

for D: $\psi_1 > S_1(0, K) - V_1(E_1, X)$

where $F'(E_1) = S_1(0, K)$.

A switch requires that at the switching surface:

$$\dot{\psi}_1 > -V_{12}E_1$$

i.e. $[S_1(0, K) - V_1(E_1, X)]\delta > - V_2(E_1, X) - V_{12}(E_1, X)E_1$

$$\Rightarrow S_1(0, K) > V_1(E_1, X) .$$

It is not possible to be so specific about the conditions for a DA switch.

AE: This merely requires that at the switching surface $\psi_2 = g'(0) > 0$ which is always true and so AE is possible anywhere and EA is impossible.

AF: A switch either way involves a jump in ψ_1, for the same reasons as AB, and so no switch is optimal.

AG: The switching conditions are easily seen to be a combination of those for AD and AE. GA is impossible.

AH: Similar to AC. A switch requires $F'(0) > V_1(0, X)$ at the switching surface.

BC: For policy B, $F'(0) > S_1(0, K)$ and K is constant. The economy is either in this region or outside it and it cannot move into it from outside or vice versa. Thus no switch is optimal.

BD: For B: $\psi_1 \geqslant F'(E_2) - V_1(0, X)$;

for D: $\psi_1 < F'(E_2) - V_1(0, X)$

where $F'(E_2) = S_1(E_2, K)$ at the switching surface. Now a BD switch requires

$$\dot{\psi}_1 < 0$$

$$\text{i.e.} \quad \psi_1 < 0$$

$$\text{i.e.} \quad F'(E_2) < V_1(0, X) \quad \text{where} \quad F'(E_2) = S_1(E_2, K) \;.$$

BE: A switch between B and E necessarily involves a jump in ψ_1. Hence this switch is non-optimal.

BF: A switch requires that $\psi_2 > 0$ at the surface

$$\text{i.e.} \quad g'(0)\delta > -S_2(E_2, K)$$

$$\text{where} \quad F'(E_2) = S_1(E_2, K) \;.$$

<u>BG</u>: The conditions here are a combination of the BF and BD switching conditions.

<u>BH</u>: This is ruled out for the same reason as a switch between B and C was judged non-optimal.

<u>CD</u>: Ruled out for same reason as BC and BH.

<u>CE</u>: For C: $\psi_1 \geqslant F'(0) - V_1(0, X)$;

for E: $\psi_1 < F'(0) - V_1(0, X)$.

A switch requires $\dot{\psi}_1 < 0 \Leftrightarrow \psi_1 < 0$

$$\Leftrightarrow F'(0) < V_1(0, X).$$

<u>CF</u>: A switch must occur at $F'(0) = S_1(0, K)$. FC is ruled out since rising K causes the economy to leave policy C, not enter it.

<u>CG</u>: A combination of CE and CF conditions. Clearly GC is non-optimal.

<u>CH</u>: HC merely requires $\psi_1 > 0$ at the surface.

i.e. $g'(0) > 0$, which is immediately satisfied.

CH is clearly non-optimal.

The remaining conditions are derived similarly to the above.

CHAPTER 6

INTERNATIONAL TRADE

I. INTRODUCTION

In this chapter we relax the assumption that the economy we are considering is closed, and allow it to both import and export the resource good. Once we move into this broader framework the way is opened for a series of questions beyond the basic one of how a country should distribute exploitation of an exhaustible asset over time. In particular we may also ask:

(a) How much of this asset should be exported (or imported) and how these exports should change over time;

(b) What should be the pattern of specialization in production of the country in question;

(c) how the optimal time path of production and consumption relates to the levels of production and consumption in the standard static competitive trade model; and

(d) what tax policy will bring the competitive model into line with the centrally controlled system.

To this end a planning model of a two sector economy (with a resources sector and a manufacturing sector) is formulated. This model is based on the formulation developed at the end of Section I, Chapter 4. Sections II - IV of the chapter are based on the assumption that the economy is in balance of payments equilibrium at every point in time while in the remaining sections this assumption is replaced by an intertemporal balance of payments constraint. For the first of these two models the findings may be summarized as follows: It turns out that for sufficiently short planning horizons and sufficiently large initial resource endowments the optimal solution for the centralized model coincides with the solution for the static competitive model, while for longer time horizons and lower initial resource endowments the optimal level of extractive production will be less than the static optimum and will decline over time until the resource is exhausted. More particularly it is shown that the optimal policy sequence for the economy depends crucially on its terms of trade and the properties of its social welfare function. Depending on these two factors the optimal policy sequence will be one of the following:

(a) Produce both goods and export some of the resource
  → import the resource
  → import the resource at the maximal rate
  → complete specialization in manufacturing production and import the resource at a maximal rate;

(b) Produce both goods and export some of the resource
  → complete specialization in manufacturing production and no trade;

(c) Produce both goods and export some of the resource
  → import the resource
  → complete specialization in manufacturing production and consume both goods.

Sequence (a) is optimal if consumption of the resource good is intrinsically preferred or if it is relatively cheap.

Sequence (b) arises when the relative preference is for consumption of the manufacturing good or if the resource good is relatively expensive.

Sequence (c) coincides with the case where there is no clear relative preference for either good.

For the subsequent model, where there is allowance made for foreign borrowing, the results appear qualitatively similar to those obtained when no borrowing is allowed.

Throughout the paper it will be assumed as a first approximation that the terms of trade facing the country in question are given and constant. It is only on this basis that we can compare results with those of the traditional trade model. In addition, to keep the analysis manageable, we are forced at this stage to ignore resource stocks in the rest of the world. We are effectively assuming that they are relatively abundant and are therefore imposing no pressure on world prices.

II. THE BASIC MODEL

We again consider an economy which possesses a finite stock (X) of a depletable resource. It extracts this resource at a rate E, so that:

$$\dot{X} = -E \quad (6.1)$$

The economy uses a single mobile factor (labour) to produce two goods, a resource good, E (extraction) and a manufacturing good, Z. The quantities of these outputs producible with a given labour supply are related by a transformation surface of the conventional form (derived for a more general case in Chapter 4, equation (4.36)):

$$Z = \phi(E) \qquad \phi' < 0, \quad \phi'' < 0 .$$

$\bar{E}$ is defined as the maximum extraction possible at any point in time (i.e. $\phi(\bar{E}) = 0$). We let C_1 denote the consumption of the resource good and C_2 the consumption of the manufacturing good. Either good may be exported or alternatively the domestic consumption of each may be augmented by means of imports. Thus,

$$C_1 = E + M - S \tag{6.2}$$

$$C_2 = \phi(E) + M' - S' \tag{6.3}$$

where $M \equiv$ imports of good 1 $\quad M' \equiv$ imports of good 2

$S \equiv$ exports of good 1 $\quad S' \equiv$ exports of good 2 .

In order to avoid a situation where a commodity is simultaneously imported and exported it is assumed that $MS = 0$ and $M'S' = 0$ (i.e. M and S cannot be simultaneously positive, etc.).

The country's international payments are assumed to be in balance:

$$p(M - S) + (M' - S') = 0 \tag{6.4}$$

where $p \equiv$ the world price of the resource good in terms of the manufacturing good. p is assumed constant over time and the country in question is assumed to be a price taker (although the world market in the resource good may not be competitive).

Social welfare (u) is given by a strictly concave, separable utility function

$$u = u(C_1, C_2) \qquad \left.\begin{matrix} u_1 > 0 & u_{11} < 0 \\ u_2 > 0 & u_{22} < 0 \\ u_{12} = u_{21} = 0 & \end{matrix}\right\} . \tag{6.5}$$

Using (6.2), (6.3) and (6.4) we may write (6.5) as

$$u = u(E + M - S, \phi(E) - p(M - S)) .$$

Before stating the main problem to be solved it is convenient to define another piece of notation. We denote $\hat{E}$ as the solution to the problem:

$$\max_{E} \; (pE + \phi(E))$$

$$\text{s.t.} \qquad E \geqslant 0$$

$$\bar{E} - E \geqslant 0 \; .$$

This is the problem which will be solved collectively by producers in a competitive system.

The basic problem which is to be considered in this chapter is that of selecting a terminal stock of the resource, X(T), and time paths for production (E), imports (M) and exports (S) of the resource good so as to maximize the present value of the stream of utility from consumption of the two goods from time 0 to time T (fixed).

Stated algebraically we wish to:

$$\underset{X(T)\; E(t),\, M(t),\, S(t)}{\text{Max}} \int_0^T u(E + M - S,\; \phi(E) - p(M - S))e^{-\delta t}dt$$

$$\delta > 0, \text{ constant },$$

subject to

$$\dot{X} = -E \qquad X(0) = X_0$$

$$E - S \geqslant 0$$

$$\phi(E) - pM \geqslant 0$$

$$E \geqslant 0$$

$$M \geqslant 0$$

$$S \geqslant 0$$

$$\bar{E} - E \geqslant 0$$

$$\dot{X} = 0 \quad \text{when } X = 0 \; .$$

A solution to the above problem must satisfy the following necessary conditions:

$\exists$ a continuous function of time, ψ, such that

$$\mathcal{L} = u(E + M - S, \phi(E) - p(M - S)) - \psi E + \lambda_1 E + \lambda_2 M$$
$$+ \lambda_3 S + \lambda_4(E - S) + \lambda_5(\phi(E) - pM) + \lambda_6(\bar{E} - E) - \lambda_7 E$$

(6.6) $\dot{\psi} = \psi\delta - \frac{\partial \mathcal{L}}{\partial X} \Leftrightarrow \dot{\psi} = \psi\delta$

(6.7) $\frac{\partial \mathcal{L}}{\partial E} = 0 \Leftrightarrow \psi = u_1 + u_2\phi' + \lambda_1 + \lambda_4 + \lambda_5\phi' - \lambda_6 - \lambda_7$

(6.8) $\frac{\partial \mathcal{L}}{\partial M} = 0 \Leftrightarrow u_1 - pu_2 + \lambda_2 - p\lambda_5 = 0$

(6.9) $\frac{\partial \mathcal{L}}{\partial S} = 0 \Leftrightarrow u_1 - pu_2 - \lambda_3 + \lambda_4 = 0$

(6.10) $\psi(T)\, X(T) = 0$

(6.11)
$$\begin{cases} \lambda_1 \geqslant 0 & E \geqslant 0 & \lambda_1 E = 0 \\ \lambda_2 \geqslant 0 & M \geqslant 0 & \lambda_2 M = 0 \\ \lambda_3 \geqslant 0 & S \geqslant 0 & \lambda_3 S = 0 \\ \lambda_4 \geqslant 0 & E - S \geqslant 0 & \lambda_4(E - S) = 0 \\ \lambda_5 \geqslant 0 & \phi(E) - pM \geqslant 0 & \lambda_5(\phi(E) - pM) = 0 \\ \lambda_6 \geqslant 0 & \bar{E} - E \geqslant 0 & \lambda_6(\bar{E} - E) = 0 \\ \lambda_7 \geqslant 0 & X \geqslant 0 & \lambda_7 X = \lambda_7 \dot{X} = 0 . \end{cases}$$

A number of policies are available to the economy. They are enumerated in Table 6.1. Two possible situations are considered, the first being that for which the static trade optimum would involve production of both goods and the second where static optimization would require complete specialization in production of the manufacturing good.

Table 6.1

List of possible policies

Policy	Control Variable		
	E	M	S
A	> 0	> 0	0
B	> 0	$\phi(E)/p$	0
C	> 0	0	> 0
D	> 0	0	E
E	$\bar{E}$	0	> 0
F	$\bar{E}$	0	$\bar{E}$
G	0	> 0	0
H	0	$\phi(0)/p$	0
I	> 0	0	0
J	$\bar{E}$	0	0
K	0	0	0

III. THE OPTIMAL PATH FOR THE ECONOMY

1: $-\phi'(0) < p < -\phi'(\bar{E})$:

From equation (6.6) it is easily seen that ψ, the marginal social cost associated with present depletion of the resource, will either be always positive, always zero or always negative. Policies A, B, C, D, G, H, I and K are consistent with $\psi > 0$. In addition it can be shown that for policies A, B, C, D and I, sgn ψ = sgn$(p + \phi'(E))$. In other words, when there is a positive social cost attached to depletion, the relevant domain of operation for the economy is to the left of $\hat{E}$ (Figure 6.1). A zero social cost of depletion means that the resource is regarded as being economically the same as a non-depletable asset and in such a case $E = \hat{E}$ (for $X > 0$). $\psi < 0$ for policies A, B, C, D, E, F and J $(X > 0)$ as well as G, H and K when $X = 0$. When $X > 0$ we see that

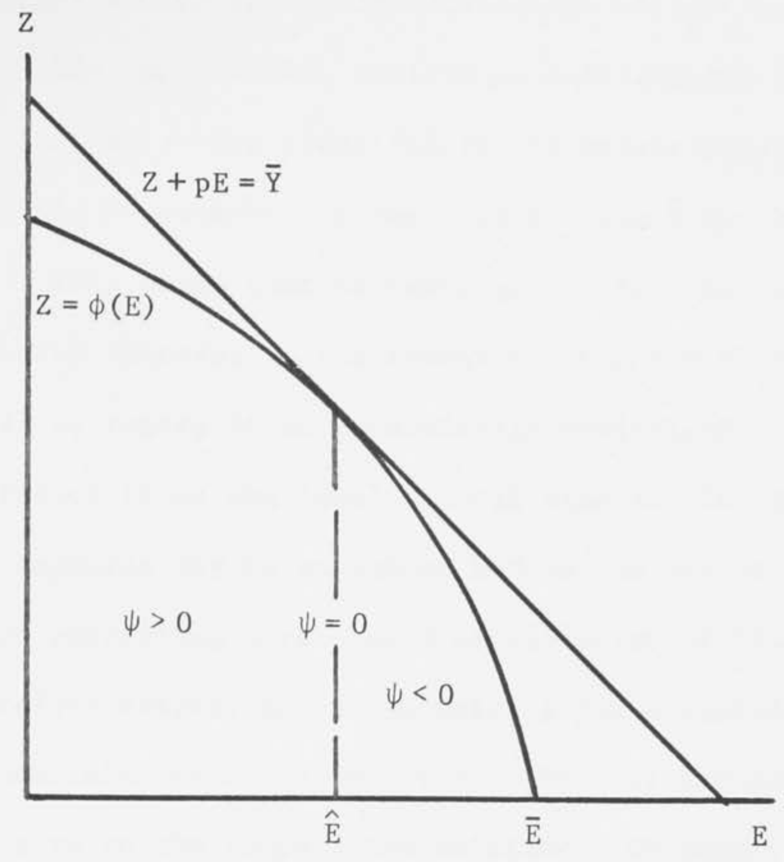

Figure 6.1

a negative social cost of depletion involves more rapid extraction than in the static competitive case (there is a loss associated with deferment of some extraction towards the future ($\psi < 0$) and so more rapid extraction in the present is appropriate). Our task is to determine which of the above three situations ($\psi \gtreqless 0$) applies. The easiest one to dismiss is $\psi < 0$. There is nothing inherent in the model (as there may be in a microeconomic model of exploitation of a common property resource with threat of entry) which would make it attractive to produce at higher rates than the competitive optimum. On the other hand the valuation placed on the future combined with the scarcity of the resource may make it optimal to produce at a level lower than the competitive optimum ($\psi > 0$). There are two situations in which the intertemporal social production optimum coincides with $\hat{E}$ for the whole programme. If either the planning horizon is sufficiently short or if the resource is initially very plentiful (X_0 is sufficiently large) then it would be possible to extract the resource at rate $\hat{E}$ for the whole planning period. This means that in terms of the time period involved there is no scarcity inherent in the resource (thus $\psi = 0$) and it is therefore natural to regard it as economically equivalent to any other output and to produce it at the level $\hat{E}$ until time T. For larger T or smaller X_0, the resource may be exhausted before the end of the programme without extracting more than $\hat{E}$ at any point of time. Its scarcity is therefore evident and it is natural for a social user cost ($\psi > 0$) to be associated with its depletion. Mineral exploitation will be retarded relative to the competitive solution. Obviously the borderline between the cases $\psi = 0$ and $\psi > 0$ is marked by the relationship $T\hat{E} = X_0$, which describes the situation in which exploitation at rate $\hat{E}$ for the whole programme results in the resource just being exhausted at time T. The results hinted at in the preceding discussion are stated formally as follows (they are proven in Appendix 6.3):

$$\text{(i)} \quad E(t) \leqslant \hat{E} \qquad \forall_t \in [0, T]$$

$$\text{(ii)} \quad E(t) = \hat{E} \qquad \forall_t \in [0, T] \quad \text{for} \quad T \leqslant X_0/\hat{E}$$

$$\text{(iii)} \quad E(t) < \hat{E} \qquad \forall_t \in [0, T] \quad \text{for} \quad T > X_0/\hat{E} \; .$$

Now, because $\psi \geqslant 0$ along the optimal path we have no further interest in policies E, F and J. The necessary conditions for the various switches to take place between the remaining policies are summarized in Table 6.2. Some of the more difficult derivations of these conditions are contained in Appendix 6.1. The synthesis of the optimal solution depends crucially on

(a) the relative price, p, and

(b) the properties of the utility function.

To clarify the relevance of these two factors we distinguish three cases:

<u>Case 1</u>: $u_1\left(\frac{\phi(0)}{p}\right) > pu_2(0)$

This states that the value of the marginal utility (VMU) associated with maximum imports of the resource good (no home production) exceeds the VMU of zero consumption of the manufacturing good. The situation is one either of intrinsic consumer preference in favour of the resource good or a low relative price of the resource good.

<u>Case 2</u>: $pu_2(\phi(0)) > u_1(0)$

Here the VMU of consuming maximal production of the manufacturing good ($pu_2(\phi(0))$) exceeds the VMU of zero consumption of the resource good and there is either a relative preference for the manufacturing good or the manufacturing good sells relatively cheaply.

<u>Case 3</u>: $u_1(0) > pu_2(\phi(0))$; $pu_2(0) > u_1\left(\frac{\phi(0)}{p}\right)$

The two inequalities state that the VMU associated with zero consumption of each commodity exceeds the VMU associated with the maximal consumption possible of the other good when $E = 0$. In this case there is no clear preference for either commodity which either is intrinsic or arises from the structure of relative prices.

Table 6.2

Necessary conditions for policy switches

Switches from \ Switches to	A	B	D	G	H	I	K
A	•	$\psi = pu_1(0)\chi_1$	0	$\psi = u_1(M_0)\bar{\chi}$	$\psi = pu_2(0)\bar{\chi}$	0	0
B	0	•	0	0	$\psi = u_1\left(\frac{\phi(0)}{p}\right)\bar{\chi}$	0	0
C	0	0	$\psi = u_1(0)\chi_2$	0	0	$\psi = u_1(E_0)\chi_0$	0
D	0	0	•	0	0	0	$\psi = pu_2(\phi(0))\bar{\chi}$
I	$\psi = u_1(E_0)\bar{\chi}$	0	0	0	0	•	0

(i) E_0 is solution of $u_1(E_0) = pu_2(\phi(E_0))$

(ii) E_1 is solution of $u_1\left[E_1 + \frac{\phi(E_1)}{p}\right] = pu_2(0)$

(iii) E_2 is solution of $u_1(0) = pu_2(\phi(E_2) + pE_2)$

(iv) M_0 is solution of $u_1(M_0) = pu_2(\phi(0) - pM_0)$

(v) $\chi_0 = 1 + \frac{\phi'(E_0)}{p}$

(vi) $\chi_1 = 1 + \frac{\phi'(E_1)}{p}$

(vii) $\chi_2 = 1 + \frac{\phi'(E_2)}{p}$

(viii) $\bar{\chi} = 1 + \frac{\phi'(0)}{p}$

For each of these three cases necessary conditions for certain policy switches are violated and it is possible to work out which policies will be operative in the respective cases. This analysis is presented in Appendix 6.2. The phase diagrams showing the Pontryagin paths for Cases 1, 2 and 3 are shown in ψ-X space ($\psi \geqslant 0$) in Figures 6.2, 6.3 and 6.4 respectively.

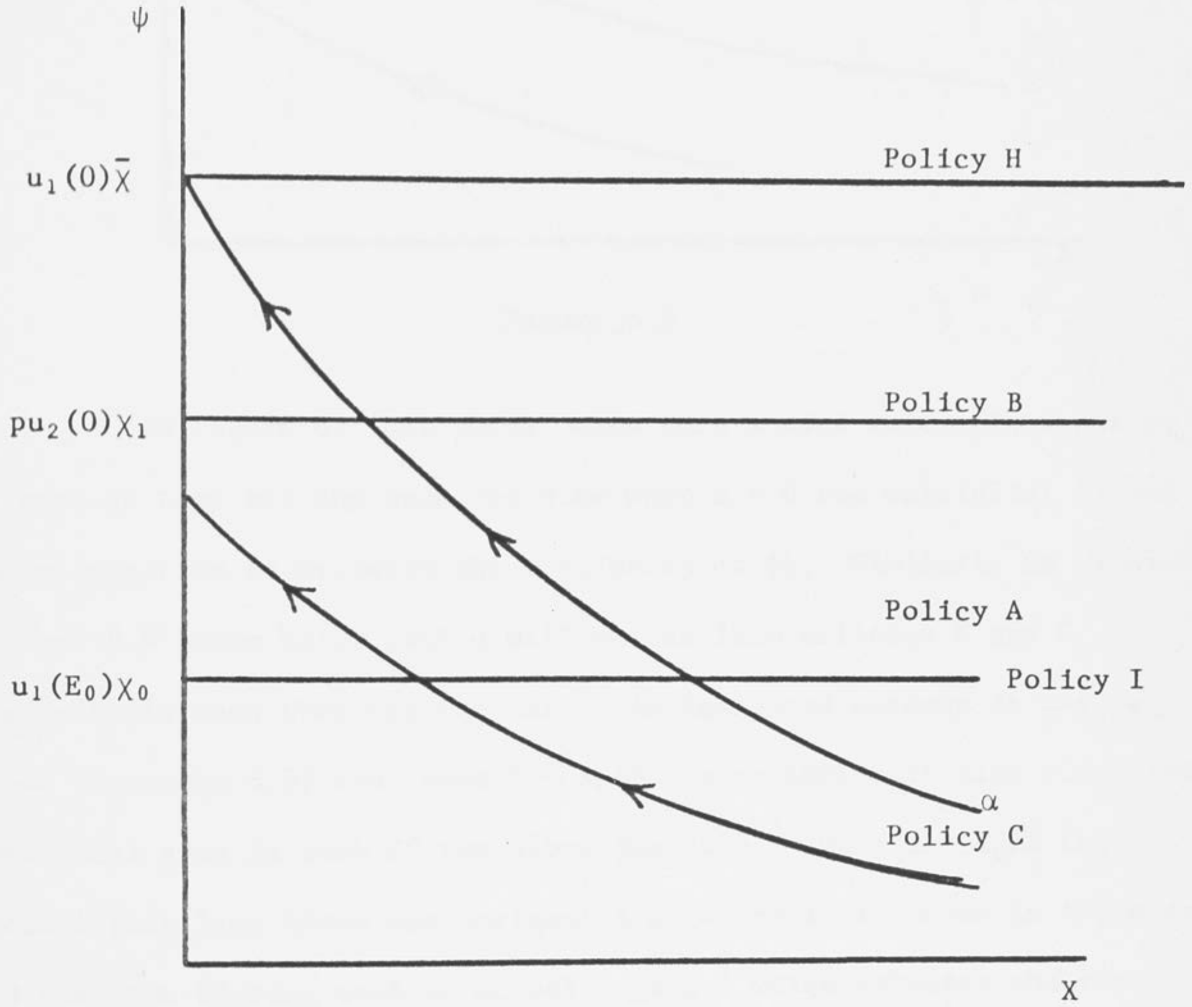

Figure 6.2 [1]

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1 Figures 6.2 and 6.4 are drawn on the particular assumption that χ_0 and χ_1 are both positive. There does not seem to be anything inherent in the definition of either χ_0 or χ_1 to prevent one or both of them being negative, so that it is possible that in Case 1, it will be optimal to commence with policy B and in Case 3, with policy A. $\bar{\chi}$ is always positive.

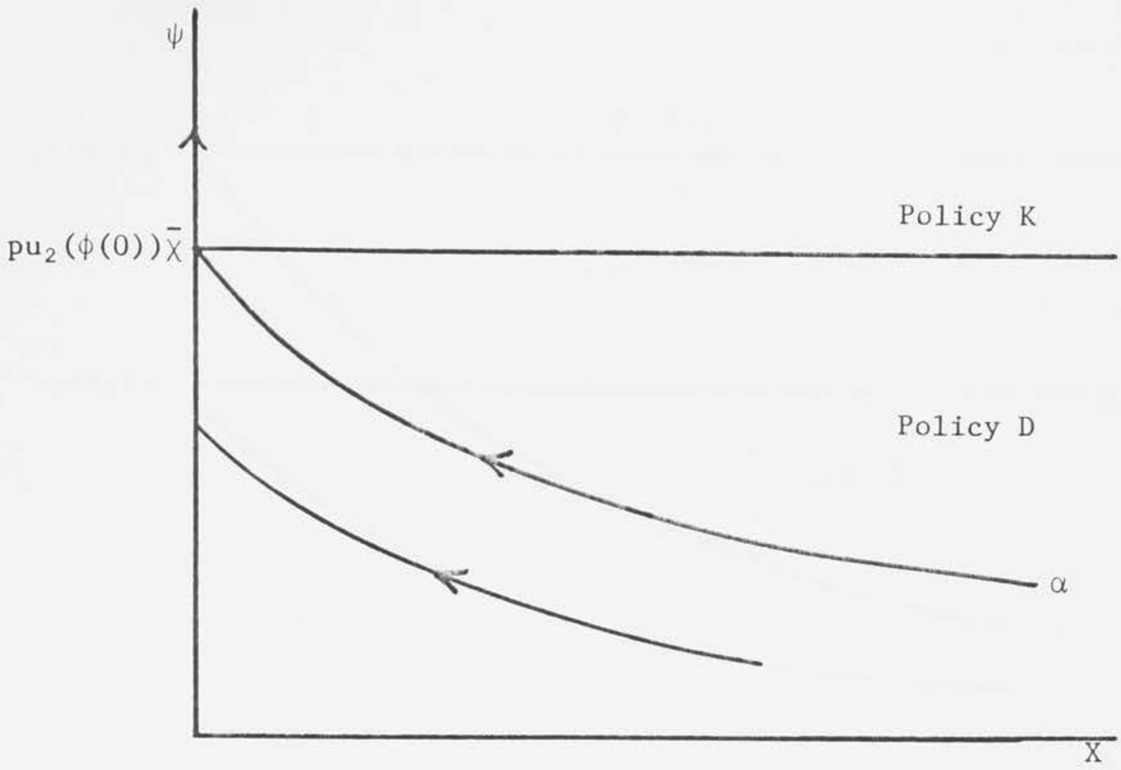

Figure 6.3

In Figure 6.2 all paths below path α will switch into policy H as soon as they hit the axis (because when X = 0 the multiplier λ_7 can become positive to preserve the continuity of ψ). Similarly in Figures 6.3 and 6.4 paths below path α will switch into policies K and G respectively when they hit the axis. As indicated already it may be shown (Appendix 6.3) that when $T \leqslant X_0/\hat{E}$ the optimal path lies along the horizontal axis in each of the above diagrams.[2] When $T > X_0/\hat{E}$ the optimal path lies above the horizontal axis and it is shown in Appendix 6.3 that the highest path on or below path α which exhausts the resource is optimal.

---

2 When the optimal path lies along the horizontal axis it is clearly possible to have $X(T) > 0$ (since $\psi(T) = 0$ ensures that the transversality condition (6.10) is satisfied) and this is a likely outcome for a short planning horizon and/or a large initial resource endowment (i.e. $T < X_0/\hat{E}$).

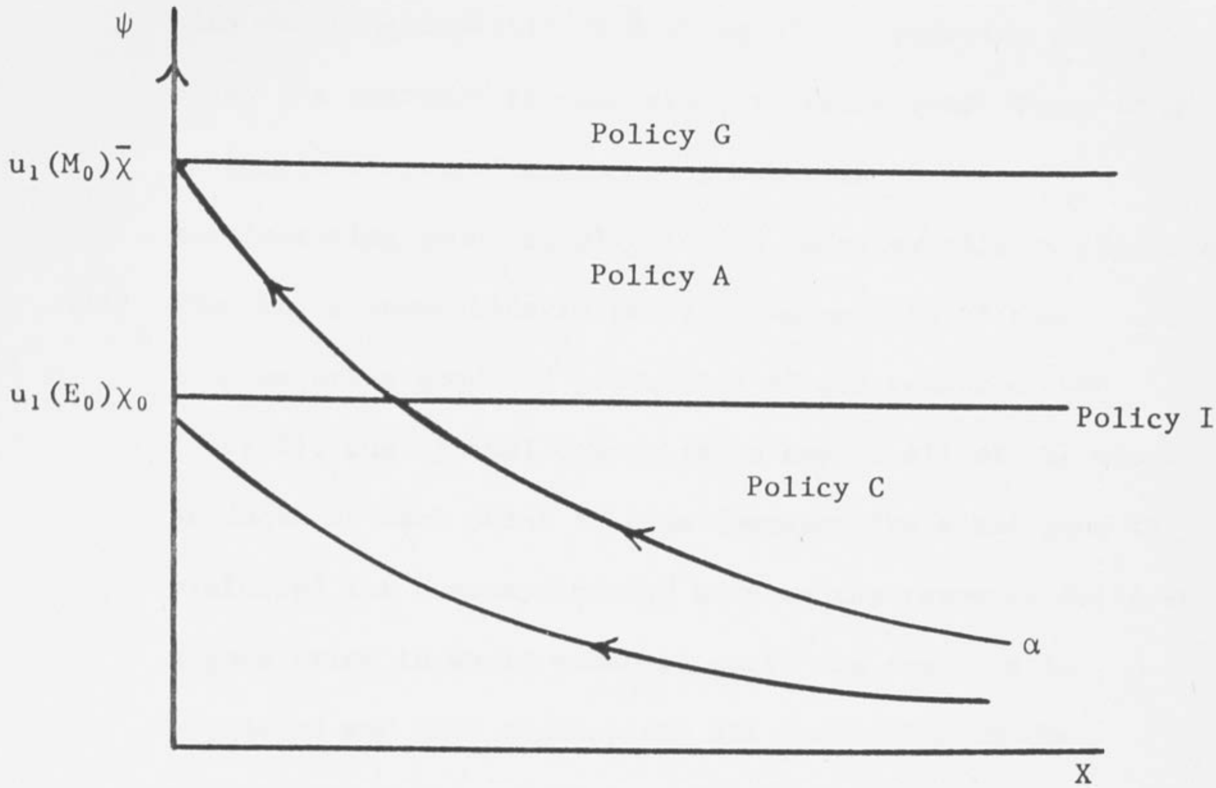

Figure 6.4

The results contained in Figures 6.2 - 6.4 may be interpreted as follows:

(i) For Case 1, where the economy has a clear preference for *consumption* and against *production* of the resource good (this latter because the terms of trade are adverse to production of the resource), the economy may find it optimal initially to export the resource for a short while (policy C), but at an early stage of the programme[3] it should switch into importing the resource (still domestically producing

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[3] At the surface $\psi = u_1(E_0)\left[1 + \frac{\phi'(E_0)}{p}\right]$ where E_0 satisfies $u_1(E_0) = pu_2(\phi(E_0))$. E_0 will be quite high (and hence ψ quite low, possibly negative) since:

$$u_1(E_0) = pu_2(\phi(E_0)) < pu_2(0) < u_1\left(\frac{\phi(0)}{p}\right) \Rightarrow E_0 > \frac{\phi(0)}{p} .$$

it), and ultimately should import as much as is consistent with existing domestic production of the resource $\left(\frac{\phi(E)}{p}\right)$ until the resource is exhausted, at which point there will be complete specialization in production of the manufacturing good and all mineral supplies will be imported.

(ii) When the economy clearly prefers consumption of the manufacturing good and production of the resource good (Case 2), the optimal course is to export all of the mineral produced at each point in time (because the other good is preferred for consumption *and* because the resource sells at a good price in world markets) until the resource is exhausted and then concentrate all production in the manufacturing sector (with no international trade).

(iii) For Case 3 it is optimal to export some of the minerals while the resource is plentiful (reflected in a relatively low ψ); as ψ rises and X becomes increasingly scarce, a switch into importing a part of domestic mineral supplies will prove optimal, and ultimately when all domestic deposits of the resource have been exhausted all resource supplies will be imported. However, unlike Case 1 some of domestic production of the manufacturing good will be consumed domestically. There is no overwhelming preference for either commodity so that even after exhaustion of the resource domestically both goods will be consumed.

2: $p < -\phi'(0)$:

In the static production model this would be the case of complete specialization in the production of the manufacturing good and we find that this is also the case in the dynamic model being considered here. The sufficiency proof in Appendix 6.3 is extended to cover this case and shows that the optimal course will be one in which E = 0 (and $\psi \leqslant 0$) for the entire programme. For example, in Case 1, the optimal course will be to follow policy H for the entire programme exporting all manufacturing output to supply imports of the resource good (which is essentially preferred and cheaply available via imports). In Cases 2 and 3 the respective policies followed will be K and G.

IV. PUBLIC POLICY

We have seen that for long time horizons and for $-\phi'(0) < p < -\phi'(\bar{E})$, the socially optimal path for extraction will diverge from the static competitive solution and it is logical to enquire: what public policies will bring the two solutions into line? Certainly, in the context of this model, an optimum tariff policy is not relevant (or, to put it another way, since the elasticity of the foreign offer curve is assumed to be infinite, the optimum tariff on minerals is zero). On the other hand if we concern ourselves with a tax on mineral production it is easy to determine a tax formula which will induce the social optimum in a competitive system. There is obviously no need to impose a tax once the resource has been exhausted so that a tax is only relevant during the period for which policies A, B, C, D and I would be operative in the socially optimal programme.

Suppose that a tax is imposed on mineral production at rate τ. Then in a competitive system the problem which will be solved by producers will be that of selecting E to maximize:

$$(6.12) \qquad (p - \tau)E + \phi(E)$$
$$\text{s.t.} \quad E \geqslant 0$$
$$\bar{E} - E \geqslant 0 \ .$$

The solution to this problem will satisfy the following conditions:

$$(6.13) \qquad \tau = p + \phi'(E) + \mu_1 - \mu_2$$

$$(6.14) \qquad \left.\begin{array}{lll} \mu_1 \geqslant 0 & E \geqslant 0 & \mu_1 E = 0 \\ \mu_2 \geqslant 0 & \bar{E} - E \geqslant 0 & \mu_2(\bar{E} - E) = 0 \end{array}\right\} .$$

For the case we are dealing with here it is obvious that $\mu_1 = \mu_2 = 0$ so that

$$(6.15) \qquad \tau = p + \phi'(E) \ .$$

The information linking (6.15) to the necessary conditions for a social optimum ((6.6) - (6.11)) and the optimum tax formulae for policies A, B, C, D, and I are contained in Table 6.3.

Table 6.3

Policy	Condition for Social Production Optimum	Optimum Tax Formula
A	$\frac{p\psi}{u_1} = \frac{\psi}{u_2} = p + \phi'$	$\tau = \frac{p\psi}{u_1} = \frac{\psi}{u_2}$
B	$\frac{p\psi}{u_1} = p + \phi'$	$\tau = \frac{p\psi}{u_1}$
C	$\frac{p\psi}{u_1} = \frac{\psi}{u_2} = p + \phi'$	$\tau = \frac{p\psi}{u_1} = \frac{\psi}{u_2}$
D	$\frac{\psi}{u_2} = p + \phi'$	$\tau = \frac{\psi}{u_2}$
I	$\frac{p\psi}{u_1} = \frac{\psi}{u_2} = p + \phi'$	$\tau = \frac{p\psi}{u_1} = \frac{\psi}{u_2}$

The tax rate which induces a competitive system to produce at socially optimal levels will also lead to the socially optimal solution for consumption of both goods. It may be easily seen that, provided E is the same in both models, the competitive and centralized solutions for imports and exports will be the same.[4] Thus the tax formulae summarized in Table 6.3 will be optimal and should be followed in the sequence suggested in Figures 6.2 - 6.4. Thus, when a particular policy is socially optimal the tax formula associated with that policy should be followed. From (6.15) it is easily shown that the tax rate should rise over time; this is what we would expect for we have already seen that social and competitive solutions diverge over time, so long as there is a positive stock of the resource left.

V. FOREIGN BORROWING

In this section we replace the assumption that international payments are always balanced with the assumption that payments are balanced over the duration of the planning period. The economy may now borrow and lend foreign exchange on world markets. This involves us writing the constraint as:

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4 For example for policy A: $M > 0$, $S = 0$ and $u_1(E + M) = pu_2(\phi(E) - pM)$ for both competitive and centralized models, and thus there is a common solution for M once E is determined. For policy B: $M = \frac{\phi(E)}{p}$, $S = 0$, and if E is the same for both models M will also have a common value. This equivalence between the necessary conditions for the consumers' problem in both the static and dynamic models is not surprising. The only source of divergence between the two models is the scarcity of the resource and this is essentially part of the production side of the models. There is no reason why the two consumers' problems should lead to different sets of conditions.

$$(6.16) \qquad \int_0^T [(pY_1 + Y_2) - (pC_1 + C_2)]e^{-rt}dt = -D_0 \ ,$$[5]

where D_0 is the initial level of debt facing the economy and r is the (constant) interest rate for both foreign borrowing and lending. We wish to:

$$\max_{E(t),C_1(t),C_2(t),X(T)} \int_0^T u(C_1, C_2)e^{-\delta t}dt \qquad \delta > 0, \text{ constant}$$

$$\text{s.t.} \qquad \dot{X} = -E$$

$$E \geqslant 0$$

$$C_1 \geqslant 0$$

$$C_2 \geqslant 0$$

$$\int_0^T [p(E - C_1) + (\phi(E) - C_2)]e^{-rt}dt = -D_0 \ .$$

---

5 This form of the constraint may be seen to be equivalent to the requirement that debt at time T be zero. If we let $B \equiv (pY_1 + Y_2) - (pC_1 + C_2)$ and $D(t)$ denote debt at time t, we have that:

$$\dot{D} = B + rD$$

$$(6.17) \qquad \int_0^T (B + rD)dt = D(T) - D_0 = -D_0 \ .$$

But it is also true that:

$$\int_0^T \dot{D}e^{-rt}dt = \left[De^{-rt}\right]_0^T + \int_0^T rDe^{-rt}dt \ .$$

Thus

$$\int_0^T Be^{-rt}dt = \left[De^{-rt}\right]_0^T = D(T)e^{-rT} - D_0 = -D_0 \ ,$$

so that (6.17) is equivalent to (6.16).

This is an optimal control problem with an isoperimetric side constraint (6.16) and the interested reader may check the method of solution by referring to Bryson and Ho [7].

The necessary conditions for a solution to the above problem are that there exist a continuous function of time, $\psi(t)$, a constant multiplier, μ, and multipliers λ_i (i = 1-5), satisfying:

$$\dot{\psi} = \psi\delta \tag{6.18}$$

$$\psi = \lambda_1 - \lambda_4 - \lambda_5 + \mu e^{(\delta-r)t}(p + \phi'(E)) \tag{6.19}$$

$$\mu_1 + \lambda_2 = p\mu e^{(\delta-r)t} \tag{6.20}$$

$$\mu_2 + \lambda_3 = \mu e^{(\delta-r)t} \tag{6.21}$$

$$\psi(T)\ X(T) = 0 \tag{6.22}$$

$$\left\{\begin{array}{lll} \lambda_1 \geqslant 0 & E \geqslant 0 & \lambda_1 E = 0 \\ \lambda_2 \geqslant 0 & C_1 \geqslant 0 & \lambda_2 C_1 = 0 \\ \lambda_3 \geqslant 0 & C_2 \geqslant 0 & \lambda_3 C_2 = 0 \\ \lambda_4 \geqslant 0 & \bar{E} - E \geqslant 0 & \lambda_4(\bar{E} - E) = 0 \\ \lambda_5 \geqslant 0 & X \geqslant 0 & \lambda_5 X = \lambda_5 \dot{X} = 0\ . \end{array}\right. \tag{6.23}$$

There are twelve possible policies open to the economy (Table 6.4).

In what follows we shall assume that $-\phi'(0) < p < \phi'(\bar{E})$ (incomplete specialization in the conventional static two-sector trade model). This implies in particular that $\psi < 0$ for policies B, D, F and H. It will constitute part of the optimality proof in Appendix 6.5 to show that $\psi \geqslant 0$ along the optimal path. With this additional knowledge and the fact that ψ never changes sign (from (6.18)) we can ignore policies B, D, F and H and confine our attention to the remaining eight policies. We can note the other general properties of the optimal path. Firstly, the transversality condition (6.22) tells us that either the

Table 6.4

Policy	Control Variable		
	C_1	C_2	E
A	0	0	$0 < E < \bar{E}$
B	0	0	$\bar{E}$
C	> 0	0	$0 < E < \bar{E}$
D	> 0	0	$\bar{E}$
E	0	> 0	$0 < E < \bar{E}$
F	0	> 0	$\bar{E}$
G	> 0	> 0	$0 < E < \bar{E}$
H	> 0	> 0	$\bar{E}$
I	0	0	0
J	0	> 0	0
K	> 0	0	0
L	> 0	> 0	0

resource is exhausted ($X(T) = 0$) or $\psi(T) = 0$, in which case $\psi(t) = 0$ $\forall_t \in [0, T]$. This means, for example, that if it is impossible to exhaust the resource before the end of the planning period, a zero user cost will be associated with the resource's depletion at all points in time. Secondly, it is shown in Appendix 6.4 that the higher the initial value of the costate (or equivalently, the higher the value of the constant multiplier, μ) the lower will be the initial level of extraction and the slower will be the process of depletion of the resource. In Appendix 6.5 (for the case $-\phi'(0) < p < -\phi'(\bar{E})$) the slowest extraction path for $\psi \geqslant 0$ is shown to be optimal. This is in keeping with results already obtained in Chapters 3 and 4 and section III of this chapter. We also find that the results proven earlier in this chapter for a static balance of payments constraint are valid for the less binding constraint assumed here. More precisely, the following results are demonstrated in Appendix 6.5:

(i) $E(t) \leqslant \hat{E}$ [6] $\quad \forall_t \in [0, T]$

(ii) $E(t) = \hat{E} \quad \forall_t \in [0, T] \quad \text{for} \quad T \leqslant X_0/\hat{E}$

(iii) $E(t) < \hat{E} \quad \forall_t \in [0, T] \quad \text{for} \quad T > X_0/\hat{E}$.

As before, the rate of extraction declines over time, so that, if we look at the optimal course for extraction in terms of the conventional two sector trade diagram (Figure 6.5), either the economy should locate at the static competitive optimum ($\hat{E}$) for the entire programme or move along the production frontier to the left of $\hat{E}$ towards the vertical axis.

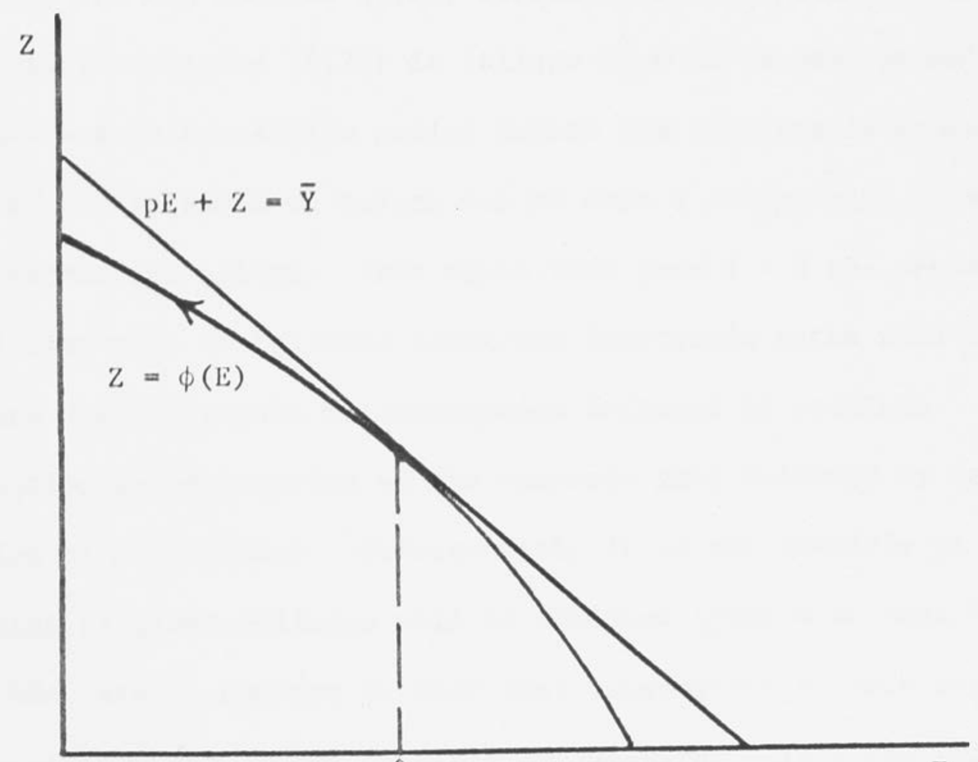

Figure 6.5

6 $\hat{E}$ is as defined in section II of this chapter.

Other properties of the optimal path will depend on the terms of trade and the form of the utility function. However an additional consideration here is the relationship between the social discount rate and the rate of interest on foreign borrowing and lending, and we must analyse separately the cases $\delta > r$, $\delta = r$ and $\delta < r$.

1: $\delta > r$:

We are able to divide this case further into two subcategories, $u_1(0) > pu_2(0)$ and $u_1(0) < pu_2(0)$. We consider these in order.

(i) $u_1(0) > pu_2(0)$:

Among other things this assumption eliminates policies E and J from consideration. Table 6.5 gives the necessary conditions which must be satisfied for the various policy switches to take place. From the transversality condition (6.22) it follows that it is non-optimal to switch into a zero extraction policy before the resource is exhausted since it is not possible to switch out of such a policy back into a positive extraction policy. This means that when $X > 0$ the sequence in which policies will be followed along the Pontryagin paths will be GCA (incomplete specialization in consumption followed by complete specialization in consumption of the resource good followed by zero consumption of both goods). Unfortunately it is not possible to say a priori which of these policies will be followed (just G or just A or GC or CA or GCA, etc.), however we know that consumption of both goods is falling over time and if the resource is exhausted before the economy can switch into policy A, for a sufficiently long time horizon policy I will eventually be optimal. Furthermore, if the marginal utility of zero consumption of both goods is very large, then C_1 and C_2 will be

Table 6.5

Switching table*

$\delta > r \qquad u_1(0) > pu_2(0)$

		Switches into				
		A	C	I	K	L
Switches out of	A		0^{+}	$\psi = \alpha(t)[p + \phi'(0)]$	0	0
	C	$p\alpha(t) = u_1(0)$ $\psi = \frac{u_1(0)}{p}[p + \phi'(E)]$		$u_1(0) = p\alpha(t)$ $\psi = \frac{u_1(0)}{p}[p + \phi'(0)]$	$\psi = \alpha(t)[p + \phi'(0)]$	0
	G	0	$u_2(0) = \alpha(t)$ $\psi = u_2(0)[p + \phi'(E)]$	0	$u_2(0) = \alpha(t)$ $\psi = u_2(0)[p + \phi'(0)]$	$\psi = \alpha(t)[p + \phi'(0)]$
	K	0	0	$u_1(0) = p\alpha(t)$		0
	L	0	0	0	$u_2(0) = \alpha(t)$	

* Condition given is that to be satisfied *at the switching surface.*

\+ A zero entry indicates that no switch is possible.

$\alpha(t) \equiv \mu e^{(\delta - r)t}$.

positive for large values of μ and ψ_0 (and correspondingly low E_0) with the result that policy G is optimal until the resource is exhausted. Certainly in the extreme case where $u_1(0) = u_2(0) = \infty$, C_1 and C_2 are both positive for the whole programme and in particular for the pre-exhaustion part of the plan.

Because E is declining and $[p + \phi'(E)] > 0$ it follows that $pE + \phi(E)$ is falling. But $pC_1 + C_2$ is also falling over time so we cannot generalize about what is happening to the balance of payments over time. Certainly for $T \leqslant X_0/\hat{E}$, E is constant over the course of the programme and C_1 and C_2 are both declining so the surplus is increasing. The economy will move from deficit to surplus over the course of the programme. This result certainly accords with our intuition and with the conclusion reached with more rigid assumptions in a related model (see [5], p.103) and it is perhaps surprising that it does not automatically carry over to the case $T > X_0/\hat{E}$. The reason it does not is actually quite simple. As a first approximation it would seem that if the social discount rate exceeds the rate of interest on foreign borrowing it would pay the country in question to borrow heavily in the early stages of the plan, taking advantage of the relatively low rate for borrowing and thus commence with a deficit which would be offset by a later surplus. However the economy also has an interest in obtaining the highest present value from its stream of consumption and because of the form of the balance of payments constraint this means (approximately) maximizing the present value of the stream of national production (discounted at rate r) and, for long time horizons, requires the level of extraction to fall over time. With production as well as consumption concentrated in the present the natural tendency to commence with a deficit and run to surplus later may be offset. On the other hand for short time horizons the resource is effectively not scarce, is thus economically equivalent to an ordinary good, and in such cases ($T \leqslant X_0/\hat{E}$) first impressions are correct.

(ii) $u_1(0) < pu_2(0)$:

This case leads to similar results to 1(i) above, with the exception that policies C and K are replaced by E and J respectively and the sequence of pre-exhaustion policies becomes GEA. Again when $u_1(0)$ and $u_2(0)$ are large one may expect G to be operative up to the time of exhaustion. The domestic consumption preference in this case is for the second good (either intrinsic preference reflected in a relatively high value for $u_2(0)$ or a favourable terms of trade (p) for the mineral good) and it is consequently optimal to run consumption of the resource good to zero first (if, in fact, there is time to run consumption of either good to zero). The sequence in which post-exhaustion policies become relevant is LJI, although again it is impossible to say which of these policies should operate without further information about the functions of the model. If $u_1(0)$, $u_2(0)$ and T are high then the economy would run through all three policies in the sequence given. The switching tables for this and the subsequent cases are essentially similar to Table 6.5 and are therefore omitted.

2: $\delta < r$:

(i) $u_1(0) > pu_2(0)$:

Here the sequence of pre-exhaustion policies of 1(i) is reversed to (ACG) although if $u_1(0)$ and $u_2(0)$ are large enough G will still be the only pre-exhaustion policy along the optimal path.

Consumption of both goods is growing over time and extraction is constant or falling. This implies that the balance of payments surplus is declining over time and the economy finds it optimal to move from surplus to deficit. The bias of the production time profile towards the present reinforces the bias of consumption towards the future, so that the result suggested by intuition for an ordinary two sector trading economy is strengthened when one of the goods is a resource.

<u>(ii) $u_1(0) < pu_2(0)$</u>:

The sequence in which policies may become operative is AEG. Apart from this modification the other conclusions obtained for 2(i) carry over unchanged.

<u>3: $\delta = r$</u>:

<u>(i) $u_1(0) > pu_2(0)$</u>:

When the discount rate equals the borrowing rate the relevant system of differential equations becomes autonomous and is simpler to analyse. Consumption of both goods is constant over time but extraction is falling for $T > X_0/\hat{E}$, causing the balance of payments surplus to fall over time. It is optimal to commence with a surplus, running to a deficit later in the plan. This is at variance with the result obtained from Bardhan's model in [5] to the effect that when the discount rate equals the borrowing rate the balance of payments is balanced at all points in time. This conclusion does not hold here for the simple reason that the scarcity of the resource makes it optimal to tilt the production of the resource towards the present away from a constant rate of production. Bardhan's conclusion does however hold here for $T \leqslant X_0/\hat{E}$, because for relatively short time horizons the resource is economically equivalent to an ordinary commodity.

Because consumption of both goods is constant for the entire plan it is impossible to switch between policies A, C, E and G or between I, J, K and L. The policy sequence will be either AI or CK (only if $u_1(0) > pu_2(0)$) or EJ (only if $u_1(0) < pu_2(0)$) or GL. High values of ψ_0 (and therefore μ) will be associated with low values of E, C_1 and C_2. The highest such value which initiates a path leading to exhaustion of the resource will be optimal. Whether this path is associated with zero consumption of either or both goods will depend on the properties of the utility function and the production frontier. Again, if $u_1(0)$ and $u_2(0)$ are large the economy should follow GL.

The following conclusions emerge from this model:

(i) The optimal intertemporal structure of production in the model of this chapter closely resembles that in the more restrictive models presented in Chapter 3 and Sections II - IV of this chapter. For long time horizons extraction of the resource should decline steadily over time and be spread as thinly as is consistent with exhaustion of the resource. The level of extraction in such cases will always be less than the static competitive optimum of the standard two-sector trade model. For short time horizons the static and intertemporal production optima coincide and extraction is constant for the entire programme.

(ii) An effect of having one of the two sectors in the economy producing from an exhaustible resource is to increase the tendency of the economy to run at a surplus in the early stages of the optimal plan. This is a simple consequence of the optimal time profile of resource exploitation being concentrated in the present.

(iii) When the discount rate exceeds the borrowing rate, optimal consumption of both goods will decline over time. When the borrowing rate exceeds the discount rate consumption will rise and when the two rates are equal consumption is constant. Along the optimal path consumption is determined independently of production.

APPENDIX 6.1

DERIVATION OF SWITCHING CONDITIONS

We will analyse a selected number of switches to illustrate the techniques involved.

AD: For policy A, $\psi = pu_2(\phi(E) - pM)\left(1 + \frac{\phi'(E)}{p}\right)$

$$= u_1(E + M)\left(1 + \frac{\phi'(E)}{p}\right)$$

For policy D, $\psi = pu_2(\phi(E) + pE)\left(1 + \frac{\phi'(E)}{p}\right)$

$$\geqslant u_1(0).\left(1 + \frac{\phi'(E)}{p}\right) .$$

Suppose that a switch from A to D is possible; i.e. suppose that ψ is continuous at the switching surface. Then if the value of E for policy i is denoted E_i,

$$\psi_A = pu_2(\phi(E_A) - pM)\left(1 + \frac{\phi(E_A)}{p}\right)$$

and $$\psi_D = pu_2(\phi(E_D) + pE_D)\left(1 + \frac{\phi'(E_D)}{p}\right)$$

and as $A \to D$, $\psi_A \to \psi_D$.

Then for given E_A immediately prior to a switch, E_D (immediately after the switch) will be strictly less than E_A, or $E_A > E_D + \varepsilon$, $\varepsilon > 0$.

But

$$\psi_D \geqslant u_1(0)\left(1 + \frac{\phi'(E_D)}{p}\right)$$

$$> u_1(0)\left(1 + \frac{\phi'(E_A)}{p}\right) + \eta\ , \qquad \eta > 0$$

$$> u_1(E_A + M)\left(1 + \frac{\phi'(E_A)}{p}\right) + \eta$$

$$= \psi_A + \eta\ .$$

Thus there is still a jump in ψ in the transition from A to D so we have a contradiction. Hence no AD switch is possible. Similarly, no DA switch is possible.

AH: For policy A: $\qquad \psi < pu_2(0)\left(1 + \frac{\phi'(0)}{p}\right)$

For policy H: $\qquad \psi \geqslant pu_2(0)\left(1 + \frac{\phi'(0)}{p}\right)$.

For a switch to take place we need $\dot{\psi} > 0$. This will be satisfied for AH. On the other hand an HA switch requires $\dot{\psi} < 0$ which is never true. Thus an AH switch may occur at $\psi = pu_2(0)\left(1 + \frac{\phi'(0)}{p}\right)$, while an HA switch is not possible.

AI: For policy A: $\qquad \psi < u_1(E)\left(1 + \frac{\phi'(E)}{p}\right)$

For policy I: $\qquad \psi = u_1(E)\left(1 + \frac{\phi'(E)}{p}\right)$.

For a switch to take place we must have

$$(6.24) \qquad \dot{\psi} > \left\{\left(1 + \frac{\phi'}{p}\right)u_{11} + \frac{u_1\phi''}{p}\right\}\dot{E} \qquad \text{prior to a switch.}$$

For policy A:

$$\dot{E} = \frac{\psi\delta(p^2u_{22} + u_{11})}{u_{11}u_{22}p^2\left(1 + \frac{\phi'}{p}\right)^2 + \phi''(u_1u_{22}p + u_2u_{11})}\ .$$

Then (6.24) becomes

$$(6.25) \qquad u_{11}\cdot\left(1 + \frac{\phi'}{p}\right)(p\phi' u_{22} - u_{11}) > 0 \ .$$

But the L.H.S. of (6.25) is always < 0. Hence no AI switch can occur. On the other hand IA is a feasible switch.

GI: For policy G:

$$\psi \geqslant u_1(M)\left(1 + \frac{\phi'(0)}{p}\right)$$

$$> u_1(M)\left(1 + \frac{\phi'(M)}{p}\right)$$

$$u_1(M) = pu_2(\phi(0) - pM)$$

$$\Rightarrow M \equiv \text{constant} \ .$$

For policy I:

$$\psi = u_1(E)\left(1 + \frac{\phi'(E)}{p}\right)$$

$$u_1(E) = pu_2(\phi(E))$$

$$\Rightarrow E \equiv \text{constant} \ .$$

It is possible to show $M < E$. Suppose $M \geqslant E$ and consider

$$f(E) \equiv \phi(E) + pE - \phi(0)$$

$$f'(E) = \phi'(E) + p > 0 \ , \quad \text{by assumption and}$$

$$f(0) = 0 \ .$$

Hence $f(E) > 0$ for $E > 0$. But if

$$M \geqslant E \ , \quad u_1(M) \leqslant u_1(E)$$

$$\Rightarrow \phi(0) - pE \geqslant \phi(0) - pM \geqslant \phi(E)$$

$$\Rightarrow f(E) \leqslant 0 \quad \text{for some } E > 0$$

$$\Rightarrow \text{contradiction.}$$

Hence $M < E$, in which case

$u_1(M) > u_1(E) + \varepsilon , \quad \varepsilon > 0$

$\Rightarrow \psi_G > \psi_I + \varepsilon$

$\Rightarrow$ no switch in either direction (discontinuity in ψ).

GK: For policy G: $M > 0$ and

(6.26) $$u_1(M) = pu_2(\phi(0) - pM) .$$

For policy K: $M = 0$ and

(6.27) $$u_1(0) \leqslant pu_2(\phi(0)) .$$

Given that (6.26) holds and that

$$\frac{d}{dM} [u_1(M) - pu_2(\phi(0) - pM)] = u_{11} + p^2 u_{22} < 0$$

we have that $u_1(0) > pu_2(\phi(0))$ which contradicts (6.27).

APPENDIX 6.2

<u>Case 1</u>: $u_1(0) > u_1\left(\frac{\phi(0)}{p}\right) > pu_2(0) > pu_2(\phi(0))$:

AB: A switch, if it takes place, will occur at

$$\psi = pu_2(0)\left(1 + \frac{\phi'(E)}{p}\right)$$

where

$$u_1\left(E + \frac{\phi(E)}{p}\right) = pu_2(0) \ . \qquad (6.28)$$

Since

$$pu_2(0) < u_1\left(\frac{\phi(0)}{p}\right)$$

and

$$u_1\left(E + \frac{\phi(E)}{p}\right) < u_1\left(\frac{\phi(0)}{p}\right) \qquad \forall_E > 0$$

so an AB switch is feasible in this case (i.e. a solution to (6.28) for E exists).

AG:

$$\psi = u_1(M)\left(1 + \frac{\phi'(0)}{p}\right)$$

$$u_1(M) = pu_2(\phi(0) - pM)$$

at the switching surface. In this case

$$u_1(M) > u_1\left(\frac{\phi(0)}{p}\right) > pu_2(0) > pu_2(\phi(0) - pM) \ .$$

Thus AG is not feasible in this case.

AH: A switch requires $u_1\left(\frac{\phi(0)}{p}\right) = pu_2(0)$ at the switching surface. Here $u_1\left(\frac{\phi(0)}{p}\right) > pu_2(0)$, thus ruling out this switch.

BH: At the switching surface we need $u_1\left(\frac{\phi(E)}{p}\right) \geqslant pu_2(0)$, which is not violated here, so that BH is admitted as a feasible switch.

CD: A switch requires

$$\psi = u_1(0)\left(1 + \frac{\phi'(0)}{p}\right)$$

where

$$u_1(0) = pu_2(\phi(E) + pE) .$$

Here

$$u_1(0) > pu_2(0) > pu_2(\phi(E) + pE) .$$

Here no switch can take place.

CI: A switch requires

$$\psi = u_1(E_0)\left(1 + \frac{\phi'(E_0)}{p}\right)$$

where

$$u_1(E_0) = pu_2(\phi(E_0)) .$$

Here

$$pu_2(\phi(E_0)) < pu_2(0) < u_1\left(\frac{\phi(0)}{p}\right)$$

and for a switch to be feasible we need $E_0 > \frac{\phi(0)}{p}$. However such a value for E_0 may be found and so CI is feasible.

DK: At the switching surface we require $u_1(0) \leqslant pu_2(\phi(0))$.

Here $u_1(0) > pu_2(\phi(0))$ so no switch is possible.

IA: A switch requires

$$\psi = u_1(E_0)\left(1 + \frac{\phi'(E_0)}{p}\right) .$$

This is feasible for the same reason CI is infeasible.

Case 2: $pu_2(0) > pu_2(\phi(0)) > u_1(0) > u_1\left(\frac{\phi(0)}{p}\right)$:

AB:

$$pu_2(0) > u_1\left(\frac{\phi(0)}{p}\right) > u_1\left(E + \frac{\phi(E)}{p}\right)$$

$\Rightarrow$ no switch.

AG:

$$pu_2(\phi(0) - M) > u_1(0) > u_1(M)$$

$\Rightarrow$ no switch.

AH:

$$pu_2(0) > u_1\left(\frac{\phi(0)}{p}\right)$$

$\Rightarrow$ no switch.

BM: Require

$$u_1\left(\frac{\phi(0)}{p}\right) \geqslant pu_2(0)$$

$\Rightarrow$ no switch.

CD:

$$u_1(0) < pu_2(\phi(0)) < pu_2(\phi(E) + pE)$$

$\Rightarrow$ no switch.

CI:

$$u_1(E) < pu_2(\phi(0)) < pu_2(\phi(E))$$

$\Rightarrow$ no switch.

DK: Need $u_1(0) < pu_2(\phi(0))$; this is satisfied and so a switch may occur.

Case 3: $u_1(0) > pu_2(\phi(0))$; $pu_2(0) > u_1\left(\frac{\phi(0)}{p}\right)$:

AB:

$$u_1\left(E + \frac{\phi(E)}{p}\right) < u_1\left(\frac{\phi(0)}{p}\right) < pu_2(0)$$

$\Rightarrow$ no switch.

AG: $$u_1(M) > u_1\left(\frac{\phi(0)}{p}\right) > pu_2(\phi(0))$$

$$< pu_2(\phi(0) - pM) .$$

Thus AG is feasible for M sufficiently large.

AH: No switch (as for cases 1 and 2).

BH: No switch (as for cases 1 and 2).

CD: $$u_1(0) > pu_2(0) > pu_2(\phi(E) + pE)$$

$\Rightarrow$ no switch.

CI: $$u_1(E) < u_1(0); \quad pu_2(\phi(E)) > pu_2(\phi(0))$$

$\Rightarrow$ CI is feasible.

DK: $$u_1(0) < pu_2(\phi(0)) \quad \text{is violated}$$

$\Rightarrow$ no switch.

APPENDIX 6.3

PROOF OF OPTIMALITY OF PATH IN MODEL OF SECTION III

1: $-\phi'(0) < p < -\phi'(\bar{E})$:

Let the path which is claimed to be optimal have its variables denoted with asterisk superscripts. Then comparing the present value of the stream of utility along this path (P*) with the corresponding present value (P) along any other path yields (using strict concavity of u and ϕ).

$$P^* - P = \int_0^T [u(E^* + M^* - S^*, \phi(E^*) - p(M^* - S^*)) - u(E + M - S, \phi(E) - p(M - S))]e^{-\delta t}dt$$

$$> \int_0^T u_1^* \, e^{-\delta t}[(E^* - E) + (M^* - S^* - (M - S))]dt + \int_0^T u_2^* \, e^{-\delta t}[\phi'(E^*)(E^* - E) - p(M^* - S^* - (M - S))]dt$$

$$= \int_0^T (u_1^* + \phi' u_2^*)(E^* - E)e^{-\delta t}dt + \int_0^T (u_1^* - pu_2^*)e^{-\delta t}(M^* - S^* - (M - S))dt$$

$$= \int_0^T (\psi^* - \lambda_1^* - \lambda_4^* - \lambda_5^*\phi' + \lambda_6^* + \lambda_7^*)(E^* - E)e^{-\delta t}dt + \int_0^T (p\lambda_5^* - \lambda_2^* - \lambda_4^*)(M^* - S^* - (M - S))e^{-\delta t}dt$$

$$= \int_0^T \psi_0^*(E^* - E)dt + \int_0^T [\lambda_1^* E + \lambda_4^*(E + M - S)$$

$$+ \lambda_5^*(\phi(E) - p(M - S)) + \lambda_6^*(\bar{E} - E) - \lambda_7^* E]e^{-\delta t}dt$$

$$\geqslant \psi_0^* \int_0^T (E^* - E)dt - \int_0^T \lambda_7^* E dt$$

$$= \psi_0^*[X(T) - X^*(T)] - \int_0^T \lambda_7^* E dt \ .$$

It is claimed that the optimal path will be the one which entails the lowest level of extraction at any point in time among those paths satisfying conditions (6.6) - (6.11). To prove this assertion we consider separately the two possibilities identified in Section III, Part 1:

(a) $T \leqslant X_0/\hat{E}$:

In this case our assertion amounts to saying that $\psi^* = 0$ represents the optimal path ($\psi > 0$ will not exhaust the resource and so violates the transversality condition (6.10)). Trivially, $\psi_0^*[X(T) - X^*(T)] = 0$ and $E = 0$ along any other path before λ_7^* can become positive (since paths for $\psi < 0$ use up the resource more rapidly) so that $\lambda_7^* E = 0$. Hence $P^* - P > 0$ and $\psi^*(t) \equiv 0$ is optimal. Thus $E(t) = \hat{E}$ $\forall_t \in [0, T]$ is optimal.

(b) $T > X_0/\hat{E}$:

If $\psi^*(t) > 0$ $\forall_t$, $X^*(t) = 0$ and the path chosen is that which takes longest to exhaust the resource (the highest path on or below path α in Figures 6.2 - 6.4) then E will have reached zero along other paths before $X^* = 0$ and thus before $\lambda_7^* > 0$. Thus we again have $\lambda_7^* E = 0$ and optimality is established.

2. $p \leqslant -\phi'(0)$:

Using similar reasoning to the above

$$\psi_0 \leqslant 0 \quad (\Rightarrow \psi(t) \leqslant 0 \quad \forall_t)$$

and

$$X^*(T) = X_0(E(t) = 0 \quad \forall_t) \quad \Rightarrow P^* - P > 0 \ .$$

APPENDIX 6.4

CAUSALITY IN THE MODEL OF SECTION V

The balance of payments constraint may be used to show how E_0, μ and ψ_0 are related. We can write the constraint in the form:

$$\int_0^T [pE + \phi(E)]e^{-rt}dt = \int_0^T (pC_1 + C_2)e^{-rt}dt \ . \tag{6.29}$$

Define $E_T \equiv \lim_{t \to T} E(t)$. Then L.H.S. of (6.29) may be written:

$$\text{L.H.S.} = - \left. \frac{[pE + \phi(E)]e^{-rt}}{r} \right|_0^T + \frac{1}{r} \int_0^T [p + \phi'(E)]e^{-rt} \frac{dE}{dt}\, dt$$

$$= - \frac{[pE_T + \phi(E_T)]}{r} + \frac{pE_0 + \phi(E_0)}{r}$$

$$+ \frac{1}{r} \int_0^T \left(\frac{\psi - \lambda_1 + \lambda_4 + \lambda_5}{\mu} \right) e^{-\delta t} \frac{dE}{dt}\, dt$$

Now the third term in the last expression equals

$$\frac{1}{\mu r} \int_0^T \psi_0 dE + \frac{1}{\mu r} \int_0^T (\lambda_4 + \lambda_5 - \lambda_1)e^{-\delta t} \frac{dE}{dt}\, dt \ .$$

Either $0 < E < \bar{E}$, in which case $\lambda_1 = \lambda_4 = \lambda_5 = 0$, or $E = 0$ or $\bar{E}$, in which case $\frac{dE}{dt} = 0$ and so the second integral in the last expression vanishes, and:

$$\text{L.H.S.} = \frac{pE_0 + \phi(E_0)}{r} - \frac{[pE_T + \phi(E_T)]e^{-rT}}{r} + \frac{\psi_0}{\mu r}(E_T - E_0)$$

$$= \frac{pE_0 + \phi(E_0)}{r} - \frac{[pE_T + \phi(E_T)]e^{-rT}}{r} + \frac{[p + \phi'(E_0)](E_T - E_0)}{r}$$

$$= \frac{\phi(E_0) - E_0\phi'(E_0)}{r} - \frac{e^{-rT}[\phi(E_T) - E_T\phi'(E_T)]}{r} .$$

Now, E_T is determined by

$$e^{-rT}[p + \phi'(E_T)] = [p + \phi'(E_0)]$$

when

$$X(t) = X_0 - \int_0^t E(\tau)d\tau > 0 \quad \forall_t < T .$$

When $X(t) = 0$ for some $t < T$, $E_T = 0$. In either case, we may write L.H.S. $= f(E_0)$ where it may be shown that $f'(E\ ) > 0$. We now consider the right hand side of equation (6.28):

$$\text{R.H.S.} = \int_0^T (pC_1 + C_2)e^{-rt}dt .$$

We have that

$$pC_1 = \alpha(\mu, t)$$

and

$$C_2 = \beta(\mu, t)$$

from (6.20) and (6.21). For $C_1 > 0$:

$$\frac{\partial C_1}{\partial \mu} = \frac{pe^{(\delta-r)t}}{u_{11}} ,$$

i.e. $\alpha_\mu < 0 \qquad (\alpha_\mu = 0 \text{ for } C_1 = 0)$.

Similarly for $C_2 > 0$:

$$\beta_\mu < 0 \qquad (\beta_\mu = 0 \text{ for } C_2 = 0) .$$

Thus

$$\text{R.H.S.} = \int_0^T h(\mu, t)dt \equiv g(\mu) ,$$

where

$$h(\mu, t) = [\alpha(\mu, t) + \beta(\mu, t)]e^{-rt}$$

$$g'(\mu) = \int_0^T h_\mu(\mu, t)dt$$

$$= \int_0^T (\alpha_\mu + \beta_\mu)e^{-rt}dt < 0 .$$

Thus, the balance of payments constraint may be written in the form:

$$f(E_0) = g(\mu)$$

with the result that:

$$\frac{dE_0}{d\mu} < 0 .$$

Now:

$$\psi_0 = \mu(p + \phi'(E_0)) .$$

So

$$\frac{d\psi_0}{d\mu} = (p + \phi'(E_0)) + \mu\phi'' \frac{dE_0}{d\mu}$$

$$> 0 \qquad \text{for } \psi_0 \geqslant 0 .$$

Thus higher initial values of the costate ($\Leftrightarrow$ higher initial values of the constant multiplier, μ) imply lower extraction and thus slower depletion of the resource.

APPENDIX 6.5

PROOF OF OPTIMALITY FOR SECTION V

We will consider only the case where $\phi'(0) < p < \phi'(\bar{E})$.

$$P^* - P = \int_0^T u(C_1^*, C_2^*) - u(C_1, C_2)]e^{-\delta t}dt$$

$$> \int_0^T [u_1(C_1^*)(C_1^* - C_1) + u_2(C_2^*)(C_2^* - C_2)]e^{-\delta t}dt$$

$$= \int_0^T [p\mu^* e^{(\delta-r)t} - \lambda_2^*)(C_1^* - C_1)$$

$$+ (\mu^* e^{(\delta-r)t} - \lambda_3^*)(C_2^* - C_2)]e^{-\delta t}dt$$

$$= \mu^* \int_0^T [p(C_1^* - C_1) + (C_2^* - C_2)]e^{-rt}dt$$

$$+ \int_0^T \lambda_2^* C_1 e^{-\delta t}dt + \int_0^T \lambda_3^* C_2 e^{-\delta t}dt$$

$$\geqslant \mu^* \int_0^T \{[pE^* + \phi(E^*)] - [pE + \phi(E)]\}e^{-rt}dt$$

$$> \mu^* \int_0^T [p(E^* - E) + \phi'(E^*)(E^* - E)]e^{-rt}dt$$

$$= \mu^* \int_0^T (E^* - E)(p + \phi'(E^*))e^{-rt}dt$$

$$= \int_0^T (E^* - E)(\psi^* + \lambda_4^* + \lambda_5^* - \lambda_1^*)e^{-\delta t}dt$$

$$\geqslant \psi_0^*(X(T) - X^*(T)) - \int_0^T \lambda_5^* E e^{-\delta t}dt \ .$$

This last expression will be positive under the following circumstances:

(i) $T \leqslant X_0/\hat{E}$: $E^*(t) = \hat{E} \quad \forall_t \in [0, T]$

(ii) $T > X_0/\hat{E}$: $E^*(t) < \hat{E} \quad \forall_t \in [0, T]$

and the path chosen as optimal is that which takes longest to exhaust the resource.

CHAPTER 7

CONCLUSION

It has been the aim of this thesis to develop models which throw light on some of the basic questions associated with the intertemporal use of exhaustible resources. As foreshadowed in the introduction the main preoccupation has been with problems of "conservation", the trade-off between the present and the future being central to such problems.

At the most basic level (Chapter 3) it was possible to see how the optimal time profile of exploitation of an exhaustible resource is affected by the form of the utility and production functions, the size of the discount rate and the length of the planning period. It was found that:

(a) For an infinite time horizon a positive discount rate is needed to ration the stock of the resource between generations;

(b) If there is a region of increasing returns to the resource input, it will be optimal to operate beyond that region whenever the resource is being exploited (the case of class II and III u-functions) — consequently exhaustion of the resource in such a case will generally be more rapid than when there are always diminishing returns to the resource (a class I u-function;

(c) The existence of a conservation motive and/or a depletion effect in extraction of the resource opens up the possibility that it is not optimal to exhaust the resource (conditions under which the resource should be exhausted were derived).

In addition the social analogue of the Hotelling "rule" that net price should rise at the rate of discount was found to hold when conservation motives and depletion effects were ignored. However, once these aspects of the problem were acknowledged the rule collapsed and there were found to be several situations where the social price of the resource should decline in the later stages of the period of exploitation. This was because the marginal net social value of the resource is eroded as depletion effects (for example) force up the costs of extraction.

Chapter 3 was concerned with the most basic form of scarcity associated with exhaustible resources — that arising from the finite nature of their stock. Chapter 4 went on to incorporate a more conventional kind of economic scarcity into the problem — the limited supply of the variable factor of production (labour) which was to be used both in conjunction with the resource and in the extraction of the resource. The emphasis in the first part of Chapter 4 was on determining the best intertemporal way of allocating a given finite labour supply between sectors, given that some of it was to be used to extract the resource, and the rest was to be used to produce one or more consumption goods. It was possible to identify the following forces at work in the determination of a time path for the allocation.

(a) When the only consumption good in the economy uses the resource as an input, consumption should decline over time and the economy should move towards the labour allocation which maximizes consumption at a point in time.

(b) The labour allocated to resource extraction should ultimately decline over time for very long time horizons, however when the planning period is short the movement of this variable is ambiguous — in order to approach the myopic rule as T is approached the labour allocated to extraction must rise, however the depletion effect will (because of the falling marginal product of labour in the resources sector) have the reverse effect; when there is no depletion effect, the labour in the resources sector should fall over time.

(c) If the economy produces a second consumption good *without the resource*, then the relative preferences for the two consumption goods become important; without a depletion effect the labour allocated to resource extraction should still fall over time, however the time paths of labour going towards production of the two consumption goods will be the net outcome of two influences: (i) the innate preference for one consumption good (as embodied in the utility function) will tend to prevent the labour allocated to production of that good from falling, and (ii) the scarcity of the resource will work to lower the allocation of labour to the resource based consumption good over time.

The second part of Chapter 4 extended the basic Chapter 3 model by considering the "relative scarcity" of two resource deposits. These deposits differed in both size and cost. A generalization of the "Ricardian" rule that resource deposits should be extracted in order of ascending costs was proven and it was noted that the presence of depletion effects makes it possible to manipulate these costs and (for a while) maintain a situation where the marginal (and average) costs of extraction are the same for both deposits and both deposits are exploited simultaneously. The final part of Chapter 4 was devoted to an extension of the conservation motive model of Chapter 3 and a generalization of the exhaustion/non-exhaustion conditions obtained there. The conditions obtained for the disaggregated model are very similar to those obtained for a single resource.

Having analysed the different types of scarcity which are relevant to the exhaustible resource planning problem the next step was to examine some ways in which the basic scarcity might be mitigated.

Chapter 5 analysed the process whereby an increasingly expensive exhaustible resource is replaced by a substitute which is essentially inexhaustible. A comparative dynamic analysis revealed that the availability of cheaper such substitutes will make it optimal to use *less* of the exhaustible resource in the earlier stage of a plan, *if depletion effects are present*. In the absence of depletion effects, the prevailing intuition that "cheaper substitutes means conserve less now" was supported.

The possibility of lowering the cost of using the backstop technology by investing in capital equipment and/or knowledge was then examined. It was found that:

(a) Unless the planning period is sufficiently long, it would not be worthwhile to initiate such a development project; the economy is better off remaining with the original resource.

(b) If it is worthwhile to invest in development of the substitute (i.e. the time horizon is long enough), such investment should date from the beginning of the plan, regardless of the initial stock of the resource.

If there is no investment required for development of the substitute, but there is uncertainty concerning the date at which it will become available, it was found that the optimal amount of the exhaustible resource left unexploited should be higher the higher the probability that the substitute will eventually become available.

Another way in which an economy may choose to relieve (or benefit from) resource scarcity is by importing (or exporting) the resource good. Chapter 6 presented a model in which a country facing a fixed terms of trade plans its pattern of trade and specialization over time so as to maximize the present value of its stream of returns. The nature of the optimal path was dependent on the economy's terms of trade and set of preferences. It was shown that for short time-horizons, the economy should operate at the static competitive optimum for a trading country while for longer time horizons the level of extraction should be less than this static optimum and falling. A set of taxes was devised to bring the competitive optimum into line with the social optimum. When allowance was made for foreign borrowing, the introduction of an exhaustible resource into the conventional foreign borrowing model worked to increase any tendency the economy might have to move from surplus into deficit and offset any opposite tendency.

Beyond the problems outlined above and discussed in the thesis there are a whole range of other interesting questions. Some of them (recycling, market failure, bias in certainty-equivalent planning, uncertainty about the size of resource deposits) are already being given close scrutiny by a number of economists. Others (exploration, tax policies, imperfect futures, markets, etc. and intertemporal resource price determination) have been generally neglected. It is to be expected that the urgency of most of these problems will encourage more research in the future. When that happens, economics might finally be justly called a "science of scarcity".

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