ON THE THEORY OF FINANCE AND INVESTMENT

BY

FRANK MILNE

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In compliance with the requirements relating to Admission to Examination, and Submission of Theses, for the Degree of Doctor of Philosophy of the Australian National University, it is affirmed that, unless otherwise stated, the work that follows is my own.
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This thesis is a theoretical study of financial and investment decisions by consumers and producers in a market economy.

The study takes as its point of departure, the certainty, competitive equilibrium theory of Arrow-Debreu, and demonstrates some well-known propositions on the irrelevance of financial decisions and the unambiguous role of value-maximization for production decisions.

Because a complete market system for all future contingencies is unrealistic, the study proceeds to examine incomplete market systems. Using a simple model developed by Diamond (1967), there is a detailed and rigorous analysis of production, leverage and short-selling decisions, given the possibility of default risk.

In subsequent chapters, corporate taxation, marketing costs for securities, and price uncertainty are introduced.

One of the conclusions of the study is that finance theory should be considered within the framework of a Temporary Equilibrium system, where current markets for commodities and securities depend upon the expectations of economic agents.
ABSTRACT

This thesis is a theoretical study of financial and investment decisions by consumers and producers in a market economy.

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CHAPTER 1

INTRODUCTION

Finance theory is a special branch of the general economic theory of capital. In particular finance theory is concerned with the way consumers and producers allocate resources over time, through the operation of capital (or financial) markets. Because, (almost invariably) the future is uncertain, the theory must incorporate a systematic treatment of economic behaviour under uncertainty.

The theorist is faced with a formidable task if he is to produce models that are manageable and yet instructive. Because of the complexity of the problem much of the existing theory deals with severely restricted views of the world. Progress in the theory has come about by relaxing assumptions, and generalizing and extending existing ideas. It is the modest aim of this thesis to contribute to this process of generalization and extension. We will approach the problem through the competitive general equilibrium theory of Arrow (1953) and Debreu (1959). Although some readers may find our arguments overly formal, it is our belief that a rigorous approach minimizes the possibility of ambiguity. (Indeed, poorly defined constructions and ambiguity bedevilled much of the early work in finance theory).

In several places the reader is assumed to have a reasonable knowledge of the techniques and arguments contained in Debreu (1959), (1962). For instance, we have quoted theorems directly from Debreu, or we have sketched modifications to Debreu's theorems in adapting them to our requirements. Nevertheless, the reader without this prerequisite knowledge should
not to be discouraged, because the technical details can be omitted without impairing an understanding of the central argument.

Each Chapter has been written as an independent statement and examination of a specific problem. Therefore, each Chapter can be read without a detailed knowledge of the preceding discussion. Inevitably, there is some overlap and repetition, although we have tried to keep such repetition to a minimum. Given the independence between Chapters, we must stress that the Chapters do follow a natural progression of ideas. This natural progression should become clear in the following summary of the contents of the Thesis.

In Chapter 2, we have given a brief discussion of the important contributions to finance and investment theory. This survey is not intended to be an exhaustive summary of contributions to the theory, but more in the nature of a thumb-nail sketch of the major theories underlying modern finance theory.

In Chapter 3\(^1\) we have presented the certainty, competitive general equilibrium model of Debreu, showing the irrelevance of corporate financing decisions, and the unanimity of shareholders with the choice of the value-maximizing production plan. By defining commodities over states of the world, the theory can be re-translated into an idealized uncertainty theory. Using Arrow's (1953), (1963) normalization procedure over states, it is possible to obtain a set of primitive securities that provide a complete set of insurance markets. Finally, we consider Diamond's (1967) model, where, in a restricted framework, there are fewer securities than states of nature. By considering patterns of returns across states of the world, as commodities, the model can be considered isomorphic to the certainty theory. Therefore, the irrelevance of corporate financing decisions and the clear-cut nature of production decisions are a feature of all these models.

\(^1\) This Chapter is to appear in the Economic Record.
Although these models are elegant, and provide a number of insights, they are strictly limited as descriptive theories. In the subsequent Chapters we have attempted to remove some of the more unrealistic features (or implications) of these models.

Because in Chapter 3, we simply sketched the main features of the Diamond model, we have provided a rigorous general equilibrium treatment in Chapter 4. Furthermore, we have investigated the possibility of the producer introducing a new security, through his production decisions. It is shown that this production decision will depend upon all the shareholders preferences, in a similar fashion to a public good problem. For the validity of the argument, we require set-up costs or monopolistic elements in the creation of a security market.

The general argument of Chapter 4 is repeated in Chapter 5, where asset creation is assumed to take place via financial leverage decisions of corporations. By means of this argument, we hope to remove a source of conflict between the claims of Modigliani and Miller (1958) on the one hand, and Stiglitz (1972b) and Smith (1970), (1972a) on the other.

In Chapter 5 there arose a case where consumers held all the risky securities in a single mutual fund. Cass and Stiglitz (1970) have provided necessary and sufficient conditions on preferences if such a mutual fund is to be formed. We have given an independent (and somewhat simpler) proof of their results in the Appendix to Chapter 5.

Chapter 6 considers the possibility of consumer short-sales. By a careful examination of the implications of default, we have been able to include short-selling into the general equilibrium treatment of Diamond's model. Again, we have considered the implications of asset creation; but

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2 This Chapter is to appear in the *Journal of Financial Economics*. 
this time through the medium of a defaulting short-sale. The Chapter concludes with a discussion of the relationship between our results and the work of other writers on the single proprietor firm.

In Chapters 4, 5 and 6 we have been concerned with the monopolistic aspects of asset creation within the Diamond model. The next three Chapters are attempts to introduce additional complications into this basic theory.

The government and taxation is introduced in Chapter 7. The government sector is treated in a summary fashion so that we can concentrate our attention on the impact of corporate taxation on financial and investment decisions. It is shown that, with proportional taxation on different financial instruments (e.g., debt, equity), production decisions continue to be determined by the value-maximization rule; but tax-avoidance, through the optimal choice of financial instruments, can be formulated as a linear programming model. The Chapter concludes with a number of examples.

To obtain the monopolistic aspect of asset creation, discussed in earlier Chapters, we argued that a sufficient condition would be set-up costs in the introduction of a new security. In Chapter 8 we consider a special case of transaction costs - marketing costs. Based upon the theory developed by Foley (1970a), we are able to provide a theory rationalizing the inactivity of many security markets. Nevertheless, because the marketing technology is assumed to be convex, competitive assumptions are quite consistent with different borrowing and lending rates, and a modified form of value-maximization rule for production decisions. Also, we obtain a modified version of the leverage indifference theorem of Modigliani and Miller (1958).

The Diamond model is limited in its restriction to one physical commodity. This assumption is removed in Chapter 9, where finance theory considerations and capital markets are introduced into a temporary equilibrium model, similar to that developed by Hicks (1939), Grandmont (1974), Green (1973)
and Sondermann (1974) and others.

Finally, in Chapter 10, we provide some thoughts on possible further developments to the theory.
CHAPTER 2

A BRIEF SURVEY OF FINANCE AND INVESTMENT LITERATURE

The theory of capital and investment has a long and important history in the development of economic theory. The literature is voluminous with contributions by most of the great economic theorists. But until comparatively recently, these theories assumed a world of certainty, so that any discussion of securities was limited to the role of the riskless bond. Although numerous asides appeared in the literature commenting upon risky securities, a formal theory was lacking. In response to the practical needs of accountants, financial controllers and businessmen, there developed a body of finance theory, which attempted to graft uncertainty onto the certainty models of capital theory. Although there was some exchange of ideas between capital theorists and finance theorists, there was an unfortunate tendency for economists, in general, to treat finance theory as a poor relation. Fortunately this attitude is disappearing with the developing interest in economic theory, in the problems of uncertainty and the organization of securities markets. The problems of consumer portfolio choice, corporate control and financial policy, that were once the domain of the finance theorists, have emerged as real problems in asset market models under uncertainty.
By way of a brief, and by no means exhaustive survey of the literature, we hope to trace the main threads of the development of finance and investment theory. For expositional purposes we will divide our discussion into three sections. In Section 1 we will consider some of the major elements in the development of capital theory under certainty. In Section 2 we will sketch the progress of finance theory up until the present time. Finally, in Section 3, we will discuss the recent work in economic theory on uncertainty and the operation of security markets, and its relation with the existing finance theory literature.

1. CAPITAL THEORY

1.1 One of the unfortunate aspects of capital theory is its reputation as a difficult and confusing body of theory. The early literature, especially that of the Austrian School, was filled with special examples and a terminology that was often obscure. Although the work of Böhm-Bawerk (1889), Wicksell (1934), Walras (1877) and other writers added significantly to the growth of capital theory, it was Irving Fisher (1930) who first introduced the essential ingredients for a discussion of an agent's capital-theoretic problem. Although his theory was a partial equilibrium treatment, Fisher's discussion of the consumer's intertemporal choice problem provides the important foundation stone for more ambitious general equilibrium treatments.

The basis of Fisher's theory is that the apparatus of the consumer's static choice problem can be reinterpreted to include time, by the simple device of considering income today and income tomorrow as separate commodities. Thus investment, and saving decisions, and interest rates appear as reinterpretations of the static theory.

1 For a brief exposition of the Austrian theory of capital (especially Böhm-Bawerk (1889) and Wicksell (1934)) the reader is referred to Lutz (1968).
Fisher illustrated his argument by means of a simple diagram. Consider a consumer whose objects of choice are income today $x_1$, and income tomorrow $x_2$. The consumer has preferences over the set of possible incomes $X = \mathbb{R}^2_+$. These preferences can be represented by the usual quasi-concave utility indicator. The consumer faces given market opportunities for trading incomes across periods by borrowing or lending at the market rate of interest $r$. Given incomes in both periods $Y_1, Y_2$ respectively, the consumer's budget constraint, or market opportunity locus is

$$x_1 + (1+r)^{-1}x_2 = Y_1 + (1+r)^{-1}Y_2.$$ 

Furthermore, the consumer has available a production technology for transforming first-period income into second-period income. Assuming that the technology can be represented by a convex set, we can represent all these elements in Figure 1. Given his income endowment $(Y_1, 0)$ (in this case) and the market opportunity locus $DY_1$, with slope $-(1+r)$, the consumer reaches the highest indifference curve $I_1$ with the marginal rate of time preference equal to the market rate of exchange. In fact, the consumer lends $LY_1$ in return for a total return $OM$ in the second period.

Given his endowment and production opportunities $GY_1$, the consumer can reach the indifference curve $I_2$ without resort to the market. But, by trading in the market as a producer, so as to maximize $y_1 + (1+r)^{-1}y_2$, where $(y_1, y_2)$ are chosen from the production locus, the consumer can maximize the value of intertemporal wealth; and using market opportunities, attain the indifference curve $I_3$.

Because the consumer faces a fixed interest rate, his convex outward indifference curves, and non-satiety, imply that he desires more wealth to less. Therefore, the choice of the productive optimum (at $H$), requires the
Figure 1
consumer to maximize the value of production - which is the intertemporal version of profit maximization. One of Fisher's important contributions was his formulation of an investment rule that was derived from consumer preferences and market opportunities.

Fisher extended his theory to a finite number of periods, so that the preceding argument generalized in an obvious way. Also he observed, that with many periods, the theory could contain a term structure of interest rates.

1.2 Hicks (1939) took the obvious step in generalizing the Fisherian theory, by introducing the existence of markets for dated commodities. By introducing a "Futures Economy", Hicks provided the formal framework for market-orientated capital theory models. This model has attained its most rigorous formulation in the model of Debreu (1959).

Far from being an arid general system, the Futures Economy provides a flexible and powerful frame within which all consistent, certainty, capital-theoretic problems can be examined. By imposing restrictions on the production technology, it is possible to include the main features of the Austrian theory of capital; or to discuss durability or scrapping policies as derivatives of the value maximization rule, and market prices for machine services. This generality has been emphasized in the work of Malinvaud (1953) (1961), (1972).

Although the Futures Economy is an elegant theory, it requires an embarrassingly rich set of forward markets: the history of the economy is fixed-up at the beginning of time, and the evolution of time merely unrolls the plan. Hicks was dissatisfied with the theory, and in response he attempted to formulate a model with a sparse set of forward markets. This Temporary Equilibrium model was an early attempt to formulate a theory of value containing uncertainty about the future. Using intuitive notions of risk premiums, Hicks was able to sketch out his theory, but without rigorous underpinnings, the theory was little more than a sketch.
Many of Hick's notions about risk premia, and the certainty equivalent approach found their way into the early finance theory literature. We will discuss this literature in the next section.

2. FINANCE THEORY

2.1 Finance Theory developed from the desire of businessmen and accountants to formalize the process of finance and investment planning. The early theory reflected the interests of the accounting profession in the approach and application of techniques. Nevertheless, in order to formulate investment rules, finance theory drew upon the established theory of the firm, so that by the early 1950's, there had developed a reasonable body of theory using capital-theoretic concepts and constructions. Although the rigour of capital theory was acknowledged, finance theorists found the restriction to a certain world inadequate. Most of the theory in the next decade applied certainty-equivalent arguments to introduce uncertainty.

Unfortunately, there is a dangerous tendency in the literature of this period to provide a superficial treatment of certainty capital theory before plunging into more "realistic" theories incorporating uncertainty. This superficial coverage of capital theory is perhaps the most important factor contributing to the numerous controversies in the literature of that period. The major controversies revolved about the questions:

(i) What was the objective of the corporation; and what was the appropriate investment rule?
(ii) Was there an optimal debt-equity ratio? and
(iii) Was there an optimal dividend policy?

2 For a good example of the state of the art at that time, see Dean (1951).
We will deal with each of these questions in turn.

2.2 Speculation about the corporate objective function was widespread; and, unfortunately, equally widespread were the misinterpretations and fallacies about the economist's theory of the firm and appropriate investment rules. In a timely article, Hirshleifer (1958) presented a clear exposition of the Fisherian theory, demonstrating that profit (or value) maximization was a deduction from competitive consumer behaviour under certainty.

Although Hirshleifer's contribution was significant, and removed some of the fallacies surrounding the certainty theory, there remained confusion over the cost of capital under uncertainty. Until the formulation of rigorous uncertainty models was achieved in the mid 1960's, the concept of the cost of capital was open to abuse. The best writers merely adapted the certainty model to uncertainty, by adopting the device of appropriate relabelling - even if the new labels were somewhat vague and ambiguous.

2.3 Most writers of the period acknowledged that the distinction between debt and equity vanished in the certain capital theory model. Because no formal uncertainty theory existed, there was considerable debate over the affect of variations in the degree of leverage (i.e., debt-equity ratio) on the value of the corporation. The traditional answer, based more on casual empiricism than on a formal theory, argued that the cost of capital was U-shaped with respect to leverage. Although a reasonable case could be made for a decline in the cost of capital with increases in leverage, when there is tax-deductability of interest payments on debt, no serious, formal arguments were produced to justify the U-shaped relationship.³

³ See Solomon, (1963) for a discussion and outline of the traditional argument.
Modigliani and Miller (1958) attacked the traditional argument by proving that the value of the corporation is invariant to leverage. Their proof consisted of an arbitrage argument, such that the consumer-shareholder could always undo the corporate leverage position by private anti-leverage. Uncertainty was introduced through the somewhat elusive notion of a risk-class. The ensuing debate was long and confusing. But if one reads the debate, it becomes clear that one of the chief difficulties with the Modigliani-Miller argument was its lack of a rigorous and completely well-defined theory of uncertainty.

2.4 Finally, the traditional theorists had argued that dividend policy altered the value of the corporation. It was asserted that, other things equal, a corporation with a low retention rate, would sell at a premium, compared with a corporation with a higher retention rate. Apart from some reasonable arguments based upon taxation considerations concerning differential tax-rates on income and capital gains, the so-called proofs of the proposition were obscure and fallacious. Intimately connected to the retention problem, was the debate over what the corporation capitalized - dividends or earnings?

Again, Modigliani and Miller (1961) launched an attack on the traditional position by spelling out the implications of the certainty, capital theory model. They proved

(i) that the corporation maximizes its present value; and

(ii) that the time-profile of an income stream is irrelevant to consumer-shareholders - only the present value has any significance.

Archer and D'Ambrosio (1967) contains the major contributions in the controversy.
Of course, they acknowledged divergences from their proposition introduced by tax considerations and market imperfections. Again a debate ensued; and again the lack of a robust uncertainty theory played a significant role in the charges and counter-charges of the protagonists. Until a satisfactory theory of asset market valuation under uncertainty could be formulated, the theoretical explorations had to give way to empirical investigations.

2.5 Although finance theorists were familiar with the work of Tobin (1958) and Markowitz (1959), on portfolio theory there was an uneasy co-existence between the traditional risk-premium approach and the new techniques of portfolio management. But in the mid-1960's, in the work of Sharpe (1964), Lintner (1965) and Mossin (1966), portfolio theory was integrated into a market equilibrium theory of asset prices. This work had a profound impact on finance theory, because simultaneously it introduced

(a) a rigorous uncertainty theory; and

(b) a theory which was simple enough in specification to be applied directly in empirical work.

Rather than reproduce a theory that has achieved text-book status, we will simply sketch its main features. Consider a two-period world, with uncertainty in the second period. Consumers have preferences over wealth in the first period, and over uncertain wealth in the second period. These preferences can be represented by the von Neumann-Morgenstern (1944) axioms, and the derived expected utility function. By assuming either

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5 See Archer and D'Ambrosio (1967) for the main articles in the controversy.

6 The theory is reproduced in detail in Fama and Miller (1972) and Mossin (1973).
(i) suitable restrictions on the utility function, or
(ii) restrictions to two parameter probability distributions 
    (e.g. the Normal distribution) for the returns to assets,

the preferences can be reduced to a preference over the mean and variance of 
a distribution. Assuming all consumers perceive the same return distributions for assets, and there exist competitive markets, then it is easy to set up an exchange model of asset equilibrium.

Producers can be added to the model by assuming that they produce securities. Because securities can be identified by their mean-variance characteristics, and consumers are identical with respect to their treatment of risky securities, it is not surprising that relatively simple relationships exist between the price of an asset and the mean-variance characteristics of the feasible set of assets. Furthermore, producers maximize profit (treating securities as "output"); and Lintner showed that the Modigliani-Miller leverage proposition follows as an easy corollary from the theory.

This basic asset model has been extended and modified in attempts to remove restrictive assumptions and conclusions. The literature is extensive, and expanding, with much current research underway.7

Although this theory has been an important step forward, its stringent assumptions have become an impediment to further theoretical exploration.8 This criticism should not be dismissed lightly, because a more general and robust theory for theoretical exploration has been developed by economists. The development of this theory will be the subject of the next section.

7 The book edited by Jensen (1972) contains an interesting selection of empirical and theoretical work, testing, elaborating and extending the basic asset model.

8 Criticism of the mean-variance asset model as a vehicle for theoretical exploration has been relatively common. For example see Hirshleifer (1970). Our judgement is based upon the belief that a theory of asset price determination should be quite general; and be constructed without recourse to overly restrictive assumptions that impede or distract the theorist from an understanding of the logical structure of the theory.
3. STATE-PREFERENCE THEORY

3.1 In a pioneering paper Arrow (1953), (1964), (1970) introduced the notion of describing commodities according to the state of world that will occur in the future. The set of states of the world can be thought of as the set of feasible histories of the world, where the scenario is beyond the control of economic agents. Therefore, the agents can trade in commodities, contingent upon the occurrence of a state of the world. Given this interpretation, consumers will have subjective probabilities over the set of states, but the derived indifference map over contingent commodities (given risk-averse behavior) will have the same properties as the certainty theory.

Arrow was able to derive also, the notion of primitive securities, that pay one unit of account in a given state and nothing otherwise. With a full set of primitive securities derived by suitable normalizations on the contingent commodity prices, Arrow had derived a simple, but potentially fruitful, financial asset model.

Subsequently, Debreu (1959) generalized the theory to include production. Given market prices for the contingent commodities, producers simply maximized profit over the set of production strategies. With Debreu's introduction of production possibilities, it was clear that capital theory arguments could be extended to capture an idealized form of uncertainty.

Unfortunately, the theory was ignored for some years, until Hirshleifer (1965), (1966) provided a straightforward exposition of the theory, and indicated the usefulness of the theory for finance and investment problems. In particular, he provided a very simple proof of the Modigliani-Miller leverage proposition, by considering security returns as linear combinations of the Arrow primitive securities.
3.2 The state-preference theory of Arrow-Debreu was certainly elegant, but it required an unrealistically large number of markets. In an effort to construct a security market model with fewer securities than states, Diamond (1967) presented a simple one-period, one-commodity model. The model has the same structure as Arrow's theory, except that securities are fixed linear combinations of primitive securities (or patterns of returns). Because there is a fixed set of securities with dimension less than the set of states, Diamond's model has production and consumption decisions restricted to a linear subspace of the potential primitive security space. By treating the securities as commodities, Diamond was able to construct a competitive market system. Furthermore, he was able to prove the Modigliani-Miller leverage proposition within the context of his model.

The Diamond model was a useful alternative to the mean-variance formulation of the finance theorists, because it did not require restrictive assumptions or preferences or subjective probability distributions. Because Diamond had some unfortunate ambiguities in his presentation, there was some subsequent discussion over the appropriate objective function for the producer, and the notion of constrained Pareto Optimality.9

Nevertheless, the model was an important step forward in stimulating economists to consider the problems introduced by incomplete forward markets. At the same time as the issue of incomplete forward markets in uncertainty models was being raised by Diamond, the deterministic growth models, with heterogeneous capital goods, were raising similar questions.10 Without a complete set of forward markets, the determination of asset prices, and the subsequent evolution of a market economy, must depend upon the expectations


10 For example see the discussion in Shell and Stiglitz (1967).
of economic agents. Once this point of view is accepted, a number of important questions need to be considered.

One interesting problem concerns the determination of active security markets. Whereas Diamond took the set of markets as fixed a priori, some writers have argued that market activity should be an outcome of a market system. One important source for market inactivity could be transaction costs in setting up and trading in securities. Hahn (1971), Foley (1970), Kurz (1974) and other writers have made a start in this direction.11

Because of the important role played by expectations, other writers (e.g. Grandmont (1974), Green (1973), Sondermann (1974)) have concentrated attention upon the short-run Temporary Equilibrium of Hicks (1939). At last a rigorous discussion of the Temporary Equilibrium model is under way.

3.3 The literature on asset markets under uncertainty (and uncertainty economics in general) has expanded rapidly in the last few years; and we have simply sketched some of the important contributions. The state of knowledge on the topic is changing rapidly, and we have not attempted an evaluation of the most recent contributions.12 Indeed, such an evaluation would probably be premature.

Concluding Comments:

We have attempted an outline of the main contributions to the theory of finance and investment. Although specialization has had the effect of erecting barriers between finance theorists and capital theorists, we have tried to show that these barriers are artificial: both groups of theorists are dealing with asset-price determination models.

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11 Also, this literature has attempted to produce a systematic account of the factors underlying a transactions demand for money. See Hahn (1971) for a discussion of these issues.

12 The reader is referred to two stimulating symposia edited by Dreze (1974) and Balch, McFadden and Wu (1974).
The purpose of this chapter is to present a critical analysis of corporate investment and finance theory in competitive general equilibrium. Our discussion has its basis in the general equilibrium theory developed by Arrow (1953) and Debreu (1959), and subsequently discussed by Hirshleifer (1970). It is well known that there exists a set of theorems in the certainty general equilibrium model, that show the irrelevance of corporate financial decisions for the real system, and the unambiguous nature of investment decisions. We will give general, rigorous proofs of these theorems: although once they are understood, they become simple (or even trivial) corollaries of the existence of competitive equilibrium. It is generally acknowledged that for finance theory to have any useful content, the theory must incorporate uncertainty. There are in fact very few uncertainty general equilibrium models that are directly comparable to the Arrow-Debreu theory. Nevertheless, we will show that the most commonly used of these models are nothing more than iso-morphisms on the Arrow-Debreu model. That is, by a careful definition of the commodity variable in these uncertainty models, they can be shown to be economically equivalent to the A-D theory. Given this result, the reappearance of the certainty finance and investment theorems in uncertainty models is easy to understand.

For the sake of exposition we have divided the chapter into five sections. In Section 1, we have assumed the certainty model as presented in Debreu. Given the assumptions ensuring the existence of equilibrium, we will
show the following propositions:

(i) In a private-ownership economy, corporate profit-maximization is consistent with consumer utility-maximization in that production decisions can be decided independently of the preferences of consumer-shareholders. (This result is sometimes called the Fisher Separation Theorem).  

(ii) Modigliani-Miller Theorem I: The value of the corporation is independent of its debt-equity ratio (or corporate leverage).

(iii) Modigliani-Miller Theorem II: The value of the corporation is independent of its dividend policy.

(iv) The amalgamation of existing corporations results in a value equal to the sum of the values of the constituent corporations.

These results imply that corporate finance theory is an irrelevant veil over the real system of consumption and production. The repackaging of wealth into fixed interest securities, equity, dividends, or through corporate amalgamations, does not change the consumption possibility sets of consumer-shareholders.

In Section 2, we will follow Arrow in defining commodity contracts contingent on a state of nature. Given this new definition of a commodity, and with appropriate modifications to consumer preferences, the contingent theory can be considered formally identical with the certainty model of Section 1. That is, the propositions (i)-(iv) can be translated into contingent commodities rather than certain commodities.

1 The first statement of the separation theorem was developed in the context of an intertemporal model by Irving Fisher (1930), but it has had recent reinterpretation and extension in a series of papers by Jack Hirshleifer. In a recent publication Hirshleifer (1970) has presented a summary statement of his work.
In Section 3, we will derive Arrow-Debreu securities (or Primitive securities) from the contingent commodity model. These securities are defined to pay a unit of account if and only if an agreed state of the world materializes. With a full set of these primitive securities, economic agents have a complete insurance market against the occurrence of natural phenomena. Again propositions (i)-(iv) apply with reinterpretation, but in addition we can incorporate some additional structure on the model. In particular:

(a) the possibility of bankruptcy, or default on corporate debt can be introduced;
(b) one can define unambiguous measures of certainty-equivalent rates of return, or costs of capital;
(c) the consumer can be seen to have a non-trivial portfolio problem in primitive securities; and
(d) given a set of corporate securities that span the state space, we can show the equivalence between holding primitive securities and a derived portfolio of corporate securities.

In Section 4, we will consider a model developed by Peter Diamond (1967), where there are fewer securities than states. By a judicious selection of assumptions, the model can be made formally equivalent to the Arrow-Debreu model. The key assumption required to produce this equivalence is the postulate that there exist competitive markets for a pre-determined set of patterns of returns. By further restrictions on the utility functions of consumers and their subjective probability distributions, the model collapses to the production version of the Sharpe-Lintner model.

Finally, in Section 5 we present a brief critique of this sequence of models. It should be clear that the common denominator in all these models, is the existence of a set of competitive markets for objects (or derived objects)
of choice. If this assumption fails, i.e., there is some monopoly power in the creation of securities, or commodities, propositions (i)-(iv) may no longer hold.

Before passing to the body of the Chapter, the reader should observe that each of the models discussed is a competitive equilibrium (suitably defined) so that corresponding to each, we can attach the theorem that a competitive equilibrium achieves a Pareto Optimum, (see Debreu Chapter 6). Notice that the Pareto Optimum depends upon consumer's subjective probabilities across states, and the definition of a commodity in each model.

1. CERTAINTY COMPETITIVE EQUILIBRIUM

1.1 Consider the competitive economy as developed in Debreu. Debreu defines a commodity in $\mathbb{R}^k$ space as a good or service completely specified physically, temporally and spatially. By using the device of dated commodities, the system is defined over $T$ discrete time periods. Associated with each commodity $h$, is a price $p_h$, so that one can represent a price system as a vector $p$ in $\mathbb{R}^k$. Note that price should be interpreted as the amount paid now for the transaction at time $t$. Within the commodity space one can define the action of an agent as a point in $\mathbb{R}^k$. The set of agents will be assumed to be partitioned into the set of consumers, $i = 1, \ldots, m$, and the set of producers, $j = 1, \ldots, n$.

Consider the $i$th consumer. He chooses his consumption $x_i$ from a non-empty subset of $\mathbb{R}^k$ - his consumption set $X_i$. The following assumptions are made on the consumption set $X_i$:

(a) $X_i$ is closed, convex, and has a lower bound for $\leq$. 
(b) Define on $X_i$ a complete preordering $\preceq_i$ called a preference preordering with the following properties:

(i) **Non-Satiety:** For any $x_i \in X_i$, $\exists x'_i \in X_i$ such that $x'_i \succ_i x_i$.

(ii) **Continuity:** For every $x'_i \in X_i$, the sets 
$\{x_i \in X_i | x_i \preceq_i x'_i\}$ and $\{x_i \in X_i | x_i \succeq_i x'_i\}$
are closed.

(iii) **Convexity:** If $x'_i, x''_i \in X_i$, such that $x'_i \neq x''_i$ and 
$t \in \mathbb{R}$ in $[0, 1]$, then $x''_i \preceq_i x'_i \Rightarrow tx'_i + (1-t)x''_i$.

Now the consumer facing a price system $p$, and given his wealth $w_i (w_i \in \mathbb{R})$, chooses his consumption $x_i \in X_i$ so that expenditure $p.x_i$ satisfies $p.x_i \leq w_i$.

To ensure that the intersection of the half-space defined by the wealth hyperplane and the consumption set, is non-empty, define the following two sets.

Define $S_i = \{p, w_i \in \mathbb{R}^{l+m} | \text{there is an } x_i \in X_i$ such that $p.x_i \leq w_i\}$.

Define the correspondence $\sigma_i : S_i \rightarrow X_i$ by

$\sigma_i(p, w_i) = \{x_i \in X_i | p.x_i \leq w_i; (p, w_i) \in S_i\}$.

Clearly, $\sigma_i(p, w_i)$ is non-empty.

Now define an equilibrium consumption for consumer $i$, for the price-wealth pair $(p, w_i)$, as the greatest element in the set $\sigma_i(p, w_i)$ for his preference ordering $\preceq_i$. To ensure that an equilibrium exists define
24.

\[ S'_i = \{ (p, w_i) \in S_i \mid \sigma_i (p,w) \text{ has a greatest element for } \leq_i \} \]

Although one requires a number of assumptions on the set of feasible production for producers to prove that a general equilibrium exists, they can be omitted for the partial equilibrium separation theorem. Define \( Y_j \), the set of feasible productions, as a non-empty subset of \( \mathbb{R}^k \), for the producer \( j \).

Define the profit of the \( j \)th producer in a price system \( p \), as \( py_j \). Now define the set

\[ T'_j = \{ p \in \mathbb{R}^k \mid p.Y_j \text{ has a maximum} \}. \]

In a private-ownership economy profits are distributed to consumers as shareholders of the production units. If it is assumed that consumers are endowed with an initial bundle of resources \( w_i \in \mathbb{R}^k \), as well as the returns from firms, the \( i \)th consumer's wealth can be written as

\[ w_i = p.\omega_i + \sum_{j=1}^{n} \theta_{ij} p.y_j, \]

where

\[ 0 \leq \theta_{ij} \leq 1, \quad \text{and} \quad \sum_{i=1}^{m} \theta_{ij} = 1. \]

With the aid of a lemma from Debreu the separation theorem can be proved for the \( m \)th consumer who holds an entitlement of \( \theta_{nn} > 0 \) of the \( n \)th firm's profits. The theorem takes the price system as given, and is a partial equilibrium result.
Theorem 1.1: (Fisher Separation Theorem)

Assume a private-ownership economy with an associated price vector \( \tilde{p} > 0 \), such that \( \tilde{p} \in T_j \cap S_i \). Assume a consumer with consumption set and preferences that satisfy assumptions (a), (b) ((i), (ii), (iii)). Then the \( m \)-th consumer will prefer the consumption equilibrium \( x^* \) associated with the profit-maximizing production plan \( y^* \) for firm \( n \) (for which \( m \) holds a profit claim \( a_{mn} > 0 \)), over any other consumption equilibrium \( x^0 \), associated with production plans, \( y_n^0 \in Y_n \), \( x^0_n \neq y^*_n \).

Proof:

From the definition of wealth in a private-ownership economy, and the assumption that \( y^* \) maximizes profit given \( \tilde{p} \), it follows that

\[
\tilde{p} \cdot w^* = \tilde{p} \cdot \omega_m + \sum_{j=1}^{m-1} \tilde{p}_j \cdot y_n^0 + \sum_{j=1}^{m-1} \tilde{p}_j \cdot y_n^0 > \tilde{p} \cdot \omega_m + \sum_{j=1}^{m-1} \tilde{p}_j \cdot y_n^\circ + \sum_{j=1}^{m-1} \tilde{p}_j \cdot y_n^\circ = \omega^0_m.
\]

Note that it has been assumed implicitly that the production plans \( y^*_j \), \( j = 1, \ldots, m-1 \) are given.

It should be clear that the intersection of the half-spaces

\[
A = \{ x_m \mid p \cdot x_m > \omega^0_m \} \text{ and } B = \{ x_m \mid p \cdot x_m < \omega^* \}
\]

is non-empty.

Now consider a result from Debreu.

Lemma (Debreu 4.9(2'))

Under the assumption of convexity, (b) (iii) for \( \geq_m \); given \( (p, w) \in S_m \) such that \( x'_m \) is a greatest element of \( g_i(p, w) \) for \( \geq_m \); and assuming non-satiety (b) (i), then \( p \cdot x'_m = \omega_m \).
Figure 1
From the lemma it follows that $p.x^*_m = w^*_m$ and $p.0^*_m = w^*_m$, but because $A \cap B \neq \emptyset$, it is implied that $0^*_m$ is interior to $B$. But the lemma states that the most preferred point is on the boundary of $B$. Therefore, $x^*_m \supset 0^*_m$. 

The proposition can be illustrated in $\mathbb{R}^2$. (Figure 1 is an adaptation of Debreu, Chapter 4, Figure 7). 

This theorem is a partial equilibrium proposition, and says nothing about the clearance of markets or the determination of a general equilibrium price system $p^*$. By examining the definition of an equilibrium of the private-ownership economy it is obvious that the separation theorem holds for every consumer $i$, with respect to every firm $j$.

Definition: (Debreu 5.5)

An equilibrium of the private-ownership economy $\mathcal{E}$ is an $(m + n + \ell)$-tuple $((x_i^*), (y_j^*), p^*)$ of points of $\mathbb{R}^\ell$ such that:

(a) $x_i^*$ is a greatest element of

$$\{x_i \in x_i \mid p^* x_i \leq p^* \omega_i + \sum_{j=1}^{n} \delta_{ij} p^* y_j\}$$

for $\leq \mathcal{L}_i$, for every $i$.

(b) $y_j^*$ maximizes profit relative to $p^*$ on $Y_j$, for every $j$.

(c) $x^* - y^* = \omega$.

The conditions required for a private-ownership economy to have an equilibrium are stated in theorem 5.7(1) of Debreu: they are the conditions on the consumption set (a), (b) ((i), (ii), (iii)), plus conditions on the production sets, and a requirement on initial endowments. Because the same
consumption conditions appear in the separation theorem and in the general equilibrium theorem, one could rewrite the separation theorem for an equilibrium price vector $p^*$, rather than a general $\bar{p}$. Also note that the lemma holds for the full equilibrium, so that the wealth constraints hold for equality for all consumers.

1.2 The existence of the separation theorem ensures that by following the rule of maximizing the value of the corporation, the controllers of production decisions will be operating in the best interests of the shareholders. We should make it clear that the expression for corporate value, $p_{y_j}$ is quite general and can be manipulated to obtain the more familiar discounting rules. To derive these rules, we need to transform the price vector to obtain the full structure of interest rates.\(^2\)

The "own interest rate" for good $h$, from period $t$ to $t+1$, can be defined as the solution to

$$p_{h,t+1} = [1 + i_{h,t}]^{-1} p_{h,t}.$$  

Similarly, the own discount factor $\beta_{ht}$, can be defined by

$$\beta_{h,t+1} = [1 + i_{ht}]^{-1} \beta_{ht}.$$  

Clearly, the own discount rates/factors are not necessarily equal, for different commodities. The usual notion of "the" rate of interest applies to a numeraire commodity - let us assume it to be commodity 1 in period 1, i.e., $p_{11} = 1$.

\(^2\) This section draws on the discussion in Chapter 10 of Malinvaud (1972), which in turn is drawn from Malinvaud's earlier paper (1953).
Thus define the discount factor $\beta_t$ by $P_{1t} = \beta_t$. Now define a set of undiscounted prices $\tilde{p}_t$ which are proportional to $p_t$, and where $P_{1t} = 1$, such that $p_t = \beta_t \tilde{p}_t$. This relationship gives us forward and spot prices in terms of the discount factor of the numeraire commodity.

Having derived the interest rate structure from the price vector ruling in the certainty futures markets, we can rewrite the value of the corporation in the familiar form

$$\begin{align*}
PY^*_j &= \sum_{t=1}^{T} \beta_t \tilde{p}_t y^*_j \\
&= \sum_{t=1}^{T} [1 + i_t]^{-1} p_t y^*_j
\end{align*}$$

where

$$y^*_j \in Y_j.$$

Although this formulation takes on the familiar discounted form, it is quite general in its specification of undiscounted profits: obviously, exponential growth of undiscounted profits, or other common formulations, are special cases of the optimal investment policy. By making one more simple, but significant assumption, we can obtain some further results.

Consider the production decisions of a firm over two successive periods $t$, and $t+1$. Total inputs in period $t$ we denote by the vector $a_{jt}$; and total outputs in period $t+1$ by $b_{jt+1}$. These production vectors are defined such that

$$v_{jt} = b_{jt} - a_{jt}; \quad b_{jl} = 0; \quad a_{jt} = 0.$$
The vector $a_{jt}$ should be interpreted as the input of all factors of production, including second-hand machinery, work-in-progress, etc. The vector $b_{jt}$ of outputs, should be thought of as the production of all outputs, defined in a similar way to inputs. Thus, for example, a piece of equipment used as an input, is transformed by the production process into a new commodity appearing in the output vector one period later. Assuming prices exist for all inputs and outputs, we can rewrite corporate value as

$$py^*_j = \sum_{t=1}^{T-1} (p_{t+1}^* b_{jt+1}^* - p_t^* a_{jt}^*)$$

Notice that the input cost includes an interest component to make it comparable with the value of output received one period later. Now it is easy to see that maximizing the value of the corporation is equivalent to maximizing the individual intertemporal profits $\Pi_t$. Because all inputs and outputs have market prices, production decisions take on a myopic character. In this particular case, maximization of short-run profit produces identical production decisions as long-run profit maximization. Because prices exist for all commodities in the production process, the corporate accountant's problem of computing values for assets, vanishes, and his task collapses to that of recording market values.

To take another interpretation of the formulation, we can consider the firm to be a separate entity from the $jt+1$ firm, that is, corporations exist only for one period. Because production decisions between periods are

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3 The myopic property was observed by Malinvaud (1953), and has received a more extensive discussion by Arrow (1964a).
independent, we can consider the number of corporations to be expanded to \( n(T-1) \). Redefining the vector \( \theta_i \) to include elements \( \theta_{ij} \), we have a system formally identical to the original model, except that firms are defined intertemporally.

1.3 By suitable interpretation of price and commodity vectors we can demonstrate that financing decisions are quite independent of production decisions. Indeed, we will show that they are rules for dividing up wealth, and are a matter of indifference for economic agents.

Consider first the question of financial leverage. A considerable literature exists arguing for and against the proposition that the proportion of debt to equity finance does not effect the value of the corporation. The original proof of the assertion was given by Franco Modigliani and Merton Miller (1958) and involved an arbitrage argument, given corporate returns and values. We can give a transparent proof of this proposition in terms of the Arrow-Debreu model. (The following proof is based on the argument in Hirshleifer (1970), Chapter 9).

Theorem 1.3: (Modigliani-Miller Theorem 1)

Given a competitive equilibrium, the value of the corporation is invariant to the proportion of debt to equity financing.

Proof:

Given the equilibrium vectors \( y^*_1 \), \( p \), define \( y^I_j < 0 \) and \( y^0_j > 0 \) such that

\[
y^*_j = y^I_j + y^0_j,
\]
i.e. partition the production vector into an input vector and an output vector. Further, partition the inputs into debt $d_j^I < 0$, and equity $e_j^I < 0$ inputs.

$$y_j^I = d_j^I + e_j^I$$

Similarly, define debt $d_j^0 \geq 0$ and equity $e_j^0 \geq 0$ outputs.

$$y_j^0 = d_j^0 + e_j^0$$

The value of debt financing $D$ and equity financing $E$ can be obtained by summing the inner products of appropriately defined price and commodity vectors

$$D = p^I . d_j^I + p^0 . d_j^0$$

$$E = p^I . e_j^I + p^0 . e_j^0$$

$$\therefore D + E = p^I (d_j^I + e_j^I) + p^0 (d_j^0 + e_j^0)$$

$$\therefore D + E = py_j^*.$$ 

The last line proves the assertion. ||

The statement and proof of the theorem has minimal structure. It is not difficult to incorporate interest rates to obtain more familiar expressions; but these add nothing to the proof. We should observe that in a certainty world, debt and equity are indistinguishable, and in that sense the proof is not surprising.
In a later paper Modigliani and Miller (1961) proposed that, in equilibrium, the value of the corporation was invariant to dividend policy. That is, given the production plan $y^*_j$ that maximizes corporate value, dividends could be retained to finance next period's investment, or paid out with a simultaneous capital raising, without altering the corporate value. In our general equilibrium framework the proof is easy.

**Theorem 1.4: (Modigliani-Miller Theorem 2)**

Given a competitive equilibrium the value of the corporation is invariant to dividend policy.

**Proof:**

For simplicity, consider the case of pure equity financing. Define the value of the corporation at time $T$ as

$$V_{jT} = \sum_{t=1}^{T-1} \beta^t P_t y^*_t = \sum_{t=T}^{T-1} v_{jt}.$$  

Defining $D_{jT}$ as the payment of dividends, and $Z_{jT}$ as the raising of new equity finance, we must have in equilibrium

$$V_{jT} = D_{jT} + V_{jT+1} - Z_{jT}.$$  

But the cash flow should satisfy the constraint

$$v_{jT} + Z_{jT} = D_{jT}.$$  

Therefore substituting for $D_{jT}$ we obtain
\[ V_{jt} = V_{jt} + V_{jt+1} \]

which is independent of the choice of dividend policy.

Although the notation is different, this theorem has the same economic content as Theorem 1.3. The consumer is not interested in the packaging of wealth, because his sole interest is in total wealth. Even though the pattern of returns for his assets may be quite bizarre, they are perfectly exchangable at market prices for his chosen pattern of consumption.

Theorems 1.3, 1.4 apply to the formation of asset combinations within a single firm, but the same principle can be applied across firms to demonstrate that the formation of conglomerates provides no economic advantages: the whole is simply the sum of the parts.\(^4\) To see this, consider a new firm \( c \), formed by summing over existing firms 1, ..., \( n' \), where \( n' \leq n \). The conglomerate firm's problem becomes

\[
\text{Max}_{\{y_c \in Y_c \}} \sum_{j=1}^{n'} p Y_j = \text{Max}_{\{y_c \in Y_c \}} \sum_{j=1}^{n'} p Y^*_j, \quad \text{where } Y_c = \sum_{j=1}^{n'} Y_j.
\]

But this problem is equivalent to

\[
\text{Max}_{\{y_j \in Y_j \}} \sum_{j=1}^{n'} p Y_j = \text{Max}_{\{y_j \in Y_j \}} \sum_{j=1}^{n'} p Y^*_j.
\]

Thus

\[
p Y^*_c = \sum_{j=1}^{n'} p Y^*_j.
\]

\(^4\) This result has been shown by Debreu (3.4(i)) for the case of all firms.
Because all agents are price-takers, the only possible justification for corporate mergers is the creation of new production opportunities by amalgamation. We can formalize this process by assuming that

\[ Y_c = \bigcup_{j=1}^{n'} Y_j, \quad \text{and} \]

\[ Y_c = \bigcup_{j=1}^{n'} Y_j \cup A, \quad A \neq \emptyset; \]

and given the usual assumptions on production sets \( Y_j, A \) (see Debreu Chapter 3) it is easy to show that

\[ PY_c^* \geq \sum_{j=1}^{n'} PY_j^* \]

The discussion in this section can be summarized by:

\textbf{Lemma 1.5}

The amalgamation of existing corporations results in a value equal to the sum of the values of the constituent corporations.

2. CONTINGENT COMMODITY EQUILIBRIUM

2.1 In Section 1 we considered a general equilibrium model in which the environment of the economy was known by all agents, for all time \( t = 1, \ldots, T \). We cannot, of course, perfectly foresee the future, but one can prepare for possible contingencies. By expanding the commodity space to include in the definition of a commodity the occurrence of a state of nature, uncertainty can be introduced into the analysis.

5 Presumably the new production opportunities are derived from a more fundamental production set exhibiting initial scale economies.
Consider commodities defined as the physical commodity $h$, at time $t$, given the occurrence of the natural state $k$. As time unrolls, the progress of the economy can be recorded as a time-state sequence $\{e_{tk}\}$. Assuming all agents agree on the set of possible time-state sequences, we can compress the sequence notion into the summary "states of the world", $b \in S$, where $S$ is the set $\{1, \ldots, s\}$ of all possible environmental histories of nature. Notice, we have been careful to define the states of the world as acts of nature beyond the control of economic agents. Economic agents are restricted to actions defined within the commodity-space.

The price of a commodity must be redefined as the present payment for a contingent commodity. That is, commodities should be viewed as contracts specifying the delivery or acceptance of a specific number of units of a physical commodity, if and only if, a particular state of the world occurs. Thus, in the first time period when all decisions are made, contracts are made in terms of commodities contingent on the occurrence of a particular state of the world.

2.2 The producer's problem can be interpreted to be formally identical with the certainty analysis of Section 1. Because the prices of contingent commodities are determined in the market, the producer maximizes profit by choosing a contingent production plan from the production space $Y_j$. The production set should be considered to be contained in the contingent commodity space. Observe that the producer does not have to take account of uncertainty, nor have any attitude to risk.

2.3 The consumer's problem is also identical with the certainty formulation: the consumption set is expanded in dimension to accommodate contingent commodities; and the preference map has contingent commodities as objects of
choice. Although the uncertainty preference map has a superficial resemblance to the certainty preference map, it must incorporate two elements characteristic of the uncertainty problem:

(a) The consumer must have some subjective notion of the probability or expected frequency of different states of the world; and

(b) He must have some attitude to uncertainty in the sense that he prefers more or less risky situations.

These notions have been formalized in the axiomatic foundation of the von Neumann-Morgenstern Utility theorem. Given the axioms, the consumer's preferences can be summarized by an expected utility function of the form

\[ U_i = \sum_s U_i(x_{is}) \pi_{is}; \quad \sum_s \pi_{is} = 1, \]

where \( U_i(x_{is}) \) is a function of planned consumption in state \( s \); and \( \pi_{is} > 0 \) is the subjective probability of the occurrence of the \( s \)th state of the world.

By specifying the \( \pi_{is} \), \( \forall_{s} \), the consumer's subjective probabilities have been attached to each state of the world. Also it is well known that attitudes to risk have implications for the concavity or convexity of the function \( U_i(x_{is}) \). Assuming risk aversion (which implies concavity of \( U_i(x_{is}) \)),

---

6 Since the original contribution (1944), there has been a number of derivations of the theorem. For example see Essay 2, Arrow (1970), or the discussion in Hirshleifer (1970).

7 Notice that the utility function is common for all states of the world. This is a direct result of (what Hirshleifer calls) the "Uniqueness Postulate". For a discussion of its implications see Hirshleifer, (1970), pp.220-1.

the expected utility function \( U_i \) is quasi-concave - or equivalently, the preference map in \( x_i \) has convex indifference surfaces. Formally, we have:

**Theorem 2.3:**

Given the von Neumann-Morgenstern utility function

\[
\sum_{s \in S} U_i(x_{is})
\]

concavity of \( U_i(x_{is}) \) implies convex indifference surfaces.

**Proof:**

The proof is an easy consequence of two well known propositions, (see Berge (1963) Chapter VIII):

(a) If \( f_1, f_2, \ldots, f_m \) are concave functions in the convex set \( C \subseteq \mathbb{R}^n \), and if \( \phi \) is an increasing concave function in \( \mathbb{R}^m \), then the function

\[
g(x) = \phi[f_1(x), \ldots, f_m(x)]
\]

is concave in \( C \).

(b) A concave function defined in a convex set \( C \subseteq \mathbb{R}^n \) is also quasi-concave in \( C \).

Because an increasing linear combination is a special case of the function \( \phi \), the proof follows directly. ||
Figure 2
We can illustrate the theorem for the case of a simple commodity, two states of the world, and given expected utility $U_1$, as in Figure 2.

2.4 Thus, given risk aversion, the contingent commodity model has a formal identity with the theory of Section 1. Not only does an equilibrium exist, but so do all the Theorems relating to investment and financing. This result should not be surprising, because if one could imagine a complete set of markets effectively insuring against natural uncertainties, the essential theory is unchanged.

3. SECURITIES MARKETS

3.1 Arrow (1970) observed that by taking a transformation of contingent prices, he was able to form a special type of security market, where there were as many securities as states of nature. Each security was defined to pay a unit of account in one state of nature, if and only if, that state occurred. Thus with a full set of securities, agents could partition their decisions into two:

(a) the agent would decide on the number of state securities to be held; and

(b) he would decide on consumption/production plans given the occurrence of any state. The choice over state securities (or Arrow-Debreu securities) is a special case of a more general portfolio problem, where agents manipulate assets rather than commodity claims. We will show the relationship between the underlying objects of choice (commodity claims) and the instruments of exchange (assets).
3.2 Consider the contingent price vector \( \hat{p} \) from the previous section. Partitioning the vector \( \hat{p} \) over states, take \( \hat{p}_s \) as the contingent price vector for state \( s \). Now consider the following transformation and normalization,

\[
\hat{p}_s = p_s \hat{p}_s, \quad \forall s; \quad \text{and}
\]

\[
\sum_s p_s = 1; \quad p_s \geq 0, \quad \forall s.
\]

We can think of \( \hat{p}_s \) as the price of commodities given the occurrence of state \( s \): it is a conditional price vector. The price \( p_s \) can be thought of as the price of a security returning a unit of wealth if and only if, state \( s \) occurs.

3.3 The producer's problem can be rewritten as

\[
\max_{\{y_{sj} \in Y_{sj}\}} \sum_s p_s \hat{p}_s y_{sj} = \sum_s p_s \hat{p}_s y^*_{sj}.
\]

As we have indicated in 3.1, the producer can divide his problem into two sub-problems. Given the conditional prices \( \hat{p}_s \) for each state \( s \), the producer chooses a production strategy yielding a conditional profit \( \hat{p}_s y^*_{sj} \). To obtain the value of the corporation, the conditional profits must be weighted by the market evaluation of wealth in each state. Alternatively we can think of the firm as supplying \( \hat{p}_s y^*_{sj} \) of the \( s \)th security, across states. The value of the corporation becomes the market value of the state securities it supplies.

Because the prices \( p_s, \hat{p}_s \) are market ruling prices, subjective evaluations are ruled out of production decisions. One could think of the set of security prices \( \{p_s\} \) as the market "probability" estimate of different
states (the set \{p_s\}, by construction, has the same properties as a probability
distribution) so that the corporate value is a market expected value. To see
the divergence between the market probability distribution and the subjective
probability distribution, observe that for consumer i, the expected value of
the corporation j is

\[ E_i(p_s y^*) = \sum_p \Pi_{is} \hat{p}_s y^* \cdot \]

We can relate the market value and the expected value by

\[ \sum_s p_s \cdot \hat{p}_s y^* = \alpha_i E_i(p_s y^*) \]

where \( \alpha_i \) is the \( i^{th} \) consumer's risk discount factor required to equate his
evaluation of expected returns to market valuation. Reflection reveals that
\( \alpha_i \approx \frac{1}{\Pi_{is}} \), depending upon the divergence between \( p_s \) and \( \Pi_{is} \). The relationship is
in the general form of a required rate of return-cost of capital formulation,
but it should be clear that \( \alpha_i \) is a derivative measure depending upon the \( i^{th} \)
consumer's subjective probability distribution.\(^9\)

3.4 We can summarize the consumer's problem by

\[ \text{Max} \sum_s U_i(x_{si}) \Pi_{is} \]

\[ \{x_s\} \]

\[ \text{s.t.} \sum_s p_s \cdot \hat{p}_s x_{si} \leq \sum_s p_s (\hat{p}_s w_i + \sum_j \hat{p}_j y_{sj}) = W_i \]

\(^9\) For a discussion and critique of certainty equivalent measures and convent-
ional valuation formulae see Myers (1968).
Because the utility function is additive we can divide the consumer's problem into the choice of conditional consumption in state $s_j$ and into the choice of securities across states. Defining $q_{si}$ as the conditional wealth in state $s$, consider the sub-problem

$$\begin{align*}
\text{Max } & U_i(x_{si}), \\
\{x_s\} & \\
\text{s.t. } & \hat{p}_s x_{si} \leq q_{si}.
\end{align*}$$

Replacing $p, w_i$ in the set definitions of Section 1 by $\hat{p}_s, q_{si}$, we can apply a result by Debreu.\(^{10}\) In essence it says that, assuming convexity of preferences, the demand correspondence $x_{si} = x_{si}(\hat{p}_s, q_{si})$, is upper semi-continuous, and the indirect utility function $v_i(\hat{p}_s, q_{si})$ is continuous. From Theorem 1 we know that $v_i$ is monotonically increasing in $q_{si}$; and by assumption, $U_i(x_{si})$, is strictly concave. Therefore, $v_i$ is an increasing, strictly concave function of $q_{si}$.

The general problem can be rewritten in terms of securities as

$$\begin{align*}
\text{Max } & \sum_s v_i(\hat{p}_s, q_{si}) w_i, \\
\{q_s\} & \\
\text{s.t. } & \sum_s \hat{p}_s q_{si} \leq w_i.
\end{align*}$$

Notice that the securities problem bears a close resemblance to the usual simple statement of Arrow-Debreu securities in terms of "wealth". If the conditional prices are the same for all states (i.e. $\hat{p}_s = \hat{p}, V_s$), we can drop any reference to the equilibrium conditional price vector, and write the

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\(^{10}\) See Debreu (4.10(i)).
problem in the familiar form, where \( v_i(p, q_{s1}) = v_i(q_{s1}) \) can be considered as
a von Neumann-Morgenstern utility function exhibiting risk-aversion. One can
derive further results on the relationship between security demands and wealth
through the addition of restrictions on \( v_i \).

If, on the other hand, conditional prices are not equal across states,
the objective function cannot be considered as a von Neumann-Morgenstern
utility function. Because conditional prices vary across states, there is an
ambiguous definition of wealth between states. This point is closely related
to one of the axioms (either explicitly or implicitly) stated in deriving the
expected utility theorem. It is assumed that preferences are defined over
consequences which are independent of their state labelling. Because pre-
ferences are defined over contingent commodities, which are assumed independent
of state labelling, one can derive preferences over wealth, if and only if,
relative prices are equal across states, so that exchange conditions within
states are directly comparable. In general, the usual measures of risk based
upon the elasticity of the utility function will be a function of relative
conditional prices.

3.5 It follows easily that the derived securities markets clear as a
direct result of equilibrium in contingent markets. From market clearance, we
obtain

\[
x_S = \omega_S + \sum_j y_{s_j}^*, \quad \forall_S
\]

---

11 See Cass and Stiglitz (1970) for necessary and sufficient conditions on
   utility for portfolio separation.

12 See footnote 7.

13 See Stiglitz (1969b), for a discussion of how prices enter as parameters
   into the usual measures of risk aversion.
That is, securities held by consumers, $q_s$, must equal the supply of securities produced by firms and original endowments.

Because the securities market is a transparent derivative of the contingent commodity model of Section 2 (and in turn, a reinterpretation of the certainty model of Section 1) the theorems relating to finance and investment continue to apply with appropriate reinterpretation. For example, the leverage theorem can be adapted to incorporate the possibility of bankruptcy. The following simple proof is due to Stiglitz (1969a).

Partition the set of states such that

\[ S = \{ s \mid \hat{p}_s y^* > r_j B_j \}, \]

and

\[ S' = \{ s \mid \hat{p}_s y^* < r_j B_j \}, \]

where $r_j$ is the contractual return on the value of bonds $B_j$. The value of bonds is given by

\[ B_j = \sum_{s \in S} p_s r_j B_j + \sum_{s \in S'} \hat{p}_s B_j y^*; \]
That is, securities held by consumers, $q_s$, must equal the supply of securities produced by firms and original endowments.

Because the securities market is a transparent derivative of the contingent commodity model of Section 2 (and in turn, a reinterpretation of the certainty model of Section 1) the theorems relating to finance and investment continue to apply with appropriate reinterpretation. For example, the leverage theorem can be adapted to incorporate the possibility of bankruptcy. The following simple proof is due to Stiglitz (1969a).

Partition the set of states such that

$$S = \{s \mid \hat{p}_{s,s} y^* \geq r_j B_j \},$$

and

$$S' = \{s \mid \hat{p}_{s,s} y^* < r_j B_j \},$$

where $r_j$ is the contractual return on the value of bonds $B_j$. The value of bonds is given by

$$B_j = \sum_{s \in S} p_{s,s} r_j B_j + \sum_{s \in S'} \hat{p}_{s,s} y^*. $$
and the value of equity by

\[ E_j = \sum_{s \in S} p_s (\hat{p}_s y^*_s - r_j B_j). \]

Thus

\[ E_j + B_j = \sum_{s} p_s \hat{p}_s y^*_s \equiv V_j. \]

3.6 Cass and Stiglitz (1970) have shown that Arrow-Debreu securities can be derived from an asset model where there are as many independent securities as states. In other words, both sets of securities form a basis for the

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It should be clear from the bond valuation equation that when \( S' = \phi \), then \( r_j = 1 \), because by assumption

\[ \sum_{s \in S} p_s = 1. \]

Also, we can show from the bond valuation equation that with bankruptcy, \( r_j \) is a non-trivial function of \( B_j \). To show this, consider the extreme case where the set of states is the closed interval \([s, \bar{s}] \subset \mathbb{R}\). Thus subscripts become arguments of functions: if we assume continuity and differentiability, summations become Riemann integrals. (For a similar formulation see Diamond). The valuation equation can be written as

\[
B_j (1 - \int_{s^*}^{s} p(s) r_j ds) - \int_{s^*}^{s} p(s) \hat{p}(s) y_j(s) ds = 0
\]

where \( s^* \) is the solution to \( \hat{p}(s) y_j(s) - r_j B_j = 0 \). Differentiating with respect to \( B_j \) we deduce

\[ \frac{dr_j}{dB_j} > 0. \]

That is, assuming \( s^* \in (s, \bar{s}) \), the contractual return is a positive function of leverage. When \( s^* = \bar{s} \), then \( B_j = V_j \) and \( E_j = 0 \).
primary security space. From Section 3.5 we showed that security markets cleared, and therefore we can write

\[ \sum_s p_s q_s = \sum_s \bar{p}_s \bar{q}_s + \sum_j \sum_s p_s \bar{q}_{sj}. \]

Defining the value of the \( j \)th firm as

\[ V_j = \sum_s p_s \bar{q}_{sj} \]

we can write in matrix notation \( V = p^t [q_s] \), where \( [q_s] \) is the \( s \times n \) matrix with typical element \( q_{sj} \). Now assuming that \( n = \bar{s} \), and that firms offer patterns of independent returns across the states, we know that \( [q_s] \) is invertable. Therefore, by substitution we obtain

\[ V[q_s]^{-1} q_s = V[q_s]^{-1} \bar{q}_s + Ve, \]

where \( e \) is the unit vector.

**Now define the asset vector \( \bar{A} \) as the unique solution to \( [q_s] \bar{A} = \bar{q}_s \); and the endowed asset vector \( \bar{A} \) as the unique solution to \( [q_s] \bar{A} = \bar{q}_s. \)** Thus, we obtain

\[ VA = V(\bar{A} + e). \]

---

15 In this particular context, the inclusion of an endowment vector may appear somewhat artificial; but we have retained it to keep conformity with the Debreu model, within which the endowment vector plays a crucial role in the existence proof. See Debreu Chapter 5.
Within this framework, the consumer's problem can be redefined (assuming conditional prices $p_s$ are equal across states) as

$$\max \sum_{s} v_s (\{q^*_s \} A_i)_{i s}$$

subject to $V A_i = V^0 I e + V A_i$

where $A_i$ is the vector of corporate securities held by the $i^{th}$ consumer; $V^0_i$ is this initial endowment of corporate securities; and $I$ the unit matrix.

The problem is now of the general asset form where the choice variables are assets offering pattern of returns across the state space. Nevertheless, observe that this is a special case which implies that a unique choice of Arrow-Debreu securities implies a unique choice of corporate securities for each consumer. If there are more corporate securities than states ($n > s$) then a unique Arrow-Debreu portfolio maps onto an indeterminate number of corporate portfolios. Alternatively, if $n < s$, the Arrow-Debreu portfolio will not, in general, be achievable by holding corporate securities. Nevertheless, there is a special case where the Arrow-Debreu framework can be applied, and where there are fewer assets than securities. We will examine this case in the next section.

16 The relationship between the asset model, the Arrow-Debreu model, and the derivation of Slutsky equations for assets has been discussed by Fischer (1972).
4. SECURITIES WITH FIXED PATTERNS OF RETURNS

4.1 In an important contribution, Diamond (1967) formulated a model with fewer securities than states of nature. He assumes a one-commodity model with only one market day before technological uncertainty is resolved. Firms can be considered to produce patterns of returns across states. Because consumers cannot trade directly in contingent commodities, but are constrained in their trades to linear combination of firms assets, their problem becomes the choice of an optimal portfolio of corporate assets. In the model's most general form, the objective function of the firm is not well-defined independently of consumer's preferences. That is, the Fisher Separation Theorem fails to apply. Nevertheless Diamond investigated a special case where separation does occur: the objective function is independent of preferences when the firm cannot change the pattern of its returns, but only its scale. As Diamond observes, the key to this result is the realization that the fixed patterns of returns can be regarded as a set of commodities, so that the model is equivalent to the Arrow-Debreu model. It is this interpretation that we wish to explore in more detail.

4.2 Consider the contingent commodity model of Section 2, with the added restrictions, that there are only two periods, one physical commodity, and \( s \) possible states of the world in period two. If markets exist for all contingent commodities, we are back in the analysis of Section 2. (Notice that the model has a commodity space restricted to dimension \( s + 1 \); and the dividend theorem (1.4) is trivially vacuous). Conversely, assume that contingent markets do not operate, and trade is limited to assets with non-negative independent, predetermined patterns of returns, \( z_1, \ldots, z_k \).
The producer's problem can be rewritten as

$$\text{Max} \quad P_c \cdot Y_{lj} + P_a \cdot a_j = P_c \cdot y_j^* + P_a \cdot a_j^*$$

where $P_c$ is the price of the consumption good in period one;

$P_a$ is the price vector of assets;

$Y_{lj}$ is the input of the first period commodity by firm $j$; and

$a_j$ is the production of $l$ assets by firm $j$.

We can summarize the production technology by assuming

$$\begin{bmatrix} Y_{lj} \\ \vdots \\ a_j \end{bmatrix} \in A_j.$$

Clearly, the producer's problem is equivalent to that outlined in Section 1.

The consumer's problem is only marginally more complicated. His budget constraint is re-translated as

$$P_c x_{li} + P_a a_i = P_c \omega_c + P_a \omega_a + \sum_j \delta_{ij} (P_c y_j^* + P_a a_j^*).$$

To translate the consumer's preference map from the original contingent commodities to assets, consider the linear mapping defined by

$$z = [z_1, \ldots, z_k].$$
Because linear mappings are continuous, and map convex sets into convex sets of equal or less dimension, the consumer's preferences can be redefined in terms of the new \( \mathcal{Q} \) "commodities", and the first period consumption good. We can write this more explicitly in the following way. The original consumer's utility in terms of contingent commodities is

\[
U_i = \sum_s U_i(x_{1i}', x_{2si})_{is}.
\]

But the consumer chooses an asset vector \( a_i \) to obtain contingent commodities \( x_{2i} \) by way of the linear transform

\[
x_{2i} = za_i.
\]

Substituting directly for \( x_{2i} \), we obtain the utility function

\[
U_i(x_{1i}', a_i) = \sum_s U_i(x_{1i}', za_i)_{is},
\]

where \( Z_s \) is a row vector of \( Z \).

4.3 Given that economic agents cannot alter the patterns of returns, the model is equivalent to the Arrow-Debreu model. Thus all the results of Section 1 (with the exception of the dividend theorem which is vacuous in a two-period model) apply. Notice that the firm is not limited to the production of a single pattern of returns, but can operate up to \( \mathcal{Q} \) "processes" depending upon the profitability of each individual process. Also observe that, apart
from the restriction of risk-aversion on the utility function, the form of that function as well as the subjective probability distributions are quite general. Thus, by placing stronger assumptions on preferences to obtain special results, we are not altering the basic theorems.

For example, if we assume that the utility function is restricted to a form that yields portfolio separation (e.g., the quadratic function), and that all consumers have identical subjective probability distributions, then the model collapses to the Sharpe (1964) - Lintner (1965) formulation. The gain from these strong assumptions is the extra result that, given a riskless asset, all consumers will divide their portfolios between the riskless asset and a market portfolio of the risky assets. It should be clear from our discussion that the investment and financing theorems are quite independent of the market portfolio result.

4.4 In 4.3 we prefaced our comments by the condition that agents could not alter the patterns of returns. But if by trading or production decisions, agents can create new assets, it is no longer obvious that the finance and investment theorems are applicable. To see this point, we will consider the attainable consumption set implied by the available securities. We will demonstrate how the creation of securities by

(i) production decisions;
(ii) portfolio and leverage decisions with a risky bond;

alters the attainable consumption sets.

17 With production decisions, this restriction on the pattern of returns, is quite straightforward; but with portfolio and leverage choice the situation is more complicated. By admitting limited liability relationships, corporate leverage and personal portfolio decisions can create new patterns of returns. For example, if the private investor takes a sufficiently short position in a security, with a positive probability of bankruptcy, he is creating a different security from the original security he is selling short. These questions are discussed in detail in Chapters 5 and 6.
For our analysis we require two results by Fischer (1972).

**Theorem 4.4:**

The returns matrix $Z$ of an asset problem with fewer assets than states of nature, defines a unique attainable consumption set in the contingent commodity space.

**Proof:**

Given $x_2 = Z a$, partition $Z$ into the $l \times l$ matrix $[Z_l]$ and the $r \times l$ matrix $[Z_r]$, where $r = s - l$. Similarly, partition the vector $x_2$; and after rearranging we obtain

$$[- [Z_r] [Z_l]^{-1} : I] x_2 = 0.$$  

We can define the obtainable consumption set as

$$C = \{ x_2 \mid [- [Z_r] [Z_l]^{-1} : I] x_2 = 0 \}.$$  

By defining suitable transformations on the vectors $Z_h$, the same attainable set can be achieved. In particular we find:

**Lemma 4.4**

The attainable consumption set $C$ is invariant if

(i) $Z_h$ is replaced by $\alpha Z_h$, where $\alpha \neq 0$;

(ii) $Z_h$ is replaced by $Z_h + \beta Z_h'$, $h \neq h'$; and $Z_h, Z_h' \in \{ Z_1, \ldots, Z_s \}$.  


Proof:

The result follows easily from a well known result in matrix algebra.

From this lemma we can gain direct insight into the effects of production portfolio and leverage decisions on the attainable consumption set. In 4.2 we restricted production decisions to the choice of scale of production patterns. From result (i) of the lemma, scale expansion does not alter the set C, so it is not surprising that the consumer is not directly concerned with production decisions, except to maximize his wealth. We can construct an analogous argument for the leverage problem. For convenience, consider the first asset to the riskless asset or bond, paying \( \hat{\beta} \) across all states, i.e., \( Z_1 = \hat{\beta} u \).

Now define the equity return \( Z_e = Z_h - Z_b \geq 0 \); where \( Z_b = Z_{h'} \), \( h' \neq h \).

Our formulation covers the case of risky bonds that are perfect substitutes for existing assets. Clearly, the case of the riskless bond is a special example of this more general proposition. From (ii) of the lemma it follows directly that C is invariant. Thus, if leverage is restricted such that the bond return always has a perfect substitute asset, then the consumer's consumption opportunities are unaltered and he is indifferent to leverage.

Conversely, consider production and leverage decisions that change the consumption set. Consider a simple production decision which involves a choice between \( Z_h \) and \( Z_{H'} \). The \( Z_h \) production decision implies the consumption set C, and \( Z_{H'} \) implies the set C'. By assumption C \( \neq \) C', so that the consumer's preferences will, in general, dictate a choice over the sets. This is the essence of Diamond's result when production patterns are not independent of scale.

\[18\] Because the creation of securities by private short sales with default risk is similar in principle to the corporate leverage problem, we will omit the discussion of the portfolio decision to avoid repetition.
Similarly, if the risky bond creates new patterns of returns, the consumption set will be altered so that consumers will have preferences over different degrees of leverage.\footnote{For a discussion of the choice of an optimal degree of corporate leverage see Chapter 4; and for an examination of the portfolio problem see Chapter 5.} Although, the leverage and production decision are similar in this respect, there is a subtle difference: leverage creates another asset so that the asset set is expanded to

\[
Z'' = [Z_1', \ldots, Z_n', Z_{h+1}', \ldots, Z_k']
\]

with the associated consumption set

\[
C'' = \{x_2 \mid [z''_k [z''_{k+1} ]^{-1} : I] x_2 = 0\}.
\]

Because \( Z_h = Z_e + Z_b \), it is not hard to see that \( C \subset C'' \).

Now there is a special case where this expansion of the consumption set does not lead to an increase in consumer welfare. In 4.3 we referred to the Sharpe-Lintner model, where all risky assets were held in a market portfolio. Yet \( x^*_{2i}, a^*_{1} \) be the \( i^{th} \) consumer's optimal consumptions and asset holdings respectively. Because the consumer holds the same proportion of each risky asset, then

\[
x^*_{2i} = Z'' a^*_{1}, \text{ where } a^*_{1} = [a^*_{12}, a^*_{13}, \ldots, a^*_{1n}]^T
\]

has \( k + 2 \) terms. Thus \( Z a_{1} + Z_b a_{1} = Z a_{1} \) implies that \( x^*_{2i} = Z a^*_{1} \); where \( a^*_{1} \) has \( k + 1 \) terms; and the consumer's optimum is always attainable in the original consumption set \( C \).
This result illustrates the danger inherent in models that restrict preferences or returns to give this strong mutual fund result. Effectively, consumers are sufficiently "alike" in their preferences or expectations to hold all the risky securities in a single mutual fund. Thus any redistribution rules for splitting a risky return are automatically undone by the proportionality rule for asset holdings.

5. A CRITIQUE

5.1 A theory of finance based upon the Arrow-Debreu model or one of its equivalents has very strong implications: the corporation's objective function is always well-defined in terms of market prices, so that all agents agree unanimously on the optimal production plan; and dividend and leverage policies, as well as take-overs, are matters of indifference to shareholders. Indeed finance theory degenerates to trivial propositions of transferring values from one hand to the other, and finance becomes an inessential fifth wheel to the theory of value.

It has been widely acknowledged that the Arrow-Debreu model of Sections 1-3, is a very unrealistic description of real world markets. We do not see a complete set of futures and contingent markets. A number of possible explanations for the non-operation of these markets have been suggested. Perhaps the most important of these is the assertion that market transactions consume real resources, so that the non-operation of contingent markets may reflect the fact that gains from trading are outweighed by the costs of operating that market.

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20 A number of writers have considered sufficient reasons for the non-existence of a complete set of contingent markets. For a discussion see Arrow (1970). Also, see Chapter 8.
5.2 Given that complete markets do not exist, then the theory of value becomes much more complicated.\textsuperscript{21} Even in the simple model of Section 4, the investment and finance theorems require a pre-determined set of returns. When there is an element of choice, or monopoly power, in creating new securities, investors can alter their attainable consumption sets. It is no longer obvious that investors will be indifferent to financing portfolio and investment decisions because they are able to create either through their own portfolio decisions or through the action of their agent - the corporation - new opportunities for consumption across states.

Because a particular agent is able to create new securities which are not perfect substitutes for existing securities, the model is no longer strictly competitive. If we assume that new assets can be created costlessly, then clearly the system will expand to the dimensions of the Arrow-Debreu model of Section 3, where there is a security for each state; or equivalently, a set of securities that span the state space. Of course, the investment and finance theorems of Section 1 are easy corollaries of this model. If we are to restrict the number of securities to be less than the number of states, we must assume that there are costs associated with the creation of securities. But this assumption is not sufficient in itself to produce a competitive market in patterns of returns. If, for example, different agents face different transaction costs for the creation of a particular security, we could imagine monopolistic or oligopolistic markets in particular securities. For a competitive market we require no barriers to entry. It should be obvious that the basic issues involved are identical to the problem of the creation of differentiated commodities.

\textsuperscript{21} Without a complete set of futures markets, the economy must progress via a sequence of temporary equilibria. Unfortunately it is difficult to construct such a model without making very strong assumptions about expectations, bankruptcy and corporate objective functions. For a readable discussion of the serious conceptual and logical problems involved see Radner (1970). For an attempted formulation, see Chapter 9.
It has been said that for the Modigliani-Miller finance theorems to be true, there must exist "perfect markets" in assets. Once one has grasped the essential point that a perfect market must imply explicit (or transparently derivable) markets for a given set of returns, these theorems follow easily because the asset model is no more than a thinly disguised Arrow-Debreu model.

22 In their text Fama and Miller (1972) stress the importance of the perfect market assumption. Unfortunately, I do not think they have defined their term "perfect market" precisely enough to reveal the essential nature of assets that are perfect substitutes. It is curious that they consider Hirshleifer's discussion of leverage in the Arrow-Debreu model, to be a partial equilibrium approach.
CHAPTER 4

CHOICE OVER ASSET ECONOMIES: CORPORATE PRODUCTION DECISIONS AND CONTROL

In a complete competitive market system for commodities, production decisions are decided unanimously by shareholders. The notion of corporate control is completely vacuous in a world where each investor agrees upon the optimal production plan. This powerful result follows directly from the so-called Fisher Separation Theorem which can be stated in the following way:

Given a complete set of market prices for the objects of choice, and competitive price-taking behaviour, then the i-th consumer-stockholder will prefer the consumption plan $x_i^*$ associated with the j-th producer's profit-maximizing plan $y_j^*$, over any other feasible consumption plan $x_i$ associated with plan $y_j 
eq y_j^*$.

Thus, it should be clear why economists have opted for profit-maximization (or in the intertemporal version - value-maximization) as the producer's objective function: shareholders will prefer the profit-maximizing solution over the solutions obtained by other corporate objective functions.$^2$

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1 By a complete market system, we mean a system where there exists a market and price for every object of choice by economic agents. The best-known example of such a system, is the Arrow-Debreu theory (1953), (1959).

2 For a full discussion and proof of the Fisher Theorem see the definitive statement by Hirshleifer (1970); and also Chapter 3.
At an intuitive level, the proof reduces to the simple statement that by maximizing the stockholder's wealth obtained from holding claims on the producer's profits, the stockholder's budget constraint defines the largest feasible consumption set.

Although the theorem holds for the usual competitive model, it is not obvious that the theorem follows, if the complete market assumption is removed. This point has been demonstrated by Diamond (1967) in a simple one-period, one-commodity model, where consumers could invest in a firm which produced a pattern of returns drawn from a feasible set of patterns of returns. Because the firm could choose the commodity space (which was a proper subset of the Arrow-Debreu commodity space) optimality required that the production plan depended upon the preferences and expectations of the firm's shareholders. Diamond observed that there existed an exceptional case, where the production function was decomposable— that is, the pattern of returns was fixed but the scale of production was open to choice. This case could be interpreted as a thinly disguised version of the certainty general equilibrium model, where the commodity set was defined as a pattern of returns across the set of feasible states of the world.  

In a more recent paper, Stiglitz (1972a) took up the optimality question within the context of the mean-variance general equilibrium model developed by Sharpe (1964) and Lintner (1965). Although he acknowledged that the Separation Theorem may not hold for an incomplete market system with choice over the set of patterns of returns, he proposed to explore the implications of the value-maximization rule.  

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3 For a discussion see Chapter 3, and also the discussion below.

4 In the subsequent literature there appears to be some confusion over Stiglitz's application of the value-maximization rule. Nevertheless, it would appear that he was well aware of its limitations: see (1972a) footnote 1, page 30.
that violated the assumption of a fixed commodity space, it was not surprising that he demonstrated the value-maximization implied a non-optimal investment allocation. (Actually, Stiglitz's results are more explicit than this, because he shows a systematic bias in the allocation between the safe and risky industries. Presumably, these results follow from the special nature of the mean-variance formulation).

Similar problems arise with choice over potential patterns of returns when the firm can alter its debt-equity ratio, and the corporate bond, (through the possibility of default) creates a new pattern of returns. Stiglitz (1969a), (1972b) has discussed this problem, showing particular cases where the consumer shareholders are indifferent to leverage. In (1972b) he has given a risk-neutral model showing that corporate leverage and production decisions were interdependent.

In this Chapter, we will attempt to generalize these results by giving a general model based upon choice over patterns of returns. This choice, in turn, implies a choice over a set of feasible asset economies. Except under special circumstances, agents will not agree upon the asset economy to be chosen, and there is latitude for conflict and bargaining. Before beginning the analysis proper, the reader may find it helpful if we outline the contents of the Chapter.

In Section 1 we have sketched a special case of the Arrow-Debreu model, where there are two periods, and one physical commodity. The existence proof and optimality proof are by Debreu (1959) and have not been reproduced here. In Section 2 we have assumed that fewer securities than states can be traded in asset markets, and by some simple lemmas it is easy to show that the asset economy is isomorphic to the Debreu economy. Therefore the asset economy has the properties of the Debreu economy - including existence and optimality proofs.
Section 3 contains a discussion of the choice over asset economies by consumers. Because consumers have preferences over consumption in each state of the world; and because the asset economy restricts the consumer to a subset of that preference map, then as the asset economy is transformed into another asset economy by changing the pattern of returns, the consumer will be restricted to another subset of his preference map. In this way, a sequence of asset economies can be generated with corresponding equilibrium consumption for each consumer. The consumers have preferences over these consumptions, so that they have preferences over the asset economies. In general, the preferences of consumers will not lead to a unanimous choice of an asset economy.

To illustrate the theory, in Section 4, we have presented an extended example based upon the Diamond model. Because the production technology can create different patterns of returns, there exists the possibility of conflict, between shareholders and risky bondholders, over the production plan. If the bondholders hold riskless bonds, then they have no direct interest in production decisions. The competitive unanimity result appears as a special case, when the pattern of returns is fixed, i.e., Diamond's decomposable case. We have mentioned also, the possibility of conflict over the choice of corporate leverage decisions, when bond-financing results in a new pattern of returns.5

Finally, in Section 5, we offer some concluding comments on the model. First we point out that the theory is consistent with proxy fights and battles for corporate control. In the case where control of production decisions is vested in the hands of a shareholder-entrepreneur, struggles for control are absent by construction, but the other shareholders are free to change their portfolios in response to different production plans. Therefore, the hypothesis that shareholders can always vote with their portfolio choice, misses the point

5 This issue is discussed at length in Chapter 5.
that control may imply a choice over asset economies. Secondly, the theory could be modified to include the elements of moral hazard in the implementation of production technique. Thus control would include a supervisory role in the production process, as well as a part in the initial planning phase. We conclude by discussing some of the limiting assumptions of the model.

1. THE CONTINGENT COMMODITY MODEL

1.1 Consider an economy that exists for two periods: in the first period, the state of the world is known with certainty by all economic agents, but in the second period there is a set of all possible states of the world $S = \{1, ..., s\}$. It is important to observe that these states are natural outcomes beyond the control of economic agents. Let there exist one physical commodity for both periods, so that the potential commodity space is $\mathbb{R}^{s+1}$. Denote the potential contingent commodity space to be $\mathbb{R}^s$, and the first period commodity space $\mathbb{R}$. Partition the set of economic agents into the set of consumers $i = 1, ..., m$, and the set of producers $j = 1, ..., n$.

1.2 Now consider the $j$th producer's production possibility set $(y_{0j}, y_{sj}) \in Y_j \subset \mathbb{R}^{s+1}$. In particular, let $y_{0j}$ be the input of the first period commodity, and let $y_{sj}$ be the resultant output for the second period across the set of possible states of the world $S$. The set $Y_j$ is assumed to have the properties:

- $A_1$ $0 \in Y_j$;
- $A_2$ $Y_j$ is closed and convex;
- $A_3$ $Y_j \cap (-Y_j) \subseteq \{0\}$, i.e., irreversibility;
- $A_4$ $Y_j \supseteq \mathbb{R}_{s+1}^s$, i.e., free disposal.
(Denote $\mathbb{R}_r^+$ $\equiv \{r \in \mathbb{R}^| r \geq 0\}$, and $\mathbb{R}_r^-$ $\equiv \{r \in \mathbb{R}^| r \leq 0\}$). It is obvious from the definitions of inputs and outputs and the irreversibility assumption that:

(i) $Y_{s_j} \cap \mathbb{R}_r^S(+) \neq \emptyset$, where $Y_{s_j}$ is the projection of $Y_j$ on $\mathbb{R}_r^S$; and

(ii) $y_{s_0j} \in \mathbb{R}_r^S(-)$.

1.3 Consider the $i^{th}$ consumer's consumption possibility set $(x_{0i}, x_{s_1}) \in X_i \subseteq \mathbb{R}_r^{S+1}$. The set $X_i$ has the following restrictions:

B1 $X_i$ is closed, convex, and has a lower bound for $x_i$.

B2 Define on $X_i$ a complete pre-ordering $\leq_i$ called a preference pre-ordering with the following conditions:

(a) Non-Satiety: For any $x_i, x_i' \in X_i$, $\exists x_i'' \in X_i$ such that $x_i'' \not\leq_i x_i'$.

(b) Continuity: For every $x_i', x_i'' \in X_i$ the sets

\[ \{x_i \in X_i | x_i \geq_i x_i'\} \quad \text{and} \quad \{x_i \in X_i | x_i \leq_i x_i''\} \]

are closed.

(c) Convexity: If $x_i', x_i'' \in X_i$ such that $x_i' \neq x_i''$, and $t \in (0, 1)$, then

\[ x_i'' \not\leq_i x_i' \Rightarrow tx_i'' + (1-t)x_i' \not\leq_i x_i' \]

Although assumptions B1, B2 are sufficient for a discussion of the existence and optimality of competitive equilibrium there does exist a more restrictive set of assumptions, which will be useful in the example considered below. These assumptions can be specified as:
B3 \[ X_i = \mathbb{R}^{s+1}(+) \].

B4 Assume the axiomatic foundations of the von Neumann-Morgenstern Utility Theorem,\(^6\) such that consumers preferences can be represented by the expected utility function

\[ U_i = \sum_{s \in S} u_i(x_{0i}, x_{si}) \pi_{si}, \]

where (i) \( \pi_{si} \in \{ \pi_{si} \in \mathbb{R}^+ | \pi_{si} > 0, \sum_{s} \pi_{si} = 1 \} \) is the \( i \)th consumer's subjective probability distribution on the set \( S \); and (ii) \( u_i(x_{0i}, x_{si}) \) is a real valued, concave utility function defined on \( \mathbb{R}^{s+1} \).

It is not difficult to show that B3, B4 have the properties assumed under the weaker conditions B1, B2.\(^7\)

1.4 Consider the initial commodity endowment \( \omega \in \mathbb{R}^{s+1} \). That is the endowment consists of some first period commodity \( \omega_0 \), and endowments of the commodity for each state of the world \( \omega_s, s \in S \). Given these initial endowments, and consumer and producer possibility sets, it is possible to define the economy \( E \):

Definition 1:

An economy \( E \) is defined by: non-empty subsets \( X_i \subseteq \mathbb{R}^{s+1} \) completely pre-ordered by \( \preceq_i \) for \( i = 1, \ldots, m \); non-empty subsets \( Y_j \subseteq \mathbb{R}^{s+1} \) for \( j = 1, \ldots, n \); and a resource endowment point \( \omega \in \mathbb{R}^{s+1} \).

---

\( ^6 \) For a general discussion and proof of the theorem, see Arrow (1970), essay 2.

\( ^7 \) See Arrow (1970), essay 4.
If consumers own resources, and control producers such that they receive a proportion of the producer's profits, then define:

Definition 2:

A private ownership economy \( E \) is defined by

(a) an economy \( E \);
(b) for every \( i \), there exists \( \omega_i \in \mathbb{R}^{S+1} \) such that \( \sum_{i} \omega_i = \omega \).
(c) for every \((i, j)\), there exists

\[ \theta_{ij} \in \{ \theta_{ij} \in \mathbb{R} \mid \theta_{ij} > 0, \sum_{i} \theta_{ij} = 1 \} . \]

If there exists competitive markets for the first period commodity, and each of the contingent commodities, then define:

Definition 3:

An equilibrium of the private ownership economy \( E \) is an \((m + n + 1)\) tuple \((x^*, (y^*_j), p^*)\) of points of \( \mathbb{R}^{S+1} \) such that:

(a) \( x^*_i \) is a greatest element of \( \{ x_i \in X_i \mid p^*x_i \leq p^*\omega_i + \sum_j \theta_{ij}p^*y^*_j \} \)
   for \( i = 1, \ldots, m \);
(b) \( y^*_j \) maximizes \( p^*y_j \) for \( y_j \in Y_j \), for \( j = 1, \ldots, n \);
(c) \( x^* - y^* = \omega \), where \( x = \sum_i x_i \), \( y = \sum_j y_j \).

Sufficient conditions for the existence of such an equilibrium have been given by Debreu (1959):

Theorem 1.1:

The private ownership economy \( E \) has an equilibrium if:

(a) \( X_i \) has the restrictions \( B1, B2 \) for \( i = 1, \ldots, m \);
(b) there exists $x_i^0 \in X_i$ such that $x_i^0 << w_i$ for $i = 1, \ldots, m$;

(c) $Y_j$ has the restrictions A1-A4, for $j = 1, \ldots, n$.

Proof:

See Debreu, 5.7.8

Because assumptions B3, B4 satisfy the weaker restrictions B1, B2, it follows as a trivial corollary of the theorem that an equilibrium exists for the von Neumann-Morgenstern formulation of preferences.

Finally, it is well-known that a competitive equilibrium achieves a Pareto Optimal allocation of resources. For the sake of formal completeness.

**Theorem 1.2:**

An equilibrium of a private ownership economy $E$ is an optimum.

Proof:

See Debreu, 6.3.

2. THE ASSET MODEL

2.1 In the preceding section competitive markets and prices were introduced for the first period commodity and the contingent commodities to obtain a simple version of the Arrow-Debreu contingent commodity model (1953), (1959).

Alternatively let us dispense with the complete market assumption, and assume instead that there are fewer securities than states of the world. That is, the economy is restricted to trading in linear combinations of contingent commodities.

Debreu's proof is more general, because it assumes the total production set $Y$ is closed, convex, irreversible and includes free disposal. Clearly A1-A4 is a special case of this more general condition.
Before proceeding any further, we will attempt an informal justification of this restriction on the number of operational markets. One possible rationalization involves the existence of transaction and set-up costs in the trading of securities. In particular consider the existence of set-up costs in informing the market of the characteristics of a security: we can think of these costs as the input of first period commodity required to issue a prospectus. Unfortunately, set-up costs introduce non-convexities into the analysis, and non-convexities create problems for the usual existence proofs. Rather than attempt such a formulation, which would produce conditions for the existence of active asset markets, we will take a short cut.\footnote{The theory would be more elegant and satisfying if the operative asset markets were determined by the model. See Chapter 8 for a discussion of these issues.}

Assume that the pattern of consumer endowments and transaction costs is such that securities can be issued by producers (i.e., coalitions of consumers), but not by individual consumers. Furthermore, each producer is restricted in the number of assets he can issue. The collection of assets issued by producers provides the available set of securities for consumer portfolios. Now we could allow for consumers and/or intermediaries in the issuance of securities, but this would complicate the analysis without adding significantly to our discussion of production and financing decisions.

It is feasible to impose restrictions on the possible patterns of returns for assets.\footnote{In the analysis below, these constraints are omitted, but they could be imposed without altering the argument.} For example, there may be legal restrictions requiring patterns of returns to conform to certain model securities (e.g., debt, equity). Of course, one might conjecture that a possible reason for the existence of such model securities arises from an attempt to reduce informational and computational costs in the description and promotion of securities.
2.2 It should be clear from the informal discussion in the previous section that the asset economy is derived from the contingent commodity economy by restricting trades to linear combinations of contingent commodities. This derivation can be formalized in the following manner.

Define $R^L$ to be the asset space, where $L \geq s$. The "production" of assets is obtained by taking linear combinations of contingent commodities from the aggregate producer's set.

Define the pattern of returns for an asset as a vector $z \in R^s$; the market set of returns as a semipositive matrix $Z$; and define the linear mapping

$$
\Gamma : R^{L+1} \rightarrow R^{S+1}
$$

by

$$
\Gamma(y_0, a) = (y_0, Z a).
$$

Because $\Gamma$ is linear, it is continuous. The range of $\Gamma$ is the vector subspace $G$ of dimension $L+1$. By restricting the mapping onto the range, i.e.,

$$
\Gamma : R^{L+1} \rightarrow G,
$$

then it is obvious that there exists a linear, inverse mapping

$$
\Gamma^- : G \rightarrow R^{L+1}.
$$

Now consider $R_j = G \cap Y_j$. The asset production set $Y^A_j$ can be constructed by the mapping $\Gamma^- : R^L_j \rightarrow R^{L+1}$. That is $Y^A_j = \Gamma^- (R_j)$. Having
constructed the set $Y^A_j$, we can prove the following lemma:

**Lemma 2.2:**

If the commodity production set $Y_j$ has the properties:

- $A_1$  $0 \in Y_j$;
- $A_2$  $Y_j$ is closed and convex;
- $A_3$  $Y_j \cap (-Y_j) \subseteq \{0\}$;
- $A_4$  $\forall n \in \mathbb{R}^{+1}$

then $Y^A_j$ has the properties of $A1-A4$.

**Proof:**

$A_1$  Because $0 \in G$ and $0 \in Y_j$, then $0 \in R_j$. But $\Gamma^-$ is linear: therefore $0 \in Y^A_j$.

$A_2$  (i)  Because $G$ is a vector subspace, it is convex; and the intersection $R_{\gamma}$ of the convex sets $Y_j$ and $G$ is also convex. Linear mappings map convex sets into convex sets: therefore $Y^A_j$ is convex.

          (ii)  Because $G$ is a vector subspace, it is closed; and the intersection $R_{\gamma}$ of the closed sets $Y_j$ and $G$ is also closed. Now $\Gamma$ is continuous, and by a well-known topological theorem, continuous mappings pull closed sets back into closed sets: therefore $Y^A_j$ is closed.

$A_3$  $Y_j \cap (-Y_j) \subseteq \{0\} \implies R_j \cap (-R_j) \subseteq \{0\}$. Then, $\Gamma^-(R_j \cap (-R_j)) \subseteq \{0\}$. Now $\Gamma^-(R_j) \equiv Y^A_j$, and because $\Gamma^-$ is linear, $\Gamma^-(R_j) = -Y^A_j$. 

Figure 1
Thus $Y_j^A \cap (-Y_j^A) \subseteq \{0\}$.

$A^4$ If has the property that because $Y_j \supseteq \mathbb{R}^{s+1}$, then $Y_j^A \supseteq \mathbb{R}^{L+1}$. Q.E.D.

The lemma can be illustrated for $s = 2$ and $L_j = 1$ as in Figure 1.

2.3 The $i^{th}$ consumer has available the market pattern of returns represented by the semi-positive $s \times L$ matrix $Z$. To obtain the $i^{th}$ consumption set in terms of assets the argument proceeds in a similar fashion to the analysis in 2.2. To simplify the analysis, the $i^{th}$ consumer will be restricted to non-negative holdings of assets, or, in other words, short-sales are disallowed.

Given this assumption, define the linear mapping

$$
\Lambda : \mathbb{R}^{L+1}_+ \rightarrow \mathbb{R}^{s+1}_-
$$

by

$$
\Lambda(x_{0i}, \beta_i) = (x_{0i}, Z\beta_i).
$$

The mapping $\Lambda$ is linear, and onto the range $H$, which is a convex polyhedral cone of dimension $L+1$. Define $T_i \equiv H \cap X_i$. The asset consumption set for consumer $i$ is $X_i^A$, and it is constructed by the inverse mapping

$$
\Lambda^- : T_i \rightarrow \mathbb{R}^{L+1}_+,
$$

which is linear. Now we can prove the following lemma.

**Lemma 2.3:**

If the commodity consumption set $X_i$ has the properties:
B1 \( X_i \) is closed, convex and has a lower bound \( \leq \alpha \);

B2 There is a preference pre-ordering \( \preceq_i \) on \( X_i \) with the properties:

- (a') Non-Satiety. For any \( x_i \in T_i \), \( \exists x_i' \in T_i \) such that \( x_i' \succ_i x_i \);\(^{11}\)

- (b') Continuity;

- (c') Convexity.

Then \( X_i^A \) has the properties B1, B2.

**Proof:**

B1 The proofs of closedness and convexity are identical to those given in Lemma 2.2. The lower boundedness of \( X_i^A \) follows from the assumption that \( X_i^A \subset R^{L+1} \').\(^{12}\)

B2 \( \Lambda \) is a representation of \( T_i \) in \( R^{L+1} \).\(^{13}\) Thus \( X_i^A \) is endowed with a preference pre-ordering. Closedness and convexity on preferences follows along similar lines to that in B1. Non-satiety is implied by the observation that \( \Lambda \) is a representation of \( T_i \). Q.E.D.

\(^{11}\) This assumption is a little stronger than the usual non-satiety assumption on \( X_i \).

\(^{12}\) By assuming no short-sales we are able to introduce a lower bound on the asset-consumption set, and the Debreu existence proof is applicable. But if short-sales are allowed, then the analysis becomes more complicated, because the consumption set does not appear to have a lower bound. An effective bound can be obtained by observing that if a consumer issues an asset (short-sells) then for it to be a perfect substitute to the existing assets he must offer the same pattern of returns. If default occurs, the pattern is altered. Therefore, no default implies a non-negative constraint on terminal wealth in every state. See Chapter 6.

\(^{13}\) A function \( f \) from \( S \) to \( T \) is said to be a representation of \( S \) in \( T \), if for \( x, x' \in S \), then \( x \preceq_S x' \Rightarrow f(x) \preceq_T f(x') \) and \( x \succ_S x' \Rightarrow f(x) \succ_T f(x') \).
Corollary 2.3:

If $B_3 \quad x_i = \mathbb{R}_{+}^{L+1}$,

B4 Preferences are represented by the von Neumann-Morgenstern axioms, with the additional assumption of risk-aversion, then, $x_i^A$ has the properties $B_3, B_4$.

Proof:

The proof follows trivially from the observation that $B_3, B_4$ are special cases of $B_1, B_2$.

2.4 Consider the initial commodity endowment $\omega \in H$, i.e., the commodity endowment can be translated via the inverse mapping $h^{-1}$ into an asset endowment. Given this asset endowment $\omega^A$, and the results of Lemmas 2.2 and 2.3, it is obvious that the asset economy has the same structure as the commodity economy; and the results reported in 1.4 apply for the asset economy. For formal completeness consider:

Definition 1':

An asset economy $E^A$ is defined by: non-empty subsets $x_i^A \subset \mathbb{R}^{L+1}$ completely pre-ordered by $\leq_i$, for $i = 1, \ldots, m$; non-empty subsets $y_j^A \subset \mathbb{R}^{L}$ for $j = 1, \ldots, n$; and a resource endowment point $\omega^A \in \mathbb{R}^{L+1}$.

Definition 2':

A private ownership asset economy $E^A$ is defined by:

(a) An asset economy $E^A$;

(b) For every $i$, there exists $\omega_i^A \in \mathbb{R}^{L+1}$ such that

$$\sum \omega_i^A = \omega^A.$$
(c) For every \((i, j)\), there exists
\[ \theta_{ij} \in \{ \theta_{ij} \in \mathbb{R} \mid \theta_{ij} \geq 0; \sum_{i} \theta_{ij} = 1 \}. \]

**Definition 3':**

An equilibrium of the private ownership economy \(E^A\) is an \((m + n + 1)\) tuple \((x^*_i, y^*_j, p^*_A)\) of points of \(\mathbb{R}^{L+1}\) such that:

(a) \(x^*_i\) is a greatest element of
\[ \{ x^A \in x^i \mid \rho^A x^A \leq \rho^A \omega_i + \sum_j \theta_{ij} p^A y^j \} \]
for \(i = 1, \ldots, m; \)

(b) \(y^*_j\) maximizes \(\rho^A y^j\) for \(y^A \in y^j\), for \(j = 1, \ldots, m; \)

(c) \(x^*_i \cdot y^*_j = \omega^A\).

**Theorem 1.1':**

The private ownership asset economy \(E^A\) has an equilibrium if:

(a) \(x^A_i\) has the restrictions \(B1, B2\) for \(i = 1, \ldots, m; \)

(b) There exists \(x^A_i \in x^i\) such that \(x^A_i \preceq \omega^A_i\), for \(i = 1, \ldots, m; \)

(c) \(y^A_j\) has the restrictions \(A1-A4\) for \(j = 1, \ldots, n.\)

**Theorem 1.2':**

An equilibrium of a private ownership asset economy \(E^A\) is an optimum.
2.5 By a simple transformation, the asset economy defined above can be reformulated in terms of the total market value of assets, and the proportions consumed or produced of these assets. That is, define for the $l^{th}$ asset:

$$A^l = \sum_i \omega_i^l + \sum_j y_{kj}^l \theta_{kj}^l;$$

$$a^l_i = x_{ki}^l / A^l; \quad a^l = (\omega_i^l / A^l) + (\sum_j \theta_{kj}^l y_{kj}^l / A^l); \quad V^l = p_k^l A_k^l.$$

Although the transformation adds nothing to the analysis above, it is familiar to portfolio theorists.

3. A SEQUENCE OF ASSET ECONOMIES

3.1 It should be clear from the discussion in 2.4 that the definitions and theorems on equilibrium and optimality presuppose the existence of a particular pattern of returns matrix. That is, let the equilibrium $(m+n+1)$ tuple associated with $Z^k$ be denoted by $((x_i^k), (y_j^k), p^k)$. Because assets were defined by a linear mapping in terms of the return per asset then, without loss of generality, the return per unit asset vectors $z^l_k$ can be normalized so that $z^l_k \in A_k^l \equiv \{z^l_k \in \mathbb{R}_{(+)}^k \mid z^l_k e = 1\}$, where $e$ is the unit vector. Thus the choice of the matrix $Z^k$ implies a choice $(z^l_1, \ldots, z^l_k, \ldots, z^l_L)$ from the set $A = \bigcap_k A_k^l$. Clearly $A_k^l$ is compact; and by Tychonoff's theorem, $A$ is compact.

Observe that by Theorem 1.1' there must exist at least one equilibrium associated with $Z^k$, but there may exist more than one equilibrium. The possibility of multiple equilibria introduces ordering difficulties when we wish to order the

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14 Other normalizations are possible, but this one is convenient and well-known.
asset economies by consumer preferences. Therefore, at this point, it will be necessary to introduce the following strong assumption:

C. Let there exist sufficient conditions on consumption sets, preferences, and production sets such that for each \( Z^k \in A \),

the associated equilibrium \((x^k_i), (y^k_j), (p^k)\) is unique.\(^{15}\)

Given assumption C, then there exists a single-valued mapping between the set of feasible patterns of returns and the set of asset equilibria. Therefore, for consumer \( i \), the equilibrium asset consumption can be translated into an equilibrium commodity consumption by the mapping \( Z^k x^k_i = x^k_i \). But, in turn, this implies that there exists a composite mapping \( \Phi: A \rightarrow x^k_i \). Denoting \( x^k_i \) as the range of \( \Phi \), then it is easy to construct a representation of \( X^k_i \) in \( A \). Therefore each consumer \( i \) has preferences over the non-empty, compact set \( A \), and the greatest element \( A^*_i \) of \( A \) for his preference pre-ordering \( \preceq_i \), may not coincide with \( A^*_i \), for consumer \( i' \neq i \).

3.2 Although the general formulation implies that unanimity in the choice of patterns of return is not necessary, there do exist special cases that give sufficient conditions for unanimity. For example, consider the following cases:

Case 1:

The first example is quite trivial: assume that all consumers have identical consumption sets, preferences and wealth. It is obvious that the m consumers can be aggregated into a single representative consumer, so that unanimity follows trivially.

\(^{15}\) A range of sufficient conditions exist - see Arrow and Hahn (1971) Chapter 9. Although these conditions imply quite strong restrictions on excess demand functions, they add nothing to our central theme, and so we will omit any further discussion of them.
Case 2:

Consider the matrix $Z_j = Z = I$, where $I$ is of dimension $s$. That is, let the $i$th security pay one unit of the commodity in the $i$th state of the world. The asset economy is of full dimension spanning the potential commodity-state space. Necessarily, choice over patterns of returns is trivial. The reader should recognize this case as the Arrow-Debreu security model.

Case 3:

Finally, consider the case where the production pattern of returns is restricted to $Z_j^1$ such that the convex polyhedral cone $C_j$ generated by $Z_j^1$, is contained in the asymptotic cone of $Y_j$, i.e., $C_j \subseteq \text{As}(Y_j)$. Because each firm $j$ is restricted to $Z_j^1$, then $A$ becomes the singleton $\{Z_j^1\}$. This formulation is a generalization of Diamond's decomposable case, where each producer is restricted to producing a fixed pattern of returns.

These three cases are not meant to provide an exhaustive listing of well-defined situations of unanimity over the sets of possible patterns of returns. Rather than pursue further examples of unanimity (or possibly indifference) we will consider an extended example where there is conflict over production plans. This example provides a useful exercise because

(i) it presents a simple case where conflict exists;
(ii) it reveals the two main sources of conflict when there is choice over the pattern of returns; and
(iii) by taking special security form, i.e., debt and equity, further results can be obtained on the role of bondholders in production decisions.
4. AN EXAMPLE

4.1 In the previous sections, the pattern of returns decision was made in isolation from the production or financing decisions. But in Diamond's model, the production decision and the pattern of returns decision were determined jointly. Our example will be based on a variant of that model, including some slight, but nevertheless interesting, extensions. Although there appears to be some differences in formulation of this example as compared to the analysis above, the reader should satisfy himself that the differences are purely superficial.16

4.2 The analysis is restricted to one period, and one physical commodity, so that uncertainty is technological. For simplicity let there be only j = 1, 2 producers. Let producer 1 have degenerate technological uncertainty such that the commodity input $I_1$ generates a total return $rI_1$ for all states of the world $s \in S$. Producer 2 suffers from technological uncertainty such that input $I_2$ generates total returns $f(I_2, t, s)$. To use Diamond's homely example, consider the physical commodity to be corn, and the function $f$ to be a production function depending upon the input of corn $I_2$, the technique of production $t$ (e.g. the use of different crop cultivation methods), and the amount of rainfall in each state of the world $s \in S$. Assuming $S$ to be a closed interval of the real line $[\underline{s}, \bar{s}] \subseteq \mathbb{R}$; and (continuing the analogy) that the rainfall stimulates production, then if $f$ is differentiable in its arguments, we have $f_s > 0$.

16 For example, Diamond restricts his theory to one period: but our more general theory can be modified to accommodate this case by dropping first period consumption from the consumption sets. Secondly, consumer utility functions have been treated as integrals rather than sums. By interpreting the integrals in the Lebesgue-Stieltjes sense, the two approaches are equivalent when there is a finite set of states.
Now consider the arguments $I_2$ and $t \in T$. If it is assumed that the production is restricted to a fixed plot of land, and there are some initial scale economies, then we can rationalize the following sign patterns for marginal and average production:

\[
\begin{align*}
I_2 &> 0, \quad \text{for } I_2 \in (0, \infty); \\
\frac{\partial (f/I_2)}{\partial I_2} &< 0 \quad \text{as } I_2 \rightarrow I_2',
\end{align*}
\]

where $I_2'$ is positive and finite. Let the set of possible production techniques $T$ be a closed interval of the real line $[t, T] \subseteq \mathbb{R}$. For the choice of technique to be non-trivial, we require that a change in the method of production

(a) lowers returns over states $s \in S_1$;
(b) does not alter returns over states $s \in S_2$; and
(c) increases returns over states $s \in S_3$, where $\{S_k\}$ partitions $S$.

For simplicity, consider the special case, where:

\[
\begin{align*}
f_t &< 0 \quad \text{for } s \in [s, s']; \\
f_t &= 0 \quad \text{for } s = s'; \\
f_t &> 0 \quad \text{for } s \in (s', s];
\end{align*}
\]

and where $s'$ may be a function of $I_2$. 
Now consider the notion of debt and equity financing by assuming that total investment $I_2$ is the sum of equity $E$ and debt $B$, i.e., $I_2 = E + B$. It will be convenient to define a measure of leverage: define $\alpha \equiv (B/I_2)$, and also $(1 - \alpha) \equiv (E/I_2)$, where $\alpha \in [0, 1]$. The debt pays a total return $r_B$ for those states where there is no default; and a total return $f(I_2, t, s)$ when default occurs. Equity pays a total return $f(I_2, t, s) - r_B$ for those states where there is no default; and a total return of zero where default occurs. Default occurs at the critical state of the world $s^*$, which is defined as the unique root of

$$f(I_2, t, s) - r_B = 0.$$ 

Observing that $B = \alpha I_2$, then $s^* = s^*(I_2, t, \alpha)$.

4.3 Let there be $i = 1, \ldots, m$ consumers. Assume that consumer $i$ maximizes his von Neumann-Morgenstern utility function $u_i \in C^2$, defined over consumption $x_i(s)$ in the state of the world $s$, and given his subjective probability distribution function $G_i(s)$. That is, the $i$th consumer's objective function is

$$U_i = \int_{s} u_i(x_i(s))dG_i(s).$$

(1)

Assume that all consumers are risk-averse, i.e., $u_i' > 0$, $u_i'' < 0$. Apart from this assumption on preferences, we do not require any other restrictions on preferences or expectations. By placing more restrictive assumptions on preferences and expectations further results can be obtained, but they do not invalidate the conclusions below. (For example, assuming a quadratic utility
function for all investors, and common expectations, the model can be formulated in the familiar Sharpe-Lintner (1964), (1965) way, where all risky securities are held in a mutual fund).

Now assume that consumer $i$ has an initial endowment of the commodity $W_i > 0$, and he can purchase proportions $\theta_{i\xi}$ of the values of the assets $V_{\xi}$, $\xi = 1, B, E$. Normalize the assets in commodity units $A_{\xi} \in \{I_1', B, E\}$, so that with prices $p_{\xi}$, we have $V_1 = p_1 I_1'$, $V_B = p_B B$, $V_E = p_E E$. Let the asset market be competitive in asset prices, so that the $i$th consumer's budget constraint becomes (assuming non-satiety)

$$\sum_{\xi} \theta_{i\xi} p_{\xi} A_{\xi} = W_i', \quad V_i'. \quad (2)$$

The proportions of the assets purchased, entitle the $i$th consumer to proportions of the physical returns: therefore, the consumption possibility set is defined by

$$x_i(s) = \begin{cases} 
\theta_{i1} I_1 r + \theta_{iB} f(I_2', t, s); & \text{for } s \in [s_1, s^*], \\
\theta_{i1} I_1 r + \theta_{iB} B r + \theta_{iE} [f - r B]; & \text{for } s \in [s^*, s].
\end{cases} \quad (3)$$

Finally, restrict the consumer's choice to $\theta_{i\xi} > 0$: that is, we rule out the possibility of short-sales. Substituting (3) into (1), the consumer's problem becomes, (observing that $B = I_2 a$, $E = (1 - a)I_2$)
Max $U_1 ((\theta, \sigma), I_1, I_2, \alpha, t)$

\[
\begin{align*}
&= \int_{s^*} u_1 (\theta_1, I_1, r + \theta_2 f(\bar{I}_2', t, s)) dG_1(s) \\
&= \int_{s^*} - \int_{s} u_1 (\theta_1, I_1, r + \theta_2 f(\bar{I}_2', t, s)) dG_1(s) \\
&+ \int_{s^*} u_1 (\theta_1, I_1, r + \theta_2 \bar{I}_2 + \theta_{BE} (I - \bar{I}_2 \alpha)) dG_1(s) \\
\end{align*}
\]

subject to

\[
\sum_{i} \theta_{iB} \lambda_i = \omega_i; \quad \theta_{iB} \geq 0.
\]

Forming the Lagrangian expression with $\lambda_i$ the multiplier associated with the budget constraint, the first order conditions for a maximum are (assuming a maximum is obtained other than at discontinuity points)

\[
\begin{align*}
&= \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \lambda_{iB} I_1 + \mu_1 = 0 \\
&= \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \lambda_{iB} I_1 + \mu_1 = 0 \\
\end{align*}
\]

\[
\begin{align*}
&= \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \lambda_{iB} I_1 + \mu_1 = 0 \\
&= \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \lambda_{iB} I_1 + \mu_1 = 0 \\
&= \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \int_{s^*} \int_{s} u_1' I_1 r dG_1(s) + \lambda_{iB} I_1 + \mu_1 = 0 \\
\end{align*}
\]
\[
\int_{s_*}^{s} \left( u_i'(f - rB)dG_i(s) + \lambda^i_P E + \nu E = 0 \right. \\
\left. \quad 0 \leq \nu E = 0, \quad 0 < \nu \leq 0, \quad \nu_e \geq 0. \right.
\]

Conditions (5), (6), (7) are in the standard form for the consumer's portfolio problem. Given the parameters \( P = \{ I_1, I_2, t, \alpha \} \), the pattern of returns is fixed, so that by the argument of Section 2, we can consider the \( i^{th} \) consumer embedded in a general equilibrium asset economy with competitive asset prices \( P_1, P_B, P_E \). In general, any variation in the parameter \( P \) will alter the pattern of returns available to the asset economy, and will result in a change in asset prices. Assuming the existence of a unique asset equilibrium, then the prices \( (p_k) \) will be functions of \( P \). As we asserted in Section 4.1, consumer choice over the parameter set is effectively a choice over asset equilibria generated by different patterns of returns.

4.4 Our first task will be to examine the conditions produced by the \( i^{th} \) consumer's utility maximization over the set of feasible investment programmes.

Observe that investment decisions involve a choice \( I_1, I_2 \), given the total available endowment \( \bar{I}, \) i.e., \( I_1 + I_2 = \bar{I} \). Therefore, replacing \( I_1 \) by \( \bar{I} - I_2 \) in the set of parameters, we can eliminate \( I_1 \). Let consumer \( i \) maximize utility over the set of asset equilibria associated with different investment divisions, so that the first order condition is
\[ - \theta_{il} \left( \int \frac{s^*}{s} u'_1 r dG_1(s) + \int \frac{s}{s^*} u'_1 r dG_1(s) \right) \]

\[ + \theta_{iB} \left( \int \frac{s^*}{s} u'_1 f_1 I_2 dG_1(s) + \int \frac{s}{s^*} u'_1 r dG_1(s) \right) \]

\[ + \theta_{iE} \int \frac{s^*}{s} u'_1 (f_1 - r a) dG_1(s) \]

\[ + \lambda^i [- \theta_{il} P_1 + \theta_{iB} P_B + \theta_{iE} P_{12} (1 - \alpha)] \]

\[ + \lambda^i \lambda^* \prod \frac{3P}{3I_2} = 0. \]

Substituting from (5), (6), (7) and eliminating \( \lambda^i \), we obtain

\[ \begin{align*}
&\left\{ \frac{0^*}{0^*} \int \frac{s^*}{s} u'_1 [f_1 - (f/I_2)] dG_1(s) + \frac{\theta_{iE}}{0^*} \int \frac{s^*}{s} u'_1 [f_1 - (f/I_2)] dG_1(s) \right. \\
&\left. \frac{\int \frac{s^*}{s} u'_1 r dG_1(s) + \frac{s^*}{s} u'_1 r dG_1(s)}{ \int \frac{s^*}{s} u'_1 r dG_1(s) + \frac{s^*}{s} u'_1 r dG_1(s)} \right\} \right. \\
&\left. - (p_1)^{-1} \sum_{k} \frac{0^*}{0^*} \frac{3P}{3I_2} = 0. \right. \]

Observe that \( \theta^*_{iB}, \theta^*_{iE} \) are the optimal proportions associated with the equilibrium \( (p^*, I^*_2) \). Equation (9) has a straightforward interpretation:
a variation in $I_2$ has an impact on preferences from two sources:

(a) from changes in the pattern of returns, given the holdings $(\theta_{\alpha i}^*, \theta_{\beta i}^*)$; and

(b) from changes in the budget constraint brought about by variations in the asset prices $P_k$.

(Notice that $\lambda^i$ is the marginal utility of wealth, so that the last term is the impact of changes in wealth). The solution of equation (9) is $I_2^{*i}$. Because utility functions and subjective probability distributions may vary over the set of consumers, it would be fortuitous if $I_2^{*i'} = I_2^{*i''}$, for $i' \neq i''$. Thus in general there is a conflict over the scale of the corporation; but, as we will show below, there does exist a set of Pareto optimal investment decisions which must be reached by extra-market arrangements.

Although we have shown that conflict is possible on quite general grounds, we can generate special cases where conflict is absent. In 3.2 we gave examples where unanimity prevailed. Leaving aside the first two cases, consider the third case, where the patterns of returns are independent of the scale of investment, i.e., the production function is said to be decomposable where

$$f(I_2', t, s) = g(I_2)h(t, s).$$  \hspace{1cm} (10)

Substituting (10) into (a), we obtain

$$[g' - (g/I_2)]\beta - (P_1)^{-1} \sum \theta_{i\alpha}^* \lambda^i \frac{\partial P_k}{\partial I_2} = 0,$$  \hspace{1cm} (11)
where
\[ \beta = \frac{\int_{s} u^1h(t, s)dG_i(s)}{\int_{s} u^1rdG_i(s)} \]

Now consider two special cases of (11).

(a) Let \( g(I_2) = I_2 \). It is easy to see from the derivation of \( s^* \), that the default state of the world is independent of \( I_2 \), but dependent on \( a \). Therefore, the pattern of returns are unaltered with variations in \( I_2 \), so that the wealth term is identically zero because \( \frac{\partial p_r}{\partial I_2} = 0 \). Furthermore \( [g' - (g/I_2)] = 0 \), such that (11) is satisfied identically.

(b) Let \( a \) be set so that default does not occur. Then again the pattern of returns are unaltered with variations in \( I_2 \). Therefore, the wealth term is identically zero, and (11) collapses to:

\[
[g - (g/I_2)] \left\{ \begin{array}{c} \int_{s} u^1h(t, s)dG_i(s) \\ \int_{s} u^1rdG_i(s) \end{array} \right\} = 0 \tag{12}
\]

Because \( h(t, s), r > 0 \), then the term \(...\) is strictly positive, and (12) is satisfied by \( [g' - (g/I_2)] = 0 \), which is independent of \( i \). Shareholders and bondholders are unanimous in their choice of production scale, because the model is a thinly distinguished Arrow-Debreu economy.
If the economy has degenerate technological uncertainty, i.e., all consumers agree that only one state of the world is feasible, then it is easy to show, assuming \( h = 1 \), that \( (g/I_2)/p_2 = r/p_1 \). Therefore the investment decision must satisfy the familiar competitive-certainty result \( g'/p_2 = r/p_1 \). I.e., the marginal rate of return equals the rate of interest.

Consider the role of bondholders in the investment decision. Given a level of leverage \( a \), low enough to ensure no default for all choices of investment (i.e., \( s^* \leq s \)), then condition (9) becomes

\[
\frac{\theta^*_i}{1} \int_{s^*}^{s} u'_i[f - (f/I_2)]dG_i(s) \leq \frac{1}{1 - \frac{s}{s^*}} \int_{s}^{s} u'_i r dG_i(s)
\]

for consumer \( i \) who holds positive equity \( (\theta^*_{iE} > 0) \); and

\[
(p_1)^{-1} \sum_k \theta^*_i \lambda^*_i \frac{\lambda^*_{iE}}{\lambda^*_{iE}} = 0 \tag{13}
\]

for consumer \( i' \) who holds no equity \( (\theta^*_{i'E} = 0) \).

Clearly the corporate bond is riskless, and therefore we know that \( p_B = p_1 \) if the corporate bond and the safe asset are held simultaneously. With a riskless corporate bond, the bondholder's interest in the investment decision enters only indirectly through the wealth effect produced by the change in asset prices. If the wealth effect is negligible (which is assumed in partial equilibrium analysis) then the riskless bondholder is indifferent to the investment decision.
On the other hand, if default is possible, then the bondholder has a direct interest in the production decision through changes in the patterns of returns. For example, if consumer i holds corporate bonds \( (\theta_{1B} > 0) \) but not equity \( (\theta_{1E} = 0) \) then the condition (a) becomes

\[
\begin{align*}
\theta_{1B} & \int_{S} u_i'[f_{I_2} - (f/I_2)]dG_i''(s) \\
& \int_{S} u_i'[rdG_i''(s)] + \int_{S} u_i'[rdG_i''(s)] \\
& + (p_1)^{-1} \sum_{k} \theta_{1E} \frac{\partial p_k}{\partial I_2} = 0. 
\end{align*}
\tag{13}
\]

Thus, if the consumer i the equity holder, and consumer i", the bondholder, have different preferences and expectations, then there is a conflict over the investment plan. This result would be consistent with empirical observations that debt-holders are not necessarily indifferent to the investment decisions of enterprises.17

4.5 In the previous section we considered the possibility that the scale of production could alter the pattern of returns generated by the risky producer. Alternatively, the producer can alter the pattern by changing his technique of production \( t \in T \). Assuming an interior solution, the necessary condition for maximum utility for consumer i is given by

---

17 Given the agricultural analogy used in introducing the model, it is interesting to observe that the historical evidence on share-cropping, leasing, etc., shows the landholder (equivalent to our bondholder) taking an effective and detailed interest in the production plan. See Reid (1973).
\[
\begin{align*}
&\phi'(t) \left\{ \begin{array}{c}
\delta^* \int_{iB} u'_1 f dG_1(s) + \theta^* \int_{iE} u'_1 f dG_1(s) \\
\int_{iB} u'_1 r dG_1(s) + \int_{iE} u'_1 r dG_1(s)
\end{array} \right\} \\
&+ (p_1)^{-1} \sum_{k} \delta^* \beta^* \frac{\partial p_k}{\partial t} = 0. \tag{14}
\end{align*}
\]

Notice that (14) has the same form as (9) where the term \([f - (f/I_2)]\) has been replaced by \(f_t\), and therefore the comments made on the scale of investment apply, with equal validity, for the choice of technique. For example, consider the analogue of the decomposable production function, i.e.,

\[
\begin{align*}
&f(I_2', t, s) = \phi(t) \psi(I_2', s) \\
\end{align*}
\]

where \(\phi(t)\) is strictly concave, and \(\psi'(t) = 0\) has a unique root \(t^*\) positive and finite. Substituting (15) into (14) we obtain

\[
\begin{align*}
&\phi'(t) \left\{ \begin{array}{c}
\delta^* \int_{iB} u'_1 \psi dG_1(s) + \theta^* \int_{iE} u'_1 \psi dG_1(s) \\
\int_{iB} u'_1 r dG_1(s) + \int_{iE} u'_1 r dG_1(s)
\end{array} \right\} \\
&+ (p_1)^{-1} \sum_{k} \delta^* \beta^* \frac{\partial p_k}{\partial t} = 0. \tag{16}
\end{align*}
\]

For unanimity to prevail, we require that there is no default: in that case (16) becomes
which is satisfied by $t^*$ independently of $i$. This result is obvious in a one commodity model, because the choice of technique (when patterns of returns are fixed) degenerates to a trivial choice where one technique dominates all others.

4.6 The remaining parameter $a$ - the measure of producer leverage - has been analysed in detail elsewhere, and so we will give a brief summary discussion here for completeness. There has been a long debate over the impact of changes in corporate leverage on the value of the corporation. Traditional theorists in corporate finance had argued that leverage changed corporate value, but their theory was imprecise in specification and often confusing. In an important contribution, Modigliani and Miller (1958) showed that in perfect markets, consumer-shareholders are indifferent to the choice of leverage. Given their assumptions, the M-M invariance theorem is impeccable. But more recently, Stiglitz (1969a), (1972b) and Smith (1972a) have argued that when corporate default is feasible, and there does not exist complete markets, then it is possible to have an optimal choice of leverage. From our discussion in sections 2 and 3, this result follows when we observe that risky leverage decisions alter the pattern of returns available to consumers. There are some special cases where indifference does prevail, even with incomplete markets.

\[ \frac{\int u_1' \psi G_1(s)}{\phi'(t)} = 0, \]  

18 See Chapter 5.
but they require strong assumptions upon preferences and expectations.

If our assertion that there exists consumer preferences over leverage is accepted, then there is a joint production - financing plan for the producer preferred by each consumer-shareholder.

4.7 To obtain the necessary conditions for Pareto optimal production decisions, consider the standard formulation used by Diamond:

\[
\begin{align*}
\text{Max } & U_1 = \int_{s^*} u_1(\theta_1I_1r + \theta_2f(I_2, t, s))dG_1(s) \\
& + \int_{s^*} u_1(\theta_1I_1r + \theta_2I_2ar + \theta_3I_2f - I_2ar)dG_1(s) \\
\text{subject to} & \quad (a) \quad U_i = \bar{U}_i, \text{ for } i = 2, \ldots, m. \\
& \quad (b) \quad \sum_i \theta_{ik} = 1, \quad \theta_{ik} \geq 0. \\
& \quad (c) \quad I_1 + I_2 = \bar{I}.
\end{align*}
\]

Forming the Lagrangean, and associating the multipliers \( \delta_i \) with constraints (a) (setting \( \delta_i = 1 \)); \( \lambda_2 \) with constraints (b), and \( \lambda_2 \) with constraint (c); then assuming interior solutions, the first order conditions are:

\[
\begin{align*}
\delta_1 \left\{ \int_{s^*} u_1'I_1rdG_1(s) + \int_{s^*} u_1'I_1rdG_1(s) \right\} + \lambda_1 = 0
\end{align*}
\]
Conditions (19)-(21) can be manipulated to give the equality between marginal rates of substitution between the two investment assets, i.e.,

\[
\delta_i \left\{ \begin{array}{l}
\int \frac{u_i f dG_i(s)}{s^*} + \int \frac{u_i I x_2 dG_i(s)}{s^*} \\
\end{array} \right\} + \lambda_B = 0 
\] (20)

\[
\delta_i \left\{ \begin{array}{l}
\int \frac{u_i (f - I x_2 a r) dG_i(s)}{s^*} \\
\end{array} \right\} + \lambda_E = 0 
\] (21)

\[
\frac{\sum_i \delta_i \left\{ \begin{array}{l}
\int \frac{u_i \theta I dG_i(s)}{s^*} + \int \frac{u_i \theta I x_2 dG_i(s)}{s^*} \\
\end{array} \right\} + \lambda_2 = 0 }{s^*} 
\] (22)

\[
\frac{\sum_i \delta_i \left\{ \begin{array}{l}
\int \frac{u_i \theta I x_2 dG_i(s)}{s^*} + \int \frac{u_i \theta I x_2 (f + \theta I x_2 - a r) dG_i(s)}{s^*} \\
\end{array} \right\} + \lambda_2 = 0 }{s^*} 
\] (23)

\[
\frac{\sum_i \delta_i \left\{ \begin{array}{l}
\int \frac{u_i \theta I x_2 f dG_i(s)}{s^*} + \int \frac{u_i \theta I x_2 f dG_i(s)}{s^*} \\
\end{array} \right\} }{s^*} = 0. 
\] (24)

\[
\frac{\int \frac{u_i f dG_i(s)}{s^*} + \int \frac{u_i f dG_i(s)}{s^*}}{s^*} = \frac{\lambda_B}{\lambda_1} + \frac{\lambda_E}{\lambda_1} , \forall i. 
\] (25)
Clearly (25) holds for the portfolio equations (5)(a)-(7)(a) where consumers face competitive asset prices. This result is a special case of the more general Theorem 1.2' of Section 2.4.

Now consider the investment decision equations (22), (23), substituting from (19)-(21) we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{iB} \frac{s^*}{s} \left( u_1^i \left( f/1_2 \right) dG_1(s) \right) + \int_{iE} \frac{s^*}{s} \left( u_1^e \left( f/1_2 \right) dG_1(s) \right) \\
\int_{iB} \frac{s}{s^*} \left( u_1^i r dG_1(s) \right) + \int_{iE} \frac{s}{s^*} \left( u_1^e r dG_1(s) \right)
\end{array} \right. \\
= 1 - \int_{iB} \frac{s^*}{s} \left( u_1^i \left( f/1_2 \right) dG_1(s) \right) + \int_{iE} \frac{s^*}{s} \left( u_1^e \left( f/1_2 \right) dG_1(s) \right)
\end{align*}
\]

which is a generalization of Diamond's condition (12). Thus the optimal investment plan must satisfy a weighted sum of ratios of expected marginal utility (obtained from the two production processes), where the weights are the proportions of bonds and equity held corresponding to the investment plans.

Now let us consider two special cases of (26).

(a) Consider the situation where default is absent. Clearly (26) collapses to the form:

\[
\begin{align*}
\left\{ \begin{array}{l}
\int \frac{s^*}{s} \left( u_1^i \left( f/1_2 \right) dG_1(s) \right) \\
\int \frac{s}{s^*} \left( u_1^i r dG_1(s) \right) 
\end{array} \right. \\
= 1 - \int \frac{s^*}{s} \left( u_1^i \left( f/1_2 \right) dG_1(s) \right)
\end{align*}
\]
Substituting from (25), we find
\[
\begin{align*}
\sum_{i=1}^{n} \left[ \begin{array}{c}
\int_{s}^{\infty} u_1'(f_I^2 - (f/I^2))dG_1(s) \\
\sum_{i=1}^{n} u_1'rdG_1(s)
\end{array} \right] &= 1 - \left[ \begin{array}{c}
\int_{s}^{\infty} u_1'(f/I^2)dG_1(s) \\
\sum_{i=1}^{n} u_1'rdG_1(s)
\end{array} \right].
\end{align*}
\]

where optimal investment plans depend solely upon equity holders. This condition is equivalent to Diamond's result (12).

(b) If the production function is decomposable, i.e., has the form (10), then

(28) reduces further to
\[
\begin{align*}
\sum_{i=1}^{n} \left[ \begin{array}{c}
\int_{s}^{\infty} u_1'f_2dG_1(s) \\
\sum_{i=1}^{n} u_1'rdG_1(s)
\end{array} \right] &= 1, \quad (29)
\end{align*}
\]

Because the indexing of consumers is arbitrary, equation (29) says that optimality implies the equating of the marginal rate of substitution (of asset patterns of returns) with the marginal rate of transformation.

Consumers are unanimous in their choice of the profit-maximizing investment.
We can perform a similar analysis for the choice of technique $t$, obtaining the condition

$$
\begin{align*}
\mathbb{E}_B \left[ \sum_{i=1}^{s^*} u_i^t dG_i(s) + \sum_{i=1}^{s^*} u_i^t dG_i(s) + \sum_{i=1}^{s^*} u_i^r dG_i(s) + \sum_{i=1}^{s^*} u_i^r dG_i(s) \right] = 0.
\end{align*}
$$

By analogy with the investment optimum we can consider special cases where the choice of technique is a unanimous decision by all consumers. But when the technique of production alters the pattern of returns, the optimal technique depends upon the preferences, expectations and asset holdings of all consumers.

This concludes our simplified example.

5. CONCLUDING COMMENTS

5.1 In the previous section we considered the Pareto conditions as an optimality criterion for production decisions. But such conditions are very weak, and are only a partial guide to the outcome of any bargaining process. If the costs of bargaining (or forming coalitions) are zero, or negligible, then rational bargaining implies a solution satisfying the Pareto conditions. But if there are costs associated with forming coalitions, then the outcome may not satisfy the Pareto criteria, and without further assumptions the result is indeterminate. Nevertheless, there are some simple cases where determinate results can be obtained – for example, consider the case of the monopolist.
Choose one consumer $e$, who we will call the entrepreneur of the risky corporation $E$. Assume that the costs of forming coalitions against $e$ are so great that the other consumers simply go along with the production plan chosen by $e$. Therefore, assuming that $e$ requires a positive holding of equity ($\theta^*_{eE} > 0$), and assuming that he holds no risky bonds ($\theta^*_{eB} = 0$), then the scale of production is determined by

$$\begin{align*}
\theta^*_{eE} \int_{s^*}^s u'[f_{eI_2} - (f/I_2)]dG_e(s) \\
- \int_{s^*}^s u'r dG_e(s) + \int_{s^*}^s u'r dG_e(s)
\end{align*}$$

$$\left(\frac{1}{p_1}\right)^{-1} \sum_{k} \theta^*_{eE} \frac{\partial p_k}{\partial I_2} = 0.$$ 

Notice that this solution does not preclude

(i) other consumers changing their portfolios, or

(ii) asset prices changing, as $I_2$ varies over the feasible investment allocations.

Although other consumers are free to trade in asset markets, the monopolistic entrepreneur chooses the asset market in which trading is to be carried out.

We could give other examples, but they would add little to the central point that the issue of control becomes important when producers can create new securities not already traded in asset markets. This model (simple as it is) does give results consistent with the real world observations of proxy fights and struggles for corporate control.
5.2 In section 2.1 we argued that transaction and set-up costs could be introduced as a justification for incomplete markets. But there is another plausible explanation: Arrow [1] has stressed the role of "moral hazard" in limiting the extent of complete insurance markets. By moral hazard is meant a situation where an agent has difficulty in distinguishing between outcomes due to natural states of the world, and the actions of other agents. In the discussion in preceding sections this element was absent because it was assumed that the states of the world were defined independently of the actions of economic agents, and all agents knew the production functions defined over the states. This assumption implies that all agents have detailed knowledge of production processes - unlike the complete market situation, where unanimity ensures that an economic agent can be trusted to choose the optimal plan.

Even though all agents may have the detailed information on which to decide the bargained plan, they must see that plan implemented. If there is no supervision or checks on agents supervising the plan, then there may exist positive incentives for the supervisory agent to alter the plan during implementation.

For example, consider the model discussed in Section 4. Before bargaining, consumer 1 may prefer the technique $t^1$ over all competing techniques, but bargaining produces the technique $t^* \neq t^1$, with associated prices ($p^*_1$), investment $I^*_2$, and portfolios ($\theta^*_1$). If consumer 1 is chosen as the agent to supervise the production technique, then he will want to implement the technique $t'$, which is the solution to

$$
\begin{align*}
\theta^*_1 & \int_{I^*_B} u'_1 f_t dG_1(s) + \theta^*_1 \int_{I^*_L_1} u'_1 f_t dG_1(s) \\
\int_{S} u'_1 rG_1(s) + \int_{S^*} u'_1 rG_1(s) &= 0.
\end{align*}
$$
Clearly the controlling agent can gain by operating upon the ignorance of other agents. Also, one could make a case for the other agents anticipating such action against their wishes, and instituting checks on their agent's actions. Therefore, we could rationalize the control over the implementation of a plan as means of avoiding moral hazard. The question of moral hazard has received little attention in the theoretical finance literature, although from casual observation one observes that "reputation" and "past dealings" do enter as factors in the calculations of lenders and stockholders.

5.3 Finally, we turn to some of the other restrictive assumptions of the model. The restrictions to one commodity and two periods are taken from the Diamond model. The two period assumption eliminates the possibility of speculation on future asset prices, and reduces the investment decision to a static once-and-for-all choice. If more periods were allowed the model would require an expectation-forming function for asset prices as the economy progressed via a sequence of temporary equilibria. In a deterministic model it has been shown that such behaviour may lead to speculative booms.

The restriction to one commodity eliminates price uncertainty in the second period commodity prices. If more than one commodity is assumed for the second period, then the consumer must form expectations about the second period commodity prices, and what is more, the expectations of the prices would, in general, be a function of current asset and commodity prices. The additional relationship between current prices and expected prices complicates the analysis.

---

19 Recalling the historical evidence cited in footnote 17, it was noted by Reid, op. cit. that landholders not only made detailed contracts on production plans, but also supervised them closely.

20 See Shell and Stiglitz (1967).

21 Price uncertainty is introduced in Chapter 9.
The other assumption, which is peculiar to our model, is the implicit assertion that each agent can calculate the equilibrium prices associated with the different patterns of returns. This assumption was required to obtain the rankings over the security patterns. If we assume, on the other hand, that agents cannot compute these prices with certainty (and objectivity) but have expectations about such prices, the theory becomes much more complicated. Presumably, the model would have similar features to the price uncertainty model described above.

Although the three assumptions we have discussed are overly restrictive, it is not obvious to me that their relaxation will alter the general thrust of the argument that conflict is possible over the creation of patterns of returns by producers.
CHAPTER 5

CHOICE OVER ASSET ECONOMIES: DEFAULT RISK AND CORPORATE LEVERAGE

In 1958 Modigliani and Miller (1958) proved a classic theorem in the theory of finance: they showed that in a perfect capital market, in equilibrium, corporate value is invariant to the debt-equity ratio. This proposition (now known as the M-M Theorem) introduced a long and often confused debate over the assumptions necessary for the proof. During the debate alternative proofs of the M-M theorem appeared as corollaries of more general systems, but these contributions compounded the confusion, because each model was based upon assumptions peculiar to itself, and it was not clear what crucial assumptions were required for the invariance theorem. Subsequently, Stiglitz (1969a) published an important paper surveying the literature, and discussing necessary conditions for the M-M theorem. His paper was important also, because it contained one of the first systematic discussions of the effects of bankruptcy on the leverage issue. In a later paper, Stiglitz (1972b), went further:

Previous studies have shown that under very general conditions, if there is no chance of bankruptcy, then financial policy has no effect on the value of the firm; there is no optimal debt-equity ratio. Under certain very restrictive conditions, the no bankruptcy condition may be removed. We show that when these restrictive conditions are not satisfied, and when there is a real possibility of bankruptcy, if the firm issues too much debt, the firm's valuation will depend on its debt-equity ratio ...1

1 See page 458, Stiglitz (1972b).
A similar position was taken by Smith (1970), (1972a), who asserts the following proposition:

If a corporation can invest at stochastic constant returns to scale, and the default risk on its bonds is positive, then an investor's optimal portfolio will have the property that he will prefer the corporation to increase, leave unchanged, or decrease its debt-equity ratio according as (corporate leverage is greater than, equal to, or less than private anti-leverage).2

Although in a recent paper Ben-Zion and Balch (1973) have shown that Smith has an inconsistency in his paper (i.e. the corporate bond rate is independent of leverage), they have not shown that the removal of the error reintroduces the invariance result. Therefore, it might appear that the Stiglitz-Smith (S-S) results are in direct conflict with the claim3 that the M-M theorem holds under quite general conditions.

The purpose of this Chapter is to show that there is no conflict once some of the implicit assumptions made by S-S are made explicit. The central point we wish to make is that whereas the M-M theorem is derived from a given commodity space, and associated equilibrium, S-S are comparing a sequence of commodity spaces with associated competitive equilibria. The key to this interpretation is contained in an aside by Stiglitz,4 where he observed that when the firm issues enough bonds to the point of default, so that a new security is created, then the consumption opportunity set of consumer-shareholders is not independent of the leverage decision. Of course, when the commodity space is altered, a new competitive equilibrium is obtained with

2 See page 71, Theorem 2, Smith (1972a).
3 For the most general statement of the M-M position we have referred to the standard text, Fama and Miller (1972).
4 See footnote 12, page 790, Stiglitz (1969a).
different security prices. If consumers know the prices in the new equilibrium, they can compare their equilibrium consumption in the old and new equilibria, and choose which one they prefer. Because consumers differ in their preference maps, then, in general, the ranking will not correspond to indifference.

Before we begin the analysis proper, the reader may find it beneficial if we sketch some of the proofs of the M-M theorem given by previous writers. Our purpose is not to reproduce the mechanics of their proofs, but to focus on the important assumptions.

1. THE LITERATURE

1.1 The M-M Proof:

Any discussion of the M-M theorem should begin with the original proof. The proof assumes the existence of an equilibrium for firms in a given "risk-class". That is, the value of the firm equals the expected return divided by a risk-adjusted discount-rate appropriate for that risk-class. Assuming that shareholders can borrow on the same terms as the levered firm, then they can always undo corporate leverage by private anti-leverage, so that by arbitrage the value of the firm is independent of leverage.

In a recent book, Fama and Miller (1972), (F-M) have generalized the proof (by omitting the restrictive risk-class assumption) to "perfect" capital markets. Their definition of a perfect market is crucial if the theorem is to be understood. They define a perfect market in the conventional, competitive sense, with the added proviso that any security issued by a firm has a perfect substitute in existence. (We shall see that it is this last assumption of a perfect substitute security where S-S and M-M differ). Given these assumptions, the M-M theorem can be proved quite easily.
1.2 Arrow-Debreu Markets:

A simple, elegant proof of the M-M theorem was found by Hirshleifer (1970) who deduced it from the equilibrium conditions of the Arrow-Debreu (1953), (1959), uncertainty model, where uncertainty is introduced by defining commodities in terms of possible states of the world. In this formulation, one can show that financial decisions are irrelevant: dividend policy and leverage are a matter of indifference to shareholders. Because objective market prices exist for all contingent commodities, production and consumption decisions can be separated so that the producer by maximizing profit, maximizes the wealth of the consumer-shareholder.5

These results are a direct consequence of the assumption that there exists a complete set of Arrow securities. It has been argued that these markets do not exist in reality because of transaction costs, and the problems associated with moral hazard.6 Without complete markets, one must cast around for a model that introduces uncertainty, and yet produces determinate results. Fortunately, there exist two models that fit these criteria, but they are severely restricted by a tight set of assumptions.

1.3 The Diamond Model:

In 1967 Peter Diamond published an important paper that has influenced much subsequent work on security market equilibrium. Diamond was careful to state explicitly the limitations of his model. He restricted his analysis to a one commodity, one period situation, where consumers could trade in a fixed set of securities. The securities represented claims on production processes that experienced technological uncertainty. Assuming no corporate bankruptcy

5 This result has come to be known as the Fisher Separation Theorem. For a discussion and references to the earlier literature see Hirshleifer (1970); and for a general proof in an Arrow-Debreu framework see Chapter 2.

6 Arrow (1970) has stressed the moral hazard problem, although there appears to be very little formal analysis of the problem.
he showed that corporate value was independent of leverage. But this is a side issue to a more important discovery that for a competitive equilibrium to exist, the production processes had to be decomposable, i.e., firms must be restricted to selling fixed patterns of returns across the states of nature. He observed that when the set of pattern of returns was fixed for the economy, the model was equivalent to the Arrow-Debreu theory with patterns of returns defined as commodities.

1.4 The Mean-Variance Model:

Sharpe (1964) and Lintner (1965), extended the portfolio theory of Tobin-Markowitz to a general equilibrium asset theory. The model assumes that all investors have identical values for the mean and variance of returns for each security, and that investors have utility functions with the mean and variance as arguments. The M-M theorem has been proved using this model. It has been realized that this model is a special case of Diamond's model, where preferences and expectations have been restricted to obtain portfolio separation and the formation of a mutual fund of risky assets.7

1.5 The Unifying Element:

In all these models, the claim is made that the market is competitive. It is not hard to show that each model can be considered as isomorphic to the Arrow-Debreu model, by taking an appropriate definition of a commodity. For example, in an asset model of the Diamond-type, the commodity is defined as a pattern of returns. The Fama-Miller assumption that a security always has a

7 By portfolio separation is meant that the consumer's choice of assets can be replicated by a risky mutual fund and the safe asset. For necessary and sufficient conditions on consumer preferences to obtain this result, see Cass and Stiglitz (1970). If all consumers agree on the subjective probability distribution, and they are sufficiently alike (for a precise statement see section 3.5) there exists a market mutual fund of the risky assets. See also the Appendix to this Chapter.
perfect substitute implies that agents must always choose patterns of returns existing in the market commodity (patterns of returns) space.\footnote{Alternatively, Ekern and Wilson (1974) have said that any security will be drawn from a set spanned by the available market securities. We consider a spanning set in section 4 below.}

At this point we should clear up a confusion that may arise. It is said sometimes that in a perfect capital market, any security created by a firm can be replicated by consumer-shareholders. In existing competitive equilibrium theory, the notion of creating a new commodity cannot be accommodated except in the two following ways:

(i) the creation of a commodity can be considered in the trivial sense of allowing agents to take non-zero values for the commodity, given a market price; and

(ii) let there be two equilibria - one without the commodity in the commodity space, and one with the extra commodity adjoined. The second equilibrium captures the creation of the new commodity. In proving the M-M theorem, the creation of securities is defined in the first sense, whereas we argue that S-S defined the creation of a new security in the second sense.

2. THE MODEL

2.1 Consider a one-good, one-period world with $i = 1, \ldots, m$ consumers and $j = 1, \ldots, n$ firms. For expository ease, we will limit the analysis initially to two firms, but in a subsequent section we will introduce more firms to obtain a more general result. Because we have restricted ourselves to a single commodity (it could be money with fixed relative prices) and a single
period, price uncertainty is absent; and we will assume that uncertainty enters via technological uncertainty. To emphasize the importance of risk aversion consider the first firm to be safe, and the second firm to be risky. Formally, consider the set of states of the world $S$ to be an interval of the real line, i.e., $[\underline{s}, \bar{s}] \subseteq \mathbb{R}$. Because firm 1 has degenerate technological uncertainty then the commodity input $I_1$ generates a total return $r_I$ (\(r > 0, \text{ but finite}\)) for all states $s \in S$. Firm 2 suffers from technological uncertainty; but for simplicity assume that the uncertainty is independent of scale, so that for a commodity input of $I_2$ the total return is $h(s)I_2$, $s \in S$. For ease, consider $h$ to be a monotonically increasing function of $s$, and $h(s) > r$.

To introduce debt and equity financing, assume that the total investment $I_2$ is the sum of equity $E$, and debt $B$, i.e., $I_2 = E + B$. For convenience define a measure of leverage as $\alpha \equiv (B/I_2)$; and also $(1-\alpha) \equiv (E/I_2)$, where $\alpha \in [0, 1]$. Assume that debt pays a nominal physical return $r_B$ for those states for which there is no default; and a total return $h(s)I_2$ when default occurs. Equity pays a total return $h(s)I_2 - r_B$ for those states where there is no default; and zero when default occurs. The firm defaults on its bonds at the critical state $s^*$, which is defined as the unique root (if $r > h(s)$) of

$$h(s)I_2 - r_B = I_2(h(s) - r\alpha) = 0.$$ 

Therefore we have $s^* = s^*(\alpha)$; and from the assumption that $h(s)$ is monotonically increasing, $s^*(\alpha)$ is single valued.

---

9 This is a simplifying assumption that conforms to the M-M notion of a risk-class; and it has been a common enough assumption in the finance literature. Nevertheless Diamond gave a brief discussion of the case where the pattern of returns was not independent of scale, and in turn this has led to a debate on investment rules. For a partial bibliography on the debate, and a general discussion of choice over asset economies see Chapter 4. This Chapter includes an example identical to the model used here, except that leverage is fixed, and the patterns of returns can be altered by production decisions.
2.2 Assume that the $i$th consumer maximizes his von Neumann-Morgenstern utility function $u_i \in C^2$, defined over consumption $x_i(s)$ in the state of the world $s$, and given his subjective probability distribution function $G_i(s)$.

The consumer, therefore, has an objective function (where the integral is a Lebesgue-Stieltjes integral),

$$U_i = \int_{s} u_i(x_i(s))dG_i(s)$$

(1)

which we will assume exhibits risk aversion, i.e., $u'_i > 0$, $u''_i < 0$. At this stage we will not place any further restrictions on the objective function.

Assume that the consumer has an initial endowment of commodity $W_i > 0$, and he can purchase proportions $\theta_{i\lambda}$ of the values of the assets $V_\lambda$, $\lambda = 1, B, E$. Normalizing the assets in commodity units $A_\lambda$, and with prices $p_\lambda$, we have $V_1 = p_1 I_1$, $V_B = p_B B$, $V_E = p_E E$. Let the asset market be competitive so that the $i$th consumer’s budget constraint (assuming non-satiety) is

$$\sum_{\lambda} \theta_{i\lambda} p_\lambda A_\lambda = W_i, \quad V_i.$$  

(2)

The proportions of the assets purchased entitle the consumer to a proportion of the physical returns: therefore, the consumption set is defined by

$$x_i(s) = \begin{cases} 
\theta_{i1} I_1 + \theta_{iB} h(s) I_2; & \text{for } s \in [s, s^*] \\
\theta_{i1} I_1 + \theta_{iB} r I_2 + \theta_{iE} I_2[h(s) - r \alpha]; & \text{for } s \in [s^*, s].
\end{cases}$$

(3)
Assuming no default, \( x_i(s) \geq 0 \), and substituting (3) into (1), we obtain

\[
\begin{align*}
\text{Max} & \quad u_i(\theta_i(I_1 + I_2, a)) \\
\text{subject to} & \quad \sum_{k} \theta_i p_k = w.
\end{align*}
\]

Forming the Lagrangian expression \( L_i \) with \( \lambda_i \) the multiplier associated with the budget constraint, the conditions for an interior maximum are (assuming hereafter that the maximum is achieved other than on points of discontinuity):

\[
\begin{align*}
\left\{ \begin{array}{l}
\text{Max} \quad u_i(\theta_i I_1 r + \theta_i h(s) I_2) dG_i(s) \\
\text{subject to} \quad \sum_{k} \theta_i p_k = w_i.
\end{array} \right.
\end{align*}
\]

10 The no-default assumption is important in ensuring that the consumer does not issue a security which has an actual pattern of returns different from its nominal pattern of returns. For a complete discussion, including its importance in equilibrium existence proofs see Chapter 6.
Conditions (5), (6) and (7) are easily recognizable as general conditions for the consumer's portfolio problem. Along with the market clearing conditions, and value-maximizing conditions for investment,

\[ \sum_{i} \theta_{i\lambda} = 1 \quad \lambda = I, B, E; \]
\[ I_{1} + B + E = \sum_{i} W_{i}; \]
\[ \frac{\partial}{\partial I_{j}} (V_{j} - I_{j}) = 0, \quad j = 1, 2. \]

they form a system of \(3(M + 1) + 3\) equations. But from Walras' Law we know that

\[ \sum_{i} \sum_{\lambda} \theta_{i\lambda} A_{i\lambda} = I_{1} + I_{2} = W = \sum_{i} W_{i}, \]

so that we have \(3m + 5\) independent equations to determine \(3m\) portfolio holdings \((\theta^{*}_{i\lambda})\), 3 relative prices, (the commodity price is 1), and the investment allocation \(I_{1}, I_{2}\).

Now this equilibrium depends upon the parameter \(\{\alpha\}\). Because we have assumed that production uncertainty is of the special multiplicative form with respect to scale, then variations in the scale of inputs do not affect the patterns of returns. That is, the investment decisions are consistent with a single asset-commodity space. To see this, observe that for an interior solution to the portfolio problem, the necessary conditions are independent of \(I_{1}, I_{2}\). On the other hand, variations in \(\alpha\), if there is default, will result
in a change in the patterns of returns available to consumers. Therefore, variations in a effectively generate a choice over asset equilibria produced by different patterns of returns.

It is important to notice that we have restricted the actions of consumers to a passive role of not creating their own securities. For simplicity, assume that consumers wish to choose between the asset equilibria generated by the financing arrangements of their agent, the firm, and they are far-sighted enough to predict with certainty the price vector associated with each asset equilibrium.\footnote{The assumption of perfect foresight of equilibrium asset prices is very strong. But if this assumption is not made, we would need to postulate a much more complicated price uncertainty model, which would complicate the simple point we wish to make.}

3. CHOICE OVER ASSET EQUILIBRIA

3.1 To reveal consumer i's preferences over the set $a \in [0, 1]$, consider the problem

$$\max_{\{a\}} U_i(\theta_{ik}(a), I_{ik}(a), a)$$

subject to

$$\sum_j \theta_{ik}(a) A_{ik}(a) p_i(a) = W_i$$

where $(\theta_{ik}(a), I_{ik}(a), p_i(a))$ is an equilibrium vector corresponding to the asset economy generated by the commodity space associated with $a \in [0, 1]$. The condition for an interior maximum is,
\[
\frac{dL^i}{da} = \sum_l \left\{ \frac{\partial L^i}{\partial \theta_{il}} \frac{d\theta_{il}}{da} + \frac{\partial L^i}{\partial p_k} \frac{dp_k}{da} \right\} + \sum_j \frac{\partial L^i}{\partial \theta_{lj}} \frac{d\theta_{lj}}{da} + \frac{\partial L^i}{\partial \alpha} = 0. \tag{10}
\]

Now from the portfolio conditions (5)-(7) it is clear that \( \frac{\partial L^i}{\partial \theta_{il}} = 0 \), and also it is easy to show that \( \frac{\partial L^i}{\partial \theta_{lj}} = 0 \), therefore we find

\[
\frac{dL^i}{da} = I_2 \left[ \theta_{iB} - \theta_{iE} \right] \int_{s^*} u_i^r dG_i(s) \]

\[
+ \lambda ^i \left[ \theta_{iB} - \theta_{iE} \right] I_2 + \lambda ^i \sum_k \theta_{ik} A \frac{dp_k}{da} = 0. \tag{11}
\]

Substituting from (6) into (11)

\[
[\theta_{iE} - \theta_{iB}] \int_{s^*} u_i^r (h(s)/a) dG_i(s) + \lambda ^i \theta_{iE} (p_B - p_E) + (\lambda ^i / I_2) \sum_k \theta_{ik} A \frac{dp_k}{da} = 0. \tag{11'}
\]

The first term of (11') can be thought of as the gain in utility from a change in the pattern of returns independently of wealth, and the last two terms are wealth effects. (Notice that \( \lambda ^i \) is the negative of the marginal utility of wealth). Eliminating \( \lambda ^i \) by (5) we obtain a further variant of (11)
Although (ll") adds little to the previous conditions it will provide a useful comparison with the conditions derived below for the optimal choice of \( \alpha \).

Now consider cases which will satisfy (ll).

### 3.2 Shareholder and Bondholder: Leverage-Preference:

Because we have assumed very weak restrictions upon preferences and expectations of consumers, there is no obvious reason why the leverage chosen by consumer \( i' \), \( \alpha_{i'} \), should correspond with that chosen by consumer \( i'' \), \( \alpha_{i''} \) (\( i' \neq i'' \)). This conflict cannot be resolved in the existing market set-up, but it is possible that extra market arrangements may achieve a determinate result (see section 5 below). It is in this sense that S-S argue that shareholders are not indifferent to leverage. Notice that Smith’s results correspond to the first term of (ll’), but he has omitted any effects due to changes in asset prices - or equivalently - in rates of return.

The condition (ll) is quite compatible with solutions for portfolio holdings, where a consumer may be a bondholder, but not a stockholder, or vice versa. Stiglitz (1972b), by assuming risk-neutral consumers, and differences in expectations, obtains this division between stock and bondholders. Needless to say, the argument is not changed in any essential way.
Although we have shown the conflict is possible over the choice of the asset economy generated by the leverage decision, there are several interesting cases where all investors are indifferent to leverage - that is, where the M-M result is obtained.

3.3 No Default:

It should be clear from our discussion above that if there is no default by the corporate bond, then the corporate bond and the safe asset are perfect substitutes, and in equilibrium they have the same price. To see this, consider an interior solution for the consumer's portfolio problem i.e., conditions (5)-(7). No default implies that the interval \([s, s^*]\) is empty because \(h(s) > or\). Therefore conditions (5), (6) collapse to

\[
\int_{s}^{s^*} u_i'(r) dG_i(s) + \lambda_i p_i = 0
\]

\(\int_{s}^{s^*} u_i'(r) dG_i(s) + \lambda_i p_B = 0\)

which imply that \(p_i = p_B\). Furthermore, we have from (7) that

\[
\int_{s}^{s^*} u_i'(h(s) - or) dG_i(s) + \lambda_i p_E (1 - \alpha) = 0.
\]

Now for \(r < h(s)\) it is easy to show that for consumer \(i\) to hold positive amounts of equity, \(\lim_{\alpha\to1} p_E = +\infty\). This extreme result depends upon the
assumption that the $i^{th}$ consumer is "small" in his holdings of assets, compared with the aggregate asset holdings. (Because he holds an $\varepsilon$-proportion of the total input to the risky asset and reaps a finite return over some states, then his physical rate of return goes to infinity). Of course, this behaviour is limiting behaviour and we will exclude the point $a = 1$ for the next few sections. (A similar argument holds for $a = 0$).

If $r \leq h(s)$ then default is absent from our discussion. Alternatively consider $r > h(s)$ so that default occurs for some $a^* \in (0, 1)$. With $a$ restricted to the no-default interval $(0, a^*)$, then (12), (13), (14) continue to hold. By summing (13) and (14), consider

$$\int \limits_{-\infty}^{\infty} u' h(s) dG_1(s) + \lambda \ p_2 = 0$$

where $p_2 = \alpha p_1 + (1-\alpha)p_E$. The price $p_2$ is the price of the (commodity) pattern of returns $h(s)$, which is independent of $a$, and because $p_1$ is also independent of $a$, we obtain

$$\frac{\int \limits_{-\infty}^{\infty} u' h(s) dG_1(s)}{\int \limits_{-\infty}^{\infty} u' r dG_1(s)} = \frac{p_2}{p_1},$$

which is completely analogous to the usual marginal rate of substitution equated to the price ratio condition, from elementary price theory. We have proved also, the M-M theorem, because the value of the corporation is independent of $a \in (0, a^*)$. 


In this simple case, we can check by examining the leverage preference condition \((11')\). Now no default for the corporate bond implies the emptiness of the interval \([s, s^*]\), and therefore the integral term vanishes. The remaining two terms can be simplified by observing that \(p_B = p_1\) is independent of leverage, whereas \(p_E = (p_2 - a p_1)/(1-a)\). Thus

\[
\theta_iE(p_1 - p_E) + \theta_iE(1-a)(p_2 - p_1)/(1-a)^2 = 0
\]

or

\[
\theta_iE(p_1 - p_E) - \theta_iE(p_1 - p_E) = 0.
\]

Therefore \(\frac{dL}{da}\) is identically zero, when there is no default, implying that consumer \(i\) is indifferent to leverage. Because the argument holds for all consumers, then the M-M theorem follows. This proof reveals the importance of the assumption that there is a perfect substitute asset for the corporate bond. If leverage was pushed high enough to force default, so that the corporate bond was not a perfect substitute for the riskless asset, then the proof fails. But if default occurs, and the risky bond has a perfect substitute (i.e., there is another security in existence) then again, the M-M proof holds. The latter proof will be considered in section 4.

### 3.4 Identical Investors:

Another special case that will give the M-M result, is the situation where all investors are identical. That is, they have identical expectations, utility and wealth. The proof of this proposition is straightforward. Writing \(u_i = u, G_i(s) = G(s), \) and \(W_i = (W/m)\), the portfolio conditions (5)-(7), along
with market clearing (8) imply that \( \theta_{1}^{i} = \theta_{2}^{i} = \theta \), i.e., all consumers hold the same proportion \( \theta \) of each asset \( \lambda \). For the choice of \( a \), the condition (11') becomes (observing that \( P_{1}^{t}, P_{2}^{t} \) are independent of \( a \)),

\[
(\theta - \theta) \int_{s} u'(h(s)/a)dG(s) + \lambda \theta(P_{B} - P_{B}^{t}) - \lambda \theta(P_{B} - P_{E}^{t}) = 0.
\]

Therefore \( \frac{dL}{da} \) is identically zero, and the replicated consumer is indifferent to leverage. The result is quite trivial because the set of consumers acts as a single consumer who must hold all the assets. Therefore, the representative consumer must hold bonds and equity in the same proportions as issued by the risky firm; and that implies that the market pattern of returns always conforms to the original production patterns.

### 3.5 Mutual Funds and Leverage:

We have just seen that if all consumers are identical, then the M-M theorem holds, because all assets are held by all consumers in the same proportion. We observed in the introductory survey, section 1, that the mean-variance formulation implied that all investors would hold securities in a single mutual fund, and, of course, that implied consumer indifference to leverage. Thus, if we can find necessary and sufficient conditions for the existence of a single mutual fund, composed of all the risky assets, then we will have isolated an important class where the M-M theorem holds.

In a recent contribution Cass and Stiglitz have found necessary and sufficient conditions for portfolio separation: i.e., conditions under which the consumer's opportunity set can be reproduced by two mutual funds partitioning the set of assets. A special case of this more general property is "monetary" separation, where one mutual fund contains all the risky assets and
the other is the safe asset. Using the Cass-Stiglitz results on separation, it is not difficult to show\(^\text{12}\) that all investors will purchase risky assets in the same proportion if:

(i) all investors have the same subjective probability distribution over the states of the world; and

(ii) all investors have utility functions of the form:

\[
\begin{align*}
\text{(a)} & \quad u_i(x_i) = -\left(\frac{x_i}{A_i}\right)^B, \quad B = 0; \\
\text{(b)} & \quad u_i(x_i) = \ln(A_i + x_i), \quad B = 1; \\
\text{(c)} & \quad u_i(x_i) = \left[\left(\frac{B-1}{B}\right)^2 - B \right]\left[\frac{A_i + Bx_i}{(B-1)/B}\right], \quad (B \neq 0, 1),
\end{align*}
\]

and \(B\) is a common value for all investors.

Given these conditions it follows that for each consumer \(i\),

\[
\theta_i^B = \theta_i^E = \theta_i^M, \quad \text{when the corporate bond is a risky asset. Again, as with the case of the identical consumers, each consumer holds a share } \theta_i \text{ of the risky mutual fund, which has a total return } h(s) \text{ and a price } p_2.\(^\text{13}\) Because } p_1 \text{ and } p_2
\]

\(^{12}\) For a proof see Rubinstein (1974), where it is referred to as the Universal Portfolio Separation Theorem. The interested reader is directed also to Mossin's book (1973) for the special case of a quadratic utility function.

We should point out that Portfolio separation can be obtained also if the subjective probability distribution is Pareto-Levy, or Normal if the variance is finite. These distributions are not strictly applicable to our problem because the pattern of returns are bounded below by zero, and are therefore not symmetric. Furthermore, for the formulation of a market mutual fund of risky assets, the consumers must agree upon the probability distributions across the states.

\(^{13}\) The similarity in the proofs between identical consumers and the mutual fund case, follows because for the mutual fund to be formed, consumers must be identical in their pattern of demands for risky assets. Or, in the space of risky asset preferences, all consumers have identical homothetic preferences, except for increasing transformations.
are independent of \( u \), the condition \((11')\) is identically zero, and our assertion is proved. (Of course, if the corporate bond is not risky, leverage indifference is proved as in 3.3).

An alternative way of obtaining this result, is to observe the following argument. Consider the equilibrium generated by the riskless pattern of returns and the risky market portfolio pattern of returns. Because of the portfolio separation property of preferences, consumers are indifferent to any set of risky patterns of returns that have a vector sum equal to the risky portfolio pattern of returns. Therefore, variations in leverage alter the components of the risky set, but not the vector sum; and thus the competitive equilibrium is not disturbed by variations in \( u \). (The same argument applies to 3.4).

4. CORPORATIONS AND PERFECT SUBSTITUTE ASSETS

4.1 The special cases we examined in 3.4 and 3.5 require severe restrictions on the preferences and expectations of consumers: in fact, the restrictions are so severe, they cannot be considered particularly realistic especially when they imply the formation of a single mutual fund, or at least a single mutual fund of risky assets. On the other hand, the M-M theorem has been asserted to exist under quite general conditions - independently of consumer preference restrictions. We can prove that assertion if there exists a perfect substitute asset, or a perfect substitute portfolio, for the risky bond issued by the corporation. There are two propositions we can prove using the perfect substitute approach:

(a) the first proposition is a restricted invariance result which includes the analysis of section 3.3 as a special case;
(b) the second proposition is a global univalence theorem which corresponds to the conventional statement of the M-M theorem (see F-M).

4.2 We will consider the first proposition. Let there exist \( n \geq 2 \) firms, where the first firm remains as our riskless producer, but the remaining firm \( j = 2, \ldots, n \) are risky. In particular, let the pattern of returns for the second firm be described as above, and for firms \( j = 3, \ldots, n \) summarize their output patterns as \( r_j(s)I_j \), where \( I_j \) is the input of physical commodity into the \( j \)-th firm. We will find it necessary to consider a finite-dimensional state space \( s \in \{s_1, \ldots, s_s\} = S \). Assume that the available patterns of returns \( r, h(s), r_3(s), \ldots, r_n(s) \) are independent, so that the matrix of available patterns \( Z \) has rank \( \tilde{n} \leq s \), where \( \tilde{n} \leq n \), consider the finite-dimensional analogue of problem (4),

\[
\begin{align*}
\text{Max } & \quad U_1(\theta_{1i}, I_j, a) \\
& = \sum_{s_1} u_1(\theta_{1i} I_1 r + \theta_{1B} h(s)I_2 + \sum_{\lambda=3}^{\tilde{n}} \theta_{1\lambda} r_{\lambda}(s)I_{\lambda}) \pi_s \\
& \quad + \sum_{s''} u_1(\theta_{1i} I_1 r + \theta_{1B} h(s)I_2 + \theta_{1E} I_3[h(s) - ar] + \sum_{\lambda=3}^{\tilde{n}} \theta_{1\lambda} r_{\lambda}(s)I_{\lambda}) \pi_{s''} \\
& \quad \text{subject to} \\
& \quad \sum_{\lambda} \theta_{1\lambda} P_{k\lambda} h_k = W_1, \quad \lambda = 1, B, E, 3, \ldots, \tilde{n}.
\end{align*}
\]

Define the sets \( S' = \{s_1, \ldots, s_{k}\} \) and \( S'' = \{s_{k+1}, \ldots, s_{\tilde{s}}\} \), where \( s_k, s_{k+1} \) are chosen such that \( h(s_k) - ar < 0 \), and \( h(s_{k+1}) - ar > 0 \). Because \( h \) is strictly increasing in \( s_k \), the sets are well-defined and partition \( S \). The analogue
of the portfolio equations (5)-(7) are

\[ \sum_{S_i} u_i I_1 r_{i|S} + \sum_{S''} u_i I_1 r_{i|S''} + \lambda^i P_j I_1 = 0 \]  \hspace{1cm} (19)

\[ \sum_{S} u_i h(s) I_2 r_{i|S} + \sum_{S''} u_i I_2 r_{i|S''} + \lambda^i P_B I_2 a = 0 \]  \hspace{1cm} (20)

\[ \sum_{S''} u_i I_2 [h(s) - ar]_{i|S''} + \lambda^i P_E I_2 (1-a) = 0 \]  \hspace{1cm} (21)

\[ \sum_{S} u_i I_j r(s)_{i|S} + \sum_{S''} u_i I_j r(s)_{i|S''} + \lambda^i P_j I_j = 0, \quad \text{for} \quad j = 3, \ldots, n. \]  \hspace{1cm} (22)

To reveal consumer i's preferences over the set \( \alpha \in [0, 1] \) consider the analogue of condition (11)

\[ \frac{dL}{d\alpha} = I_2 [\theta_{iB} - \theta_{iE}] \sum_{S} u_i r_{i|S} + \lambda^i I_2 [\theta_{iB} - \theta_{iE} P] \]

\[ + \lambda^i \sum_{j=3}^{n} \theta_{ij} A_j \frac{dp_j}{d\alpha} = 0. \]  \hspace{1cm} (23)

It is easy to show that the results obtained when the number of assets was restricted to \( \ell = 1, B, E \), continue to hold for the expanded set of assets. Nevertheless, we can prove a generalization of the result discussed in section 3.3, i.e., when the corporate bond is riskless. Indifference to leverage in the no default region was shown because the corporate bond had a perfect substitute - the riskless asset. But if the corporation's risky bond, or equity, has a perfect substitute, then the argument is perfectly analogous.
Consider the case where the corporate leverage is restricted to the range \( \alpha \in (h(s_k)/r, h(s_{k+1})/r) \), so that \( S', S'' \) are invariant to changes in leverage. Let there exist two assets \( j_1, j_2 \) with the pattern of returns

\[
\begin{align*}
     r_{j1} &= \begin{cases} 
        h(s) & \text{for } s \in S' \\
        0 & \text{for } s \in S''
    \end{cases}, \\
     r_{j2} &= \begin{cases} 
        0 & \text{for } s \in S' \\
        r & \text{for } s \in S''
    \end{cases}
\end{align*}
\]

and competitive prices \( p_{j1} \) and \( p_{j2} \) respectively. The portfolio conditions for an interior solution for these assets are

\[\sum_{S'} u_i h(s) \pi_{is} + \lambda \pi_{i}^{j1} = 0 \tag{24}\]

\[\sum_{S''} u_i r \pi_{is} + \lambda \pi_{i}^{j2} = 0. \tag{25}\]

But from (20), and by summing (24) and (25) multiplied by \( \alpha \), we find

\[\sum_{S'} u_i h(s) \pi_{is} + \sum_{S''} u_i r \pi_{is} = -\lambda (p_{j1} + p_{j2}) = -\lambda p_B \alpha. \tag{26}\]

Therefore, \( p_B = (p_{j1}/\alpha) + p_{j2} \). Similarly, by adding (20) and (21)

\[\sum_{S'} u_i h(s) \pi_{is} + \sum_{S''} u_i h(s) \pi_{is} = -\lambda (\alpha p_B + (1-\alpha)p_B) + \lambda p_2. \tag{27}\]

Observe that \( p_2 \) is independent of \( \alpha \), because the left-hand side of (27) is independent of \( \alpha \), (\( \lambda \) is independent of \( \alpha \) by (19)). Therefore, we have shown
the M-M theorem for the restricted range of leverage for which a perfect substitute combination of assets exists. Notice that the corporate asset has a value composed of a linear combination of composite asset prices, i.e.,

$$P_2 = P_{j1} + \alpha p_{j2} + (1-\alpha)p_E.$$ 

For completeness, it is easy to substitute into condition (23) and show that \( \frac{dL_i}{du} \) is identically zero for restricted leverage.

If there exists a continuous chain of substitutes for the risky bond, over the full range of leverage, \( \alpha \in [0, 1] \) then the global invariance result can be obtained by linking a sequence of restricted invariance results. There is one important instance where this occurs - a market with a "spanning set" of securities. A special case of a "spanning set" is Arrow-Debreu securities.

4.3 To introduce Arrow-Debreu securities into our model, let there be \( s \) securities; one for each state, such that the return for the \( k \)th A-D security is

$$r_k = \begin{cases} 
1 & \text{for } s = s_k \\
0 & \text{for } \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_s\}, 
\end{cases}$$

\( k \in \{1, \ldots, s\} \equiv s, \)

and the competitive price is \( p_k \). From an interior solution of the portfolio conditions (19)-(22), it follows that
(i) \[ P_1 = r \sum_S P_k \]

(ii) \[ P_B = \left( \sum_{S'} h(s_k) P_k / a + \sum_{S''} P_k r \right) \]

(iii) \[ P_E = \sum_{S''} [h(s_k) - ra] P_k / (1-a) \]

Therefore, it follows from (ii) and (iii) that

\[ P_B + P_E = \sum_S h(s_k) P_k = P_2' \]

which proves the global M-M theorem, as shown by Stiglitz (1969a) and others.\(^{14}\)

But the set of Arrow-Debreu securities is just one of many sets of securities that span the positive orthant (i.e., the consumption set). Indeed, it is well known,\(^{15}\) that with short-sales, a spanning set of securities can be converted into A-D securities by an appropriate choice of weights. Therefore we can generalize the M-M proof above, to spanning sets of securities, by prefacing the proof with the remark that a spanning set can be converted into A-D securities. Clearly this spanning set of securities is what F-M define as a "perfect" capital market.

\(^{14}\) It is easy to show that \(P_E, P_B\) are non-trivial functions of \(a\) i.e.,

\[ \frac{dp_B}{da} = -\left( \sum_{S'} P_k h(s_k) \right) / a^2 < 0, \text{ a.e.} \]

\[ \frac{dp_E}{da} = \sum_{S''} [P_k h(s_k) - r] / (1-a)^2 > 0, \text{ a.e.} \]

Observe that the nominal rate of interest \((r/P_B)\) is an increasing function of \(a\), when there is default.

4.4 The All Debt Firm:

In section 3.3 above we excluded the extreme points \( a = 0, 1 \), corresponding to a complete equity and complete debt firm. Now it is sometimes mentioned in asides in the literature that an all-debt firm with default risk is equivalent to an all-equity firm. In our formulation, this can be shown easily enough in the following way. By market clearance it follows that for \( a = 0 \), we must have \( \theta_{1B} = 0 \); therefore the objective function of (18) becomes

\[
\sum_{S} u_{i} (\theta_{1L} r + \theta_{1E} h(s)) + \sum_{k=3}^{n} \theta_{iB} r_{k} (s) I_{i} \pi_{is}'. \tag{18'}
\]

Similarly, if \( a = 1 \) the objective function becomes

\[
\sum_{S} u_{i} (\theta_{1L} r + \theta_{1E} h(s)) + \sum_{k=3}^{n} \theta_{iB} r_{k} (s) I_{i} \pi_{is}'. \tag{18''}
\]

as long as \( r \geq h(s) \). This assumption is required to ensure that as leverage becomes complete, the risky bond mirrors the corporate pattern of returns. Clearly (18'), (18'') are identical and lead to the conclusion that an all-debt, and all-equity firm are formally identical in this type of model. Indeed, the whole notion of a nominal interest rate, \( (r/P_{b}) \) is irrelevant to the analysis. If \( r < h(s) \), then we must restrict our discussion to leverage \( a < 1 \), because the problem is not well-defined at \( a = 1 \) (although we can discuss limiting behaviour).
5. OPTIMAL LEVERAGE

5.1 In the previous section we discussed cases where the M-M theorem may, or may not be applicable. Now we propose to investigate the determinates of the optimal choice of leverage. We will show that when there is a choice over asset spaces generated by variations in leverage, the choice of leverage depends upon the preferences and expectations of all holders of the corporation's securities. Indeed, the public good nature of the creation of a new security market will become obvious, when our optimal conditions are compared to the sort of conditions produced by public good problems.

5.2 The Pareto conditions are derived from the following problem.

\[
\begin{align*}
\text{Max } U_1 &= \sum_{S} u_1(\theta_1 I_1 r + \theta_1 h(s) I_2 + \sum_{i=3}^{n} \theta_i r_i(s) I_i)_{is} \\
&+ \sum_{S''} u_1(\theta_1 I_1 r + \theta_2 I_2 r + \theta_3 I_3 h(s) - \theta_1) \\
&+ \sum_{i=3}^{n} \theta_i r_i(s) I_i)_{is} \\
\text{subject to} \\
(i) & \quad U_i = \bar{U}_i, \quad i = 2, \ldots, m; \\
(ii) & \quad \sum_{i}^{n} \theta_i = 1, \quad \ell = 1, B, E, 3, \ldots, \bar{n}; \\
(iii) & \quad \sum_{j=1}^{\bar{n}} \bar{I}_j = \bar{I}.
\end{align*}
\]

(28)
We can form a Lagrangian expression, associating multipliers with the constraints. The first order conditions for an interior (i.e., \( x_i(s) > 0 \)) solution imply the following condition for optimal \( \alpha \),

\[
\sum_i \left\{ \left( \theta_{iB} - \theta_{iE} \right) \frac{ \sum_s u_i'(s) \pi_i s }{ \sum_s u_i' r_i s + \sum_s u_i' r_i is} \right\} = 0. \tag{29}
\]

We observed in 3.2 that consumers with different preferences, expectations and wealth may not be indifferent to leverage if there was no perfect substitute asset for the risky corporate bond. Although consumers may not be able to satisfy their private choice of leverage, there does exist a Pareto solution which determines an optimal leverage set for all consumers. This is a weak criterion, because if consumers undertake extra-market bargaining, where coalitions may form and reform costlessly, then we cannot deduce to whom most of the benefits will flow in Pareto bargain. On the other hand, if the formation of coalitions involve communication costs, then some consumers may be able to exercise monopoly power. For example, consider an extreme case where one consumer - we can call him the entrepreneur - can exploit other bond and shareholders because the cost of forming coalitions against him is too costly compared with the feasible gains from bargaining. The entrepreneur will choose the leverage according to the solution of condition (11) (i.e., his private leverage choice), and attract the anticipated clientele. Of course, as the entrepreneur varies leverage over the feasible range, the other consumers change their portfolios; but this variation does not alter the point that they may have preferences over the set of feasible degrees of leverage.
5.3 In sections 3 and 4 we discussed cases where variations in leverage left the competitive equilibrium undisturbed - and therefore, consumers were indifferent to variations in leverage. It is well-known that a competitive equilibrium achieves a Pareto optimum, so therefore the indifference set of leverage ratios is also an optimum set.

CONCLUSION

In this chapter we have tried to reconcile the arguments of M-M and S-S, and show that their arguments are not in conflict, but depend upon different assumptions. The models make different assumptions about the dimensions of the asset space, and differ over the concept of asset creation. For M-M, the creation of an asset implies the taking of a non-zero holding of an asset (commodity), given an associated market price. On the other hand, S-S consider the creation of an asset as an addition to the commodity space, and the creation of a new asset equilibrium.

One of the unsatisfactory features of the S-S type theory is the assumption that asset markets are restricted in dimension, and it is only firms that can increase the dimension - and then only to a limited extent. Implicit in this restriction is a much more complicated theory involving the existence of markets and transaction costs. A satisfactory theory would introduce these elements into the model explicitly, and produce conditions for the existence or non-existence of markets. Furthermore, the theory would need to demonstrate that coalitions of consumers (in our case firms) may be able to "open" a new asset market, whereas isolated consumers may not. Presumably the argument would require set-up costs which were large compared to individual consumer wealth, but could be surmounted by the coalitions.
Alternatively, the M-M argument presumes that the asset market is of fixed dimension for the decisions of economic agents. It would appear from F-M's discussion of a perfect asset market that transaction costs are not important and the "creation" of a new asset by a firm can be duplicated by an consumer.\(^16\) If this statement is taken to mean that the firm or any consumer can create any pattern of returns for which there exists a market price (or a derivable market prices, because it is a perfect substitute for a convex combination of existing assets) then the asset economy must be equivalent to an Arrow-Debreu economy. A number of writers have objected to the reasonableness of complete Arrow-Debreu markets,\(^17\) although positivists could counter this objection by the argument that a theory should be judged by its testable implications and predictive ability, and not by the realism, or otherwise, of its assumptions.

Our reconciliation of the S-S and M-M arguments also offers a possible insight into the implicit theorizing of the more traditional finance literature.\(^18\) The traditional theorists argued that corporate value would increase with increases in leverage from zero to some positive level \(a'\), remain constant over some interval \([a', a'']\), and decline for "unreasonable" levels of leverage \([a'', 1]\). The initial phase was rationalized by the tax-deductability of debt interest payments; and, in general, there appears to be little dispute over this point.\(^19\) Because we have omitted taxes in our

\(^{16}\) See, p.155, F-M.

\(^{17}\) For example, see Arrow (1970).

\(^{18}\) From a careful reading of the traditional position (for a statement see Solomon (1963)). I think it would be fair to say that their conclusions on the relationship between corporate value and leverage, were derived from casual empiricism, rather than from any rigorous model. Although their theoretical constructions leave much to be desired, we are more interested in their conclusions, which they thought mirrored the market-place. I am not suggesting that this argument is what the traditional theorists had in mind, nor that they would agree with this ex poste, but that it is consistent with their conclusions.

\(^{19}\) See F-M, Chapter 4, section III.D.
discussion, we can take it that the initial interval \([0, a')\) is empty, and
corporate value is invariant to leverage up to the "unreasonable" interval
\([a", 1]\). Now it might be possible to rationalize this relationship in the
following way. For low levels of leverage, the corporate bond is riskless and
is a perfect substitute for the riskless asset (say the government bond); and
even with moderate risk of default, the corporate bond has substitutes in the
market; but with extreme leverage, the asset patterns - especially the equity
patterns - may not have substitutes, because they have "unreasonable" patterns.

By "unreasonable" would have to be meant patterns of returns that gave positive
returns in states that were considered by all consumers to have virtually zero
probability of occurrence, so that the gains from opening markets would not
cover the set-up costs. Of course, from our general equilibrium treatment,
we cannot say anything about corporate value over the sequence of asset
economies, nor does it make any sense, because all asset prices are presumed
to vary. The use of corporate value as a surrogate for consumer preferences
fails, because the Fisher Theorem is no longer applicable.\(^{20}\) Nevertheless,
the point the early theorists were trying to make about consumer preferences
at high levels of leverage, might be rationalized by the non-existence of markets
for extreme patterns of returns.

Finally, we should mention the factor of "moral hazard" - or the
confusion of outcomes generated by states of the world and the actions of agents.
The model we have used has technological uncertainty, but does not allow any
role for agents altering the pattern of returns by production decisions, or
even more crudely, entrepreneurial agents absconding with the funds! As
Hirshleifer (1970) has observed, the concept of uncertainty is very precise,

\(^{20}\) For the Fisher Theorem to apply as the firm introduces new assets, we would
require a partial equilibrium analysis, so that all asset values are held
constant, except the firm's equity and debt. Clearly it is no longer
necessary that corporate value is invariant to leverage, but it would
require further assumptions to show that the value declined with very
high leverage. (Implicitly, we must assume \(r < h(s)\), see section 4.4).
and eliminates the possibility of some agents exploiting the ignorance of others. I suspect that one could make a case for the proposition that the incentives for deviating from the contractual production by the equity-holding entrepreneur, are positively related to leverage, as long as the bondholders have difficulty in distinguishing between the outcome of actions by the entrepreneur, and outcomes produced by states of the world. Of course, these assumptions would violate the perfect market assumptions made by M-M.

21 Unfortunately, there has been very little formal analysis of moral hazard, so that there is a danger of it becoming a catch-all for behaviour that might deviate from the standard market theory. For a simple example of moral hazard in the implementation of production techniques, see Chapter 4.
In the analysis of the consumer's portfolio problem under uncertainty, it has been commonplace to assume sufficient conditions on preferences, probability distributions and patterns of returns for assets to ensure portfolio separation. That is, the consumer's problem can be reduced in dimension from the choice over a set of securities $K = \{1, \ldots, K'\}$, to a choice over a set of mutual funds $H = \{1, \ldots, H'\}$, where $H' < K'$. Usually writers have restricted their attentions to one, or, at most, two mutual funds, where one of the funds is a safe asset.\(^1\) Most writers have been content with sufficient conditions for portfolio separation;\(^2\) but in an important contribution Cass and Stiglitz (1970) provided necessary and sufficient conditions on preferences and asset returns for portfolio separation.

Using a somewhat simpler approach, we will provide necessary and sufficient conditions for portfolio separation. Not only are the proofs simpler than those provided by Cass-Stiglitz, they have the added advantage that they have a close association with a substantial body of literature investigating homotheticity of consumer preferences. Indeed, one of the key steps in our proofs, is provided by a well-known result due to Bergson (1936).

Our exposition has been divided into three sections. In Section one we set out the consumer portfolio problem and define portfolio separation. In

\[^1\] For example see the discussion in Arrow (1970).

\[^2\] For example see Rubinstein (1974).
Section two we prove a series of theorems providing necessary conditions on preferences for portfolio separation under different assumptions about returns. And finally, in Section three we give a brief discussion of the relationships between portfolio separation and commodity aggregation; and between portfolio separation and aggregation over consumer preferences.

1. THE CONSUMER'S PROBLEM

1.1 Consider a one-commodity, one-period world, where decisions must be made before uncertainty is resolved. Let there be a set of states of the world \( S = \{1, \ldots, s\} \). Therefore, we can consider the single physical commodity to be differentiated by states, so that the set of potential commodities has been expanded to dimension \( s \). Assume that the consumer has a consumption set \( \mathcal{X} = \mathbb{R}^s \). Although we can define a preference pre-ordering over the set \( \mathcal{X} \), with the usual properties of closedness and convexity of indifference surfaces, we will assume instead that the preference pre-ordering satisfies the von Neumann-Morgenstern axioms, and so therefore there exists a utility function

\[
U = \sum_{s \in S} u(x_s) \pi_s
\]

where \( x \in \mathcal{X} \) and \( \pi_s \in \{\pi_s \in \mathbb{R}^s \mid \sum_{s \in S} \pi_s = 1\} \). Furthermore, we will assume that the function \( u() \in \mathbb{C}^2 \); and that the preferences exhibit risk-aversion, i.e., \( u'(\cdot) > 0, u''(\cdot) < 0 \).

Now we assume that the consumer is restricted in his trades to the exchange of securities that offer patterns of returns (of the physical commodity)

---

By necessity, the discussion in Section 1.1 is brief. But for a fuller discussion and proofs of the assertions see Chapters 6 and 9.
across the states of the world. Assuming that the security returns are semi-positive, we can represent the relationship between the vector of assets \( a \in \mathbb{R}^{K'} \) and the vector of consumption \( x \in X \) by

\[
x = Za,
\]

where \( Z \) is an \( s \times K' \) semi-positive matrix. If defaulting securities are completely anticipated by the market, we can allow short-selling by imposing the condition \( x \geq 0 \). In turn, this condition, along with the assumption that \( X \) is the positive orthant, and the nature of the linear mapping between assets and commodities, implies that the asset portfolio set \( A \) is a convex polyhedral cone.

Because the utility function is a representation of an underlying preference pre-ordering, it is easy to show via the linear mapping (2) that these preferences induce a preference pre-ordering over the asset set \( A \).

(The easiest way to see this is to substitute (2) into (1), i.e.,

\[
U = \sum_{s \in S} u(Z_s a)p_s
\]

where \( Z_s \) is the \( s \)th row vector of \( Z \).

Finally, assume that the consumer is restricted to his choice of assets by a budget constraint,

\[
p a = w_0,
\]

where \( p \in \mathbb{R}^{K'}_{(+)\text{+}} \) is a vector of prices, and \( w_0 > 0 \) is his initial wealth.
1.2 If the consumer maximizes (3) with (4) as a constraint, we can derive

demand correspondences of assets as functions of prices and wealth, given
preferences, probability estimates over states, and the patterns of returns
of the set of assets. Now assume that the demand correspondence $a(p, W_0)$
conforms to the restriction of portfolio separation, i.e.

$$a(p, W_0) = \sum_{h \in H} a_h(p, W_0) \delta_h(p)$$

where $\delta_h \in \Delta = \{\delta_h \in \mathbb{R}^K \mid \sum_{k \in K} \delta_{hk} = 1\}$, and

$a_h \in \mathbb{R}$.

From the budget constraint (4) we obtain,

$$pa = \sum_{h \in H} a_h p_h^0 = W_0.$$ 

That is, the consumer's expenditure on portfolio $h$ is $a_h p_h^0$, so that

the elasticity of demand for each security in the portfolio with respect to
expenditure on the portfolio is unity. Because this result holds for any
admissible $p$, it follows that the underlying preferences for $\delta_h$ must be homone-
thetic. That is, denoting indifference by $\approx$, we have for

$$\delta'_h, \delta''_h \in \Delta \text{ and } \delta'_h \approx \delta''_h \Rightarrow a\delta'_h \bigcirc a\delta''_h$$

for $a > 0$. Now we are in a position to prove a (trivial) lemma. First, define

---

4 Equivalently, we can say that the Engel curves (with respect to changes in
the expenditure on the portfolio) are straight lines through the origin.
Considering securities in portfolios to be different commodities, we can
say that the security preferences are homogeneously separable; see Green
(1964).
\( x^h = Z^h \) for \( h \in H \), such that

\[
x = \sum_{h \in H} a_h x^h = \sum_{h \in H} a_h Z^h, \quad \text{and} \quad x^h > 0 \quad \text{for} \quad h \in H. \quad (5)
\]

**Lemma:**

The consumer has homothetic preferences over \( Z^h \), if, and only if, he has homoethetic preferences over \( x^h \).

**Proof:**

\[
Z^h (\bigcirc) Z^h \iff x^h (\bigcirc) x^h
\]

Therefore \( x^h \) is endowed with homothetic preferences.

Given this lemma, we have been able to transform the problem to one in which our interest centres on the homothetically of the portfolio returns \( x^h \).

The implications of this observation are examined in the next section.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR PORTFOLIO SEPARATION

2.1 Substituting \( x^h \) into (1), we obtain

\[
U = \sum_{s \in S} u(\sum_{h \in H} a_h x^h)^{\pi_s}. \quad (5)
\]

Furthermore, from the lemma, we know that \( x^h \) is homothetic. To begin let us

---

5 We can dispense with the assumption \( x^h > 0 \) without altering our results; but we have included it because of the interpretation that a mutual fund has limited liability.
consider the simple case of a single, risky mutual fund, i.e., \( H = \{1\} \) and \( x^1 \neq a e \), where \( a \in \mathbb{R} \), and \( e \) is the unit vector. For this to be the case, we will show that the utility function \( u(\cdot) \) must be of the Bergson class.

The following theorem, with slight modifications, is taken from Katzner (1970), Chapter 2.

**Theorem 1:**

Given a utility function of the form (5) with

(i) \( H = \{1\} \);
(ii) \( u'(\cdot) > 0, u''(\cdot) < 0 \);
(iii) \( x^1 \neq a e \);

then the underlying preferences are homothetic on \( x^1 \), if and only if there exists a real number \( b < 1 \), such that

\[
u(x_s) = \begin{cases} 
\lambda (x_s)^b + \gamma, & \text{if } b \neq 0 \\
\lambda (\log x_s) + \gamma, & \text{if } b = 0
\end{cases}
\]

where \( \lambda, \gamma \) are constants, \( b\lambda > 0 \) if \( b \neq 0 \), and \( \lambda > 0 \) if \( b = 0 \).

**Proof:**

From the assumption of homotheticity, and for any \( s', s'' \in S \), and \( a > 0 \) we have

\[
\frac{u'(x_{s'})}{u'(x_{s''})} = \frac{u'(ax_{s'})}{u'(ax_{s''})}.
\]
Differentiating with respect to \( \alpha \),

\[
\frac{u''(ax_s')x_s'}{u'(ax_s')} = \frac{u''(ax_s)x_s''}{u'(ax_s'')}. 
\]

Multiplying through by \( \alpha \), and noting that \( x_s = ax_s' \), then

\[
\frac{u''(x_s')x_s'}{u'(x_s')} = b - 1. 
\]

Because \( u' > 0 \), \( u'' < 0 \) and \( x_s' > 0 \), then \( (b - 1) < 0 \); or \( b < 1 \). Writing \( W_s = \log u'(x_s') \), and \( y_s = \log x_s' \), then

\[
\frac{dW_s'}{dy_s'} = \frac{u''(x_s')x_s'}{u'(x_s')} = b - 1, 
\]

whose unique solution is

\[
W_s' = (b - 1)y_s' + \beta_s'. 
\]

i.e.

\[
\log u'(x_s') = (b - 1) \log (x_s') + \beta_s'. 
\]

Therefore,

\[
u'(x_s') = \beta_s'(x_s')^{b-1}. 
\]
Because $u(\cdot)$ is a function independent of the state, we have

$$u'(x_s) = \beta'(x_s)^{b-1}, \quad s \in S.$$ 

The unique solution of this equation is

$$u(x_s) = \begin{cases} 
\lambda(x_s)^b + \gamma, & \text{if } b \neq 0, \\
\lambda(\log x_s) + \gamma, & \text{if } b = 0.
\end{cases}$$

The restrictions on the parameters follow from the restriction $u'(\cdot) > 0$. The sufficiency proof is left as an easy exercise for the reader. ||

**Comment:**

From the condition

$$\frac{u''(x_s) x_s}{u'(x_s)} = b - 1,$$

the reader should recognize that the utility function exhibits Arrow (1970) - Pratt (1964) constant relative risk aversion.

2.2 We can generalize Theorem 1 to the case where there are $n'$ risky mutual funds. The proof is very similar to that in Theorem 1.

---

6 Arrow (1970) has argued that, because the utility function must be bounded (by construction), it cannot exhibit constant relative risk aversion over its entirety. Nevertheless, we can employ one of the answers to the St. Petersburg paradox by assuming that the consumer is embedded in an economy where the set of attainable states are bounded, and that the utility function is monotone over the set of attainable consumptions. For such an argument see Chapter 9.
Theorem 2:

Given a utility function of the form (5) with

(i) $H = \{1, \ldots, H'\}$;

(ii) $u'(x) > 0, u''(x) < 0$;

(iii) $x^h \neq a_h e$;

then the underlying preferences are homothetic on $x^h, h \in H$, if, and only if, there exists a real number $b < 1$, such that

$$u(x^h) = \begin{cases} 
\lambda(x_s)^b + \gamma, & \text{if } b 
eq 0, \\
\lambda(\log x_s) + \gamma, & \text{if } b = 0,
\end{cases}$$

where $\lambda, \gamma$ are constants, $b \lambda > 0$ if $b \neq 0$, and $\lambda > 0$ if $b = 0$.

Proof:

Now

(a) $x_s = \sum_{h \in H} a_h x^h$.

From the homotheticity of preferences over $x^h$, and for $\eta > 0, a_h \neq 0$, we have

$$u'(\eta \sum_{h \in H} a_h x^h) = u'(\sum_{h \in H} x^h) \cdot u'(\eta \sum_{h \in H} a_h x^h).$$

Differentiating with respect to $a_h$ and $\eta$, and setting $\eta = 1$,
(b) \[
\frac{u''(x^h_s)x^h_s}{u'(x^h_s)} = b^h, \quad h = 1, \ldots, H',
\]

(c) \[
\frac{u''(x^h_s)x^h_s}{u'(x^h_s)} = b - 1.
\]

Clearly \(b^h, (b - 1)\) are negative, because \(u'(\cdot) > 0, u''(\cdot) < 0\) and \(x^h, x > 0\) by assumption. Substituting for \(x^h_s\) from (b) into (a), we obtain

\[
\frac{u''(x^h_s)x^h_s}{u'(x^h_s)} = \sum_{H}^{h} \alpha^h_b.
\]

But from (c), we obtain immediately

\[
b - 1 = \sum_{H}^{h} \alpha^h_b,
\]

so that the proof proceeds in an identical manner to that of Theorem 1. \(\|

2.3 \quad \text{Finally, we will introduce what Cass-Stiglitz call "monetary separation", where the portfolio } h = 1 \text{ pays amounts } \rho e, \rho \in \mathbb{R}, e \text{ is the unit vector; and } x^h \neq \alpha_h e, \alpha_h \in \mathbb{R}, h \in \{2, \ldots, H'\} = H' \text{ are the returns from the risky portfolios. The monetary separation case is well-known to the readers of the portfolio theory, and so we will not review its history here.} \quad 7 \quad \text{Using similar techniques to those applied above, we can prove:}

7 \quad \text{For a survey and bibliography see Jensen (1972).}
Theorem 3:  

Given a utility function of the form (5) with 

(i) \( H = \{ 1, \ldots, H' \}, \ H'_R = \{ 2, \ldots, H' \} \);  
(ii) \( u'( ) > 0, u''( ) < 0 \);  
(iii) \( x^l = a_{\perp} e, x^h \neq a_{n} e, h \in H'_R \);  

then the underlying preferences are homothetic on \( x^h \in h \in H \), if, and only if, 

\[
\begin{align*}
\text{u}(x) &= \begin{cases} 
\lambda (A + B x)^h + \gamma, & \text{if } b \neq 0, \\
\lambda \log (A + B x) + \gamma, & \text{if } b = 0,
\end{cases} \\
\end{align*}
\]

where \( b, \lambda, \gamma, A, B \) are constants such that \( (A + B x)(b - 1) < 0 \); and \( Bb (A + B x)^{b-1} > 0 \) for \( b \neq 0 \) and \( b \lambda > 0 \) for \( b = 0 \); or 

\[
\text{u}(x) = c \exp(b x^h) + \lambda,
\]

where \( \lambda, b, c \) are constants such that \( b < 0, c < 0 \).  

Proof:  

From the assumptions on portfolio returns, we have  

(a) \( x_s = \sum_{h \in H} a_{h} x^h + a_{\perp} \rho = x^R \rho + a_{\perp} \rho \).  

Now we can distinguish three cases depending upon the variability or constancy of \( a_{\perp}, a_{h}^h, h \in H'_R \) respectively.
Case (i):

Assume that $a_h$, $h \in H$ is variable. Therefore homotheticity of preferences over $x^h$, with $n > 0$, $a_h \neq 0$ implies

$$\frac{u'(n \left( \sum H^h x^h_{s'} + a_1 \rho \right))}{u'(n \left( \sum H^h x^h_{s''} + a_1 \rho \right))} = \frac{u'(\sum H^h x^h_{s'} + \rho)}{u'(\sum H^h x^h_{s''} + \rho)}$$

Differentiating with respect to $a_h$, and $n$, the proof proceeds identically to that in Theorem 2. Therefore,

$$u(x) = \begin{cases} \lambda(x)^b + \gamma, & \text{if } b \neq 0, \\ \lambda(\log x) + \gamma, & \text{if } b = 0. \end{cases}$$

with restrictions on the constants as in Theorem 2.

Case (ii):

Assume $a_1$ is a constant, but $a_h$, $h \in H_R$ variable. Homotheticity over $x^h$, with $n > 0$, $a_h \neq 0$ implies

$$\frac{u'(n \left( \sum H^h x^h_{s'} + a_1 \rho \right))}{u'(n \left( \sum H^h x^h_{s''} + a_1 \rho \right))} = \frac{u'(\sum H^h x^h_{s'} + \rho)}{u'(\sum H^h x^h_{s''} + \rho)}$$
Differentiating with respect to $\alpha_{h}$, $h \in H_{R}$ and $\eta$, and setting $\eta = 1$, results in,

\[ \frac{u''(x_{s},)}{u'(x_{s},)} x_{s} = b^{h}, \quad h = 2, \ldots, H' \]

and

\[ \frac{u''(x_{s},)}{u'(x_{s},)} x_{s} = b. \]

Substituting (b) into (a) for $x_{s}$, we find

\[ \frac{u''(x_{s},)}{u'(x_{s},)} (x_{s} - \alpha_{1} \rho) = \sum_{H_{R}}^{*} a_{h} b^{h}. \]

But $x_{s}^{R} = x_{s} - \alpha_{1} \rho$, and from (c) it follows that

\[ b^{R} = \sum_{H_{R}}^{*} a_{h} b^{h}. \]

Multiplying through (d) by a constant $B \neq 0$,

\[ \frac{u''(x_{s},)}{u'(x_{s},)} (A + Bx_{s}) = Bb^{R} = b - 1, \]

where $A = -B_{1} \rho$; $b - 1 = Bb^{R} = B \sum_{H_{R}}^{*} a_{h} b^{h}$.

From risk-aversion, it follows that $(A + Bx_{s})(b - 1) < 0$. By familiar arguments we obtain
\[ u(x_s) = \begin{cases} 
\lambda (A + Bx_s)^b + \gamma, & \text{if } b \neq 0, \\
\lambda \log (A + Bx_s) + \gamma, & \text{if } b = 0,
\end{cases} \]

where \(\lambda, \gamma\) are constant such that \(Bb\lambda (A + Bx_s)^{b-1} > 0\) for \(b \neq 0\); and \(\lambda B > 0\) if \(b = 0\).

Case (iii):

Finally, assume that \(a_h\) is constant for all \(h \in H_R\). By similar arguments to those above, we obtain

\[ \frac{u''(x_{s1})}{u'(x_{s1})} = b^1 = b. \]

From risk-aversion, \(b < 0\). The unique solution to this differential equation is given by

\[ u(x_s) = c \exp(bx_s) + \gamma, \]

where \(\gamma, c\) are constants, such that \(c < 0\) (because \(u'(\cdot) > 0\)). Sufficiency proofs are well-known for this class of utility functions.

Comment:

The reader should recognize that the utility functions are of the hyperbolic absolute risk-averse (HARA) class. This class contains the familiar quadratic utility function employed in some of the earliest portfolio analysis. Observe that the restriction for certain parameter values, may imply restrictions on the feasible set \(X\), if monetary separation is to be possible.

\(^8\) See Hakansson (1971).
PORTFOLIO SEPARATION AND AGGREGATION

3.1 Portfolio separation is simply a means by which the choice of consumer asset holdings can be reduced to the choice over mutual funds, or groups of the original assets. For the standard classical consumer problem, there exists a significant body of literature dealing with necessary conditions over consumer preferences to ensure that the consumer's problem can be reduced to a two-stage maximization process: given prices for groups of commodities, the optimal proportions in the group are determined; and given a composite price index for each group, expenditure is determined for each group. Usually, this literature assumes that the groups are formed by partitioning the set of commodities. Portfolio separation, on the other hand, allows for the possibility that a particular asset may be held in all groups (mutual funds).

3.2 Because of the role of homotheticity in portfolio separation, it is possible also to aggregate over individual consumers to obtain a representative consumer. For example, if we take the hypothesis of Theorem 1, that consumer $i \in I = \{1, \ldots, M\}$ has homothetic preferences such that he holds a single mutual fund, and all consumers have identical homothetic preferences over assets (so that the same mutual fund is held by all consumers), then there exists a representative or composite consumer whose demand function is the same as the sum of the individual consumer market demands, and whose wealth is the sum of the individual consumer wealths.\(^{10}\) That is

$$\sum_{i \in I} a_i(p, W_{0i}) = a(p, \sum_{i \in I} W_{0i})$$

\(^9\) See Green (1964) for a survey on aggregation. See also Chipman (1974).

\(^{10}\) This theorem is taken from Chipman (1974).
and

$$a_i(p, W_{0i}) = a(p, W_{0i}).$$

For consumers to have identical homothetic preferences in our model, we require that they have the same probability estimates ($\pi_{i^*} = \pi_{i^*}', i', i'' \in I$) and common parameters $b$, $\lambda$.

In the case of monetary separation and two mutual funds, Rubinstein (1974) has provided some further conditions on aggregation. For example, it is easy to see that we can aggregate over consumers if the same mutual funds are chosen, and the mutual funds are held in the same proportion.

Concluding Comments:

We have attempted to obtain a straightforward, but general proof for portfolio separation restrictions on preferences. As Cass and Stiglitz acknowledge these utility functions are very restrictive. Although portfolio separation may prove to be a useful hypothesis for empirical work, there is a danger that the assumption of portfolio separation may obscure the fact that general portfolio theory has the same structure as conventional demand theory.

11 For example see the survey by Jensen (1972), where it is concluded that econometric testing of the mean-variance (portfolio-separation) capital-market models has not been very encouraging.
 CHAPTER 6

BORROWING, SHORT-SALES AND CONSUMER DEFAULT
IN AN ASSET ECONOMY

In the Sharpe-Lintner (1964), (1965) asset-pricing model it is usual to assume that consumers are able to borrow or lend, without restriction, at the riskless rate of interest. This assumption has been recognized to involve an inconsistency in the theory, because with personal borrowing, there is always a positive probability of default, so that the consumer's bond is not a perfect substitute for the safe asset.¹ To avoid this problem, some writers have assumed either

(i) that all securities are held long (i.e., in non-negative amounts); or

(ii) borrowing on short-sales are restricted to exclude the possibility of default.²

Although the implications of default for short-selling have received scant

¹ For example, in the standard finance text of Fama and Miller (1972), p.294, it is acknowledged that riskless borrowing is "somewhat artificial". Indeed it is more than "artificial", it is an inconsistency in the model. See footnote 27 below.

² Arrow (1970) excludes short-selling, although this is not implied as a criticism of his contribution which is concerned with developing risk-aversion measures, rather than looking at all the implications of portfolio decisions.

Diamond (1967) includes short-sales and borrowing but excludes bankruptcy. Unfortunately, he does not include this restriction explicitly, although it is not difficult to formulate these constraints as we show below.
attention\(^3\), the closely related issue of corporate leverage with default risk on the corporate bond has attracted much more investigation.\(^4\)

Our discussion falls into two parts. The first part focuses on proving the existence of an equilibrium in an asset economy with short-sales. The usual existence proofs of competitive equilibrium assumes that consumption sets are bounded below (see Debreu (1959)), but with short-selling this assumption is no longer valid in the asset space. If consumer default is ignored, then it might appear that the existence of equilibrium will depend upon restrictions on the asset preferences of consumers.\(^5\) These conditions need to be introduced, it might be argued, because the attainable state of the economy is unbounded, and consumers may take unbounded portfolio positions, so that the compactification arguments used in employing the fixed point theorems fail. Nevertheless, if consumers are sufficiently similar in their asset preferences, then an equilibrium exists. We can give some sort of intuitive explanation of this result from the existing asset theory. If all consumers are identical in preferences and wealth, then all assets are held long (in identical portfolios). Another instance of this type of result is obtained from the Sharpe-Lintner model where short-sales (of risky securities) never occurs, because all consumers hold them in the same portfolio proportions. Now we conjecture that by changing the asset preferences by an \(\epsilon\)-amount from the representative consumer, the

\(^3\) An exception is section II in Smith's paper (1970).

\(^4\) For example, see the references in Fama and Miller (1972). Also see Chapter 5.

\(^5\) For results along these lines in a temporary equilibrium model see Green (1973). Also notice his comments in the conclusion on collateral loans and the unboundedness of short-selling. Hart (1974) has considered a securities market version of Green's paper. In another paper, Hart (1973) considers no-default in a multi-good model. His existence proof is different from ours.
individual portfolios will change continuously, if there is sufficient convexity present. The essence of the argument is to eliminate cases of unbounded portfolios - by restraining preferences to be not too dissimilar to the representative consumer.

This result is not very satisfying because the existence of equilibrium depends tenuously upon similarity of consumer asset preferences. But this is not the only problem with the argument: it ignores default and treats the consumer's defaulting short-sale (bond) as a perfect substitute for the nominal security. Unless one argues that the holders of the defaulting security cannot distinguish it from the nominal security (which may be plausible to a partial extent in an imperfectly informed market), the fully-informed consumer will never hold the defaulting security long, and markets will not clear. If default is introduced explicitly so that a defaulting short-sale is not considered a perfect substitute for the nominal security, then a partial restriction is introduced on short-sales. Although the asset consumption set is unbounded below, we can employ the co-ordinate-free existence proof of Debreu (1962) to show a competitive equilibrium exists independently of similarity restrictions on preferences. The proof is given in Section 1 of the paper.

This formulation is sufficiently general to include defaulting bonds which have perfect substitutes in the market. The key assumption is that the consumer is constrained to trading in market securities with market prices.

In Section 2 we consider an extension of the argument in Section 1. We have shown that an equilibrium exists for a given set of securities, but by defaulting, the consumer may create a security not contained in the original asset set. If we assume that the consumer has this power of asset creation, but other consumers do not (say because of set-up costs in the creation or set-up costs in the creation or

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6 By a perfect substitute security, we mean a security that has the same pattern of returns across states of the world.
issuance of new securities), and when the asset is created it can be traded freely by all agents in a competitive market, then a new asset equilibrium will be obtained. By creating new securities by default, the consumer can generate a conceptual sequence of asset-economy equilibria from which he must choose the most preferred. This concludes our equilibrium discussion of the first part.

The second part of the paper, in Section 3, concerns a partial equilibrium treatment of the defaulting short-sale. Rather than treat the new security market as being competitive, it might be plausible to consider a situation where the defaulting consumer is the sole supplier of the new security. By setting other asset prices constant, and with some other simplifying but inessential assumptions, we are able to derive an offer curve for the defaulting security. This analysis has some interesting features.

(i) By way of a simplified example we can obtain some additional insights into the equilibrium model of Section 1. In particular
(a) we derive a two-asset case illustrating the consumer's asset feasibility set; and
(b) we illustrate the role of wealth on the extent of short-selling and borrowing behaviour.

(ii) We show that conventional demand theory tools can be applied when a new security is introduced. In particular, we obtain generalized Slutsky terms.
1. **EQUILIBRIUM IN AN ASSET ECONOMY WITH SHORT-SALES**

1.1 The asset model will be based upon Diamond's (1967) formulation. Consider an economy that exists for two periods: in the first period the state of the world is known with certainty by all economic agents; but in the second period there is a set of possible states of the world \( S = \{s_1, \ldots, s_s\} \). Let there be one physical commodity so that the first period commodity space is \( \mathbb{R} \), and the second period potential commodity space is \( \mathbb{R}^S \), making the total potential commodity space \( \mathbb{R}^{S+1} \). We say a "potential" commodity space, because we will treat an asset economy where second period actions by agents are restricted to trades in assets that offer linear combinations of contingent commodities. Rather than treat the contingent commodities explicitly, it will be much easier to consider the objects of choice to be the assets. The trick is to show that the asset economy can be made isomorphic to the certainty equilibrium theory.

1.2 Let us proceed by defining the commodity space in the asset economy to be the space \( \mathbb{R}^{L+1} \), where there are \( L \) assets and the first period commodity. As far as production is concerned we can omit any mention of contingent commodities and consider production sets in \( \mathbb{R}^{L+1} \).

Define the \( j \)th producer's production set to be \( (y_{0j}, y_{Lj}) \in \mathbb{R} \subset \mathbb{R}^{L+1} \). And define the aggregate production set \( \mathbb{Y}^A \) to be \( \mathbb{Y}^A = \sum_j y_j^A \).

Now consider the following assumptions on the production sets \( \mathbb{Y}_j^A \).

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7 For a more detailed discussion of the relationship between asset and state-preference claims in production, see Chapter 4.
Assumptions A1, A3, A4 are usual assumptions on aggregate production; and A2 requires that the first period commodity is an input, and the second period assets are outputs. These assumptions are very general, and allow the producer to produce more than one security.\footnote{Production should be interpreted in the widest possible sense to include the issuing of different classes of securities; e.g., bonds, equity, etc.}

1.3 The analysis for the $i$th consumer is a little more complicated because his preference map for assets must be derived from the more fundamental preferences for contingent commodities. Our strategy will be to make quite general assumptions on the contingent commodity consumption set, and derive the asset consumption set via the linear mapping provided by asset returns.

Consider the $i$th consumer's consumption possibility set $x_i \in X_i \subseteq \mathbb{R}^{S+1}$. Consider the following assumptions on the consumption possibility set:

\begin{enumerate}
  \item[(B1)] $X_i = \mathbb{R}^{S+1}_+$, $\forall i$
  \item[(B2)] Define on $X_i$ a complete pre-ordering $\leq_i$ called a preference pre-ordering with the following conditions:
     \begin{enumerate}
       \item[(a)] Non-Satiety: For any $x_i \in X_i$, $\exists x_i' \in X_i$ such that $x_i' \not\preceq_i x_i$.
     \end{enumerate}
\end{enumerate}
Continuity: For every $x_i' \in X_i$, the sets
\[
\{x_i \in X_i \mid x_i \supseteq x_i'\} \text{ and } \{x_i \in X_i \mid x_i \supseteq x_i'\}
\]
are closed.

(c) If $x_i', x_i'' \in X_i$ such that $x_i' \neq x_i''$, and $t \in (0, 1)$, then
\[
x_i'' \supseteq x_i' \implies tx'' + (1-t)x_i' \supseteq x_i'.
\]

Given these assumptions we wish to construct the $i^{th}$ consumer's asset consumption set and preferences. Now all consumers have available the aggregate pattern of returns from assets, represented by the semi-positive $s \times L$ matrix $Z$.

Define the linear mapping
\[
\Lambda : \mathbb{R}^s \rightarrow \mathbb{R}^{L+1}, \text{ by }
\]
\[
Z'\beta = \alpha, \text{ where } \beta \in \mathbb{R}^{L+1}, \alpha \in \mathbb{R}^s, \text{ and }
\]
\[
Z' = \begin{pmatrix}
1 & 0 \\
0 & Z
\end{pmatrix}
\]
is an $(s+1) \times (L+1)$ semi-positive matrix.

The mapping $\Lambda$ is linear, and onto the range $H$, which is a vector sub-space of dimension $L+1$. Define $T_i = H \cap X_i$. Clearly $T_i \neq \emptyset$, because $H \cap X_i \supseteq \{0\}$. The asset consumption set for $i$ is $X_i^A$, and it is obtained by the inverse mapping
\[
\Lambda^- : T_i \rightarrow \mathbb{R}^{L+1}, \text{ which is linear.}
\]

At this point, the reader should observe that because $X_i$ has been assumed to be the positive orthant, the inverse mapping $\Lambda^-$ implies that only non-defaulting securities will be issued by the consumer. That is, the consumer is constrained in his short-sales so that the nominal pattern of returns he promises to pay is in fact feasible for him. Or, the consumer is constrained
to fulfill his contractual obligations. This assumption is important for our
existence proof as we will see below.

Now we can prove the following lemma:

**Lemma 1.1:**

If the commodity consumption set $X_i$ satisfies conditions B1, B2 (b) (c), and the condition for any $x_i \in T_i$, $\exists x'_i \in T_i$ such that $x'_i \succ_i x_i$, then

- **B1**: $X_i$ is closed and convex;
- **B2**: There is a preference pre-ordering $\preceq_i$ on $X_i$ with the properties:
  - (a) Non-satiety;
  - (b) Continuity;
  - (c) Convexity.

**Proof:**

The proof is straightforward.\(^9\)

In the lemma, the asset set $X_i^A$ has all the properties of the set $X_i$ except lower boundedness. Nevertheless, we can obtain a weaker restriction on $X_i^A$. Because $X_i = \mathbb{R}^{s+1}$ and $H$ is a vector sub-space of dimension $L+1 \leq s+1$, then $T_i = H \cap X_i$ is a convex polyhedral cone in the positive orthant, and because $A^-$ is linear, $X_i^A$ is a convex polyhedral cone. Furthermore, because $Z'$ is semi-positive, it follows that $X_i^A \succeq \mathbb{R}^{L+1}$. We can summarize these simple results in

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\(^9\) See Chapter 4.
Lemma 1.2:

\[ X_i^A \text{ is a convex polyhedral cone with } X_i^A \supseteq \mathbb{R}^{L+1}. \]

Furthermore, we have for the aggregate consumption set \( X^A = \bigcap_{i} X_i^A \).

Lemma 1.3:

\[ X^A \text{ is a convex polyhedral cone with } X^A \supseteq \mathbb{R}^{L+1}. \]

Proof:

The proof follows from the observation \( X_i = \mathbb{R}^{S+1}, \forall i \), and the asset mapping is independent of \( i \).

This concludes our discussion of the consumption sets in the asset economy.

1.4 Now let there be initial commodity endowments \( \omega_i \in H \) for each consumer so that by the inverse mapping \( \Lambda^{-1} \), we can obtain asset endowments \( \omega_i^A \). Given these endowments we are in a position to define the notions of an asset economy, an equilibrium of an asset economy, and given sufficient conditions for the existence of such an equilibrium. The strategy of our proof will be to use the very general formulation of Debreu (1962).

Definition 1:

An asset economy \( E^A \) is defined by: non-empty sub-sets \( X_i^A \), completely pre-ordered by \( \preceq_i \), for \( i = 1, \ldots, m \); non-empty sub-sets \( Y_j^A \) for \( j = 1, \ldots, n \); and a resource endowment point \( \omega^A \), where \( X_i^A, Y_j^A \subseteq \mathbb{R}^{L+1} \) and \( \omega^A \in \mathbb{R}^{L+1} \).

If consumers own resources, and control producers such that they receive a proportion of the producer's profits, then define:
Definition 2:

A private-ownership asset economy $\mathcal{E}^A$ is defined by:

(a) an asset economy $E^A$;

(b) for every $i$, there exists $\omega_i^A \in \mathbb{R}^{L+1}$ such that $\sum_i \omega_i^A = \omega^A$;

(c) for every $(i, j)$, there exists $\theta_{ij} \in \{\theta_{ij} \in \mathbb{R} \mid \theta_{ij} > 0, \sum_i \theta_{ij} = 1\}$.

If there exists competitive markets for the first period commodity and each of the contingent commodities, the define:

Definition 3:

A quasi-equilibrium of the private ownership asset economy $\mathcal{E}^A = (\mathcal{X}_i^A, \mathcal{Y}_j^A, \omega_i^A, (\theta_{ij}))$ is an $(m + n + 1)$ tuple $((x_i^*), (y_j^*), (p^*))$ of points of $\mathbb{R}^{L+1}$ such that:

(a) for every $i$, $x_i^*$ is the greatest element of

$$\{x_i \in x_i^A \mid p^* x_i \leq p^* \omega_i^A + \sum_i \theta_{ij} y_j^* \}$$

and/or

$$p^* x_i = p^* \omega_i^A + \sum_i \theta_{ij} p^* y_j^* = \min p^* x_i^A;$$

(b) for every $j$, $p^* y_j^*$ is the greatest element of

$$(p^* y_j, y_j \in y_j^A);$$

(c) $\sum_i x_i^* - \sum_j y_j^* = \sum_i \omega_i^A$;

(d) $p^* \neq 0$. 

Now to prove the existence of a quasi-equilibrium (and subsequently an equilibrium) we need the following definition:

**Definition 4:**

If $x_i^A$ is the $i$th consumer's attainable asset consumption set, let $D$ be the smallest cone with vertex 0 owning all the points of the form

$$\bigcup_{i} (x_i^A - \omega_i^A)$$

where $x_i^A \geq x_i^A, \forall i$.

Now consider the following theorem drawn from Debreu (1962).

**Theorem 1:**

The private ownership economy $\mathbb{A}$ has a quasi-equilibrium if

(a.1) $\text{As}(X^A) \cap (-\text{As}(X^A)) = \{0\}$;\(^{10}\)

(a.2) $x_i^A$ is closed and convex, $\forall i$;

(b.1) for $x_i^A \in x_i^A$, $\exists x_i^A' \in X_i^A$ such that $x_i^A' \geq_i x_i^A$;

(b.2) for every $x_i^A' \in X_i^A$, the sets \(\{x_i^A \in X_i^A | x_i^A' \geq_i x_i^A\}\) are closed;

(b.3) for every $x_i^A' \in X_i^A$, the set \(\{x_i^A \in X_i^A | x_i^A' \geq_i x_i^A\}\)

is convex;

---

10 By $\text{As}(X)$ we mean the asymptotic cone of a set $X$. For scalar $k \geq 0$, let $X_k$ be the smallest closed cone with vertex 0 containing all points in $X$ with norm at least $k$; then

$$\text{As}(X) = \bigcap_{k \geq 0} X_k.$$  

The asymptotic cone is sometimes called the recession cone.
(c.1) \((w^A_j + Y^A_j) \cap X^A_i \neq \emptyset\); 

(c.2) there is a closed, convex production set \(Y^A_i\), such that for every \(i\),

\[\{w^A_i\} + As(Y^A_i) - D \cap X^A_i \neq \emptyset;\]

(d.1) \(0 \in Y^A_j, \forall j;\)

(d.2) \(As(X^A_i) \cap As(Y^A_j) = \{0\}.\)

These sufficient conditions are very general and cover most of the usual assumptions in the literature. Nevertheless we need to demonstrate that the assumptions we made above in A1-A4 for asset production sets, and in B1, B2 for consumption sets (and therefore for the derived asset consumption sets) satisfy the general conditions. This result can be demonstrated in the following theorem.

**Theorem 2:**

If a private ownership economy \(\mathcal{E}^A\) satisfies:

(i) asset production sets have the restrictions A1-A4;

(ii) consumption sets satisfy the restrictions B1, B2;

(iii) the asset pattern of returns \(Z\) is semi-positive, and the rank of \(Z, \rho(Z) = L \leq \tilde{s};\)

(iv) \(w^A_i \in \mathbb{R}^{L+1}_{+}, \forall i;\)

11 It is usually acknowledged that these conditions are the most general available for the existence of a competitive equilibrium.
then $E^A$ has a quasi-equilibrium. \textsuperscript{12}

Proof:

The proof will proceed by showing that (i)-(iv) satisfy the conditions (a.1) through (d.2).

(a.1) By lemma 1.3, $X^A$ is a convex polyhedral cone $\Rightarrow A_0(X^A) = X^A$.

Now,

$$X^A = \{ x^A \in \mathbb{R}^{L+1} \mid z'x^A \geq 0, z' \text{ semipositive} \},$$

and therefore,

$$-X^A = \{ x^A \in \mathbb{R}^{L+1} \mid z'x^A \leq 0, z' \text{ semipositive} \}.$$

If $x^A' \in X^A \cap (-X^A)$, then $z'x^A' = 0$; but $p(Z') = L+1$ $\Rightarrow x^A' = 0$.

(a.2) $X^A$ is closed and convex by Lemma 1.1 or Lemma 1.2.

(b.1) Non-satiation, Continuity and Convexity follow from Lemma 1.1.

(b.2) $A(X)$ is closed and convex by Lemma 1.1.

(c.1) Because $\omega^A \in \mathbb{R}^{L+1}$ and $0 \in Y^A$, $\forall j$, then $\{(\omega^A) + Y^A \supseteq \{0\}$, and $x^A \supseteq \{0\}$ by Lemma 1.3, therefore $\{(\omega^A) + Y^A \cap X^A \supseteq \{0\}$.

\textsuperscript{12} We have assumed that there is only one commodity in the first period; but it is a trivial exercise to show that more than one commodity in the first period can be accommodated. Furthermore, in the first period, the consumption set can include negative values (outputs), as long as it conforms to the conditions of the theorem. On the other hand, more than one commodity in the second period creates additional complications. See Chapter 9.
(c.2) $Y^A$ is closed and convex by A3; and

$$\left( \{ \omega_i^A \} + \text{As}(Y^A) - D \right) \cap X_i^A \neq \emptyset,$$

because:

(i) $\omega_i^A \in \mathbb{R}_{(+)}$;

(ii) $\text{As}(Y^A) \supset \{ 0 \}$;

(iii) $D \supset \{ 0 \}$ (by definition);

(iv) $X_i^A \supset K^{L+1}$ by Lemma 1.2.

(d.1) $0 \in Y_j$ by Al.

(d.2) Consider $a \in \mathbb{R}^{L+1}$ such that

$$a_0 = 1, \ a_i = 0, \ i = 1, \ldots, L.$$

Define:

$$K^+ = \{ v \in \mathbb{R}^{L+1} \mid av \geq 0 \};$$

$$K^- = \{ v \in \mathbb{R}^{L+1} \mid av \leq 0 \};$$

$$K = \{ v \in \mathbb{R}^{L+1} \mid av = 0 \}.$$

Now it is obvious that $X^A \subset K^+$, and from A2, A4 that $Y^A \subset K^-$ (and because $\text{As}(Y^A) \subset Y^A$, then $\text{As}(Y^A) \subset K^-$. Also A2, A4 $\Rightarrow Y^A \cap K = \{ 0 \}$; and $X^A \cap K = \{ 0 \}$. Therefore $X^A \cap \text{As}(Y^A) = \{ 0 \}. \ | |$

Having shown that the asset economy has a quasi-equilibrium, the next step is to consider sufficient conditions which ensure that the quasi-equilibrium is an equilibrium. For Debreu (1962) we have:
Lemma 1.4:

If $\mathcal{E}^A$ has a quasi-equilibrium, then $\mathcal{E}^A$ has an equilibrium if

$$p^* x_i = p^* \omega_i + \sum_j \theta^* p^* y_j = \min p^* x_i$$

occurs for no consumer $i$. | 

A number of sufficient conditions can be introduced to exclude the "exceptional case" where the consumer's demand correspondence is no longer upper semi-continuous.\(^{13}\)

To illustrate the kind of arguments that are needed to ensure the existence of an equilibrium, consider the following two examples:

Example 1:

We know from footnote 7 of Debreu (1962) that the quasi-equilibrium price $p^* A$ is polar $(A_\infty (Y) - D)$. Although the existence proof produces $p^* A \neq 0$, it is not obvious from our assumptions that $p^* A \geq 0$. Nevertheless, if we modify the assumptions on $Y^A$, so that we have free disposal, i.e., $Y^A \supseteq R^{L+1}_{(-)}$, then clearly, profit maximization implies $p^A \geq 0$. With semi-positive prices, we can introduce the strong Debreu (1959) conditions that each consumer's endowment is strictly greater than some point in his consumption set. That is:

Lemma 1.5:

If $\mathcal{E}^A$ has a quasi-equilibrium and,

(i) $Y^A \supseteq R^{L+1}_{(-)}$;

(ii) there exists $x^A_i \in X_i$ such that $x^A_i \triangleq \omega^A_i$, $\forall i$, then

$\mathcal{E}^A$ has an equilibrium. |

\(^{13}\) This problem is well-known as "the exceptional case". See Debreu (1959), (1962).
Proof:

Because $Y^A \supseteq K_{(-)}$, then $p^* \geq 0$. But (ii) implies (for $p^* \geq 0$) that $p^* A^* > \min p^* A^* = 0$. The last equality follows because $Y^A$ is convex polyhedron with $X_i \subseteq R^{I+1}_{(+)}$.

Example 2:

Some formulations of the asset model dispense with the free disposal assumption, and therefore other assumptions are required. Consider the modification to assumption B2, i.e.,

**B2'** Assume the axiomatic foundations of the von Neumann-Morgenstern Utility Theorem, and assume risk-aversion such that

$$U_i = \sum_{S} u_{i}(x_{0i}, x_{si}) \pi_{si}$$

where:

(i) $u_{i}$ is real-valued, strictly increasing in $x_{si}$ concave and defined on $R^{S+1}_{(+)}$.

(ii) $\pi_{si} \in \{\pi_{si} \in R \mid \pi_{si} > 0 \sum_{S} \pi_{si} = 1\}$. | |

It is easy to show that B2' is a special case of B2. But we can obtain a further result. Because $u_{i}$ is strictly increasing in $x_{si}$, and $Z$ is semi-

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14 For example see the formulation used by Hart (1973).

15 This is not strictly correct, because for the existence of a von Neumann-Morgenstern utility indicator, $u_{i}$ must be bounded above. See Arrow (1970). But we can avoid this difficulty by assuming that $u_{i}$ is strictly increasing on the attainable consumption set, which is closed and bounded.
positive, it follows that $x_{kl}^A > x_{kl}^B$ for $k = 1, \ldots, L$, $l = 1, \ldots, A$. Therefore, if $p_i^* < 0$, then there exists an unbounded sequence $\{x_i^q\}$ satisfying $x_i^q$'s budget constraint such that

$$u_i(x_i^q) > u_i(x_i^q'),$$

which violates the assumption of equilibrium. Therefore, $p_i^* > 0$, for $i = 1, \ldots, L$. By a similar argument on profit maximization, it follows that $p_0^* > 0$ (i.e., $y_0$ is an always desired input). Therefore we have the lemma:

**Lemma 1.6:**

If $\mathcal{E}^A$ has a quasi-equilibrium and,

(i) let preferences satisfy B2', and

(ii) $\omega_{ki} > 0$ for some $k \in \{0, 1, \ldots, L\}$, $\forall_i$

then $\mathcal{E}^A$ has an equilibrium. |

**Proof:**

Clearly $p^* \omega_i x_i^A > \min p^* x_i^A = 0$, which satisfies Lemma 1.4.16

1.5 Before closing this section we should make the following comments.

First, the special case of Arrow-Debreu securities (1953), (1959) can be incorporated by letting $Z = I$, the unit matrix. In this case $x_1^A = x_1^A$, and therefore $x_i^A$ is bounded below by assumption B1. Of course, the standard Debreu (1959) proof which involves lower boundedness of consumption sets is

16 This case is sufficiently general to include the case where initial endowments are restricted to the first period commodity. A stronger condition along the same lines can be found in Debreu (1959), i.e.,

$$\exists x_i^A \in X_i^A \text{ such that } x_i^A \ll \omega_i^A, \forall_i.$$
applicable now. Nothing has been said about preferences, because, as long as they conform to the general assumption B1, B2, existence is ensured. That is, even if \( \pi_{is'} = 0 \) for \( s' \in S \), then \( i^{th} \) consumer is bounded in his short-sale activities by the bankruptcy constraint - indeed, short-sales cannot occur at all with Arrow-Debreu securities.

Secondly, observe that the resulting equilibrium is an optimum by the standard arguments of Chapter 6 of Debreu (1959): clearly, the theory is a special case of the standard certainty competitive equilibrium and its associated optimality. The optimum is defined with the pattern of returns \( Z \) as a parameter of the system (rather in the same way as production technology) and therefore the optimum coincides with Diamond's (1967) constrained optimum.

Thirdly, there appears to be a limitation in the existence theorem to sets of independent security returns. This restriction arose to ensure part a.1 of the existence proof; and, essentially, it is required to provide boundedness for the set of attainable portfolios for consumers. (For a discussion of attainable states, see Debreu (1959) Section 5.4). However, the introduction of dependent security returns, does not alter the equilibrium, for the independent securities can be derived easily by taking linear combinations of quantities and prices of the independent securities. Therefore, a consumer may take on unbounded portfolio position in dependent securities, although the attainable set of independent securities are bounded. Clearly, there are a large number of dependent security equilibria that can be derived from an independent security equilibrium.

Furthermore, these dependent security equilibria form an indifference class so far as consumers and producers are concerned, because the derivation of securities is a non-essential grouping of independent securities.
Fourthly, we stress the generality of the model in encompassing defaulting securities that have perfect substitutes in the market: that is, the consumer short-sells a security "a" to the point of default, but that security with default has a perfect substitute "b". Because security "a" and security "b" have different patterns of returns they should, in general, have different prices. To illustrate our point, consider the extreme case of an Arrow-Debreu security economy, with prices $p^*_A = (p^*_1, \ldots, p^*_S)$. A riskless bond b, paying "r" units of commodity in each state of the world will have a price $p^*_b$, such that in equilibrium

$$p^*_b = \sum_{S} r_{S} p^*_S.$$ 

Let the bond default such that for $s \in S'$ it pays $r_s < r$, and for $s \in S''$ it pays $r$, where $S', S''$ partition $S$. Therefore in equilibrium, the value of the defaulting bond $p^*_b$ will be

$$p^*_b = \sum_{S'} r_{S} p^*_S + \sum_{S''} r_{S} p^*_S < p^*_b.$$ 

It should be obvious that as the bond becomes riskier (i.e., the payment in at least one state declines) the price of the bond falls. Therefore, the formulation we have adopted incorporates a situation where the consumer faces a spectrum of bond prices associated with increasing default risk. The range of bond prices has nothing to do with market imperfection - the bond prices differ because they have different patterns of returns. In the next Section we will deal with a different situation where the consumer has the opportunity
to offer a security that does not have a perfect substitute in the market. Because the consumer will have some monopoly power in the introduction of a new security (or commodity) the analysis above is no longer applicable.

2. CHOICE OVER ASSET ECONOMIES

2.1 In the previous section we have given quite general conditions for the existence of an equilibrium in an asset economy with short sales. Therefore, for any particular pattern of returns $z^q$ there exists an equilibrium $(m + n + 1)^A_A^A$ tuple $((x^q_i), (y^q_j), p^q)$. Because $z^q$ is a matrix of returns per asset, consider the normalization obtained by choosing the vectors $z_{q_k}$ from the unit simplex. By Tychonoff's theorem, $z^k$ will be drawn from a compact set $A$.

Now we propose to use this sequence of asset economies to investigate the notion that a consumer may issue a defaulting short-sale which is not a perfect substitute for any of the existing securities. In effect, the consumer is creating a new asset economy with an additional commodity, and the resulting consumptions, productions and prices will change depending upon

(i) his choice of pattern of returns of the new security; and

(ii) the restrictions (if any) on the supply of the new security.

The distinction between the choice of the pattern of returns, and the supply of assets result from the distinction between monopoly power in the creation of patterns of returns, and monopoly power in the supplying of a particular asset. We will exploit this distinction in two models:
2.2 The first model assumes that one consumer has power to introduce a new pattern of returns not in existence, but once it is introduced, the market for the new security becomes competitive; and

(b) the second model assumes that the consumer not only has power in the introduction of the new security, but also a monopoly on its supply.

We will attempt to justify the consumer's monopoly power in introducing the new security by assuming that there are fixed costs in informing the market of the characteristics of a security. Because of the pattern of endowments and transaction costs our consumer can introduce a new security, but once that security is introduced and traded in the market, its characteristics become public knowledge, ensuring a competitive market. By defaulting on his short-sale(s) the consumer is de facto issuing a new security, so that he must choose an asset economy from the possible sequence of asset economies generated by his decision.

If we assume that the competitive equilibrium is unique, then it is easy to construct a preference pre-ordering (from the consumer's maximal consumptions and the linear mapping $Z^q$) over the set $\{Z^q\}$. Therefore, the consumer will have a preference pre-ordering over the non-empty, compact set $A$, for which there exists a greatest element $Z^*$. The choice of $Z^*$ by our consumer, of course, implies a decision about the new security to be introduced.

One of the limitations of the model is that it assumes the consumer can compute all the prices associated with the asset equilibria. Nevertheless, we think that it is a greatly simplifying assumption that eliminates unnecessary complication. Another objection may take the form that the transactions cost
argument is far-fetched, and new securities can be created costlessly. This argument must imply that there are complete markets in the Arrow-Debreu sense: thus any security issued by an agent is a perfect substitute for a convex combination of existing assets. If this argument is accepted, the discussion in Section 1 is sufficient unto itself.17

2.3 The second model we will present removes the asymmetry between the monopoly power of the consumer in the introduction of the security and its subsequent competitive market. Instead assume that any security traded in the market requires a transaction fixed-cost (for example, the issuing of a prospectus). Now assume that there exists a core set of asset markets which are competitive, but the consumer can issue a new security for which he is the sole supplier, because new entrants to that security market are prohibited by the relatively high issuance costs. We could formulate the monopolistic market as a sector of a competitive general equilibrium model;18 but we will find it more profitable to break with the general equilibrium approach and consider a partial equilibrium model. We will take up this argument in the next section.

3. SHORT-SELLING, BORROWING AND DEFAULT IN PARTIAL EQUILIBRIUM

3.1 Although the analysis in this section invokes a number of simplifications it retains the essential features that enable comparison with the general equilibrium construction of Section 1. For simplicity, consider an exchange model with two market securities - a safe asset and a risky asset. Taking the one commodity-states of the world approach and eliminating the first period

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17 I suspect that the construction assumed by Fama-Miller (1972) is of this type.

18 We have in mind the type of construction where the monopolist has an inverse demand function generated by the competitive sector. See Fitzroy (1974).
in a similar way to Diamond's model (1967), let there be a continuum of states \( s \in [\underline{s}, \bar{s}] \subseteq \mathbb{R} \), where \( r(s) \) and \( Z(s) \) are continuous return functions for the safe and risky asset respectively, such that \( r(s) = r \), and \( Z(s) \in [\underline{Z}, \bar{Z}] \), \( r, \underline{Z}, \bar{Z} \) non-negative and finite, and \( Z(s) \) monotonically increasing. If the initial endowment of the riskless asset is \( \omega_1 \), and of the risky asset \( \omega_2 \), then aggregate terminal wealth is

\[
Y(s) = r\omega_1 + Z(s)\omega_2.
\]

Let there be \( \{b, 1, 2, \ldots, m\} \) consumers. We will assume that consumer \( b \) can sell short or borrow without restriction but consumer \( i = 1, 2, \ldots, m \) is restricted to holding all securities long. (This is not a substantial assumption, but it does simplify our analysis). Assume that all consumers maximize their von Neumann-Morgenstern utility functions, defined over consumption in different states of the world, and subject to their budget constraints on feasible portfolios. To begin, consider the problem facing consumer \( b \). Given his initial endowment \( W_b \), and the market prices \( P_1, P_2 \), his budget constraint is (assuming non-satiety)

\[
P_1 x_{1b} + P_2 x_{2b} = W_b. \tag{1}
\]

Assuming limited liability, consumer \( b \)'s terminal wealth (or consumption) \( Y_b(s) \) must be non-negative, i.e.,

\[
Y_b(s) = rx_{1b} + Z(s)x_{2b} \geq 0. \tag{2}
\]

19 The continuum of states is a convenient simplification that enables us to use the standard Riemann integral. For a finite set of states, and/or more general utility and probability distributions, the integral can be interpreted in the Lebesgue-Stieltje's sense. Consequently, derivatives and derived functions are defined almost everywhere.
Substituting for $x_{1b}$ from (1) into (2) we obtain

$$Y_b(s) = (\frac{P_1}{W_b} - x_{2b}) (r/P_1) + p_2 x_{2b} (\frac{Z(s)}{p_2}).$$

(3)

Observe that $(r/P_1)$ and $(Z(s)/p_2)$ are rates of return on the assets, and that to avoid dominance, $P_1, P_2$ should have values such that

$$(Z/P_2) < (r/P_1) < (Z/P_2).$$

Otherwise, consumers would hold only one security in absolute preference to the other, and market clearing conditions would be violated. Now assume that $P_2 = 1$, and consider (3) when $Y_b(s) = 0$. Solving for $Z^*$, the default return, we find

$$Z^* = \{ (x_{2b} - W_b) / x_{2b} \} (r/P_1).$$

(4)

For $x_{2b} > 0$, we find

(i) $x_{2b} = W_b \Rightarrow Z^* = 0;$

(ii) $\lim_{x_{2b} \to 0^+} Z^* = -\infty;$

(iii) $\lim_{x_{2b} \to +\infty} Z^* = (r/P_1).$
Figure 1
For \( x_{2b} < 0 \), we find

\[
\text{(i)} \quad \lim_{x_{2b} \to 0} z^* = +\infty; \\
\text{(ii)} \quad \lim_{x_{2b} \to -\infty} z^* = \left(\frac{r}{p_1}\right). 
\]

Defining \( x_{2b}^* \) and \( x_{2b}^* \) as the solutions to (4) for \( Z \) and \( \bar{Z} \) respectively, we can graph (4) as in Figure 1. We can define the interval \([x_{2b}^*, x_{2b}^*]\) as the interval of "no bankruptcy". Notice that it contains three distinct regions:

(a) \([x_{2b}^*, 0]\) where the risky security is held short and the safe asset is held long;

(b) \([0, W_b]\) where both assets are held long; and

(c) \([W_b, x_{2b}^*]\) where the risky asset is held long and the consumer borrows (or the safe asset is held short).

It is easy to see that \( x_{2b}^* \) and \( x_{2b}^* \) are functions of \( W_b \) and \( p_1 \) such that

\[
\frac{\partial x_{2b}^*}{\partial W_b} > 0; \quad \frac{\partial x_{2b}^*}{\partial p_1} > 0; \\
\frac{\partial x_{2b}^*}{\partial W_b} < 0; \quad \frac{\partial x_{2b}^*}{\partial p_1} > 0. 
\]

The effect of increasing wealth on the interval is intuitive: greater wealth allows increased absolute borrowing or short-sales before bankruptcy can occur, because default is a function of the proportion of borrowing (short-sales) to initial wealth, i.e., \( Z^* = (1 - e_b)(r/p_1) \) where
can be expressed as $e_b = (W_b / x_{2b})$. An increase in the price of the riskless bond increases $x_{2b}$ because the bond is less expensive for borrowing (i.e., the rate of interest is lower); but decreases the amount of no-default short-sales because the risky asset is now relatively dearer.

So far we have limited our discussion to the boundaries of the no-bankruptcy interval. In the interior of this interval the consumer $b$ trades with existing patterns of returns available in the market. Thus, even if he borrows or short-sells, his bond (or short-sale) is a perfect substitute for the market security. But if there are states of the world where he defaults, the security issued by $b$ will not be a perfect substitute for the market security. For example, let $b$ issue enough bonds to take a position $x_{2b}^* > x_{2b}$ such that $Z^* > Z$. Now for states of the world where $Z \in [Z^*, Z]$ consumer $b$'s bond gives a return $r$, but for $Z \in [Z, Z^*]$ the consumer's bond returns less than $r$. Clearly the risky bond is dominated by the market riskless bond if both have the same price $P_1$, so that markets would not clear. Therefore, for market clearance, we require that the consumer's risky bond $x_{3b}$ to have a price $P_3 < P_1$.

We conclude, that for $x_{2b} \notin [x_{2b}^*, x_{2b}]$, the consumer $b$ creates new patterns of returns, and his bonds or short-sales will sell at a discount because of the possibility of default. Outside the interval $[x_{2b}^*, x_{2b}]$, the function $Z^*(x_{2b}, P_1, P_2)$ does not apply because $b$ faces different prices. Before we move on to $b$'s portfolio problem, notice the following simple result: if $Z, \bar{Z}$ collapses about $r$, then assuming neither security dominates, we have $P_1 = P_2 = 1$. The interval $[x_{2b}^*, x_{2b}]$ now coincides with $[\mathcal{R}]$, because both securities are perfect substitutes. In a certainty world, all securities are perfect substitutes.

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20 Although this assertion is intuitive enough, we will derive it as a by-product of the analysis in 3.3.
identical (trivially) with the riskless rate of interest, and bankruptcy vanishes from the analysis.

3.2 The Borrower’s Problem:

Consider consumer b’s portfolio problem allowing for borrowing, short-sales and default:

(i) for $x^{2b} \leq \bar{x}^{2b}$, consumer b’s bond price $p_{3} = p_{1}$,
and for $x^{2b} > \bar{x}^{2b}$, $p_{3} \leq p_{1}$, such that $p_{3}(x^{2b})$
is continuous, differentiable and $p_{3}'(x^{2b}) < 0$.

(ii) for $x^{2b} \geq \bar{x}^{2b}$ consumer b’s short-sale price $p_{4} = p_{2} = 1$,
and for $x^{2b} < \bar{x}^{2b}$, $p_{4} < p_{2}$, such that $p_{4}(x^{2b})$ is
continuous, differentiable and $p_{4}'(x^{2b}) > 0$.

The first thing to notice about (i) and (ii) is their symmetry: this is not surprising because the securities differ only in their pattern of returns across states. Secondly, we have introduced the assumption that b faces demand curves for his securities. We will attempt a justification for this assumption in 3.3. Now given that b wishes to maximize his von Neumann-Morgenstern utility over terminal wealth, and given limited liability, b’s problem becomes:

$$\max_{\{x^{2b}\}} u^{b} = \delta \left\{ \int_{\bar{Z}} u_{b}(0)dG_{b}(Z) + \int_{Z^{*}} u_{b}([W_{b} - x^{2b}]/(r/p_{3}) + x^{2b}Z)dG_{b}(Z) \right\}$$

$$+ (1-\delta) \left\{ \int_{\bar{Z}} u_{b}([(W_{b} - p_{4}x^{2b})]/(r/p_{1}) + x^{2b}Z)dG_{b}(Z) + \int_{Z^{*}} u_{b}(0)dG_{b}(Z) \right\}^{21}$$

21 We will take the integral to be Riemann. See footnote 19.
where \( Z^* = \{ [P_4 x_{2b} - W_{2b}]/x_{2b} \}(r/P_3) \); 

\( u_b(Y_b) \) is positive and finite; 

\( u_b \in C^2 \), such that \( u'_b > 0 \), \( u''_b < 0 \) (i.e. risk-aversion); and 

\( u'_b, u''_b \) finite a.e., and \( \delta \) is the Kronecher delta such that 

\[
\delta = \begin{cases} 
1 & \text{if } x_{2b} > 0 \\
0 & \text{if } x_{2b} < 0.
\end{cases}
\]

Clearly, the portfolio problem is symmetrical for borrowing or short-sales. Let us assume that (5) takes its maximum in the region \( x_{2b} > 0 \), that is, consumer \( b \) borrows, or holds short the riskless asset, and holds long the risky asset.

The first order condition for a maximum of (5) is

\[
\frac{\partial}{\partial Z} \left[ u'_b[Z - (r/P_3) - p'_3 B(r/P_3)] \delta_b(Z) \right] = 0, \tag{6}
\]

where \( B \equiv (W_{2b} - x_{2b})/P_3 \). (Of course \( B = x_{1b} \) in the no default interval).

The second order condition for a maximum is

\[
\frac{\partial}{\partial Z} \left[ \frac{\partial}{\partial Z} \left[ u''_b[Z - (r/P_3) - B(r/P_3)] \delta_b(Z) \right] \right] < 0. \tag{7}
\]
If the optimal portfolio $x^*_b < \bar{x}_b$, then $Z^* < \bar{Z}$. $P_3 = P_1', P_3' = 0$, and conditions (6) and (7) collapse to the familiar portfolio conditions

$$
\int_Z u'_b [Z - r/P_1] dG_b (Z) = 0, \tag{6'}
$$

and

$$
\int_Z u''_b [Z - r/P_1]^2 dG_b (Z) < 0. \tag{7'}
$$

Returning to (7) notice that the second term is unambiguously negative, and a sufficient condition for the last term to be negative is $P_3'' < 0$. The first term, on the other hand, is unambiguously positive, because

(a) $\frac{\partial Z^*}{\partial x_b} > 0$,

(b) $[Z^* - (r/P_3) - B(r/P_3)P_3'] < 0$ from (6),

(c) $u'_b|_{Z^*} = u'_b (0)$.

Thus we require that the first term to be dominated by the other two terms if the second order condition is to be satisfied. Now a sufficient condition for $x_b < \bar{x}_b$ is that $G_b (Z)$ be strictly increasing (i.e., $g_b (Z) > 0$) and

$$
\lim_{y_b \to 0} u'_b (y_b) = +\infty.
$$

It is easy to show that (7) cannot be satisfied for $Z^* \geq \bar{Z}$, or $x_b < \bar{x}_b$. The intuitive reason for this is that $b$ has an extreme preference for positive
consumption, and therefore he will never default in any state of the world. By the symmetry of the borrowing-short-selling argument, consumer b would always choose a portfolio in the no-default interval, and our discussion of new assets would be irrelevant. In the argument below, we will assume that such a situation does not necessarily arise.

Also, a sufficient condition for a finite optimal portfolio $x_{2b}'$, is that for a sufficiently large, finite $x_{2b}'$, we have $r/p_3(x_{2b}') = Z$. The intuitive meaning of this is straightforward. As the borrower goes increasingly into debt to finance his holdings of the risky asset, his bond price falls (or his nominal interest rate rises) to such an extent that his optimal portfolio is bounded above by $x_{2b}'$. This proposition requires the assumption that the borrower's bond price is a decreasing function of his borrowing (with default), and falls sufficiently to bound the portfolio of even the most optimistic borrower. We will attempt a justification for the assumption on the sign of the derivative of the bond price in the next section.

3.3 The Lender's Problem:

Recall from 3.1 that we postulated that consumers $i = 1, \ldots, m$ hold all securities long. We could have assumed borrowing and short-selling, without default for this group, but the argument would be a little messier, although essentially unchanged. Given that all securities are held long, then consumer i's terminal wealth is

22 The reader may recognize the Inada-like condition sometimes used in optimal growth models to bound consumption away from zero. The assumption can be justified on the grounds that the consumer has an absolute preference for survival.

23 Assuming risk-neutrality, we will show below that the borrower must expect a higher return on $x_2$ than the set of lenders. In this sense, we say that the borrower is more optimistic. With risk-aversion, portfolio choice is a function of expectations and preferences and we cannot make a distinction based upon expectations alone.
Given the $i$th consumer's budget constraint,

$$P_{1}x_{1i} + P_{2}b_{i} + x_{2i} = W_{i},$$

and assuming he is a price taker for all securities, we can write

$$- (B_{i}/B)x_{2b}Z = B_{i}(rZ/Z^*).$$

Therefore, the consumer's problem becomes

$$\begin{align*}
\text{Max} & \quad U_{i}^{*} = \int_{Z} u_{i}([W_{i} - P_{2}b_{i} - x_{2i}](r/P_{1}) + B_{i}r(Z/Z^*) \\
& \quad + x_{2i}Z) dG_{i}(Z) + \int_{Z^*} u_{i}([W_{i} - P_{2}b_{i} - x_{2i}](r/P_{1}) \\
& \quad + B_{i}r + x_{2i}Z) dG_{i}(Z),
\end{align*}$$

where

$$r_{b} = \begin{cases} r & \text{for } Z \in [Z^*, \bar{Z}] \\
- B^{-1}x_{2b}Z & \text{for } Z \in [\underline{Z}, Z^*], \end{cases}$$

$$Z^* = \{(x_{2b} - W_{b})/x_{2b}(r/P_{3})\},$$

and

$$\sum_{i=1}^{l} B_{i} = -B.$$
subject to
\[ W_i - P_3 B_i - x_{z_i} \geq 0, \]
\[ B_i \geq 0, \]
\[ x_{z_i} > 0. \]

The first order conditions for an interior maximum are

\[ U_{B_i}^{z_i} = \int_{Z^*}^{\bar{Z}} u_1'(z - (r P_3 / P_1)) dG_1(z) + \int_{Z^*}^{\bar{Z}} u_1'[z - (r P_3 / P_1)] dG_1(z) = 0 \quad (9) \]

\[ U_{x_{z_i}}^{z_i} = \int_{Z^*}^{\bar{Z}} u_1'[z - r/P_1] dG_1(z) + \int_{Z^*}^{\bar{Z}} u_1'[z - r/P_1] dG_1(z) = 0. \quad (10) \]

In (9) and (10), \( p_3, B \) enter as parameters of the derived demand equations for \( x_{z_i}, B_i, x_{l_i} \). If \( p_3, B \) are constrained so that \( Z^* \leq Z \), then (9), (10) collapse to the form where

\[ \int_{Z}^{\bar{Z}} u_1'[r - (r P_3 / P_1)] dG_1(z) = 0, \quad (9') \]

\[ \int_{Z}^{\bar{Z}} u_1'[z - (r/P_1)] dG_1(z) = 0. \quad (10') \]
Because B is now a riskless bond, we require $P_3 = P_1$, or otherwise $B$ or $x_1$ would be held short depending upon $P_3 - P_1 > 0$. Thus the distinction between $B$ and $x_1$ vanishes so that consumer b’s bond is a perfect substitute for the riskless asset.

Conversely, if $B, P_b$ are such that $Z^* > Z$, then it is easy to see that $P_3 < P_1$ as asserted in 3.1. To justify the assumption $p'_b(B) > 0$ is more difficult: but the analysis is rewarding because we show that ordinary demand theory considerations carry over into the demand for assets.

Now differentiating equations (9) and (10) with respect to $P_3$ we obtain the system

$$
\begin{align*}
\begin{pmatrix}
U^i_{B_1B_1} & U^i_{B_1x_21} \\
U^i_{x_21B_1} & U^i_{x_21x_21}
\end{pmatrix}
\begin{pmatrix}
\frac{dB}{dp_3} \\
\frac{dx_{21}}{dp_3}
\end{pmatrix} &=
\begin{pmatrix}
- U^i_{B_1P_3} \\
- U^i_{x_21P_3}
\end{pmatrix}
\end{align*}
$$

or

$$Mv = y.$$

From the second order conditions for a maximum we know that

$$U^i_{B_1B_1} U^i_{x_21x_21} < 0, \quad |M| = \Delta = \begin{vmatrix}
U^i_{B_1B_1} & U^i_{B_1x_21} \\
U^i_{x_21B_1} & U^i_{x_21x_21}
\end{vmatrix} > 0;$$

and from Cramer’s Rule we find
Similarly, we can repeat the analysis for B, and we derive

\[
\frac{dB_1}{db} = A^{-1} \begin{vmatrix}
-i U_{1,1} B_1 & i U_{1,1} B_1 X_2 \\
-i U_{2,1} B_1 & i U_{2,1} B_1 X_2 
\end{vmatrix}.
\] (13)

Although the derivations are tedious, there is some reward for evaluating the terms appearing in the determinants, because they can be written in the form

\[
\frac{i}{U_{1,1} B_1} = \alpha' \frac{\partial Z^*}{\partial B} ; \quad \frac{i}{U_{2,1} B_1} = \beta' \frac{\partial Z^*}{\partial B} ;
\]

\[
\frac{i}{U_{1,1} P_3} = \alpha + \alpha' \frac{\partial Z^*}{\partial P_3} ; \quad \frac{i}{U_{2,1} P_3} = \beta + \beta' \frac{\partial Z^*}{\partial P_3} ;
\]

where

\[
(i) \quad \alpha' \equiv \left[ \frac{r/(Z^*)}{Z} \right]^2 \int \frac{Z^*}{Z} \left\{ u'' \left[ -B_1 Z (r/Z^*) - (r P_3/P_1) \right] \right\} \]

\[
- u_1' Z) dG_1 (Z) \ll 0.
\]
(ii) \[ \beta' \equiv U_{x_2}^i Z_* = \frac{1}{(Z^*)^2} \int \frac{Z^*}{Z} u''[\frac{-B_1 Z(Z - (r/P_1))]dG_1(Z) \geq 0. \]

(iii) \[ \frac{3Z_*}{\partial B} = - p_3[(r/p_3) - Z_*][W_1 - p_3B]^{-1} < 0. \]

(iv) \[ \frac{3Z_*}{\partial P_3} = -(Z^*)^2/r < 0. \]

(v) \[ U_{B_1 X_2}^i = U_{X_2 B_1}^i = \frac{Z^*}{Z} \int \frac{Z^*}{Z} u''[Z - (r/P_1)] \frac{r}{Z^*} - \frac{(rP_3/P_1)}{Z^*} dG_1(Z) \]

\[ + \int \frac{Z^*}{Z} u''[Z - (r/P_1)] \frac{r}{Z^*} - \frac{(rP_3/P_1)}{Z^*} dG_1(Z) \geq 0. \]

(vi) \[ \alpha \equiv \int \frac{Z^*}{Z} \left\{ u''[\frac{-B_1 r/P_1]}{Z(Z^*)} - \frac{(rP_3/P_1)}{Z^*} \right\} dG_1(Z) \]

\[ + \int \frac{Z^*}{Z} \left\{ u''[\frac{-B_1 r/P_1]}{Z} - \frac{(rP_3/P_1)}{Z^*} \right\} dG_1(Z) \geq 0. \]

(vii) \[ \beta \equiv \int \frac{Z^*}{Z} u'[-B_1 r/P_1][Z - (r/P_1)] dG_1(Z) \]

\[ + \int \frac{Z^*}{Z} u'[-B_1 r/P_1][Z - (r/P_1)] dG_1(Z) \geq 0. \]
Given these expansions, we can rewrite (12) and (13) in the following way:

\[
\frac{dB_i}{dP_3} = \Delta^{-1}(A_1 + A_2 \frac{\partial^2 }{\partial P_3})
\]

(12')

\[
\frac{dB_i}{dB} = \Delta^{-1}(A_2 \frac{\partial^2 }{\partial B})
\]

where

\[
A_1 = \begin{bmatrix}
-\alpha & U_i^{B_1} x_{2i} \\
-\beta & U_i^{x_{2i} x_{2i}}
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-\alpha' & U_i^{B_1} x_{2i} \\
-\beta' & U_i^{x_{2i} x_{2i}}
\end{bmatrix}
\]

Now it is easy to see that \( \Delta^{-1} A_1 \) is the Slutsky term for \( B_i \), which combines the negative own substitution term and the ambiguously-signed income effect. We will proceed by making the usual assumption that \( B_i \) is a normal good (i.e., \( \Delta^{-1} A_1 < 0 \).)

---

24 This result follows easily from the observation that we can treat patterns of returns as characteristics of commodities. For a more extensive discussion see Fischer (1972).

25 Different sufficient conditions can be employed to produce this result. For example, if we assume constant absolute risk-aversion (i.e. \( A = u''/u' \)) then \( B_i \) is a normal good. That is, substituting into (9) and (10) we obtain:

\[
\int_{Z}^{Z^*} u''[r(Z)/Z^*] - (rP_3/P_1) dG_i(Z) + \int_{Z^*}^{Z} u''[r - (rP_3/P_1) dG_i(Z) = 0 \tag{9''}
\]

\[
\int_{Z}^{Z^*} u''[Z - (r/P_1)] dG_i(Z) + \int_{Z^*}^{Z} u''[Z - (r/P_1)] dG_i(Z) = 0. \tag{10''}
\]

But (9''), (10'') \( \Rightarrow \beta = 0 \) and \( \alpha < 0 \Rightarrow \Delta^{-1} A_1 < 0 \).
The other term $\Delta^{-1}A_2$ provides the impact of a change in the pattern of returns produced by the risky bond, brought about by either

(i) a change in the price of the bond, or

(ii) a change in the amount of the risky bond issued by the borrower.

Again, without imposing restrictions on our general formulation, we cannot sign $A_2$. Nevertheless, there is a close connection between a change in the pattern of returns of a security (and its effects on the demand for that security) and changes in point-prices in the problem of "point-rationing". This problem is discussed in Samuelson (1947) where he shows that point-rationing constraints can be added to the usual consumer problem, and generalized Slutsky conditions can be obtained. Diamond and Yaari (1972) have exploited these results by showing that the consumer's asset problem (with a finite state-space) is isomorphic to the point-rationing problem. Therefore, changes in the pattern of returns for an asset are isomorphic to changes in the point-prices for a particular commodity. Without further exploration we will assert that the term $\Delta^{-1}A_2$ has similar properties to a Slutsky term - with a compensated own substitution term, and a generalized income effect. The own substitution term will be negative, and the income effect will depend upon the superiority or inferiority of the asset.26

Although indeterminateness should be expected from the generalized income effects, we can obtain clear-cut results in two cases:

(a) when the holding of $B_1$ is "small", and

(b) when there is risk-neutrality.

26 We mean generalized inferiority or superiority in the sense employed by Diamond and Yaari (1972).
(a) It is well-known that the Slutsky term will have a negative own-effect if the commodity under investigation comprises a sufficiently small amount of the total budget; that is, if $B_1 \to 0$, the income effect, no matter the sign, will be dominated by the negative substitution effect. In our formulation, we find

$$A_1^0 = \lim_{B_1 \to 0} A_1 = \lim_{B_1 \to 0} \left\{ \frac{Z^*}{Z} \int u'_1(r/P_1) dG_i(Z) + \int u'_1(r/P_1) dG_i(Z) \right\} < 0.$$

Similarly we find

$$A_2^0 = \lim_{B_1 \to 0} A_2 = \lim_{B_1 \to 0} \left\{ \frac{Z^*}{Z} \int u'_1 r/(Z^*)^2 \frac{Z}{Z} \int u'_1 Z dG_i(Z) \right\} < 0.$$

But observe that $\frac{\partial Z^*}{\partial P_3} = - (Z^*)^2 / r < 0$ from (14)(iv), and therefore $A_2^0 \frac{\partial}{\partial P_3} > 0$.

Now combining the Slutsky term and the pattern of returns effect, we obtain

$$\frac{dB_1^0}{dp_3} = \lim_{B_1 \to 0} \frac{dB_1}{dp_3} = - \lim_{B_1 \to 0} \left\{ \frac{Z^*}{Z} \int u'_1 \left[ Z - (r/P_1) \right] dG_i(Z) + \int u'_1 (r/P_1) dG_i(Z) \right\} < 0,$$

because from (10) we deduce that the term {\ldots} is negative.

From continuity, we can strengthen our result to the following:

there exists an $\varepsilon$-neighbourhood of $B_1 = 0$, such that $N(0, \varepsilon) = \{B_1 \in [0, \varepsilon)\}$,

$$\frac{dB_1}{dp_3} < 0 \text{ for } B_1 \in N(0, \varepsilon).$$

Because $A^0 < 0$, we have immediately,
and again we can apply continuity to extend the argument to an $\epsilon$-neighbourhood.

For the final step in the argument, observe that

$$
\sum_{i=1}^{n} B_i + B = 0,
$$

and therefore

$$
\left( \sum \frac{3B_i}{3B} + 1 \right) dB + \left( \sum \frac{3B_i}{3P_3} \right) dP_3 = 0,
$$

or

$$
\frac{dP_3}{dB} = - \left( \sum \frac{3B_i}{3B} + 1 \right) \left( \sum \frac{3B_i}{3P_3} \right)^{-1}.
$$

If $B_i \in N(0, \epsilon)$ then $\frac{dP_3}{dB} > 0$ as asserted in 3.2.

In this section, we have tried to keep the discussion as general as possible, but clearly, the existence of ambiguously-signed income effects and pattern of return effects create difficulties in obtaining signs for comparative static analysis.

(b) If we are prepared to make the strong assumption of risk-neutrality the problem reduces to almost trivial dimensions; but nevertheless, it is instructive because it reveals a possible source of borrowing and short-selling behaviour.
Given risk-neutrality (9) and (10) become

$$
\Phi = [r/Z^*] \mathbb{E}(Z \leq Z^*) + r[1 - G(Z^*)] - (rP_3/P_1) = 0
$$

(15)

$$
\mathbb{E}(Z) - (r/p_1) = 0,
$$

(16)

where $\mathbb{E}(\cdot)$ is the expectations operator. Notice that we have been forced to drop the subscript $i$, because we have assumed all $i$ consumers hold securities long, and that in turn must require identical probability distributions.

Alternatively, we know that risk-neutrality implies linear indifference curves (in a finite state-space) so that for all securities to be held long, the indifference curves between two assets must have the same slope for all consumers.

Differentiating (15) with respect to $B$ and $P_3$,

$$
\Phi_B = \left[-\left(\frac{r}{Z^*}\right)^2\right] \mathbb{E}(Z \leq Z^*) \frac{3Z^*}{dB} \equiv F \frac{3Z^*}{dB},
$$

(17)

$$
\Phi_{P_3} = - (r/p_1) + \frac{3Z^*}{dp_3}.
$$

(18)

Clearly $F < 0$, so that we have $\Phi_B > 0$. We can sign $\Phi_{P_3}$ by observing that

$$(r/p_1) = \mathbb{E}(Z)$$

from (16). Substituting into (18) we find

$$
\Phi_{P_3} = - \mathbb{E}(Z) + \mathbb{E}(Z \leq Z^*) < 0.
$$

(19)

Thus $\frac{dp_3}{dB} = \frac{\Phi_{P_3}}{\Phi_B} > 0$ as required. We can treat the borrower's problem in a
similar manner. It is easy to see from our discussion of the borrower's problem that the expected return for \( b \), \( \bar{\mathbb{E}}_B(z) \) is greater than the rate of return for the safe security \( (r/p_1) \). But notice that for the lender to hold all securities long (equation (16)) we must have \( \bar{\mathbb{E}}(z) = (r/p_1) \). Thus, in the case of risk-neutrality, borrowing requires \( \bar{\mathbb{E}}_B(z) > \bar{\mathbb{E}}(z) \). In the sense of expected values, the borrower can be said to be more optimistic about returns from asset \( x_2 \), than the set of lenders. This result is neither new nor surprising, for it is widely acknowledged that divergent expectations are a possible source of borrowing and short-selling behaviour.

3.4 In the previous section we saw that if consumers had risk-neutral preferences, then divergent expectations were sufficient to generate short-holdings of securities. It is well-known also that common expectations and variations in preferences and wealth are sufficient to imply borrowing or short-sales. The best-known example of this phenomenon is the Sharpe-Lintner model, where all risky securities are held in the same proportion by all consumers (thus ruling out short-sales), but borrowing may occur.\(^{27}\) If the restriction on preferences is strengthened to the requirement that all consumers are identical with respect to preferences and wealth, then market clearance implies the trivial result that all securities are held long. Therefore, borrowing and short-sales require a divergence in consumer's preferences and/or expectations.

Another interesting implication of our simple model, is the role played by the borrower's wealth. We observed in 3.1, that the default rate of return \( Z^* \) could be written as a function of the proportion of the risky asset

\(^{27}\) Unfortunately most discussions ignore default as an important element in the analysis, even though it is assumed that \( q_1(0) > 0 \) - which implies a positive probability of default for any level of borrowing.
held in the portfolio in the determination of the pattern of returns created by the borrower. Therefore, in the lender's problem, we could have replaced B by \((B/W_b)\), without altering the qualitative results on the response of the lender. Because the demand curve for the borrower's bond is a function of the proportion \((B/W_b)\), it follows that the demand curve will be more elastic the greater the wealth \(W_b\). The reason for this is simple: the consumer's initial wealth acts as a buffer between low returns on the risky asset and the constant interest rate payable on borrowed funds. The greater the initial wealth of the borrower, the more protection he provides the lender against default on a given bond issue. This argument provides a plausible explanation for the observation that borrowing terms, for a given loan size, become easier the wealthier the borrower.\(^{28}\) We should stress that the general relationship between wealth and borrowing terms applies also to the competitive asset model of Section 1. This can be seen in the simple example of this section, where no-default interval is a function of the consumer's initial wealth. Because there are two market securities, the no-default interval is a representation of the consumer's feasibility set. If more assets are added, the more general representation of Section 1 is required, and the no-default interval becomes a multi-dimensional set.

4. CONCLUDING COMMENTS

4.1 Our analysis is analytically very similar to the formulation of a single proprietor firm. We will give a quick sketch of that literature, indicating its relationship to our discussion.

\(^{28}\) For example, in passing, Keynes (1936) observed (p.145) "... if a venture is a risky one, the borrower will require a wider margin between his expectation of yield and the rate of interest at which he will think it worth while to borrow, whilst the very same reason will lead the lender to require a wider margin between what he charges and the pure rate of interest in order to induce him to lend (except when the borrower is so strong and wealthy that he is in a position to offer an exceptional margin of security)". Our italics.
In 1937, Kalecki (1937) produced a verbal argument asserting that the scale of the firm could be bounded above by the increasing risk associated with financial leverage on a fixed amount of entrepreneurial capital. His argument was intended to provide a limit to the firm's scale, when constant returns to scale and competitive pricing conditions implied an apparent embarrassment for received theory. Kalecki argued that the scale of the firm would be limited by two related forces:

(i) given the fixed entrepreneurial capital, increases in borrowing would result in higher interest rates brought about by the increased probability of default; and

(ii) the entrepreneur's own attitude toward risk, given the increasing probability of default.

These two strands of the argument were amalgamated under the general title of the "Principle of Increasing Risk". By a simple translation our theory of the consumer can be presented as a representation of the single proprietor firm, exhibiting the characteristics described by Kalecki.

Now Kalecki admits29 that his theory has a weakness: his proposition is based on the assumption that the amount of entrepreneurial capital is bounded above by the entrepreneur's own wealth; but he acknowledges that the joint-stock company with its access to equity capital markets would appear to invalidate his argument. Furthermore, his discussion does not mention the reasons for the break-down of profit-maximization and unanimity in production decisions, which are the outcomes of conventional theory. We could dismiss his theory because of these difficulties. Nevertheless, Kalecki in his defence introduced corporate control as major limiting factor in the use of equity

29 See the revised version of his paper included as a chapter in Kalecki (1971).
financing. In the classical theory, control is vacuous, because production decisions are unanimous. But it can be shown that control and conflict appear when there are incomplete markets for time-state claims, and the firm can create new securities by production decisions.\textsuperscript{30} Perhaps Kalecki's proposition could be rescued by an argument along these lines.

4.2 The single proprietor firm has reappeared in recent discussions of capital rationing.\textsuperscript{31} This problem concerns the determination of borrowing terms between a single proprietor firm and a bank. The agents' objective functions are virtually identical to those given in Section 3 of this paper. The divergence comes in the assumption of relative marketpower of the two agents. We have assumed the borrower faces a large number of lenders, whereas the banking problem stresses the market power of the bank over the borrowers. In particular, Smith (1972b) and Jaffee (1972) consider a situation of bilateral monopoly with its associated indeterminancy of borrowing terms. We should observe that to obtain capital rationing, arbitrary restrictions on interest rates (e.g. usury laws) have to be imposed. Under these restrictions rationing is the usual price-theoretic result, and that outcome has nothing to do with default or riskiness of the loan. This concludes our discussion of the single proprietor firm literature.

4.3 One of the aims of this Chapter has been to show that standard price theoretic notions can be applied to asset theory. By defining a commodity by its pattern of returns characteristic, we have been able to deal with default risk in general equilibrium and partial equilibrium frameworks. Of course, we

\textsuperscript{30} See Chapter 4.

\textsuperscript{31} See Jaffee (1972), Smith (1972b) and references quoted in their bibliographies.
have made simplifying assumptions - the one commodity - one period assumption needs to be relaxed. But there is another assumption that is restrictive, and it bears further investigation. We have assumed that agents can recognize objectively, the pattern of returns for any security. In his production model, Diamond (1967) recognized that stockholders needed to know, objectively, the pattern of returns generated by a production process. But with default and short-selling, agents are required to know the pattern of returns associated with a short-sold security, and this requires a detailed knowledge of the issuer's portfolio. In a world with costs associated with the transmission of information, one would suspect that the offer curve would reflect the degree of information about the issuer's portfolio, and there may be a place for the vaguer notions of "trust" and "reputation".\(^{32}\)

\(^{32}\) The problem of introducing information into an equilibrium model has been discussed by Radner (1968). For a discussion of the related problem of information and product quality, with some reference to securities markets, see Akerlof (1970).
In a recent paper, Stiglitz (1973) analysed the impact of personal and corporate taxes on the financing and investment policies of corporations. He derived conditions for optimal leverage and retention decisions depending upon the relative magnitudes of corporate and personal tax rates: but his more general conclusion was that the corporate tax structure acted as a lump-sum tax on corporate profits, and the real investment decisions were invariant - even though financial decisions may be affected strongly. It is well-known\(^1\) that in a competitive equilibrium without taxes, financial policy is irrelevant to the real decisions of economic agents, so that real and financial decisions can be separated. Therefore, Stiglitz's argument was that this essential separation property was preserved with the introduction of taxation on financial instruments, but the choice of financial instruments was determined by the desire to maximize the wealth of the shareholder.

Whereas Stiglitz couched his argument in terms of a partial equilibrium treatment with a sole proprietorship,\(^2\) we will present a general equilibrium treatment. This approach is revealing because it demonstrates clearly

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\(^1\) For a discussion and guide to the literature see Chapter 3.

\(^2\) Stiglitz does have a section where he discusses widely-held producers, but most of his analysis is in terms of a sole-proprietorship.
the separation of financial-tax-avoidance decisions and real decisions. Furthermore, given that the government acts passively in setting tax rates before the actions of agents is determined, then shareholders and bondholders in each firm will form coalitions against the government. For particular simple tax structure cases there will exist a unanimous decision on financing policy for the set of private agents. Of course, unanimity includes the coalition's choice of a unique optimal financial structure, or an optimal set of financial structures, or complete indifference to the set of all feasible financial structures. But for more realistic tax structures with differential taxes on private agents, there will exist a game-theoretic situation in the choice of financing instruments. By a selection of reasonable assumptions on the relative magnitudes of different taxes, it is possible to generate results consistent with everyday observation. Nevertheless we suggest that it would be an over-simplification to attribute the importance of financial decisions solely to tax effects, because the underlying real model assumes a complete market theory. We know that a complete forward market system does not exist in observable market economies, and it is no longer clear that the simple real market-financial market dichotomy continues to apply.\(^3\)

In Section 1 of the chapter we sketch the general equilibrium model showing the relationship between the choice of a financing-tax policy and the resulting distribution of wealth. Given a particular distribution of wealth, the model is identical to the Arrow-Debreu theory, and with the usual assumptions on production sets, consumption sets, etc., existence of an equilibrium is ensured. It follows as an easy corollary that the usual investment rules apply. Assuming that the holders of financial securities of the firm consider themselves negligible tax-payers, there is a well-defined game situation.

\(^3\) For a more complete discussion of these issues see Radner (1970).
In Section 2 we consider a number of examples to illustrate particular solutions of the more general formulation. We consider the leverage and retention problem as considered by Stiglitz; the depreciation problem, the creation of tax-havens and clientele affects; and finally, in the context of a complete Arrow-Debreu security market, the introduction of bankruptcy charges through the administration of the legal system.

1. THE GENERAL EQUILIBRIUM MODEL

1.1 Recall the competitive economy as developed in Debreu (1959). Debreu defines a commodity in $\mathbb{R}^k$ as a good or service completely specified physically, temporally and spatially. Let there be two sets of economic agents: $i = 1, \ldots, m$ consumers, and $j = 1, \ldots, n$ producers.

Consumer $i$ chooses his consumption $x_i$ from a non-empty subset $X_i$, his consumption set, where $X_i \subseteq \mathbb{R}^k$. The consumption set will have the usual properties:

(a) $X_i$ is closed, convex and bounded below for $\preceq$;

(b) Define on $X_i$ a complete pre-ordering $\preceq_i$ called a preference pre-ordering with the following properties:

(i) **Non-Satiety**: For any $x_i \in X_i$, $\exists x'_i \in X_i$ such that $x'_i \succ_i x_i$.

(ii) **Continuity**: For every $x'_i \in X_i$, the sets

$\{x_i \in X_i \mid x_i \succeq_i x'_i\}$ and $\{x_i \in X_i \mid x_i \preceq_i x'_i\}$

are closed.

(iii) **Convexity**: If $x'_i, x''_i \in X_i$, such that $x'_i \neq x''_i$ and $t \in \mathbb{R}$ in $(0, 1)$, then $x'_i \bigcirc_i x''_i \Rightarrow tx'_i + (1-t)x''_i$.  


(c) Producer \( j \) chooses his production \( y_j \) from his production
set \( Y_j \subseteq \mathbb{R}^k \). The total production set for the economy
is \( Y = \sum_j Y_j \). The production sets will have the properties:

(d) (i) \( 0 \in Y_j \);

(ii) \( Y \) is closed and convex;

(iii) \( Y \cap (-Y) \subseteq \{0\} \);

(iv) \( Y \supseteq \mathbb{R}^k \).

Finally, consider each consumer \( i \) to have an initial endowment
\( \omega_i \in \mathbb{R}^k \). In particular, consider the endowment to be such that

(e) \( \exists x_i^0 \in X_i \), such that \( x_i^0 \ll \omega_i \).

Now consider two definitions:

**Definition 1:**

A private ownership economy \( \mathcal{E} \) is defined by: an economy
\( ((X_i, \leq_i), (Y_j), (\omega)) \); for each \( i \), a point \( \omega_i \in \mathbb{R}^k \) such that \( \sum_i \omega_i = \omega \), for
each \( (i, j), \theta_{ij} \geq 0 \) such that \( \sum_j \theta_{ij} = 1 \), for \( j \).

**Definition 2:**

An equilibrium of the private ownership economy is an \((m + n + 1)\)
tuple \((x^*_i), (y^*_j), (p^*)\) of points of \( \mathbb{R}^k \) such that:

(a) \( x^*_i \) is a greatest element of

\[
\{ x_i \in X_i \mid p^* x_i \leq p^* \omega_i + \sum_j \theta_{ij} p^* y^*_j \}
\]

for \( \leq_i \), for \( i \);

(b) \( y^*_j \) maximizes \( p^* y_j \), for \( y_j \in Y_j \), for \( j \);

(c) \( x^* - y^* = \omega \).
(c) Producer \(j\) chooses his production \(y_j\) from his production set \(Y_j \subseteq \mathbb{R}^l\). The total production set for the economy is \(Y = \bigcup_j Y_j\). The production sets will have the properties:

(d) (i) \(0 \in Y_j\);
(ii) \(Y\) is closed and convex;
(iii) \(Y \cap (-Y) \subseteq \{0\}\);
(iv) \(Y \supseteq \mathbb{R}^l\).

Finally, consider each consumer \(i\) to have an initial endowment \(\omega_i \in \mathbb{R}^l\). In particular, consider the endowment to be such that

(e) \(\exists x_i^0 \in X_i\), such that \(x_i^0 \ll \omega_i\).

Now consider two definitions:

**Definition 1:**

A private ownership economy \(E\) is defined by: an economy \(((X_i, \preceq_i), (Y_j), (\omega))\); for each \(i\), a point \(\omega_i \in \mathbb{R}^l\) such that \(\sum_i \omega_i = \omega\), for each \((i, j)\), \(\theta_{ij} \geq 0\) such that \(\sum_j \theta_{ij} = 1\), \(\forall j\).

**Definition 2:**

An equilibrium of the private ownership economy is an \((m + n + 1)\) tuple \(((x^*_i), (y^*_j), (p^*))\) of points of \(\mathbb{R}^l\) such that:

(a) \(x_i^*\) is a greatest element of

\[
\{x_i \in X_i \mid p^*x_i \leq p^*w_i + \sum \theta_{ij}p^*y_j^*\}
\]

for \(\preceq_i\), \(\forall i;\)

(b) \(y_j^*\) maximizes \(p^*y_j\), for \(y_j \in Y_j\), \(\forall j;\)

(c) \(x^* - y^* = \omega\).
Given these definitions, the following theorem can be proved:

**Theorem 1: (Debreu):**

The private ownership economy $E = \{(X_i, \{Y_i\}, \omega_i)\}$ has an equilibrium if assumptions (a)-(d) hold.

Furthermore, it is well-known (see Debreu Chapter 6) that an equilibrium is a Pareto optimum relative to a price system. This brief outline of the competitive model should suffice for our discussion below.

1.2 In economy $E$ it is straightforward to show that all consumers wish the producers to maximize profit. This result has come to be known as the Fisher Separation Theorem. Also financial policy is a matter of indifference to shareholders, because dividend and leverage policy simply partition the value of the corporation in a non-essential way. By a similar argument, simple mergers do not effect real investment decisions, and the merged value is the sum of the constituent parts. Therefore, in a complete, competitive market system (sometimes called a perfect market system), financial theory becomes an irrelevant veil.

In the following sections we will show that the introduction of taxes on financial instruments may invalidate these results.

1.3 Consider an economy, partitioned into a private sector and a government sector. We will define this economy as a mixed economy. The government sector will be treated in a summary fashion so that its responses to prices in competitive markets will have the characteristics of a private consumer. Its wealth, or revenue, will be derived from its own endowment of commodities.

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4 For proofs of the propositions in this Section see Chapter 3.
proportional taxes on consumer endowments, and proportional taxes on producer profits, and this wealth will be partly consumed, and partly redistributed to consumers. By introducing the government in this way it will be possible to retain the essential characteristics of the economy described in 1.1.5.

Consumer \(i\) is assumed to have a consumption set and preferences \((X_i, S_i)\) satisfying conditions (a) and (b) of 1.1. His budget constraint will contain taxes and the government redistribution payment. In particular, let there be proportional taxes \(t_i, t_{ij} \in [0, 1]\) on endowment \(w_i\), and on the profit \(\pi_j\) going to \(i\), respectively, and a commodity redistribution \(r_i \in \mathbb{R}^\mathbb{m}\), such that we can write the \(i\)th budget constraint as

\[
px_i \leq (1-t_i)p w_i + pr_i + \sum_j (1-t_{ij}) \pi_j + W_i, \quad i = 1, \ldots, \mathbb{m}. \tag{1}
\]

Notice that the profits tax \(t_{ij}\) may include as special cases \(t_{ij} = t\) for all \((i, j)\); or \(t_{ij} = t_j\) for all \((i, j)\), \(t_j \neq j\). The discrimination between recipients of profits is an attempt to introduce quasi-progressive taxes without destroying the homogeneity of degree zero of the demand response. The commodity redistribution \(r_i\), in a perfect market system incorporates redistribution of value as well as physical commodity redistribution.

We will assume that the government chooses consumption \(x_G \in X_G\), and commodity redistribution \((r_i) \in R \prod_i R_i\), where the consumption and redistribution sets satisfy

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5 Our model is similar to that discussed in Shoven (1974). His major interest is in ad valorem taxes, whereas we wish to concentrate on wealth taxes. In the final section of his paper, Shoven sketches the way in which wealth and profit taxes could be incorporated in his model.
proportional taxes on consumer endowments, and proportional taxes on producer profits, and this wealth will be partly consumed, and partly redistributed to consumers. By introducing the government in this way it will be possible to retain the essential characteristics of the economy described in 1.1.5.

Consumer $i$ is assumed to have a consumption set and preferences $(X_i, \sigma_i)$ satisfying conditions (a) and (b) of 1.1. His budget constraint will contain taxes and the government redistribution payment. In particular, let there be proportional taxes $t_i, t_{ij} \in [0, 1]$ on endowment $\omega_i$, and on the profit $\pi_j$ going to $i$, respectively, and a commodity redistribution $r_i \in \mathbb{R}^k$, such that we can write the $i$th budget constraint as

$$pX_i \leq (1-t_i)p\omega_i + pr_i + \sum_j (1-t_{ij})\theta_{ij}pY_j = W_i, \quad i = 1, \ldots, m. \quad (1)$$

Notice that the profits tax $t_{ij}$ may include as special cases $t_{ij} = t$ for all $(i, j)$; or $t_{ij} = t_j$ for all $(i, j)$, $t_j \neq t_{ij}$. The discrimination between recipients of profits is an attempt to introduce quasi-progressive taxes without destroying the homogeneity of degree zero of the demand response. The commodity redistribution $r_i$, in a perfect market system incorporates redistribution of value as well as physical commodity redistribution.

We will assume that the government chooses consumption $x_G \in X_G$, and commodity redistribution $(r_i) \in R \equiv \bigcap_i R_i$, where the consumption and redistribution sets satisfy

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5 Our model is similar to that discussed in Shoven (1974). His major interest is in ad valorem taxes, whereas we wish to concentrate on wealth taxes. In the final section of his paper, Shoven sketches the way in which wealth and profit taxes could be incorporated in his model.
Government consumption $x_G$ can be regarded as a convenient summary for government activity in the private market. It would not be difficult to incorporate additional structure in the government sector to include the production and distribution of public goods and services; but this would lead us into a field irrelevant to our tax problem. Assuming the absence of monopsony, or monopoly power on the part of the government in private markets, G's budget constraint can be written as:

$$\mathbb{E} x_G + \sum_i p_i r_i \leq \sum_i \{ t_i p_{i1} + \sum_j t_j \theta_i \lambda_j \} + \sum_i p_i W_i = W,$$  \hspace{1cm} (2)

where $\omega$ is the commodity endowment for G. Because we wish to keep our formulation as general as possible, we will avoid specifying G's preferences or their properties over $x_G \otimes R$, except to require that the choice $(x_G, (r_i))$ depends upon prices in a well-behaved way. Taking

$$P = \{ p \in \mathbb{N}^k \mid \sum_h p_h = 1 \},$$

then we assume:

(i) Let $x_G(p)$, be an upper semi-continuous correspondence $\phi_G: P \to X_G$, such that for $p' \in P$, $x_G(p')$ is non-empty and convex;

6 For example, Foley (1970a) uses the notion of the core in a treatment of an economy with public goods.
(ii) Let \( p_i(p) \), \( i = 1, \ldots, m \), be continuous functions of \( p, f_i : P \rightarrow \mathbb{R} \);

(iii) Let \( x_G(p), r_i(p) \) satisfy the budget constraint (2) with equality;\(^7\)

(iv) \( x_G(p), r_i(p) \) are homogeneous of degree zero, i.e.,

\[
x_G(p) = x_G(tp); \quad r_i(p) = r_i(tp), \quad t > 0.
\]

Given these preliminaries, we can proceed by giving formal definitions of a mixed economy and its equilibrium in an analogous fashion to the concept of a private-ownership economy and its equilibrium.

**Definition 3:**

A mixed economy \( EM \) is defined by: an economy \( (X_1, \leq_1, (Y_i), w, x_G, R) \); for each \( i \) a point \( w_i \in \mathbb{R}^x \), and a point \( w_G \in \mathbb{R}^y \), such that

\[
\sum_i w_i + w_G = \omega,
\]

for each \( (i, j), \theta_{ij} > 0 \) such that \( \sum_i \theta_{ij} = 1, \forall j \), for each \( i \), \( t_i \in [0, 1] \), and each \( (i, j), t_{ij} \in [0, 1] \).

**Definition 4:**

An equilibrium of a mixed economy \( EM \) is an \((2m + n + 2)\) tuple \((x^*_1, (y^*_j), (r^*_i), (x^*_G), p^*)\) of points of \( \mathbb{R}^x \) such that:

---

\(^7\) Presumably, the government has a non-satiation-type assumption to satisfy the budget constraint with equality.
(a) $x_i^*$ is a greatest element of 

$$\{x_i \in X_i \mid p^* x_i \leq (1-t_i)p^* w_i + p^* r_i + \sum_j (1-t_{ij}) \theta_{ij} p^* y_j\}$$

for $i \geq 1$, $V_i$;

(β) $y_j^*$ maximizes $p^* y_j^*$, $y_j \in Y_j$;

(δ) $x_G^*(p^*), r_i^*(p^*)$ satisfy

$$p^* x_G + \sum_i p^* r_i = \sum_i \{t_i p^* w_i + \sum_j (1-t_{ij}) \theta_{ij} p^* y_j\} + p^* w_G;$$

(γ) $x^* + x_G^* - y^* = \omega$.

To prove the existence of this equilibrium, we require an analogous condition on endowments to (c). Therefore we assume

(c') $\exists x_i^0 \in X_i$, such that

$$x_i^0 \ll (1-t_i)w_i + r_i(p),$$

for $p \in P$ and $i = 1, \ldots, m$.

This assumption can be rationalized along the lines that G's redistribution scheme is restricted to ensure that all consumers have a minimal standard of living.\(^8\)

The existence proof for the equilibrium is a straightforward variation of Theorem 1.

\(^8\) Of course, the assumption fulfills an important technical requirement in ensuring that demand correspondences are upper semi-continuous for $p \in P$. 
Lemma 1:

The mixed economy $\mathcal{E}^M$, has an equilibrium, given the assumptions (a), (a'), (b), (c'), (d), (e).

Proof:

(i) The producer's problem is identical to that in the private-ownership economy $\mathcal{E}$.

(ii) Because $p_r(p)$ is a continuous function of $p$, then it is easily established that $W_i$ is also a continuous function of $p$. Given (a), (b), (c'), the consumer's problem is a trivial generalization of that in $\mathcal{E}$.

(iii) Writing $G$'s budget constraint as $px_G \leq W_G - \sum p_r$, then it should be clear that, given (e), $G$'s response $x_G(p)$ has all the characteristics of the consumer response in $\mathcal{E}$.

(iv) Because commodity transfers cancel for the aggregate economy, the excess demand $z = x + x_G - y - \omega$; and in equilibrium $x + x_G - y = \omega$.

Therefore, because $\mathcal{E}^M$ has the same essential characteristics as $\mathcal{E}$, the existence of an equilibrium follows from Theorem 1.

We could define a private optimum along analogous lines to the optimum of a private-ownership economy, by treating the government taxation and redistribution schemes as a datum, not unlike the production technology.

---

9 This is a sketch proof and assumes knowledge of Debreu's arguments.
1.4 In (i) of the proof of Lemma 1, we observed that producers maximized profit (or value) without regard to the taxes \( t_{ij} \). Therefore, the classical investment rules would apply, and, furthermore, the investment allocation would not be distorted by differential tax rates on profits. There is nothing mysterious about this result once it is realized that profit-taxes are equivalent to a situation where the government is a shareholder of the producer in the same way as any other consumer.\(^{10}\) Of course, we must be careful to distinguish efficiency arguments from wealth distributions issues. Notice that our treatment of the government sector allows us to generate a sequence of equilibria not only by a sequence of tax-redistribution schemes, but also by an equivalent sequence of endowment-share schemes.

1.5 Although production decisions are invariant to lump sum corporate taxes, the financial structure will not be invariant if there are differential taxes on financial instruments. In this section we will incorporate more structure into the corporate finance and taxation sector of the economy \( E^M \).

Needless to say all the results we obtained for that economy continue to hold, given the wealth distribution implied by the financial-tax decisions of corporate shareholders.

Let there be \( k = 1, \ldots, K \) financial instruments. For all \((j, k)\) let \( v_{jk} \) be the pre-tax value of the \( k \)th instrument issued by producer \( j \); and let \( t_{ijk} \in [0, 1] \) for all \((i, j, k)\) be the tax-rate associated with the instrument \((j, k)\) and held by consumer \( i \). Also, let \( t_i \in [0, 1] \) be the tax-rate on \( i \)'s commodity endowment. The values of the financial instruments must

\(^{10}\) This is an interesting interpretation, because in the limiting case, where all profits are claimed by taxation, the model can be made to correspond to the market socialist economy of Lange (1936) and others. Our formulation is sufficiently general to include socialist restrictions on consumer trade in capital goods and natural resources, saving-investment decisions, and so on. Whereas the private and mixed economies are presumed to achieve equilibrium by Walrasian tâtonnement, the socialist system may reach equilibrium by government-set prices.
obey the budget constraints

\[ px_i \leq (1-t_i)p\omega_i + pr_i + \sum_j \theta_j \sum_k (1-t_{ijk})v_{jk} \quad i = 1, \ldots, m, \quad (3) \]

and the definitional requirements

\[ \sum_k v_{jk} = py_j, \quad j = 1, \ldots, n. \quad (4) \]

Now let there be a constraint set, imposed by the intertemporal profit relations and the tax authorities, upon the financial structure of producers. In particular, let the proportions of financial instruments issued,

\[ \phi_j = v_j (py_j)^{-1}, \]

be constrained by \( \phi_j \), which is a closed, convex subset of the unit simplex.

By such a restriction we could include constraints on the permissible leverage for the tax-deductability of interest payments, constraints on share repurchases, etc.

By the Fisher Separation Theorem, we know that the consumer desires to maximize his wealth, i.e.,

\[
\begin{align*}
\max \quad W &= \sum_i \theta_i \sum_j (1-t_{ijk})\phi_j (py_j), \\
\{ \phi_j \in \phi_j; y_j \in Y_j \}
\end{align*}
\]

That is, the sum of the present-value profit for every period equals the value of the corporation.
It is easy to see that problem (5) can be decomposed into two independent problems. That is

\[
\begin{align*}
\text{(5a)} \quad & \text{Max } \sum_{y_j \in Y_j} p y_j = p y_j^* \\
\text{(5b)} \quad & \text{Max } \sum_{\phi_j \in \Phi_j} \left\{ (1-t_{ijk}) \phi_{jk} \right\}
\end{align*}
\]

It should be clear that the optimum production $y_j^*$ is independent of the consumer $i$. This result is a verification of our assertion in 1.4, that production decisions are decided unanimously by shareholders.

We have assumed implicitly that the consumer’s objects of choice are chosen independently of any reactions through the government redistribution scheme. This would appear to be a reasonable assumption, except for the trivial situation where the government is an irrelevant intermediary paying all taxes back to the original contributors (and the irrelevancy of the government is recognized by the taxpayers). In this trivial case, financial structure is irrelevant.

Because $\phi_j$ is a convex polyhedron, and the objective function is linear, problem (5b) is a linear programming problem. Therefore, the optimal solution will have the usual unique (vertex), or non-unique properties, and although the computation of an optimal solution is tedious, the principles are straightforward. The first two examples in Sections 2.1 and 2.2 illustrate very simple examples of this arguments.

Another important property of the problem (5b) is that the optimal solution $\phi_j^*$ is dependent upon the taxes facing consumer $i$. In the special case where all shareholders are treated equally by the tax authorities (i.e.,
\( t_{ijk} = t_{jik} \), then \( \phi^j_1 = \phi^j_2 \). The choice of the financial structure becomes a unanimous decision by all the shareholders. In effect, the shareholders have formed a syndicate against the taxing authority, manipulating the financial structure to their mutual benefit to avoid taxation. On the other hand, with differential tax structures for shareholders, this simple solution fails, and the choice of the financial structure becomes a public good problem, and the financial structure cannot be determined by any simple rule independently of shareholder preferences.

A partial solution to this problem may be achieved by allowing shareholders to trade in shares. In a tax-less world, the initial portfolio allocations of consumers, in equilibrium, are a matter of indifference if they produce the same wealth distribution.\(^{12}\) But with differential taxation between consumers, it might be profitable to trade in shares, so that consumers with similar tax-structures will hold securities in the same producers. We will give a simple example of this clientele effect in Section 2.4 below.

Before we move on to some specific examples of our formulation, we stress the central nature of our result: when taxes are levied upon financial instruments that are not tied directly to the real economic decisions of the producer, then the shareholders will form a coalition to produce a financial structure to minimize the tax payable, leaving the real production decisions unchanged.

\(^{12}\) That is, every portfolio allocation that gives rise to the same set of consumer wealths form an indifference class. The reader can prove the assertion easily by taking \( i = j = 2 \) and \( p^*y^*_1 = p^*y^*_2 \).
2. SOME EXAMPLES

2.1 The Leverage Problem:

Consider the problem of the choice of corporate leverage, where the corporate tax allows interest payments on corporate bonds to be deducted. To begin, assume the absence of personal taxes; and the existence of an upper bound on leverage, imposed by the taxing authority. Formalizing the problem in the notation of problem (5b), let \( \phi_{jb} \) be the proportion of the value of bonds to total value; \( \phi_{je} = (1 - \phi_{jb}) \) be the proportion of equity, \( t_{jb} \), \( t_{je} \) be the tax rates respectively; and the leverage constraint \( \phi_{je} \geq \alpha \in (0, 1] \). Given the taxes \( t_{jb} = 0, t_{je} > 0 \), it is clear that the optimum is achieved when \( \phi_{je} = \alpha \); that is, leverage is pushed to its upper limit.

The introduction of personal taxation leaves the argument unchanged. To see this, consider a personal tax rate on income (wealth) of \( t_i > 0 \). The problem (5b) becomes

\[
\max_{\{\phi, \epsilon\}} (1-t_i) \phi \sum_k (1-t_{jk}) \phi_k.
\]

Clearly, the solution \( \phi_{je} = \alpha \) continues to hold independently of \( t_i \), \( \theta_{ij} > 0 \), so that the shareholders reach a unanimous decision on the financial structure.

2.2 The Dividend Retention Problem:

To investigate the dividend-retention problem we require some additional structure incorporating time into our formulation.\(^{13}\) Treating prices at present value prices and dating commodities, we obtain

\[\text{For a more extensive discussion see Malinvaud (1972), Chapter 10. Also, see Chapter 3.}\]
In general, \( v_{jt} \) may be positive or negative, depending upon the outflow or inflow of funds to shareholders. Ignoring debt-financing, let \( v_{jtf} \) be the raising of new equity, \( v_{jte} \) the repayment of equity from accumulated new equity raisings, \( v_{jtg} \) capital gains from retentions, and \( v_{jtd} \) the payment of dividends. By an accounting identity, we have

\[
\begin{align*}
\sum_{\tau=1}^{T} P_{\tau} y_{j\tau}^* = \sum_{\tau=1}^{T} v_{j\tau}^*.
\end{align*}
\]

where \( v_{jtk} > 0 \) for all \( \tau \) and \( k = d, e, f, g \); and \( \tau' \in \{1, \ldots, T \mid v_{jT} > 0\} = \tau' \) \( \tau'' \in \{0, \ldots, T \mid v_{jT} < 0\} = \tau'' \).

The constraint set will contain the identity in proportional form (by dividing through by \( v_{jT} \)), as well as the repayment constraint, and any other constraints imposed by the taxAuthorities (e.g., a certain proportion of profit must be paid as a dividend). Assuming associated tax-rates on the instruments \( t_{ijk} \), \( k' = d, e, g \), and for \( i = 1, \ldots, m \) and a full loss offset corporate tax \( t_j \), consumer \( i \)'s tax-wealth-maximizing problem becomes

\[
\begin{align*}
\text{Max} & \quad (1-t_j) \sum_{i=1}^{m} \sum_{k'=d}^{g} (1-t_{ijk'}) \phi_{jtk'} - \sum_{\tau''} \phi_{j\tau''} f_j \\
\{\phi, \emptyset\} & \quad \quad (7)
\end{align*}
\]

for \( j = 1, \ldots, n \). Clearly the corporate tax is redundant in the choice of \( \phi_j \); but the choice of instruments \( k' \) will depend upon \( t_{ijk'} \) and \( \phi_j \). To illustrate a possible solution, consider the simple case where \( T'' = \phi \), and therefore,
Let the tax authorities impose a minimum dividend policy \( j_{td} \geq a > 0 \). It is easy to show that the problem has the same properties as a sequence of leverage problems (as considered in 2.1). If capital gains taxes are less than income taxes, then \( j_{td} = a \) (retentions will be at a maximum).

Debt can be included in the problem, but this complicates the analysis without altering the simple nature of the linear programming problem.

2.3 Depreciation Policy:

In the discussion above we have considered the taxation and distribution of pure profit, i.e., the rent after the payment of all inputs at market prices; but the time-distribution of depreciation tax-payments has similar properties. Now there appears to have been a confused debate (for example, see the discussion in Samuelson (1964) over the appropriate set of tax-allowable depreciation schemes that will not distort production plans from the pre-tax situation. In a complete competitive equilibrium, there is an unambiguous solution to the problem.

Consider the vector \( y_{jt} \) to be partitioned into the vector of inputs \( a_{jt} \) and the vector of outputs \( b_{jt} \). Assuming competitive markets, undiscounted prices (distinguished by bars), and a discount factor \( \beta_t \), we obtain

\[
P_y j = \sum_{t=1}^{T-1} \beta_t \left[ \bar{p}_{t+1} b_{jt+1} - (1 + \rho_t) \bar{p}_t a_{jt} \right]
\]

where \( \rho_t \) is the rate of interest at time \( t \).

Now partition the output vector \( b_{jt+1} \) into "normal" output \( q_{jt+1} \) and used machines \( m_{jt+1} \), and the input vector \( a_{jt} \) into machines \( m_{jt} \) and variable

\[14 \text{ For an elaboration on this formulation see the reference in Footnote 13.}\]
input $l_{jt}$. Therefore we have

$$F Y_j = \sum_{T=1}^{T-1} \beta T^{JT} \{ - \gamma T^{JT} + \mu T^{JT+1} - (1 + \rho T) \left( \frac{m T^{JT}}{m T^{JT+1}} + \frac{\kappa T^{JT}}{\kappa T^{JT+1}} \right) \}. \quad (9)$$

The user cost of machines, $V_{jt+1}$ can be defined as the sum of economic depreciation $r_{jt+1}$, and the interest cost on machines used, $c_{jt+1}$, i.e.,

$$V_{jt+1} = \left( \frac{m T^{JT}}{m T^{JT+1}} - \frac{m T^{JT+1}}{m T^{JT+1}} \right) + \rho T \frac{m T^{JT}}{m T^{JT+1}}$$

$$= r_{jt+1} + c_{jt+1}. \quad (10)$$

The present value of user cost is

$$D_j = \sum T \beta T \left( r_{jt} + c_{jt} \right). \quad (11)$$

Therefore, given a uniform corporate tax, with full loss-offset provisions, we have

$$(1-\tau)py_j = (1-\tau) \left( \sum T \beta T \left( - \gamma T^{JT} + \mu T^{JT+1} - (1 + \rho T) \frac{\kappa T^{JT}}{\kappa T^{JT+1}} \right) - D_j \right). \quad (12)$$

But any scheme $d_{jt}$ such that $\sum T \beta T d_{jt} = D_j$ will,

(a) provide the same wealth to shareholders and the taxation authority; and

(b) leave the optimal production plan $y^*_j$ unchanged.
We should point out that the immediate expensing method of depreciation (with capital gains treated as profit) is one of the non-distortionary schemes. This conclusion follows directly from an interpretation of equation (9). Clearly the expensing method is an obvious candidate for an easily applied, but non-distortionary, depreciation allowance.

Unfortunately, many depreciation schemes, imposed by governments, are distortionary. They produce an effective change in the relative private prices of variable input and physical capital, by the imposition of a tax or subsidy on capital.

2.4 Clientele Effects and Tax-Havens:

In the previous sections we have considered examples where the share allocations and the producer entity are taken as given. We will give a couple of simple examples where these parameters are under the control of shareholders, and optimal choices are made before taxes are collected.

In a world without taxes we have shown that there is an infinite number of share-allocations \( (\theta_{ij}) \) which provide the same consumer wealths. But with the introduction of a tax-structure that discriminates between the shareholders of a producer, and also between producers, it is possible that shareholders will exchange shares so that particular producers will have clientele shareholders. For example, consider a small economy with two producers \( j', j'' \) with identical production sets, and two consumers \( i', i'' \).

Let the initial endowment shares be \( \theta_{ij} = 1/2 \), and the tax-rates be \( t_{i'j'} < t_{i'j''} \), \( t_{i''j''} < t_{i''j'} \). Now the consumers can enter into a share-trade such that they both increase their wealth, and which results in an allocation

\[
(\theta_{i'j'}, \theta_{i'j''}) = (1, 0), \text{ and } (\theta_{i''j'}, \theta_{i''j''}) = (0, 1).
\]
Clearly, the producers have sole proprietors that have been induced into their shareholding by avoiding tax.

Our second example concerns the setting up of dummy companies, or tax-havens. This case can be included in our formal model by including in the definition of a financial instrument, the creation of a "company" or legal entity which is recognized by the tax authorities. If differential taxes are charged for companies, it will pay the shareholders to create companies with the lowest tax-rate. A well-known example of this behaviour is the existence of tax-havens in some countries. Our formal model includes this case if we define commodities and agents internationally, and sum over the set of national governments. Of course, national governments place legal constraints on such behaviour in attempts to reduce tax avoidance by these means.

2.5 Bankruptcy Charges:

Our last example concerns bankruptcy charges introduced by the working of the legal system. We assume that the legal system is a public service provided by the government, and that corporate bankruptcy requires a legal process for which a legal tax is imposed. Because our formal model can be interpreted as an economy with

(a) complete Arrow-Debreu markets; or

(b) a fixed set of securities in which trade takes place,\(^{15}\)

we can consider bankruptcy and uncertainty in a restricted context.

\(^{15}\) See Chapters 3 and 5 for this interpretation. We are assuming that a perfect substitute always exists for the risky bond.
Consider the economy described in 2.1. Now for the riskless bond (i.e., for leverage below the point at which default occurs) the problem is as in 2.1. But for leverage greater than this point a new security is created with a bankruptcy tax imposed. Whether or not leverage is chosen to be the maximum no-default level, or complete leverage, will depend upon the relative taxes. If the government judges a defaulting bond to be equity attracting, the corporate tax, then leverage, will be bounded by the no-default upper bound.16

3. CONCLUDING COMMENTS

We have shown that with complete markets for the real objects of choice, the corporate tax problem corresponds to a redistribution of wealth. Because the taxing authority prescribes taxes on financial instruments, it is possible that an optimal financial structure exists for private shareholders where tax payments are minimized in some game-theoretic sense. Ruling out a progressive personal tax, it would appear that most corporate tax systems and their accompanying legal constraints can be characterized by a linear programming approach.

The taxing authority has been formulated as a passive body, setting tax rules for the optimizing shareholders to choose the best tax-avoidance strategy. Nevertheless, it would appear that the constraints imposed by the authority are a means for partially closing tax loopholes, given the structure of taxes on financial instruments. Because, in a more realistic theory, tax avoidance absorbs real resources, one cannot but conclude that the tax structure should be greatly simplified.

16 Stiglitz discusses these issues in more detail, but we have omitted any formal proof because it is a relatively straightforward application of the techniques developed above.
Finally, we should caution the reader that these simple results have been derived from a complete market model. In incomplete markets, it is possible for the financial structure to be a relevant consideration in real decisions; and the simple profit maximization rule may not be applicable or well-defined.

Even so, we suspect that these are added complications that do not upset our fundamental conclusions.
CHAPTER 8

ON THE OPERATION OF FINANCIAL MARKETS WITH MARKETING COSTS FOR SECURITIES

The evolution of the theory of financial markets (or share markets) has been somewhat tortuous, but it has become evident that the well-defined financial models have the same formal structure as the competitive equilibrium theory of Arrow (1953) and Debreu (1959) (hereafter A-D). These financial models are very restrictive: they assume one commodity and two periods, so avoiding the problems of price uncertainty and multiple period planning.\(^1\) Inter-temporal trades are carried out via the issuance of claims (shares) as the output of producers in the second period. Uncertainty enters through technological uncertainty in the second period. By defining the claims with associated patterns of returns, the model can be incorporated into the A-D framework. In the special case, where short-sales are permitted and there are sufficient claims to span the primitive security space, the financial market becomes a trivial translation of a complete A-D primitive security economy.

A number of well-known finance and investment propositions have been deduced from this model. In particular, we wish to concentrate upon:

**Proposition I:** Profit maximization is a unanimous rule adopted by shareholders for production decisions.

\(^1\) Although some recent attempts have been made to remove these assumptions, the restricted model has become the stock-in-trade of finance theorists. For the removal of price uncertainty, see Chapter 9.
Proposition II: Shareholders are indifferent to the degree of financial leverage.²

Finance writers (e.g. Fama and Miller (1972)) regard the financial market model as a "perfect" market model. Furthermore, there has been a common insistence (e.g. Hirshleifer (1970), Fama and Miller) that transaction costs are a market imperfection that destroy the simple, clear-cut results of the perfect market. An opposing position has been taken by Stigler (1967) who argued that transaction costs are not necessarily a market imperfection, but can be accommodated into the competitive model by suitable redefinitions. Unfortunately, Stigler did not provide a formal model to back his verbal argument.

Economists, also, have been grappling with the transaction cost problem. The recent literature has been stimulated by the observations that

- (a) a complete set of A-D primitive securities (or an equivalent spanning set) is an unrealistic fiction (for example see Radner (1970)); and
- (b) no adequate treatment exists for the transactions demand for money.

A number of different formulations exist for treating transaction costs;³ but perhaps the most accessible for our purposes is the model proposed by Foley (1970b). By simple modifications to the definition of traded commodities, and to production and consumption sets, he was able to retain the formal structure

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² For a general discussion of these propositions and the relationship between A-D and financial market models see Chapter 3. The finance theorist will recognize Proposition I as the Fisher Separation Theorem; and Proposition II as the Modigliani-Miller Theorem. The latter proposition is discussed in detail in Chapter 5.

³ For example, see Hahn (1971), Foley (1970b) and Kurz (1974), for early excursions into a rapidly expanding area.
of the A-D theory and yet incorporate a form of transaction cost - marketing cost.

By amalgamating the financial and marketing models we are able to provide a formal proof of Stigler's arguments. We will show that Proposition I remains unscathed. Proposition II also remains true, but under more restrictive assumptions. These assumptions have nothing to do with market imperfection, but are the result of market efficiency conditions. In addition we are able to provide

(i) sufficient conditions for non-trivial brokerage and intermediation behaviour; and
(ii) plausible conditions for inactivity of some security markets.

This chapter falls into three sections: Section one provides a brief summary of the Foley model; Section two outlines the financial market model; and Section three draws the two models together, discussing implications and limitations of the resulting theory.

1. THE A-D THEORY WITH COSTLY MARKETING

1.1 Foley considers a special case of transaction costs, where the operation of a commodity market requires real input per unit commodity traded. He justifies this assumption as a "marketing" cost which includes the costs of informing market participants of prices and supply and demand, distribution costs, etc. To incorporate the marketing cost, Foley doubles the dimension of the commodity space, partitioning commodities into buyer's goods and seller's goods. The difference between their associated prices is the cost per unit for marketing.
1.2 Consider the set of consumers $I = \{1, \ldots, m\}$. Consumer $i \in I$ has a consumption set $X_i \subseteq \mathbb{R}^l$, where $l$ is the dimension of the commodity space. Let $X = \bigoplus_i X_i$, and $\hat{X}_i$ be the attainable consumption set for $i \in I$. Assume the consumption sets have the following properties: For all $i \in I$,

A.1 $X_i$ is closed and convex;

A.2 Define on $X_i$ a complete pre-ordering $\lesssim_i$ called a preference pre-ordering which satisfies:

(a) Non-Satiety: For any $x_i, x_{i}' \in \hat{X}_i$, $\exists x_{i}'' \in X_i$ such that $x_i'' \gtrsim_i x_{i}'$;

(b) Continuity: For every $x_i' \in X_i$ the sets

\[ \{x_i \in X_i \mid x_i \lesssim_i x_i'\} \text{ and } \{x_i \in X_i \mid x_i \gtrsim_i x_i'\} \]

are closed;

(c) Convexity: For every $x_i' \in X_i$, $\{x_i \in X_i \mid x_i \gtrsim_i x_i'\}$ is convex;

and for the aggregate consumption set:

A.3 $\hat{X}(X) \cap (-\hat{X}(X)) = \{0\}$.\(^5\)

A.4 $0 \in X_i$, $i \in I$.

Now consider the division between bought and sold commodities for the consumer, by defining for $k \in K = \{1, \ldots, l\}$.

---

\(4\) For a rigorous definition, see Debreu (1962). Intuitively, it means the feasible consumption of $i$, if the endowments and productive capacity of the economy were placed at the personal disposal of $i$.

\(5\) The asymptotic cone of $X$, $\hat{X}(X)$ is defined as $\bigcap_{k} K_k$, where $K_k$ is drawn from the set of closed cones containing $\{x \in X \mid |x| < k\}$, where $k > 0$.
\[
B = \max \{0, x_{ik}\}, \quad k \in K,
\]
\[
S = \min \{0, x_{ik}\}, \quad k \in K,
\]

and therefore \( x_i = x_i^B + x_i^S \). A new consumption set \( x_i^M \subset \mathbb{R}^{2\ell} \) and preferences \( \preceq_i^M \) can be defined by

(a) \( x_i^M = \{(x_i, v_i) \mid x_i \in x_i^B, v_i \geq \max \{0, x_{ik}\}, k \in K\} \)

(b) if \((x_i, v_i) \in x_i^M \) and \((x_i', v_i') \in x_i^M \), then

\[
(x_i, v_i) \preceq_i^M (x_i', v_i') \iff x_i \preceq_i x_i'.
\]

It is easy to show that \( x_i^M \) satisfies the restrictions A.1-A.4.6

If \( B, S \) are the prices associated with buying and selling, respectively; \( \gamma_j \) is the profit of the \( j \)th producer; and \( \theta_{ij} \) is the share held by \( i \) in \( j \), then the budget constraint for \( i \) is

\[
P^B x_i^B + P^S x_i^S \leq \sum_j \theta_{ij} \gamma_j = \gamma_i.
\] (1)

Assuming \( B \geq S \geq 0 \) (otherwise consumer demand will be unbounded by a dominance argument), then define the buying premium \( \Pi = P^B - P^S \geq 0 \). The budget constraint (1) becomes

\[
P^S x_i + \Pi v_i \leq \gamma_i.
\] (2)

---

6 See Foley. We have replaced the lower boundedness of \( x_i \) by A.3; but this is a trivial extension.
Therefore consider the new consumer problem with $x_i^M = (x_i, v_i) \in X_i^M$, and $X_i^M$ endowed with the preference pre-ordering $\preceq_i^M$, where the budget constraint is

$$\mathbf{p}^M x_i^M \leq w_i, \quad \mathbf{p}^M = (p^S, \Pi).$$  \hspace{1cm} (2')

1.3 The producer's problem can be analysed in a similar manner. Consider the set of producers $j \in J = \{1, \ldots, n\}$. Producer $j$ has a production set $Y_j^M \subset \mathbb{R}^{2k}$ where $y_j^M = (y_j^I, y_j^B) \in Y_j^M$. Given the price vector $p^M$, the producer $j$ wishes to maximize his profit, i.e.,

$$\max_{y_j \in Y_j^M} \mathbf{p}^S y_j^* + \Pi y_j^B. \hspace{1cm} (3)$$

Let $\sum_j Y_j^M = Y^M$ be the aggregate production set; and assume that the production sets satisfy the following conditions:

B.1 $0 \in Y_j^M$;

B.2 $Y_j^M$ is closed and convex;

B.3 $Y_j^M \supset \mathbb{R}^{2k}$;

B.4 $Y_j^M \cap \mathbb{R}^{2k} = \{0\}$.

---

7 This assumption could be weakened to $Y^M$ is closed and convex.
By way of interpretation, consider $y_j$ as the net transaction undertaken at the selling price, $p^s_j$; and $y_j^B$ as the vector of inputs or outputs that attract the premium $n$.  

1.4 Having assembled the consumer and producer problems we turn to the existence of an equilibrium with marketing costs. Define such an equilibrium as:

**Definition 1:**

An equilibrium of a private-ownership marketing economy

$$\mathcal{E}^M = ((x^M_i, \leq^M_i), y^M_j, \theta^M_{ij})$$

is an $(m + n + 1)$ tuple $((x^M_i), (y^M_j), (p^M))$ of points of $\mathbb{R}^{2L}$, such that:

(a) $x^M_i$ is the greatest element of $\{x^M_i \in x^M_i | \forall x^M_i \leq \sum \theta^M_{ij} p^M y^M_j\}$ for $\leq^M_i$, $i \in I$;  

(b) $y^M_j \in y^M_j$ maximizes $p^M y^M_j$, for $j \in J$;  

(c) $x^M_i - y^M_i = 0$.

Before giving sufficient conditions for the existence of an equilibrium, we follow Debreu (1962) in considering a quasi-equilibrium.

**Definition 2:**

A quasi-equilibrium of the private ownership marketing economy

$$\mathcal{E}^M = ((x^M_i, \leq^M_i), (y^M_j), \theta^M_{ij})$$

---

Note that $y_j$ may have positive and negative components.
is an \((m + n + 1)\) tuple of points \((x_1^M, y_j^M, p_j^M)\) of \(\mathbb{R}^{2^0}\) such that

(a) \(x_1^M\) is the greatest element of

\[
\{x_1^M \in X_1^M \mid p_j^M x_1^M \leq \sum_{j \in J} \theta_j p_j^M y_j^M\}
\]

for \(\leq^M_i\) and/or

\[
p_j^M x_1^M = \sum_{j \in J} \theta_j p_j^M y_j^M = \min_p^M x_1^M
\]

for \(i \in I;\)

(b) \(y_j^M \in Y_j^M\) maximizes \(p_j^M y_j^M\) for \(j \in J;\)

c) \(x^M - y^M = 0;\)

d) \(p^M \neq 0.\)

Given the definition of a quasi-equilibrium we quote Debreu's theorem providing sufficient conditions for the existence of a quasi-equilibrium.

**Theorem 1:** (Debreu):

The private ownership marketing economy \(\varepsilon^M\) has a quasi-equilibrium if:

(a.1) \(X_1\) is closed and convex;

(a.2) \(x_1^M \in X_1^M\) \(\exists x_1^M' \in X_1^M\) such that \(x_1^M' \succ^M_i x_1^M.\)

(a.3) for every \(x_1^M' \in X_1^M\), the sets \(\{x_1^M \in X_1^M \mid x_1^M \succ^M_i x_1^M'\}\)

and \(\{x_1^M \in X_1^M \mid x_1^M \preceq^M_i x_1^M'\}\) are closed;
(a.4) for every \( x_i^M \in X_i^M \), the set \( \{ x_i^M \in X_i^M \mid \bigcup_i x_i^M \geq x_i^M \} \)

is convex;

(a.5) \( \text{ As}(X^M) \cap (-\text{ As}(X^M)) = \{0\} \);

(b.1) \( Y^M \cap X^M \neq \emptyset \);

(b.2) Let \( Y^M \) be closed and convex such that for \( i \in I \),

\[
\text{ As}(Y^M) - D \cap X_i^M \neq \emptyset ;\]

(b.3) \( \text{ As}(X^M) \cap \text{ As}(Y^M) = \{0\} \);

(c.1) \( 0 \in Y_j^M \) for \( j \in J \).

It is easy to show that the conditions imposed in Sections 1.2 and 1.3 are included in the conditions above, and that a quasi-equilibrium exists for the market economy. To make a quasi-equilibrium, an equilibrium it is required that

\[
P^* x^*_i = \sum_j i_j p_j^* y^*_j = \min p_j^* x^*_j \text{ occurs for no } i \in I. \quad (4)
\]

A number of well-known sufficient conditions exist to ensure that (4) is obtained.\(^9\) For example,\(^{11}\) assume that we can add an endowment vector

\[^9\] Define \( D \) as the smallest cone with vertex 0, owning all the points of the form \( \sum_i x_i^M \) where

\[
x_i^M > x_i^M, \quad i \in I.
\]

\[^{10}\] In fact this condition is required to ensure the upper semi-continuity of the demand correspondence. For a discussion, see Debreu (1959), (1962).

\[^{11}\] This is Debreu's (1959) condition.
\[ M \in \mathbb{R}^{2^k} \]
to the budget constraint of \( i \in I \), then assuming there exists
\[ x_i^{M'} \leq \omega_i \quad \text{for all } i \in I, \]
we have \( p^* M^* x_i^* \geq \min p^* M^* x_i^* \).

Finally by using a theorem by Foley,\(^{12}\) we can translate the consumer's demand \( x_i^{M^*} \) back into \( (x_i^*, x_i^*) \) so that we have a final interpretation in the two-price model.

2. THE ASSET ECONOMY

2.1 We will set up the asset economy so that it is conformable with the discussion in Section 1. We will sketch the argument, although a more detailed treatment exists in Chapters 4 and 6. Consider an economy that exists for two periods: in the first period the state of the world is known with certainty by all economic agents; but in the second period there is a set of possible states of the world \( S = \{1, \ldots, s\} \). Let there be one physical commodity so that the first period commodity space is \( \mathbb{R} \), and the second period potential commodity space \( \mathbb{R}^S \). We say a "potential" commodity space, because we will treat an asset economy where second period actions by agents are restricted to trades in assets that are associated with linear combinations of contingent commodities. We will treat the assets as objects of choice.

To formalize the argument, consider the independent patterns of returns generated by the \( L \) assets to be represented by a semi-positive \( S \times L \) matrix \( Z \). Define the linear mapping

\[
\Lambda : \mathbb{R}^{(+) \times L} \rightarrow \mathbb{R}^{(+)^S}
\]

\[ Z'\beta = u, \text{ where } \beta \in \mathbb{R}^{S+1}, u \in \mathbb{R}^{S+1}, \text{ and } \]

\(^{12}\) See Foley, Theorem 2.2.
\[
Z' = \begin{pmatrix}
1 & 0 \\
0 & Z
\end{pmatrix}
\] is \((s+1) \times (L+1)\) semi-positive matrix.

Let \(H\) be the range of \(A\) in \(\mathbb{R}^{s+1}\). Our strategy will be to consider the consumption and production sets, and preferences in the potential commodity space, and derive asset sets and preferences via the inverse mapping \(A^{-1}\). In this way we can construct an asset economy with all the characteristics of an A-D model.

2.2 Consider the set of producers \(J = \{1, \ldots, n\}\). The \(j\)th producer has a production possibility set \(Y_j \subset \mathbb{R}^{s+1}\). Defining \(Y = \bigcup_j Y_j\) let the production set have the properties:

\[(a.1)\] \(0 \in Y_j;\)

\[(a.2)\] \(Y_j\) is closed and convex;

\[(a.3)\] \(Y \cap \mathbb{R}^{s+1}_{(+)} = \{0\}.

Now defining \(G_j\) (resp. \(G\)) = \(H \cap Y_j\) (resp. \(H \cap Y\)), we can construct the asset production sets \(Y_j^A, Y_j^A\) via the inverse mapping \(A^{-1}: G(j) \rightarrow \mathbb{R}^{L+1}\).

It is not difficult to show\(^{13}\) that the asset production sets have the properties:

\[(a'.1)\] \(0 \in Y_j^A;\)

\[(a'.2)\] \(Y_j^A\) is closed and convex;

\[(a'.3)\] \(Y_j^A \cap \mathbb{R}^{L+1}_{(+)} = \{0\}.

\(^{13}\) See Chapter 4.
2.3 Consider the set of consumers \( I = \{i, \ldots, m\} \). The \( i \)th consumer's consumption set is \( X_i \subset \mathbb{R}^{s+1} \), and we impose the following restrictions on \( X_i \):

(β.1) \( X_i = \mathbb{R}^{s+1}_{(+)} \) for \( i \in I \);

(β.2) Define an \( X_i \) a complete pre-ordering \( \preceq_i \) called a preference pre-ordering which satisfies the conditions:

(a) Non-Satiety;
(b) Continuity;
(c) Convexity.

By considering the inverse mapping \( \lambda^* : K \to \mathbb{R}^{l+1} \), where \( K = H \cap X_i \), one can generate asset consumption sets \( X_i^A \) and associated preferences \( \preceq_i^A \). It is important to realize that the inverse mapping allows for short-selling, but incorporates the constraint that the nominal pattern of returns promised by the short-seller is in fact feasible. It can be shown\(^{14}\) that the consumer's asset consumption set and his preferences have the properties:

(β' .1) \( X_i^A \) is closed, convex polyhedral cone with

\[ X_i^A \cap (- X_i^A) = \{0\} \]

(β' .2) The preference pre-ordering on \( X_i^A \), i.e., \( \preceq_i^A \)

has the properties:

(a) Non-Satiety;
(b) Continuity;
(c) Convexity.

---

\(^{14}\) See Chapter 6.
2.4 Now the discussion of an asset economy $E^A$ equilibrium is relatively straightforward. We can define a quasi-equilibrium for a private ownership asset economy in the following manner.

**Definition 3:**

A quasi-equilibrium of the private ownership asset economy

\[ E^A = ((x^A_i, \leq^A_i, y^A_j), \theta_{ij}, p^*_A) \]

is an $(m + n + 1)$ tuple $((x^*_A), (y^*_A), p^*_A)$ of points of $[R^{L+1}]$ such that:

(a) for every $i \in I$, $x^*_A$ is the greatest element of

\[ \{ x^*_i \in x^A_i \mid p^*_A x^*_i \leq \sum_j \theta_{ij} p^*_A y^*_j \} \]

for \( \leq^A_i \), and/or

\[ p^*_A x^*_i = \sum_j \theta_{ij} p^*_A y^*_j = \text{Min}_{x^*_i} p^*_A x^*_i; \]

(b) for $j \in J$, $p^*_A y^*_j = \text{Max}_{y^*_j} p^*_A y^*_j$;

(c) $x^*_A - y^*_A = 0$;

(d) $p^*_A \neq 0$.

Given our assumptions $a'.1-a'.3$, $b'.1$, $b'.2$, then using Debreu's sufficient conditions outlined in Theorem 1 of Section 1, it can be shown that a quasi-equilibrium exists for our asset economy. Furthermore, by similar arguments to that in Section 1, we can introduce some minor restrictions to ensure that a quasi-equilibrium is a full equilibrium.\[^{15}\]

\[^{15}\] See Chapter 6 for two sufficient conditions.
3. AN ASSET ECONOMY WITH COSTLY MARKETING OF ASSETS

3.1 In the two previous sections we outlined an A-D theory with costly marketing, and an asset economy with A-D characteristics. Our task now is to amalgamate these two models into a single model where there are costs in the marketing of assets. The formal discussion of this amalgamation is undertaken in Section 3.2. In Section 3.3, we explore possible interpretations of our results including the role of intermediation; and in Section 3.4 we sketch some results by Heller (1972) incorporating non-convexities into the marketing technology.

3.2 In our discussion of the asset economy, we assumed that the asset space was of finite dimension L. Because the characteristic pattern of returns for an asset is only defined up to a positive constant of proportionality, we can limit our discussion of the potential asset patterns of returns space to the unit simplex,

$$S^- = \{ z \in \mathbb{R}^L_+ \mid ze = 1 \},$$

where $e$ is the unit vector. Thus the pattern of returns matrix $Z$ is composed of $L$ vectors $Z_k \in S^-$, $k \in \{i, \ldots, L\}$. Although $S^-$ is compact, it is uncountable. That is, if we were to treat $S^-$ as the potential asset patterns of returns space, we would require a general equilibrium treatment in infinite dimensional commodity space. Such theories exist, but they require advanced techniques. Instead, we will apply a finite approximation that enables us to use the finite dimensional constructions of the preceding sections.

16 For example, see Bewley (1972).
Taking the usual metric, \( S_s \) can be considered as a compact subset of a metric space \( \mathbb{R}^S \). Consider

Definition 4:

Let \( X \) be a metric space. For \( \varepsilon > 0 \), define \( B \subset X \) as an \( \varepsilon \)-net, if \( B = \{b_1, \ldots, b_n\} \) and for \( \varepsilon \)-neighbourhoods \( N(b_i, \varepsilon) \), we have

\[
X \subset \bigcup_{b_i \in B} N(b_i, \varepsilon).
\]

Lemma 1:

If \( X \) is a compact metric space, then there exists an \( \varepsilon \)-net for any \( \varepsilon > 0 \).

Proof: See Nikaido (1968).

Consider the potential asset patterns of returns as an \( \varepsilon \)-net on \( S_s \) i.e., \( B = \{b_1, \ldots, b_L\} \subset S_s \). Therefore, for any security \( k, z_k \in S_s \), we can find an approximate security \( i, b_i \in B \), such that the metric \( d(b_i, z_k) < \varepsilon \).

We could rationalize this approximation by the observation that agents cannot perceive minor variations in patterns of returns about a representative pattern of returns.\(^{17}\)

Given a finite set of representative asset patterns of returns we can proceed as in Section 2 to construct an asset economy. Now consider introducing marketing costs into the trading of assets. Because an asset economy has the same structure as an \( A-D \) economy, the analysis of a private

\(^{17}\) Any rigorous argument along these lines would be somewhat complicated. Our formulation is a crude attempt to capture its essential features.
ownership asset-marketing economy is a straightforward amalgamation of the two models. For formal completeness consider an asset economy with marketing cost for assets (but not for the first period commodity).\textsuperscript{18}

Definition 5:

An equilibrium of a private ownership asset marketing economy

\[ E^{AM} = \left( (X_i^{AM}, \leq_i^{AM}), Y_j^{AM}, \theta_{ij} \right) \]

is an \((m + n + 1)\) tuple \((X_i^{AM}, Y_j^{AM}, P^{AM})\) of points of \(\mathbb{R}^{2L+1}\), such that,

(a) \(X_i^{AM}\) is the greatest element of

\[ \left\{ X_i^{AM} \mid P^{AM} X_i^{AM} \leq \sum_{j} \theta_{ij} P^{AM} Y_j^{AM} \right\} \]

for \(\leq_i^{AM}\), \(i \in I;\)

(b) \(Y_j^{AM}\) maximizes \(P^{AM} Y_j^{AM}, j \in J;\)

(c) \(X_i^{AM} - Y_j^{AM} = 0.\)

Sufficient conditions for the existence of this equilibrium can be obtained by superimposing the sufficient conditions given in Section 1 onto the sufficient conditions in Section 2.

3.3 We turn now to an interpretation of the economy \(E^{AM}\).

The issuance of securities requires expenditure of the first period commodity in marketing the issue, i.e., informing asset holders of the pattern of returns the asset will yield across the set of states. Producers may issue securities.

\textsuperscript{18} Of course, the theory is general enough to include marketing costs for the first-period commodity; but we wish to focus upon the interpretation that the first-period commodity is required as a marketing input for securities.
securities with claims upon their second period production; and consumers may issue securities by short-selling (borrowing) using their portfolios as collateral. Now the marketing process can be undertaken by the issuing agent (through the services of a wholly-owned marketing company), or through a market specialist in security flotation. The existence of market specialists or brokers, could be rationalized on the usual grounds of specialization. Their efficiency over other agents arises presumably from their knowledge and contacts in the security markets. We stress that our assumptions require the marketing industry to be competitive, otherwise our analysis is inappropriate (this question is examined in more detail in the next section).

Observe that the transactions technology is sufficiently general to include differential costs for different securities. We have in mind the idea that the transmission of complicated information is more costly than the transmission of simple messages. Such an argument would provide sufficient grounds for the existence of simple rules for describing patterns of returns (e.g., bonds, equity).

Now we turn to a discussion of the level of activity in any particular asset market. Inactivity is defined as a situation where every agent has a zero equilibrium demand (supply) in a particular asset market (say asset k'). Although inactivity is a possibility in the ordinary A-D market without transactions costs, the addition of these costs makes inactivity more probable. For example, consider Foley's discussion of the case of an exchange economy with marketing costs. The first order condition for consumer i's utility maximization provide us with
Clearly, if \( U_{i k'}^i \in (P_{k'}^s, P_k^B) \), then consumer \( i \) will neither hold nor short sell the \( k' \)th security. If \( \Delta_{k'} \) is the marginal marketing cost of the first unit of asset \( k' \), then a sufficient condition for all consumers to take a zero position in the asset, is \( \Delta_{k'} \geq U_{i k'}^i (\lambda_1)^{-1} \), for all \( i \in I \). We can interpret this condition as saying that if the marginal cost of marketing is large compared to the marginal utility of trading in the asset, then no trade will result. This condition could be obtained if \( \Delta_{k'} \) is large, and the pattern of returns \( z_{ik'}^i \), and the subjective probabilities \( \Pi_{is} \) are such that

\[
\left( \sum_{s} z_{ik'}^i \Pi_{is} \right) \rightarrow 0
\]

for all \( i \in I \). That is, the expected returns are considered to be low by all consumers, compared with the marketing cost.

The producer also has a straightforward computation based upon the profit maximizing rule. He will not issue the asset if the selling price is exceeded by the marginal cost of the first unit. Therefore, the inactivity
of an asset market is the result of normal utility and profit maximizing calculation of economic agents.

We have stressed Foley's argument that the marketing model has the same form as the A-D theory. Then it follows that Proposition I holds. That is, shareholders are unanimous in their choice of the profit-maximizing plan over any other feasible production plan. This argument appears to contradict the analysis of Hirshleifer (1970) and others who have asserted that the Proposition is invalidated with different borrowing (buying) and lending (selling) rates (prices). By treating bought and sold commodities as different commodities, Foley has avoided confusion over the relevant buying or selling price. For example, in the case of the firm marketing securities, the securities bought and sold are treated as different commodities.\(^{19}\)

Finally, we examine the implications of marketing costs for the question of corporate leverage. We will show that leverage is an irrelevant detail when the firm issues securities with active markets. Our proof provides an extension of Proposition II when there are transaction costs. To prove our assertion consider the following argument. Let there be three securities, 1, 2, 3 which have active markets. From the consumer's first order conditions (1) it is easy to show that if \(z_1 = a_2 z_2 + a_3 z_3\) then \(p_1^B = a_2 p_2^B + a_3 p_3^B.\)\(^\text{20}\)

Similarly, the producer faces security prices \(p_k^S, k = 1, 2, 3.\) Assuming that producers can repackage assets by linear combinations of patterns of returns (in the same way as consumers) then profit-maximization and market activity imply \(p_1^S = a_2 p_2^S + a_3 p_3^S.\) The last implication proves our assertion that the value of the firm is invariant to variations in leverage in active markets.

\(^{19}\) To modify Hirshleifer's (1970) familiar diagram (see Chapter 7), the dimensions would have to be increased by one to incorporate a three-dimensional production surface relating the first-period input to the bought and sold commodity.

\(^{20}\) In our formal analysis we restricted our treatment to independent securities. But given an equilibrium associated with basic securities, it is easy, by forming linear combinations of basic securities, to generate derived securities and their associated prices. On this point see Chapter 6.
Our proof, of course, proves more than this: it provides us with further implications of the market activity assumption. Because \( \Pi_k = P_k^B - P_k^S \), then we have \( \Pi_1 = a_2 \Pi_2 + a_3 \Pi_3 \). This implies restrictions on the equilibrium marketing costs of portfolios with the same patterns of returns. If marketing technologies differ for different securities, the range of active markets for nominal substitute portfolios (judged by their patterns of returns) may be restricted.21 Nevertheless, the shareholders choice is unanimous, because the decision to market a security in an inactive market is, by definition, uneconomic, and the value of the firm would be reduced unambiguously.

3.4 One of the limitations of the Foley marketing model is the assumption of convex marketing technology. It is widely acknowledged22 that there are significant set-up costs in the flotation of securities, so that the marketing technology exhibits a non-convexity. Non-convexities create difficulties for the usual existence proofs of equilibrium. Nevertheless, using some results by Starr (1969), Heller (1972) has been able to extend the Foley model to include non-convexities and obtain an approximate market equilibrium. His strategy was to define a measure of the non-convexity of a set \( S \) as

\[
\rho(S) = \sup_{x \in \text{Con} S} \inf_{y \in S} |x - y|
\]

Assuming \( \rho \) is bounded for each production set, he considered an equilibrium

21 That is, restricted in the sense that some feasible leverage choices are not profit-maximizing strategies.

22 For a general discussion of transaction costs, and some evidence of set-up costs, on the New York stock-exchange, see Demsetz (1968).
for the economy with the production sets replaced by their convex hulls.\(^2\)

Then he showed the existence of a market pseudo-equilibrium where a set of subsidies ensure profit-maximizing production, and aggregate excess demand is less than a bound given by \(\rho\) and the number of commodities. Therefore, as the scale of the economy (in terms of \(|J|, |I|\) or aggregate endowment) becomes large, the pseudo-equilibrium approximates a full equilibrium.

Heller admits that, in the absence of subsidies, and if set-up costs are substantial, then we have a classic possibility for imperfect competition; and the price-taker assumption (based upon the implicit notion of large numbers) is no longer valid. Indeed it may be the case that some security markets are dominated by single suppliers, or only a few suppliers, because of the large flotation costs.\(^2\)

Finally, we should emphasize the limitations of marketing costs as a special case of a more general transaction cost theory. Foley's construction assumes that market institutions exist with objective buying and selling prices, even though there is zero activity in those markets. Conversely, if the operation of markets include price dissemination as a costly activity, the activity of agents will depend upon their conjectures of prices that would be obtained with the creation of new markets.

Also, the marketing cost model ignores the possibility of transaction costs specific to any agent. We have in mind the notion of time and trouble that the act of transacting costs any agent. In the case of financial securities the agent-specific transaction technology could include the costs of demonstrating trustworthiness in fulfilling contracts. Given a past

\(^2\) The convexified economy has the same properties as the theory of Section 1.

\(^2\) The monopoly aspect of asset creation and supply is discussed in Chapters 4, 5 and 6. If the Hirshleifer argument is interpreted as a monopoly problem of issuing securities, then there appears to be a reasonable case for shareholder conflict. On this point see Chapter 6.
history of information for the economy, it would be reasonable to expect that "established, respected" agents would face lower costs than the late arrival.

Such an agent-specific transaction technology has been formalized, and introduced into the A-D theory by Kurz (1974). As we have tried to indicate briefly above, the model could be interpreted profitably, but we will not attempt any further analysis here.
The conventional, one-period model of resource allocation can be extended to incorporate intertemporal considerations, by the simple device of dating commodities. Such an argument will be familiar to readers of Hick's *Value and Capital*, (1939), or Debreu's *Theory of Value*, (1959). In this complete market system there is a full set of operational forward markets covering all commodities traded during the planning period. After the initial market day, when all prices, and production and consumption plans are determined, the economy proceeds according to the initial plans (or strategies). As a descriptive model of a market economy, the model has a number of obvious limitations.¹

Given these limitations, and prompted by the recent publication of Keynes' *General Theory*, Hicks introduced the idea of a Temporary Equilibrium, where clearance of commodity and bond markets was allowed in the current period, but future periods entered only through the price expectations used in the production and consumption plans of economic agents. Although this method of analysis has been used extensively in aggregate Keynesian models, its micro-economic foundations have been investigated only recently in the work of Grandmont (1974), Sondermann (1974 and Green (1973) and others. In

¹ For a full discussion see Radner (1970).
particular, these models incorporate a rigorous discussion of uncertainty, in the planning decisions of agents, and therefore provide a richer framework than the certainty equivalent formulation of earlier writers.

Throughout the 1960s, finance theorists had been developing stock-market models which produced simple results for finance and investment decisions of corporations. The earliest of the models were based upon the well-known mean-variance formulation of portfolio theory, but, as the models became more general, the finance-investment rules were found to be independent of the restrictive assumptions on consumer preferences. The most current version of this model comprises a two-period, one physical commodity analysis, where uncertainty enters through technological uncertainty. The effective commodity space includes the physical commodity in the first period, and a set of assets with associated patterns of returns to be paid in the second period. By imposing suitable restrictions, the model can be considered as a reinterpretation of the static competitive equilibrium. This isomorphic property implies that the deductions of the static model apply to the stock-market model, and so the standard result on profit-maximization, and the irrelevance of financial policy follow directly.

This model has a number of well-known limitations: in this Chapter we will be content to remove the restriction of one physical commodity. The introduction of many commodities in the second period will be assumed to introduce price uncertainty in the same way as a Temporary Equilibrium System.

2 The best known of these models are contained in the papers by Sharpe (1964) and Lintner (1965). See also the discussion in Fama and Miller (1972).

3 For example see Diamond (1967).

4 See the discussion in Chapters 3, 4 and 5.
Our formulation will parallel the stock-market model in producing profit-maximization, and the irrelevance of financial policy for corporations. Furthermore, the theory will include short-selling and the possibility of default on securities. By a careful definition of the meaning of the characteristics of an asset, we are able to provide:

(i) a more robust proof of the existence of equilibrium with short-sales, than that provided by Green; and
(ii) a situation where profit-maximization is well-defined (and where the Fisher Separation Theorem applies) rather than using corporate utility maximization as employed by Sondermann.

The Chapter is divided into two sections: part one includes the formal model and equilibrium existence proof; part two consists of a discussion of possible extensions and implications of the model.

1. THE MODEL

1.1 Consider economic activity to occur over two time periods: in period one decisions are taken by agents on spot and financial markets; and period two occurs in the future where the consequences of actions, taken in period one, will be observed. Let there be \( k_1 \) commodities in period one, and \( k_2 \) commodities in period two, such that \( k = k_1 + k_2 \). For simplicity, at this

---

5 The Fisher Separation Theorem states that in an economy of price-takers, all consumer-shareholders will prefer the profit-maximizing production plan over any other feasible plan. For a proof see Chapter 3.

6 Much of the notation has been borrowed from Green; and our treatment, especially of the consumer's problem, owes a heavy debt to Green and Sondermann.
stage,\(^7\) we will assume one state of the world in the second period, so that uncertainty enters purely through price uncertainty.

1.2 In this section we will discuss the consumer's problem. Let there be a set of consumers \(I = \{1, \ldots, m\}\). Consumption of commodities will be considered as a vector \(x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2, i \in I\). Consumer \(i\)'s consumption set \(X_i (x_i \in X_i)\) will be taken as the positive orthant, i.e., \(X_i \in \mathbb{R}^2_+\);\(^8\) and he has a commodity endowment \(\omega_i = (\omega_{i1}, \omega_{i2})\), which is known with certainty. Finally, let \(X = \sum_i X_i\) be the aggregate consumption set.

We turn now to assumptions concerning consumer \(i\)'s preferences over his commodity space. Assume that these preferences can be represented by a von Neumann-Morgenstern utility function \(u_i : \mathbb{R}^2_+ \rightarrow \mathbb{R}\). (To be formally correct, we should say that the consumer conforms to the Savage axioms where probabilities are subjective). In particular, assume that \(u_i\) is continuous, concave, strictly monotone increasing and bounded.\(^9\) We can summarize these assumptions on the consumer as:

\[
\begin{align*}
A.1 & \quad X_i = \mathbb{R}^2_+ \\
A.2 & \quad u_i : \mathbb{R}^2_+ \rightarrow \mathbb{R} \text{ is a mapping such that,} \\
& \quad (a) \quad u_i \text{ is continuous;} \\
& \quad (b) \quad u_i \text{ is concave in } x_i; \\
& \quad (c) \quad u_i \text{ is strictly monotone increasing;} \\
& \quad (d) \quad |u_i| \leq H, \text{ where } H \text{ is positive and finite.}
\end{align*}
\]

\(^7\) This assumption is removed in Section 2.1 below.

\(^8\) This is a simplifying assumption. We could assume, as in Debreu (1959) that \(X_i\) is merely bounded below. This complicates the analysis a little without altering our results on existence etc.

\(^9\) Of course, this utility function can be considered as a representation of a preference pre-ordering on \(X_i\). Also, concavity implies risk-aversion.
Now in period one, the consumer is restricted to trading in first-period commodities and the set $K = \{1, \ldots, K'\}$ of financial securities. That is, the action of the consumer is $a_i = (x_{1i}, b_i) \in \mathbb{R}^G$, where $G = \mathbb{R}_1 + K'$. These markets will be assumed to be competitive, so that the consumer faces given market prices $p = (p_1', p_b) \in \mathbb{R}^G$, where $p_1 \in \mathbb{R}^1$ is the price vector for period one commodities, and $p_b \in \mathbb{R}^{K'}$ the price vector for financial securities. Because we will assume free disposal, and our interest is in a relative price system, let

$$p \in \Delta^G = \{p \in \mathbb{R}^G_{(+)} \mid \sum_h p_h = 1\}.$$ 

In period two, the consumer anticipates that he will face commodity prices $q \in \Delta^2 = \{q \in \mathbb{R}^2_{(+)} \mid \sum_h q_h = 1\}$. Because these prices are not determined in the first period equilibrium, consumers have expectations about the possible second-period prices when those markets open. It is reasonable to presume that consumers will be influenced by the first period prices (and earlier prices in the history of the economy — although they are fixed, and therefore, are irrelevant to our analysis here) in determining their expectations about future prices. Thus, we can represent consumer $i$'s subjective beliefs about second-period prices, as a mapping

$$\psi_i = \Delta^G \rightarrow \mathcal{M}(\Delta^2, \mathcal{B}),$$

where $\mathcal{M}(\ldots)$ is the set of probability measures on $\Delta^2$ with associated Borel sets $\mathcal{B}$. (Notice that the measurable space $(\Delta^2, \mathcal{B})$ is also a metric space). 10

10 For a discussion of these concepts see Kingman and Taylor (1966).
As a short-hand we will write the mapping as $\Psi_i(p)$.

Given these preliminaries, we can treat the consumer's choice problem as a dynamic programming problem. Consider the consumer to take an action $a_i = (x_{1i}', b_i')$; then in the second period he must solve the problem

$$\begin{align*}
\text{Max} & \quad u_1(x_{1i}', x_{2i}) \\
\{ (x_{1i}', x_{2i}) \in X_1 \} \\
\text{subject to} & \\
q x_{2i} \leq q \omega_{2i} + r(q)b_i' \equiv W_{2i}.
\end{align*}$$

Let $r(q)$ be a $K'$ vector of independent security returns depending continuously upon the second-period price $q$. For security $k \in K$ let the security return $r_k(q)$ be non-negative and homogeneous of degree one; i.e., $r_k(q) \geq 0$ and $tr_k(q) = r_k(tq)$, for $t > 0$. Our formulation is sufficiently general to include quite complicated returns. For example, we could include contracts to pay a certain sum, if and only if, a certain set of second-period prices eventuate. Another example allows for forward trading in commodities, or market baskets of commodities. That is, let $Z$ be an $\ell_2 \times K'$ semi-positive matrix of security commodity returns, in the sense that $r(q) = qZ$. Clearly such a security is the promise to deliver a bundle of commodities, no matter what second-period price eventuates. If $Z$ is taken to be the unit matrix of dimension $\ell_2$, then we have Green's forward trading in individual commodities.

Furthermore, notice that for a commodity bundle $Z_k = e$, where $e$ is the unit vector, we find $qZ_k = r_k(q) = 1$, for $q \in \Delta^2$. That is, the security that promises a unit of each commodity, can be considered a riskless security.
For formal completeness, we can consider the set of security
returns \( r(q) \), to be a continuous mapping \( r: \Delta^2 \rightarrow \mathbb{R}^k \).\(^{11}\)

Because we have allowed short-selling we run into an apparent
problem concerning the unboundedness below of trades in securities. But this
is an apparent difficulty only, because, as has been demonstrated by Debreu
(1962), a consumption set unbounded below does not necessarily destroy an
existence proof for equilibrium. Indeed, the lower boundedness was used in
an earlier Debreu (1959) proof, as a relatively strong condition to ensure
boundedness of the attainable states for the economy. We will show that,
with reasonable assumptions, short-selling does not destroy the boundedness
of the attainable states.

The next step in the argument is to establish the importance of
default in determining the feasible set of portfolios for the consumer. When
a security is short-sold we will assume that the consumers must have non-
negative wealth, in the second period, to cover the promise to deliver. Other-
wise, we would be forced to assume that the market has imperfect information
about the feasible returns offered by agents. Notice that our formulation is
sufficiently general to allow for defaulting securities, in the sense that the
default on the nominal interest rate is fully recognized by all agents.

The argument can be formalized in the following way. Although non-
negative wealth, \( W_{21} \geq 0 \), is a sufficient restriction; for simplicity, let us
assume a slightly stronger condition, \( r(q)b_1 \geq 0 \). We can rationalize this
assumption by the story that the endowment \( w_{21} \) includes an element of govern-
ment redistribution ensuring that the consumer does not starve, even though

\(^{11}\) In the usual finite-dimensional, one-commodity model, the security return
is a finite dimensional vector; and there arises the possibility of a
finite spanning set of securities. (See Eken and Wilson (1974)).
The possibility of finite spanning is removed with an uncountable set
of states (i.e., \( \Delta^2 \)).
his portfolio may give a zero return. Assuming \( \omega_{2i} > 0 \), then \( W_{2i} > 0 \).

Define the feasible security set \( B_1 \) by

\[
B_1 = \{ b_i \in \mathbb{R} | r(q)b_i \geq 0; q \in \Delta^2 \}.
\]

Observe that \( B_1 \) is the same for all \( i \in I \). Clearly \( B_1 \) is closed, convex, and \( 0 \in B_1 \).

Now we need to derive consumer preferences over the set \( B_1 \), from the original preferences assumed over the second-period commodities. Define

\[
S_{2i} = \{ (W_{2i}, q) \in \mathbb{R} \times \Delta^2 \mid \exists x_{2i} \in X_1, \text{s.t.} qx_{2i} \leq W_{2i} \}.
\]

Because \( W_{2i} > 0 \), \( q \in \Delta^2 \), \( X_1 = \mathbb{R}_+ \), then \( S_{2i} \neq \emptyset \).

Note: Until the end of this section assume that \( B_1, X_1 \) are bounded.

Define

\[
\gamma_{2i}(W_{2i}, q) = \{ (x_{2i}', x_{2i}) \in X_1 \mid qx_{2i} \leq W_{2i}; (W_{2i}, q) \in S_{2i} \}.
\]

From Debreu (1959: 4.10) we know that \( \gamma_{2i}(W_{2i}, q) \) is continuous, non-empty, compact and convex. But \( W_{2i} = qW_{2i} + r(q)b_i \) and therefore we can write

\[
W_{2i} : B_1 \times \Delta^2 \rightarrow \mathbb{R} \text{ as } W_{2i}(b_i, q). \text{ This is a linear affine function in } b_i.
\]

Thus we obtain

12 Imagine that consumers are endowed with a government pension that always ensures a positive level of consumption. For a discussion along these lines, incorporating a government sector, see Chapter 7.
\[ \gamma_{2i}^i(b_1, q) = \gamma_{2i}(\nu_{2i}(b_1, q), q), \]

where \( \gamma_{2i}^i \) is continuous, convex, non-empty and compact. Now defining the indirect utility function,

\[ \phi_i(x_i', b_i, q) = \max_{(x_{1i}', x_{2i}') \in \gamma_{2i}^i} u_i(x_i', \gamma_{2i}(b_1, q)), \]

and the demand correspondence,

\[ \zeta_i(x_i', q) = \{ (x_{1i}', x_{2i}) \in X \mid (x_{1i}', x_{2i}) \in \phi_i^{-1}(x_i', b_i, q) \}, \]

we can prove the following lemma:

**Lemma 1:**

(i) \( \phi_i \) is continuous;

(ii) \( \zeta_i \) is upper semi-continuous.

**Proof:**

The proof follows from a direct application of:

**Maximum Theorem (Berge (1963)):**

If \( f \) is a continuous numerical function in the topological space \( Y \), and \( \Gamma \) is a continuous mapping of the topological space \( X \) into \( Y \) such that, for each \( x \), \( \Gamma x \neq \phi \), then the numerical function \( M \) defined by \( M(x) = \max \{ f(y) \mid ye \Gamma x \} \)
is continuous in \( X \), and the mapping \( F \) defined by \( F(x) = \{ y \mid y \in M(x) \} \) is an upper semi-continuous mapping of \( X \) into \( Y \).

Define \( a_i \in A_i = \mathbb{X}^1 \times B_i \), where \( \mathbb{X}^1 \) is the projection of \( X_i \) on \( \mathbb{R}^1 \).

Clearly \( A_i \) is closed and convex; and from the boundedness assumption, it follows that \( A_i \) is bounded. Also it is easy to show (assuming an independent set of security returns)

\[
A_i \cap (-A_i) = \{0\}.
\]

Now we can prove the following lemma on the indirect utility function.

**Lemma 2:**

\( \phi_i(a_i, q) \) is concave and strictly monotone increasing on \( A_i \) for \( q \in \Delta^2 \).

**Proof:**

Because \( u_i \) is strictly monotone increasing, the budget constraint holds with equality. Therefore, \( \phi_i \) is strictly monotone increasing, because \( u_i \) and \( W_{2i} \) are strictly monotone increasing. Concavity follows because \( W_{2i} \) is linear affine in \( b_i \); and \( u_i \) is concave.

Given the derivation of the indirect utility function for assets and first-periods commodities, the consumer's first-period problem can be written as:

\[13 \text{ For a proof see Chapter 6.}\]
Before proving the main theorem of this section, we require an assumption to ensure that the consumer's subjective beliefs on expected prices are well-behaved.

A.3 \( \psi_i(p) \) is continuous with respect to \( p \).

Now consider the derived utility function

\[
v_i : A_i \otimes \Delta^G \rightarrow \mathbb{R} \text{ defined by}
\]

\[
v_i(a_i, p) = \int_{\Delta_2} \phi_i(a_i, p) d\psi_i(p).
\]

We can prove the following theorem:\(^{14}\)

**Theorem 1:**

The derived utility function \( v_i \) has the properties:

(i) \( v_i \) is bounded, i.e. \( |v_i| \leq H \), where \( H \) is positive and finite;

---

\(^{14}\) An almost identical theorem has been proved by Sondermann; but our proof differs in applying a mathematical theorem directly.
(ii) \( v_i \) is continuous;

(iii) \( v_i \) is strictly monotone increasing with respect to \( a_i \);

(iv) \( v_i \) is concave with respect to \( a_i \in A_i \), for given \( p \in L^G \).

Proof:

(i) Boundedness of \( \phi_i \) follows directly from A.2(d). But because \( v_i \) is a probability measure, we have

\[
|v_i| \leq \int_A H d\psi_i = H.
\]

(ii) To prove continuity, we require a standard proposition from mathematical analysis.

Proposition (Royden (1968)):

Let \((X, \mathcal{B})\) be a measurable space and \(\mu^n\) a sequence of measures on \(\mathcal{B}\) which converge set-wise to \(\mu\). Let \(f^n\), \(g^n\) be two sequences of measurable functions which converge pointwise to \(f\) and \(g\). Suppose \(|f^n| \leq g^n\) and

\[
\lim \int g^n d\mu^n = \int g d\mu < \infty.
\]

Then

\[
\lim \int f^n d\mu^n = \int f d\mu.
\]

We can apply this proposition to our problem by choosing

\(\mu^n = H\), \(f^n = \phi_i^n\), \(g^n = \psi_i^n\). (Notice that continuity
implies measurability). Now by Lemma 1 and A.3, we know that for \((a_1', p)^n\) in \(A_1 \times \Delta^G\), such that \((a_1', p)^n \to (a_1, p)\), then \(\phi_i^n \to \phi_i\) and \(\psi_i^n \to \psi_i\). But we have satisfied the conditions of the proposition, so that our assertion is proved; i.e., \(v_i^n \to v_i\).

(iii) Strict monotonicity follows easily from Lemma 2 and elementary properties of the integral;

(iv) Concavity follows from Lemma 2, and the fact that the integral is a linear operator.

Before closing this Section on the consumer, we will summarize our results by formulating the derived consumer's problem in axiomatic form.

Using the indirect utility obtained from securities, we have the derived problem:

\[
\begin{aligned}
\text{Max} & \quad v_i(a_1', p) \\
\{a_1 \in A_1\} & \quad \text{subject to} \\
& \quad pa_1 \leq \tilde{w}_{i'}
\end{aligned}
\]

where:

B.1 \(A_1\) is a closed, convex cone;

B.2 \(A_1 \cap (-A_1) = \{0\}\);

B.3 \(v_i : \Delta^G \to \mathbb{R}\) is
(i) continuous and bounded; and
(ii) strictly monotone increasing and concave with respect to \( a \in \mathbb{A} \).

Finally, let the aggregate set \( \mathbb{A} = \bigcup \mathbb{A}_i \), where \( a \in \mathbb{A} \).

1.3 We could proceed, as does Sondermann and others, by postulating that the producer maximizes expected utility of the market value. But such an objective is unsatisfactory, because it leave open the question of whose preferences are to be imposed upon the firm. In the traditional Fisherian theory, profit-maximization can be deduced as an optimal rule agreed upon by all consumer-shareholders. We will follow that tradition here. By making some restrictive assumptions upon the underlying production sets for commodities, we are able to provide the production manager with a derived problem in terms of first-period commodities and securities. Therefore, the first-period problem affords a direct comparison with the certainty general equilibrium model.

Let there be a set of \( J = \{1, \ldots, n\} \) producers. Consider the underlying commodity production sets \( Y_j \subseteq \mathbb{R}^l_j, j \in J \); and \( Y = \bigcup_j Y_j \).

Emphasising the intertemporal nature of the production sets, let

\[
Y_j = (Y_{1j}, Y_{2j}) \in Y_j \text{ such that } Y_{1j} \in \mathbb{R}^{l_1}, Y_{2j} \in \mathbb{R}^{l_2}. \]

If we take the idea that \( Y_{1j} \) are inputs used to produce second-period outputs, \( Y_{2j} \), then clearly

\[
Y_{1j} \in \mathbb{R}^{l_1(-)}, Y_{2j} \in \mathbb{R}^{l_2(+)}.
\]

Furthermore, we introduce a very restrictive assumption on the relationship between inputs and outputs.\(^{15}\) For \( Y_{1j}' \), define

\[ Y_{1j}' \in \mathbb{R}^{l_2(+)} \]

---

\(^{15}\) These assumptions are a generalization of Diamond's separable production function. In an unpublished paper, Oliver Hart (1973) has considered a multi-commodity space in the second period, but restricts output such that our \( D_j \) is a fixed vector in \( \mathbb{R}^{l_2(+)} \).
Assume that \( Y_{2j}(y_{1j}) \) has the form \( D_j g_j(y_{1j}) \), where \( D_j \subseteq \mathbb{R}^{k_2} \) and \( g_j \in \mathbb{R}^k \); and let

(i) \( 0 \in D_j \)

(ii) \( D_j \) be compact.

Given the input \( y_{1j} \), the second period profit can be written as

\[
q Y_{2j} = q d_j g_j(y_{1j})
\]

where \( y_{2j} \in Y_{2j}(y_{1j}) \) and \( d_j \in D_j \). Assuming that profit will be maximized, we obtain

\[
\max_{\{d_j \in D_j\}} q d_j g_j(y_{1j}) = \sum_k r_k(q) a_k \cdot b_k^{y_{1j}}
\]

where \( a_k g_j(y_{1j}) \) def \( b_k^{y_{1j}} \). That is, the second period profit can be interpreted as the production of security returns \( \sum_k r_k(q) a_k \). From assumptions (i) and (ii) on \( D_j \) it is clear that \( q d_j \geq 0 \) is continuous. We will assume that \( r_k(q) \) is an element of the independent market returns vector \( r(q) \).

Thus, in the first period, the producer chooses a vector of inputs \( y_{1j} \) and a vector \( b_j \) of securities he will issue. Assuming a competitive market for both inputs and securities, the producer faces the problem

\[
\max_{\{y_{1j}, b_j\}} (p_1 y_{1j} + p_2 b_j).
\]
Let \((y_{1j}', b_j') \in Y_j^A \subset \mathbb{R}_1^G\); and \((y_1', y_b') \in Y^A = j y_j^A\). Assume that these derived production sets have the well-known properties: \(^{16}\)

- \(B.4\) \(0 \in Y_j^A\);
- \(B.5\) \(Y_j^A\) is closed and convex;
- \(B.6\) \(y_1^A \in \mathbb{R}_1^G(-), y_b^A \in \mathbb{R}_1^G(+);\)
- \(B.7\) \(Y^A \cap \mathbb{R}_1^G(+) = \{0\}.

Before closing this section on the producer, we should point out that we have assumed very restrictive conditions on the pattern of production in the second period. Without these conditions it could be feasible for the producer to supply a security for which there was no market. Because there are a finite number of security markets, but a continuum of potential securities, the set of feasible productions must be limited to be consistent with that set of securities. These issues will be discussed in more detail in Section 2.2.

1.4 Having examined the market responses in the first period, for the set of economic agents, we are in a position to define a private-ownership financial economy \(\mathcal{E}^F\), and provide an existence proof for its equilibrium. An economy is said to be a private-ownership economy when consumers own the initial endowments \(\omega_i^A \in \mathbb{R}_1^G, i \in I\), and receive shares

\[
\{\theta_{ij}^A \in \mathbb{R}_1^G\mid \sum_i \theta_{ij}^A = 1\}
\]

from the profits of producers.

\(^{16}\) Assumptions B.4, B.5 and B.7 are discussed in Debreu (1959). Assumption B.6 is a slightly more explicit version of the irreversibility assumption \(Y \cap (-Y) \subset \{0\}\).
Definition 1:
A private-ownership financial economy $\mathcal{E}^F$ is defined by: an economy $((A_i, v_i), (y_j^A), (\omega_i^A))$; for each $i \in I$, a point $\omega_i^A \in \mathbb{N}_G$ such that $\sum_i^A \omega_i = \omega_i^A$; for each pair $(i, j)$,

$$\{\theta_{ij} \in \mathbb{R}_+^+ \mid \sum_i \theta_{ij} = 1\}$$

for $j \in J$.

Definition 2:
An equilibrium of a private-ownership financial economy $\mathcal{E}^F$ is an $(m + n + 1)$ tuple $((a_i^*), (y_j^A^*), (p^*))$ of points of $\mathbb{N}_G^J$ such that:

(a) $\{a_i \in A_i \mid p^*a_i < p^*y_j^A + \sum_j \theta_{ij}y_j^A\}$ for $i \in I$;

(b) $y_j^A^*$ maximizes $p^*y_j^A$ for $j \in J$;

(c) $a_i^* - y_j^A = \omega_i$.

To prove the existence of an equilibrium for this economy, we will apply a slightly modified version of Debreu's (1959) proof. Because the consumer's set $A_i$ is not bounded below we have to show that our assumptions on production and consumption sets are sufficient to bound the attainable actions of economic agents.

Definition 3:
A state $((a_i), (y_j^A))$ of $\mathcal{E}^A$ is attainable if $a_i \in A_i$, $i \in I$, $y_j^A \in y_j^A$, $j \in J$ and $a_i^* - y_j^A = \omega_i$.

Now Debreu [1959, Section 5.4] has a proof giving sufficient conditions on $A$ and $y_A$ to ensure boundedness of the attainable states. It is evident from his proof that these conditions can be replaced by the
co-ordinate-free set of conditions.\(^\text{17}\)

\begin{align*}
C.1 & \ As(A) \cap As(y^A) = \{0\}, \\
C.2 & \ As(A) \cap As(-A) = \{0\}, \\
C.3 & \ y^A \cap (-y^A) \subseteq \{0\}.
\end{align*}

Condition C.2 can be satisfied by our conditions B.2. It is easy to show that B.6 and B.7 imply C.3. Also we can prove that we have sufficient conditions on production and consumption sets to satisfy C.1.

Lemma 3:

Given the assumptions B.1, B.6 and B.7, then \( As(A) \cap As(y^A) = \{0\} \).

Proof:

Define \( K^+ = \{ v \in \mathbb{R}^G \mid (v_1, \ldots, v_k) = v' \geq 0 \} \)

\[ K^- = \{ v \in \mathbb{R}^G \mid v' \leq 0 \} \]

\[ K = \{ v \in \mathbb{R}^G \mid v' = 0 \}. \]

Observe that \( K^+ \cap K^- = K \). Now

(a) \( As(A) = A \subseteq K^+ \);

(b) from B.6 \( As(y^A) \subseteq y^A \subseteq K^- \);

(c) B.6 and B.7 \( \Rightarrow y^A \cap K = \{0\} \);

(d) B.2 \( \Rightarrow A \cap K \supseteq \{0\} \).

Therefore, \( A \cap As(y^A) = \{0\} \).  

\text{\( ^{17} \) Define the asymptotic cone of a set \( X \), \( As(X) \) in the following way. For \( k \in \mathbb{R}^+ \), let \( X^k = \{ x \in X \mid |x| \geq k \} \) and \( \Gamma(X^k) \) be the intersection of all closed cones (vertex 0) containing \( X^k \). \( As(X) = \bigcap_k \Gamma(X^k) \).}
We are now in a position to prove:

**Theorem 2:**

The private-ownership financial economy $\xi^F$ has an equilibrium if we assume:

B.1 - B.7;

B.8 $\exists a_i^0 \prec \omega_i^A$ for $i \in I$;

B.9 the bound $H$ has been chosen such that for any attainable

$$a_i^*, \quad |v_i(a_i^*, p)| < H$$

for $p \in \Delta^G$ and all $i \in I$.\(^\text{18}\)

**Proof:**

The proof is the same as Debreu (1959, Section 5.7) except at the following points.

(a) By Lemma 3 we have sufficient assumptions to ensure that the attainable states are bounded;

(b) From B.9, an equilibrium price $p^* \neq 0$;

(c) From the strict monotone increasing property of $v_i$ (i.e., B.3 (ii)) it follows that $p^* \gg 0$. Therefore, by B.8 we have an $a_i^0 \in A_i$ such that $p a_i^0 < p^A_i$.

(d) By an application of the Maximum Theorem (Berge (1963)), and remembering (c) above, it follows that the demand

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\(^\text{18}\) This assumption replaces the Debreu (1959) assumption of non-satiety. In Debreu (1962) this assumption is weakened to non-satiety in the attainable consumption set: our assumption is the analogue of that assumption. Because the $v_i$ are representations, up to a linear transform, of underlying preferences, $H$ can always be chosen large enough for any attainable set.
Comment:

Because, in this model, prices alter consumer preferences for assets, part (d) of the proof is non-standard, although the Maximum Theorem is sufficiently general to encompass such an extension. Arrow and Hahn (1971, Chapter 6) discuss price-influenced preferences in the context of Veblen effects. Their existence proof was somewhat different.

2. SOME EXTENSIONS AND IMPLICATIONS OF THE MODEL

2.1 The model can be extended to include more than one state of the world in the second period. This can be accomplished by re-defining second-period commodities to include the states of the world as a commodity characteristic. Also the subjective probability distributions must be re-defined for second-period prices in each state of the world, so that the consumer's expectations, of the occurrence of states, will be incorporated into the probability distribution. Given this simple extension to the model, the one-commodity model, without price-undertainty, can be deduced as a special case. If there is a continuum of states, our model corresponds to Diamond's (1967) model.

2.2 It is well-known that the complete market system equilibrium is a Pareto Optimum (see Debreu (1959) Chapter 6). But our model differs in two important aspects from the complete market system, and this requires appropriate modifications to the concept of optimality.
In the one-commodity model, the set of securities may or may not span the second-period commodity space. Given a set of securities that did not span the space, Diamond introduced the notion of a constrained optimum, which explicitly excluded other independent securities as infeasible activities. Presumably trades in these securities were inadmissible because of prohibitive transaction costs in setting up markets. Although transaction costs are difficult to introduce into general equilibrium models, there are some special cases where market activity can be deduced from economic calculation.\(^{19}\) The resulting equilibrium is an optimum. The constrained optimum could be considered as an approximation to the full optimum where market activity is an economic decision.

Thus we could use the notion of a constrained optimum in our model; or by introducing a form of transaction costs, we could use a full optimum with the activity of different asset markets being determined by market forces.

The second aspect, in which the temporary equilibrium model differs from the complete market system, concerns the functional dependence of the derived preferences in the first-period, on first-period prices. The usual notion of an optimum requires modification to allow for this dependence. Arrow and Hahn (Chapter 6) have examined this dependence and introduced the notion of *conditional Pareto efficiency*. That is, for any first-period price vector, \(p\), take the preferences as given, and define Pareto efficiency in terms of those preferences. Of course, this definition is not independent of the market system as it is the case for the usual definition of an optimum.

\(^{19}\) For a formulation using marketing costs for assets see Chapter 8. Because our temporary equilibrium financial model has the same reduced form properties as the one-commodity model, it is a simple matter to translate the discussion (in the paper cited) to cover a temporary equilibrium financial model with marketing costs for securities.
2.3 Because, as we have pointed out, our model has the same formal properties as the one-commodity model, the results on financing deduced for that model continue to apply. That is, we can show that corporate leverage is irrelevant for the value of the firm. This result follows from the observation that any combination \( \sum \alpha_k r_k(q) \) of security returns, equal to the return \( r_k(q) \), is a perfect substitute in production and consumption.\(^{20}\) Notice that default on corporate debt is allowed for in the sense that corporate debt may have a return that is non-constant for \( q \in \Delta^2 \).

Similarly, we have allowed for consumer default in short-sales (or borrowing). The important assumption required is that the market can recognize the feasibility of a preferred security return. If, on the other hand, there is imperfect information in the market, the perceived consumption sets for securities may be larger or smaller than the actual consumption sets. Such a possibility may introduce unperceived default, in the sense that the actual pattern of returns of a security may diverge from the perceived pattern of returns.

2.4 Finally, observe that we have not constrained our discussion to positive endowments of assets. The model is sufficiently general to allow for past contracts, so that the consumer may have short-selling obligations to fulfill in the first period. Clearly this is a small step on the road toward a multiperiod planning model, with a term structure of interest rates associated with an intertemporal portfolio of obligations.

\(^{20}\) In our formulation, we assumed that there was an independent set of market returns. This assumption was required to obtain B.2. But given an independent (or basic) set of market returns, and an associated equilibrium, it is easy to derive a large number of securities, formed by linear combinations of the basic securities without altering the equilibrium. See Chapter 6.
The aim of this thesis has been to give a rigorous general equilibrium treatment of finance theory. In the analysis, we have borrowed extensively from the work of Arrow and Debreu; and from some subsequent work that is derived from the A-D theory. We have tried to generalize and extend many well-known results in finance, while, at the same time, emphasizing the notion that finance theory is merely a special branch of intertemporal economics (or capital theory).

Before concluding, we will give a brief discussion of possible extensions to the theory, and also some ideas on integrating the theory with existing economic models.

An obvious extension to the theory is the addition of many planning periods to the Temporary Equilibrium model of Chapter 9. Such an extension appears feasible. It would provide a number of interesting features to the theory: the model would include an endogenously determined term-structure of interest rates under uncertainty; the current price of securities would be influenced by consumer expectations of security prices in subsequent periods - and thereby introduce an important speculative element of stock-market

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1 The author intends to work on such a model in the near future. Unfortunately time limitations preclude an examination of this extension of the theory for this thesis.

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behaviour; and finally, the theory would provide a rigorous framework for studying dividend policy in an incomplete market framework.

Associated with this model, would be the important issue of the characteristics of the time-path of a sequence of Temporary Equilibria. Clearly the relative stability of this path will depend upon a number of factors including the expectation formation process for future prices.²

Another question arising out of our discussion, concerns the role of information in capital markets. When we considered the problem of short-sales, and the role played by default risk, it was assumed that the returns of securities over states and/or future prices are known by all agents. But the transmission of such information is a costly business, so that a more reasonable theory would include search costs and learning behaviour. In fact, we suspect that some forms of behaviour (which in past literature have been labelled "market imperfections") can be attributed to informational problems.

These are but two of many possible avenues for future research and extension of the theory.

We turn now to a consideration of integrating finance theory with existing economic models.

Because the Keynesian system is a temporary equilibrium theory (albeit a highly aggregative one) capital market theory can be considered as a more refined and disaggregated version of Keynesian consumption, saving and investment behaviour. Furthermore, the theory provides a rigorous examination of non-money asset markets. In fact, the Temporary Equilibrium model of Chapter 9 has all the characteristics of the Keynesian system, except

² Shell and Stiglitz (1967) have examined a simple, deterministic example. For another statement of the deterministic problem and a discussion of its relationship with optimal planning models see Samuelson (1967). Clearly, an uncertainty model will have many of the features of the deterministic model, in terms of stability analysis, and rules for expectations formation.
(i) the absence of a transaction motive for holding cash; and
(ii) a plausible explanation for unemployment in labour markets.

Currently a number of writers are working on the transaction motive for holding cash. These theories are based upon the introduction of transaction costs into exchange. Not only are they trying to rationalize the holding of cash for transaction purposes, but also (as we saw in Chapter 8) the inactivity and non-existence of many forward markets.

By way of these brief comments, we hope to indicate the role of finance theory within a broader framework. Although finance and capital market theory have made clear advances, much remains to be done.

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3 In Grandmont (1974) the analysis of temporary equilibrium is limited to a single asset - money. Therefore, money is required as a device for transmitting purchasing power between periods. With the introduction of interest-bearing securities the demand for money will vanish unless there is a positive return to cash from price deflation.

4 Unemployment is consistent with the model, if it is defined as a situation of over-supply in some labour markets (e.g., regional labour markets) with a corresponding zero wage. Some writers, e.g., Phelps (1970) have discussed informational problems and job search as a source of unemployment. Other writers have considered unemployment as the outcome of disequilibrium behaviour, e.g., see Grandmont and Laroque (1973).

5 For a discussion see Radner (1970) and Hahn (1971).
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Royden, H.L., (1968), Real Analysis, (second edition), Macmillan.


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p.35  line 5, add the condition: "A \not\subseteq \sum_{j=1}^{n} Y_j".

p.46  Replace the statement of footnote 14 by: "To show this, observe that

\[ r_j = \frac{B_j - \sum_{i \in S} p_i s_i^* y_i^*}{\sum_{i \in S} p_i s_i} \] .

p.55  line 6: \( Z_n = [Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_i] \)

p.104 line 5: "dividend" not "divident".

p.106 line 14: New paragraph: "In proving the ....

p.126 Equation 28, first three lines should have "i" replaced by "l".

p.151 line 7: "chapter" not "paper".

p.158 line 2: insert: "... definition (see Debreu (1962) for a complete statement):"

p.158 Condition b.1: "(>)_i" rather than "(\geq)_i".

p.161 Condition C.2(i): "\( w_i \in \mathbb{R}_{(+) \mathbb{L}+1} \)".

p.163 Replace the statement of footnote 15 by: "Although \( u_i \) is strictly increasing in \( x_{si} \), it must be bounded above. See Arrow (1970)".
p.175  line 6: "(i) for $0 \leq x_{2b} \leq x_{2b}$ ..."
line 9: "(ii) for $0 \geq x_{2b} \geq x_{2b}$"

p.209  line 6: "identity" not "idientity".

p.219  last line "{$x \in X | |x| \geq k$, where $k \geq 0$}".

p.228  Condition (6) should read

"(6) for $j \in J, A^* y_j^* = Max p^* y_j^*;"

p.245  Line 10: Delete the sentence on line 10 and replace with:
"Let $X_i$ be bounded (i.e. assume the consumer realizes the finiteness of feasible consumptions) and let $B_i$ be bounded until the end of this section".

p.246  line 6: Replace line 6 with

"$\xi_i (x'_{1i}, q) = \{ (x'_{1i}, x_{2i}) \in X_i | u_i (x'_{1i}, x_{2i}) = \phi_i (x'_{1i}, b_i', q) \} \def\text{def}"

p.248  line 10: Should read:

"$u_i (a_i, p) \def\text{def} \int_2^{k} \phi_i (a_i, q) d\gamma_i (p)$".

p.251  Line 4: Should read:

"The aggregate set $A = \sum A_i = A_i \ i \in I$ by construction.

Let $a \in A$".
Condition (a) should read:

"(a) \( a^*_i \) maximizes \( u_i(a_i, p) \) where \( \{a_i \in A_i \mid \ldots \} \).

p.255 line 5: Should read:

"Because \( A = A_i \, i \in I \), then B.2 implies C.2."

p.256 Delete condition B.9. In the proof delete statement (b) and replace with "Monotonicity of \( u_i \Rightarrow p^* \neq 0."

In footnote 18 delete the statement and replace with:

"Assumptions on endowments can be weakened. See Debreu (1962)"

Attach this footnote to B.8.

p.257 line 17: "uncertainty" not "undertainty".